# MATH 60604A
# Statistical modelling
# § 6e - Random slope model

Léo Belzile

HEC Montréal
Department of Decision Sciences

## Model formulation

We consider a linear mixed model with a random slope and a random intercept for the revenge data, of the form

$$\boldsymbol{Y}_i \mid \boldsymbol{\mathcal{B}}_i = \boldsymbol{b}_i \sim \mathsf{No}_5\left(\mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\boldsymbol{b}_i, \sigma^2\mathbf{I}_5\right)$$
$$\boldsymbol{\mathcal{B}}_i \sim \mathsf{No}_2(\mathbf{0}_2, \boldsymbol{\Omega})$$

where $\mathbf{Z}_i = [\mathbf{1}_5, \texttt{time}_i]$ is a $5 \times 2$ model matrix for the random effects and $\boldsymbol{\Omega} = \left(\begin{smallmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{smallmatrix}\right)$.

The columns of $\mathbf{Z}_i$ typically include as covariates

- time or
- indicators for categorical variables (group effect).

# Random effects on the predictor variables

Suppose the matrix $\mathbf{Z}_i = [\mathbf{1}_{n_i}, \mathbf{X}_{1i}]$.

$$Y_{ij} = (\beta_0 + b_{0i}) + (\beta_1 + b_{1i})X_{ij1} + \beta_2 X_{ij2} + \cdots + \beta_p X_{ijp} + \varepsilon_{ij}.$$

- The conditional effect of the variable $X_1$ for group $i$ is $\beta_1 + b_{1i}$
- The parameter $\beta_1$ is the "slope" of $X_1$ averaged over the entire population.
- $\beta_1 + b_{1i}$ is the effect of $X_1$ specific to group $i$.

# Covariance of the response

- The covariance matrix of $Y_{ij}$ depends on the predictors in $\mathbf{Z}_i$ which have random effects.
- For example, if $\mathbf{Z}_i = [\mathbf{1}_{n_i}, \mathbf{X}_{1i}]$, the marginal variance of $Y_{ij}$ is

$$\mathrm{Var}\left(Y_{ij} \mid \mathbf{X}_i\right) = \omega_{11} + \mathrm{X}_{ij1}^2 \omega_{22} + 2\mathrm{X}_{ij1}\omega_{12} + \sigma_\varepsilon^2.$$

- With independent errors, the covariance between two observations in the same group is

$$\mathrm{Cov}\left(Y_{ij}, Y_{ik} \mid \mathbf{X}_i\right) = \omega_{11} + \mathrm{X}_{ij1}\mathrm{X}_{1ik}\omega_{22} + (\mathrm{X}_{ij1} + \mathrm{X}_{1ik})\omega_{12}.$$

- It may be difficult to estimate parameters if the errors has a complex covariance structure (not to mention computational costs).

## SAS code for random slope model

```
proc mixed data=statmod.revenge;
model revenge = sex age vc wom t
    / ddfm=kenwardroger solution;
random intercept t / subject=id type=un v=1 vcorr=1;
run;
```

The output includes information about the number of covariance parameters, the number of random effects, etc.

| Dimensions | |
|---|---|
| **Covariance Parameters** | 4 |
| **Columns in X** | 6 |
| **Columns in Z per Subject** | 2 |
| **Subjects** | 80 |
| **Max Obs per Subject** | 5 |

# Covariance matrix of response

| Covariance Parameter Estimates | | |
|---|---|---|
| **Cov Parm** | **Subject** | **Estimate** |
| **UN(1,1)** | id | 0.3064 |
| **UN(2,1)** | id | -0.05268 |
| **UN(2,2)** | id | 0.01730 |
| **Residual** | id | 0.2055 |

| Estimated V Matrix for Subject 1 | | | | | |
|---|---|---|---|---|---|
| **Row** | **Col1** | **Col2** | **Col3** | **Col4** | **Col5** |
| 1 | 0.4239 | 0.1830 | 0.1476 | 0.1122 | 0.07682 |
| 2 | 0.1830 | 0.3704 | 0.1468 | 0.1287 | 0.1106 |
| 3 | 0.1476 | 0.1468 | 0.3515 | 0.1452 | 0.1444 |
| 4 | 0.1122 | 0.1287 | 0.1452 | 0.3672 | 0.1782 |
| 5 | 0.07682 | 0.1106 | 0.1444 | 0.1782 | 0.4175 |

- The variance of the random intercept is $\omega_{11} = 0.3064$
- The variance of the random slope is $\omega_{22} = 0.01730$
- The correlation between the random effects is $-0.72$.

# Testing for correlation between random effects

- We can test whether $\mathcal{H}_0 : \omega_{12} = 0$ versus $\mathcal{H}_a : \omega_{12} \neq 0$ by fitting the model with diagonal covariance and performing a likelihood ratio test (REML, since they have the same fixed effects)
  - in SAS, change `type=un` to `type=vc` (default option)
  - the test statistic is $R = 8.98$
  - its null distribution is $\chi_1^2$ (regular problem, covariance can be negative)
  - the $p$-value is 0.002:
  - the correlation between the random effects is strongly significant.

# Model comparison

- We can do similar comparisons with the random intercept-only model,
  - this corresponds to $\mathscr{H}_0 : \omega_{12} = \omega_{22} = 0$
  - the asymptotic null distribution is $\frac{1}{2}\chi_1^2 + \frac{1}{2}\chi_2^2$
  - but this approximation is poor …probably better to use information criteria.