# Chebyshev Polynomials - Definition and Properties

The **Chebyshev polynomials** are a sequence of orthogonal polynomials that are related to De Moivre's formula. They hav numerous properties, which make them useful in areas like solving polynomials and approximating functions.

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## Chebyshev Polynomials of the First Kind

THEOREM

The $n^{\text{th}}$ **Chebyshev polynomial of the first kind**, denoted by $T_n(x)$, is defined as

$$T_n(x) = \cos\left(n \cos^{-1} x\right),$$

or equivalently

$$T_n(\cos\theta) = \cos n\theta. \square$$

Since we know that

$$\begin{aligned}\cos 0\theta &= 1\\ \cos 1\theta &= \cos\theta\\ \cos 2\theta &= 2\cos^2\theta - 1\\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta,\end{aligned}$$

we can conclude that

$$\begin{aligned}T_0(x) &= 1\\ T_1(x) &= x\\ T_2(x) &= 2x^2 - 1\\ T_3(x) &= 4x^3 - 3x.\end{aligned}$$

EXAMPLE

Find $T_4(x)$ above.

To find $T_4(x)$, we can equivalently find a function of $\cos 4\theta$ in terms of $\theta$.

Using the cosine sum formula, we get

$$\cos 4\theta = \cos\theta\cos 3\theta - \sin\theta\sin 3\theta.$$

Recall that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Then

$$\begin{aligned}\cos\theta\cos 3\theta - \sin\theta\sin 3\theta &= \cos\theta\left(4\cos^3\theta - 3\cos\theta\right) - 3\sin^2\theta - 4\sin^4\theta \\ &= 4\cos^4\theta - 3\cos^2\theta + 3\left(1-\cos^2\theta\right) - 4\left(1-\cos^2\theta\right)^2 \\ &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

Thus,

$$T_4(x) = 8x^4 - 8x^2 + 1. \square$$

EXAMPLE

Find $T_5(x)$ above.

To find $T_5(x)$, we can, just as in the previous example, find a function of $\cos 5\theta$ in terms of $\theta$.

Using the cosine sum formula again, we get

$$\cos 5\theta = \cos\theta\cos 4\theta - \sin\theta\sin 4\theta.$$

But then we have to replace $\cos 4\theta$ with $8\cos^4\theta - 8\cos^2\theta + 1$, and then manually compute $\sin 4\theta$, and then...

Forget it. This is turning into a hopeless bash; we can't be doing this for $T_6(x)$ or $T_7(x)$, and we definitely can't easily generalize this to $T_n(x)$. We could always use De Moivre's formula, but the calculation is also very extensive.

If only there were an easier way... $\square$

How would we obtain a more general formula? In answering the previous question, most people tried to expand

$$\cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta.$$

We can easily convert the first 2 terms into the $T_n$ form. However, the issue with this approach is that $\sin n\theta\sin\theta$ is not e deal with, and will (currently) require much further expansion.

Instead, we will use the fact (from trigonometric sum and product formulas) that

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta.$$

This gives us the recurrence relation:

THEOREM

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \square$$

## Coefficients of Chebyshev Polynomials of the First Kind

The following is a table of initial values of $T_n(x)$:

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x. \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1.
\end{aligned}$$

We can make the following conjectures about these coefficients:

1. Coefficients are integers.
2. Constant term is $(-1)^k$ for $n = 2k$ and 0 for $n = 2k+1$.
3. Leading term is $2^{n-1}$.
4. Linear term of $T_{2n+1}(x)$ is $(-1)^n(2n+1)$.
5. The non-zero terms have exponents with the same parity as $n$.
6. Sum of coefficients is 1.
7. Non-zero coefficients alternate in sign.
8. The coefficient of the $x^2$ term in $T_{2k}(x)$ is $(-1)^{k+1}2k^2$.
9. The sum of the absolute value of the coefficients is $\frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n$.
10. The roots of $T_n$ are $\cos\left(\dfrac{(2k+1)\pi}{2n}\right)$, where $k \in \mathbb{Z}$.

We will prove some of these conjectures. The rest are left as exercises for the reader.

EXAMPLE

Prove conjecture 2.

We will prove this using induction. First, in the two base cases, we see that $T_0(x) = 1$ and $T_1(x) = x$, which satisfies t the constant terms are $(-1)^0$ and $0$, respectively. Now we must prove that given $T_n(x)$ has a coefficient of $0$ if $n = 2k$ and $(-1)^k$ if $n = 2k$, then $T_{n+1}(x)$ also satisfies this.

Note that the constant term can be evaluated by plugging in $x = 0$. Doing so in the recurrence relation of $T_n(x)$ give:

$$T_{n+1}(0) = 2 \times 0 \times T_n(0) - T_{n-1}(0) = -T_{n-1}(0).$$

This means that if $T_{2k}(0) = (-1)^k$, then $T_{2k+2}(0) = (-1)^{k+1}$; also, if $T_{2k+1}(0) = 0$, then $T_{2k+3}(0) = 0$, completin induction. $\square$

EXAMPLE

Prove conjecture 6.

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The sum of the coefficients of $T_n(x)$ is just $T_n(1)$. Recalling that $T_n(\cos\theta) = \cos n\theta$, we see that we want to evaluate $\cos n\theta$ when $\cos\theta = 1$. But this means $\theta = 0$, so $\cos n\theta = \cos 0 = 1$.

Therefore, $T_n(1) = 1$ and we are done. $\square$

EXAMPLE

Prove conjecture 10.

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We desire to find the roots $x$ of $T_n(x) = 0$.

Substituting $x = \cos\theta$, we want to instead find the roots of

$$T_n(\cos\theta) = \cos n\theta = 0.$$

This happens at $n\theta = \dfrac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$.

Thus,

$$\theta = \frac{\pi}{2n} + \frac{k\pi}{n} = \frac{(2k+1)\pi}{2n}.$$

But this means

$$x = \cos\theta = \cos\left(\frac{(2k+1)\pi}{2n}\right),$$

so we are done. $\square$

## Problems Involving Chebyshev Polynomials of the First Kind

EXAMPLE

Show that if $r$ is a rational number such that $\cos(r\pi)$ is rational, then $\cos(r\pi) \in \{0, \pm\frac{1}{2}, \pm 1\}$.

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TRY IT YOURSELF

$$\left[\frac{d}{d(\cos x)}\left(\sum_{n=1}^{100} \cos nx\right)\right]_{x=2\pi} = ?$$

Submit your answer

## Chebyshev Polynomials of the Second Kind

Because we have a function relating $\cos\theta$ to $\cos n\theta$, it makes sense to suspect that there is such a function for $\sin\theta$ and too. Indeed, the Chebyshev polynomials of the second kind are exactly this:

THEOREM

The $n^{\text{th}}$ **Chebyshev polynomial of the second kind**, denoted by $U_n(x)$, is defined by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta} \qquad \square$$

## Coefficients of Chebyshev Polynomials of the Second Kind

## Additional Facts

1. $T_n(-x) = (-1)^n T_n(x)$
2. $T_{2n}(0) = (-1)^n$
3. $T_n(1) = 1$
4. $T_{2n+1}(0) = 0$
5. $T_n(-1) = (-1)^n$
6. Prove that

$$T_n(x) = \frac{\left(x - \sqrt{x^2-1}\right)^n + \left(x + \sqrt{x^2-1}\right)^n}{2}.$$

7. Prove that

$$T_n(x) = \frac{(-2)^n n!}{(2n)!} \sqrt{(1-x^2)} \frac{d^n}{dx^n} \left(1-x^2\right)^{\frac{n-1}{2}}.$$

8. Show that the generating function for $T_n(x)$ is given by

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=1}^{\infty} T_n(x) t^n.$$

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