Contents

Chapter 1

Introduction to the Bayesian framework

Remark 1. The topics of this block are contained in Chapters 2 and 3 of *Lambert B. (2018) A Student's Guide to Bayesian Statistics, Sage.*
Another reference is *Hoff, P. D. (2009). A first course in Bayesian statistical methods. Springer Science & Business Media. ISBN: 978-0-387-92299-7.* For this part section chapter 1 (1.1 - 1.4) and 2(2.1 - 2.6) while for remarks on important statistical distributions see pages 253-258.

1.1 Introduction to Bayesian reasoning

1.1.1 The concept of event

Definition 1.1.1 (Event)**.** An event is a logical entity which can be either true (T) or false (F).

Important remark 1. We have that:

- the event is something physical, observable, stated by a proposition
- in an experimental situation, after the experiment, it must be possible to verify whether the event has been T or F

Example 1.1.1 (A negative example)**.** The proportion of heads when tossing a coin is not an event.

1.1.2 Probability of an event

Remark 2. Two main idea/interpretation of probability are available.

Definition 1.1.2 (**Objective probability** (*logical* or *frequentist*))**.** Probability is a physical property of the event.

Important remark 2 (Critique). The definition is linked tot he concept of repeatable events but *repeatable events do non exist*: only past experiences under similar conditions do exist.

Definition 1.1.3 (**Subjective probability**)**.** Probability is the measure of plausibility that an individual assigns to an (uncertain) event.
The probability of an event E, for a certain individual (in a certain moment) is the price $P(E) = p$ that he considers right to pay to participate at a bet where he will win 1 if E occurs or 0 if it doesn't.

Remark 3. In this definition, probability:

- is not a physical property of the event, rather a formalization of the beliefs (and information) that the individual possesses about the event;
- can be different for different individuals (thus the term "subjective");

Remark 4. It is important to state that also what is *not observable* may receive a probability

Remark 5. The classical statistician does not accept subjective probability, since the last is not related to the concept of frequency.

1.1.3 Coherence of subjective probability

Definition 1.1.4 (Coherence of subjective probability)**.** A probability assessment about the n events $E_1, E_2, \ldots, E_n$ is said to be *coherent* if no combination of bets on these events allows a sure win (independently on the events E_i, $i = 1, \ldots, n$ that actually occur).

NB: These n events are alternative I think, think a partition ...

Remark 6. Necessary and sufficient condition for coherence of the subjective probability is expressed by the following theorem.

Theorem 1.1.1. *A necessary and sufficient condition for $P(E)$ coherence is that $0 \leq P(E) \leq 1$. In particular, if $P(E) = 0$ the event is impossible, while if $P(E) = 1$ the event is said certain.*

Proof. Let $p = P(E)$ the *price* and let assume to bet S about the occurrence of E . When

NB: per il gioco dobbiamo pagare p per singolo euro di vincita, se scommettiamo su S di vincita dobbiamo pagare pS

- E occurs, the gain obtained by the bet is the wager/win minus the price for the wager itself:
$$W(E) = S - pS = S(1 - p) = S(1 - P(E))$$
- E does not occur, the gain is negative (just the price paid):
$$W(\bar{E}) = -pS = -P(E) \cdot S$$

How choosing p and S in order avoiding to obtain a sure win (positive earning in any case)?

- if $p < 0$ it would be enough to bet a positive wager $S > 0$ to guarantee a sure win.
- if $p > 1$ it would be enough to bet a negative wager $S < 0$ to guarantee the sure win.

Thus it follows that to avoid a sure win it must be that $0 \leq P(E) \leq 1$.
Furthermore:

- if the event E is certain, it's payoff is $W(E) = S(1-p)$: the only way to avoid a sure win is to set $W(E) = 0$, by fixing $P(E) = 1$.
- if E is impossible, $\bar{E}$ is certain, its payoff is $W(\bar{E}) = -pS$: in order to avoid sure wins it has to be $W(\bar{E}) = 0$, from which $p = P(E) = 0$ (case of impossible events).

□

Theorem 1.1.2. *The probability of the union of many events (incompatible when considered in couples) is the sum of their probabilities.*

Proof. Omitted □

1.1.4 The axiomatic Kolmogorov framework

Remark 7. Allows any computations regarding probabilities, exploiting the analogy between probability and measure, and between mathematical expectation and integration according to Lebesgue.
It needs at least the definition of an algebra. or better, of a σ-algebra

Definition 1.1.5 (Algebra)**.** A class A of subsets of Ω is an algebra if:

1. $\Omega \in A$;
2. if $E_i \in A$ then $\bar{E}_i \in A$;
3. *finite additivity*: $\cup_{i=0}^{n} E_i \in A$ for events that are incompatible two by two

Definition 1.1.6 (σ-algebra)**.** Has the same two first properties, but the third consists of complete rather than finite additivity: $\cup_{i=0}^{\infty} E_i \in A$.

Remark 8. Complete additivity cannot be derived from finite additivity with the application of a limit.

1.1.5 Differences between classical/Bayesian statistics

Differences are resumed in table 1.1

1.1.5.1 Classical statistical inference

Inference is based on the distribution of a statistic, that varies in the set of possible sampling; inference relies on the idea the experiment may be repeated in order to obtain such distribution.
A critique to this reasoning is that decisions are taken on the basis of something that never will be observed. (Only one sample, that produces only one data set, is indeed achieved).

1.1.5.2 Likelihood based inference

The likelihood function provides all information contained in the sample and, for that specific sample, it measures the plausibility of the various alternatives for expressing how the phenomenon is.

	Classical statistics	**Bayesian statistics**
Experiment assumptions	Independence	Exchangeable series
Interpretation of probability	Relative frequency: can be applied to events that can be repeated	Degree of credibility: can be applied to unique events and series of events.
Statistical inference based on ...	Sampling distribution: a sampling space has to be specified.	Final/posterior distribution: the initial distribution has to be assigned.
Parameter estimation	Needs a theory of estimation	Needs description/synthesis of the final/posterior distribution.
Role of personal evaluations	The choice of the experiment is needed, as well as the choice of the procedures to adopt. Personal evaluations remain external (they are not quantified: the problem *appears* as dealt in an objective way.)	All knowledge can be formally incorporated in the initial distribution.

Table 1.1: Differences in bayesian vs frequentist inference

Example 1.1.2. The likelihood of the binomial distribution does not contain the binomial coefficient. Such likelihood is the same of the Bernoulli distribution. The binomial coefficient is not a part of the likelihood, since it does no contain the parameter.

1.1.5.3 Bayesian paradigm

Peculiarity of Bayesian inference: the link between likelihood and final distribution, starting from the initial/prior probability distribution.
The *main assumption* is that prior probabilities can be assigned, starting from an initial probability distribution.

1.1.5.4 Critical observations on classical statistics

- Several techniques of classical statistics do not respect the likelihood principle, that states that two experiments give the same information if the corresponding likelihood functions are inductively equivalent, i.e. differ for a multiplicative constant.
- The estimate may depend on how the experiment is developed

1.1.6 Probability distributions vs likelihoods

Example 1.1.3 (Binomial Case: Difference between Probability Distributions and Likelihoods)**.** Consider the throw of 10 coins with a given probability of head θ. Table 1.2 describes the probability of all possible outcomes (obtain

from 0 to 10 heads) using a binomial distribution for some different values of the parameter θ:

- each line contains the probability distribution of the outcomes from 0 to 10 for different values of the parameter (the values of each line sum to 1)
- each column likelihood (it is a function, not a probability distribution) of some values of the parameters for the possible different results (the sum of the probabilities of each column is not 1)

	Y=0	Y=1	Y=2	Y=3	Y=4	Y=5	Y=6	Y=7	Y=8	Y=9	Y=10	Sum
0	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000
0.1	0.349	0.387	0.194	0.057	0.011	0.001	0.000	0.000	0.000	0.000	0.000	1.000
0.2	0.107	0.268	0.302	0.201	0.088	0.026	0.006	0.001	0.000	0.000	0.000	1.000
0.3	0.028	0.121	0.233	0.267	0.200	0.103	0.037	0.009	0.001	0.000	0.000	1.000
0.4	0.006	0.040	0.121	0.215	0.251	0.201	0.111	0.042	0.011	0.002	0.000	1.000
0.5	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001	1.000
0.6	0.000	0.002	0.011	0.042	0.111	0.201	0.251	0.215	0.121	0.040	0.006	1.000
0.7	0.000	0.000	0.001	0.009	0.037	0.103	0.200	0.267	0.233	0.121	0.028	1.000
0.8	0.000	0.000	0.000	0.001	0.006	0.026	0.088	0.201	0.302	0.268	0.107	1.000
0.9	0.000	0.000	0.000	0.000	0.000	0.001	0.011	0.057	0.194	0.387	0.349	1.000
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000
Sum	1.491	0.829	0.906	0.910	0.909	0.909	0.909	0.910	0.906	0.829	1.491	

Table 1.2: Probability vs likelihood

Important remark 3. The examples concerning events are very simple/intuitive. When probabilities of events are substituted by the probabilities of random variables and the notion of statistical model is introduced, things get a little more complicated.

1.2 Bayes theorem for events

Theorem 1.2.1 (Compound probability theorem)**.** *The conditional probability of one event E_1 given another E_2 can be written as*

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_1)P(E_2|E_1)}{P(E_2)}$$

Remark 9. In the case of independence, the relationship becomes even simpler.

Remark 10. Bayes theorem starts from the compound probability theorem by changing the denominator.

Theorem 1.2.2 (Bayes theorem)**.** *Considering a finite partition $\{H_i\}$ $i = 1, \ldots, I$ of the certain event, and an event B with probability $P(B) > 0$.*
The probabilities of B conditional on the H_i are known: $P(B|H_i)$, $i = 1, \ldots, I$.
Then

$$P(H_i|B) = \frac{P(H_i)P(B|H_i)}{P(B)} = \frac{P(H_i)P(B|H_i)}{\sum_{i=1}^{I} P(H_i)P(B|H_i)}.$$

	D	$\bar{D}$
T	0.95	0.02
$\bar{T}$	0.05	0.98
	1	1

Table 1.3: Experiment and conditional distributions

1.2.1 Examples on (discrete) prior for events

1.2.1.1 Rare disease test (asymmetric/informative prior)

Remark 11. If an individual is positive to the medical test for a disease, which is the probability that she/he is sick? Much of the answer depends on the contribution of the prior/*non modifiable* situation

Example 1.2.1 (Rare disease)**.** The event D is suffering from a rare disease, with $P(D) = 0.001$ (so $P(\bar{D}) = 0.999$). The prevalence of the disease $P(D)$ is our prior/starting point, the *state of the world*: it cannot be changed (neither $P(\bar{D})$).
The event T is being positive at a medical test for that disease. Technology can provide probabilities of T that are conditional on being sick or not where hold:

$$P(T|D) + P(\bar{T}|D) = 1$$
$$P(T|\bar{D}) + P(\bar{T}|\bar{D}) = 1$$

An experiment T is performed conditional on D and results are organized in tables like 1.3 where two distributions are available (conditional on the two states of the world) in columns.
Now some naming:

- $P(\bar{T}|D)$ is the probability of false negatives: the disease is present but the test is negative (the individuals are lost for following tests). It is the first type error $\alpha = 0.05$.
- $P(T|D)$ is the *sensitivity* $(1 - \alpha) = 0.95$
- $P(T|\bar{D})$ is the probability of false positives: the result of the test is positive even if the individual has not the disease (second type error) $\beta = 0.02$.
- $P(\bar{T}|\bar{D})$ is the *specificity* (power) $(1 - \beta) = 0.98$.

The quantity $P(D|T)$ (probability of being diseased given a positive test) is computed using Bayes theorem as:

$$P(D|T) = \frac{P(D)P(T|D)}{P(T)}$$

In order to compute the denominator $P(T)$ at (i.e. the probability of being positive to the test, irrespective of the state of the world) the weighted sum of the probabilities of being sick for all possible states of the world has to be computed (weighs are $P(D)$ and $P(\bar{D})$ respectively). In this way the possible

	D	$\bar{D}$
T	0.99	0.005
$\bar{T}$	0.01	0.995
	1	1

Table 1.4: Techology improvement

states of the world have been *integrated out* of the formula.

$$\begin{aligned} P(T) &= P(D)P(T|D) + P(\bar{D})P(T|\bar{D}) \\ &= 0.001 \cdot 0.95 + 0.999 \cdot 0.02 \\ &= 0.02093 \end{aligned}$$

We thus have that

$$P(\bar{T}) = 1 - P(T) = 1 - 0.02093 = 0.9791$$

and

$$\begin{aligned} P(D|T) &= \frac{P(D)P(T|D)}{P(T)} = \frac{0.001 \cdot 0.95}{0.02093} = 0.04539 \\ P(\bar{D}|T) &= \frac{P(\bar{D})P(T|\bar{D})}{P(T)} = \frac{0.999 \cdot 0.02}{0.02093} = 0.9546 \end{aligned}$$

Note that $0.04539 + 0.9546 = 1$, coherently.

Example 1.2.2 (Techology improvements)**.** If $P(A) = 0.001$ and $P(\bar{A}) = 0.999$ remain unchanged, while technology improves as in the conditional distribution of table 1.4, the results become

$$\begin{aligned} P(D|T) &= 0.1654 \\ P(\bar{D}|T) &= 0.8346 \end{aligned}$$

So probability of being diseased given a positive test is increased due to a greater confidence in the testing system (among the positives

Example 1.2.3 (Less rare disease)**.** If disease is less rare, i.e. $P(A) = 0.01$

- under the first hypothesis on technological development we have $P(A|B) = 0.324$ and $P(\bar{A}|B) = 0.676$,
- under the second hypothesis on technological development $P(A|B) = 0.666$ and $P(\bar{A}|B) = 0.333$.

So in order to obtain $P(A|B) > 0.5$, the disease cannot be too rare and technological development must be very high.

1.2.1.2 DNA test for a crime (ignorance prior))

Example 1.2.4 (Crime and genetic tests)**.** A crime has been committed and there's a suspected: the event C is "the suspected committed the crime". Initially

	C	$\bar{C}$
Compatible DNA	0.999	0.02
Not compatible DNA	0.001	0.98
	1	1

Table 1.5: genetic test performance

no one knows whether the suspected committed the crime or not so our prior (*ignorance prior*) is

$$P(C) = P(\bar{C}) = 0.5$$

A genetic test is available and in situation like this its performance are reported in table 1.5.
What is needed is $P(C|\text{compatible DNA})$ which can be obtained via the Bayes theorem as

$$P(C|\text{compatible DNA}) = 0.9804$$

In other words, if the DNA of the suspected is compatible, the probability that the suspected committed the crime strongly increases compared to $P(C) = 0.5$.

1.2.2 Prior odd ratios, Bayes factor

Some definitions using the notation of D (disease) and T (test)

$$\text{Prior odds ratio in favor} = \frac{P(D)}{1 - P(D)} = \frac{P(D)}{P(\bar{D})}$$
$$\text{Prior odds ratio against} = \frac{P(\bar{D})}{P(D)}$$

We're interested in:

$$\text{Posterior odds ratio in favor } = \frac{P(D|T)}{P(\bar{D}|T)}$$

to compute it note first that

$$P(D|T) = \frac{P(D)P(T|D)}{P(D)P(T|D) + P(\bar{D})P(T|\bar{D})}$$
$$P(\bar{D}|T) = \frac{P(\bar{D})P(T|\bar{D})}{P(D)P(T|D) + P(\bar{D})P(T|\bar{D})}$$

Since the denominators are the same their computation is not needed and thus

$$\text{Posterior odds ratio in favor} = \frac{P(D|T)}{P(\bar{D}|T)} = \frac{P(D)P(T|D)}{P(\bar{D})P(T|\bar{D})} = \frac{P(D)}{P(\bar{D})}r$$

The ratio:

$$r = \frac{P(T|D)}{P(T|\bar{D})}$$

is known as **Bayes factor**; since we are speaking of events, it depends on data and not on priors.
So posterior odds ratio in favor can be written as prior odds ratio in favor (which contains info on the prior only) times the Bayes factor (which contains info on the data only).
Note that if $P(D) = 0.5$ (*ignorance prior*), the prior odds ratio in favor is 1 and all the decision depends on the Bayes factor/data.

Example 1.2.5 (Rare disease (continued))**.** If $P(D) = 0.001$:

$$\text{Prior odds ratio in favor} = \frac{P(D)}{P(\bar{D})} = \frac{0.001}{0.999} = 0.001001$$
$$\text{Prior odds ratio against} = \frac{P(\bar{D})}{P(D)} = \frac{0.999}{0.001} = 999$$

The Bayes factor is

$$r = \frac{P(T|D)}{P(T|\bar{D})} = \frac{0.95}{0.02} = 47.5$$

Here the Bayes factor assumes a high > 1 value, the hypothesis of disease is seconded by/after conducing the experiment; the value of the bayes factor is something similar to hypothesis testing based on only the sample as frequentist do[1].
The posterior odds ratio is

$$\frac{P(D|T)}{P(\bar{D}|T)} = \frac{0.0045}{0.955} = 0.04712$$

Example 1.2.6 (Crime and DNA test (continued))**.** We have

$$\begin{aligned}\text{Prior odds ratio in favor} &= \frac{P(C)}{P(\bar{C})} = 1\\ \text{Posterior odds ratio in favor} &= \frac{P(C)}{P(\bar{C})} \cdot r = 1\frac{P(\text{compatible DNA}|C)}{P(\text{compatible DNA}|\bar{C})}\\ &= \frac{0.999}{0.002} = 50.02.\end{aligned}$$

1.3 The statistical model as the basic element for inference

1.3.1 Definition of a statistical model

Among the basic statistical models some examples can be considered.

1. Experiment with known probability of success: Bernoulli distribution.
2. Model for repeated measures: normal distribution.

[1] Actual testing is not used so much in Bayesian statistics because Bayesian says that everything is included in the posterior distribution.

3. Time of functioning of homogeneous apparatuses: exponential distribution.
4. Non parametric models.
5. Sampling without replacement.
6. Inverse sampling.

The case of *inverse sampling* deserves some comments. It illustrates how the way of conducting the experiment may determine the statistical distribution to consider.
The proportion of a characteristic in a population can be managed via the Binomial distribution (that models X as the number of successes in n trials); also X can be the number of failures before obtaining n successes leading to the following distribution (negative binomial)

$$p(x|n,\theta) = \binom{n+x-1}{x}\theta^n(1-\theta)^x$$

Support of X: set of natural numbers that are $\geq x$. $\mathbb{N} = \{x, x+1, \ldots\}$, with $0 \leq \theta \leq 1$.

Now it turns out that n is not necessarily an integer (in negative binomial can be a real number);

- if that is the case the distribution is known as Pascal distribution (special, most famous, case of negative binomial)
- if furthermore $n = 1$ then $p(x|\theta) = \theta(1-\theta)^x$, and we have a Geometric distribution.

Remark 12. Alternative ways of writing a binomial coefficient can be found in the literature, since

$$\binom{n+x-1}{x} = \binom{n-1}{n-x-1}.$$

1.4 Probability refresher

Remark 13. Idea è mettere qui in un unico posto tutti i richiami di probabilità da usare nel seguito.

1.4.1 χ^2 as Gamma distribution

Remark 14. χ^2 distribution is a special case of Gamma; in this Section we see how one can pass from a $X \sim \chi^2_\nu$ to a $Gamma(\alpha, \beta)$.

Important remark 4 (First parametrization). In case of positive support random variable, we say that X is distributed as χ^2 with ν degrees of freedom $X \sim \chi^2_\nu$ if:

$$p(X|\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\nu/2-1}\exp\left(-\frac{1}{2}x\right)$$

where $E(X|\nu) = \nu$ and $V(X|\nu) = 2\nu$.
If we create a new variable by applying the transformation

$$Y = \frac{X}{S} = S^{-1}X$$

then $X = SY$ and $\frac{\partial X}{\partial Y} = S$ (Jacobian of the transformation). Thus

$$Y \sim S^{-1}\chi^2_\nu$$

with

$$E(Y|\nu) = S^{-1}\nu$$
$$V(Y|\nu) = S^{-2}2\nu$$

For the density of Y (we substitute X with SY and multiply by the Jacobian):

$$\begin{aligned} p(Y|\nu) &= \frac{1}{2^{\nu/2}\Gamma(\nu/2)} (Sy)^{\nu/2-1} \exp\left(-\frac{1}{2}Sy\right) S \\ &= \frac{S^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2)} (y)^{\nu/2-1} \exp\left(-\frac{1}{2}Sy\right) \end{aligned}$$

In order to recognize it as a Gamma distribution with parameters α, λ, let's set

$$S = 2\lambda \implies \lambda = S/2$$
$$\alpha = \nu/2 \implies \nu = 2\alpha$$

obtaining

$$\begin{aligned} p(Y|\alpha,\lambda) &= \frac{(2\lambda)^\alpha}{2^\alpha\Gamma(\alpha)} (y)^{\alpha-1} \exp\left(-\frac{1}{2}2\lambda y\right) \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \exp(-\lambda y) \end{aligned}$$

which is the first parametrization with, thus

$$E(Y|\alpha,\lambda) = \alpha/\lambda$$
$$V(Y|\alpha,\lambda) = \alpha/\lambda^2$$

Important remark 5 (Second parametrization). If we set $\beta = 1/\lambda$ we find the well-known two-parameters Gamma distribution $Gamma(\alpha, \beta)$

$$p(Y|\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)} y^{\alpha-1} \exp\left(-\frac{y}{\beta}\right)$$

with moments

$$E(Y|\alpha,\beta) = \alpha\beta$$
$$V(Y|\alpha,\beta) = \alpha\beta^2$$

Easily, via a simple substitution, one retrieves the moments of Y defined as a χ^2:

$$E\left(Y|\frac{\nu}{2},\frac{2}{S}\right)=\frac{\nu}{S}$$
$$V\left(Y|\frac{\nu}{2},\frac{2}{S}\right)=\frac{2\nu}{S^2}$$

The transformed variable Y coming from X is distributed as $Y\sim S^{-1}\chi^2_\nu$, in this case, since $S^{-1}=\beta/2$ and $\nu=2\alpha$ we get

$$Gamma(\alpha,\beta)=\frac{\beta}{2}\chi^2_{2\alpha}$$

We can find the moments of the χ^2 with S^{-1} and ν and those of the Gamma with α and β, obtaining the same results.

Important remark 6 (One-parameter Gamma distribution). This special case $Y\sim Ga(\alpha)$ is derived from the second parametrization, having set $\beta=\lambda=1$:

$$p(Y|\alpha)=\frac{1}{\Gamma(\alpha)}y^{\alpha-1}\exp(-y)$$

that is also $Gamma(\alpha)=\frac{1}{2}\chi^2_{2\alpha}$.

1.4.2 Inverse χ^2 distribution

TODO: to check yet

If $X\sim\chi^2_\nu$ e $Y\sim S^{-1}\chi^2_\nu$

$$p(X|\nu)=\frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\nu/2-1}\exp\left(-\frac{1}{2}x\right),\qquad X>0$$
$$p(Y|\nu)=\frac{S^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2)}y^{\nu/2-1}\exp\left(-\frac{1}{2}Sy\right),\qquad Y>0.$$

The inverse χ^2 is such that $\frac{1}{X}\sim\chi^2_\nu$ or in other words $X\sim\chi^{-2}_\nu$

$$p(X|\nu)=\frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{-\nu/2-1}\exp\left(-\frac{1}{2}x^{-1}\right),\qquad X>0$$

but also such that $\frac{1}{Y}\sim S^{-1}\chi^2_\nu$ or in other words $Y\sim S\chi_\nu^{-2}$

$$p(Y|\nu)=\frac{S^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2)}y^{-\nu/2-1}\exp\left(-\frac{1}{2}\frac{S}{y}\right),\qquad Y>0.$$

Important remark 7. Remember that in the density of $\chi^{-2}_{\nu-1}$, Y has exponent $-\frac{\nu-1}{2}-1=-\frac{\nu+1}{2}$, so we can write

$$p(Y|\nu)=\frac{S^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2)}y^{-(\nu+1)/2}\exp\left(-\frac{1}{2}\frac{S}{y}\right),\qquad Y>0.$$

that is a $\chi^{-2}_{\nu-1}$.

Chapter 2

From prior to posterior distribution

TODO - before vs after in exchangeability and introduction of choose in binary model (sufficient stat) pag 35 - kernel density e posterior density pag 38

Remark 15. The topics of this block are contained in Chapters 4, 5,6, 7 and 8 of Lambert. Other references in Hoff, chapters 3 and 5[1]

Remark 16. In this section we move from events (and their probability) to random variables (and their distribution function); the aim is to adapt all the machinery of bayesian stuff to do inference.
We need to adapt bayes theorem, the usage of prior for parameters of interest and likelihood to arrive at a posterior distribution of the parameter of interest. I have an idea of the state of the world (prior). Classical statistics is based on the fact that something fixed is in the population and how to use data to estimate that; the parameter is unknown but fixed. In Bayesian stats there's uncertainty on the state of the world/parameter, so we can assume a probability

[1]Hoff, P. D. (2009). *A first course in Bayesian statistical methods.* Springer Science & Business Media. ISBN: 978-0-387-92299-7
The notes illustrate the passage from the prior distribution of the parameter to the posterior. This argument is developed in **Chapter 3** of the textbook, where several further topics are introduced, that will be deepened later in these notes, after the present block.
In this block of notes, we focus on the cases of **a)** discrete prior and likelihood **b)** continuous prior and discrete likelihood, where the discrete likelihood is the binomial, the continuous prior is the Beta. See how the textbook sketches the topic for the binomial likelihood in *Section 3.1 - The binomial model.*
The relevance of predictive distributions is illustrated in *Section 2.3* of the textbook.
Also the normal univariate model is illustrated in this block of notes. In the textbook this topic appears in **Chapter 5**, where the case of normal continuous prior and normal continuous likelihood is developed (continuous normal posterior): *Section 5.1 - The normal model* and *Section 5.2 - Inference for the mean, conditional on the variance.*
Hoff's **Chapter 3** deals with *conjugacy* in *Section 3.1* (page 38), the whole *Section 3.3 - Exponential families and conjugate priors.*
The link between the distribution χ^2 and the two parameters Gamma, together with the case of normal gamma prior and discrete Poisson likelihood (continuous gamma posterior) are developed in *Section 3.2 - The Poisson model.*
The topic of credibility intervals is developed in the whole *Section 3.1.2 - Confidence regions* and is not developed in this block.
Important statistical distributions. For remarks on important statistical distributions see pages 253-258.

of states of the world, the prior.
After fixing the prior something is done (experiment) to see if our idea changes. However in passing from probability of events to statistical distributions something occurs/changes.

2.1 Bayes theorem for random variables

Important remark 8. We can express the passage from the prior to the posterior as:

$$h(\theta|x) = \frac{g(\theta) \cdot p(x|\theta)}{p(x)} = \frac{g(\theta) \cdot L(\theta; x)}{p(x)} \propto g(\theta) \cdot L(\theta; x)$$

where

- $g(\theta)$ is the prior probability distribution for a parameter of interest
- $p(x|\theta) = L(\theta; x)$ is the likelihood of the current sample given the value asumed by θ
- the denominator $p(x)$ is a constant that only depends on the data (it's averaged on all the possible value of the parameter θ, which are at least two[2])

Important remark 9. In the following we specify this expression for the discrete and continuous case.

2.1.1 Discrete parameter and discrete data

Here θ has p possible values, $\theta_1, \ldots, \theta_p$, each with a certain prior probability that sums to the unit:

$$\sum_{i=1}^{p} g(\theta_i) = 1$$

For a sample of size n, we have:

$$\begin{aligned} h(\theta_i|x_1, \ldots, x_n) &= \frac{g(\theta_i) \cdot L(\theta_i; x_1, \ldots, x_n)}{\sum_{i=1}^{p} g(\theta_i) \cdot L(\theta_i; x_1, \ldots, x_n)} \\ &\propto g(\theta_i) \cdot L(\theta_i; x_1, \ldots, x_n) \\ &\propto g(\theta_i) \cdot L(\theta_i; x_1) \cdot \ldots \cdot L(\theta_i; x_n) \end{aligned}$$

where in the last passage the likelihood has been rewritten as a product (of individual likelihood/densities), since we assume independence of the observation (actually we do not need observation to be independent, they could be correlated, but we do not deal with this case).
The posterior distribution is a probability distribution:

$$\sum_{i}^{p} h(\theta_i|x_1, \ldots, x_n) = 1.$$

[2]If the prior was Dirac there would not be the needs to perform an experiment

Important remark 10. We do not develop the case of discrete parameter and continuous data. This can be used when prior are expressed by experts' considerations, or to choosing/compare among a discrete set of models; each model may receive an a priori probability and can be supported by an experiment.

2.1.2 Continuous parameter and discrete data

Parameter θ follows a (prior) probability distribution $g(\theta)$ which integrates to 1. For samples of size n, we can write

$$\begin{aligned} h(\theta|x_1,\dots,x_n) &= \frac{g(\theta)\cdot L(\theta;x_1,\dots,x_n)}{\int g(\tilde{\theta})\cdot L(\tilde{\theta};x_1,\dots,x_n)d\tilde{\theta}} \\ &\propto g(\theta)L(\theta;x_1,\dots,x_n) \end{aligned}$$

Also in this case the likelihood can be written in a simpler way under the independence condition.

2.2 Exchangeability

In Bayesian statistics the concept of exchangeability (of a set of random variables) become important

NB prof: sezione completamente introdotta, sintesi di Hoff 2.7 e 2.8

Definition 2.2.1 (Exchangeability)**.** Let $Y_1,\dots,Y_n$ be a sequence of random variablew and $p(y_1,\dots,p_n)$ its joint density.
We say $Y_1,\dots,Y_n$ are exchangeable if $p(y_1,\dots,y_n) = p(y_{\pi_1},\dots,y_{\pi_n})$ for all permutations π of $\{1,\dots,n\}$.

Remark 17. Roughly speaking, $Y_1,\dots,Y_n$ are exchangeable if the subscript label convey no information about the outcomes.

Remark 18. What is the relationship between exchangeability and iid? It can be proved that

Theorem 2.2.1 (DeFinetti)**.** *The random variable $Y_1,\dots,Y_n$ are exchangeable for all n if and only if they depend on a common unknown parameter having distribution $\theta \sim p(\theta)$, but conditionally of its assumed variable are iid, that is*

$$\begin{cases} \theta \sim p(\theta) \\ Y_1,\dots,Y_n|\theta \text{ are iid} \end{cases} \iff Y_1,\dots,Y_n \text{ are exchangeable for all } n$$

Proof. Omitted □

Important remark 11. If exchangeability holds we can assume, conditionally on the parameter defining the random variable distributions, that these random variable are iid.
In the application of bayes theorem this helps in writing likelihood which can be written as product of individual density (using the common parameter), that is assuming (conditional) independence.

Important remark 12. When is the condition $Y_1,\dots,Y_n$ are exchangeable for all n reasonable?
For this condition to hold we must have *exchangeability* and *repeatability*:

- exchangeability will hold if the labels convey no informations (eg not a time)
- situations in which repeatability is reasonable include the following:
  - $Y_1, \ldots, Y_n$ are outcomes of a repeatable experiment
  - $Y_1, \ldots, Y_n$ are sampled from a finite population with relacement
  - $Y_1, \ldots, Y_n$ are sampled from an infinite population without relacement

Thus in the classical case, if $Y_1, \ldots, Y_n$ are exchangeable and sampled from a finite population of size $N > n$ without replacement, then they can be modeled as approximately being iid

Remark 19. Starting from the next session we see some one-parameter models; these areclass of sampling distributions that is indexed by a single unknown parameter such as binomial, normal and poisson models

2.3 Inference for a proportion

Remark 20. Here we see the how to apply the bayes theorem with random variable: we have a prior distribution for the parameter of interest (proportion) and we conduct an experiment; finally we obtain a posterior distribution for the parameter.
We tackle the case using different priors for the parameter (discrete or continuous) and considering an experiment which produces dichotomic data

2.3.1 Discrete prior

Important remark 13. In both case we start from a prior of four possible proprotions, with uniform probability:

$$\theta = (0.2; 0.4; 0.6; 0.8)$$
$$g(\theta) = 1/4 = 0.25$$

2.3.1.1 Sample of $n = 1$

Example 2.3.1 (One replication of the experiment)**.** The probability distribution for this unique observation, that is also the likelihood for the parameter, is the Bernoulli

$$p(X|\theta) = \theta^x (1-\theta)^{1-x}$$

If

- $X = 1$ is observed then $p(X = 1|\theta_j) = \theta_j$, for each possible value of θ.
  The passages we do to modify our prior opinion about any possible value of the parameter after observing $X = 1$ is resumed in table 2.1.
  So it turns out that a single positive answer makes 4 times more plausible the that the population proportion is 0.8 rather than 0.2.

θ_j	$g(\theta_j)$	$p(X=1\|\theta_j)$	$g(\theta_j)p(X=1\|\theta_j)$	$h(\theta\|X=1)$
0.2	0.25	0.2	0.05	0.10
0.4	0.25	0.4	0.1	0.20
0.6	0.25	0.6	0.15	0.30
0.8	0.25	0.8	0.20	0.40
Sum	1		$p(x)=0.5$	1

Table 2.1: Experiment with single unit extraction with $X = 1$.

θ_j	$g(\theta_j)$	$p(X=0\|\theta_j)$	$g(\theta_j)p(X=0\|\theta_j)$	$h(\theta\|X=0)$
0.2	0.25	0.8	0.2	0.40
0.4	0.25	0.6	0.15	0.30
0.6	0.25	0.4	0.1	0.20
0.8	0.25	0.2	0.05	0.10
Sum	1		$p(x)=0.5$	1

Table 2.2: Experiment with single unit extraction with $X = 0$.

- $X = 0$ is observed then $p(X = 0|\theta_j) = 1 - \theta_j$.
  The passages we do to modify our prior opinion about any possible value of the parameter after observing $X = 0$ is resumed in table 2.2.
  Thus a single negative answer makes 4 times less plausible the statement that the population proportion is 0.8 rather than 0.2.

2.3.1.2 Sample of $n = 20$

Example 2.3.2 (n replications of the experiment)**.** Instead of performing only one trial, we have $n = 20$. The probability distribution of the number r of successes among n observations is the binomial:

$$p(r|n,\theta) = \binom{n}{r}\theta^r(1-\theta)^{n-r}$$

In case we observe $r = 15$ successes out of $n = 20$ trials, the likelihood/data generation distribution is:

$$L(\theta_j; 15, 20) = \binom{20}{15}\theta_j^{15}(1-\theta_j)^5, \quad j = 1, 2, 3, 4$$

The passages we do to modify our prior opinion about any possible value of the parameter after observing $r = 15$ successes out of $n = 20$ trials is resumed in table **??** (where the two central columns were multiplied by 10^{-7} in order to be readable, all are very low probability values).
Note that the likelihood could not to contain the binomial coefficient (since it is simplified in the division going from column 3 to 4, being both at the numerator (column 3) and denominator (sum of column 3)) .
Again the last division of column 4 to compute $h(\theta|r,n) = L(\theta; r, n)g(\theta)$ for each θ_j make a normalization so that the posterior sums to the unit and is a proper probability distribution.
Again the posterior (conditional on the experiment) for the parameter is proportional to the product of the likelihood by the prior.

NB: prof calls likelihood without binomial coefficient "proper" likelihood since the binomial coefficient does not contains the parameter θ_j

θ_j	$g(\theta_j)$	$L(\theta_j; 15, 20) \times 10^{-7}$	$g(\theta_j)L(\theta_j; 15, 20) \times 10^{-7}$	$h(\theta_j \vert 15, 20)$
0.2	0.25	0.00	0.000	0.000
0.4	0.25	0.83	0.201	0.005
0.6	0.25	48.10	12.025	0.298
0.8	0.25	112.60	28.150	0.697
Sum	1		40.376(∗)	1

Table 2.3: Experiment with $n = 20$

Comparison with pure ML estimation For comparison's sake, let's look at the ML estimator of the proportion (using only the sample, not the prior). Here we could ignore the binomial coefficient (as done by prof) in the optimization (since it's just a constant which does not depend on θ). The derivation goes like:

NB prof: nelle note originali vi è un typo algebrico all'ultimo passaggio

$$L(\theta; r, n) = \binom{n}{r} \theta^r (1-\theta)^{n-r}$$

$$\log L(\theta; r, n) = \log \binom{n}{r} + r \log \theta + (n-r) \log(1-\theta)$$

$$\frac{\partial \log L(\theta; r, n)}{\partial \theta} = r\frac{1}{\theta} + (n-r)\frac{1}{1-\theta}(-1) = \frac{r}{\theta} - \frac{n-r}{1-\theta} = \frac{r(1-\theta) - (n-r)\theta}{\theta}$$
$$= \frac{r - r\theta - n\theta + r\theta}{\theta(1-\theta)} = \frac{r - n\theta}{\theta(1-\theta)}$$

Thus by putting $\frac{\partial \log L}{\partial \theta} = 0$ we derive the ML estimator, which as we know is:

$$\hat{\theta} = \frac{r}{n}$$

Applied to our sample is $\hat{\theta} = \frac{15}{20} = 0.75$ (the result is somewhat between the parameter having the posterior higher probability).
This value, a point estimate, differs from what will be derived when passing for priors to posteriors.

```
# riproduzione della tabella, ignorando il 10^{-7}
theta <- seq(0.2, 0.8, 0.2)
prior <- rep(0.25, 4)
lik   <- sapply(theta, function(t) dbinom(x = 15, size = 20, prob = t))
num   <- prior * lik
den   <- sum(num)
posterior <- num/den
## sum(posterior) ## ==1

round(cbind(theta, prior, lik, num, posterior), digits = 3)

##      theta prior   lik   num posterior
## [1,]   0.2  0.25 0.000 0.000     0.000
## [2,]   0.4  0.25 0.001 0.000     0.005
## [3,]   0.6  0.25 0.075 0.019     0.298
## [4,]   0.8  0.25 0.175 0.044     0.697
```

```
## probabilmente non torna nelle colonne centrali perché', a parte l'esponente,
## ha ignorato il coefficiente binomiale nei conti, sebbene lo presenti nella
## formula

## check with expected value of posterior distribution
sum(theta * posterior) ## not exactly the same as ML estimate, btw

## [1] 0.7383344
```

2.3.2 Continuous prior

Remark 21. The prior knowledge for the proportion θ of the population can be expressed as a continuous distribution using the Beta.

2.3.2.1 Beta distribution reminder

Definition 2.3.1. Characterized by two parameters, $a > 0$ and $b > 0$, defined by:

$$p(X|a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} I_{(0,1)}(x)$$

and contains the Beta function

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

which contains the Gamma function that, for a positive integers n, is defined as

$$\Gamma(n) = (n-1)!$$

Important remark 14. The first two moments of the Beta distribution are

$$E(X|a,b) = \frac{a}{(a+b)}$$
$$V(X|a,b) = \frac{ab}{(a+b+1)(a+b)^2}$$

Remark 22. The variance can be written differently, by noticing that:

$$\frac{ab}{(a+b+1)(a+b)^2} = \frac{a}{a+b}\frac{b}{a+b}\frac{1}{a+b+1} = \frac{a}{a+b}\left(1-\frac{a}{a+b}\right)\frac{1}{a+b+1}$$

NB prof: qui nell'originale due volte a al numeratore

and thus obtaining

$$V(X|a,b) = \frac{E(X|a,b)\,(1-E(X|a,b))}{(a+b+1)}. \tag{2.1}$$

Example 2.3.3. When $a = b = 1$, the $Beta(1,1)$ coincides with a $U(0,1)$

2.3.2.2 Uniform prior

Remark 23. We start with a $U(0,1)$ prior, that is also $Beta(1,1)$.[3]

Let us assume a uniform prior in the interval (0,1), i.e. $U(0,1)$:

$$g(\theta|0,1) = \begin{cases} 1 & 0 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$

for which the moments are (by adapting the Beta formulas with $a = b = 1$):

$$E(\theta|0,1) = 1/2 = 0.5 \qquad V(\theta|0,1) = 1/12 = 0.083$$

As before in the experiment we observe $r = 15$ successes out of $n = 20$ trials.

The expression of the likelihood is also the same (below the binomial coefficient remains), with continuos domain from 0 to 1:

NB prof: qui dovrebbe essere = invece di $\propto$ se si tiene il coefficiente binomiale?

$$L(\theta; 15, 20) = \binom{20}{15}\theta^{15}(1-\theta)^5$$

The posterior distribution of θ is defined in this case by having an integral over all its possible values at the denominator

$$h(\theta|r,n) = \frac{g(\theta)L(\theta;r,n)}{\int_0^1 g(\theta)L(\theta;r,n)d\theta} = \frac{g(\theta|0,1)L(\theta;15,20)}{\int_0^1 g(\theta|0,1)L(\theta;15,20)d\theta}$$

Now considering the case of $U(0,1)$ prior (for which the density is $g(\theta) = 1, \forall\theta \in [0,1]$, experiment with $r = 15$ and $n = 20$, the posterior becomes:

$$\begin{aligned} h(\theta|15,20) &= \frac{1 \cdot \binom{20}{15}\theta^{15}(1-\theta)^5}{\int_0^1 1 \cdot \binom{20}{15}\theta^{15}(1-\theta)^5 d\theta} = \frac{\theta^{15}(1-\theta)^5}{\int_0^1 \theta^{15}(1-\theta)^5 d\theta} \\ &= \frac{\theta^{16-1}(1-\theta)^{6-1}}{\int_0^1 \theta^{16-1}(1-\theta)^{6-1}d\theta} = \frac{\theta^{16-1}(1-\theta)^{6-1}}{B(16,6)} \\ &= Beta(16,6) \end{aligned}$$

Virtual and actual sample So, it turns out that if the prior is a $Beta(1,1)$ ($U(0,1)$) and in the experiment we have 15 successes and 5 failures, the posterior becomes a $Beta(16,6)$. This gives an interpretation to the parameter a, b of the distribution which can be seen as sample size (before is virtual, after experiment is virtual + actual) of the units having failures and successes (tab 2.4).
To express a prior $U(0,1)$ information, that is a $Beta(1,1)$, 2 virtual cases are enough (1 success and 1 failure); enough to model a distribution with that expected value and variance. (The concept of virtual sample will be summarized also below.)

Prior vs Posterior Now let's compare moments and shapes of the prior and posterior distributions to appreciate the knowledge benefit of the experiment: before it we knew nothing (we knew θ can be in 0,1 but no more than that). In:

[3] The example that is developed below is analogous to the *happiness data* example of Hoff, *Chapter 3 - One-parameter models*, *Section 3.1 - The binomial model.*

	Prior	Experiment	Posterior
Successes	1	15	16
Failures	1	5	6
Total	2	20	22

Table 2.4: Virtual and actual sample for this case

	Prior $Beta(1,1)$ $(U(0,1))$	Posterior $Beta(16,6)$
$E(\theta)$	$E(\theta\|1,1) = 1/2 = 0.5$	$E(\theta\|16,6) = \frac{a}{a+b} = \frac{16}{22} = 0.7272$
$V(\theta)$	$V(\theta\|1,1) = 1/12 = 0.083$	$V(\theta\|16,6) = \frac{ab}{(a+b+1)(a+b)^2} = \frac{16 \cdot 6}{(16+6+1)(16+6)^2} = 0.008624$

Table 2.5: Prior vs posterior moments in uniform(0,1)-binomial case

- table 2.5 the moments of the two distribution are compared and an improvement can be observed: the posterior variance is approximately 1/10 of the prior variance (this is due to the consideration of both the prior and the experiment information)
- fig 2.1 we can plot the distribution to shows the passage from the prior to the posterior through the experiment

Conjugacy introduction

Remark 24. [From BDA3] The property that the posterior distribution follows the same parametric form as the prior distribution (in this case the Beta) is called *conjugacy*.
Here we can say that the **beta prior** distribution is a **conjugate family** for the *binomial likelihood*.
The conjugate family is mathematically convenient in that the posterior distribution follows a known parametric form

2.3.2.3 Generic beta prior

A more generic prior $Beta(a,b)$ (with $a,b > 0$ to be choosen to have a certain expected value and variance of θ distribution) is expressed, as usual as

NB prof: qui cambio notazione per uniformarla a prima, g prior h posterior

$$\begin{aligned} g(\theta|a,b) &= \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} I_{(0,1)}(\theta) \\ &\propto \theta^{a-1}(1-\theta)^{b-1} \end{aligned}$$

The binomial likelihood is:

$$\begin{aligned} p(r|n,\theta) = L(\theta;r,n) &= \binom{n}{r} \theta^r (1-\theta)^{n-r} \\ &\propto \theta^r (1-\theta)^{n-r} \end{aligned}$$

The posterior distribution is thus

NB prof: qui qualche esplicitazione in più introdotta

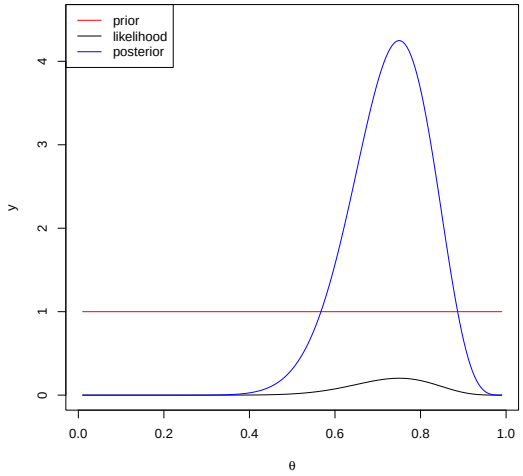


Figure 2.1: Uniform prior and binomial likelihood case

$$\begin{aligned} h(\theta|a,b,r,n) &= \frac{g(\theta|a,b)\cdot L(\theta;r,n)}{\int_0^1 g(\theta|a,b)\cdot L(\theta;r,n)d\theta} \\ &\propto g(\theta|a,b)L(\theta;r,n) \\ &\propto \theta^{a-1}(1-\theta)^{b-1}\cdot\theta^r(1-\theta)^{n-r} \\ &= \theta^{a+r-1}(1-\theta)^{b+(n-r)-1} \end{aligned}$$

Remark 25. The last term is the *kernel* of a $Beta(a+r, b+n-r)$ density (that is its numerator, ignoring the constant denominator $B(a+r, b+n-r)$).

Remark 26. [Miei ragionamenti] Abbiamo trovato che il kernel della posteriori è quello di una beta con parametri $a+r$, $b+n-r$;

- sappiamo che differisce dalla distribuzione della posteriori per una costante,
- sappiamo però che la posteriori è una distribuzione con integrale a 1

quindi la costante deve essere la costante di una beta con tali parametri e dunque la posteriori è effettivamente una beta con tali parametri

Remark 27. [From wikipedia] In statistics, especially in Bayesian statistics, the kernel of a probability density function (pdf) or probability mass function (pmf) is the form of the pdf or pmf in which any factors that are not functions of any of the variables in the domain are omitted.[1] Note that such factors may well be functions of the parameters of the pdf or pmf. These factors form part of the normalization factor of the probability distribution, and are unnecessary in many situations. For example, in pseudo-random number sampling, most sampling algorithms ignore the normalization factor. In addition, in Bayesian analysis of conjugate prior distributions, the normalization factors are generally ignored during the calculations, and only the kernel considered. At the end, the form of the kernel is examined, and if it matches a known distribution, the

normalization factor can be reinstated. Otherwise, it may be unnecessary (for example, if the distribution only needs to be sampled from).

Important remark 15. So we have seen the generalized rules for cooking up a beta prior based on a and b, and a binomial likelihood based on r, n (where the special case of uniform was tackled before), ending up in a beta posterior with parameters depeding on prior and likelihood.
The next step is just changing the prior to accomodate different hypotheses on previous knowledge.

2.3.2.4 Beta with prespecified expected value/variance

Consider another prior, based on conjectures on the expected value and the variance

$$\begin{cases} E(\theta|a,b) = 0.4 \\ V(\theta|a,b) = 0.01 \end{cases}$$

To obtain a, b we set a system of equation[4] and exploit the alternative definition of variance of the beta distribution (eq 2.1)

$$\begin{cases} E(\theta|a,b) = \frac{a}{a+b} = 0.4 \\ V(\theta|a,b) = \frac{E(\theta|a,b)(1-E(\theta|a,b))}{a+b+1} = 0.01 \end{cases} \quad \begin{cases} \frac{a}{a+b} = 0.4 \\ \frac{0.4 \cdot 0.6}{a+b+1} = 0.01 \end{cases} \quad \dots \begin{cases} a = 9.2 \\ b = 13.8 \end{cases}$$

So in order to express those prior information (in terms mean and variance) we need $a = 9.2$ successes and $b = 13.8$ failures in the prior "virtual sample" and thus the prior information corresponds to 23 virtual cases.
The experiment is kept the same as before: 20 trials and 15 successes, as well as the binomial likelihood.
Thus:

- the complete sample has 43 cases (tab 2.6)
- the posterior distribution is a Beta, with parameters

  $$a' = a + r = 9.2 + 15 = 24.2$$
  $$b' = b + n - r = 13.8 + 20 - 15 = 18.8$$

  and moments

  $$E(\theta|a',b') = E(\theta|a,b,n,r) = \frac{a'}{a'+b'} = \frac{a+r}{a+r+b+n-r} = 0.562$$
  $$V(\theta|a',b') = V(\theta|a,b,n,r) = \frac{(a' \cdot b')}{(a'+b'+1) \cdot (a'+b')^2} = 0.0056$$

- the distributions plot (fig 2.2) shows the passage from the prior to the posterior through the experiment and that the variability of the posterior is smaller that in the first case.

[4] Full development:

$$\begin{cases} \frac{a}{a+b} = 0.4 \\ \frac{0.4 \cdot 0.6}{a+b+1} = 0.01 \end{cases} \quad \begin{cases} a = 0.4a + 0.4b \\ a + b + 1 = 24 \end{cases} \quad \begin{cases} b = \frac{3}{2}a \\ a = 23 - b \end{cases} \quad \begin{cases} b = \frac{3}{2}(23 - b) \\ a = 23 - b \end{cases} \quad \begin{cases} b = \frac{69}{5} = 13.8 \\ a = 23 - 13.8 = 9.2 \end{cases}$$

	Prior	Experiment	Posterior
Successes	9.2	15	24.2
Failures	13.8	5	18.8
Total	23	20	43

Table 2.6: Sample sizes of prior (virtual) and experiment (actual)

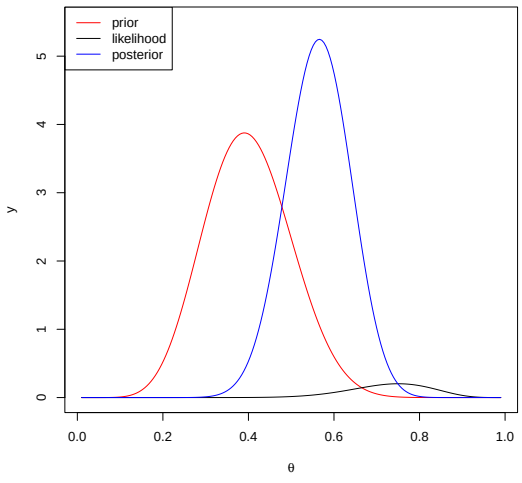


Figure 2.2: General beta-binomial example

	Prior	Experiment	Posterior
Successes	124.5	15	139.5
Failures	124.5	5	129.5
Total	249	20	269

Table 2.7: Virtual sample with precise/indifferent prior

2.3.2.5 Precise/informative, indifference, beta prior

Considering another prior, based on the following conjectures on the expected value and the (small) variance

$$\begin{cases} E(\theta|a,b) = 0.5 \\ V(\theta|a,b) = 0.001 \end{cases}$$

These hypotheses yields to the following parameters for the prior

$$\begin{aligned} a &= 124.5 \\ b &= 124.5 \end{aligned}$$

Thus the prior expected value 0.5 (and variance 0.001) is equivalent to 124.5 virtual successes out of 249 virtual cases. Adopting the same experiment/likelihood:

- the complete sample becomes composed of 269 cases (2.7)
- parameters of the posterior are

  $$\begin{aligned} a' &= 124.5 + 15 = 139.5 \\ b' &= 124.5 + 20 - 15 = 129.5 \end{aligned}$$

  so it is a $Beta(139.5; 129.5)$ with moments

  $$\begin{aligned} E(\theta|a',b') &= E(\theta|a,b,n,r) = \frac{139.5}{269} = 0.518 \\ V(\theta|a',b') &= V(\theta|a,b,n,r) = 0.000925 \end{aligned}$$

  To note that the ratio (posterior variance/prior variance) is near to 1, since the experiment adds very few cases to the virtual ones:

  $$\frac{V(\theta|a,b,n,r)}{V(\theta|a,b)} = \frac{0.000925}{0.001} = 0.99.$$

- finally, the plot (fig 2.3) shows the passage from the prior to the posterior through the experiment.

2.3.2.6 Virtual sample sum up

Important remark 16. In essence[5] , in the previous examples:

[5]This topic is also dealt in Hoff, Section 3.3.1, pages 38-39 for the binomial model, and later, in Section 3.2. for the Poisson model.

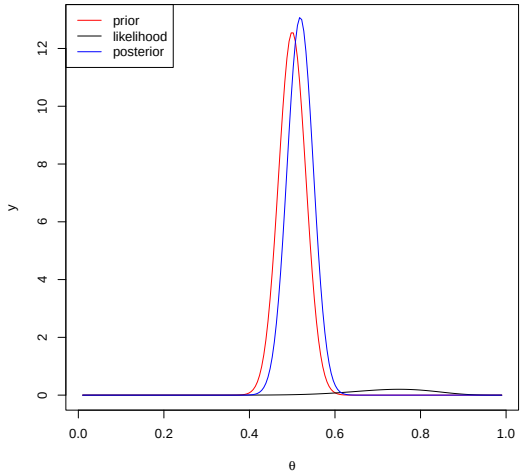


Figure 2.3: Experiment with precise uninformative prior

1. for a conjecture on the 0.5 expected value, without considering its variability, a prior of 1 case out of 2 is sufficient (tab 2.4)
2. for a conjecture on specific expected value and variances, a prior of 23 virtual cases is needed (tab 2.6)
3. a very precise conjecture on an indifference expected value needs a prior of many virtual cases (tab 2.7)

2.3.2.7 Expected values of posterior, prior and likelihood

Through the Bayes theorem, one can pass from a prior $Beta(a, b)$ to a posterior $Beta(a', b')$ with $a' = a + r$ and $b' = b + n - r$. Therefore some algebrical manipulations can highlight one fact

$$\begin{aligned} E(\theta|a', b') &= \frac{a'}{a' + b'} = \frac{a + r}{a + r + b + n - r} = \frac{a + r}{a + b + n} = \frac{a}{a + b + n} + \frac{r}{a + b + n} \\ &= \frac{a}{a + b + n}\frac{a + b}{a + b} + \frac{r}{a + b + n}\frac{n}{n} = \frac{a + b}{a + b + n}\frac{a}{a + b} + \frac{n}{a + b + n}\frac{r}{n} \\ &= \frac{a + b}{a + b + n}E(\theta|a, b) + \frac{n}{a + b + n}\bar{X} \end{aligned}$$

The last passage show how the posterior distribution expected value can be seen as a weighted mean of the prior $E(\theta|a, b)$ and the sample mean from the experiment $\bar{X}$, with weights proportional to the virtual sample and sample size respectively.

2.4 Inference for a mean

NB prof: Sezione interamente rivista nella notazione perché θ e ϕ le confondo e soprattutto di ϕ secondo me non c'è bisogno, sostituibile con σ^2.
In sintesi è un gran mischione di notazioni tra appunti e hoff che però mi risulta personalmente funzionale.

Here we consider the case of a continuous random variable for which our parameter of interest is the mean (assuming known/given variance of the variable),

with the following assumptions. For:

- the **prior**, we assume that the population mean could be approximatively normally distributed/described, with parameter θ_0 and τ_0^2 parameters.

$$\theta \sim N(\theta_0, \tau_0^2)$$
$$g(\theta) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left[-\frac{1}{2\tau_0^2}(\theta - \theta_0)^2\right] \propto \exp\left[-\frac{1}{2\tau_0^2}(\theta - \theta_0)^2\right]$$

  Note that θ_0 and τ_0^2 are *not* θ and σ^2, the mean and variance of the continuous variable itself, the first of which we're interested in

- the **likelihood** we assume that we have sampled $n > 1$ units coming from independent normals distribution with variance σ^2 known and mean θ unknown:

$$\left\{X_1, X_2, \ldots, X_n \mid \theta, \sigma^2\right\} \sim i.i.d.\ N(\theta, \sigma^2)$$

  Thus the likelihood becomes:

$$\begin{aligned} p(x_1, \ldots, x_n \mid \theta) &= \prod_{i=1}^{n} p(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x_i - \theta)^2\right] \\ &\propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2\right] \end{aligned}$$

- the **posterior** can be derived by writing only one exponent, computing the squares, transferring the terms not containing θ to the left side of the equation $\propto$:

$$\begin{aligned} h(\theta \mid x_1, \ldots, x_n) &\propto g(\theta)\, p(x_1, \ldots, x_n \mid \theta) \\ &= \exp\left[-\frac{1}{2\tau_0^2}(\theta - \theta_0)^2\right] \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2\right] \\ &= \exp\left\{-\frac{1}{2}\left[\frac{1}{\tau_0^2}(\theta^2 + \theta_0^2 - 2\theta\theta_0)^2 + \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i^2 + n\theta^2 - 2\theta\sum_{i=1}^{n} x_i\right)\right]\right\} \\ &= \exp\left\{-\frac{1}{2}\left[\frac{\theta^2}{\tau_0^2} + \frac{\theta_0^2}{\tau_0^2} - \frac{2\theta\theta_0}{\tau_0^2} + \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2} + \frac{n\theta^2}{\sigma^2} - \frac{2\theta\sum_{i=1}^{n} x_i}{\sigma^2}\right]\right\} \end{aligned}$$

  Now at this point, remembering it's a function of θ we gather terms with θ^2, θ and all the remaining stuff which will be avoided by proportionality

$$\begin{aligned} &= \exp\left\{-\frac{1}{2}\left[\theta^2 \underbrace{\left(\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}\right)}_{a} - 2\theta \underbrace{\left(\frac{\theta_0}{\tau_0^2} + \frac{n\bar{x}}{\sigma^2}\right)}_{b} + \underbrace{\frac{\theta_0^2}{\tau_0^2} + \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2}}_{c}\right]\right\} \\ &= \exp\left[-\frac{1}{2}\big(a\theta^2 - 2b\theta\big)\right] \end{aligned}$$

Now, following Hoff, let's see if $h(\theta \mid x_1, \ldots, x_n)$ takes the form of a normal density

$$\begin{aligned}
h(\theta \mid x_1, \ldots, x_n) &\propto \exp\left[-\frac{1}{2}\big(a\theta^2 - 2b\theta\big)\right] \\
&= \exp\left[-\frac{1}{2}a\left(\theta^2 - \frac{2b\theta}{a}\right)\right] \\
&= \exp\left[-\frac{1}{2}a\left(\theta^2 - \frac{2b\theta}{a} + \frac{b^2}{a^2} - \frac{b^2}{a^2}\right)\right] \\
&= \exp\left[-\frac{1}{2}a\left(\theta^2 - \frac{2b\theta}{a} - \frac{b^2}{a^2}\right) + \frac{1}{2}\frac{b^2}{a}\right] \\
&\propto \exp\left[-\frac{1}{2}a\left(\theta - \frac{b}{a}\right)^2\right] \\
&= \exp\left[-\frac{1}{2}\left(\frac{\theta - \frac{b}{a}}{1/\sqrt{a}}\right)^2\right]
\end{aligned}$$

This function is the kernel of a normal with mean b/a and $1/\sqrt{a}$ standard deviation, so $h(\theta \mid x_1, \ldots, x_n)$ being a probability distribution it will be normal.
We name the mean and variance of posterior density as θ_1 and τ_1^2, so after the experiment

$$\theta | x_1, \ldots, x_n \sim N(\theta_1, \tau_1^2)$$

where:

$$\begin{aligned}
\theta_1 &= \frac{b}{a} = \frac{\frac{1}{\tau_0^2}\theta_0 + \frac{n}{\sigma^2}\bar{x}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \\
\tau_1^2 &= \frac{1}{a} = \frac{1}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}}
\end{aligned}$$

Important remark 17. Some remarks:

1. the inverse of variance is often referred as *precision* of a random variable (that is how close we are to the centre). Let the following be the prior, sampling and posterior precisions:

$$\begin{aligned}
&\frac{1}{\tau_0^2} = \tilde{\tau}_0^2 && \text{prior precision} \\
&\frac{1}{\sigma^2} = \tilde{\sigma}^2 && \text{sampling precision} \\
&\frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2} = \tilde{\tau}_1^2 && \text{posterior precision}
\end{aligned} \tag{2.2}$$

It is convenient to think about precision as the quantify of information/-precision on an additive scale. For this normal-normal model equation 2.2 yields:

$$\tilde{\tau}_1^2 = \tilde{\tau}_0^2 + n\tilde{\sigma}^2$$

and posterior information/precision = prior info + data info.

2. we can "state" a weight as ratio between prior and posterior precision, and then develop it; we have:

$$\omega = \frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}} = \frac{\frac{1}{\tau_0^2}}{\frac{\sigma^2+n\tau_0^2}{\tau_0^2\sigma^2}} = \frac{1}{\tau_0^2}\cdot\frac{\tau_0^2\sigma^2}{\sigma^2+n\tau_0^2} = \frac{\sigma^2}{\sigma^2+n\tau_0^2}$$

The remaining weight (to sum up to 1)

$$1-\omega = \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}} = \frac{n\tau_0^2}{\sigma^2+n\tau_0^2}$$

3. the posterior variance can be rewritten as

$$\begin{aligned}\tau_1^2 &= \frac{1}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}} = \frac{1}{\frac{\sigma^2+n\tau_0^2}{\tau_0^2\sigma^2}} = \frac{\tau_0^2\sigma^2}{\sigma^2+n\tau_0^2} = \sigma^2\cdot\frac{\tau_0^2}{\sigma^2+n\tau_0^2}\\ &= \frac{\sigma^2}{n}\cdot\frac{n\tau_0^2}{\sigma^2+n\tau_0^2} = \frac{\sigma^2}{n}(1-\omega)\end{aligned}$$

So the posterior variance is smaller than the one of the ML estimator for the mean ($\frac{\sigma^2}{n}$)

4. regarding the posterior expected value, it can be rewritten as a weighted mean of prior expectation and the sample mean, with the weights above

$$\theta_1 = \frac{\frac{1}{\tau_0^2}\theta_0+\frac{n}{\sigma^2}\bar{x}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}} = \theta_0\underbrace{\left(\frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\right)}_{\omega} + \bar{x}\underbrace{\left(\frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\right)}_{1-\omega}$$

5. Thus finally:

$$\theta \mid x_1,\ldots,x_n \sim\ N\left(\omega\theta_0+(1-\omega)\bar{x}, \frac{\sigma^2}{n}(1-\omega)\right)$$

Important remark 18 (About the variance of the prior and sample sizes). If either $\tau_0^2 \to \infty$ or $n \to \infty$, then in both cases

$$\omega = \frac{\sigma^2}{\sigma^2+n\tau_0^2} \to 0$$

In this cases, therefore after the experiment:

$$\theta \mid x_1,\ldots,x_n \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

The posterior will no longer depend on the prior (the parameters of the prior disappear from the posterior) and moments of the posterior coincide with the ML estimates.

Example 2.4.1. Suppose that $\tau_0^2 < \sigma^2$, and especially $\tau_0^2 = \frac{\sigma^2}{m}$ with $m > 1$. In this case we have that

- the prior distribution of the mean is

$$\theta \sim N\left(\theta_0, \frac{\sigma^2}{m}\right)$$

Adopting the sample mean distribution, m can be seen as the number of observation which contributed to the prior distribution definition, the so-called *virtual sample*.
The virtual sample size can also be seen as

$$m = \frac{\sigma^2}{\tau_0^2}$$

i.e., as the ratio of the prior precision to the likelihood *precision*;

- the weights for the posterior moments become

$$\omega = \frac{\frac{m}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{m}{\sigma^2}} = \frac{m}{m+n}$$
$$1 - \omega = \frac{n}{m+n}$$

- expected value, variance and distribution simplifies to

$$\theta_1 = \theta_0 \frac{m}{m+n} + \bar{x}\frac{n}{m+n}$$
$$\tau_1 = \frac{\sigma^2}{n}\frac{n}{m+n} = \frac{\sigma^2}{m+n}$$
$$\theta \mid x_1, \ldots, x_n \sim N\left(\theta_0 \frac{m}{m+n} + \bar{x}\frac{n}{m+n}, \frac{\sigma^2}{m+n}\right)$$

2.5 Inference for a count

Remark 28. Some measurements (eg number of friends) have values that are whole numbers. For these phenomenon the simplest probability model of the measurement is the Poisson model.
In a bayesian context thus, the likelihood depends only on one parameter of the Poisson distribution, the mean $\lambda > 0$. Now we switch to the common notation to θ to mean λ as parameter of interest.
A prior distribution for θ that is in some sense natural is the Gamma distribution.

Important remark 19 (Gamma-Poisson model). We assume:

- a Poisson data generating process. Conditionally on unknown mean θ each exchangeable/independent rv is distributed according to

$$p(X|\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

and thus the **likelihood** for a sample of size n

$$L(\theta; x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

- a Gamma **prior** distribution for mean θ. Has positive only support (which is what we want) $\theta > 0$ and depends on two shape parameters, positive as well; two parametrization are used the following is preferred.
  The density depends on two parameters α, λ (ie $G(\alpha, \lambda)$)

$$p(\theta|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(-\lambda\theta)$$

  In this parametrization

$$E(\theta) = \frac{\alpha}{\lambda}$$
$$V(\theta) = \frac{\alpha}{\lambda^2}$$

- the **posterior** $p(\theta|\alpha, ..., x)$ is a Gamma as well and can be obtained according

$$\begin{aligned} p(\theta|\alpha, \lambda, x) &\propto e^{-n\theta}\theta^{\sum_{i=1}^n x_i} \cdot \theta^{\alpha-1}e^{-\lambda\theta} \\ &= e^{-\theta(n+\lambda)} \cdot \theta^{\alpha+\sum_{i=1}^n x_i - 1} \end{aligned}$$

  we recognize the kernel of a gamma, that is $G(\alpha', \lambda')$, where

$$\alpha' = \alpha + \sum_{i=1}^n x_i$$
$$\lambda' = \lambda + n$$

  for which

$$E(\theta|\alpha, \lambda, x) = \frac{\alpha'}{\lambda'} = \frac{\alpha + \sum_{i=1}^n x_i}{\lambda + n}$$
$$V(\theta|\alpha, \lambda, x) = \frac{\alpha'}{\lambda'^2} = \frac{\alpha + \sum_{i=1}^n x_i}{(\lambda + n)^2}$$

Remark 29. Looking at the posterior expectation we can interpret the parameters of prior and likelihood:

- λ is interpreted as number of prior observations (while n as sample size of the experiment)
- α is interpreted as sum of counts from the λ prior observations (while $\sum_{i=1}^n x_i$ is for the experiment)

Thus the posterior expected value is a weighted mean between prior mean and experiment sample mean (with weights based on number of observations).

Example 2.5.1. [Birth rate] A Survey was conducted to estimate the mean number of children women without (θ_1) and with (θ_2) bachelor degree.
Let's assume that the prior for that mean are distributed both according

$$\theta_1, \theta_2 \sim Ga(\alpha = 2, \lambda = 1)$$

with a common expected value of 2 children per woman.
The survey gathered data on number of children in the two groups ($n_1 = 111$ women without degree, $n_2 = 44$ women with), and the mean of children per woman was higher in the without bachelor degree mothers:

$$n_1 = 111, \sum_{i=1}^{n_1} Y_{i,1} = 217 \implies \bar{Y}_1 = 1.95$$
$$n_2 = 44, \sum_{i=1}^{n_2} Y_{i,2} = 66 \implies \bar{Y}_1 = 1.50$$

Assuming a Poisson model is appropriate to describe/synthetize the empirical distribution, a posterior distribution for the two parameters are easily two gammas

$$\theta_1 | \left\{ n_1 = 111, \sum_{i=1}^{n_1} Y_{i,1} = 217 \right\} \sim Ga(2 + 217, 1 + 111) = Ga(219, 112)$$
$$\theta_2 | \left\{ n_2 = 44, \sum_{i=1}^{n_2} Y_{i,2} = 66 \right\} \sim Ga(2 + 66, 1 + 44) = Ga(68, 45)$$

The posterior means for θ_1, θ_2 becomes $219/112 = 1.95$ and $68/45 = 1.51$, so we moved way far from the initial 2 mean value for the women with degree.

Remark 30 (Alternative parametrization of the Gamma). Using the second parametrization:

- for the prior density we define $\beta = \frac{1}{\lambda}$ and thus the density becomes (eg $G(\alpha, \beta)$):

$$p(\theta|\alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \theta^{\alpha-1} \exp\left(-\frac{\theta}{\beta}\right)$$

  For this parametrization

$$E(\theta) = \alpha\beta$$
$$V(\theta) = \alpha\beta^2$$

- the posterior using the second parametrization

$$\begin{aligned} p(\theta|\alpha, \beta, x) &\propto e^{-n\theta} \theta^{\sum_{i=1}^n x_i} \cdot \theta^{\alpha-1} e^{-\frac{\theta}{\beta}} \\ &= e^{-\theta(n+\frac{1}{\beta})} \cdot \theta^{\alpha+\sum_{i=1}^n x_i - 1} \end{aligned}$$

  If we substitute

$$\beta' = \left(n + \frac{1}{\beta}\right)^{-1} = \left(\frac{\beta n + 1}{\beta}\right)^{-1} = \frac{\beta}{\beta n + 1}$$

  we recognize the kernel of a Gamma, $G(\alpha', \beta')$ where

$$\alpha' = \alpha + \sum_{i=1}^n x_i$$
$$\beta' = \frac{\beta}{\beta n + 1}$$

for which

$$E(\theta|\alpha,\beta,x) = \left(\alpha + \sum_{i=1}^{n} x_i\right)\left(\frac{\beta}{\beta n+1}\right)$$
$$V(\theta|\alpha,\beta,x) = \left(\alpha + \sum_{i=1}^{n} x_i\right)\left(\frac{\beta}{\beta n+1}\right)^2$$

Important remark 20. Looking, for instance, at the expectations, the two parametrizations are equivalent since

$$\frac{\beta}{\beta n+1} = \frac{1}{n+\lambda}.$$

2.6 Natural conjugate distributions

Remark 31. We have seen, among other, that beta prior distribution and binomial sampling model lead to a beta posterior distribution.
To reflect this we say that *the class of beta priors is conjugate for the binomial sampling model.*
It is desirable to have a prior such that the posterior has a tractable form and is algebraically convenient/known.

Definition 2.6.1 (Conjugacy)**.** A class $\mathcal{P}$ of prior distributions for θ is called conjugate for a sampling model $p(x|\theta)$ if the posterior is in the same class of distributions

$$p(\theta) \in \mathcal{P} \implies p(\theta|x) \in \mathcal{P}$$

Remark 32. In other words, *conjugacy* is the property that the posterior distribution follows the same parametric form as the prior distribution.

2.6.1 Sufficient statistics

Remark 33. Sufficiency is a property of a statistic/function T (e.g. the sum of cases) computed on a sample dataset $(x_1, \ldots, x_n)$, in relation to a parametric model.
Informally speaking, a sufficient statistic contains all of the information that the dataset provides about the model parameters.

Remark 34. The following theorem provides a characterization of a sufficient statistic

Theorem 2.6.1 (Fisher-Neyman factorization theorem)**.** *A statistic t is sufficient for θ given the sample x if and only if there are functions f and g such that:*

$$p(x|\theta) = g(x)f(t|\theta)$$

Remark 35. Often we have $g(x) = 1$, so only $f(t|\theta)$ remains.

Important remark 21 (Sufficient statistic). If $t = T(x)$ is a sufficient statistic for the sample x:

$$p(\theta|x) = p(\theta|t)$$

In the bayesian framework being sufficient implies that

$$p(\theta|x) = p(\theta|t) \propto p(\theta)p(t|\theta)$$

Remark 36. Only likelihoods that admit sufficient statistics are considered.

2.6.2 One-parameter exponential family

Definition 2.6.2. A density belongs to the one-parameter exponential family if:

- for one observation, if it can be expressed in the form:

$$p(x|\theta) = L(\theta; x) = g(x)h(\theta)\exp\left[t(x)\Psi(\theta)\right]$$

- for n independent observations, if the likelihood of the sample $p(x|\theta)$ can be expressed as:

$$L(\theta; x) \propto h(\theta)^n \exp\left[\sum t(x_i)\Psi(\theta)\right]$$

  where $g(x)$ can be omitted since it is not a function of the parameter, and $\sum t(x_i)$ is a sufficient statistic for θ.

2.6.2.1 Examples of distributions belonging to this family

Remark 37. The following examples show how different distributions belong to the exponential family.

Example 2.6.1 (Normal with known variance)**.**

$$\begin{aligned} p(x \mid \theta) &= (2\pi\phi)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\phi}(x-\theta)^2\right\} \\ &= \underbrace{(2\pi\phi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^2\frac{1}{\phi}\right)}_{g(x)} \underbrace{\exp\left(-\frac{\theta^2}{2\phi}\right)}_{h(\theta)} \underbrace{\exp\left(\frac{x\theta}{\phi}\right)}_{\exp[t(x)\Psi(\theta)]} \end{aligned}$$

Example 2.6.2 (Normal with known mean)**.**

$$p(x \mid \phi) = \underbrace{(2\pi)^{-\frac{1}{2}}}_{g(x)} \underbrace{\phi^{-\frac{1}{2}}}_{h(\phi)} \underbrace{\exp\left\{-\frac{1}{2\phi}(x-\theta)^2\right\}}_{\exp[t(x)\Psi(\phi)]}$$

Example 2.6.3 (Poisson)**.**

$$p(x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

note that $\ln(\lambda^x) = x\ln\lambda$ thus $\lambda^x = \exp(x\ln x)$, thus

$$p(x \mid \lambda) = \underbrace{\frac{1}{x!}}_{g(x)} \underbrace{\exp(-\lambda)}_{h(\lambda)} \underbrace{\exp\left[x\ln\lambda\right]}_{\exp[t(x)\Psi(\lambda)]}$$

Example 2.6.4 (Binomial)**.**

$$\begin{aligned} p(x \mid \pi) &= \binom{n}{x} \pi^x (1-\pi)^{n-x} \\ &= \binom{n}{x} (1-\pi)^n \pi^x (1-\pi)^{-x} \end{aligned}$$

considering that

$$\ln\left(\pi^x (1-\pi)^{-x}\right) = x \ln \frac{\pi}{1-\pi}$$

then

$$\pi^x (1-\pi)^{-x} = \exp\left[x \ln \frac{\pi}{1-\pi}\right]$$

so that, finally

$$p(x \mid \pi) = \underbrace{\binom{n}{x}}_{g(x)} \underbrace{(1-\pi)^n}_{h(\pi)} \underbrace{\exp\left[x \ln \frac{\pi}{1-\pi}\right]}_{\exp[t(x)\Psi(\pi)]}$$

Example 2.6.5 (Exponential)**.**

$$p(x \mid \theta) = \theta \exp(-\theta x) = \underbrace{\theta}_{h(\theta)} \underbrace{\exp(-\theta x)}_{\exp[t(x)\Psi(\theta)]} \underbrace{1}_{g(x)}$$

2.6.2.2 Relationship between conjugacy and exponential family

Remark 38. There's a relation between exponential family and conjugacy

Important remark 22. If the experiment produces data belonging to a distribution of the exponential family, having thus likelihood

$$L(\theta; x) \propto h(\theta)^n \exp\left[\sum t(x_i)\Psi(\theta)\right]$$

the conjugate prior $\mathcal{P}$ belongs to the family with density

$$p(\theta) \propto h(\theta) \exp\left\{\tau \Psi(\theta)\right\}$$

where τ stresses the fact that no observation related to the experiment can be considered.

Example 2.6.6. Some examples of experiments with likelihood belonging to the one parameter exponential family (and depend on the sufficient statistic for the parameter) are reported in table 2.8

2.6.3 Two-parameters exponential family

Definition 2.6.3. A probability density belongs to the two parameters exponential family:

- for one observation, if it can be expressed in the form:

$$p(x|\theta, \varphi) = L(\theta, \varphi; x) = g(x)h(\theta, \varphi) \exp\left[t(x)\Psi(\theta, \varphi) + u(x)\chi(\theta, \varphi)\right],$$

Likelihood	Conjugate Prior	Case/Naming
Binomial	Beta	Beta-Binomial
Normal (known variance)	Normal	Normal-Normal
Normal (known mean)	Inverse-gamma	Normal-Inverse-gamma
Poisson	Gamma	Gamma-Poisson

Table 2.8: Likelihood and conjugate priors: notable examples.

- for n independent observations if the likelihood of the sample $p(x|\theta,\varphi)$ can be expressed as

$$L(\theta,\varphi;x) \propto h(\theta,\varphi)^n \exp\left[\sum t(x_i)\Psi(\theta,\varphi) + \sum u(x_i)\chi(\theta,\varphi)\right],$$

  where $g(x)$ is not considered since it is not a function of the parameter.

Remark 39. The relation between exponential family and conjugacy becomes the following

Important remark 23. In this case the family of conjugate densities has the form:

$$p(\theta,\varphi) \propto h(\theta,\varphi)^n \exp\left[\tau\Psi(\theta,\varphi) + \nu\chi(\theta,\varphi)\right]$$

Important remark 24. Here $(\sum t(x_i), \sum u(x_i))$ is a sufficient statistic for the bidimensional vector (θ,φ) given x.

Chapter 3

Interval estimation, prediction, hypothesis testing

The topics of this block are contained in Chapters 7, 10 and 18 of Lambert B. (2018) A Student's Guide to Bayesian Statistics, Sage:[1]

3.1 Credibility intervals

Remark 40. It is often desirable to identify regions of the parameter space that are likely to contain the true value of the parameter.

Remark 41. In the Bayesian framework, the information to exploit for inference is contained in the posterior distribution, where the parameter of interest θ is a random variable.
Aside expected value of the posterior distribution (which can be thought as estimate of parameter of interest), the construction of *posterior intervals* allows to communicate uncertainty on the estimate.
Such intervals are usually called **Bayesian confidence intervals** or **credibility intervals**[2]

3.1.1 Credibility vs confidence (frequentist) intervals

Remark 42. In general to construct regions of the parameter space that are likely to contain the true value of the parameter frequentist and bayesian behaves differently:

[1] Credibility Intervals (Section 7.7 Intervals of uncertainty: (7.7.1 Failings of the Frequentist confidence interval; 7.7.2 Credible intervals; 7.7.3 Reconciling the difference between confidence and credible intervals) High Posterior Density Region: Section 7.7.2 (Treasure hunting: the central posterior and highest density intervals) Hypothesis Testing Chapter 10 - Evaluation of model fit and hypothesis testing (See, in particular, Section 10.6 Marginal likelohoods and Bayes Factor).

In Hoff, Credibility intervals (often named posterior confidence intervals) are discussed throughout the textbook, starting from "Comparison to non-Bayesian methods" in Chapter 1. Special emphasis to HDPR is given in Chapter 3 (Section 3.1.2: Confidence regions). Hypothesis testing is a topic that is not developed in the textbook.

[2] The term "credibility" was introduced in Edwards, Lindman *et al.* (1963).

- bayesian after observing the sample $X = x$ (or say $x_1, \dots, x_n$) can construct an interval $[l(x), u(x)]$ such that the probability that $l(x) < \theta < u(x)$ is large
- frequentist costruct a rule that will provide a random interval which will contains the true parameter with given probability. Once constructed the parameter will belong to the interval or not

Definition 3.1.1 (Bayesian coverage)**.** An interval $[l(x), u(x)]$ based on the observed data $X = x$ (or say $x_1, \dots, x_n$) has 95% Bayesian coverage for θ if

$$\mathbb{P}\left(l(x) < \theta < u(x) | X = x\right) = 0.95$$

Remark 43. The interpretation of this interval is that it describes your information about the location of the true value of θ after we've observed $X = x$ (or say $x_1, \dots, x_n$).

Remark 44. Classical statistics arrives at similar conclusions, but builds random intervals with probability $1 - \alpha$ of containing the fixed but unknown value of the parameter θ.

Definition 3.1.2 (Frequentist coverage)**.** A random interval $[l(X), u(X)]$ has 95% frequentist coverage for θ if, before data are gathered

$$\mathbb{P}\left(L(x) < \theta < U(x) | \theta\right) = 0.95$$

Remark 45. In a sense, the frequentist and bayesian notions of coverage describe pre and post-experimental coverage respectively

Remark 46. In the frequentist approach, once observed the sample and plug the data in the confidence interval formula $[l(x), u(x)]$ is obtained and then

$$\mathbb{P}\left(l(x) < \theta < u(x) | \theta\right) = \begin{cases} 0 & \theta \notin [l(x), u(x)] \\ 1 & \theta \in [l(x), u(x)] \end{cases}$$

This hightlights the lack of post-experimental interpretation of frequentist coverage.
Although this may make the frequentist interpretation seem lacking, it's still useful in many situations. Suppose we are running a large number of unrelated experiments and creating a confidence interval for each one of them: if our intervals each have 95% frequentist coversage probability, we can expect that 95% of our intervals will contain the correct parameter value.

Remark 47. The following example shows i guess how in special cases where the computation is actually identical the interpretation differs.

Example 3.1.1. Let $(x_1, \dots, x_n | \theta)$ be a sample from $N(\theta, 1)$, if the prior for θ is *uninformative* (later in these notes), the posterior is

$$\theta | \overline{x} \sim N\left(\overline{x}, \frac{\sigma_0^2}{n}\right) = N\left(\overline{x}, \frac{1}{n}\right)$$

where $\sigma_0^2 = 1$ and $\overline{x}$ is a sufficient statistic for θ. In this situation:

- the symmetric 95% credibility interval for θ is

$$0.95 = P\left\{\overline{x} - \frac{2}{\sqrt{n}} \leq \theta \leq \overline{x} + \frac{2}{\sqrt{n}} | \overline{x}\right\}.$$

  the interpretation of this interval is *direct* since θ has a probability distribution and it is conditional to the data through $\overline{x}$; $1-\alpha$ is a probability statement made directly on the quantity of interest.

- the frequentist confidence interval would be

$$0.95 = P\left\{\overline{x} - \frac{2}{\sqrt{n}} \leq \theta \leq \overline{x} + \frac{2}{\sqrt{n}} | \theta\right\}.$$

  that is conditioned on θ. Probability statements can be done only on $\overline{x}$, that is the only observed random variable.
  The interpretation of a (frequentist) confidence interval refers to *indirect* considerations on the true value of the parameter, it is based on data that are not observed and produced by a large number of repetitions.

Remark 48. Can a confidence interval have the same Bayesian and frequentist coverage probability? Hartigan showed that (for types of intervals shown in Hoff), confidence interval procedure that fives 95% bayesian coverage will have approximately 95% frequentist coverage as well, at least asymptotically.
Particularly an interval that has 95% Bayesian coverage has frequentist coverage that is

$$\mathbb{P}\left(L(x) < \theta < U(x) | \theta\right) = 0.95 + \epsilon_n$$

where $|\epsilon_n| < \frac{a}{n}$ for some constant a, so as $n \to infty$ the term become negligible.

3.1.2 Bayesian methods for credibility intervals

Important remark 25. Two different approaches are used:

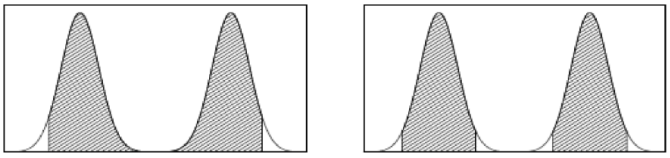
- **posterior quantiles**: the chosen extremes of the interval are the quantiles of the posterior distribution such that the posterior probabilities in both tails are $\alpha/2$ (thus way obtaining a $100(1-\alpha)\%$ level interval);
- **highest posterior density region (HPDR)**: in this case, besides containing the $100(1-\alpha)\%$ posterior probability, the interval has also to satisfy the request of containing most of the distribution and the constraint that the density within the range is never less than in the external areas. The idea is useful in the case of distributions that are strongly asymmetric and/or multimodal, as shown in figure 3.1

3.1.2.1 Quantiles interval estimation

Considering the cumulative (posterior) distribution function $F(\theta | x_1, \ldots, x_n)$, for a fixed value α, we can find an interval (a, b) such that:

$$F(b) - F(a) = P\left\{a < \theta < b | x_1, \ldots, x_n\right\} = 1 - \alpha$$

The (a, b) interval is called credibility interval for θ at credibility level α.



Hypothetical density for which the 95% central interval and 95% highest posterior density region dramatically differ: (a) central posterior interval, (b) highest posterior density region.

Figure 3.1: A mixture situation

Example 3.1.2. Suppose that in the case of example 2.5.1 we want to find the 95% credibility interval for the two parameters (θ_1 and θ_2) which have a posterior distributions $\theta_1 \sim Ga(219, 112)$, $\theta_2 \sim Ga(68, 45)$, and expected value 1.95 and 1.51 respectively. With R

```
quantiles <- c(0.025, 0.975)
qgamma(p = quantiles, 219, 112)    # without bachelor degree

## [1] 1.704943 2.222679

qgamma(p = quantiles, 68, 45)      # with bachelor degree

## [1] 1.173437 1.890836
```

3.1.2.2 Highest Posterior Density Region (HPDR)

The HPDR is a specific credibility interval (the smallest) for which the following properties hold:

1. $F(b) - F(a) = 1 - \alpha(0.95)$
2. if $h(\theta|x_1 \dots x_n)$ is the posterior density, for $a \leq \theta \leq b$, $h(\theta|x_1 \dots x_n)$ has the highest value with respect to any other interval for which property 1 holds.

Extending to more than one dimension, instead of a credibility interval we refer to a multidimensional credibility region R (region with minimum volume). The above properties become:

1. $P(\theta \in R|\mathbf{x}_1 \dots \mathbf{x}_n) = 1 - \alpha$
2. $\forall \theta_1 \in R$ and $\theta_2 \in R \;\; h(\theta_1|\mathbf{x}_1 \dots \mathbf{x}_n) \geq h(\theta_2|\mathbf{x}_1 \dots \mathbf{x}_n)$.

Theorem 3.1.1 (Box and Tiao,1973)**.** *The Highest Posterior Density Interval/Region always exists and it is unique for all the intervals/regions of level* $(1 - \alpha)$ *in which the posterior density is not uniform in any interval/region of the space of* θ *.*

Proof. omitted □

prof: Sezione inotta/riadattata da pag 234, mie prove erificare

HDPR identification in general/practice In general a close formula for HDPR can be tricky to find (unless we're in a special case as unimodality below), but if we can generate random data from the posterior distribution (Hoff example page 234), the following step (using R) allows to determine which values of θ are contained in the HDPR:

1. simulate a sample from the posterior density (eg: `r*` function)
2. compute estimates of the posterior density: if posterior is known use otherwise use `density` command
3. then normalize the density values so they sum to 1 (obtaining thus a proper distribution)
4. sort these discrete probabilities in decreasing order
5. find the first value such that the cumulative sum of the sorted values exceeds $(1 - \alpha)$; the HPD region includes all values of θ which have a discretized probability greater than this cutoff

Example 3.1.3. We calculate the HPDR for the posterior distribution of the birth rate of women without bachelor degree

```
set.seed(1)

## data
alpha <- 219
lambda <- 112

## quantile
quantiles <- c(0.025, 0.975)
qgamma(quantiles, alpha, lambda)
```

```
## [1] 1.704943 2.222679
```

```
## HDPR using known density function ("dgamma")
thetas <- rgamma(n = 1000, alpha, lambda)          # theta/posterior extraction
post_density <- dgamma(x = thetas, alpha, lambda) # density of theta
post_prob <- post_density/sum(post_density)
db <- data.frame(thetas, post_prob)
db <- db[order(db$post_prob, decreasing = TRUE), ]
db$cumprob <- cumsum(db$post_prob)
range(db$thetas[db$cumprob < 0.95])                # unimodal, we can use range
```

```
## [1] 1.767418 2.135385
```

```
## HDPR using "density" instead (kernel density estimate)
post_density <- density(thetas)                          # only changes
db2 <- data.frame(thetas = post_density$x,               # only changes
                  post_prob = post_density$y / sum(post_density$y))
db2 <- db2[order(db2$post_prob, decreasing = TRUE), ]    # same as above from now on
db2$cumprob <- cumsum(db2$post_prob)
range(db2$thetas[db2$cumprob < 0.95])
```

```
## [1] 1.699463 2.217133
```

Example 3.1.4 (Mixture/multimodal)**.** Suppose the posterior for one parameter is a mixture of two distribution. HDPR can be computed as follows and depicted in figure 3.2

```
set.seed(1)
rmixture <- function(n = 10000){
  mu1 <- 0
  mu2 <- 5
  sd  <- 1
  pi  <- 0.5
  a <- rnorm(n, mean = mu1, sd = sd)
  b <- rnorm(n, mean = mu2, sd = sd)
  ifelse(runif(n) < 0.5, a, b)
}

posterior <- rmixture()
hist(posterior, breaks = 100)

## write a function that does it all in the general case
hpdr <- function(posterior, level = 0.95){
  post_density <- density(posterior)
  db2 <- data.frame(thetas = post_density$x,
                    post_prob = post_density$y / sum(post_density$y))
  db2 <- db2[order(db2$post_prob, decreasing = TRUE), ]    # same as above from now on
  db2$cumprob <- cumsum(db2$post_prob)
  db2$in_hpdr <- db2$cumprob < 0.95
  db2 <- db2[order(db2$thetas), c("thetas", "in_hpdr")]
  # search for changes in ci belonging
  db2$change <- abs(c(0, diff(db2$in_hpdr))) > 0
  db2[db2$change, 'thetas']
}

(ci <- hpdr(posterior))

## [1] -2.017002  2.160098  2.903965  7.023845

abline(v = ci, col = 'red', lty = 'dashed')
```

Example 3.1.5 (Asimmetric posterior)**.** Supposing for a proportion a uniform prior distribution and an experiment of 2 successes among 10 unit extracted. The posterior is $\theta|suminx_i = 2 \sim \text{Beta}\,(()1+2, 1+8)$. Let's plot the distribution and compute the intervals in fig 3.3

```
succ <- 3
fail <- 9

## quantiles interval
(quantile_int <- qbeta(c(0.025, 0.975), succ, fail))

## [1] 0.06021773 0.51775585
```

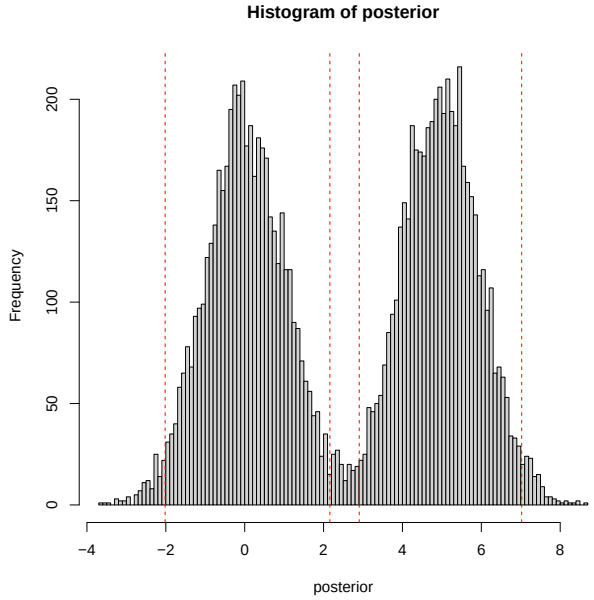


Figure 3.2: Mixture and hpdr

```
## hpdr
posterior_sim <- rbeta(10000, succ, fail)
(hpdr_int <- hpdr(posterior_sim))

## [1] 0.03949387 0.49114634

## plot
plot_fun(function(x) dbeta(x = x, succ, fail),
         from = 0, to = 0.8, cartesian_plane = FALSE,
         xlab = expression(theta), ylab = "P(theta|x)")
abline(v = quantile_int, col = 'red')
abline(v = hpdr_int, col = 'blue')
legend('topright', legend = c('quantiles', 'HPDR'),
       col = c("red", "blue"), lty = 1)
```

HPDR identification with unimodality Lets refer to the univariate and unimodal case (unimodality helps since we hope to find a single maximum within the interval (a, b)) and use method of the Lagrange multipliers.

The Lagrangian function is:

$$\mathcal{L} = (b-a) + \lambda \left\{ \int_a^b h(\theta|x_1, \ldots, x_n) d\theta - (1-\alpha) \right\}$$

where λ is the cost associated to not satisfying the constraint. The quantity within brackets is to be minimized.

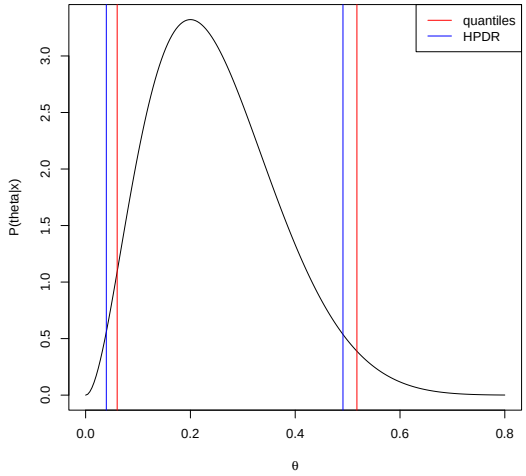


Figure 3.3: Asymmetric posterior

We know that $[H(\theta)]_a^b = H(b) - H(a)$ and that $h(\theta) = \frac{\partial H(\theta)}{\partial \theta}$. So:

$$\frac{\partial \mathcal{L}}{\partial a} = -1 - \lambda h(a|x_1, \ldots, x_n) + 0$$

$$\frac{\partial \mathcal{L}}{\partial b} = 1 + \lambda h(b|x_1, \ldots, x_n) + 0$$

In order to verify the first-order condition we set these derivatives equal to 0 and find the critical point in the interval (a, b).

$$-1 - \lambda h(a|x_1, \ldots, x_n) = 0 \quad ; \quad h(a|x_1, \ldots, x_n) = -\frac{1}{\lambda}$$

$$1 + \lambda h(b|x_1, \ldots, x_n) = 0 \quad ; \quad h(b|x_1, \ldots, x_n) = -\frac{1}{\lambda}$$

Being a probability function, $h(\cdot)$ must be always positive hence λ must be negative ($\lambda < 0$).

To understand if the critical value is a minimum or a maximum we compute the second derivatives, that are contained in a 2×2 table. The diagonal elements are:

$$\frac{\partial^2 \mathcal{L}}{\partial a^2} = -\lambda \frac{\partial h(a|x_1, \ldots, x_n)}{\partial a} \quad \text{and} \quad \frac{\partial^2 \mathcal{L}}{\partial b^2} = -\lambda \frac{\partial h(b|x_1, \ldots, x_n)}{\partial b}$$

The out-of-diagonal elements are equal to zero since:

$$\frac{\partial^2 \mathcal{L}}{\partial a \partial b} = \frac{\partial^2 \mathcal{L}}{\partial b \partial a} = 0.$$

The matrix of second derivatives is diagonal.

Since it must be $\lambda < 0$ both diagonal elements of the matrix of second derivatives of the Lagrangian function are positive, the Hessian matrix is definite positive. As an immediate consequence, the critical point identifies a minimum for the interval (a, b).

If we drop the assumption of unimodality the process of identification of the HPDR is not straightforward.

3.2 Prediction

NB prof: Sezione riorganizzata usando teoria di gelman con notazione rivista, e aggiungendo un esempio di hoff

Remark 49. In this section we study the probability distribution for a new observation; these are called *predictive distributions*.

3.2.1 Predictive distributions

Remark 50. To make inferences about a new observation we can do it basically in two moments: before or after the experiment.
This originates actually two possible distribution for this new observation, the *prior predictive distribution* and the *posterior predictive distribution*.
Now let's consider the general case where the single unknown parameter θ can assume infinite values

Definition 3.2.1 (Prior predictive distribution)**.** We want to make prediction for a new observation Z; we're actually interested in its distribution, which is

$$P(Z) \overset{(1)}{=} \int P(Z, \theta)\, \mathrm{d}\theta \overset{(2)}{=} \int P(\theta)P(Z|\theta)\, \mathrm{d}\theta$$

where:

- in (1) can be thought as marginalization of the joint distribution of Z, θ for any value of θ
- where in turn, (2), the joint distribution can be rewritten as product of prior distribution $P(\theta)$ and conditional distribution $P(Z|\theta)$

Remark 51. Things changes a bit if we set ourself in the post experiment moment and we can condition $P(Z)$ on the its information $P(Z|x_1, \ldots, x_n)$ retrieved on the observed sample. Adding the conditioning to the step above we are interested in the "marginal" of $P(Z|x_1, \ldots, x_n)$

Definition 3.2.2 (Posterior predictive distribution)**.**

$$\begin{aligned} P(Z|x_1, \ldots x_n) &\overset{(1)}{=} \int P(Z, \theta|x_1, \ldots, x_n)\, \mathrm{d}\theta \\ &\overset{(2)}{=} \int P(Z|\theta, x_1, \ldots, x_n) \cdot P(\theta|x_1, \ldots, x_n)\, \mathrm{d}\theta \end{aligned}$$

where

- (1) again we write "the marginal" as integration of the joint of Z, θ for any possible value of the added parameter θ

- (2) the joint of Z, θ can be rewritten conditioning on θ as well and weighting using the posterior distribution/probability for the new conditioning parameter θ

At this moment if we can assume *conditional independence* between Z and the sample, (that is $Z \perp\!\!\!\perp x_1, \dots, x_n | theta$, eg if we sample a new independent unit, not say a time/geographical series), then we can further have that $P(Z|\theta, x_1, \dots, x_n) = P(Z|\theta)$ and the equation simplify further

$$P(Z|x_1, \dots x_n) = \int P(Z|\theta) \cdot P(\theta|x_1, \dots, x_n) \, \mathrm{d}\theta$$

Remark 52. In general when casually speaking about "predictive distribution", one likely means the posterior predictive distribution.

3.2.2 Examples

NB prof: integrato Hoff pag 40, 47

Example 3.2.1 (Beta binomial)**.** Let $x_1, \dots, x_n$ be the outcomes from a sample of n binary random variable/observation, a beta prior for θ, and let Z be a new observation, coming from the sampe population, yet to be observed, that can be either $Z = 1$ or $Z = 0$.
We're interested in $P(Z = 1|x_1, \dots, x_n)$. We have:

$$\begin{aligned} P(Z = 1|x_1, \dots, x_n) &= \int P(Z = 1, \theta|x_1, \dots, x_n) \, \mathrm{d}\theta \\ &= \int P(Z = 1|\theta, x_1, \dots, x_n) \cdot P(\theta|x_1, \dots, x_n) \, \mathrm{d}\theta \\ &\overset{(1)}{=} \int P(Z = 1|\theta) \cdot P(\theta|x_1, \dots, x_n) \, \mathrm{d}\theta \\ &= \int \theta \cdot P(\theta|x_1, \dots, x_n) \, \mathrm{d}\theta \\ &= \mathbb{E}\left[\theta|x_1, \dots, x_n\right] = \frac{a + \sum_{i=1}^n x_i}{a + b + n} \end{aligned}$$

where in (1) we assumed $Z, x_1, \dots, x_n$ are independent.
We know as well that

$$P(Z = 0|x_1, \dots, x_n) = 1 - P(Z = 1|x_1, \dots, x_n) = 1 - \frac{a + \sum_{i=1}^n x_i}{a + b + n} = \dots = \frac{b + \sum_{i=1}^n (1 - x_i)}{a + b + n}$$

Remark 53. In this example we see how in general the posterior predictive distribution:

1. does not depend on any unknown quantity (θ is ruled out by marginalization); if it did, we would not be able to use it to make predictions
2. does depend on previously observed data, this because $x_1, \dots, x_n$ gives information on θ which in turn gives information about Z. It would be bad if Z were independent of $X_1, \dots, X_n$ it would mean that we could never infer anything about the unsampled population from the smple cases

Example 3.2.2 (Experiment of $n = 1$, discrete uniform prior)**.** Coming back to example 2.3.1, a binary experiment (with outcome $x = 1$ or $x = 0$) is performed. As we've seen, the experiment induces two different posterior distributions, each conditional to the occurred realization.
The single future observation Z may be $Z = 1$ or $Z = 0$.
The task of prediction aim to develop $p(Z|x)$ ruling out the values of the parameters θ; in the case of discrete distribution for θ

$$p(Z|x) \overset{(1)}{=} \sum_j p(Z, \theta_j|x) \overset{(2)}{=} \sum_j p(Z|\theta_j, x)p(\theta_j|x) \overset{(3)}{=} \sum_j p(Z|\theta_j)p(\theta_j|x)$$

where:

- (1) again we write the marginal on Z as a sum on the joint Z, θ_j
- (3) computations lie on the very important hypothesis that future and past observations are independent *conditionally* on the values θ_j, that is $Z \perp\!\!\!\perp x|\theta_j$, so that $P(Z|\theta_j, x) = P(Z|\theta_j)$ meaning that the distribution of a future observation is the same of the generic past observation. If that holds, in this case all the random variables in the example Z, X are ruled by the Bernoulli distribution:

$$P(Z|\theta) = \theta^z(1-\theta)^{1-z}$$

In the numerical example 2.3.1, the parameter θ has 4 possible values $(0.2; 0.4; 0.6; 0.8)$.
The probability of a future success given a success in the experiment is

$$\begin{aligned} P(Z = 1|x = 1) &= \sum_j p(Z = 1|\theta_j)p(\theta_j|x = 1) \\ &\overset{(1)}{=} \sum_j \theta_j p(\theta_j|x = 1) \\ &= 0.2 \times 0.1 + 0.4 \times 0.2 + 0.6 \times 0.3 + 0.8 \times 0.4 \\ &= 0.02 + 0.08 + 0.18 + 0.32 = 0.6 \end{aligned}$$

where in (1) remembering that the probability of a future success conditional on the values of the parameters is $P(Z = 1|\theta_j) = \theta_j$. On the other hand, the probability of a future failure given a success in the experiment is

$$\begin{aligned} P(Z = 0|x = 1) &= \sum_j p(Z = 0|\theta_j)p(\theta_j|x = 1) \\ &= \sum_j (1 - \theta_j)p(\theta_j|x = 1) \\ &= 0.8 \times 0.1 + 0.6 \times 0.2 + 0.4 \times 0.3 + 0.2 \times 0.4 \\ &= 0.08 + 0.12 + 0.12 + 0.08 = 0.4 \end{aligned}$$

Note that as expected said it must be $P(Z = 1|x = 1) + P(Z = 0|x = 1) = 1$.
Similarly it must be that (not verified/shown) $p(Z = 1|x = 0) + p(Z = 0|x = 0) = 1$

Example 3.2.3. [Gamma-poisson] As seen in section 2.5, for count data if we assume a parameter of interest with prior $\theta \sim Gamma(\alpha, \lambda)$, an experiment $x_1, \dots, x_n|\theta \sim \text{Pois}\,(()\theta)$ then the posterior of the parameter is $\theta|x_1, \dots, x_n \sim Gamma(\alpha + \sum_{i=1}^{n} x_i, \lambda + n)$.
Which is the **predictive probability distribution** for the count of a new observation Z? It turns out (see hoff pag 47) it is **negative binomial** with parameters $(\alpha + \sum_{i=1}^{n} x_i, \lambda + n)$

Example 3.2.4. [Numerical example] In the birth rate example we have the two posterior distribution which are $\theta_1 \sim Ga(219, 112)$, $\theta_2 \sim Ga(68, 45)$.
The predictive probability distribution for new observations are respectively $NBinom(219, 112)$ and $NBinom(68, 45)$.
To obtain them in R we must pay attention to the second parameter of `dnbinom` function which must be inserted as α/λ not λ. In fig **??** the posterior distribution for θ (left) and posterior predictive distribution for number of children (right)

```
par(mfrow = c(1, 2))

## plot 1
xlim <- c(0, 5)
plot_fun(f = function(x) dgamma(x = x, shape = 2, rate = 1),
         from = 0, to = 5, cartesian_plane = FALSE, ylim = c(0, 3),
         main = 'Prior and posterior distribution for theta',
         xlab = "theta", ylab = 'Density')
plot_fun(f = function(x) dgamma(x = x, shape = 219, rate = 112),
         from = 0, to = 5, add = TRUE, col = 'red', cartesian_plane = FALSE)
plot_fun(f = function(x) dgamma(x = x, shape = 68, rate = 45),
         from = 0, to = 5, add = TRUE, col = 'blue', cartesian_plane = FALSE)
legend('topright',
       legend = c("prior", "Less than BD", "BD or higher"),
       lty = "solid",
       col = c("black","red", "blue"))

## plot 2
n_children <- 0:10
round(p_nc_nobs <- dnbinom(n_children, size = 219, mu = 219/112), 3)

##  [1] 0.143 0.277 0.269 0.176 0.086 0.034 0.011 0.003 0.001 0.000 0.000

round(p_nc_bs   <- dnbinom(n_children, size = 68, mu = 68/45), 3)

##  [1] 0.224 0.332 0.249 0.126 0.049 0.015 0.004 0.001 0.000 0.000 0.000

plot(x = n_children, y = p_nc_nobs, type = 'h', ylim = c(0, 0.35),
     main = 'Posterior predictive distributions', col = "red")
points(x = n_children + 0.1, y = p_nc_bs  , type = 'h', col = 'blue')
legend('topright',
       legend = c("Less than BD","BD or higher"),
       lty = "solid",
       col = c("red","blue"))
```

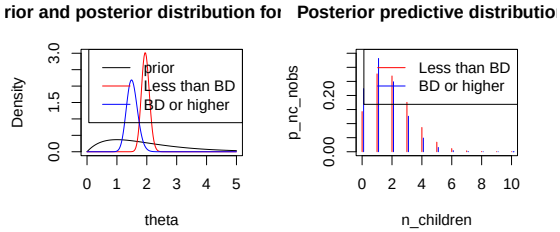


Figure 3.4: Posterior distribution of parameter and posterior predictive distributions for number of children

Remark 54. for the negative binomial with parameter $(\alpha + \sum_{i=1}^{n} x_i, \lambda + n)$ we have

$$E(Z|x_1,\dots,x_n) = \frac{\alpha + \sum_{i=1}^{n} x_i}{\lambda + n} = E(\theta|x_1,\dots,x_n)$$

$$Var(Z|x_1,\dots,x_n) = \frac{\alpha + \sum_{i=1}^{n} x_i}{\lambda + n}\frac{\lambda + n + 1}{\lambda + n} = Var(\theta|x_1,\dots,x_n)\cdot(\lambda + n + 1) = E(\theta|x_1,\dots,x_n)\cdot\frac{\lambda + n + 1}{\lambda + n}$$

Focusing on the last equation, the predictive variance, this can be seen as a measure of our uncertainty in prediction on a new unit Z from the population. Uncertainty stems from uncertainty about the population (which is not a Dirac) and variability in sampling.
For:

- large n, uncertainty about θ is small, being $\frac{\lambda+n+1}{\lambda+n} \approxeq 1$ and uncertainty about Z stems primarily from sampling variability, which for the poisson model is equal to θ.
- small n uncertainty in Z includes the uncertainty in θ and so the total uncertainty is larger than just the sampling variability, that is $\frac{\lambda+n+1}{\lambda+n} > 1$

3.3 Hypothesis testing

3.3.1 Classical hypothesis

NB prof: riorganizzato qua e la usando lee/wikipedia

Important remark 26 (History).

- First proposal: Pearson (1892)
- Fishers proposal: a non Bayesian rule based on Popper conception, relying on hypotheses falsification. According to this principle, it is not possible to establish criteria for accepting hypotheses. The hypothesis to be tested does not receive any probability (hypotheses are not random variables in the frequentist context).
  According to this viewpoint, the refusal of a null hypothesis does not favor any alternative: the decision is postponed until information is improved.

- Proposal of the p-value and its popularity in the era of computers. The p-value is the probability of observing the value that is actually obtained, or an even more extreme value, under the null hypothesis. This deserves a discussion.

- Introduction of the idea of alternative hypothesis by Neyman and Pearson. The innovation of the proposal is the computation of the ratio of the likelihood of the null hypothesis out of the likelihood of the alternative hypothesis. In this case, the power of a test can be computed.

- Also a decision based hypothesis testing has been proposed, which assesses the consequences of accepting the alternatives.

In classical hypothesis testing theory the main question is whether the alternative hypothesis has to be considered or not.

classical hypothesis testing is set on fixing hypothesis Definition of first and second type errors. Definition of a rejection region R; balancing first and second type errors when choosing the rejection region.

NB prof: copiato da wikipedia

Important remark 27 (Performing a frequentist hypothesis test in practice). The typical steps involved in performing a frequentist hypothesis test in practice are:

1. Define a hypothesis (claim which is testable using data).

2. Select a relevant statistical test with associated test statistic T.

3. Derive the distribution of the test statistic under the null hypothesis from the assumptions. In standard cases this will be a well-known result. For example, the test statistic might follow a Student's t distribution with known degrees of freedom, or a normal distribution with known mean and variance.

4. Select a significance level (α), the maximum acceptable false positive rate. Common values are 5% and 1%.

5. Compute from the observations the observed value t_{obs} of the test statistic T.

6. Decide to either reject the null hypothesis in favor of the alternative or not reject it. The Neyman-Pearson decision rule is to reject the null hypothesis H_0 if the observed value t_{obs} is in the critical region, and not to reject the null hypothesis otherwise.

Important remark 28 (Problems with frequentist methods in hypothesis testing). We can mention:

- difficulty of understanding the real meaning of preassigned significance levels (boh non chiaro, è la prob di errore che accettiamo per poter rifiutare la nulla)

- inadequacy of the frequentist method for a point null hypothesis, since a sample may be found having a so great size to induce the refusal of the null hypothesis

- p-value depends on values that have not been observed (we add the probability of a more extreme result, which actually has not been observed): thus an hypothesis that is possibly true (H_0) may be refused because it did not predict (extremer) results that actually did not occur;
- p-value depends on the sample size: if it tends to infinity, the p-value tends to 0.

3.3.2 Bayesian hypothesis testing

Regarding the value assumed by θ we can define two competing situation by defining two disjoint set, Θ_0, Θ_1, to which θ can belong

$$\Theta_0 \cup \Theta_1 = \Theta, \quad \Theta_0 \cap \Theta_1 = \emptyset$$

The different competing hypotheses than can formalized as

$$H_0 : \theta \in \Theta_0$$
$$H_1 : \theta \in \Theta_1$$

We can define the posterior probabilities of that hypotheses

$$p_0 = P(\theta \in \Theta_0 | x_1, \ldots, x_n)$$
$$p_1 = P(\theta \in \Theta_1 | x_1, \ldots, x_n)$$

and decide between H_0, and H_1 accordingly.

3.3.3 Posterior OR factorization, Bayes factor

Important remark 29. Bayesian hypothesis testing aims at measuring how much the experimental evidence makes the main hypothesis, H_0, stronger than the alternative one H_1, by comparing posterior probability using the posterior odd in favour of H_0

Definition 3.3.1 (Posterior odd in favour of H_0)**.** Defined as the ratio between the posterior probability of H_0 over the posterior of H_1

$$\frac{p_0}{p_1} = \frac{P(\theta \in \Theta_0 | x_1, \ldots, x_n)}{P(\theta \in \Theta_1 | x_1, \ldots, x_n)} = \frac{P(H_0 | x_1, \ldots, x_n)}{P(H_1 | x_1, \ldots, x_n)}$$

Important remark 30 (Bayesian decision). The decision is taken according to the value of the posterior odds in favor of H_0: if $\frac{p_0}{p_1} > 1$, then H_0 is accepted. By the fact that Bayesians assign probabilities to hypotheses H_0 and H_1: significance levels α need not to be specified to make a choice.

Remark 55. It's called odds ratio because if the probabilities of the two competing hypotheses $p_0 + p_1 = 1$ (considered that $\Theta_0 \cup \Theta_1 = \Theta$, $\Theta_0 \cap \Theta_1 = \emptyset$), the ratio of the two probabilities is said odds in favor of the numerator (H_0).

Remark 56. Now let's see which are the components of the posterior odds ratio, by exploding the posterior probabilities using Bayes theorem

Important remark 31 (Posterior odds ratio factorization).

The posterior odds ratio may be developed as follows. Considering two *mutually exclusive* alternative hypotheses H_0 and H_1, as we did, following quantities can be computed:

$$P(\theta \in \Theta_0 | x_1, \dots, x_n) = \frac{P(\theta \in \Theta_0) P(x_1, \dots, x_n | \theta \in \Theta_0)}{P(\theta \in \Theta_0) P(x_1, \dots, x_n | \theta \in \Theta_0) + P(\theta \in \Theta_1) P(x_1, \dots, x_n | \theta \in \Theta_1)}$$

$$P(\theta \in \Theta_1 | x_1, \dots, x_n) = \frac{P(\theta \in \Theta_1) P(x_1, \dots, x_n | \theta \in \Theta_1)}{P(\theta \in \Theta_0) P(x_1, \dots, x_n | \theta \in \Theta_0) + P(\theta \in \Theta_1) P(x_1, \dots, x_n | \theta \in \Theta_1)}$$

If we combine them in the posterior odds ratio we get a simplification given that the denominator is common (this is due to the fact that Θ is partitioned in Θ_1 and Θ_2):

$$\frac{p_0}{p_1} = \underbrace{\frac{P(\theta \in \Theta_0 | x_1, \dots, x_n)}{P(\theta \in \Theta_1 | x_1, \dots, x_n)}}_{\text{Posterior OR}} = \underbrace{\frac{P(\theta \in \Theta_0)}{P(\theta \in \Theta_1)}}_{\text{Prior OR}} \cdot \underbrace{\frac{P(x_1, \dots, x_n | \theta \in \Theta_0)}{P(x_1, \dots, x_n | \theta \in \Theta_1)}}_{\text{Bayes factor}}$$

where thus the posterior odds in favour of H_0 is factorized in the prior odds in favour of H_0 times the so-called *Bayes factor*, which is the ratio of likelihood under the two concurrent hypotheses. By rearranging algebrically we have another interpretation of Bayes factor:

$$\text{Bayes factor} = \frac{\text{Posterior OR}}{\text{Prior OR}}$$

which is how change the odds in favour of H_0 after the experiment. Often the prior odds ratio is posed equal to 1.

Remark 57. The Bayes factor (that is the ratio between the posterior odds and the prior odds) depends only on the sample data: it indicates how much data are in favor of a model rather than another. It is an important feature for assessing, with respect to prior evaluations, the change of opinion after an experiment.
It is employed also as a tool for model selection: since the prior odds ratio is often 1, for an objectivist Bayesian the Bayes factor is a way to perform model comparison.

Remark 58. If the competing hypotheses are more than 2 instead, the one with greatest posterior probability can be chosen. In general the hypothesis with highest posterior probability is chosen.

Remark 59. We now see some cases where hypotheses can be *simple*, for which Θ_0 (or Θ_0) is composed of a single value θ_0 (respectively θ_1) or composite (more than one possible value).

3.3.4 Simple hypotheses

Remark 60. If both the competing hypotheses are simple, it turns out that the Bayes factor = Likelihood ratio.
The likelihood ratio is the frequentist test statistic for comparing simple hypotheses (Neyman Pearson Lemma)

NB prof: fatto tutti i calcoli per esteso non considerando la statistica sufficiente per avere il caso generale (eg lee p4.4.4 pag 149)

Example 3.3.1. Supposing that we extract from a normal population $X \sim N(\theta, 1)$ with unknown parameter of interest and unitary variance. Let

$$H_0 : \theta = 0$$
$$H_1 : \theta = 1$$

so we have 2 competing hypotheses, both simple. Finally, we suppose that $P(H_0) = P(H_1) = 0.5$.
The densities under H_0 ($\mu = \theta_0 = 0, \sigma = \sigma^2 = 1$)

$$f_{H_0}(x) = \frac{1}{\sqrt{2\pi \cdot 1}} \exp\left[-\frac{1}{2}\left(\frac{x-0}{1}^2\right)\right] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right]$$

while under H_1 ($\mu = \theta_1 = 1, \sigma = \sigma^2 = 1$)

$$f_{H_1}(x) = \frac{1}{\sqrt{2\pi \cdot 1}} \exp\left[-\frac{1}{2}\left(\frac{x-1}{1}^2\right)\right] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2 - \frac{1}{2} + x\right]$$

The likelihood ratio from the sample/experiment is

$$\begin{aligned}
\frac{P(x_1,\ldots,x_n|\theta_0)}{P(x_1,\ldots,x_n|\theta_1)} = \frac{\prod_{i=1}^n f_{H_0}(x)}{\prod_{i=1}^n f_{H_1}(x)} &= \frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right]}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2 - \frac{1}{2} + x\right]} \\
&= \prod_{i=1}^n \exp\left[-\frac{1}{2}x_i^2 + \frac{1}{2}x_i^2 + \frac{1}{2} - x_i\right] \\
&= \exp\left[\sum_{i=1}^n \left(\frac{1}{2} - x_i\right)\right] = \exp\left(\frac{n}{2} - \overline{x} \cdot n\right) \\
&= \exp\left[n\left(\frac{1}{2} - \overline{x}\right)\right]
\end{aligned}$$

Note that the "baricentre" is $\frac{1}{2}$ which is halfway between θ_0 and θ_1 (if $\overline{x} < \frac{1}{2}$ the term between parenthesis positive and likelihood ratio greater than 1, showing experimental evidence in favour of H_0).
Going back to the posterior odds ratio we have:

$$\frac{P(H_0|x_1,\ldots,x_n)}{P(H_1|x_1,\ldots,x_n)} = \frac{P(H_0)}{P(H_1)} \cdot \frac{P(x_1,\ldots,x_n|\theta_0)}{P(x_1,\ldots,x_n|\theta_1)} = 1 \cdot \exp\left[n\left(\frac{1}{2} - \overline{x}\right)\right] = \exp\left[n\left(\frac{1}{2} - \overline{x}\right)\right]$$

If in the experiment with $n = 10$ we have $\overline{x} = 2$ (nearest to alternative hypothesis parameter) Bayes factor and posterior OR become:

$$\frac{P(H_0|x_1,\ldots,x_n)}{P(H_1|x_1,\ldots,x_n)} = \exp(-15)$$

so the posterior odds is a very small value, which induces to reject H_0 in favor of H_1.

Remark 61. As we'll see then, the equality

$$\text{Bayes factor} = \text{Likelihood ratio}$$

should hold only for simple null and alternative hypotheses.

3.3.5 Simple null and composite alternative

Important remark 32. In this case we have that

$$\begin{aligned}
&H_0 : \theta = \theta_0, \quad \Theta_0 = \{\theta_0\} \\
&H_1 : \theta \in \Theta_1, \quad \Theta_1 = \{\theta_1, \ldots\}
\end{aligned}$$

and at the denominator, for the likelihood of alternative hypothesis, we need to average for possible values/probabilities of each $\theta \in \Theta_1$, obtaining the posterior odds:

$$\begin{aligned}\frac{P(H_0|x_1,\ldots,x_n)}{P(H_1|x_1,\ldots,x_n)} &= \frac{P(\theta=\theta_0)\cdot P(x_1,\ldots,x_n|\theta_0)}{P(\theta\in\Theta_1)\cdot P(x_1,\ldots,x_n|\theta\in\Theta_1)} \\ &= \frac{P(\theta=\theta_0)}{P(\theta\in\Theta_1)}\cdot\frac{P(x_1,\ldots,x_n|\theta_0)}{\int_{\tilde{\theta}\in\Theta_1} g_1(\tilde{\theta})\cdot P(x_1,\ldots,x_n|\tilde{\theta})\,\mathrm{d}\tilde{\theta}}\end{aligned}$$

where

- $\tilde{\theta}$ has no particular statistical meaning (just the integration convention of not using θ when integrating by that);
- $g_1(\tilde{\theta})$ is defined to be a proper/normalized density (eg it integrate to 1 in Θ_1). Thus it's defined as:

  $$g_1(\tilde{\theta}) = \frac{P(\theta)}{P(\theta\in\Theta_1)}$$

  where at the numerator we have the prior density of θ and at denominator the total probability that θ falls in the Θ_1 region.
- we start to see the dependence of the calculation of the bayes factor from the prior of θ through $P(\theta)$ at previous point (so it's not correct to say that Bayes factor depends on experiment/sample data only)

Remark 62. Similar results can be derived even for for composite null and alternative hypothesis (but maybe this is less interesting)

3.3.6 Composite null and alternative hypotheses

Important remark 33. In the most general case we have

$$\begin{aligned}&H_0 : \theta\in\Theta_0, \quad \Theta_0 = \{\theta_0,\ldots\} \\ &H_1 : \theta\in\Theta_1, \quad \Theta_1 = \{\theta_1,\ldots\}\end{aligned}$$

$$\begin{aligned}&\Theta_0\cap\Theta_1 = \emptyset \\ &\Theta_0\cup\Theta_1 = \Theta\end{aligned}$$

Similarly to what done in the previous case, for the likelihoods of both hypothesis, we need to average for possible values/probabilities of each θ. The posterior odds become:

$$\begin{aligned}\frac{P(H_0|x_1,\ldots,x_n)}{P(H_1|x_1,\ldots,x_n)} &= \frac{P(\theta\in\Theta_0)\cdot P(x_1,\ldots,x_n|\theta\in\Theta_0)}{P(\theta\in\Theta_1)\cdot P(x_1,\ldots,x_n|\theta\in\Theta_1)} \\ &= \frac{P(\theta\in\Theta_0)}{P(\theta\in\Theta_1)}\cdot\frac{\int_{\tilde{\theta}\in\Theta_0} g_0(\tilde{\theta})\cdot P(x_1,\ldots,x_n|\tilde{\theta})\,\mathrm{d}\tilde{\theta}}{\int_{\tilde{\theta}\in\Theta_1} g_1(\tilde{\theta})\cdot P(x_1,\ldots,x_n|\tilde{\theta})\,\mathrm{d}\tilde{\theta}}\end{aligned}$$

where this time

- g_0, g_1 are again defined to be a proper/normalized density (eg they integrate to 1 in Θ_0, Θ_1 respectively). Thus as:

$$g_0(\tilde{\theta}) = \frac{P(\theta)}{P(\theta \in \Theta_0)}, \quad g_1(\tilde{\theta}) = \frac{P(\theta)}{P(\theta \in \Theta_1)}$$

  where at the numerator we have the prior density of θ and at denominator the total probability that θ falling in the two regions

- again the Bayes factor depend not only on the data but from the prior of θ through $P(\theta)$ and g_0, g_1at previous point (so it's not correct to say that Bayes factor depends on experiment/sample data only)

Chapter 4

Simulation

Remark 63. In Hoff Monte Carlo approximations (the basis of simulation methods) are developed in Chapter 4; posterior approximations via the Gibbs sampler are developed in Chapter 6; nonconjugate priors and Metropolis-Hastings algorithms are developed in Chapter 10.

4.1 Monte Carlo approximation

NB prof: ripreso da hoff

Remark 64 (Rationale). Once we've found the posteriors of our interest we may be interested in summarizing aspects of the posterior other than mean for example:

- calculate $P(\theta \in A|x_1, \ldots, x_n)$ for arbitrary set A
- interested in the distribution of a function/transformation of θ
- interested in the distribution of a function/transformation of more than one parameters: eg comparing two posterior distribution of two parameters, say looking at the distribution $\theta_1 - \theta_2$ or θ_1/θ_2

Important remark 34. Obtaining formula/distribution for these posterior quantities can be difficult/impossible; however if we can generate random sample values of the parameters θs from their posterior distributions involved, then then all these quantities of interest can be aproximated to an arbitrary precision using Monte Carlo method

4.1.1 Monte Carlo method

4.1.1.1 Single parameter

Let θ be a parameter of interest and $x_1, \ldots, x_n$ the sample from a distribution $p(x_1, \ldots, x_n|\theta)$.
Suppose we could sample S independent random values from the *posterior distribution* $p(\theta|x_1, \ldots, x_n)$:

$$\theta^{(1)}, \ldots, \theta^{(S)} \sim \text{iid } p(\theta|x_1, \ldots, x_n)$$

then, as S increases, the empirical distribution of the samples $\left\{\theta^{(1)}, \ldots, \theta^{(S)}\right\}$ would approximate $p(\theta|x_1, \ldots, x_n)$ better and better. For this reason it's called called Monte carlo approximation to $p(\theta|x_1, \ldots, x_n)$.
Furthermore let $g(\theta)$ be any function of the parameter; we know that the expected value of $g(\theta)$ is the integral (g can be identity as well):

$$E[g(\theta)|x_1, \ldots, x_n] = \int p(\theta|x_1, \ldots, x_n) \cdot g(\theta) \, \mathrm{d}\theta$$

The law of large numbers says that if $\left\{\theta^{(1)}, \ldots, \theta^{(S)}\right\}$ are iid samples from $p(\theta|x_1, \ldots, x_n)$ then, as $S \to \infty$ the sample mean of the transformed parameter converge to the expected value, that is to the value of the integral above

$$\frac{1}{S} \sum_{s=1}^{S} g(\theta^{(s)}) \to E[g(\theta)|x_1, \ldots, x_n]$$

This implies that, as $S \to \infty$, any aspect of a posterior distribution we may be interested in can be approximated arbitrarily exaclty with a large enough Monte carlo sample. Precisely:

- the monte carlo mean (where g is just the identity) converges to the expected value of the posterior:

$$\hat{\theta} = \frac{\sum_{s=1}^{S} \theta^{(s)}}{S} \to E[\theta|x_1, \ldots, x_n]$$

- sample variance converges to distribution variance

$$\hat{\sigma}^2 = \frac{\sum_{s=1}^{S} \theta^{(s)} - \hat{\theta}}{S - 1} \to Var[\theta|x_1, \ldots, x_n]$$

- the proportion of sample below a treshold converge to the probability distribution

$$\frac{\#(\theta^{(s)} \leq c)}{S} \to P(\theta \leq c|x_1, \ldots, x_n)$$

- the empirical distribution converge to the theorical one

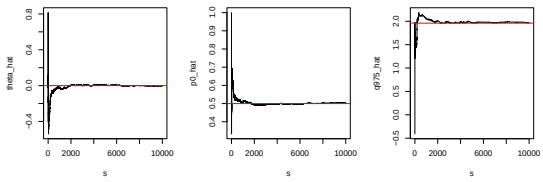
$$\left\{\theta^{(1)}, \ldots, \theta^{(S)}\right\} \to p(\theta|x_1, \ldots, x_n)$$

- α-quantile of $\left\{\theta^{(1)}, \ldots, \theta^{(S)}\right\}$ converge to corresponding θ_α

Example 4.1.1. As a simple start, let's suppose our posterior is a N(0,1) simulation of monte carlo mean (0), cumulative distribution at 0 (0.5), 97.5 quantile (1.96). We plot the monte carlo estimates of these quantities up to $S = 10000$ simulations (fig **??**)

```
S <- 10000
s <- 1:S
set.seed(452304)
```

```r
thetas <- rnorm(S)

cummean <- function(x) cumsum(x)/seq_along(x)

theta_hat <- cummean(thetas) # mean
p0_hat <- cummean(thetas < 0) # cumulative distribution: % below 0

cumf <- function(x, f){
  id <- seq_along(x)
  apply_f_up_to_i <- function(i) f(x[seq_len(i)])
  sapply(id, apply_f_up_to_i)
}

q975_hat  <- cumf(thetas, function(x) quantile(x, probs = 0.975))

## plot
par(mfrow = c(1,3))
plot(s, theta_hat, type = 'l')
abline(h = 0, col = 'red')

plot(s, p0_hat, type = 'l')
abline(h = 0.5, col = 'red')

plot(s, q975_hat, type = 'l')
abline(h = qnorm(.975), col = 'red')
```



4.1.1.2 Mean confidence bar

Furthermore, due to the Central limit theorem, for large S we have that monte carlo sample mean is normally distributed with mean $E[g(\theta)|x_1, \ldots, x_n]$ and variance $\hat{\sigma}^2/S$ thus

$$\frac{\hat{\theta} - E[g(\theta)|x_1, \ldots, x_n]}{\sqrt{\hat{\sigma}^2/S}}$$

is approximately distributed as $N(0, 1)$; this could be helpful to construct

- test regarding the value of the expected value/integral
- confidence bounds on the approximation of the expected value: eg an approximate .95 CI monte carlo confidence interval for posterir mean of θ is $\hat{\theta} \pm 1.96\sqrt{\sigma^2/S}$

- choose the number of simulation S to be large enough so that the monte carlo standard error $\sqrt{\sigma^2/S}$ is less than a specified precision if we want to report a confidence interval for $E(\theta|x_1,\dots,x_n)$.
  EG supposing we've generated a Monte carlo sample of size $S = 100$, for which the estimated $Var(\theta|x_1,\dots,x_n)$ was 0.024; if we want that the width of the confidence interval of the monte carlo mean to be less than 0.01 we would need to increase our sample size so that

$$1.96\sqrt{0.024/S} < 0.01 \implies S > 960$$

Example 4.1.2. plot the monte carlo estimate of mean with asymptotic normal confidence bar in fig 4.1

```
S <- 1000
s <- 1:S
set.seed(452304)
thetas <- rnorm(S)

cummean <- cumsum(thetas)/seq_along(thetas)
cumvar <- cumsum(thetas^2)/seq_along(thetas) - cummean^2
cumse  <- sqrt(cumvar/seq_along(thetas))

z <- qnorm(0.975)
up <- cummean + z * cumse
low <- cummean - z * cumse

plot(s, cummean, type = 'l')
lines(s, up, type = 'l', lty = 'dotted')
lines(s, low, type = 'l', lty = 'dotted')
abline(h = 0, col = 'red')
```

4.1.1.3 Multi-parameter

NB prof: integrazione lambert pag 272

Remark 65 (Multiparameter monte carlo simulation). In this way we see how a sample mean can approximate an integral of interest (expected value); the solution holds for multiple parameters/integrals of interest (not yet seen in practice). For example, let θ_1, θ_2 be two parameters of interest of a distribution: supposing we can extract from the joint posterior distribution $p(\theta_1, \theta_2|x_1,\dots,x_n)$ one way to calculate the bivariate expected value (and avoid double integration) is using Monte carlo method.
The expected value of the distribution $g\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right)$ (with $g : \mathbb{R}^2 \to \mathbb{R}^2$ I guess) is

$$E\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}|x_1,\dots x_n\right)\cdot g\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right)\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1$$

It turns out that the monte carlo sample mean converge to the expected value

$$\frac{1}{S}\sum_{s=1}^{S}\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right) \to E\left(g\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right)$$

allowing us to avoid a double integral and simply computing a (vector) mean

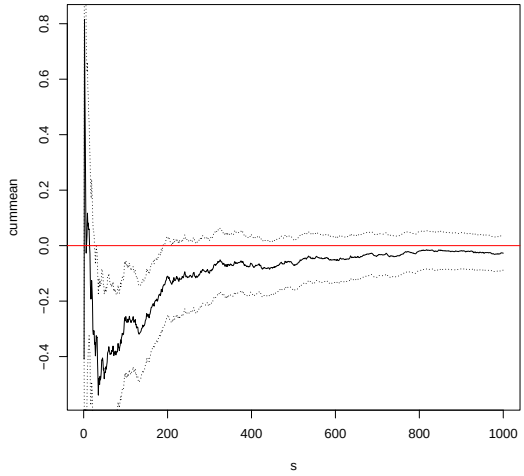


Figure 4.1: MC sim2

4.1.2 Applications

4.1.2.1 Posterior inference for arbitrary functions

Important remark 35 (Function of one parameter). Supposing we're interested in a transformation γ of the parameter coming from the posterior we have to simply transform the estractions $\gamma(\theta)$, obtaining $\left\{\gamma^{(1)} = \gamma(\theta_1), \ldots, \gamma^{(S)}\right\}$ and according to the law of large number

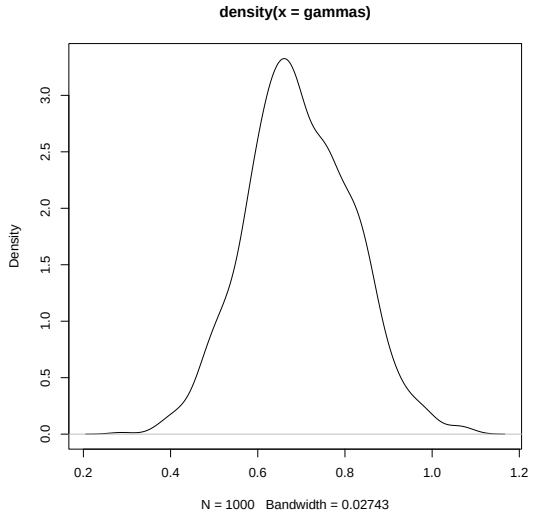
- the empirical distribution $\left\{\gamma(\theta_1), \ldots, \gamma^{(S)}\right\} \to p(\gamma|x_1, \ldots, x_n)$
- $\hat{\gamma} = \sum_{s=1}^{S} \gamma^{(s)}/S \to E(\gamma|x_1, \ldots, x_n)$
- $\sum_{s=1}^{S} (\gamma^{(s)} - \hat{\gamma})^2/(S-1) \to Var(\gamma|x_1, \ldots, x_n)$

Example 4.1.3. Suppose we've the posterior $p(\theta|x_1, \ldots, x_n)$ of a beta binomial model were θ is a proportion, suppose a $Beta(200, 100)$; if we're interested in the log-odds (instead of the proportion)

$$\gamma(\theta) = \log\left(\frac{\theta}{1-\theta}\right)$$

we procede simply as follows to plot the density of the transformation

```
thetas <- rbeta(1000, 200, 100)
gammas <- log(thetas/(1-thetas))
# distribution
plot(density(gammas))
```



Important remark 36 (Function of two parameters). We may be interested in comparing two parameters:'asis'

- we may compare them via a function, eg $\theta_1 - \theta_2$ or θ_1/θ_2
- we may be interested in computing $P(\theta_1 > \theta_2|x_1, \ldots, x_n)$

Both these quantities (distribution, probability) can be obtained with Monte carlo sampling.
The sequence of couples $\left\{(\theta_1^{(1)}, \theta_2^{(1)}), \ldots, (\theta_S^{(1)}, \theta_S^{(1)})\right\}$ and consists of S indepen-det samples from the joint posterior distribution of θ_1 and θ_2

- is an approximation of the bivariate distribution of the two parameters
- can be used to obtain univariate distributions of $\gamma = f(\theta_1, \theta_2)$ with $f : \mathbb{R}^2 \to \mathbb{R}^1$, such as difference or proportions
- can be used to obtain Monte carlo approximations such as

$$\frac{1}{S}\sum_{s=1}^{S} I(\theta_1^{(s)} > \theta_2^{(s)}) \to P(\theta_1 > \theta_2|x_1, \ldots, x_n)$$

Example 4.1.4. Supposing, in the example of birth rate per woman without (θ_1) and with (θ_2) bachelor degree, that posterior distribution after conducting a study that

$$\theta_1|x_{1,1}, \ldots, x_{n_1,1} \sim Gamma(219, 112)$$
$$\theta_2|x_{1,2}, \ldots, x_{n_2,2} \sim Gamma(68, 45)$$

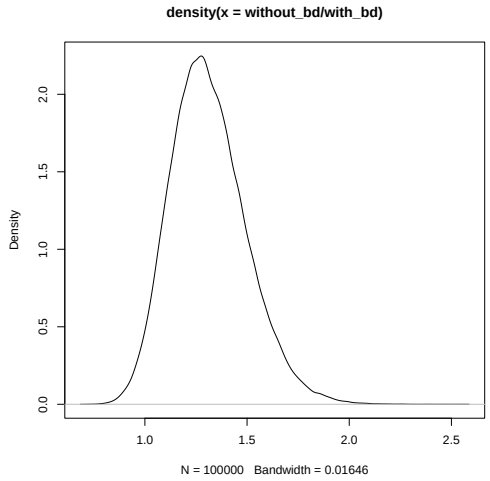
Then, if we want to

- calculate $P(\theta_1 > \theta_2|x)$

```r
S <- 1e05
without_bd <- rgamma(S, 219, 112) # theta_1
with_bd <- rgamma(S, 68, 45) # theta_2
mean(without_bd > with_bd) # this is the final result

## [1] 0.9726
```

- if we want to construct the density of the ratio θ_1/θ_2

```r
plot(density(without_bd / with_bd))
```



4.1.2.2 Sampling from predictive distributions

In section 3.2.1 we have seen how, under conditional independence (on the parameter of interest) the probability distribution of the new observation given the results of the experiment (posterior predictive distribution) can be written as

$$P(Z = z|x_1, \dots x_n) = \int P(Z = z|\theta) \cdot P(\theta|x_1, \dots, x_n)\, \mathrm{d}\theta$$

To obtain the posterior predictive probability that Z is equal to some specific value z we can apply the Monte carlo method:

- sample S thetas from experiment posterior: $\theta^{(1)}, \dots, \theta^{(S)} \sim$ iid $p(\theta|x_1, \dots, x_n)$
- use the extracted thetas to sample individual observation from the corresponsing distribution involving theta: $z^{(1)}$ $p(z|\theta^{(1)}), \dots, z^{(S)}$ $p(z|\theta^{(S)})$
- the sequence $\left\{(\theta^{(1)}, z^{(1)}), \dots, (\theta^{(S)}, z^{(S)})\right\}$ constitutes S independent samples from the joint posterior distribution of (θ, Z) while the sequence $z^{(1)}, \dots, z^{(S)}$ S independent samples from the *marginal* posterior distribution of Z, which is the posterior predictive distribution

- we can obtain an approximation of $P(Z = z|x_1, \ldots, x_n)$ by calculating the monte carlo mean $\sum_{s=1}^{S} p(z|\theta^{(s)})/S$. [1]

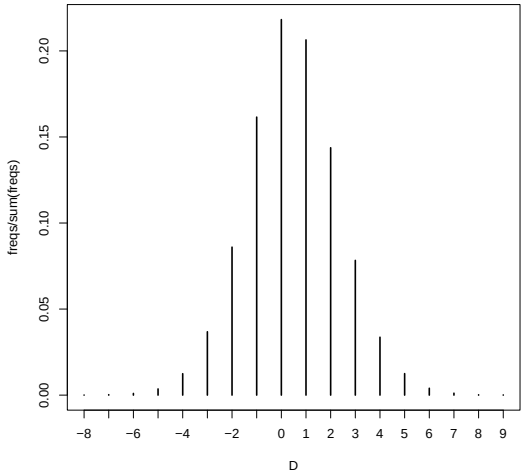
Example 4.1.5. Here we are interested in the predictive probability that an woman without college degree would have more children than one with college degree.
The posterior distribution of mean number of children for woman without and with bachelor degree are Gamma(219,112) and Gamma(68, 45) respectively

```
## create the posterior extraction of theta as before
S <- 1e05
without_bd_lambdas <- rgamma(S, 219, 112) # theta_1
with_bd_lambdas <- rgamma(S, 68, 45) # theta_2

## use all the extraction to extract a new observation, one for each theta
without_children <- rpois(S, without_bd_lambdas)
with_children <- rpois(S, with_bd_lambdas)

mean(without_children > with_children)

## [1] 0.48005
```

However, once we have generated these monte carlo samples from the posterior preictive distribution we can use to calculate other quantities of interest

```
  ## other aspects of interest
  mean(without_children) # mean children for a new obs without degree

## [1] 1.95504

  mean(with_children)

## [1] 1.51472

  ## difference D = (Z_1 - Z_2) in number of children between two individuals,
  ## one sampled from each group
  D <- without_children - with_children
  freqs <- table(D)
  plot(freqs/sum(freqs))
```

[1] This procedure will work well if $p(z|\theta)$ is discrte and we are interested in quantities that are easily computed from $p(y|\theta)$



4.1.2.3 Posterior predictive model checking

we can compare the sample data (ecdf) with posterior predictive density to

Example 4.1.6. Previously (exercise 3.2.4) we derived for two sample of the posterior distribution for women without bd is $\theta_1 \sim Gamma(219, 112)$, and thus predictive probabilities for number of children is $NBinom(219, 112)$.
We can compare this theorical distribution with empirical distribution from the sample (fig 4.2): the two distribution seem to be in conflict (eg the observed data have twice as many women with two children than one).

```
# data from https://www2.stat.duke.edu/~pdh10/FCBS/Replication/gss.RData

## load("/tmp/gss.RData")
## y1<-gss$CHILDS[gss$FEMALE==1 &  gss$YEAR>=1990  & gss$AGE==40 & gss$DEG<3 ]
## y1<-y1[!is.na(y1)]
## dput(y1)

## 111 female
data <- c(2L, 2L, 5L, 0L, 2L, 1L, 2L, 4L, 2L, 0L, 3L, 1L, 0L, 0L, 2L, 1L, 3L,
          3L, 2L, 3L, 1L, 2L, 2L, 2L, 6L, 2L, 3L, 2L, 1L, 2L, 2L, 2L, 0L, 2L,
          4L, 2L, 3L, 4L, 1L, 4L, 1L, 0L, 0L, 0L, 5L, 4L, 3L, 2L, 1L, 0L, 0L,
          3L, 2L, 1L, 0L, 2L, 1L, 4L, 2L, 1L, 2L, 0L, 1L, 3L, 4L, 0L, 0L, 4L,
          3L, 3L, 0L, 1L, 1L, 2L, 0L, 1L, 3L, 2L, 6L, 3L, 3L, 1L, 2L, 1L, 3L,
          1L, 3L, 2L, 2L, 2L, 2L, 2L, 3L, 2L, 3L, 0L, 2L, 2L, 3L, 2L, 1L, 4L,
          4L, 0L, 2L, 2L, 0L, 2L, 2L, 3L, 0L)

(edf <- table(data) / sum(table(data))) ## empirical distribution

## data
##          0          1          2          3          4          5          6
## 0.18018018 0.17117117 0.34234234 0.18018018 0.09009009 0.01801802 0.01801802
```

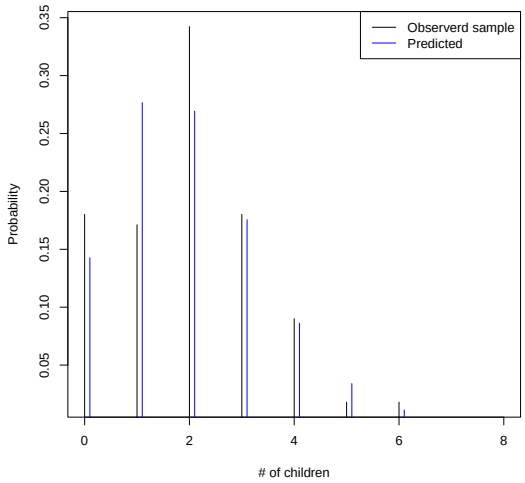


Figure 4.2: Observed vs predicted probabilities of number of children

```
pred_distr <- dnbinom(x = 0:8, size = 219, mu = 219/112) # posterior pred. distributi

## plot1
plot(x = 0:8, y = c(edf, NA, NA),
     type = "h", xlab = '# of children', ylab = 'Probability')
points(x = 0:8 + 0.1, y = pred_distr, col = 'blue', type = 'h')
legend('topright', legend = c("Observerd sample", "Predicted"), col = c("black", "blu
       lty = 1)
```

Remark 66. Some possible explanations for difference between sample and predicted distribution

1. it's the result of the sample variability for which the sample data is very different from the population: this particularly true if sample size is low (not the case with over 100 observations)

2. it's a feature of the population and the observed sample is correctly reflecting it. In contrast, the Gamma-Poisson model unable to represent this feature because there's no Poisson distribution that has such a sharp peak at $x = 2$

These explanation can be assessed numerically with Monte Carlo simulation. We may be interested in the ratio

$$t = \frac{\#\text{ mothers with two children}}{\#\text{ mothers with one children}}$$

which in the sample is $38/19 = 2$

```
table(data)

## data
##  0  1  2  3  4  5  6
## 20 19 38 20 10  2  2
```

Suppose we sample a different set of 111 women what distribution of t we would expect? This can be tackled using monte carlo; we could, for each simulation $s \in \{1, \ldots, S\}$

1. sample a θ from its posterior distribution $\theta^{(s)} \sim p(\theta|x_1, \ldots, x_n)$

2. create a sample of n new units using the posterior predictive density $Z^{(s)} = (z_1^{(s)}, \ldots, z_n^{(s)}) \sim \text{iid} p(z|\theta^{(s)})$, called *posterior predictive datasets* (each of size n)

3. compute or statistics of interest on the generated sample $t^{(s)} = t(Z^{(s)})$

Once the sample of $t^1 \ldots, t^S$ we

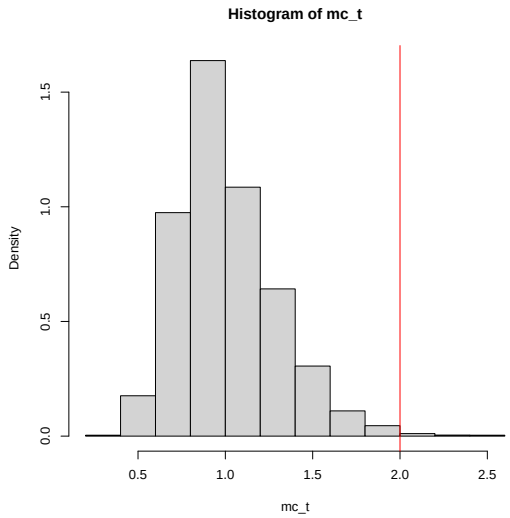
- plot the distribution of $t^1 \ldots, t^S$

- add the value which t assumes in the observed data

- calculate the probability of $t^1 \ldots, t^S$ being more extreme than the sample t

Example 4.1.7. Following the previous steps

```
S <- 10000
thetas <- rgamma(S, 219, 112)
mc_samples <- lapply(thetas, function(t) rpois(n = 111, lambda = t))
mc_t <- sapply(mc_samples, function(x) sum(x==2)/sum(x==1))

# plot ?hist
hist(mc_t, freq = FALSE)
abline(v = 2, col = 'red')
```




```
mean(mc_t >= 2)

## [1] 0.0054
```

So:

- if the gamma poisson model were correct we would obtain a observed value of the ratio of interest 0.51% cases[2]; this suggest our poisson model is flawed. It predicts that we would hardly ever see a dataset that resembled our observed one in terms of t; if we were interested in making prediction we would have to consider a more complicated model (for example, a multinomial sampling model).
- on the other hand, a simple Poisson model may suffice if we are interested only in certain aspects of predictive distribution. We should at least make sure that our model generates predictive datasets $Z^{(s)}$ that resemble the observed dataset in terms of features that are of interest.

4.2 Issues with independent sampling

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[2] These types of posterior predictive checks have given rise to a notion of posterior predictive p-values, which despite their name, do not generally share the same frequentist properties as p-values based on classical goodness-of-fit tests. This distinction is discussed in Bayarri and Berger (2000), who also consider alternative types of Bayesian goodness of fit probabilities to serve as a replacement for frequentist p-values.