Designing AC Power Grids Using Integer Linear Programming

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Abstract. Recent developments have drawn focus towards the efficient calculation of flows in AC power grids, which are difficult to solve systems of nonlinear equations. The common linearization approach leads to the well known and often used DC formulation, which has some major drawbacks. To overcome these drawbacks we revisit an alternative linearization of the AC power flow. Work on this model has already been done in the 1990s but was intractable at that time. In view of recent developments in the field of integer programming, we show that this model is computationally tractable.

1 Introduction

A power grid is a transmission network transporting electrical energy from power plants to some substations near urban or industrial centers. To reduce the loss of energy, different high voltages (at least 110kV) and alternating currents (AC) are used to transport the power. The so called AC power flow consists of two single power flows called active and reactive flow.

The network consists of different subnetworks each with its own level of voltage, whereby a single network contains nodes with a specific demand of active and reactive power which are connected through lines. The nodes may represent substations which lead to different networks of a lower voltage level or groups of customers. The lines may represent underground power cables or overhead power lines.

In this paper, we consider the design of AC power grids, including the placement of supply equipment (called generators). The potential topology is modelled as an undirected graph $N = (V, E)$, where the set V denotes the demand nodes and E the set of all possible lines between the nodes. The design problem is to find the minimum cost network which fulfills all demands. Given a selection of the lines, we need to calculate the power flow in the network. In an AC network, we have an active and a reactive power flow which periodically reverse their direction. The computation of these bidirectional power flows involves complex numbers and nonlinear functions.

The most common way to handle the nonlinearities is to use the so-called *DC model* which provides linear approximations. Although the DC modelling of the power flow has proven to be very fast, its major drawback is that information about the reactive flows is lost. As engineers depend on them, they have shifted their focus towards metaheuristics like genetic algorithms to solve power flow problems [7]. In view of recent developments in integer linear programming, we revisit a less known linearization which approximates both the active and the reactive power flow.

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This paper gives an overview of our model and some preliminary computational studies which show some promising results towards future work. We hope that by using more powerful tools like the generation of valid inequalities, our model can provide an alternative to the often used DC model.

2 The General Model

Given the potential topology $N = (V,E)$, let the set A consist of both arcs (v,w) and (w,v) for all $\{v,w\} \in E$. Additionally, we have a set $\mathcal{G}$ of possible generators with different construction and operating costs c_g and maximum power feed Ψ_g for each $g \in \mathcal{G}$. All these generators operate on the same voltage level $\mathbf{U}$. Generators can be installed at a subset $S \subseteq V$ to fulfill the power requirements.

For every potential line $e \in E$, let c_e the operating and construction costs, $\hat{c}_e$ a cost factor for the active power losses, R_e the line resistance, and X_e the reactance.

For every line, we compute the conductance $G_e = R_e/(R_e^2 + X_e^2)$ and susceptance $B_e = -X_e/(R_e^2 + X_e^2)$.

Finally let P_v and Q_v denote the active and reactive power demand of node $v \in V$.

At each node $v \in V$, we have to calculate the voltage $|U_v| \cdot e^{i\vartheta_v}$, where i denotes the imaginary unit. $|U_v|$ is called the voltage magnitude and ϑ_v the voltage angle. Therefore, we introduce continuous variables U_v and ϑ_v for each node, which are bounded by $U_{\min}$ and $U_{\max}$ and accordingly $\vartheta_{\min}$ and $\vartheta_{\max}$. In addition, let $P_{\mathrm{gen}\,v}$ and $Q_{\mathrm{gen}\,v}$ denote the active and reactive power feed at node v. For each $e \in E$ variables $x_e \in \{0,1\}$ denote whether or not the line is constructed. Variables $y_{vg} \in \mathbb{Z}_0^+$ denote how many generators of type g are installed at node v.

Let $P(a)$ and $Q(a)$ be functions (depending on U_v and ϑ_v, defined below) which model the active and reactive flow on arc $a \in A$ and $f(P(e))$ a function wich represents the power losses on line $e \in E$. We consider the following nonlinear model describing the optimal network design:

$$\min \sum_{e \in E} (\hat{c}_e \cdot f(P(e)) + c_e) \cdot x_e + \sum_{v \in V} \sum_{g \in \mathcal{G}} c_g \cdot y_{vg}$$

$$\sum_{(v,w) \in A} P((v,w)) \cdot x_{\{v,w\}} = P_v - P_{\mathrm{gen}\,v} \qquad \forall v \in V \tag{1a}$$

$$\sum_{(v,w) \in A} Q((v,w)) \cdot x_{\{v,w\}} = Q_v - Q_{\mathrm{gen}\,v} \qquad \forall v \in V \tag{1b}$$

$$\sum_{g \in \mathcal{G}} \Psi_g \cdot y_{vg} \geq P_{\mathrm{gen}\,v} + Q_{\mathrm{gen}\,v} \qquad \forall v \in S \tag{1c}$$

$$x \in X$$

Here the constraints (1a) and (1b) ensure that the active and reactive power demands are fulfilled at each node. Additionally, they conserve the flow in the network. Constraint (1c) guarantees generators are needed at a node before it can feed their power into the network. Further desired properties regarding the network topology can be modelled by a set X, e.g., two-connectivity.

If a node feeds into the network, it will have a fixed voltage magnitude $\mathbf{U}$. As the variable y_{vg} is integer we need to introduce a binary variable z_v which is set to one iff $\sum_{g\in\mathscr{G}} y_{vg} \geq 1$. For every possible feeding node $v \in S$ this leads to the constraints

$$U_v + (U_{\min} - \mathbf{U}) \cdot z_v \geq U_{\min}, \qquad z_v - \sum_{g\in\mathscr{G}} y_{vg} \leq 0,$$
$$U_v + (U_{\max} - \mathbf{U}) \cdot z_v \leq U_{\max}, \qquad -M \cdot z_v + \sum_{g\in\mathscr{G}} y_{vg} \leq 0, \tag{1d}$$

with M sufficiently large. The constraints on the left side guarantee that a node with at least one generator has a voltage magnitude of $\mathbf{U}$. The other constraints force z_v to be one iff $\sum_{g\in\mathscr{G}} y_{vg} \geq 1$.

The nonlinear functions for the AC power flow on arc $(k, j) \in A$ are [1]:

$$P((k,j)) = U_k^2 G_{kj} - U_k U_j G_{kj} \cos(\vartheta_k - \vartheta_j) - U_k U_j B_{kj} \sin(\vartheta_k - \vartheta_j), \tag{2a}$$
$$Q((k,j)) = -U_k^2 B_{kj} + U_k U_j B_{kj} \cos(\vartheta_k - \vartheta_j) - U_k U_j G_{kj} \sin(\vartheta_k - \vartheta_j). \tag{2b}$$

The difference between the two active flows on a line is called the active power loss. For a line $e = \{k, j\}$ with corresponding arcs $a_+ = (k, j)$ and $a_- = (j, k)$ it is:

$$f(P(e)) = P(a_+) + P(a_-) = G_e \cdot \left(U_k^2 + U_j^2 - 2U_k U_j \cos(\vartheta_k - \vartheta_j)\right), \tag{3}$$

Note that these nonlinear functions are multiplied with a binary variable. Although recent developments in the field of mixed integer nonlinear programming (MINLP) have yield some promising results, the resulting MINLP model is very difficult to handle, cf. [2].

3 Linear Approximations of the Power Flow Functions

As one main difficulty lies in the choice of f, P and Q, we now discuss two linear choices for these functions.

3.1 The DC Power Flow

The most common linearization of (2a) and (2b) is the so called *DC power flow model*. To linearize the model, we assume $\cos(\vartheta_k - \vartheta_j) \approx 1$, $\sin(\vartheta_k - \vartheta_j) \approx \vartheta_k - \vartheta_j$, $R_{kj} \ll X_{kj}$, and $|U_k| = |U_j| = U_0$ with U_0 a fitting constant. From $R_{kj} \ll X_{kj}$ it follows that $G_{kj} \approx 0$ and we get the DC power flow equations

$$P((k,j)) = -U_0^2 B_{kj} \cdot (\vartheta_k - \vartheta_j), \qquad Q((k,j)) = 0.$$

Notice, for $U_0 = 1$ this reduces to the well-known $P((k,j)) = (\vartheta_k - \vartheta_j)/X_{kj}$.

The above assumptions guarantee that we get a symmetric flow, meaning that $P_{kj} = -P_{jk}$ holds. Therefore, we have no active power losses ($f = 0$). This linearization is used in a variety of different integer linear programs concerning power grid problems, cf. [3, 4, 6].

One major advantage of this linear model is, its interpretation as DC power flows and not just only as an approximation of the AC power flows. The drawback, however, is the missing insight on the reactive flows Q. We like to stress the fact that the above assumptions can only be made, if the underlying network fulfills some specific properties. The checking whether these conditions hold is, however, often forgotten. We refer to [9] for a discussion of the importance of these conditions.

3.2 Approximation of the AC Power Flow

The following approach was first introduced by Moser [8] and improved by Braun [5], but to our knowledge no further work has been conducted since then. For each arc $a = (k, j) \in A$, we introduce new variables $\Delta U_a := U_k - U_j$ and $\Delta \vartheta_a := \vartheta_k - \vartheta_j$.

Like in the DC linearisation we assume $\cos(\Delta \vartheta_{kj}) \approx 1$, $\sin(\Delta \vartheta_{kj}) \approx \Delta \vartheta_{kj}$, and $|U_k| = |U_j| = U_0$, but without setting $\Delta U_{kj} = 0$ and $G_{kj} = 0$. This leads to the following linearisation of the AC power flow equations:

$$\begin{aligned} P((k,j)) &= |U_k|^2 G_{kj} - |U_k||U_j|G_{kj}\cos(\Delta \vartheta_{kj}) - |U_k||U_j|B_{kj}\sin(\Delta \vartheta_{kj}) \\ &\approx |U_k|G_{kj}(|U_k| - |U_j|) - |U_k||U_j|B_{kj}\Delta \vartheta_{kj} \\ &= U_0 G_{kj}\Delta U_{kj} - U_0^2 B_{kj}\Delta \vartheta_{kj} \end{aligned} \tag{4a}$$

$$\begin{aligned} Q((k,j)) &= -|U_k|^2 B_{kj} + |U_k||U_j|B_{kj}\cos(\Delta \vartheta_{kj}) - |U_k||U_j|G_{kj}\sin(\Delta \vartheta_{kj}) \\ &\approx |U_k|B_{kj}(-|U_k| + |U_j|) - |U_k||U_j|G_{kj}\Delta \vartheta_{kj} \\ &= -U_0 B_{kj}\Delta U_{kj} - U_0^2 G_{kj}\Delta \vartheta_{kj} \end{aligned} \tag{4b}$$

This approximation leads to a symmetric power flow as well (and therefore $f = 0$), but it allows the computation of a reactive flow.

4 Linearization of the Network Design Model

In this section, we construct a mixed integer linear program to determine the optimal network design with respect to the AC power flow. Even by using the linear power flow equations (4a) and (4b), we have to overcome the fact that the constraints (1a) and (1b) are nonlinear. Therefore, we substitute these constraints by

$$\sum_{(v,w)\in A} P((v,w)) = P_v - P_{\mathrm{gen}\,v} \quad \forall v \in V \tag{5a}$$

$$\sum_{(v,w)\in A} Q((v,w)) = Q_v - Q_{\mathrm{gen}\,v} \quad \forall v \in V. \tag{5b}$$

To guarantee that $P((v,w)) = Q((v,w)) = 0$ if $x_{\{v,w\}} = 0$, we force $\Delta U_{(v,w)} = \Delta \vartheta_{(v,w)} = 0$ iff $x_{\{v,w\}} = 0$. For this, we introduce for each $a = (v,w)$ complementary variables $\Delta \tilde{U}_a$ and $\Delta \tilde{\vartheta}_a$ satisfying

$$\Delta \tilde{U}_a = U_v - U_w - \Delta U_a \quad \text{and} \quad \Delta \tilde{\vartheta}_a = \vartheta_v - \vartheta_w - \Delta \vartheta_a. \tag{5c}$$

The requirement now holds iff for each $a = (v,w)$ and $e = \{v,w\}$ the constraints

$$\begin{aligned} -\Delta U_{\max} \cdot x_e &\leq \Delta U_a \leq \Delta U_{\max} \cdot x_e, \\ -\Delta \vartheta_{\max} \cdot x_e &\leq \Delta \vartheta_a \leq \Delta \vartheta_{\max} \cdot x_e, \\ -\Delta U_{\max} \cdot (1 - x_e) &\leq \Delta \tilde{U}_a \leq \Delta U_{\max} \cdot (1 - x_e), \\ -\Delta \vartheta_{\max} \cdot (1 - x_e) &\leq \Delta \tilde{\vartheta}_a \leq \Delta \vartheta_{\max} \cdot (1 - x_e), \end{aligned} \tag{5d}$$

are satisfied, with appropriate values for $\Delta U_{\max}$ and $\Delta \vartheta_{\max}$, computed from the given bounds for U and ϑ.

5 Computational Experience and Further Remarks

Note that in the above model there are no explicit bounds on the power flows given. However, the bounds on the voltage drop imply bounds on the maximal flow per line. To reduce the running time, we focused our studies on a medium sized 110 kV network with 28 nodes and 67 possible lines. We used the above model with both approximations for the power flow equations.

The DC model has proven to be very fast, as the problem was solved within seconds. As the DC model operates under the assumption of a constant voltage magnitude, the bounds on the voltage magnitudes are meaningless. Therefore, the optimal solution of the DC model was the minimum cost circuit of the network.

The overall performance of the linearized AC model was acceptable (but significantly slower than the DC model) and the optimal designs produced have the expected shape, meaning a couple of intersected circuits.

For both solutions, we calculated nonlinear power flows for the computed topologies using Newton's method. We observed that the calculated AC power flow approximation yields a good starting solution for the method. The error margin of the approximation is small, e.g., less than 6% for the voltage magnitude. Furthermore, the nonlinear solution fulfills the bounds of the voltage drop. In contrast, the nonlinear power flow solution of the topology given by the DC model does not satisfy these bounds, and therefore the topology is not valid for the problem setting.

In the future we like to derive more information from the very fast DC model and use it for the AC approximation. In addition, much work still has to be done to enhance the performance of the AC model. We hope that by deriving valid inequalities the performance of these problems can be signficantly improved.

Our next focus lies on the calculation of an approximation of the active power loss. Although our linearization forces the power loss to be zero, we can use a different approach to successfully approximate the power loss. This model is way more complex than the above, but surprisingly it seems to outperform it significantly regarding computation time.

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