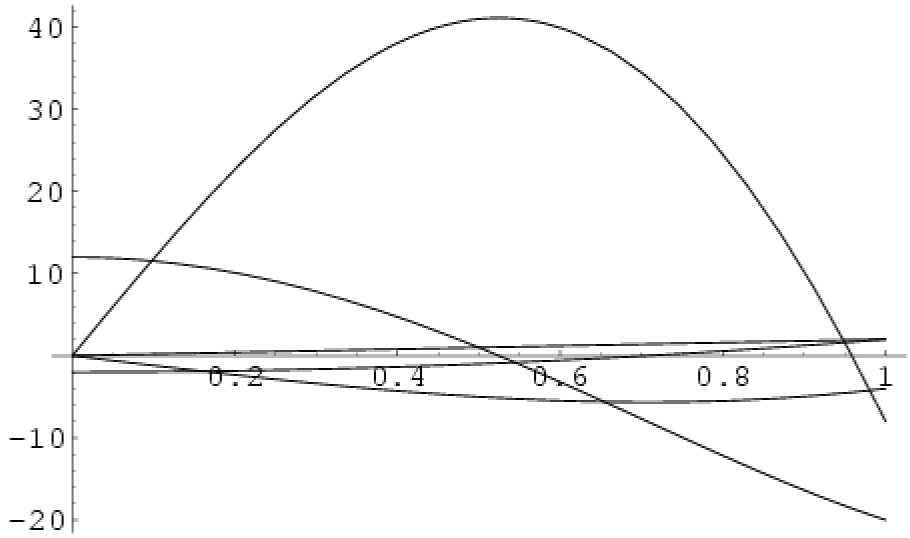
Hermite Polynomial



A set of Orthogonal Polynomials. The Hermite polynomials $H_n(x)$ are illustrated above for $x \in [0,1]$ and $n = 1$, 2, ..., 5.

The Generating Function for Hermite polynomials is

$$\exp(2xt - t^2) \equiv \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}. \tag{1}$$

Using a Taylor Series shows that,

$$\begin{aligned} H_n(x) &= \left[\left(\frac{\partial}{\partial t}\right)^n \exp(2xt - t^2)\right]_{t=0} \\ &= \left[e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\right]_{t=0}. \end{aligned} \tag{2}$$

Since $\partial f(x-t)/\partial t = -\partial f(x-t)/\partial x$,

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2}\left[\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\right]_{t=0} \\ &= \end{aligned} \tag{3}$$

$$(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Now define operators

$$\bar{O}_1 \equiv -e^{x^2} \frac{d}{dx} e^{-x^2} \tag{4}$$

$$\bar{O}_2 \equiv e^{x^2/2} \left(x - \frac{d}{dx} \right) e^{-x^2/2}. \tag{5}$$

It follows that

$$\bar{O}_1 f = -e^{x^2} \frac{d}{dx}[f e^{-x^2}] = 2xf - \frac{df}{dx} \tag{6}$$

$$\begin{aligned} \bar{O}_2 f &= e^{x^2/2} \left(x - \frac{d}{dx} \right) [f e^{-x^2/2}] \\ &= xf + xf - \frac{df}{dx} = 2xf - \frac{df}{dx}, \end{aligned} \tag{7}$$

so

$$\bar{O}_1 = \bar{O}_2, \tag{8}$$

and

$$-e^{x^2} \frac{d}{dx} e^{-x^2} = e^{x^2/2} \left(x - \frac{d}{dx} \right) e^{-x^2/2}, \tag{9}$$

which means the following definitions are equivalent:

$$\exp(2xt - t^2) \equiv \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} \tag{10}$$

$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{11}$$

$$H_n(x) \equiv e^{x^2/2} \left(x - \frac{d}{dx} \right) n e^{-x^2/2}. \tag{12}$$

The Hermite Polynomials are related to the derivative of the Error Function by

$$H_n(z) = (-1)^2 \frac{\sqrt{\pi}}{2} e^{z^2} \frac{d^{n+1}}{dz^{n+1}} \operatorname{erf}(z). \tag{13}$$

They have a contour integral representation

$$H_n(x) = \frac{n!}{2\pi i}\int e^{-t^2+2tx}t^{-n-1}\,dt. \tag{14}$$

They are orthogonal in the range $(-\infty,\infty)$ with respect to the Weighting Function e^{-x^2}

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \delta_{mn}2^n n!\sqrt{\pi}. \tag{15}$$

Define the associated functions

$$u_n(x) \equiv \sqrt{\frac{a}{\pi^{1/2}n!2^n}}\,H_n(ax)e^{-a^2x^2/2}. \tag{16}$$

These obey the orthogonality conditions

$$\int_{-\infty}^{\infty} u_n(x)\frac{du_m}{dx}\,dx = \begin{cases} a\sqrt{\frac{n+1}{2}} & m=n+1 \\ -a\sqrt{\frac{n}{2}} & m=n-1 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

$$\int_{-\infty}^{\infty} u_m(x)u_n(x)\,dx = \delta_{mn} \tag{18}$$

$$\int_{-\infty}^{\infty} u_m(x)xu_n(x)\,dx = \begin{cases} \frac{1}{a}\sqrt{\frac{n+!}{2}} & m=n+1 \\ \frac{1}{a}\sqrt{\frac{n}{2}} & m=n-1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\int_{-\infty}^{\infty} u_m(x)x^2u_n(x)\,dx = \begin{cases} \frac{2n+1}{2a^2} & m=n \\ \frac{\sqrt{(n+1)(n+2)}}{2a^2} & m=n+2 \\ 0 & m\neq n\neq n\pm 2 \end{cases}$$

(20)

$$\int_{-\infty}^{\infty} e^{-x^2}H_\alpha H_\beta H_\gamma\,dx = \sqrt{\pi}\frac{2^s\alpha!\beta!\gamma!}{(s-\alpha)!(s-\beta)!(s-\gamma)!},$$

(21)

if $\alpha+\beta+\gamma=2s$ is Even and $s\geq\alpha$, $s\geq\beta$, and $s\geq\gamma$. Otherwise, the last integral is 0 (Szegö 1975, p. 390).

They also satisfy the Recurrence Relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{22}$$

$$H_n'(x) = 2nH_{n-1}(x). \tag{23}$$

The Discriminant is

$$D_n = 2^{3n(n-1)/2} \prod_{\nu=1}^{n} \nu^\nu \tag{24}$$

(Szegö 1975, p. 143).

An interesting identity is

$$\sum_{\nu=0}^{n} \binom{n}{\nu} H_\nu(x) H_{n-\nu}(y) = 2^{n/2} H_n[2^{-1/2}(x+y)]. \tag{25}$$

The first few Polynomials are

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120 \\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x \\
H_8(x) &= 256x^8 - 3594x^6 + 13440x^4 - 13440x^2 + 160 \\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x \\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.
\end{aligned}$$

A class of generalized Hermite Polynomials $\gamma_n^m(x)$ satisfying

$$e^{mxt-t^m} = \sum_{n=0}^{\infty} \gamma_n^m(x) t^n \tag{26}$$

was studied by Subramanyan (1990). A class of related Polynomials defined by

$$h_{n,m} = \gamma_n^m \left(\frac{2x}{m}\right) \tag{27}$$

and with Generating Function

$$e^{2xt-t^m} = \sum_{n=0}^{\infty} h_{n,m}(x) t^n \tag{28}$$

was studied by Djordjevic (1996). They satisfy

$$H_n(x) = n! h_{n,2}(x). \tag{29}$$

A modified version of the Hermite Polynomial is sometimes defined by

$$\mathrm{He}_n(x) \equiv H_n\left(\frac{x}{\sqrt{2}}\right). \tag{30}$$

See also Mehler's Hermite Polynomial Formula, Weber Functions

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