

# KERNEL METHODS

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## INTRODUCTION

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## Linear Model

1. Simple and Well Studied
2. Computationally Feasible

## Real-world Problems

1. Non Linear
2. Sophisticated Interdependencies

**Question** Can we apply **Linear Model** on **Real-world Problems**?

## How about **Feature Transform**?

Mapping features to higher dimensional space. For example, the quadratic transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^6$

$$\Phi : (x_1, x_2) \mapsto (x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2}x_1, \sqrt{2}x_2, 1).$$

Line in  $\mathbb{R}^6$  corresponds to quadratic curve in  $\mathbb{R}^2$ . Common techniques in the machine learning world.

**Learning Stage** Fit linear model  $h : \mathbb{R}^6 \rightarrow \{0, 1\}$  on  $\Phi(\mathbf{x})$ .

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**Learning Stage** Fit linear model  $h : \mathbb{R}^6 \rightarrow \{0, 1\}$  on  $\Phi(\mathbf{x})$ .

**Testing Stage**  $h(\Phi(\mathbf{x}_{test}))$  gives the prediction.

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- Cons** Extra computational costs, e.g., inner products – the higher the dimension, the harder the computation.

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1. Mathematical shortcut for preventing computing inner product directly.
2. Combine the favors of Linearity and Non-Linearity.

## Kernel Trick

Consider the inner product in the previous example

$$\begin{aligned}\langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle &= \Phi(\mathbf{x})^T \Phi(\mathbf{y}) \\ &= x_1^2 y_1^2 + x_2^2 y_2^2 + 2x_1 x_2 y_1 y_2 + 2x_1 y_1 + 2x_2 y_2 + 1 \\ &= (\mathbf{x}^T \mathbf{y} + 1)^2 = (\langle \mathbf{x}, \mathbf{y} \rangle + 1)^2\end{aligned}$$

It implies we can define a special function

$$K(\mathbf{x}_1, \mathbf{x}_2) = \langle \Phi(\mathbf{x}_1), \Phi(\mathbf{x}_2) \rangle,$$

which is the so-called kernel function.

## COMMON KERNEL

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## Polynomial Kernel

Suppose we use the polynomial feature transformation  $\Phi_d(x)$ , which collects all the possible coefficients in the  $d$ -degree polynomial.

We then have the polynomial kernel

$$K_{\Phi_d}(x, y) = (\langle x, y \rangle + \alpha)^d$$

# POLYNOMIAL KERNEL CONT.

## Quadratic Transformation as An Special Case

Let  $d = 2$ , the transformation

$$\Phi_2(x) = (x_n^2, \dots, x_1^2, \sqrt{2}x_nx_{n-1}, \dots, \sqrt{2}x_nx_1, \sqrt{2}x_{n-1}x_{n-2}, \dots, \\ \sqrt{2}cx_n, \dots, \sqrt{2}cx_1, c)$$

And the kernel function

$$\begin{aligned} K(x, y) &= \left( \sum_{i=1}^n x_i y_i + c \right)^2 \\ &= \sum_{i=1}^n x_i^2 y_i^2 + \sum_{i=2}^n \sum_{j=1}^{i-1} (\sqrt{2}x_i x_j)(\sqrt{2}y_i y_j) + \sum_{i=1}^n (\sqrt{2}cx_i)(\sqrt{2}cy_i) + c^2 \end{aligned}$$

## Gaussian Kernel

The Gaussian Kernel is defined as

$$K(x, y) = \exp(-\gamma \|x - y\|^2), \text{ with } \gamma > 0$$

## As An Extreme Case of Polynomial Kernel

For  $\gamma = 1$

$$\begin{aligned}K(x, y) &= \exp(\|x - y\|^2) \\&= \exp(-x^2) \exp(-y^2) \exp(2xy) \\&= \exp(-x^2) \exp(-y^2) \left( \sum_{i=0}^{\infty} \frac{(2xy)^i}{i!} \right) \\&= \sum_{i=0}^{\infty} \left( \exp(-x^2) \sqrt{\frac{2^i}{i!}} x^i \right) \left( \exp(-y^2) \sqrt{\frac{2^i}{i!}} y^i \right) \\&= \Phi(x)^T \Phi(y)\end{aligned}\tag{1}$$

It implies  $\Phi(x) = \exp(-(x)^2) (1, \sqrt{\frac{2}{1!}}x, \sqrt{\frac{2^2}{2!}}x^2, \dots)$ .

## KERNEL MACHINES

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## Soft Margin SVM dual

$$\max_{\forall \alpha \geq 0, \forall \beta \geq 0} \left( \min_{b, w} \frac{1}{2} w^T w + C \sum_{n=1}^N \xi_n + \sum_{n=1}^N \alpha_n (1 - \xi_n - y_n (w^T z_n + b)) + \sum_{n=1}^N \beta_n (-\xi) \right)$$

by KKT condition

$$\max_{\forall 0 \leq \alpha \leq C, \beta_n = C - \alpha_n} -\frac{1}{2} \sum_{n=1}^N \|\alpha_n y_n x_n\|^2 + \sum_{n=1}^N \alpha_n$$

## Soft Margin SVM dual

Unfold formula and multiply -1.

$$\min_{\alpha} \frac{1}{2} \sum_{n=1}^N \sum_{m=1}^N \alpha_n \alpha_m y_n y_m \mathbf{x}_n^T \mathbf{x}_m - \sum_{n=1}^N \alpha_n$$

$$\text{subject to } \sum_{n=1}^N y_n \alpha_n = 0; \alpha_n \geq 0, \forall n \in [0, N], n \in \mathbb{N}$$

$$\text{implicitly } w = \sum_{n=1}^N \alpha_n y_n \mathbf{x}_n; \beta_n = C - \alpha_n, \forall n \in [0, N], n \in \mathbb{N}$$

We can do kernel method on red side  $\mathbf{x}_n^T \mathbf{x}_m$

## Ridge Regression

Minimize the quadratic cost as well as the regularization term

$$C(\mathbf{w}) = \frac{1}{2} \sum_i (y_i - \mathbf{w}^T \mathbf{x}_i)^2 + \frac{1}{2} \lambda \|\mathbf{w}\|^2$$

We have analytic solution

$$\mathbf{w} = (\lambda \mathbf{I} + \sum_i \mathbf{x}_i \mathbf{x}_i^T)^{-1} (\sum_j y_j \mathbf{x}_j)$$

# KERNEL RIDGE REGRESSION CONT.

Plug the Kernel Function In

Note that

$$(P^{-1} + B^T R^{-1} B)^{-1} B^T R^{-1} = P B^T (B P B^T + R)^{-1}$$

Therefore,

$$\begin{aligned} \mathbf{w} &= (\lambda I + \sum_i \phi(\mathbf{x}_i) \phi(\mathbf{x}_i)^T)^{-1} (\sum_j y_j \phi(\mathbf{x}_j)) \\ &= (\lambda I + \Phi \Phi^T)^{-1} \Phi \mathbf{y} \\ &= \Phi (\Phi^T \Phi + \lambda I)^{-1} \mathbf{y} \end{aligned}$$

## Prediction

In the prediction stage, we first transform  $\phi(\mathbf{x}_{test})$

$$y = \mathbf{w}^T \phi(\mathbf{x}_{test}) = \mathbf{y}^T (\Phi^T \Phi + \lambda I)^{-1} \Phi^T \phi(\mathbf{x}_{test})$$

All the calculation can be done in terms of inner product, and hence we can apply the kernel trick.

## CONCLUSION

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# SUMMARY

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- Linear to Non-linear
- Computationally Feasible
- Ability to Fit Arbitrarily Complicated Model

QUESTIONS?