Lecture 2: Games in normal form

Self interested agents

What does it mean to say what an agent is self-interested?

- It does not necessarily mean that they want to cause harm to each other
- It does not necessarily mean that they care only about themselves
- It means that each agent has his own description of which sates of the world he likes, and acts based on this description

The dominant approach to modeling an agent's interest is utility theory

- Aims to quantify an agent's degree of preference across a set of available alternatives
- Aims to understand how these preferences change when an agent faces uncertainty about which alternatives he will receive
- We assume that agents actions are consistent with utility –theoretic assumptions.

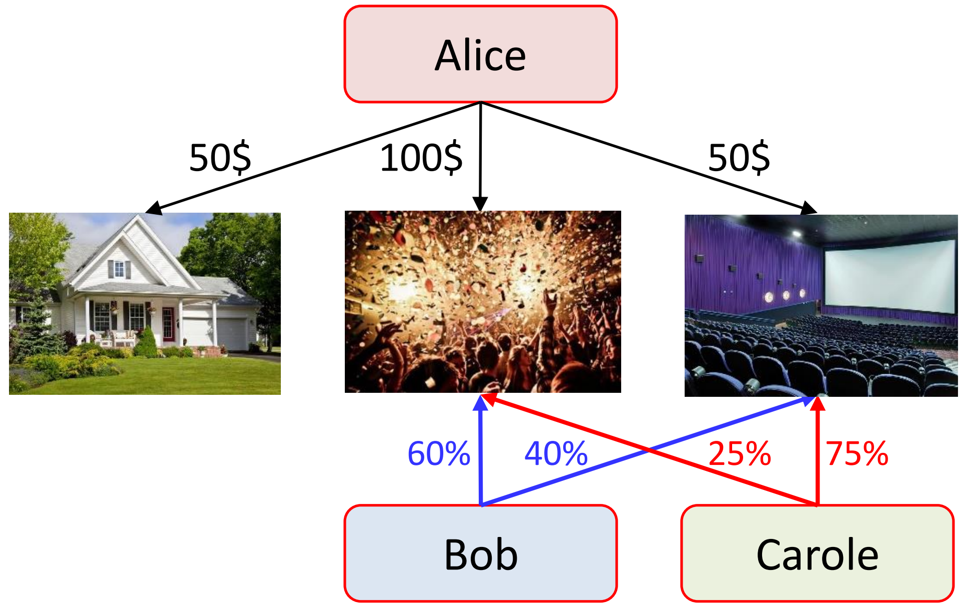
A utility function is a mapping from states of the world to real numbers

- These numbers are interpreted as agent's level of happiness in the given states
- Confronting uncertainties, utility is defined as the expected value of his utility function with respect to the appropriate probability distribution over states

Utility function can be used as a basis for making decision

- Agents simply need to choose the course of action that maximizes expected utility

- When the world contains two or more utility-maximizing agents whose sections can affect each others' utility, things become complicated



Alice Hates Bob

Alice likes Carole

- When Alice see Bob in the club, utility will be decreased to 10
- When Alice see Bob in the movie, utility decreases to 10

- When Alice see Carole in any place, her utility will be increase by a factor 1.5

Utility function can be used as a basis for making decision

	$B = c$	$B = m$
$C = c$	50	50
$C = m$	50	50

$A = h$

	$B = c$	$B = m$
$C = c$	15	150
$C = m$	10	100

$A = c$

	$B = c$	$B = m$
$C = c$	50	10
$C = m$	75	15

$A = m$

- It will be easier to determine Alice's best course of action if we list Alice's utility for each possible state of the world.
- There are 12 outcomes that can occur: Bob and Carol can each be in either the club or the movie theater, and Alice can be in the club, the movie theater, or at home.
- Alice has a baseline level of utility for each of her three actions, and this baseline is adjusted if either Bob, Carol, or both are present.
- Following the description of our example, we see that Alice's utility is always 50 when at home

Utility function can be used as a basis for making decision

$A = h$

	0.6 $B = c$	0.4 $B = m$
0.25 $C = c$	50	50
0.75 $C = m$	50	50

$A = c$

	0.6 $B = c$	0.4 $B = m$
0.25 $C = c$	15	150
0.75 $C = m$	10	100

$A = m$

	0.6 $B = c$	0.4 $B = m$
0.25 $C = c$	50	10
0.75 $C = m$	75	15

- So how should Alice choose among her three activities?
- To answer this question we need to combine her utility function with her knowledge of Bob and Carol's randomized entertainment habits.
  - $\bar{u}(A = h) = 50$
  - $\bar{u}(A = c) = 0.25(0.6 \cdot 15 + 0.4 \cdot 150) + 0.75(0.6 \cdot 10 + 0.4 \cdot 100) = 51.75.$
  - $\bar{u}(A = m) = 0.25(0.6 \cdot 50 + 0.4 \cdot 10) + 0.75(0.6(75) + 0.4(15)) = 46.75.$
- Thus, Alice prefers to go to the club (even though Bob is often there and Carol rarely is) and prefers staying home to going to the movies

Preferences and utility

- why should a single-dimensional function be enough to explain preferences over an arbitrarily complicated set of alternatives?
- why should an agent's response to uncertainty be captured purely by the expected value of his utility function, rather than also depending on other properties of the distribution such as its standard deviation or number of modes?
- Utility theorists respond to such questions by showing that the idea of utility can be grounded in a more basic concept of preferences.
- We need a way to talk about how preferences interact with uncertainty about which outcome will be selected.
- In utility theory this is achieved through the concept of lottery.
  - A lottery is the random selection of one of a set of outcomes according to specified probabilities
  - Formally, a lottery is a probability distribution over outcomes written $[p_1 : o_1, \ldots, p_k : o_k]$, where each $o_i \in O$, each $p_i > 0$ and $\sum_{i=1}^{k} p_i = 1$

Axioms of utility theory

- **Axiom 1 (Completeness)** $\forall\, o_1, o_2,\ \ o_1 \succ o_2\ \ or\ \ o_2 \succ o_1\ \ or\ \ o_1 \sim o_2$

- **Axiom 2 (Transitivity)** $if\ \ o_1 \succcurlyeq o_2\ and\ \ o_2 \succcurlyeq o_3,\ \ then\ \ o_1 \succcurlyeq o_3$

- **Axiom 3 (Substitutability)**
  $if\ \ o_1 \sim o_2\ then\ for\ all\ sequences\ of\ one\ or\ more\ outcomes\ o_3, \dots, o_k\ and\ sets\ of$
  $probabilities\ p, p_3, \dots, p_k\ for\ which\ p + \sum_{i=3}^{k} p_i = 1,$
  $$[p{:}\, o_1, p_3{:}\, o_3, \dots, p_k{:}\, o_k] \sim [p{:}\, o_2, p_3{:}\, o_3, \dots, p_k{:}\, o_k]$$

- **Axiom 4 (Decomposability)** $if\ \forall o_i \in O,\ \ P_{l_1}(o_i) = P_{l_1}(o_i)\ then\ l_1 \sim l_2.$

- **Axiom 5 (Monotonicity)** $if\ o_1 \succ o_2\ \ and\ \ p \succ q\ then\ [p{:}\, o_1, 1 - p{:}\, o_2] \succ [q{:}\, o_1, 1 - q{:}\, o_2]$

- **Lemma 1** $if\ a\ preference\ relation\ \ \succcurlyeq$
  $satisfies\ the\ axioms\ completeness, transitivity, decomposabiity, and\ monotoniciy,$
  $and\ if\ o_1 \succ o_2\ \ and\ o_2 \succ o_3,\ \ then\ there\ eixists\ some\ probability\ p\ such\ that\ for\ all$
  $p' < p,\ \ o_2 \succ [p'{:}\, o_1, 1 - p'{:}\, o_3], and\ for\ all\ p'' > p, [p''{:}\, o_1, 1 - p''{:}\, o_3] \succ o_2.$

- **Axiom 6 (Continuity)** $if\ o_1 \succ o_2\ \ and\ o_2 \succ o_3,\ \ then\ \exists p \in [0,1]\ such\ that$
  $o_2 \sim [p{:}\, o_1, 1 - p{:}\, o_3].$

Axioms of utility theory

Theorem (Von Neumann and Morgenstern, 1944)

$if\ a\ preference\ relation \succcurlyeq satisfies\ the\ axioms\ completeness, transitivity,$
$substiutability, decomposabiity, monotonicity, and\ continuity, then\ there\ exists$
$a\ function\ u: O \longmapsto [0, 1]\ with\ the\ properties\ that$

1. $u(o_1) \geq u(o_2)\ iff\ o_1 \succcurlyeq o_2, and$
2. $u([p_1: o_1, \ldots, p_k: o_k]) = \sum_{i=1}^{k} p_i u(o_i)$

Key ingredients of a game

Players: who are the decision makers?

- People? Robots? Governments? Companies? Employees?

Actions: what can the players do?

- Enter a bid in an auction? Decide whether to start up a company? Decide when to buy car? Decide to sell a stock? Decide how to vote?

Payoffs: what motivates players?

- Do they care about some profit? Do they care about other players?

Defining Games – Two standard representations

- Normal Form (a.k.a. Matrix Form, Strategic Form) List what payoffs get as a function of their actions
  - It is as if players moved simultaneously
  - But strategies encode many things...
- Extensive Form Includes timing of moves (later in course)
  - Players move sequentially, represented as a tree
  - Chess: white player moves, then black player can see white's move and react...
  - Keeps track of what each player knows when he or she makes each decision
  - Poker: bet sequentially – what can a given player see when they bet?

- Above classification is not based on the type of game, but on the way how we represent it!
- Because most other representations of interest can be reduced to it, the normal – form representation is arguably the most fundamental in game theory

Normal form game

Definition (Normal-form game)

Finite, n-person normal form game: $\mathrm{G}(N, A, u)$:

- $N = \{1, \dots, n\}$ is a finite set of n, indexed by i
- $A = A_1 \times \ldots \times A_n$, where A_i is a finite set of actions available to plyer i
  - Each vector $a = (a_1, \dots, a_n) \in A$ is an action profile
- $u = (u_1, \dots, u_n)$, where $u_i : A \mapsto \mathbb{R}$ is real-valued utility (or payoff) function for player i

Normal form game – the standard matrix representation

- Writing a 2-player game as a matrix:
  - "row" player is player 1,
  - "column" player is player 2
  - rows correspond to actions $a_1 \in A_1$
  - columns correspond to actions $a_2 \in A_2$
  - cells listing utility or payoff values for each player:
  (the row player first, then the column)

- Prisoner's Dilemma game can be represented as

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

C: Cooperate D: Defect

- $N = \{1, 2\}$
- $A = A_1 \times A_2$, where $A_1, A_2 = \{C, D\}$
- $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$ are given as numbers

Prisoner's Dilemma

	C	D
C	a, a	b, c
D	c, b	d, d

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma
- Why Dilemma?

Prisoner's Dilemma

	C	D
C	a, a	b, c
D	c, b	d, d

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma

- Why Dilemma?

Prisoner's Dilemma

		C	D
Player 1	C	$a,$	$b,$
	D	$c,$	$d,$

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma
- Why Dilemma?
  - For player 1, playing D is always better!

Prisoner's Dilemma

Player 2

	C		$\boldsymbol{D}$
C	$, a$	$<$	$, c$
D	$, b$	$<$	$, d$

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma

- Why Dilemma?
  - For player 1, playing D is always better!
  - For player 2, playing D is also always better!

Prisoner's Dilemma

	C	D
C	$\boldsymbol{a}, \boldsymbol{a}$	b, c
D	c, b	$\boldsymbol{d}, \boldsymbol{d}$

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma

- Why Dilemma?
  - For player 1, playing D is always better!
  - For player 2, playing D is also always better!
  - However, the outcomes (d, d) of playing (D, D) is dominated by the outcomes (a, a) of playing (C, C)

Common-payoffs games

Definition (Common-payoff game)

A common-payoff game is a game in which for all action profiles $a \in A_1 \times \cdots \times A_n$ and any pair of agents i, j, it is the case that $u_i(a) = u_j(a)$

- Represents **pure coordination**
- Sometimes called pure coordination games or team games
- The agents have no conflicting interests;
  - their sol challenge is to coordinate on an action that is maximally beneficial to all

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

<Example: Traffic game>

Zero-sum games

Definition (Constant-sum game)

A two-player normal-form game is constant-sum if there exists a constant c such that for each strategy profile $a \in A_1 \times A_2$ it is the case that $u_1(a) + u_2(a) = c$.

- Represents **pure competition**
- In general, $c = 0$ and the game is called zero-sum game
  - Positive affine transformations can always make a general sum G into zero-sum G.

	Head	Tails
Head	1, -1	-1, 1
Tails	-1, 1	1, -1

<Matching Pennies game>

	Rook	Paper	Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

<Rock, Paper, Scissors game>

Battle of the sexes

- In general games includes elements of both **coordination** and **competition**

Player 2 (Husband)

Player 1 (wife)		TF	LA
	TF	$\boxed{2, 1}$	0, 0
	LA	0, 0	$\boxed{1, 2}$

- A husband and wife wish to go to the movies, and they can select among two movies
  - They prefer to go together rather than to separate movies
  - Wife prefer "Transformer" (TF), the husband prefers "LALALAND" (LA)

Strategies in normal-form games

- **Pure strategy**:
  - Select a single action and play
  - Call a set of pure strategy for each agent a pure-strategy profile
- **Mixed strategy:**
  - Randomizing over the set of available actions according to some probability distribution
  - In a multiagent setting, the role of mixed strategy is critical
- **Why do we need a mixed strategy?**
  - It would be a pretty bad idea to play any deterministic strategy in matching pennies game or Rock-Paper-Scissor game

	Head	Tails
Head	1, -1	-1, 1
Tails	-1, 1	1, -1

Mixed strategy

Definition (Mixed strategy)

Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then, the set of mixed strategies for player i is $S_i = \Pi(A_i)$

Definition (Mixed strategy profile)

The set of mixed-strategy profile is simply the Cartesian product of the individual mixed-strategy sets, $S = S_1 \times \ldots \times S_n$.

- $s_i(a_j)$ denote the probability that an action a_j will be played under mixed strategy s_i
  - For example, $A = \{\text{Rook, Paper,Scissors}\}$, $s_i(\mathrm{R}) = 0.2$, $s_i(\mathrm{P}) = 0.3$, $s_i(\mathrm{S}) = 0.5$

Definition (Support)

The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

- A pure strategy is a special case of randomized strategy, in which the support is a single action
- A strategy is fully mixed if it has full support (i.e., if it assigns every action a nonzero probability)

Mixed strategy

	Rook	Paper	Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

$$s_1 = \{\overset{\text{Rook}}{0.2}, \overset{\text{Paper}}{0.3}, \overset{\text{Scissors}}{0.5}\}$$

$$s_2 = \{\overset{\text{Rook}}{0.3}, \overset{\text{Paper}}{0.7}, \overset{\text{Scissors}}{0}\}$$

- Support for s_1 is {Rook, Paper, Scissors} and s_1 is fully mixed strategy
- Support for s_2 is {Rook, Paper}

Mixed strategy

Definition (Expected utility of a mixed strategy)

Given a normal-form game (N, A, u), the expected utility u_i for player i of the mixed strategy profile $s = (s_1, \dots, s_n)$ is defined as

$$u_i(s) = \sum_{a \in A} u_i(a) \Pr(a|s)$$

where $\Pr(a|s)$ is probability that action $a = (a_1, \dots, a_n)$ is selected given strategy s. That is,

$$\Pr(a|s) = \prod_{j=1}^{n} s_j(a_j) = s_1(a_1) \times \ldots \times s_n(a_n)$$

	C	D
C	-1, 1	-4, 0
D	0, -4	-3, -3

$$s = (s_1, s_2)$$

$$s_1 = \{\overset{C}{0.3}, \overset{D}{0.7}\} \quad s_2 = \{\overset{C}{0.6}, \overset{D}{0.4}\}$$

$$\begin{aligned} u_1(s) = {} & u_1(C,C)s_1(C)s_2(C) \\ & + u_1(C,D)s_1(C)s_2(D) \\ & + u_1(D,C)s_1(D)s_2(C) \\ & + u_1(D,D)s_1(D)s_2(D) \end{aligned} \qquad \begin{aligned} u_1(s) = {} & -1 \times 0.3 \times 0.6 \\ & -4 \times 0.3 \times 0.4 \\ & +0 \times 0.7 \times 0.6 \\ & -3 \times 0.7 \times 0.4 \end{aligned}$$

- It can be represented compactly as $u_1(s) = s_1^T U_1 s_2$

with $U_1 = \begin{bmatrix} u_1(C,C) & u_1(C,D) \\ u_1(D,C) & u_1(D,D) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 0 & -3 \end{bmatrix}$