OPTIMAL SIZING OF CAPACITORS PLACED ON A RADIAL DISTRIBUTION SYSTEM

Mesut E. Baran *Felix F. Wu*

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley
Berkeley, CA 94720

Abstract - The capacitor sizing problem is a special case of the general capacitor placement problem. The problem is to determine the optimal size of capacitors placed on the nodes of a radial distribution system so that the real power losses will be minimized for a given load profile. This problem is formulated as a nonlinear programming problem. The ac power flow model of the system, constraints on the node voltage magnitudes, and the cost of capacitors are explicitly incorporated in the formulation. A solution algorithm, based on a Phase I - Phase II feasible directions approach, is proposed. Also presented are a new formulation of the power flow equations in a radial distribution network and a numerically robust, computationally efficient solution scheme. This solution scheme is used as a subroutine in the optimization algorithm. Test results for both the capacitor sizing problem and the power flow solution schemes are presented.

I. INTRODUCTION

Capacitors are widely used in distribution systems for reactive power compensation to achieve power and energy loss reduction, voltage regulation, and system capacity release. The extent of these benefits will depend upon how the capacitors are placed on the system, i.e., (ii) the number and location, (i) the type (fixed or switched), (iii) the size, and (iv) the control scheme of the capacitors.

Therefore, the capacitor placement problem in distribution feeders consists of determining the place (number and location), type, size, and control scheme of capacitors to be installed such that the benefits mentioned above be weighted against the fixed and running costs of the capacitors and their accessories.

The early analytical methods for capacitor placement are mainly developed by Neagle and Samson [1]. The problem is defined as determining the location and size of a given number of fixed type capacitors to minimize the power loss for a given load level. The results, the simplest of which is the celebrated 2/3 rule, were based on a simplified, voltage independent system model obtained by: (i) considering only the main feeder with uniform load distribution and wire size, (ii) taking into account only the loss reduction due to the change in the reactive component of the branch currents, (iii) ignoring the changes in the node voltages, and (iv) neglecting the cost of capacitors. Later, a more comprehensive derivation of these rules was given by Bae, [5]. Fawzi et. al. [11] incorporated the released substation kVA and the voltage raise at light load conditions into the model.

These analytical methods are also integrated with some heuristics to obtain computer oriented solution methodologies and thus to overcome some of their shortcomings. Cook [2] extended the formulation of the problem to include peak power and energy loss reduction and proposed such a solution method to determine the number as well as location and the size of capacitors. Chang [4] considered voltage limits in his solution scheme.

Duran [3], by exploiting the discrete nature of the capacitor sizes, used a more realistic model for the feeder with many sections of different wire sizes and concentrated loads and proposed a dynamic programming solution method. Ponnavaiko and Prakaso [12] considered load growth as well as system capacity release and voltage raise at light load conditions and used a local optimization technique called the method of local variables by treating capacitor sizes as discrete variables.

Recently, interests in Distribution Automation and Control (DAC) have generated quite a lot of new studies in capacitor placement problem. Grainger and Lee pioneered in reformulating the problem as a nonlinear programming problem and in developing computationally efficient solution methods. In [6], they generalized Cook's formulation for a radial feeder, regarded the capacitor sizes as continuous variables, and developed a computationally simple, iterative solution scheme. They later introduced switched type capacitors with simultaneous switching [7] and a voltage dependent model for loss reduction [8]. In [9], together with El-Kib, they proposed a solution method to determine the optimal design and real-time control scheme for switched capacitors with non-simultaneous switching under certain assumptions. Grainger and Civanlar considered the optimal design and control scheme for continuous capacitive compensation case [10] and introduced a voltage dependent solution scheme [14]. El-Kib et. al., [16] extended the methodologies developed in [6]-[9] to encompass unbalanced three-phase feeders. Kaplan [13] extended the formulation for multi lateral feeders and proposed the use of heuristics for the solution. Grainger and Civanlar [15] combined capacitor placement and voltage regulator problems for a general distribution system and proposed a decoupled solution methodology.

The basic idea in our formulation of the general problem is similar to that of Grainger et. al., i.e., formulating it as a nonlinear programming problem. We will consider a capacitor placement problem that (i) incorporates directly the ac power flows as system model, (i) enforces the voltage constraints. We develop a solution methodology by using *decomposition techniques*. At the lowest level of the decomposition, we have special capacitor placement problems, called the ***base problems***. A base problem is a sizing type problem which is used to find the optimum sizes of a set of selected capacitors placed on the nodes of the distribution system and therefore, it is of practical interest in its own right. The solution algorithm for the sizing problem is a special Phase I - Phase II Feasible Directions method which takes into account the structure of the problem. We present the formulation of the sizing problem and the solution algorithm developed for it in this paper. The solution of the general problem is presented in another paper [17].

Also in this paper, a new formulation of the ac power flow equations for the radial distribution systems is introduced and a computationally efficient and numerically robust solution methodology, called DistFlow, is presented. Such a solution methodology is needed because of its repeated use in the optimization algorithm.

In section 2, the new power flow equations for radial distribution systems are presented. The sizing problem is stated in section 3. Section 4 describes the general approach for the solution. The structure of the problem is studied in section 5 and the proposed solution algorithm is presented in section 6. Section 7 contains the test runs and conclusions are given in section 8.

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II. DISTRIBUTION SYSTEM POWER FLOW

Power flows in a distribution system obey physical laws (Kirchoff laws and Ohm law), which become part of the constraints in the capacitor placement problem. In our proposed solution algorithm for the capacitor placement problem, the distribution system power flow solution is to be used as a subroutine in every iteration. Therefore, it is essential to have a computationally efficient and numerically robust method for solving the distribution system power flow.

In this section, we present the new power flow equations for radial distribution systems. The formulation is conducive to efficient solution methods. For pedagogic convenience, we first consider a special case where there is only one main feeder. The general case for any radial distribution system is considered next. To simplify the presentation, the system is assumed to be balanced 3-phase system.

Special Case : Radial Main Feeder

Consider a distribution system consists of a radial main feeder only. The one-line diagram of such a feeder comprising n branches / nodes is shown in Fig.1.

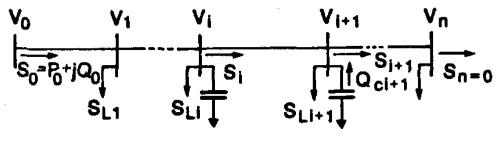

Fig.1 : One line diagram of a main distribution feeder

In the figure, V_0 represents the substation bus voltage magnitude and is assumed to be constant. Lines are represented by a series impedance $z_l = r_l + jx_l$ and loads are treated as constant power sinks, $S_L = P_L + jQ_L$. Shunt capacitors to be placed at the nodes of the system will be represented as reactive power injections.

With this representation, the network becomes a ladder network with nonlinear shunt loads. If the power supplied from the substation, $S_0 = P_0 + jQ_0$ is known, then the power and the voltage at the receiving end of the first branch can be calculated as follows.

$$S_1 = S_0 - S_{loss\,1} - S_{L1} = S_0 - z_1|S_0|^2/V_0^2 - S_{L1}$$

$$V_1\angle\theta_1 = V_0 - z_1 I_0 = V_0 - z_1 S_0^*/V_0$$

Repeating the same process yields the following recursive formula for each branch on the feeder.

$$P_{i+1} = P_i - r_{i+1}(P_i^2+Q_i^2)/V_i^2 - P_{Li+1} \quad (1.i)$$

$$Q_{i+1} = Q_i - x_{i+1}(P_i^2+Q_i^2)/V_i^2 - Q_{Li+1} + Q_{ci+1} \quad (1.ii)$$

$$V_{i+1}^2 = V_i^2 - 2(r_{i+1}P_i+x_{i+1}Q_i) + (r_{i+1}^2+x_{i+1}^2)(P_i^2+Q_i^2)/V_i^2 \quad (1.iii)$$

Where,

P_i, Q_i : real and reactive power flows into the sending end of branch $i+1$ connecting node i and node $i+1$,

V_i : bus voltage magnitude at node i,

Q_{ci} : reactive power injection from capacitor at node i.

Eq.(1), called the *branch flow equation*, has the following form

$$\mathbf{x}_{0i+1} = \mathbf{f}_{0i+1}(\mathbf{x}_{0i}, u_{i+1}) \quad (2)$$

Where, $\mathbf{x}_{0i} = [P_i\ Q_i\ V_i^2]^T$ and $u_{i+1} = Q_{ci+1}$.

Note that if there is no capacitor at node i, then u_i does not appear in Eq.(2). By abusing notation, we will simply use $\mathbf{u}$ as an nc dimensional vector containing the nc capacitors to be sized i.e.,

$$\mathbf{u}^T = [u_1 \cdots u_{nc}] = [Q_{cn_1} \cdots Q_{cn_{nc}}]$$

Note also that we have the following terminal conditions:

(i) at the substation; let the given substation voltage be V^{sp} , then

$$x_{00_3} = V_0^2 = V^{sp\,2} \quad (3.i)$$

(ii) at the end of the main feeder;

$$x_{0n_1} = P_n = 0 \quad ; \quad x_{0n_2} = Q_n = 0 \quad (3.ii)$$

The $3n$ branch flow equations of (2) together with the 3 boundary conditions of (3) constitute the system equations and will be referred to as *DistFlow Equations*. They are of the form

$$\mathbf{G}(\mathbf{x}_0, \mathbf{u}) = 0 \quad (4)$$

Where, $\mathbf{x}_0 = [\mathbf{x}_{00}^T \cdots \mathbf{x}_{0n}^T]^T$ is the branch variables and $\mathbf{u}$ is the capacitor sizes. For a given *load profile* (i.e., set of demands P_{Li}, Q_{Li} ; $i = 1, \ldots, n$) and capacitor sizes *(control variables)*, $\mathbf{u}$, $3(n+1)$ DistFlow equations can be used to determine the *operating point*, $\mathbf{x}_0$ of the system. Such a solution algorithm is developed in the appendix.

General Case

The DistFlow equations can be generalized to include laterals. Consider a lateral branching out from the main feeder as shown in Fig.2. For notational simplicity, the lateral branching out from node k will be referred to as the *lateral k* and the node *k* will be referred to as the *branching node*.

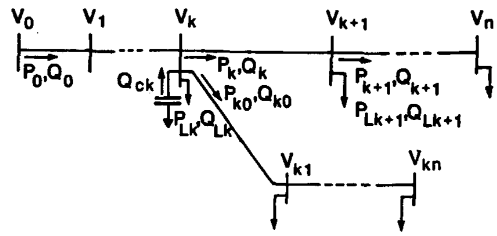

Figure 2 : One line diagram of a section of a feeder

For lateral k with nk branches, the same process of formulation applied to the main feeder can be repeated for the lateral by using the line flow equations of (1) and the new terminal conditions, $V_{k0} = V_k$, $P_{kn} = 0$, $Q_{kn} = 0$. Where, we have used the dummy variable V_{k0} for notational simplicity. As a result, we have the following $3(nk+1)$ equations.

$$\begin{aligned} &\mathbf{x}_{ki+1} = \mathbf{f}_{ki+1}(\mathbf{x}_{ki}, u_{ki+1}) \quad i = 0, \ldots, nk-1 \\ &x_{k0_3} = V_{k0}^2 = V_k^2 = x_{0k_3} \\ &x_{kn_1} = P_{kn} = 0 \quad ; \quad x_{kn_2} = Q_{kn} = 0 \end{aligned} \quad (5)$$

Hence, in general, for a distribution network of n branches and l laterals, there are $3(n+l+1)$ DistFlow equations (ac power flow equations) comprising of Eq.(5) for each lateral $k = 0, \ldots, l$, including the main feeder as the 0'th lateral. They are of the form

$$\mathbf{G}(\mathbf{x}, \mathbf{u}) = 0 \quad (6)$$

Where, $\mathbf{x} = [\mathbf{x}_1^T \cdots \mathbf{x}_l^T\ \mathbf{x}_0^T]^T$, $\mathbf{x}_k = [\mathbf{x}_{k0}^T \cdots \mathbf{x}_{kn}^T]^T$.

DistFlow equations can be used to determine the operating point, $\mathbf{x}$ of the system if the capacitor sizes, $\mathbf{u}$ are given. We prefer to use DistFlow equations over conventional ac power flow equations because the special structure of the DistFlow equations can be utilized to develop a computationally efficient and numerically robust solution algorithm. The details of the derivation of the solution algorithm are presented in the appendix. Other distribution system load flow methods have been proposed, see [18] and the references cited therein.

III. SIZING PROBLEM IN CAPACITOR PLACEMENT

The capacitor placement problem is to determine the location, type, and size of the capacitors to be placed on a distribution system. The objectives are to reduce the losses on the system and to maintain the desired voltage profile while keeping the cost of the capacitor addition at a minimum. In this general form, the problem includes the following factors: (i) the location of the capacitors, (ii) the type of capacitors -fixed or switched, (iii) load variations, and (iv) the cost of capacitors. A solution scheme for this general problem is proposed in another paper [17]. The proposed approach calls for the solution of a *Sizing Problem* as a subroutine.

The sizing problem is a special case of the capacitor placement problem; it assumes that

- the capacitors are placed (their number and places are given),
- there is only one load profile,

• the cost of a capacitor is a differentiable function of its size.

Therefore, the sizing problem determines the optimal sizes of the capacitors once they are placed. This special case was considered by Lee and Grainger in [8]. However, in this proposed formulation, the *voltage profile* of the system (voltages at all nodes) is required to lie within the acceptable limits.

To formulate the sizing problem as a nonlinear programming problem, consider a radial distribution system with n branches, l laterals, and nc capacitors placed at the nodes of the system and a load profile, P_{Li} , Q_{Li} , $i = 1, \ldots, n$. The objective comprises two terms; the savings due to real power loss reduction in the system as a result of capacitor placement, and the cost of capacitors. The real power losses in the network can be calculated as the sum of the i^2r loss on each branch, i.e.,

$$p(\mathbf{x}) = \sum_{k=0}^{l}\sum_{i=0}^{kn-1} r_{ki+1}\frac{P_{ki}^2+Q_{ki}^2}{V_{ki}^2} \tag{7}$$

The other term in the objective function, namely the capacitor cost function, is to be assumed linear with a constant marginal cost r_c*\$/kvar* for simplicity. Then the objective can be written as

$$f_o = k_p p(\mathbf{x}) + \sum_{j=1}^{nc} r_{cj} u_j \tag{8}$$

Where, k_p is the weighting factor for the losses.

There are three sets of constraints.

(i) The power flow equations : $\mathbf{G}(\mathbf{x},\mathbf{u}) = 0$

(ii) Voltage limits : $V_{ki}^{min\,2} \le \mathbf{x}_{ki_3} = V_{ki}^2 \le V_{ki}^{max\,2}$

(iii) Capacitor Limits : $0 \le \mathbf{u} \le \mathbf{u}^{max}$

To summarize, the sizing problem in capacitor placement is

$$PS \qquad \min\ f_o = k_p p(\mathbf{x}) + \sum_{j=1}^{nc} r_{cj} u_j$$

$$\begin{aligned} s.t. \qquad & \mathbf{x}_{ki+1} = \mathbf{f}_{ki+1}(\mathbf{x}_{ki}, u_{ki+1}) && k = 0 \cdots l \\ & \mathbf{x}_{k0_3} = \mathbf{x}_{0k_3} \qquad \mathbf{x}_{kn_1} = \mathbf{x}_{kn_2} = 0 && i = 0 \cdots nk-1 \\ & V_{ki}^{min\,2} \le \mathbf{x}_{ki_3} = V_{ki}^2 \le V_{ki}^{max\,2} \\ & 0 \le \mathbf{u} \le \mathbf{u}^{max} \end{aligned}$$

This is a *discrete optimal control* type problem; branch flow equations acts like the difference equations and the terminal conditions constitute mixed boundary conditions. We first discuss the general solution approach to this problem and then the special properties of the problem. An efficient solution algorithm is proposed by taking advantage of these properties.

IV. FEASIBLE DIRECTIONS METHODS

There are basically two types of methods for solving the sizing problem type nonlinear programming problems [21] - [23]; the ones that use only first order derivatives - the first order methods - and the ones that require the computation of the second order derivatives - the second order methods.

For power system applications, the first order methods are easy to implement; in particular, the power flow solution can be used as a subroutine in the optimization. However, early experience with the reduced gradient-penalty function method for optimal power flow problems,(OPF) revealed occasional convergence problems due to zigzaging of solution points from iteration to iteration [19]. It is believed that this is caused by the fact that the objective function is rather flat near the optimum.

The selection of the solution method for the sizing problem, PS is based on the following considerations.

(i) PS has quite a large number of equality constraints due to DistFlow equations. These equality constraints can be eliminated by noting the fact that they can be used to solve $\mathbf{x}$ in terms of $\mathbf{u}$ and we have a very efficient solution method, DistFlow to implement it. But then the objective and the inequality constraints of PS become implicit functions of $\mathbf{u}$; therefore, the second order derivatives become difficult to compute.

(ii) Objective function of PS does not seem to be flat near the optimum as in the OPF case.

Based on these considerations, a first order method, called *Phase I - Phase II Feasible Directions Method*, is selected for the solution of the PS problem. The method obtains the solution in two steps. The first step, Phase I, is applied to get a feasible point if the initial point is infeasible. In the second step, Phase II, the solution is improved iteratively until the solution converges to the optimal point. This is an appealing property of the method since it allows one to stop earlier in the solution process and yet get an approximate and feasible solution. Also to improve the convergence, the method incorporates the functional inequality constraints directly in the direction of movement, i.e., *search direction*, unlike the reduced gradient-penalty function method in which these constraints are treated as penalties.

The first step in applying this method to PS involves the transcription of the problem into a nonlinear programming problem with inequality - constraints - only form by eliminating the equality constraints of DistFlow equations, (6). Since, as shown in the appendix, the Jacobian of the system equations, $\frac{\partial \mathbf{G}}{\partial \mathbf{x}}$ is well defined, it seems reasonable to assume that the conditions for the implicit function theorem hold and there is a solution $\mathbf{x}(\mathbf{u})$ for all $0 \le \mathbf{u} \le \mathbf{u}^{max}$ such that $\mathbf{G}(\mathbf{x}(\mathbf{u}),\mathbf{u}) = 0$, i.e., one can solve $\mathbf{x}$ in terms of $\mathbf{u}$ using Eq.(6). Then the problem reduces to the following inequality constrainted-type problem.

$$PC \qquad \min_{\mathbf{u}} \ \{\ f_o(\mathbf{u}) \ \mid \ f_j(\mathbf{u}) \le 0 \quad j = 1, \ldots, n \ \}$$

A Phase I - Phase II Feasible Directions Method can be applied to solve this problem. A comprehensive review of these methods are given in [20]. The basic factor in determining the computational efficiency of these methods is the number of gradients of the constraints used in the calculation of the *search direction* $\mathbf{h}$; for example for phase I,

$$\mathbf{h} = -\mathit{argmin}\ \{\ \tfrac{1}{2} \mid \sum_{j\in M} \mu_j \nabla f_j \mid^2 \ \mid \ \sum_{j\in M} \mu_j = 1\ ,\ \mu_j \ge 0\ \}$$

where, the set M is supposed to contain all the violated constraints.

Computationally, it is desirable to reduce the size of M as much as possible. The ideal case would be to consider only one constraint at a time, for example, the constraint which is violated the most. But then it can be shown that if the solution follows such an approach, it may jam up around a sharp corner of the feasible region as shown in Fig.3. The use of only one gradient at a time ($\mathbf{h}_1$ in the figure) will lead to a zigzagging solution trajectory. On the other hand, using a convex combination of two gradients, ($\mathbf{h}$ in the figure), will lead to a much faster convergence. Therefore, the incorporation of only one constraint at a time in the calculation of the search direction should be used only if sharp corners will not be encountered in the feasible region defined by the inequality constraints of the problem.

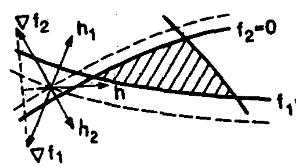

Figure 3 : A feasible region with a "sharp corner".

V. GEOMETRY OF THE FEASIBLE REGION

In this section, we are going to investigate whether we can simplify the calculations involved in Feasible Directions Methods by selecting only a few gradients from the violated constraints. As mentioned earlier, the problem boils down to checking whether the feasible region has sharp corners or not.

In order to have an idea about the existence of sharp corners in the feasible region defined by the voltage constraints, we simplify the branch flow equations of (1) by neglecting the branch loss terms $r_{i+1}(P_i^2+Q_i^2)/V_i^2$. Then the branch flow equations become

$$P_{i+1} = P_i - P_{Li+1} \tag{9.i}$$

$$Q_{i+1} = Q_i - Q_{Li+1} + Q_{ci+1} \tag{9.ii}$$

$$V_{i+1}^2 = V_i^2 - 2(r_{i+1}P_i + x_{i+1}Q_i) \tag{9.iii}$$

Note that the real power flow is decoupled and can be dropped from calculations. This simplified DistFlow model is similar to the current model adopted in literature [6]. The appealing properties of these equations are that they are linear and can be solved directly. The general form of the voltage equation is as follows.

$$V_j^2 = (V_j^0)^2 + 2\sum_{k=1}^{n} \xi_{jk} Q_{ck} \tag{10}$$

Where, ξ_{jk} is the total line reactance of the section of the network which corresponds to intersection of two paths extending between the nodes 0-j and 0-k.

Therefore, the upper and the lower voltage constraints become hyperplanes and the feasible region defined by these constraints will be a polytope. To illustrate such a feasible region, consider the small system shown in Fig.4.

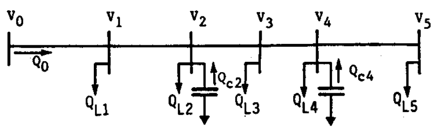

Figure 4 : A small main feeder

Some of the voltage equations are

$$V_1^2 = (V_1^0)^2 + 2x_1Q_{c2} + 2x_1Q_{c4}$$

$$V_3^2 = (V_3^0)^2 + 2(x_1 + x_2)Q_{c2} + 2(x_1 + x_2 + x_3)Q_{c4}$$

$$V_5^2 = (V_5^0)^2 + 2(x_1 + x_2)Q_{c2} + 2(x_1 + x_2 + x_3 + x_4)Q_{c4}$$

The corresponding feasibility region is shown in Fig.5.

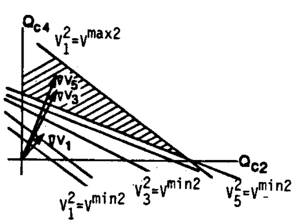

Figure 5 : Feasible region defined by voltage constraints

Now consider the hyperplanes that corresponds to the lower bound constraints. For the example, they are stacked on top of each other in almost parallel fashion as depicted in Fig.5. Therefore, the intersection of two hyperplanes of the lower bound constraints will not create sharp corners. A sharp corner may be created by the hyperplanes of constraints $V_5^2 = V^{min2}$ and $V_1^2 = V^{max2}$ as shown in the figure.

The foregoing observation on the example leads us to assume that in general:

- the constraints of the same type (either the lower voltage limit constraints or the upper voltage limits constraints) will not create sharp corners in the feasible region,
- sharp corners may be created by the constraints of different types.

This assumption will be used in the next section to simplify the calculations involved in a Phase I - Phase II type solution algorithm.

VI. SOLUTION ALGORITHM

In this section, we will present an efficient solution algorithm for the *Sizing Problem*. The algorithm involves modification of the Phase I - Phase II feasible directions method specifically for the radial distribution system capacitor placement problem.

From the analysis of Sec.V, it is reasonable to assume that: (i) the constraints of the same type do not create sharp corners in the feasible region, and (ii) sharp corners are created only when constraints of different types are binding. Therefore, we modify the feasible directions by including only one or two gradients from the violated constraints. If only lower voltage limits are violated, we include the gradient of the most violated constraint (corresponding to the lowest bus voltage) in the feasible direction. If both lower voltage limit and upper voltage limit violations are present, we include the gradient of the most violated lower voltage constraint and the gradient of the most violated upper voltage constraint in the feasible direction.

To be more specific, the selection of the feasible direction **h** is as follows.

1. **Phase I** (to get a feasible point)

Case 1: There are only lower voltage limit violations. Then, we consider only the smallest voltage V_l below the lower bound limit $V^{\min}$, i.e.,

$$V_l \le V_j \quad \forall\ j \qquad \text{and} \qquad V_l < V^{\min}$$

and select the feasible direction as $\mathbf{h} = \nabla\hat{v}_l$.

Case 2 : There are only upper voltage limit violations. Then, we consider only the greatest voltage V_m above the upper bound limit $V^{\max}$, i.e.,

$$V_m \ge V_j \quad \forall\ j \qquad \text{and} \qquad V_m > V^{\max}$$

and select the feasible direction as $\mathbf{h} = -\nabla\hat{v}_m$.

Case 3 : There are both lower voltage limit violations and upper voltage limit violations. We consider only the smallest voltage, V_l below $V^{\min}$ and the greatest voltage, V_m above $V^{\max}$ as defined above and calculate the feasible direction as

$$\mathbf{h} = -\underset{\mu_l,\mu_m}{argmin}\ \{\tfrac{1}{2}|\mathbf{h}|^2 \mid \mathbf{h} = -\mu_l\nabla\hat{v}_l + \mu_m\nabla\hat{v}_m\ ;\ \mu_l,\mu_m \ge 0\ ;\ \mu_l + \mu_m = 1\}$$

2. **Phase II** (to get the optimal point)

Case 1: " ε_v active constraints" are lower voltage limits only, i.e,

$$V_l \le V_j \quad \forall\ j \qquad \text{and} \qquad V_l < V^{\min} + \varepsilon_v$$

Then we select the search direction as

$$\mathbf{h} = -\underset{\mu_0,\mu_l}{argmin}\ \{\tfrac{1}{2}|\mathbf{h}|^2 \mid \mathbf{h} = \mu_0\nabla f_o - \mu_l\nabla\hat{v}_l\ ;\ \mu_0,\mu_l \ge 0\ ;\ \mu_0 + \mu_l = 1\}$$

Case 2 : " ε_v active constraints" are upper voltage limits only, i.e.,

$$V_m \ge V_j \quad \forall\ j \qquad \text{and} \qquad V_m > V^{\max} - \varepsilon_v$$

Then we select the search direction as

$$\mathbf{h} = -\underset{\mu_0,\mu_m}{argmin}\ \{\tfrac{1}{2}|\mathbf{h}|^2 \mid \mathbf{h} = \mu_0\nabla f_o + \mu_m\nabla\hat{v}_m\ ;\ \mu_0,\mu_m \ge 0\ ;\ \mu_0 + \mu_m = 1\}$$

Case 3 : There are both ε_v active lower and upper voltage limits. We consider only V_l and V_m as defined above and calculate the search direction as

$$\mathbf{h} = -\underset{\mu_0,\mu_l,\mu_m}{argmin} \left\{ \tfrac{1}{2}|\mathbf{h}|^2 \;\middle|\; \begin{array}{l} \mathbf{h} = \mu_0\nabla f_o - \mu_l\nabla\hat{v}_l + \mu_m\nabla\hat{v}_m \\ \mu_0,\ \mu_l,\ \mu_m \ge 0\ ;\ \mu_0 + \mu_l + \mu_m = 1 \end{array} \right\}$$

The overall solution algorithm is presented below

step 0 : initialization

choose $\varepsilon_v, \varepsilon_c > 0\ ;\ \alpha, \beta \in (0,1)\ ;\ \mathbf{u}_0$

step 1 : feasibility check

if $\exists\ V_l \le V^{\min}$ or $\exists\ V_m \ge V^{\max}$

then phase = 1 else phase = 2

step 2 : search direction

Calculate the search direction **h** as explained above

step 3 : step size

- calculate $\Delta\bar{u}$ such that $\mathbf{u}_i + \Delta\bar{u}\,\mathbf{h} \le \mathbf{u}^{\max}$
- calculate the step size λ_i by **Armijo Rule** i.e.,

if phase = 1 then

$$\lambda_i = \max_{k=0\ldots K} \left\{ \lambda_i = \beta^k \Delta\bar{u} \;\middle|\; \begin{array}{c} \psi(\mathbf{u}_i + \lambda_i\mathbf{h}) - \psi(\mathbf{u}_i) \ge -\lambda_i\alpha|\mathbf{h}|^2 \\ G(\mathbf{x}_{i+1}, \mathbf{u}_i + \lambda_i\mathbf{h}) = 0 \end{array} \right\}$$

Where, $\psi(\mathbf{u}_i) = \max\{(V^{min2} - \hat{v}_l(\mathbf{u}_i)),\ (\hat{v}_m(\mathbf{u}_i) - V^{max2})\}$

if phase = 2 then

$$\lambda_i = \max_{k=0...K} \left\{ \lambda_i = \beta^k \Delta\overline{u} \;\middle|\; \begin{array}{c} f_o(\mathbf{u}_i+\lambda_i \mathbf{h}) - f_o(\mathbf{u}_i) \le -\lambda_i \alpha |\mathbf{h}|^2 \\ G(\mathbf{x}_{i+1}, \mathbf{u}_i+\lambda_i \mathbf{h}) = 0 \\ V^{min\,2} \le \hat{v}_j(\mathbf{x}_{i+1}, \mathbf{u}_i+\lambda_i \mathbf{h}) \le V^{max\,2} \end{array} \right\}$$

step 4 : update

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \lambda_i \mathbf{h}$$

step 5 : convergence check

If $\lambda_i |\mathbf{h}| > \varepsilon_c$ then go to step 1 else stop

Step size calculations in step 3 involves the solution of system equations, $G(\mathbf{x}_{i+1}, \mathbf{u}_i+\lambda_i \mathbf{h}) = 0$ every time β^k is updated to get the corresponding state variables $\mathbf{x}_{i+1}(\mathbf{u}_{i+1})$. Therefore, computationally it is important to have a quick search. One way is to limit the search interval by putting an upper limit, $\Delta Q_c^{\lim}$ on the maximum allowable change in control variables, $\Delta\overline{u} |h_k|^{\max}$. This will speed up the search in Phase II since the step size will get smaller as solution approaches to the optimal point. Also an upper bound K has to be defined on the number of searches when implementing the search.

Another improvement usually made on the above algorithm, [20] involves the modification of feasible direction for Phase I by combining phase I direction $\mathbf{h}_I$ and the phase II direction ∇f_o as follows.

$$\mathbf{h} = (1/|\mathbf{h}_I|)\mathbf{h}_I + (\upsilon / |\nabla f_o|)\nabla f_o$$

This way the solution will be pushed towards the optimal point in the feasible region and hence the convergence will improve in phase II.

The algorithm can start from any initial point, $\mathbf{u}_0$. However, a good initial point can easily be obtained by employing the approximate method of [6] with simplified DistFlow equations of (9).

VII. TEST RESULTS

The proposed algorithm has been implemented in Fortran-77 on both VAX 11/750 and IBM PC-AT. In the implemented algorithm, the optimization parameters are set as follows:

$\alpha = 0.4$, $\beta = 0.6$, $\upsilon = 0.3$, $\varepsilon_c = 0.6$, $\varepsilon_v = 0.04$, $K = 5$, $\Delta Q_c^{\lim} = 3.0$

We will present three capacitor sizing test runs and sample power flow solutions for two test systems.

The first test system, TS1 is a 9-branch main feeder developed by Grainger et. al. [8]. We also follow Grainger et. al. in setting $k_p = 168$ *\$/kW*, $r_c = 4.9$ *\$/kvar* and placing three capacitors Q_{c6}, Q_{c5}, Q_{c1} on the nodes 6, 5, 1 of the feeder respectively. The second test system, TS2 is a 69 branch, 9-lateral feeder derived from a portion of the PG&E distribution system. The network data of this system is given in [17]. There are three capacitors placed on the system; one on the main feeder, Q_{c18}, and two on a lateral, Q_{c47}, Q_{c52}.

A summary of the power flow solutions for the test systems before the capacitor placement is presented in Table 2. The solutions are obtained by the DistFlow solution scheme mentioned in the appendix. They converge in 4 and 3 iterations for TS1 and TS2 respectively. These results indicate that the convergence of DistFlow is fast. Considering the fact that the feeders have branches with diverse r/x ratio (ratio varies between 3 to 0.02), and TS1 has a low voltage and high loss profile, the results also indicate the method's numerical robustness. The run time figures shown in Table 2 are obtained by running the program on a VAX 11/750.

Table 2 : Power Flow results for the test systems by DistFlow

System	Substation		Total Loss		$V_{\min}$	Run Time	
	$P_0(kW)$	$Q_0(kvar)$	$P_{ls}(kW)$	$Q_{ls}(kvar)$	V_0	CPU	I/O
TS1	13 151.77	5 222.47	783.77	1 036.47	.837	0.14	0.31
TS2	4 027.10	2 796.77	225.00	102.16	.909	0.75	1.50

Test results of the proposed method for capacitor sizing are presented below.

Test System 1 (TS1)

Two test runs were conducted for TS1.

Test run A : In this test run, we (i) neglect the cost of capacitors, (ii) ignore the voltage constraints. Therefore, the solution is an unconstrainted optimal point and gives the maximum loss reduction. As the initial starting point, the approximate solution given in [8] is used. The resulting solution trajectory is given in Fig.7. A summary of the solution and the computation time are presented in Table 3. The following observations are made from Fig.7 and the comparison of Table 2 and Table 3.

- The convergence is achieved in about 7 iterations (2 times the number of capacitors to be placed).
- The approximate solution is not close to the actual solution as pointed out in [8].
- At the unconstrainted optimal point, the power loss reduction is maximum (100 kW and 4890 kvar) and the voltage profile is better (the minimum voltage raises form .837 p.u. to .882 p.u. and all the node voltages are still below the substation voltage).

Test run B : In this test run, cost of capacitors are considered and the lower and upper limits on all bus voltages are set to 0.9 p.u. and 1.1 p.u. respectively. The resulting solution trajectory is shown in Fig.8. The trajectory starts with Phase I since the initial point is infeasible (lower voltage limit violation). In the forth iteration a feasible point is obtained and the program switches to Phase II. The convergence is obtained in the 12'th iteration. A summary of the results and the computation time are presented in Table 3. We have the following observations about the test.

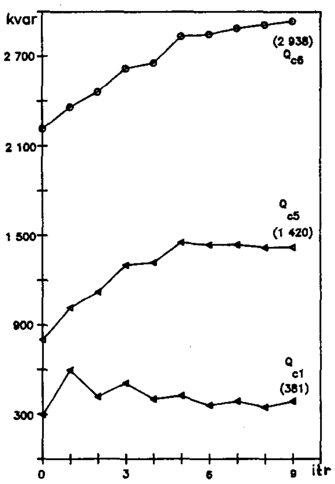

Figure 7 : Solution Trajectory for TS1 Test A

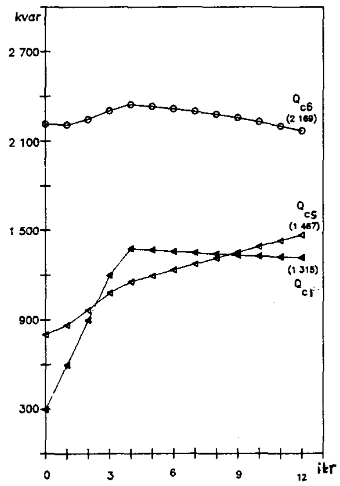

Figure 8 : Solution Trajectory for TS1 Test B

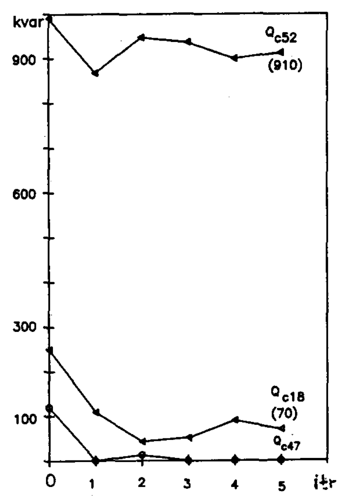

Figure 9 : Solution Trajectory for TS2

- The convergence in Phase I is fast; the number of iterations is roughly equal to the number of capacitors.
- The convergence in Phase II gets slower since the solution trajectory moves along the boundary of a lower limit voltage limit (the total number of iterations is roughly equal to 2 to 3 times the number of capacitors.
- The incremental changes of capacitors gets smaller as iteration increases.
- The results show the feasibility of capacitor placement for both loss reduction and voltage regulation (minimum voltage is raised to .901 p.u. while still achieving 64.32 kW loss reduction).

Table 3 : Summary of test results for TS1.

Test Run	Substation		Loss Reduction		V_{min}	Run Time	
	$P_0(kW)$	$Q_0(kvar)$	$\Delta P_0(kW)$	$\Delta Q_0(kvar)$	V_0	CPU	I/O
Test A	13 050.87	329.62	100.90	4 892.85	.882	4.38	1.3
Test B	13 087.45	279.43	64.32	4 943.04	.901	6.08	1.9

Test System 2 (TS2)

In this test run, capacitor costs and the voltage constraints are considered also. The initial point is obtained by the application of the approximate method of [6] with the simplified DistFlow equations presented in section 5. The corresponding solution trajectory is given in Fig.9. Since the the system without any capacitors is feasible, the test run uses Phase II only. A summary of the solution and the computation time are presented in Table 4. The following observations are made.

- The convergence is obtained in 4 iterations (less than 2 times the number of capacitors).
- The test result, $Q_{c15} = 0$ indicates that the capacitor on bus 15 is not economical to install.
- Considerable loss reduction is obtained through optimal capacitor placement (67.55 kW and 1018.39 kvar).

Table 4 : Summary of test results for TS2.

Substation		Loss Reduction		V_{min}/V_0	Run Time	
$P_0(kW)$	$Q_0(kvar)$	$\Delta P_0(kW)$	$\Delta Q_0(kvar)$		CPU	I/O
3 959.55	1 778.38	67.55	1 018.39	.925	10.91	0.92

The overall test results indicate that the proposed Phase I - Phase II solution algorithm has an acceptable convergence rate, it is computationally efficient, and it allows one to get approximate and feasible solutions.

Analysis of the Results

Consider Fig.8 for TS1. In phase I part of the solution, the objective is to find a solution with voltage profile within the limits (feasibility). The initial point (0-th iteration in Fig.8) gives a voltage profile that is too low. The most effective way to boost the voltage profile is to place the capacitor at the node where the voltage is lowest. In this case, it is the node #1 at the far end of the feeder. However, the solution at the end of phase I results in reactive power flowing in reverse direction from the substation, hence high losses. In phase II of the solution process, the capacitance in node 1 is shifted to node 5 in order to reduce the losses.

In TS2, the biggest load is at node 50. The solution shown in Fig.9 has the capacitor at node 52, which is close to node 50, a rather high value. There is not much load around the node 18 and the capacitance placed there is also rather small. This shows that the reactive power compensation is done mostly locally. Note also that node 52 is at lower voltage than that node 47, therefore it is more effective to place capacitor on node 52 than on node 47 for voltage boost.

It should also be pointed out that the loads in TS1 have high power factors, whereas the loads in TS2 have low power factors. Therefore, as predicted, the amount of loss reduction in TS2 is much higher in percentage (30% vs 13%).

VIII. CONCLUSIONS

In this paper, a capacitor sizing problem for capacitors placed on a radial distribution system is formulated as a non-linear programming problem and a solution algorithm is developed. The problem finds the optimal size of the capacitors so that the power losses will be minimized for a given load profile while considering the cost of the capacitors. The formulation also incorporates the ac power flow model for the system and the voltage constraints.

The solution algorithm developed for the capacitor sizing problem is based on a Phase I - Phase II feasible directions approach. This general method is made computationally more efficient by exploiting the special structure of the problem.

Another contribution of the paper is the introduction of new power flow equations and a solution method, called *DistFlow*, for radial distribution systems. The method is computationally efficient and numerically robust, especially for distribution systems with large r/x ratio branches. The DistFlow is used repeatedly as a subroutine in the optimization algorithm for the capacitor sizing problem.

The test results of the proposed algorithm for the capacitor sizing problem are also presented. They indicate that the method is computationally efficient and has good convergence characteristics. They demonstrate the feasibility of capacitor placement for both loss reduction and voltage regulation purposes.

The capacitor sizing problem considered in this paper is a special case of the general capacitor placement problem. The general problem determines the place and the type of capacitors to be installed as well as their sizes. The load variations as a function of time are also incorporated in the general problem in the calculation of the the energy losses. The general problem is considered in another paper [17]. The solution algorithm developed in [17] uses the algorithm developed here as a subroutine.

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Appendix : Solution of DistFlow Equations

The DistFlow equations of a radial distribution system, as shown in section 2, comprises branch flow equations and the associated terminal conditions for each lateral including the main feeder which is treated as the 0'th lateral. They are of the following form.

$$\mathbf{x}_{ki+1} = \mathbf{f}_{ki+1}(\mathbf{x}_{ki}, u_{ki+1}) \qquad k = 0,1,\ldots,l \tag{a.1.i}$$

$$\mathbf{x}_{k0} = \mathbf{x}_{0k_3} \qquad \mathbf{x}_{kn} = \mathbf{x}_{kn_2} = 0 \qquad i = 0,1,\ldots,nk-1 \tag{a.1.ii}$$

We can use these equations to determine the operating point, $\mathbf{x}_{ki} = [P_{ki} Q_{ki} V_{ki}^2]$ of the system for a given, $\mathbf{u}$. But rather then using the above equations to solve for $\mathbf{x}_{ki}'s$ directly, we are going to conceptually reduce the number of equations first and then propose an efficient solution algorithm.

Reduction of DistFlow Equations

Note that for a given lateral k, we need only to know the variables at the beginning of the lateral, $\mathbf{x}_{k0} = [P_{ko} Q_{k0} V_{k0}^2]$; since then the rest of the variables, $\mathbf{x}_{ki}$ can be calculated by using the branch flow equations successively. To be more specific, let us consider two cases again, namely first the main feeder and then the general case.

Special Case : Main Feeder Only

Since the substation voltage, V_o is given, the only variables need to be determined are $\mathbf{z}_{00} = [P_0 Q_0]^T$, i.e., the power supplied from the substation. Therefore, $\mathbf{z}_{00}$ constitutes the "state" of the system. To eliminate the other variables, $\mathbf{x}_{0i}$ from the Eq.(a.1), we use the terminal conditions at the end of the feeder, i.e.,

$$P_n = \hat{p}_{0n}(\mathbf{x}_{0n-1}) = 0 \quad ; \quad Q_n = \hat{q}_{0n}(\mathbf{x}_{0n-1}) = 0 \tag{a.2}$$

and eliminate $\mathbf{x}_{0i}'s$ by substituting the branch flow equations, Eq.(a.1.i) recursively for $i = n-1, n-2, \ldots, 1$. As a result, we get two equations of the following form.

$$\hat{p}_{0n}(\mathbf{z}_{00}, \mathbf{u}) = 0 \quad ; \quad \hat{q}_{0n}(\mathbf{z}_{00}, \mathbf{u}) = 0 \tag{a.3}$$

We will use these two equations to solve $\mathbf{z}_{00}$ for a given $\mathbf{u}$.

General Case : Feeder with Laterals

For simplicity, it will be assumed that the system consists only of a main feeder and l primary laterals (laterals branching out from the main). We generalize the procedure of reducing the system equations as follows. For lateral k, we choose two new variables, P_{k0} and Q_{k0} - the real and the reactive power flows into the lateral respectively - as the extra state variables $\mathbf{z}_{ko} = [P_{ko} Q_{ko}]^T$. Then the same process of reduction applied to the main feeder can be repeated for the lateral by using the branch equations of (a.1) and the associated terminal conditions, $P_{kn} = 0$, $Q_{kn} = 0$. This will give two new equations of the following form.

$$\hat{p}_{kn}(\mathbf{z}_{10}, \ldots, \mathbf{z}_{ko}, \mathbf{z}_{00}, \mathbf{u}) = 0 \tag{a.4.i}$$

$$\hat{q}_{kn}(\mathbf{z}_{10}, \ldots, \mathbf{z}_{ko}, \mathbf{z}_{00}, \mathbf{u}) = 0 \tag{a.4.ii}$$

Hence, in general, for a distribution network of n branches and l laterals, reduced DistFlow equations comprise Eq.(a.4) for each lateral $k = 1, \ldots, l$ together with the equations for the main feeder, Eq(a.3) which can now be rewritten as,

$$\hat{p}_{0n}(\mathbf{z}_{10}, \ldots, \mathbf{z}_{lo}, \mathbf{z}_{00}, \mathbf{u}) = 0 \tag{a.5.i}$$

$$\hat{q}_{0n}(\mathbf{z}_{10}, \ldots, \mathbf{z}_{lo}, \mathbf{z}_{00}, \mathbf{u}) = 0 \tag{a.5.ii}$$

Solution of DistFlow Equations

The reduced DistFlow equations are of the form

$$\mathbf{H}(\mathbf{z}, \mathbf{u}) = \mathbf{0} \tag{a.6}$$

and can be used to solve for the "state variables" $\mathbf{z} = [\mathbf{z}_{10}^T \cdots \mathbf{z}_{lo}^T \mathbf{z}_{00}^T]^T$.

Let's consider solving (a.6) for $\mathbf{z}$ by Newton-Raphson Method (NR) for a given $\mathbf{u}$. From an estimated value of $\mathbf{z}^j$, an iteration step of NR involves three steps:

Step 1 : calculation of the mismatches $\mathbf{H}(\mathbf{z}^j)$

Step 2 : construction of the system Jacobian matrix

$$\mathbf{J}(\mathbf{z}^j) = \frac{\partial \mathbf{H}}{\partial \mathbf{z}} \bigg|_{\mathbf{z}=\mathbf{z}^j} \tag{a.7}$$

Step 3 : solution of the following system of equations to update the states $\mathbf{z}$.

$$\mathbf{J}(\mathbf{z}^j) \Delta \mathbf{z}^j = -\mathbf{H}(\mathbf{z}^j) \tag{a.8}$$

We consider the special case - the main feeder only - first. The mismatches for this case can be calculated as follows. At iteration j, the substation bus voltage V_0 is given and the updated values of the substation powers P_0, Q_0 are available. Therefore, the recursive branch flow equations of (a.1.i) can be employed to update the variables $\mathbf{x}_{0i} = [P_i, Q_i, V_i^2]^T$ $i = 1, \ldots, n$ successively along the feeder starting from the substation and proceeding towards the end of the feeder. When we reach the end of the feeder, the updated variables P_n , Q_n will be the mismatches as indicated by (a.2). We'll refer to this procedure as a *forward sweep*.

The elements of system Jacobian can be calculated by using the Chain Rule. For the main feeder case, the Jacobian is a 2x2 matrix of the following form.

$$\mathbf{J}(\mathbf{z}_{00}) = \begin{bmatrix} \dfrac{\partial \hat{p}_{0n}}{\partial P_0} & \dfrac{\partial \hat{p}_{0n}}{\partial Q_0} \\ \dfrac{\partial \hat{q}_{0n}}{\partial P_0} & \dfrac{\partial \hat{q}_{0n}}{\partial Q_0} \end{bmatrix} \tag{a.9}$$

By using the branch flow equations, (a.1.i) and applying the Chain Rule, $\mathbf{J}(\mathbf{z}_{00})$ can be constructed as follows.

$$\mathbf{J} = \begin{bmatrix} \partial \hat{p}_{0n}/\partial \mathbf{x}_{0n-1} \\ \partial \hat{q}_{0n}/\partial \mathbf{x}_{0n-1} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{x}_{0n-1}}{\partial \mathbf{x}_{0n-2}} \end{bmatrix} \cdots \begin{bmatrix} \dfrac{\partial \mathbf{x}_{0i}}{\partial \mathbf{x}_{0i-1}} \end{bmatrix} \cdots \begin{bmatrix} \dfrac{\partial \mathbf{x}_{01}}{\partial \mathbf{x}_{00}} \end{bmatrix} \tag{a.10}$$

Where,

$$\mathbf{J}_i = \begin{bmatrix} \dfrac{\partial \mathbf{x}_{0i}}{\partial \mathbf{x}_{0i-1}} \end{bmatrix} = \begin{bmatrix} 1-2r_i\dfrac{P_{i-1}}{V_{i-1}^2} & -2r_i\dfrac{Q_{i-1}}{V_{i-1}^2} & r_i\dfrac{(P_{i-1}^2+Q_{i-1}^2)}{V_{i-1}^4} \\ -2x_i\dfrac{P_{i-1}}{V_{i-1}^2} & 1-2x_i\dfrac{Q_{i-1}}{V_{i-1}^2} & x_i\dfrac{(P_{i-1}^2+Q_{i-1}^2)}{V_{i-1}^4} \\ -2(r_i-z_i^2\dfrac{P_{i-1}}{V_{i-1}^2}) & -2(x_i-z_i^2\dfrac{Q_{i-1}}{V_{i-1}^2}) & 1-z_i^2\dfrac{(P_{i-1}^2+Q_{i-1}^2)}{V_{i-1}^4} \end{bmatrix}$$

$\mathbf{J}_i$ is the Jacobian of branch flow equation i and will be referred as the *branch Jacobian*. Therefore, calculation of the 2x2 Jacobian in Eq.(a.9) can easily be conducted by the multiplication of 2x3 and 3x3 matrices in (a.10).

The third step of the solution procedure involves the solution of a 2x2 matrix equation.

This calculation procedure can be generalized for general radial distribution feeders. Again, we consider a system having a main feeder and l primary laterals only.

The mismatches can be calculated by generalizing the forward sweep procedure as follows. For the lateral k, given the estimated power injected into lateral, P_{k0}, Q_{k0}, and the voltage at the branching node (substation for the main), V_k, the forward sweep method of the main feeder case can be applied to update the variables $\mathbf{x}_{ki} = [P_{ki}, Q_{ki}, V_{ki}^2]^T$ along lateral and get the mismatches corresponding to this lateral: P_{kn} and Q_{kn}. To update all the variables and thus obtain all the mismatches, above procedure is applied to all the laterals in an order such that the main feeder is updated before the laterals branching out from it.

System Jacobian, $\mathbf{J}$ for the general case will be a 2(l+1)x(l+1) matrix of the following form.

$$\mathbf{J} = \begin{bmatrix} J_{11} & & & & J_{10} \\ J_{21} & J_{22} & & & J_{20} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdots & & \cdot \\ \cdot & \cdot & & & \cdot \\ J_{l1} & J_{l2} & & J_{ll} & J_{ln} \\ J_{01} & J_{02} & & J_{0l} & J_{00} \end{bmatrix} \tag{a.12}$$

Where,

$$J_{ki} = \begin{bmatrix} \dfrac{\partial \hat{p}_{kn}}{\partial P_{i0}} & \dfrac{\partial \hat{p}_{kn}}{\partial Q_{i0}} \\ \dfrac{\partial \hat{q}_{kn}}{\partial P_{i0}} & \dfrac{\partial \hat{q}_{kn}}{\partial Q_{i0}} \end{bmatrix}$$

and can be calculated by Chain Rule similar to (a.10).

The third step involves the solution of 2(l+1)x2(l+1) matrix equation of the form (a.8).

Note that the mismatches and the Jacobian matrix in the proposed method involve only the evaluation of simple algebraic expressions and no trigonometric functions as opposed to the standard load flow case. Thus computationally the proposed method is efficient. In the following tow subsections, we show that the method has very good numerical properties and it can be modified to make it computationally even more efficient.

Numerical Properties

The DistFlow method described above is numerically robust and the solution scheme is insensitive to system parameters, in particular, the branch r/x ratios. This can be seen as follows.

When the elements of the branch jacobians, $\mathbf{J}_i$ are calculated in per unit, p.u., $\mathbf{J}_i$ has the following structure

$$\mathbf{J}_i \approx \begin{bmatrix} 1 & \varepsilon & \Delta P_i \\ \varepsilon & 1 & \Delta Q_i \\ \varepsilon & \varepsilon & 1 \end{bmatrix} \tag{a.13}$$

Where, $|z_{i+1}| \approx |\varepsilon| \approx |V_{i+1}^2 - V_i^2| \ll 1$, $\Delta P_i \approx P$ loss on branch i, and $\Delta Q_i \approx Q$ loss on branch i. Therefore, $det(\mathbf{J}_i) \approx 1$; indicating that $\mathbf{J}_i$'s are very well conditioned.

As a result of (a.13), the system Jacobian, $\mathbf{J}$ is well conditioned for the main feeder only case.

For the general case, it can easily be shown that, the diagonal elements, J_{kk} of the system Jacobian has the same properties as that of the main feeder, J_{oo} and the system Jacobian is block diagonally dominant, i.e.,

$$\det(J_{kk}) \approx 1 \gg \det(J_{ki}) \quad \forall\ i \neq k\ ,\ k \neq 0 \tag{a.14}$$

Therefore, the system Jacobian, $\mathbf{J}$ is well conditioned, i.e.,

$$\det(\mathbf{J}) \approx 1 \tag{a.15}$$

and this is true independent of the line parameters.

These results indicate that the NR solution algorithm will be numerically stable and robust. This very desirable property is one of the advantages of the new formulation over the conventional one where convergence problems has been experienced when there are branches with high r/x ratios in the system [18].

Furthermore, these numerical properties can be exploited to modify the DistFlow solution scheme to make it even more efficient computationally.

Enhancement of Computational Efficiency

We use the numerical properties and the special structure of the Jacobian matrix, $\mathbf{J}$ in simplifying its construction and in solving the update equations of (a.8).

Since by (a.14), the off diagonal terms J_{ki}, $i \neq k$ $k = 1, \ldots, l$ are much smaller than the diagonal terms, J_{kk}, $k = 1, \ldots, l$, off diagonal terms can be dropped in the Jacobian except the ones in the last row, i.e., $\mathbf{d}^T = [J_{01} \cdots J_{0l}]$. This basically decomposes the update equations into l+1 equations as follows.

$$J_{kk}\Delta\mathbf{z}_{k0} = \mathbf{H}_k(\mathbf{z}) \qquad k = 1, \ldots, l \tag{a.16.i}$$

$$J_{00}\Delta\mathbf{z}_{00} = \mathbf{H}_0(\mathbf{z}) - \mathbf{d}^T\mathbf{r}_l \tag{a.16.ii}$$

Where, $\mathbf{H}_k = [\hat{p}_{kn}(\mathbf{z})\ \hat{q}_{kn}(\mathbf{z})]^T$, $\mathbf{r}_l = [\Delta\mathbf{z}_{10}^T \cdots \Delta\mathbf{z}_{l0}^T]^T$. Therefore, the states that correspond to the laterals, $\mathbf{z}_{i0}, i = 1, \ldots, l$ can be updated independently first by Eq.(a.16.i), then the state for the main feeder, $\mathbf{z}_{00}$ is updated by (a.16.ii).

Furthermore, since J_{kk}'s are well conditioned and almost constant, as indicated by (a.13), the Jacobian needs to be constructed only once and then it can be used in all the iterations. Therefore, updating step of NR reduces to the solution of (a.16).

This modified solution scheme brings down the overall cost of computation to about $8n$+8(l+1) multiplications for each iteration; $8n$ multiplications to calculate the mismatches and 8(l+1) multiplications to solve the update equations of (a.16). This amount of computation is comparable to $6n$ multiplications involved in just solving the update equations (like Eq.(a.16)) of conventional fast decoupled power flow solution scheme.

Mesut E. Baran received his B.S. and M.S. from Middle East Technical University, Turkey. He is currently a Ph.D. student at the University of California, Berkeley.

Felix F. Wu received his B.S. from National Taiwan University, M.S. from the University of Pittsburgh, and Ph.D. from the University of California, Berkeley. He is a professor of Electrical Engineering and Computer Sciences at the University of California, Berkeley.

Discussion

A. Chandrasekaran and **R. P. Broadwater** (Center for Electric Power, Tennessee Technological University, Cookeville, TN): The authors are to be complimented for bringing under a single mathematical framework all the various problems of capacitor placement, sizing, time-varying loads, and other special features of radial power distribution systems. A lot of rigor has gone into the analysis of transmission systems. However, radial distribution systems have received very little, and mostly sporadic and desultory, mathematical treatment. The paper under discussion fulfills a long-felt need in distribution analysis. We request the authors to comment on the following observations.

1) Recognition of the minimum (state?) variables required to solve the distribution systems as the branch flows in the source branch and the first branches in the radials in the DistFlow algorithm is of special interest. In [D1] we proposed a load flow formulation based on branch losses and voltages and the resulting Jacobean structure is similar. What is the authors' opinion about the convergence characteristics when the variables chosen are the losses?

2) Even though much is talked about the difficulties of convergence in radial systems, most of the practical algorithms in use are based on the simple ladder technique of Kersting [D2] and appear to work very well. Are there documented cases that the authors are aware of where convergence was a problem? Is it possible to construct meaningful examples of radial systems depicting nonconvergence with simple methods?

3) The upper voltage constraints used in Test Run B is 1.1 pu, and if the source voltage is 1 pu, does this mean leading power factor currents can flow in some parts of the system? Are there "optimal" limits for permissible upper and lower voltages?

4) The cost factors k_p and r_{cj} used in (8) are oversimplified. What will be the effect on the solution if more complex factors are used?

We congratulate the authors once again for a very interesting paper.

References

[D1] A. Chandrasekaran and R. P. Broadwater, "A New Formulation of Load Flow Equations in Balanced Radial Distribution Systems," *Canadian Electrical Engineering Journal*, Vol. 12, No. 4, October 1987, pp. 147–151.

[D2] W. H. Kersting and D. L. Mendive, "An Application of Ladder Network Theory to the Solution of Three-Phase Radial Load Flow Problems," Presented at the IEEE Winter Power Meeting, New York, 1976.

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N. Vempati and **R. R. Shoults** (The University of Texas at Arlington, Arlington, TX): The authors are to be commended for their excellent paper. Their approach of rewriting the branch flow equations in a recursive format enables an efficient algorithm. This is under the assumption of a radial distribution system. When loops are encountered, similar equations can be formulated. Would the authors comment on the robustness and speed of the algorithm in the presence of loops.

Loads and capacitors are considered to constant power devices in the paper. It is now well known that reactive power load is not a constant power type load, rather it is more like a constant susceptance. Would the authors comment on the impact, if any, on accuracy if voltage dependent load models are not used. It has been our experience with various optimization procedures that significant differences result when different load modeling assumptions are used.

The authors allude to the fact that second-order methods are also available in addition to first-order methods for the solution of nonlinear optimization problems. They also present their rationale for implementing the first-order method. In optimal power flow (OPF) problems, it has been shown that first-order methods can give misleading results. The convergence rate can be too slow and sometimes results in a suboptimal solution. The second-order Newton method has been found to be more reliable. It would be interesting to see comparison cases between their first-order method and a second-order Newton type method for their problem formulation. Would the authors please comment.

We again congratulate the authors for a very fine paper, which is very characteristic of their proven abilities.

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M. E. Baran and F. F. Wu : The authors appreciate the interest shown in this paper by the discussers and thank them for their comments.

Concerning the first question by Mr. Vempati and Dr. Shoults about the extension of the DistFlow algorithm to solve networks with loops, two possible approaches seem workable. The first approach would be to put DistFlow equations corresponding to the radial system together with additional equations due to looping branches and to apply Newton Raphson algorithm to solve the resulting equations. In this case, the structure of the system Jacobian can still be exploited to reduce computational burden and hence retain the robustness and the speed of the algorithm. However, computations for each iteration will increase as the number of loops increases since the system Jacobian will get fuller. An alternative approach would be opening of the the loops at certain points of the looping branches and putting current injections at these points to simulate the effect of the loops. An iterative scheme can be developed to obtain the solution as proposed in [18]. In each iteration, currents passing through the opened points on the looping branches are estimated and then DistFlow equations are solved to determine the power flow in the radial system with extra injections. The robustness of the method would still be valid because of the usage of the branch powers as variables. However, the speed of convergence and hence the computational efficiency of the method would probably suffer as the number of loops increases.

DistFlow method can also be generalized to represent the voltage dependent loads and capacitors as shunt susceptances. Modification for voltage dependent load representation will entail making the real and reactive power components of load, P_{Li+1}, Q_{Li+1} in branch flow equations (1.i) and (1.ii) a function of voltage, (i.e. $P_{Li+1} = \hat{p}_{Li+1}(V_{i+1})$, $Q_{Li+1} = \hat{q}_{Li+1}(V_{i+1})$). Similarly, reactive power injections by capacitors, Q_{ci+1} in Eq.(1.ii), can be changed as $Q_{ci+1} = Q^o_{ci+1}V^2_{i+1}$, where Q^o_{ci+1} corresponds to the nominal capacity of the capacitor, to represent the capacitors as shunt susceptances. We intend to incorporate voltage dependent loads and capacitors into our program.

Because of the fact that starting point of the algorithm can be selected as the solution of the approximate method (see comment at the last paragraph of Sec.VI), we have not encountered the problem of slow convergence and suboptimal solutions that are known to exist in the first order OPF methods. The success of second order OPF is due largely to the exploitation of the structure of the OPF equations. We believe the development of a successful second order solution method for the capacitor sizing problem would be an interesting research project. Here, we want also to point out another additional advantage of the method which is not mentioned in the paper. The method used can easily be modified to include different load/capacitor models as pointed above.

We thank Mr. Chandrasekaran and Dr. Broadwater for directing our attention to reference [D1]. The use of branch losses and voltages as state variables is an interesting idea. The convergence characteristics of the same method (e.g. Newton-Raphson) applied to two different formulations of the same problem may be different. We do not have experience on the method using losses as state variables to comment on its convergence characteristics.

Reference [18] pointed out the convergence problems for distribution power flows. In addition to convergence, different methods may exhibit different computational efficiency. The so-called simple methods may be convergent in most cases, but the convergence is usually slow.

Optimal sizing of capacitors may result in leading power factor currents in the branches depending on the load profile, places of capacitors and the voltage limits. This is more likely the case when capacitors are used for voltage regulation purposes (like TS1 TB in the paper). The permissible upper and lower voltages are set by the load demand requirements and system operating requirements. The optimality seems in practice to be dependent on local condition.

Representing the cost of a capacitor by a fixed cost and a linear capacity cost is an approximation. The capacity cost in practice will be a staircase curve. A linear cost curve is the simplest approximation of the staircase curve as a differentiable function. A quadratic function, or any other nonlinear function may be used for a better approximation. The increase in computation as result is minimal. However the effect, if any, on the overall convergence is not known.

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