UNIT 14 LEGENDRE POLYNOMIALS

Structure

14.1 INTRODUCTION

You are now familiar with Laplace's equation as it occurred in your different physics courses, viz. PHE-05, PHE-07. For instance, the temperature distribution over a spherical or a cylindrical shell during the steady state satisfies Laplace's equation. The gravitational potential due to a mass and electrostatic potential due to a charge distribution obey Laplace's equation at points away from the source. The Laplace's equation in spherical polar co-ordinates (r,θ,ϕ) admits solutions expressible in Legendre polynomials. Similarly, in order to obtain the solution of the (θ,ϕ) part of the Schrödinger equation for an electron, we need to know the Legendre polynomials, also called Legendre functions of the first kind. Further, though the description of an atom in terms of the principal, orbital, azimuthal and spin quantum numbers became possible from an experimental study of the atomic spectra, its theoretical explanation on the basis of Schrödinger equation required a knowledge of the properties of Legendre polynomials. In Unit 3, Block 1 of the Physics Elective Course PHE-05 : Mathematical Methods in Physics – II, you have learnt to solve Legendre's differential equation using the power series method. Recall that these solutions could not be expressed in terms of known elementary functions — sine, cosine, exponential, etc. In fact, these solutions turned out to be new functions with very interesting properties.

In this Unit we shall revisit the solution of Legendre's differential equation and obtain the Legendre polynomials in two different ways: By solving the differential equation and from the generating function. (You should refresh your knowledge by studying Unit 3 of PHE-05 course.) You will then learn the properties of Legendre polynomials; the most conspicuous among these is the orthogonality property. We have discussed many applications of Legendre polynomials. We expect you to study these examples carefully and link them up with the relevant topics.

It is important to study Legendre's associated differential equation but we shall not go into the mathematical rigor of the properties of the associated Legendre polynomials. Moreover, you will not be examined for these.

Objectives

After studying this unit, you should be able to:

- identify Legendre's and associated Legendre's differential equations;
- obtain Legendre polynomials from the solutions of Legendre's differential equation;
- obtain Legendre polynomials from the generating function as well as Rodrigues' formula;
- derive the recurrence relations for Legendre polynomials;
- derive the orthogonality relation for Legendre polynomials; and
- solve problems related to electrostatic and gravitational potentials.

14.2 LEGENDRE'S DIFFERENTIAL EQUATION AND LEGENDRE POLYNOMIALS

You first encountered **Legendre's differential equation** in Examples 1 and 3 of Unit 3 in Block 1 of PHE-05 course. Let us rewrite the equation:

$$(1-x^2)y''-2xy'+n(n+1)y=0 \tag{14.1}$$

The solution of this equation has been worked out in the margin. It is

$$y=a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4-\ldots\right]$$

$$+a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5-\ldots\right] \tag{14.2}$$

For an even integer $(n \geq 0)$, the first bracketed term in this series (with even powers of x) terminates leading to a polynomial solution. For an odd integer ($n > 0$), the latter term in the series (with odd powers of x) terminates and gives a polynomial solution. That is to say, for any integer ($n \geq 0$), Legendre's equation has a polynomial solution. For $n = 0,1,2,\ldots$, Eq. (14.2), respectively, leads to

$$y=a_0, \qquad n=0 \tag{14.3a}$$

$$y=a_1 x, \qquad n=1 \tag{14.3b}$$

$$y=a_0(1-3x^2), \qquad n=2 \tag{14.3c}$$

$$y=a_1\left(\frac{3x-5x^3}{2}\right), \qquad n=3 \tag{14.3d}$$

These expressions (of y) are, apart from a multiplication constant, the Legendre polynomials $P_n(x)$. The multiplicative constant is chosen so that $P_n(1) = 1$.

You will recall that Eq. (14.2) can be rewritten as

$$y = a_0 y_1(x) + a_1 y_2(x) \tag{14.4}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent. You should note that Eq. (14.4) does not give the general solution of Legendre's differential equation. To understand the general solution we have to reconsider the recursion formula given in Eq. (iii) in the margin:

$$a_{k+2}=-\frac{(n-k)(n+k+1)}{(k+1)(k+2)}a_k \tag{14.5}$$

You will note that the coefficients a_{k+2} will vanish when (i) $n = k$ and/or (ii) $n = -(k+1)$. $a_{k+2} = 0$ is indicative of the series getting terminated with C_k as the last non-zero coefficient. That is how we get polynomial solutions. While obtaining the polynomial solutions $P_n(x)$, we considered the polynomial solution with $n = k$ only. When we take $n = -(k+1)$, the series obtained diverges for $x = \pm 1$. This solution is unbounded and is called a **Legendre function of second kind** and is denoted by $Q_n(x)$. Thus the most general solution of Legendre's differential equation can be written as

$$y = A_1 P_n(x) + A_2 Q_n(x) \tag{14.6}$$

We rewrite Eq. (14.1) as

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n-1)y.$$

Let us now make the substitution $x = \cos\theta$. By Chain rule, we have

$$\frac{d}{dx}=\frac{d\theta}{dx}\cdot\frac{d}{d\theta}$$

$$=-\frac{1}{\sin\theta}\frac{d}{d\theta}$$

and

$$1-x^2=\sin^2\theta$$

Hence, Legendre's equation takes the form

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin^2\theta\left(-\frac{1}{\sin\theta}\right)\frac{dy}{d\theta}\right]$$

$$+n(n+1)y=0$$

or

$$+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right)+n(n+1)y=0$$

This is the θ-part of the Laplace's equation in spherical polar coordinates.

You will recognise that $x = \pm 1$ are regular singular points of Legendre's differential equation. We therefore solve it in the range $-1 < x < 1$ using power series method and write

$$y(x)=\sum_{k=0}^{\infty}a_k x^k \tag{i}$$

Substituting this and its derivatives $y'(x)$ and $y''(x)$ in Legendre's differential equation, we get

$$(1-x^2)\sum_{k=2}^{\infty}k(k-1)a_k x^{k-2}$$

$$-2x\sum_{k=1}^{\infty}ka_k x^{k-1}$$

$$+m\sum_{k=0}^{\infty}a_k x^k=0 \tag{ii}$$

where $m = n(n+1)$. In expanded form, this is rewritten as

$$(2a_2+ma_0)+(6a_3-2a_1+ma_1)x$$

$$+(12a_4-2a_2-4a_2+ma_2)x^2+\ldots=0$$

Equating the coefficient of each power of x to zero, we get the recursion relation

$$a_{k+2}=\frac{k(k+1)-m}{(k+1)(k+2)}a_k$$

$$=-\frac{(n-k)(k+n+1)}{(k+1)(k+2)}a_k \tag{iii}$$

...s tells us that we can express the ...cients with even subscripts in terms ...and those with odd subscripts in ...ns of a_1. On inserting the values for the ...ficients we get Eq. (14.2).

Here we shall restrict ourselves to Legendre polynomials of the first kind, that is $P_n(x)$. Let us now derive the expressions for $P_n(x)$.

From Eq. (14.5) we note that if $k = n$, $a_{n+2} = 0$ and, by induction, $a_{n+4} = 0 = a_{n+6} \ldots$

To continue our discussion, we invert Eq. (14.5) to obtain $a_k = -\dfrac{(k+1)(k+2)}{(n-k)\times(n+k+1)} a_{k+2}$.

By taking $k = n-2$, $n-4$,...we get

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)} a_n$$

This yields the polynomial solution

$$y = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)} x^{n-4} - \ldots \right] \qquad (14.7)$$

The Legendre polynomials $P_n(x)$ are defined by choosing

$$a_n = \frac{(2n-1)(2n-3)\ldots 3!}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

This choice of the coefficients a_n ensures that $P_n(1) = 1$ when $n = 0$. Thus

$$\boxed{P_n(x) = \frac{(2n)!}{2^n (n!)^2} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)} x^{n-4} - \ldots \right]} \qquad (14.8)$$

We now work out a simple example to enable you to fix these concepts.

To show the equivalence of expressions in Eq. (14.8), we note that even terms are not occurring in the numerator defining the coefficients a_n. So we multiply the numerator and denominator by $2n(2n-2)(2n-4)\ldots 4\times 2$. This gives

$$a_n = \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots 4\times 3\times 2\times 1}{n!(2n)(2n-2)(2n-4)\ldots\times 4\times 2}$$

$$= \frac{2n!}{n!\,2n(2n-2)(2n-4)\ldots\times 4\times 2}$$

Now $(2n)(2n-2)(2n-4)\ldots\times 4\times 2$.

$$= (2n)\times 2(n-1)\times 2(n-2)\ldots\times 2(2)\times 2(1)$$

$$= 2^n n!$$

so that the expression for a_n reduces to

$$a_n = \frac{2n!}{2^n (n!)^2}$$

Example 1: Expressions for first few Legendre polynomials

Starting from Eq. (14.8), prove that

a) $P_0(x) = 1$
b) $P_1(x) = x$
and
c) $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

Solution

From Eq. (14.8), we have

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4(2n-1)(2n-3)} x^{n-4} - \ldots \right]$$

a) Since $0! = 1$, and $x^0 = 1$, we readily obtain $P_0(x) = 1$.
b) Insert $n = 1$. Then

$$P_1(x) = \frac{2!}{2^1 \times (1!)^2} x = \frac{2x}{2} = x$$

c) Similarly, on inserting $n = 2$, we get

$$P_2(x) = \frac{4!}{2^2 \times (2!)^2}\left[x^2 - \frac{2\times(2-1)}{2\times(4-1)}x^0\right]$$

$$= \frac{24}{4\times 4}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}\left(3x^2 - 1\right)$$

Proceeding in this way, you can obtain expressions for higher order Legendre Polynomials. To give you some practice, you should solve the following SAQ.

Spend 2 min

SAQ 1

Prove that $P_3(x) = ½\,(5x^3 - 3x)$

Refer to Fig 14.1. It shows plots of $P_n(x)$ for $n = 2, 3, 4, 5$. You may have to refer to these plots and the values of $P_n(x)$ while solving physical problems. Note that Legendre polynomials are categorised as **special functions**. (In this Block you will come across three other special functions — Hermite, Laguerre and Bessel functions.) This nomenclature has genesis in the fact that these functions are far more complex than elementary functions.

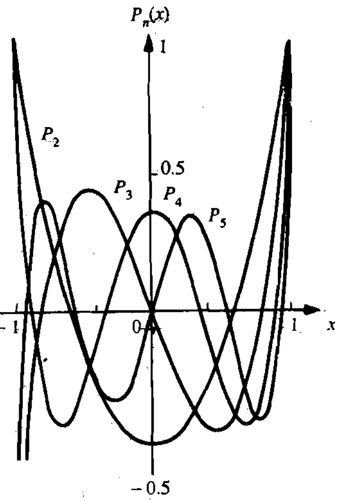

Fig.14.1: Plot of $P_n(x)$ versus x for $n = 2,3,4,5$.

So far we have obtained expressions for Legendre polynomials by solving Legendre's differential equation. These expressions can also be generated from a function of two variables, say x and t. When expanded in powers of t, the x-dependent coefficients define the Legendre polynomials. This forms the subject matter of discussion for the following section.

14.3 GENERATING FUNCTION

Consider the function

$$g(x,t) = \left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n \qquad (14.9)$$

We expand $\left(1 - 2xt + t^2\right)^{-1/2} \equiv \{1 - (2x - t)t\}^{-1/2}$ in powers of t for $|t| < 1$ using binomial expansion:

The binomial expansion of $(1-x)^{-n}$ is

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

$$[1 - t(2x-t)]^{-1/2} = 1 + \left(\frac{1}{2}\right)t(2x-t) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)t^2(2x-t)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)t^3(2x-t)^3 + \ldots$$

$$= 1 + \frac{t(2x-t)}{2} + \left(\frac{1\times 3}{2\times 4}\right)t^2(2x-t)^2 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)t^3(2x-t)^3 + \ldots$$

$$= 1 + \frac{1}{2}(2xt - t^2) + \frac{3}{8}t^2\left(4x^2 + t^2 - 4xt\right) + \frac{5}{16}t^3\left(8x^3 - t^3 - 12x^2t + 6xt^2\right) + \ldots$$

$$= 1 + xt - \frac{1}{2}t^2 + \frac{3}{2}x^2t^2 + \frac{3}{8}t^4 - \frac{3}{2}xt^3 + \frac{5}{2}x^3t^3 - \frac{5}{16}t^6 - \frac{15}{4}t^4x^2 + \ldots$$

$$= 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \frac{1}{2}(5x^3 - 3x)t^3 + \ldots \qquad (14.10)$$

On equating the coefficients of t^n for $n = 0,1,2$, and 3 in Eqs. (14.9) and (14.10), we find that the first few Legendre polynomials are given by

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

You should note that these expressions are the same as obtained in Example 1.

The coefficient of t^n in the above expansion will be

$$\frac{1\times3\times5\ldots\times(2n-1)}{2\times4\times6\ldots\times2n}(2x)^n - \frac{1\times3\times5\ldots\times(2n-3)}{2\times4\times6..\times(2n-2)}\cdot\frac{(n-1)}{1!}(2x)^{n-2} + \frac{1\times3\times5\ldots\times(2n-5)}{2\times4\times6\ldots\times(2n-4)}\cdot\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

This may be rewritten as

$$\frac{1\times3\times5\ldots\times(2n-1)}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times4(2n-1)(2n-3)}x^{n-4} - \ldots\right\}$$

You can readily identify this with Eq. (14.8). So we may conclude that Eq. (14.9) with $|t| < 1$ signifies the generating relation for Legendre polynomials and $g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}$ is called the **generating function** for Legendre polynomials.

We shall now obtain some special values using the generating function.

For $x = 1$, Eq. (14.9) gives

$$\left(1 - 2t + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

The left hand side now reduces to $\left(1-t\right)^{-1}$, which has series representation as $\sum_{n=0}^{\infty} t^n$. Then the above expression takes the form

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

On comparing the coefficients of t^n, we get $P_n(1) = 1$ so that the expression of the generating function gets validated.

The generating function is useful in solving physical problems involving the potential associated with any inverse square force. To illustrate this we consider an electric charge q placed on the z-axis at $z = a$ (Fig 14.2). From Unit 4 of PHE-07 course on Electric and Magnetic Phenomena you will recall that the electrostatic potential at a non-axial point due to this charge is given by

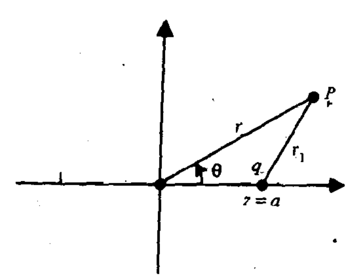

Fig.14.2: Electrostatic potential due to a charge q displaced from origin.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1} \qquad (14.11)$$

From the properties of a triangle, we can write

$$r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$$

On inserting this expression in Eq. (14.11), we get

$$V = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$$

$$= \frac{q}{4\pi\varepsilon_0 r}\frac{1}{\sqrt{1+\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta}} \tag{14.12}$$

For $r > a$, the expression under the radical sign may be written as $(1 - 2xt + t^2)^{-1/2}$ where $x = \cos\theta$ and $t = \frac{a}{r}$; $|t| < 1$. From Eq. (14.10) we note that when $(1 - 2xt + t^2)^{-1/2}$ is expanded in powers of t for $|t| < 1$, the coefficient of t^n can be identified with $P_n(x)$.

On inserting this result in Eq. (14.12), we get

$$V = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n \tag{14.13}$$

Example 2: Derivations from the generating function

Calculate the values of $P_{2n}(0)$ and $P_{2n+1}(0)$.

Solution

Putting $x = 0$, in Eq. (14.12), we get

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n$$

Using binomial expansion, we can write

$$(1+t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \ldots + (-1)^n \frac{1\times3\times5\ldots\times(2n-1)}{2^n n!}t^{2n} + \ldots = P_0(0) + P_1(0)t + P_2(0)t^2 + \ldots + P_{2n}(0)t^{2n}$$

On comparing the powers of t^{2n}, we get

$$P_{2n}(0) = (-1)^n \frac{1\times3\times5\ldots\times(2n-1)}{2^n n!}$$

Since in the expansion of $(1+t^2)^{-1/2}$ we only get even power of t, we have

$$P_{2n+1}(0) = 0$$

You may now like to solve an SAQ on generating functions.

SAQ 2

Spend 5 min

Prove that

(a) $P_n(-1) = (-1)^n$ and

(b) $P_n(-x) = (-1)^n P_n(x)$

We shall now use the generating function to obtain the recurrence relations or recursion relations for Legendre polynomials. This nomenclature stems from the fact that expressions for higher order polynomial can be derived from a knowledgeof the expressions for lower order polynomials.

14.4 RECURRENCE RELATIONS

The generating relation for Legendre polynomials is

$$g(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

To obtain two primary recurrence relations, we differentiate $g(x,t)$ partially with respect to t and x, respectively. Differentiating partially with respect to t, we get

$$\frac{\partial g}{\partial t} = \left(-\frac{1}{2}\right)\frac{(-2x+2t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

We rewrite this result as

$$\frac{(x-t)}{\sqrt{1-2xt+t^2}} = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

and use the generating relation to write

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$\left(1-2xt+t^2\right)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

On re-arrangement, we obtain

$$\sum_{n=0}^{\infty}\left[(n+1)P_n(x)t^{n+1} - (2n+1)xP_n(x)t^n + nP_n(x)t^{n-1}\right] = 0$$

To collect the coefficients of t^n, we replace n by $n-1$ in the first term and by $n+1$ in the last term. Then equating the resultant expression to zero, we get

$$nP_{n-1}(x) - (2n+1)xP_n(x) + (n+1)P_{n+1}(x) = 0$$

so that

$$\boxed{(2n+1)x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x)} \qquad (14.14)$$

This is one of the two important recurrence relations and correlates any Legendre polynomial with its adjoining polynomials. For example, by putting $n = 1$ in Eq. (14.14), we get

$$3x\,P_1(x) = 2\,P_2(x) + P_0(x)$$

But we know that $P_0(x) = 1$, and $P_1(x) = x$. Hence, we find that

$$3x^2 = 2P_2(x) + 1$$

or

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Similarly, $P_3(x)$ can be obtained using expressions for $P_1(x)$ and $P_2(x)$; $P_4(x)$ can be obtained using expressions for $P_2(x)$ and $P_3(x)$, and so on.

To derive the second recurrence relation, we differentiate $g(x, t)$ partially with respect to x:

$$\frac{\partial g}{\partial x} = \frac{\left(-\frac{1}{2}\right)(-2t)}{t(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\,t^n$$

so that

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\,t^n$$

or

$$\frac{t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\,t^n$$

On combining this result with Eq. (14.12), we get

$$t\sum_{n=0}^{\infty} P_n(x)\,t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\,t^n$$

Raise the index of the first term by one and lower the index of second terms by one of the RHS.

On rearranging terms, we obtain

$$\sum_{n=0}^{\infty}\left[2x\,P_n'(x) + P_n(x)\right]t^{n+1} = \sum_{n=0}^{\infty}(1+t^2)P_n'(x)t^n = \sum_{n=0}^{\infty} t^n P_n'(x) + \sum_{n=0}^{\infty} P_n'(x)t^{n+2}$$

As before, on collecting coefficients of t^{n+1} from both sides and equating them, we get the second recurrence relation:

$$\boxed{2x\,P_n'(x) + P_n(x) = P_{n+1}'(x) + P_{n-1}'(x)} \qquad (14.15)$$

Another very useful recurrence relation is obtained by combining Eqs. (14.14) and (14.15). To this end we first differentiate Eq. (14.14) with respect to x and multiply the result by 2. This gives

$$2(2n+1)P_n(x)+2(2n+1)x\,P_n'(x)=2(n+1)P_{n+1}'(x)+2n\,P_{n-1}'(x)$$

We rewrite it as

$$2(2n+1)x\,P_n'(x)=2(n+1)P_{n+1}'(x)+2nP_{n-1}'(x)-2(2n+1)P_n(x)$$

Next, we multiply Eq. (14.15) by $(2n+1)$. This leads to

$$2(2n+1)x\,P_n'(x)+(2n+1)\,P_n(x)=(2n+1)P_{n+1}'(x)+(2n+1)\,P_{n-1}'(x)$$

Substituting for $2(2n+1)x\,P_n'(x)$ in this relation, we obtain

$$2(n+1)P_{n+1}'(x)+2n\,P_{n-1}'(x)-2(2n+1)\,P_n(x)$$
$$+2(2n+1)\,P_n(x)=(2n+1)P_{n+1}'(x)+(2n+1)P_{n-1}'(x)$$

On simplification, we obtain

$$\boxed{P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)\,P_n(x)} \qquad (14.16)$$

Eqs. (14.14) and (14.16) are the prime recurrence relations for the Legendre polynomials. Using these recurrence relations and Legendre's differential equation, you can obtain the following relations:

$$P_{n+1}'(x)=(n+1)\,P_n(x)+x\,P_n'(x) \qquad (14.17a)$$

$$P_{n-1}'(x)=x\,P_n'(x)-n\,P_n(x) \qquad (14.17b)$$

$$(1-x^2)P_n'(x)=n\,P_{n-1}(x)-n\,x\,P_n(x) \qquad (14.17c)$$

and

$$(1-x^2)P_n'(x)=(n+1)x\,P_n(x)-(n+1)\,P_{n+1}(x) \qquad (14.17d)$$

You should note that recurrence relations are identities in x and simplify proofs and derivations.

SAQ 3

Spend 15 min

Prove the relations (14.17a-d)

14.5 ORTHOGONALITY RELATIONS

Two functions $A(x)$ and $B(x)$ are said to be **orthogonal** on the interval (a,b) if they satisfy the relation

$$\int_a^b A(x)B(x)\,dx=0$$

From Block 2 of your PHE-05 course you may recall that sine and cosine functions are orthogonal in the interval $(-1,1)$. A function whose norm is unity, i.e., $\int f^2dx=1$ is said to be **normalised**. A system of normalised functions whose any two different functions are orthogonal is said to be **orthonormal**.

One of the important characteristics of Legendre polynomials is that they are orthogonal. This property enables us to express a given function defined on the interval $(-1,1)$ in a series of Legendre polynomials. It is therefore important for you to master its applications. The m th and the n th order Legendre polynomials $P_m(x)$ and $P_n(x)$ respectively satisfy the equations

$$(1-x^2)\,P_m''-2x\,P_m'+m(m+1)\,P_m(x)=0 \qquad (14.18a)$$

and

$$(1-x^2) P_n'' - 2x P_n' + n(n+1) P_n(x) = 0 \qquad (14.18b)$$

Multiply the first equation by $P_n(x)$ and the second equation by $P_m(x)$. Next subtract the latter product from the former. This will lead to the relation

$$(1-x^2)(P_n P_m'' - P_m P_n'') - 2x(P_n P_m' - P_m P_n') = [n(n+1) - m(m+1)]\, P_m P_n$$

You can verify quite easily that the left hand side is equal to $\frac{d}{dx}\left\{(1-x^2)\,(P_n P_m' - P_m P_n')\right\}$ so that we can write

$$\frac{d}{dx}\left\{(1-x^2)\,(P_n P_m' - P_m P_n')\right\} = [n(n+1) - m(m+1)] P_m P_n$$

On integrating both sides over x from $x = -1$ to $x = +1$, we obtain

$$[n(n+1) - m(m+1)] \int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = (1-x^2)[P_n P_m' - P_m P_n'] \Big|_{-1}^{+1}$$

You should note that $(1-x^2)$ vanishes for both the limits $(x = \pm 1)$ implying that the right hand side will be zero always. Further, when $m \neq n$, for the above relation to hold we must have

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0 \qquad (14.19)$$

which means that the scalar product of Legendre polynomials of different orders (in the range $-1 \leq x \leq 1$) is zero. Eq. (14.19) constitutes the **orthogonality relation for Legendre polynomials**. Let us now evaluate the integral $\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx$ for $m = n$. In other words, we have to evaluate the integral $\int_{-1}^{+1} [P_n(x)]^2\, dx$.

From the generating relation we recall that

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Similarly, we can write

$$\frac{1}{\sqrt{1 - 2sx + s^2}} = \sum_{m=0}^{\infty} P_m(x) s^m$$

Multiplying these equations and integrating with respect to x, between $x = -1$ and $x = +1$, we get

$$\int_{-1}^{+1} \frac{dx}{\left(\sqrt{1 - 2tx + t^2}\right) \times \left(\sqrt{1 - 2sx + s^2}\right)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} P_m(x)\, P_n(x)\, dx \right\} s^m t^n$$

From Eq. (14.19) we understand that the RHS survives for terms with $m = n$ only. We also observe that when we consider terms for which $m = n$, the notations t and s become identical and we can write

$$\int_{-1}^{+1} \frac{dx}{1-2tx+t^2} = \sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} [P_n(x)]^2\, dx \right\} t^{2n}$$

To evaluate the integral on the LHS, we put $1-2\,tx + t^2 = u$ so that $-2t\, dx = du$ or $dx = -\dfrac{du}{2t}$.
When $x = -1$, the limit of integration changes to $u = 1 + 2t + t^2 = (1 + t)^2$. Similarly, when $x = +1$, we have $u = 1 - 2t + t^2 = (1 - t)^2$. Hence

$$I = \int_{(1+t)^2}^{(1-t)^2} -\frac{du}{2tu} = \frac{1}{2t} \int_{(1-t)^2}^{(1+t)^2} \frac{du}{u}$$

$$= \frac{1}{2t} \ln |u| \Big|_{(1-t)^2}^{(1+t)^2}$$

$$= \frac{2}{2t} \ln\left(\frac{1+t}{1-t}\right) = \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)$$

We now recall the series expansion of a logarithmic function:

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots$$

and

$$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \ldots$$

so that

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \frac{1}{t}\{\ln(1+t) - \ln(1-t)\}$$

$$= \frac{1}{t}\left(2t + \frac{2t^3}{3} + \frac{2t^5}{5} + \ldots\right)$$

$$= 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \ldots\right) = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}$$

Thus

$$\sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} [P_n(x)]^2\, dx \right\} t^{2n} = \sum_{n=0}^{\infty} \left(\frac{2}{2n+1}\right) t^{2n}$$

On equating the coefficients of t^{2n}, we get

$$\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1} \qquad (14.20)$$

We can combine Eqs. (14.19) and (14.20) to write

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn} \qquad (14.21)$$

In Vector Analysis you have learnt that the scalar product of a unit vector with itself is unity and the scalar product of one unit vector with another is zero. Writing symbolically.

$$\left.\begin{aligned}\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1\\ \text{and}\qquad\\ \hat{i}.\hat{j} = \hat{j}.\hat{i} = 0\\ \hat{j}.\hat{k} = \hat{k}.\hat{j} = 0\\ \hat{k}.\hat{i} = \hat{i}.\hat{k} = 0\end{aligned}\right\}$$

These properties are utilised for writing a three-dimensional vector in terms of the linear combination of its components in the three directions because $\hat{i}, \hat{j}$ and $\hat{k}$ form a complete set of orthogonal vectors in 3-D:

$\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$

Using the properties of scalar product for unit vectors, we have

$A_1 = \mathbf{A}.\hat{\mathbf{i}}, A_2 = \mathbf{A}.\hat{\mathbf{j}}$ and $A_3 = \mathbf{A}.\hat{\mathbf{k}}$

where δ_{mn} is the *Kroneckers' delta*, defined as

$$\left.\begin{aligned}\delta_{mn} &= 0, \text{ when } m \neq n\\ &= 1, \text{ when } m = n\end{aligned}\right\}$$

Eq. (14.21) tells us that the scalar product of a Legendre polynomial of a particular order with that of another order is zero, whereas it is non-zero for the product of a Legendre polynomial with itself (in the range $-1 \leq x \leq 1$). This suggests that Legendre polynomials of different orders are orthogonal.

Completeness of Legendre polynomials

In vector analysis, we define a set of orthogonal basis vectors as complete if there is no other vector orthogonal to them all in the number of dimensions under consideration. By analogy, we define a set of orthogonal functions as complete if there is no other function orthogonal to all of them. In Unit 7, Block 2 of your PHE-05 course, you learnt to use an infinite series of sine and cosine terms to express a function, temperature distribution say, in a Fourier series on $(-\pi,\pi)$. Now you will learn to expand a function in a series of Legendre polynomials, which form a complete orthogonal set on $(-1,1)$. This completeness means that any well-behaved function $f(x)$ can be approximated to any desired accuracy by a series of $P_k(x)$ through the relation

$$f(x) = \sum_{k=0}^{\infty} A_k P_k(x) \qquad -1 \leq x \leq 1 \tag{14.22}$$

To obtain the coefficient A_k, we multiply both sides by $P_m(x)$ and integrate the resultant expression in the range -1 to $+1$:

$$\int_{-1}^{+1} P_m(x) f(x)\, dx = \sum_{k=0}^{\infty} A_k \int_{-1}^{+1} P_m(x) P_k(x)\, dx$$

Using orthogonality relation (Eq. (14.21)), we can write

$$\begin{aligned}\int_{-1}^{+1} P_m(x) f(x) dx &= \sum_{k=0}^{\infty} A_k \left(\frac{2}{2m+1}\delta_{km}\right)\\ &= \frac{2}{2m+1}\sum_{k=0}^{\infty} A_k \delta_{km}\\ &= \frac{2A_m}{2m+1}\end{aligned}$$

since $\delta_{km} = 0$ except for $k = m$ and $\delta_{mm} = 1$.

Hence

$$A_m = \frac{2m+1}{2}\int_{-1}^{+1} P_m(x) f(x)\, dx$$

or

$$A_k = \frac{2k+1}{2}\int_{-1}^{+1} P_k(x) f(x)\, dx \tag{14.23}$$

We shall now illustrate this with the help of an example.

Example 3: Expansion of a function in terms of Legendre polynomials

Expand the function $f(x)=\begin{cases}1, & 0<x<1\\ 0, & -1<x<0\end{cases}$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)\,dx$.

Solution

From Eq. (14.23), we have

$$A_k = \frac{2k+1}{2}\int_{-1}^{+1} P_k(x) f(x)\,dx$$

$$= \frac{2k+1}{2}\left[\int_{-1}^{0} P_k(x) f(x)\,dx + \int_{0}^{1} P_k(x) f(x)\,dx\right]$$

We are given that

$$f(x) = \begin{cases}0 \text{ for } -1<x<0\\ 1 \text{ for } \;\; 0<x<1\end{cases}$$

On inserting these values in the expression for A_k, we get

$$A_k = \frac{2k+1}{2}\int_{0}^{1} P_k(x)\,dx$$

Now, refer to Example1and SAQ 1. We recall that $P_0(x)=1, P_1(x)=x$,
$P_2(x)=\frac{(3x^2-1)}{2}, P_3(x)=\frac{5x^3-3x}{2}$, and so on. Therefore

$$A_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

$$A_1 = \frac{3}{2}\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{4}$$

$$A_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \frac{5}{4}\int_0^1 \left(3x^2-1\right)dx = 0$$

and

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{4}\int_0^1 \left(5x^3-3x\right)dx = -\frac{7}{16}$$

Proceeding in this way, you will find that $A_4 = 0, A_5 = \frac{11}{32}$ and so on.

Thus, we can write

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \ldots$$

You may now like to do an SAQ.

SAQ 4

Spend 10 min

a) In Example 3 you must have observed that $A_k = 0$ for even $k \neq 0$. Re-establish this result using the recurrence relation given by Eq. (14.16).

b) Expand $f(x) = x^2$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

You have learnt to obtain Legendre polynomials from the generating function. There is another simple way of arriving at the Legendre polynomials. This is through Rodrigues' Formula, which we discuss now.

14.6 RODRIGUES' FORMULA

Let us consider the function

$$v = (x^2 - 1)^n$$

and differentiate it with respect to x. This gives

$$\frac{dv}{dx} = n(x^2 - 1)^{n-1} \times 2x = 2nx(x^2 - 1)^{n-1}$$

so that

$$(x^2 - 1)\frac{dv}{dx} = 2nx(x^2 - 1)^n = 2nxv$$

That is, v satisfies the differential equation

$$(1 - x^2)\frac{dv}{dx} + 2nxv = 0$$

Differentiating it again with respect to x, we get

$$(1 - x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nx\frac{dv}{dx} + 2nv = 0$$

or

$$(1 - x^2)\frac{d^2v}{dx^2} + 2(n-1)x\frac{dv}{dx} + 2nv = 0 \tag{14.24}$$

If you differentiate it again with respect to x, you can write the resultant expression as

$$(1 - x^2)\frac{d^{2+1}v}{dx^{2+1}} - 2x\frac{d^{1+1}v}{dx^{1+1}} + 2(n-1)\frac{dv}{dx} + 2(n-1)x\frac{d^{1+1}v}{dx^{1+1}} + 2n\frac{dv}{dx} = 0.$$

This can be re-arranged and written in a compact form as

$$(1 - x^2)\frac{d^{2+1}v}{dx^{2+1}} + 2x(n-1-1)\frac{d^{1+1}v}{dx^{1+1}} + (1+1)(2n-1)\frac{dv}{dx} = 0$$

After r differentiations, you will get

$$(1 - x^2)\frac{d^{2+r}v}{dx^{2+r}} + 2x(n-r-1)\frac{d^{1+r}v}{dx^{1+r}} + (r+1)(2n-r)\frac{d^rv}{dx^r} = 0$$

When $r = n$, we get the n th derivative of Eq. (14.24):

$$(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} - 2x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n} = 0$$

This equation can be rewritten as

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^n v}{dx^n}\right) - 2x\frac{d}{dx}\left(\frac{d^n v}{dx^n}\right) + n(n+1)\frac{d^n v}{dx^n} = 0$$

On comparing it with Legendre's differential equation

$$(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

you will note that $d^n v / dx^n$ satisfies the Legendre's equation. So can write

$$P_n(x) = C\frac{d^n v}{dx^n}$$

That is, $P_n(x)$ is a constant multiplier times $\frac{d^n v}{dx^n}$. On substituting for v, we obtain

$$P_n(x) = C\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

where C is a constant. To determine this constant, we have to consider terms with the highest power of x on both sides. From Eq. (14.8) we recall that the term with the highest power of x ir the expression for $P_n(x)$ is $\frac{(2n)!}{2^n (n!)^2}x^n$. Hence,

$$\frac{(2n)!}{2^n (n!)^2}\cdot x^n = C\frac{d^n}{dx^n}x^{2n} = C.2n(2n-1)(2n-2)...[2n-(n-1)]x^n$$

$$= C\frac{(2n)!}{n!}x^n$$

On comparing the coefficients of x^n on both sides, we get

$$C = \frac{1}{(2^n)n!}$$

Hence, we can write

$$\boxed{P_n(x) = \frac{1}{(2^n)n!}\frac{d^n}{dx^n}(x^2-1)^n} \qquad (14.25)$$

This is known as the **Rodrigues' formula** for $P_n(x)$. We now consider an application of this formula.

Example 4: Rodrigues' formula

Obtain the value of $P_3(x)$ using Rodrigues' formula.

Solution:

From Eq. (14.25) we can write

$$P_3(x) = \frac{1}{(2^3)3!}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \left(\frac{1}{48}\right)\frac{d^3}{dx^3}\left(x^6-3x^4+3x^2-1\right)$$

Differentiating the expression in the small brackets with respect to x, you will get

$$\frac{d}{dx}\left(x^6-3x^4+3x^2-1\right)=6x^5-12x^3+6x$$

Again differentiate the resultant expression with respect to x:

$$\frac{d^2}{dx^2}\left(x^6-3x^4+3x^2-1\right)=\frac{d}{dx}\left(6x^5-12x^3+6x\right)=30x^4-36x^2+6$$

and

$$\frac{d^3}{dx^3}\left(x^6-3x^4+3x^2-1\right)=\frac{d}{dx}\left(30x^4-36x^2+6\right)=120x^3-72x$$

$$\therefore \quad P_3(x)=\left(\frac{1}{48}\right).24\left(5x^3-3x\right)=\frac{1}{2}\left(5x^3-3x\right)$$

You may now like to do an SAQ.

Spend 5 min

SAQ 5

Obtain $P_4(x)$ using Eq. (14.25).

We have discussed the basic operations involving Legendre polynomials. We now intend to discuss their applications in physics. The most instructive applications arise while solving Laplace's equation in spherical polar co-ordinates for potential and temperature related problems. You should go through the following examples carefully as you can learn a lot of good physics.

Example 5: The potential due to a point charge

We have discussed this problem in Sec. 14.3. Let us take another look at it. Refer to Fig. 14.3.

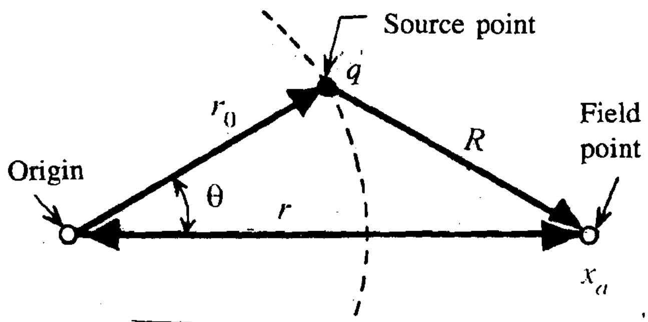

Fig 14.3: Polar co-ordinates for potential due to a point charge

By referring to the Δ OPQ, we can write

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos\theta}$$

so that

$$\frac{1}{R} = \left(r^2 + r_0^2 - 2rr_0 \cos\theta\right)^{-1/2}$$

This can be expressed in two ways depending on whether or not $r > r_0$. We express it in powers of (r/r_0) when $r < r_0$ and in powers of r_0/r when $r > r_0$:

$$\frac{1}{R} = \frac{1}{r_0}\left[1 - 2\left(\frac{r}{r_0}\right)\cos\theta + \left(\frac{r}{r_0}\right)^2\right]^{-1/2} \quad \text{for } r < r_0$$

and

$$\frac{1}{R} = \frac{1}{r}\left[1 - 2\left(\frac{r_0}{r}\right)\cos\theta + \left(\frac{r_0}{r}\right)^2\right]^{-1/2} \quad \text{for } r > r_0 .$$

If we identify $\cos\theta$ with μ and (r/r_0) or (r_0/r) with t, the terms on the RHS of the above expression become analogous to the generating function (Eq. 14.12). Then we may write

$$\frac{1}{R} = \frac{1}{r_0}\sum_{n=0}^{\infty}\left(\frac{r}{r_0}\right)^n P_n(\mu), \quad \text{for } r < r_0$$

and

$$\frac{1}{R} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n P_n(\mu), \quad \text{for } r > r_0$$

Since the potential at a non-axial point is given by $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$, we have

$$\underset{(r<r_0)}{V_1} = \frac{q}{4\pi\varepsilon_0 r_0}\sum_{n=0}^{\infty}\left(\frac{r}{r_0}\right)^n P_n(\mu)$$

and

$$\underset{(r>r_0)}{V_2} = \frac{q}{4\pi\varepsilon_0 r_0}\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^{n+1} P_n(\mu)$$

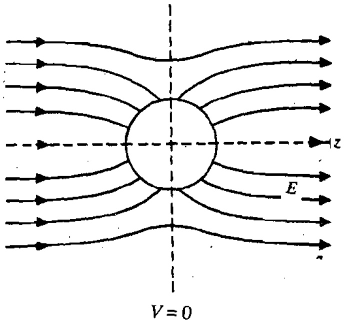

Fig.14.4: A conducting sphere in an electric field

We assume a trial solution

$$R = r^p$$

Then

$$\frac{dR}{dr} = p r^{p-1}$$

and

$$\frac{d^2R}{dr^2} = p(p-1)r^{p-2}$$

Substituting in (i), we get

$$p(p-1)r^p + 2pr^p - \ell(\ell+1)r^p = 0$$

or

$$p(p+1) = \ell(\ell+1)$$

Thus two solutions for p are $p = \ell$ and $p = -(\ell+1)$ and radial part of Laplace's equation has the solution

$$R_\ell(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

Example 6: Conducting sphere in a uniform electric field

Let us consider a conducting sphere placed in a uniform electric field of strength E_0, as shown in Fig. 14.4 and calculate the electrostatic potential at a point outside the sphere. To this end, we note that the potential function will be independent of the azimuthal angle ϕ. From Unit 6 TQ 1, of Block 2 of PHE-05 course on Mathematical Methods in Physics-II, you will recall that Laplace's equation can be split into the following differential equations:

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \ell(\ell+1)R = 0 \qquad \text{(i)}$$

and

$$\frac{1}{\sin}\frac{d}{d\theta} + \left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\,\Theta = 0 \qquad \text{(ii)}$$

where $\ell(\ell+1)$ is the separation constant. The former of these equations is the radial part of Laplace's equation and admits solutions of the form

$$R_\ell(r) = A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

You will recall that (ii) is Legendre's equation, which admits Legendre Polynomials as solutions. Hence, the solution of Laplace's equation with azimuthal symmetry (no ϕ-dependence) is given by

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \qquad \text{(iii)}$$

To determine the constants A_n and B_n, we have to impose boundary conditions. Since the original uniform electric field (before the sphere is placed in electric field) is E_0, we must have

$$V(r\to\infty) = -E_0 z = -E_0 r\cos\theta \qquad (\because z = r\cos\theta)$$

or

$$V = -E_0 r\, P_1(\cos\theta) \qquad (\because P_1(x) = x) \qquad \text{(iv)}$$

For $r\to\infty$, the second term in the square brackets on the right side of (iii) will disappear. Thus, on comparing (iii) and (iv), we get

$$A_0 = 0,\ A_n = 0 \quad \text{for all } n > 1$$

and

$$A_1 = -E_0 \qquad \text{(v)}$$

For the second boundary condition, we choose the surface of the sphere to be at zero potential. Thus from (iii) we have

$$V(r=a) = \left(A_0 + \frac{B_0}{a}\right) + \left(\frac{B_1}{a^2} - E_0 a\right)P_1(\cos\theta) + \sum_{\ell=1}^{\infty} B_\ell \frac{P_\ell(\cos\theta)}{a^{\ell+1}} = 0$$

For this equality to hold for all values of θ, each coefficient of $P_\ell(\cos\theta)$ must vanish. Hence, we have

$$A_0 = B_0 = 0$$

$$B_n = 0 \quad \text{for } n \geq 2$$

and

$$B_1 = E_0\, a^3$$

Inserting these results in the above expression, we find that the electrostatic potential is given by

$$V = -E_0\, r\, P_1(\cos\theta) + \frac{E_0\, a^3}{r^2} P_1(\cos\theta)$$

$$= -E_0\, r \cos\theta \left(1 - \frac{a^3}{r^3}\right) \qquad \text{(vi)}$$

In this case, the potential outside the sphere is only relevant as potential inside a conducting sphere is zero.

We have here discussed the problem of a conducting sphere in a uniform electric field. You should now solve the problem of a dielectric sphere in a uniform electric field.

SAQ 6

Spend 15 min

A dielectric sphere of radius a and dielectric constant ε_1 is placed in a uniform electric field $\mathbf{E}_0$. The sphere is surround by a medium of dielectric constant ε_2. Calculate the potentials inside and outside the sphere.

Hint: This problem is an extension of Example 6. So begin with expression for potential function given by (vi). You will have to apply the boundary conditions

$$\varepsilon_1 E_{1r} = \varepsilon_2 E_{2r} \quad \text{and} \quad E_{1t} = E_{2t}$$

where E_t and E_r are tangential and radial (i.e. normal) components of electric field:

$$E_r = -\frac{\partial V}{\partial r} \quad \text{and} \quad E_t = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

In electrodynamics and gravitation you have come across problems of determination of potential due to a ring or a disc at an axial point. In the following example you will note that for the determination of potential at an off-axial point also we require a knowledge of Legendre functions.

Example 7: Potential due to a charged ring

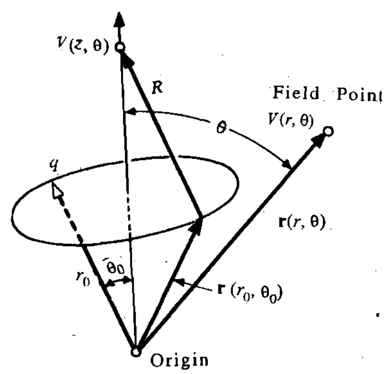

Fig.14.5: Potential due to a charged ring at an off-axial point

Refer to Fig. 14.5. The point P is an off-axial point at a distance r from the origin of the coordinate system and angle θ with the axis of symmetry. We wish to calculate the potential at P. If the distance from the origin along the polar axis is denoted by z, the potential along the polar axis, as calculated in Example 6, can be written as

$$V_{r<r_0}(z,0) = \frac{q}{4\pi\varepsilon_0 r_0} \sum_{n=0}^{\infty} \left(\frac{z}{r_0}\right)^n P_n(\cos\theta_0)$$

and

$$V_{r>r_0}(z,0) = \frac{q}{4\pi\varepsilon_0 r_0} \sum_{n=0}^{\infty} \left(\frac{r_0}{z}\right)^{n+1} P_n(\cos\theta_0)$$

The potential at an off-axial point, not lying on the polar axis, in obtained by multiplying the nth term in the series by $P_n(\cos\theta)$ and writing r for z. The resultant expressions are

$$V_{r<r_0}(r,\theta) = \frac{q}{4\pi\varepsilon_0 r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(\cos\theta_0) P_n(\cos\theta)$$

and

$$V_{r>r_0}(r,\theta) = \frac{q}{4\pi\varepsilon_0 r_0} \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^{n+1} P_n(\cos\theta_0) P_n(\cos\theta)$$

Example 8: Temperature distribution in a sphere

We wish to know steady-state temperature inside a sphere of radius a whose upper hemisphere is maintained at a temperature $T = T_0$ and the lower hemisphere is maintained at temperature $T = -T_0$. To this end, we solve Laplace's equation in spherical coordinates subject to the given boundary conditions. Equation (iii) in Example 6 gives the solution of azimuthally symmetric Laplace's equation in spherical coordinates. Since temperature distribution is independent of ϕ, we can write

$$T(r,\theta) = \sum_{m=0}^{\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)P_m(\cos\theta) \qquad \text{(i)}$$

Since temperature will be finite at the centre of the sphere ($r = 0$), we must have $B_m = 0$ for all m; otherwise solution will diverge. Hence expression in (i) reduces to

$$T(r,\theta) = \sum_{m=0}^{\infty} A_m r^m \, P_m(\cos\theta) \qquad \text{(ii)}$$

In view of the given conditions, the temperature distribution on the surface of the sphere can be written as

$$f(x) = \begin{cases} -T_0 & -1 < x < 0 \\ T_0 & 0 < x < 1 \end{cases} \qquad \text{(iii)}$$

where $x = \cos\theta$. Applying the boundary conditions to Eq. (ii), we get

$$f(x) = \sum_{m=0}^{\infty} A_m a^m \, P_m(x) \qquad \text{(iv)}$$

To determine the constants A_m, we use the orthogonality relation. To this end, we multiply both sides of this equation by $P_\ell(x)$ and integrate the resultant expression over x in the range -1 to $+1$. This yields

$$\int_{-1}^{1} P_\ell(x) f(x)\,dx = \sum_{m=0}^{\infty} A_m a^m \int_{-1}^{1} P_\ell(x) P_m(x)\,dx \qquad \text{(v)}$$

Using the orthogonality relation for the Legendre Polynomials, we get

$$A_\ell = \frac{2\ell+1}{2a^\ell}\int_{-1}^{1} P_\ell(x) f(x)\,dx$$

$$= \frac{2\ell+1}{2a^\ell} T_0\left[-\int_{-1}^{0} P_\ell(x)\,dx + \int_0^1 P_\ell(x)\,dx\right]$$

$$= \left(\frac{2\ell+1}{2}\right)\left(\frac{T_0}{a^\ell}\right)\left[\int_0^1 P_\ell(x)\,dx - \int_0^1 P_\ell(-x)\,dx\right] \qquad \text{(vi)}$$

Since $P_\ell(-x) = (-1)^\ell \, P_\ell(x)$, the above expression simplifies to

$$A_l = \begin{cases} (2\ell+1)(T_0/a^\ell)\int_0^1 P_\ell(x)\,dx & \ell = \text{odd} \\ 0 & \ell = \text{even} \end{cases} \qquad \text{(vii)}$$

From this you can readily write the values of first few coefficients:

$$A_1 = \frac{3T_0}{a}\int_0^1 P_l(x)\,dx = \frac{3T_0}{a}\int_0^1 x\,dx = \frac{3T}{2a}$$

$$A_3 = \frac{7T_0}{2}\int_0^1 P_3(x)\,dx = \frac{7T_0}{a^3}\int_0^1\left(\frac{5x^3-3x}{2}\right)dx = -\frac{7T_0}{8a^3}$$

Hence the temperature distribution inside the given sphere is given by

$$T(r,\theta) = T_0\left[\frac{3r}{2a}\cos\theta - \frac{7r^3}{16a^3}\left(5\cos^3\theta - 3\cos\theta\right) + \ldots\right] \qquad \text{(viii)}$$

Example 9: Electric potential inside a sphere

We consider a sphere of radius a such that $V(r,\theta)\big|_{r=a} = V_0\cos^3\theta$ and assume that there are no charges at the origin. Since V must satisfies Laplace's equation and the boundary condition has no ϕ dependence, the solution will be obtained in terms of Legendre Polynomials. The general solution can be written as

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \qquad \text{(i)}$$

From this form of the solution, we note that V can be finite at the origin only if $B_n = 0$ for all n. Then expression in (i) reduces to

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n\, r^n P_n(\cos\theta) \qquad \text{(ii)}$$

On applying the given boundary condition, we have

$$V(r,\theta) = V_0\cos^3\theta = \sum_{n=0}^{\infty} A_n\, a^n P_n(\cos\theta) \qquad \text{(iii)}$$

To solve for A_n, we rewrite $\cos^3\theta$ in terms of Legendre Polynomials. To this end, we recall that $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$ and we can write

$$\cos^3\theta = \frac{2}{5}P_3(\cos\theta) + \frac{3}{5}\cos\theta$$

Since $\cos\theta = P_1(\cos\theta)$, we find that

$$\cos^3\theta = \frac{2}{5}P_3(\cos\theta) + \frac{3}{5}P_1(\cos\theta)$$

Inserting this result in (iii) we obtain

$$\frac{1}{5}\left[2V_0 P_3(\cos\theta) + 3V_0 P_1(\cos\theta)\right] = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) \qquad \text{(iv)}$$

Using the orthogonality property of Legendre polynomials, you can easily see that $A_1 = 3V_0/5a$ and $A_3 = 2V_0/5a^3$. Hence

$$V(r,\theta) = \frac{3}{5}V_0(r/a)P_1(\cos\theta) + \frac{2}{5}V_0(r/a)^3 P_3(\cos\theta)$$

From the PHE-11 course on Modern Physics, you will recall that for the study of structure of atom, it is essential to understand the hydrogen atom problem. For this, we solve the time-independent Schrödinger equation. The solution of Schrödinger equation requires a knowledge of associated Legendre polynomials (Appendix A). This combined with the solution of ϕ-part gives what is known as the spherical harmonics. The spherical harmonics forms the base of many important concepts related to the structure of atom. You will study these in your post-graduate studies in physics.

14.7 SUMMARY

- **Legendre's differential equation** is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

- The **polynomial solution** of Legendre's differential equation is given by

$$y = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\times 4\times(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

with

$$a_n = \frac{(2n-1)(2n-3)\ldots\times 3\times 1}{n!}$$

- The **generating function** for Legendre polynomials is given by

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |t|<1$$

- The **recurrence relations** for Legendre polynomials are:

$$(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + n\,P_{n-1}(x)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

$$P'_{n+1}(x) = (n+1)\,P_n(x) + x\,P'_n(x)$$

$$P'_{n-1}(x) = x\,P'_n(x) - n\,P_n(x)$$

$$(1-x^2)P'_n(x) = n\,P_{n-1}(x) - n\,x\,P_n(x)$$

$$(1-x^2)P'_n(x) = (n+1)\,x\,P_n(x) - (n+1)\,P_{n+1}(x)$$

- The **orthogonality relation** for Legendre polynomials is given by

$$\int_{-1}^{+1} P_m(x).P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}.$$

where

$$\delta_{mn} = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n \end{cases}$$

- Any function $f(x)$ can be expanded in terms of $P_k(x)$ as

$$f(x) = \sum_{k=0}^{\infty} A_k P_k(x), \qquad -1 \le x \le 1$$

where

$$A_k = \frac{2k+1}{2} \int_{-1}^{+1} P_k(x) f(x)\, dx$$

- The **Rodrigues' formula** for Legendre polynomials is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

14.8 TERMINAL QUESTIONS *Spend* 25 *min*

1. Evaluate the integrals

   (a) $\int_{-1}^{1} x P_{n-1}(x) P_n(x)\, dx$; and (b) $\int_{-1}^{1} x P_n(x)\, dx$

2. Calculate the potential V at a point (a) inside and (b) outside a hollow sphere of unit radius if upper-half of its surface is charged to potential V_0 and the lower-half is at zero potential (Fig.14.6).

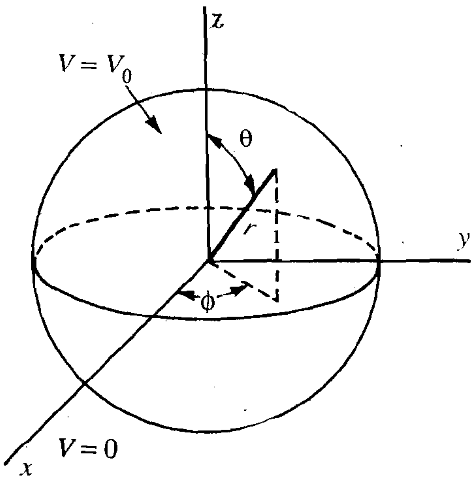

Fig.14.6: A sphere of unit radius: upper-half of its surface is at finite potential V whereas lower-half is at zero potential.

3. Calculate the gravitational potential at a point (r, θ, ϕ) in space due to a uniform ring of radius a and mass per unit length m_l (Fig.14.7). It is given that the potential at the point P (at a distance z from the centre of the ring along its axis) is given by

$$V_p = \frac{2\pi a\, m_l}{(a^2 + z^2)^{1/2}}$$

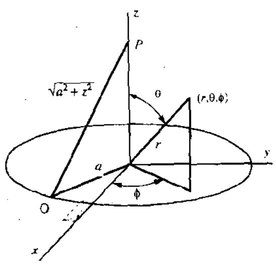

Fig.14.7: A uniform ring

14.9 SOLUTIONS AND ANSWERS

Self-assessment Questions

1.
$$P_3(x) = \frac{6!}{2^3(3!)^2}\left(x^3 - \frac{3\times 2}{2\times 5}x\right)$$

$$= \frac{1\times 2\times 3\times 4\times 5\times 6}{8\times 6\times 6}\left(\frac{5x^3 - 3x}{5}\right) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_3(1) = \frac{1}{2}(5-3) = 1$$

2. (a) Putting $x = -1$ in Eq. (14.12), we get

$$\left(1 + 2t + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1)t^n$$

or

$$\left(1 + t\right)^{-1} = \sum_{n=0}^{\infty} P_n(-1)t^n$$

$$\therefore \quad 1 - t + t^2 - \ldots + (-1)^n t^n - \ldots = \sum_{n=0}^{\infty} P_n(-1)t^n$$

$$\therefore \quad P_n(-1) = (-1)^n$$

(b) We observe that the generating function remains unchanged even if we replace x by $-x$ and t by $-t$. Thus

$$g(t, x) = g(-t, -x)$$

$$= \left[1 - 2(-t)(-x) + (-t)^2\right]^{-1/2}$$

$$= \sum_{n=0}^{\infty} P_n(-x)(-t)^n = \sum_{n=0}^{\infty} (-1)^n P_n(-x)t^n$$

so that

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} (-1)^n P_n(-x)t^n$$

$$\therefore \quad P_n(x) = (-1)^n P_n(-x)$$

or

$$P_n(x) = (-1)^n P_n(x)$$

3. By adding Eqs. (14.15) and (14.16), we get

$$2P'_{n+1}(x) = 2(n+1)P_n(x) + 2x P'_n(x)$$

$$\therefore \quad P'_{n+1}(x) = (n+1)P_n(x) + x P'_n(x)$$

which is same as Eq. (14.17a).

On subtracting Eq. (14.16) from Eq. (14.15), we get

$$2P'_{n-1}(x) = -2n\,P_n(x) + 2x\,P'_n(x)$$

$$\therefore \quad P'_{n-1}(x) = xP'_n(x) - n\,P_n(x)$$

which is identical with Eq. (14.17b).

In Eq. (14.17a), we replace n by $(n-1)$. This gives

$$P'_n(x) = n\,P_{n-1}(x) + x\,P'_{n-1}(x)$$

Now, multiplying Eq. (14.17b) by x, we get

$$xP'_{n-1}(x) = x^2\,P'_n(x) - n\,x\,P_n(x)$$

On comparing the two above equations, we get

$$\left(1-x^2\right)P'_n(x) = n\,P_{n-1}(x) - n\,x\,P_n(x)$$

which is Eq.(14.17c).

By replacing n by $(n+1)$ in this expression, we get

$$\left(1-x^2\right)P'_{n+1}(x) = (n+1)\,P_n(x) - (n+1)x\,P_{n+1}(x)$$

Using Eq.(14.17a), we get

$$\left(1-x^2\right)(n+1)P_n(x) + x\left(1-x^2\right)P'_n(x) = (n+1)\,P_n(x) - (n+1)x\,P_{n+1}(x)$$

or

$$-(n+1)x^2\,P_n(x) + x\left(1-x^2\right)P'_n(x) = -(n+1)x\,P_{n+1}(x)$$

As $x \neq 0$ (in general), we get

$$\left(1-x^2\right)P'_n(x) = (n+1)x\,P_n(x) - (n+1)\,P_{n+1}(x)$$

which is the result contained in Eq. (14.17d).

4. (a) $$A_k = \frac{2k+1}{2}\int_0^1 P_k(x)\,dx$$

Using Eq.(14.16), we can write

$$A_k = \frac{1}{2}\int_0^1 \left[P'_{k+1}(x) - P'_{k-1}(x)\right]dx$$

$$= \frac{1}{2}\left|P_{k+1}(x) - P_{k-1}(x)\right|_0^1$$

$$= \frac{1}{2}\{P_{k+1}(1) - P_{k-1}(1) + P_{k+1}(0) - P_{k-1}(0)\}$$

$$= \frac{1}{2}\{P_{k+1}(0) - P_{k-1}(0)\}$$

Hence, for even k (other than $k = 0$), $A_k = 0$.

(b) $$f(x) = x^2 = \sum_{k=0}^{\infty} A_k P_k(x)$$

We have to find A_k, $k = 0,1,2,3,\ldots$ such that

$$x^2 = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 P_3(x) + \ldots$$

$$= A_0(1) + A_1(x) + A_2\left(\frac{3x^2-1}{2}\right) + A_3\left(\frac{5x^3-3x}{2}\right) + \ldots$$

Since left hand side is a polynomial of degree 2, we must have $A_3 = 0, A_4 = 0, A_5 = 0$ and so on.

$$\therefore \quad x^2 = \left(A_0 - \frac{A_2}{2}\right) + A_1 x + \frac{3}{2}A_2 x^2$$

$$\Rightarrow \quad A_0 - \frac{A_2}{2} = 0, \quad A_1 = 0, \text{ and } \quad \frac{3}{2}A_2 = 1$$

Thus

$$A_0 = \frac{1}{3}, A_1 = 0 \text{ and } \quad A_2 = \frac{2}{3}$$

Alternatively, you can calculate A_k's using the relation

$$A_k = \frac{2k+1}{2}\int_{-1}^{+1} x^2 P_k(x)\,dx$$

5. $$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}\left(x^2-1\right)^4$$

$$= \frac{1}{16\times 24}\frac{d^4}{dx^4}\left(x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right)$$

$$= \frac{1}{16\times 24}\left(8\times 7\times 6\times 5x^4 - 4\times 6\times 5\times 4\times 3x^2 + 6\times 4\times 3\times 2\times 1\right)$$

$$= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

6. Refer to Fig. 14.8. You will note that we have to consider two separate potential functions V_1 (for inside the sphere) and V_2 (for outside the sphere). Guided by Eq.(vi) of Example 7, we can write

$$V_1 = B_1 r\cos\theta + B_2 r^{-2}\cos\theta \qquad \text{(i)}$$

$$V_2 = -E_0 r\cos\theta + A_2 r^{-2}\cos\theta \qquad \text{(ii)}$$

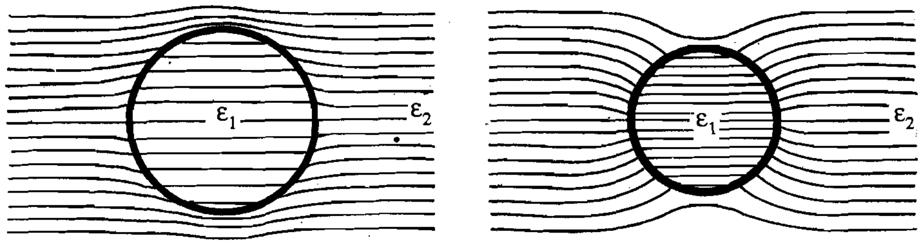

Fig.14.8: The lines of electric displacement due to a dielectric sphere of dielectric constant ε_1 placed in a uniform electric field in a medium of dielectric constant ε_2.

From physical considerations, we should not have $V_1 \to \infty$ for $r \to 0$. Therefore, we must have $B_2 = 0$.

Again, the potential function must be continuous at the boundary. Therefore, we have

$$V_1 = V_2 \quad \text{at} \quad r = R$$

(This condition is equivalent to $E_{1t} = E_{2t}$.)

$$\therefore \quad B_1 R\cos\theta = -E_0 R\cos\theta + A_2 R^{-2}\cos\theta$$

or

$$B_1 = A_2 R^{-3} - E_0 \qquad \text{(iii)}$$

Now,

$$\varepsilon_1 E_{1r} = \varepsilon_2 E_{2r} \quad \text{at} \quad r = R$$

or

$$-\varepsilon_1\left(\frac{\partial V_1}{\partial r}\right)_{r=R} = -\varepsilon_2\left(\frac{\partial V_2}{\partial r}\right)_{r=R}$$

$$\therefore \quad \varepsilon_1(-B_1\cos\theta) = \varepsilon_2\left(E_0\cos\theta + 2A_2R^{-3}\cos\theta\right)$$

or $$-B_1 = (\varepsilon_2/\varepsilon_1)\left(E_0 + 2A_2R^{-3}\right) \qquad \text{(iv)}$$

From Eqs. (iii) and (iv), we get

$$B_1 = \left(\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right)E_0$$

and

$$A_2 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right)R^3 E_0$$

So the potential inside and outside the sphere are given by

$$V_1 = -\left(\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right)E_0 r\cos\theta \qquad \text{(v)}$$

and

$$V_2 = -\left(1 - \frac{R^3}{r^3}\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right)E_0 r\cos\theta \qquad \text{(vi)}$$

Terminal Questions

1. (a) We start with the Recurrence relation

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$$

We multiply both sides by $P_{n-1}(x)$ and integrate the terms with respect to x between the limits -1 and $+1$. This leads to the relation

$$(n+1)\int_{-1}^{1} P_{n+1}(x)\,P_{n-1}(x)\,dx-(2n+1)\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx+n\int_{-1}^{1} P_{n-1}^2(x)\,dx=0$$

From Eq. (14.21), we recall that

$$\int_{-1}^{1} P_{n+1}(x)\,P_{n-1}(x)\,dx=0$$

and

$$\int_{-1}^{1} P_{n-1}^2(x)\,dx=\frac{2}{2(n-1)+1}=\frac{2}{2n-1}$$

$$\therefore\quad 0-(2n+1)I_a+\frac{2n}{2n-1}=0$$

where $I_a=\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx$. On rearranging terms, we get

$$(2n+1)I_a=\frac{2n}{2n-1}$$

so that

$$I_a=\frac{2n}{(2n-1)(2n+1)}$$

(b) To evaluate the integral $\int_{-1}^{1} xP_n(x)\,dx$, we note that $x=P_1(x)$, so that $I_b=\int_{-1}^{1} P_1(x)P_n(x)\,dx$. Therefore orthogonality relation for Legendre polynomials (Eq. 14.21) implies that $I_b=0$ for $n\neq 1$ and $I_b=\frac{2}{2\times 1+1}=\frac{2}{3}$ for $n=1$

2. Refer to Fig. 14.6 again. The potential has azimuthal symmetry so that V is independent of ϕ. Thus

$$V=V(r,\theta)=\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\mu) \qquad \text{(i)}$$

where $\mu=\cos\theta$ and we have the conditions

$$V(1,\theta)=\begin{cases} V_0,\ \text{for } 0<\theta<\pi/2, & \text{i.e. } 0<\mu<1 \\ 0,\ \text{for } \pi/2<\theta<\pi, & \text{i.e. } -1<\mu<0 \end{cases}$$

(a) Potential at an interior point ($0 \le r < 1$)

As V should not tend to infinity for $r \to 0$, we must have $B_n = 0$ for all n. Thus the expression in (i) reduces to

$$V(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\mu)$$

so that

$$V(1,\theta) = \sum_{n=0}^{\infty} A_n P_n(\mu) \qquad \text{(ii)}$$

Now, from Eq. (14.22), we can write

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} V(1,\theta)\, P_n(\mu)\, d\mu$$

$$= \left(\frac{2n+1}{2}\right) V_0 \int_0^1 P_n(\mu)\, d\mu \qquad \text{(iii)}$$

On inserting the values of $P_n(\mu)$ for $n = 1,2,3,\ldots$; and evaluating the resultant integral, we will obtain

$$A_0 = \frac{1}{2}V_0,\ A_1 = \frac{3}{4}V_0,\ A_2 = 0, A_3 = -\frac{7}{16}V_0,\ A_4 = 0,\ A_5 = \frac{11}{32}V_0,\ldots \qquad \text{(iv)}$$

Thus

$$V(r,\theta) = \frac{V_0}{2}\left[1 + \frac{3r}{2}P_1(\cos\theta) - \frac{7}{8}r^3 P_3(\cos\theta) + \frac{11}{16}r^5 P_5(\cos\theta) - \ldots\right] \qquad \text{(v)}$$

(b) Potential at a point outside the sphere ($1 < r < \infty$)

Since V is bounded for $r \to 0$, we must have $A_n = 0$ for all n; otherwise the function given by (i) will diverge.

$$\therefore \quad V(r,\theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\mu)$$

and

$$V(1,\theta) = \sum_{n=0}^{\infty} B_n P_n(\mu) \qquad \text{(vi)}$$

Thus $B_n = A_n$ of part (a) and we can write

$$V(r,\theta) = \frac{V_0}{2r}\left[1 + \frac{3}{2r}P_1(\cos\theta) - \frac{7}{8r^3}P_3(\cos\theta) + \frac{11}{16r^5}P_5(\cos\theta) + \ldots\right]$$

3. Refer to Fig. 14.7. As in the case of TQ 2, we can write

$$V = \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\mu) \qquad \text{(i)}$$

corresponding to the regions $0 \le r < a$ and $r > a$, we have to consider two cases.

(a) $0 \le r < a$

In this case $B_n = 0$ as otherwise the solution will become unbounded at $r = 0$. So, we have

$$V = \sum_{n=0}^{\infty} A_n r^n P_n(\mu) \qquad \text{(ii)}$$

For $\theta = 0$ (i.e. $\mu = 1$), this must reduce to the potential along the z-axis, so that $r = z$. So we have

$$\frac{2\pi a\, m_\ell}{\sqrt{a^2 + z^2}} = \sum_{n=0}^{\infty} A_n z^n \qquad \text{(iii)}$$

In order to obtain A_n, we have to expand the left side as a power series in z. This leads to

$$\frac{2\pi a\, m_\ell}{\sqrt{a^2 + z^2}} = 2\pi\, m_\ell \left(1 + \frac{z^2}{a^2}\right)^{-1/2}$$

$$= 2\pi\, m_\ell \left[1 - \frac{1}{2}\left(\frac{z}{a}\right)^2 + \frac{1\times 3}{2\times 4}\left(\frac{z}{a}\right)^4 - \frac{1\times 3\times 5}{2\times 4\times 6}\left(\frac{z}{a}\right)^6 + \ldots\right] \qquad \text{(iv)}$$

On comparing Eqs. (iii) and (iv), we get

$$A_0 = 2\pi\, m_\ell,\ A_1 = 0,\ A_2 = -\frac{2\pi\, m_\ell}{2a^2},\ A_3 = 0,\ A_4 = \frac{2\pi\, m_\ell \times 1\times 3}{2\times 4\, a^4}, \ldots$$

Thus, we have

$$V = 2\pi\, m_\ell \left[P_0(\cos\theta) - \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) + \frac{1\times 3}{2\times 4}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) - \ldots\right]$$

(b) $r > a$

In this case $A_n = 0$, as otherwise the solution will become unbounded for $r \to \infty$. So, we have

$$V = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\mu)$$

As in case (a) this expression must reduce to the potential on z-axis for $\theta = 0$ and $r = z$, i.e.

$$\frac{2\pi a\, m_\ell}{\sqrt{a^2 + z^2}} = \sum_{n=0}^{\infty} \frac{B_n}{z^{n+1}}$$

As in case (a), we have to expand the left hand side but we have to keep in mind that we have to compare with terms having $\frac{1}{z^{n+1}}$. So we express the left side as

$$\frac{2\pi a\, m_\ell}{\sqrt{a^2+z^2}} = \frac{2\pi a\, m_\ell}{z}\left(1+\frac{a^2}{z^2}\right)^{-1/2}$$

$$= \frac{2\pi a\, m_\ell}{z}\left[1-\frac{1}{2}\left(\frac{a}{z}\right)^2+\frac{1\times 3}{2\times 4}\left(\frac{a}{z}\right)^4-\frac{1\times 3\times 5}{2\times 4\times 6}\left(\frac{a}{z}\right)^6+\ldots\right] \quad \text{(v)}$$

On comparing we get

$$B_0 = 2\pi a\, m_\ell,\ B_1 = 0,\ B_2 = -2\pi a\, m_\ell\left(\frac{1}{2}a^2\right),\ B_3 = 0,\ B_4 = 2\pi a\, m_\ell\left(\frac{1\times 3}{2\times 4}a^4\right),\ldots$$

Thus, we get

$$V = \frac{2\pi a\, m_\ell}{r}\left[P_0(\cos\theta)-\frac{1}{2}\left(\frac{a}{r}\right)^2 P_2(\cos\theta)+\frac{1\times 3}{2\times 4}\left(\frac{a}{r}\right)^4 P_4(\cos\theta)-\ldots\right]$$

APPENDIX A: ASSOCIATED LEGENDRE POLYNOMIALS

Laplace's equation in spherical polar co-ordinates (r,θ,ϕ) is:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{A.1}$$

To solve this equation we use the method of separation of variables:

$$V = R(r)\,\Theta(\theta)\,\Phi(\phi) \tag{A.2}$$

From Unit 6, Block 2 of PHE-05 course, you will recall that on inserting this substitution in Eq. (A.1) and rearranging the resultant expression, we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{dr}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}$$

Since LHS depends only on r and RHS depends on θ and ϕ, it follows that each side must be a constant, which we put equal to $-\ell(\ell+1)$. This ensures physically meaning solution for Θ, which is one of the factors determing the potential function. Now, we write

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\ell(\ell+1) \tag{A.3}$$

and

$$\frac{1}{\Theta\sin\theta}\frac{d}{dr}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -\ell(\ell+1) \tag{A.4}$$

We can rewrite Eq. (14.28) as

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \ell(\ell+1)R = 0$$

or

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \ell(\ell+1)R = 0$$

As shown earlier, the solution of the radial part of Laplace's equation is of the form

$$R(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \tag{A.5}$$

Let us now consider the angular part of Laplace's equation. On multiplying Eq. (A.4) by $\sin^2\theta$ and rearranging terms, you can write

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \ell(\ell+1)\sin^2\theta$$

Since left hand side is a function of ϕ only and the right side involves only θ, we put each side equal to a constant, say $-m^2$. Thus, the azimuthal part of Laplace's equation can be written as

...u will recall that if a vector quantity A is irrotational (i.e. curl $\mathbf{A} = 0$, it can be expressed as $\mathbf{A} = -\nabla V$, where V is ...alar potential. If over and above A is solenoidal, then

$$\nabla.\mathbf{A} = 0$$
$$\nabla.\nabla V = 0$$
$$\nabla^2 V = 0$$

which is Laplace's equation. In other words, if a vector quantity is solenoidal as well as irrotational, the scalar potential from which the vector is derived satisfies Laplace's equation.

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

This equation admits solutions of the form

$$\Phi = B_1 \cos m\phi + B_2 \sin m\phi \tag{A.6}$$

where B_1 and B_2 are constants.

Putting the separation constant equal to $-m^2$ means that the constant is a negative definite quantity (or zero) on being real. Had it been otherwise, we would have obtained exponential solution for Φ which is not physically meaningful.

The Θ part can now be rewritten as

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \left(m^2 - \ell(\ell+1)\sin^2\theta\right)\Theta = 0$$

or

$$\sin^2\theta \frac{d^2\Theta}{d\theta^2} + \sin\theta\cos\theta \frac{d\Theta}{d\theta} - \left(m^2 - \ell(\ell+1)\sin^2\theta\right)\Theta = 0 \tag{A.7}$$

This equation is called the **Legendre's associated differential equation**. If we set $m = 0$, i.e. there is no ϕ dependence, then Eq. (A.7) reduces to Legendre's differential equation. We can transform Eq. (A.7) by introducing a change of variable:

$$x = \cos\theta$$

so that $\sin^2\theta = 1-x^2$ and by Chain rule

$$\frac{d}{d\theta} = \frac{d}{dx}\cdot\frac{dx}{d\theta}$$
$$= -\sin\theta \frac{d}{dx}$$

$$\therefore \quad \sin\theta\cos\theta \frac{d\Theta}{d\theta} = -\sin^2\theta\, x \frac{d\Theta}{dx} = -x\left(1-x^2\right)\frac{d\Theta}{dx}$$

Similarly, we can write

$$\frac{d^2\Theta}{d\theta^2} = \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) = -\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{dx}\right)$$
$$= -\cos\theta \frac{d\Theta}{dx} - \sin\theta \frac{d^2\Theta}{dx^2}\frac{dx}{d\theta}$$

so that

$$\frac{d^2\Theta}{d\theta^2} = -\cos\theta \frac{d\Theta}{dx} + \sin^2\theta \frac{d^2\Theta}{dx^2}$$
$$= \left(1-x^2\right)\frac{d^2\Theta}{dx^2} - x\frac{d\Theta}{dx}$$

On inserting these results in Eq. (A.7), we get

$$\left(1-x^2\right)\left\{\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - x\frac{d\Theta}{dx}\right\} - x\left(1-x^2\right)\frac{d\Theta}{dx} - \left[m^2 - \ell(\ell+1)\left(1-x^2\right)\right]\Theta = 0$$

Dividing throughout by $\left(1-x^2\right)$, we get Legendre's associated differerential equation in x:

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\left(\ell(\ell+1)-\frac{m^2}{1-x^2}\right)\Theta=0 \tag{A.8}$$

It may be pointed out here that solutions of Legendre's associated differential equation are labelled by both parameters ℓ and m and are written $P_\ell^m(x)$.

For non-negative integral values of m and ℓ, the general solution of Eq. (A.8) is given by

$$\Theta = C_1 P_\ell^m(x) + C_2 Q_\ell^m(x)$$

$$= C_1 P_\ell^m(\cos\theta) + C_2 Q_\ell^m(\cos\theta) \tag{A.9}$$

Note that $P_\ell^m(x)$ is finite for $-1 \le x \le 1$ (which conforms with $x = \cos\theta$) and $Q_\ell^m(x)$ is unbounde for $x = \pm 1$. For this reason, as a solution of the Laplace's equation, we consider $P_\ell^m(\cos\theta)$ only, and that too with integral values of m and n.

The associated Legendre polynomials are connected with Legendre polynomials through the relation·

$$P_\ell^m(x) = \left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_\ell(x) \tag{A.10}$$

where $P_\ell(x)$ is the ℓth Legendre polynomial. You must note that if $m > \ell, P_\ell^m(x) = 0$.

Orthogonality

Like Legendre polynomials, the associated Legendre polynomials $P_n^m(x)$ are also orthogonal in the range $-1 < x < 1$. Mathematically, we write

$$\int_{-1}^{1} P_n^m(x)\, P_k^m(x)\, dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{kn} \tag{A.11}$$

As before, we can expand $f(x)$ in a series of the form

$$f(x) = \sum_{k=0}^{\infty} A_k P_k^m(z) \tag{A.12a}$$

where

$$A_k = \frac{(2k+1)(k-m)!}{2(k+m)!}\int_{-1}^{1} f(x)\, P_k^m(x)\, dx \tag{A.12b}$$