Choosing the threshold in extreme value analysis
EVA 2025

Léo Belzile, HEC Montréal (joint work with Sonia Alouini and Anthony Davison)

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Motivation: peaks over threshold

In the simplest applications:

1. we choose a high threshold u

   - equivalently, choose the number of upper order statistics n_u with threshold $u = X_{(n-n_u)}$.

2. we fit the limiting generalized Pareto distribution with scale σ_u and shape ξ to exceedances $X - u$ over threshold u.

   - if the shape $\xi > 0$, we can use Hill's estimator

3. we use the resulting model for extrapolation beyond u.

Threshold selection

Bias and variance trade-off:

- taking u too high will mean that the number of exceedances n_u is small (increase uncertainty)

- taking u too low increases the risk of biased extrapolation.

Since the limit holds as u increases to the upper endpoint of the support of X, we must use a so-called intermediate sequence $n_u/n \to 0$ as $n_u \to \infty$ for consistency.

Limiting distribution and threshold stability

Limiting distribution of threshold exceedances is generalized Pareto. **If** the limit was exact and $X - u \sim \mathrm{GPD}(\sigma_u, \xi)$

- For a higher threshold $v > u$, then

$$X - v \mid X > v \sim \mathrm{GPD}\{\sigma_v = \sigma_u + \xi(v - u), \xi\}.$$

- Provided $\xi > -1$, the expected value of exceedances is

$$\mathsf{E}(X - u) = \sigma_u/(1 + \xi).$$

Why another survey paper?

There are earlier reviews of the topic, but the literature keeps increasing…

- The most comprehensive reviews are Scarrott and MacDonald (2012), Caeiro and Gomes (2016) and Langousis et al. (2016).

- Selective numerical comparisons in Gomes and Oliveira (2001), Murphy, Tawn, and Varty (2025) and Schneider, Krajina, and Krivobokova (2021), among others.

Some problems with (most methods)

Conditional model: only consider data above the threshold.

Consider candidate thresholds $u_1 < \cdots < u_K$.

Resulting problems:

- Dependence between estimates, test statistics, P-values due to sample overlap.

- Non-nested models.

- Multiple testing problem.

- Potential for automation.

- Non-stationarity (dependence on covariate, change over time).

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Objective

We provide an extensive review of threshold selection mechanisms for peaks over threshold analysis, including

- visual diagnostics,

- extended generalized Pareto models,

- goodness-of-fit tests,

- semiparametric methods based on Hill's estimator (not covered in this presentation due to time constraints),

- etc.

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How to benchmark methods?

What makes a threshold procedure good? In practice, we care about the extrapolation, often

- a high quantile (return level), or

- a probability of exceedance.

Benchmarking the method based on proximity with the asymptotic shape parameter is **not** a good point of reference.

Scale and shape parameters are negatively correlated.

Penultimate effects

Smith (1987) show that a better approximation is obtained by letting the shape vary with u, with $\xi_t = r'(u)$, where

$$r(t) = \{1 - F(t)\}/f(t)$$

is the reciprocal hazard function.

mev package: `smith.penult(family = "norm", method = "pot", qu = c(0.9, 0.95, 0.99))`

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Illustration of penultimate effects: normal example

sampling distribution of shape for MLE based on 1000 exceedances of standard normal

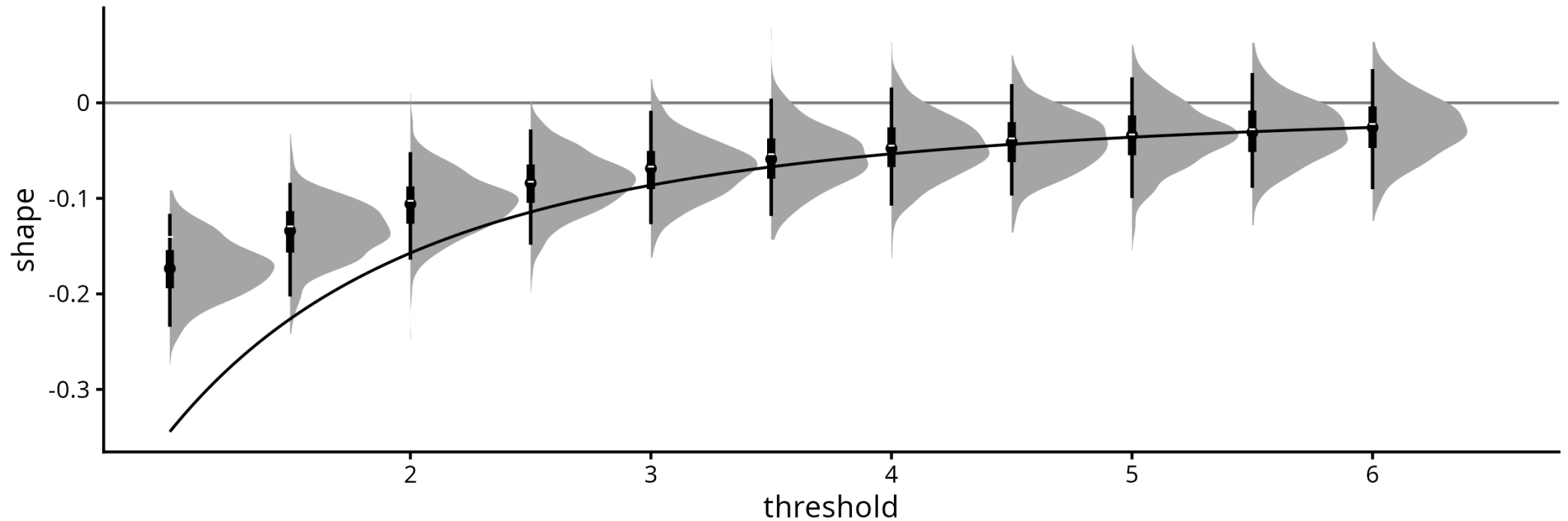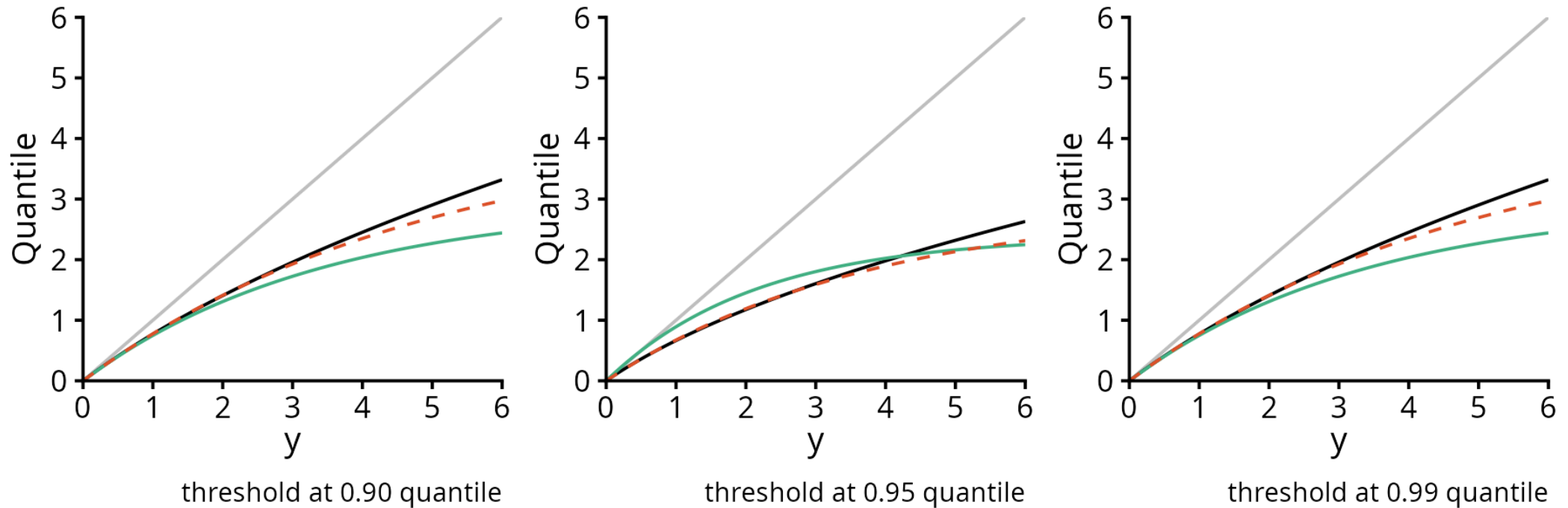# Illustration of penultimate effects: normal quantiles

threshold at 0.90 quantile   threshold at 0.95 quantile   threshold at 0.99 quantile

Conditional quantiles above threshold, rescaled. $45°$ line shows limiting exponential with true conditional (black), fitted GPD with 1M observations (dashed red) and penultimate (green).

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Visual diagnostics on Padova data

Mean residual life (Davison and Smith 1990), parameter stability and Hill plots.
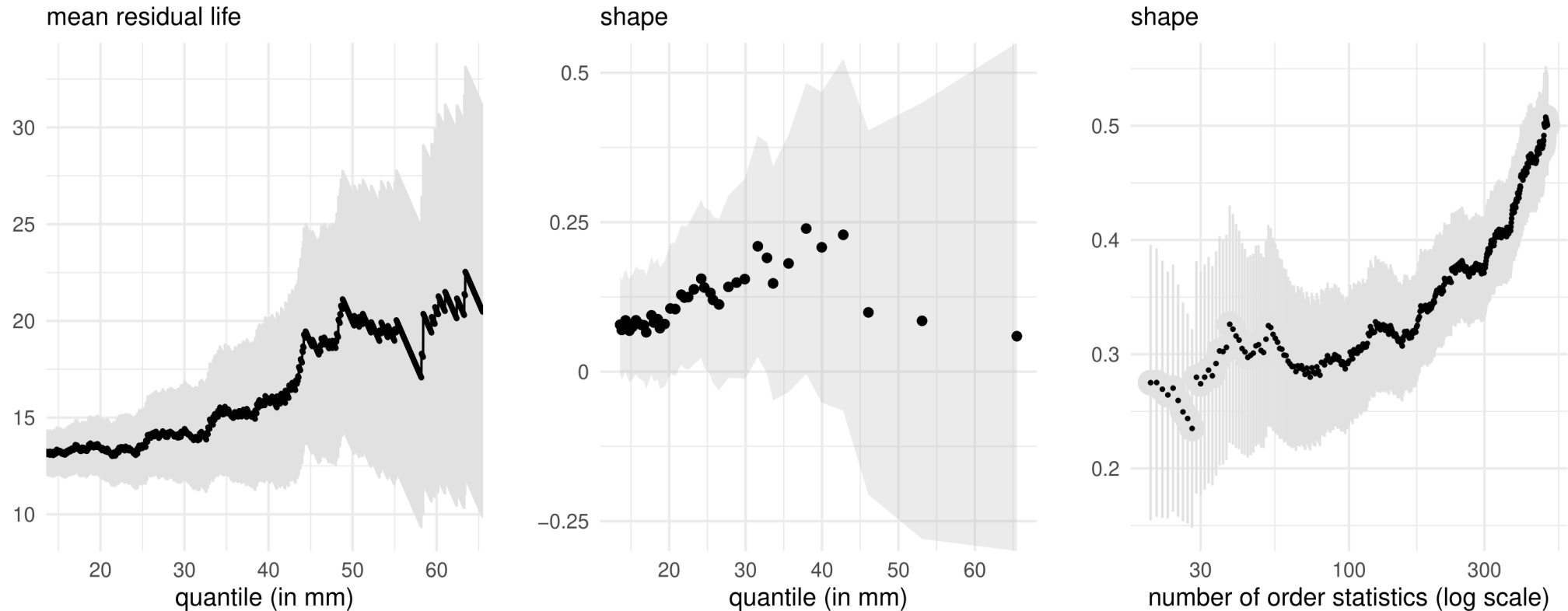
Which threshold would you choose?

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Caveats

- Difficulty in automating selection (need visual inspection); proposals in Langousis et al. (2016) or Danielsson et al. (2019), but both fall short.

- Problems: pointwise confidence intervals.

- Underlying assumptions affected by penultimate effects.

- The plots say nothing about goodness-of-fit!

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Stability?
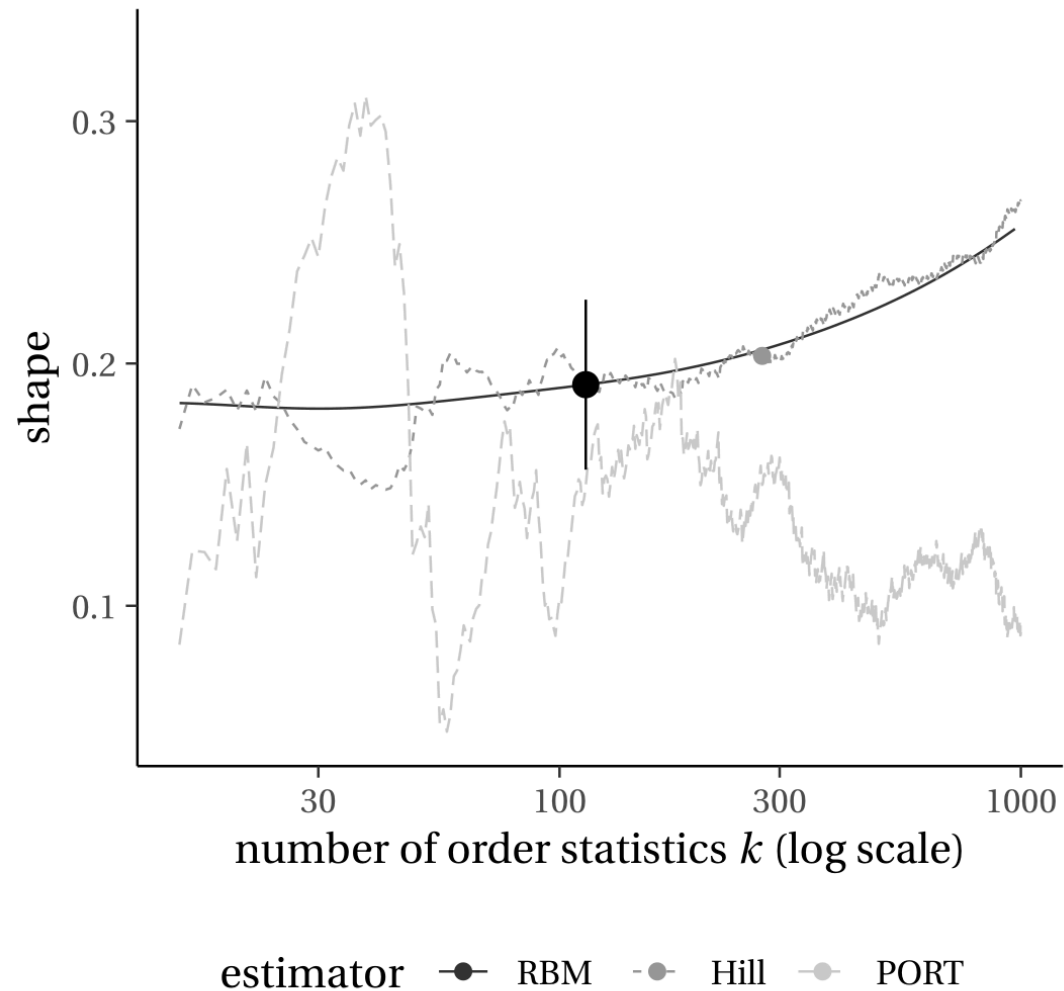
Hill, PORT and random block maxima ([Wager 2014]) estimates as a function of n_u.

Generalized Pareto model extensions

Build extended models with additional parameters (with continuity constraints) and test for equality of shape.

Some model are inspired by penultimate approximations (GPD-like, but with varying shape).

Northrop and Coleman (2014): piecewise generalized Pareto model coupled with score tests.

- tests are dependent because the data are re-used; no control of overall error rate.

- the power of the test depends on the choice of thresholds and particularly on the largest threshold u_K.

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Extended generalized Pareto models

Embed generalized Pareto $F(x; \sigma, \xi)$ in a more flexible model using a continuous distribution function G_κ on $[0, 1]$. The EGP(σ, ξ, G_κ) distribution function is then

$$\Pr(X \leq x) = G_\kappa\{F(x; \sigma, \xi)\}.$$

Choose G to keep the tail properties. See the chapter of P. Naveau (2025) for a recent review.

- Papastathopoulos and Tawn (2013): models imply the density at the origin is zero.

- Gamet and Jalbert (2022) propose two models, but one leads to non-regular asymptotics (restriction on boundary of the parameter space).

- Philippe Naveau et al. (2016) additional models with two parameters.

Extended generalized Pareto models

Test for restriction to generalized Pareto sub-model using likelihood ratio tests (profile).

- Could allow for a bit more data to be included, at the expense of additional parameters to estimate (and potentially more variability).

- Same problems as parameter stability plots (sequential tests, overlapping data).

Splicing models

Glue a distribution for the bulk with one for the tail using a mixture of disjoint components below u (bulk) and above u (generalized Pareto).

See Scarrott and MacDonald (2012) and Hu and Scarrott (2018) for reviews.

If we build u as a parameter of the model, then

- it leads to sample contamination, as fit of the GPD may be driven by bulk.

- the profile likelihood for threshold u is non-monotone.

- Bayesian version with random threshold (Nascimento, Gamerman, and Lopes 2012) accounts for uncertainty.

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Goodness-of-fit measures

1. Fit a generalized Pareto distribution at each candidate threshold.

2. Compute either

   - a suitable statistic which indicates departure from the posulated distribution.

   - a measure of discrepancy between empirical distribution of exceedances above u and generalized Pareto model (via Kolmogorov–Smirnov, Cramér–von Mises, etc.)

3. Perform tests sequentially until rejection, or select the "best" threshold according to the criterion.

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Some proposals

- Idea dates back to Pickands (1975).

- Choulakian and Stephens (2001), Bader, Yan, and Zhang (2018) (using ForwardStop).

- Recent proposals using L-moment estimators including Kiran and Srinivas (2021), Solari et al. (2017), Silva Lomba and Fraga Alves (2020).

- Some *ad hoc* proposals (Thompson et al. 2009) work well in practice.

Note: goodness-of-fit tests null distributions require adjustment for rounded values (estimate null via Monte Carlo).

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Sequential analysis

Wadsworth (2016) obtains asymptotic joint distribution of MLE from a superposition of Poisson processes.

Build independent increments of shape to form a white noise sequence $\xi_i^* = (\hat{\xi}_{u_{i+1}} - \hat{\xi}_{u_i})/\{(I_{u_{i+1}}^{-1} - I_{u_i}^{-1})_{\xi,\xi}^{1/2}\}$.

- Parameter stability plots with simultaneous confidence intervals!

- White noise test for ξ_i^*, with alternative hypothesis $\mathscr{H}_a : \xi_i^* \sim \mathsf{Normal}(\beta, \sigma)(i = 1, \ldots, j-1)$ and $\xi_i^* \sim \mathsf{Normal}(0, 1)$ for $j, \ldots, k-1$ motivated by results on model misspecification (White 1982).

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Comments on Wadsworth (2016)

Fails 17% of the time in our simulations with equally spaced quantiles.

- Problems with positive definiteness of information matrices for shape increments.

- Method sensitive to rounding (add noise?)

- Must choose the threshold sequence carefully.

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Predictive distribution

Northrop, Attalides, and Jonathan (2017) propose a Bayesian method based on leave-one-out cross validation with a binomial-generalized Pareto (BGP) model and a single validation threshold $v > u_k$ above which we assess the model performance.

The measure of goodness-of-fit proposed is an estimate of the negated Kullback–Leibler divergence,

$$\widehat{T}_v(u_j) = \sum_{i=1}^{n} \log \widehat{f}_v(x_r \mid \boldsymbol{x}_{-r}, u_j).$$

The selected threshold is the one maximizing this diagnostic.

Can use Bayesian model averaging to account for the uncertainty originating from threshold selection.

Metric-based diagnostic

Build quantile-quantile (QQ-) plot, with pointwise confidence intervals obtained using a parametric bootstrap.

For each bootstrap sample $b = 1, \ldots, B$, estimate the empirical quantile function F_b and evaluate it at the plotting position $p_i = i/(n+1)$ for $i = 1, \ldots, n$.

Varty et al. (2021) and Murphy, Tawn, and Varty (2025) propose repeating this with simulated iid data from F_0 (tolerance intervals).
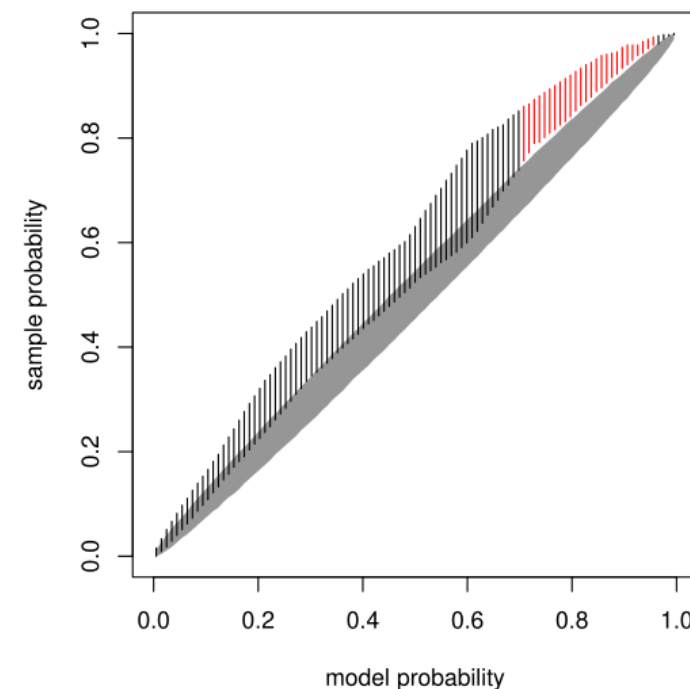

Figure from Varty et al. (2021)

Metric-based adjustment

Build metric based on exponential (or generalized Pareto) **quantile** $F_0^{-1}(p_i)$ against $F_b^{-1}(p_i)$ with mean absolute difference or mean squared difference.

Pick the threshold with the smallest average distance.

- amenable to different sampling schemes (censoring, non-identically distributed data, time-varying thresholds).

- computationally-intensive by design.

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Simulation study: comparison of methods

We considered 13 different distributions from simulation studies in Choulakian and Stephens (2001) and Schneider, Krajina, and Krivobokova (2021).

- Consider 1000 replications of IID data, data with serial correlation in the tail, rounded observations, with different tail behaviour.

- Data of size $n \in \{1000, 2000\}$ with candidate thresholds at the sample $\{0.8, 0.81, \ldots, 0.99\}$ quantiles, keeping a minimum of 20 exceedances.

- Evaluate bias, variance, RMSE of point estimator for 0.999 quantile.

- Compare to oracle (model with the closest quantile estimate among candidates).

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Which quantile level on average?

threshold selection method across all simulation scenarios

mean percentile level
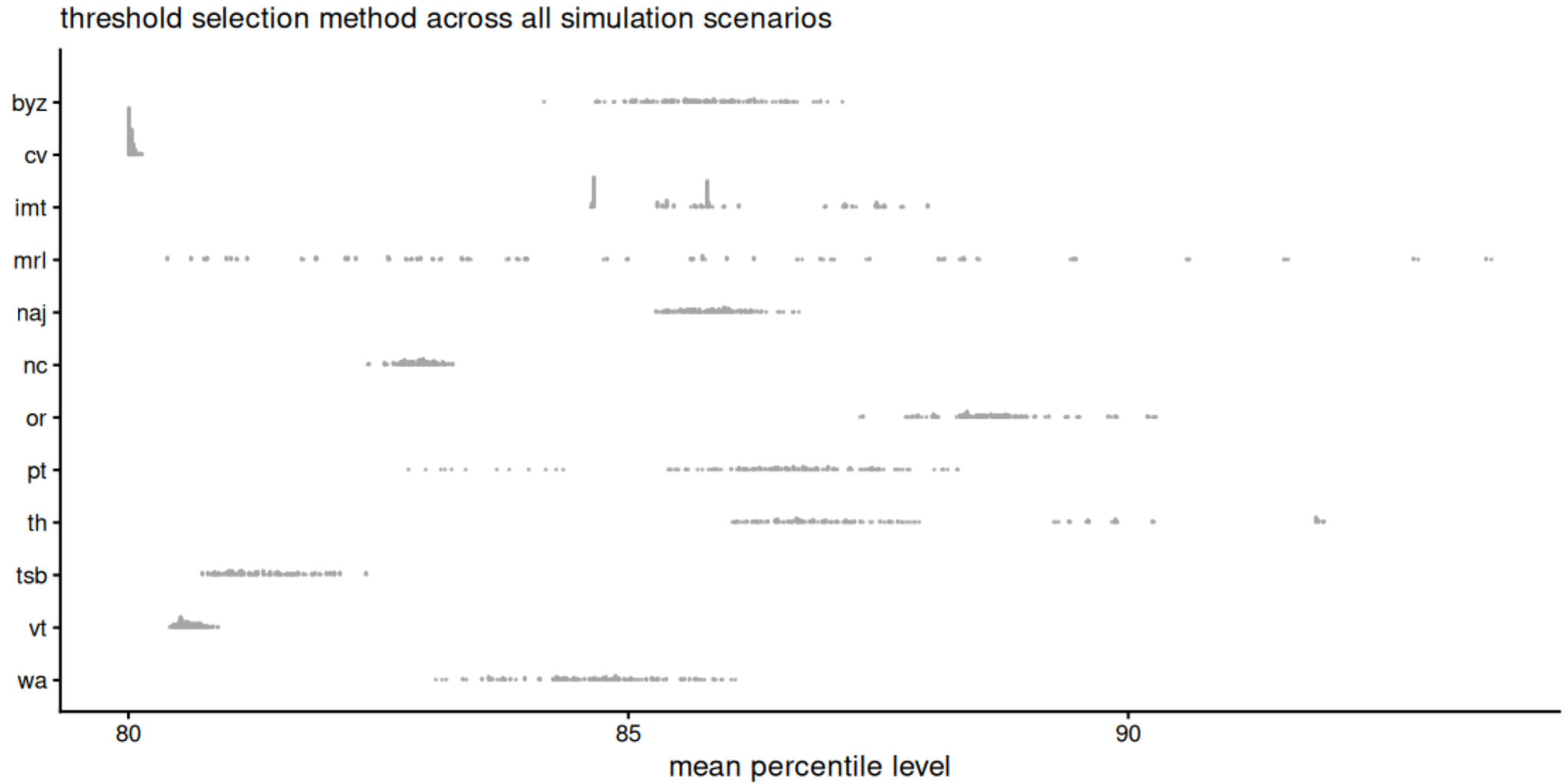
Findings

- No universally better method.

- Many of the automated procedures return the lowest possible threshold.

- Using Forward stop method to account for multiple testing leads to thresholds that are much lower.

More comments

- The oracle method returns average threshold around the 87% and the 90% quantile.

- Mean residual life plots are uniformly scattered.

- Varty et al. (2021)'s metric diagnostic leads to very small thresholds.

- Northrop, Attalides, and Jonathan (2017) and Thompson et al. (2009) lead to a greater variability of selected quantile levels for the thresholds but are variable.

- Wadsworth's sequential testing performs best with heavy tailed distributions, but otherwise is not competitive in the rankings.

Conclusions and future work

Is the problem well-formulated? There is no "correct" threshold, so are we barking up the wrong tree?

Some alternatives:

- Weighting with different threshold choices to account for uncertainty (Stein 2023).

- Should we be fitting sub-asymptotic models to much more data?

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Question period

Thank you for your attention and thanks to the conference organizers.

Preprint coming (hopefully) soon on arXiv.

See you in Montréal, July 5th-9th, 2027!

Thanks to NSERC for funding

Slides at `lbelzile.github.io/EVA2025-choosing-threshold`

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Semiparametric methods

The paper also compares 18 different semiparametric methods using Hill-type estimators for heavy-tailed data.

- Careful: many numerical implementations don't specify a minimum sample size!

- Primer: best methods include Caeiro and Gomes (2016), Gomes, Figueiredo, and Neves (2012), Wager (2014).

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Don't use the following

- Methods based on minimization of the asymptotic mean squared error can break down catastrophically for particular data sets.

- Methods by Gomes et al. (2013) and Hall and Welsh (1985) behave erratically with small shape parameters: these procedures lead to strongly biased shape parameter estimates. This is due to them keeping more than 15% of the data for inference.

- Drees and Kaufmann (1998) bias-reduction method leads to large width of confidence intervals (unwanted variability), an

- Many other methods are extremely variable.

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Testing using maximum likelihood distribution

Thompson et al. (2009) propose using constant values

$$\tau_j = \widehat{\sigma}_j - \hat{\xi}_j u_j$$

and performing Pearson's test of normality for the differences $\tau_{j+1} - \tau_j$ ($j = 1, \ldots, k-1$), stopping whenever the hypothesis is rejected at level $\alpha = 0.2$.

Ad hoc proposal... but works well in simulations.

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