

MATRIX TWO-PERSON GAMES

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Outline

👉 Matrix Two-Person Games

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↳ Matrix Two-Person Games

↳ Solution

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↳ Solution

↳ NASH equilibrium

Outline

☛ Matrix Two-Person Games

☛ Solution

☛ NASH equilibrium

☛ Symmetric Games

Introduction

JOHN VON NEUMANN

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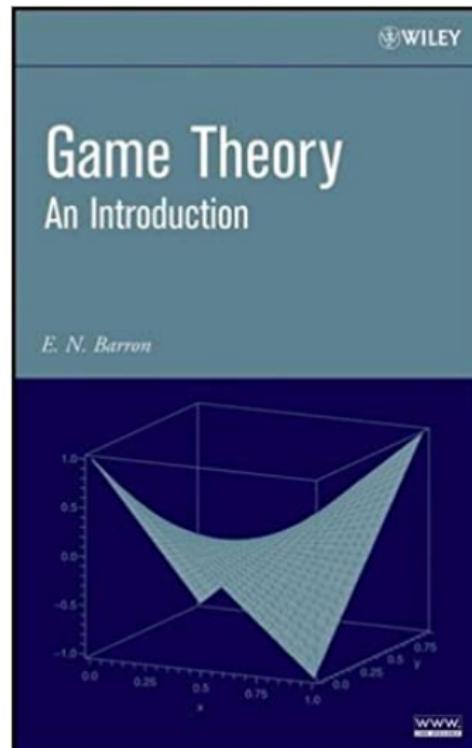
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- 👉 Player II \rightarrow strategy j , $j = 1, \dots, m$

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MATRIX Two-PERSON GAME (**Zero sum game**)

Player I chooses strategy i , Player II chooses strategy j

then Player I get a_{ij} and Player II get $-a_{ij}$.

Both players want to **maximize** their individual payoffs.

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- Player II \rightarrow strategy $j, j = 1, \dots, m$

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GAME MATRIX $A = (a_{ij})_{n \times m}$

Player I wants to have a strategy $i_* \in I := \{1, \dots, n\}$ such that

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Put $v^- = \min_j \max_i a_{ij}$ $v^+ = \max_i \min_j a_{ij}$

$$v^- \leq v^+ (\textcolor{red}{Exercise!})$$

Saddle points in pure strategies

👉 Player I → row player

Saddle points in pure strategies

- 👉 Player I → row player
- 👉 Player II → column player

Saddle points in pure strategies

👉 Player I → row player

👉 Player II → column player

We call a pair of row i_* and a column j_* a *saddle point in pure strategies* of the game if

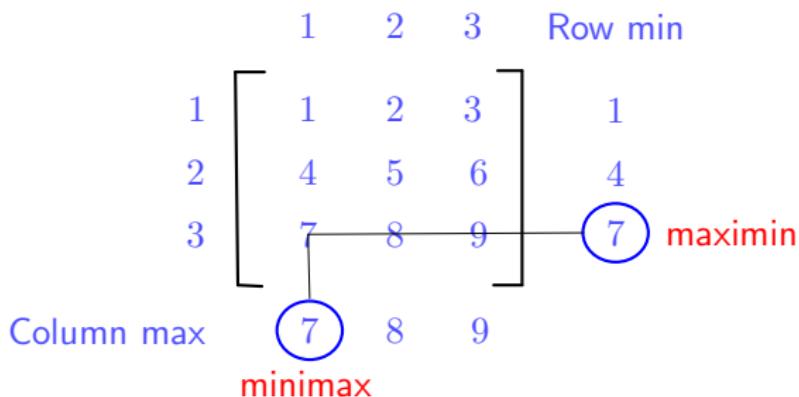
$$a_{ij_*} \leq a_{i_*j_*} \leq a_{i_*j} \quad \forall i = 1, \dots, n; j = 1, \dots, m.$$

👉 A game has a saddle point in pure strategies if and only if

$$v^- = \min_j \max_i a_{ij} = \max_i \min_j a_{ij} = v^+$$

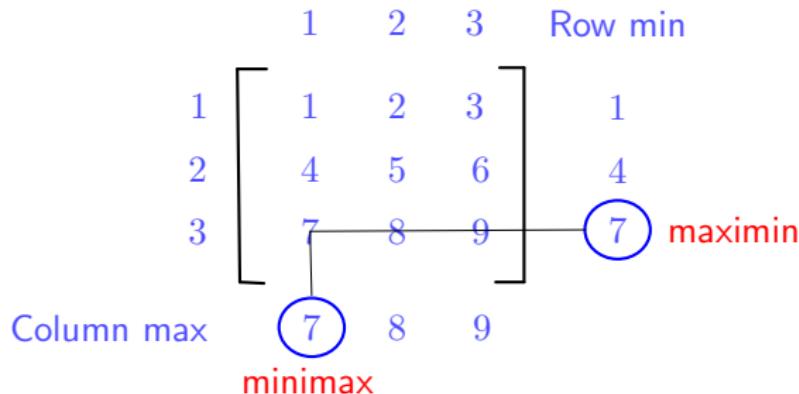
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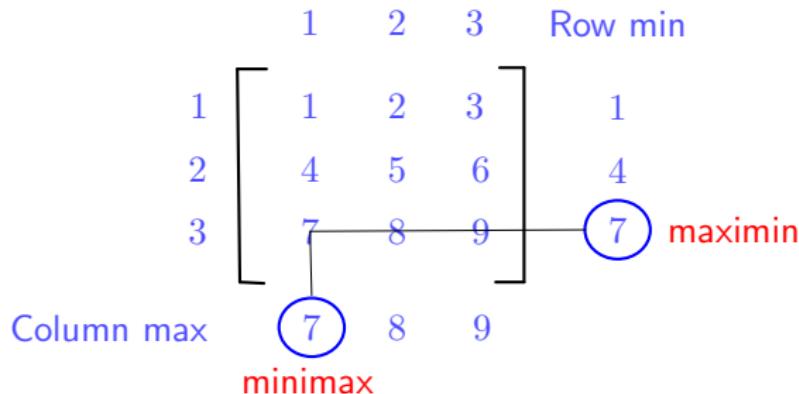
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👉 There may exist more than one solution (saddle point) in pure strategies

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👉 $v = v^- = v^+$ is called the *value of the game*

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	1	2	3	4	Row min
1	5	-3	3	4	-3
2	-4	5	4	5	-4
3	4	-4	-3	3	-4
Column max	5	5	4	5	
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👉 A solution (saddle point) in pure strategies *may not exist*

John Nash proved that in any game where a finite number of players each has a finite number of choices, there is at least one position from which no single player alone can improve his/her position by changing strategy.

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 NASH EQUILIBRIUM

A mixed strategy

Suppose that the players play the game many times. A *mixed strategy* for player I is a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with

$$x_i \geq 0 \quad \forall i \in I, \quad \sum_{i \in I} x_i = 1. \quad (1)$$

Denote by S_n the set of $x = (x_1, \dots, x_n)$ satisfying (1). The choice of a mixed strategy $x \in S_n$ of player I means that this player selects strategy $i \in I$ with the *probability* x_i . Similarly, a mixed strategy for player II is a vector $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ with

$$y_j \geq 0 \quad \forall j \in J, \quad \sum_{j \in J} y_j = 1. \quad (2)$$

Let S_m be the set of $y = (y_1, \dots, y_m)$ satisfying (2). The choice of a mixed strategy $y \in S_m$ of player II means that the player selects strategy $j \in J$ with the *probability* y_j .

If Player I abides by a mixed strategy $x \in S_n$ and Player II abides by a mixed strategy $y \in S_m$, then the *expected average payoff* to Player I of the game is

$$\begin{aligned} E(x, y) &= \sum_{i \in I} \sum_{j \in J} a_{ij} \text{Prob}(Player\ I\ uses\ i\ and\ Player\ II\ uses\ j) \\ &= \sum_{i \in I} \sum_{j \in J} a_{ij} \text{Prob}(Player\ I\ uses\ i) \cdot \text{Prob}(Player\ II\ uses\ j) \\ &= x^T A y. \end{aligned}$$

Saddle points

Minimax Theorem

Saddle point in mixed strategies

👉 A *saddle point in mixed strategies* is a pair (\bar{x}, \bar{y}) of probability vectors $\bar{x} \in S_n$ and $\bar{y} \in S_m$ satisfying

$$E(x, \bar{y}) \leq E(\bar{x}, \bar{y}) \leq E(\bar{x}, y) \quad \forall (x \in S_n, y \in S_m). \quad (3)$$

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Put $f(x, y) = x^T A y$. Then, (3) means

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall (x \in S_n, y \in S_m).$$

👉 Game has a saddle point in mixed strategies *if and only if*

$$\min_{y \in S_m} \max_{x \in S_n} f(x, y) = \max_{x \in S_n} \min_{y \in S_m} f(x, y)$$

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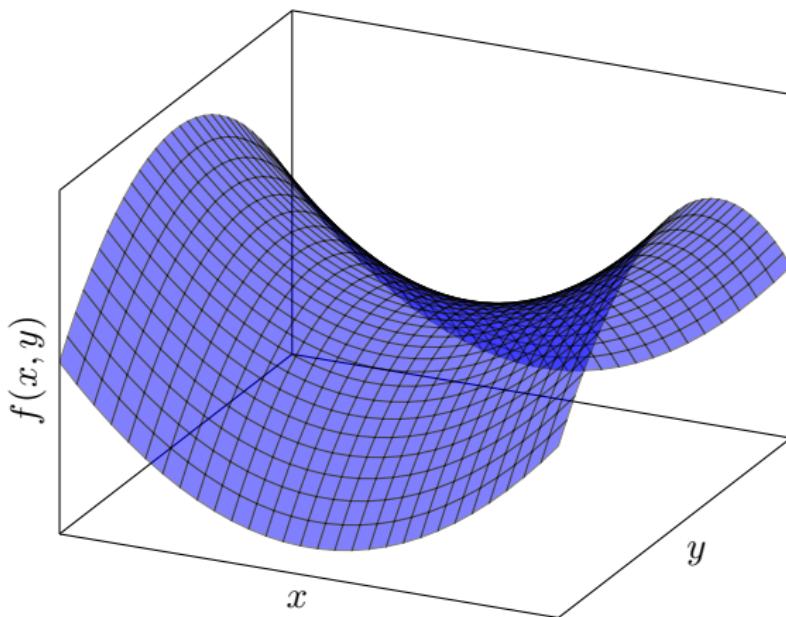
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Hanoi

