APPENDIX C

CHEBYSHEV POLYNOMIALS

This appendix reviews basic properties of the Chebyshev polynomials, which find a variety of applications in classical numerical analysis.

DEFINITION. The **Chebyshev polynomials** are the functions generated by the following recursion:

$$T_0(z) = 1,$$

$$T_1(z) = z,$$

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z).$$

This recursion gives rise to several equivalent representations. For example,

$$T_n(z) = \tfrac{1}{2}\left[\left(z + \sqrt{z^2 - 1}\right)^n + \left(z - \sqrt{z^2 - 1}\right)^n\right], \quad n = 0, 1, 2, \ldots;$$

$$T_n(z) = \cos\left(n \cos^{-1} z\right), \quad -1 \leqslant z \leqslant 1.$$

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The first few Chebyshev polynomials are as follows:

$$T_0(z) = 1,$$

$$T_1(z) = z,$$

$$T_2(z) = 2z^2 - 1,$$

$$T_3(z) = 4z^3 - 3z,$$

$$T_4(z) = 8z^4 - 8z^2 + 1,$$

$$T_5(z) = 16z^5 - 20z^3 + 5z.$$

In general, the nth Chebyshev polynomial has leading coefficient 2^{n-1}.

The nth Chebyshev polynomial T_n has n real zeros. The next proposition gives more specific information.

THEOREM C.1 (ZEROS OF CHEBYSHEV POLYNOMIALS). *The Chebyshev polynomial T_n has n zeros in the interval $(-1, 1)$ and $n+1$ local extrema in the interval $[-1, 1]$. At the local extrema, $|T_n(z)| = 1$.*

PROOF: We use the representation $T_n(z) = \cos(n \cos^{-1} z)$. Notice that $\cos(n\theta)$ vanishes for

$$\theta = \frac{(2N+1)}{n} \frac{\pi}{2},$$

where N ranges over the integers. Letting $\theta = \cos^{-1} z$ shows that $T_n(z) = 0$ for

$$z = \cos\left(\frac{2N+1}{n} \frac{\pi}{2}\right), \qquad N = 0, 1, 2, \ldots, n-1.$$

These are the n zeros lying in the interval $[-1, 1]$. Also, $\cos(n\theta)$ has local extrema at the points $\theta = N\pi/n$, so setting $\theta = \cos^{-1} z$ shows that $T_n'(z) = 0$ for $z = \cos(N\pi/n)$, $N = 0, 1, 2, \ldots, n$. At these points $T_n(z) = (-1)^N$. ■

Figure C.1 depicts the graph of

$$T_8 = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1.$$

Notice that the polynomial is relatively well behaved in the interval $[-1, 1]$, the function values being confined to the range $[-1, 1]$. Intuitively, this controlled behavior inside $[-1, 1]$ occurs at the expense of the behavior outside the interval, where the polynomial rapidly shoots off toward infinity.

The next theorem asserts that, in a sense, the controlled behavior inside $[-1, 1]$ is the best that we can expect for a polynomial of specified degree.

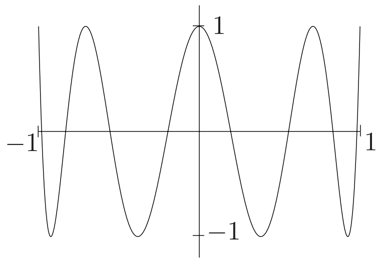

Figure C.1 The Chebyshev polynomial $T_8(z)$.

THEOREM C.2 (MINIMAX PROPERTY OF CHEBYSHEV POLYNOMIALS). *Of all polynomials p having degree exactly n and leading coefficient 2^{n-1}, T_n possesses the smallest value of $\|p\|_\infty := \sup_{z\in[-1,1]} |p(z)|$.*

(As we have seen, $\|T_n\|_\infty = 1$.)

PROOF: We argue by contradiction. Assume that $p \neq T_n$ is a polynomial having degree exactly n and leading coefficient 2^{n-1} and that $\|p\|_\infty < \|T_n\|_\infty$. Let $z_0, z_1, \ldots, z_n$ denote the extrema of T_n, ordered so that the points $z_0, z_2, z_4, \ldots$ are local maxima and $z_1, z_3, z_5, \ldots$ are local minima. We have

$$p(z_0) < T_n(z_0), \quad p(z_1) > T_n(z_1), \quad p(z_2) < T_n(z_2), \quad \ldots$$

Thus the nonzero polynomial $p - T_n$ changes signs n times in the interval $(-1, 1)$, which implies that $p - T_n$ has n roots in $(-1, 1)$. It follows that $p - T_n$ has degree at least n. But p and T_n both have degree n and possess the same leading coefficient, so $p - T_n$ has degree at most $n - 1$. This is a contradiction. ∎