Lecture 11-Auctions and competitive biding

Motivations

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- Many economic transactions are conducted through auctions
  - Treasury bills
  - Foreign exchange
  - Publicly owned companies
  - Mineral rights
  - Airwave spectrum rights
  - artwork
  - antiques
  - cars
  - houses
  - Government contracts
- Also can be thought of as auctions
  - Takeover battles
  - Queues
  - Wars of attrition
  - Lobbying contests

Types of Auctions

- **Open bid auctions**
  - **Ascending-bid Auction**
    - ✓ Known as English auction
    - ✓ Price is raised until only one bidder remains, who wins and pays the final price
  - **Descending-bid Auction**
    - ✓ Known as Dutch auction
    - ✓ Price is lowered until someone accepts, who wins and pays the current price
- **Sealed bid auctions**
  - **First price auction**
    - ✓ Highest bidder wins and pays his bid
  - **Second price auction**
    - ✓ Highest bidder wins and pays the second highest bid

Open Bid		Sealed Bid
English Auction	~	Second Price
Dutch Auction	~	First Price

Types of Auctions

- **Auctions also differ with respect to the valuation of the bidders**

  - Private value auctions **(IPV)**
    - ✓ Each bidder knows only his own value
    - ✓ Valuations are independent across bidders
    - ✓ Bidders have beliefs over other bidder's values
    - ✓ Risk neutral bidders
      - ➢ If the winner's value is θ_i and pays p, her payoff is $\theta_i - p$
    - ✓ Artwork, antiques, memorabilia
    - ✓ $u_i(a_1, a_2, \ldots, a_n; \theta_i)$

  - Common value auctions
    - ✓ Actual value of the object is the same for everyone
    - ✓ Bidders have different private information about that value
    - ✓ Oil filed auctions, company takeovers
    - ✓ $u_i(a_1, a_2, \ldots, a_n; \theta_1, \theta_2, \ldots, \theta_n)$

Second Price Sealed Auctions

- Set of players $N = \{1,2,\dots,n\}$
- Type set $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$, $\underline{\theta}_i \geq 0$
- Actions set $A_i = \mathbb{R}_+$
- Beliefs
  - ✓ Opponent's valuations are independent draws from a distribution function F
  - ✓ F is strictly increasing and continuous
- Strategy for player i is a function $s_i \colon \Theta_i \to A_i$
- Payoff function

$$u_i(a_i, a_{-i}; \theta_i) = \begin{cases} \theta_i - a_j^*, & \text{if } a_j \leq a_i \text{ for all } j \neq i \text{ and } a_j^* \equiv \max_{j \neq i}\{a_j\} \\ 0, & \text{if } a_j > a_i \text{ for some } j \neq i \end{cases}$$

a_j^* is the price paid by the winner if the bid profile is a

- Payoff of player i with valuation θ_i as a function of his own bid a_i and the strategies used by the other players $s_j(\cdot), j \neq i$:

$$E_{\theta_{-i}}[u_i(a_i, s_{-i}(\theta_{-i}); \theta_i) | \theta_i] = \Pr\{i \text{ wins and pays } p\} \times (\theta_i - p) + \Pr\{i \text{ loses}\} \times 0$$

Second Price Sealed Auctions

Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies

Region L	Region M	Region H

- There are three possible cases of interest with respect to the other $n - 1$ bids:

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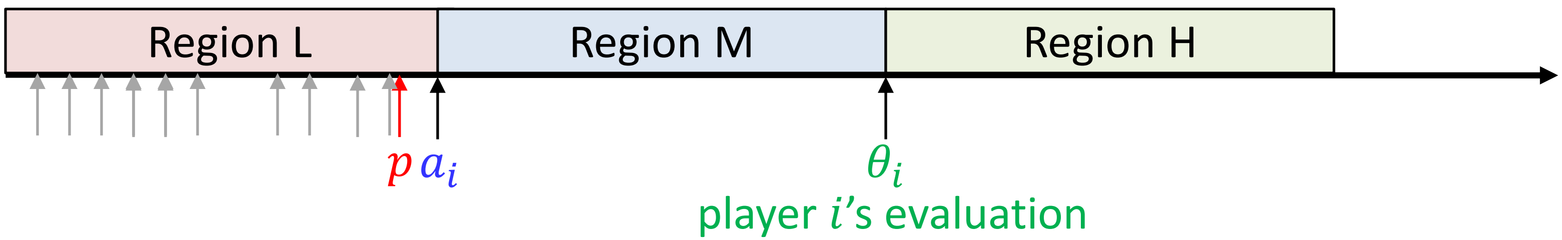

- There are three possible cases of interest with respect to the other $n-1$ bids:

Case 1:

- Player i is the highest bidder a_i, in which case i wins and pays price $p < a_i$.
- This corresponds to the situation in which all the other $n-1$ bids are in region L, including the second highest bid p.
- If instead of bidding a_i, player i would have bid θ_i then he would still win and paly the same price, so in case 1, bidding his valuation is as good as bidding a_i

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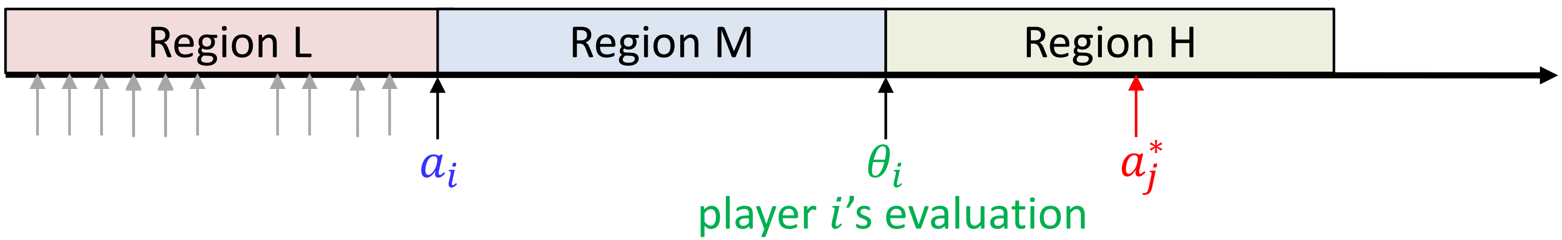

- There are three possible cases of interest with respect to the other $n-1$ bids:

Case 2:

- The highest bidder j bids $a_j^* > \theta_i$, in which case i loses
- This corresponds to the situation in which the winning bid is in region H
- If instead of bidding a_i, player i would have bid θ_i then he would still loses to a_j^*, so in case 2 bidding his valuation is as good as bidding a_i

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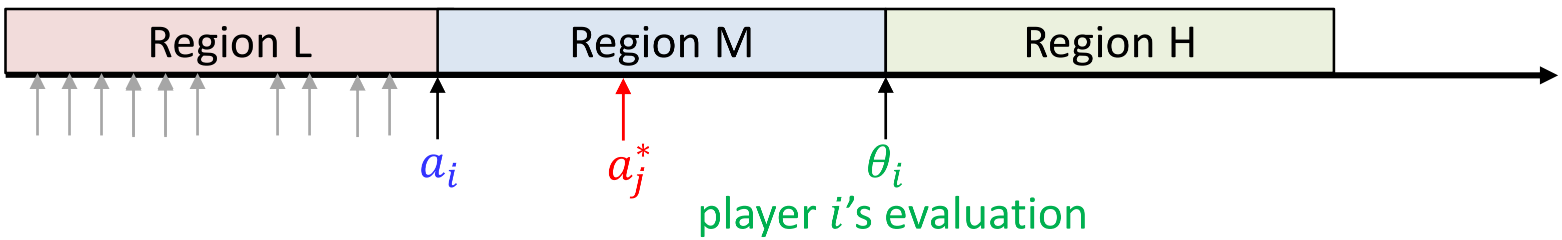
Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies



- There are three possible cases of interest with respect to the other $n-1$ bids:

Case 3:

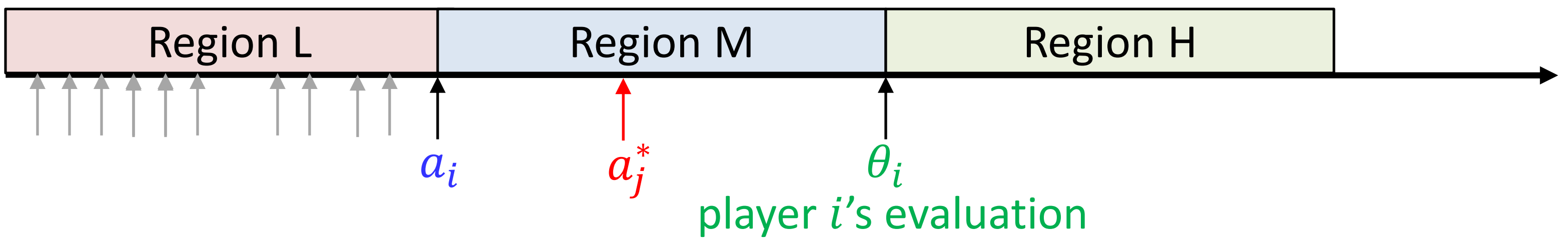
- The highest bidder j bids $a_i < a_j^* < \theta_i$, so that the highest bid is in region M and i does not win
- If player i would have bid θ_i, he would have won the auction and received a payoff of

$$u_i = \theta_i - a_j^*$$

making this a profitable deviation, so in case 3, bidding his valuation θ_i is strictly better than bidding a_i

Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies



- There are three possible cases of interest with respect to the other $n-1$ bids:
- Since **cases 1-3** cover all the relevant situations, we conclude that bidding θ_i weakly dominates any lower bid because it is never worse and sometimes better
- A similar argument shows that a bid $a_i > \theta_i$ will also be weakly dominated by bidding θ_i

Second Price Sealed Auctions

Notes:

- The fact that every player has a weakly dominant strategy, $s_i(\theta_i) = \theta_i$, implies that each player bidding his valuation is a Bayesian Nash equilibrium in weakly dominant strategies

- This result is **noteworthy** not only because its simple prescription, but also

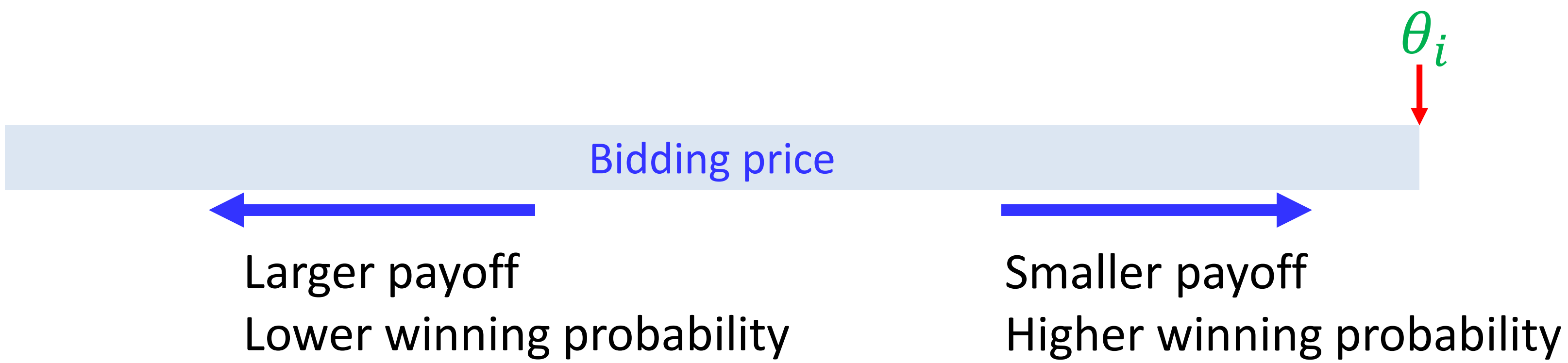
  ✓ In the private value setting, bidders in a second-price sealed-bid auction do not caser about the probability distribution over their opponents' type
    - ✓ The assumption of common knowledge of the distribution of types can be relaxed
    - ✓ We can apply this result even when we have no idea about their opponents' value

  ✓ Even if types are correlated but values are private, then it is a weakly dominant strategy to bid truthfully

  ✓ In the private value setting, the outcome of a second-price sealed-bid auction is Pareto optimal because the person who values the good most will be the one who gets it

First-Price Sealed-Bid Auctions

- Highest bidder wins and pays his bid
  - Would you bid your value?
  - What happens if you bid less than your values?



  - Bidding less than your value is known as bid shading

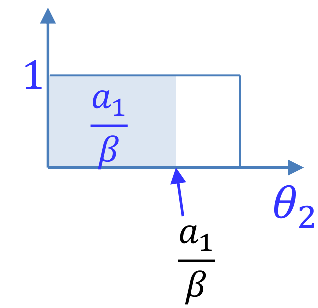
Bayesian Equilibrium for First Price Auctions

Simple scenario

(Simple assumptions)

- Only 2 bidders
- You are player 1 and your value is $\theta_1 > 0$
- You believe the other bidder's value is uniformly distributed over $[0,1]$
- You believe the other bidder uses strategy $s_1(\theta_2) = \beta\theta_2$: proportional to his vale θ_2
- Your expected payoff if you bid a_i

$$u_1 = (\theta_1 - a_1)\,\mathrm{Pr}(\text{you win}) = (\theta_1 - a_1)\,\mathrm{Pr}(a_1 > \beta\theta_2)$$
$$= (\theta_1 - a_1)\,\mathrm{Pr}\left(\theta_2 < \frac{a_1}{\beta}\right)$$
$$= (\theta_1 - a_1)\frac{a_1}{\beta}$$



- Maximizing implies first derivative with respect to a_1 equal to zero

$$-\frac{a_1}{\beta} + \frac{(\theta_1 - a_1)}{\beta} = 0$$

- Solving for a_1

$$a_1 = \frac{\theta_1}{2}$$

- Bidding half the value is a Bayesian equilibrium

Bayesian Equilibrium for First Price Auctions

Little bit completed

- n bidders
- You are player 1 and your value is $\theta_1 > 0$
- You believe the other bidder's value θ_i is uniformly distributed over $\theta_i \sim U[0,1]$
- You believe the other bidder uses strategy $s_i(\theta_i) = \beta\theta_i$
- Your expected payoff if you bid a_i

$$
\begin{aligned}
u_1 = (\theta_1 - a_1)\,\mathrm{Pr}(\text{you win}) &= (\theta_1 - a_1)\,\mathrm{Pr}(a_1 > \beta\theta_2, a_1 > \beta\theta_3, \dots, a_1 > \beta\theta_n) \\
&= (\theta_1 - a_1)\,\mathrm{Pr}\left(\theta_2 < \frac{a_1}{\beta}\right) \times \mathrm{Pr}\left(\theta_3 < \frac{a_1}{\beta}\right) \times \dots \times \mathrm{Pr}\left(\theta_n < \frac{a_1}{\beta}\right) \\
&= (\theta_1 - a_1)\left(\frac{a_1}{\beta}\right)^{n-1}
\end{aligned}
$$

- Maximizing implies first derivative with respect to a_1 equal to zero

$$
-\left(\frac{a_1}{\beta}\right)^{n-1} + (n-1)\frac{(\theta_1 - a_1)}{\beta}\left(\frac{a_1}{\beta}\right)^{n-2} = 0
$$

- Solving for a_1

$$
a_1 = \frac{n-1}{n}\theta_1
$$

Bayesian Equilibrium for First Price Auctions

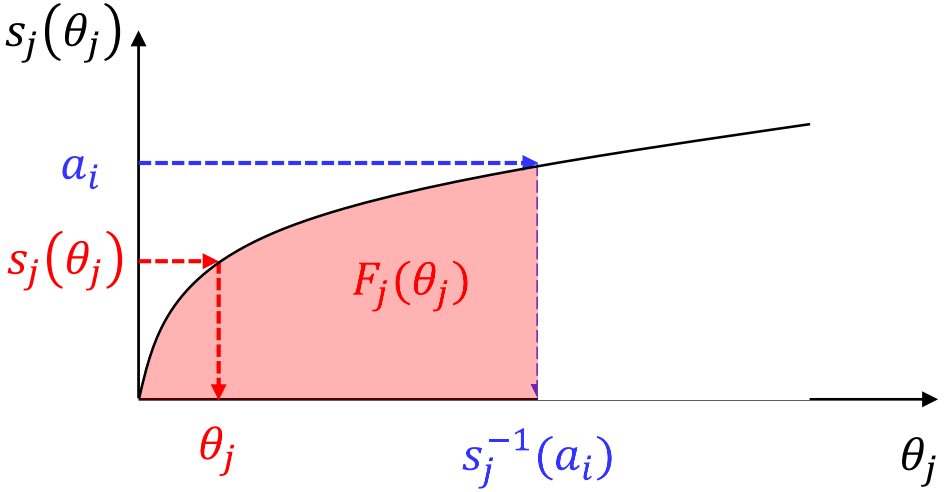
General setting

- **Assumption:** The higher a player's valuation, the higher is his bid, That is,

$$\text{if } \theta_j' > \theta_j'' \text{ then } s_j(\theta_j') > s_j(\theta_j'') \qquad \text{(more general assumption)}$$

- The probability that i's bid is higher than $j's$ bid can be simply computed as

$$\Pr\{s_j(\theta_j) < a_i\} = \Pr\{\theta_j < s_j^{-1}(a_i)\} = F_j\left(s_j^{-1}(a_i)\right)$$



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$$\Pr\{s_j(\theta_j) < a_i\} = \Pr\{\theta_j < s_j^{-1}(a_i)\} = F_j\left(s_j^{-1}(a_i)\right)$$

- Then, we can compute the expected payoff of player i is

$$E_{\theta_{-i}}[u_i(a_i, s_{-i}(\theta_{-i}); \theta_i)|\theta_i] = (\theta_1 - a_i)\Pr(\text{you win})$$
$$= (\theta_i - a_i) \times \prod_{j \neq i}\left[F_j\left(s_j^{-1}(a_i)\right)\right]$$

- We can compute the best action a_i as one that maximizes the expected payoff
- For n bidders, the Bayesian Nash equilibrium bid (strategy) function is

$$s(\theta) = \theta - \frac{\int_{\underline{\theta}}^{\overline{\theta}} [F(x)]^{n-1} dx}{[F(x)]^{n-1}}$$