2018 Spring: IE481A & IE801

Game Theory with Engineering Applications

Midterm Examination

Course Name/Number: ___________________________________

Student Name: _____________________________ Student ID: __________________

By signing below, I affirm that I have not violated any part of the Code of Academic Conduct on this exam, and that this work is honest representation of my learning from the class.

Signature_________________________________ Date _________________________

Problem	Points	Full score
1. Normal form game		20
2. Nash Equilibrium for infinite game		20
3. Extensive Form Game with Imperfect Information Game		15
4. First-Price, Sealed-bid Auction		15
5. Infinitely repeated game		15
6. Bayesian Game		15
Total		100

[20 points] Problem 1 (Normal Form Game)

Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Every quarter that both firms operate yields each a loss equal to −1, and each quarter that a firm operates alone yields a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are(−1,1) because both firms lose −1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) [5 points] Write down this game in matrix form

Answer : Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter.

		Player 2 E	T	N
Player 1	E	0,0	0,2	0,4
	T	2,0	-1,-1	-1,1
	N	4,0	1,-1	-2,-2

(b) [5 points] Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer : There are no strictly dominated strategies but there is a weakly dominated one: T. To see this note that choosing both E and N with probability $\frac{1}{2}$ each yields the same expected payoff as choosing T against E or N, and a higher expected payoff against T, and hence $\sigma_i = \left(\sigma_i(E), \sigma_i(T), \sigma_i(N)\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ weakly dominates T. The reason there is no strictly dominated strategy is that, starting with σ_i increasing the weight on E causes the mixed strategy to be worse than T against E, while increasing the weight on N causes the mixed strategy to be worse than T against N, implying it is impossible to find a mixed strategy that strictly dominates T.

(c) [5 points] Find the pure strategy Nash equilibria

Answer : Because T is weakly dominated, it is suspect of never being a best response. A quick observation should convince you that this is indeed the case: it is never a best response to any of the pure strategies, and hence cannot be part of a pure strategy Nash equilibrium. Removing T from consideration results in the reduced game:

		Player 2	
		E	N
Player 1	E	0,0	0,4
	N	4,0	-2,-2

for which there are two pure strategy Nash equilibria, (E,N) and (N,E).

(d) [5 points] Find the unique mixed strategy Nash equilibrium

Answer : We start by ignoring T and using the reduced game in part (c) by assuming that the weakly dominated strategy T will never be part of a Nash equilibrium. We need to find a pair of mixed strategies, $\left(\sigma_1(E), \sigma_1(N)\right)$ and $\left(\sigma_2(E), \sigma_2(N)\right)$ that make both players indifferent between E and N. For player 1 the indifference equation is,

$$0 = 4\sigma_2(E) - 2\left(1 - \sigma_2(E)\right)$$

which results in $\sigma_2(E) = \frac{1}{3}$, and for player 2 the indifference equation is symmetric, resulting in $\sigma_1(E) = \frac{1}{3}$.

Hence, the mixed strategy Nash equilibrium of the original game is $\left(\sigma_i(E), \sigma_i(T), \sigma_i(N)\right) = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$. Notice that at this Nash equilibrium, each player is not only indifferent between E and N, but choosing T gives the same expected payoff of zero. However, choosing T with positive probability cannot be part of a mixed strategy Nash equilibrium.

To prove this let player 2 play the mixed strategy $\sigma_2 = \left(\sigma_2(E), \sigma_2(T), \sigma_2(N)\right) = (\sigma_{2E}, \sigma_{2T}, 1 - \sigma_{2E} - \sigma_{2T})$. The strategy T for player 1 is at least as good as E if and only if,

$$0 \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2E} \geq \frac{1}{3}$.

The strategy T for player 1 is at least as good as N if and only if,

$$4\sigma_{2E} - \sigma_{2T} - 2(1 - \sigma_{2E} - \sigma_{2T}) \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2T} \leq 1 - 3\sigma_{2E}$ reduces to $\sigma_{2T} \leq 0$, which can only hold when $\sigma_{2E} = 1/3$ and $\sigma_{2T} = 0$ (which is the Nash equilibrium we found above).

A symmetric argument holds to conclude that $\left(\sigma_i(E), \sigma_i(T), \sigma_i(N)\right) = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$ is the unique mixed strategy Nash equilibrium.

[20 points] Problem 2 (Nash Equilibrium for Infinite Game)

Two candidates are competing in a political race. Each candidate i can spend $s_i \geq 0$ on ads that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability 1/2. Each candidate values wining at a payoff of $v > 0$, and the cost of spending s_i is just s_i.

(a) [4 points] Given two spend levels (s_1, s_2), write the expected payoff of a candidate i.

Answer : Player i's payoff function is

$$v_i(s_1, s_2) = \frac{s_i v}{s_1 + s_2} - s_i$$

(b) [4 points] What is the function that represents each player's best response function?

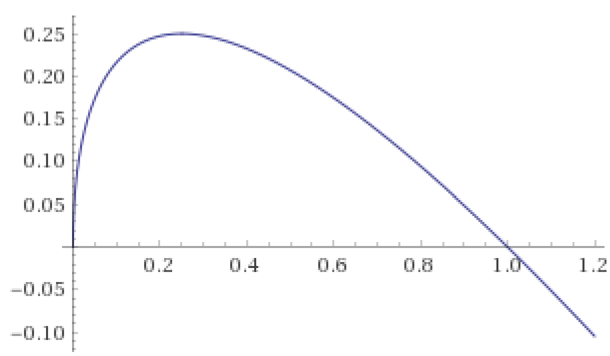
Answer : Player 1 maximizes his payoff $v_1(s_1, s_2)$ shown in (a) above and the first order optimality condition is,

$$\frac{v(s_1 + s_2) - s_1 v}{(s_1 + s_2)^2} - 1 = 0$$

And if we use $s_1(s_2)$ to denote player 1's best response function then it explicitly solves the following equality that is derived from the first-order condition,

$$[s_1(s_2)]^2 + 2s_1(s_2)s_2 + (s_2)^2 - vs_2 = 0$$

Because this is a quadratic equation we cannot write an explicit best response function (or correspondence). However, if we can graph $s_1(s_2)$ as shown in the following figure (the values correspond for the case of $v = 1$).



Similarly, we can derive the symmetric function for player 2.

(c) [4 points] Find the unique Nash equilibrium

Answer : The best response functions are symmetric mirror images and have a symmetric solution where $s_1 = s_2$ in the unique Nash equilibrium. We can therefore use any one of the two best reponse functions and replace both variables with a single variable s,

$$s^2 + 2s^2 + s^2 - vs = 0 \rightarrow s = \frac{v}{4}$$

so that the unique Nash equilibrium has $s_1^* = s_2^* = \frac{v}{4}$.

(d) [4 points] What happens to the Nash equilibrium spending levels if v increases?

Answer : It is easy to see from part (c) that higher values of v cause the players to spend more in equilibrium. As the stakes of the prize rise, it is more valuable to fight over it.

(e) [4 points] What happens to the Nash equilibrium levels if player 1 still values winning at v, but player 2 values winning at kv where $k > 1$?

Answer : Now the two best response functions are not symmetric. The best response function of player 1 remains as above, but that of player 2 will now have kv instead of v,

$$s_1^2 + 2s_1s_2 + s_2^2 - vs_2 = 0 \;\; \text{(BR1)}$$

and

$$s_2^2 + 2s_1s_2 + s_1^2 - kvs_1 = 0 \;\; \text{(BR2)}$$

Subtracting (BR2) from (BR1) we obtain,

$$ks_1 = s_2,$$

which implies that the solution will no longer be symmetric and, more over, $s_2 > s_1$, which is intuitive because now player 2 cares more about the prize. Using $ks_1 = s_2$ we substitute for s_2 in (BR1) to obtain,

$$s_1^2 + 2k(s_1)^2 + k^2(s_1)^2 - kvs_1 = 0$$

which results in,

$$s_1 = \frac{kv}{1 + 2k + k^2} < \frac{v}{1 + 2k + k^2} < \frac{v}{4}$$

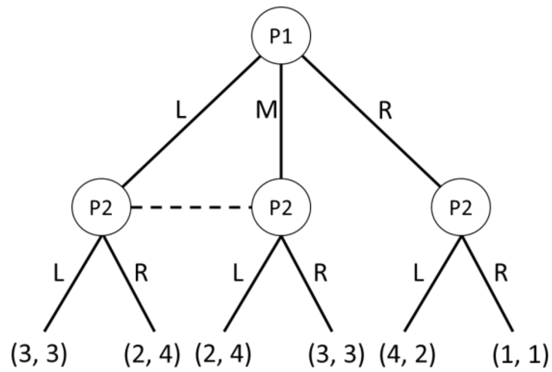
where both inequalities follow from the fact that $k > 1$. From $ks_1 = s_2$ above we have

$$s_2 = \frac{k^2v}{1 + 2k + k^2} < \frac{k^2v}{k^2 + 2k^2 + k^2} < \frac{v}{4}$$

where the inequality follows from $k > 1$.

[15 points] Problem 3 (Extensive Form Game with Imperfect Information Game)

Consider the following extensive-form game:



(a) [4 points] Give the normal form representation of this game
Answer :

	Left / Left	Left / Right	Right / Left	Right / Right
Left	3,3	3,3	2,4	2,4
Middle	2,4	2,4	3,3	3,3
Right	4,2	1,1	4,2	1,1

(b) [4 points] Give a Nash equilibrium where P1 sometimes plays Left?

Answer : P1 mixes 50/50 between Left and Middle, and P2 mixes 50/50 between Left/Right and Right/Right.

(c) [4 points] What are the subgame-perfect equilibria in this game?

Answer : P1 plays Right and then P2 plays Left. P2 can play anything in response to P1 playing Left or Middle.

(d) [3 points] Can P2 improve his expected payoff in the subgame-perfect equilibria of this game by eliminating one of his actions in an information set? If so, which action?

Answer : Yes, by eliminating the Left action in the information set where P1 goes Right. Then P1 will always prefer going either Left or Middle, resulting in a payoff of at least 3 for P2 compared to just 2 before.

[15 points] Problem 4 (First-Price, Sealed-bid Auction)
Consider the following first-price, sealed-bid auction where an indivisible good is sold. There are $n \geq 2$ buyers indexed by $i = 1,2,\dots,n$. Simultaneously, each buyer i submits a bid $b_i > 0$. The agent who submits the highest bid winds. If there are $k > 1$ players submitting the highest bid, then the winner is determined randomly among these players-each has probability $1/k$ of winning. The winner i gets the object and pays his bid b_i, obtaining payoff $v_i - b_i$, while the other buyers get 0, where $v_1,\dots,v_n$ are independently and identically distributed with probability density function f where

$$f(\mathrm{x}) = \begin{cases} 3x^2 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) [10 points] Compute the symmetric, linear Bayesian Nash equilibrium. That is, find the equilibrium of the form (compute a and c)

$$b_i = a + cv_i$$

where $c > 0$.

Answer : We look for an equilibrium of the form

$$b_i = a + cv_i$$

where $c > 0$. Then, the expected payoff from bidding b_i with type v_i is

$$U(b_i;v_i) = (v_i - b_i)\,Pr\big(b_i > a + cv_j, \qquad \forall j \neq i\big)$$
$$= (v_i - b_i)\prod_{j\neq i} Pr(\,b_i > a + cv_j)$$
$$= (v_i - b_i)\prod_{j\neq i} Pr(\,v_j < \frac{b_i - a}{c})$$
$$= \ (v_i - b_i)\prod_{j\neq i}\left(\frac{b_i - a}{c}\right)^3$$
$$= \ (v_i - b_i)\left(\frac{b_i - a}{c}\right)^{3(n-1)}$$

for $b_i \in [a, a+c]$. The first order condition is

$$\frac{\partial U(b_i;v_i)}{\partial b_i} = -\left(\frac{b_i - a}{c}\right)^{3(n-1)} + 3(n-1)\frac{1}{c}(v_i - b_i)\left(\frac{b_i - a}{c}\right)^{3(n-1)-1} = 0$$

i.e.,

$$b_i = \frac{a + 3(n-1)v_i}{3(n-1)+1}$$

Since this is an identity, we must have

$$a = \frac{a}{3(n-1)+1} \rightarrow a = 0$$

and

$$c = \frac{3(n-1)}{3(n-1)+1}$$
$$b_i = \frac{3(n-1)v_i}{3(n-1)+1}$$

(b) [5 points] What happens as $n \to \infty$?

Answer : As $n \to \infty$,

$$b_i \to v_i$$

In the limit, each bidder bids his valuation, and the seller extracts all the gains from trade.

Hint: Since $v_1, \ldots, v_n$ is independently distributed, for any $\mathrm{w}_1, \ldots, \mathrm{w}_\mathrm{k}$, we have

$$Pr(v_1 \leq w_1, \ldots, v_k \leq w_k) = Pr(v_1 \leq w_1) \ldots Pr(v_k \leq w_k)$$

[15 points] Problem 5 (Infinitely repeated game)

Consider the infinitely repeated version of the following game:

	H	D
H	1, 1	3, 0
D	0, 3	2, 2

The payoff of player i to any infinite sequence of payoffs $\{u_{it}\}$ is given by the normalized discounted sum of payoffs:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_{it}$$

where $0 < \delta < 1$

(a) [10 points] For what values of δ, if any, does it constitute a subgame perfect equilibrium when both players choose this strategy?

- Choose D in period 1
- Choose D after any history in which both players have always played D.
- Choose H after any other history

Answer :

The behavior of a player who uses this strategy depends only on whether the history is all (D,D) or not. Let's consider each type of history, and check if the strategy satisfies the one-shot deviation property. We'll consider only Player 1's problem, since the game is symmetric.

(a) Suppose the history is all (D,D). If Player 1 does not deviate, the sequence of outcomes will be (D,D), (D,D), … with payoffs 2, 2, … with discounted average of 2.

If Player 1 does a one-shot deviation, the sequence of outcomes will be (H,D), (H,H), (H,H), … with payoffs 3, 1, 1, … with discounted average $3(1-\delta)+\delta$. Not deviating is optimal if

$$2 \geq 3(1-\delta)+\delta \rightarrow \ \delta \geq 1/2$$

(b) Suppose the history is *not* all (D,D). If Player 1 does not deviate, the sequence of outcomes will be (H,H), (H, H), … with payoffs 1, 1, … with discounted average of 1.

If Player 1 does a one-shot deviation, the sequence of outcomes will be (D,H), (H,H), (H,H), … with payoffs 0, 1, 1, … with discounted average δ. This is always less than 1, therefore it is always optimal to not deviate.

Combining both conditions, this strategy can support a SPNE when $\delta \geq 1/2$.

(b) [5 points] Suppose the game is modified to have the following payoffs:

	H	D
H	0, 0	3, 1
D	1, 3	2, 2

For what values of δ, if any, does it constitute a subgame perfect equilibrium when both players choose the strategy in part (a)?

Answer :

(a) Suppose the history is all (D,D). If Player 1 does not deviate, the sequence of outcomes will be (D,D), (D,D), … with payoffs 2, 2, … with discounted average of 2.

If Player 1 does a one-shot deviation, the sequence of outcomes will be (H,D), (H,H), (H,H), … with payoffs 3, 0, 0, … with discounted average $3(1-\delta)$. Not deviating is optimal if

$$2 \geq 3(1-\delta) \rightarrow \ \delta \geq 1/3$$

(b) Suppose the history is *not* all (D,D). If Player 1 does not deviate, the sequence of outcomes will be (H,H), (H, H), … with payoffs 0, 0, … with discounted average of 0.

If Player 1 does a one-shot deviation, the sequence of outcomes will be (D,H), (H,H), (H,H), … with payoffs 1, 0, 0, … with discounted average $1-\delta$. This is always greater than 0, therefore it is never optimal to not deviate.

Combining both conditions, this strategy can support a SPNE for any $0 < \delta < 1$.

[15 points] Problem 6 (Bayesian Game: M&A game)

Firm A (the "acquirer") is considering taking over firm T (the "target"). It does not know firm T's value; it believes that this value, when firm T is controlled by its own management, is at least \$0 and at most \$100, and assigns equal probability to each of the 101 dollar values (0\$, 1\$, 2\$, …,100\$) in this range. Firm T will be worth 50% more under firm A's management than it is under its own management. Suppose that firm A bids y to take over firm T, and firm T is worth x (under its own management). Then if T accepts A's offer, A's payoff is $\frac{3}{2}x - y$ and T's payoff is y; if T rejects A's offer, A's payoff is 0 and T's payoff is x.

(a) [5 points] Model this situation as a Bayesian game in which firm A chooses how much to offer and firm T decides the lowest offer (i.e., above this value, T will accept) to accept.

Players : Firms A and T

States : The set of possible values of firm T (the integers from 0 to 100)

Actions : Firm A's set of actions is its set of possible bids (nonnegative numbers), and firm T's set of actions is the set of possible cutoffs (nonnegative numbers) above which it will accept A's offer.

Signals : Firm A receives the same signal in every state; firm T receives a different signal in every state.

Beliefs : The single type of firm A assigns an equal probability to each state; each type of firm T assigns probability 1 to the single state consistent with its signal.

Payoff functions : If firm A bids y, firm T's cutoff is at most y, and the state is x, then A's payoff is $\frac{3}{2}x - y$ and T's payoff is y. If firm A bids y, firm T's cutoff is greater than y, and the state is x, then A's payoff is 0 and T's payoff is x.

(b) [5 points] Find the Nash equilibrium (equilibria?) of this game.

To find the Nash equilibria of this game, first consider the behavior of each type of firm T. Type x is at least as well off accepting the offer y than it is rejecting it if and only if $y \geq x$. Thus type x's optimal cutoff for accepting offers is x, regardless of firm A's action.

Now consider firm A. If it bids y then each type x of T with $x < y$ accepts its offer, and each type x of T with $x > y$ rejects the offer. Thus the expected value of the type that accepts an offer $y \leq 100$ is $\frac{1}{2}y$, and the expected value of the type that accepts an offer $y > 100$ is 50. If the offer y is accepted then A's payoff is $\frac{3}{2}x - y$, so that its expected payoff is $\frac{3}{2}\left(\frac{1}{2}y\right) - y = -\frac{1}{4}y$ if $y \leq 100$ and $\frac{3}{2}(50) - y = 75 - y$ if $y > 100$. Thus, firm A's optimal bid is 0!

We conclude that the game has a unique Nash equilibrium, in which firm A bids 0 and the cutoff for accepting an offer for each type x of firm T is x.

(c) [5 points] Explain why the logic behind the equilibrium is called adverse selection.

Even though firm A can increase firm T's value, it is not willing to make a positive bid in equilibrium because firm T's interest is in accepting only offers that exceed its value, so that the average type that accepts an offer has a value of only half the offer. As A decreases its offer, the value of the average firm that accepts the offer decreases: the selection of firms that accept the offer is adverse to A's interest.