A quantum algorithm for finding the minimum*

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1 Introduction

Let T[0..N-1] be an unsorted table of N items, each holding a value from an ordered set. For simplicity, assume that all values are distinct. The minimum searching problem is to find the index y such that T[y] is minimum. This clearly requires a linear number of probes on a classical probabilistic Turing machine.

Here, we give a simple quantum algorithm which solves the problem using $\mathcal{O}(\sqrt{N})$ probes. The main subroutine is the quantum exponential searching algorithm of [2], which itself is a generalization of Grover's recent quantum searching algorithm [3]. Due to a general lower bound of [1], this is within a constant factor of the optimum.

2 The algorithm

Our algorithm calls the quantum exponential searching algorithm of [2] as a subroutine to find the index of a smaller item than the value determined by a particular threshold index. The result is then chosen as the new threshold. This process is repeated until the probability that the threshold index selects the minimum is sufficiently large.

If there are $t \geq 1$ marked table entries, the quantum exponential searching algorithm will return one of them with equal probability after an expected number of $\mathcal{O}(\sqrt{N/t})$ iterations. If no entry is marked, then it will run forever. We obtain the following theorem.

Theorem 1 The algorithm given below finds the index of the minimum value with probability at least $\frac{1}{2}$. Its running time is $\mathcal{O}(\sqrt{N})$.

We first give the minimum searching algorithm, then the proof of the probability of success.

QUANTUM MINIMUM SEARCHING ALGORITHM

- 1. Choose threshold index $0 \le y \le N-1$ uniformly at random.
- 2. Repeat the following and interrupt it when the total running time is more than $22.5\sqrt{N} + 1.4 \lg^2 N$. Then go to stage 2(2c).
 - (a) Initialize the memory as $\sum_j \frac{1}{\sqrt{N}} |j\rangle |y\rangle$. Mark every item j for which T[j] < T[y].
 - (b) Apply the quantum exponential searching algorithm of [2].
 - (c) Observe the first register: let y' be the outcome. If T[y'] < T[y], then set threshold index y to y'.

3. Return y.

By convention, we assume that stage 2(2a) takes $\lg(N)$ time steps and that one iteration in the exponential searching algorithm takes one time step. The work performed in the stages 1, 2(2c), and 3 is not counted.

For the analysis of the probability of success, assume that there is no time-out, that is, the algorithm runs long enough to find the minimum. We refer to this as the *infinite algorithm*. We start by analyzing the expected time to find the minimum.

At any moment the infinite algorithm searches the minimum among the t items which are less than T[y]. During the execution any such element will be chosen at some point as the threshold with some probability. The following lemma states that this probability is the inverse of the rank of the element and is independent of the size of the table.

Lemma 1 Let p(t,r) be the probability that the index of the element of rank r will ever be chosen when the

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 $^{^1\}mathrm{As}$ notation, we use lg for the binary logarithm and ln for the natural logarithm.

infinite algorithm searches among t elements. Then stage 2(2a) before T[y] holds the minimum is at most p(t,r) = 1/r if $r \le t$, and p(t,r) = 0 otherwise.

Proof. The case r > t is obvious. The case r < tis proven by induction on t for a fixed r: The basis p(r,r) = 1/r is obvious since the output distribution of the exponential searching algorithm is uniform. Assume that for all $k \in [r, t]$ the equation p(k,r) = 1/r holds. Then when t+1 elements are marked, y is chosen uniformly among all t+1 indices, and can hold an item of rank either r, greater than r, or less than r. Only the former two cases contribute to the summation:

$$p(t+1,r) = \frac{1}{t+1} + \sum_{k=r+1}^{t+1} \frac{1}{t+1} p(k-1,r)$$
$$= \frac{1}{t+1} + \frac{t+1-r}{t+1} \cdot \frac{1}{r}$$
$$= \frac{1}{r}.$$

The expected number of iterations used by the exponential searching algorithm of [2] to find the index of a marked item — among N items where t items are marked — is at most $\frac{9}{2}\sqrt{N/t}$ (see [2]). We can now deduce the expected time used to find the minimum:

Lemma 2 The expected total time used by the infinite algorithm before y holds the index of the minimum is at most $m_0 = \frac{45}{4}\sqrt{N} + \frac{7}{10}\lg^2 N$.

Proof. The expected total number of time steps of stage 2(2b) before T[y] holds the minimum is at most

$$\begin{split} \sum_{r=2}^{N} p(N,r) \frac{9}{2} \sqrt{\frac{N}{r-1}} &= \frac{9}{2} \sqrt{N} \sum_{r=1}^{N-1} \frac{1}{r+1} \frac{1}{\sqrt{r}} \\ &\leq \frac{9}{2} \sqrt{N} \left(\frac{1}{2} + \sum_{r=2}^{N-1} r^{-3/2} \right) \\ &\leq \frac{9}{2} \sqrt{N} \left(\frac{1}{2} + \int_{r=1}^{N-1} r^{-3/2} dr \right) \\ &= \frac{9}{2} \sqrt{N} \left(\frac{1}{2} + \left[-2r^{-1/2} \right]_{r=1}^{N-1} \right) \\ &\leq \frac{45}{4} \sqrt{N}. \end{split}$$

The expected total number of time steps of

$$\sum_{r=2}^{N} p(N,r) \lg N = (H_N - 1) \lg N$$

$$\leq \ln N \lg N$$

$$\leq \frac{7}{10} \lg^2 N,$$

where H_N denotes the Nth harmonic number.

Theorem 1 follows immediately from lemma 2 since after at most $2m_0$ iterations, T[y] holds the minimum value with probability at least $\frac{1}{2}$.

3 Final remarks

The probability of success can be improved by running the algorithm c times. Let y be such that T[y] is the minimum among the outcomes. With probability at least $1 - 1/2^c$, T[y] holds the minimum of the table. Clearly, the probability of success is even better if we run the algorithm only once with time-out $c2m_0$, because then we use the information provided by the previous steps.

If the values in T are not distinct, we use the same algorithm as for the simplified case. The analysis of the general case is unchanged, except that in lemma 1, the equation p(t,r) = 1/r now becomes an inequality, $p(t,r) \leq 1/r$. Hence, the lower bound for the success probability given in theorem 1 for the simplified case is also a lower bound for the general case.

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