# Matrix Two-Person Games

**Duong Thi Kim Huyen**

ORLab, Phenikaa School of Computing, PHENIKAA University

# Introduction

John von Neumann

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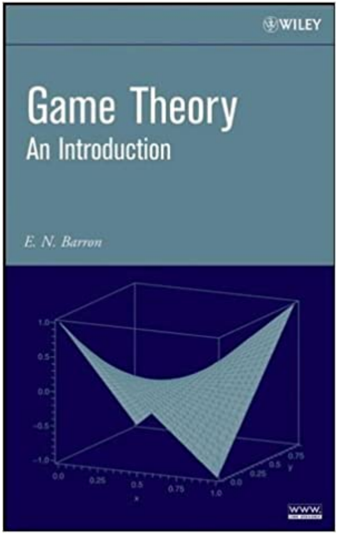

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- Player II $\longrightarrow$ strategy $j$, $j = 1, ..., m$

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MATRIX TWO-PERSON GAME (Zero sum game)

Player I chooses strategy $i$, Player II chooses strategy $j$

then Player I get $a_{ij}$ and Player II get $-a_{ij}$.

Both players want to *maximize* their individual payoffs.

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GAME MATRIX $A = (a_{ij})_{n \times m}$

Player I wants to have a strategy $i_* \in I := \{1, \ldots, n\}$ such that

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$$v^- \leq v^+ (Exercise!)$$

# Saddle points in pure strategies

☞ Player I $\longrightarrow$ row player

# Saddle points in pure strategies

- Player I $\longrightarrow$ row player
- Player II $\longrightarrow$ column player

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- Player II $\longrightarrow$ column player

We call a pair of row $i_*$ and a column $j_*$ a *saddle point in pure strategies* of the game if

$$a_{ij_*} \leq a_{i_*j_*} \leq a_{i_*j} \quad \forall\ i = 1, \ldots, n;\ j = 1, \ldots, m.$$

☞ A game has a saddle point in pure strategies if and only if

$$v^+ = \min_j \max_i a_{ij} = \max_i \min_j a_{ij} = v^-$$

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$$\begin{array}{c c l}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} & \text{Row min} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \begin{array}{l} 1 \\ 4 \\ 7 \quad \text{maximin} \end{array} \\
\text{Column max} & \begin{array}{ccc} 7 & 8 & 9 \end{array} & \\
 & \text{minimax} & 
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There may exist more than one solution (saddle point) in pure strategies

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☞ $v = v^- = v^+$ is called the *value of the game*

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☞ A solution (saddle point) in pure strategies *may not exist*

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☞ Nash equilibrium

# A mixed strategy

Suppose that the players play the game many times. A *mixed strategy* for player I is a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with

$$x_i \geq 0 \ \ \forall i \in I, \quad \sum_{i \in I} x_i = 1. \tag{1}$$

Denote by $S_n$ the set of $x = (x_1, \dots, x_n)$ satisfying (1). The choice of a mixed strategy $x \in S_n$ of player I means that this player selects strategy $i \in I$ with the *probability* $x_i$. Similarly, a mixed strategy for player II is a vector $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ with

$$y_j \geq 0 \ \ \forall j \in J, \quad \sum_{j \in J} y_j = 1. \tag{2}$$

Let $S_m$ be the set of $y = (y_1, \dots, y_m)$ satisfying (2). The choice of a mixed strategy $y \in S_m$ of player II means that the player selects strategy $j \in J$ with the *probability* $y_j$.

If Player I abides by a mixed strategy $x \in S_n$ and Player II abides by a mixed strategy $y \in S_m$, then the *expected average payoff* to Player I of the game is

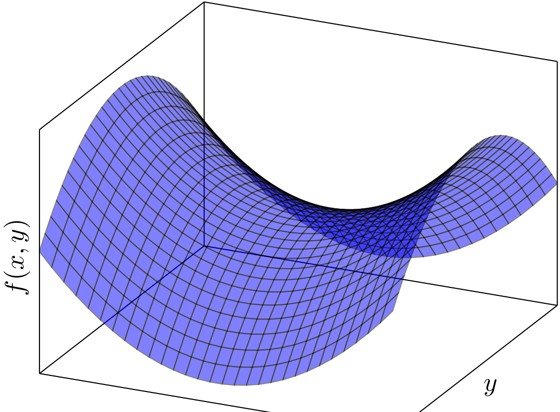
$$\begin{aligned} E(x,y) &= \sum_{i\in I}\sum_{j\in J} a_{ij} \textit{Prob(Player I uses i and Player II uses j)} \\ &= \sum_{i\in I}\sum_{j\in J} a_{ij} \textit{Prob(Player I uses i). Prob(Player II uses j)} \\ &= x^T A y. \end{aligned}$$

# Saddle points

**Definition 1**

Let $C$ and $D$ be sets. A function $f : C \times D \to \mathbb{R}$ has at least a *saddle point* $(x^*, y^*)$ with $x^* \in C$, $y^* \in D$ if

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y) \text{ for all } x \in C, y \in D.$$



# von Neumann Minimax Theorem

**Theorem 2**

*Let $f : C \times D \to \mathbb{R}$ be convex, closed, and bounded. Suppose that $x \mapsto f(x,y)$ is concave and $y \mapsto f(x,y)$ is convex. Then*

$$v^+ = \min_{y \in D} \max_{x \in C} f(x,y) = \max_{x \in C} \min_{y \in D} f(x,y) = v^-.$$

Exercise:

- Proof that $v^- \leq v^+$
- Proof that if $(x^*, y^*)$ is a saddle point of $f$ then

$$v^+ = v^- = f(x^*, y^*).$$

# Saddle point in mixed strategies

☛ *A saddle point in mixed strategies* is a pair $(\bar{x}, \bar{y})$ of probability vectors $\bar{x} \in S_n$ and $\bar{y} \in S_m$ satisfying

$$E(x, \bar{y}) \leq E(\bar{x}, \bar{y}) \leq E(\bar{x}, y) \qquad \forall\ (x \in S_n, y \in S_m). \tag{3}$$

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Put $f(x, y) = x^T A y$. Then, (3) means

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \qquad \forall\ (x \in S_n, y \in S_m).$$

☞ Game has a saddle point in mixed strategies *if and only if*

$$\min_{y \in S_m} \max_{x \in S_n} f(x, y) = \max_{x \in S_n} \min_{y \in S_m} f(x, y)$$

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☞ *If* $(\bar{x}, \bar{y})$ is a saddle point in mixed strategies *then*

$$f(\bar{x}, \bar{y}) = \min_{y \in S_m} \max_{x \in S_n} f(x, y) = \max_{x \in S_n} \min_{y \in S_m} f(x, y)$$

# Extragradient Algorithm*

Cho $Q \subset \mathbb{R}^n$, $S \subset \mathbb{R}^m$ là các tập lồi đóng khác rỗng. Giả sử rằng $\varphi(x, y)$ lồi theo $x$, lõm theo $y$, và khả vi, hơn nữa các đạo hàm riêng thỏa mãn điều kiện Lipschitz trên $Q \times S$, tức là tồn tại hằng số $\ell > 0$ sao cho:

$$\|\nabla_x \varphi(x, y) - \nabla_x \varphi(x', y')\| \leq \ell \left(\|x - x'\|^2 + \|y - y'\|^2\right)^{1/2},$$

$$\|\nabla_y \varphi(x, y) - \nabla_y \varphi(x', y')\| \leq \ell \left(\|x - x'\|^2 + \|y - y'\|^2\right)^{1/2},$$

với mọi $x, x' \in \mathbb{R}^n$ và $y, y' \in \mathbb{R}^m$.

*Korpelevich, G. M.: The extragradient method for finding saddle points and other problems, Ekonom. i Mat. Metody 12, 747–756 (1976) In Russian. (English translation in Matekon 13, 35–49 (1977))

Phương pháp Extragradient[a] để tìm điểm yên ngựa của hàm số $\varphi(x, y)$ trên $Q \times S$ được xác định bởi mối quan hệ sau:

$$\bar{x}^k = P_Q\left(x^k + \alpha \nabla_x \varphi(x^k, y^k)\right), \quad \bar{y}^k = P_S\left(y^k - \alpha \nabla_y \varphi(x^k, y^k)\right),$$

$$x^{k+1} = P_Q\left(x^k + \alpha \nabla_x \varphi(\bar{x}^k, \bar{y}^k)\right), \quad y^{k+1} = P_S\left(y^k - \alpha \nabla_y \varphi(\bar{x}^k, \bar{y}^k)\right).$$

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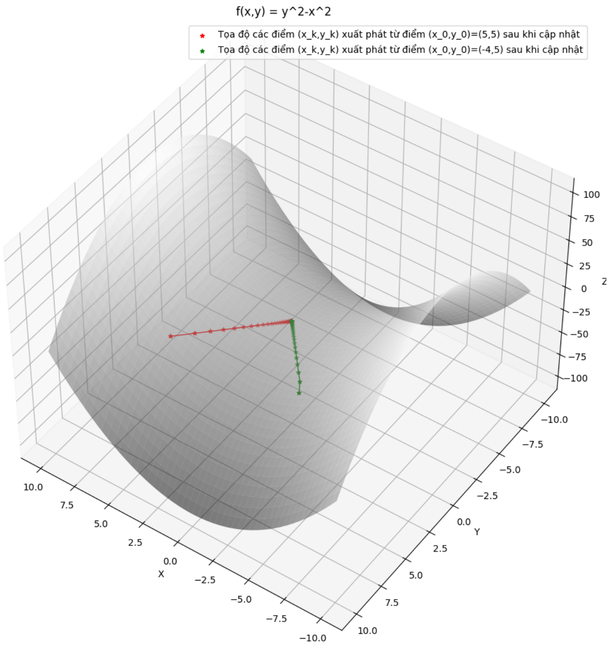
Ví dụ sau đây là một minh họa sử dụng thuật toán Extragradient để tìm điểm yên ngựa của một hàm toàn phương hai biến số mà lồi theo một biến, lõm theo biến còn lại.

**Ví dụ:** Cho $f(x,y) = y^2 - x^2$. Bằng cách giới hạn $x, y \in [-4, 4]$, chúng ta đặt $\Omega = Q = S = [-4, 4]$, chúng ta thực hiện thuật toán Extragradient tìm điểm yên ngựa của $f(x,y)$ trên $[-4,4] \times [-4,4]$.



f(x,y) = y^2-x^2
Tọa độ các điểm (x_k,y_k) xuất phát từ điểm (x_0,y_0)=(5,5) sau khi cập nhật
Tọa độ các điểm (x_k,y_k) xuất phát từ điểm (x_0,y_0)=(-4,5) sau khi cập nhật
X
Y
Z

# Extragradient Algorithm for Solving Matrix Games

Như chúng ta đã biết nghiệm của trò chơi ma trận là điểm yên ngựa của hàm số $f(x, y) = x^T Ay$. Quan sát rằng $f$ lõm theo biến $x$, lồi theo biến $y$, và $\nabla f(x, y) = (Ay, A^T x)$. Rõ ràng, $\nabla f(x, y)$ là Lipschitz với hằng số $\ell := \left(\sum_{j=1}^n \sum_{i=1}^n a_{ij}^2\right)^{1/2}$ trên $S_n \times S_m$. Do đó, thuật toán Extragradient có thể được áp dụng để tìm điểm yên ngựa của $f$ trên $S_n \times S_m$. Chọn một điểm ban đầu $(x^0, y^0) \in S_n \times S_m$ và chọn một bước nhảy $\alpha$ với $0 < \alpha < 1/\ell$.

Ta có thể xây dựng một dãy lặp hội tụ về một điểm yên ngựa của trò chơi ma trận hai người như sau:

$$\bar{x}^k = P_{S_n}\left(x^k + \alpha \nabla_x f(x^k, y^k)\right), \quad \bar{y}^k = P_{S_n}\left(y^k - \alpha \nabla_y f(x^k, y^k)\right),$$

$$x^{k+1} = P_{S_m}\left(x^k + \alpha \nabla_x f(\bar{x}^k, \bar{y}^k)\right), \quad y^{k+1} = P_{S_m}\left(y^k - \alpha \nabla_y f(\bar{x}^k, \bar{y}^k)\right).$$

Vì $\nabla_x f(x, y) = Ay$ và $\nabla_y f(x, y) = A^T x$, các công thức trên có thể viết lại thành:

$$\bar{x}^k = P_{S_n}(x^k + \alpha A y^k), \quad \bar{y}^k = P_{S_n}(y^k - \alpha A^T x^k), \tag{4}$$

$$x^{k+1} = P_{S_m}(x^k + \alpha A \bar{y}^k), \quad y^{k+1} = P_{S_m}(y^k - \alpha A^T \bar{x}^k). \tag{5}$$

# Bài tập

Nhập vào một ma trận bất kỳ, tìm điểm yên ngựa của trò chơi bằng thuật toán extragradient.

# Hanoi