Quantlib project:
Add Greeks to binomial tree engines

***Students names*:**

- ABDALLAOUI Nada
- ZRIAA Imane
- BEN AYED Marwen

IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

I. Introduction:

The Delta and Gamma of an option can be calculated numerically as:

$$\Delta(u) = (P\,(u + \delta u) - P\,(u)) \,/\, \delta u \quad , \quad \Gamma(u) = (\Delta(u) - \Delta(u - \delta u)) \,/\, \delta u$$

where $P\,(u)$ is the price of the option as a function of the underlying value u and δu is a small perturbation of such value; the current Delta and Gamma are $\Delta(u0)$ and $\Gamma(u0)$ where u0 is the current value (that is, at t = 0) of the underlying.

On a binomial tree as currently implemented, there's a single point at t = 0 as shown in figure 1(a). This makes it impossible to calculate quickly and accurately the Delta and Gamma of an option; we can either re-run the calculation with a perturbed value of the underlying asset (which is slow, since we need three full calculations for Delta and Gamma) or we can use points at the first time step after t = 0 to obtain option values for a perturbed underlying (which is not accurate, since the option value at those points would be taken at t ≠ 0; this is what the *BinomialVanillaEngine* class currently does).

The problem can be solved by having three points for t = 0 with the center point at $u0$, as shown in figure 1(b), so that we can obtain the option values for $u + \delta u$ and $u - \delta u$ to use in the formulas for Delta and Gamma. Modify the *BinomialTree* class accordingly, and modify the *BinomialVanillaEngine* class so that it calculates Delta and Gamma correctly. Do you expect the extra points at each step to have a noticeable effect of the performance of the engine? Why? Verify your intuition by timing the code before and after your changes.

Note: the tree is built based on *log u* instead of u, so you'll have two different δu for the points at the left and right side. Take this into account during the calculations.

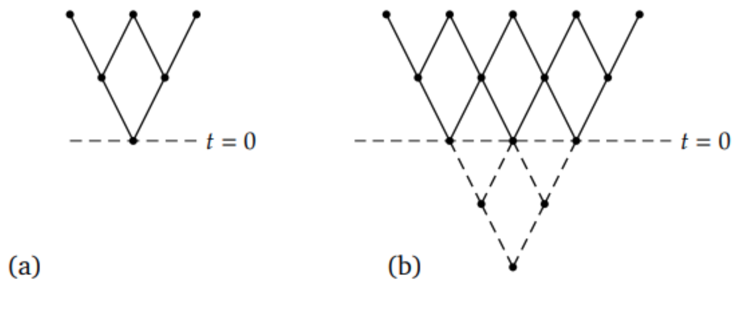

Figure 1: trees with one (a) and three (b) points at $t = 0$.

II. Solution:

A. Increasing the size of the columns:

In the class ***BinomialTree***, which is a class for all types of trees, we change the function **size(*i*)** to increase the size of the tree, so that at column ***i*** the number of nodes will be ***i*+3** instead of ***i*+1.** The nodes that we add correspond to the price perturbations required to compute ***delta*** and ***gamma*** at time step ***i***.

B. Adapting the index argument to the extra nodes:

For a given node **(*i*, *index*)**, the underlying prices depend on the values of both ***i*** and ***index***, where ***i*** is the time step number and ***index*** is the number of the nodes at the *ith* time step. With the new configuration, the ***index*** goes from 0 to *i*+2. Therefore, after increasing the size of the tree by two nodes, the real ***index*** needed for the underlying calculation was shifted by 1. So to get back to the right underlying price, as it was given by the original structure, we needed to shift back the ***index*** value by 1. We modified the ***underlying*(*i*, *index*)** function to account for the aforementioned issue. We created a ***new_index*** = ***index*** **-1** that substitutes the old ***index*** in calculating the underlying price.

C. Test:

To test all the modifications that we mentioned above, we take the example of ***EqualProbabilitiesTree***.

Using the following underlying calculation formula:

$$u_i^{index} = x_0\, e^{i.dx\, +\, new\_index.up\_}$$

We get for the first time step (***i*=0**):

$$u_0 = u_0^{\,1} = x_0\ ;\ u_0^{\,+} = u_0^{\,2} = x_0\, e^{2up\_}\ ;\ u_0^{\,-} = u_0^{\,0} = x_0\, e^{-2up\_}$$

Then,

$$delta = (p(u_0^{\,+})-p(u_0^{\,-}))\ /\ (u_0^{\,+}-u_0^{\,-}) \text{ and } gamma = (delta^{+} - delta^{-})/((u_0^{\,+}-u_0^{\,-})/2)$$

Where

$$delta^{+}= (p(u_0^{\,+})-p(u_0))\ /\ (u_0^{\,+}-u_0);\ delta^{-}=(p(u_0)-p(u_0^{\,-}))\ /\ (u_0-u_0^{\,-})$$

We tested the new model for u_0= 36 on the ***JarrowRudd*** tree, which is a derivative tree of the ***EqualProbabilitiesTree.***

We got a ***delta*** = **-0.84566** which is a very good approximation of the analytical ***delta*** **=-0.85** obtained by running the pricing engine three times with three different initializations of *u*.

The new architecture didn't affect the NPV that is equal to 4.205.

D. Summary:

We summarize below, the results obtained for each of the available tree architectures for an American Option with $\boldsymbol{u_0}$**=36**, ***strike*=40** and a date of maturity on 24 May 2021:

Tree	Original NPV	Current NPV	Original Delta	Current Delta	Original Gamma	Current Gamma	Elapsed time(s)
JarrowRudd	4.205	4.205	-0.849518	-0.84566	0.0720572	0.072163	0.000599
CoxRossRubinstein	4.20472	4.20472	-0.849587	-0.84679	0.0720719	0.072156	0.000546
AdditiveEQPBinomial	4.20559	4.20559	-0.849272	-0.84642	0.0720707	0.072175	0.000518
Trigeorgis	4.20473	4.20473	-0.849586	-0.84678	0.0720719	0.072156	0.00052
Tian	4.20488	4.20488	-0.849297	-0.84683	0.0721481	0.072078	0.003956
LeisenReimer	4.19732	4.19732	-0.850354	-0.85024	0.0727937	0.071623	0.003481
Joshi4	4.19732	4.19732	-0.850355	-0.85024	0.0727937	0.071623	0.003983

The NPV remains unchanged. We obtained a good approximation of the original greeks. Both ***Gamma*** and ***Delta*** are calculated at the same time step 0 by calling the function ***rollback*** only once, which leads to a reduction of the execution time.