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Chapter 1 Investigations

Investigation 1.1

A farmer has 110 acres of available land he wishes to plant with a mixture of potatoes, corn, and cabbage. It costs him $400 to produce an acre of potatoes, $160 to produce an acre of corn, and $280 to produce an acre of cabbage. He has a maximum of $20,000 to spend. He makes a profit of $120 per acre of potatoes, $40 per acre of corn, and $60 per acre of cabbage.

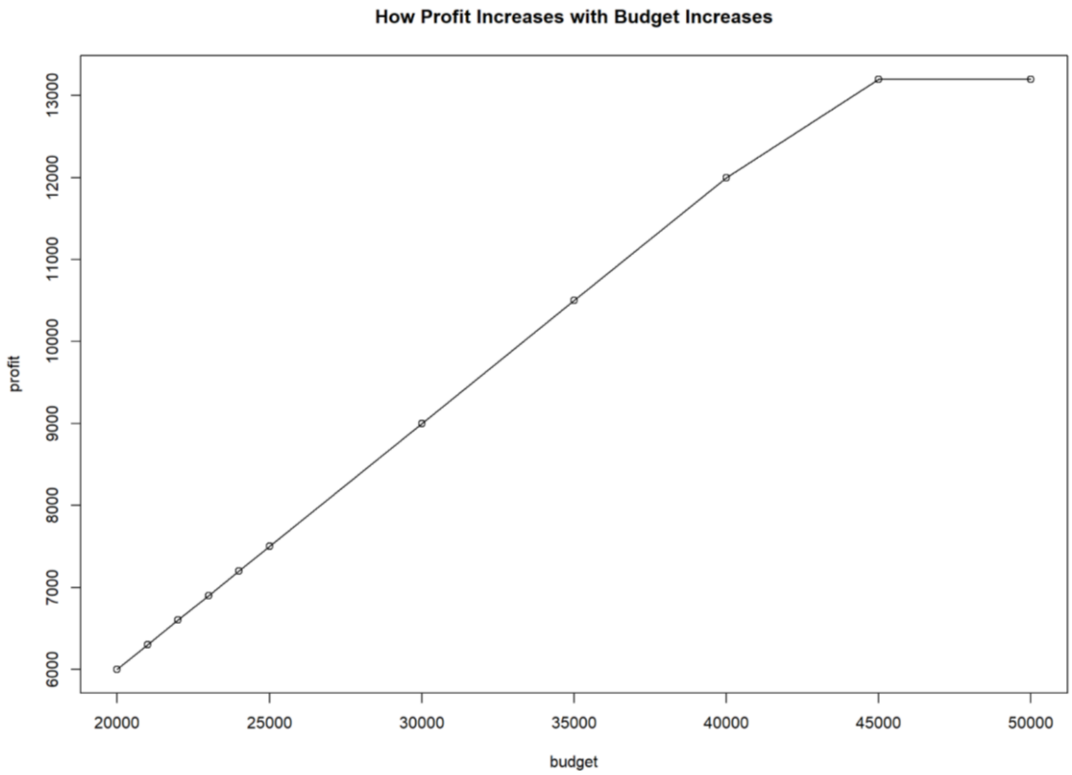
In this scenario, we will define the following variables: P: number of acres of potatoes planted, C: number of acres of corn planted, and B: number of acres of cabbage planted. We want to maximize profit, which means maximizing 120P + 40C + 60B (as these are the respective profits per acre of each crop). We have constraints with the amount of land available (110 acres) and the budget to spend to plant the crops ($20,000). Because there are only 110 acres of land available, we must have $P + C + B \leq 110$ and due to the planting budget, we must also have $400P + 160C + 280B \leq 20{,}000$ (these are the respective costs to plant a single acre of each crop). Thus, our linear programming problem is as follows:

$$\begin{array}{ll} \text{maximize} & 120P + 40C + 60B \\ \text{subject to} & P + C + B \leq 110 \\ & 400P + 160C + 280B \leq 20{,}000 \end{array}$$

Using Excel to solve our linear programming problem, we get the following results:

Decision Variables						
	Potatoes	Corn	Cabbage			
Acres	50	0	0			
Objective Function						
	Potatoes	Corn	Cabbage			
Acres	50	0	0			
Profit	120	40	60			
Total	6000					
Constraints						
	Potatoes	Corn	Cabbage	Total		
Cost	400	160	280	20000	<=	20000
Land				50	<=	110

From these results, we can see that there will be maximum profit if we plant **50 acres of potatoes (and zero acres of both corn and cabbage**). This means that if the farmer wants to maximize his profits, he will **leave 60 of his 110 acres unplanted**. Even though this seems strange, to only be planting potatoes and leaving more than half of his land unplanted, this **maximizes profit without going over the planting budget**. With a budget of $20,000, our maximum profit is $6,000. If the farmer was able to increase his budget, he would see the following growth in profit:



Calculating the profit for each increase in budget, we can also see that we will actually only recommend planting potatoes, all the way until we reach the maximum amount of land we have to plant. This does make sense because potatoes yield a significantly larger profit per acre than corn or cabbage, so if there is no cap on the number of potatoes to farm (such as market demand for potatoes, which we would not want to exceed because the excess would not yield a profit), we would never want to plant corn or cabbage instead of potatoes. From the graph above, we can see that as the farmer increases his budget (from $20,000 all the way to doubling it to $40,000), for every $1,000 increase in the farmer's budget, we can expect to see an increase of $300 in profit.

Investigation 1.2 A product can be made in three sizes, large, medium, and small, which yield a net unit profit of $12, $10, and $9, respectively. The company has three centers where this product can be manufactured and these centers have a capacity of turning out 550, 750, and 275 units of the product per day, respectively, regardless of the size or combination of sizes involved. Manufacturing this product requires cooling water and each unit of large, medium, and small sizes produced require 21, 17, and 9 gallons of water, respectively. The centers 1, 2, and 3 have 10,000, 7000, and 4200 gallons of cooling water available per day, respectively. Market studies indicate that there is a market for 700, 900, and 340 units of the large, medium, and small sizes, respectively, per day. The problem is to determine how many units of each of the sizes should be produced at the various centers in order to maximize the profit.

(a) Formulate and solve the linear programming model for this problem. How many units of each of the sizes should be produced at the various centers to maximize the profit?

To solve this linear programming problem, we will first note that we should not produce more than the market demand for each size of the product, and we cannot exceed the production capacity or water capacity at any of the centers. Ultimately, we want to maximize the profit of the company as a whole, given the constraints.

Decision Variables												
	Large, C1	Medium, C1	Small, C1	Large, C2	Medium, C2	Small, C2	Large, C3	Medium, C3	Small, C3			
Quantity	162.5	387.5	0	0	231.76471	340	0	247.05882	0			
Objective Function												
	Large, C1	Medium, C1	Small, C1	Large, C2	Medium, C2	Small, C2	Large, C3	Medium, C3	Small, C3			
Quantity	162.5	387.5	0	0	231.76471	340	0	247.05882	0			
Profit	12	10	9	12	10	9	12	10	9			
Total	13673.2352941											
Constraints												
	Large, C1	Medium, C1	Small, C1	Large, C2	Medium, C2	Small, C2	Large, C3	Medium, C3	Small, C3	Total		
Demand L										162.5	<=	700
Demand M										866.3235	<=	900
Demand S										340	<=	340
C1 Quantity										550	<=	550
C2 Quantity										571.7647	<=	750
C3 Quantity										247.0588	<=	275
C1 Water	21	17	9							10000	<=	10000
C2 Water				21	17	9				7000	<=	7000
C3 Water							21	17	9	4200	<=	4200

Using Excel's Solver function, we find that, in order to maximize profit to $13,673.24, we should produce:

Center 1:	162.5 large,	387.5 medium, and	0 small units
Center 2:	0 large,	231,76471 medium, and	340 small units
Center 3:	0 large,	247.05882 medium, and	0 small units

It should be noted that the solution above is limited by the water capacity of all three centers, and is also meeting the market demand for small products and is at the production capacity for center 1. These are the 5 of our 9 factors that are currently limiting our profits.

(b) Next, suppose that the following additional constraint is introduced: By company policy, the fraction (scheduled production)/(center's capacity) must be the same at all the centers. Using this additional information, formulate and solve a new linear programming model to maximize the profit. How has your result changed? Why do you think that this might be a policy that an actual manufacturing company might implement?

When we introduce this new constraint, we get the following result:

Center 1:	419.2941176 large,	0 medium,	0 small
Center 2:	0 large,	231,76471 medium,	340 small
Center 3:	159 large,	50.647059 medium,	0 small

This keeps the production at Center 2 exactly the same, and keeps the number of small units at Centers 1 and 3 the same (at zero). So this increases the large at Center 1, decreases the medium at Center 1, increases the large at Center 3, and decreases the medium at Center 3. Overall, each center is producing 76.2353% of its maximum production capacity.

Decision Variables												
	Large, C1	Medium, C1	Small, C1	Large, C2	Medium, C2	Small, C2	Large, C3	Medium, C3	Small, C3			
Quantity	419.2941176	0	0	0	231.76471	340	159	50.647059	0			
Objective Function												
	Large, C1	Medium, C1	Small, C1	Large, C2	Medium, C2	Small, C2	Large, C3	Medium, C3	Small, C3			
Quantity	419.2941176	0	0	0	231.76471	340	159	50.647059	0			
Profit	12	10	9	12	10	9	12	10	9			
Total	12823.6470588											
Constraints												
	Large, C1	Medium, C1	Small, C1	Large, C2	Medium, C2	Small, C2	Large, C3	Medium, C3	Small, C3	Total		
Demand L										578.2941	<=	700
Demand M										282.4118	<=	900
Demand S										340	<=	340
C1 Quantity										419.2941	<=	550
C2 Quantity										571.7647	<=	750
C3 Quantity										209.6471	<=	275
C1 Water	21	17	9							8805.176	<=	10000
C2 Water				21	17	9				7000	<=	7000
C3 Water							21	17	9	4200	<=	4200
Equality A										0.762353	=	0.762353
Equality B										0.762353	=	0.762353
Equality C										0.762353	=	0.762353

This might be a policy that an actual manufacturing company might implement to make sure that all 3 centers are being utilized proportionally. If one center is used more than others, those workers might get burnt out faster, or the workers at the other centers might complain about not being given enough hours. By adding this new policy, the company can try to normalize working conditions across its 3 centers.

(c) Which, if any, of the water capacity constraints was binding? What happens to the solution and to the overall maximum profit if you increase the water capacity at each center by 1%? 5%? 10%? Vary each water capacity one at a time, while holding the others fixed, and write a report on your findings. Include graphs and tables to illustrate your results. Then do the same thing for the other constraints, and discuss your results in detail.

Changing Water Capacities

	Large C1	Medium C1	Small C1	Large C2	Medium C2	Small C2	Large C3	Medium C3	Small C3	Total Profit
C1 Water 1%	187.5	362.5	0	0	231.76471	340	0	247.05882	0	13723.2353
C1 Water 5%	287.5	262.5	0	0	231.76471	340	0	247.05882	0	13923.2353
C1 Water 10%	412.5	137.5	0	0	231.76471	340	0	247.05882	0	14173.2353
C2 Water 1%	162.5	387.5	0	0	235.88235	340	0	247.05882	0	13714.4118
C2 Water 5%	162.5	387.5	0	0	252.35294	340	0	247.05882	0	13879.1177
C2 Water 10%	162.5	387.5	0	0	272.94118	340	6.07143	239.55882	0	14082.8571
C3 Water 1%	162.5	387.5	0	0	231.76471	340	0	249.52941	0	13697.9412
C3 Water 5%	162.5	387.5	0	0	231.76471	340	0	259.41177	0	13796.7647
C3 Water 10%	162.5	387.5	0	0	231.76471	340	0	271.76471	0	13920.2941

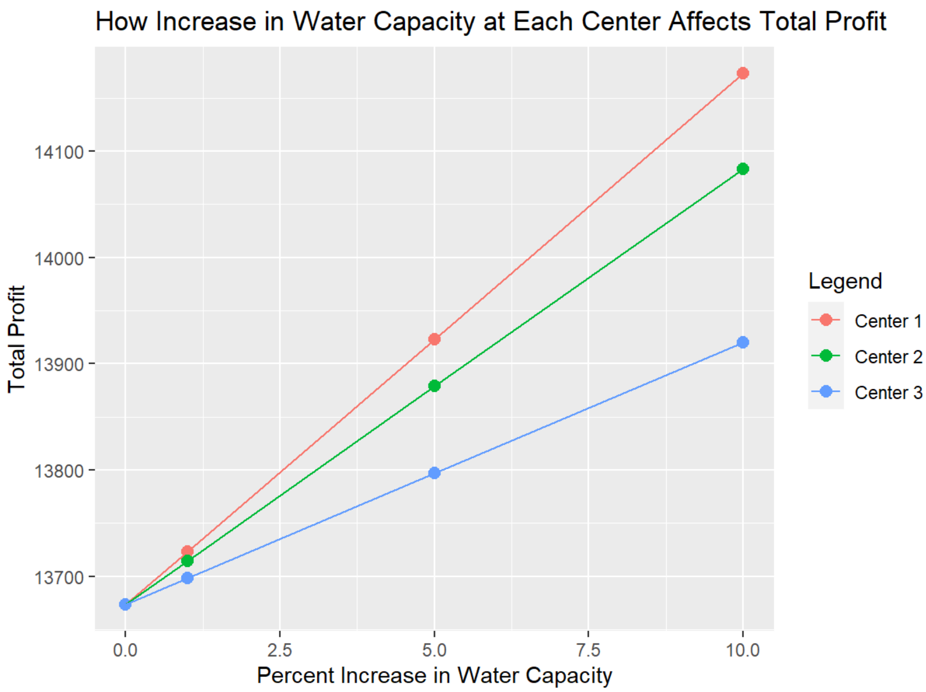
Changing Production Capacities

	Large C1	Medium C1	Small C1	Large C2	Medium C2	Small C2	Large C3	Medium C3	Small C3	Total Profit
C1 Q 1%	139.125	416.375	0	0	231.76471	340	0	247.05882	0	13681.4853
C1 Q 5%	45.625	531.875	0	0	231.76471	340	89.6131	136.36029	0	13682.8571
C1 Q 10%	0	588.23529	0	0	231.76471	340	135.238	80	0	13682.8571
C2 Q 1%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
C2 Q 5%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
C2 Q 10%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
C3 Q 1%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
C3 Q 5%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
C3 Q 10%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353

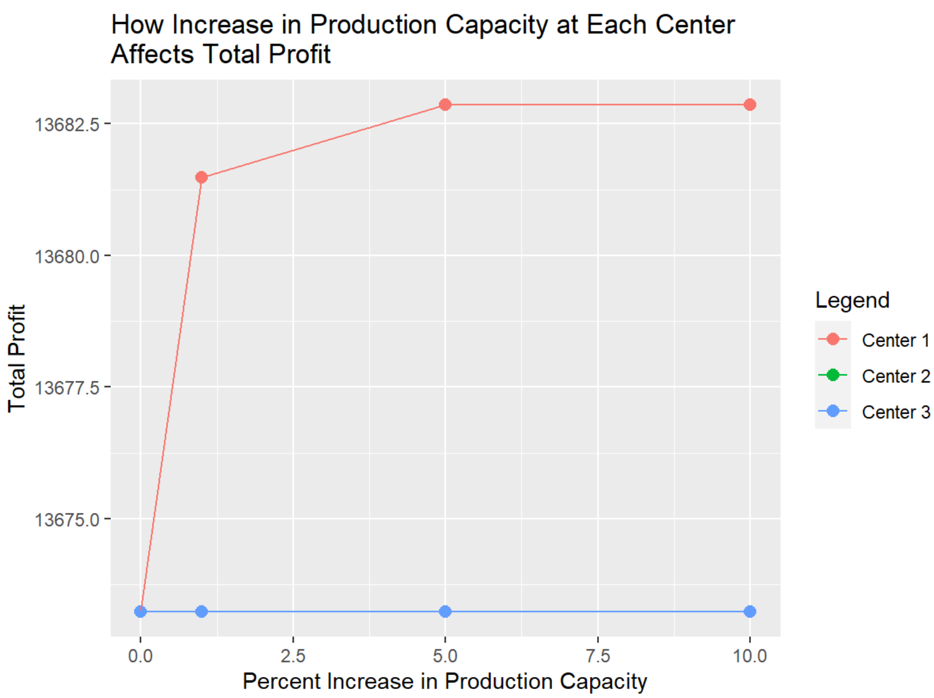
Changing Market Demand

	Large C1	Medium C1	Small C1	Large C2	Medium C2	Small C2	Large C3	Medium C3	Small C3	Total Profit
Large 1%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
Large 5%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
Large 10%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
Medium 1%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
Medium 5%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
Medium 10%	162.5	387.5	0	0	231.76471	340	0	247.05882	0	13673.2353
Small 1%	162.5	387.5	0	0	229.96471	343.4	0	247.05882	0	13685.8353
Small 5%	162.5	387.5	0	0	222.76471	357	0	247.05882	0	13736.2353
Small 10%	162.5	387.5	0	0	213.76471	374	0	247.05882	0	13799.2353

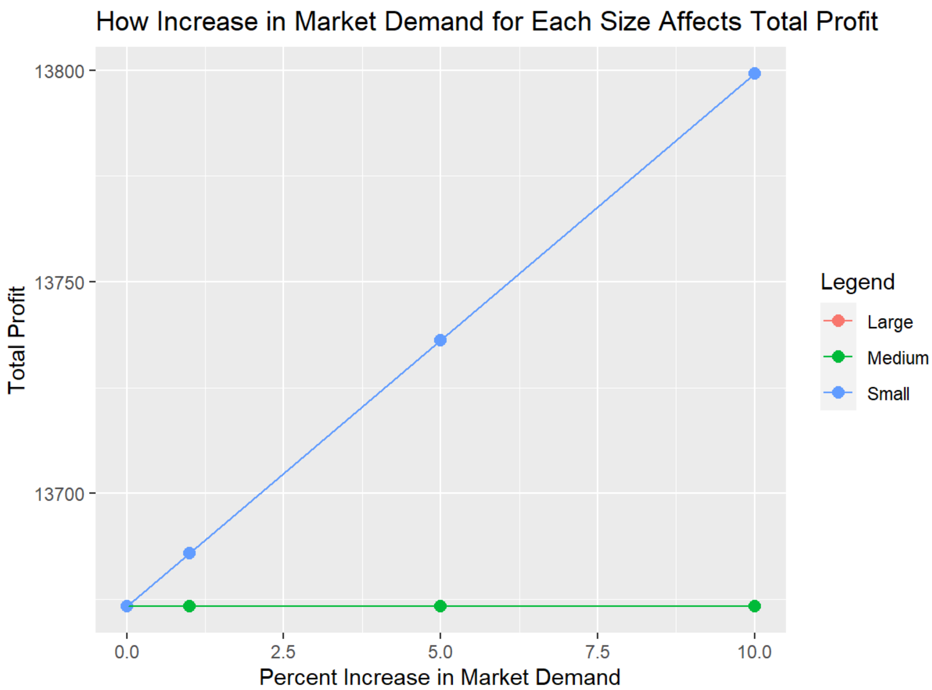
From the tables above, we can see that all of the water capacities are binding - an increase in any of the water capacities will lead to an increase in profits. In contrast, increasing the production capacities of any of the 3 centers has negligible to no effect on total profit. An increase in market demand for large or medium products will not change our profit, only an increase in market demand for the small product would lead to a minimal profit increase (less than 1% increase in profit for a 10% increase in market demand). The graphs below show how increasing each of the water capacities and production capacities for each of the three centers, and market demands for each of the three sizes, would impact total profit.



The graph above shows how increasing water capacity at each center affects the total profit. A 0% increase is equivalent to our original conditions. We can see that the water capacity at each center is binding, and increasing the water capacity at Center 1 has the largest effect on total profit. This is because Center 1 had the largest Water Capacity to being with. So a 5% increase at Center 1 is more than a 5% increase at Center 2 or 3.



The graph above shows how increasing production capacity at each center affects the total profit. We can see from the graph that the only production capacity that is binding is that of Center 1. Increasing the production capacity of Center 2 or 3 has no effect on the total profit. Only increasing the production capacity at Center 1 yields and increase in total profit, but it is important to note that this is not linear like our other scenarios. Increasing production capacity at Center 1 will only increase total profit to a certain extend, and then will flatline and not have an effect on total profit.



We can see from the graph above that the only market demand that is binding is that of the Small size of our product. Increasing the market demand by any amount for the Large or Medium sizes does not change our overall total profit.

Investigation 1.3 Despite unprecedented industry competition in 1983 and 1984, United Airlines managed to achieve substantial growth with service to 48 new airports. In 1984, revenues increased 6 percent (as compared to 1983), while costs grew less than 2 percent. A substantial component of this success was the implementation of linear programming to improve the scheduling and utilization of personnel at the airline's reservations offices and airports. At the time, United Airlines employed about 4,000 reservations sales representatives and personnel at its 11 reservations offices and about 1,000 customer service agents at its 10 largest airports. Some employees were part-time, working shifts ranging from two to eight hours; some employees were full-time, working 8- or 10-hour shifts. Shifts started at several different times, and the number of employees required at each location to provide the required level of service varied greatly during the 24-hour day, often including considerable changes even in a single half-hour period. Trying to design the work schedules for all the employees at a given location to meet these requirements most efficiently is a combinatorial nightmare. The application of mathematical modeling and linear programming to solve this problem was reported to have had "an overwhelming impact not only on United's management and members of the project team but also for many who had never before heard of management science or mathematical modeling."

In this investigation, you will explore a simplified version of this type of application of linear programming. Consider an airline personnel scheduling problem with the following constraints and requirements. First, an analysis has been made of the minimum number of customer service agents that need to be on duty at different times of the day. These numbers are illustrated in the table below.

Time Period	Minimum Number of Agents Needed
6:00 am to 8:00 am	48
8:00 am to 10:00 am	79
10:00 am to 12:00 noon	65
12:00 noon to 2:00 pm	87
2:00 pm to 4:00 pm	64
4:00 pm to 6:00 pm	73
6:00 pm to 8:00 pm	82
8:00 pm to 10:00 pm	43
10:00 pm to 12:00 midnight	52
12:00 midnight to 6:00 am	15

Next, one of the provisions in the company's contract with the union that represents the employees is that each agent works an eight-hour shift, and the authorized shift times are as follows:

- **Shift 1: 6:00 am to 2:00 pm**
- **Shift 2: 8:00 am to 4:00 pm**
- **Shift 3: 12:00 noon to 8:00 pm**
- **Shift 4: 4:00 pm to 12:00 midnight**
- **Shift 5: 10:00 pm to 6:00 am**

Finally because some shifts are less desirable than others, the wages specified in the contract differ by shift. For each shift, the daily compensation (including benefits) is shown in the table below.

Shift	**Daily Cost Per Agent**
Shift 1	**$170**
Shift 2	**$160**
Shift 3	**$175**
Shift 4	**$180**
Shift 5	**$195**

The objective is to determine how many agents should be assigned to the respective shifts each day to minimize the total personnel costs while meeting (or exceeding) the minimum service requirements.
We want to figure out how many workers should be assigned to each shift, such that each of the timeslots has at least the minimum number of necessary staff members, but minimizes the total cost of the labor in total. To minimize the overall cost of labor at $30,610, given the daily cost per agent per shift and the minimum number of agents needed at each time, we would recommend hiring 48 workers for shift 1, 31 workers for shift 2, 39 workers for shift 3, 43 workers for shift 4, and 15 workers for shift 5. This ensures that all of the criteria are met.

Formulate and solve the linear programming model for this problem.

Decision Variables								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	39	43	15			
Objective Function								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	39	43	15			
Pay	170	160	175	180	195			
Total	30610							
Constraints								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
6-8AM	48					48	>=	48
8-10AM	48	31				79	>=	79
10AM-Noon	48	31				79	>=	65
Noon-2PM	48	31	39			118	>=	87
2-4PM		31	39			70	>=	64
4-6PM			39	43		82	>=	73
6-8PM			39	43		82	>=	82
8-10PM				43		43	>=	43
10PM-mid				43	15	58	>=	52
mid-6AM					15	15	>=	15

Once you've solved this linear programming problem, consider a revised scenario in which management is in negotiations for a new contract with the union. In each case listed below, management would like to know if the proposed change will result in a new minimum total cost as compared to the original optimal solution.

(a) The daily cost per agent for shift 2 changes from $160 to $165.

Decision Variables								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	39	43	15			
Objective Function								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	39	43	15			
Pay	170	165	175	180	195			
Total	30765							
Constraints								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
6-8AM	48					48	>=	48
8-10AM	48	31				79	>=	79
10AM-Noon	48	31				79	>=	65
Noon-2PM	48	31	39			118	>=	87
2-4PM		31	39			70	>=	64
4-6PM			39	43		82	>=	73
6-8PM			39	43		82	>=	82
8-10PM				43		43	>=	43
10PM-mid				43	15	58	>=	52
mid-6AM					15	15	>=	15

If the daily cost per agent for shift 2 changes from $160 to $165, our recommended staffing for each of the shifts does not change. We will still recommend hiring 48 workers for shift 1, 31 workers for shift 2, 39 workers for shift 3, 43 workers for shift 4, and 15 workers for shift 5. This will result in a new total cost of $30,765, which is an increase of $155 per day (this is a $5 increase in pay for each of the 31 shift 2 workers). Agreeing to this change will **increase** the airline's total cost.

(b) The daily cost per agent for shift 4 changes from $180 to $170.

Decision Variables								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	33	49	15			
Objective Function								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	33	49	15			
Pay	170	160	175	170	195			
Total	30150							
Constraints								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
6-8AM	48					48	>=	48
8-10AM	48	31				79	>=	79
10AM-Noon	48	31				79	>=	65
Noon-2PM	48	31	33			112	>=	87
2-4PM		31	33			64	>=	64
4-6PM			33	49		82	>=	73
6-8PM			33	49		82	>=	82
8-10PM				49		49	>=	43
10PM-mid				49	15	64	>=	52
mid-6AM					15	15	>=	15

If the daily cost per agent for shift 4 changes from $180 to $170, this will affect the number of agents staffed for shifts 3 and 4, and will also decrease the total cost. We will still recommend hiring 48 workers for shift 1, 31 workers for shift 2, and 15 workers for shift 5. However, now, with Shift 4 workers costing less per day than shift 3 workers, we will recommend hiring 33 workers for shift 3 and 49 workers for shift 4. This will still meet the minimum requirement for each of the timeslots throughout the day, but our new total cost will be $30,150, which is $460 less than before this decrease in cost for shift 4 workers. Agreeing to this change will **decrease** the airline's total cost.

(c) The changes in parts (a) and (b) both occur.

Decision Variables								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	33	49	15			
Objective Function								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	33	49	15			
Pay	170	165	175	170	195			
Total	30305							
Constraints								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
6-8AM	48					48	>=	48
8-10AM	48	31				79	>=	79
10AM-Noon	48	31				79	>=	65
Noon-2PM	48	31	33			112	>=	87
2-4PM		31	33			64	>=	64
4-6PM			33	49		82	>=	73
6-8PM			33	49		82	>=	82
8-10PM				49		49	>=	43
10PM-mid				49	15	64	>=	52
mid-6AM					15	15	>=	15

If the changes in parts (a) and (b) both occur, we will keep the same number of workers per shift that we recommended in part (b), but the total cost will be $30,305. This is $155 higher than the scenario in part (b), due to the increase in cost for shift 2 workers that came from part (a) above. However, it should be noted that this scenario is still cheaper than our original total cost, as it is $305 less expensive than not having these changes occur. Agreeing to this change will **decrease** the airline's total cost.

(d) The daily cost per agent increases by $4 for shifts 2, 4, and 5, but decreases by $4 for shifts 1 and 3.

Decision Variables								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	39	43	15			
Objective Function								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
Workers	48	31	39	43	15			
Pay	166	164	171	184	199			
Total	30618							
Constraints								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
6-8AM	48					48	>=	48
8-10AM	48	31				79	>=	79
10AM-Noon	48	31				79	>=	65
Noon-2PM	48	31	39			118	>=	87
2-4PM		31	39			70	>=	64
4-6PM			39	43		82	>=	73
6-8PM			39	43		82	>=	82
8-10PM				43		43	>=	43
10PM-mid				43	15	58	>=	52
mid-6AM					15	15	>=	15

Somewhat surprisingly, if we increase the daily cost per agent by $4 for shifts 2, 4, and 5 but decrease the daily cost per agent by $4 for shifts 1 and 3, our recommendation for the number of agents per shift does not change, and the total cost would be $30,618, which is only $8 more expensive than not making this change. While this change might make a difference for the workers, the difference is negligible for the company. Staffing recommendations would stay the same during all times of the day, and the total cost is practically the same. Agreeing to this change will **decrease** the airline's total cost (but notably, by a negligible amount).

(e) The daily cost per agent increases by 2 percent for each shift.

Decision Variables					
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5
Workers	48	31	39	43	15

Objective Function					
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5
Workers	48	31	39	43	15
Pay	173.4	163.2	178.5	183.6	198.9
Total	31222.2				

Constraints								
	Shift 1	Shift 2	Shift 3	Shift 4	Shift 5			
6-8AM	48					48	>=	48
8-10AM	48	31				79	>=	79
10AM-Noon	48	31				79	>=	65
Noon-2PM	48	31	39			118	>=	87
2-4PM		31	39			70	>=	64
4-6PM			39	43		82	>=	73
6-8PM			39	43		82	>=	82
8-10PM				43		43	>=	43
10PM-mid				43	15	58	>=	52
mid-6AM					15	15	>=	15

If the daily cost per agent increases by 2% for each shift, we would still recommend hiring 48 workers for shift 1, 31 workers for shift 2, 39 workers for shift 3, 43 workers for shift 4, and 15 workers for shift 5. It makes sense that these numbers will not change, as the cost for each of the shifts has been raised by the same amount across the board. With this increase in cost per shift, the new total cost is $31,222.20, which is $612.20 more than our original total cost of $30,610. This is equivalent to just increasing the total cost by 2 percent. The important aspect of this scenario is that if we increase the cost of each shift by the same proportion (not the same amount, but the same percentage), the staffing recommendation will not change, and the total cost will just increase by the same percentage that the shifts were increased by. For example, if the daily cost per agent increased by 3.5 percent for each shift, we would keep our staffing recommendation the same, and the total cost would simply increase by 3.5 percent. Overall, agreeing to this change will **increase** the airline's total cost.