Lecture 15. evolutionarily games and replicator dynamics

K-beauty contest

- **The rules of the basic beauty-contest game.**
  - Each person of ***N-players*** is asked to choose a (***real*** or ***integer***) number from the ***interval 0 to 100***.
  - The winner is the person whose choice is closest to ***p*** times the ***mean*** of the choices of all players (where p is, for example, ***2/3***).
  - The winner gets a ***fixed*** prize of ***$20***. In case of a tie the prize is ***split*** amongst those who tie.
  - The same game may be repeated ***several periods.*** Subjects ***are informed*** of the ***mean, 2/3 mean*** and ***all choices*** after each period

What would you play?
What is a Nash equilibrium?

K-beauty contest

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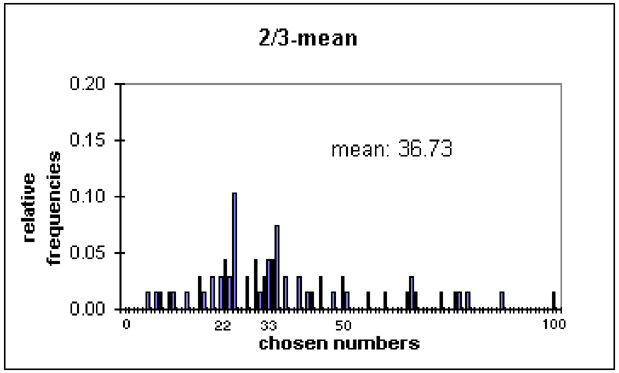
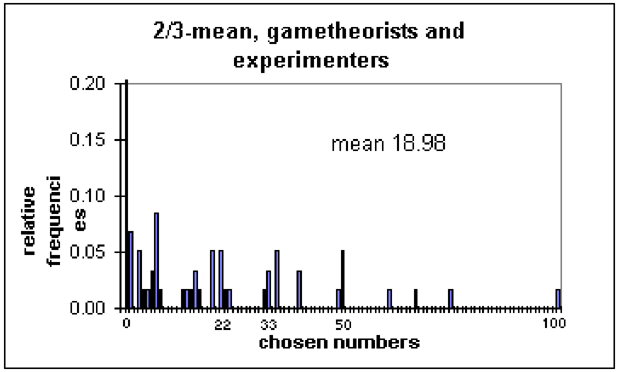
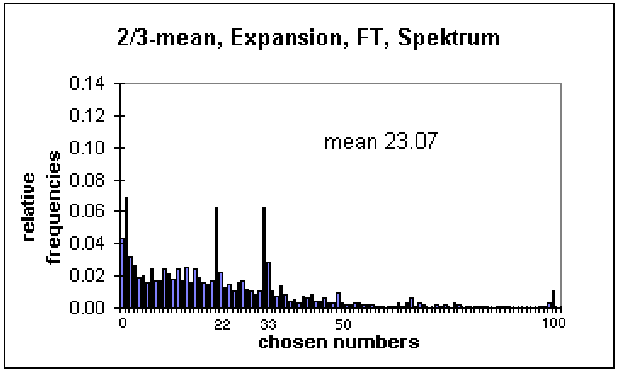






Nagel (1995)

- Most models of economic behavior are based on the assumption of rationality of economic agents and common knowledge of rationality.
- Do people play Nash equilibrium?
- In the context of the k-beauty game, we saw that even very smart game theorists do not play the unique Nash equilibrium (or the unique strategy profile surviving iterated elimination of strictly dominated strategies).
- Why?
  - Either because in new situations, it is often quite complex to work out what is “best”.
  - Or more likely, because, again in new situations, individuals are uncertain about how others will play the game.
- If we played the k-beauty game several more times, behavior would have approached or in fact reached the Nash equilibrium prediction.

- This reasoning suggests the following:
- Perhaps people behave using simple rules of thumb; these are somewhat "myopic," in the sense that they do not involve full computation of optimal strategies for others and for oneself.
- But they are also "flexible" rules of thumb in the sense that they adapt and respond to situations, including to the (actual) behavior of other players.
- What are the implications of this type of adaptive behavior?
- Two different and complementary approaches:
  - evolutionary game theory
  - Learning in games.

Evolution and game theory

- The theory of evolution goes back to Darwin's classic, The Origins of Species (and to Wallace)
- Darwin focused mostly on evolution and adaptation of an organism to the environment in which it was situated. But in The Descent of Man, in the context of sexual selection, he anticipated many of the ideas of evolutionarily game theory.
- evolutionary game theory was introduced by John Maynard Smith in Evolution and the Theory of Games, and in his seminal papers, Maynard Smith (1972) "Game Theory and the Evolution of Fighting" and Maynard Smith and Price (1973) "The Logic of Animal Conflict".
- The theory was formulated for understanding the behavior of animals in game-theoretic situations (to a game theorist, all situations). But it can equally well be applied to modeling "myopic behavior" for more complex organisms—such as humans.

Evolution and game theory

- In its simplest form the story goes like this: each organism is born programmed to play a particular strategy.
- The game is the game of life—with payoffs given as fitness (i.e., expected number of off springs). If the organism is successful, it has greater fitness and more offspring, also programmed to play in the same way. If it is unsuccessful, it likely dies without offspring.
- Mutations imply that some of these offspring will randomly play any one of the feasible strategies.

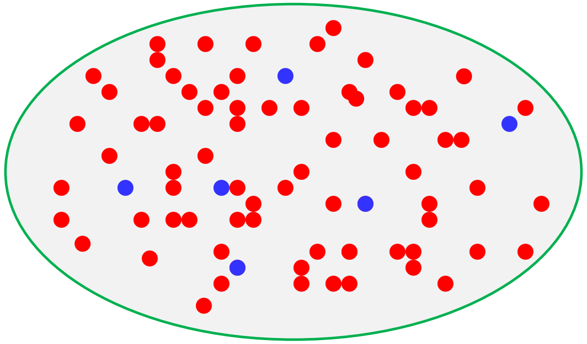
Two approaches to evolutionary Game Theory

- There are two approaches to evolutionarily game theory
  - The evolutionarily stable strategy:
    - Static property of the game
    - Equilibrium concept to analysis evolutionarily game
  - The evolutionary dynamics:
    - Dynamic property of the game
    - An explicit model of the process by which the frequency of strategies change in the population

The evolutionarily stable strategy:

The setting

Large population of agents



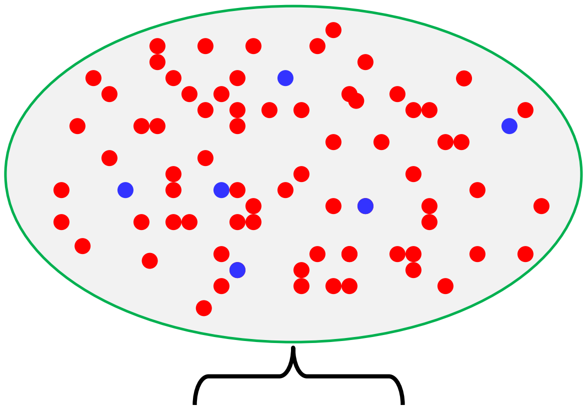
- Consider a large population of agents (organisms, animals, humans).
- Agents are designed to play a certain way
  - Incumbents (다수) play I
  - Mutants (돌연변이) play M (a small fraction)

- At each instant, each agent is randomly matched with one other agent from the population, whose distribution is modeled by the mixed strategy

- **Which strategy is evolutionally stable or not?**

The setting

Large population of agents



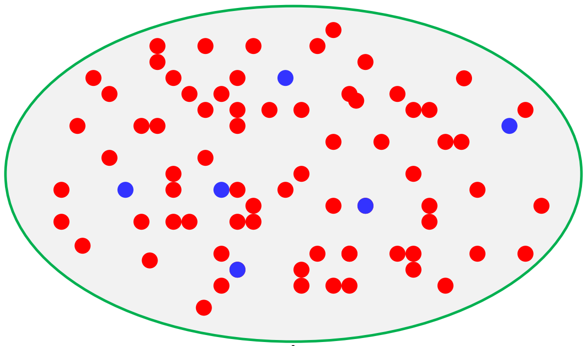
	$1-\epsilon$ I	ϵ M
• I	$3, 3$	$0, 4$
• M	$4, 0$	$1, 1$

- At each instant, each agent is randomly matched with one other agent from the population, and they play a symmetric strategic form game.
- The population is fictitiously represented by the column player
- The population's distribution is modeled by the mixed strategy $(1-\epsilon, \epsilon)$
  - Mutant: a small portion of ϵ of mutants plays M
  - Incumbent: the rest of the population $(1-\epsilon)$ play I

- The interaction between each pair of organisms can be modeled as a two player sym. game $G(N, A, u)$ with a common set A of actions and a payoff function u.
  - $u(a, a')$ is the expected reward, representing fitness
  - The fitness can indicates expected number of offspring

The setting

Large population of agents

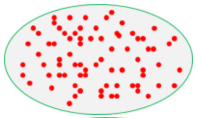
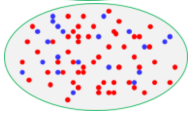


$1-\epsilon$ ϵ

	I	M
• I	$3,3$	$0,4$
• M	$4,0$	$1,1$

$$u(I, Population) = u(I,I)(1-\epsilon) + u(I,M)\epsilon$$

$$u(M, Population) = u(M,I)(1-\epsilon) + u(M,M)\epsilon$$

- Strategies with higher payoffs expand and those with lower payoffs contract
  - If $u(I, Population) > u(M, Population)$: Mutants will distinguish 
  - If $u(M, Population) > u(I, Population)$: Mutants will expand 

Motivating example

	Populations $1-\epsilon$ C	ϵ D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

<Prisoner's dilemma as an evolutionarily game>

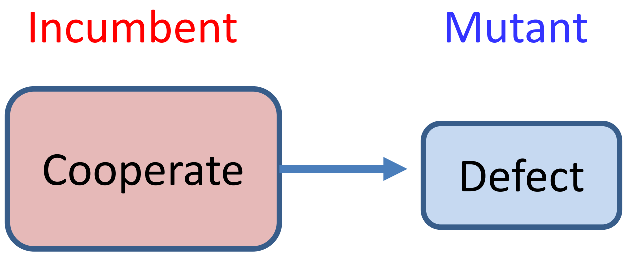
- Case 1: Incumbent (playing C) vc. the population:
  - C vc. $[(1-\epsilon)C + \epsilon D]$
  - The expected payoff : $(1-\epsilon)3 + \epsilon 0 = 3(1-\epsilon)$
- Case 2: Mutant (playing D) vc. the population:
  - D vc. $[(1-\epsilon)C + \epsilon D]$
  - The expected payoff : $(1-\epsilon)4 + \epsilon 1 = 4(1-\epsilon) + \epsilon$
- We need to compare the expected payoffs of incumbents and mutants when involved in random matchings with other individuals
  - Since $4(1-\epsilon) + \epsilon > 3(1-\epsilon)$, mutant performs better than the incumbent
    - ➢ Cooperation (Incumbent) is not a an evolutionarily stable strategy

Evolutionarily stable strategy – Intuition

A strictly dominated strategy is not an evolutionarily stable

Populations

	C ($1-\epsilon$)	D (ϵ)
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

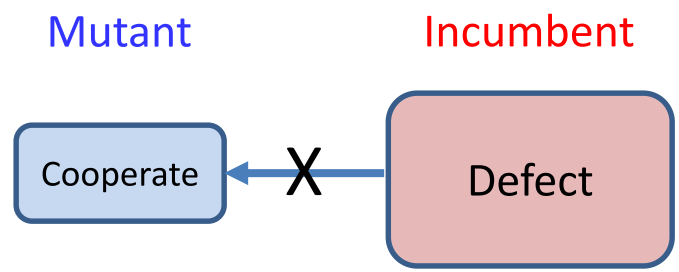


- **Cooperation,** which is as strictly dominated strategy, is not evolutionarily stable

A strictly dominated strategy is not an evolutionarily stable

Populations

	ϵ	$1-\epsilon$
	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

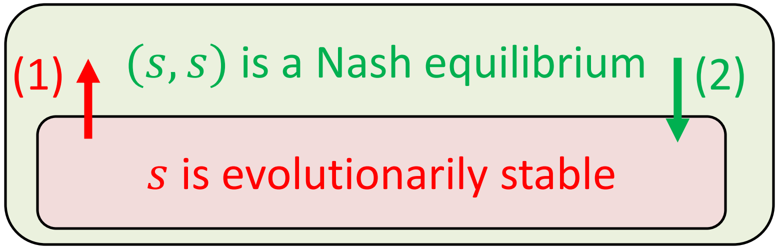


- Case 1: Mutant (playing C) vc. the population:
  - C vc. $[\epsilon C + (1-\epsilon)D]$
  - The expected payoff : $(1-\epsilon)0 + \epsilon 3 = 3\epsilon$
- Case 2: Incumbent (playing D) vc. the population:
  - D vc. $[\epsilon C + (1-\epsilon)D]$
  - The expected payoff : $\epsilon 4 + (1-\epsilon)1 = (1-\epsilon) + 4\epsilon$

The incumbent is more successful than the mutant on random matchings.

- Implies that mutations from D tend to extinguish

Evolutionarily stable strategy – Intuition



From evolutionarily stable strategy to symmetric Nash equilibrium

	ϵ b	$1-\epsilon$ c
b	$0, 0$	$1, 1$
c	$1, 1$	$0, 0$

(s, s) is a Nash equilibrium

(1) $\uparrow$

s is evolutionarily stable

- Case 1: Mutant (playing b) vc. the population:
  - b vc. $[\epsilon b + (1-\epsilon)c]$
  - The expected payoff : $\epsilon 0 + (1-\epsilon)1 = (1-\epsilon)$
- Case 2: Incumbent (playing c) vc. the population:
  - c vc.$[\epsilon b + (1-\epsilon)c]$
  - The expected payoff : $\epsilon 1 + (1-\epsilon)0 = \epsilon$
- Mutant (playing b) performs better than the incumbent (playing c)
  - ➢ c is not a symmetric NE, because turning to b is strictly profitable
  - ➢ **Strategy c is not an evolutionarily stable strategy**

- If a strategy s is not a symmetric NE, then it is not a evolutionarily stable
- If s is evolutionarily stable $\Rightarrow$ (s, s) is Nash equilibrium

Evolutionarily stable strategy – Intuition

A Nash equilibrium strategy is not necessarily an evolutionarily stable strategy

	ϵ a	$1-\epsilon$ b
a	$\boxed{1,1}$	$0,0$
b	$0,0$	$\boxed{0,0}$

(s,s) is a Nash equilibrium

(2)

s is evolutionarily stable

- Case 1: Mutant (playing a) vc. the population:
  - a vc. $[\epsilon a + (1-\epsilon)b]$
  - The expected payoff : $\epsilon 1 + (1-\epsilon)1 = \epsilon$
- Case 2: Incumbent (playing b) vc. the population:
  - b vc.$[\epsilon a + (1-\epsilon)b]$
  - The expected payoff : $\epsilon 0 + (1-\epsilon)0 = 0$
- Mutant (playing a) performs better than the incumbent (playing b)
  - ➢ **Strategy b is not an evolutionarily stable strategy**
  - ➢ This is true despite the symmetric profile (b,b) being a Nash equilibrium

Formal definition (Biology)

- Evolutionarily stable strategies have also been studied by **evolutionarily Biologist**

Definition (evolutionarily stable strategy (ESS))

Given a symmetric two-player normal-form game $G = (N, S, u)$, a strategy $s^* \in S$ is an evolutionarily stable strategy if there exists $\bar{\epsilon} > 0$ such that for any $s \neq s^*$ and for any $\epsilon < \bar{\epsilon}$, we have

$$\underbrace{u(s^*, \overbrace{\epsilon s + (1-\epsilon)s^*}^{\text{population}})}_{\text{Payoff to incumbent } s^* \text{ (다수)}} > \underbrace{u(s, \overbrace{\epsilon s + (1-\epsilon)s^*}^{\text{population}})}_{\text{Payoff to mutant } s \text{ (소수)}}$$

- Due to the property of expectation, the above equation can be written as

$$\epsilon u(s^*, s) + (1-\epsilon)u(s^*, s^*) > \epsilon u(s, s) + (1-\epsilon)u(s, s^*)$$

- Two interpretations:
  - We can state that the incumbents perform better than the mutants on random matchings
  - The strategy s^* cannot be invaded by s

Formal definition (Economics)

- Evolutionarily stable strategies have also been studied by **Economists**

Definition (evolutionarily stable strategy (ESS))

A mixed strategy $s^* \in S$ is evolutionarily stable if for any $s \in S$ the following two conditions hold:

(a) $u(s^*, s^*) \geq u(s, s^*)$

(b) if $u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

s^*: Incumbent strategy
s: Mutant strategy

- Condition (a) states essentially that the symmetric profile (s^*, s^*) is a Nash equilibrium
  - $u(s^*, s^*) > u(s, s^*)$: a strict Nash equilibrium
  - $u(s^*, s^*) \geq u(s, s^*)$: not-a-strict Nash equilibrium

- Condition (b) says that if the symmetric profile(s^*, s^*) is not a strict Nash equilibrium, then the mutant must perform poorly when playing against another mutant.

Equivalence of the Two definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (First definition → second definition)

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$

First definition

(a) $u(s^*, s^*) \geq u(s, s^*)$

(b) if $u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

second definition

- Since the first definition holds for any $\epsilon < \bar{\epsilon}$, as $\epsilon \to 0$,

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*) \text{ implies}$$

$$u(s^*, s^*) \geq u(s, s^*)$$

Thus establishing part 1 of the second definition

Equivalence of the Two definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (First definition → second definition)

- To establish part 2, suppose that $u(s^*, s^*) = u(s, s^*)$. Recall that u is linear in its arguments (since it is expected utility), so definition 1, $u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$, can be written as

$$\epsilon u(s^*, s) + (1-\epsilon)u(s^*, s^*) > \epsilon u(s, s) + (1-\epsilon)u(s, s^*)$$

- Since $u(s^*, s^*) = u(s, s^*)$, this is equivalent to

$$\epsilon u(s^*, s) > \epsilon u(s, s)$$

Since $\epsilon > 0$, part 2 of the second definition, $u(s^*, s) > u(s, s)$, follows

Equivalence of the Two definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (Second definition → First definition)

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$

First definition

↑

(a) $u(s^*, s^*) \geq u(s, s^*)$

(b) $if\ u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

second definition

Equivalence of the Two definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (Second definition → First definition)

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$

First definition

$\Uparrow$

(a) $u(s^*, s^*) \geq u(s, s^*)$

(b) $if\ u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

second definition

- We have two possibilities in $u(s^*, s^*) \geq u(s, s^*)$
  - $u(s^*, s^*) > u(s, s^*)$
  - $u(s^*, s^*) = u(s, s^*)$

Equivalence of the Two definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (Second definition → First definition)

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$

First definition

↑

(a) $u(s^*, s^*) \geq u(s, s^*)$

(b) $if\ u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

second definition

- If $u(s^*, s^*) > u(s, s^*)$, then the condition in the first definition

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$

is satisfied for $\epsilon = 0$, and hence for sufficiently small ϵ as well

Equivalence of the Two definitions

Theorem

The two definitions of evolutionarily stable strategies are equivalent.

Proof: (Second definition → First definition)

$$u(s^*, \epsilon s + (1-\epsilon)s^*) > u(s, \epsilon s + (1-\epsilon)s^*)$$ First definition

$$\Uparrow$$

(a) $u(s^*, s^*) \geq u(s, s^*)$

(b) $if\ u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

second definition

- If $u(s^*, s^*) = u(s, s^*)$, then the second definition implies

$$u(s^*, s) > u(s, s)$$

$$\Rightarrow \epsilon u(s^*, s) > \epsilon u(s, s)$$

Since $\epsilon > 0$

$$\Rightarrow \epsilon u(s^*, s) + (1-\epsilon)u(s^*, s^*) > \epsilon u(s, s) + (1-\epsilon)u(s, s^*)$$

Since $(1-\epsilon)u(s^*, s^*) = (1-\epsilon)u(s, s^*)$

then the condition in first definition holds.

evolutionarily Stability and Nash Equilibrium

Definition (Evolutionarily stable strategy (ESS))

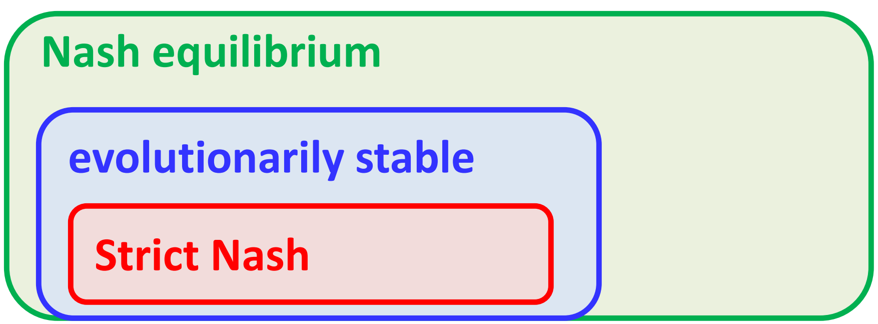
A mixed strategy $s^* \in S$ is evolutionarily stable if for any $s \in S$ the following two conditions hold:

(a) $u(s^*, s^*) \geq u(s, s^*)$
(b) if $u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

Theorem

- A strict (symmetric) Nash equilibrium of a symmetric game is an evolutionarily stable strategy.
- An evolutionarily stable strategy is a Nash equilibrium

- Their converses are not true, however.



A non-strict Nash equilibrium can be an evolutionarily stable strategy

	a	b
a	$1, 1$	$1, 1$
b	$1, 1$	$0, 0$

- We show that a non strict Nash equilibrium can be evolutionarily stable

s^* is *evolutionariliy stable* if $u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

- The symmetric profile (a, a) is a symmetric Nash equilibrium, but not a strict one
  - $u_1(a, a) = u_1(b, a) = 1$

- Strategy a is evolutionarily stable or not?

  ➢ Stable because

$u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

$u_1(a, a) = u_1(b, a)$, then $u_1(a, b) > u_1(b, b)$

1 1 1 0

Evolution of social convention

	L	R
L	$2, 2$	$0, 0$
R	$0, 0$	$1, 1$

- This example deals with a game that presents **multiple** evolutionarily stable strategies

- The profiles (L, L) and (R, R) are two strict Nash equilibrium solutions
  - Both L and R are involuntarily stable strategies
  - These strategies need not be equally good

Battle of the sexes

	L	R
L	$0, 0$	$2, 1$
R	$1, 2$	$0, 0$

- There exist non symmetric pure Nash equilibrium solutions
  - means that we have no monomorphic (단일형) population
- There may exist evolutionarily stable mixed strategies
  - Strategies correspond to genes
  - Mixed strategies correspond to a polymorphic (다형성) population
  - In the parlance of evolutionarily biology, this is a population with multiple genes
- The solution $(s^*, s^*) = \left[\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right]$ is a symmetric mixed-strategy Nash equilibrium, which yields a polymorphic population
- The mixed strategy s^* is ESS

Mixed NE is always week Nash

  - ➢ s^* is *evolutionariliy stable* if $u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$
  - ➢ $u(s^*, L) = \frac{1}{3} > u(L, L) = 0, \qquad u(s^*, R) = \frac{4}{3} > u(R, R) = 0,$

L and R are supports for any s

Monomorphic and Polymorphic Evolutionarily Stability

$vs.$

	$Hawk$	$Dove$
$Hawk$	$\left(\frac{V-C}{2}, \frac{V-C}{2}\right)$	$(V, 0)$
$Dove$	$(0, V)$	$\left(\frac{V}{2}, \frac{V}{2}\right)$

- We develop a game characterized by monomorphic evolutionarily stable strategies
- V is the prize of victory and C is the cost of fight
- If $V > C$, then the game can be assimilated to the Prisoner's dilemma
  - Exist a unique strict Nash equilibrium (Hawk, Hawk)
  - Consequently, Hawk is an evolutionarily stable strategy
- evolutionarily interpretation:
  - ➢ all individuals will end up selecting an aggressive behavior, namely strategy Hawk

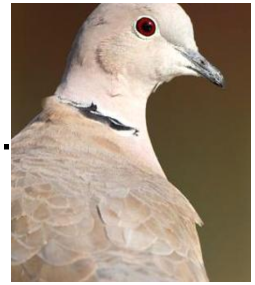
Polymorphic and Polymorphic Evolutionarily Stability

 $vs.$ 

	$Hawk$	$Dove$
$Hawk$	$\left(\frac{V-C}{2}, \frac{V-C}{2}\right)$	$(V, 0)$
$Dove$	$(0, V)$	$\left(\frac{V}{2}, \frac{V}{2}\right)$

- What will happen if $V < C$?
- Then the profiles $(Hawk, Dove)$ and $(Dove, Hawk)$ are two non symmetric NEs
- We also have a mixed Nash equilibrium, which is given by $(s^*, s^*) = \left[\left(\frac{V}{C}, 1 - \frac{V}{C}\right), \left(\frac{V}{C}, 1 - \frac{V}{C}\right)\right]$
- Is this mixed strategy $s^* = \left(\frac{V}{C}, 1 - \frac{V}{C}\right)$ Evolutionarily stable?

Polymorphic and Polymorphic Evolutionarily Stability

 $vs.$ 

	$Hawk$	$Dove$
$Hawk$	$\left(\frac{V-C}{2}, \frac{V-C}{2}\right)$	$(V, 0)$
$Dove$	$(0, V)$	$\left(\frac{V}{2}, \frac{V}{2}\right)$

- What will happen if $V < C$?
- Then the profiles $(Hawk, Dove)$ and $(Dove, Hawk)$ are two non symmetric NEs
- We also have a mixed Nash equilibrium, which is given by $(s^*, s^*) = \left[\left(\frac{V}{C}, 1-\frac{V}{C}\right), \left(\frac{V}{C}, 1-\frac{V}{C}\right)\right]$
- Is this mixed strategy $s^* = \left(\frac{V}{C}, 1-\frac{V}{C}\right)$ Evolutionarily stable?

- The mixed strategy s^* is ESS
  - s^* is *evolutionariliy stable* if $u(s^*, s^*) = u(s, s^*)$, then $u(s^*, s) > u(s, s)$

$$u(s^*, s^*) = u(D, s^*), u(s^*, s^*) = u(D, s^*) \rightarrow \begin{array}{l} u(s^*, D) > u(D, D) \\ u(s^*, H) > u(H, H) \end{array}$$

- As V increases, more players playing Hawk are in the evolutionarily stable strategy

Nash equilibrium does not imply ESS

- Consider the modified rock-paper-scissors game:

	Rook	Paper	Scissors
Rook	γ, γ	-1, 1	1, -1
Paper	1, -1	γ, γ	-1, 1
Scissors	-1, 1	1, -1	γ, γ

- Here $0 \leq \gamma < 1$ (If $\gamma = 0$, this is the standard rock-paper-scissors game)
- For all such γ, there is a unique mixed strategy equilibrium $s^* = (1/3, 1/3, 1/3)$, with expected payoff $u(s^*, s^*) = \frac{\gamma}{3}$.
- But for $\gamma > 0$, this is not ESS. For example, $s = R$ would invade, since

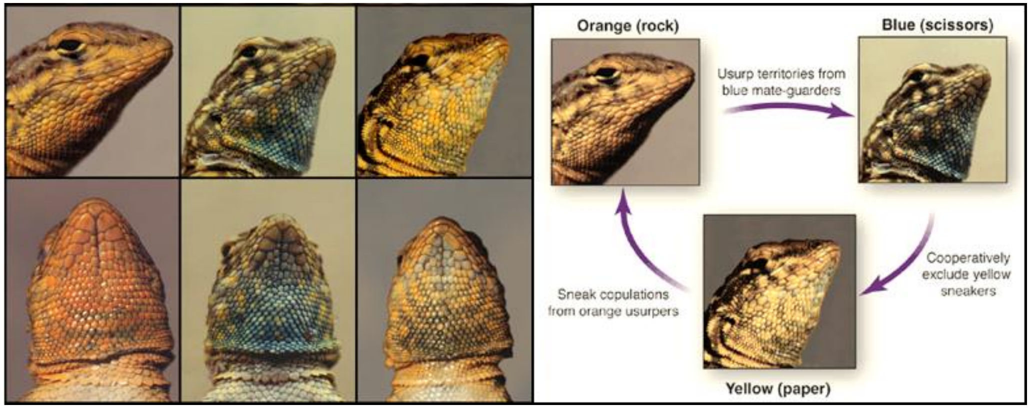
$$u(s^*, s) = \gamma \times \frac{1}{3} - 1 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{\gamma}{3} < u(s, s) = \gamma$$

Under ESS: if $\underline{u(s^*, s^*) = u(s, s^*)}$, then $u(s^*, s) > u(s, s)$

Hold when one player is plying a mixed Nash: $u(s^*, s^*) = u(R, s^*) = u(P, s^*) = \mathrm{u}(S, s^*)$

- This also shows that ESS doesn't necessarily exist

Do Animas play games?



- Side-blotched lizards seem to play a version of the Hawk-Dove game. Three productive strategies for male lizards with distinct throat colors (that are genetically determined):
  - orange color: very aggressive and defend large territories;
  - blue color: less aggressive defense smaller territories;
  - yellow color: not aggressive, opportunistic mating behavior.
- Tails seem to be as follows:
  - when all are orange, yellow does well; when all are yellow, blue does well; and when all are blue, orange does well.
- This is similar to the modified rock-paper-scissors pattern, and in nature, it seems that there are fluctuations in composition of male colorings as we should expect on the basis that the game does not have any evolutionarily stable strategies.