

# Complete factorial designs

## Session 5

MATH 80667A: Experimental Design and Statistical Methods  
HEC Montréal

# Outline

**Factorial designs and interactions**

**Model formulation**

# Factorial designs and interactions

# Complete factorial designs?

## **Factorial design**

study with multiple factors (subgroups)

## **Complete**

Gather observations for every subgroup

# Motivating example







**Response:**  
retention of information  
two hours after reading a story

**Population:**  
children aged four

**experimental factor 1:**  
ending (happy or sad)

**experimental factor 2:**  
complexity (easy, average or hard).

# Setup of design

complexity	happy	sad
complicated		
average		
easy		

**Factors are crossed**

# Efficiency of factorial design

**Cast problem  
as a series of one-way ANOVA  
vs simultaneous estimation**

**Factorial designs requires  
fewer overall observations**

**Can study interactions**

# Interaction

**Definition:** when the effect of one factor depends on the levels of another factor.

Effect together  
 $\neq$   
sum of individual effects

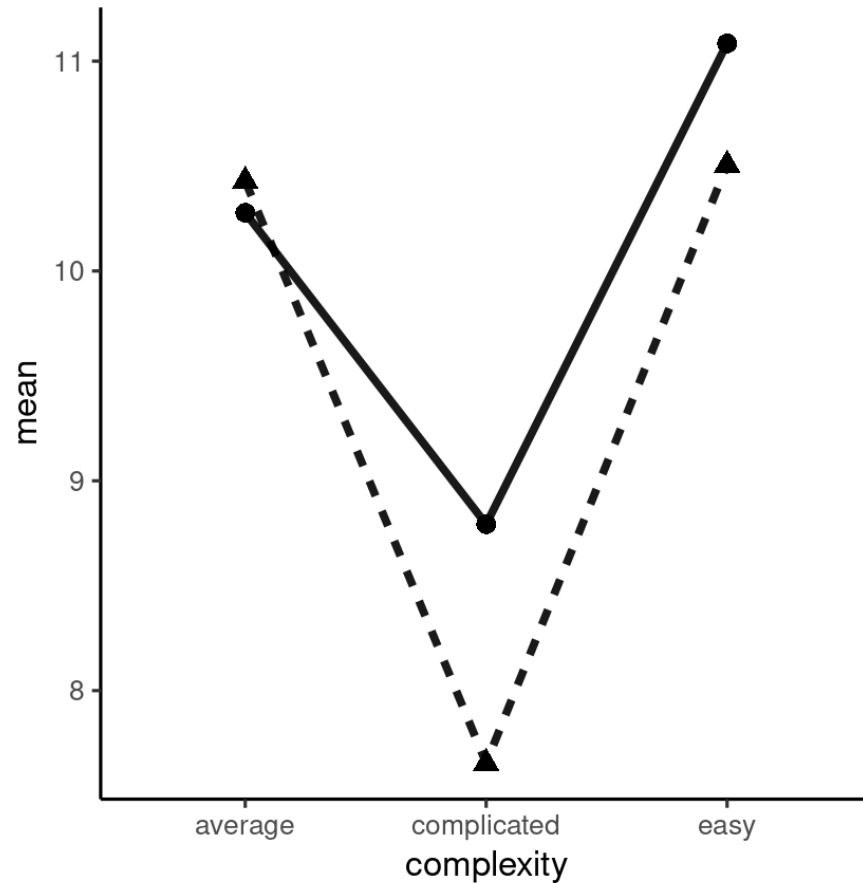


# Interaction or profile plot

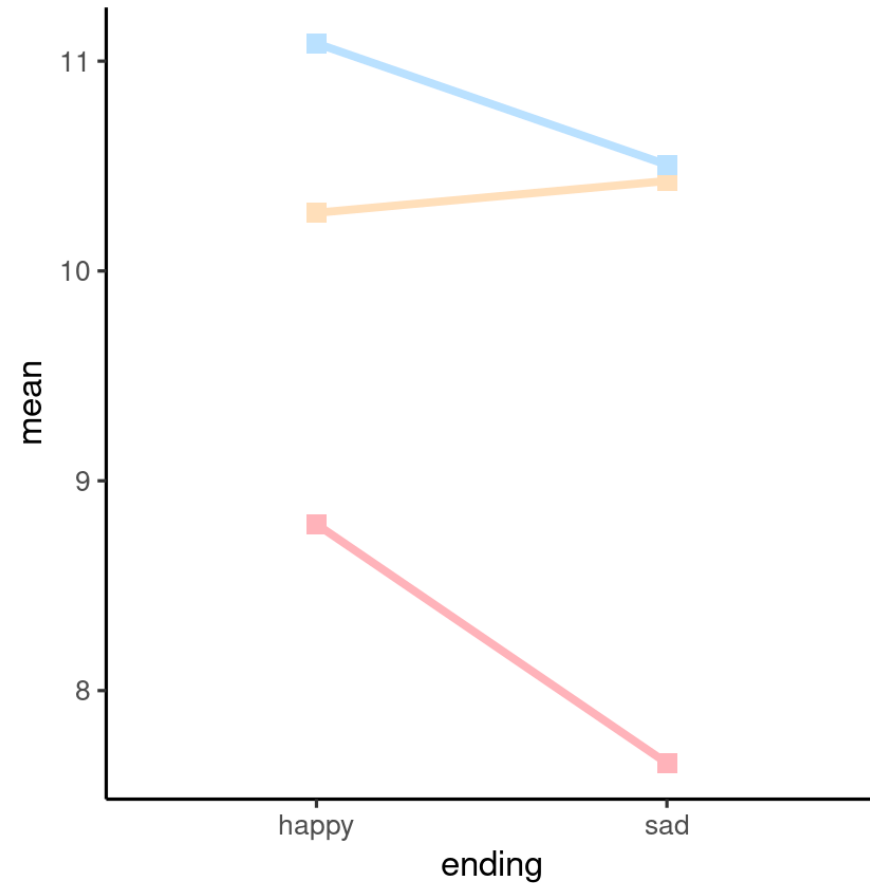
**Graphical display:  
plot sample mean per category**

**with uncertainty measure  
(1 std. error for mean  
confidence interval, etc.)**

# Interaction: lines are not parallel

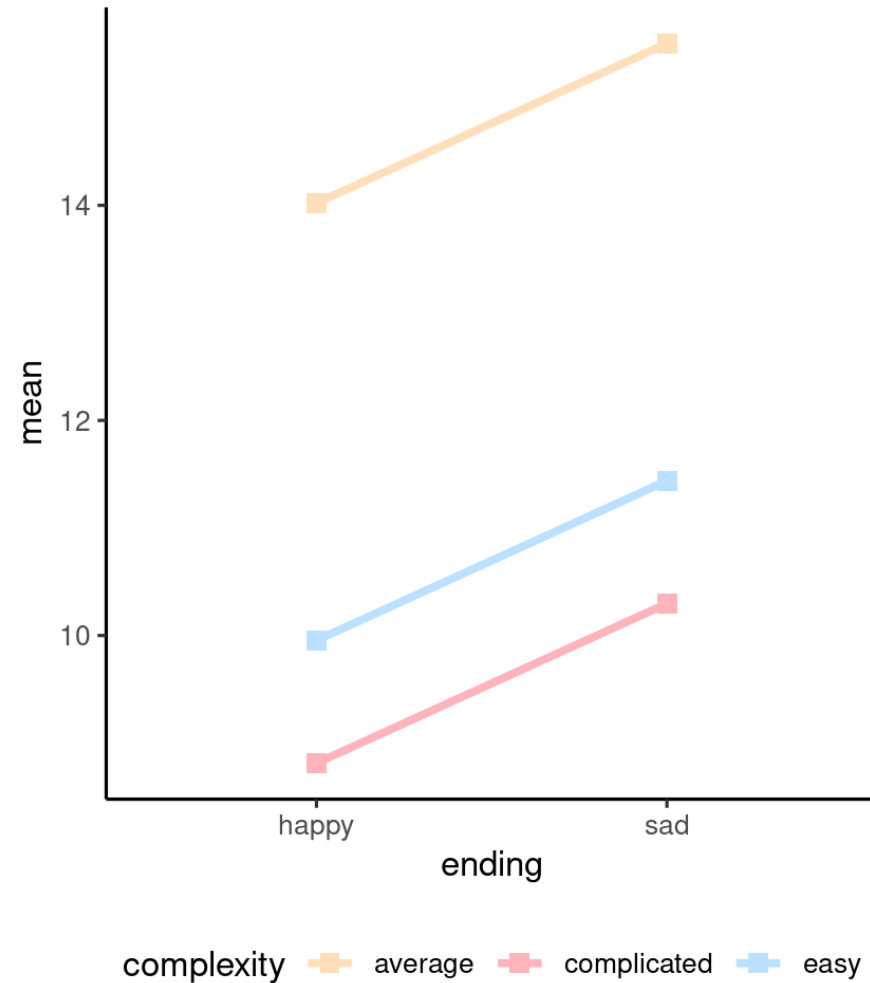
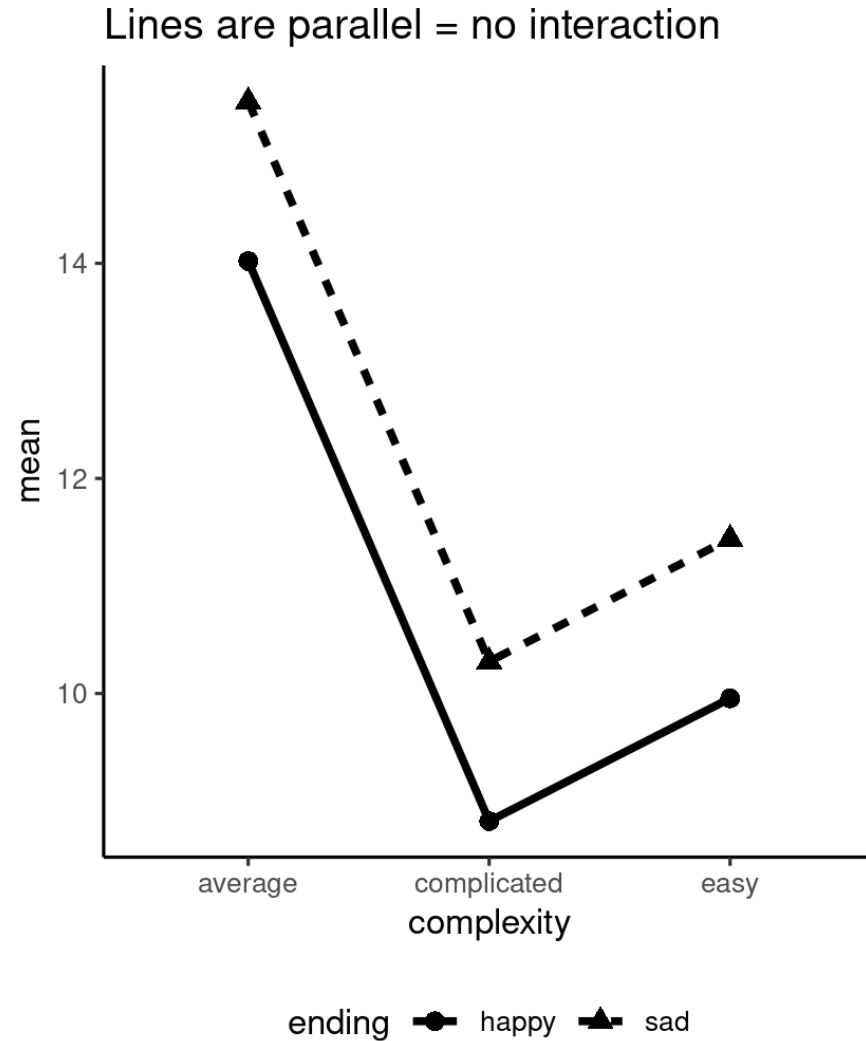


ending ● happy ▲ sad

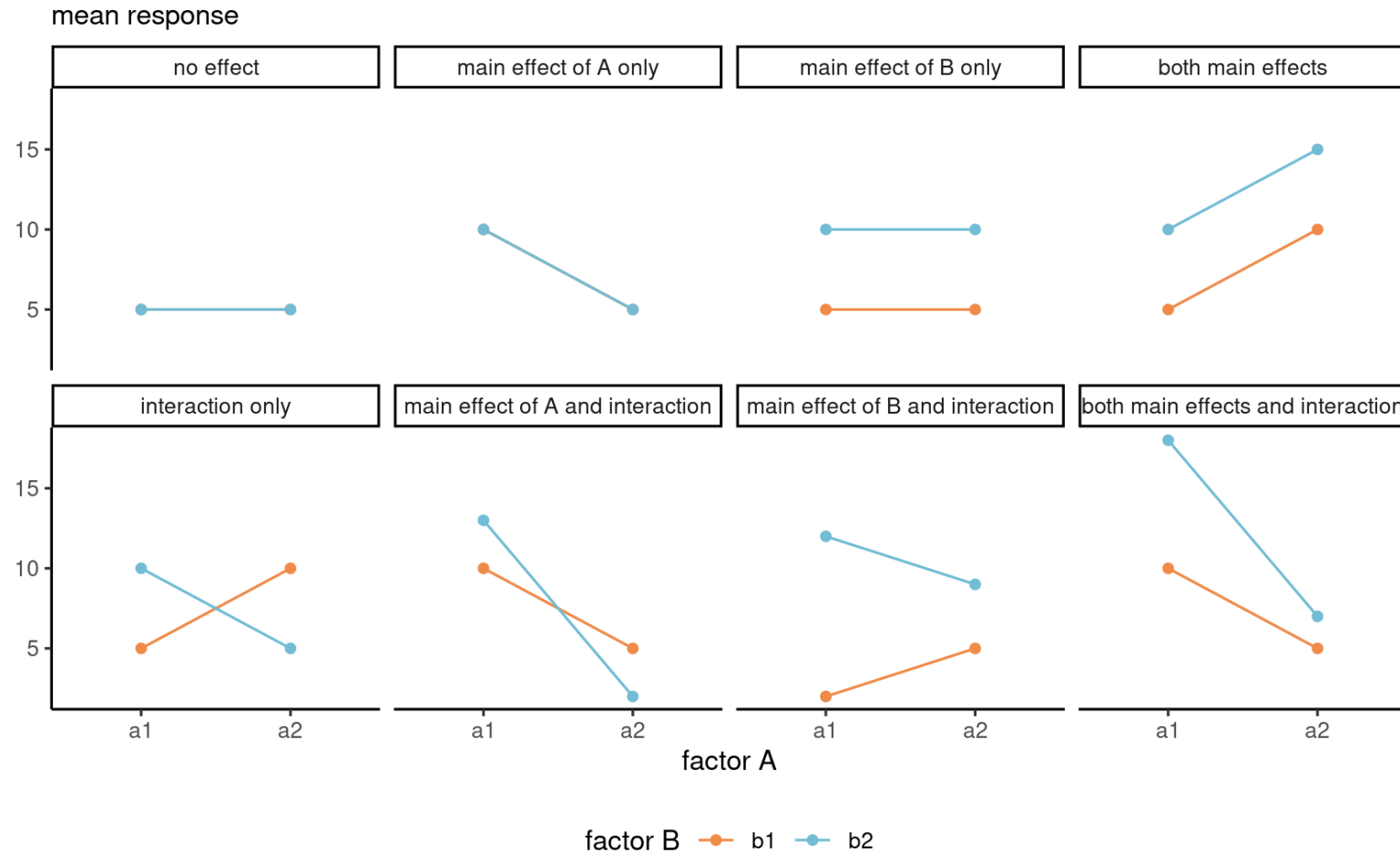


complexity — average — complicated — easy

# No interaction: parallel lines



# Interaction plot for 2 by 2 design



# Model formulation

# Formulation of the two-way ANOVA

Two factors:  $A$  (complexity) and  $B$  (ending) with  $n_a$  and  $n_b$  levels.

Write the average response  $Y_{ijr}$  of the  $r$ th measurement in group  $(a_i, b_j)$  as

$$\begin{array}{ccccc} Y_{ijr} & = & \mu_{ij} & + & \varepsilon_{ijr} \\ \text{response} & & \text{subgroup mean} & & \text{error term} \end{array}$$

where

- $Y_{ijr}$  is the  $r$ th replicate for  $i$ th level of factor  $A$  and  $j$ th level of factor  $B$
- $\varepsilon_{ijr}$  are independent error terms with mean zero and variance  $\sigma^2$ .

# One average for each subgroup

$B$ ending $A$ complexity	$b_1$ (happy)	$b_2$ (sad)	<b>row mean</b>
$a_1$ (complicated)	$\mu_{11}$	$\mu_{12}$	$\mu_{1.}$
$a_2$ (average)	$\mu_{21}$	$\mu_{22}$	$\mu_{2.}$
$a_3$ (easy)	$\mu_{31}$	$\mu_{32}$	$\mu_{3.}$
<i>column mean</i>	$\mu_{.1}$	$\mu_{.2}$	$\mu$

# Row, column and overall average

- Mean of  $A_i$  (average of row  $i$ ):

$$\mu_{i.} = \frac{\mu_{i1} + \cdots + \mu_{in_b}}{n_b}$$

- Mean of  $B_j$  (average of column  $j$ ):

$$\mu_{.j} = \frac{\mu_{1j} + \cdots + \mu_{n_a j}}{n_a}$$

- Overall average (overall all rows and columns):

$$\mu = \frac{\sum_{i=1}^{n_a} \sum_{j=1}^{n_b} \mu_{ij}}{n_a n_b}$$



# Vocabulary of effects

## Definitions

- **simple effects**: difference between levels of one in a fixed combination of others (change in difficulty for happy ending)
- **main effects**: differences relative to average for each condition of a factor (happy vs sad ending)
- **interaction effects**: when simple effects differ depending on levels of another factor

## What it means relative to the table

- **simple effects** are comparisons between cell averages within a given row or column
- **main effects** are comparisons between row or column averages
- **interaction effects** are difference relative to the row or column average

# Marginal effects

	happy	sad
column means	$\mu_{.1}$	$\mu_{.2}$

complexity	row means
complicated	$\mu_{1.}$
average	$\mu_{2.}$
easy	$\mu_{3.}$

# Simple effects

	happy	sad
means (easy)	$\mu_{1.}$	$\mu_{2.}$

	mean (happy)
complexity	
complicated	$\mu_{11}$
average	$\mu_{21}$
easy	$\mu_{31}$

# Contrasts

Suppose the order of the coefficients is factor  $A$  (complexity, 3 levels, complicated/average/easy) and factor  $B$  (ending, 2 levels, happy/sad).

test	$\mu_{11}$	$\mu_{12}$	$\mu_{21}$	$\mu_{22}$	$\mu_{31}$	$\mu_{32}$
main effect $A$ (complicated vs average)	1	1	-1	-1	0	0
main effect $A$ (complicated vs easy)	1	1	0	0	-1	-1
main effect $B$ (happy vs sad)	1	-1	1	-1	1	-1
interaction $AB$ (comp. vs av, happy vs sad)	1	-1	-1	1	0	0
interaction $AB$ (comp. vs easy, happy vs sad)	1	-1	0	0	-1	1

# Hypothesis tests for main effects

Generally, need to compare multiple effects at once

**Main effect of factor  $A$**

$\mathcal{H}_0: \mu_{1.} = \cdots = \mu_{n_a.}$  vs  $\mathcal{H}_a$ : at least two marginal means of  $A$  are different

**Main effect of factor  $B$**

$\mathcal{H}_0: \mu_{.1} = \cdots = \mu_{.n_b}$  vs  $\mathcal{H}_a$ : at least two marginal means of  $B$  are different.

# Equivalent formulation of the two-way ANOVA

Write the model for a response variable  $Y$  in terms of two factors  $a_i, b_j$ .

$$Y_{ijr} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijr}$$

where

- $\alpha_i = \mu_{i.} - \mu$ 
  - mean of level  $a_i$  minus overall mean.
- $\beta_j = \mu_{.j} - \mu$ 
  - mean of level  $b_j$  minus overall mean.
- $(\alpha\beta)_{ij} = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu$ 
  - the interaction term for  $a_i$  and  $b_j$ .

# One average for each subgroup

$B$ ending $A$ complexity	$b_1$ (happy)	$b_2$ (sad)	<b>row mean</b>
$a_1$ (complicated)	$\mu + \alpha_1 + \beta_1 + (\alpha\beta)_{11}$	$\mu + \alpha_1 + \beta_2 + (\alpha\beta)_{12}$	$\mu + \alpha_1$
$a_2$ (average)	$\mu + \alpha_2 + \beta_1 + (\alpha\beta)_{21}$	$\mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22}$	$\mu + \alpha_2$
$a_3$ (easy)	$\mu + \alpha_3 + \beta_1 + (\alpha\beta)_{31}$	$\mu + \alpha_3 + \beta_2 + (\alpha\beta)_{32}$	$\mu + \alpha_3$
column mean	$\mu + \beta_1$	$\mu + \beta_2$	$\mu$

**More parameters than data cells!**

The model in terms of  $\alpha$ ,  $\beta$  and  $(\alpha\beta)$  is overparametrized.

# Sum-to-zero parametrization

Too many parameters!

Impose sum to zero constraints

$$\sum_{i=1}^{n_a} \alpha_i = 0, \quad \sum_{j=1}^{n_b} \beta_j = 0, \quad \sum_{j=1}^{n_b} (\alpha\beta)_{ij} = 0, \quad \sum_{i=1}^{n_a} (\alpha\beta)_{ij} = 0.$$

which imposes  $1 + n_a + n_b$  constraints.



# Why use the sum to zero parametrization?

- Testing for main effect of  $A$ :

$$\mathcal{H}_0 : \alpha_1 = \dots = \alpha_{n_a} = 0$$

- Testing for main effect of  $B$ :

$$\mathcal{H}_0 : \beta_1 = \dots = \beta_{n_b} = 0$$

- Testing for interaction between  $A$  and  $B$ :

$$\mathcal{H}_0 : (\alpha\beta)_{11} = \dots = (\alpha\beta)_{n_a n_b} = 0$$

In all cases, alternative is that at least two coefficients are different.

# Seeking balance

**Balanced sample**  
(equal nb of obs per group)

With  $n_r$  replications per subgroup,  
total sample size is  $n = n_a n_b n_r$ .

# Why balanced design?

With equal variance, this is the optimal allocation of treatment unit.

**maximize power**

Estimated means for main and total effects correspond to marginal averages.

**equiweighting**

Unambiguous decomposition of effects of  $A$ ,  $B$  and interaction.

**orthogonality**

# Rewriting observations

$$\begin{aligned} (y_{ijr} - \hat{\mu}) &= (\hat{\mu}_{i.} - \hat{\mu}) \\ \text{obs vs grand mean (total)} &\quad \text{row mean vs grand mean}(A) \\ &+ (\hat{\mu}_{.j} - \hat{\mu}) \\ &\quad \text{col mean vs grand mean}(B) \\ &+ (\hat{\mu}_{ij} - \hat{\mu}_{i.} - \hat{\mu}_{.j} + \hat{\mu}) \\ &\quad \text{cell mean vs additive effect}(AB) \\ &+ (y_{ijr} - \hat{\mu}_{ij}) \\ &\quad \text{obs vs cell mean (resid)} \end{aligned}$$

# Decomposing variability

Constructing statistics as before by decomposing variability into blocks.

We can square both sides and sum over all observations.

With balanced design, all cross terms cancel, leaving us with the **sum of square** decomposition

$$SS_{\text{total}} = SS_A + SS_B + SS_{AB} + SS_{\text{resid}}.$$

# Sum of square decomposition

The sum of square decomposition

$$SS_{\text{total}} = SS_A + SS_B + SS_{AB} + SS_{\text{resid}}.$$

is an estimator of the population variance decomposition

$$\sigma_{\text{total}}^2 = \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma_{\text{resid}}^2.$$

where  $\sigma_A^2 = n_a^{-1} \sum_{i=1}^{n_a} \alpha_i^2$ ,  $\sigma_{AB}^2 = (n_a n_b)^{-1} \sum_{i=1}^{n_a} \sum_{j=1}^{n_b} (\alpha\beta)_{ij}^2$ , etc.

Take ratio of variability (effect relative to residual) and standardize numerator and denominator to build an  $F$  statistic.

# Analysis of variance table

term	degrees of freedom	mean square	$F$
$A$	$n_a - 1$	$MS_A = SS_A / (n_a - 1)$	$MS_A / MS_{\text{res}}$
$B$	$n_b - 1$	$MS_B = SS_B / (n_b - 1)$	$MS_B / MS_{\text{res}}$
$AB$	$(n_a - 1)(n_b - 1)$	$MS_{AB} = SS_{AB} / \{(n_a - 1)(n_b - 1)\}$	$MS_{AB} / MS_{\text{res}}$
residuals	$n - n_a n_b$	$MS_{\text{resid}} = SS_{\text{res}} / (n - ab)$	
total	$n - 1$		

Read the table backward (starting with the interaction).

- If there is a significant interaction, the main effects are **not** of interest and potentially misleading.

# Intuition behind degrees of freedom

$B$ ending $A$ complexity	$b_1$ (happy)	$b_2$ (sad)	<b>row mean</b>
$a_1$ (complicated)	$\mu_{11}$	X	$\mu_{1.}$
$a_2$ (average)	$\mu_{21}$	X	$\mu_{2.}$
$a_3$ (easy)	X	X	X
<i>column mean</i>	$\mu_{.1}$	X	$\mu$

Terms with X are fully determined by row/column/total averages



# Multiplicity correction

With equal sample size and equal variance, usual recipes for ANOVA hold. Correction depends on the effect: e.g., for factor  $A$ , the critical values are

- Bonferroni:  $1 - \alpha / (2m)$  quantile of  $\text{St}(n - n_a n_b)$
- Tukey: Studentized range ( $q_{\text{tukey}}$ )
  - level  $1 - \alpha / 2$ ,  $n_a$  groups,  $n - n_a n_b$  degrees of freedom.
- Scheffé: critical value is  $\{(n_a - 1) f_{1-\alpha}\}^{1/2}$ 
  - $f_{1-\alpha}$  is  $1 - \alpha$  quantile of  $F(\nu_1 = n_a - 1, \nu_2 = n - n_a n_b)$ .

Software implementations available in `emmeans` in **R**.