Homework 1

Problem 1 (Sharing game)

Two people are trying to share a 100$. They each simultaneously announce a real number from 0 to 100$. If the sum of the numbers is less than or equal to 100$, each person receives the number he announced. If the sum is greater than 100$, then the player who announced the smaller number, say x, receives x while the other person receives $100 - x$. If the sum is greater than 100 and both numbers are the same, then each receives 50$.

(1) Formulate this situation a strategic game (i.e., define players, actions, payoff functions)

(2) Show that (50, 50) is a Nash equilibrium

(3) Can you find more Nash equilibria? Prove your answer

Sol)

(1)

- Players: the two people
- Actions: Each player's action is a real number a_i, $0 \leq a_i \leq 100$
- Preferences: Each player's preferences can be represented by the following payoff functions

$$u_1(a_1, a_2) = \begin{cases} a_1 & \text{if} \quad a_1 + a_1 \leq 100 \\ a_1 & \text{if} \quad a_1 + a_1 > 100, a_1 < a_2 \\ 100 - a_2 & \text{if} \quad a_1 + a_1 > 100, a_1 > a_2 \\ 50 & \text{if} \quad a_1 + a_1 > 100, a_1 = a_2 \end{cases}$$

$$u_2(a_1, a_2) = \begin{cases} a_2 & \text{if} \quad a_1 + a_1 \leq 100 \\ a_2 & \text{if} \quad a_1 + a_1 > 100, a_2 < a_1 \\ 100 - a_1 & \text{if} \quad a_1 + a_1 > 100, a_2 > a_1 \\ 50 & \text{if} \quad a_1 + a_1 > 100, a_1 = a_2 \end{cases}$$

(2)

Suppose $a_2 = 50$

- If player 1 plays $a_1 = 50$, payoff is 50.
- If $a_1 < 50$, payoff is less than 50.
- If $a_1 > 50$, payoff is $100 - a_2 = 50$.

$a_1 = 50$ is a best response to $a_2 = 50$, and vice versa. Therefore, $(50,50)$ is a Nash equilibrium.

(3)

The best response correspondence of player i is:

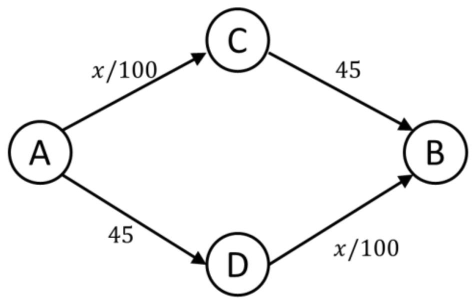
$$B_i(a_j) = \begin{cases} \{a_i | 100 - a_j \leq a_i \leq 100\} & \text{if} \ a_j \leq 50 \\ \emptyset & \text{if} \ a_j > 50 \end{cases}$$

The only intersection of B_1 and B_2 is at $(50,50)$

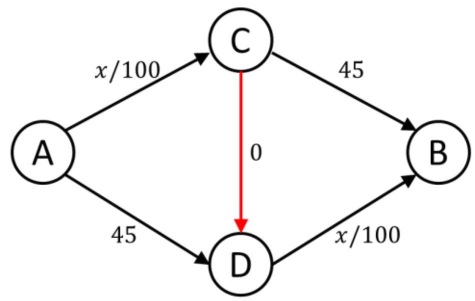
Problem 2 (Nash equilibrium route for reaching a destination)

Consider the following traffic network. You need to reach destination B starting from A. The number on the edge represents the travel time required to move between two nodes. For example, the travel time from node C to B is 45 (minutes).

- The travel time for the routes A-C and D-B depends on the number of car x using that route.
- The travel time for the routes A-D and C-B are fixed as 45.



(a) Current



(a) New road

Now, assume that 4000 cars want to move from A to B as part of the morning commute. The two possible routes that each car can choose are: the upper route through C and the lower route through D.

(1) Formulate this problem as a game by 4000 players

- $N = \{1, \dots, 4000\}$ set of players
- $A = A_1 \times \dots \times A_{4000}$ with $a_i \in A_i = up, down\}$
- $u = (u_1, \dots, u_{4000})$ with $u_i(a_1, \dots, a_{4000}) = \begin{cases} \sum_{j=1}^{4000} \frac{\mathbb{I}(a_j == up)}{100} + 45 & \text{if } a_j = up \\ 45 + \sum_{j=1}^{4000} \frac{\mathbb{I}(a_j == down)}{100} & \text{if } a_j = down \end{cases}$

(2) What is a Nash equilibrium? Describe the conditions where each car is not willing to deviate from. (express this condition using x, the number of cars moving upper route A-C-B)

Any list of strategies in which the drivers balance themselves evenly between the two routes (2000 on each) is Nash equilibrium, and these are the only NE.

- Why the equal balance is NE? with an even balance, no driver has an incentive to switch over other route because it will increase the time
- Consider a list of strategies in which x drivers use the upper rout and the remaining $4000 - x$ drivers use the lower route. Then if x is not equal to 2000, the two routes will have unequal travel times, and any driver on the slower route would have an incentive to switch to the faster one. Hence any list of strategies in which $x = 2000$ can be Nash equilibrium

(3) What will happen if the city constructs a new really fast highway between C and D. Assume the travel time between C and D is 0. How will this new road affect the commuting time?

There is a unique Nash equilibrium in this new highway network, but is leads to a worse travel time for everyone. At equilibrium, every driver uses the route through both C and D; as a result, the travel time for every driver is 80 (since $4000/100 + 0 + 4000/100 = 80$).

- To see why this is an equilibrium, note that no driver can benefit by changing their route: with traffic snaking through C and D the way it is, any other route would now take 85 minutes
- To see why it's the only equilibrium, you can check that the creation of the edge from C to D has in fact made the route through C and D a dominant strategy for all drivers: regardless of the current traffic pattern, you gain by switching your route to go through C and D.

Problem 3 (Normal form game with three-players)

Consider the 3-player normal form game bellows:

	L	R
T	$(5,5,5)$	$(2,6,2)$
B	$(6,2,2)$	$(3,3,-1)$
N		

	L	R
T	$(2,2,6)$	$(-1,3,3)$
B	$(3,-1,3)$	$(0,0,0)$
F		

Here each player has two strategies:

- (T,B) for player 1,
- (L,R) for player 2
- (N,F) for player 3.

In this notation, player 3 gets to select left or right table, player 2 selects the column, and player 1 selects the row. For example, if they play $(\mathrm{T},\mathrm{L},\mathrm{F})$ they each get (2,2,6) respectively.

(1) List all of the pure strategy Nash equilibrium profiles of this game.

$(\mathrm{B},\mathrm{R},\mathrm{F})$ is the only Nash equilibrium, since these strategies are strictly dominant for all 3 players. As many people noticed, this game is a direct generalization of Prisoners' Dilemma to 3 players.

(2) Compute the maxmin value (or security level) for player 1.

The value is 0, since player 1 can guarantee himself at least as much by playing B

(3) Suppose it is common knowledge that player 3 will play N with probability 0.8 and F with probability 0.2. This induces a game just between player 1 and player 2. Show the Normal Form of this induced game (include payoffs for all three players).

	L	R
T	$(4.4,4.4,5.2)$	$(1.4,5.4,2.2)$
B	$(5.4,1.4,2.2)$	$(2.4,2.4,-0.8)$

An easy way to compute this was to notice that payoffs for players 1, 2 differ exactly by 3 between two matrices, and payoffs for player 3 differ by 1.

(4) What are the pure strategy Nash Equilibria in the induced game?

(B,R) is still the only (dominant strategy) Nash equilibrium. Now this is exactly the 2-player prisoner's dilemma.

Problem 4 (Extension of Cournot Duopoly game to multiple firms)

In Cournot Duopoly game, assume there are n firms.

- Each firm can choose the production quantity $q_i \geq 0$ for $i = 1, \dots n$
- The market price decrease as the firms' total production $Q = \sum_{i=1}^{n} q_i$ increases as:

$$P(Q) = \begin{cases} a - Q & \text{if } Q \leq a \\ 0 & \text{if } Q > a \end{cases}$$

- The cost function of each firm i is $C_i(q_i) = cq_i$ with $c < a$
- The firm i's profit, equal to its revenue minus its cost, is

$$u_i(q_1, \dots, q_n) = q_i P(Q) - C_i(q_i)$$

(1) Find the best response function of each firm and set up the conditions for $(q_1^*, \dots, q_n^*)$ to be a Nash equilibrium

Suppose there are n firms with identical unit cost c. As before, the best response of firm 1 will be

$$b_1(q_2, \dots, q_n) = \begin{cases} q_1(\alpha - c - q_2 - \cdots - q_n) & if\ q_1 + \cdots + q_n \leq \alpha \\ -cq_1 & if\ q_1 + \cdots + q_n \leq \alpha \end{cases}$$

There are n equations in the conditions for Nash equilibrium:

$$q_1^* = b_1(q_2^*, \dots, q_n^*)$$

$$q_1^* = b_1(q_2^*, \dots, q_n^*)$$

$$\dots$$

$$q_1^* = b_1(q_2^*, \dots, q_n^*)$$

which becomes (setting $Q = q_1 + q_2 + \cdots + q_n$)

$$q_1^* = \frac{1}{2}(\alpha - c - Q + q_1)$$

$$q_2^* = \frac{1}{2}(\alpha - c - Q + q_2)$$

$$\dots$$

$$q_n^* = \frac{1}{2}(\alpha - c - Q + q_n)$$

We could use linear algebra to solve this system of equations, but a simple way is to rearrange each equation so that

$$q_1^* = \alpha - c - Q$$

This shows that all firms choose the same level of output in equilibrium. Total output $Q = nq_i = \frac{n}{n+1}(\alpha - c)$ and $q_i = \frac{\alpha-c}{n+1}$. The market price is:

$$P(Q) = \alpha - Q = \frac{\alpha}{n+1} + \frac{n}{n+1}c$$

(2) How the price $P(Q)$ changes as the number of firms n increases

Taking the limit as $n \to \infty$, the first term goes to 0 and the second term goes to c. Therefore, as the number of firms becomes very large, the Nash equilibrium becomes the same as the perfectly competitive equilibrium outcome.

Problem 5 (Mixed Nash strategy for animals)

Two animals are fighting over some prey. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome when its opponent is passive to that in which its opponent is aggressive. The following is the conditions you can use to formulate the strategic behaviors of the animal

- take each player's payoff to (Aggressive, Aggressive) to be 0
- take each player's payoff to (Passive, Aggressive) to be 1
- Each player is indifferent between the outcome (Passive, Passive) and the lottery that assigns probability 0.5 to (Aggressive, Aggressive) and probability 0.5 to (Aggressive, Passive)
- Each player is indifferent between the outcome (Passive, Aggressive) and the lottery that assigns probability 2/3 to the outcome (Aggressive, Aggressive) and probability 1/3 to the outcome (Passive, Passive).

(1) Represent this game as a normal form game

(2) Plot the best response curves for the two animals

(3) Find all Nash equilibria (including pure and mixed strategies)

Sol)

Denote by u_i a payoff function whose expected value represents player i's preferences. The conditions in the problem imply that for player 1 we have

$$u_1(Passive, Passive) = \frac{1}{2}u_1(Passive, Passive) + \frac{1}{2}u_1(Aggressive, Passive)$$

$$u_1(Passive, Aggressive) = \frac{2}{3}u_1(Aggressive, Aggressive) + \frac{1}{3}u_1(Passive, Passive)$$

Given $u_1(Aggressive, Aggressive) = 0$ and $u_1(Passive, Aggressive) = 1$, we have

$$u_1(Passive, Passive) = \frac{1}{2}u_1(Aggressive, Passive)$$

and

$$1 = \frac{1}{3}u_1(Passive, Passive)$$

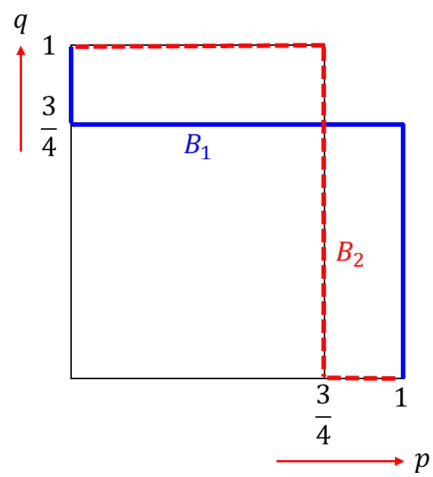
so that

$$u_1(Passive, Passive) = 3 \text{ and } u_1(Aggressive, Passive) = 6$$

similarly,

$$u_2(Passive, Passive) = 3 \text{ and } u_2(Passive, Aggressive) = 6$$

	$Aggressive$	$Passive$
$Aggressive$	$(0,0)$	$(6,1)$
$Passive$	$(1,6)$	$(3,3)$



Thus the game is given in the matrix form. The players' best response functions are shown in the above plot. The game has three mixed strategy Nash equilibria:

$$\big((0,1),(1,0)\big), \left(\left(\frac{3}{4},\frac{1}{4}\right),\left(\frac{3}{4},\frac{1}{4}\right)\right), \text{and } \big((1,0),(0,1)\big)$$

Problem 6 (Nash and Correlated equilibria for a two player game)

Consider the following two player game:

	L	R
T	$6, 6$	$2, 8$
B	$8, 2$	$0, 0$

(1) Find all pure strategy Nash equilibria and the payoff vector for each one

There are two pure Nash equilibria: (T, R) with payoff vector $(2, 8)$, and (B, L) with payoff vector $(8, 2)$.

(2) Find all mixed strategy Nash equilibria and the payoff vector for each one

So we seek a pair of mixed strategies $(p, 1-p)$, $(q, 1-q)$ so that either actin is a best response for either player. That requires $6p + 2(1-p) = 8p$ and $6q + 2(1-q) = 8q$, or $(p, q) = (0.5, 0.5)$ which has payoff vector $(4,4)$.

(3) Identify all correlated equilibria by specifying the set of inequalities that they need to satisfy

Each correlated equilibrium corresponds to a probability distribution (a, b, c, d) over the possible pairs of actions, $\{(T, L), (T, R), (B, L), (B, R)\}$. The conditions needed to be correlated equilibrium, in addition to (a, b, c, d) being a probability distribution, are

$$(T \to B) \quad 6a + 2b \geq 8a + 0b$$

$$(B \to T) \quad 8c + 0d \geq 6c + 2d$$

$$(L \to R) \quad 6a + 2c \geq 8a + 0c$$

$$(R \to L) \quad 8b + 0d \geq 6b + 2d$$

where, for example, the equation for $(T \to B)$ insures that the first player would not receive a higher expected payoff by using B whenever told to play T. The equations reduce to (a, b, c, d) is a probability vector such that $a \leq b, a \leq c. d \leq b,$ and $d \leq c$.

(4) Among all the correlated equilibria, provide one with the largest sum of payoffs

The payoff vector for a given choice of (a, b, c, d) is $(6a + 2b + 8c,\ 6a + 8b + 2c)$. The sum of payoffs is $12a + 10(b + c)$. To maximize the sum of payoffs, we clearly should let $d = 0$. Then $b + c = 1 - a$ and the sum of payoffs is $10 + 2a$. The largest a can be is $1/3$, so the correlated equilibrium with the maximum sum of payoffs is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$, and the payoff vector is $(5.333 \ldots,\ 5.333 \ldots)$