Lecture 3: Analyzing games

Solution concept

- A solution concept is a method of analyzing games with the objective of restricting the set of all possible outcomes to those that are more reasonable than others.

- A solution concept results in one or more strategy profiles, which we call equilibrium

- An equilibrium is prediction emerged by applying a solution concept to a target game

$$equilibrium_1 = f_1(G)$$
$$equilibrium_2 = f_2(G)$$

$$equilibrium_n = f_n(G)$$

$f_1, \ldots, f_n$ are solution concepts

- To set up the background for equilibrium analysis, it is useful to summarize the assumptions that we will be using in the lecture

  - **Players are "rational"**: A rational player is one who chooses his strategy $s_i \in S_i$, to maximize his payoff consistent with his beliefs about what is going on in the game

  - **Players are "intelligent"**: An intelligent player knows everything about the game: the actions, the outcomes, and the preferences of all the players.

  - **Common knowledge** : The fact that players are rational and intelligent is common knowledge among the players of the game

  - **Self-enforcing:** Any prediction (or equilibrium) of a solution concept must be self-enforcing
    - Core of our analysis and at the heart of non-cooperative game theory
    - Each player is in control of his own actions, and he will stick to an action only if he finds the action is in his best interest

- Pareto optimality
- Nash equilibrium
- Maximin and minmax strategies
- Minimax regret
- Removal of dominated strategies
- Rationalizability
- Correlated equilibrium
- Trembling-hand perfect equilibrium
- Etc.
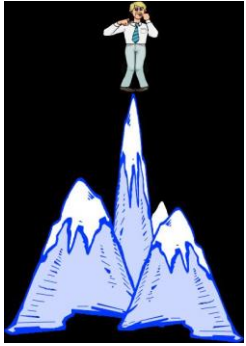
- **Single agent decision making:**

  - Optimal strategy is one that maximizes the agent's expected utility for a given environment
  - Uncertainties arose from stochastic environment, partially observable states, uncertain rewards, etc., which can be dealt with probability concepts.

$$a^* = \underset{a}{\operatorname{argmax}}\, E_s[u(a, s)]$$

- **Multiagents decision making:**

  - The environment includes other agents, each of which tries to maximize its own utility
  - Thus the notion of an optimal strategy for a given agent is not meaningful because *the best strategy depends on the choices of others*
  - We need to identify certain subsets of outcomes, called solution concepts
  - Two of the most fundamental solution concepts are
    - *Pareto optimality*
    - *Nash equilibrium*

$$u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \;\; \forall a_1$$
$$u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2) \;\; \forall a_2$$

Pareto optimality

- We've defined some canonical games and thought about how to play them.

- Now let's examine the games from the outside:

  - From the point of view of an outside observer, can some outcomes of a game be said to be better than others?

  - Can we say that one agent's interests are more important than another's

  - Imagine trying to find the revenue-maximizing outcome when you don't know what currency is used to express each agent's payoff

    - Are there ways to still prefer one outcome to another?

Pareto optimality

<div align="center">

Outcome of strategy s   **Outcome of strategy s'**

</div>

Agent 1's utility : 10 unit of currency x Agent 1's utility : 20 unit of currency x

Agent 2's utility : 500 unit of currency y Agent 2's utility : 10 unit of currency y

- Can we insist that the outcome of strategy s is better than that of strategy s'?

Pareto optimality

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

Agent 2's utility : 10 unit of currency y

- Can we insist that the outcome of strategy s is better than that of strategy s'?
  - No, because we cannot say that one agent's utility is more important than the other's

- Is there any situation that we can be sure that one outcome is better than another ?

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x
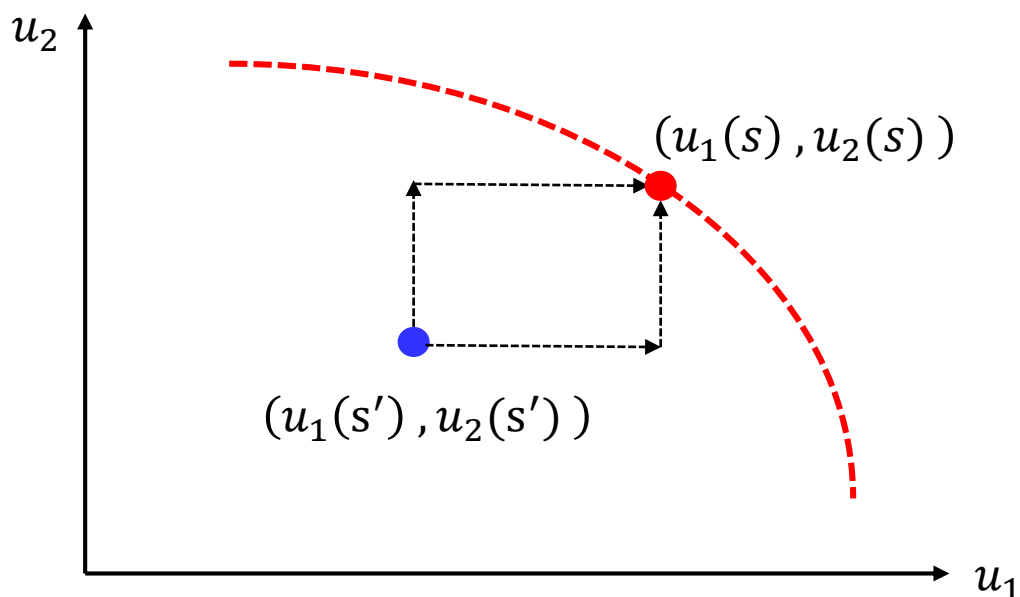
Agent 2's utility : 1000 unit of currency y

- The outcome of s' is always better than the outcome of s

Definition (Pareto domination)

Strategy profile s Pareto dominates strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.


$(u_1(s), u_2(s))$

$(u_1(s'), u_2(s'))$

- In other words, in a Pareto-dominated strategy profile some players can be made better off without making any other player worse off

- We cannot generally identify a single "best" outcomes; instead we may have a set of non-comparable optima

Definition (Pareto optimality)

Strategy profile s is Pareto optimal, or strictly Pareto efficient, if there does not exist another strategy profile $s' \in S$ that Pareto dominates s
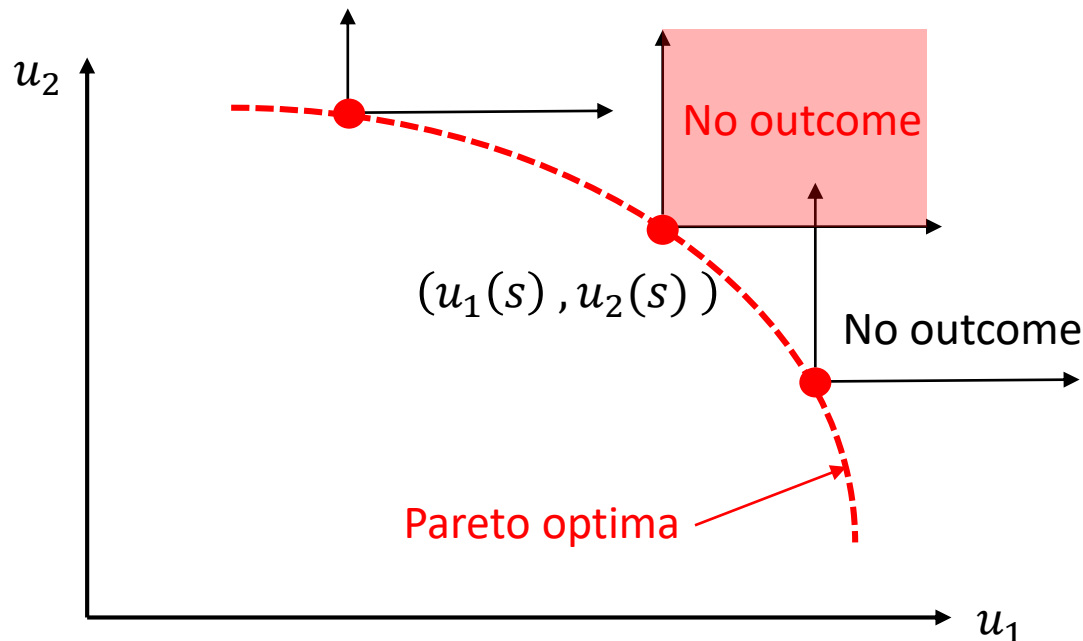

- Every game must have at least one Pareto optimal strategy profile, and there must always exist at least one such optimum in which all players adopt pure strategies.

- Some agent will have multiple optima
  (for example, in zero-sum games, all strategy profiles are strictly Pareto efficient. **Why?**)

Pareto optimal outcomes in various games

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

Coordination game

$u_2(a_1, a_2)$

(Left, Left) or (Right, Right)

1

1 $u_1(a_1, a_2)$
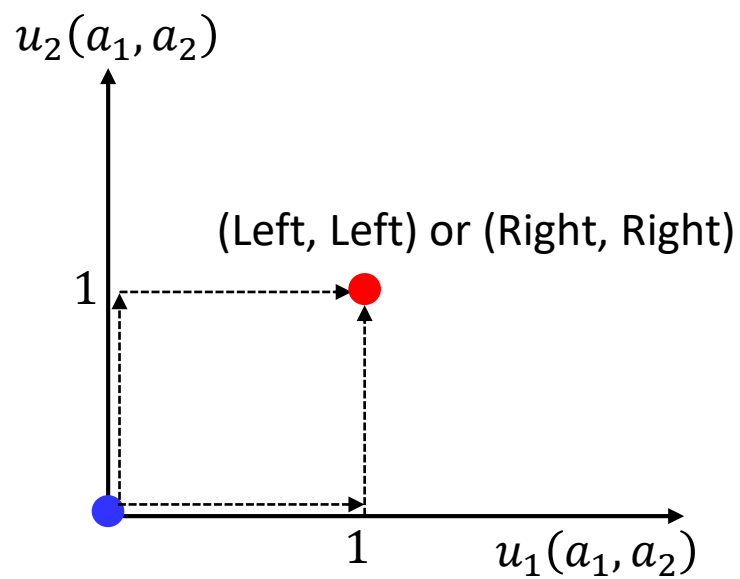
	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

Battle of Sex game

$u_2(a_1, a_2)$

2

(LA, LA)

1

(TF, TF)

1

2

$u_1(a_1, a_2)$

(LA, TF)
(TF, LA)
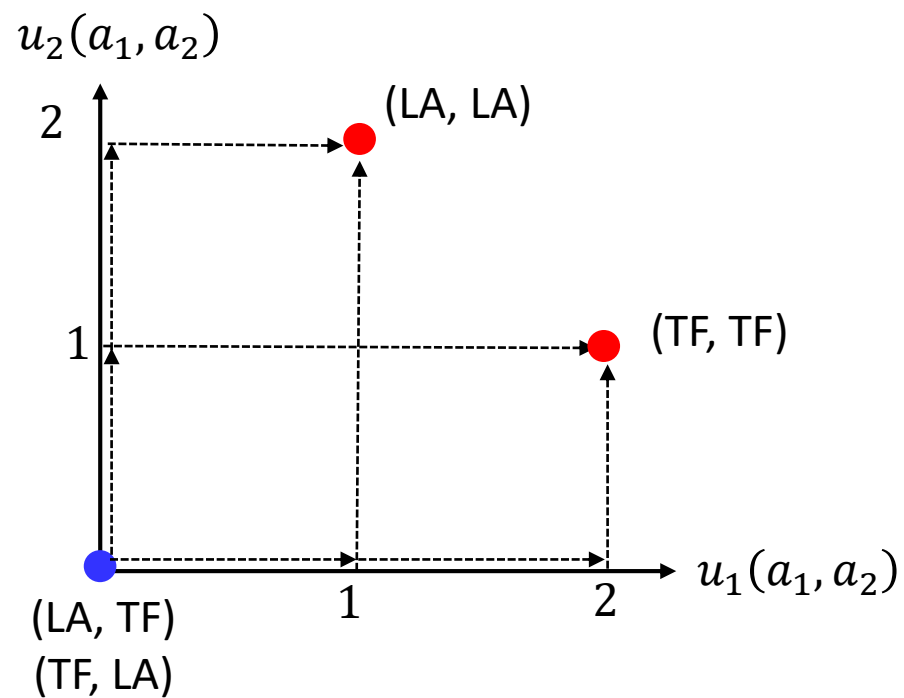
Pareto optimal outcomes in various games

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game
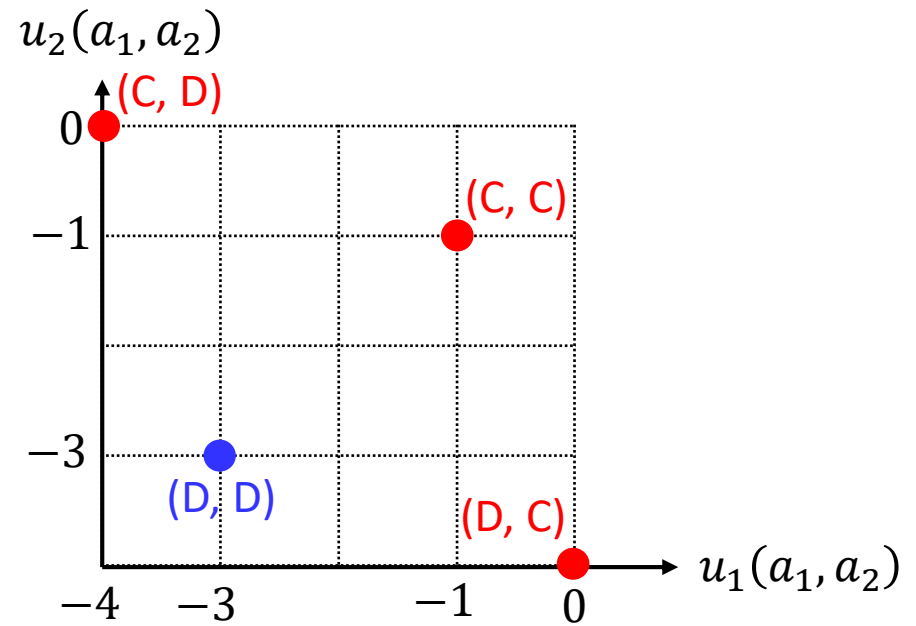
Best response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ to be strategy profiles of other agents (all agents except i)
  - then, $s = (s_i, s_{-i})$

Definition (Best response)

Player i's best response to the strategy profile s_{-i} is a mixed strategy $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$

$$s_i^* \in BR(s_{-i})$$

- The best response is not necessarily unique
- Indeed, except in the extreme case in which there is a unique best response that is a pure strategy, the number of best responses is always infinite.
- When the support of a best response s_i^* includes two or more actions, any mixture of these actions must also be a best response
- If there are two pure strategies that are individually best responses, any mixture of the two is necessarily also a best response

Nash equilibrium

- Really, no agent knows what the others will do
- What can we say about which actions will occur ?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies s_i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i, s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR(s_{-i}^*)$

- A Nash equilibrium is a stable strategy profile:
  - no agent would want to change his strategy if he knew what strategies the other agents were following

Definition (Strict Nash)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a strict Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$.

Definition (Week Nash)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a week Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$, and s^* is not a strict Nash equilibrium.

- Mixed-strategy Nash equilibria are necessarily week

- Pure-strategy Nash equilibria can be either strict or week, depending on the game.

Pure-strategy Nash equilibria in the Battle of the Sexes game

Player 2

		TF	LA
	TF	2, 1	0, 0
Player 1	LA	0, 0	1, 2

- We immediately see that it has two pure-strategy Nash equilibria

Pure-strategy Nash equilibria in the Battle of the Sexes game

Player 2

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

Player 1

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\text{TF}, \text{TF})$$

$$u_1(\text{TF}, \text{TF}) > u_1(\text{LA}, \text{TF})$$
$$u_2(\text{TF}, \text{TF}) > u_2(\text{TF}, \text{LA})$$

Pure-strategy Nash equilibria in the Battle of the Sexes game

Player 2

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

Player 1

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\mathrm{LA}, \mathrm{LA})$$

$$u_1(\mathrm{LA}, \mathrm{LA}) > u_1(\mathrm{TF}, \mathrm{LA})$$
$$u_2(\mathrm{LA}, \mathrm{LA}) > u_2(\mathrm{LA}, \mathrm{TF})$$

Nash equilibrium examples

How to easily find pure Nash equilibria?

Player 2

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

Player 1

How to easily find pure Nash equilibria?

Player 2

		TF	LA
	TF	2, 1	0, 0
Player 1			
	LA	0, 0	1, 2

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

How to easily find pure Nash equilibria?

Player 2

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

Player 1

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

- **Step 2**: For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs

Nash equilibrium examples

How to easily find pure Nash equilibria?

Player 2

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

Player 1

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

- **Step 2**: For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs

- **Step 3**: If any matrix entry has both an under- and an over line, it is the outcome of a **Nash equilibrium in pure strategies**

How to easily find pure Nash equilibria?

Player 2

	L	C	R
U	7, 0	4, 2	1, 8
M	2, 4	5, 5	2, 3
D	8, 1	3, 2	0, 0

Player 1 M

Find pure Nash equilibria by yourself

How many did you get?

$u_2(a_1, a_2)$

(C, D)

(C, C)

(D, D)

(D, C)

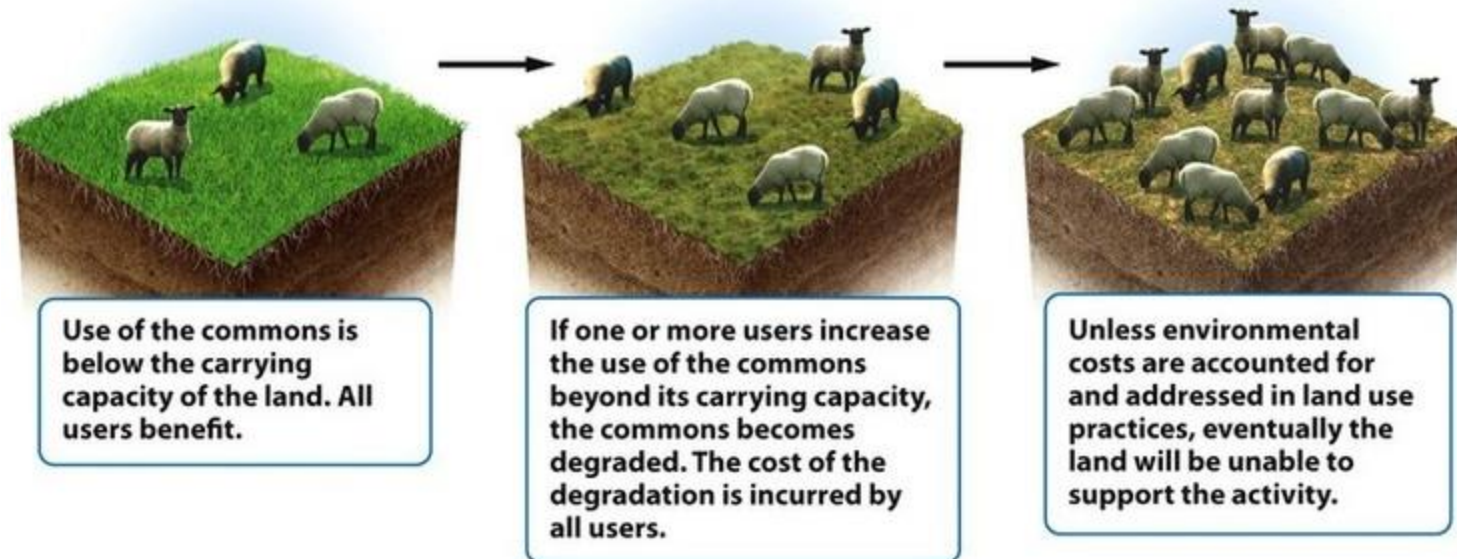$u_1(a_1, a_2)$

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game
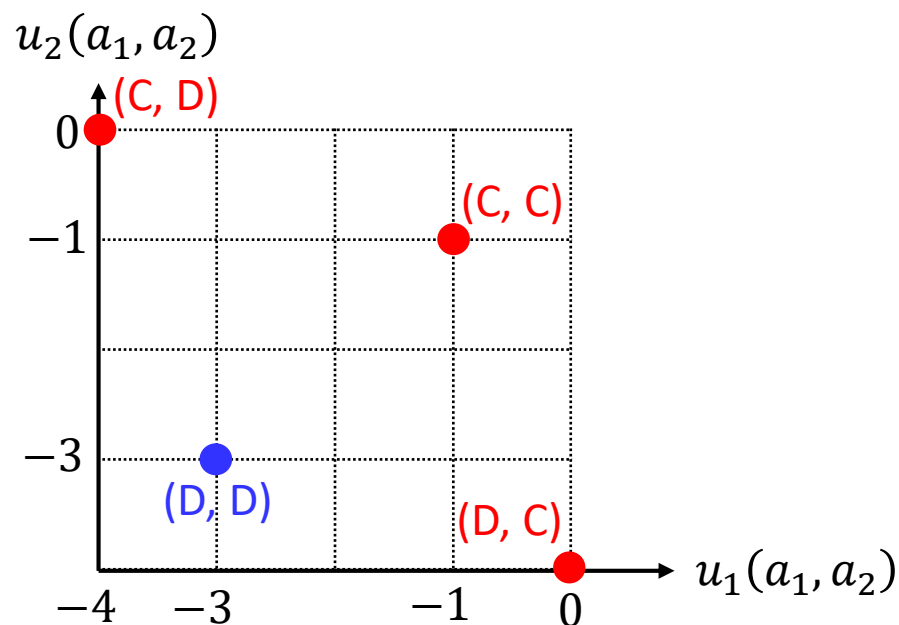
- As seen in Prisoner's Dilemma game, Nash equilibrium does not guarantee Pareto optimality

- People in many situations will do what is best for them, *at the expense of social efficiency*

- The solution concepts took the game as given, and they impose rationality and common knowledge of rationality to try to see what players would choose to do.

- **If each player seeks to maximize their individual well-being then the players may hinder their ability to achieve socially optimal outcomes**

Use of the commons is below the carrying capacity of the land. All users benefit.

If one or more users increase the use of the commons beyond its carrying capacity, the commons becomes degraded. The cost of the degradation is incurred by all users.

Unless environmental costs are accounted for and addressed in land use practices, eventually the land will be unable to support the activity.

- There are n players, say firms, in the world, each choosing how much to produce

- Their production activity in turn consumes some of the clean air that surrounds our planet

- There is a total amount of clean air equal to K, and any consumption of clean air comes out of this common resource

- Each player i chooses his own consumption of clean air for production, k_i

- The clean air left is $K - \sum_{i=1}^{n} k_i$

- The payoff for player i from the choice $k = (k_1, k_2, \ldots, k_n)$ is equal to

$$u_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right)$$

The benefit of consuming individual air consumption

The benefit of consuming the remainder of the clean air

- To solve for a Nash equilibrium, we need to find some profile of choices $k^* = (k_1^*, k_2^*, \dots, k_n^*)$ for which $k_i^* = BR_i(k_{-i}^*)$ for all $i \in N$

- Then we have a system of n equations, on for each player's best-response function, with n unknowns, the choices of each player.

- For example, to get player i's best-response function, the following first-order condition of his payoff function should be satisfied

$$\frac{\partial u_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^{n} k_j} = 0$$
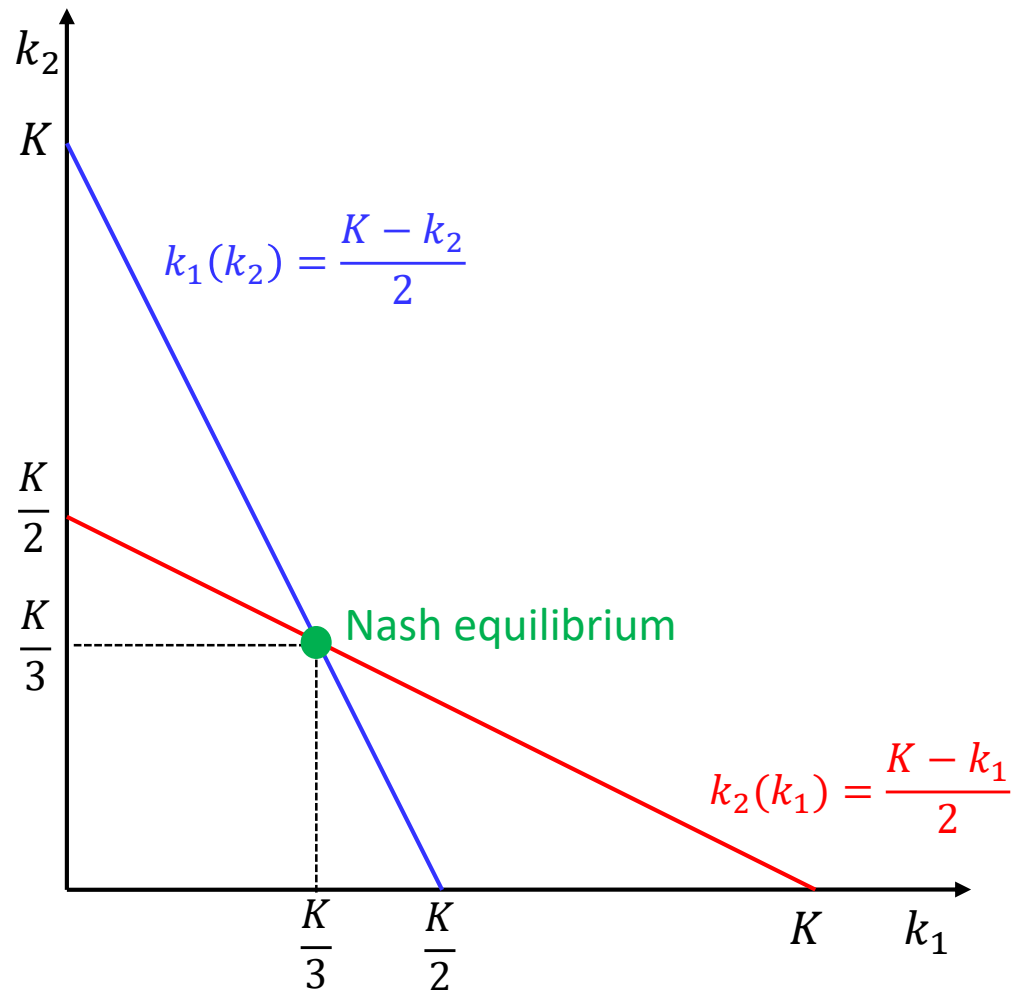
which gives player i's best response function,

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}$$

- In case there are two firms, we have two best-response equations:

$$k_1(k_2) = \frac{K - k_2}{2} \quad \text{and} \quad k_2(k_1) = \frac{K - k_1}{2}$$

Nash equilibrium examples : The Tragedy of the Commons

- Now we can ask whether this two-player behave to make the society better
  - Is consuming $K/3$ for each player too much or too little?
  - Can we find another consumption profile that will make everyone better off?

- We will maximize the sum of all the payoff functions, which we can think of as the "world's payoff function $w(k_1, k_2)$

- We can maximize

$$\max_{k_1, k_2} w(k_1, k_2) = \sum_{i=1}^{2} u_i(k_1, k_2) = \sum_{i=1}^{2} \left\{ \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j \right) \right\}$$

- The first-order conditions of this problem are

$$\frac{\partial w(k_1, k_2)}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0$$

$$\frac{\partial w(k_1, k_2)}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0$$

- The solution for this is $k_1 = k_2 = \frac{K}{4}$, that gives $u_1 = u_2 = \ln\frac{K}{4} + \ln\frac{K}{2} = \ln(\frac{K^2}{8})$
  - which is larger that $u_1 = u_2 = \ln\frac{K}{3} + \ln(\frac{K}{3}) = \ln(\frac{K^2}{9})$ for Nash equilibrium

- Two identical firms, players 1 and 2, produce some good

- Firm i produce quantity q_i

- Cost for production is $c_i(q_i) = c_i q_i$

- Price is given by $d = a - b(q_1 + q_2)$

- The profit of company i given its opponent chooses quantity q_j is

$$u_i(q_i, q_j) = (a - bq_i - bq_j)q_i - c_i q_i = -bq_i^2 + (a - c_i)q_i - bq_j q_i$$

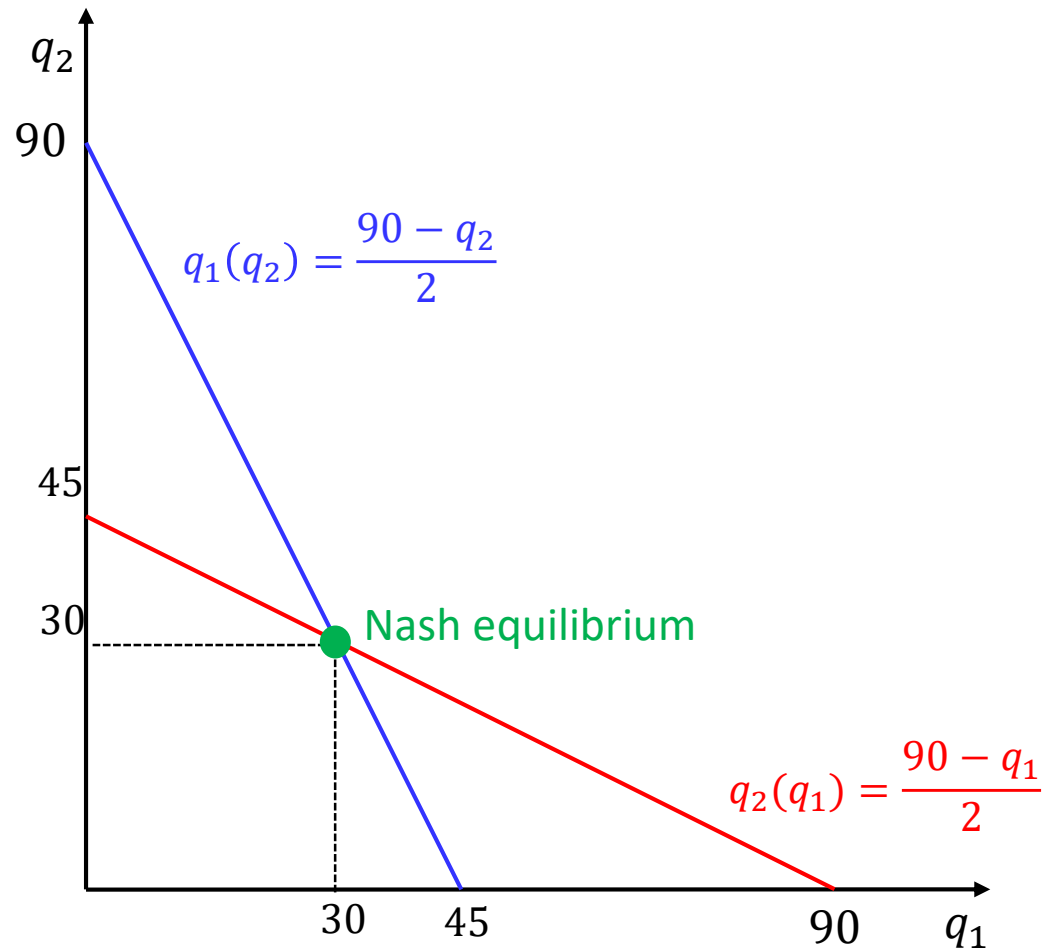- The best-response function for each firm is given by the first-order condition

$$BR_i(q_j) = \frac{a - bq_j - c_i}{2b}$$

Nash equilibrium examples: Cournot Duopoly

- In case there are two firms, we have two best-response equations:

$$q_1 = \frac{a - bq_2 - c_1}{2b} \quad \text{and} \quad q_2 = \frac{a - bq_1 - c_2}{2b}$$

$a = 100, b = 1, c_1 = c_2 = 10$

Mixed strategy Nash equilibrium

- Why anyone would wish to randomize between actions?

- We will see mixed (stochastic) strategies turns out to be an important type of behavior to consider , with interesting implications and interpretations.

- No pure strategy Nash equilibria exists for the following Matching Pennies game

Player 2

	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

Player 1

- Nash equilibrium will indeed exist if we allow players to choose random strategies

Definition (Mixed strategy)

Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then, the set of mixed strategies for player i is $S_i = \Pi(A_i)$

Definition (Mixed strategy profile)

The set of mixed-strategy profile is simply the Cartesian product of the individual mixed-strategy sets, $S = S_1 \times \ldots \times S_n$.

- $s_i(a_j)$ denote the probability that an action a_j will be played under mixed strategy s_i
  - For example, $A = \{$Rook, Paper, Scissors$\}$, $s_i(\text{R}) = 0.2$, $s_i(\text{P}) = 0.3$, $s_i(\text{S}) = 0.5$

Definition (Support)

The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

$A_1 = \{L, R\}$
$S_1 = \Pi(A_1) = \{(s_1(L), s_1(R)): s_1(L), s_1(R) \geq 0, s_1(L) + s_1(R) = 1\}$
$s_1 \in S_1, \text{i.e.,} s_1 = (q, 1 - q)$

Beliefs and mixed strategies

- Introducing probability distributions not only enriches the set of actions from which a player can choose but also allows us to enrich the beliefs that players can have

Definition (Belief)

A belief for player i is given by a probability distribution $\pi_i \in \Pi(A_{-i})$ over the actions of his opponents. We denote by $\pi_i(a_{-i})$ the probability player i assigns to his opponents playing $a_{-i} \in A_{-i}$

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies $s_i \in \Pi(A_i)$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Nash Equilibrium if, for all agents i, s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR(s_{-i}^*)$

- We can think of s_{-i}^* as the belief of player i about his opponents, π_i, which captures the idea that player i is uncertain of his opponent's behavior

  - The profile of mixed strategies s_{-i}^* thus captures this uncertain belief over all of the pure strategies that player i's opponent can play

  - Rationality requires that a player play a best response given his belief (Nash equilibrium requires that these beliefs are correct, i.e., a system of equations should be satisfied)

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

- Imagine that the Nash equilibrium profile s_i^* contains more that one pure strategy -say a_i and a_i' as supports.

- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*)?

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

- Imagine that the Nash equilibrium profile s_i^* contains more that one pure strategy -say a_i and a_i' as supports.

- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*)?

if s_i^* is a Nash equilibrium, and if a_i and a_i' are in the support of s_i^*, then

$$u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

Proof:
- assume $u_i(a_i, s_{-i}^*) > u_i(a_i', s_{-i}^*)$ and a_i and a_i' are support of s_i^*
- Adjusting the mixed strategy $s_i = \{s_i(a_i), s_i(a_i')\} \to \{s_i(a_i) + s_i(a_i'), 0\}$ will increase u_i
- s_i^* could not have been a best response to s_{-i}^*
- Therefore, by contradiction, $u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*)$

if s_i^* is a Nash equilibrium, and both if a_i and if a_i' are in the support of s_i^*, then

$$u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$
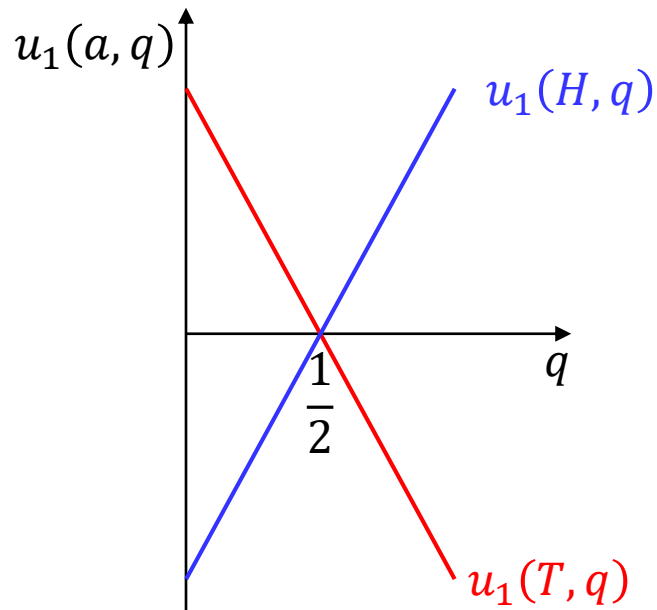
- This result will play an important role in computing mixed-strategy Nash equilibria

  ➤ If a player is playing a mixed strategy then he must be indifferent between the actions he is choosing with positive probability (i.e., actions in the support)

- One player's indifference will impose restrictions on the behavior or other players

  ➤ This restriction will help us find the mixed-strategy Nash equilibrium

Finding mixed Nash equilibria : Matching Pennies

	q H	$1-q$ T
p H	$1, -1$	$-1, 1$
$(1-p)$ T	$-1, 1$	$1, -1$

$u_1(H, q) = q \times 1 + (1-q) \times (-1) = 2q - 1$
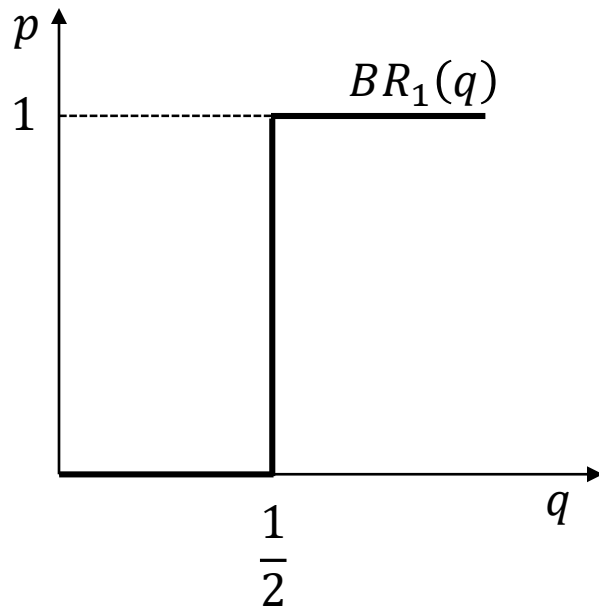$u_1(T, q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$

$$BR_1(q) = \begin{cases} p = 0 & if \ q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \ q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \ q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

		q H	$1-q$ T
p	H	$1, -1$	$-1, 1$
$(1-p)$	T	$-1, 1$	$1, -1$

$u_1(H, q) = q \times 1 + (1-q) \times (-1) = 2q - 1$
$u_1(T, q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$

$$BR_1(q) = \begin{cases} p = 0 & if \ q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \ q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \ q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

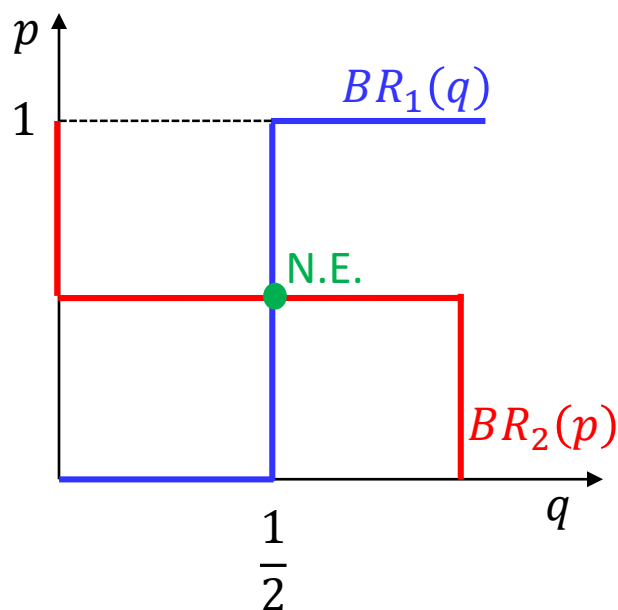## Finding mixed Nash equilibria : Matching Pennies

	q H	$1-q$ T
p **H**	$1,-1$	$-1,1$
$(1-p)$ **T**	$-1,1$	$1,-1$

$$u_1(H,q) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$u_1(T,q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$$

$$u_2(H,p) = p \times (-1) + (1-p) \times 1 = 1 - 2p$$
$$u_2(T,p) = p \times 1 + (1-p) \times (-1) = 2p - 1$$

$$BR_1(q) = \begin{cases} p = 0 & if \ q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \ q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \ q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & if \ p < 1/2 & \text{(Playing } H\text{)} \\ q \in [0,1] & if \ p = 1/2 & \text{(Indifferent)} \\ q = 0 & if \ p > 1/2 & \text{(Playing } T\text{)} \end{cases}$$

The intersections of two best response curve→ Nash equilibria
To find Nash equilibrium, make other player indifferent between some of his pure actions

Mixed-strategy Nash equilibria in the **Matching Pennies game**

Player 2

	p Heads	$1-p$ Tails
q Heads	$1, -1$	$-1, 1$
$1-q$ Tails	$-1, 1$	$1, -1$

Player 1

No pure strategy Nash equilibria exists

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions → otherwise, player 2 would play the action that is superior than the other

- Let us assume Player 1' strategy is to play Heads with probability q and Tails with $1 - q$

$$u_2(\text{Heads}) = u_2(\text{Tails})$$
$$-1 \times q + 1 \times (1 - q) = 1 \times q - 1 \times (1 - q)$$
$$q = \frac{1}{2}$$

Mixed-strategy Nash equilibria in the **Matching Pennies game**

Player 2

	p Heads	$1-p$ Tails
q Heads	$1, -1$	$-1, 1$
$1-q$ Tails	$-1, 1$	$1, -1$

Player 1

No pure strategy Nash equilibria exists

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions → otherwise, player 1 would play the action that is superior than the other

- Let us assume Player 2' strategy is to play Heads with probability p and Tails with $1-p$

$$u_1(\text{Heads}) = u_1(\text{Tails})$$
$$1 \times p - 1 \times (1-p) = -1 \times p + 1 \times (1-p)$$
$$p = \frac{1}{2}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

Player 2

	p TF	$1-p$ LA
q TF	2, 1	0, 0
$1-q$ LA	0, 0	1, 2

Player 1

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions → otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play TF with probability q and LA with $1-q$

$$u_2(\text{TF}) = u_2(\text{LA})$$
$$1 \times q + 0 \times (1-q) = 0 \times q + 2 \times (1-q)$$
$$q = \frac{2}{3}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

Player 2

	p TF	$1 - p$ LA
q TF	2, 1	0, 0
$1 - q$ LA	0, 0	1, 2

Player 1

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions → otherwise, player 1 would play the action that is superior than the other

- Let us assume Player 2' strategy is to play TF with probability p and LA with $1 - p$

$$u_1(\text{LA}) = u_1(\text{TF})$$
$$2 \times p + 0 \times (1 - p) = 0 \times p + 1 \times (1 - p)$$
$$p = \frac{1}{3}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

Player 2

		1/3 TF	2/3 LA
Player 1	2/3 TF	2, 1	0, 0
	1/3 LA	0, 0	1, 2

- Now, we can confirm that we have indeed found an equilibrium:
  - Both players play in a way that makes the other indifferent, they are both best responding to each other
- Expected payoff for both agents is 2/3 in this equilibrium
  - Each of the pure-strategy equilibria Pareto-dominates the mixed strategy equilibrium
- **This mixed strategy, as all other mixed strategies, is a week Nash equilibrium**

$$u_1(s_i^*, s_{-i}^*) \geq u_1(s_i, s_{-i}^*) \qquad u_1\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right) \geq u_1\left((x, 1-x), \left(\frac{1}{3}, \frac{2}{3}\right)\right) \text{ For any } 0 \leq x \leq 1$$

Mixed-strategy Nash equilibria in the Rock-Paper-Scissor game

	p_R Rook	p_P Paper	p_S Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

$u_1(\text{R}) = u_1(\text{P}) = u_1(\text{S})$
$\Rightarrow 0p_R + (-1)p_P + 1p_S = 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_S$
$\Rightarrow 0p_R + (-1)p_P + 1p_S = 1p_R + 0p_P + (-1)p_S \Rightarrow 2p_S = p_R + p_P$
$\Rightarrow 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_S \quad \Rightarrow 2p_R = p_S + p_P$
$\Rightarrow p_R = p_P = p_S \quad (1)$

$p_R + p_P + p_S = 1 \qquad (2)$

- Due to (1) and (2), $p_R = p_P = p_S = 1/3$ (Mixed strategy Nash Equilibrium)
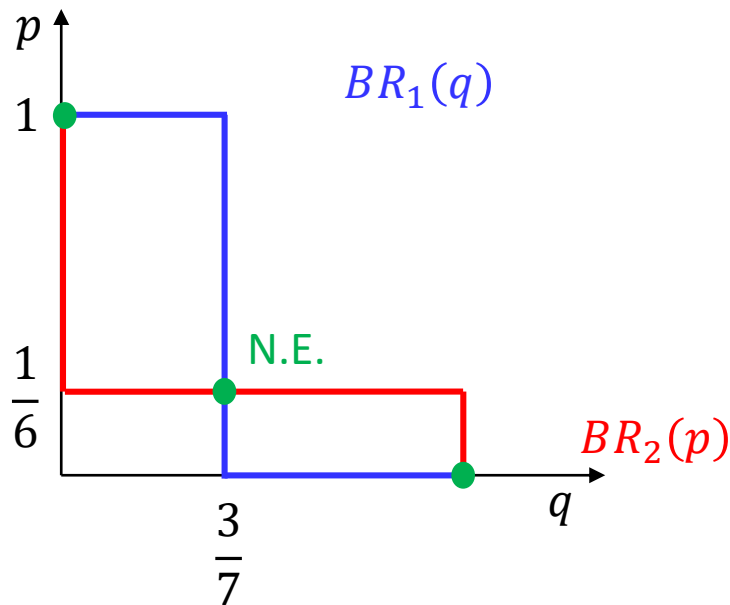
Multiple mixed strategies

		q H	$1-q$ T
p	**H**	$0, 0$	$3, 5$
$(1-p)$	**T**	$4, 4$	$0, 3$

$u_1(H, q) = q \times 0 + (1-q) \times (3) = 3q - 3$
$u_1(T, q) = q \times (4) + (1-q) \times 0 = 4q$

$u_2(H, p) = p \times (0) + (1-p) \times 4 = 4 - 4p$
$u_2(T, p) = p \times 5 + (1-p) \times (3) = 2p + 3$

Nash equilibriums are $\left\{ (1,0), \left(\frac{1}{6}, \frac{3}{7} \right), (0,1) \right\}$

The meaning of playing mixed-strategy

- Randomize to confuse your opponent
  - consider the matching pennies example

- Randomize when uncertain about the other's action
  - consider battle of the sexes

- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit

- Mixed strategies describe population dynamics:
  - agents chosen from a population have deterministic strategies.
  - Mixed strategies gives the probability of getting each pure strategies.

The existence of Nash equilibria

Theorem (Nash, 1951)

Every game with a finite number of players and action profiles has at least one Nash equilibrium