4. $B_i^n(t) = B_{n-i}^n(1-t)$

5. $B_i^n(t) \geq 0$

6. B_i^n has exactly one maximum

7. $\{B_i^n, i = 0 \ldots, n\}$ is a basis of $\mathcal{P}_n([0,1])$

Due to the last property we can write every polynomial in $\mathcal{P}_n([t_{\min}, t_{\max}])$ as a linear combination of Bernstein polynomials:

$$p(t) = \sum_{i=0}^{n} b_i \widehat{B}_i^n(t)$$

where the b_i are called *Bézier points* and

$$\widehat{B}_i^n(t) := B_i^n\big(\frac{t - t_{\min}}{t_{\max} - t_{\min}}\big),$$

with $t_{\min} := \min_i t_i$ and $t_{\max} := \max_i t_i$.
The Bézier points for the interpolating polynomial are given as the solution of the linear system

$$\begin{pmatrix} \widehat{B}_0^n(t_0) & \widehat{B}_1^n(t_0) & \cdots & \widehat{B}_n^n(t_0) \\ \vdots & \vdots & & \vdots \\ \widehat{B}_0^n(t_n) & \widehat{B}_1^n(t_n) & \cdots & \widehat{B}_n^n(t_n) \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}.$$

Note, the entries of the governing matrix are all in $[0,1]$ by construction.

1.2.6 Chebyshev Polynomials

We consider Theorem 10 again and try to state the interpolation task in such a way that the interpolation error is minimized.
The only way to minimize the error in the error expression above is to minimize $\max |\omega_{n+1}(t)|$ by optimally placing the nodes. In many direct application of interpolation there is often no freedom in choosing the interpolation points (e.g. the time when measurements are made), but when designing more complex numerical methods which include interpolation as a substask, one often considers optimal placing of the interpolation points to optimize the method's accuracy. The classical example for this is Gaussian quadrature formula, see Sec. 1.4.2

Definition 14 *The polynomials*

$$T_n(t) = \cos(n \arccos t) \qquad t \in [-1, 1]$$

are called Chebyshev-Polynomials

To see that these are indeed polynomials, set $t := \cos\alpha$ and consider

$$\cos n\alpha = 2\cos\alpha\cos(n-1)\alpha - \cos(n-2)\alpha$$

which gives the *three term recursion*

$$T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t), \qquad T_0(t) = 1$$

Consequently the T_i are polynomials of degree i. In 1.4 some examples for Chebyshev polynomials are depicted.

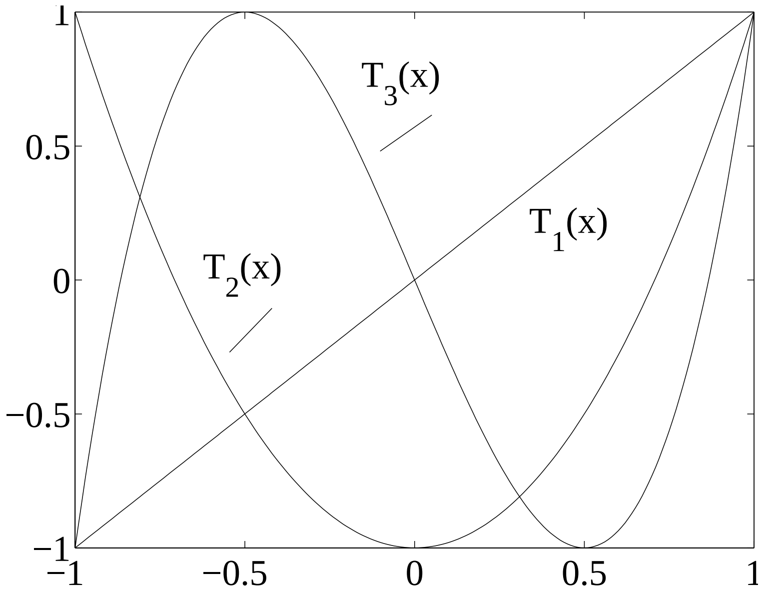

Figure 1.4: Chebyshev Polynomials

Chebyshev polynomials have special properties, which make them useful for our purposes:

- The T_i have integer coefficients.
- The leading coefficient is $a_n = 2^{n-1}$.
- T_{2n} is even, T_{2n+1} is odd.

- $|T_n(t)| \leq 1$ for $x \in [-1,1]$ and $|T_n(t)| = 1$ for $t_k := \cos(k\pi/n)$.
- $T_n(1) = 1, T_n(-1) = (-1)^n$
- $T_n(\bar{t}_k) = 0$ for $\bar{t}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for $k = 1, \ldots, n$

Furthermore Chebyshev polynomials have an important minimal property which we want to prove now:

Theorem 15

1. *Let $P \in \mathcal{P}_n([-1,1])$ have a leading coefficient $a_n \neq 0$. Then there exists a $\xi \in [-1,1]$ with*
$$|P(\xi)| \geq \frac{|a_n|}{2^{n-1}}.$$
2. *Let $\omega \in \mathcal{P}_n([-1,1])$ have a leading coefficient $a_n = 1$. Then the scaled Chebychev polynomials $T_n/2^{n-1}$ have the minimal property*
$$\|T_n/2^{n-1}\|_\infty \leq \min_\omega \|\omega\|_\infty$$

Proof:([DH95])
The first part will be proven by contradiction:
Let $P \in \mathcal{P}_n$ be a polynomial with leading coefficient $a_n = 2^{n-1}$ and $|P_n(t)| < 1$ for all $x \in [-1,1]$. Then, $P - T_n \in \mathcal{P}_{n-1}$ as both polynomials have the same leading coefficient. We consider now this difference at $t_k := \cos\frac{k\pi}{n}$:

$$\begin{aligned} T_n(t_{2k}) = 1 \land P(t_{2k}) < 1 \quad &\Rightarrow \quad P(t_{2k}) - T_n(t_{2k}) < 0 \\ T_n(t_{2k+1}) = -1 \land P(t_{2k+1}) > -1 \quad &\Rightarrow \quad P(t_{2k+1}) - T_n(t_{2k+1}) > 0. \end{aligned}$$

Thus, the difference polynomial changes its sign at least n times in the interval $[-1,1]$ and has consequently n roots in that interval. This contradicts the fact $P - T_n \in \mathcal{P}_{n-1}$. By this we showed that for each polynomial $P \in \mathcal{P}_n$ be a polynomial with leading coefficient $a_n = 2^{n-1}$ there exists a $\xi \in [-1,1]$ with $|P_n(\xi)| \geq 1$. By scaling we finally see that for a general polynomial with $a_n \neq 0$ there exists a $\xi \in [-1,1]$ with $|P_n(\xi)| \geq \frac{|a_n|}{2^{n-1}}$.
The second part of the theorem then follows directly. □

We apply this theorem to the result on the approximation error (cf. Th. 10) of polynomial interpolation and conclude for $[a,b] = [-1,1]$:

The approximation

$$f(t) - p(f|t_0, \ldots, t_n)(t) = \frac{1}{(n+1)!} f^{(n+1)}(\tau) \cdot \omega_{n+1}(t)$$

error is minimal if $\omega_{n+1} = T_{n+1}/2^n$, i.e. if the t_i are roots of the $n+1^{\text{st}}$ Chebychev polynomial, so-called *Chebychev points.*
In case of $[a,b] \neq [-1,1]$ we have to consider the map:

$$[a,b] \to [-1,1] \qquad t \mapsto \tau = 2\frac{t-a}{b-a} - 1$$

and

$$[-1,1] \to [a,b] \qquad \tau \mapsto t = \frac{1-\tau}{2}a + \frac{1+\tau}{2}b$$

1.2.7 Three-term recursion and orthogonal polynomials

We saw that Chebyshev polynomials can be generated by a three term recursion. In this chapter we want to characterize in more details the class of polynomials generated by three term recursions.
First we introduce an *inner product* (scalar product) in function spaces:

$$< f, g >_w := \int_a^b w(t) f(t) g(t) dt \tag{1.12}$$

with a weight function $w(t) : (a,b) \to \mathbb{R}^+$.
By using inner products we can define orthogonality:

Definition 16

- *Two functions f, g are called orthogonal with respect to the inner product $< \cdot, \cdot >_w$ if*

$$< f, g >_w = 0 \tag{1.13}$$

- *A sequence of polynomials $p_k \in \mathcal{P}_k$ is called orthogonal if for all k:*

$$< p_k, g >_w = 0 \quad \forall g \in \mathcal{P}_{k-1}$$

Orthogonal polynomials and three-term recursions are related to each other by the following theorem:

Theorem 17
There exists a unique sequence of orthogonal polynomials with leading coefficient one

$$p_k(t) = t^k + \pi_{k-1}(t) \quad \pi_{k-1} \in \mathcal{P}_{k-1}.$$

It obeys the three-term recursion

$$p_{k+1}(t) = (t - \beta_{k+1})p_k(t) - \gamma_{k+1}^2 p_{k-1}(t)$$

with $p_{-1}(t) := 0$, $p_0(t) := 1$ *and*

$$\beta_{k+1} := \frac{< tp_k, p_k >_w}{< p_k, p_k >_w} \quad \gamma_{k+1}^2 := \frac{< p_k, p_k >_w}{< p_{k-1}, p_{k-1} >_w}$$

Proof:
The proof is by induction. Let us assume that $p_0, \dots, p_{k-1}$ have already been constructed. They form an orthogonal basis of $\mathcal{P}_{k-1}$. If p_k is a polynomial with leading coefficient one, then $p_k(t) - tp_{k-1}(t) \in \mathcal{P}_{k-1}$. Thus, there exist coefficients c_j such that

$$p_k(t) - tp_{k-1}(t) = \sum_{j=0}^{k-1} c_j p_j(t) \tag{1.14}$$

with

$$c_j = \frac{< p_k - tp_{k-1}, p_j >_w}{< p_j, p_j >_w}$$

(why?).
As $< p_k - tp_{k-1}, p_j >_w = < p_k, p_j >_w - < tp_{k-1}, p_j >_w$ we obtain when requiring that p_k is orthogonal to all lower degree polynomials

$$c_j = -\frac{< tp_{k-1}, p_j >_w}{< p_j, p_j >_w} = -\frac{< p_{k-1}, tp_j >_w}{< p_j, p_j >_w}$$

which results in $c_0 = \ldots = c_{k-3} = 0$ and

$$c_{k-1} = -\frac{< tp_{k-1}, p_{k-1} >_w}{< p_{k-1}, p_{k-1} >_w}$$

and

$$c_{k-2} = -\frac{< p_{k-1}, tp_{k-2} >_w}{< p_{k-2}, p_{k-2} >_w}.$$

As $tp_{k-2} = p_{k-1} +$ (lower degree polynomial) we can get

$$c_{k-2} = -\frac{< p_{k-1}, tp_{k-2} >_w}{< p_{k-2}, p_{k-2} >_w} = -\frac{< p_{k-1}, p_{k-1} >_w}{< p_{k-2}, p_{k-2} >_w}.$$

From (1.14) we then obtain

$$p_k(t) = (t + \underbrace{c_{k-1}}_{-\beta_k})p_{k-1} + \underbrace{c_{k-2}}_{-\gamma_k^2} p_{k-2}$$

which completes the proof. □

Example 18
The Chebyshev polynomials are orthogonal polynomials on $[-1, 1]$ *with repect to the weight function* $w = (1 - t^2)^{-1/2}$.

Example 19
For $a = -1, b = 1$ *and* $w(t) = 1$ *we obtain the* Legendre polynomials P_k, *which can be constructed e.g. by the following MAPLE code:*

```
p_m:=0; p_0:=1;p_1:=t;
beta_2:=int(t*p_1*p_1,t=-1..1)/int(p_1*p_1,t=-1..1);
gamma2_2:=int(p_1*p_1,t=-1..1)/int(p_0*p_0,t=-1..1);
p_2:=(t-beta_2)*p_1-gamma2_2*p_0;
beta_3:=int(t*p_2*p_2,t=-1..1)/int(p_2*p_2,t=-1..1);
gamma2_3:=int(p_2*p_2,t=-1..1)/int(p_1*p_1,t=-1..1);
p_3:=(t-beta_3)*p_2-gamma2_3*p_1;
```

Chebyshev polynomials have their importance in approximation theory. We saw their importance for optimally placing interpolation points. Legendre polynomials give optimal integration (quadrature) formulas as will be seen in Section 1.4. In order to show this we need to show some more properties of orthogonal polynomials.

Theorem 20
Let $p_k \in \mathcal{P}_k$ *be orthogonal to all* $p \in \mathcal{P}_{k-1}$.
Then p_k *has* k *simple* real *roots in the open interval* (a, b).

Proof:
Let $t_0, \ldots, t_{m-1}$ be m distinct points in (a, b) where p_k changes sign.
Then $Q_m(t) := (t - t_0)(t - t_1) \ldots (t - t_{m-1})$ changes sign at the same points.
Thus, $wQ_m p_k$ does not change sign in (a, b) and we get

$$< Q_m, p_k >_w = \int_a^b w(t) Q_m(t) p_k(t) \mathrm{d}t \neq 0.$$

Since the p_k are orthogonal polynomials the degree of Q_m has to be k. Thus, p_k has exactly k simple real roots in (a, b). □

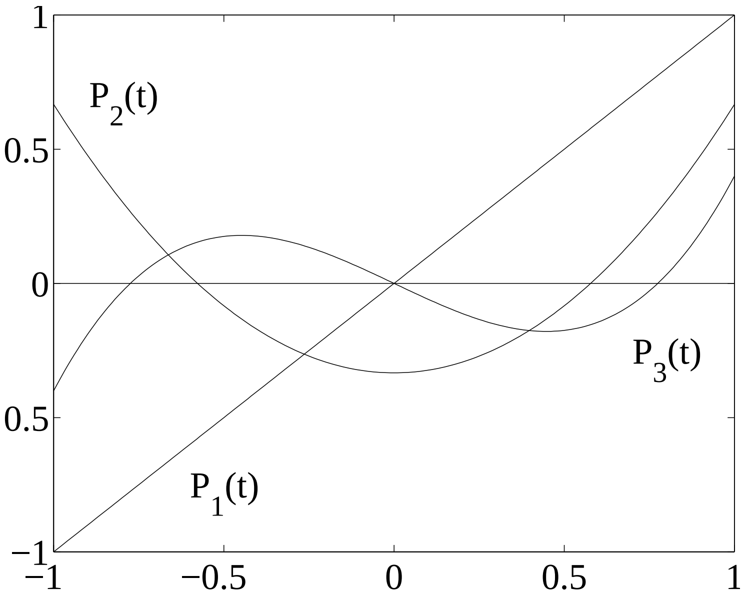

Figure 1.5: Legendre polynomials

1.3 Bézier Curves

We leave now (for a while) the interpolation topic and study the principal ideas of how to use polynomials and splines for curve design[2]. Recall the definition of a parametric and non parametric curve, cf. p. 2.
We construct curves by iterated *linear interpolation*. For this end, we need first some definitions and conventions.

1.3.1 Some notations and definitions

Definition 21
We consider barycentric combinations *of points:*

$$b = \sum_{i=0}^{n} \alpha_i b_i \quad \text{with } b_i \in \mathbb{E}^2, \alpha_i \in \mathbb{R} \text{ and } \sum_{i=0}^{n} \alpha_i = 1$$

where $\mathbb{E}^2$ *is the space of all points in* $\mathbb{R}^2$ *(to be exact:* $\mathbb{E}^2$ *is an affine linear space over* $\mathbb{R}^2$*.)*

Special cases of barycentric combinations are *convex combinations*, where $\alpha_i \geq 0$.

[2] Much more detail on this topic can be found in [Far88].