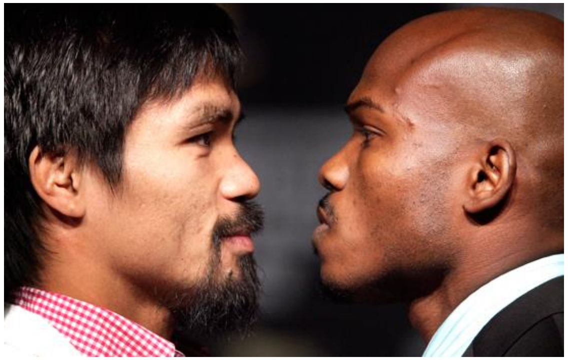
Lecture 13 Signaling Game

Introduction

- In games of incomplete information, there is at least one player who is uninformed about the type of another player.
- In some instances, it will be beneficial for players to reveal their types to their opponents

“I am strong and hence you should not waste time and energy fighting me



- Of course even a weak player would like to try to convince his opponent that he is strong

- There has to be some credible means, beyond such “cheap talk” through which the player can signal his type and make his opponent believe him.

Signaling game procedure

- Nature chooses a type for player 1 that player 2 does not know, but cares about (common values)
- Player 1 has a rich action set in the sense that there are at least as many actions as there are types, and each action imposes a different cost on each type
- Player 1 chooses an action first, and player 2 then responds after observing player 1's choice
- Given player 2's belief about player 1's strategy, player 2 updates his belief after observing player 1's choice. Player 2 then makes his choice as a best response to this updated beliefs.

Signaling game procedure

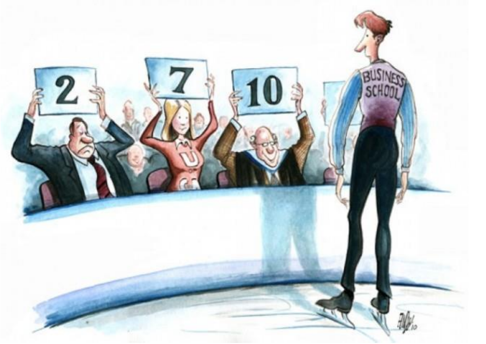
- **Two important classes of perfect Bayesian equilibria**

  - **Pooling equilibria**
    - All the types of player 1 chose the same action
    - Reveals nothing to player 2
    - Player 2's beliefs must be derived from Bayes' rule only in the information sets that are reached with positive probability.
    - All other information sets are reached with probability zero, player 2 must have beliefs that support his own strategy
    - The sequential rational strategy of player 2 given his beliefs is what keeps player 1 from deviating from his pooling strategy

  - **Separating equilibria**
    - Each type of player 1 chooses a different action
    - Reveals his type in equilibrium to player 2
    - Player 2's beliefs are thus well defined by Bayes' rule in all the information sets that are reached with positive probability
    - If there are more actions than types for player 1, the player 2 must have beliefs in the information sets that are not reached, which in turn must support the strategy of player 2 and player 2's strategy support the strategy of player 1.

The MBA game

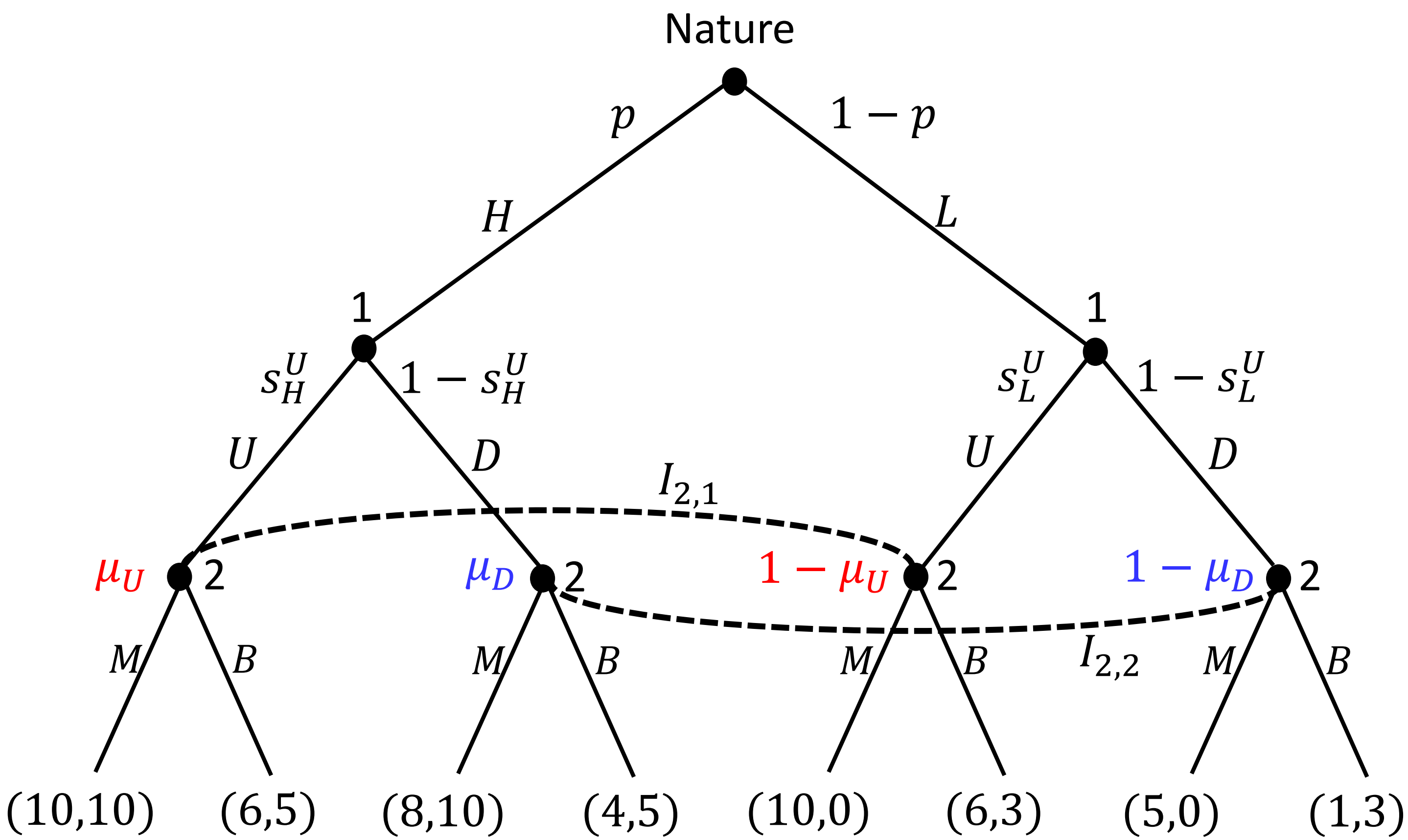
- Nature choose player 1's skill (productivity at work)
  - $\theta_1 \in \Theta_1 = \{H, L\}$
  - $\Pr\{\theta_1 = H\} = p > 0$
- After player 1 learns his type, he can choose whether to get an MBA degree (D) or be contend with his undergraduate degree (U)
  - $a_1 \in A_1 = \{D, U\}$
  - The cost for MBA are
    - $c_H = 2$ for high-skilled type
    - $c_L = 5$ for low-skilled type
- Player 2 is an employer, who can assign player 1 to one of two jobs
  - Manager (M)
  - Blue-color worker (B)
  - $a_2 \in A_2 = \{M, B\}$
  - The market wages for two jobs are
    - $w_M = 10$ for Manager
    - $w_B = 6$ for Blue-color worker



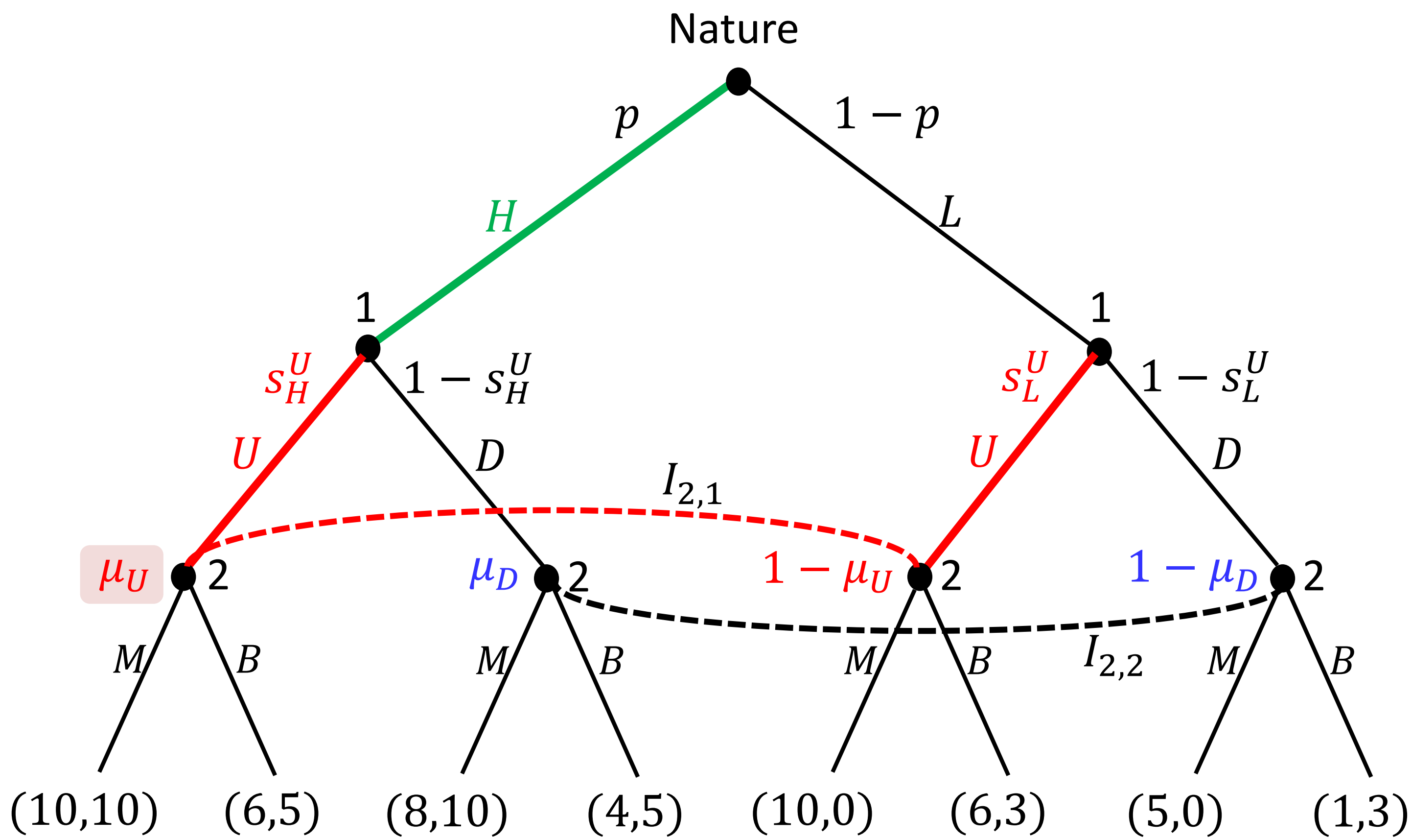
- Player 2's payoff is determined by the combination of skill and job assignments (It is assumed that MBA degree adds nothing to productivity)

	M	B
H	10	5
L	0	3

The MBA game


Nature
p
$1-p$
H
L
1
1
s_H^U
$1-s_H^U$
s_L^U
$1-s_L^U$
U
D
U
D
$I_{2,1}$
μ_U
2
μ_D
2
$1-\mu_U$
2
$1-\mu_D$
2
M
B
M
B
M
B
$I_{2,2}$
M
B
(10,10)
(6,5)
(8,10)
(4,5)
(10,0)
(6,3)
(5,0)
(1,3)

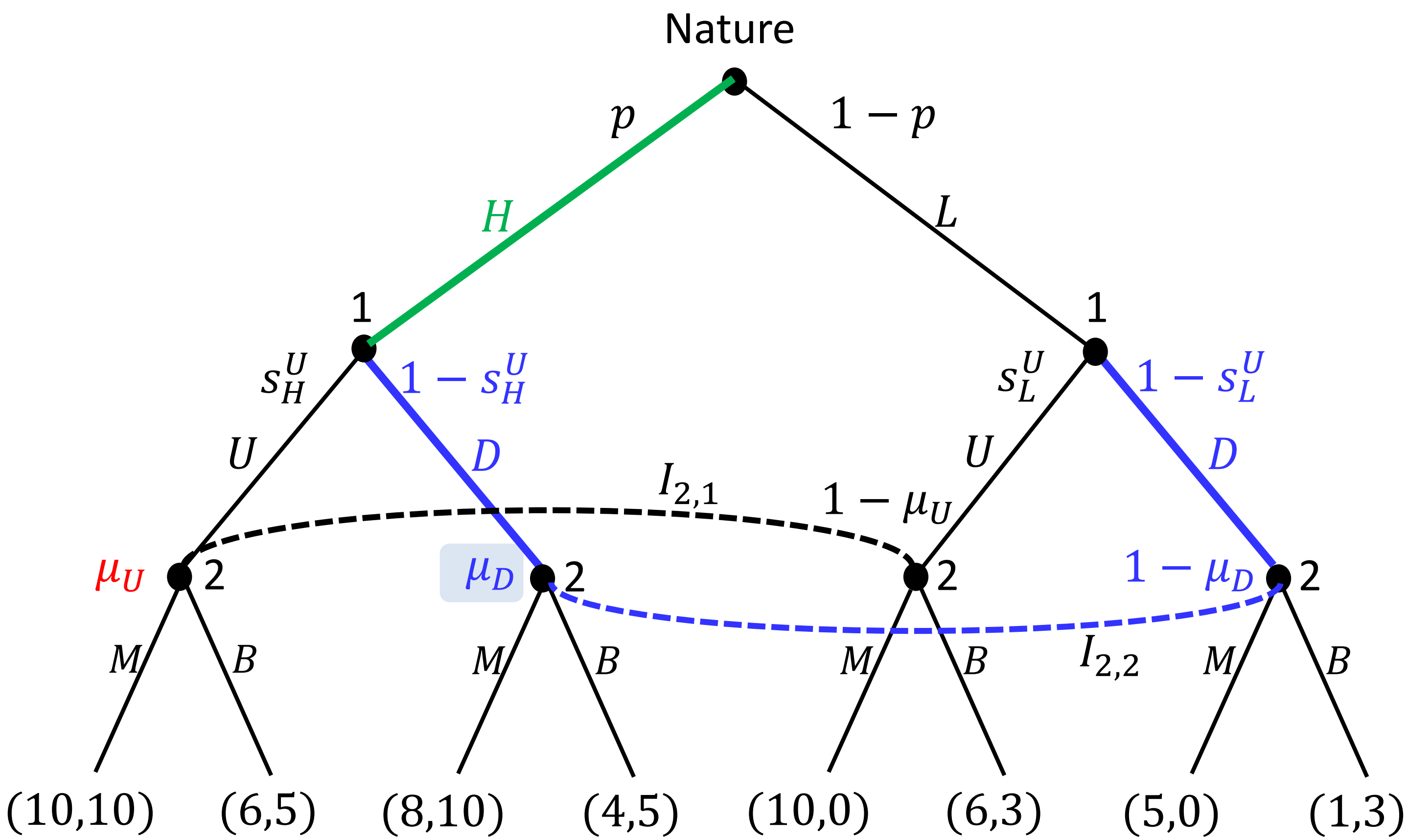
The MBA game



$$\mu_U = P(H|U) = \frac{P(H \cap U)}{P(U)} = \frac{P(H)P(U|H)}{P(H)P(U|H) + P(L)P(U|L)} = \frac{ps_H^U}{ps_H^U + (1-p)s_L^U}$$

- If $s_H^U = s_L^U = 1$, then beliefs are defined only for $I_{2,1}$ ($\mu_U = p$)
- We have freedom for μ_D because information set $I_{2,2}$ is not reached

The MBA game



$$\mu_D = P(H|D) = \frac{P(H \cap D)}{P(D)} = \frac{P(H)P(D|H)}{P(H)P(D|H) + P(L)P(D|L)} = \frac{p(1-s_H^U)}{p(1-s_H^U) + (1-p)(1-s_L^U)}$$

- If $s_H^U = s_L^U = 0$, then beliefs are defined only for $I_{2,1}$ ($\mu_D = p$)
- We have freedom for μ_U because information set $I_{2,1}$ is not reached

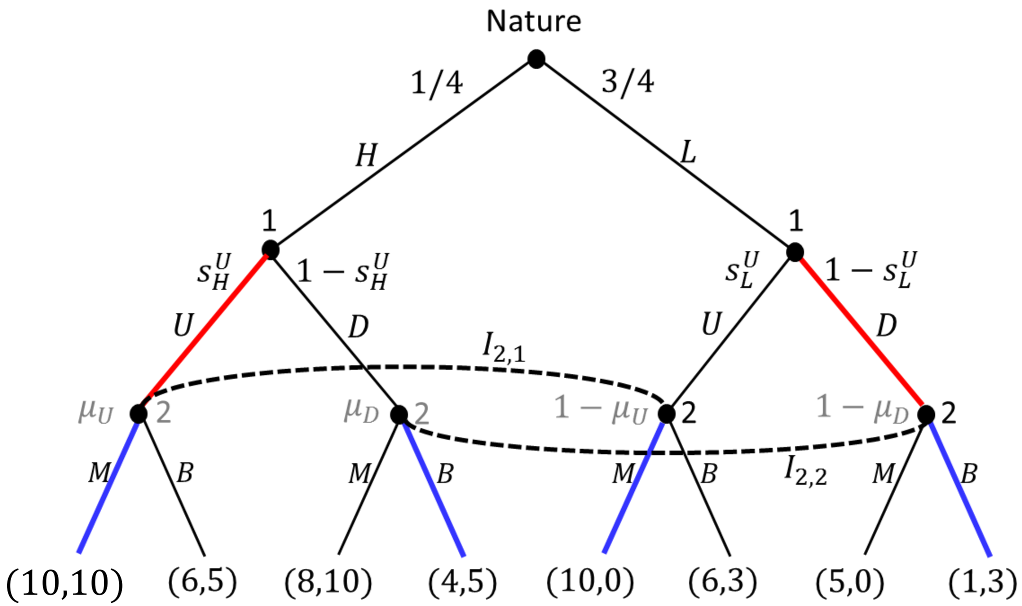
The MBA game

- Now, we are ready to proceed to find the perfect Bayesian equilibria in the Master's Game

- Each player has two information sets with two actions in each of these sets
  - $s_1 = a_1^H a_1^L \in A_1 = \{UU, UD, DU, DD\}$
    - where a_1^H is the action taken when Nature chooses H
    - where a_1^L is the action taken when Nature chooses L
  - $s_2 = a_2^U a_2^D \in A_2 = \{MM, MB, BM, BB\}$
    - where a_2^U is the action taken when player 1 takes U
    - where a_2^D is the action taken when player 1 takes D

The MBA game

- We can convert the game into the following normal form game ($p = 1/4$)



$u_1(UD, MB) = \frac{1}{4}10 + \frac{3}{4}1 = 3.25$ $\qquad$ $u_2(UD, MB) = \frac{1}{4}10 + \frac{3}{4}3 = 4.75$

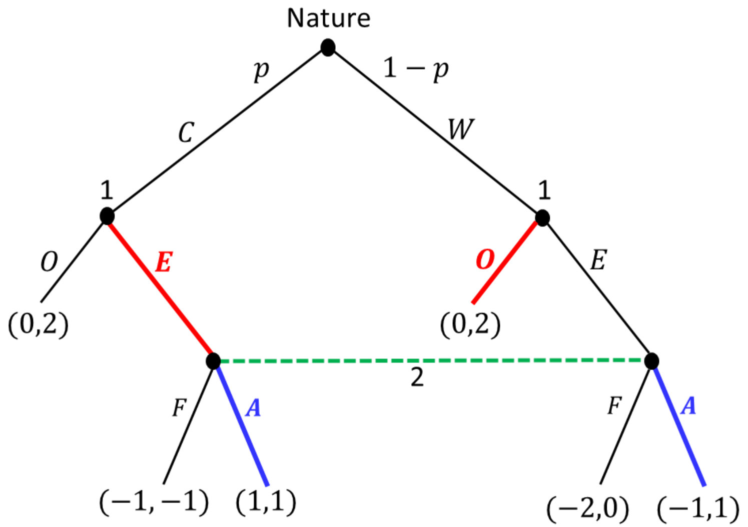
	MM	MB	BM	BB
UU	$10, 2.5$	$10, 2.5$	$6, 3.5$	$6, 3.5$
UD	$6.25, 2.5$	$3.25, 4.75$	$5.25, 1.25$	$2.25, 3.5$
DU	$9.5, 2.5$	$8.5, 1.25$	$6.5, 4.75$	$4.5, 3.5$
DD	$5.75, 2.5$	$1.75, 3.5$	$5.75, 2.5$	$1.75, 3.5$

How to compute Perfect Bayesian (Nash) Equilibrium

- First find all the profiles of strategies in the Bayesian game that are Bayesian Nash equilibria
- Then, we can systemically check for each Bayesian Nash equilibrium to see whether we can find a system of beliefs so that together they constitute a perfect Bayesian equilibrium

Proposition (Perfect Bayesian (Nash) Equilibrium)

If a profile of (possibly mixed) strategies $s^* = (s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium of a Bayesian game Γ, and if s^* induces all the information sets to be reached with positive probability, the s^*, together with the belief system μ^* uniquely derived from s^* and the distribution of types, constitutes a perfect Bayesian equilibrium for Γ

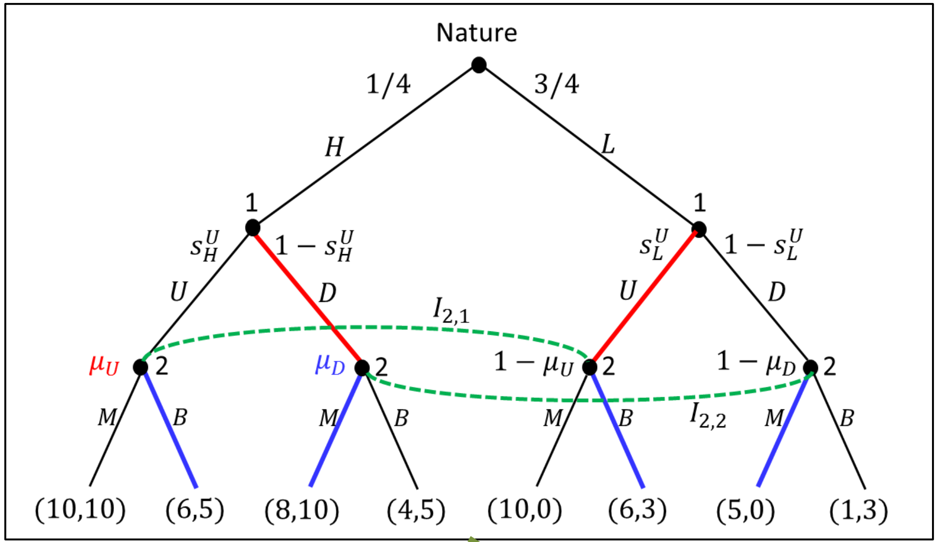


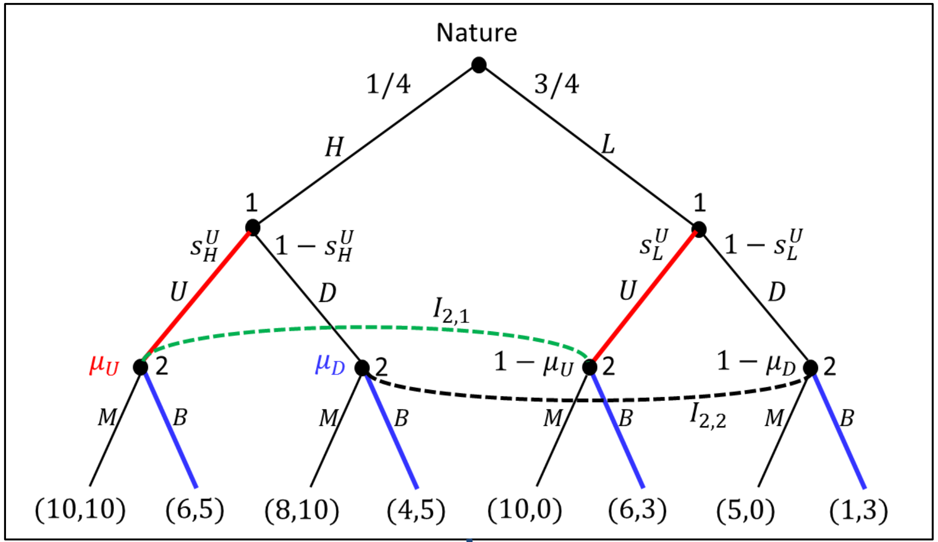
$s^* = \big((EO), A\big)$ with $\mu_E = 1$ is PBNE because

- It is Bayesian Nash equilibrium
- All the information sets are reached with positive probability
- $\mu_E = 1$ is consistent with s^*

The MBA game

- We can convert the game into the following normal form game ($p = 1/4$)





- There are two pure strategies Bayesian Nash equilibria

	MM	MB	BM	BB
UU	10, 2.5	10, 2.5	6, 3.5	6, 3.5
UD	6.25, 2.5	3.25, 4.75	5.25, 1.25	2.25, 3.5
DU	9.5, 2.5	8.5, 1.25	6.5, 4.75	4.5, 3.5
DD	5.75, 2.5	1.75, 3.5	5.75, 2.5	1.75, 3.5

The MBA game

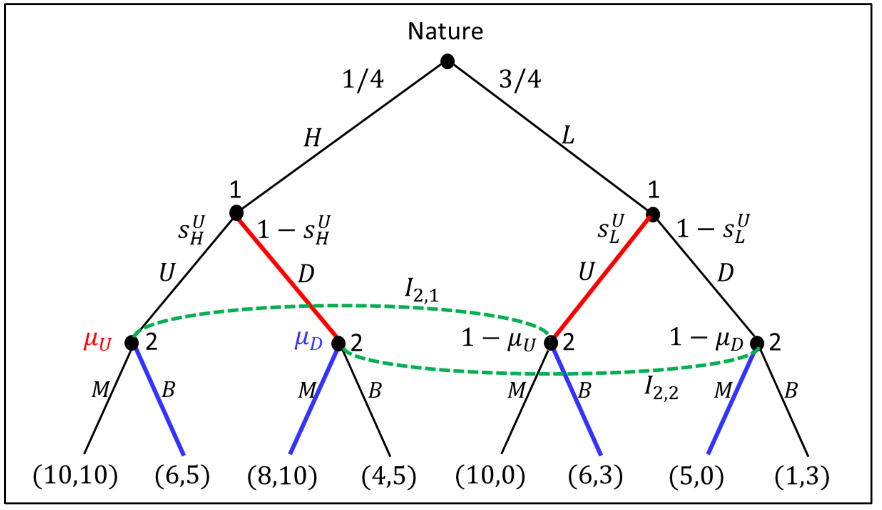
- $s = (DU, BM)$ is **the perfect Bayesian equilibrium** because
  - ✓ All of the information sets are reached with positive probabilities
  - ✓ The derived beliefs from (DU, BM) are $\mu_U = 0$ and $\mu_D = 1$

$$\mu_U = P(H|U) = \frac{P(H)P(U|H)}{P(H)P(U|H) + P(L)P(U|L)} = \frac{p s_H^U}{p s_H^U + (1-p) s_L^U} = \frac{\frac{1}{4} \times 0}{\frac{1}{4} \times 0 + \frac{3}{4} \times 1} = 0$$

$$\mu_D = P(H|D) = \frac{P(H)P(D|H)}{P(H)P(D|H) + P(L)P(D|L)} = \frac{p\left(1 - s_H^U\right)}{p(1 - s_H^U) + (1-p)p(1 - s_L^U\ )} = \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times 0} = 1$$

  - ✓ Each player are best responding to these beliefs as seen from the induced normal form game or the extensive form game (already satisfied because we start from Bayesian eq.)

	MM	MB	BM	BB
UU	10, 2.5	10, 2.5	6, 3.5	6, 3.5
UD	6.25, 2.5	3.25, 4.75	5.25, 1.25	2.25, 3.5
DU	9.5, 2.5	8.5, 1.25	6.5, 4.75	4.5, 3.5
DD	5.75, 2.5	1.75, 3.5	5.75, 2.5	1.75, 3.5



The MBA game

- What about $s = (UU, BB)$?
  - ✓ Information set $I_{2,1}$ is reached with positive prob. Thus, unique beliefs are derived as

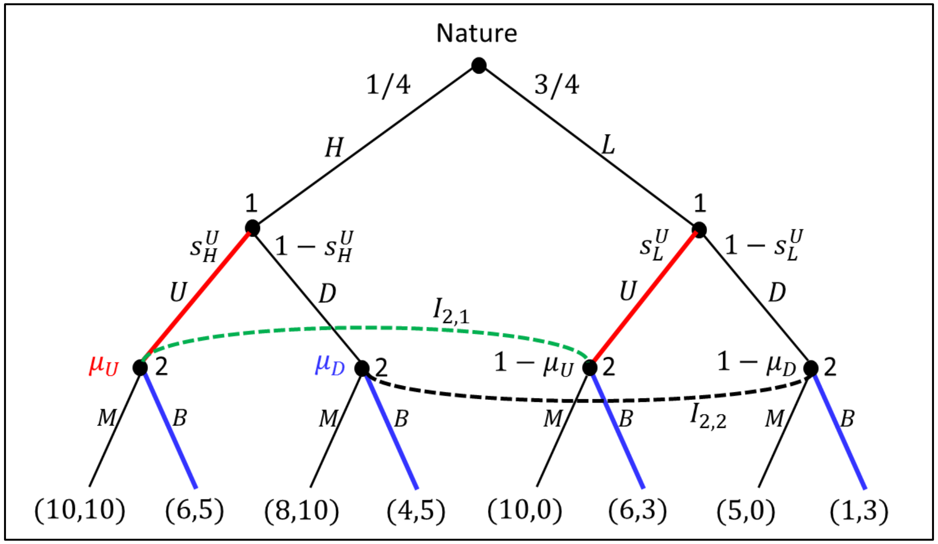
$$\mu_U = P(H|U) = \frac{P(H)P(U|H)}{P(H)P(U|H) + P(L)P(U|L)} = \frac{ps_H^U}{ps_H^U + (1-p)s_L^U} = \frac{\frac{1}{4}\times 1}{\frac{1}{4}\times 1 + \frac{3}{4}\times 1} = \frac{1}{4}$$

  - ✓ At the information set $I_{2,1}$, for player 2 to play B is the best response because

$$u_2(UU, M) = \frac{1}{4}10 + \frac{3}{4}0 < u_2(UU, B) = \frac{1}{4}5 + \frac{3}{4}3$$

  - ✓ Similarly, for player 1 to play UU is the best response because

$$u_1(UU, BB) = \frac{1}{4}6 + \frac{3}{4}6 > u_1(UD, BB), u_1(DU, BB)u_1, (DD, BB)$$

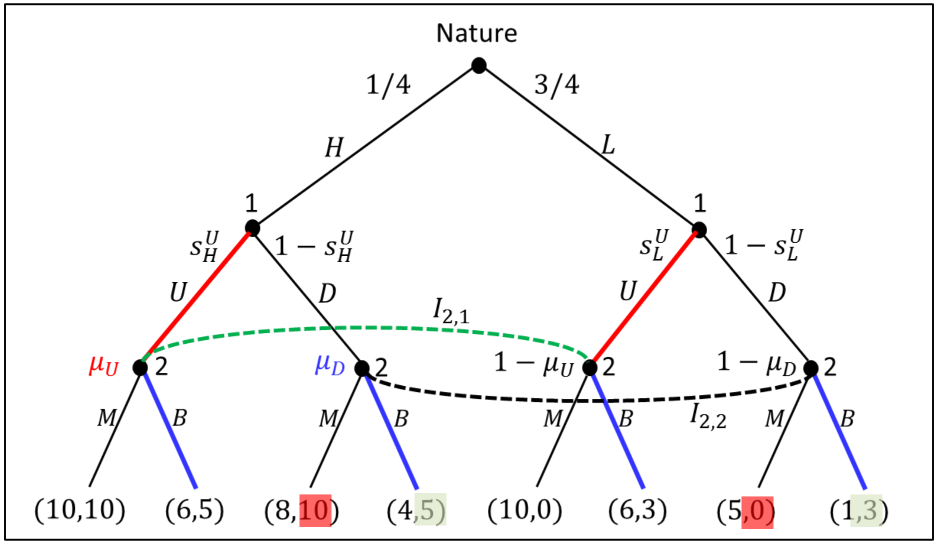


The MBA game

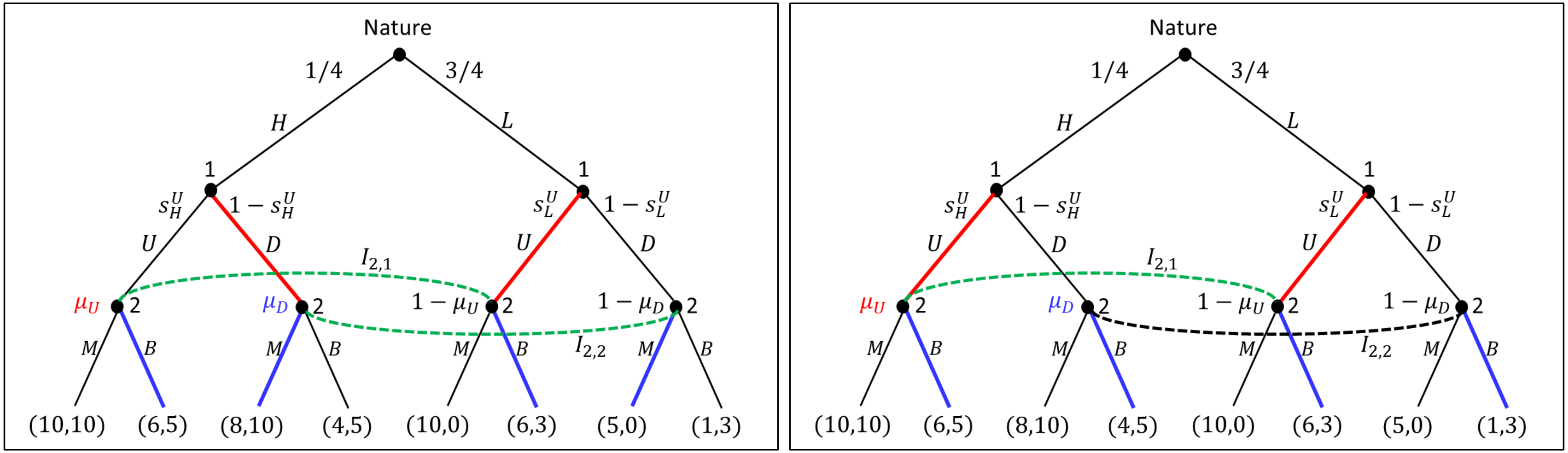
- What about $s = (UU, BB)$?
  - ✓ Information set $I_{2.2}$ is **not reached** with positive prob. Thus, no unique beliefs is made

$$\mu_D = P(H|D) = \frac{P(H)P(D|H)}{P(H)P(D|H) + P(L)P(D|L)} = \frac{p(1 - s_H^U)}{p(1 - s_H^U) + (1-p)p(1 - s_L^U\ )} = \frac{\frac{1}{4} \times 0}{\frac{1}{4} \times 0 + \frac{3}{4} \times 0} = ?$$

  - ✓ We need to check if there are beliefs μ_D that support for player 2 to play B as a best response for player 2 in this information set $I_{2.2}$.
    - when $u_2(s_1, B) = 5\mu_D + 3(1 - \mu_D) \geq 10\mu_D + 0(1 - \mu_D)$, playing B is Best res.
    - Thus, $\mu_D \in \left[0, \frac{3}{8}\right]$ is valid belief for supporting for player 2 to play B
  - ✓ Therefore, $s = (UU, BB)$ with $\mu_U = 1/4$ and $\mu_D \in \left[0, \frac{3}{8}\right]$ constitutes a **perfect Bayesian equilibrium**



Summary



- The first perfect Bayesian equilibrium with strategies (DU, BM)
  - Different types of player chose different actions, thus using their actions to reveal to player 2 their true types
  - This is a **separating** perfect Bayesian equilibrium

- The Second perfect Bayesian equilibrium with strategies (UU, BB)
  - Both types of player do the same thing, thus player 2 learns nothing from player 1's action
  - This is a **pooling** perfect Bayesian equilibrium