Linear Three-Phase Power Flow for Unbalanced Active Distribution Networks with PV Nodes

Yi Wang, *Student Member, IEEE*, Ning Zhang, *Member, IEEE*, Hai Li,
Jingwei Yang, *Student Member, IEEE*, and Chongqing Kang, *Fellow, IEEE*

***Abstract*—High penetration of distributed renewable energy promotes the development of an active distribution network (ADN). The power flow calculation is the basis of ADN analysis. This paper proposes an approximate linear three-phase power flow model for an ADN with the consideration of the ZIP model of the loads and PV nodes. The proposed method is not limited to radial topology and can handle high R/X ratio branches. Case studies on the IEEE 37-node distribution network show a high accuracy and the proposed method is applicable to practical uses such as linear or convex optimal power flow of the ADN.**

***Index Terms*—Active distribution network, linear power flow, PV nodes, unbalanced distribution systems, ZIP model.**

I. INTRODUCTION

HIGH penetration of distributed renewable energy promotes the development of an active distribution network (ADN). Power flow calculation is the basis of ADN analysis. Basic power flow equations are nonlinear which poses great challenges for the optimization of an ADN, especially an unbalanced network. Traditional DC power flow for transmission networks may cause considerable error when applied to an unbalanced distribution network due to the high R/X ratio of the feeders. Two linear three-phase power flow models for a distribution network are proposed in [1] and [2] respectively based on the current injection method and high accuracy can be achieved. However, PV nodes cannot be effectively considered in these current injection methods since the unknown reactive power of the PV nodes cannot be modeled by the current injection method.

In an ADN, there will be more and more PV nodes, such as nodes with a reactive compensator, distributed generation (DG) under constant voltage mode, and a voltage source inverter (VSI). This paper proposes the formulation of a linear power flow model for an unbalanced three-phase ADN with the consideration of PV nodes. The contributions of this paper are as follows: 1) a linear three-phase power flow for unbalanced distribution systems is proposed; 2) besides PQ nodes, PV nodes can also be modeled; 3) the linearization of ZIP characteristics of the loads is proposed.

Manuscript received February 19, 2017; revised May 17, 2017; accepted June 22, 2017. Date of publication September 30, 2017; date of current version August 15, 2017. This work was supported in part by the National Key R&D Program of China (No. 2016YFB0900100), and the National Science Foundation of China (No. 51325702, 51677096).

Y. Wang, N. Zhang, H. Li, J. Yang, and C. Kang (corresponding author, e-mail: cqkang@tsinghua.edu.cn) are with the State Key Laboratory of Power Systems, Department of Electrical Engineering, Tsinghua University, Beijing 100084, China.

DOI: 10.17775/CSEEJPES.2017.00240

II. METHODOLOGY

A. Basic Power Flow Equations

The basic three-phase power flow equations can be formulated by taking the mutual inductance and interphase capacitance among different phases into consideration. It is shown as follows:

$$\begin{aligned} P_i^\alpha &= V_i^\alpha \sum_{j=1}^{\mathrm{N}} \sum_{\beta=\mathrm{a,b,c}} V_j^\beta (G_{ij}^{\alpha\beta}\cos\theta_{ij}^{\alpha\beta} + B_{ij}^{\alpha\beta}\sin\theta_{ij}^{\alpha\beta}) \\ Q_i^\alpha &= V_i^\alpha \sum_{j=1}^{\mathrm{N}} \sum_{\beta=\mathrm{a,b,c}} V_j^\beta (G_{ij}^{\alpha\beta}\sin\theta_{ij}^{\alpha\beta} - B_{ij}^{\alpha\beta}\cos\theta_{ij}^{\alpha\beta}) \end{aligned} \tag{1}$$

where N denotes the number of nodes of the ADN; i and j are node indexes; α and β are phase indexes; P_i^α and Q_i^α denote the active and reactive injections of phase α and node i; V_i^α denotes the voltage magnitude and phase angle of the phase α and node i; $G_{ij}^{\alpha\beta}$, $B_{ij}^{\alpha\beta}$, and $\theta_{ij}^{\alpha\beta}$ denote the conductance, susceptance, and angle difference between phases α and β, and nodes i and j respectively.

B. Linear Approximations

Three linear approximations are applied to linearize the relationship among active and reactive power, and voltage magnitude and phase angle.

Approximation 1: Since the angle difference $\theta_{ij}^{\alpha\alpha}$ of each branch in the ADN is very small, the trigonometric sines and cosines of it can be approximated as: $\cos\theta_{ij}^{\alpha\alpha} \approx 1$ and $\sin\theta_{ij}^{\alpha\alpha} \approx \theta_{ij}^{\alpha\alpha}$. $\theta_{ij}^{\alpha\beta}$, the angle difference of the different phases, is near $2/3\pi$ or $-2/3\pi$, the trigonometric functions of which can also be linearized similarly near $2/3\pi$ or $-2/3\pi$.

Approximation 2: The productions of the two voltages are linearized as follows by neglecting the second-order small quantities [3].

$$V_k(V_k-V_j)=(V_k-V_j)+(1-V_k)^2-(1-V_k)(1-V_j)\approx(V_k-V_j) \tag{2}$$

Approximation 3: The ZIP model of the loads describing the load variation with respect to the voltage is (taking active power as an example):

$$P(V) = (F_{\mathrm{Z}}V^2 + F_{\mathrm{I}}V + F_{\mathrm{P}})P_0 \tag{3}$$

where F_{Z}, F_{I}, and F_{P} are the constant coefficients corresponding to impedance, current, and power contributions; P_0 denotes the active power of the load under normal voltage.

The second-order small quantities ΔV^2 is neglected to obtain the approximated IP model of the loads.

$$\begin{aligned} P(V) &= (F_{\mathrm{Z}}V^2 + F_{\mathrm{I}}V + F_{\mathrm{P}})P_0 \\ &= (F_{\mathrm{Z}}(1+\Delta V)^2 + F_{\mathrm{I}}(1+\Delta V) + F_{\mathrm{P}})P_0 \\ &\approx (F_{\mathrm{P}} - F_{\mathrm{Z}})P_0 + [(2F_{\mathrm{Z}} + F_{\mathrm{I}})P_0]\, V \end{aligned} \tag{4}$$

C. Linear Power Flow Model

The first two approximations linearize the nonlinear terms and trigonometric functions in the right side of (1) by first order approximation. The third approximation transforms the ZIP model into the PI model of the loads in the left side of (1) by removing the production of two voltages. Thus, the relationship between voltages and injections described in (1) is fully linearized.

Equation (1) can be viewed as the summation of two parts:

$$A = V_i^{\alpha} \sum_{j=1}^{n} \sum_{\beta=a}^{c} V_j^{\beta} G_{ij}^{\alpha\beta} \cos\theta_{ij}^{\alpha\beta} \tag{5}$$

$$B = V_i^{\alpha} \sum_{j=1}^{n} \sum_{\beta=a}^{c} V_j^{\beta} B_{ij}^{\alpha\beta} \sin\theta_{ij}^{\alpha\beta}. \tag{6}$$

To illustrate our derivation process, the first part of (1), which is denoted as A, is provided to demonstrate how the linear relationship is derived. The rest of (1) can be linearized accordingly.

According to Approximation 1, $\cos\theta_{ij}^{\alpha\beta}$ can be linearized as follows:

$$\begin{aligned} A &= V_i^{\alpha} \sum_{j=1}^{n} \sum_{\beta=a}^{c} V_j^{\beta} G_{ij}^{\alpha\beta} \cos\theta_{ij}^{\alpha\beta} \\ &\approx \sum_{j=1}^{n} V_j^{\alpha} G_{ij}^{\alpha\alpha} + \sum_{j=1}^{n} \sum_{\beta\neq\alpha} V_j^{\beta} G_{ij}^{\alpha\beta} \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\theta_{ij}^{\alpha\beta}\right). \end{aligned} \tag{7}$$

By disassembling the second sum term and according to Approximation 2, (7) can be further approximated:

$$\begin{aligned} A \approx\ & \frac{3}{2}\sum_{j=1}^{n} V_j^{\alpha} G_{ij}^{\alpha\alpha} - \frac{1}{2}\sum_{j=1}^{n}\sum_{\beta=\mathrm{a}}^{\mathrm{c}} V_j^{\beta} G_{ij}^{\alpha\beta} \\ & + \frac{\sqrt{3}}{2}\sum_{j=1}^{n} G_{ij}^{\alpha\alpha_{\mathrm{lead}}}\theta_{ij}^{\alpha\alpha_{\mathrm{lead}}} - \frac{\sqrt{3}}{2}\sum_{j=1}^{n} G_{ij}^{\alpha\alpha_{\mathrm{lag}}}\theta_{ij}^{\alpha\alpha_{\mathrm{lag}}} \end{aligned} \tag{8}$$

where α_{lead} and α_{lag} denote the phases that lead or lag to phase α respectively.

Then, the original three-phase power equations can be neatly presented in a matrix form:

$$\begin{bmatrix} \boldsymbol{P}^{\mathrm{A}} \\ \boldsymbol{P}^{\mathrm{B}} \\ \boldsymbol{P}^{\mathrm{C}} \\ \boldsymbol{Q}^{\mathrm{A}} \\ \boldsymbol{Q}^{\mathrm{B}} \\ \boldsymbol{Q}^{\mathrm{C}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_{PV}^{\mathrm{AA}} & \boldsymbol{J}_{PV}^{\mathrm{AB}} & \boldsymbol{J}_{PV}^{\mathrm{AC}} & \boldsymbol{J}_{P\theta}^{\mathrm{AA}} & \boldsymbol{J}_{P\theta}^{\mathrm{AB}} & \boldsymbol{J}_{P\theta}^{\mathrm{AC}} \\ \boldsymbol{J}_{PV}^{\mathrm{BA}} & \boldsymbol{J}_{PV}^{\mathrm{BB}} & \boldsymbol{J}_{PV}^{\mathrm{BC}} & \boldsymbol{J}_{P\theta}^{\mathrm{BA}} & \boldsymbol{J}_{P\theta}^{\mathrm{BB}} & \boldsymbol{J}_{P\theta}^{\mathrm{BC}} \\ \boldsymbol{J}_{PV}^{\mathrm{CA}} & \boldsymbol{J}_{PV}^{\mathrm{CB}} & \boldsymbol{J}_{PV}^{\mathrm{CC}} & \boldsymbol{J}_{P\theta}^{\mathrm{CA}} & \boldsymbol{J}_{P\theta}^{\mathrm{CB}} & \boldsymbol{J}_{P\theta}^{\mathrm{CC}} \\ \boldsymbol{J}_{QV}^{\mathrm{AA}} & \boldsymbol{J}_{QV}^{\mathrm{AB}} & \boldsymbol{J}_{QV}^{\mathrm{AC}} & \boldsymbol{J}_{Q\theta}^{\mathrm{AA}} & \boldsymbol{J}_{Q\theta}^{\mathrm{AB}} & \boldsymbol{J}_{Q\theta}^{\mathrm{AC}} \\ \boldsymbol{J}_{QV}^{\mathrm{BA}} & \boldsymbol{J}_{QV}^{\mathrm{BB}} & \boldsymbol{J}_{QV}^{\mathrm{BC}} & \boldsymbol{J}_{Q\theta}^{\mathrm{BA}} & \boldsymbol{J}_{Q\theta}^{\mathrm{BB}} & \boldsymbol{J}_{Q\theta}^{\mathrm{BC}} \\ \boldsymbol{J}_{QV}^{\mathrm{CA}} & \boldsymbol{J}_{QV}^{\mathrm{CB}} & \boldsymbol{J}_{QV}^{\mathrm{CC}} & \boldsymbol{J}_{Q\theta}^{\mathrm{CA}} & \boldsymbol{J}_{Q\theta}^{\mathrm{CB}} & \boldsymbol{J}_{Q\theta}^{\mathrm{CC}} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}^{\mathrm{A}} \\ \boldsymbol{V}^{\mathrm{B}} \\ \boldsymbol{V}^{\mathrm{C}} \\ \boldsymbol{\theta}^{\mathrm{A}} \\ \boldsymbol{\theta}^{\mathrm{B}} \\ \boldsymbol{\theta}^{\mathrm{C}} \end{bmatrix} \tag{9}$$

where $\boldsymbol{P}^{\alpha}$, $\boldsymbol{Q}^{\alpha}$, $\boldsymbol{V}^{\alpha}$, and $\boldsymbol{\theta}^{\alpha}$ denote the N-dimensional vectors for active power, reactive power, voltage magnitude and angle of phase α. $\boldsymbol{J}_{(PQ)(V\theta)}^{\alpha\beta}$ is the $\mathrm{N}\times\mathrm{N}$ dimensional matrix describing the coupling relationship between active (or reactive) power and voltage magnitude (or angle) of phases α and β. This matrix uncovers how the injections of the nodes effect the nodal voltages.

The matrix $\boldsymbol{J}_{(PQ)(V\theta)}^{\alpha\beta}$ can be calculated using a three-phase admittance matrix. For example, $\boldsymbol{J}_{PV}^{\mathrm{AB}} = -\frac{1}{2}\boldsymbol{G}^{\mathrm{AB}} + \frac{\sqrt{3}}{2}\boldsymbol{B}^{\mathrm{AB}}$, $\boldsymbol{J}_{PV}^{\mathrm{AA}} = \boldsymbol{G}^{\mathrm{AA}}$, where $\boldsymbol{G}^{\alpha\beta}$ and $\boldsymbol{B}^{\alpha\beta}$ denote the conductance and susceptance matrices of phases α and β. If the ZIP model of the loads is considered, the vectors $\boldsymbol{P}^{\alpha}$, $\boldsymbol{Q}^{\alpha}$ and matrices $\boldsymbol{J}_{PV}^{\alpha\alpha}$, $\boldsymbol{J}_{QV}^{\alpha\alpha}$ should be revised accordingly by (4). Finally, the vectors $\boldsymbol{V}^{\alpha}$ and $\boldsymbol{\theta}^{\alpha}$ can be solved by the Gauss elimination using (9).

D. Discussions

The proposed approach has four advantages:

1) *It is capable of modeling unbalanced networks using the three-phase admittance matrix*: The unbalance of the distributed energy resource allocation aggravates the unbalance of the active distribution network. By extending the single phase power flow equations and using a three-phase admittance matrix, the unbalance of the distribution system can be modeled.
2) *Both PQ, and PV nodes can be taken into consideration*: The PQ nodes and PV nodes can be directly modeled into the original unbalanced three-phase power flow model. The PI node means that there is a constant current injection into the node,
3) *The ZIP model of the loads can be linearly embedded*: Approximation 3 shows that the ZIP characteristics of the load can be linearized as an IP model. Thus, the active and reactive power injected to the node is a linear function of the node voltage. By revising the vectors $\boldsymbol{P}^{\alpha}$, $\boldsymbol{Q}^{\alpha}$ and matrices $\boldsymbol{J}_{PV}^{\alpha\alpha}$, $\boldsymbol{J}_{QV}^{\alpha\alpha}$, the ZIP model can be taken into account.
4) *It is applicable to both a radial and meshed network*: It should be noted that there is not any assumption and approximations for the topology of the distribution network in our proposed method. On the contrary, the meshed network has less voltage drop compared with radial network, which will improve the performance of our proposed method.

III. Case Study

The proposed method is applied to the IEEE 37-node test feeder with unbalanced loads [4]. For simplicity, the voltage dependencies of all loads are assumed to be identical ($F_{\mathrm{Z}} = 0.7$, $F_{\mathrm{I}} = 0.2$, $F_{\mathrm{P}} = 0.1$) for both active and reactive power. Nodes #21 and #27 are specified as PV nodes with the same voltage magnitudes as the original system in the case with PV nodes. The voltages obtained by the back-forward sweep algorithm are used as "real" values. The error of the proposed method is defined as the absolute difference between the calculated value and real value. Fig. 1 shows the real and calculated voltage magnitudes with (W) and without

(O) PV nodes. The dotted line (calculated) is close to, or even overlapped with the solid line (real value). Without PV nodes, the maximum and average errors are 1.49×10^{-3} and 3.70×10^{-4}. Such errors meet the needs of many practical applications. Fig. 2 presents the errors of the three-phase voltage magnitudes calculated by our proposed method with PV nodes. The maximum and average errors are 2.91×10^{-4} and 8.79×10^{-5} respectively, which are comparable with the method proposed in [1]. The introduction of PV nodes can also largely reduce the error.

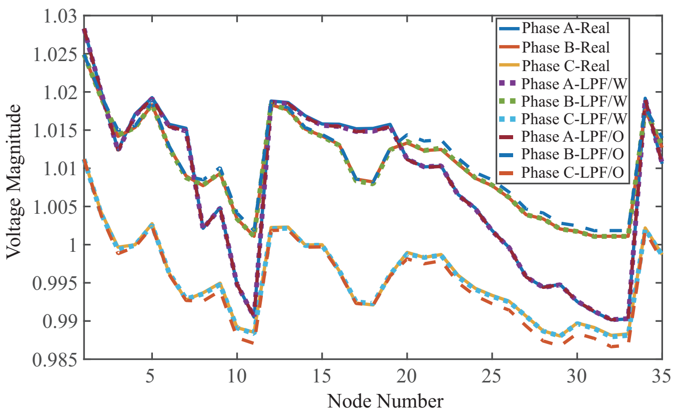

Fig. 1. Voltage magnitudes of all nodes calculated by AC power flow and the proposed method with (W) and without (O) PV nodes.

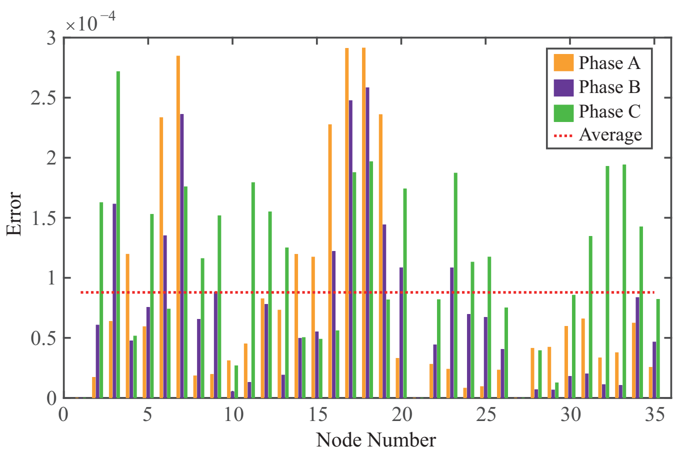

Fig. 2. Error of the three-phase voltage magnitudes between AC power flow and the proposed method with PV nodes.

In rural distribution networks, voltage percent variations may be higher than 5%. To illustrate that our proposed method is applicable for larger voltage variations, the original load is expanded by 150%, 200%, and 300%. The average error of the calculated voltage magnitudes are 1.00×10^{-3}, 1.40×10^{-3}, and 2.80×10^{-3}, respectively. The calculated voltage magnitude of all nodes when the loads are tripled is shown in Fig. 3. It can be seen that even through the node voltage can be as low as 0.91, the calculated voltages can obtain relative high accuracy. In addition, the reactive power injections of the PV nodes (Nodes #21 and #27) have been calculated. The relative errors are 0.013% and 0.014% respectively, which also has acceptable accuracy.

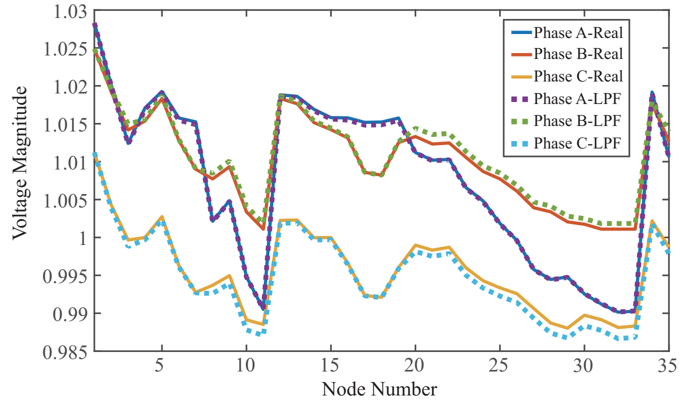

Fig. 3. Voltage magnitudes of all nodes calculated by AC power flow and the proposed method with loads tripled.

IV. Conclusion

This paper proposes a linear three-phase power flow model for unbalanced an ADN. The ZIP model of the loads and PV nodes is effectively considered in the proposed model. It can achieve high accuracy especially for the ADN with PV nodes. The error is so small that the proposed linearized power flow has great potential in probabilistic load flow analysis, linear or convex optimal power flow, and volt-VAR control.

References

[1] A. Garces, "A Linear Three-Phase Load Flow for Power Distribution Systems," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 827–828, Jan. 2016.

[2] H. Ahmadi, J. R. Martı´, A. von Meier, "A Linear Power Flow Formulation for Three-Phase Distribution Systems," *IEEE Transactions on Power Systems*, vol. 31, no. 6, pp. 5012-5021, Nov. 2016.

[3] J. W. Yang, N. Zhang, C. Q. Kang, and Q. Xia, "A State-Independent Linear Power Flow Model with Accurate Estimation of Voltage Magnitude," *IEEE Transactions on Power Systems*, vol. 32, no. 5, pp. 3607-3617, Sept. 2017.

[4] W. H. Kersting, "Radial distribution test feeders," *IEEE Transactions on Power Systems*, vol. 6, no. 3, pp. 975-985, Aug 1991.

Yi Wang (S'14) received the B.S. degree from the Department of Electrical Engineering in Huazhong University of Science and Technology (HUST), Wuhan, China, in 2014.

He is currently pursuing Ph.D. degree in Tsinghua University. He is also a visiting student researcher at the University of Washington, Seattle, WA, USA. His research interests include data analytics in smart grid and multiple energy systems.

Ning Zhang (M'12) received both a B.S. and Ph.D. from the Electrical Engineering Department of Tsinghua University in China in 2007 and 2012, respectively.

He is currently an Assistant Professor at the same university. His research interests include multiple energy system integration, renewable energy, and power system planning and operation.

Hai Li received the B.S. degree from the Department of Electrical Engineering in Huazhong University of Science and Technology (HUST), Wuhan, China, in 2017.

He is currently pursuing Ph.D. degree in Tsinghua University. His research interests include optimal power flow and renewable energy.

Jingwei Yang (S'15) received the B.S. degree from the Electrical Engineering Department of Tsinghua University in China in 2015.

He is currently pursuing Ph.D. degree in Tsinghua University. His research interests include optimal power flow, renewable energy and multiple energy system integration.

Chongqing Kang (M'01–SM'08–F'17) received the Ph.D. degree from the Department of Electrical Engineering in Tsinghua University, Beijing, China, in 1997.

He is currently a Professor in Tsinghua University. His research interests include power system planning, power system operation, renewable energy, low carbon electricity technology and load forecasting.