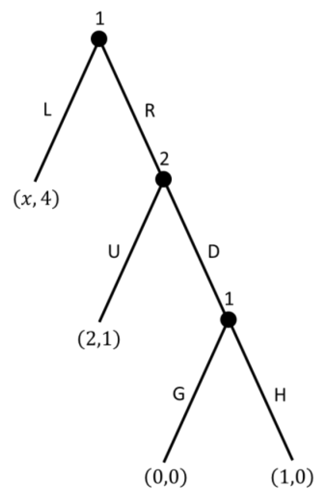
Homework 2

Problem 1 (Extensive form game with perfect information)

Consider the following extensive form game:



(1) when $x = 1$, find the all pure strategy Nash equilibria and subgame perfect Nash equilibria

Using backward induction, we find that the unique SPNE is (RH, U). The normal form of the game is given by the matrix

	U	D
LG	1,4	1,4
LH	1,4	1,4
RG	2,1	0,0
RH	2,1	1,0

The set of NE are: (LG, D), (LH, D), (RG, U), (RH, U)

(2) Find the range of x for which (R, U) is the unique subgame perfect NE outcome

The unique SPNE of the subgame beginning after history R is (U, H) with a payoff of $(2, 1)$. Therefore, at the beginning of the game, R is the unique optimal choice if $x < 2$.

(3) Find the range of x for which L is a Nash equilibrium outcome

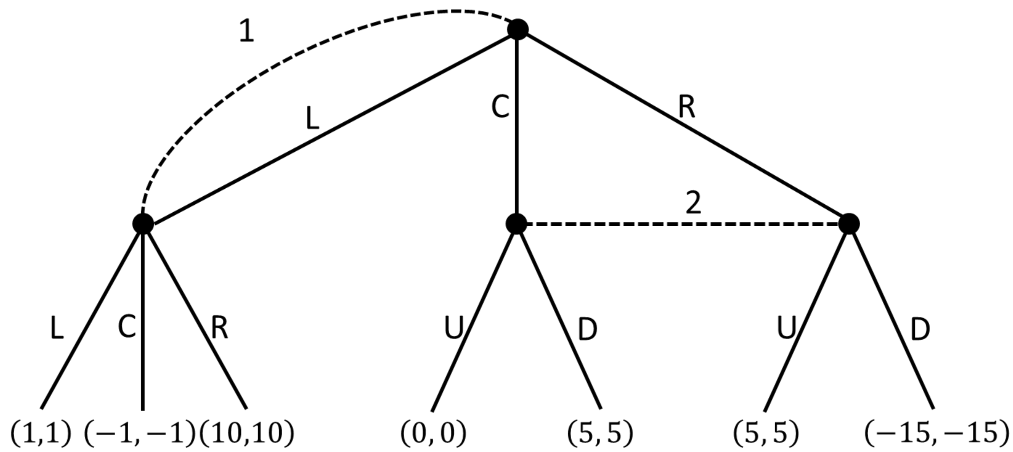
The normal of the extensive form game given x is:

	U	D
LG	x, 4	x, 4
LH	x, 4	x, 4
RG	2, 1	0, 0
RH	2, 1	1, 0

We want to find the range of x that makes at least one of (LG, U), (LG, D), (LH, U), (LH, G) a NE; then L is a NE outcome. If $x < 1$, then none of these are NE. Therefore, the range of x is $x \geq 1$.

Problem 2 (Imperfect information games, Pure vs. Mixed vs. Behavioral)

Consider the two-player game of imperfect information given in Figure 1. It is a common-value game, so the value at a leaf defines the payoff of both players. You must justify your answers.



(1) List all pure strategy Nash equilibria ("none exists" is a possible answer).

Note that if we are only considering pure strategies (or mixed strategies for that matter), the choices C and R at player 1's second choice point do not matter. The only pure Nash equilibria are: (C, D) and (R, U). (L, U) and (L, D) are not equilibria because Player 1 could improve his score by switching from L.

(2) Now we include mixed (but not behavioral) strategies. List all of the Nash equilibria (excluding any that you already found in part (1)). As there could be an infinite number of mixed Nash equilibria, you should use variables and give ranges over which strategy profiles constitute Nash equilibria.

First, we note that if player 1 plays a mixture of L/C or L/R, then player 2's only best response is to play a pure strategy of D (against L/C) or U (against L/R). But then it is clear that Player 1 is not playing a best response, since he could improve by shifting out of L entirely. Now we may consider a general mixture of $L/C/R$. As we learned above, C and R must both be in Player 1's support, so we may say that player 1 plays R with probability q $(0 < q < 1)$, plays C with probability p $(0 < p < 1)$, and plays L with the remainder, i.e. $1 - p - q$.

In order for such a mixture to work in an equilibrium, player 1 must be indifferent between his expected utility after playing C and his expected utility of playing R, for otherwise he could improve by playing pure C or pure R. Thus, we calculate player 2's mixture to make player 1 indifferent between C and R. This mixture turns out to be $([4/5]U/[1/5]D)$, since the expected utility of both C and R is now 1.

We now see that player 1 is now indifferent between L, C, and R, since his expected utility of playing L is 1 as well. Any mixture of $L/C/R$ is thus a best response to $([4/5]U/[1/5]D)$. To find the equilibria, it only remains to find the set of player 1 mixtures which will make player 2 indifferent between U and D, thus making $([4/5]U/[1/5]D)$ a best response, We note that player 2 expects $5q$ utility from playing U, and $5p - 15q$ utility from playing D. Setting these equal, $5p = 20q$, so $p = 4q$. Thus, as long as player 1 is four times more likely to play C than R, player 2 is indifferent between U and D.

We are now prepared to fully describe the equilibria:

- Player 1: $([1-5q]L/[4q]C/[q]R)$ for q in $[0,0.2]$
- Player 2: $([4/5]U/[1/5]D)$

(3) What is the highest expected payoff obtainable if the players use behavioral strategies?

Answer:

Let player 1 behaviorally play L with probability p, and R with probability $1-p$. Player 2 plays pure U. The expected payoff is then $p2 + (p)(1-p)(10) + (1-p)(5) = 5 + 5p - 9p^2$. Maximizing this function, we see that the maximum occurs when $p = 5/18$. Plugging 5/18 in for p, we get back our final answer of $205/36 \approx 5.69$

Problem 3 (An Infinitely repeated game: Tit-for-Tat strategy)

Consider the following infinitely repeated version of the following game:

	C	D
C	4, 4	0, 6
D	6, 0	1, 1

The payoff of player i to any infinite sequence of payoffs $\{u_i^t\}$ is given by the normalized discounted sum of payoffs:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i^t$$

where $0 < \delta < 1$

(1) For what values of δ, if any, does it constitutes a subgame perfect equilibrium when both player choose this strategy?

- Choose C in period 1.
- Choose C after any history in which the previous periods' outcome was (C, C)
- Choose D after any other history

Sol)

- On the equilibrium path
    - If both players do not deviate: outcome path is $(C, C), (C, C)$,… and the payoffs are $(4, 4), (4, 4)$,… thus both players' discounted average payoffs are 4.
    - Suppose player 1 does a one-shot deviation. Then the outcome path is $(D, C), (D, D), (D, D)$, … and the payoffs are $(6, 0), (1, 1), (1,1)$,…thus player 1's discounted average payoff is $(1-\delta)(6+\delta+\delta^2+\cdots) = 6(1-\delta)+\delta$
    - Not deviating is optimal if $4 \geq 6(1-\delta)+\delta$, or if $\delta \geq \frac{2}{5}$.

- Off the equilibrium path
    - Outcome path is $(C, C), (C, C)$,…and the payoffs are $(1, 1), (1, 1)$,…thus both players' discounted average payoffs are 1.
    - Suppose player 1 does one-shot deviation. Then the outcome path is $(C, D), (D, D), (D, D)$, … and the payoffs are $(0, 6), (1, 1), (1, 1)$,…
    - Player 1's discounted average payoff is always smaller than 1 regardless of δ

- Therefore, this is a SPNE if $\delta \geq \frac{2}{3}$

(2) For what values of δ, if any, does it constitutes a subgame perfect equilibrium when both player choose this strategy?

- Choose C in period 1.
- Do whatever our opponent did in the previous period.

Which is called Tit-for-Tat strategy.

- Ending in (C, C): not deviating gives outcome path $(C, C), (C, C)$,..., which results in a discounted average payoff of 4 to both players. Suppose player 1 does a one shot deviation, which gives outcome path $(D, C), (C, D), (D, C), (C, D)$,... Player 1 gets payoffs 6, 0, 6, 0,... with a discounted average of $\frac{6}{1+\delta}$. Not deviating is optimal if $4 \geq \frac{6}{1+\delta}$, or if $\delta \geq \frac{1}{2}$

- Ending in (C, D): not deviating gives outcome path $(D, C), (C, D), (D, C), (C, D)$... Player 1 gets payoff sequence 6, 0, 6, 0,... with a discounted average of $\frac{6}{1+\delta}$. If player 1 does a one-shot deviation, the outcome path is $(C, C), (C, C)$,...with a discounted average payoff of 4. Not deviating is optimal $4 \leq \frac{6}{1+\delta}$, or if $\delta \leq \frac{1}{2}$

- Ending in (D, C): not deviating gives outcome path $(C, D), (D, C), (C, D), (D, C)$, ... Player 1 gets payoff sequence 0, 6, 0, 6, 0,... with a discounted average of $\frac{6\delta}{1+\delta}$. If player 1 does a one-shot deviation, the outcome path is $(D, D), (D, D)$,...with a discounted average payoff of 1. Not deviating is optimal $\frac{6\delta}{1+\delta} \geq 1$, or if $\delta \geq \frac{1}{5}$.

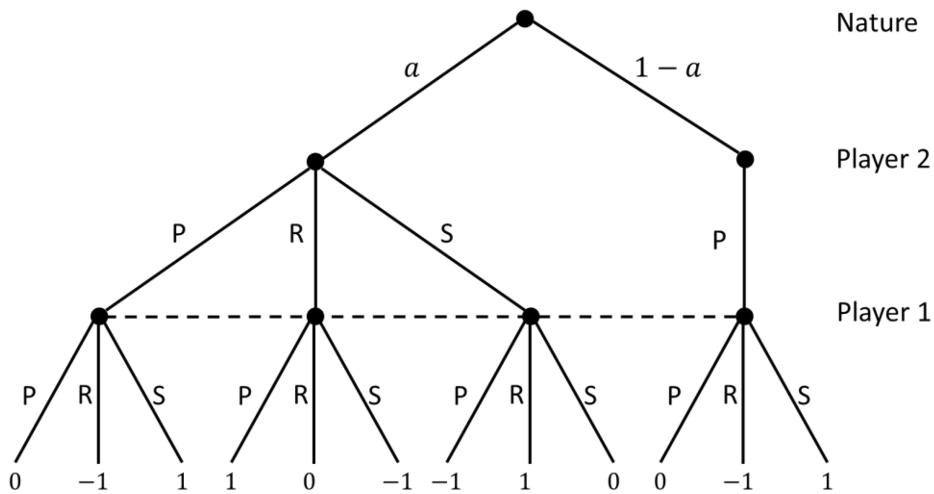
- Ending in (D, D): not deviating gives outcome path $(D, D), (D, D)$, ... Player 1 gets a discounted average payoff of 1. If player 1 does a one-shot deviation, the outcome path is $(C, D), (D, C), (C, D), (D, C)$, ...Player 1 gets payoff sequence 0, 6, 0, 6,... with a discounted average of $\frac{6\delta}{1+\delta}$. Not deviating is optimal if $1 \geq \frac{6\delta}{1+\delta}$, or $\delta \leq \frac{1}{5}$.

  There is no δ that satisfies all these conditions, so this strategy profile is not a SPNE.

Problem 4 (Extensive form games)

Consider the following Rock (R), Paper (P), Scissors (S) game. Suppose that, with probability a, player 1 faces a rational opponent (who believes there to be common knowledge of rationality) and, with probability $1 - a$, she faces an opponent who will play P for sure. That is, before the game, Nature selects player 2's type and player 1 does not see Nature's choice.

(1) Draw the extensive form of this Bayesian game:



Payoff for player 1 only are displayed. As this is a zero-sum game, player 2's payoffs are the negation of player 1's payoffs.

(2) Denote strategies for player 2 as $(P, P), (P, R), (P, S)$. Complete the following normal form representation of the game: (Since this is a zero-sum game, simply write player 1's expected payoff in that situation in each cell.)

	PP	PR	PS
P	0	a	$-a$
R	-1	$a-1$	$2a-1$
S	1	$1-2a$	$1-a$

(3) Find all equilibria of this game for $a = 1/3$.

If we substitute 1/3 for a, the table from (2) becomes:

	PP	PR	PS
P	0	$1/3$	$-1/3$
R	-1	$-2/3$	$-1/3$
S	1	$1/3$	$2/3$

After elimination of strictly dominated strategies we are left with the following 2X2 normal form game:

	PR	PS
P	1/3	$-1/3$
S	1/3	$2/3$

It is clear that in any equilibrium player 2 must play (P, R), otherwise player 1 will strictly prefer playing S in which case player 2 will strictly prefer playing (P, R). Thus, in any equilibrium player 1, who is indifferent to his strategies if player 2 plays P, R, must preserve player 2's preference to play P, R. If we assign p to be player 1's probability of playing P this mean the following inequalities must be satisfied.

$$\frac{1}{3} \leq -\frac{\mathrm{p}}{3} + \frac{2(1-p)}{3}$$
$$\frac{1}{3} \leq -p + \frac{2}{3}$$
$$-\frac{1}{3} \leq -p$$
$$p \leq 1/3$$

Therefore, all Nash equilibria are of the form $(P, R, (pP, (1-p)S))$ for $p \leq 1/3$.

Problem 5 (Folk Theorem)

Consider the following the following game:

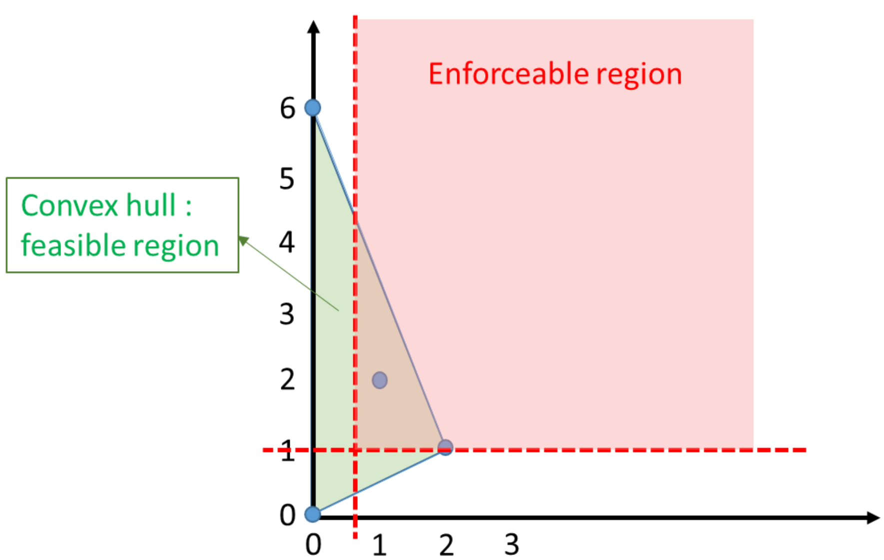
	L	R
U	0, 6	1, 2
D	2, 1	0, 0

(1) What is the **minmax** payoff for each player and what strategy does the other player have to play to force that payoff (i.e. the other player's minimax strategy)?

In this game, player 2 can force player 1's maximin payoff by making player 1 indifferent. It is easy to see that this happens when player 2 plays $\left(\frac{1}{3}L, \frac{2}{3}\mathrm{R}\right)$. The best payoff player 1 can hope to achieve agains this payoff is 2/3. Player 2, however, has a dominant strategy (L). The lowest payoff player 2 can get by playing this strategy is 1 and this happens when player 1 plays (D).

(2) Which of the following payoffs are feasible and enforceable for this game? (check if a strategy satisfies the two conditions)

	Yes	No
1. $(0, 0)$		V
2. $(0, 6)$		V
3. $(1, 2)$	V	
4. $(1, 3.5)$	V	
5. $(1, 0.5)$		V
6. $(2, 2)$		V



(3) Consider the infinitely repeated version of this game with a limit average reward, describe a Nash equilibrium of the repeated game that achieves a payoff of (1,3).

Player 1 plays the following sequence of strategies repeatedly (U, U, D). Player 2 plays (L, R, L). If player 2 deviates from this sequence at any point, player 1 pulls the trigger by playing D forever. If player 1 deviates, player 2 likewise pulls the trigger by playing $\left(\frac{1}{3}\mathrm{L}, \frac{2}{3}\mathrm{R}\right)$ forever. This pair of strategies will evenly play the actions UL, UR, and DL, which achieves a payoff vector of (1,3). Furthermore, it is a Nash equilibrium because the payoff of deviating is 2/3 and 1 for players 1 and 2 respectively which is not a best response.

Problem 6 (Battle of Sex game with incomplete information)

Consider a variant of the situation modeled by Battle of Sex in which player 1 is unsure whether player 2 prefers to go out with her or prefers to avoid her, whereas player 2 knows player 1's preferences. Specifically, suppose player 1 thinks that with probability 1/2 player 2 wants to got out with her, and with probability 1/2 player 2 wants to avoid her. That is, player 1 thinks that with probability 1/2 she is playing the game on the left of the figure below and with probability 1/2 she is playing the game on the right. Find Bayesian Nash equilibria

1

prob. $\frac{1}{2}$

2

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

2 wishes to meet 1

prob. $\frac{1}{2}$

2

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

2 wishes to avoid 1

$$u_1(B, BB) = \frac{1}{2}u_1(B, B, \theta_1) + \frac{1}{2}u_1(B, B, \theta_2) = 2$$

$$\vdots$$

The Bayesian game can be converted to the following normal form game, from which the BNE can be founded as $(\boldsymbol{B}, \boldsymbol{BS})$

	BB	BS	SB	SS
B	2, 0.5	**1, 1.5**	1, 0	0, 1
S	0, 0.5	0.5, 0	0.5, 1.5	1, 1

Problem 7 (Cournot's duopoly game with incomplete information)

Two firms compete in selling a good; one firm does not know the other firm's cost function. How does the imperfect information affect the firms' behavior? Assume that both firms can produce the good at constant unit cost. Assume also that they both know that firm 1's unit cost is c, but only firm 2 knows its own unit cost; firm 1 believes that firm 2's cost is c_L with probability θ and c_H with probability $1 - \theta$, where $0 < \theta < 1$ and $c_L < c_H$.

The firms' payoffs are their profits; if the actions chosen are $(q1, q2)$ and the state is I (either L or H) then firm 1's profit is $q_1(P(q_1 + q_2) - c)$ and firm 2's profit is $q_2(P(q_1 + q_2) - c_I)$, where

$$P(q1 + q2) = \begin{cases} a - (q_1 + q_2) & if\ a \geq (q_1 + q_2) \\ 0 & \text{otherwise} \end{cases}$$

is the market price when the firms' outputs are q_1 and q_2.

(1). For values of c_H and c_L close enough that there is a Nash equilibrium in which all outputs are positive, find this equilibrium.

The expected payoff for player 1 is

$$u_1(q_1, q_2^L q_2^H) = \theta[(a - q_1 - q_2^L - c)q_1] + (1 - \theta)[(a - q_1 - q_2^H - c)q_1]$$

From the first order optimality condition, $du_1/dq_1 = 0$, we have the best response curve

$$q_1 = BR_1(q_2^L, q_2^H) = \begin{cases} \frac{1}{2}[a - c - (\theta q_2^L + (1 - \theta)q_2^H)] & \text{if } \ \theta q_2^L + (1 - \theta)q_2^H \leq a - c \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we can compute the best response curve for player 2 as

$$q_2^I = BR_2^I(q_1) = \begin{cases} \frac{1}{2}(a - c_I - q_1) & \text{if } \ q_1 \leq a - c_I \\ 0 & \text{otherwise} \end{cases}$$

for the type $I \in \{H, L\}$

The three equations that define a Nah equilibria are

$$q_1^* = BR_1(q_2^{L*}, q_2^{H*}),\ q_2^{L*} = BR_2^L(q_1^*),\ q_2^{H*} = BR_2^H(q_1^*)$$

Solving these equations under the assumption that they have a solution in which all three outputs are positive, we obtain

$$q_1^* = \frac{1}{3}[a - 2c + \theta c_L + (1 - \theta)c_H]$$

$$q_2^{L*} = \frac{1}{3}(a - 2c_L + c) - \frac{1}{6}(1 - \theta)(c_H - c_L)$$

$$q_2^{H*} = \frac{1}{3}(a - 2c_H + c) + \frac{1}{6}\theta(c_H - c_L)$$

(2) Compare this equilibrium with the Nash equilibrium of the game in which firm 1 knows that firm 2's unit cost is c_L, and with the Nash equilibrium of the game in which firm 1 knows that firm 2's unit cost is c_H.

If both firms know that the unit costs of the two firms are c_1 and c_2 then in a Nash equilibrium the output of firm i is

$$q_1^* = \frac{1}{3}(a - 2c_i + c_j)$$

- If firm 1 knows firm 2's unit cost c_L:
    - $q_1^* = \frac{1}{3}(a - 2c + c_L)$
    - $q_2^* = \frac{1}{3}(a - 2c_L + c)$
- If firm 1 knows firm 2's unit cost c_H:
    - $q_1^* = \frac{1}{3}(a - 2c + c_H)$
    - $q_2^* = \frac{1}{3}(a - 2c_H + c)$

If firm 1 knew that firm 2's cost was high, then it would produce a relatively large output

if it knew this cost were low, then it would produce a relatively small output.

Problem 8 (Signaling game)

A signaling game is a two-player game in which Nature selects a game to be played according to a commonly known distribution (as in a Bayesian game), player 1 is informed of Nature's choice, then player 2 chooses an action knowing player's action but not Nature's choice. Consider the signaling game where Nature chooses between the following two games with equal probability.

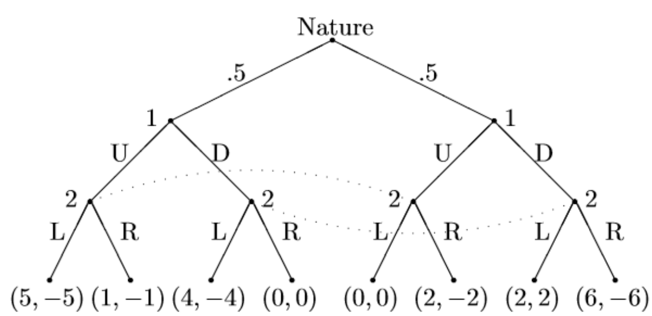
	L	*R*
U	5, -5	1, -1
D	4, -4	0, 0

	L	*R*
U	0, 0	2, -2
D	2, -2	6, -6

(1) What is player 1's optimal play in each subgame? If player 1 uses this strategy in the signaling game, what will player 1's expected payoff be if player 2 best responds?

Answer: In both subgames, player 1 has a dominant strategy and therefore asking for player 1's optimal play is well defined. For the first game player 1 would choose U and in the second player 1 would choose D. When player 2 best response with R and L respectively (which can be done even in the context of the signaling game), player 1's expected payoff is (1 + 2)/2 = 1.5.

(2) Draw an extensive-form game (with Nature) of this signaling game (in a form of an extensive-form game with Nature). Make sure to include the distribution that Nature follows.



	LL	*LR*	*RL*	*RR*
UU	(2.5,-2.5)	(2.5,-2.5)	(1.5,-1.5)	(1.5,-1.5)
UD	(3.5,-3.5)	(5.5,-5.5)	(1.5,-1.5)	(3.5,-3.5)
DU	(2,-2)	(0,0)	(3,-3)	(1,-1)
DD	(3,-3)	(3,-3)	(3,-3)	(3,-3)

(3) Find a Nash Equilibrium of the induced extensive-form game. Does it differ from the strategy specified in part (1)? Why?

It is easier to solve this game using the induced normal form of this game (Table 5). From this table it is easy to see that the only pure strategy is (DD, RL).

Player 1's strategy in any of these equilibria (DD) differs from the strategy specified in part (a) because in this game there is value in not giving player 2 any information about Nature, which can be done by discarding the information about Nature that player 1 knows. In this case, the benefit of player 2's lack of knowledge offsets the handicap of not being able to act on Nature.