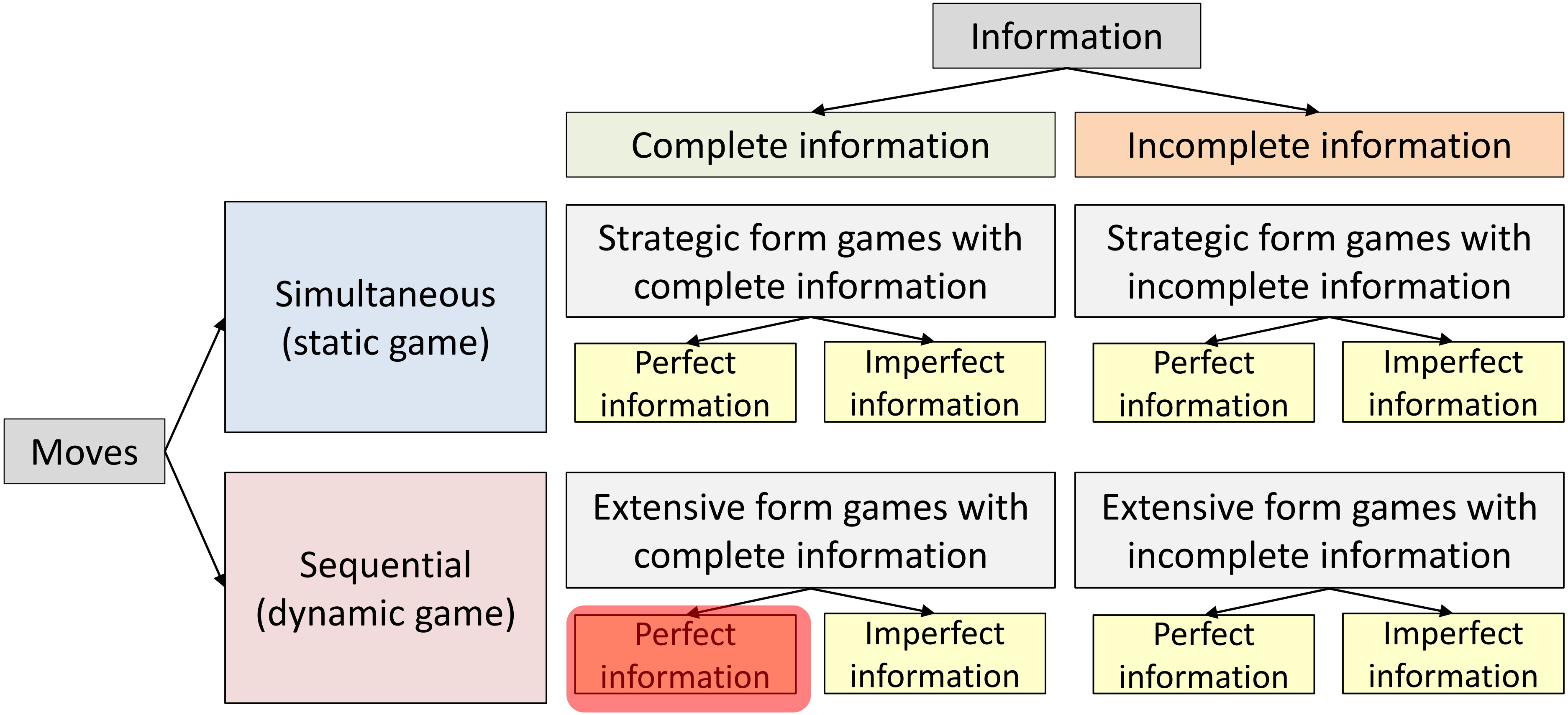
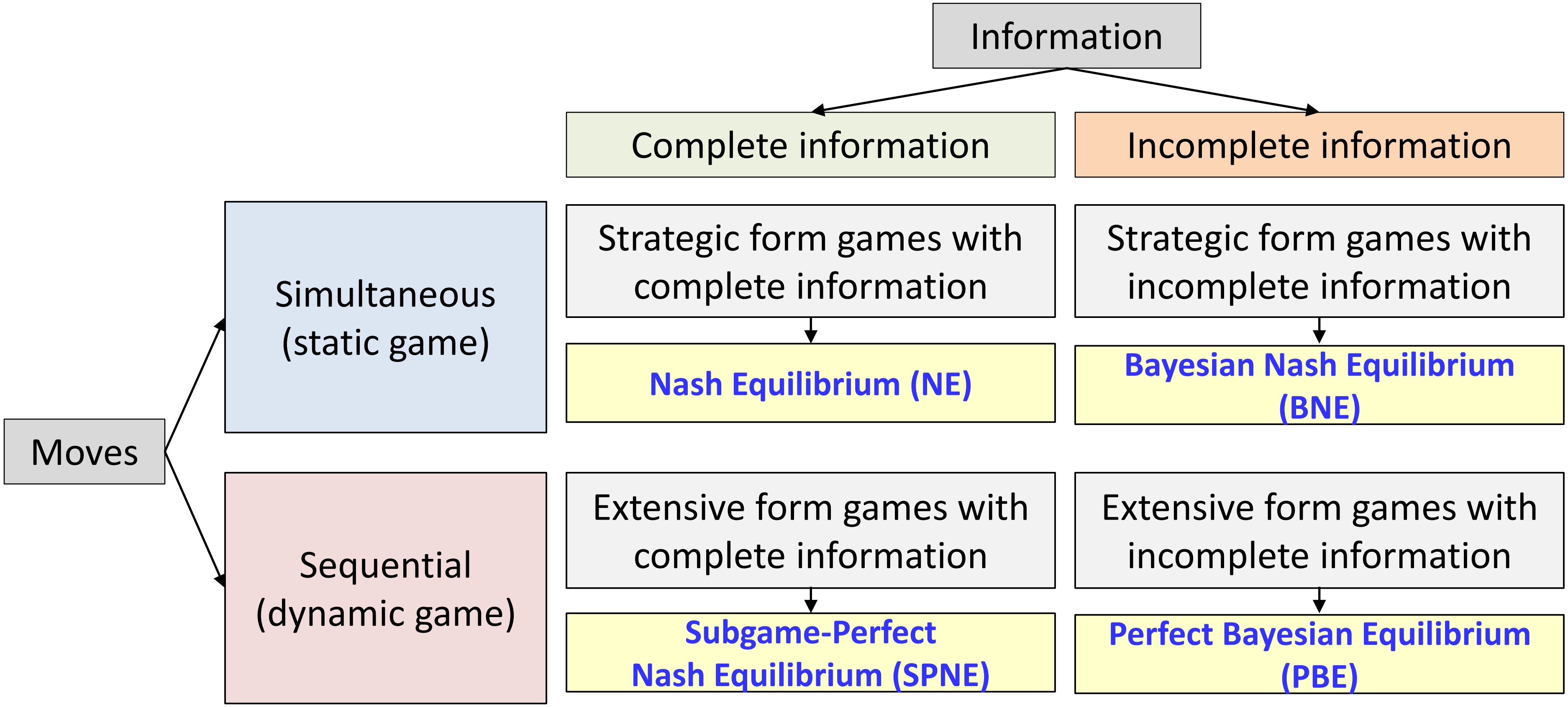
Lecture 6: Perfect information extensive-form game

Scope of this lecture



- **Incomplete Information**: Players do not have complete information about the game being played (regarding payoffs)
- **Imperfect Information**: Players do not perfectly observe the actions of other players or forget their own actions (regarding actions)
- **We will study, how each of these four classes of games can be mathematically described**

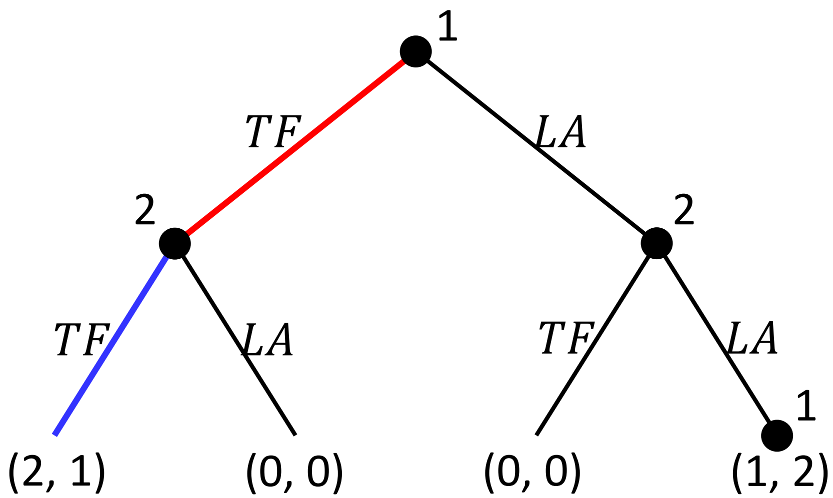
Scope of this lecture



- Corresponding to these four games, we will discusses four notions of equilibrium
- We will also study how these four classes of games used to model various problems

Motivations

		Player 2 (Husband)	
		TF	LA
Player 1 (wife)	TF	2, 1	0, 0
	LA	0, 0	1, 2



- Assumes that
  - ✓ Wife finishes work at 3:00 p.m. while husband finish work at 5:00 p.m.
- Wife can go to a movie theater for TF and call husband to come to the theater
  - ✓ If husband accept to come, they will get (2,1), otherwise they will get (0,0)
- The fundamental difference of this game with comparing to the previous simultaneous version is that
  - ✓ when husband moves he know what wife have done! (she choses what she like to see)
  - ✓ Furthermore, wife knows, by common knowledge of rationality, that husband will choose to follow her decision because it is his best response to do so (no choice….)

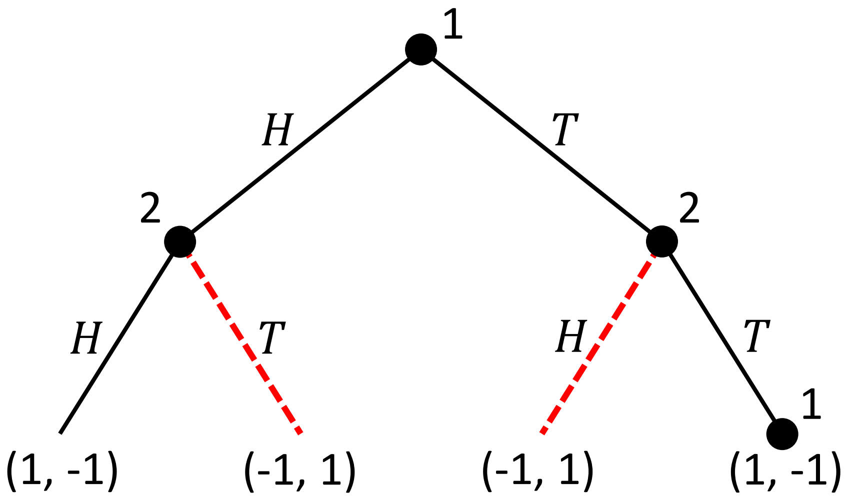
Motivations

- Is moving first always better?

- Is moving first always better?

No,

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$



- Moving first, in some case, reveals player's strategy given that other player can see first mover's action

Motivations

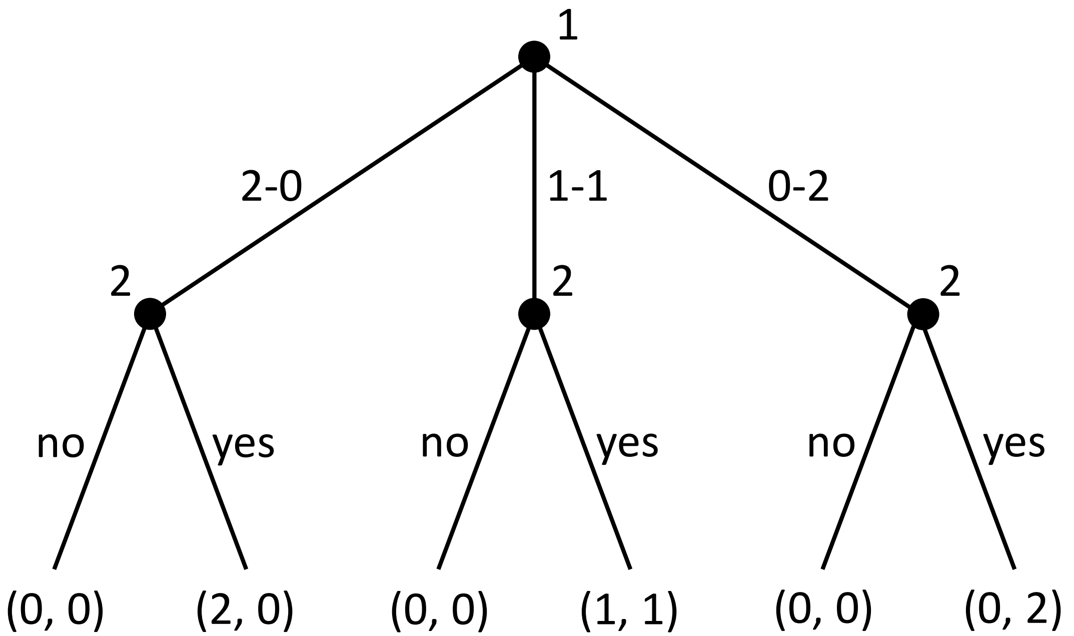
- The normal form game representation does not incorporate any notion of sequence, or time, of the actions of the players
  - Used for simultaneous game
  - Preserves game-theoretic properties
- The extensive form is an alternative representation that makes the temporal structure explicit
  - More precisely, it is not the chronological order of play that matter, but what players know when they make their choices
- Two variants:
  - perfect information extensive-form games
  - imperfect-information extensive-form games
- We will restrict our discussion to finite games, that is, to games represented as finite trees

Motivations

- **Perfect Information**
  - All players know the game structure (complete information).
  - Each player, when making any decision, is perfectly informed of all the events that have previously occurred.

- **Imperfect Information**
  - All players know the game structure (complete information).
  - Each player, when making any decision, may not be perfectly informed about some (or all) of the events that have already occurred.

Perfect-information game



The sharing games

- A perfect-information game in extensive form (or, simply, a perfect information game) is a tree in the sense of graph theory
  - Each node represents the choice of one of the players
  - Each edge represents a possible action
  - Leaves represent final outcomes over which each player has a utility function

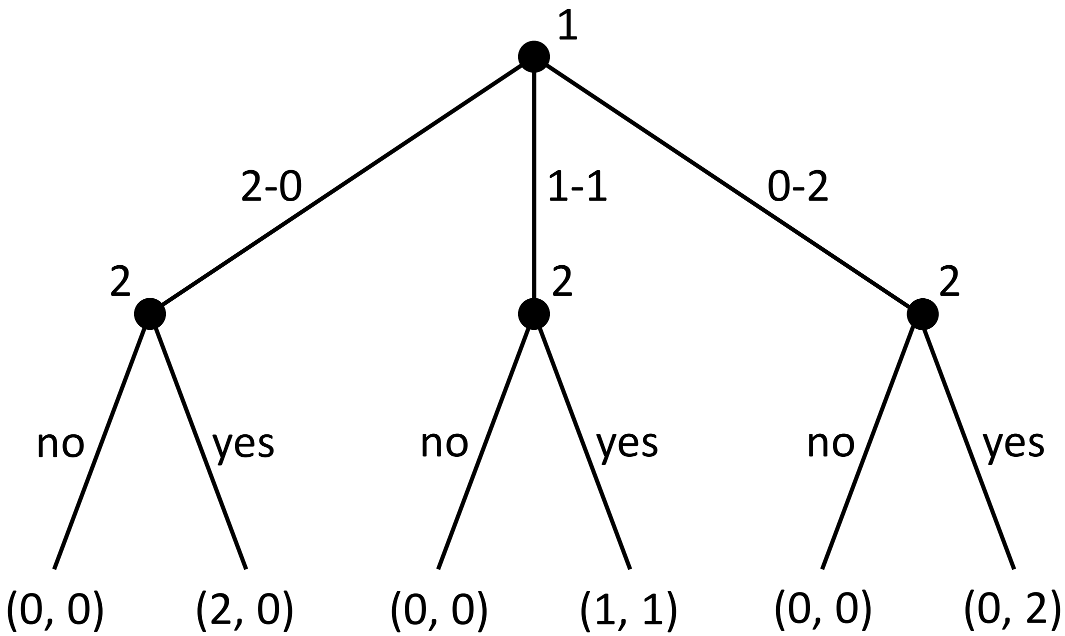
Definition

Definition (Perfect-information game)

A (finite) perfect-information game (in extensive form) is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$:

- N is a set of n players;
- A is a (single) set of actions;
- H is a set of nonterminal choice nodes;
- Z is a set of terminal nodes, disjoint from H;
- $\chi: H \longmapsto A$ is the action function, which assigns to each choice node a set of possible actions
- $\rho: H \longmapsto N$ is the player function, which assigns to each nonterminal node a player $i \in N$ who choose an action at that time
- $\sigma: H \times A \longmapsto H \cup Z$ is the successor function, which maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$; and
- $u = (u_1, \dots, u_n)$, where $u_i: Z \longmapsto \mathbb{R}$ is a real-valued utility function for player i on the terminal node Z

Pure strategies



The sharing games

- A pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player

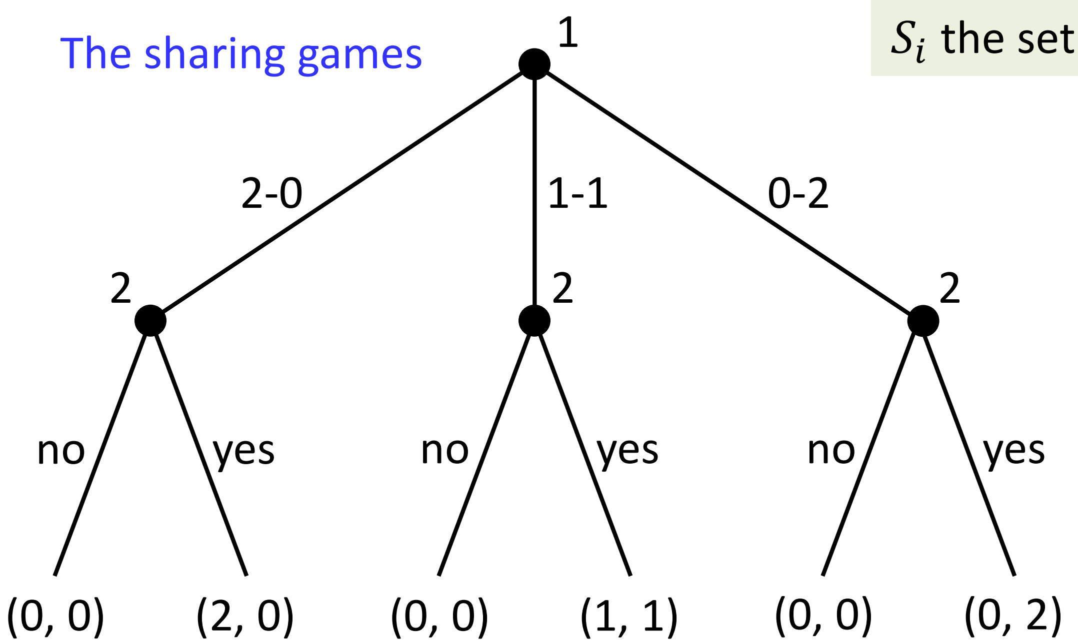
Pure strategies

Definition (Pure strategy in a perfect information game)

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player i consists of the Cartesian product $\prod_{h \in H, \rho(h)=i} \chi(h)$

- A pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player
- An agent's strategy requires a decision at each choice node, regardless of whether or not it is possible to reach that node given the other choice nodes

The sharing games



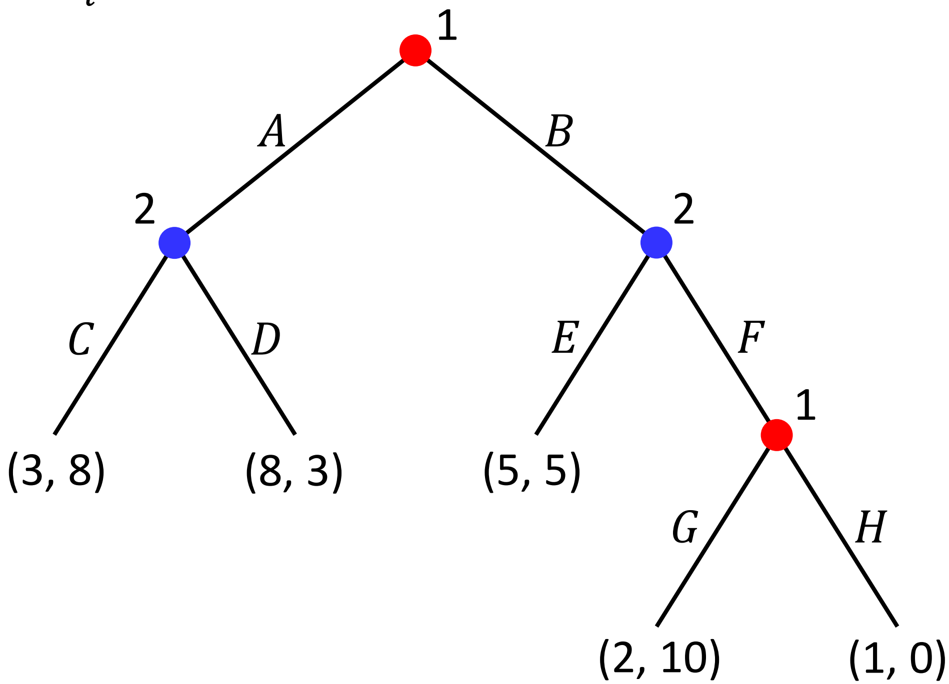
S_i the set of all pure-strategies mappings $s_i \in S_i$

$$S_1 = \{2-0, 1-1, 0-2\}$$

$$S_2 = \{ (yes, yes, yes), (yes, yes, no), (yes, no, yes), (yes, no, no), (no, yes, yes), (no, yes, no), (no, no, yes), (no, no, no) \}$$

Pure strategies example

- In summary, a pure strategy for player i is a mapping $s_i: H_i \rightarrow A_i$ that assigns an action $s_i(h_i) \in A_i(h_i)$ for every node $h_i \in H_i$ for player i. We denote by S_i the set of all pure-strategy mappings $s_i \in S_i$.



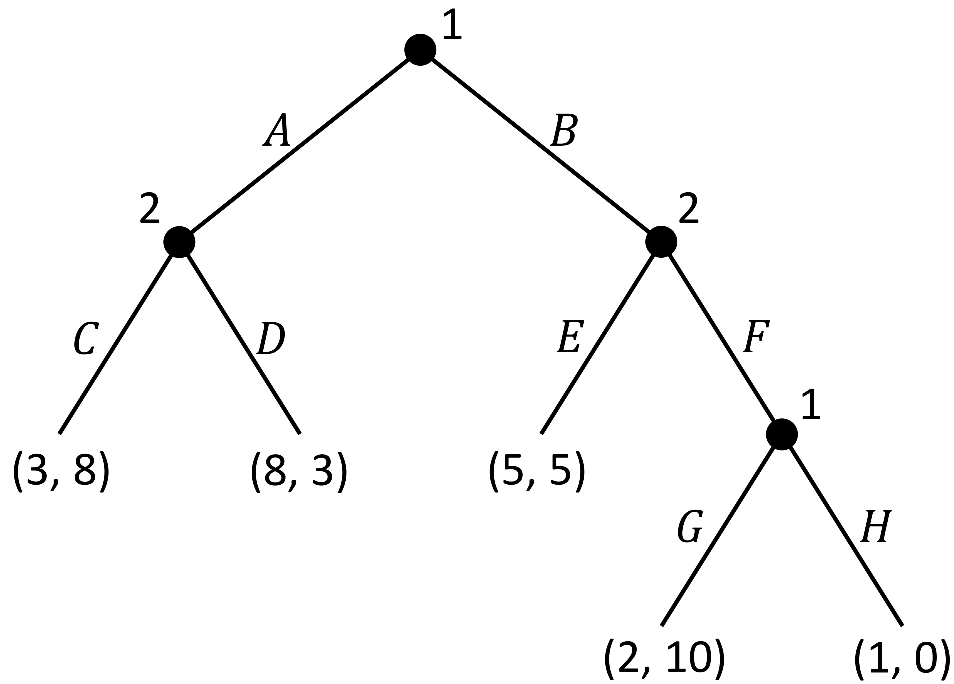
- In order to define a complete strategy for this game, each of the players must choose an action at each of his two choice node:
  - $S_1 = \{(A, G), (A, H), (B, G), (B, G)\}$
  - $S_2 = \{(C, E), (C, F), (D, E), (D, F)\}$
- It is important to note that we have to include the strategies (A, G) and (A, H) even though the choice between G and H is not available conditional on taking A

Mixed strategies definition

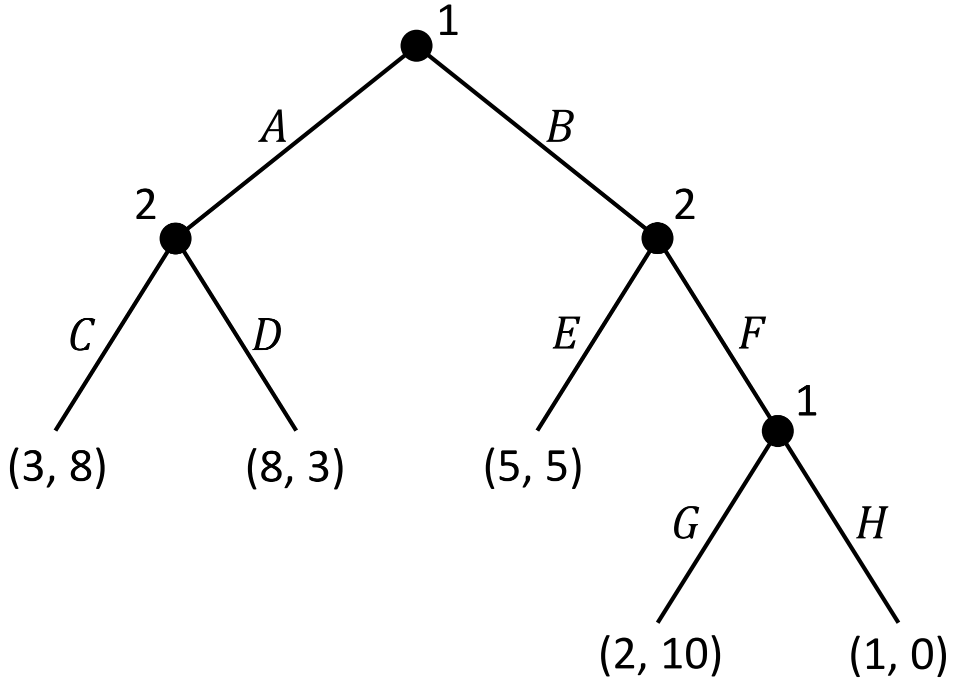
Definition (Mixed strategy in a perfect information game)

A mixed strategy for player i is a probability distribution over his pure strategies $s_i \in S_i$.

- When a mixed strategy is used, the player selects a plan randomly before the game is played and then follows a particular pure strategy
- A mixed strategy for player 2 is a probability distribution over his pure strategies $S_2 = \{(C,E),(C,F),(D,E),(D,F)\}$, for example, $\{0.15(C,E), 0.35(C,F), 0.15(D,E), 0.35(D,F)\}$
- After one of pure strategies is selected form mixed strategy (probability distribution), the player is choosing a pure plan of action



Behavior strategy motivation



- Is " If player 1 chooses A then I will play C, while if he plays B then I will mix and play E with probability 1/3" possible?

- That is, a player can randomize their action at each node they encounter as the game unfolds

Behavior strategy

Definition (Behavioral strategy)

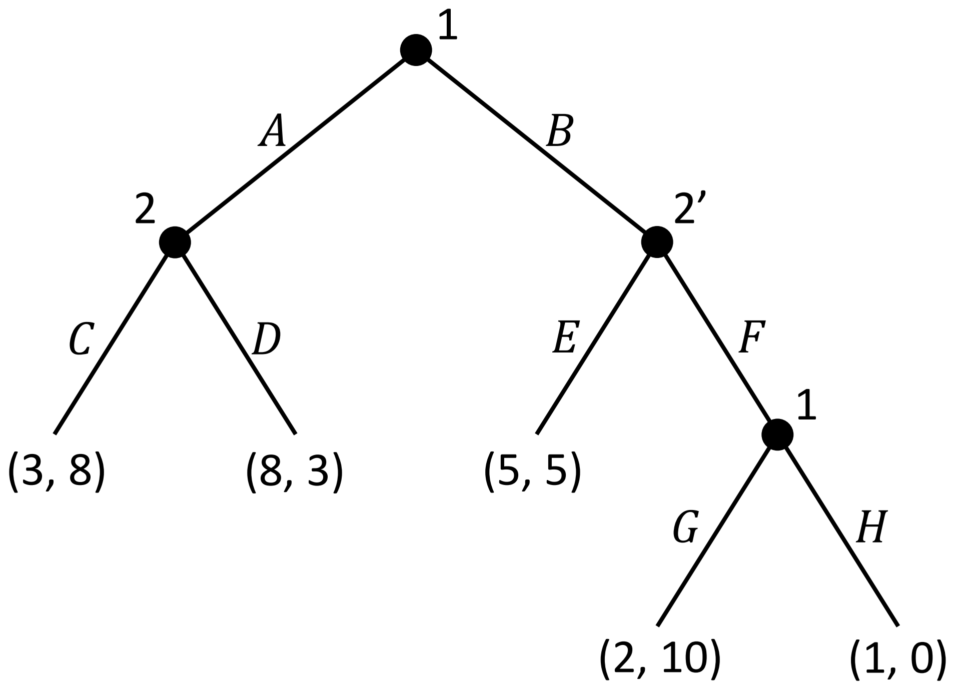
A behavioral strategy specifies for each decision node $h_i \in H_i$ an independent probability distribution over $A_i(h_i)$, action available for agent i at node h_i, and is denoted by $b_i: H_i \rightarrow \Delta A(h_i)$ where $b_i(a_i(h_i))$ is the probability that player i plays action $a_i(h_i) \in A_i(h_i)$ in node h_i.

- A behavioral strategy is more in tune with the dynamic nature of the extensive-form game.
- When using such a strategy, a player mixes among his actions whenever he is called to play

A behavioral strategy

-at node 2 : $\{0.5C, 0.5D\}$

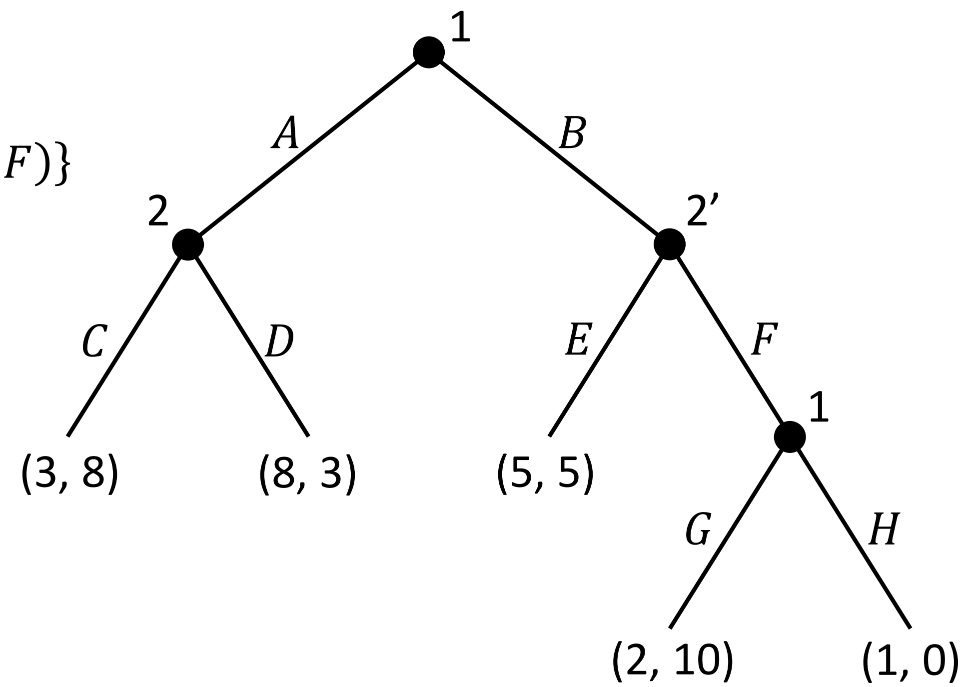
-at node 2’: $\{0.3E, 0.7F\}$



Behavior strategy

A mixed strategy for player 2,
$\{0.15(C,E), 0.35(C,F), 0.15(D,E), 0.35(D,F)\}$

A behavioral strategy
-at node 2 : $\{0.5C, 0.5D\}$
-at node 2’: $\{0.3E, 0.7F\}$



Definition (Perfect recall game)

A game of perfect recall is one in which no player ever forgets information that he previously knew

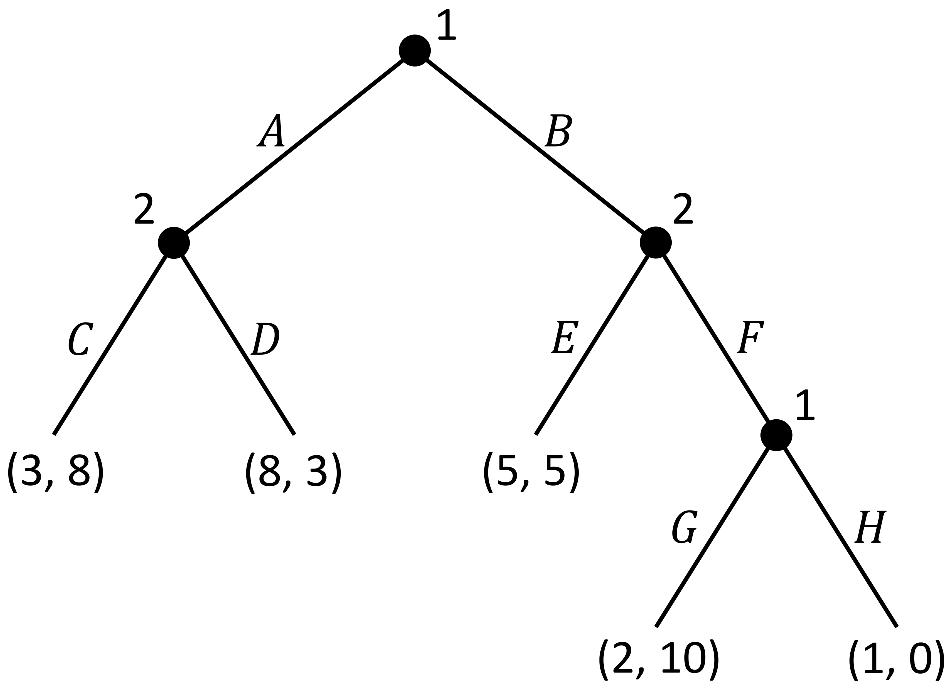
- For the class of perfect-recall games, mixed and behavioral strategies are equivalent, in the sense that given strategies of i’s opponents, the same distribution over outcomes can be generated by either a mixed or behavioral strategy of player i.

Nash equilibria

Theorem

Every (finite) perfect-information game in extensive form has a pure-strategy Nash equilibrium (PSNE).

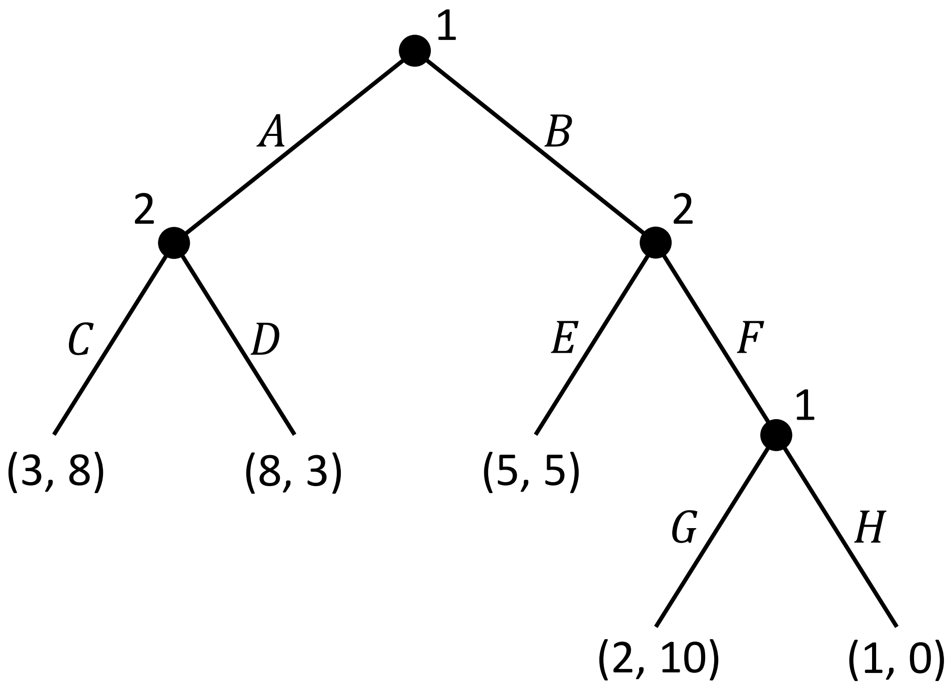
Induced normal form



	CE	CF	DE	DF
AG	$3,8$	$3,8$	$8,3$	$8,3$
AH	$3,8$	$3,8$	$8,3$	$8,3$
BG	$5,5$	$2,10$	$5,5$	$2,10$
BH	$5,5$	$1,0$	$5,5$	$1,0$

- Every perfect-information game there exists a corresponding normal-from game as bellow:
- The temporal structure of the extensive form presentation can result in a certain redundancy within the normal form game
    - E.g., we write down 16 payoff pairs instead of 5 in normal form game
    - It can result in an exponential blowup of the game representation
- While we can write any extensive-from game as a normal form game, we can't do the reverse
    - For example, matching pennies cannot be written as perfect information extensive form

Nash equilibria



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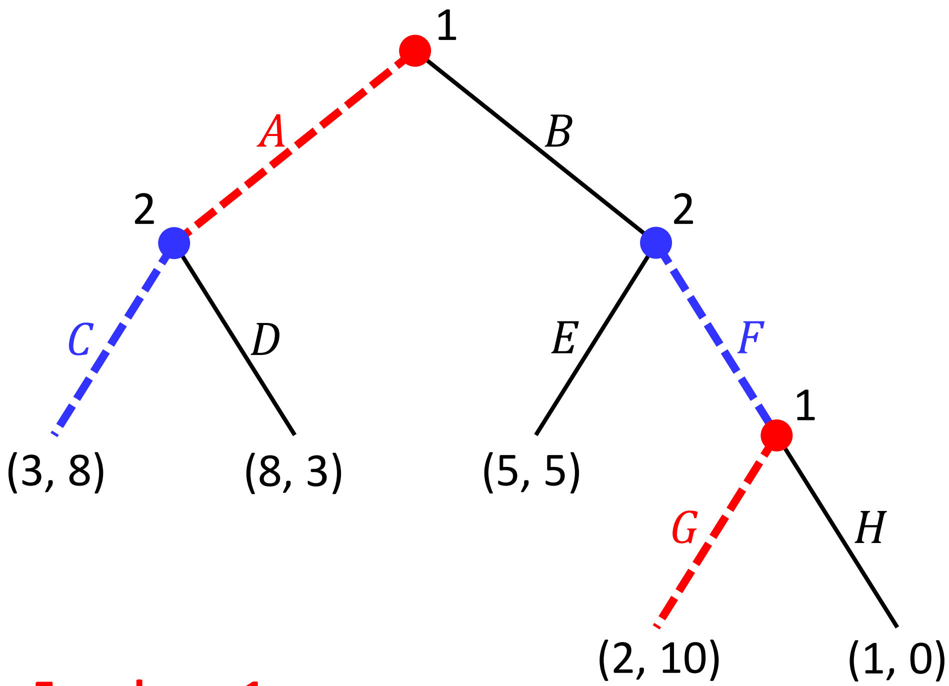
- What are the (three) pure-strategy equilibria?
  - $(A,G),(C,F)$
  - $(A,H),(C,F)$
  - $(B\ H),(C,E)$

Nash equilibria



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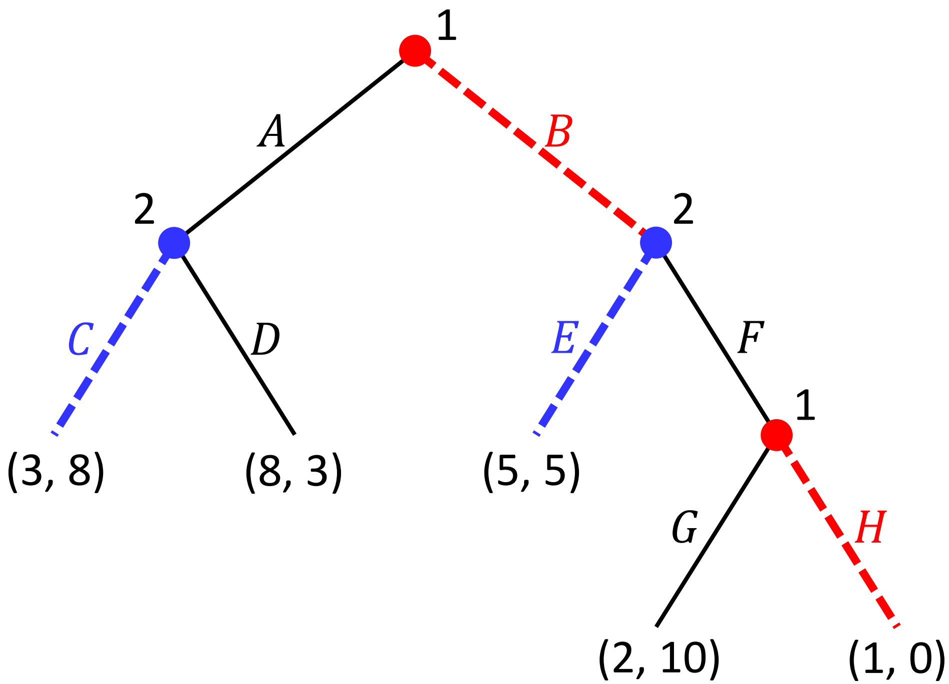
BH	$5, 5$	$1, 0$	$5, 5$	$1, 0$

- **For player 1:**
  - If player 1 plays B rather than A at the first node, he will get a payoff 2 instead of 3; thus, there is no incentive to change the action
  - If player 1 plays H rather than G at the second node, there is no change in his payoff

- **For player 2:**
  - If player 2 plays D rather than C at the first node, he will get a payoff 3 instead of 8; thus, there is no incentive to change the action
  - If player 2 plays E rather than F at the second node, there is no change in his payoff

Thus, $\{(A, G), (C, F)\}$ is an equilibrium

Nash equilibria



	CE	CF	DE	DF
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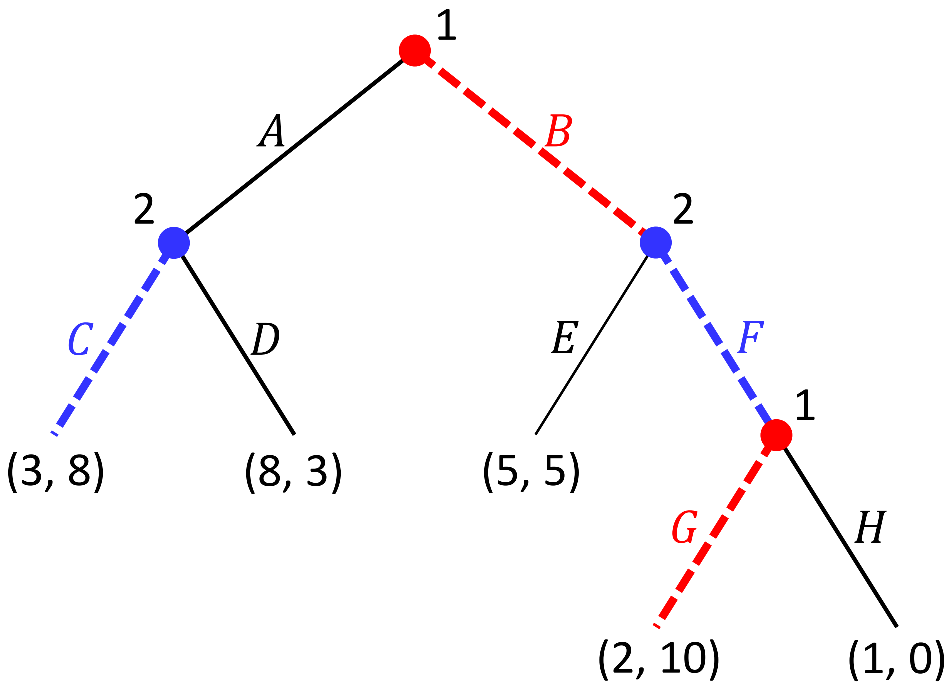
- What will happen if player 1 choose to paly BG instead BH?

Nash equilibria



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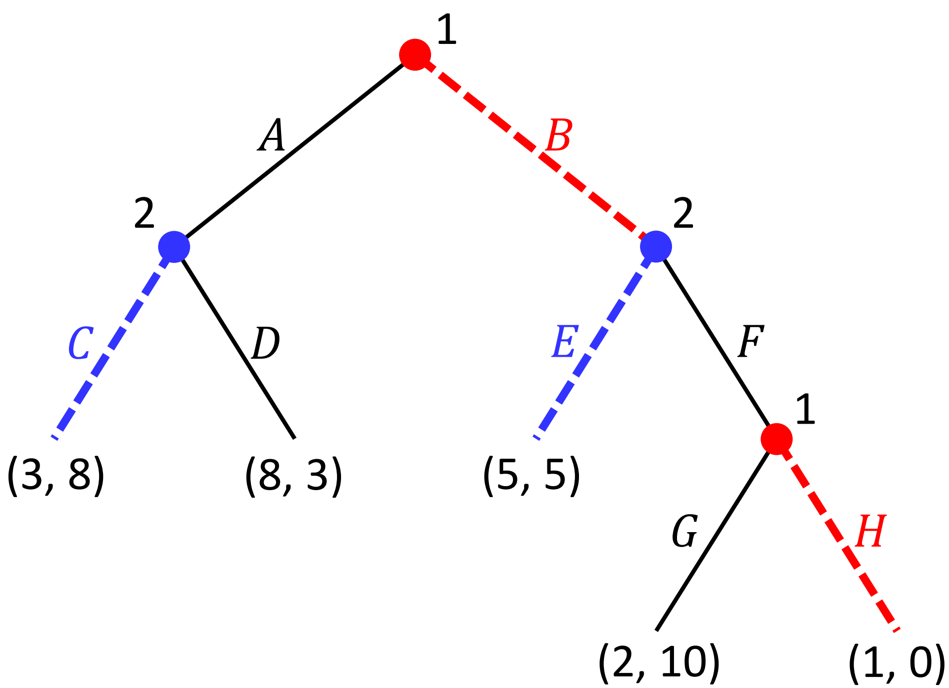
BH	$\boxed{5, 5}$	$1, 0$	$5, 5$	$1, 0$

- What will happen if player 1 choose to paly BG instead BH?
  - Player 2 will choose to play CF instead CE to get a payoff 10 instead 5
  - As a result, player 1 will get a payoff 2 instead of 5 (bad for player 1)

Nash equilibria



	CE	CF	DE	DF
AG	3, 8	3, 8	8, 3	8, 3
AH	3, 8	3, 8	8, 3	8, 3
BG	5, 5	2, 10	5, 5	2, 10
BH	$\boxed{5, 5}$	1, 0	5, 5	1, 0

- That is, player 1 is threatening player 2 to choose E by playing H by committing to choose an action that is harmful to player 2 in this second decision node
- If player 2 choose to play F, then would player 1 really follow through on his threat and play H?
  - This action is not reasonable because choosing H instead of G will reduce his payoff given that player 1 reaches that decision node

We need to define an equilibrium **refinement concept** that does not suffer from this issue

Sequential rationality

- We will insist that a player use strategies that are optimal at every node (stage) in the game tree
- We call this principle *sequential rationality,* because it implies that players are playing rationally at every strategy in the sequence of the game, whether it is on or off the equilibrium path of play

Definition (Sequentially rational)

Given strategies $s_{-i} \in \prod A_{-i}$ of i's opponents, we say that s_i is sequentially rational if and only if player i is playing a best response to s_{-i} in each of his decision node.

Subgame perfection

Definition (Subgame of G rooted at h)

Given a perfect-information extensive-form game G, the subgame of G rooted at node h is the restriction of G to the descendants of h.

Definition (Subgames of G)

The set of subgames of G consists of all of subgames of G rooted at some node in G

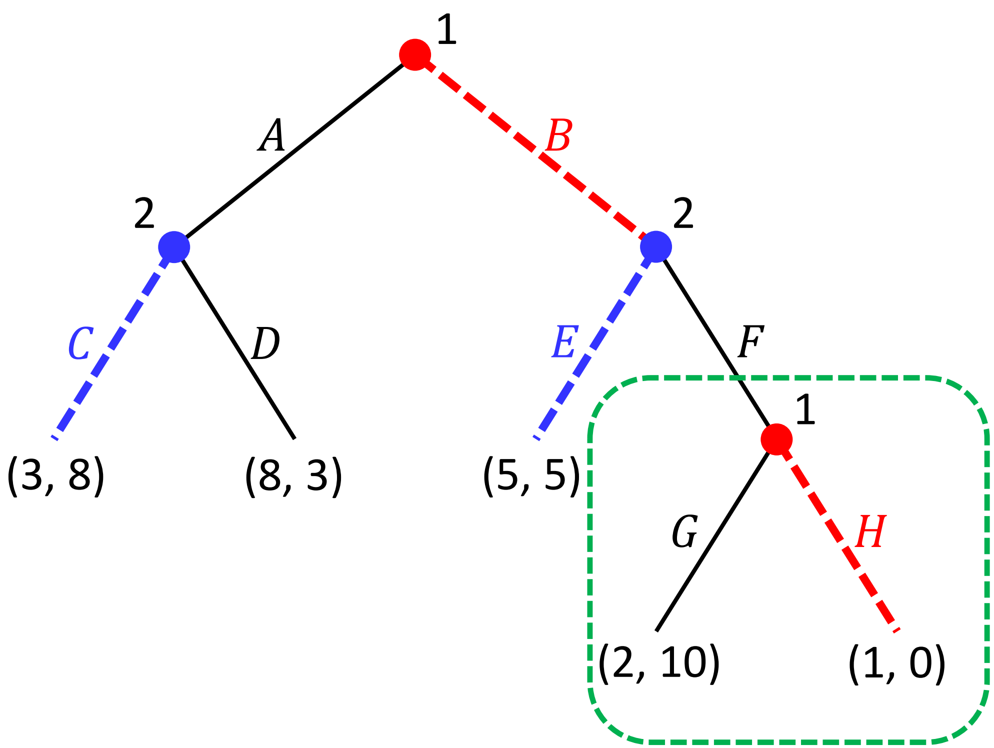
Subgame perfection

Definition (Subgame-perfect equilibrium)

The subgame-perfect equilibria (SPE) of a game G are all strategy profiles s such that for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.

- SPE is a refinement of the Nash equilibrium in perfect-information games in extensive form
- Since G is its own subgame, every SPE is also Nash equilibrium
- SPE is a stronger concept than Nash equilibrium
  (i.e., every SPE is a NE, but not every NE is a SPE)
- Every perfect-information extensive-form game has at least one subgame-perfect equilibrium
- The concept of SPE rules out unwanted Nah equilibria with "nonedible threats"
- SPE requires not only that a Nash equilibrium profiles of strategies be combination of best responses on the equilibrium path, which is a necessary condition of a Nash equilibrium, but also that the profile of strategies consist of mutual best Reponses off the equilibrium path

Subgame perfection



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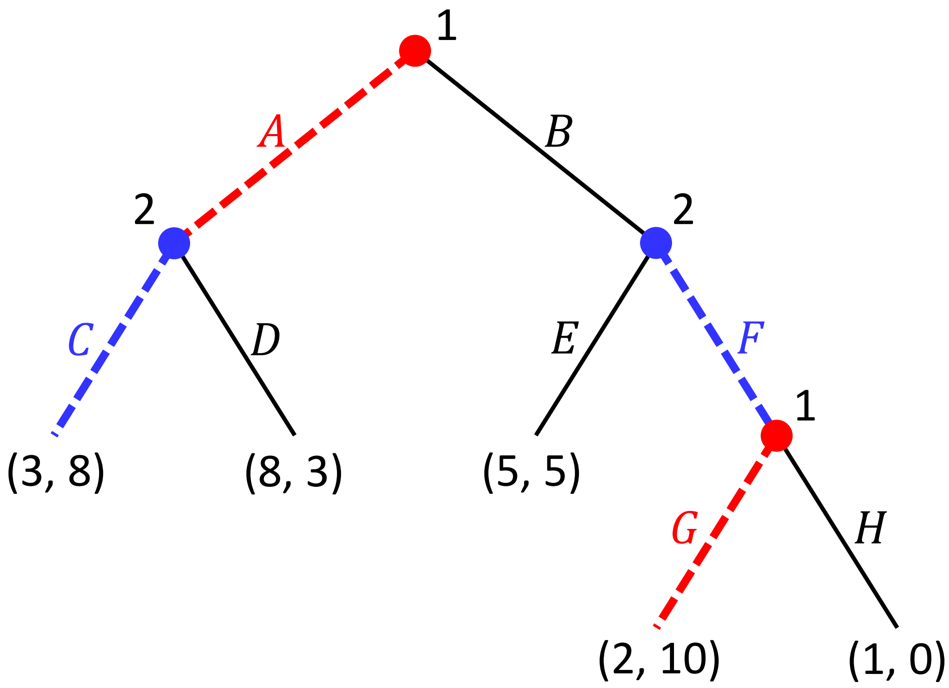
Subgame rooted player 1's second decision node

- The unique Nash equilibrium for this subgame is for paler 1 to paly G
- Thus, the action H, the restriction of the strategies (B, H) to this subgame, is not optimal in this subgame
  - (B, H) cannot be part of a subgame perfect equilibria

Subgame perfection example



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- The only SPE for this game is $\{(A, G), (C, F)\}$

Computing subgame perfect equilibria

- The principle of backward induction:
  - One identifies the equilibria in the “bottom-most” subgame tress, and assumes that those equilibria will be played as one backs up and considers increasingly large tree
  - Guarantee a subgame perfect equilibrium and simple computation
  - Can be implemented as a single depth-first traversal of the game tree and thus requires time linear in the size of the game representation
- The Nash equilibrium for a general sum game:
  - Finding Nash equilibria of general games require time exponential in the size of the normal form.
  - In addition, the induced normal form of an extensive-form game is exponentially larger than the original representation.

Computing subgame perfect equilibria

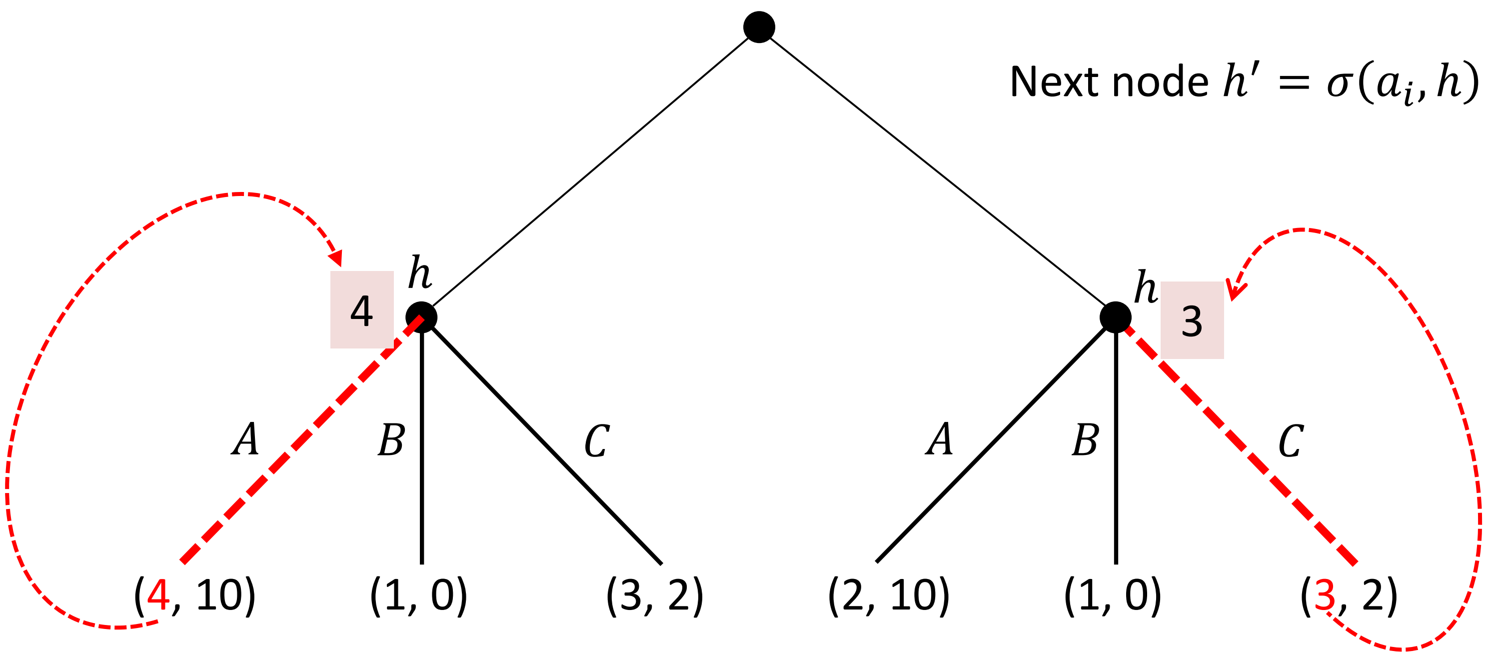
- Procedure for finding the value of a sample (subgame-perfect) Nash equilibrium of a perfect-information extensive-form game

- Every time a given player i has the opportunity to act at a given node $h \in H$ $(i.e., \rho(h) = i)$:

$$a_i^* = \underset{a_i \in \chi(h)}{\operatorname{argmax}}\, u_i(\sigma(a_i, h))$$

Available action $= \chi(h) = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$

Next node $h' = \sigma(a_i, h)$



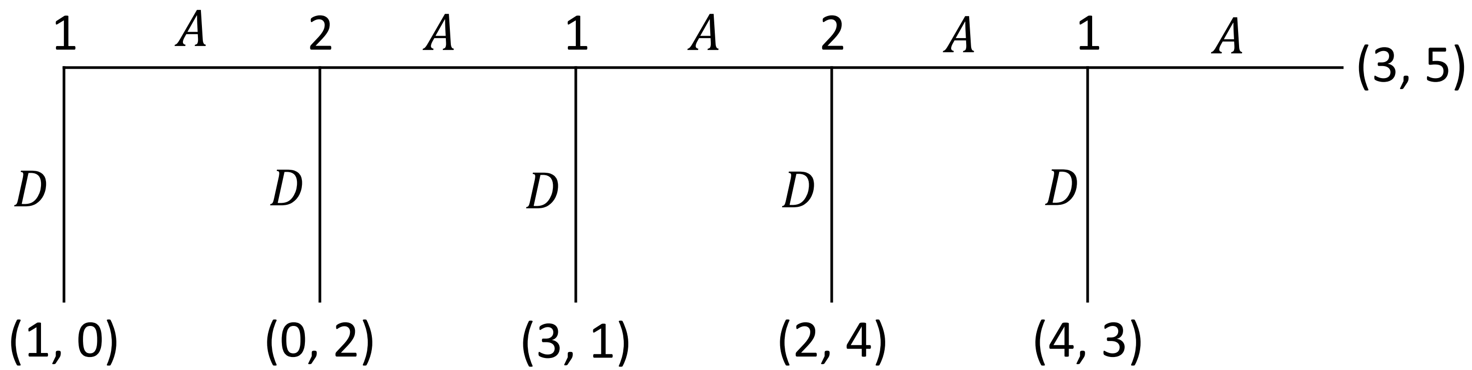
Backward induction algorithm

- Procedure for finding the value of a sample (subgame-perfect) Nash equilibrium of a perfect-information extensive-form game

```
function BackwardInduction (node h) returns u(h)
if h ∈ Z then
    return u(h)
best_util ← −∞
forall a ∈ χ(h) do
    util_at_child ← BackwardInduction(σ(h, a))
    if util_at_child_ρ(h) > best_util_ρ(h) then
        best_util ← util_at_child
return best_util
```

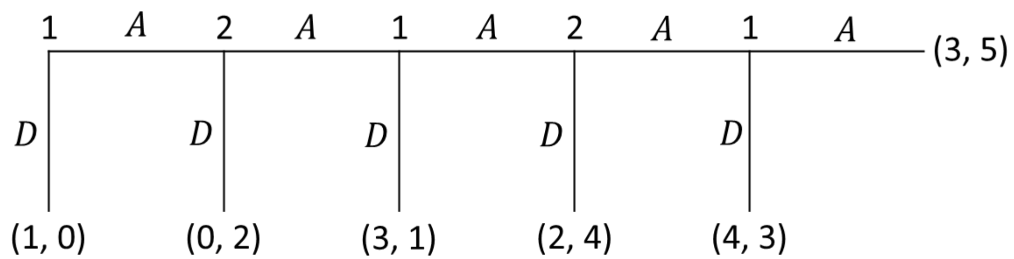
- *util_at_child* is a vector denoting the utility of each player
- The procedure doesn't return an equilibrium strategy, but rather labels each node with a vector of real numbers
  - This labeling can be seen as an extension of the game's utility function to the non-terminal nodes H
  - The equilibrium strategies: take the best action at each node.

SPE example: Centipede Game



Let’s play this game as a fun

SPE example: Centipede Game



- What happens when we use this procedure on Centipede?
  - In the only equilibrium, player 1 goes down in the first move.
  - However, this outcome is Pareto-dominated by all but one other outcome

- Two considerations:
  - Practical: human subjects don't go down right away
  - Theoretical: what should you do as player 2 if player 1 doesn't go down?
    - SPE analysis says to go down. However, that same analysis says that P1 would already have done down. How do you update your beliefs upon observation of a measure zero event?
    - But if player 1 knows that you will do something else, it is rational for him not to go down anymore… a paradox
    - There's a whole literature on this question

SPE example: Stackelberg Duopoly

- Two identical firms, players 1 and 2, produce some good
- Firm i produce quantity q_i
- Cost for production is $c_i(q_i) = c_i q_i$
- Price is given by $d = a - b(q_1 + q_2)$
- The profit of company i given its opponent chooses quantity q_j is

$$u_i(q_i, q_j) = (a - bq_i - bq_j)q_i - c_i q_i = -bq_i^2 + (a - c_i)q_i - bq_j q_i$$

- The best-response function for each firm is given by the first-order condition

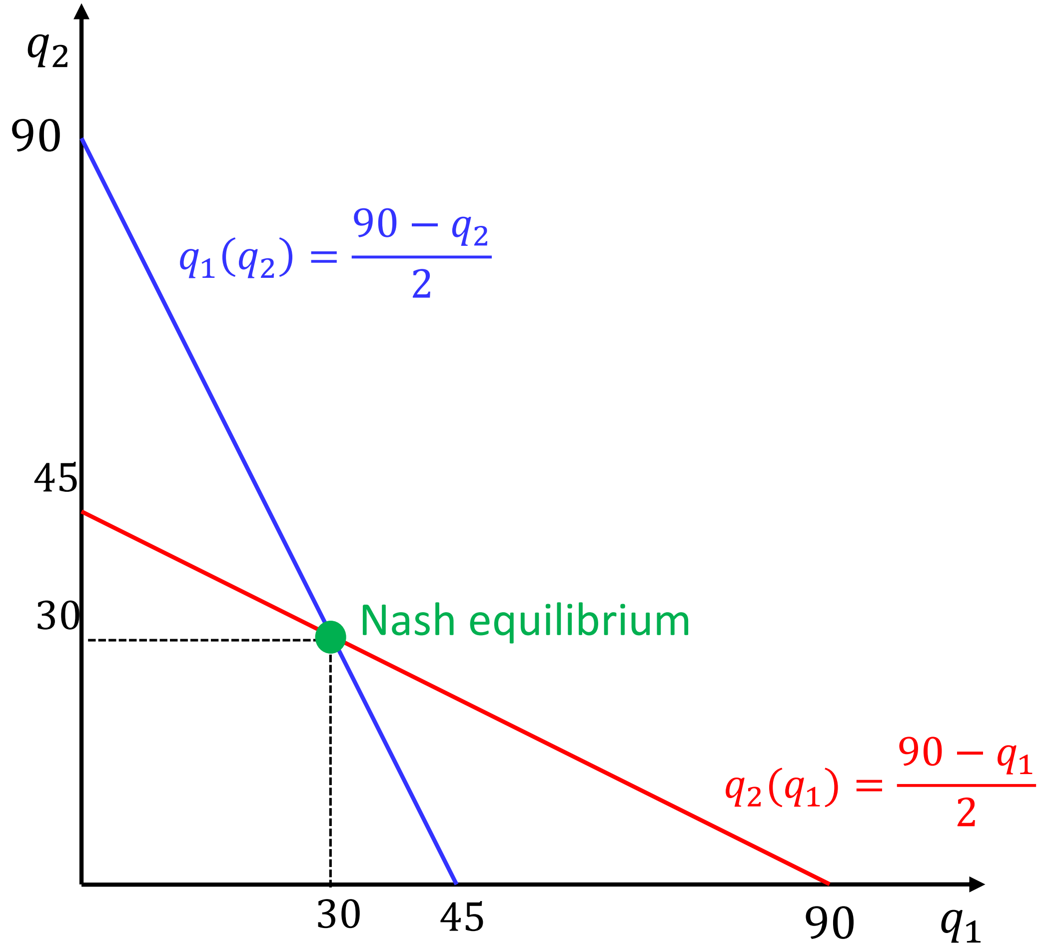
$$BR_i(q_j) = \frac{a - bq_j - c_i}{2b}$$

SPE example: Stackelberg Duopoly

- In case there are two firms, we have two best-response equations:

$$q_1 = \frac{a - bq_2 - c_1}{2b} \quad \text{and} \quad q_2 = \frac{a - bq_1 - c_2}{2b}$$

$a = 100, b = 1, c_1 = c_2 = 10$



SPE example: Stackelberg Duopoly

- Now, assume player 1 will choose q_1 first and player 2 will observe the choice made by player 1 before it makes its choice of q_2

- Assume player 1 choose q_1, then player 2 will best respond to it

$$q_2(q_1) = \frac{90 - q_1}{2}$$

- Assuming common knowledge of rationality, what should player 1 do?
  - ✓ Player 1 know exactly how a rational player 2 would respond to tits choice of q_1
  - ✓ Thus, player 1 will replace the fixed q_2 in its profit function with the best response of firm 2 and choose q_1 to solve

$$\max_{q_1} [100 - q_1 - q_2] q_1 - 10 q_1$$
$$\Rightarrow \max_{q_1} \left[100 - q_1 - \left(\frac{90 - q_1}{2}\right)\right] q_1 - 10 q_1$$
$$\Rightarrow 100 - 2q_1 - 45 + q_1 - 10 = 0$$
$$\Rightarrow q_1 = 45, q_2 = 22.5$$

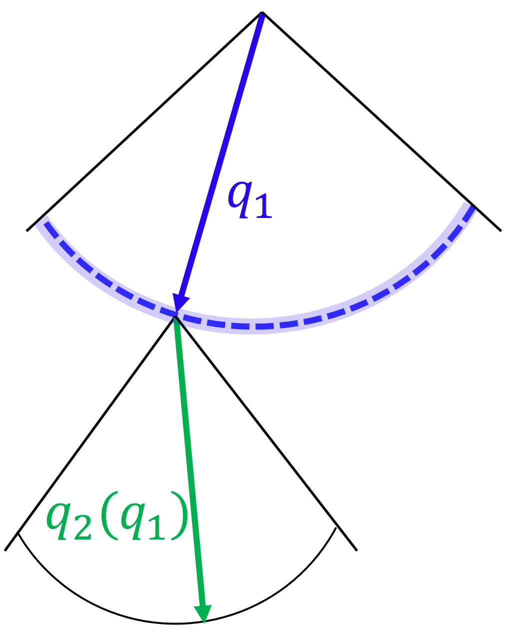
- $u_1(45{,}22.5) = (100 - 45 - 22.5) \times 45 - 10 \times 45 = 1012.5 > 900\ (\mathrm{NE})$
- $u_2(45{,}22.5) = (100 - 22.5 - 45) \times 22.5 - 10 \times 22.5 = 506.25 < 900 (\mathrm{NE})$

First-mover advantage

SPE example: Stackelberg Duopoly

- If a profile of strategies survives backward induction then this profile is also a subgame-perfect equilibrium (SPE), and in particular Nash equilibrium (NE)

- Be careful to specify the strategies of player 2:
  - Player 2 has a continuum of information sets, each being a particular choice of q_1
  - We must specify q_2 for every information set, each of which correspond to an action q_1 chosen by player 1



$$\text{SPE} = \left(q_1, q_2(q_1)\right) = \left(45, \frac{90 - q_1}{2}\right)$$

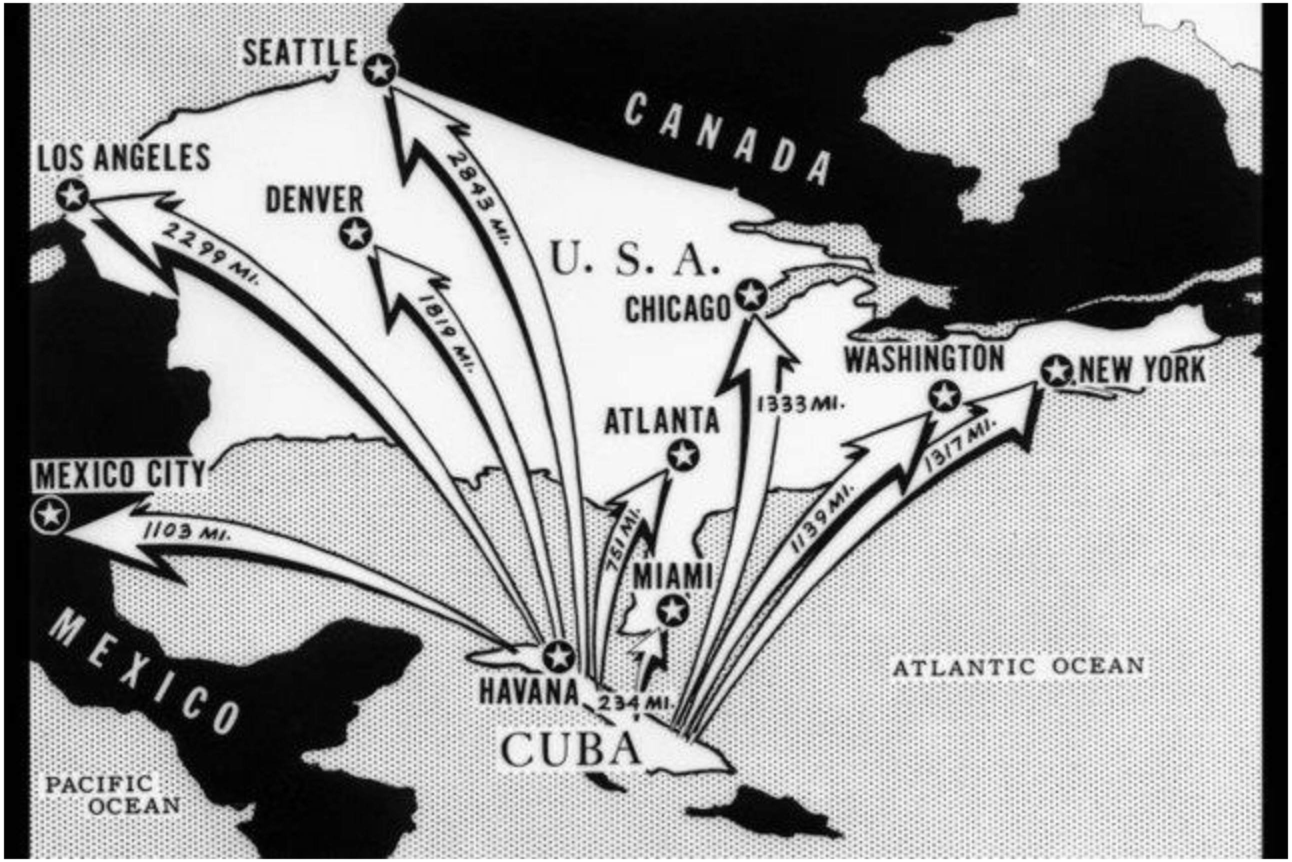
$$\cancel{(45, 22.5)}$$

$$u_1(q_1, q_2) = (100 - q_1 - q_2)q_1 - 10q_1$$

$$u_2(q_1, q_2) = (100 - q_1 - q_2)q_2 - 10q_2$$

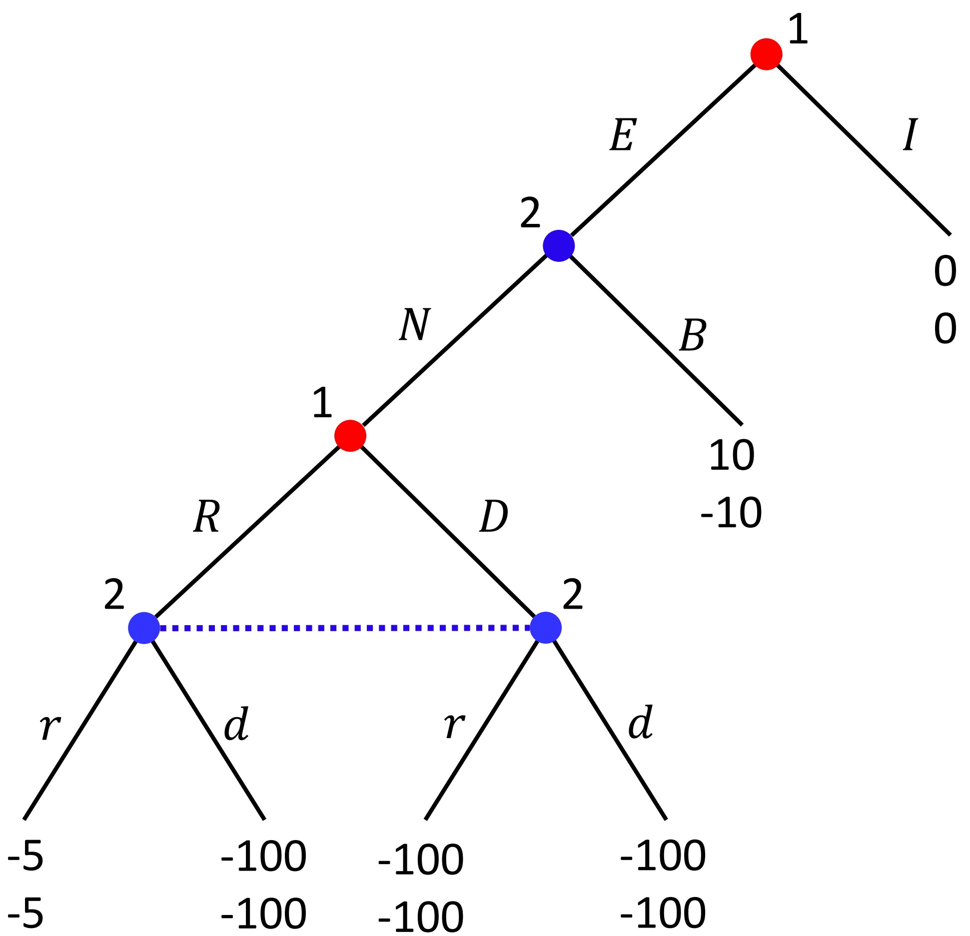
SPE example: Mutually Assured Destruction

Cuban missile crisis of 1962



SPE example: Mutually Assured Destruction

Cuban missile crisis of 1962



Player 1 (U.S.):

- Ignore incident (I)
- Escalate situation (E)

Player 2 (USSR.):

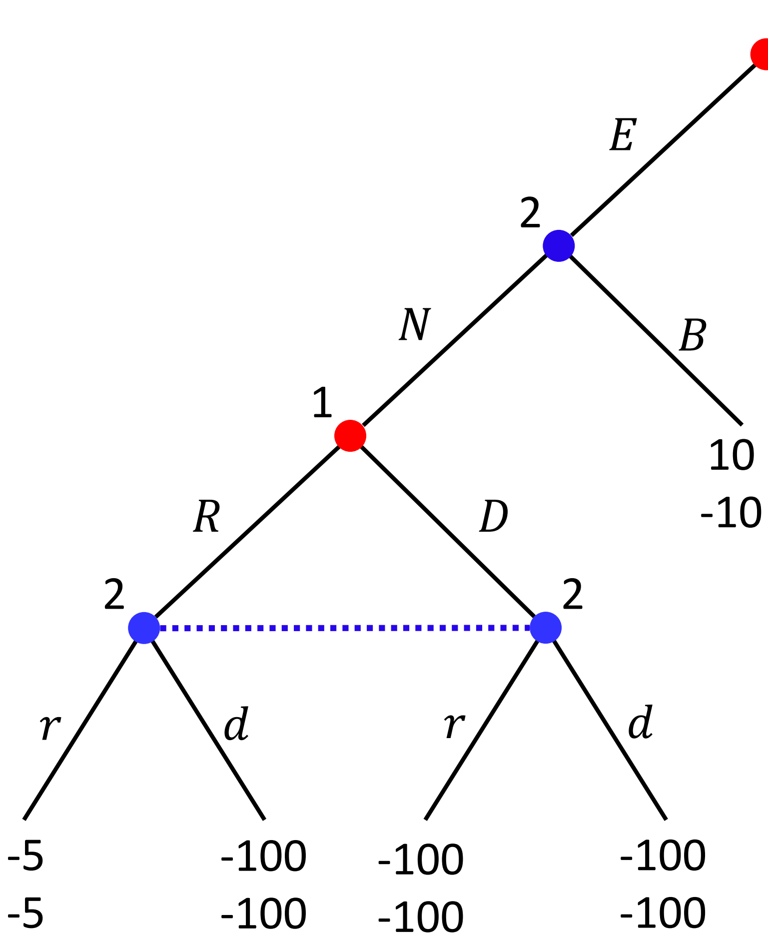
- Nuclear confrontation(N)
- Back down (B)

Player 1 & Player 2 (War game):

- Retreat (R for PL1 and r for PL2)
- Doomsday (D for PL1 and d for PL2)

SPE example: Mutually Assured Destruction

- Convert the extensive form game into a normal form game



	Br	Bd	Nr	Nd
IR	0, 0	0, 0	0, 0	0, 0
ID	0, 0	0, 0	0, 0	0, 0
ER	10, -10	10, -10	-5, -5	-100, -100
ED	10, -10	10, -10	-100, -100	-100,-100

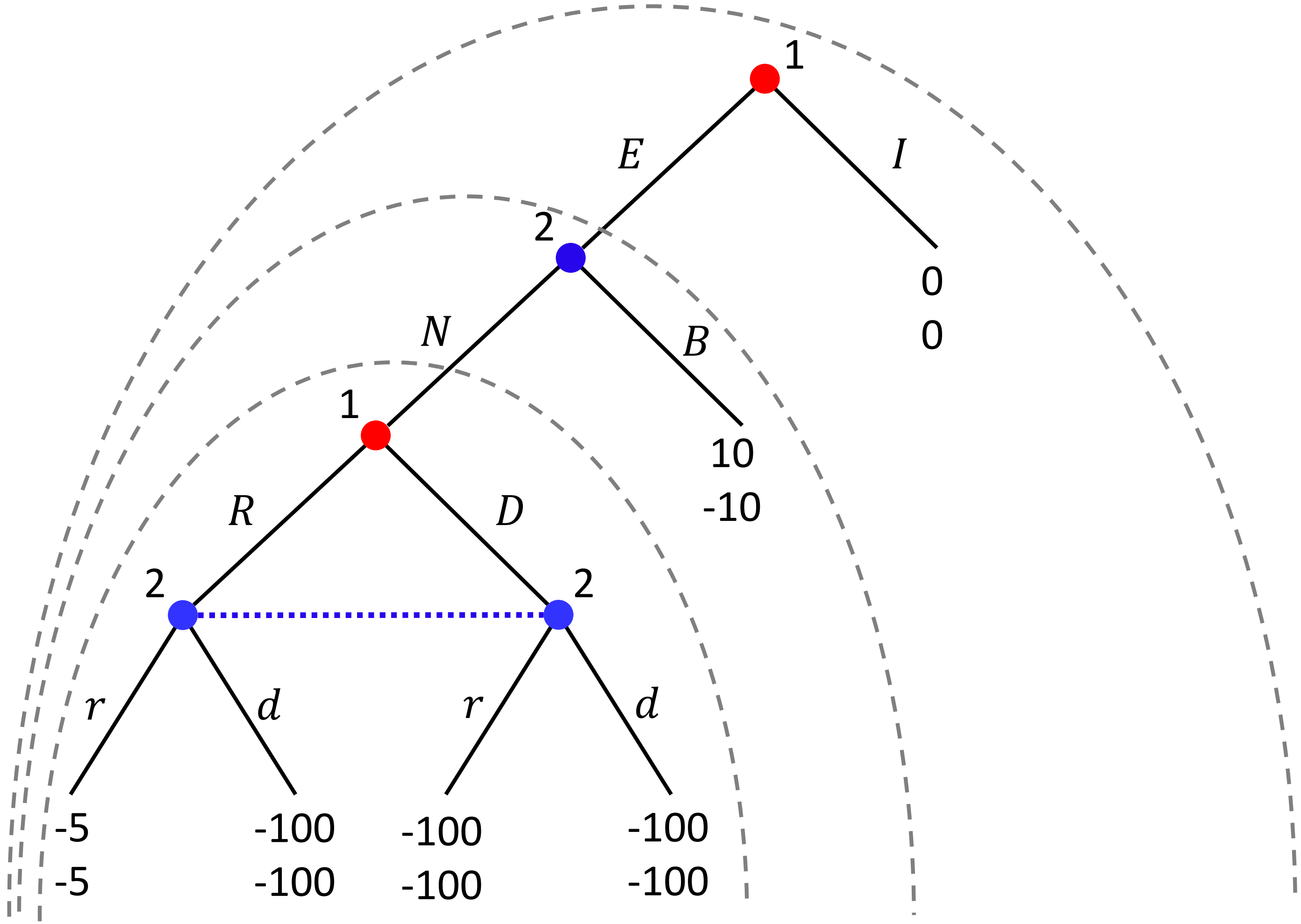
Are they subgame perfect equilibrium?

- There are six pure strategy Nash equilibria:

$$\text{NE}s = \{(IR, Nr), (IR, Nd), (ID, Nr), (ID, Nd), (ED, Br), (ED, Bd)\}$$

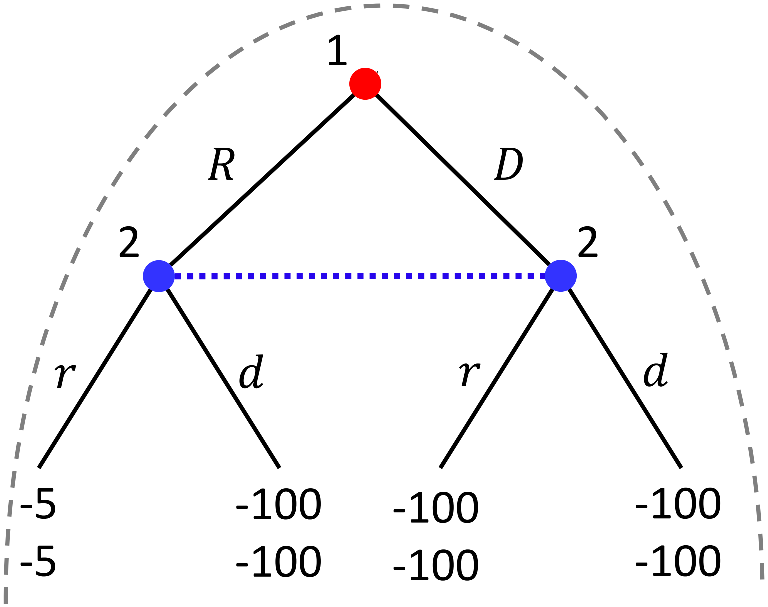
SPE example: Mutually Assured Destruction

- Find the subgame-perfect equilibria using backward induction



SPE example: Mutually Assured Destruction

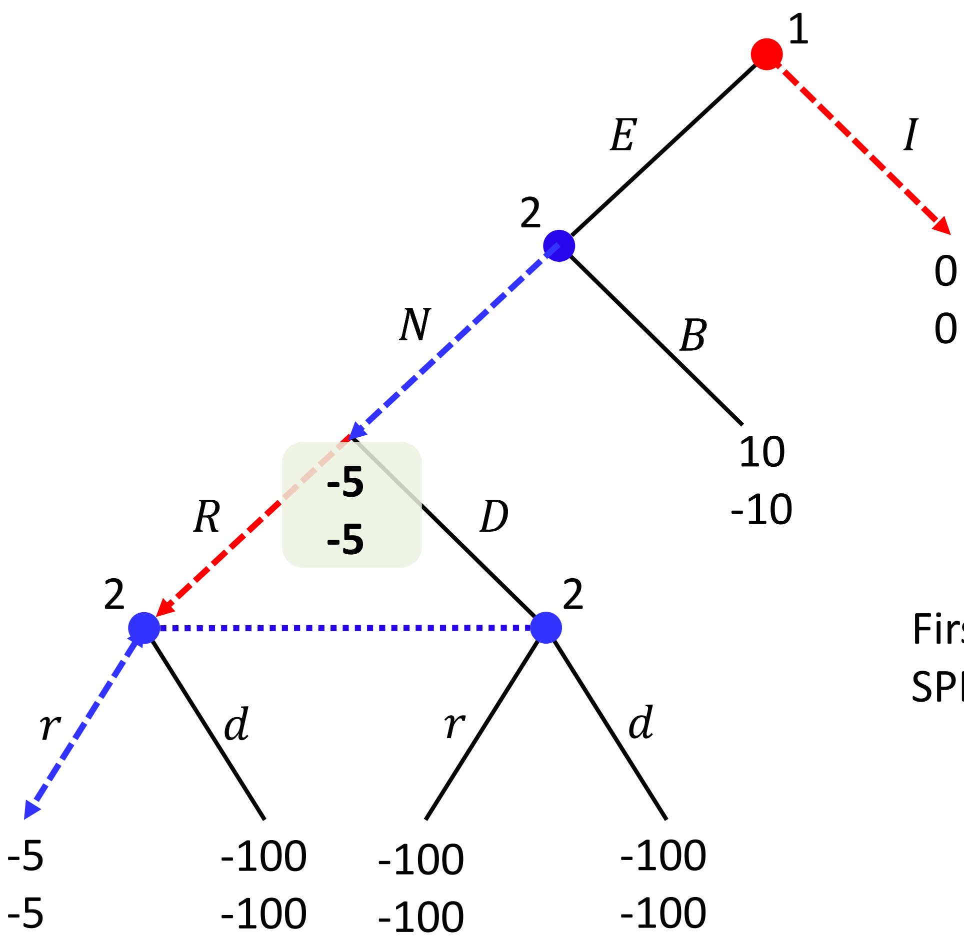
- Find the subgame-perfect equilibria using backward induction



	r	d
R	$\mathbf{-5, -5}$	$-100, -100$
D	$-100, -100$	$\mathbf{-100, -100}$

SPE example: Mutually Assured Destruction

- Find the subgame-perfect equilibria using backward induction

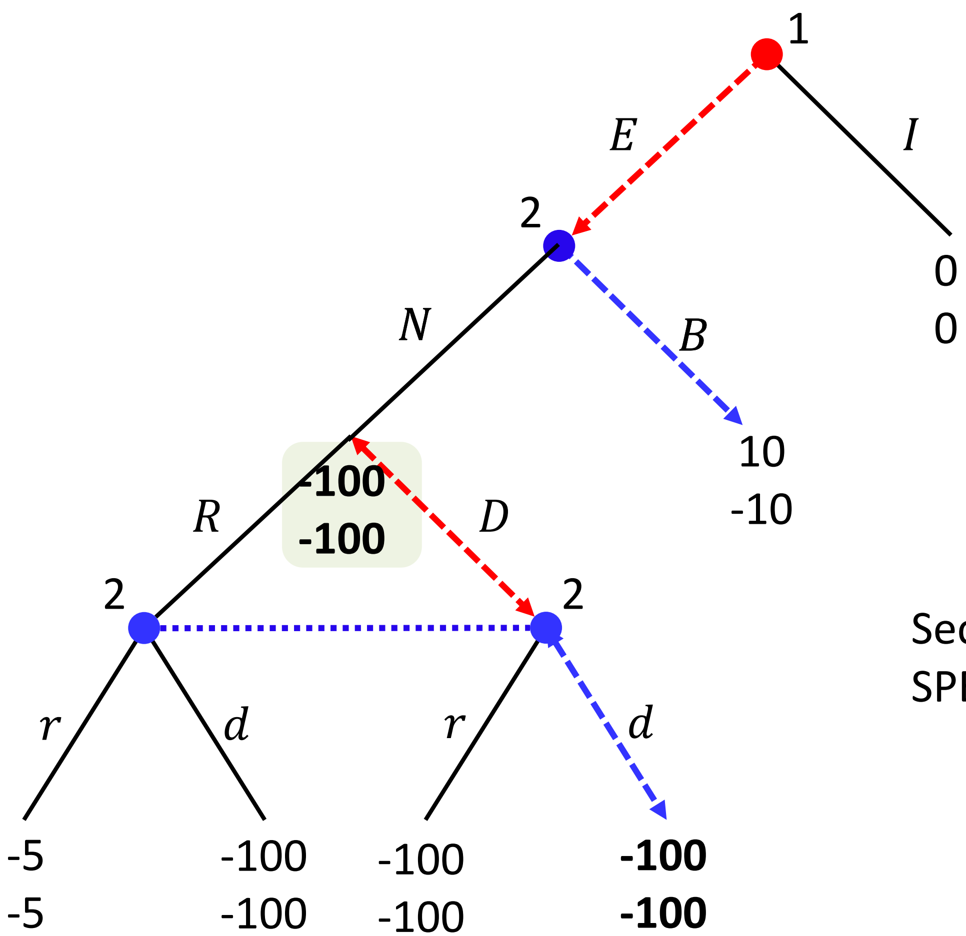


First subgame perfect equilibrium is SPE1 =(IR, Nr)

Case 1: NE1=(R, r) is used at the last subgame

SPE example: Mutually Assured Destruction

- Find the subgame-perfect equilibria using backward induction



Second subgame perfect equilibrium is SPE2 $=(ED, Bd)$

Case 2: NE2=(D, d) is used at the last subgame