Lecture5-Computing Solution Concepts for Normal Form Games

Motivations

- So far, we have ignored the issues of computation for finding equilibriums
- How hard is it to compute the Nash equilibria of a game?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
 $dom \leftarrow true$
 for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i **do**
  if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
   $dom \leftarrow false$
   break
  end if
 end for
 if $dom = true$ **then return** $true$
end for
return $false$

- **We will discuss the computation methods for:**
  - Nash equilibria of two-player, zero-sum game
  - Nash equilibria of two-player, general-sum game
  - Nash equilibria of n-player, general-sum game
  - maximin and minmax strategies for two-player, general-sum games
  - Computing correlated equilibria

Linear Programming (LP)

- Mathematical optimization problem can be expressed as

$$\begin{array}{ll} \text{minmize} & f_o(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \dots, m \end{array}$$

- $x = (x_1, \dots, x_n)$: optimization variables
- $f_0 \colon \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i \colon \mathbf{R}^n \to \mathbf{R}$, $i = 1, \dots, m$: constraint functions

- A linear program is defined by:
  - a set of real-valued variables
  - a linear objective function
    - a weighted sum of the variables
  - a set of linear constraints
    - the requirement that a weighted sum of the variables must be greater than or equal to some constant

$$\begin{array}{ll} \text{minmize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \dots, m \end{array}$$

Computing Nash equilibria of two-player, zero-sum game

Theorem (Minmax theorem by von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

- Consider a two-player, zero-sum game $G = \left(\{1,2\}, A_1 \times A_2, (u_1, u_2)\right)$.
- Let $U_1^* = -U_2^*$
- By the minmax theorem, U_1^* holds constant in all equilibria and that it is the same as the value that player 1 achieves under a minmax strategy by player 2

$$\bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2) \quad = \quad \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minmax strategy by player 2

Computing Nash equilibria of two-player, zero-sum game

- Standard form convex optimization problem can be converted into epigraph form:

Using slack variables

$$\begin{array}{ll} \text{minimize (over } x,\ s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \geq 0, \quad i = 1, \ldots m \end{array}$$

Standard convex optimization from

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Epigraph form

$$\begin{array}{ll} \text{minimize (over } x,\ t) & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$
\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \le U_1^* & \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \ge 0 & \forall a_2 \in A_2
\end{aligned}
$$

- First, identify the variables:
  - U_1^* is the expected utility for player 1
  - $s_2^{a_2}$ is player 2's probability of playing action a_2 under his mixed strategy
- each $u_1(a_1, a_2)$ is a constant
- Decision variables are U_1^* and $s_2^{a_2}$ for $\forall a_2 \in A_2$
- The LP will choose player 2's mixed strategy in order to minimize U_1^*

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

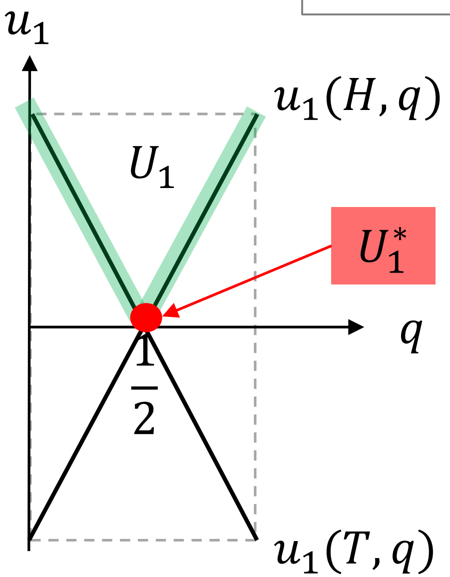
$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minimize $\quad U_1^*$

subject to $\quad \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \qquad \forall a_1 \in A_1$

$$\sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_2^{a_2} \geq 0 \qquad \forall a_2 \in A_2$$



- Player 2's minmax strategy:

$$\underline{s}_2 = \operatorname*{argmin}_{s_2} \max_{s_1} u_1(s_1, s_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$\begin{array}{lll}
\text{minimize} & U_1^* & \\
\text{subject to} & \displaystyle\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \le U_1^* & \forall a_1 \in A_1 \\
& \displaystyle\sum_{a_2 \in A_2} s_2^{a_2} = 1 & \\
& s_2^{a_2} \ge 0 & \forall a_2 \in A_2
\end{array}$$

- For every pure strategy j of player 1, his expected utility for playing any action $j \in A_1$ given player 2's mixed strategy s_2 is at most U_1^*
  - Those pure strategies for which the expected utility is exactly U_1^* will be in player 1's best response set

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minimize $\quad U_1^*$

subject to $\quad \displaystyle\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \qquad \forall a_1 \in A_1$

$$\sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_2^{a_2} \geq 0 \qquad \forall a_2 \in A_2$$

- Player 2 plays the mixed strategy that minimizes the utility player 1 can gain by playing his best response

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \qquad \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 \qquad \forall a_2 \in A_2
\end{aligned}$$

- s_2 is a valid probability distribution

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* & \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 & \forall a_2 \in A_2
\end{aligned}$$

Introduce slack variables $r_1^{a_1}$ for every $a_1 \in A_1$

$$\begin{aligned}
\text{minimize} \quad & U_1^* \\
& \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^* & \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 & \forall a_2 \in A_2 \\
& r_1^{a_1} \geq 0 & \forall a_1 \in A_1
\end{aligned}$$

Computing Nash equilibria of two-player, zero-sum game

For player 1's strategy

$$U_1^* = \bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

$$
\begin{array}{lll}
\text{maximize} & U_1^* & \\
\text{subject to} & \displaystyle\sum_{a_1 \in A_1} u_1(a_1, a_2) \times s_1^{a_1} \geq U_1^* & \forall a_2 \in A_2 \\
& \displaystyle\sum_{a_1 \in A_1} s_1^{a_1} = 1 & \\
& s_1^{a_1} \geq 0 & \forall a_1 \in A_1
\end{array}
$$

- First, identify the variables:
    - U_1^* is the expected utility for player 1
    - $s_1^{a_1}$ is player 1's probability of playing action a_1 under his mixed strategy
- each $u_1(a_1, a_2)$ is a constant
- Decision variables are U_1^* and $s_1^{a_1}$ for $\forall a_1 \in A_1$
- The LP will choose player 1's mixed strategy in order to maximize U_1^*

Computing Nash equilibria of two-player, general-sum game

- The problem of finding a Nash equilibrium of a two-player, general-sum game cannot be formulated as a linear programming
    - The two players' interests are no longer directly opposed
    - We cannot state our problem as an optimization problem: one player is not trying to minimize the other's utility

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \quad \forall a_2 \in A_2$$

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^*, \forall a_1 \in A_1, r_1^{a_1} \geq 0$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} u_1(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_1^*, \forall a_2 \in A_2, r_2^{a_2} \geq 0$$

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^*, \forall a_2 \in A_2, r_1^{a_1} \geq 0$$

$$s_1^{a_1} > 0 \rightarrow r_1^{a_1} = 0; s_1^{a_1} \times r_1^{a_1} = 0$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} u_1(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_1^*, \forall a_2 \in A_2, r_2^{a_2} \geq 0$$

$$s_2^{a_2} > 0 \rightarrow r_2^{a_2} = 0; s_2^{a_2} \times r_2^{a_2} = 0$$

Computing Nash equilibria of two-player, general-sum game

Linear complementarity problem (LCP) formulation for two-player, general-sum game

$$u_1(a_1, s_2) \leq u_1(a_1^*, s_2) \quad \forall a_1 \in A_1$$

$$u_2(s_1, a_2) \leq u_2(s_1, a_2^*) \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \ s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \qquad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \qquad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \ s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

Linear complementarity problem (LCP) formulation for two-player, general-sum game

$$u_1(a_1, s_2) \leq u_1(a_1^*, s_2) \quad \forall a_1 \in A_1$$

$$u_2(s_1, a_2) \leq u_2(s_1, a_2^*) \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \ s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^* \qquad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_2^* \qquad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \ s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$r_1^{a_1} \geq 0, \ r_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

- The slack variables are introduced to convert inequality constraints to equality constrains

Issues

- The variables U_1^* and U_2^* would be insufficiently constrained
  - We want these values to express the expected utility that each player would achieve by playing his best responses to the other player's chosen mixed strategy

Computing Nash equilibria of two-player, general-sum game

Linear complementarity problem (LCP) formulation for two-player, general-sum game

$$u_1(a_1, s_2) \le u_1(a_1^*, s_2) \;\; \forall a_1 \in A_1$$

$$u_2(s_1, a_2) \le u_2(s_1, a_2^*) \;\; \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \; \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \ge 0, \;\; s_2^{a_2} \ge 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^* \qquad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_2^* \qquad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \; \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \ge 0, \;\; s_2^{a_2} \ge 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$r_1^{a_1} \ge 0, \;\; r_2^{a_2} \ge 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$r_1^{a_1} \cdot s_1^{a_1} = 0, \;\; r_2^{a_2} \cdot s_2^{a_2} = 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

- Add the nonlinear constraints, called the complementarity condition (non-linear programing)
- This constraint requires that whenever an action is played by a given player with positive probability (supports for a strategy) then the corresponding slack variable must be zero
  - It capture the fact that, in equilibrium, all strategies that are played with positive probability must yield the same expected payoff
  - all strategies that lead to lower expected payoffs are not played

Computing Nash equilibria of two-player, general-sum game

- LCP problem can be formulated in a Quadratic programming that can be solved using an optimization solver (for this class, we can use a library for LCP solver)

- Classical algorithm to solve LCP is Lemke-Howson algorithm, which is similar to simplex method for Linear Programming (LP)

Computing Nash equilibria of n-player, general-sum game

- For n-player games where $n \geq 3$, the problem of finding a Nash equilibrium can no longer be represented even as an LCP
  - Hopelessly impractical to solve exactly
- Textbook discusses how to formulate the problem to find NEs using heuristic methods

Computing maximin and minmax strategies for two-player, general-sum games

- Let's say we want to compute a maxmin strategy for player 1 in an arbitrary 2-player game G
  - Create a new game G' where player 2's payoffs are just the negatives of player 1's payoffs.
  - The maxmin strategy for player 1 in G does not depend on player 2's payoffs
    - Thus, the maxmin strategy for player 1 in G is the same as the maxmin strategy for player 1 in G'
  - By the minmax theorem, equilibrium strategies for player 1 in G' are equivalent to a maxmin strategies
  - Thus, to find a maxmin strategy for G, find an Nash equilibrium strategy for G'

$$G = \big(\{1,2\}, A_1 \times A_2, (u_1, u_2)\big) \longrightarrow G' = \big(\{1,2\}, A_1 \times A_2, (u_1, \boldsymbol{-u_1})\big)$$

Computing correlated equilibria

- A sample correlated equilibrium can be found in polynomial time using a linear programming formulation
- Every game has at least one correlated equilibrium in which the value of the random variable can be interpreted as a recommendation to each agent of what action to play, and in equilibrium the agents all follow these recommendations.
- Thus, we can find a sample correlated equilibrium if we can find a probability distribution over pure action profiles satisfying

$$\sum_{a_{-i} \in A_{-i}} \pi(a) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi(a) u_i(a_i', a_{-i}) \qquad \forall i \in N, \forall a_i, a_i' \in A_i \qquad (1)$$

$$\pi(a) > 0 \qquad \forall a \in A \qquad (2)$$

$$\sum_{a \in A} \pi(a) = 1 \qquad (3)$$

- Variables: $\pi(a)$, constants: $u_i(a)$
- Constraint (1) requires player i must be better off playing action a_i when he is told to do so than playing any other action a_i', given that other players play their prescribed action
- Constraint (2) and (3) requires p is a valid probability distribution

Computing correlated equilibria

- One can select a desired correlated equilibrium by adding an objective function to the linear program.
  - For example, the problem maximizes the sum of the agents' expected utilities by adding the objective function (social-welfare maximizing CE)

$$
\begin{aligned}
&\text{maximize} && \sum_{a\in A} \pi(a) \sum_{i\in N} u_i(a) && \\
&\text{subject to} && \sum_{a_{-i}\in A_{-i}} \pi(a) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \pi(a) u_i(a_i', a_{-i}) && \forall i \in N, \forall a_i, a_i' \in A_i \quad (1)\\
& && \pi(a) > 0 && \forall a \in A \quad (2)\\
& && \sum_{a\in A} \pi(a) = 1 && (3)
\end{aligned}
$$

- Utilitarian equilibrium: an equilibrium which maximizes the sum of the expected payoffs of the players
- Libertarian i equilibrium: an equilibrium which maximizes the expected payoff of Player i
- Egalitarian equilibrium: an equilibrium which maximizes the minimum expected payoff of a player is called an.

Computing correlated equilibria : Example

	C	F
C	$2, 5$	$0, 0$
F	$0, 0$	$5, 2$

- Formulate LP to find the Libertarian 1 equilibrium (do it by your self):

Difference between Nash and Correlated equilibrium?

Why are CE easier to compute than NE?

- Intuitively, correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.

- To change this program so that it finds NE, the first constraint would be

$$\boxed{\sum_{a_{-i}\in A_{-i}} \pi(a)u_i(a_i,a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \pi(a)u_i(a_i',a_{-i}) \qquad \forall i \in N, \forall a_i, a_i' \in A_i}$$

$$\boxed{\sum_{a\in A}\left(\prod_{j\in N} s_j(a_j)\right)u_i(a_i,a_{-i}) \geq \sum_{a\in A}\left(\prod_{j\in N} s_j(a_j)\right)u_i(a_i',a_{-i}) \qquad \forall i \in N, a_i' \in A_i}$$

The constrain is non-linear!

$\pi(a_1^1, a_2^1)$	$\pi(a_1^1, a_2^2)$
$\pi(a_1^2, a_2^1)$	$\pi(a_1^2, a_2^2)$

Joint distribution

$s_1(a_1^1)s_2(a_2^1)$	$s_1(a_1^1)s_2(a_2^2)$
$s_1(a_1^2)s_2(a_2^1)$	$s_1(a_1^2)s_2(a_2^2)$

independent distribution