# Complete factorial designs

**Session 6**

MATH 80667A: Experimental Design and Statistical Methods
HEC Montréal

# Outline

Unbalanced designs

Multifactorial designs

# Unbalanced designs

# Premise

So far, we have exclusively considered balanced samples

**balanced = same number of observational units in each subgroup**

Most experiments (even planned) end up with unequal sample sizes.

# Noninformative drop-out

Unbalanced samples may be due to many causes, including randomization (need not balance) and loss-to-follow up (dropout)

If dropout is random, not a problem

- Example of Baumann, Seifert-Kessel, Jones (1992):

> Because of illness and transfer to another school, incomplete data were obtained for one subject each from the TA and DRTA group

# Problematic drop-out or exclusion

If loss of units due to treatment or underlying conditions, problematic!

Rosensaal (2021) rebuking a study on the effectiveness of hydrochloriquine as treatment for Covid19 and reviewing allocation:

> Of these 26, six were excluded (and incorrectly labelled as lost to follow-up): three were transferred to the ICU, one died, and two terminated treatment or were discharged

Sick people excluded from the treatment group! then claim it is better.

Worst: "The index [treatment] group and control group were drawn from different centres."

# Why seek balance?

Two main reasons

1. Power considerations: with equal variance in each group, balanced samples gives the best allocation
2. Simplicity of interpretation and calculations: the interpretation of the $F$ test in a linear regression is unambiguous

# Finding power in balance

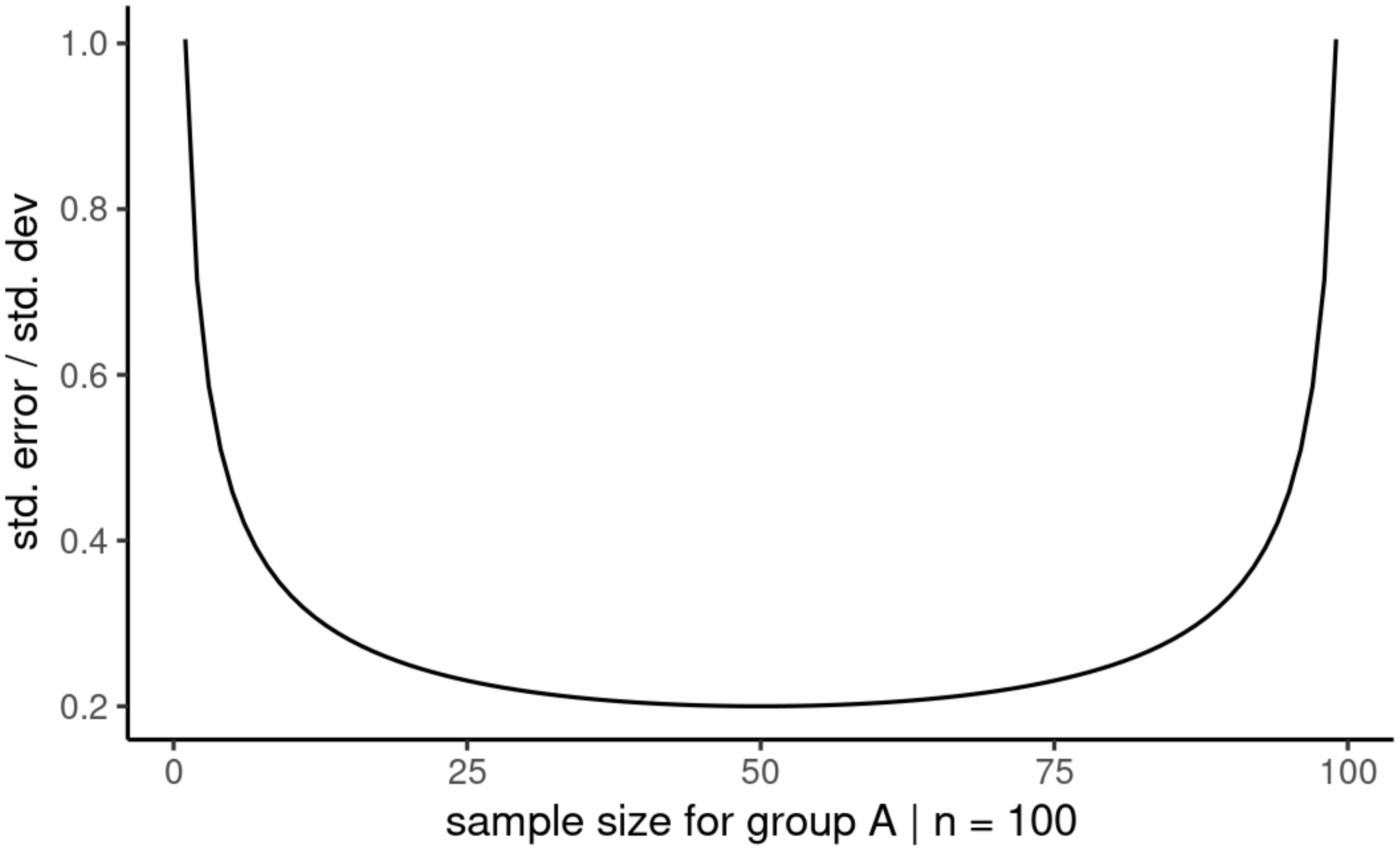
Consider a t-test for assessing the difference between treatments $A$ and $B$ with equal variability

$$t = \frac{\text{estimated difference}}{\text{estimated variability}} = \frac{(\widehat{\mu}_A - \widehat{\mu}_B) - 0}{\mathsf{se}(\widehat{\mu}_A - \widehat{\mu}_B)}.$$

The standard error of the average difference is

$$\sqrt{\frac{\text{variance}_A}{\text{nb of obs. in } A} + \frac{\text{variance}_B}{\text{nb of obs. in } B}} = \sqrt{\frac{\sigma^2}{n_A} + \frac{\sigma^2}{n_B}}$$

# Optimal allocation of ressources




The allocation of $n = n_A + n_B$ units that minimizes the std error is $n_A = n_B = n/2$.

# Example: tempting fate

We consider data from Multi Lab 2, a replication study that examined Risen and Gilovich (2008) who

> explored the belief that tempting fate increases bad outcomes. They tested whether people judge the likelihood of a negative outcome to be higher when they have imagined themselves [...] tempting fate [...] (by not reading before class) or not [tempting] fate (by coming to class prepared). Participants then estimated how likely it was that [they] would be called on by the professor (scale from 1, not at all likely, to 10, extremely likely).

The replication data gathered in 37 different labs focuses on a 2 by 2 factorial design with gender (male vs female) and condition (prepared vs unprepared) administered to undergraduates.

Load data | Check balance | Marginal means

- We consider a 2 by 2 factorial design.
- The response is `likelihod`
- The experimental factors are `condition` and `gender`
- Two data sets: `RS_unb` for the full data, `RS_bal` for the artificially balanced one.

Load data | **Check balance** | Marginal means

```
summary_stats <-
  RS_unb |>
  group_by(condition) |>
  summarize(nobs = n(),
            mean = mean(likelihood))
```

Summary statistics

| condition | nobs | mean |
|---|---|---|
| unprepared | 2192 | 4.606 |
| prepared | 2241 | 4.060 |

Load data | Check balance | **Marginal means**

```r
# Enforce sum-to-zero parametrization
options(contrasts = rep("contr.sum", 2))
# Anova is a linear model, fit using 'lm
# 'aov' only for *balanced data*
model <- lm(
  likelihood ~ gender * condition,
  data = RS_unb)
library(emmeans)
emm <- emmeans(model,
               specs = "condition")
```

## Marginal means for condition

| condition | emmean | SE |
| --- | --- | --- |
| unprepared | 4.504 | 0.0540 |
| prepared | 4.022 | 0.0535 |

Note unequal standard errors.

# Explaining the discrepancies

Estimated marginal means are based on equiweighted groups:

$$\widehat{\mu} = \frac{1}{4}(\widehat{\mu}_{11} + \widehat{\mu}_{12} + \widehat{\mu}_{21} + \widehat{\mu}_{22})$$

where $\widehat{\mu}_{ij} = n_{ij}^{-1} \sum_{r=1}^{n_{ij}} y_{ijr}$.

The sample mean is the sum of observations divided by the sample size.

The two coincide when $n_{11} = \cdots = n_{22}$.

# Why equal weight?

- The ANOVA and contrast analyses, in the case of unequal sample sizes, are generally based on marginal means (same weight for each subgroup).
- This choice is justified because research questions generally concern comparisons of means across experimental groups.

# Revisiting the $F$ statistic

Statistical tests contrast competing **nested** models:

- an alternative model, sometimes termed "full model"
- a null model, which imposes restrictions (a simplification of the alternative model)

The numerator of the $F$-statistic compares the sum of square of a model with (given) main effect, etc., to a model without.

# What is explained by condition?

Consider the $2 \times 2$ factorial design with factors $A$: `gender` and $B$: `condition` (prepared vs unprepared) without interaction.

What is the share of variability (sum of squares) explained by the experimental condition?

# Comparing differences in sum of squares (1)

Consider a balanced sample

```
anova(lm(likelihood ~ 1, data = RS_bal),
      lm(likelihood ~ condition, data = RS_bal))
# When gender is present
anova(lm(likelihood ~ gender, data = RS_bal),
      lm(likelihood ~ gender + condition, data = RS_bal))
```

The difference in sum of squares is 141.86 in both cases.

# Comparing differences in sum of squares (2)

Consider an unbalanced sample

```
anova(lm(likelihood ~ 1, data = RS_unb),
      lm(likelihood ~ condition,
         data = RS_unb))
# When gender is present
anova(lm(likelihood ~ gender, data = RS_unb),
      lm(likelihood ~ gender + condition,
         data = RS_unb))
```

The differences of sum of squares are respectively 330.95 and 332.34.

# Orthogonality

Balanced designs yield orthogonal factors: the improvement in the goodness of fit (characterized by change in sum of squares) is the same regardless of other factors.

So effect of $B$ and $B \mid A$ (read $B$ given $A$) is the same.

- test for $B \mid A$ compares $\mathsf{SS}(A, B) - \mathsf{SS}(A)$
- for balanced design, $\mathsf{SS}(A, B) = \mathsf{SS}(A) + \mathsf{SS}(B)$ (factorization).

We lose this property with unbalanced samples: there are distinct formulations of ANOVA.

# Analysis of variance - Type I (sequential)

The default method in **R** with `anova` is the sequential decomposition: in the order of the variables $A$, $B$ in the formula

- So $F$ tests are for tests of effect of
  - $A$, based on $\mathsf{SS}(A)$
  - $B \mid A$, based on $\mathsf{SS}(A, B) - \mathsf{SS}(A)$
  - $AB \mid A, B$ based on $\mathsf{SS}(A, B, AB) - \mathsf{SS}(A, B)$

**Ordering matters**

Since the order in which we list the variable is **arbitrary**, these $F$ tests are not of interest.

# Analysis of variance - Type II

Impact of

- $A \mid B$ based on $\mathsf{SS}(A, B) - \mathsf{SS}(B)$
- $B \mid A$ based on $\mathsf{SS}(A, B) - \mathsf{SS}(A)$
- $AB \mid A, B$ based on $\mathsf{SS}(A, B, AB) - \mathsf{SS}(A, B)$
- tests invalid if there is an interaction.
- In **R**, use `car::Anova(model, type = 2)`

# Analysis of variance - Type III

Most commonly used approach

- Improvement due to $A \mid B, AB$, $B \mid A, AB$ and $AB \mid A, B$
- What is improved by adding a factor, interaction, etc. given the rest
- may require imposing equal mean for rows for $A \mid B, AB$, etc.
    - (**requires** sum-to-zero parametrization)
- valid in the presence of interaction
- but $F$-tests for main effects are not of interest
- In **R**, use `car::Anova(model, type = 3)`

# ANOVA for unbalanced data

```r
model <- lm(
  likelihood ~ condition * gender,
  data = RS_unb)
# Three distinct decompositions
anova(model) #type 1
car::Anova(model, type = 2)
car::Anova(model, type = 3)
```

ANOVA (type I)

| | Df | Sum Sq | F value |
|---|---|---|---|
| gender | 1 | 164.94 | 29.1 |
| condition | 1 | 332.34 | 58.7 |
| gender:condition | 1 | 36.55 | 6.5 |
| Residuals | | 442925086.33 | |

ANOVA (type II)

| | Df | Sum Sq | F value |
|---|---|---|---|
| gender | 1 | 166.33 | 29.4 |
| condition | 1 | 332.34 | 58.7 |
| gender:condition | 1 | 36.55 | 6.5 |
| Residuals | | 442925086.33 | |

ANOVA (type III)

| | Df | Sum Sq | F value |
|---|---|---|---|
| gender | 1 | 167.71 | 29.6 |
| condition | 1 | 227.88 | 40.2 |
| gender:condition | 1 | 36.55 | 6.5 |
| Residuals | | 442925086.33 | |

# ANOVA for balanced data

```r
model2 <- lm(
  likelihood ~ condition * gender,
  data = RS_bal)
anova(model2) #type 1
car::Anova(model2, type = 2)
car::Anova(model2, type = 3)
# Same answer - orthogonal!
```

ANOVA (type I)

| | Df | Sum Sq | F value |
|---|---|---|---|
| condition | 1 | 141.86 | 24.1 |
| gender | 1 | 121.69 | 20.6 |
| condition:gender | 1 | 37.88 | 6.4 |
| Residuals | 2500 | 14733.84 | |

ANOVA (type II)

| | Df | Sum Sq | F value |
|---|---|---|---|
| condition | 1 | 141.86 | 24.1 |
| gender | 1 | 121.69 | 20.6 |
| condition:gender | 1 | 37.88 | 6.4 |
| Residuals | 2500 | 14733.84 | |

ANOVA (type III)

| | Df | Sum Sq | F value |
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| condition:gender | 1 | 37.88 | 6.4 |
| Residuals | 2500 | 14733.84 | |

# Recap

- If each observation has the same variability, a balanced sample maximizes power.
- Balanced designs have interesting properties:
  - estimated marginal means coincide with (sub)samples averages
  - the tests of effects are unambiguous
  - for unbalanced samples, we work with marginal means and type 3 ANOVA
  - if empty cells (no one assigned to a combination of treatment), cannot estimate corresponding coefficients (typically higher order interactions)

# Practice

From the OSC psychology replication

> People can be influenced by the prior consideration of a numerical anchor when forming numerical judgments. [...] The anchor provides an initial starting point from which estimates are adjusted, and a large body of research demonstrates that adjustment is usually insufficient, leading estimates to be biased towards the initial anchor.

Replication of Study 4a of Janiszewski & Uy (2008, Psychological Science) by J. Chandler

# Multifactorial designs

# Beyond two factors

We can consider multiple factors $A$, $B$, $C$, $\ldots$ with respectively $n_a$, $n_b$, $n_c$, $\ldots$ levels and with $n_r$ replications for each.

The total number of treatment combinations is

$$n_a \times n_b \times n_c \times \cdots$$

**Curse of dimensionality**

# Full three-way ANOVA model

Each cell of the cube is allowed to have a different mean

$$\underbrace{Y_{ijkr}}_{\text{response}} = \underbrace{\mu_{ijk}}_{\text{cell mean}} + \underbrace{\varepsilon_{ijkr}}_{\text{error}}$$

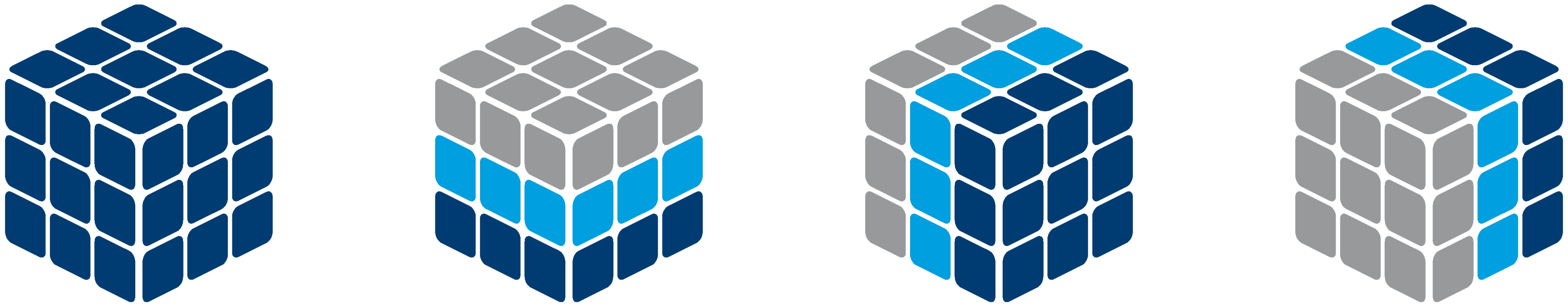
with $\varepsilon_{ijkt}$ are independent error term for

- row $i$
- column $j$
- depth $k$
- replication $r$

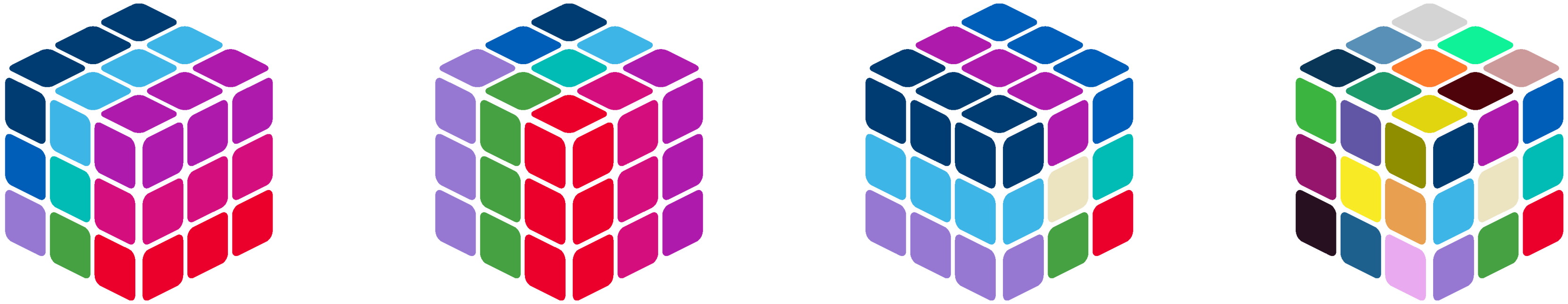
# Parametrization of a three-way ANOVA model

With the **sum-to-zero** parametrization with factors $A$, $B$ and $C$, write the response as

$$
\begin{aligned}
\underset{\text{theoretical average}}{\mathsf{E}(Y_{ijkr})} &= \underset{\text{global mean}}{\mu} \\
&\quad + \underset{\text{main effects}}{\alpha_i + \beta_j + \gamma_k} \\
&\quad + \underset{\text{two-way interactions}}{(\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}} \\
&\quad + \underset{\text{three-way interaction}}{(\alpha\beta\gamma)_{ijk}}
\end{aligned}
$$



global mean, row, column and depth main effects



row/col, row/depth and col/depth interactions and three-way interaction.

# Example of three-way design

Petty, Cacioppo and Heesacker (1981). Effects of rhetorical questions on persuasion: A cognitive response analysis. Journal of Personality and Social Psychology.

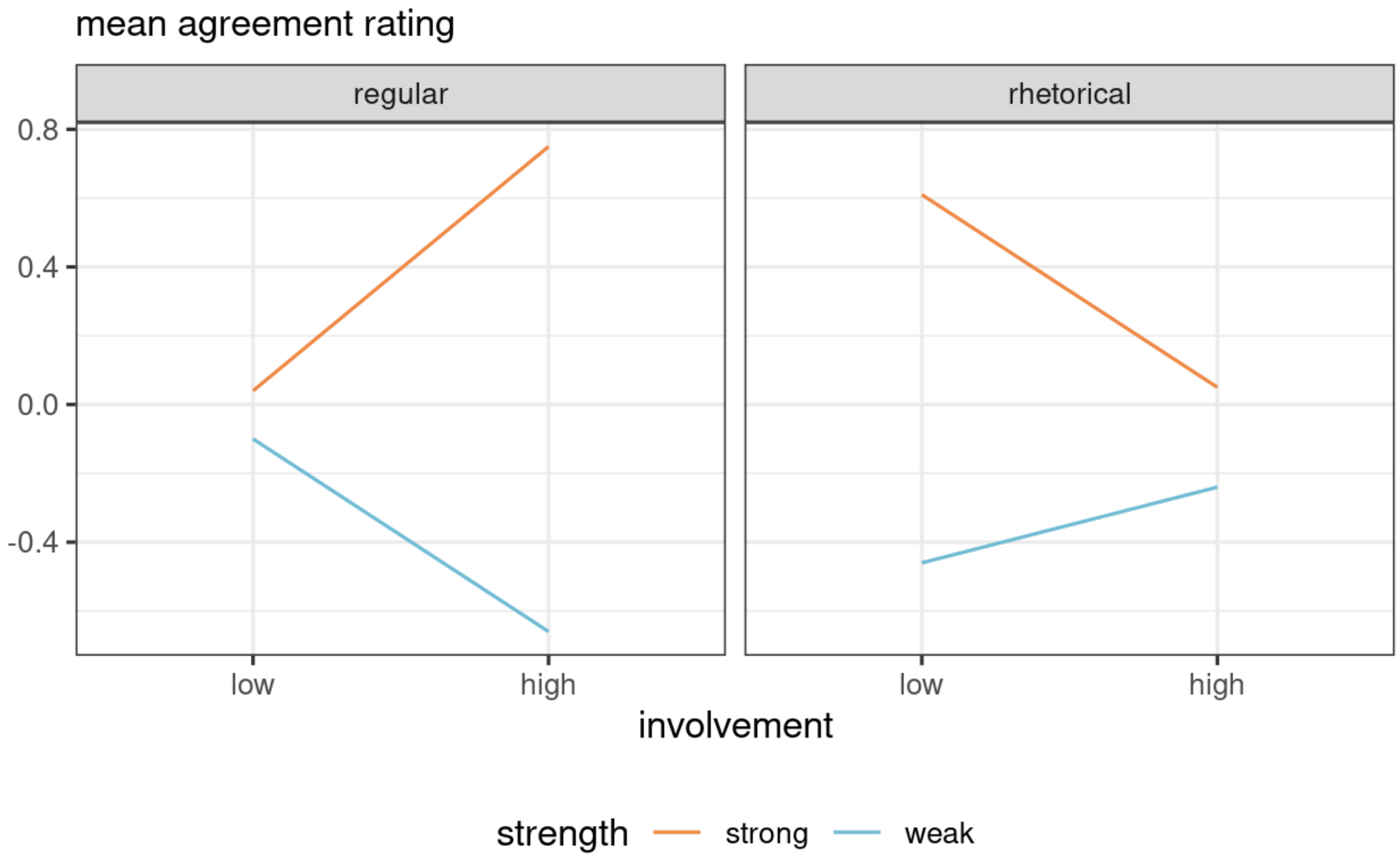
A $2 \times 2 \times 2$ factorial design with 8 treatments groups and $n = 160$ undergraduates.

Setup: should a comprehensive exam be administered to bachelor students in their final year?

- **Response** Likert scale on $-5$ (do not agree at all) to $5$ (completely agree)
- **Factors**
- $A$: strength of the argument (`strong` or `weak`)
- $B$: involvement of students `low` (far away, in a long time) or `high` (next year, at their university)
- $C$: style of argument, either `regular` form or `rhetorical` (Don't you think?, ...)

# Interaction plot

Interaction plot for a $2 \times 2 \times 2$ factorial design from Petty, Cacioppo and Heesacker (1981)



# The microwave popcorn experiment

What is the best brand of microwave popcorn?

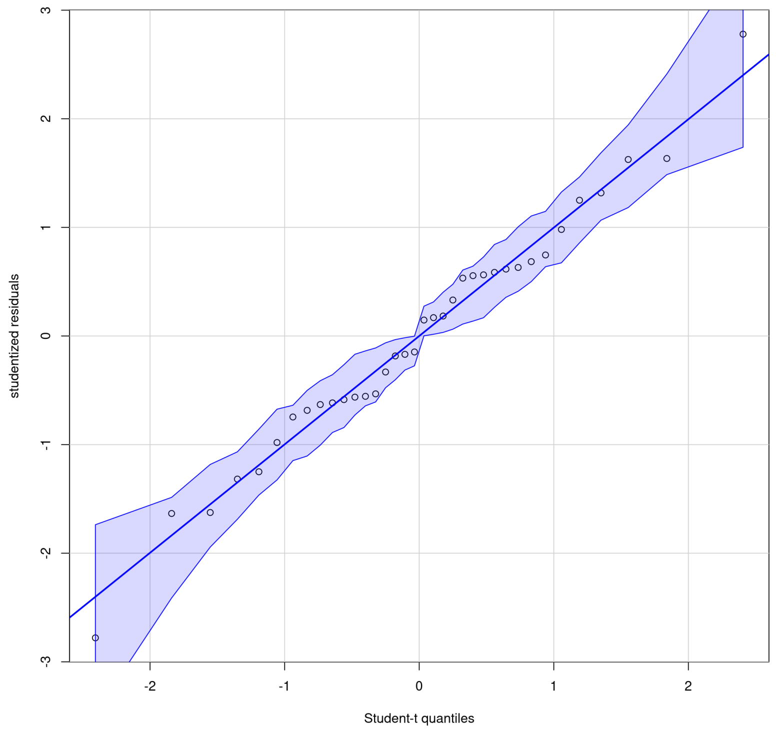
- **Factors**
- brand (two national, one local)
- power: 500W and 600W
- time: 4, 4.5 and 5 minutes
- **Response**: ~~weight~~, ~~volume~~, ~~number~~, percentage of popped kernels.
- Pilot study showed average of 70% overall popped kernels (10% standard dev), timing values reasonable
- Power calculation suggested at least $r = 4$ replicates, but researchers proceeded with $r = 2$...

ANOVA QQ-plot R code Interaction plot

```r
data(popcorn, package = 'hecedsm')
# Fit model with three-way interaction
model <- aov(percentage ~ brand*power*time,
             data = popcorn)
# ANOVA table - 'anova' is ONLY for balanced designs
anova_table <- anova(model)
# Quantile-quantile plot
car::qqPlot(model)
```

Model assumptions: plots and tests are meaningless with `(n_r=2)` replications per group...

ANOVA | QQ-plot | R code | Interaction plot

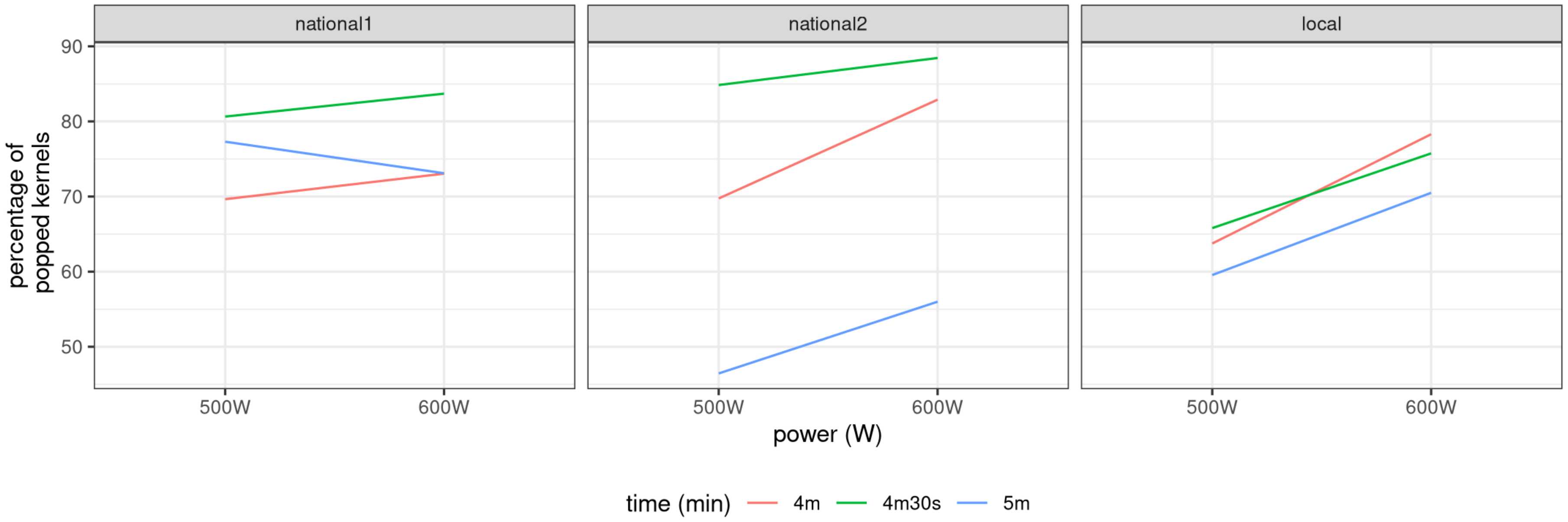



All points fall roughly on a straight line.

ANOVA QQ-plot **R code** Interaction plot

```
popcorn |>
   group_by(brand, time, power) |>
   summarize(meanp = mean(percentage)) |>
ggplot(mapping = aes(x = power,
                     y = meanp,
                     col = time,
                     group = time)) +
  geom_line() +
  facet_wrap(~brand)
```

ANOVA | QQ-plot | R code | **Interaction plot**




No evidence of three-way interaction (hard to tell with $r = 2$ replications).

# Analysis of variance table for balanced designs

| terms | degrees of freedom |
|---|---|
| $A$ | $n_a - 1$ |
| $B$ | $n_b - 1$ |
| $C$ | $n_c - 1$ |
| $AB$ | $(n_a - 1)(n_b - 1)$ |
| $AC$ | $(n_a - 1)(n_c - 1)$ |
| $BC$ | $(n_b - 1)(n_c - 1)$ |
| $ABC$ | $(n_a - 1)(n_b - 1)(n_c - 1)$ |
| residual | $n_a n_b n_c (R - 1)$ |
| total | $n_a n_b n_c n_r - 1$ |

## Analysis of variance table for microwave-popcorn

| | Degrees of freedom | Sum of squares | Mean square | F statistic | p-value |
|---|---|---|---|---|---|
| brand | 2 | 331.10 | 165.55 | 1.89 | 0.180 |
| power | 1 | 455.11 | 455.11 | 5.19 | 0.035 |
| time | 2 | 1554.58 | 777.29 | 8.87 | 0.002 |
| brand:power | 2 | 196.04 | 98.02 | 1.12 | 0.349 |
| brand:time | 4 | 1433.86 | 358.46 | 4.09 | 0.016 |
| power:time | 2 | 47.71 | 23.85 | 0.27 | 0.765 |
| brand:power:time | 4 | 47.33 | 11.83 | 0.13 | 0.967 |

# Omitting terms in a factorial design

The more levels and factors, the more parameters to estimate (and replications needed)

- Costly to get enough observations / power
- The assumption of normality becomes more critical when $r = 2$!

It may be useful not to consider some interactions if they are known or (strongly) suspected not to be present

- If important interactions are omitted from the model, biased estimates/output!

# Guidelines for the interpretation of effects

Start with the most complicated term (top down)

- If the three-way interaction $ABC$ is significative:
  - don't interpret main effects or two-way interactions!
  - comparison is done cell by cell within each level
- If the $ABC$ term isn't significative:
  - can marginalize and interpret lower order terms
  - back to a series of two-way ANOVAs

# What contrasts are of interest?

- Can view a three-way ANOVA as a series of one-way ANOVA or two-way ANOVAs...

Depending on the goal, could compare for variable $A$

- marginal contrast $\psi_A$ (averaging over $B$ and $C$)
- marginal conditional contrast for particular subgroup: $\psi_A$ within $c_1$
- contrast involving two variables: $\psi_{AB}$
- contrast differences between treatment at $\psi_A \times B$, averaging over $C$.
- etc.

See helper code and chapter 22 of Keppel & Wickens (2004) for a detailed example.

# Effects and contrasts for microwave-popcorn

Following preplanned comparisons

- Which combo (brand, power, time) gives highest popping rate? (pairwise comparisons of all combos)
- Best brand overall (marginal means marginalizing over power and time, assuming no interaction)
- Effect of time and power on percentage of popped kernels
- pairwise comparison of time $\times$ power
- main effect of power
- main effect of time

# Preplanned comparisons using emmeans

Let $A$=brand, $B$=power, $C$=time

Compare difference between percentage of popped kernels for 4.5 versus 5 minutes, for brands 1 and 2

$$\mathscr{H}_0 : (\mu_{1.2} - \mu_{1.3}) - (\mu_{2.2} - \mu_{2.3}) = 0$$

```
library(emmeans)
# marginal means
emm_popcorn_AC <- emmeans(model,
                          specs = c("brand","time"))
contrast_list <-
  list(
    brand12with4.5vs5min = c(0, 0, 0, 1, -1, 0, -1, 1,0))
contrast(emm_popcorn_AC,   # marginal mean (no time)
         method = contrast_list) # list of contrasts
```

# Preplanned comparisons

Compare all three times (4, 4.5 and 5 minutes)

At level 99% with Tukey's HSD method

- Careful! Potentially misleading because there is a `brand * time` interaction present.

```
# List of variables to keep go in `specs`: keep only time
emm_popcorn_C <- emmeans(model, specs = "time")
pairs(emm_popcorn_C,
      adjust = "tukey",
      level = 0.99,
      infer = TRUE)
```