Chapter 7

Laguerre Polynomials

by

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Module 2: Recurrence relations and Orthogonal property of Laguerre polynomials.

1. Recurrence Relation

In trying to find a formula for some mathematical sequence, a common intermediate step is to find the n^{th} term, not as a function of n, but in terms of earlier terms of the sequence. Such relations are called recurrence relations. In mathematics, a recurrence relation is an equation that recursively defines a sequence or multidimensional array of values.

Recurrence Relation I: $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$.

Proof: The generating function of Laguerre polynomial gives

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{exp\{-xt/(1-t)\}}{1-t}$$

We differentiate both sides with respect to t to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_n(x).nt^{n-1} &= \frac{1}{(1-t)^2}\ e^{\frac{-xt}{1-t}} - \frac{1}{1-t} \times e^{\frac{-xt}{1-t}} \times \frac{x}{(1-t)^2} \\ &= \frac{1}{1-t}\sum_{n=0}^{\infty} L_n(x)t^n - \frac{x}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)t^n\end{aligned}$$

Multiplying both sides by $(1-t)^2$ and simplifying, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} nL_n(x)t^{n-1} - 2\sum_{n=0}^{\infty} nL_n(x)t^n + \sum_{n=0}^{\infty} nL_n(x)t^{n+1} &= \sum_{n=0}^{\infty} L_n(x)t^n \\ &- \sum_{n=0}^{\infty} L_n(x)t^{n+1} - x\sum_{n=0}^{\infty} L_n(x)t^n \qquad (1)\end{aligned}$$

We now equate the coefficients of t^n from both sides in (1) to get

$$(n+1)L_{n+1}(x) - 2nL_n(x) + (n-1)L_{n-1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x)$$

$$\Rightarrow \quad (n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$$

Recurrence Relation II: $xL_n^{'}(x) = nL_n(x) - nL_{n-1}(x)$**.**

Proof: The generating function of Laguerre polynomial gives

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{exp\{-xt/(1-t)\}}{1-t} \tag{2}$$

We now differentiate both sides of (2) with respect to x, to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n^{'}(x)t^n &= \frac{1}{1-t}.exp\left[-\frac{xt}{1-t}\right]\cdot\left[\frac{-t}{1-t}\right] \\
\sum_{n=0}^{\infty} L_n^{'}(x)t^n &= \frac{-t}{1-t}\sum_{n=0}^{\infty} L_n(x)t^n, \quad \text{(by (2)}
\end{aligned}$$

$$\begin{aligned}
(1-t)\sum_{n=0}^{\infty} L_n^{'}(x)t^n &= -t\sum_{n=0}^{\infty} L_n(x)t^n \\
\sum_{n=0}^{\infty} L_n^{'}(x)t^n - \sum_{n=0}^{\infty} L_n^{'}(x)t^{n+1} &= \sum_{n=0}^{\infty} L_n(x)t^{n+1} \qquad (3)
\end{aligned}$$

Equating the coefficients of t^n from both sides, (3) gives

$$\begin{aligned}
L_n^{'}(x) - L_{n-1}^{'}(x) &= -L_{n-1}(x) \\
\Rightarrow L_{n-1}^{'}(x) &= L_n^{'}(x) + L_{n-1}(x) \qquad (4) \\
\Rightarrow L_n^{'}(x) = L_{n+1}^{'}(x) &+ L_n(x) \quad \text{(replacing n by n+1)} \\
\Rightarrow L_{n+1}^{'}(x) &= L_n^{'}(x) - L_n(x) \qquad (5)
\end{aligned}$$

Recurrence Relation I:

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \tag{6}$$

We differentiate (6) with respect to x to obtain

$$(n+1)L_{n+1}^{'}(x) = (2n+1-x)L_n^{'}(x) - L_n(x) - nL_{n-1}^{'}(x) \tag{7}$$

Substituting the values of $L_{n-1}^{'}(x)$ and $L_{n+1}^{'}(x)$ from (4) and (5) in (7), we get

$$\begin{aligned}
(n+1)[L_n^{'}(x) - L_n(x)] &= (2n+1-x)L_n^{'}(x) - L_n(x) - n[L_n^{'}(x) + L_{n-1}(x)] \\
xL_n^{'}(x) &= nL_n(x) - nL_{n-1}(x), \quad \text{(on simplification)}
\end{aligned}$$

Example 1. Show that $L_n^{'}(x) = -\sum_{r=0}^{n-1} L_r(x)$.

Solution The generating function of Laguerre polynomial gives

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{exp\{-xt/(1-t)\}}{1-t} \tag{8}$$

We differentiate both sides of (8) with respect to x to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^{'}(x)t^n &= \frac{1}{1-t}.exp\left[-\frac{xt}{1-t}\right].\left[\frac{-t}{1-t}\right] \\ \Rightarrow \left[\frac{-t}{1-t}\right]\sum_{r=0}^{\infty} L_r(x)t^r &= -t(1-t)^{-1}\sum_{r=0}^{\infty} L_r(x)t^r \text{ (using (8))}. \\ &= -t\sum_{s=0}^{\infty} t^s \sum_{r=0}^{\infty} L_r(x)t^r, \text{ (by binomial expansion)}\end{aligned}$$

$$\Rightarrow \sum_{n=0}^{\infty} L_n^{'}(x)t^n = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} L_r(x)t^{r+s+1} \tag{9}$$

It is clearly seen that the coefficient of t^n on the left hand side of (9) is $L_n^{'}(x)$. To obtain the coefficients of t^n on right hand side of (9), we put $r+s+1=n$ so that $s=n-r-1$. Now, for a fixed value of r, the coefficient of t^n on right hand side of (9) is $-L_r(x)$. Since, $s \geq 0 \Rightarrow n-r-1 \geq 0 \Rightarrow r \leq n-1$, the all possible values of r are 0,1,2,...,n-1. For all these values of r, $-L_r(x)$ is the coefficient of t^n. Therefore, the total coefficients of t^n on right hand side of (9) is given by

$$-\sum_{r=0}^{n-1} L_r(x)$$

Thus, equating the coefficients of t^n from both sides of (9), we get

$$L_n^{'}(x) = -\sum_{r=0}^{n-1} L_r(x)$$

Example 2. Using

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t}\exp\left[\frac{-tx}{1-t}\right],$$

show that

(i) $L_n^{'}(x) = n\left[L_{n-1}^{'}(x) - L_{n-1}(x)\right]$

(ii) $xL_n^{'}(x) = nL_n(x) - n^2 L_{n-1}(x)$

Solution (i): We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} e^{\frac{-tx}{1-t}} \tag{10}$$

We differentiate both sides of (10) with respect to x, to obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{'}(x) &= \frac{1}{1-t} e^{\frac{-tx}{1-t}} \left[-\frac{t}{1-t} \right] \\
\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{'}(x) &= -\frac{t}{1-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x), \text{ (using (10))} \\
\Rightarrow (1-t) \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{'}(x) &= = -t \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) \\
\Rightarrow \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{'}(x) &= \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n^{'}(x) - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n(x) \\
\Rightarrow \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n^{'}(x) &= \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n^{'}(x) - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n(x) \qquad (11)
\end{aligned}$$

Equating the coefficients of t^n on both sides of (11), we have

$$\begin{aligned}
\frac{1}{n!} L_n^{'}(x) &= \frac{1}{(n-1)!} L_{n-1}^{'}(x) - \frac{1}{(n-1)!} L_{n-1}(x) \\
\Rightarrow L_n^{'}(x) &= n \left[L_{n-1}^{'}(x) - L_{n-1}(x) \right]
\end{aligned}$$

Solution (ii): Recurrence relation I:

$$L_{n+1}(x) = (2n+1-x) L_n(x) - n L_{n-1}(x) \tag{12}$$

We differentiate both sides of (12) with respect to x, to obtain

$$L_{n+1}^{'}(x) = (2n+1-x) L_n^{'}(x) - L_n(x) - n L_{n-1}^{'}(x) \tag{13}$$

$$\begin{aligned}
\text{Already proved in (i) } L_n^{'}(x) &= n \left[L_{n-1}^{'}(x) - L_{n-1}(x) \right] \qquad (14) \\
\Rightarrow L_{n+1}^{'}(x) &= (n+1) \left[L_n^{'}(x) - L_n(x) \right] \qquad (15)
\end{aligned}$$

(replacing n by n+1)

Again, from (14), we get

$$n L_{n-1}^{'}(x) = L_n^{'}(x) + n L_{n-1}(x) \tag{16}$$

Using (15) and (16), (13) reduces to

$$\begin{aligned}
(n+1) \left[L_n^{'}(x) - L_n(x) \right] &= (2n+1-x) L_n^{'}(x) - L_n(x) - n \left[L_n^{'}(x) + n L_{n-1}(x) \right] \\
x L_n^{'}(x) &= n L_n(x) - n^2 L_{n-1}(x) \text{ (on simplification)}
\end{aligned}$$

2. Orthogonality properties of Laguerre polynomials.

If $L_m(x)$ and $L_n(x)$ are Laguerre's polynomials (m,n being positive integers), then,

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\ dx = \delta_{mn}$$

where

$$\delta_{mn} = \begin{cases} 0 & \text{m}\neq\text{n} \\ 1 & \text{m=n} \end{cases}$$

Proof: The generating function for Laguerre's polynomial gives

$$\begin{aligned} \sum_{n=0}^{\infty} L_n(x) t^n &= \frac{exp\left\{-\frac{xt}{1-t}\right\}}{1-t} \\ \sum_{m=0}^{\infty} L_m(x) s^m &= \frac{exp\left\{-\frac{xs}{1-s}\right\}}{1-s} \\ \Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_n(x) L_m(x) t^n s^m &= \frac{e^{-x\{t/(1-t)+s/(1-s)\}}}{(1-t)(1-s)} \end{aligned} \tag{17}$$

We now multiply both sides of (17) by e^{-x} and integrate both sides from 0 to ∞ with respect to x, which gives

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left\{\int_0^\infty e^{-x} L_n(x) L_m(x)\ dx\right\} t^n s^m \\ &= \frac{1}{(1-t)(1-s)} \int_0^\infty e^{-x\{1+t/(1-t)+s/(1-s)\}}\ dx \\ &= \frac{1}{(1-t)(1-s)} \left| \frac{e^{-x\{1+t/(1-t)+s/(1-s)\}}}{-\{1+\frac{t}{1-t}+\frac{s}{1-s}\}} \right|_0^\infty \\ &= \frac{1}{(1-t)(1-s)} . \frac{1}{1+\frac{t}{1-t}+\frac{s}{1-s}} = \frac{1}{1-st} \\ &= (1-st)^{-1} = 1 + st + (st)^2 + (st)^3 + ... = \sum_{n=0}^{\infty} s^n t^n \end{aligned}$$

$$\text{Therefore,} \quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left\{\int_0^\infty e^{-x} L_n(x) L_m(x)\ dx\right\} t^n s^m = \sum_{n=0}^{\infty} s^n t^n \tag{18}$$

When $m \neq n$, equating coefficients of $t^n s^m$ on both sides of (18) gives

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\ dx = 0.$$

When m = n, equating coefficients of $t^n s^n$ from both sides of (18) gives

$$\int_0^\infty e^{-x}[L_n(x)]^2 \ dx = 1$$

Combining we get

$$\int_0^\infty e^{-x} L_n(x) L_m(x) \ dx = \delta_{mn}$$

where

$$\delta_{mn} = \begin{cases} 0 & \text{m}\neq\text{n} \\ 1 & \text{m=n} \end{cases}$$

Example 3. Prove that $\int_x^\infty e^{-y} L_n(y) \ dy = e^{-x}[L_n(x) - L_{n-1}(x)]$.

Solution: We integrate the given integral by parts, taking e^{-y} as second function.

$$\begin{aligned} \int_x^\infty e^{-y} L_n(y) \ dy &= [-e^{-y} L_n(y)]_x^\infty - \int_x^\infty (-e^{-y}) L_n'(y) \ dy \\ &= e^{-x} L_n(x) + \int_x^\infty (e^{-y}) L_n'(y) \ dy \end{aligned}$$

Using the property $L_n'(y) = -\sum_{r=0}^{n-1} L_r(y)$, we get,

$$\begin{aligned} \int_x^\infty e^{-y} L_n(y) \ dy &= e^{-x} L_n(x) + \int_x^\infty (e^{-y}) \left\{ -\sum_{r=0}^{n-1} L_r(y) \right\} \ dy \\ &= e^{-x} L_n(x) - \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y) \ dy \end{aligned}$$

$$\int_x^\infty e^{-y} L_n(y) \ dy + \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y) \ dy = e^{-x} L_n(x) \tag{19}$$

$$\Rightarrow \sum_{r=0}^{n} \int_x^\infty e^{-y} L_r(y) \ dy = e^{-x} L_n(x) \tag{20}$$

Subtracting (19) from (20), we get

$$\sum_{r=0}^{n} \int_x^\infty e^{-y} L_r(y) \ dy - \int_x^\infty e^{-y} L_n(y) \ dy - \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y) \ dy = 0$$

$$\begin{aligned} \Rightarrow \int_x^\infty e^{-y} L_n(y) \ dy &= \sum_{r=0}^{n} \int_x^\infty e^{-y} L_r(y) \ dy - \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y) \ dy \\ &= e^{-x} L_n(x) - e^{-x} L_{n-1}(x) \quad \text{(using 20)} \\ &= e^{-x}[L_n(x) - L_{n-1}(x)] \end{aligned}$$