Lecture 9. Repeated games

Lessons from multi-stages game

- First, when players play a sequence of games over time, it will be to their benefit to use conditional strategies in later stage-games to support desirable behavior in early stage-games
- Second, the future that the players face must be important enough to support these dynamic incentives as self-enforcing
    - Using so-called reward-and-punishment strategies to sustain static non-best-response behavior is possible only if the palyers do not discout the futre too heavily

Repeated games

- A repeated game is simply a multistage game in which the same stage-game is being player at every stage

- A repeated game is a special case of multistage games

- A repeated game can be used to model interactions occurring more than once:

    - Firms in a marketplace
    - Political alliances
    - Friends (favor exchange...)
    - Workers (team production...)

- How to model such repeated conflicts and find an equilibrium strategy?

Examples

- **OPEC: Oil Prices**
  - 20$/bbl or less from 1930-1973 (2008 dollars)
  - 50$/bbl by 1976
  - 90$/bbl by 1982
  - 40$/bbl or less from 1986 to 2002
  - 100$/bbl by late 2008 ..

- Cooperative Behavior: Cartel is much like a repeated Prisoner's Dilemma
  - Need to easily observe each other's plays and react (quickly) to punish undesired behavior
  - Need patient players who value the long run (wars don't help!)
  - Need a stable set of players and some stationarity helps
    - constantly changing sources of production can hurt, but growing demand can help...

https://blogs.cornell.edu/info2040/2015/09/10/game-theory-and-oil-prices/

Setups

- Questions we'll need to answer before analyzing games:
  - what will agents be able to observe about others' play?
  - how much will agents be able to remember about what has happened?
  - what is an agent's utility for the whole game?

- Some of these questions will have different answers for
  - finitely-repeated games
  - infinitely-repeated games.

Finitely Repeated Games

Definition (Finitely repeated game)

Given a stage game G, $G(T,\gamma)$ denotes the finitely repeated game in which the stage-game G is played T consecutive times, and γ is the common discount factor.

Finitely Repeated Games

- Everything is straightforward if we repeat a game a finite number of times
- We can write the whole thing as an extensive-form game with imperfect information
  - at each round players don't know what the others have done; afterwards they do
  - overall payoff function is additive: sum of payoffs in stage games

	C	D
C	$-1,-1$	$-4,0$
D	$0,-4$	$-3,-3$

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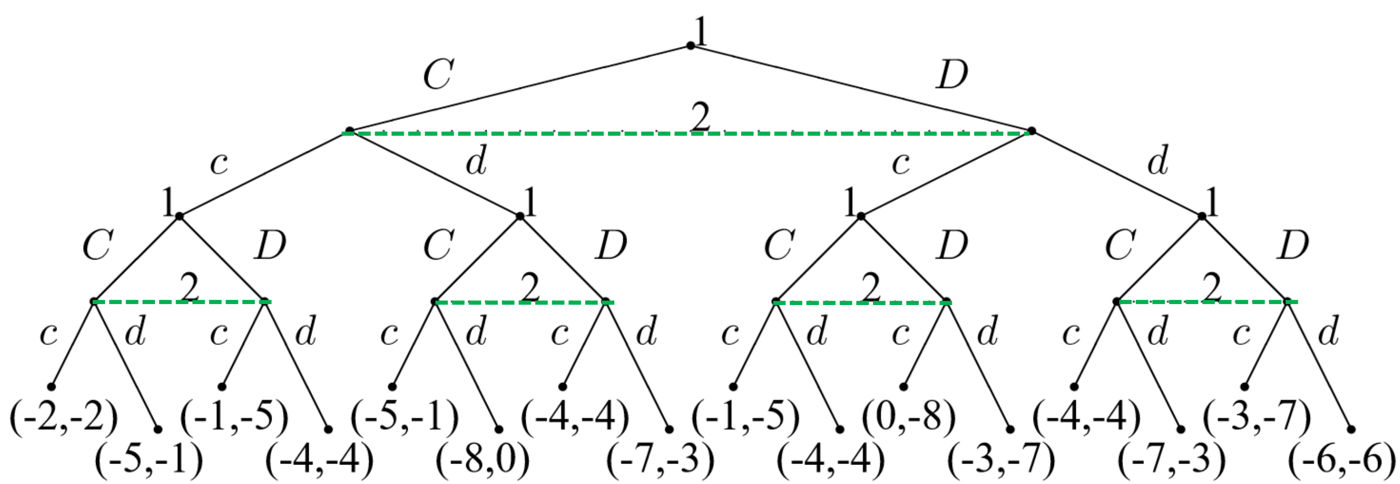

Finitely Repeated Games: Example

	m	f	r
M	4, 4	-1, 5	0, 0
F	5, -1	**1, 1**	0, 0
R	0, 0	0, 0	**3, 3**

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- There are two pure-strategy Nash equilibria, (R, r) and (F, f)
  - (R, r) can serves as "carrot"
  - (F, f) can serves as "stick"

  can be used to discipline first-period behavior
  - Implies that for a high enough discount factor, we may be able to find SPE

Finitely Repeated Games: Example

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- Convicne yourself that for a discount factor $\gamma \geq 1/2$, the following strategies constitutes SPE
  - Player 1:
    - stage 1: Play M
    - stage 2: play R if (M, m) was played in stage 1, and play F if anything but (M, m) was played in stage 1
  - Player 2:
    - stage 1: Play m
    - stage 2: play r if (M, m) was played in stage 1, and play f if anything but (M, m) was played in stage 1

Finitely Repeated Games: Example

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Answer:

- This repeated game has nine outcomes at the first game
- The strategy for player 1 is

$$s_1^* = \left(s_1^1, s_1^2(h_1)\right)$$

$$\text{where } s_1^1 = M \text{ and } s_1^2(h_1) = \begin{cases} R & \text{if } h_1 = (M, m) \\ F & \text{if } h_1 \neq (M, m) \end{cases}$$

- To show this strategy is SPE, we need to show
    - In the second stage the players are clearly playing a Nash equilibrium regardless of the history of play
    - Players would not want to deviate from M in the first stage of the game:

$$u_1(M, s_2) = 4 + \gamma 3 \geq u_1(F, s_2) = 5 + \gamma 1 \text{ when } \gamma \geq \frac{1}{2}$$

carrot punishment

Finitely Repeated Games: Example

- The difference between this example and the Prisoner-Revenge Game of the previous chapter is
  - The same game is repeated twice
- It is the multiplicity of equilibria in the stage-game that is giving the players the leverage to use conditional second-stage strategies of the reward-and-punishment kind.

Proposition

If the stage-game of a finitely repeated game has a unique Nash equilibrium, then the finitely repeated game has a **unique** subgame-perfect equilibrium.

- The proof is the same for multi-stage games
- Illustrative example:
  - Repetition of Prisoner’s Dilemma game 500 times **(this is a finite number !!)**
  - What is PSE?

Finitely Repeated Games: Example

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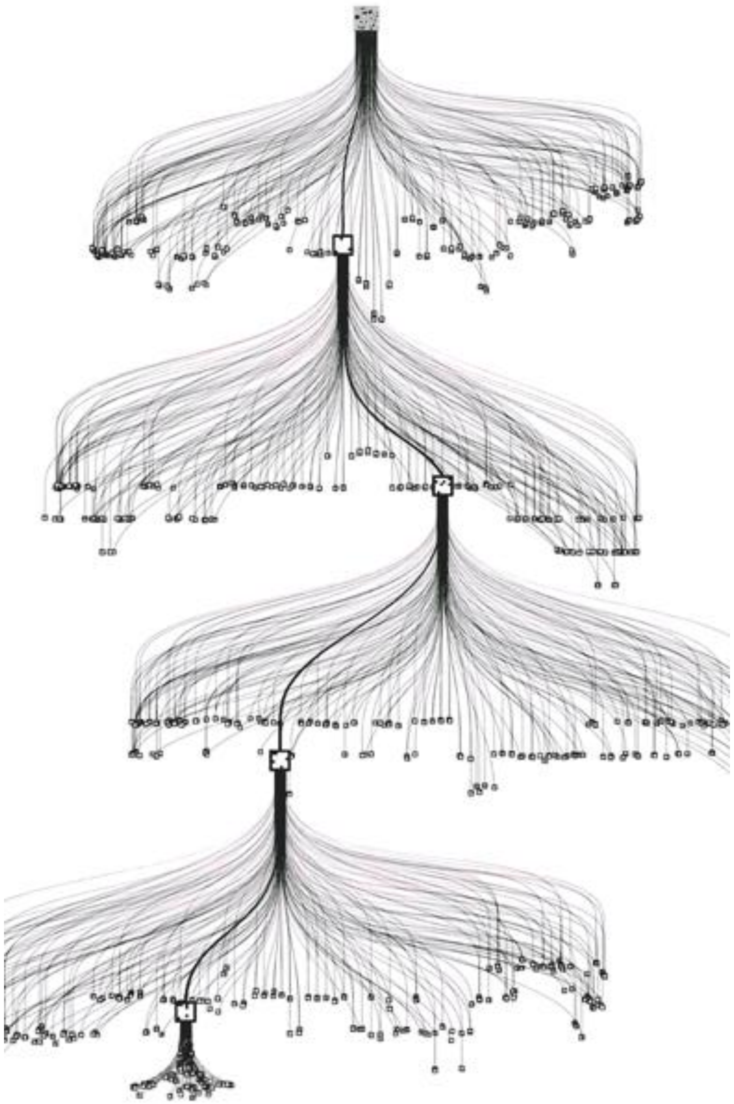
- The proof is the same for multi-stage games
- Illustrative example:
  - Repetition of Prisoner's Dilemma game 500 times **(this is a finite number !!)**
  - What is PSE?

- To give an incentives to cooperate, the players must be able to construct **reward-and-punishment continuation** equilibrium strategies
- These continuation strategies themselves must be equilibrium strategies and hence must rely on **multiple equilibria** in the continuation of the repeated game

Infinitely repeated Games: Motivations



Infinitely repeated games

Proposition

If the stage-game of a finitely repeated game has a unique Nash equilibrium, then the finitely repeated game has a **unique** subgame-perfect equilibrium.

- What would happen if we assume the game does not have a final period? That is, what would happen if players find that there is always a "long" future ahead of them
- This slight but critical modification allows player to chose an strategy that is not a static Nash equilibrium of the stage-game even when the stage-game has a unique Nash equilibrium
  - ➢ The players will have the freedom to support a wide range of behaviors that are not consistent with a static Nash equilibrium in the stage game.

Negative aspects:

- Consider an infinitely repeated game in extensive form:
  - an infinite tree!
- Thus, payoffs cannot be attached to terminal nodes, nor can they be defined as the sum of the payoffs in the stage games (which in general will be infinite).

Definition (Future discounted reward)

Given an infinite sequence of payoffs $u_i^1, u_i^2, \ldots$ for player i and discount factor γ with $0 \leq \gamma \leq 1$, i's future discounted reward is

$$u_i = \sum_{t=1}^{\infty} \gamma^{t-1} u_i^t.$$

- The interpretations for future discounted reward
  - the agent cares more about his well-being in the near term than in the long term
  - the agent cares about the future just as much as the present, but with probability $1 - \gamma$ the game will end in any given round.

Definition (Average reward 1)

Given an infinite sequence of payoffs $u_i^1, u_i^2, \ldots$ for player i, the average reward of i is

$$\bar{u}_i = (1 - \gamma) \sum_{t=1}^{\infty} \gamma^{t-1} u_i^t.$$

- The average payoff from a sequence is a normalization of the net present value
  - We are scaling down the net present value by a factor of $(1 - \gamma)$
  - This is convenient because of the following mathematical property

$$\bar{u}_i = (1-\gamma)\{v + \gamma v + \gamma^2 v + \cdots\} = (1-\gamma)\frac{v}{1-\gamma} = v$$

➢ i.e., the average payoff of an infinite fixed sequence of some value v is itself equal to v

Infinitely repeated games

Definition (Average reward 2)

Given an infinite sequence of payoffs $u_i^1, u_i^2, \ldots$ for player i, the average reward of i is

$$\lim_{k \to \infty} \frac{1}{k} \sum_{t=1}^{k} u_i^t$$

Strategy space

- What is a pure strategy in an infinitely-repeated game?
  - Player’s strategy is a complete contingent play that specifies what the player will do in each information set

- For the extensive form representation of an infinitely repeated game
  - Game expand in “length” and “depth”
  - Infinite number of information set
  - Which requires an infinite number of actions!

- Representing a pure strategy using conventional way is impossible

Strategy space

- Every information set of each player is identified by a unique path of play or history that was played in the previous sequence
- For example, we play Prisoner's Dilemma four times, there will be 64 unique information sets, each of which corresponds to a unique path of play, or history, in the first three stages
  - There is one-to-one relationship between information sets and histories of play
- Let's define history more formally

Definition (history)

Consider an infinitely repeated game. Let H_t denote the set of all possible histories of length t, $h_t \in H_t$, and let $H = \cup_{t=1}^{\infty} H_t$ be the set of all possible histories. A pure strategy for player i is a mapping $s_i: H \to S_i$ that map histories into actions of the stage-game.

- Some famous strategies (for repeated PD):
  - Tit-for-tat: Start out cooperating. If the opponent defected, defect in the next round. Then go back to cooperation.
  - Grim trigger: Start out cooperating. If the opponent ever defects, defect forever.

Subgame-perfect equilibria for infinitely repeated games

Definition (Subgame-perfect equilibria for infinitely repeated games)

A profile of pure strategies $\left(s_1^*(\cdot), s_2^*(\cdot), \dots, s_n^*(\cdot)\right)$, $s_i: H \rightarrow S_i$ for all $i \in N$, is a subgame-perfect equilibrium if the restriction of $\left(s_1^*(\cdot), s_1^*(\cdot), \dots, s_n^*(\cdot)\right)$ is a Nash equilibrium in every subgame. That is, for any history of the game h_t, the continuation play $\left(s_1^*(\cdot), s_1^*(\cdot), \dots, s_n^*(\cdot)\right)$ is a Nash equilibrium.

- How could we check that a profile of strategies is a Nash equilibrium for any history h_t?
- Let's take a simple and familiar case

Proposition

Let $G(\gamma)$ be an infinitely repeated game, and let $(a_1^*, a_2^*, \dots, a_n^*)$ be a (static) Nash equilibrium strategy profile of the stage-game G. Define the repeated-game strategy for each player i to be the history-independent Nash strategy, $s_i^*(h) = a_i^*$ for all $h \in H$. Then, $\left(s_1^*(\cdot), s_1^*(\cdot), \dots, s_n^*(\cdot)\right)$ is a subgame-perfect equilibrium in the repeated game for any $\gamma < 1$

- That is, keep playing a static Nash equilibrium of a stage game is SPE for the whole game (infinitely repeated game), regardless of histories uncounted during the game.

Subgame-perfect equilibria for infinitely repeated games

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…

Playing the static Nash equilibrium

→ subgame-perfect equilibrium

Subgame-perfect equilibria for infinitely repeated games

- The more interesting question that remains is whether or not we can support other types of behavior as part of a subgame-perfect equilibrium

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Playing the static Nash equilibrium strategy

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→ subgame-perfect equilibrium?

Playing the static Nash equilibrium strategy

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Playing non Nash equilibrium strategies

Subgame-perfect equilibria for infinitely repeated games

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⋯ → subgame-perfect equilibrium

Playing the static Nash equilibrium strategy

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⋯ → subgame-perfect equilibrium?

Playing the static Nash equilibrium strategy

+

Playing non Nash equilibrium strategies

Answer is yes.

- By applying reward-and punishment strategies

SPE for infinitely repeated games: Examples

		Player 2	
		c	d
Player 1	C	$4, 4$	$-1, 5$
	D	$5, -1$	$1, 1$

- Consider the path of play in which the two players choose (C, c) in every period
  - The average payoffs are $(\bar{u}_1, \bar{u}_2) = (4, 4)$

- Is this path of play can be supported as subgame-perfect equilibrium?
  - No, because deviation from C to D will give a higher payoff 5 which is larger than 4
  - Deviation is profitable

- To make playing (C, c) continuously to be equilibrium, we need to find some way to "punish" deviation

SPE for infinitely repeated games: Examples

		Player 2	
		c	d
Player 1	C	$4, 4$	$-1, 5$
	D	$5, -1$	$1, 1$

- Consider the following strategy
  - **Player 1:**
    - stage 1: $s_1^1 = C$
    - for any stage $t > 1$: $s_1^t(h_{t-1}) = \begin{cases} C & \text{iff } h_{t-1} = \{(C,c),(C,c),\dots,(C,c)\} \\ D & \text{iff } h_{t-1} \neq \{(C,c),(C,c),\dots,(C,c)\} \end{cases}$
  - **Player 2:**
    - stage 1: $s_2^1 = c$
    - for any stage $t > 1$: $s_2^t(h_{t-1}) = \begin{cases} c & \text{iff } h_{t-1} = \{(C,c),(C,c),\dots,(C,c)\} \\ d & \text{iff } h_{t-1} \neq \{(C,c),(C,c),\dots,(C,c)\} \end{cases}$
- For any deviation from cooperation in the past, the players will revert to playing defect, and by the definition of the strategies they will stick to defect thereafter (forever)
- This strategy is referred to as grim-trigger strategies
- Is this strategy SPE?

SPE for infinitely repeated games: Examples

- To verify that the grim-trigger strategy pair is a subgame-perfect equilibrium we need to check that their is no profitable deviation in any subgame.

Proposition

In an infinitely repeated game $G(\gamma)$, a profile of strategies $s^* = (s_1^*, \dots, s_n^*)$ is subgame-perfect equilibrium if and only if there is no player i and no single history h_{t-1} for which player i would gain from deviation from $s_i(h_{t-1})$

- The definition may not seem helpful because there are still an infinite number of histories
- There is hope! Whenever we are trying to support one kind of behavior forever with the threat of resorting to another kind of behavior,
  - we have two "states" in which the players can be
    - On the equilibrium path
    - Off the equilibrium path
  - We need only to check that they would not want to deviate from each of these states

SPE for infinitely repeated games: Examples

- To verify that the grim-trigger strategy pair is a subgame-perfect equilibrium we need to check that their is no profitable deviation in any subgame.

- **On the equilibrium path**
  - Category of histories that are consecutive sequences of (C, c)
  - That is, $h_{t-1} = \{(C, c), (C, c), \dots, (C, c)\}$
  - If a player chooses to play C, his average payoff is

$$\bar{u}_i = 4 + \gamma 4 + \gamma^2 4 + \cdots = \underbrace{4}_{\text{Today's payoff}} + \underbrace{\frac{4\gamma}{1-\gamma}}_{\text{future's payoff}}$$

  - If a player chooses to play d, he gets 5 instead of 4 in the immediate stage of deviation, followed by his continuation payoff, which is infinite sequence of 1s

$$\bar{u}_i' = 5 + \gamma 1 + \gamma^2 1 + \cdots = 5 + \frac{1\gamma}{1-\gamma}$$

- We can conclude that a player will not deviate from the equilibrium path if

$$\bar{u}_i = 4 + \frac{4\gamma}{1-\gamma} > 5 + \frac{1\gamma}{1-\gamma} = \bar{u}_i'$$

$$\Rightarrow \gamma \geq \frac{1}{4}$$

SPE for infinitely repeated games: Examples

- To verify that the grim-trigger strategy pair is a subgame-perfect equilibrium we need to check that their is no profitable deviation in any subgame.

- **Off the equilibrium path**
  - Category of histories that are **not** consecutive sequences of (C, c)
  - That is, $h_{t-1} \neq \{(C, c), (C, c), \ldots, (C, c)\}$

- In any subgame that is off the equilibrium path, the proposed strategies recommend that the players play (D, d)

- Then, we need to show that no player would want to choose C (or c) instead of D (or d) because
  - ✓ Given his belief that his opponent will play defect, such deviation from defect to cooperation will cause him a loss of -2 at the current and subsequent stages.

Player 2

Player 1	c	d
C	$4, 4$	$-1, 5$
D	$5, -1$	$1, 1$

SPE for infinitely repeated games: Examples

Some notes

- Basic logic:
  - Play something with relatively high payoffs, and if anyone deviates
  - Punish by resorting to something that
    - Has lower payoffs (at least for that player)
    - and is credible: It is an equilibrium in the subgame

- We see the value of patience. If the players are sufficiently patient, so that the future carries a fair amount of weight in their preference, then there is a reward-and-punishment strategy that will allow them to cooperate forever.

- Recall the following property in multi-stage game (finite repetition):

> In multi-stage game (finite repletion) These continuation strategies themselves must be equilibrium strategies and hence must rely on **multiple equilibria** in the continuation of the repeated game

  - Where are the multiple equilibria coming from in the current example?
    - This is where the infinite repetition creates "magic" through bootstrapping
    - From the unique equilibrium of the stage-game, we get multiple equilibria of the repeated game

SPE for infinitely repeated games: Tacit Collusion example

- **Tacit collusion** occurs where firms undergo actions that are likely to minimize a response from another firm, e.g. avoiding the opportunity to price cut an opposition.

- Put another way, two firms agree to play a certain strategy without explicitly saying so.

Nash equilibrium examples: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good
- Firm i produces quantity q_i
- Cost for production is $c_i(q_i) = 10q_i$
- Price is given by $d = 100 - q = (100 - q_i - q_j)$
- The profit of company i given its opponent chooses quantity q_j is

$$u_i(q_i, q_j) = (100 - q_i - q_j)q_i - 10q_i = -q_i^2 + 90q_i - q_j q_i$$

- Nash-Cournot equilibrium
  - $q_1^*, q_2^* = 30;$
  - $p = 100 - (30 + 30) = 60$
  - $u_1^N = u_1(q_1^*, q_2^*) = u_2^N = 900$

- Monopoly
  - $q_1^* + q_2^* = 45;$
  - $p = 100 - (45) = 55$
  - $u_1(q_1^*, q_2^*) + u_2(q_1^*, q_2^*) = 2025;$

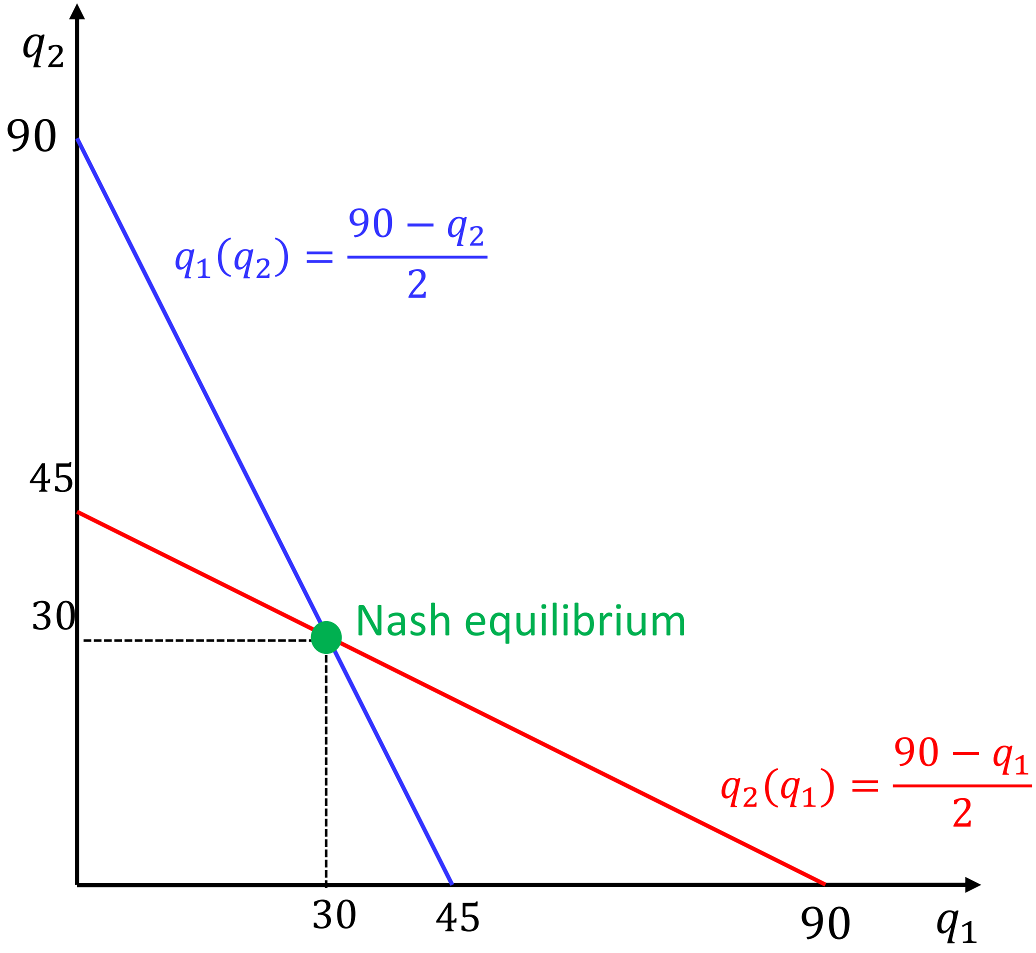
- If the two firms agree on producing $q_1^* + q_2^* = 45$, they can make more money in total
- Depending on how they set their production, q_1^* and q_2^*, various way of splitting profit become possible
  - $u_1(q_1^*, q_2^*) = u_2(q_1^*, q_2^*) = 1012.5 \text{ when } q_1^* = q_2^* = 22.5$

Nash equilibrium examples: Cournot Duopoly

- In case there are two firms, we have two best-response equations:

$$q_1 = \frac{a - bq_2 - c_1}{2b} \quad \text{and} \quad q_2 = \frac{a - bq_1 - c_2}{2b}$$

$a = 100, b = 1, c_1 = c_2 = 10$



SPE for infinitely repeated games: Tacit Collusion example

- We now proceeds to see how reward-and-punishment strategies will allow our firms to coordinate on monopoly profit in self–enforcing subgame perfect equilibrium without the need to use binding contract

- First, we have to decide how the firms will cooperate to split the monopoly profit
  - $q_1^c + q_2^c = 45$, which results in $d = 55$
  - Let's assume $q_1^c = 22, q_2^c = 23$
  - $u_1^c = u_1(q_1^c, q_2^c) = (100 - 45)q_1^c - 10q_1^c = 990$
  - $u_2^c = u_2(q_1^c, q_2^c) = (100 - 45)q_2^c - 10q_2^c = 1035$

- Second, Specifies firms' strategies, following the logic of the infinitely repeated PD game
  - **Firm 1:**
    - stage 1: $q_1^1 = q_1^c$
    - for any stage $t > 1$: $q_1^t(h_{t-1}) = \begin{cases} q_1^c \text{ iff } h_{t-1} = \{(q_1^c, q_2^c), (q_1^c, q_2^c), \dots, (q_1^c, q_2^c)\} \\ q_1^N \text{ iff } h_{t-1} \neq \{(q_1^c, q_2^c), (q_1^c, q_2^c), \dots, (q_1^c, q_2^c)\} \end{cases}$
  - **Firm 2:**
    - stage 1: $q_2^1 = q_2^c$
    - for any stage $t > 1$: $q_2^t(h_{t-1}) = \begin{cases} q_2^c \text{ iff } h_{t-1} = \{(q_1^c, q_2^c), (q_1^c, q_2^c), \dots, (q_1^c, q_2^c)\} \\ q_2^N \text{ iff } h_{t-1} \neq \{(q_1^c, q_2^c), (q_1^c, q_2^c), \dots, (q_1^c, q_2^c)\} \end{cases}$

  Where (q_1^N, q_2^N) is Nash equilibrium

SPE for infinitely repeated games: Tacit Collusion example

- Third, we need to check that no firm wants to deviate from the proposed strategies

$$(q_1^c = 22, q_2^c = 23\ )$$

- **On the equilibrium path**
  - Category of histories that are consecutive sequences of (q_1^c, q_2^c)
  - That is, $h_{t-1} = \{(q_1^c, q_2^c), (q_1^c, q_2^c), \dots, (q_1^c, q_2^c)\}$
  - If a player i chooses to play q_i^c, his average payoff is

$$u_i^c + \gamma u_i^c + \gamma^2 u_i^c + \cdots = u_i^c + \frac{\gamma u_i^c}{1-\gamma}$$

  - If a player chooses to play $q_i^d = BR(q_j^c)$, he gets the following average payoff

$$\underbrace{u_i^d}_{\text{Immediate increase}} + \underbrace{\gamma u_i^N + \gamma^2 u_i^N + \cdots}_{\text{Continuous reduced payoff}} = u_i^d + \frac{\gamma u_i^N}{1-\gamma}$$

- We can conclude that a player will not deviate from the equilibrium path if

$$u_i^c + \frac{\gamma u_i^c}{1-\gamma} > u_i^d + \frac{\gamma u_i^N}{1-\gamma} \Rightarrow \gamma \geq \frac{u_i^d - u_i^c}{u_i^d - u_i^N}$$

- For example, if player 1 deviate from $q_1^c = 22.5$ to $q_1^d = BR(q_2^c = 23) = \frac{90-23}{2} = 33.5.$
- $u_1^d = u_1\left(u_1^d = 33.5, q_2^c\right) = (100 - 45)33.5 - 10 \times 33.5 = 1125.25$
- $\gamma_1 \geq \frac{u_i^d - u_i^c}{u_i^d - u_i^N} = \frac{1122.5-990}{1122.5-900} = 0.595$

SPE for infinitely repeated games: Examples

- To verify that the grim-trigger strategy pair is a subgame-perfect equilibrium we need to check that their is no profitable deviation in any subgame.

- **Off the equilibrium path**
  - Category of histories that are **not** consecutive sequences of(q_1^c, q_2^c)
  - That is, $h_{t-1} \neq \{(q_1^c, q_2^c), (q_1^c, q_2^c), \dots, (q_1^c, q_2^c)\}$

- In any subgame that is off the equilibrium path, the proposed strategies recommend that the players play $(q_1^N, q_1^N) = (30,30)$

- Then, we need to show that no player would want to choose $q_i \neq q_i^N$ instead of q_i^N
- because
  - ✓ Given his belief that his opponent will play q_j^N, such deviation will only decrease his payoff due to the definition of Nash equilibrium

Motivation for Folk Theorem

- With an infinite number of equilibria, what can we say about Nash equilibria?
  - Nash's theorem only applies to finite games
  - we won't be able to construct an induced normal form and then appeal to Nash's theorem to say that an equilibrium exists
- Furthermore, with an infinite number of strategies, there could be an infinite number of pure-strategy equilibria!
- It turns out we can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate the equilibria.

Folk Theorem

- Consider any n-player game $G = (N, A, u)$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.
- Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$
  - i's minmax value is the amount of utility i can get when $-i$ play a minmax strategy against him

Definition (Enforceable payoff)

A payoff profile r is **enforceable** if $r_i \geq v_i$.

Definition (Feasible payoff)

A payoff profile r is **feasible** if there exist rational, non-negative values α_a such that for all i, we can express r_i as $\sum_{a \in A} \alpha_a u_i(a)$ with $\sum_{a \in A} \alpha_a = 1$.

- Feasible: a convex, rational combination of the outcomes in G.

Theorem (Folk Theorem)

Consider any n-player game G and any payoff vector $(r_1, r_2, \dots, r_n)$.

1. If r is the payoff in any Nash equilibrium of the infinitely repeated G with average rewards, then for each player i, r_i is enforceable.
2. If r is both feasible and enforceable, then r is the payoff in some Nash equilibrium of the infinitely repeated G with average rewards

Payoff in Nash ⇒ enforceable

- Suppose r is not enforceable, i.e. $r_i < v_i$ for some i.
- Then consider an alternative strategy for i: playing $BR(s_{-i}(h))$, where $s_{-i}(h)$ is the equilibrium strategy of other players given the current history h and $BR(s_{-i}(h))$ is a function that returns a best response for i to a given strategy profile s_{-i} in the (unrepeated) stage game G
- By definition of a minmax strategy, player i will receive a payoff of at least v_i in every stage game if he adopts $BR(s_{-i}(h))$
- So i's average reward is also at least v_i.
- Thus, if $r_i < v_i$ then s cannot be a Nash equilibrium.

(strategy s_i can be deviated to $BR(s_{-i}(h))$ in order to get a higher pay off v_i than r_i)

Feasible and Enforceable ⇒ Nash

- Since r is a feasible payoff profile, we can write it as $r_i = \sum_{a \in A} \left(\frac{\beta_a}{\gamma}\right) u_i(a)$ where β_a and γ are non-negative integers with $\gamma = \sum_{a \in A} \beta_a$
  - Recall that if feasible $r_i = \sum_{a \in A} \alpha_a u_i(a)$ with $\sum_{a \in A} \alpha_a = 1$
- We're going to construct a strategy profile that will cycle through all outcomes $a \in A$ of G with cycles of length γ, each cycle repeating action a exactly β_a times.
- Let (a^t) be such a sequence of outcomes. Let's define a strategy s_i of player i to be a trigger version of playing(a^t):
  - if nobody deviates, then s_i plays a_i^t in period t.
  - However, if there was a period t' in which some player $j \neq i$ deviated, then s_i will play minmax strategy to minimize u_j such that
    - $v_j = \min_{s_{-j} \in S_{-j}} \max_{s_j \in S_j} u_j(s_{-i}, s_i)$
    - Player i will player $\left(s_{-j}\right)_i$
- First observe that if everybody plays according to s_i, then, by construction, player i receives average payoff of r_i (look at averages over periods of length γ).

$$u_i(a^1), u_i(a^2), u_i(a^2), u_i(a^3), u_i(a^3), u_i(a^3) \rightarrow r_i = \frac{1}{7} u_i(a^1) + \frac{2}{7} u_i(a^2) + \frac{3}{7} u_i(a^3),$$

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Feasible and Enforceable ⇒ Nash

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- **Second**, this strategy profile is a Nash equilibrium.
  - Suppose everybody plays according to s_i, and player j deviates at some point.
  - Then, forever after, player j will receive his minmax payoff $v_j \leq r_j$, rendering the deviation unprofitable.