BOUNDARY VALUE PROBLEMS

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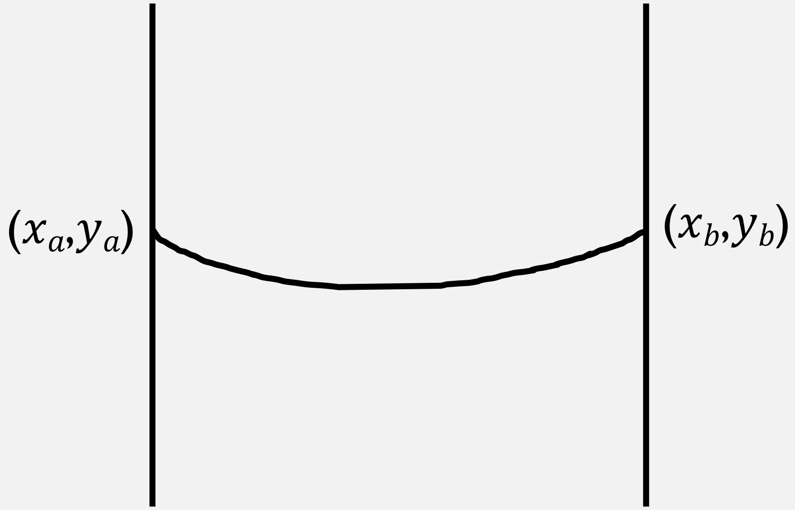
CHAIN WITH FIXED POINTS

- Suspend a chain (or rope or cord) of length L from its ends, which are fixed to the points (x_a, y_a) and (x_b, y_b) in the x-y plane. What is the curve $y(x)$ taken by the chain?



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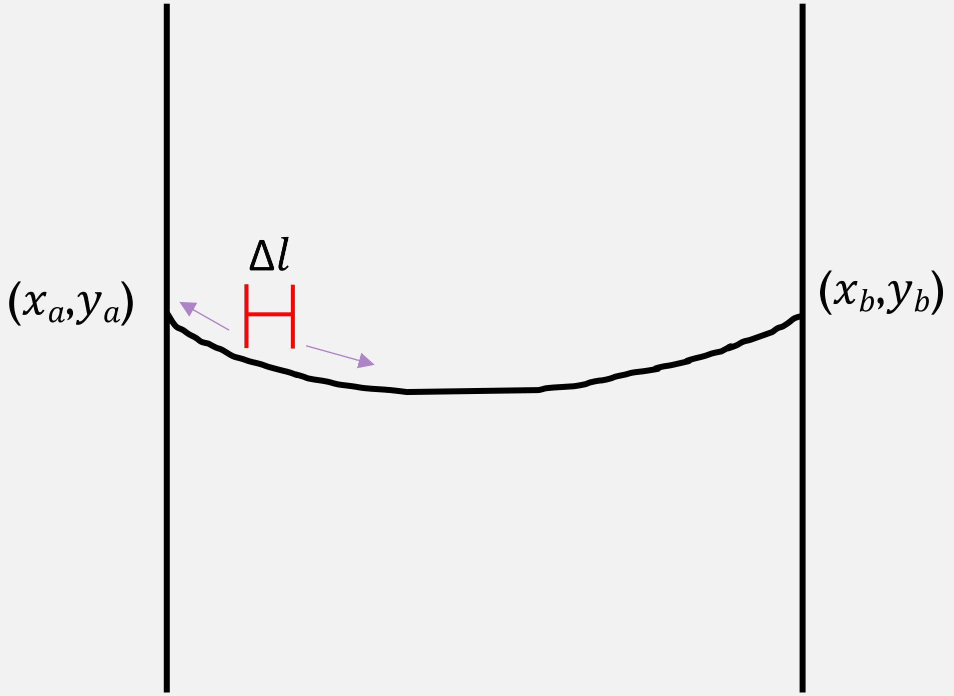

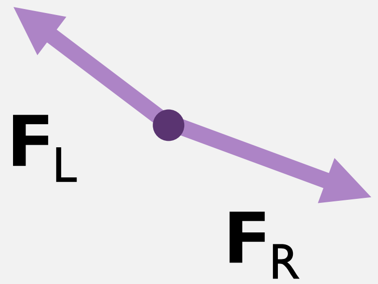

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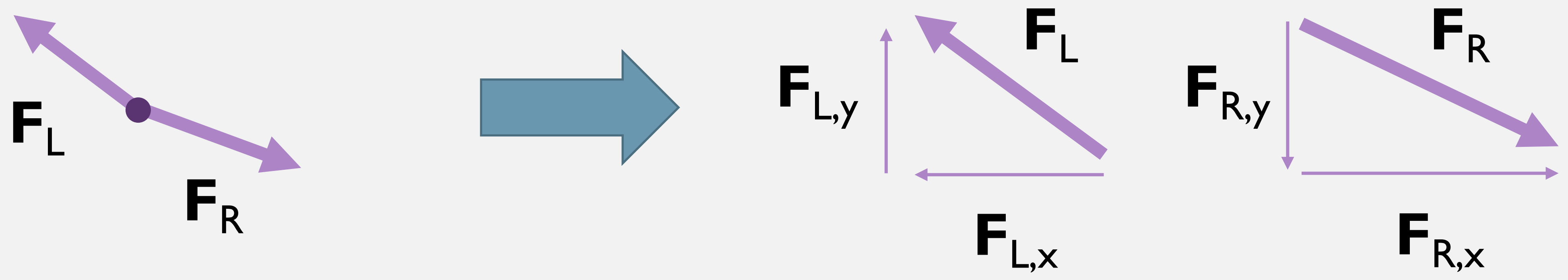

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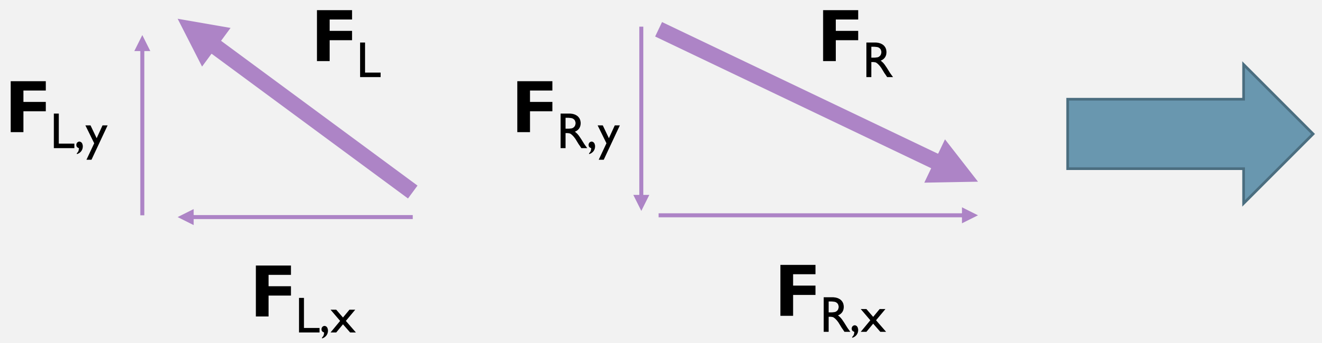

$$\begin{cases} F_{L,x} + F_{R,x} = 0 \\ F_{L,y} + F_{R,y} = mg \end{cases}$$

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- At the ends, the force vectors are tangent to the chain, giving:

$$\frac{F_{R,y}}{F_{R,x}} = \frac{dy}{dx}|_R$$

$$\frac{F_{L,y}}{F_{L,x}} = \frac{dy}{dx}|_L$$

- Combining, we get:

$$\left(\frac{dy}{dx}|_R - \frac{dy}{dx}|_R\right) F_{R,x} = mg$$

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- Using mass per unit length (μ = m/ Δℓ), Δℓ = $\sqrt{\Delta x^2 + \Delta y^2}$, and letting k = $\mu g/F_{r,x}$, we see:

$$\frac{1}{\Delta x}\left(\frac{dy}{dx}|_R - \frac{dy}{dx}|_L\right) = k\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

- Let the limit as Δℓ gives:

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

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CHAIN WITH FIXED POINTS

- Solution curve to this differential equation is a catenary!
- We need to use numerical approximation methods to solve this with code

$$\left(\frac{dy}{dx}\right)_i \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

$$\left(\frac{d^2 y}{dx^2}\right)_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y_i = \frac{1}{2}(y_{i+1} + y_{i-1}) - \frac{kh^2}{2}\sqrt{1 + (y_{i+1} - y_{i-1})^2/(2h)^2}$$

CHAIN WITH FIXED POINTS

Must use a trial solution

This is a guess for each value of y_i that satisfies the boundary conditions y_0=y_a and y_N=y_b

Step through the grid of possible values, applying our ODE to each of the interior steps (aka nodes) s.t. i=1,…,$N-1$.

THIS is a *relaxation sweep*

Result of one sweep: an improved set of values y_i

Can repeat as many times as needed until the y_i values stop changing by a significant amount, a good approximate solution

$$y_i = \frac{1}{2}(y_{i+1} + y_{i-1}) - \frac{kh^2}{2}\sqrt{1 + (y_{i+1} - y_{i-1})^2/(2h)^2}$$

WORK THROUGH A PROBLEM

Talk to your neighbors about what you're doing and why. Comment code lines that don't have comments to explain what is going on. I'll come around to check for each step.

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