Statistical modelling

#1.b Central limit theorem

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Null distribution

When we perform an hypothesis test, we need to know the behaviour of the statistic under the null hypothesis in order to draw a conclusion (reject/fail to reject $\mathscr{H}_0$)

The test statistic is often

- a Wald statistic (mean or maximum likelihood estimator)

Under regularity conditions and for n sufficiently large, the null distribution is approximately normal. Why?

Central Limit Theorem (informal)

Let $Y_1, \ldots, Y_n$ be a random sample from a distribution with

- expectation μ,
- (finite) variance σ^2.

If n is large, the mean $\overline{Y}_n$ approximately follows a normal distribution with mean μ and variance σ^2/n.

$$\overline{Y}_n \stackrel{\cdot}{\sim} \mathsf{No}(\mu, \sigma^2/n)$$

Central Limit Theorem (formal)

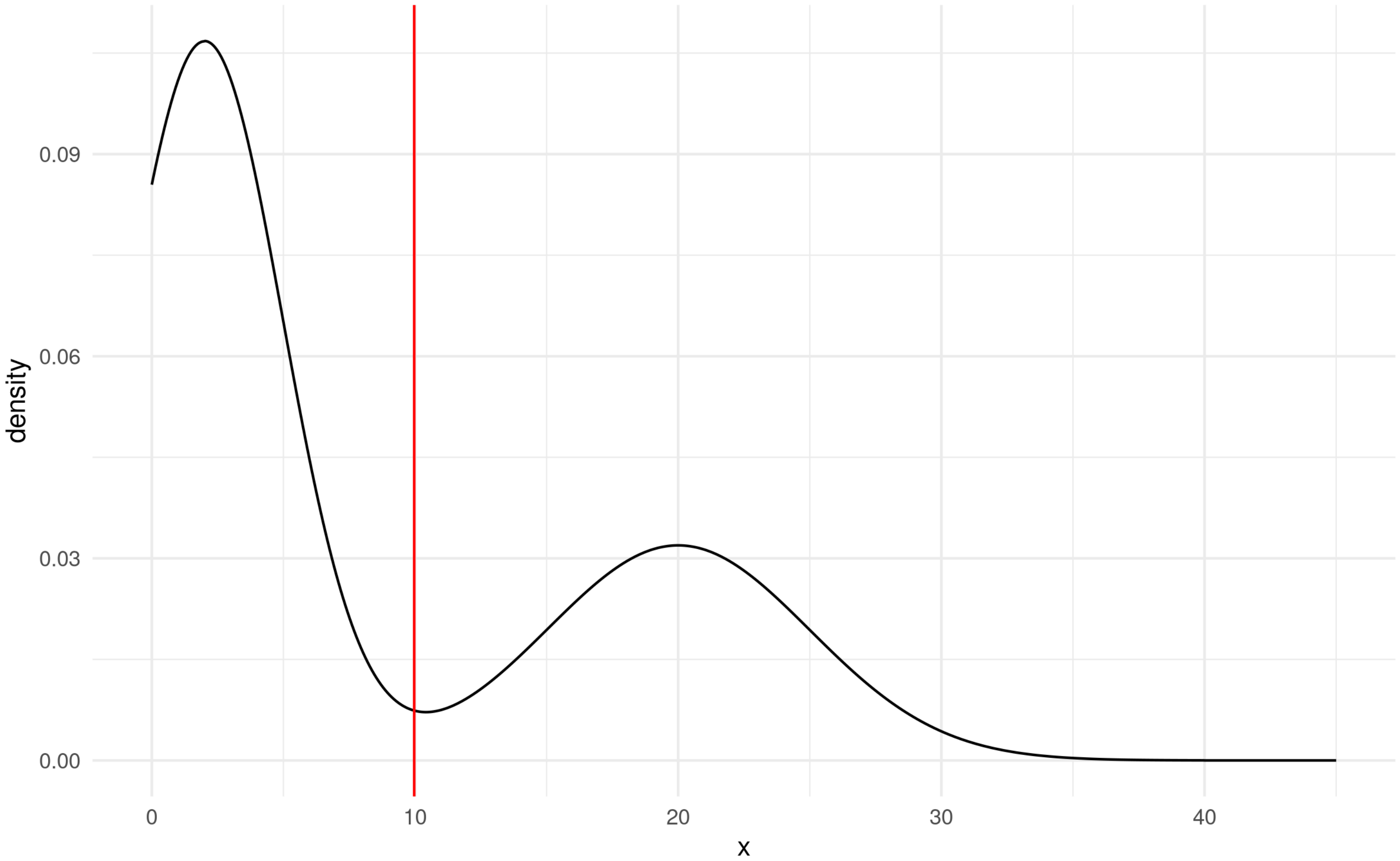
Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables with distribution F and finite variance and let $\overline{Y}_n = n^{-1} \sum_{i=1}^n Y_i$ denote the sample mean.

For any $y \in \mathbb{R}$, the mean converges in distribution to a normal distribution, i.e.,

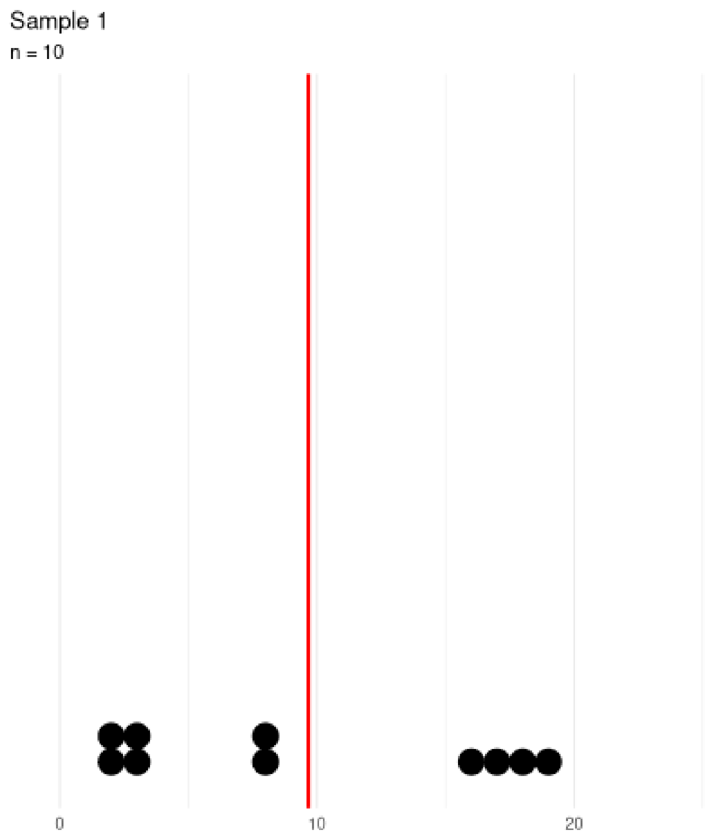
$$\lim_{n \to \infty} \mathsf{P}\left(\sqrt{n}\frac{\overline{Y}_n - \mu}{\sigma} \leq y\right) = \Phi(y)$$

where $\Phi(y)$ is the distribution function of $\mathsf{No}(0,1)$.

Let's represent graphically the central limit theorem by drawing samples repeatedly from the following distribution (left truncated, multimodal, etc.)

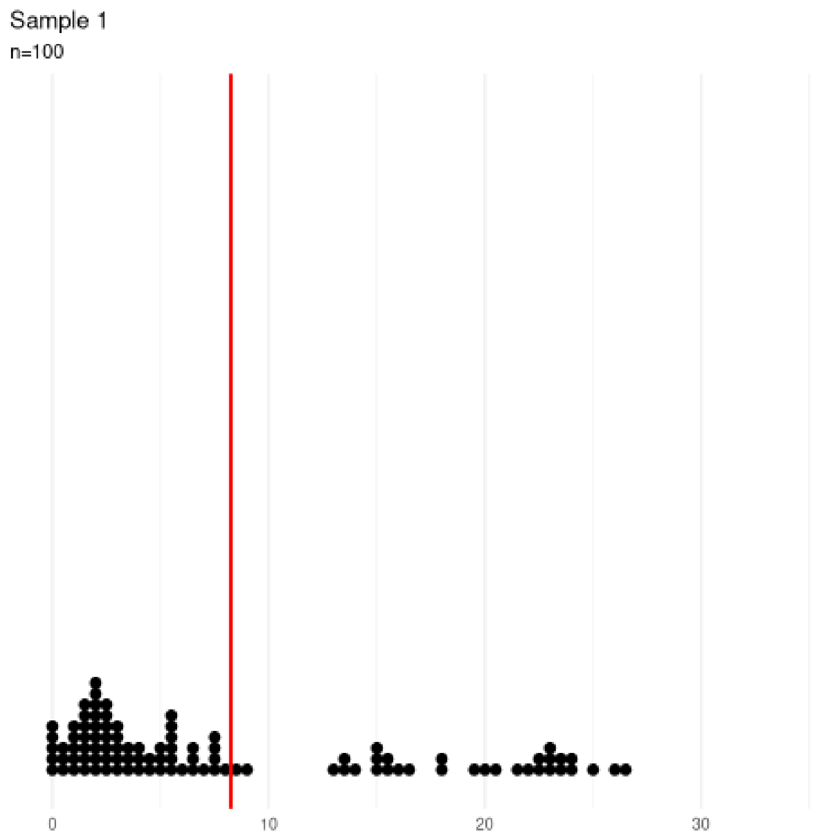


Let's draw $\mathbf{20}$ random samples of size $\boldsymbol{n} = \mathbf{10}$.



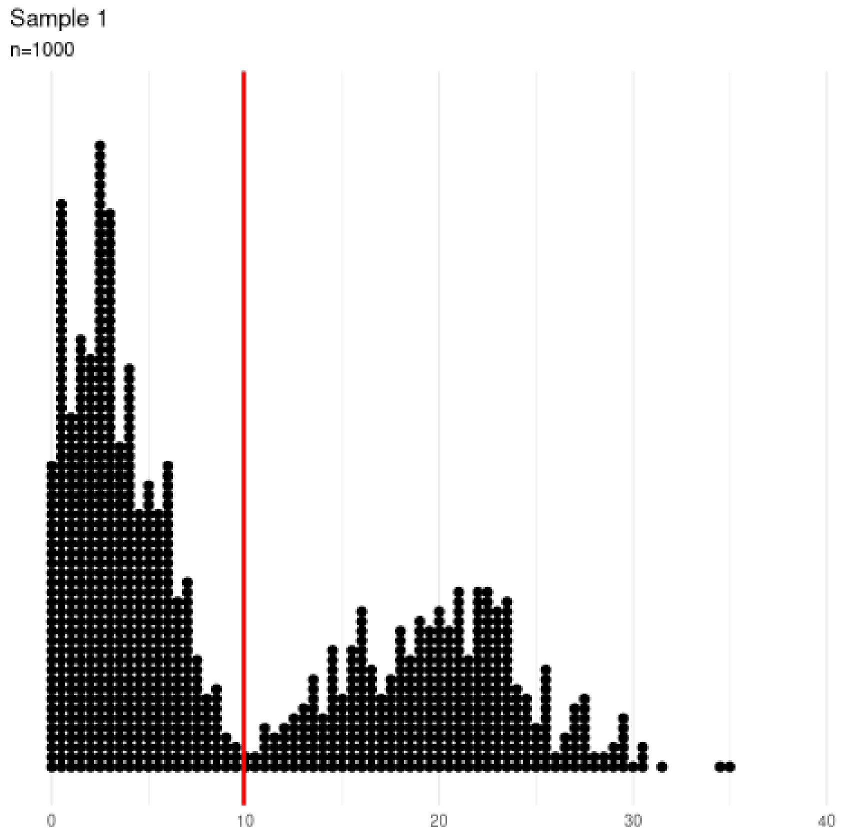
Dot plot of random sample of size $\boldsymbol{n} = \mathbf{10}$ and sample mean (vertical red line)

If we increase the sample size to $n = 100$, the variability of the sample mean decreases.



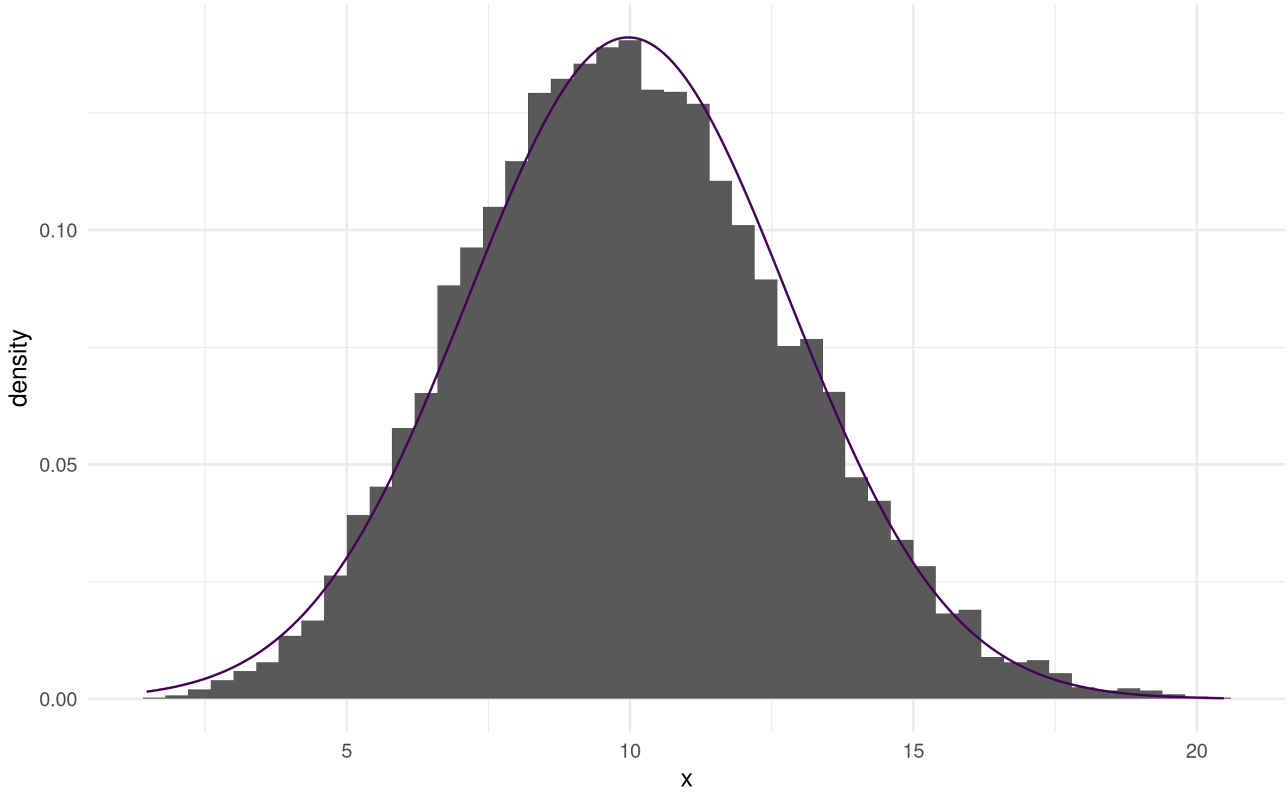
Dot plot of random sample of size $n = 100$ and sample mean (vertical red line)

Same thing, this time with $n = 1000$ observations per sample.



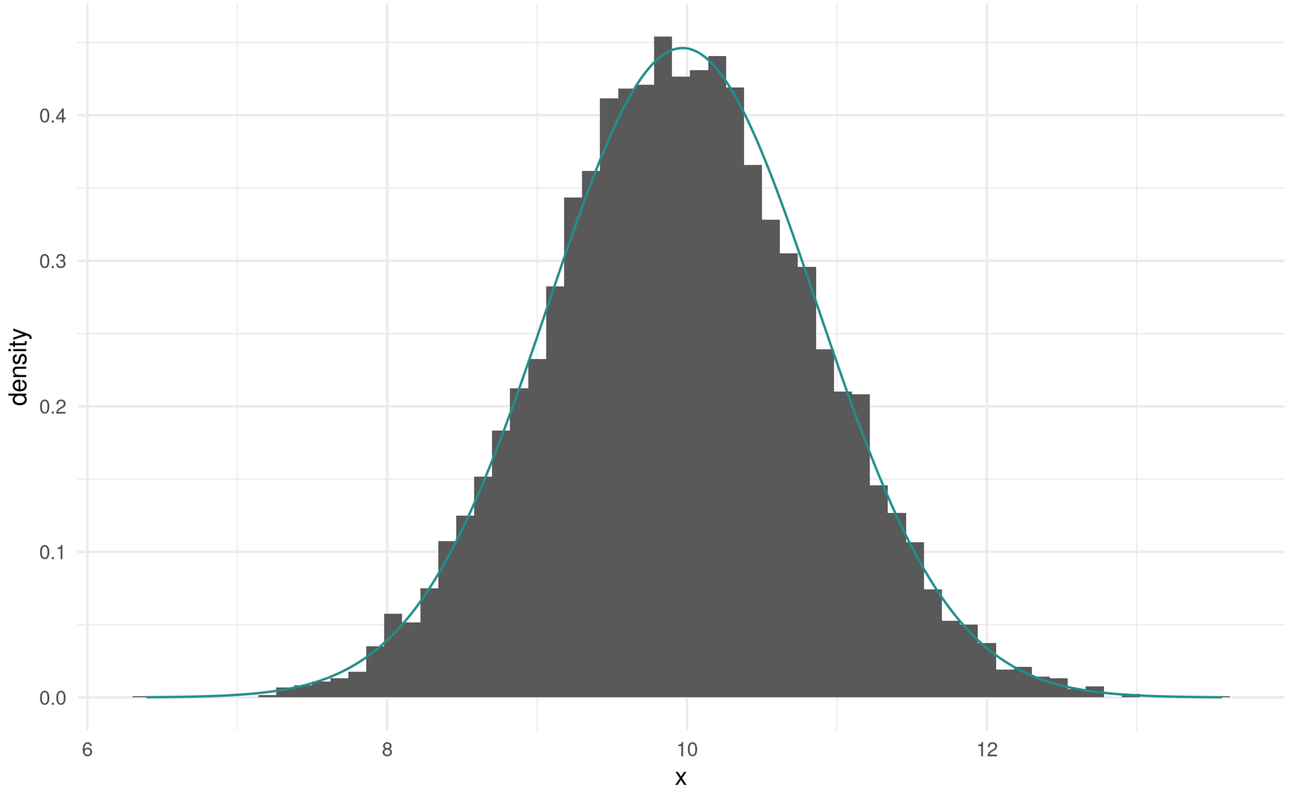
Dot plot of random sample of size $n = 1000$ and sample mean (vertical red line)

If we draw an histogram of the means (vertical red lines), what do we obtain?



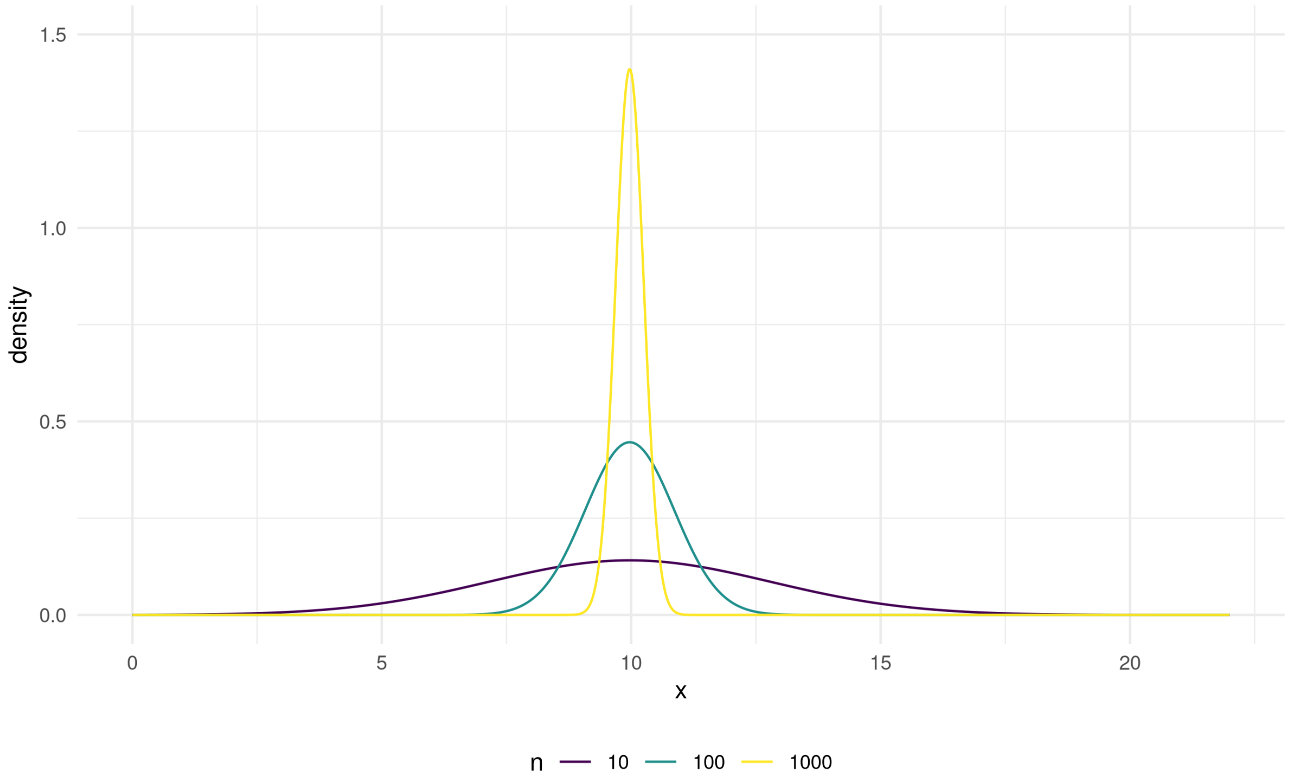
Histogram of the empirical distribution of sample means of $n = 10$ observations and CLT normal approximation.

The quality of the CLT approximation improves when the sample size n increases



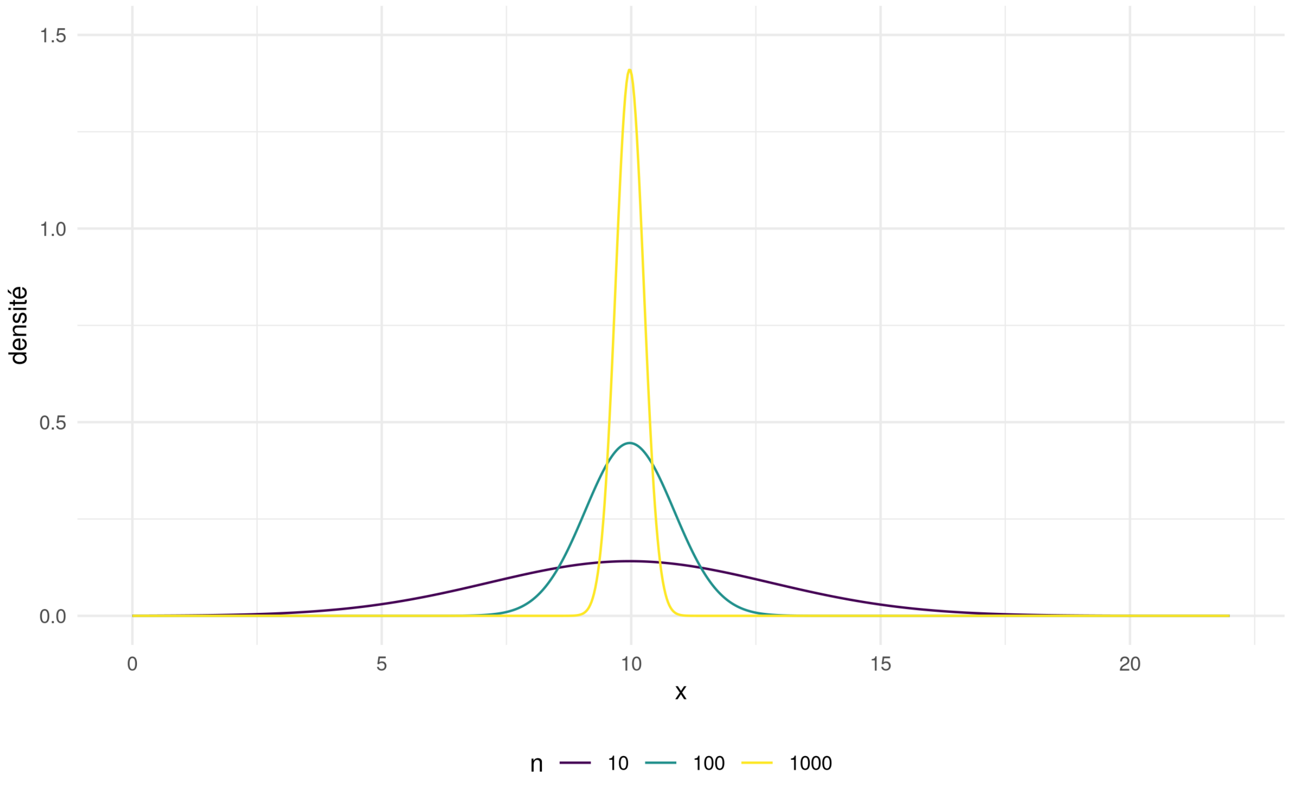
Histogram of the empirical distribution of sample means of $n = 100$ observations and CLT normal approximation.

Convergence is faster near the mean than in the tails of the distribution.



Histogram of the empirical distribution of sample means of $n = 1000$ observations and CLT normal approximation. observations.

The variance of the sample mean $\overline{Y}_n$ when $\mathsf{Va}(Y_i) = \sigma^2 (i = 1, \ldots, n)$ is roughly σ^2/n.



Normal approximation of the mean for different sample sizes.