KERNEL METHODS

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INTRODUCTION

Motivation

Linear Model

1. Simple and Well Studied
2. Computationally Feasible

Real-world Problems

1. Non Linear
2. Sophisticated Interdependencies

Question Can we apply Linear Model on Real-world Problems?

Motivation Cont.

How about **Feature Transform**?

Mapping features to higher dimensional space. For example, the quadratic transformation from $\mathbb{R}^2$ to $\mathbb{R}^6$

$$\Phi : (x_1, x_2) \mapsto (x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2}x_1, \sqrt{2}x_2, 1).$$

Line in $\mathbb{R}^6$ corresponds to quadratic curve in $\mathbb{R}^2$. Common techniques in the machine learning world.

Learning Stage Fit linear model $h : \mathbb{R}^6 \to \{0, 1\}$ on $\Phi(\mathrm{x})$.

Motivation Cont.

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Testing Stage $h(\Phi(\mathbf{x}_{test}))$ gives the prediction.

Motivation Cont.

However …

Pros Fitting complicated underlying rules – the higher the dimension, the more powerful the model.

Motivation Cont.

However ...

Pros Fitting complicated underlying rules – the higher the dimension, the more powerful the model.

Cons Extra computational costs, e.g., inner products – the higher the dimension, the harder the computation.

Kernel Trick

Luckily, **Kernel Trick** comes to rescue ...

1. Mathematical shortcut for preventing computing inner product directly.

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1. Mathematical shortcut for preventing computing inner product directly.
2. Combine the favors of Linearity and Non-Linearity.

Kernel Trick Cont.

Kernel Trick

Consider the inner product in the previous example

$$\begin{aligned}\langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle &= \Phi(\mathbf{x})^T \Phi(\mathbf{y}) \\ &= x_1^2 y_1^2 + x_2^2 y_2^2 + 2x_1 x_2 y_1 y_2 + 2x_1 y_1 + 2x_2 y_2 + 1 \\ &= (\mathbf{x}^T \mathbf{y} + 1)^2 = (\langle \mathbf{x}, \mathbf{y} \rangle + 1)^2\end{aligned}$$

It implies we can define a special function

$$K(\mathbf{x}_1, \mathbf{x}_2) = \langle \Phi(\mathbf{x}_1), \Phi(\mathbf{x}_2) \rangle,$$

which is the so-called kernel function.

COMMON KERNEL

Polynomial Kernel

Suppose we use the polynomial feature transformation $\Phi_d(x)$, which collects all the possible coefficients in the d-degree polynomial.

We then have the polynomial kernel

$$K_{\Phi_d}(x, y) = (\langle x, y \rangle + \alpha)^d$$

Polynomial Kernel Cont.

Quadratic Transformation as An Special Case

Let $d = 2$, the transformation

$$\begin{aligned}\Phi_2(x) = (&x_n^2, \ldots, x_1^2, \sqrt{2}x_n x_{n-1}, \\ &\ldots, \sqrt{2}x_n x_1, \sqrt{2}x_{n-1}x_{n-2}, \ldots, \\ &\sqrt{2c}x_n, \ldots, \sqrt{2c}x_1, c)\end{aligned}$$

And the kernel function

$$\begin{aligned}K(x, y) = (\sum_{i=1}^{n} x_i y_i + \alpha)^2 = \sum_{i=1}^{n} (x_i^2)(y_i^2) + \\ \sum_{i=2}^{n} \sum_{j=1}^{i-1} (\sqrt{2}x_i x_j)(\sqrt{2}y_i y_j) + \sum_{i=1}^{n} (\sqrt{2c}x_i)(\sqrt{2c}y_i)\end{aligned}$$

Gaussian Kernel

Gaussian Kernel

The Gaussian Kernel is defined as

$$K(x, y) = \exp(-\gamma \|x - y\|^2), \text{with } \gamma > 0$$

Gaussian Kernel Cont.

As An Extreme Case of Polynomial Kernel

For $\gamma = 1$

$$\begin{aligned}
K(x,y) &= \exp(\|x-y\|^2) \\
&= \exp(-(x)^2)\exp(-(y)^2)\exp((2xy)) \\
&\overset{*}{=} \exp(-(x)^2)\exp(-(y)^2)(\sum_{i=0}^{\infty}\frac{(2xy)^i}{i!}) \\
&= \sum_{i=0}^{\infty}(\exp(-(x)^2\sqrt{\frac{2^i}{i!}})(x)^i)(\exp(-(y)^2\sqrt{\frac{2^i}{i!}})(y)^i) \\
&= \Phi(x)^T\Phi(y)
\end{aligned} \tag{1}$$

It implies $\Phi(x) = \exp(-(x)^2)(1, \sqrt{\frac{2}{1!}}x, \sqrt{\frac{2^2}{2!}}x^2, \ldots)$.

KERNEL MACHINES

CONCLUSION

MTHEME

The *mtheme* is a Beamer theme with minimal visual noise inspired by the HSRM Beamer Theme by Benjamin Weiss.

Enable the theme by loading

```
\documentclass{beamer}
\usetheme{m}
```

Note, that you have to have Mozilla's *Fira Sans* font and XeTeX installed to enjoy this wonderful typography.

Sections

Sections group slides of the same topic

```
\section{Elements}
```

for which the *mtheme* provides a nice progress indicator …

ELEMENTS

Typography

`The theme provides sensible defaults to \emph{emphasize} text, \alert{accent} parts or show \textbf{bold} results.`

becomes

The theme provides sensible defaults to *emphasize* text, accent parts or show **bold** results.

Lists

Items

- Milk
- Eggs
- Potatos

Enumerations

1. First,
2. Second and
3. Last.

Descriptions

PowerPoint Meeh.

Beamer Yeeeha.

Animation

- This is important

Animation

- This is important
- Now this

Animation

- This is important
- Now this
- And now this

Animation

- This is really important
- Now this
- And now this

Figures

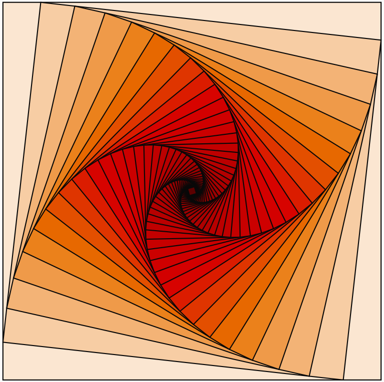


Figure: Rotated square from texample.net.

Tables

Table: Largest cities in the world (source: Wikipedia)

City	Population
Mexico City	20,116,842
Shanghai	19,210,000
Peking	15,796,450
Istanbul	14,160,467

Blocks

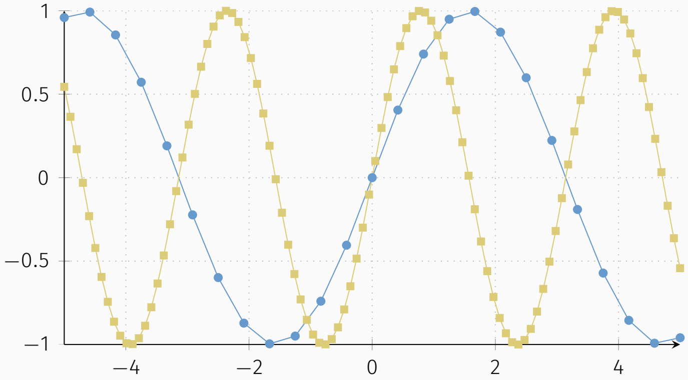
This is a block title

This is soothing.

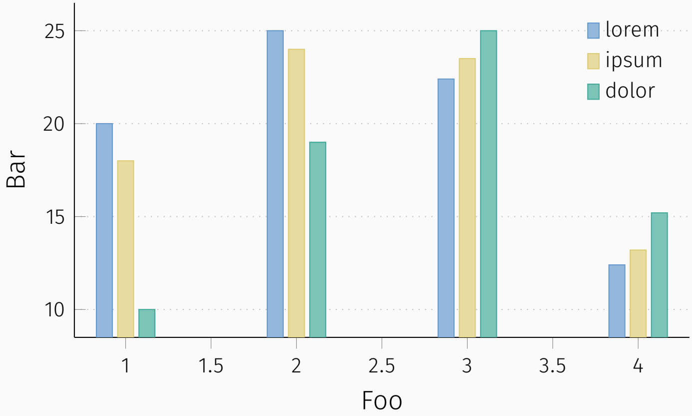
Math

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Line plots



Bar charts



Quotes

> *Veni, Vidi, Vici*

CONCLUSION

Summary

Get the source of this theme and the demo presentation from

`github.com/matze/mtheme`

Questions?