1 Operators

$$a|b :\Leftrightarrow \exists c\ b = ac \text{ for } a \neq 0$$
$$a \equiv_m b :\Leftrightarrow m|(a-b) \text{ i.e. } \exists r \in \mathbb{Z}\ a = b + rm$$

2 Propositions

Implication:	$A \to B :\Leftrightarrow \neg A \vee B$
Two-sided Implication:	$A \leftrightarrow B :\Leftrightarrow A \equiv B$
Associativity:	$(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$
Distributive Law:	$(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$
	$(A \vee B) \wedge (C \vee D) \equiv (A \wedge C) \vee (A \wedge D) \vee (B \wedge C) \vee (B \wedge D)$
Idempotence:	$F \wedge F \equiv F$
Absorption:	$F \wedge (F \vee G) \equiv F$
de Morgan's Law:	$\neg(A \wedge B) \equiv (\neg A \vee \neg B)$

3 Proofs

To prove a sentence (either true or false) means to show that it's a tautology. The following **proof patterns** may be used.

3.0.1 Direct Proof of an Implication

Example: $F \to G$

A **direct proof of an implication** works by assuming F and then deriving G from F.

$$F \Rightarrow ... \Rightarrow ... \Rightarrow ... \Rightarrow G$$

3.0.2 Indirect Proof of an Implication

Example: $F \to G$

An **indirect proof of an implication** proceeds by assuming $\neg G$ and deriving $\neg F$ under this assumption.

$$\neg G \Rightarrow ... \Rightarrow ... \Rightarrow ... \Rightarrow \neg F$$

3.0.3 Composition of Implications

Example: $F \to G$ and $G \to H$

1. Prove the statement F
2. Prove the implications $F \Rightarrow G$ and $G \Rightarrow H$

3.0.4 Case Distinction

1. Define a complete list of cases
2. Prove the statement for each case separately

3.0.5 Proof by Contradiction

Assume that the sentence F is false and derive a false statement from it.

$$\neg F \Rightarrow ... \Rightarrow ... \Rightarrow ... \Rightarrow \bot$$

3.0.6 Existence Proof

Example: $\exists x\ P(x)$

Either find a variable which satisfies the sentence (**constructive**) or proof the existence of such a variable without exhibiting it (**non-constructive**).

3.0.7 Proof by Counterexample

Example: $\neg\forall x\ P(x)$

Find a variable such that the sentence is wrong.

3.0.8 Proof by Induction

Example: $\forall n\ P(n)$

1. **Basis step**: Prove $P(0)$
2. Assume $P(n)$
3. **Induction step**: Prove $P(n+1)$

4 Predicate Logic

4.1 Rules

1. $\forall x\ P(x) \land \forall x\ Q(x) \Leftrightarrow \forall x\ (P(x) \land Q(x))$
2. $\exists x\ (P(x) \land Q(x)) \Rightarrow \exists x\ P(x) \land \exists x\ Q(x)$
3. $\neg\forall x\ P(x) \Leftrightarrow \exists x\ \neg P(x)$
4. $\neg\exists x\ P(x) \Leftrightarrow \forall x\ \neg P(x)$
5. $\exists y \forall x\ P(x,y) \Rightarrow \forall x \exists y\ P(x,y)$
6. $\forall x\ (\exists x\ P(x) \land P(x)) \lor P(\underline{x})$, where $\underline{\text{x}}$ is free

5 Sets

$$A \subseteq B :\Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$$
$$A = B \Leftrightarrow A \subseteq B \land B \subseteq A$$
$$P(A) := \{S \mid S \subseteq A\}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \times B = \{(a,b) \mid a \in A \land b \in B\}, |A \times B| = |A| * |B|$$
$$\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}(teilmenge)|\mathcal{P}| = 2^{|A|}$$

$Ordered\_sets := (a,b) := \{\{a\},\{a,b\}\}$

A teilmenge of a set is the set itself, every element of the set and the null element.

6 Relations

6.1 Reflexive

Formula: $a\ \rho\ a$
Set: $id \subseteq \rho$
Matrix: Diagonal is all 1
Graph: Every vertex has a loop
Examples: $\leq$, $\geq$, $|$, $\equiv_m$ on $\mathbb{Z}$

6.2 Transitive

Formula: $a\ \rho\ b\ \wedge\ b\ \rho\ c \Rightarrow a\ \rho\ c$
Set: $\rho^2 \subseteq \rho$
Examples: $\leq$, $\geq$, $|$, $<$, $>$, $\equiv_m$ on $\mathbb{Z}$

6.3 Symmetric

Formula: $a\ \rho\ b \Leftrightarrow b\ \rho\ a$
Set: $\rho = \hat{\rho}$
Matrix: Matrix is symmetric
Graph: Undirected graph, possibly with loops
Examples: $\equiv_m$ on $\mathbb{Z}$

6.4 Antisymmetric

Formula: $a\ \rho\ b \wedge b\ \rho\ a \Rightarrow a = b$
Set: $\rho \cap \hat{\rho} \subseteq id$
Graph: No cycle of length 2
Examples: $\leq$, $\geq$ on $\mathbb{Z}$ and $|$ on $\mathbb{N}$

6.4.1 Relations as Sets

$a\ \rho\sigma\ b$**:**	$\exists b \in B : (a\ \rho\ b\ \wedge\ b\ \sigma\ c)$
$a\ (\rho \cup \sigma)\ b$**:**	Either $a\ \rho\ b$ or $a\ \sigma\ b$
$a\ (\rho \cap \sigma)\ b$**:**	$a\ \rho\ b$ and $a\ \sigma\ b$
The empty set $\emptyset$:	symmetric and transitive

6.4.2 Transitive closure

$$p^\star = \cup_{n=1}^{\infty} p^n \tag{1}$$

6.4.3 Equivalence Relation

Example: $=$
A relation that is reflexive, symmetric, and transitive.

The set of elements in A that equivalent to $a \in A$ according to the equivalence relation θ is called the **equivalence class** of a.

$$[a]_\theta := \{b \in A \mid b\ \theta\ a\}$$

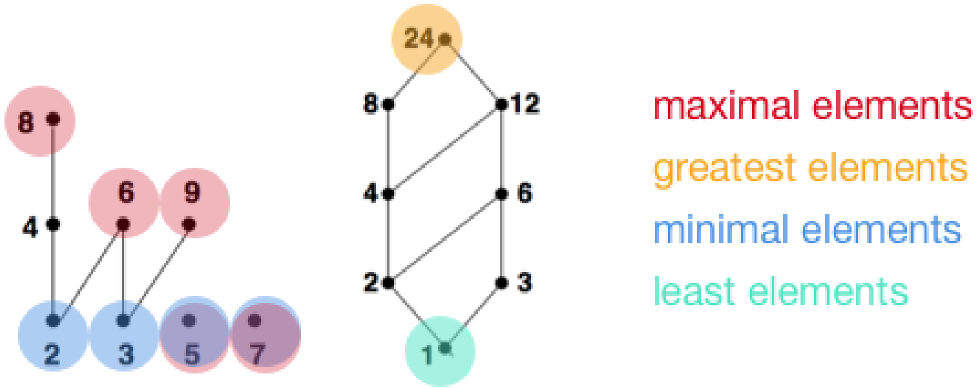
The set A/θ of equivalence classes of θ on A is a **partition**.
If you are looking for a equivalence relation give the matrix with the above properties.
Hint:
$a\ \theta\ b \Rightarrow [a] = [b]$
$a\ \not\theta\ b \Rightarrow [a] \cap [b] = \emptyset$

6.5 Partial Order = ordnungsrelation

Example: $\leq$ and $\geq$ on $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, $=$ on $\mathbb{N}$
A relation that is reflexive, antisymmetric, and transitive.
If every two elements in a poset are comparable than the Set is **totally ordered**.

Special elements in a poset $(A, \preceq)$ with a subset $S \subseteq A$:

minimal (maximal) element:	$a \in S$ if there exists no $b \in S$ with $b \prec a$ $(b \succ a)$



Hasse Diagram of the Poset $(\{2,3,4,5,6,7,8,9\};|)$ and $(\{1,2,3,4,6,8,12,24\};|)$

least (greatest) element: $a \in S$ if $a \preceq b$ $(a \succeq b)$ for all $b \in S$
lower (upper) bound: $a \in A$ if $a \preceq b$ $(a \succeq b)$ for all $b \in S$
greatest lower (least upper) bound: $a \in A$ if a is the greatest (least) element of the set of all lower (upper) bounds of S

6.6 Function

injective: no collisions. $\forall h_1, h_2 \epsilon A \; h_1 \neq h_2 \Rightarrow f(h_1) \neq f(h_2)$.
surjective: every value in the codomain is taken on for some argument
bijective: one-to-one mapping (injective and surjective)

bereinigter pranexform is when you write a formula with first all the quantifiers and then the formula.

7 Combinatorics

	with repetition	**without repetition**
ordered	n^k	$\frac{n!}{(n-k)!}$
	A passcode of length n with k different digits	How many ways can k places be awarded to n people
unordered	$\binom{n+k-1}{k}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
	Choose at most k scoops of ice cream from n different flavours	Select k from n objects

Hint: $\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = \binom{n}{n-1} = n$

7.1 Countability

same cardinality $A \sim B$: There exists a bijection $A \to B$
B has at least the cardinality of A $A \preceq B$: $A \sim C$ for some subset $C \subseteq B$
B dominates A $A \prec B$: $A \preceq B \wedge A \not\sim B$
countable: $A \preceq \mathbb{N}$

Hint:
The set $\{0,1\}^* := \{0,1,00,01,...\}$ of **finite binary sequences** is countable.
The set $\{0,1\}^\infty$ is uncountable (Cantor's diagonalization argument).
The set A^n **of n-tuples** over A is countable.
The **union of a countable list** of countable sets is countable (can be considered as tuples).
$\mathbb{N}$, $\mathbb{Z}$ und $\mathbb{Q}$ are countable $\mathbb{P}(\mathbb{Z})$ ist uberabzahlbar(=not countable).

7.2 Inclusion/Exclusion Principle

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

7.3 Double counting principal

We want to count the subset of A×B(which is a relation). We can aϵA the number m_a of bϵB such that (a,b)ϵS. Or the same for B(equal to the sum of ones in the matrix representation).

7.4 Pigeon hole principal

If a set of n objects is partitioned into k < n sets, then at least one of these sets contains at least $\frac{n}{k}$ objects

7.5 Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

8 Graph Theory

8.1 Special Graphs

complete graph K_n	n vertices, $\frac{n(n-1)}{2}$ edges, (n)-regular (every vertex has the same degree n) .
k regular graph	Each vertex has k neighbours.
compete bipartite graph $K_{m,n}$	$m+n$ vertices, mn edges, with two vertex subsets $|V_b| = m$ and $|V_w| = n$
tree:	undirected, connected graph with no cycles and $n-1$ edges.
hypercube Q_d:	*d-regular* with 2^d vertices and $2^{d-1}d$ edges
mesh $M_{m,n}$:	mn vertices

8.2 Traversals

walk:	sequence of vertices such that consecutive vertices are connected
tour:	a walk with distinct edges
circuit:	a tour that ends where it started
Hamiltonian cycle:	a circuit that visits all vertices

8.3 Planar Graphs

For *connected, planar* graphs, the following equations hold:

$r = |E| - |V| + 2$ (number of regions)
$\Sigma_{v \in V} deg(v) = 2|E|$ (sum of the degrees of the regions)
if $|V| \geq 3 \Rightarrow |E| \leq 3|V| - 6$

$$\text{if } |V| \geq 3 \text{ and bipartite} \Rightarrow |E| \leq 2|V| - 4$$
$$K_n \text{ is planar if and only if } n \leq 4\text{i}$$

Rules to prove **non-planarity**

- deletion of edges
- deletion of singleton vertices
- merging neighboring vertices

8.4 Isomorphism

Two graphs are **isomorphic** (denoted $\cong$) if there exists a bijection $\pi : V \mapsto V'$ such that

$$\{u, v\} \in E \Leftrightarrow \{\pi(u), \pi(v)\} \in E'$$

Hint: Look for cycles with vertices that have a distinct number of degrees. If the graph in question doesn't contain that specific cycle, it can't be isomorph.

1. $|E| + |E^-| = 2|E|$ ist gleich der Maximalen Anzahl kanten in Graph.
2. $\frac{|v|(|v|-1)}{2}$ ist gleich der Maximalen Anzahl kanten in Graph.

8.5 Trees

A tree is an undirected connected graph with no cycles. A forest is an undirected graph with no cycles. A leaf is a vertex with degree one. A tree has n-1 edges.

9 Number Theory

9.1 Division

Hint: Every non-zero integer is a divisor of 0. 1 and -1 are divisors of every integer.

9.2 Greatest Common Divisor

For integers a and b (not both 0), an integer d is called a $gcd(a, b)$ if d divides both a and b and if every common divisor of a and b divides d.

$$d|a \text{ and } d|b \text{ and } c|a \wedge c|b \Rightarrow c|d$$
$$gcd(a, b) :\Leftrightarrow \exists u, v \; ua + vb$$

9.3 Chinese Remainder Theorem

$$\begin{aligned}
\text{given} \quad & z \equiv_{b_1} c_1 \\
& z \equiv_{b_2} c_2 \\
& z \equiv_{b_3} c_3 \\
\text{then} \quad & z = B_1 x_1 c_1 + B_2 x_2 c_2 + B_3 x_3 c_3 \\
\text{where} \quad & B_i = \frac{B}{b_1} \text{ with } B = b_1 b_2 b_3 \text{ and } B_i x_i \equiv_{b_i} 1 \\
\text{To find different } z \text{ to satisfy certain constraints} \quad & z' = z \pm n \cdot B, \; n \in \mathbb{N}
\end{aligned}$$

9.4 Extended Euclidean Algorithm

$$\begin{aligned}
\text{given} \qquad x &= gcd(888, 54) \\
\text{then} \qquad 888 &= 54(16) + 24 \\
54 &= 24(2) + \underline{6} \\
24 &= 6(4) + 0 \\
\text{to find 6 = u(888) + v(54):} \qquad 6 &= 54 - 24(2) \\
&= 54 + 24(-2) \\
&= 54 + (888 - 54(16))(-2) \\
&= 54 + (888 + 54(-16))(-2) \\
&= 54 + 888(-2) + 54(32) \\
&= (-2)888 + (33)54
\end{aligned}$$

9.5 Ideal

$$\begin{aligned}
(a, b) &:= \{ua + vb | u, v \in \mathbb{Z}\} \\
(a) &:= \{ua | u \in \mathbb{Z}\}
\end{aligned}$$

For $a, b \in \mathbb{Z}$ there exists $d \in \mathbb{Z}$ such that $(a, b) = (d)$. This is implies that d is the **gcd** of a and b.

9.6 Least Common Multiple

$l = lcm(a, b)$ is the common multiple of a and b which divides every common multiple of a and b.

$$a|l' \text{ and } b|l' \ \Rightarrow l|l'$$

It follows:

$$gcd(a, b) \cdot lcm(a, b) = ab$$

9.7 Modular Arithmetic

$$\begin{aligned}
R_m(a + b) &= R_m(R_m(a) + R_m(b)) \\
R_m(ab) &= R_m(R_m(a) \cdot R_m(b)) \\
R_m(a^{bc}) &= R_m(R_m(a^b)^c)
\end{aligned}$$

9.8 Multiplicative Inverses

The **congruence equation** has a solution $x \in \mathbb{Z}_m$ if and only if $gcd(a, m) = 1$. The solution is unique.

$$ax \equiv_m 1$$

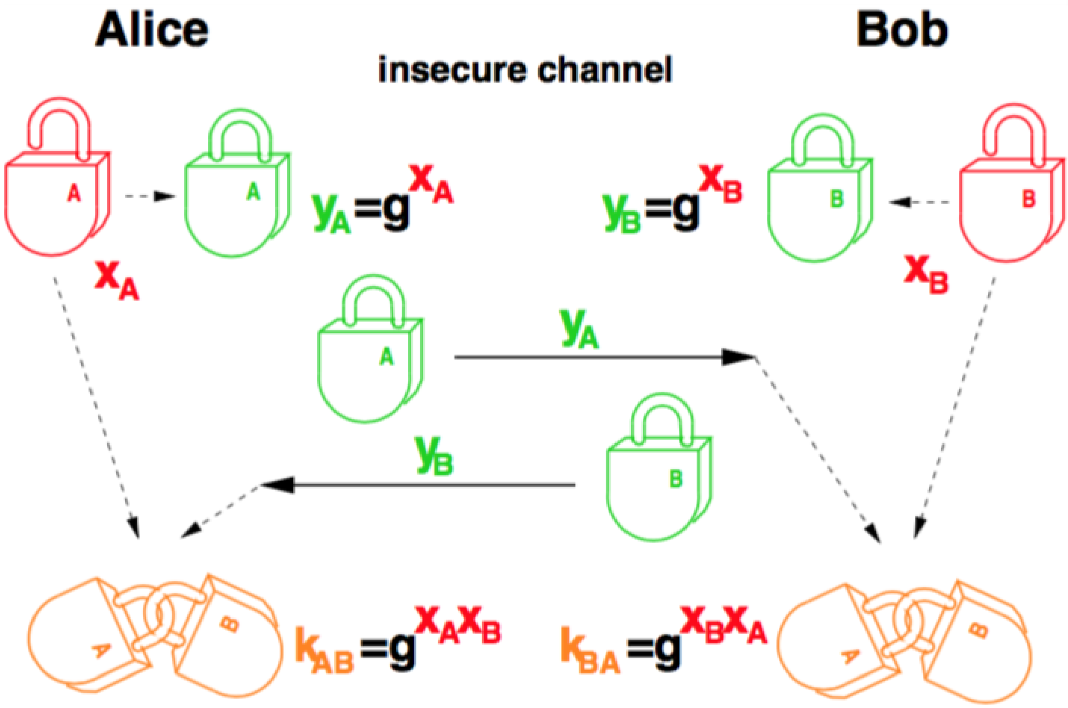
The x satisfying the equation is called the **multiplicative inverse of a modulo m (a Einheit of the Ring)** ($x \equiv_m a^{-1}$ or $x \equiv_m \frac{1}{a}$).

9.9 Diffie-Hellmann Key-Agreement Protocol

The Diffie-Hellmann protocol is based on the **discrete logarithm problem**. Basically, while $y = R_p(g^x)$ can be computed efficiently, it can't be solved for x.

Alice	**insecure channel**	**Bob**
select x_A at random from $\{0, \dots, p-2\}$		select x_B at random from $\{0, \dots, p-2\}$
$y_A := R_p(g^{x_A})$		$y_B := R_p(g^{x_B})$
	$\xrightarrow{y_A}$	
	$\xleftarrow{y_B}$	
$k_{AB} := R_p(y_B^{x_A})$		$k_{BA} := R_p(y_A^{x_B})$

$$k_{AB} \equiv_p y_B^{x_A} \equiv_p (g^{x_B})^{x_A} \equiv_p g^{x_A x_B} \equiv_p k_{BA}$$



9.10 RSA

A finite group needs G needs to be chosen. Usually, the group $\mathbb{Z}_n^*$ where $n = pq$ is the product of two secret prime numbers. Then d is equal to

$$d \equiv_{|G|} e^{-1} \equiv_{(p-1)(q-1)} e^{-1}$$

where d is the **private key** and the tuple (n, e) is the **public key**.
Hint: It's not possible to calculate d without knowing G's order.

Alice	insecure channel	Bob
Generate primes p and q $n = p \cdot q$ $f = (p-1)(q-1)$		
select e $d \equiv_f e^{-1}$	$\xrightarrow{n, e}$	plaintext $m \in \{1, \ldots, n-1\}$
$m = R_n(y^d)$	$\xleftarrow{y}$	ciphertext $y = R_n(m^e)$

10 Algebra

10.1 Special Properties

Some special properties of an algebra $\langle S; *, e \rangle$ are

neutral element:	$e \in S$ such that $e * a = a * e = a$
associativity:	$*$ is associative if $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$
inverse element:	b is the inverse of a if $b * a = a * b = e$
commutative/abelian:	$a * b = b * a$ for all $a, b \in S$

The **neutral** and **inverse element** can have a left and right version. E.g. $e * a = a$ is the left neutral element. However, there is *always only one* neutral/inverse element.

10.2 Special Algebras

	Notation	**Axioms**	**Examples**
Semigroup	$\langle S; *\rangle$	$*$ is associative	
Monoid	$\langle M; *, e\rangle$	$*$ is associative	
		e is the neutral element	
Group	$\langle G; *, \hat{}, e\rangle$	$*$ is associative	$\langle \mathbb{Z}; +, -, 0\rangle$, $\langle \mathbb{Q} - \{0\}; \cdot, ^{-1}, 1\rangle$, $\langle \mathbb{R}; +, -, 0\rangle$
		e is the neutral element	
		every $a \in G$ has an inverse element	
Ring	$\langle R; +, -, 0, \cdot, 1\rangle$	$\langle R; +, -, 0\rangle$ is a commutative group	$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ (commutative)
		$\langle R; \cdot, 1\rangle$ is a monoid	
		$a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in R$	
Integral Domain (korper)		cummutative	$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
		no zerodividers $ab = 0 \Rightarrow a = 0 \vee b = 0$	
Field	$GF(p) \equiv \mathbb{Z}_p$	$\langle F - \{0\}; \cdot, ^{-1}, 1\rangle$ is a commutative ring	$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$
		every nonzero element is a unit (has an inverse)	

Hint: In order to prove a specific algebra, prove its axioms and that the set is closed in correspondence to its operations.

10.3 Groups

10.3.1 Direct Product

The **direct product of** n **groups** $\langle G_1; *_1\rangle, ..., \langle G_n; *_n\rangle$ is the group

$$\langle G_1 \times ... \times G_n, \star\rangle$$

where the operation $\star$ is component-wise:

$$(a_1, ..., a_n) \star (b_1, ..., b_n) = (a_1 *_1 b_1, ..., a_n *_n b_n)$$

10.3.2 Homomorphism

A function ψ from a group $\langle G; *, \hat{}, e\rangle$ to a group $\langle H; \star, \hat{}, e'\rangle$ is a group homomorphism if, for all a and b

$$\psi(a * b) = \psi(a) \star \psi(b)$$

Furthermore, ψ is an **isomorphism** if it's a bijection.
A group homomorphism satisfies:

$$\psi(e) = e'$$
$$\psi(\hat{a}) = \widehat{\psi(a)}$$

10.4 Subgroup

A subset $H \subseteq G$ of a group $\langle G; *, \hat{}, e\rangle$ is called a subgroup if $\langle H; *, \hat{}, e\rangle$ is *closed* with respect to all operations.

$$\begin{aligned}
&a * b \in H \text{ for all } a, b \in H \\
&e \in H \\
&\hat{a} \in H \text{ for all } a \in H
\end{aligned}$$

The smallest subgroup of a group G containing the element $g \in G$ is the **group generated** by g:

$$\langle g\rangle := \{g^n | n \in \mathbb{Z}\}$$

where the resulting group is called **cyclic**.

Hint: The order of a subgroup of a finite group divides its enclosing group's order i.e. $|H|$ divides $|G|$. A subgroup of size two contains e and an element a, a has to be it's own invers: aa=e. A subgroup with a prime order is cyclic and therefore isomorph to $\mathbb{Z}_n, \bigoplus$ and because of $\bigoplus$ is also kommutativ.

10.4.1 Cyclic Group

A **cyclic group** of order n is isomorphic with $\langle \mathbb{Z}_n; \oplus\rangle$.

Hint: Every group of prime order is cyclic, and in such a group every element except the neutral element is a generator.
Hint: $\mathbb{Z}_p^*$ is cyclic if and only if $m = 2$, $m = 4$, $m = p^e$ or $m = 2p^e$, where p is a prime and $e \geq 1$

10.4.2 Order

of a finite group: $|G|$ is the order of G

of an element of G: The order of $a \in G$ is the least $m \geq 1$ such that $a^m = e$ if such an m exists, and $ord(a) = \infty$ otherwise.
Hint: $ord(e) = 1$. If $ord(a) = 2$, then $a^{-1} = a$.

10.5 Group $\mathbb{Z}_m^*$ and Euler's Function

$\langle \mathbb{Z}_m^*; \odot, ^{-1}, 1\rangle$ is a group with the set

$$\mathbb{Z}_m^* := \{a \in \mathbb{Z}_m \mid gcd(a, m) = 1\}$$

The **Euler function** is defined as follows:

$$\varphi(m) = |\mathbb{Z}_m^*| = (p-1)(q-1) \text{ with } n = pq$$

where p and q are prime.
Hint: If p is a prime, then $\mathbb{Z}_p^* = \{1, ..., p-1\} = \mathbb{Z}_p - \{0\}$

Fermat, euler

$$a^{\varphi(m)} \equiv_m 1 \tag{2}$$

for every prime p and every a not divisible by p

$$a^{p-1} \equiv_p 1 \tag{3}$$

10.6 Polynomials over Fields

$R[x]$ denotes a **polynomial ring**, a set of polynomials over R.
A polynomial is called **monic**, if its first coefficient is 1.
The polynomial $a(x) \in F[x]$ is called **irreducible** if it is divisible only by constants and by constant multiples of $a(x)$. Moreover, $\alpha \in F$ is a **root** $\Leftrightarrow$ $(x-\alpha)$ divides $a(x)$.

Hint: Every polynomial of degree 2 except x^2+x+1 is reducible. Every irreducible polynomial of degree ≥ 2 has no roots.

Example: Polynomial Division on $GF(2)$:

$$
\begin{aligned}
&(x^4+x+1) : (x^2+x+1) = x+2 \\
&\underline{-(x^3+2x)} \\
&\quad -2x^2-2x+5 \\
&\quad \underline{-(2x^2+4)} \\
&\qquad -2x+1
\end{aligned}
$$

Example: Polynomial Interpolation

$$
\begin{array}{ll}
\text{given} & a(x) \text{ with } a(3)=2, a(4)=6, a(5)=7 \\
\text{then} & a(x) = 2\dfrac{(x-4)(x-5)}{(3-4)(3-5)} + 6\dfrac{(x-3)(x-5)}{(4-3)(4-5)} + 7\dfrac{(x-3)(x-4)}{(5-3)(5-4)}
\end{array}
$$

10.6.1 Finite fields

There exists a finite field with q elements if and only if q is a power of a prime. The prime is called the charakteristik of the field. Moreover, any two finite fields of the same size q are isomorphic.

10.7 Error-Correcting Codes

A **(k,n)-error-correcting code** $\mathcal{C}$ over the alphabet $\mathcal{A}$ with $|\mathcal{A}| = q$ is a subset of cardinality q^k of $\mathcal{A}^n$ i.e. one element is of length n, with q^k different elements.

Hint: Usually, $\mathcal{A} = \{0,1\}$ with $q=2$ is being considered

The **Hamming distance** between two codewords is the number of positions at which the two codewords differ.
The **minimum distance** of an error-correcting code $\mathcal{C}$ is the minimal Hamming distance between any two codewords.
A code $\mathcal{C}$ with minimum distance d can correct t errors if and only if $d \geq 2t+1$.

11 Logic

11.1 Proof System

A **proof system** is a quadruple $\Pi = (\mathcal{S}, \mathcal{P}, \tau, \phi)$ with the following components:

set of statements $\mathcal{S}$: every $s \in \mathcal{S}$ is either $true$ or $false$
set of proofs $\mathcal{P}$: e.g. strings over some alphabet
truth function τ: defines the meaning (*semantics*) of objects in $\mathcal{S}$
verification function ϕ: $\phi(s,p) = 1$ means that p is a valid proof for the statement s

The proof system $\Pi = (\mathcal{S}, \mathcal{P}, \tau, \phi)$ is

sound if no false statement has a proof
$\phi(s,p) = 1 \Rightarrow \tau(s) = 1$
complete if every true statement has a proof
$\tau(s) = 1 \Rightarrow \exists p \in (P)\ \phi(s,p) = 1$

11.2 Syntax and Semantics

	Description	Notation
Syntax	alphabet of allowed symbols and which strings are valid	
Interpretation	an assignment to all variable symbols	$\mathcal{A}(A) = \{0,1\}$
Semantics	a function σ assigning to each formula F and each suitable interpretation $\mathcal{A}$ a truth value	$\sigma(F,\mathcal{A}) = \{0,1\}$, $\mathcal{A}(F)$
Model	an interpretation $\mathcal{A}$ for which F is true	$\mathcal{A} \models F$

Hint: $F \models G$ means that every model for F is also a model for G.

11.2.1 Structure

A **structure** is a tuple $\mathcal{A} = (U, \phi, \psi, \xi)$ with the following components:

universe U: nonempty set
function ϕ: assigns to each function symbol a function $U^k \mapsto U$
function ψ: assigns to each predicate symbol a function $U^k \mapsto \{0,1\}$
function ξ: assigns to each variable symbol a value in U

11.3 Calculi

A **derivation rule** is a rule for deriving a formula from a set of formulas. G can be derived from the set $\{F_1, ..., F_k\}$ by rule R:

$$\{F_1, ..., F_k\} \vdash_R G$$

A **calculus** K is a finite set of derivation rules $K = \{R_1, ..., R_m\}$. It is

sound/correct if and only if every derivation rule is correct
complete if F is a logical consequence of M, then F can be derived from M using K

11.4 Normal Forms

11.4.1 Conjunctive Normal Form (CNF)

$$F = (L_{11} \vee ... \vee L_{1m_1}) \wedge ... \wedge (L_{n1} \vee ... \vee L_{nm_n})$$

11.4.2 Disjunctive Normal Form (DNF)

$$F = (L_{11} \wedge ... \wedge L_{1m_1}) \vee ... \vee (L_{n1} \wedge ... \wedge L_{nm_n})$$

Hint: Every formula is equivalent to a formula in **CNF** and **DNF**.

11.5 Resolution Calculus

Given a Formula F in CNF, one can transform it into a set of clauses:

$$\mathcal{K}(F) = \{\{L_{11}, ..., L_{1m_1}\}, ..., \{L_{n1}, ..., L_{nm_1}\}\}$$

A clause K is then a **resolvent** of clauses K_1 and K_2 if there is a literal L such that $L \in K_1$ and $\neg L \in K_2$

$$K = (K_1 - \{L\}) \cup (K_2 - \{\neg L\})$$

This derivation is denoted as follows:

$$\{K_1, K_2\} \vdash_{res} K$$

Hint: A set M of formulas is unsatisfiable if and only if $\mathcal{K}(M) \vdash_{res} \emptyset$

11.5.1 Prenex Form

In order to bring a formula into prenex form

1. Resolve all name collisions
2. Pull the quantifiers to the front by inverting them every time they surpass a $\neg$.