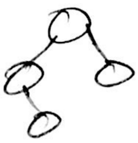
① Let N_h be the minimum number of nodes forming an AVL tree T of height h. One of T's subtrees will have height $h-1$ and for N_h to be minimal, the other subtree will have height $h-2$. So T has $N_{h-1}+N_{h-2}+1$ nodes.



is base case with $N_h > 2N_{h-2}$

$N_h = N_{h-1} + N_{h-2} + 1$ and $N_{h-1} = N_{h-2} + N_{h-3} + 1$

so $N_h = N_{h-2} + N_{h-3} + 1 + N_{h-2} + 1$ thus $N_h > 2N_{h-2} > 2(2N_{h-4})$ $> 2(2)(2N_{h-6})$ $h - 2(\frac{h}{2}) = 0$ so 2 is subtracted from the subscript $h/2$ times to get 0. A tree with height $h/2$ has minimum of $2^{h/2}$ nodes so $N_h > 2^{h/2}$ so $\lg N_h > \lg 2^{h/2}$ thus $2\lg N_h > h$ so $\boxed{h = O(\lg N_h)}$

② For an unsuccessful search, any key not already stored is equally likely to hash to any of the m slots. the expected time to search for a key is the expected time to search an red-black tree plus O(1) for the hash function. The height of a red-black tree with n internal nodes is $O(2\lg(n+1))$ because a subtree rooted at any node x has at least $2^{bh(x)}-1$ internal nodes which follows from induction with base case $bh=0$ so $2^0-1=0$ internal nodes since the root is a leaf. for the inductive step we use a root x with $h>0$ with 2 children which must have $bh(x)$ or $bh(x)-1$. So each child has at least $2^{bh(x)-1}-1$ nodes from inductive hypothesis. So subtree rooted at x has at least $(2^{bh(x)-1}-1)+(2^{bh(x)-1}-1)+1 = 2^{bh(x)}-1$ internal nodes. At least half the nodes on any path from root to leaf must be black. so $bh \geq h/2$ thus $n \geq 2^{h/2}-1$ so $h \leq 2\lg(n+1)$. The expected time for an unsuccessful search is the expected time to reach a leaf in the tree plus O(1) which is $\boxed{O(1+2\lg(n+1))}$

The number of elements visited during a successful search is one more than the number of levels visited before the element being searched for.

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New elements can appear anywhere in the tree so the average number of levels visited is $n/2 + 1$. This doesn't reduce the asymptotic complexity so a successful search is still bounded by $O(1 + 2\lg(n+1))$

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