# Lecture 12 Sequential rationality with incomplete information

## (Dynamic Bayesian game)

## Motivation

- Many situations of incomplete information cannot be represented as static or strategic form games.
- Instead, we need to consider extensive form games with an explicit order of moves—or dynamic games.
- In this case, as mentioned earlier in the lectures, we use information sets to represent what each player knows at each stage of the game.
- Since these are dynamic games, we will also need to strengthen our Bayesian Nash equilibria to include the notion of perfection—as in subgame perfection.
- The relevant notion of equilibrium will be Perfect Bayesian Equilibria, or Perfect Bayesian Nash Equilibria.

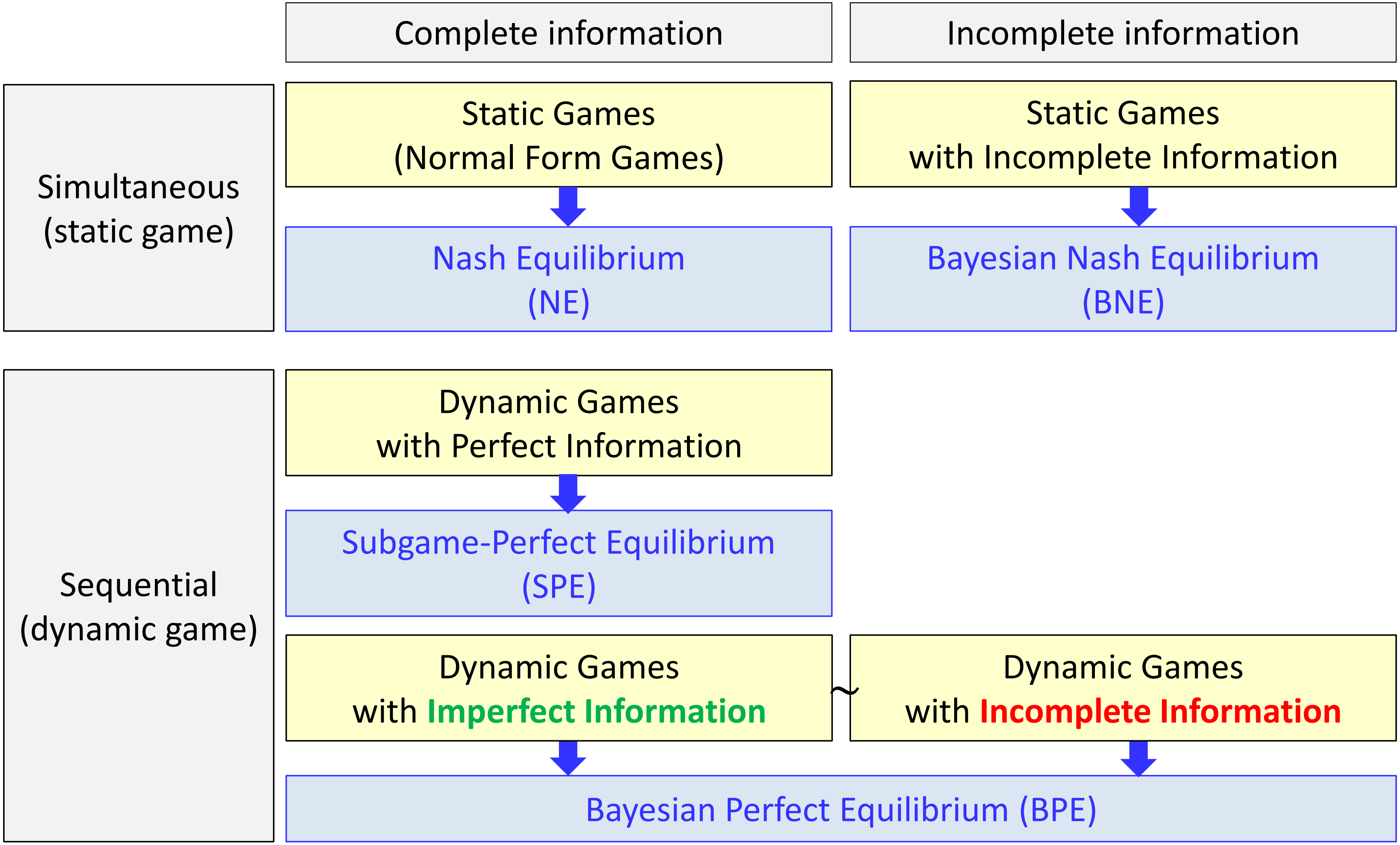
# Introduction

| | Complete information | Incomplete information |
|---|---|---|
| Simultaneous<br>(static game) | Static Games<br>(Normal Form Games)<br>↓<br>Nash Equilibrium<br>(NE) | Static Games<br>with Incomplete Information<br>↓<br>Bayesian Nash Equilibrium<br>(BNE) |
| Sequential<br>(dynamic game) | Dynamic Games<br>with Perfect Information<br>↓<br>Subgame-Perfect Equilibrium<br>(SPE) | Dynamic Games<br>with Incomplete Information<br>↓<br>Bayesian Perfect Equilibrium<br>(BPE) |

- We have defined a subgame perfect equilibrium to include the notion of perfection (sequential rationality) in dynamic games with complete information

- We need to strengthen our Bayesian Nash equilibria to include the notion of perfection—as in subgame perfection

# Introduction



Complete information

Incomplete information

Simultaneous
(static game)

Static Games
(Normal Form Games)

Nash Equilibrium
(NE)

Static Games
with Incomplete Information

Bayesian Nash Equilibrium
(BNE)

Sequential
(dynamic game)

Dynamic Games
with Perfect Information

Subgame-Perfect Equilibrium
(SPE)

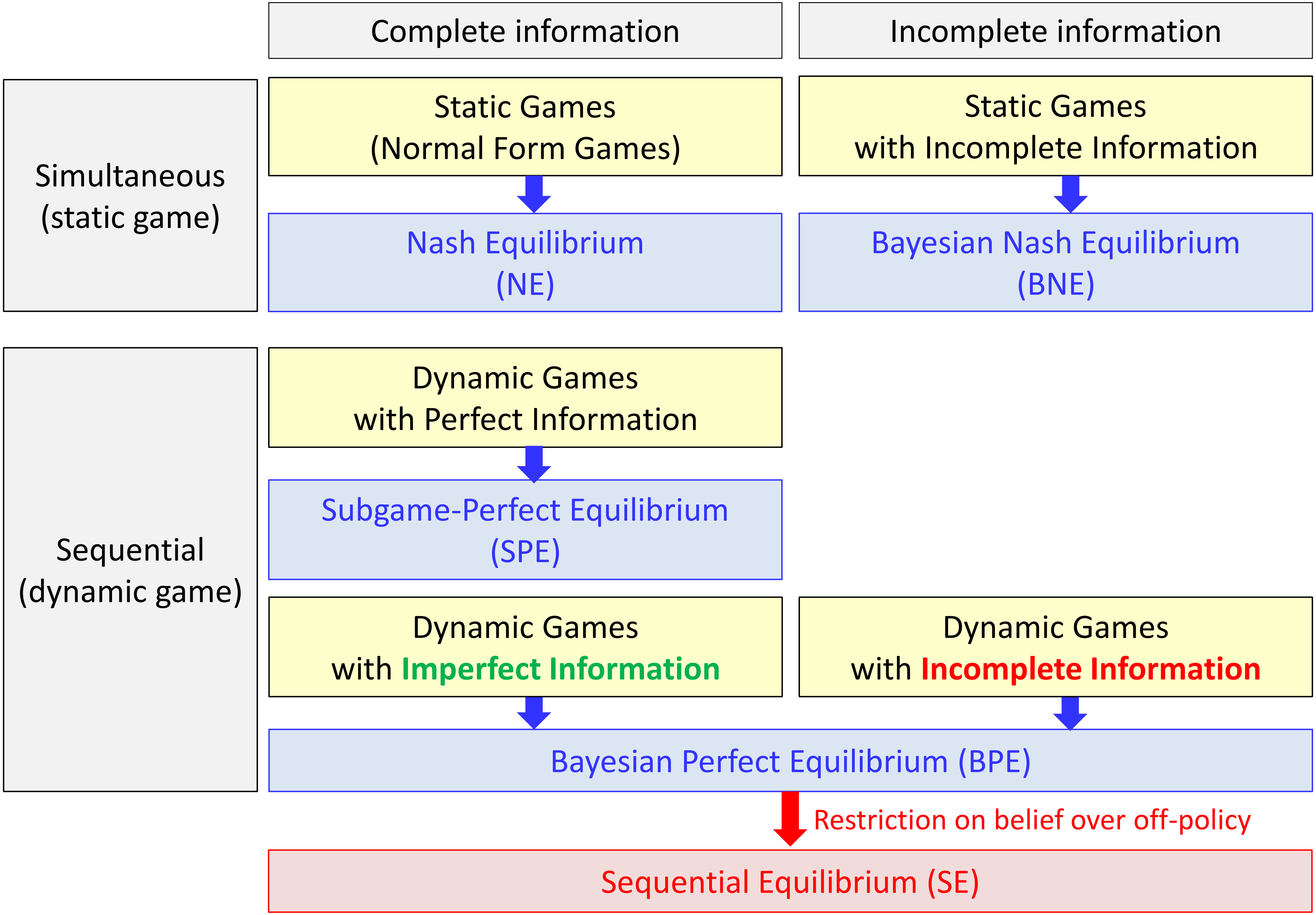
Dynamic Games
with **Imperfect Information**

~

Dynamic Games
with **Incomplete Information**

Bayesian Perfect Equilibrium (BPE)

# Introduction



Complete information

Incomplete information

Simultaneous (static game)

Static Games (Normal Form Games)

Nash Equilibrium (NE)

Static Games with Incomplete Information

Bayesian Nash Equilibrium (BNE)

Sequential (dynamic game)

Dynamic Games with Perfect Information

Subgame-Perfect Equilibrium (SPE)

Dynamic Games with **Imperfect Information**

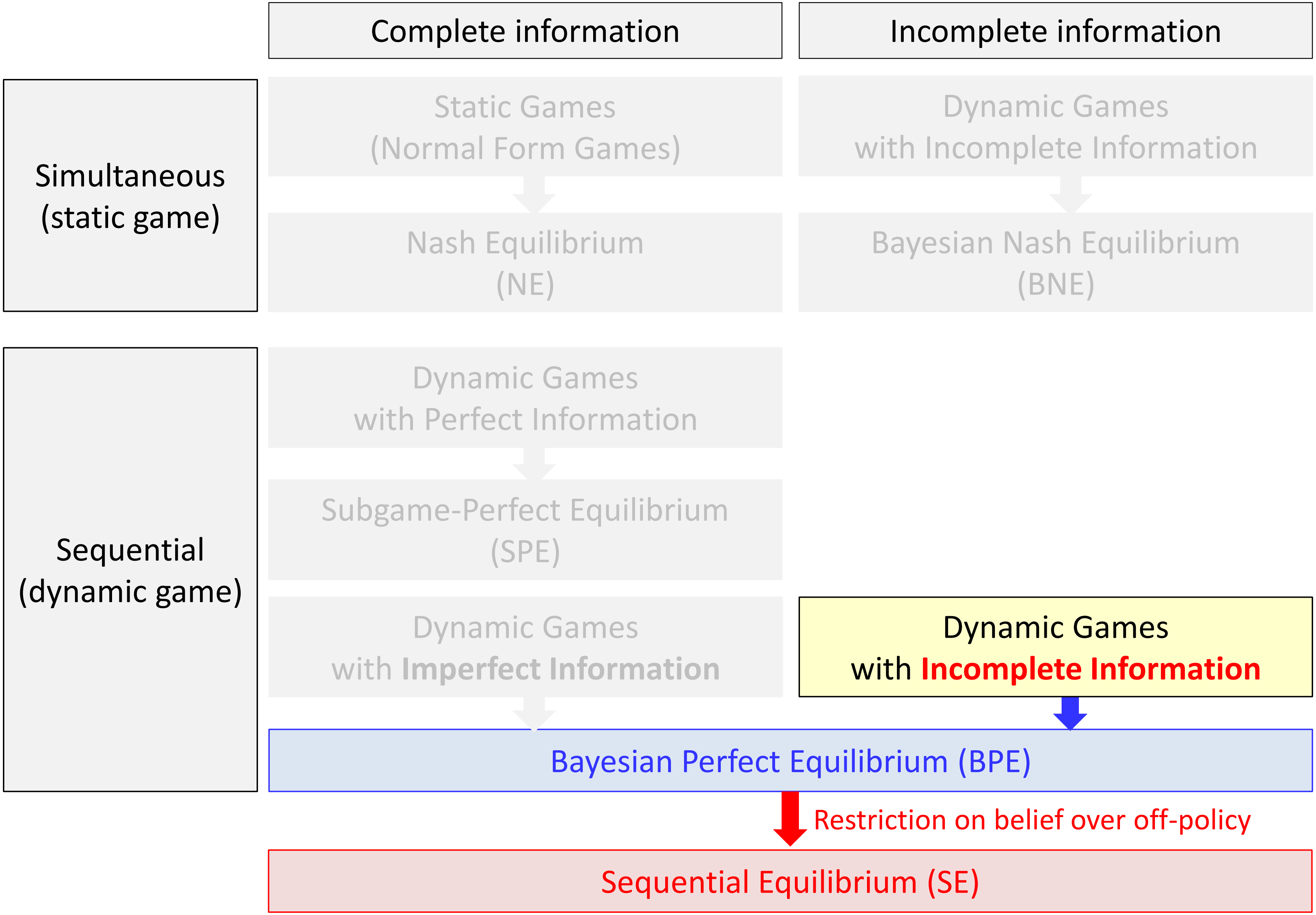
Dynamic Games with **Incomplete Information**

Bayesian Perfect Equilibrium (BPE)

Restriction on belief over off-policy

Sequential Equilibrium (SE)

# Introduction



Complete information

Incomplete information

Simultaneous (static game)

Static Games (Normal Form Games)

Nash Equilibrium (NE)

Dynamic Games with Incomplete Information

Bayesian Nash Equilibrium (BNE)

Sequential (dynamic game)

Dynamic Games with Perfect Information

Subgame-Perfect Equilibrium (SPE)

Dynamic Games with **Imperfect Information**

Dynamic Games with **Incomplete Information**

Bayesian Perfect Equilibrium (BPE)

Restriction on belief over off-policy

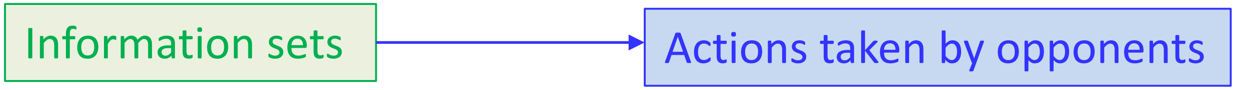
Sequential Equilibrium (SE)

## Introduction

- This chapter applies the idea of **sequential rationality** to dynamic games of incomplete information (Bayesian Games)

- In Bayesian Games, we have shown that some players will have information sets that correspond to the set of types that their opponents may have
  - Opponent players’ types are resulted by Nature’s choice
  - Belief concept was devised to capture uncertainties over the type of others

Dynamic games with **imperfect** information



Information sets → Actions taken by opponents

Dynamic games with **incomplete** information

Information sets → Types of opponents

## Introduction

- This chapter applies the idea of **sequential rationality** to dynamic games of incomplete information (Bayesian Games)

- In Bayesian Games, we have shown that some players will have information sets that correspond to the set of types that their opponents may have
  - Opponent players' types are resulted by Nature's choice
  - Belief concept was devised to capture uncertainties over the type of others

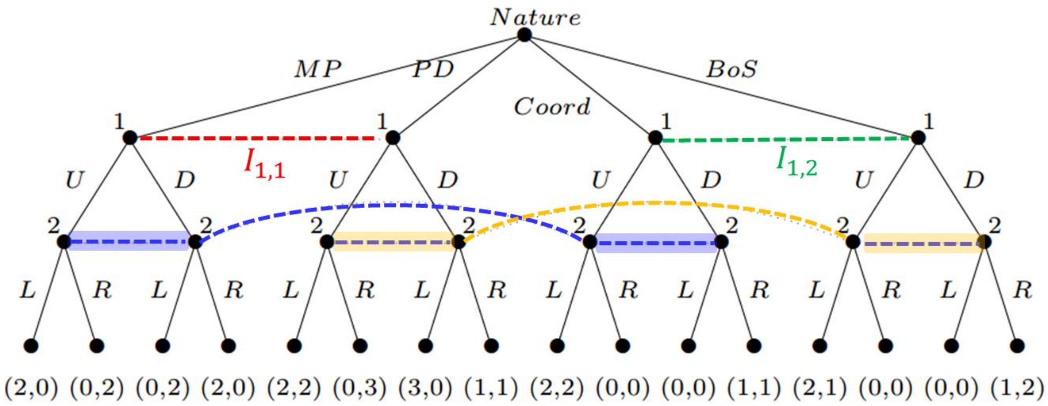
- We will discuss two aspects in defining an equilibrium concept for Bayesian Game
  1. Sequentially rational with regard the belief set
  2. The consistency of the beliefs with respect to
     - ✓ the environment (Nature)
     - ✓ the strategies of all other players

- We want to focus attention on equilibrium play in which players play best-response actions both
  - On the equilibrium path
  - Off the equilibrium path (points in the game that are not reached)

# Expressing Bayesian Dynamic Games

- In Bayesian games (static game with incomplete information), we have discussed three representations for the games:
  - Information sets
  - Extensive form game with imperfect information set with Nature
  - Epistemic types

- We will use "Extensive form game with imperfect information set with Nature" representation because
  - Easy to expand to sequential (dynamic) game setting
  - We can use the solution concepts for "Extensive form game with imperfect information"



# Bayesian games represented by epistemic types

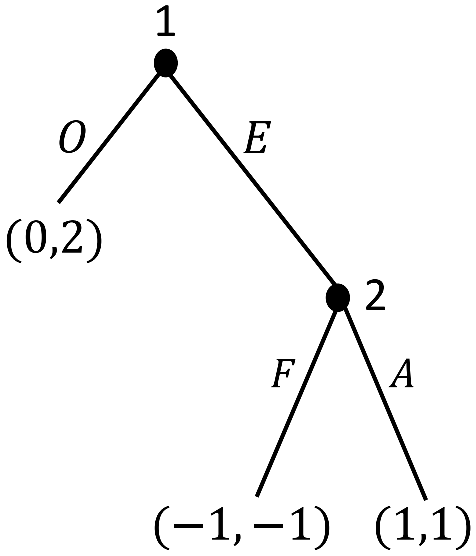
**Definition (Dynamic Bayesian game)**

A Bayesian game is a tuple $(N, A, \Theta, p, u)$ where:

- $N$ is a set of agents
- A sequence of histories $H^t$ at the $t$-th stage of the game, each history assigned to one of the players (or to Nature)
- $A = A_1 \times \cdots \times A_n$, where $A_i$ is the set of actions available to player $i;$
- $\Theta = \Theta_1 \times \cdots \times \Theta_n$, where $\Theta_i$ is a set of types for each player $i : \theta_i \in \Theta_i$
- $p : \Theta \longmapsto [0,1]$ is a common prior over types
- $I = (I_1, \dots, I_n)$, where $I_i = \left(I_{i,1}, \dots, I_{i,k_i}\right)$, is an information partition, which determine which of the histories assigned to a player are in the same information set
- $u = (u_1, \dots, u_n)$, where $u_i : A \times \Theta_i \longmapsto \mathbb{R}$ is the utility function of player $i$, which is type dependent, i.e.,
  - $u_i(s, \theta_i)$ is the utility function with a type $\theta_i \in \Theta_i$
  - $u_i(s, \theta)$ is the utility function with $\theta = (\theta_1, \dots, \theta_n) \in \Theta$

- The assumption is that all of the above is common knowledge among the players, and that each agent knows his own type

# The  problem with subgame perfection



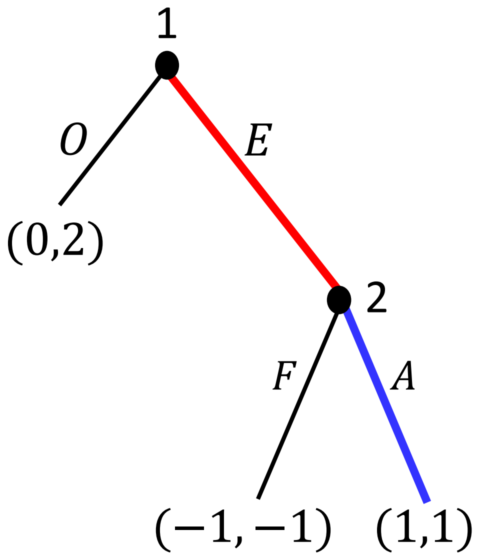
(A simple entry game)

- **Player 1**: A potential entrant to an industry that has a monopolistic incumbent, player 2
  - Can decide to enter the market (Enter)
  - Can decide not to enter (Stay out)
- **Player 2**: If player 1 enters the market, player 2
  - Can Fight with player 1
  - Can Accommodate with player 1

Extensive form game
Subgame-Perfect equilibrium

# The problem with subgame perfection

- **Player 1**: A potential entrant to an industry that has a monopolistic incumbent, player 2
  - Can decide to enter the market (Stay out)
  - Can decide not to enter (Enter)
- **Player 2**: If player 1 enters the market, player 2
  - Can Fight with player 1
  - Can Accommodate with player 1



(A simple entry game)

| | $F$ | $A$ |
|---|---|---|
| $O$ | $0, 2$ | $0, 2$ |
| $E$ | $-1, -1$ | $1, 1$ |

Nash equilibria = $\{(O, F)\ , (E, A)\ \}$

Subgame perfect equilibrium = $(E, A)$

Extensive form game
Subgame-Perfect equilibrium

# The problem with subgame perfection

- Now, consider "incomplete information"

- Imagine that the entrant may be one of two types
  - Competitive (C): have a technology that is as good as that of the incumbent
  - Weak (W) : have a inferior technology

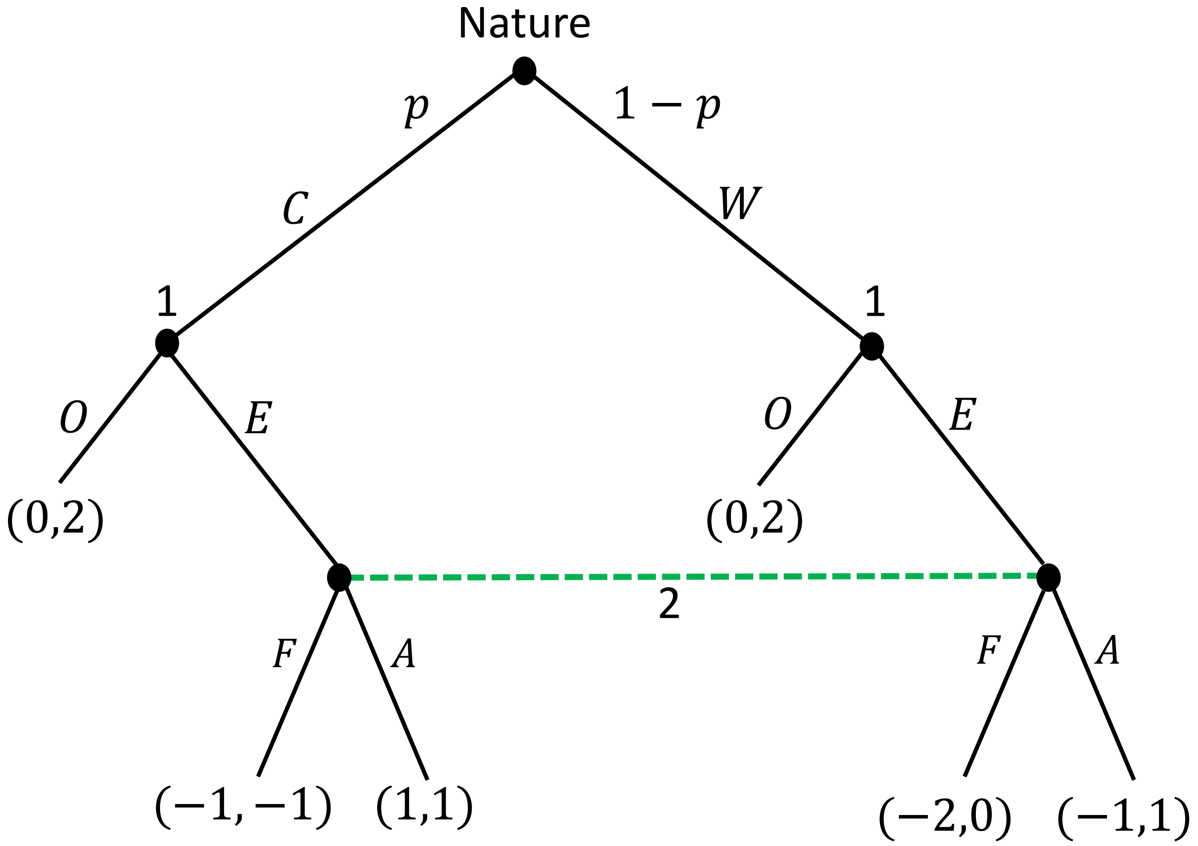
- A particular case of this story can be captured by the following sequence of events:

1. Nature chooses the entrant's type, which can be weak (W) or competitive (C), so that $\theta_1 \in \{W, C\}$, and let $P\{\theta_1 = C\} = p$. The entrant knows his own type but the incumbent knows only the probability distribution over types (common prior)

2. The entrant chooses between $E$ and $O$ as before, and the incumbent observes the entrant's choice

3. After observing the action of the entrant, and it if the entrant enters, the incumbent can choose between $A$ and $F$

# The problem with subgame perfection



Extensive form game
Subgame-Perfect equilibrium

+

Bayesian game
Bayesian Nash equilibrium

=

Dynamic Bayesian game
Perfect Bayesian Nash equilibrium

**Sequentially rationality**

**Rationality based on belief**

**Sequential rationality with consistent belief**

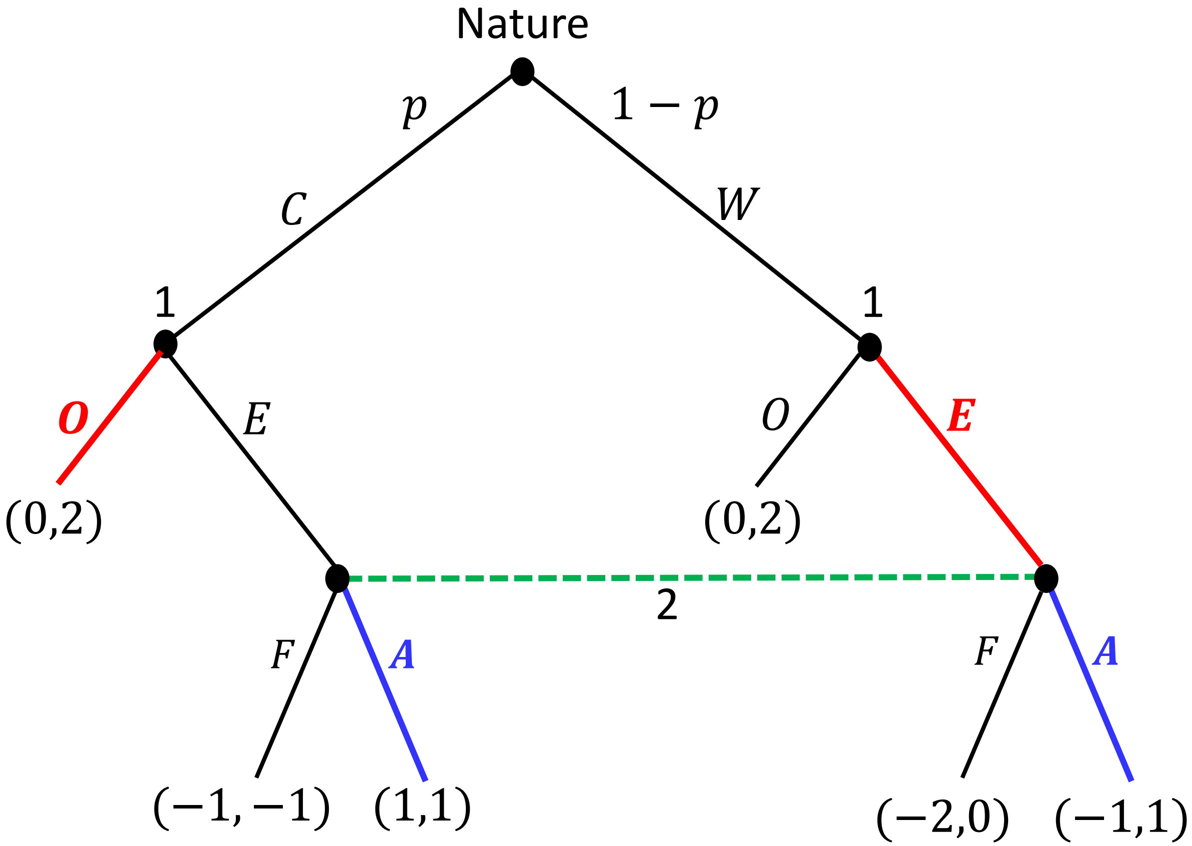
# The problem with subgame perfection

- Let’s convert the game into a normal form.

- Player 1 has four pure strategies
  - Two different types $\theta_1 \in \{W, C\}$
  - $s_1(\theta_1)$ is the action chosen by player 1 when the type is $\theta_1$
  - For each type, two possible actions $s_1(\theta_1) \in \{O, E\}$ :
  - Thus, a pure strategy $s_1 = \big(s_1(\theta_1 = C), s_1(\theta_1 = W)\big) \in S_1 = \{OO, OE, EO, EE\}$

- Player 2 has two pure strategies
  - Only 1 information set that follows entry
  - Two actions are available in that information set $s_2 \in S_2 = \{A, F\}$

# The problem with subgame perfection



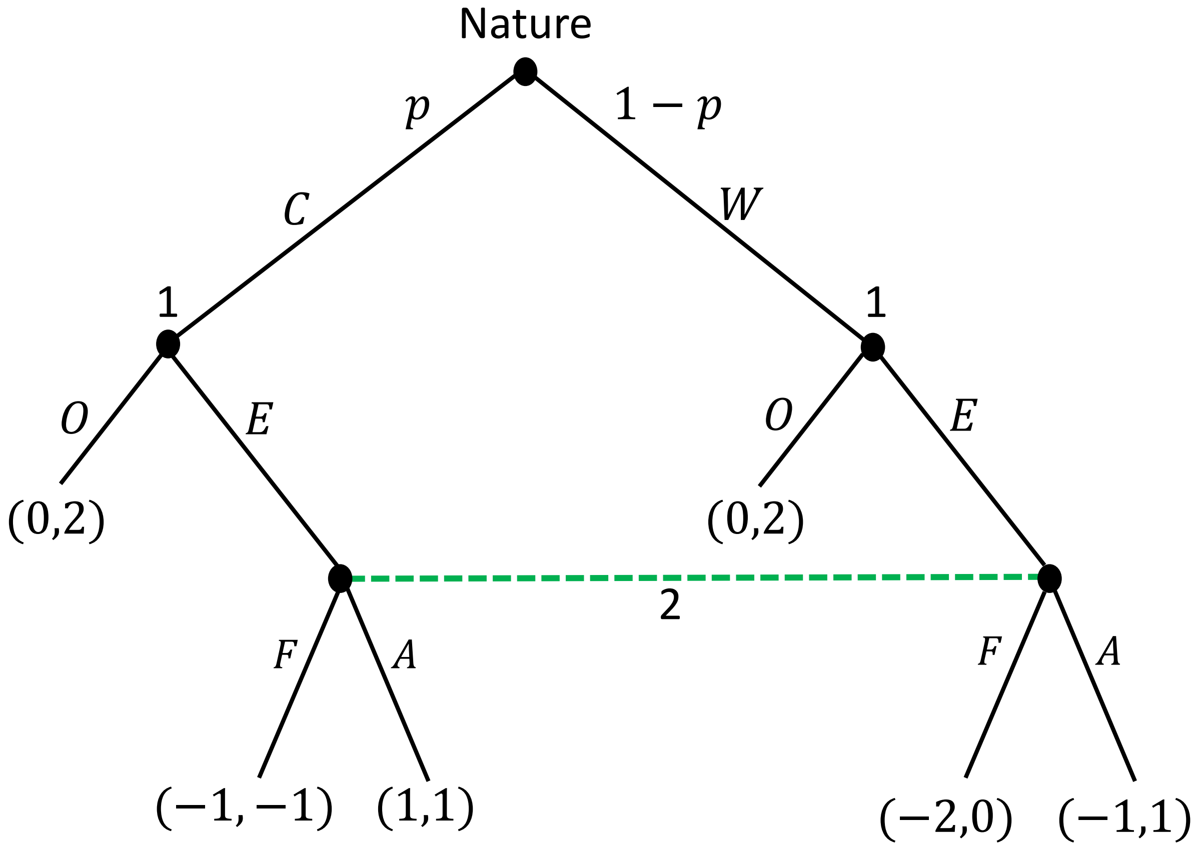
| | | **Player 2** | |
|---|---|---|---|
| | | $F$ | $A$ |
| **Player 1** | $OO$ | $0, 2$ | $0, 2$ |
| | $OE$ | $-1, 1$ | $-\frac{1}{2}, \frac{3}{2}$ |
| | $EO$ | $-\frac{1}{2}, \frac{1}{2}$ | $\frac{1}{2}, \frac{3}{2}$ |
| | $EE$ | $-\frac{1}{2}, -\frac{3}{2}$ | $1, 0$ |

$$p = \frac{1}{2}$$

- To convert the game into normal form, an expected payoff should be computed
- The expectation is over the randomizations caused by Nature (Ex Ante). For example,

$$E[u_1(s_1, s_2)] = E\big[u_1\big((OE), A\big)\big] = p \times 0 + (1-p) \times (-1) = p - 1$$
$$E[u_2(s_1, s_2)] = E\big[u_2\big((OE), A\big)\big] = p \times 2 + (1-p) \times 1 = p + 1$$

# The problem with subgame perfection

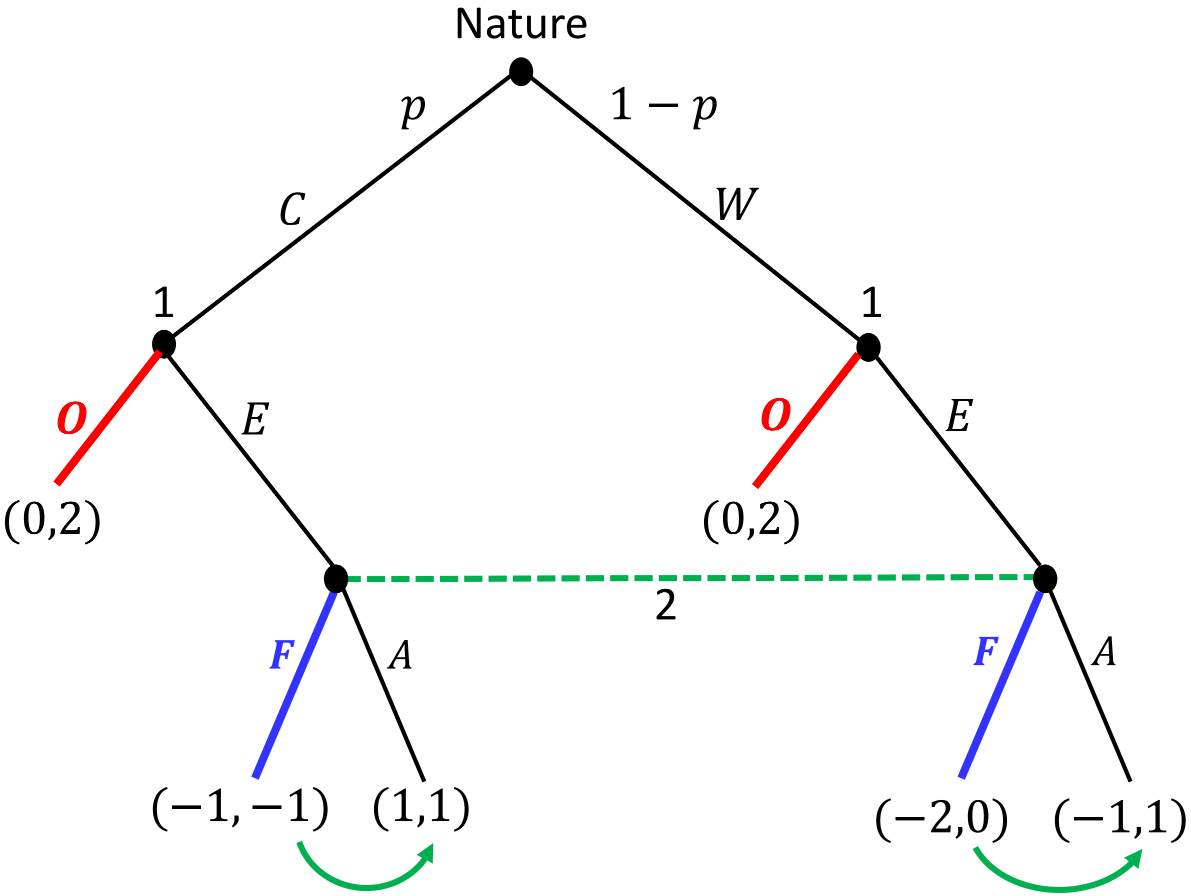


| | | **Player 2** | |
|---|---|---|---|
| | | $F$ | $A$ |
| **Player 1** | $OO$ | $0,2$ | $0,2$ |
| | $OE$ | $-1,1$ | $-\frac{1}{2},\frac{3}{2}$ |
| | $EO$ | $-\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{3}{2}$ |
| | $EE$ | $-\frac{1}{2},-\frac{3}{2}$ | $1,0$ |

when $p=\frac{1}{2}$

- Pure strategy Bayesian Nash equilibria:
  - $\{(OO\ ,F),(EO\ ,A)\}$
- Which of these two equilibria survives as a subgame-perfect equilibrium in the expensive-form game?
  - Both BNE survives because there is **only a single subgame**, the game itself!

# The problem with subgame perfection

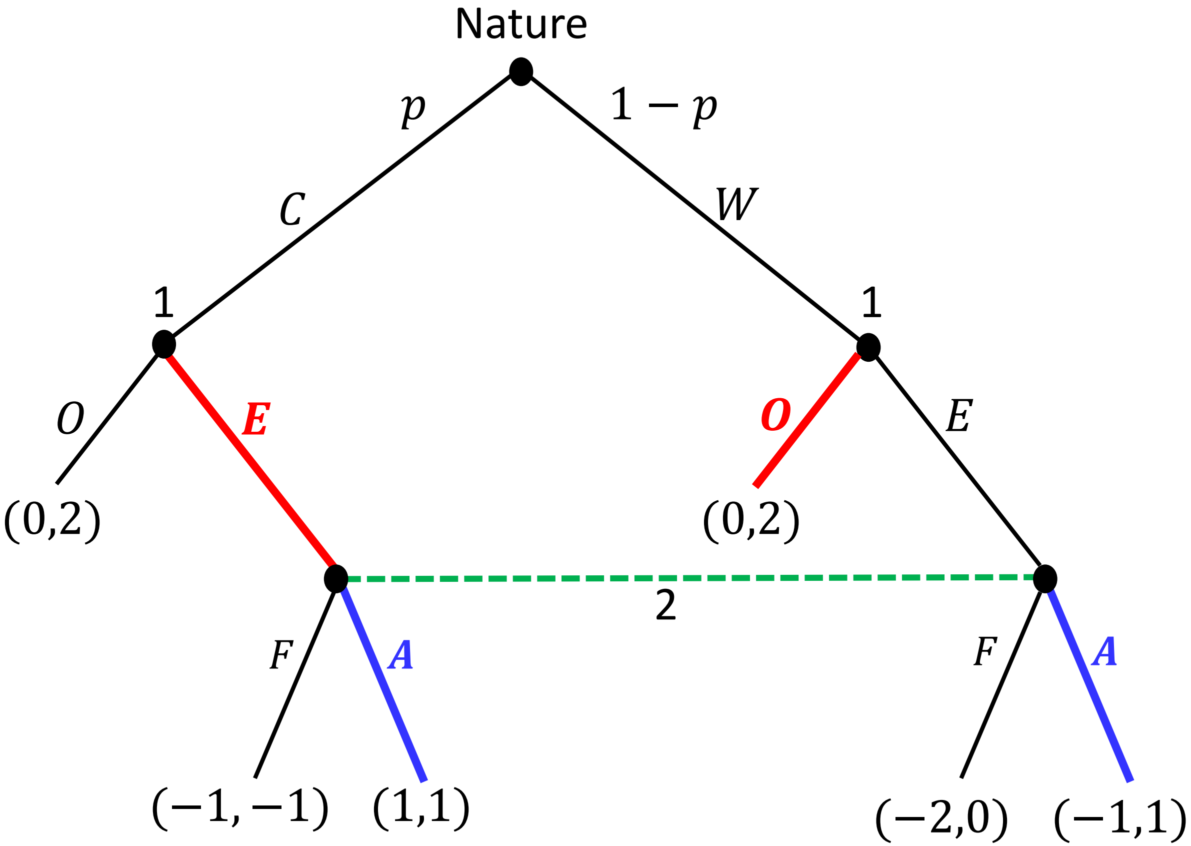


| | | **Player 2** | |
|---|---|---|---|
| | | $F$ | $A$ |
| **Player 1** | $OO$ | $0, 2$ | $0, 2$ |
| | $OE$ | $-1, 1$ | $-\frac{1}{2}, \frac{3}{2}$ |
| | $EO$ | $-\frac{1}{2}, \frac{1}{2}$ | $\frac{1}{2}, \frac{3}{2}$ |
| | $EE$ | $-\frac{1}{2}, -\frac{3}{2}$ | $1, 0$ |

$$p = \frac{1}{2}$$

- First Bayesian Nash Equilibrium $(OO\ , F)$
  - Player 2 threatens to fight, but if he finds himself in the information set that follows entry, he has a strict best response which is to accommodate
  - Thus the Bayesian Nash equilibrium $(OO\ , F)$ involves non-credible behavior of player 2 that is not sequentially rational

# The problem with subgame perfection



**Player 2**

| **Player 1** | $F$ | $A$ |
|---|---|---|
| $OO$ | $0, 2$ | $0, 2$ |
| $OE$ | $-1, 1$ | $-\frac{1}{2}, \frac{3}{2}$ |
| $EO$ | $-\frac{1}{2}, \frac{1}{2}$ | $\frac{1}{2}, \frac{3}{2}$ |
| $EE$ | $-\frac{1}{2}, -\frac{3}{2}$ | $1, 0$ |

$$p = \frac{1}{2}$$

- Second Bayesian Nash Equilibrium $(EO, A)$
  - The Bayesian Nash equilibrium $(EO, A)$ is a **Perfect Bayesian Nash Equilibrium**
  - Perfect Bayesian Nash Equilibrium requires more rigorous structure so that sequential rationality to be well defined
  - We will describes the requirements for **Perfect Bayesian Nash Equilibrium**

# Perfect Bayesian (Nash) Equilibrium

- In the previous game, we need to make statements about the sequential rationality of player 2 within each of his information sets even though the information set is not itself the first node of a proper subgame

- We need to be able to make statements like "in this information set player 2 is playing a best response, and therefore his behavior is sequentially rational."

- To describe a player's best response within his information set, we will have to ask what the player is playing a best response to
    - We must include beliefs in the analysis

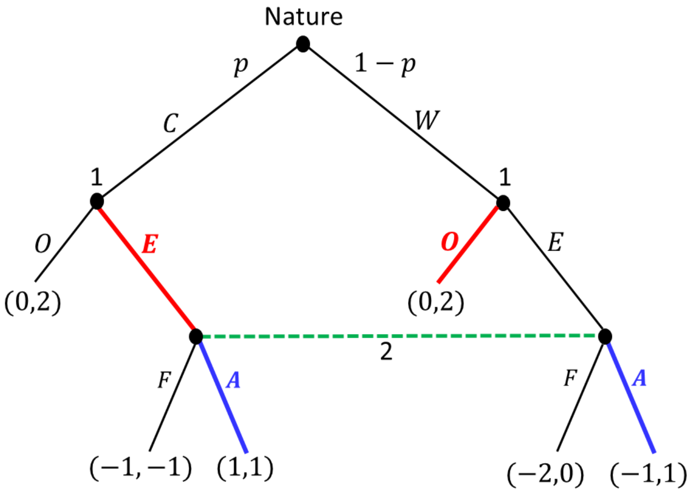
- In conclusion, we need to consider the beliefs of player 2 in his information sets and then analyze his best response to these beliefs

# Perfect Bayesian (Nash) Equilibrium

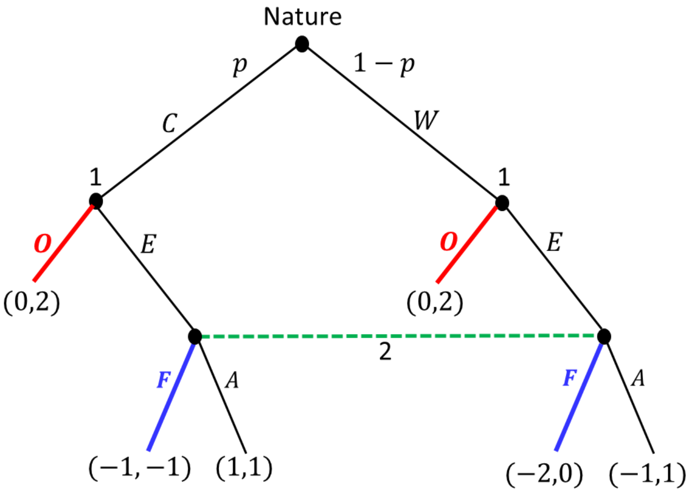
## Definition (On & Off the equilibrium)

Let $s^* = (s_1^*, \dots, s_n^*)$ be a Bayesian Nash equilibrium profile of strategies in a game of incomplete information.

- We say that an information set is on the equilibrium path if given $s^*$ and given the distribution of types, it is reached with positive probability.
- We say that an information set is off the equilibrium path if given $s^*$ and the distribution of types, it is reached with zero probability



Player 1's information set (singleton) is always reached
Player 2's information set is reached with probability $p$



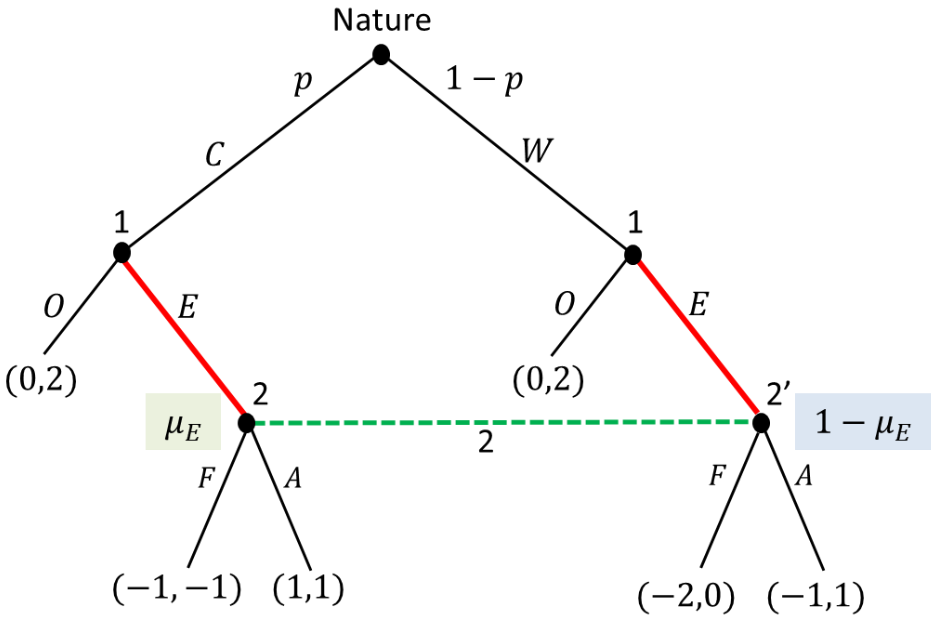
Player 1's information set is always reached
Player 2's information set is never reached

# Perfect Bayesian (Nash) Equilibrium

## Definition (A system of beliefs $\mu$)

A **system of beliefs** $\mu$ of an extensive-form game assigns a probability distribution over decision nodes to every information set. That is, for every information set $I$ and every decision node $h \in I$, $\mu(h) \in [0,1]$ is the probability that player $i$ who moves in information set $I$ assigns to his being at $h$, where

$\sum_{h \in I} \mu(h) = 1$ for every $I$



$\mu_E$: Player 2's belief that he is at the node corresponding to player 1 being competitive $(\mathrm{C})$ and playing $E$

$1-\mu_E$: Player 2's belief that he is at the node corresponding to player 1 being Week $(\mathrm{W})$ and playing $E$

# Perfect Bayesian (Nash) Equilibrium

**Requirement 1** for Perfect Bayesian Nash Equilibrium

Every player will have a well-defined belief over where he is in each of his information sets. That is, the game will have a system of beliefs

# Perfect Bayesian (Nash) Equilibrium

**Requirement 2** for Perfect Bayesian Nash Equilibrium

Let $s^* = (s_1^*, \ldots, s_n^*)$ be a Bayesian Nash equilibrium profile of strategies. We require that in all information sets beliefs that are on the equilibrium path be consistent with Bayles's rule.

- How should the beliefs in a system of beliefs be determined?
  - Recall that for Nash equilibrium, the beliefs of players about the strategies of their opponents to be correct

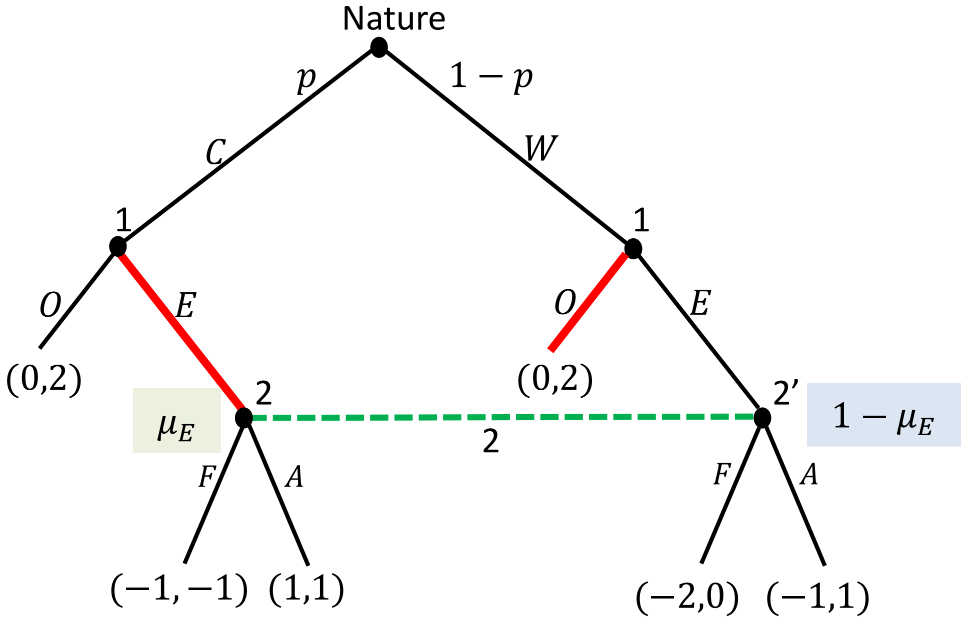
- In games of incomplete information, we require similar requirements. Two constrains will influence whether a player’s beliefs are correct
  - Endogenous constraint on beliefs
    - Constrained by the behavior of the **other players**
    - Which are the variables that players can control
  - Exogenous constraint on beliefs
    - Constrained by the choice of **Nature**
    - Which is not something that the players control but rather part of the environment

# Perfect Bayesian (Nash) Equilibrium

**Requirement 2** for Perfect Bayesian Nash Equilibrium

Let $s^* = (s_1^*, \dots, s_n^*)$ be a Bayesian Nash equilibrium profile of strategies. We require that in all information sets beliefs that are on the equilibrium path be consistent with Bayles's rule.

**Pure strategy case:**



- Imagine that player 1 is playing a pure strategy $(EO)$

$$\mu_E = P(\theta_1 = C|\text{plyaer 1 coose } E); \qquad 1 - \mu_E = P(\theta_1 = W|\text{plyaer 1 coose } E)$$

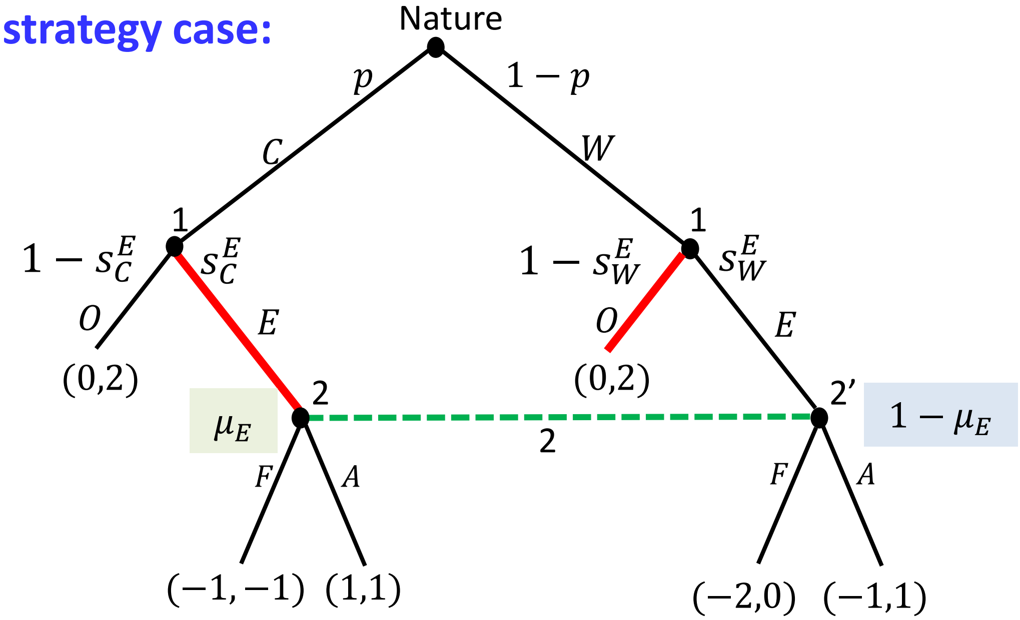
- The only consistent belief is $\mu_E = 1$

# Perfect Bayesian (Nash) Equilibrium

**Requirement 2** for Perfect Bayesian Nash Equilibrium

Let $s^* = (s_1^*, \dots, s_n^*)$ be a Bayesian Nash equilibrium profile of strategies. We require that in all information sets beliefs that are on the equilibrium path be consistent with Bayles's rule.

**Mixed (Behavioral) strategy case:**



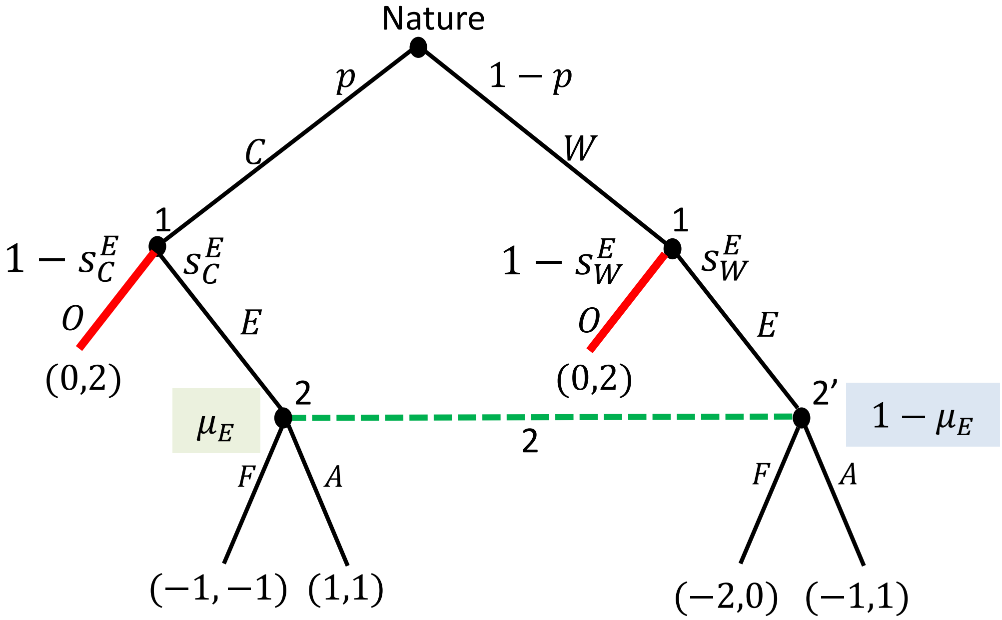
$$\mu_E = P(C|E) = \frac{P(C \cap E)}{P(E)} = \frac{P(C)P(E|C)}{P(C)P(E|C) + P(W)P(E|W)} = \frac{ps_C^E}{ps_C^E + (1-p)s_W^E}$$

The pure strategy $EO$ is just a special case with $s_C^E = 1$ and $s_W^E = 0 \Rightarrow \mu_E = \frac{p\times 1}{p\times 1+(1-p)\times 0} = 1$

# Perfect Bayesian (Nash) Equilibrium

**Requirement 3** for Perfect Bayesian Nash Equilibrium

At information sets that are off the equilibrium path, **any belief** can be assigned to which Bayes' rule does not apply



$$\mu_E = P(C|E) = \frac{P(C \cap E)}{P(E)} = \frac{P(C)P(E|C)}{P(C)P(E|C) + P(W)P(E|W)} = \frac{ps_C^E}{ps_C^E + (1-p)s_W^E} = \frac{p \times 0}{p \times 0 + (1-p) \times 0}$$

- Bayes' rule does not apply because given the suggested strategy both the numerator and the denominator are zero →Setting $\mu_E$ can be any number in the interval $[0,1]$

# Perfect Bayesian (Nash) Equilibrium

**Requirement 4** for Perfect Bayesian Nash Equilibrium

Given their beliefs, players' strategies must be sequentially rational. That is, in every information set players will play a best response to their beliefs.

- Consider player $i$ with beliefs over information sets derived from the beliefs system $\mu$, given player $i$'s opponents playing $s_{-i}$.

- Above requirement says that if $I_i$ is an information set for player $i$, then it must be true the he is playing a strategy $s_i$ that satisfies

$$E[u_i(s_i, s_{-i}, \theta_i)|I_i, \mu] \geq E[u_i(s_i', s_{-i}, \theta_i)|I_i, \mu] \text{ for all } s_i' \in S_i$$

  - ✓ Where expectations are given over the beliefs of player $i$ using $\mu$

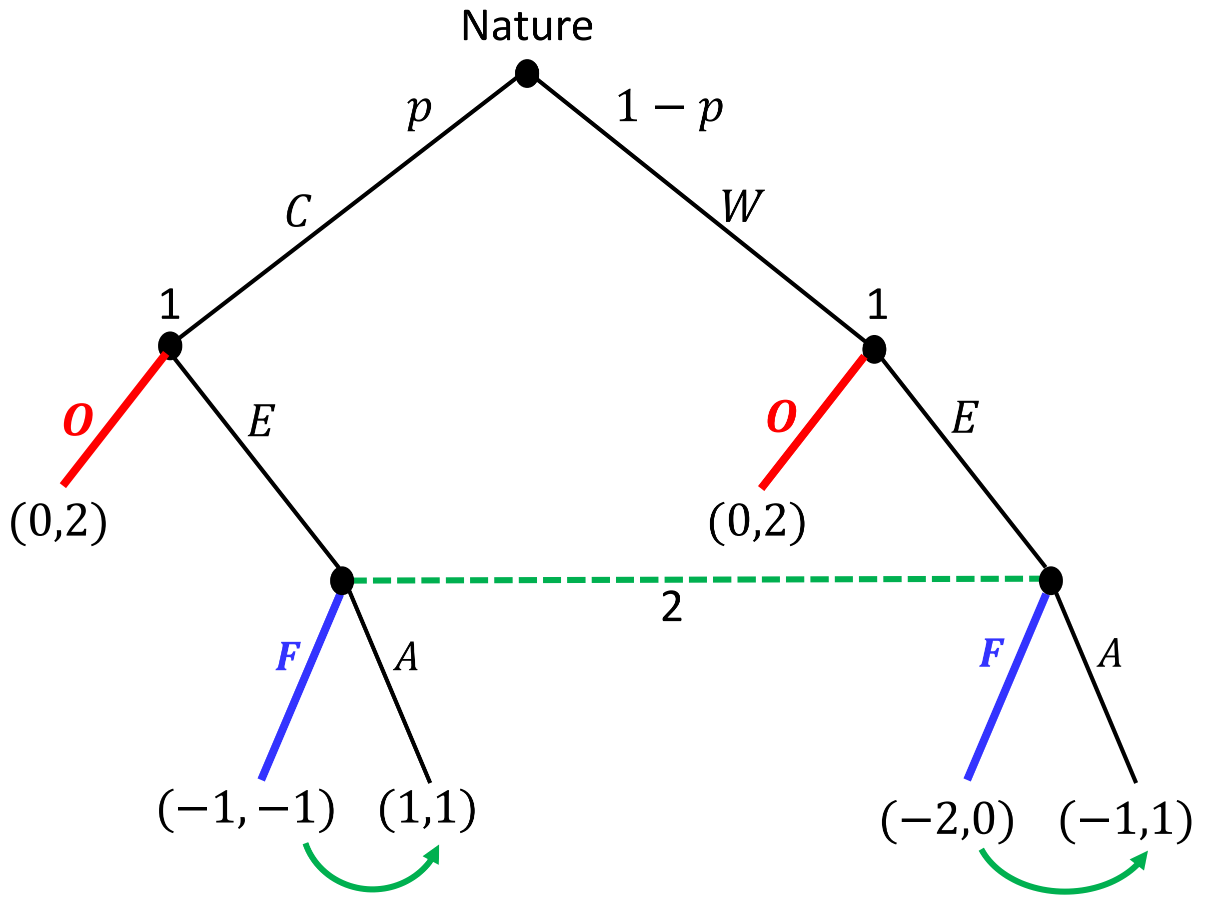
# Perfect Bayesian (Nash) Equilibrium

**Definition (Perfect Bayesian (Nash) Equilibrium)**

A Bayesian Nash equilibrium profile $s^* = (s_1^*, \ldots, s_n^*)$ together with a system of beliefs $\mu$ constitutes a perfect Bayesian equilibrium for an $n$ −player game if they satisfy requirements 1-4

- This definition puts together our four requirements in a way that will guarantee **sequentially rationality**

# Perfect Bayesian (Nash) Equilibrium



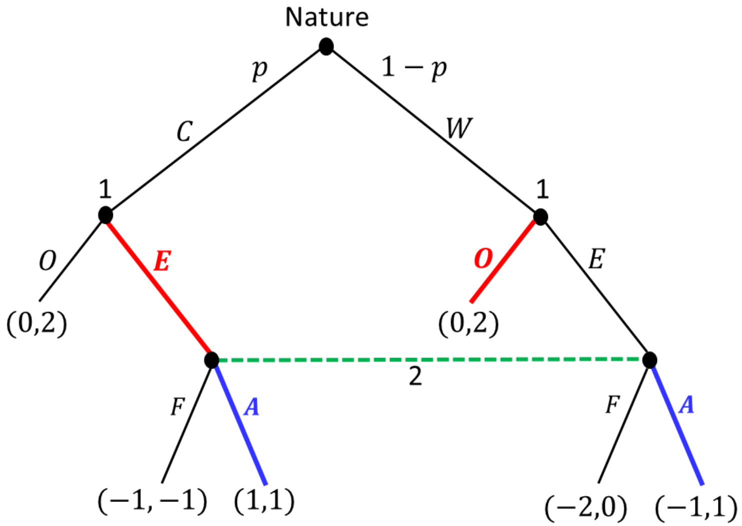
- Because player 2’s information set is off the equilibrium, we can arbitrarily assign probability distribution on the information set for player 2.
- It contradicts requirement 4: Playing F is not best response!

## How to compute Perfect Bayesian (Nash) Equilibrium

- First find all the profiles of strategies in the Bayesian game that are Bayesian Nash equilibria
- Then, we can systemically check for each Bayesian Nash equilibrium to see whether we can find a system of beliefs so that together they constitute a perfect Bayesian equilibrium

### Proposition (Perfect Bayesian (Nash) Equilibrium)

If a profile of (possibly mixed) strategies $s^* = (s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium of a Bayesian game $\Gamma$, and if $s^*$ induces all the information sets to be reached with positive probability, the $s^*$, together with the belief system $\mu^*$ uniquely derived from $s^*$ and the distribution of types, constitutes a perfect Bayesian equilibrium for $\Gamma$



$s^* = \big((EO), A\big)$ with $\mu_E = 1$ is PBNE because

- It is Bayesian Nash equilibrium
- All the information sets are reached with positive probability
- $\mu_E = 1$ is consistent with $s^*$

# Sequential Equilibrium

**Motivations:**

- Perfect Bayesian equilibrium has become the most widely used solution concept for dynamic games with incomplete information

- There are, however, examples of games in which the perfect Bayesian equilibrium solution concept allows for equilibria that seem unreasonable

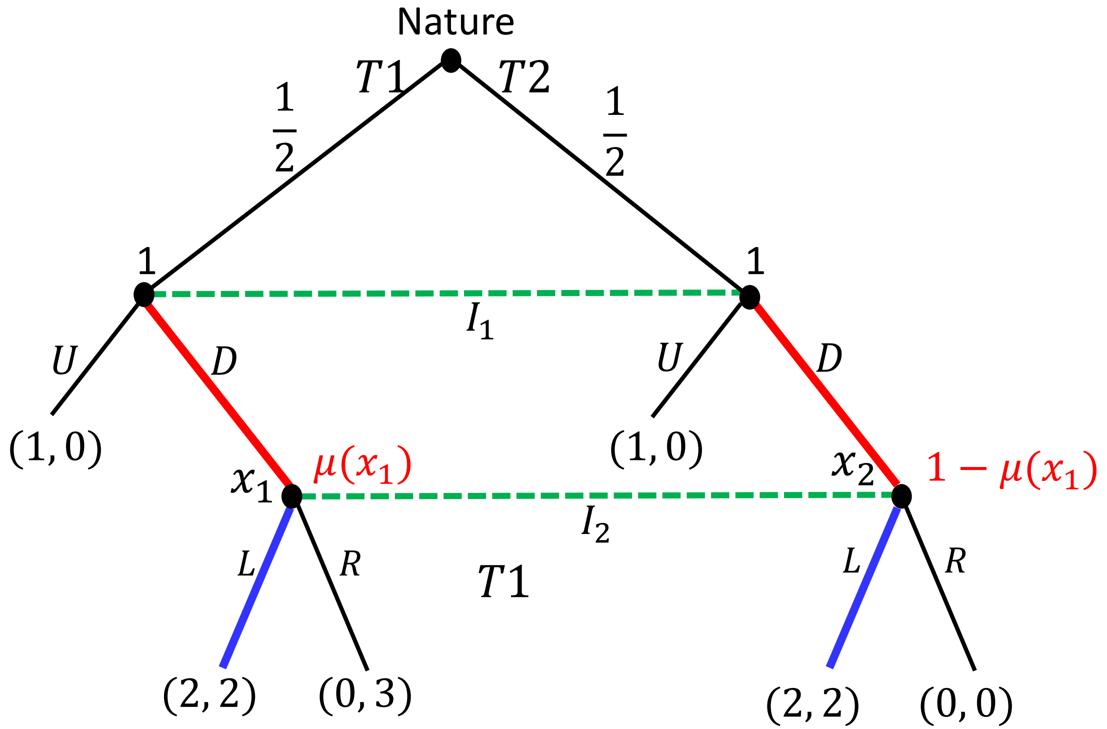
  - The reason for this is that requirement 3 of the perfect Bayesian equilibrium concept places no restrictions on beliefs that are off the equilibrium path

**Requirement 3** for Perfect Bayesian Nash Equilibrium

At information sets that are off the equilibrium path, **any belief** can be assigned to which Bayes’ rule does not apply

## Sequential Equilibrium

**Examples:**



- If player 1 plays $D$ with a positive probability then by requirement 2, the beliefs of player 2 are completely determined by Bayes' rule

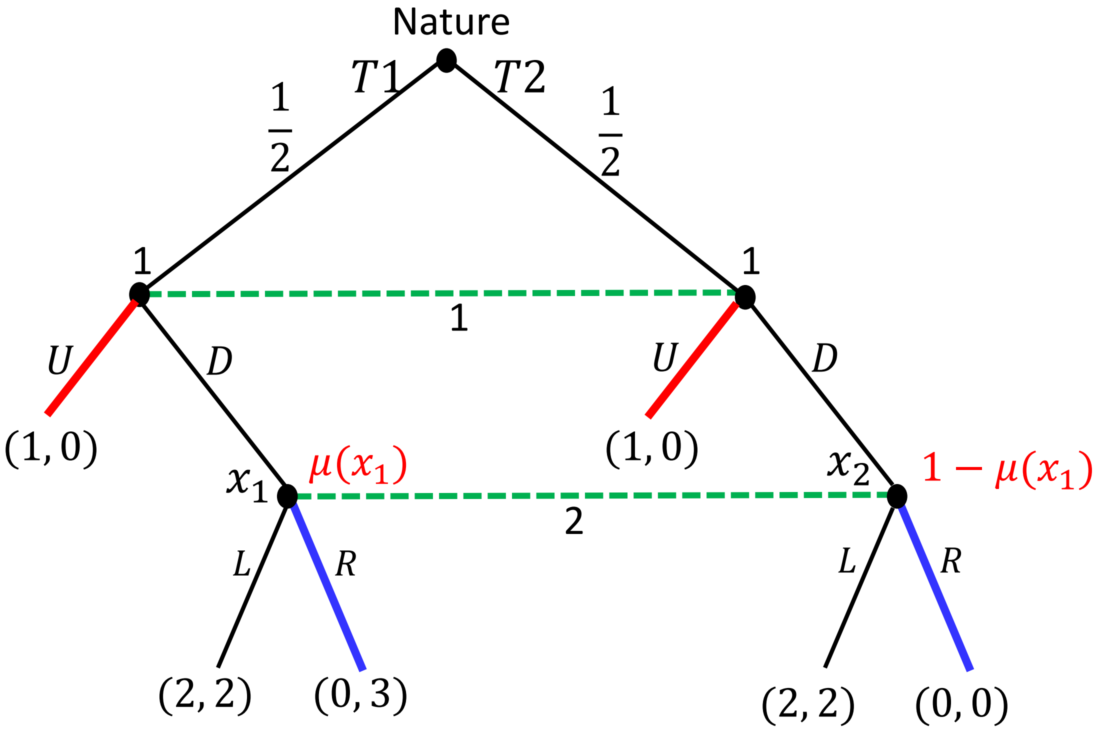
$$\mu(x_1) = P(T1|D) = \frac{P(T1 \cap D)}{P(D)} = \frac{P(T1)P(D|T1)}{P(T1)P(D|T1) + P(T2)P(D|T2)} = \frac{\frac{1}{2} s_{T_1}^{D}}{\frac{1}{2} s_{T_1}^{D} + \frac{1}{2} s_{T_2}^{D}} = \frac{1}{2}$$

$$(s_{T1}^{D} = s_{T2}^{D})$$

- With these beliefs player 2 must play $L$
- If player 2 play $L$ then player 1's best response is to play $D$
- Thus, a pair of strategies $(D, L)$ together with the implied beliefs $\mu_2(x_1) = \mu_2(x_2) = 1/2$ is PBNE

# Sequential Equilibrium

**Examples:**

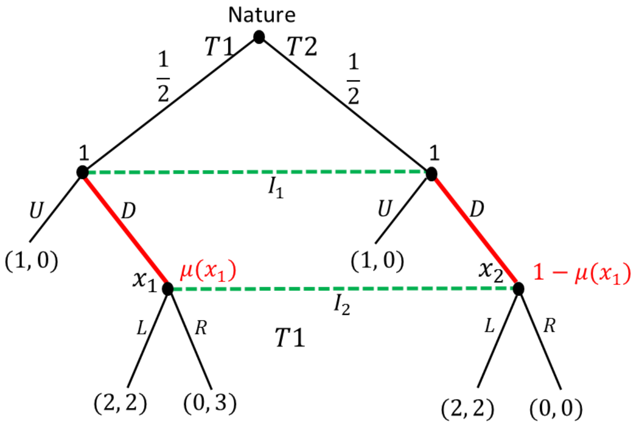


- Note the pair of strategies $(U, R)$ can also be supported as a perfect Bayesian equilibrium
- If player 1 play $U$, by requirements 2 and 3, beliefs are not restricted in player 2's information set
  - Player 2 can have any belief, e.g., $\mu_D = \mu_2(x_1) > 2/3$
  - In this case he believes play $R$ is best response
  - As a result, plying $U$ for player 1 is also a best response
- We need a more strong equilibrium refinement
  - ➢ Put restrictions on the sorts of beliefs that players can hold in information set that are off the equilibrium path

# Sequential Equilibrium

## Definition (Consistency)

A Profile strategies $s^* = (s_1^*, \ldots, s_n^*)$, together with a system of beliefs $\mu$, is **consistent** if there exists a sequence of nondegenerate mixed strategies, $\{s^k\}_{k=1}^{\infty}$, and a sequence of beliefs that are derived from each $s^k$ according to Bayes' rule, $\{\mu^k\}_{k=1}^{\infty}$, such that $\lim_{k \to \infty} (s^k, \mu^k) = (s^*, \mu^*)$



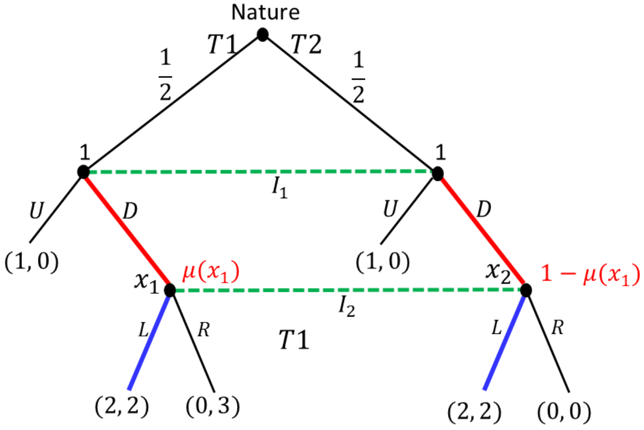
- The only consistent beliefs for player 2 are $\mu(x_1) = \mu(x_2) = \frac{1}{2}$
- the requirement that $\{s^k\}_{k=1}^{\infty}$ be a sequence of nondegenerate mixed strategies, which implies that each player is mixing among all his actions with positive probability.
- Then, every information set can reached with a positive probability, the beliefs $\mu(x_1) = 1/2$ can be derived from Bayes' rule
- So, any sequence of the form required by consistency the limit of beliefs must be $\mu(x_1) = \mu(x_2) = \frac{1}{2}$

# Sequential Equilibrium

**Definition (Sequential equilibrium)**

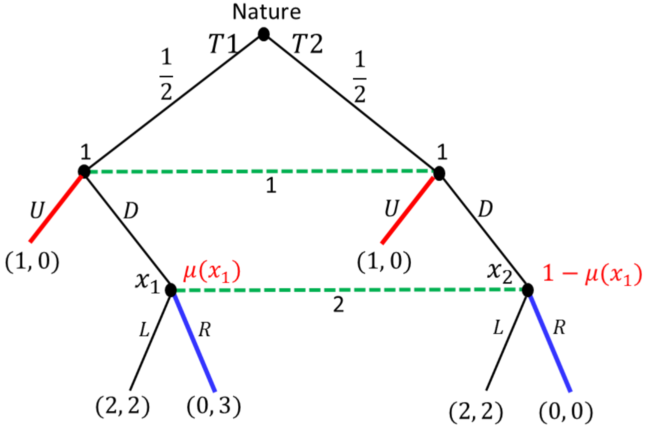
A Profile strategies $s^* = (s_1^*, \dots, s_n^*)$, together with a system of beliefs $\mu^*$, is a **sequential equilibrium** if consistent if $(s^*, \mu^*)$ is a consistent perfect Bayesian equilibrium

- Every sequential equilibrium is a perfect Bayesian equilibrium, but the revers is not ture.



$$\mu_2(x_1) = \mu_2(x_2) = 1/2$$

SE thus PBNE



$$\mu_D = \mu_2(x_1) > 2/3$$

PBNE but not SE