MATH 60604A
Statistical modelling
§ 6c -Linear mixed models

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Introduction to random effects models

Random effects give another way of accounting for within-group correlation and allows prediction of group-level effects in addition to population-level effects.

- The main characteristic of the linear mixed model is to allow certain variables to have random effects, i.e., to have parameters that vary from one group to another (from one person to another in repeated measures data).
- While each group is allowed an individual effect, the overall average of these effects is zero.

Random effects models

- When an explanatory variable is modeled with a random effect, we assume that the total effect of this variable is a combination of
  1. a common effect for the entire population and
  2. a within-group effect.
- For example, when considering repeated measures from the same individuals, the effect of a variable can be split into a common effect for all individuals in the population, and a unique effect for each individual.
- In the example of worker motivation, the effect of years of service could be split up into a common effect for all employees (in all units) and a unique effect in each unit for employees.

Linear mixed model as hierarchical model

The linear mixed effect model is

$$\boldsymbol{Y}_i \mid \boldsymbol{\mathcal{B}}_i = \boldsymbol{b}_i \sim \mathsf{No}_{n_i}\left(\mathbf{X}_i\beta + \mathbf{Z}_i\boldsymbol{b}_i, \mathbf{R}_i\right)$$
$$\boldsymbol{\mathcal{B}}_i \sim \mathsf{No}_q(\mathbf{0}_q, \boldsymbol{\Omega})$$

- The response for group i, $\boldsymbol{Y}_i$ follows a multivariate normal distribution given **random effects $\boldsymbol{b}_i$**.
- We term the coefficients β associated to the model matrix $\mathbf{X}_i$ **fixed effects**.

Linear mixed models: fixed effects

We can write the linear mixed model as

$$[\boldsymbol{Y}_i \mid \boldsymbol{\mathcal{B}}_i = \boldsymbol{b}_i] = \mathbf{X}_i\beta + \mathbf{Z}_i\boldsymbol{b}_i + \varepsilon_i, \qquad i = 1, \ldots, m.$$

where

- $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{in_i})^\top$ is the n_i vector of responses of group i.
- $\mathbf{X}_i$ is the $n_i \times (p+1)$ matrix of explanatory variables for group i, whose ith row is $\mathbf{X}_{ij} = (1, \mathrm{X}_{ij1}, \ldots, \mathrm{X}_{ijp})^\top$.
  - The first column correspond to the intercept and all its entries are one.
  - The other columns of $\mathbf{X}_i$ each represent an explanatory variable.
- β is a $(p+1)$ vector of **fixed** effects parameters.

Linear mixed models: random effects

We can write the model

$$[\boldsymbol{Y}_i \mid \boldsymbol{\mathcal{B}}_i = \boldsymbol{b}_i] = \mathbf{X}_i\beta + \mathbf{Z}_i\boldsymbol{b}_i + \varepsilon_i, \qquad i = 1, \ldots, m.$$

where

- $\mathbf{Z}_i$ is a $n_i \times q$ matrix consisting of a subset of the columns of $\mathbf{X}_i$.
  - The columns of $\mathbf{Z}_i$ are those of the variables with **random effects**.
  - If there are no random effects, $q = 0$ and we retrieve the linear model.
- $\boldsymbol{\mathcal{B}}_i = \boldsymbol{b}_i$ is a q vector of random effects for group i
- ε_i is the n_i vector of errors of group i.

General form: random effects models

In the linear mixed model, both $\mathcal{B}_i$ and ε_i are random vectors and

- the random effects $\mathcal{B}_i$ and $\mathcal{B}_j$ $(i \neq j)$ are independent.
- the random effects are independent from the errors
- the ε_i are independent from one another and don't depend on the explanatory variables
- both $\mathcal{B}_i$ and ε_i have mean zero, meaning

$$\mathsf{E}\left(\mathcal{B}_i\right) = \mathbf{0}_{n_i}, \qquad \mathsf{E}\left(\varepsilon_i \mid \mathbf{X}_i\right) = \mathbf{0}_{n_i}$$

Conditional and marginal mean and variance

We specify covariance models for the random effects and the errors,

$$\mathsf{Cov}\left(\mathcal{B}_i\right) = \boldsymbol{\Omega}, \quad \mathsf{Cov}\left(\varepsilon_i\right) = \mathbf{R}_i, \quad i = 1, \ldots, m$$

The conditional mean and variance of $\boldsymbol{Y}_i$ are

$$\mathsf{E}\left(\boldsymbol{Y}_i \mid \mathbf{X}_i, \mathcal{B}_i = \boldsymbol{b}_i\right) = \mathbf{X}_i\beta + \mathbf{Z}_i\boldsymbol{b}_i, \qquad \mathsf{Cov}\left(\boldsymbol{Y}_i \mid \mathbf{X}_i, \mathcal{B}_i = \boldsymbol{b}_i\right) = \mathbf{R}_i$$

whereas the marginal mean and variance of $\boldsymbol{Y}_i$ are

$$\mathsf{E}\left(\boldsymbol{Y}_i \mid \mathbf{X}_i\right) = \mathbf{X}_i\beta, \qquad \mathsf{Cov}\left(\boldsymbol{Y}_i \mid \mathbf{X}_i\right) = \boldsymbol{\Sigma}_i = \mathbf{Z}_i\boldsymbol{\Omega}\mathbf{Z}_i^\top + \mathbf{R}_i.$$

Random effects models

The parameters of the models which we will **estimate** are

- the vector of coefficients of the fixed effects, β
- the parameters ψ of the marginal covariance $\boldsymbol{\Sigma}$ of $\boldsymbol{Y}$, which arises from the covariance structure of the errors and of the random effects.

Random group effects

- With a linear mixed model, the conditional mean $\mathsf{E}\left(Y_{ij} \mid \mathbf{X}_i, \boldsymbol{b}_i\right)$ can be thought of as a prediction of the value of Y_{ij} after accounting for the group-specific effects.
- when we add a random effect for the group variable, we can still estimate effects of variables that are fixed within a group.

Prediction

- We can predict $\mathcal{B}_i$ by its conditional mean given $\boldsymbol{Y}_i$.
- If the parameters (β, ψ) were known,

$$\mathsf{E}\left(\mathcal{B}_i \mid \boldsymbol{Y}_i\right) = \boldsymbol{\Omega}\mathbf{Z}_i^\top \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{Y}_i - \mathbf{X}_i\beta)$$

  where

$$\boldsymbol{\Sigma}_i = \mathbf{Z}_i\boldsymbol{\Omega}\mathbf{Z}_i^\top + \mathbf{R}_i.$$

- We can plug-in parameters estimates $(\widehat{\beta}, \widehat{\psi})$ to obtain predictions of the random effect,

$$\widehat{\boldsymbol{b}}_i = \widehat{\boldsymbol{\Omega}}\mathbf{Z}_i^\top \widehat{\boldsymbol{\Sigma}}_i^{-1}(\boldsymbol{Y}_i - \mathbf{X}_i\widehat{\beta})$$

Fixed or random effect?

- There is no universal definition of fixed and random effects…
- Loosely speaking, the main difference between fixed and random effects is
    - **fixed effects** for group are used when we have few groups and lots of replicates and we care about the effect of the group (small m, large n_i).
    - **random effects** are used when there are enough levels of the factor group to estimate the variance σ_b^2 reliably; we are not interested in the effects per say (large m, small n_i).