1) $\int x^3 \cdot e^{-x^2} dx = \int \frac{x^{3\,2}}{-2x} d\left(e^{-x^2}\right) =$

$= -\frac{x^2}{2} e^{-x^2} - \int e^{-x^2} d\left(-\frac{1}{2}x^2\right) =$

$= -\frac{x^2}{2} e^{-x^2} - \int e^{-x^2} (-x)\, dx =$

$= -\frac{x^2}{2} e^{-x^2} - \int \frac{x}{2x} d\left(e^{-x^2}\right) =$

$= -\frac{x^2}{2} e^{-x^2} - \frac{1}{2} \int d\left(e^{-x^2}\right) =$

$= -\frac{x^2}{2} e^{-x^2} + \frac{1}{2} e^{-x^2} = -e^{-x^2} \frac{(x^2-1)}{2};$

2) $\int \frac{x^3+1}{x^3-5x^2+6x} dx =$ VI. 1.7

$$\begin{array}{l|l} x^3+1 & x^3-5x^2+6x \\ \hline x^3-5x^2+6x & 1 \\ \hline 5x^2-6x+1 & \end{array}$$

$= \int \left(1 + \frac{5x^2-6x+1}{x^3-5x^2+6x}\right) dx = \int \left(1 + \frac{5x^2-6x+1}{x(x-2)(x-3)}\right) dx =$

$= \int \left(1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)}\right) dx =$

$$\frac{5x^2-6x+1}{x(x-2)\cdot(x-3)} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-3)} =$$

$$= \frac{A(x-2)(x-3) + Bx(x-3) + Cx(x-2)}{x(x-2)\cdot(x-3)} =$$

$$= \frac{A(x^2-5x+6) + B(x^2-3x) + C(x^2-2x)}{x(x-2)\cdot(x-3)} =$$

$$= \frac{(A+B+C)x^2 + (-5A-3B-2C)x + 6A}{x(x-2)\cdot(x-3)},$$

$$\begin{cases} A+B+C=5, \\ -5A-3B-2C=-6, \\ 6A=1 \end{cases} \Rightarrow A=\frac{1}{6};$$

$$1+4B+3C=11$$

$$4B+3C=10;$$

$$B=\frac{5}{2}-\frac{3C}{4};$$

$$\frac{1}{6}+\frac{5}{2}-\frac{3C}{4}+C=5$$

$$\frac{1}{4}C = \overset{16}{5} - \overset{1}{\frac{1}{6}} - \overset{13}{\frac{5}{2}}$$

$$\frac{1}{4}C = \frac{\cancel{8}-16}{6}; \quad \Big| \cdot 4$$

$$C = -\frac{\cancel{11}}{\cancel{6}_3} \cdot \cancel{4}^2 = -\frac{22}{3};$$

$$C = \frac{28}{3}$$

$$-\frac{9}{4}$$

$$= x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C;$$

3) $$\int\left(\frac{x}{x^2-3x+2}\right)^2 dx = \int \frac{x^2}{(x-1)^2\cdot(x-2)^2}\,dx =$$

$$=$$

$$\frac{x^2}{(x-1)^2\cdot(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} = \frac{A(x-1)(x-2)^2 + B(x-2)^2 + C(x-1)^2(x-2) + D(x-1)^2}{(x-1)^2\cdot(x-2)^2} =$$

$$= \frac{A(x-1)\cdot(x^2-4x+4) + B(x^2-4x+4) + C(x-2)\cdot(x-2x+1) + D(x^2-2x+1)}{(x-1)^2\cdot(x-2)^2} =$$

$$= \frac{(A+C)x^3 + (-5A+B-4C+D) + (8A-4B+5C-2D) + 4A+4B-2C-D}{(x-1)^2\cdot(x-2)^2};$$

$$\begin{cases} A+C=0, \\ -5A+B-4C+D=1, \\ 8A-4B+5C-2D=0, \\ 4A+4B-2C-D=0 \end{cases}$$

$C = -A;$

$$A + B + D = 1$$
$$3A - 4B - 2D = 0$$
$$6A + 4B - D = 0$$

$$\begin{cases} 5A + 5B = 1, \\ -4A - 2B = 1 \end{cases} \qquad A + B = \frac{1}{5}$$

$$A = \frac{1}{5} - B;$$

$$-4\left(\frac{1}{5} - B\right) - 2B = 1$$

$$2B = 1 + \frac{4}{5}$$

$$B =$$

$$A = \frac{4}{5}; \quad B = -\frac{3}{5}, \quad C = -\frac{4}{5}; \quad D = \frac{6}{5}$$

$$= \int \left(\frac{4}{5(x-1)} - \frac{3}{5(x+1)^2} - \frac{4}{5(x-2)} + \frac{6}{5(x-2)^2} \right) dx =$$

$$= \frac{4}{5} \ln|x-1| + \frac{3}{5} \cdot \frac{1}{(x+1)} - \frac{4}{5} \ln|x-2| + \frac{6}{5(x-2)} + C =$$

$$= \frac{4}{5} \ln\left|\frac{x-1}{x-2}\right| + \frac{3}{5} \cdot \frac{1}{(x+1)} + \frac{6}{5(x-2)} + C;$$

4) $\int \frac{x}{x^3-1}\, dx =$

$$\frac{x}{x^3-1} = \frac{x}{(x-1)\cdot(x^2+x+1)} =$$

$$= \frac{A}{x-1} + \frac{Bx + C}{x^2+x+1}$$

$$\int \frac{2x+3}{\underbrace{x^2+x+1}_{2\cdot x\cdot \frac{1}{2}}}\,dx = \int \frac{2x+3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx =$$

$$\left(x+\frac{1}{2}\right)^2-\frac{1}{4}$$

$$= \int \frac{2\left(x+\frac{1}{2}\right)+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = 2\int \frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(x+\frac{1}{2}\right)$$

$$+ 2\int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2} = \ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)+$$

$$+ \frac{2\cdot 2}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}};$$

$$\int \frac{t\,dt}{t^2+a} = \frac{1}{2}\ln\left(t^2+a\right)+C;$$

$$\int \frac{dt}{t^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{t}{a}+C;$$

VI.1.7 3) iki galo

7)

9) 13)

8 Apibrėžtinis integralas

Sakykime, $f: [a, b] \to R$;

Pažymėkime P intervalo $[a; b]$ padalinima taškais $x_0 = a < x_1 < x_2 < \ldots < x_n = b$. ir n natūrinai lygių dalių.

Tegul $\Delta x_1 = x_1 - x_0$;

$\Delta x_2 = x_2 - x_1$;

$\ldots\ldots$

$\Delta x_n = x_n - x_{n-1}$;

Intervaluose $[x_{i-1}; x_i]$ pasirinkam taškus ξ_i, t.y.

$$\xi_i \in [x_{i-1}; x_i] \quad i=1,\ldots,n$$

1) Suma:

$$\sigma(P; \xi) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i;$$

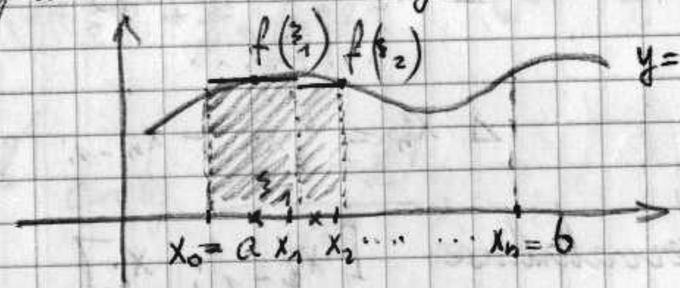
vadinama funkcijos $f: [a, b] \to R$ integraline suma, atitinkamai $[a, b]$

padalinimą P ir turi pavadinti;

Jis apibrėžiamas tašku, kad integralinė suma priklausoma nuo:

1) funkcijos
2) intervalo
3) intervalo padalinimo;
4) taškų ξ parinkimo;

Integralinės sumos geometrinė reikšmė:



$\sigma(P, \xi)$ - sutraukiamosios figūros plotas;

Integralinių sumų pavyzdžiai:

1) $f(x) = x^2$ $[a, b] = [0; 1]$

$$P = \left\{0; \frac{1}{n}; \frac{2}{n}; \dots, \frac{n-1}{n}, 1\right\};$$

$$\left[\frac{i-1}{n}; \frac{i}{n}\right],$$

$$\xi_i = \frac{\frac{i-1}{n} + \frac{i}{n}}{2} = \frac{i}{n} - \frac{i}{2n} =$$

$$= \frac{2i-1}{2n}; \qquad i = 1, 2, \dots, n$$

Šiuo atveju:

$$\sigma(P; \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i =$$

$$= \sum_{i=1}^{n} \xi_i^2 \left(\frac{i}{n} - \frac{i-1}{n}\right) =$$

$$= \sum_{i=1}^{n} \left(\frac{2i-1}{2n}\right)^2 \frac{1}{n} = \frac{1}{4n^3} \sum_{i=1}^{n} (2i-1)^2 =$$

$$= \frac{1}{4n^3}\left(1^2 + 3^2 + 5^2 + \dots + (2n-1)^2\right);$$

Tačiau,

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2-1)}{3};$$

Vadinasi,

$$\sigma(P, \xi) = \frac{1}{4n^3} - \frac{n(4n^2-1)}{3} =$$

$$= \frac{4n^2-1}{12n^2} = \frac{1}{3} - \frac{1}{12n^2};$$

2) $f(x) = x, \quad [a; b] = [0; 3]$

$$\mathcal{P} = \left\{ 0, \frac{3}{n}, \frac{6}{n}, \frac{9}{n}, \dots, \frac{3(n-1)}{n}, \frac{3n}{n} = 3 \right\};$$

$$i = \left[\frac{3(i-1)}{n}; \frac{3i}{n} \right],$$

$$\xi_i = \frac{3(n-1)}{n}; \qquad i = 1, 2, \dots, n;$$

Sumo atveju:

$$G(\mathcal{P}, \xi) = \sum_{i=1}^{n} f(\xi_i) \Delta_i =$$

$$= \sum_{i=1}^{n} \xi_i \left(\frac{3i}{n} - \frac{3(i-1)}{n} \right) =$$

$$= \sum_{i=1}^{n} \frac{3(i-1)}{n} \cdot \frac{3}{n} = \frac{9}{n^2} \sum_{i=1}^{n} (i-1) =$$

$$= \frac{9}{n^2} \left(1 + 2 + 3 + \dots + (n-1) \right) =$$

$$= \frac{9}{n^2} \left(\frac{1 + (n-1)}{2} \cdot (n-1) \right) =$$

$$= \frac{9}{2n^2} \left(n^2 - n \right) = \frac{9}{2} - \frac{9}{2n};$$

Pažymėkime $\Delta(\mathcal{P}) = \max_i \Delta x_i =$

$$= \max_i (x_i - x_{i-1});$$

2 Ap. Sakome, kad skaičius I yra funkcijos f integralinė suma reiškia, kai $\Delta(\mathcal{P}) \to 0$, jeigu $\forall \varepsilon > 0 \ \exists \delta > 0$ toks, kad

$$| I - G(\mathcal{P}, \xi) | < \varepsilon$$

visoms padalinimams $\mathcal{P}$ ir visoms taškų ξ padalinimams, kuriems

$$\Delta(\mathcal{P}) < \varepsilon.$$

Žymėsime:

$$I = \lim_{\Delta(\mathcal{P}) \to 0} G(\mathcal{P}; \xi);$$

3 Ap. Funkcija $f: [a; b] \to R$ vadinama integruojama (Rimano prasme)

intervale $[a; b]$, jeigu egzistuoja baigtinė riba

$$\lim_{\Delta(\mathcal{P}) \to 0} G(\xi, \mathcal{P}) = I,$$

Ribą I vadiname funkcijos $f(x)$ apibrėžtinu integralu intervale $[a; b]$ ir žymime

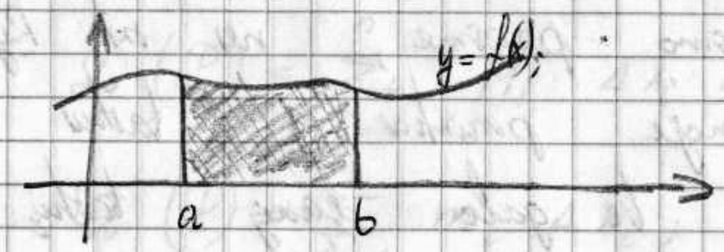
$$I = \int_a^b f(x)\,dx;$$

galime įsitikinti, kad:

$$\int_0^1 x^2 dx = \lim_{n \to \infty} \left(\frac{1}{3} - \frac{1}{12n^2} \right) = \frac{1}{3},$$

$$\int_0^3 x\,dx = \lim_{n \to \infty} \left(\frac{9}{2} - \frac{9}{2n} \right) = \frac{9}{2};$$

Jeigu $f(x)$ teigiama, tai $\int_a^b f(x)\,dx$ geometrinė prasmė yra kreivinės trapecijos plotas;



Jeigu $f(x)$ teigiama kuro greitis, tai $\int_a^b f(x)\,dx$ yra kelias nuveikas nuo taško a iki b;

§ Darbu sumos

Nustatyti ar funkcija integruojama. Rymano prasme arba rasti integralą remiantis vien apibrėžimu laikai sumin, kad rezultatas būtų lygus intervalą $[a; b]$ vienodomis ir parinktis ξ taškus.

Natūralu iškelti paprastą klausimą, kuriuo funkcijos $f(x)$ integramumo sąlygomis. Jeigu f neaprėžta, tai jos integralas neegzistuoja Rymano prasme, ne int. Rym. sumoje paimkim ξ taškus atsiras be galo daug taškų.

$f: [a; b] \to R$, aprėžta,

Sakykime, $P = \{ x_0 = a < x_2 < x_3 < \dots < x_n = b \}$ yra intervalo $[a; b]$ padalinimas.

Pažymėkime:

$$M_i = \sup \{ f(x) : x \in [x_{i-1}; x_i] \}$$

$$m_i = \inf \{ f(x) : x \in [x_{i-1}; x_i] \}$$

$$\max_{n \in N} \{ 1 - \frac{1}{n} \} \quad \nexists$$

$$\sup_{n \in N} \{ 1 - \frac{1}{n} \} = 1 ;$$

1 Ap. Sumos

$$\mathcal{D}(P) = \sum_{i=1}^{n} M_i \Delta x_i ;$$

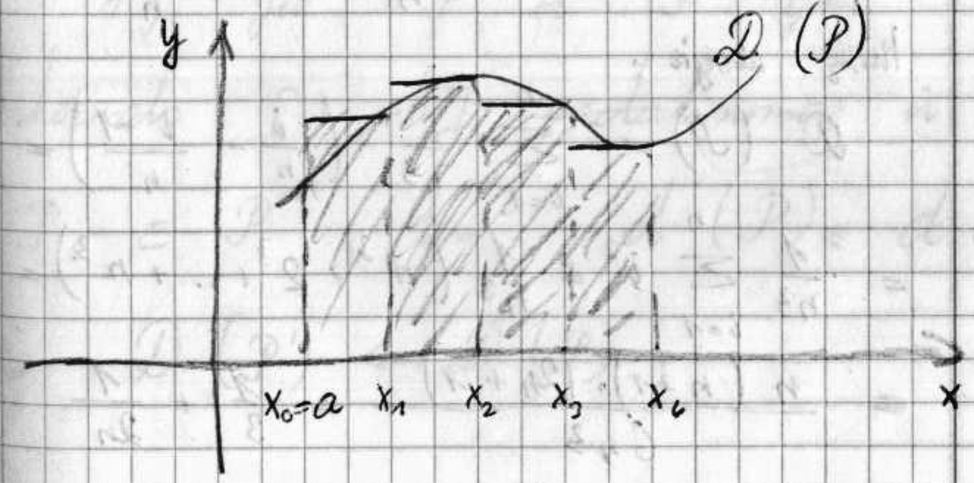
$$d(P) = \sum_{i=1}^{n} m_i \Delta x_i ;$$

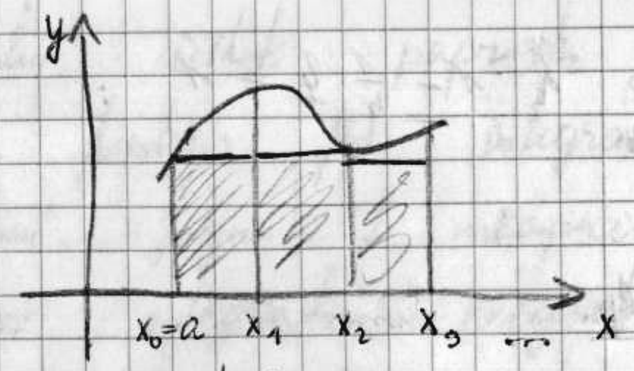
funkcijos f atitinkamai sumomis, atitinkamai padalinima P;

$\mathcal{D}$ – viršutinė Darbu suma;

d – apatinė Darbu suma;

Geometrinė Darbu sumų vaizdas





$d(\mathcal{P})$

Pvz. 1) $f(x) = x^2$; $[a; b] = [0; 1]$

$$\mathcal{P} = \left\{ 0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < 1 \right\}$$

$$[x_{i-1}; x_i] = \left[\frac{i-1}{n}; \frac{i}{n} \right]$$

$$M_i = \sup \left\{ x^2 : \frac{i-1}{n} \leq x \leq \frac{i}{n} \right\} = \frac{i^2}{n^2};$$

$$m_i = \inf \left\{ x^2 : \frac{i-1}{n} \leq x \leq \frac{i}{n} \right\} = \frac{(i-1)^2}{n^2};$$

Todėl aprėpia:

$$\mathcal{D}(\mathcal{P}) = \sum_{i=1}^{n} \frac{i^2}{n^2} \left(\frac{i}{n} - \frac{i-1}{n} \right) =$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \left(1^2 + 2^2 + \dots + n^2 \right) =$$

$$= \frac{n(n+1)\cdot(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} +$$

$$+ \frac{1}{6n^2};$$

$$d(\mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \left(\frac{1}{n} - \frac{i-1}{n} \right) =$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{1}{n^3} \left(1^2 + 2^2 + \dots + (n-1)^2 \right) =$$

$$= \frac{1}{n^3} \left(\frac{(n-1)n(2n-1)}{6} \right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2};$$

Aišku, kad bet kuriam intervalo $[a; b]$ padalinimui

$$d(\mathcal{P}) \leq C(\mathcal{P}, \xi) \leq \mathcal{D}(\mathcal{P});$$

Darbu sumų savybės:

1) Jei $\mathcal{P}$ ir $\mathcal{P}'$ yra intervalo $[a; b]$ padalinimai ir $\mathcal{P} \subset \mathcal{P}'$, tai $d(\mathcal{P}) \leq d(\mathcal{P}')$ ir $\mathcal{D}(\mathcal{P}') \leq \mathcal{D}(\mathcal{P})$;

2) Jeżeli P' i P'' są bez końca intervalo $[a, b]$ podziałami, tai $d(P') \leq D(P'')$;

3) $\sup\limits_{P} d(P) \leq \inf\limits_{P} D(P)$;

4) $\sup\limits_{P} d(P) = \lim\limits_{\Delta(P) \to 0} d(P)$;

$\inf\limits_{P} D(P) = \lim\limits_{\Delta(P) \to 0} D(P)$;

Środkiem 1 - ej serii:

Pokażemy najpierw atwiej, kai P' tai vienu tashu dangiem niegu P;

Tegu

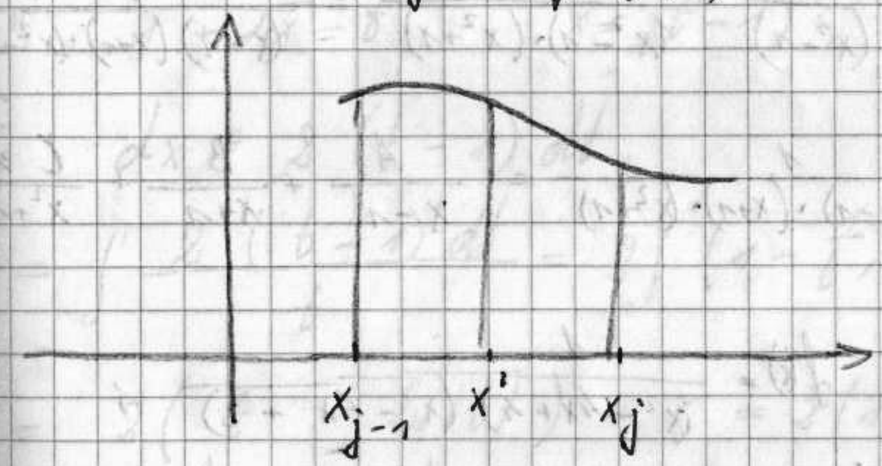
$P = \{ x_0 < x_1 < \ldots < x_{j-1} < x_j < \ldots < x_n = b \}$;

$P' = \{ x_0 < x_1 < \ldots < x_{j-1} < x' < x_j < \ldots < x_n = b \}$;

Pažymime $M_j = \sup\{ f(x), x \in [x_{j-1}; x_j] \}$

$M_j' = \sup\{ f(x), x \in [x_{j-1}; x'] \}$

$M_j'' = \sup\{ f(x), x \in [x'; x_j] \}$.



kadangi $M_j' \leq M_j$

ir $M_j'' \leq M_j$, tai

$D(P') - D(P) = M_j'(x' - x_{j-1}) + M_j''(x_j - x') - M_j(x_j - x_{j-1}) =$

$$= (M'_j - M_j)\cdot(x' - x_{j-1}) +$$

$$+ (M''_j - M_j)\cdot(x_j - x') \le 0$$

tarpi

$$\mathcal{D}(\mathcal{P}') \le \mathcal{D}(\mathcal{P});$$

1) $f(x) = \dfrac{1}{(x^4-1)}$; $E = R \setminus \{-1, 1\}$

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)\cdot(x^2+1)} = \frac{1}{(x-1)\cdot(x+1)\cdot(x^2+1)}$$

$$\frac{1}{(x-1)\cdot(x+1)\cdot(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C\,x^{+D}}{x^2+1}$$

2) $f(x) = \dfrac{1}{(x^2-4x+4)\cdot(x^2-4x+3)}$;

$E = R \setminus \{2\}$,

$$\frac{1}{(x-2)^2\cdot(x^2-4x+5)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} +$$

$$+ \frac{Cx + D}{x^2-4x+5};$$

Racionalus integralai

VI. 1. 8

1) $\displaystyle\int \frac{dx}{1+\sqrt{x}} =$

I $\sqrt{x} = t$, $x = t^2$, $dx = 2t\,dt$;

$$= \int \frac{2t\,dt}{1+t} = \dots$$

II $1 + \sqrt{x} = t$; $x = (t-1)^2$,

$dx = 2(t-1)\,dt$;

$$= \int \frac{2(t-1)\,dt}{t} = 2\int\left(1 - \frac{1}{t}\right)dx =$$

$$= 2(t - \ln|t|) + C = 2(1+\sqrt{x} -$$

$$- \ln|1+\sqrt{x}|) + C;$$

$$(= 2(\sqrt{x} - \ln|1+\sqrt{x}|) + C);$$

2) $\displaystyle\int \frac{dx}{(1+\sqrt[4]{x})^3\sqrt{x}} =$

$\sqrt[4]{x} = t$, $x = t^4$,

$dx = 4t^3\,dt$;

$$= \int \frac{4t^3\,dt}{(1+t^3)\cdot t^2} = 4\int \frac{t\,dt}{(1+t)^3} =$$

$$= 4\int \frac{t+1-1}{(1+t)^3}\,dt =$$

$$= 4\int \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3}\right) d(t+1) =$$

$$= 4\left(-\frac{1}{1+t} + \frac{1}{2(1+t)^2} + C\right) =$$

$$\int \frac{dy}{y^3} = \int y^{-3}\,dy = \frac{y^{-2}}{-2} + C =$$

$$= -\frac{1}{2y^2} + C;$$

$$= 4\left(-\frac{1}{1+\sqrt[4]{x}} + \frac{1}{2(1+\sqrt[4]{x})^2}\right) + C;$$

3) $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})} =$

$\sqrt[6]{x} = t, \quad x = t^6, \quad dx = 6t^5\,dt$

$$= \int \frac{6t^5\,dt}{t^6 + 2t^9 + t^8} = \int \frac{6t^5\cdot dt}{t^5(t+2t^4+t^3)} =$$

$$= \int \frac{dt}{t(2t^3+t^2+1)} =$$

$$\begin{array}{l|l} -\,2t^3 + t^2 + 1 & t+1 \\ \hline 2t^3 + 2t^2 & 2t^2 - t + 1 \\ \hline -t^2 + 1 & \\ -t^2 - t & \\ \hline -t + 1 & \\ t + 1 & \\ \hline 0 & \end{array}$$

$$= 6\int \frac{dt}{t(t+1)\cdot(2t^2-t+1)} =$$

$$\frac{1}{t(t+1)\cdot(2t^2-t+1)} = \frac{A}{t} + \frac{B}{t+1} +$$

$$+ \frac{Cx + D}{2t^2-t+1}$$ pakaigt

$\sqrt{ax^2+bx+c}$ $(\ldots > 0)$

$D < 0$; $t = \sqrt{ax^2+bx+c} - \sqrt{a}\,x$;

$D > 0$; $t = \sqrt{\dfrac{a(x-x_1)}{x-x_2}}$;

x_1, x_2 – dang šaknų ;

4) $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} =$

$t = \sqrt{x^2+x+1} - x$;

$(x+t)^2 = \left(\sqrt{x^2+x+1}\right)^2$

$x^2 + 2xt + t^2 = x^2 + x + 1$

$2xt - x = -t^2 + 1$

$x(2t-1) = -t^2+1 \quad |:(2t-1)$

?! $x = -\dfrac{t^2+1}{2t-1}$

$\sqrt{x^2+x+1} = x + t = -\dfrac{t^2+1}{2t-1} + t =$

$= \dfrac{-t^2-1+2t^2-t}{2t-1} = \dfrac{t^2-t-1}{2t-1}$;

$dx = -\dfrac{(t^2+1)'\cdot(2t-1)-(t^2+1)\cdot(2t-1)'}{(2t-1)^2}\,dt =$

$= \dfrac{2t^2-2t+2}{(2t-1)^2}\,dt$;

$= \displaystyle\int \frac{(2t^2-2t+2)\,dt}{(2t-1)^2\cdot\left(\frac{t^2+1}{2t-1} + \frac{t^2-t-1}{2t-1}\right)} =$

$= 2\displaystyle\int \frac{(t^2-t+1)\,dt}{(2t-1)\cdot(2t^2-t)} =$

$= 2\displaystyle\int \frac{(t^2-t+1)\,dt}{t(2t-1)^2} =$

$= \dfrac{A}{t} + \dfrac{B}{2t-1} + \dfrac{C}{(2t-1)^2}$;

$$\int R(\sin x, \cos x)\,dx$$

$$\sin x = \frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}},$$

$$\cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}, \quad tg\frac{x}{2} = t,$$

$$2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}$$

$$x = 2\,arctg\,t;$$

$$dx = \frac{2}{1+t^2}\,dt;$$

$$tg\,x = \frac{2tg\frac{x}{2}}{1-tg^2\frac{x}{2}} = \frac{2t}{1-t^2};$$

5) $$\int \frac{\sin x\cos x}{\sin x+\cos x}\,dx = \int \frac{\frac{2t}{1+t^2}\cdot\frac{1-t}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt =$$

$$= 4\int \frac{t(1-t^2)}{(1+t^2)^2\cdot(2t+1-t^2)}\,dt =$$

$$= 4\int \frac{t\cancel{(t-1)}\cdot(t+1)}{(1+t^2)^2\cdot(t-1)^{\cancel{2}}}\,dt =$$

$$= 4\int \frac{t(t+1)}{(t-1)\cdot(1+t^2)^2}\,dt;$$

$$\int \sin^n x\,\cos^m x\,dx$$

6) $$\int \sin^3 x\,\cos^2 x\,dx = -\int \sin^2 x\,\cos^2 x\,d(\cos x) =$$

$$= -\int (1-\cos^2 x)\,\cos^2 x\,d(\cos x) =$$

$$\cos x = t$$

$$= \int (t^2-1)\,t^2\,dt = \frac{t^5}{5} - \frac{t^3}{3} + C,$$

$$t = \cos x;$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C;$$

?! 7) $\int \sin^2 x \cos^2 x \, dx =$

$$= \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} \, dx =$$

$$= \frac{1}{4}\int \left(1-\cos^2 2x\right) dx = \frac{1}{4}\int \left(1 - \frac{1+\cos 4x}{2}\right) dx;$$

8) $$\int \frac{\sin^2 x}{1+\sin^2 x} \, dx \overset{\text{reikia}}{=} \int \frac{\left(\frac{2t}{1+t^2}\right)^2 \cdot 2 \, \frac{1}{1+t^2}}{1+\left(\frac{2t}{1+t^2}\right)^2} \, dt =$$

$t = \operatorname{tg}\frac{x}{2}$

$= \ldots$

$$= \int \frac{\sin^2 x + 1 - 1}{1+\sin^2 x} \, dx = \int \left(1 - \frac{1}{1+\sin^2 x}\right) dx$$

$$= \int \frac{\frac{1-\cos 2x}{2}}{1+\frac{1-\cos 2x}{2}} \, dx = \int \frac{1-\cos 2x}{3-\cos 2x} \, dx$$

$$= \int -\frac{\operatorname{tg}^2 x}{\frac{1}{\cos^2 x}+\operatorname{tg}^2 x} \, dx = \int \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x+\operatorname{tg}^2 x} \, dx =$$

$\frac{1}{\cos^2 x} = 1+\operatorname{tg}^2 x;$

$t = \operatorname{tg} x;$

$$= \int \frac{\operatorname{tg}^2 x \, dx}{1+2\operatorname{tg}^2 x} = \int \frac{t \, \frac{1}{1+t^2}}{1+2t^2} \, dt;$$

Balandžio 16;

Kontrolinis 13 h - 00 min; VI_18 VI_19

Sudėtis

5 uždaviniai:

1) eilutė; 20 tšk.
2) integralas su kaitmu; 20 tšk.
3) integralas su dalinės integravimo formulės taikymu; 20 tšk.
4) integralas racionalusis; 20 tšk.
5) integralas iracionalusis; 20 tšk.

$= \ldots$ 2 savybės įrodymas.

▷ Tegu $P = P' \cup P''$

Aišku, kad P yra mažesnis intervalo $[a; b]$ padalinimas.

Be to $P' \subset P$

$P'' \subset P$

Iš 1-os savybės turime

$$d(P') \leq d(P);$$

ir $D(P'') \geq D(P)$

todėl

$$d(P') \leq d(P) \leq D(P) \leq D(P'')$$

3 savybė. $\sup\limits_{P} d(P) \leq \inf\limits_{P} D(P)$;

▷ Tegu P' yra koks nors intervalo $[a, b]$ padalinimas; Pagal 2 savybę bet kokiems padalinimams P

$$d(P) \leq D(P')$$

Todėl ir

$$\sup_{P} d(P) \leq D(P');$$

nes $D(P')$ yra aibės $\{d(P)\}$ viršutinis rėžis, o tikslus viršutinis rėžis $\sup\limits_{P} d(P)$ yra nedidesnis už bet kurį viršutinį rėžį;

Kadangi P' yra bet koks intervalo $[a, b]$ padalinimas, tai

$\sup\limits_{P} d(P)$ yra apatinis aibės $\{D(P')\}$ rėžis. Kadangi tikslus apatinis rėžis yra mažesnis už bet kurį apatinį rėžį, tai

$$\sup_{P}(P) \leq \inf_{P} D(P');$$

S^d Būtina ir pakankama integruojamo pagal Rymana sąlyga

Sakykime $f(x) : [a; b] \to R$ yra aprėžta realaus argumento funkcija;

Teorema. $\int_a^b f(x)\,dx$ egzistuoja (Rymano prasme) tada ir tik tada, kai

$$\lim_{\Delta(P)\to 0} \left(D(P) - d(P) \right) = 0 ;$$

Esant pakankamoms teoremą sąlygoms:

$$\int_a^b f(x)\,dx = \lim_{\Delta(P)\to 0} D(P) - \lim_{\Delta(P)\to 0} d(P);$$

Teoremos įrodymas remiasi 4 Darbu sąlygomis;

Naudojantis šiuo kriterijumi galima nustatyti dangelio konkrečių integralų egzistavimą.

Pvz. Nustatykime ar $\exists \int_0^1 x^2\,dx$ pagal Rymaną;

Tegul $P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ yra bet kokio intervalo $[0; 1]$ padalinimas. Tada

$$D(P) - d(P) = \sum_{i=1}^{n} \sup\{x^2;$$

$$x^2 \in [x_{i-1}; x_i]\} (x_i - x_{i-1}) -$$

$$- \sum_{i=1}^{n} \inf\{x^2, x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} x_i^2 (x_i - x_{i-1}) - \sum_{i=1}^{n} x_{i-1}^2 (x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} \left(x_i^2 - x_{i-1}^2 \right) \cdot \left(x_i - x_{i-1} \right) \leq$$

$$\leq \max_i \left(x_i^2 - x_{i-1}^2 \right) \sum_{i=1}^{n} \left(x_i - x_{i-1} \right) =$$

$$= \max_i \left(x_i^2 - x_{i-1}^2 \right) \cdot \left(x_n - x_0 \right) =$$

$$= \max_i \left(x_i^2 - x_{i-1}^2 \right) = \max_i \left(x_i - x_{i-1} \right) \cdot$$

$$\cdot \overbrace{\left(x_i + x_{i-1} \right)}^{i \leq 2} \leq 2 \max_i \left(x_i - x_{i-1} \right) =$$

$$= 2 \Delta (P)$$

Taigi

$$0 \leq D(P) - d(P) \leq 2 \Delta (P),$$

bet kuriem padalinimam P;

Vadinas, $\lim\limits_{\Delta (P) \to 0} \left(D(P) - d(P) \right) = 0$;

Pagal teoremą (Rymano teoremą)

$$\int_0^1 x^2 dx \quad \exists$$

Taigi galim teigti, kad

$$\int_0^1 x^2 dx = \lim_{n \to \infty} \left(\frac{1}{3} - \frac{1}{12n^2} \right) = \frac{1}{3};$$

5 skyr. Integruojamų funkcijų klasė

1 Teorema (Kantoras). Tolydi uždarame intervale $[a; b]$ funkcija yra jame integruojama.

2 Teorema. Monotonika intervale $[a; b]$ funkcija yra jame integruojama.

▷ Nagrinėsime atvejį, kai $f(x)$ yra nemažėjanti. Kitas atvejis, kai $f(x)$ yra nedidėjanti nagrinėjamas analogiškai.

Sakykime P - yra intervalo $[a; b]$ padalinimas

$$\Delta (P) = \max_i \left(x_i - x_{i-1} \right);$$

Nesunku pastebėti, kad

$$\mathcal{D}(\mathcal{P}) - d(\mathcal{P}) = \sum_{i=1}^{n} \sup\{f(x);$$

$$x \in [x_{i-1}; x_i]\} \Delta x_i -$$

$$- \sum_{i=1}^{n} \inf\{f(x), x \in [x_{i-1}; x_i]\} \Delta x_i =$$

$$= \sum_{i=1}^{n} f(x_i) \Delta x_i - \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i =$$

$$= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1}) \right) \Delta x_i \leq$$

$$\leq \max_i \Delta x_i \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1}) \right) =$$

$$= \Delta(\mathcal{P}) \left(f(x_1) - f(x_0) \right) =$$

$$= \Delta(\mathcal{P}) \left(f(b) - f(a) \right);$$

Taigi tai turam

$$0 \leq \mathcal{D}(\mathcal{P}) - d(\mathcal{P}) < \Delta(\mathcal{P}) \left(f(b) - f(a) \right);$$

Iš gautos nelygybės aišku, kad

$$\lim_{\Delta(\mathcal{P}) \to 0} \left(\mathcal{D}(\mathcal{P}) - d(\mathcal{P}) \right) = 0;$$

Pagal § 4.3 teoremą funkcija $f(x)$ integruojama intervale $[a; b]$.

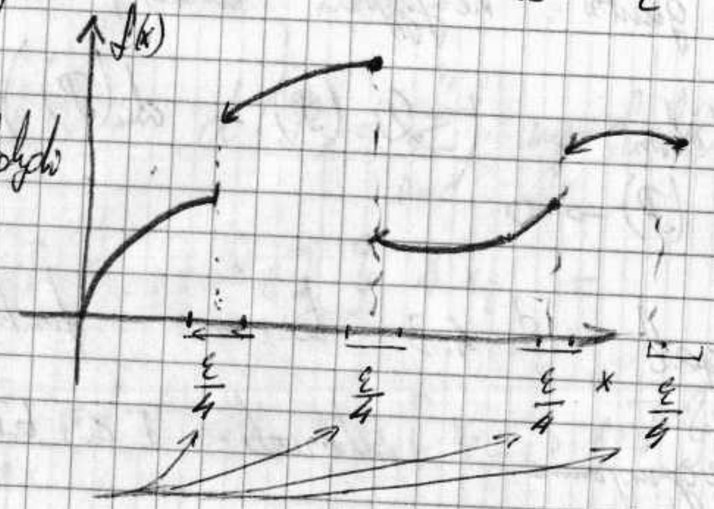
3 Teorema. (Lebesgue). Sakykime $f(x)$ yra aprėžta intervale $[a; b]$; $f(x)$ integruojama Rymano prasme, t. ir t. d., kai $f(x)$ beveik visur intervale $[a; b]$ tolydi.

Pastaba. f yra vadinama tolydžiu intervale $[a; b]$ beveik visur, jeigu $\forall \varepsilon > 0$ funkcijos f trūkio taškus galima uždengti intervalų sąjungos, kurios

tendus długo nachodzimy $w \geq \varepsilon$.

b. r. tylko

bez ich równe



długa suma $= \varepsilon$ b. r. tylko

Pagal 3 teorema gauname, kad sąlyga neintegruojamos Rymano prasme funkcijai pakanka gana sunku. Funkcija bet turėti labai daug trūkių.

Neintegruojamos funkcijos pvz.

$r : [0; 1] \to R$

$$r(x) = \begin{cases} 1, & x \text{ racionalus} \\ 0, & x \text{ iracionalus} \end{cases}$$

Ji kiekvienu bet kuriuo $[0; 1]$

padalinimui P: $P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$

$$D(P) - d(P) = \sum_{i=1}^{n} \sup\{r(x),\ x \in [x_{i-1}; x_i]\} \Delta x_i - \sum_{i=1}^{n} \inf\{r(x),\ x \in [x_{i-1}; x_i]\} \Delta x_i =$$

$$= \sum_{i=1}^{n} \Delta x_i - 0 = x_n - x_0 = 1 \not\to 0$$

Pagal §' 4.3 teorema

$$\int_0^1 r(x)\,dx \ \nexists;$$

§' Apibrėžtinio integralo savybės

Sakykime $f : [a; b] \to R$;

$g : [a; b] \to R$

1 Savybė. Jei f integruojama intervale $[a; b]$, tai

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx;$$

2. Savybė. Jei f integruojama ant $[a; b]$, tai f integruojama bet kuriame $[c; d] \subset [a; b]$.

Be to, bet kuriam $c \in (a; b)$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

3. Savybė. Jei f integruojama, tai $c \cdot f$ irgi integruojama tame intervale.

Be to:

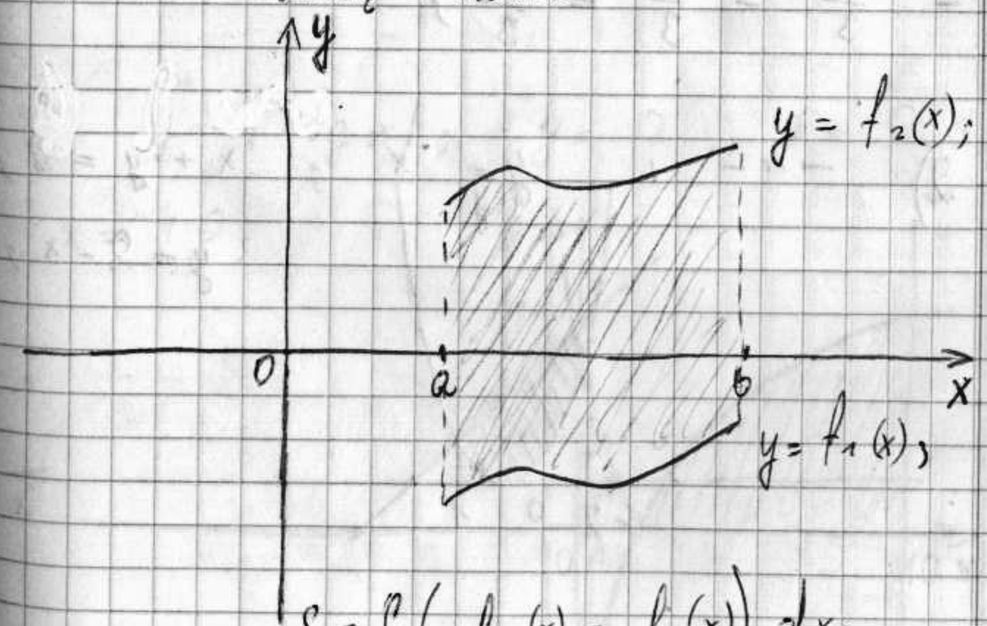
$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx;$$

4. Savybė. Jeigu f ir g yra integruojamos ant, tai $f + g$ irgi integruojama tame intervale. Be to:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$

5. Savybė. Jeigu f ir g yra ant intervale $[a; b]$, tai $f \cdot g$ irgi integruojama tame intervale;

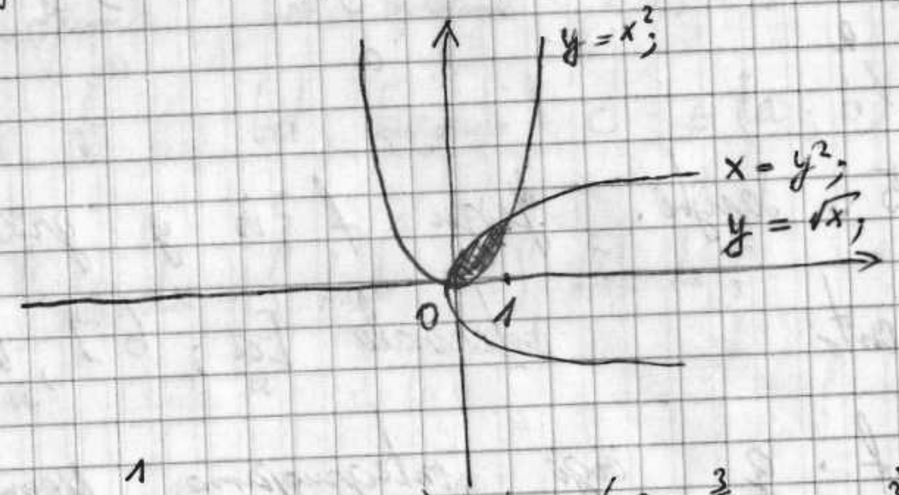
Namų darbas



$$S = \int (f_2(x) - f_1(x))\,dx;$$

1) Raskti figūros, ap. brėžiamos

$y = x^2$; $x = y^2$ plotą;

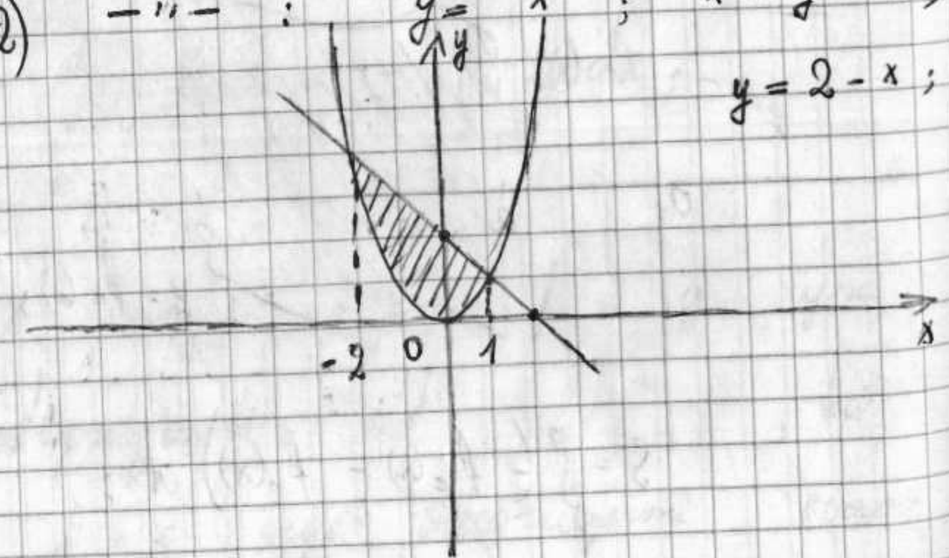


$$S = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right)\Big|_0^1 =$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3};$$

2) —"— : $y = x^2$; $x + y = 2$,

$y = 2 - x$;



$$\begin{cases} y = x^2 \\ x + y = 2 \end{cases},$$

$x^2 + x - 2 = 0$ $\quad D = 1 + 8 = 9$;

$(x-1)\cdot(x+2) = 0$ $\quad x_{1,2} = \frac{-1 \pm 3}{2} = \left[\begin{array}{l} -2 \\ 1 \end{array}\right.$

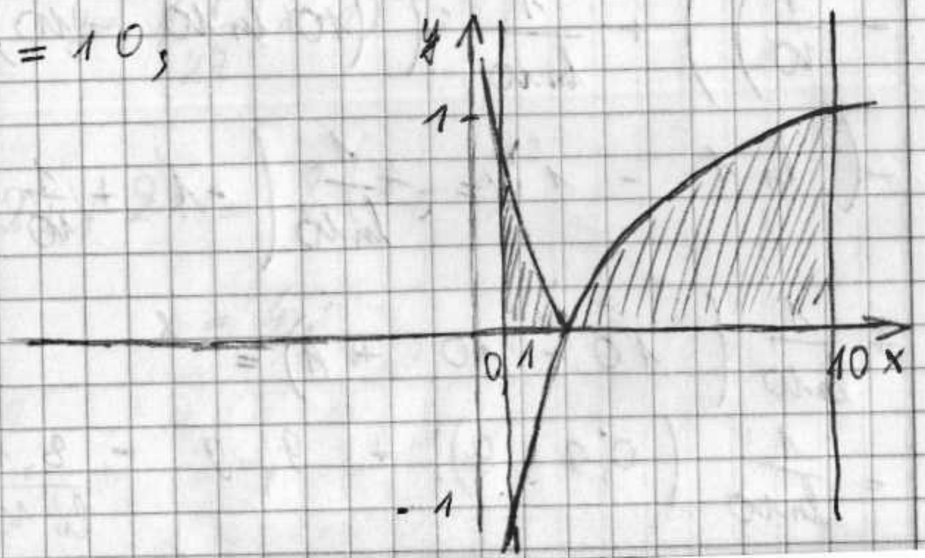
$$S = \int_{-2}^{1} \left(2 - x - x^2\right) dx = \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^{1} =$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) =$$

$=$

3) $y = |\lg x|$; $y = 0$; $x = 0{,}1$;

$x = 10$;



$$S = \int_{0,1}^{1} (-\lg x)\,dx + \int_{1}^{10} (\lg x)\,dx =$$

$$= -\frac{1}{\ln 10}\left(x\ln x - x\right)\Big|_{0,1}^{1} + \frac{1}{\ln 10}\left(x\ln x - x\right)\Big|_{1}^{10} =$$

$$\int (\lg x)\,dx = \int \frac{\ln x}{\ln 10}\,dx = \frac{1}{\ln 10}\int \ln x\,dx;$$

$$\int \ln x\,dx = x\ln x - \int x\,d(\ln x) =$$

$$= x\ln x - \int x\cdot\frac{1}{x}\,dx =$$

$$= x\ln x - x;$$

неправильно?!

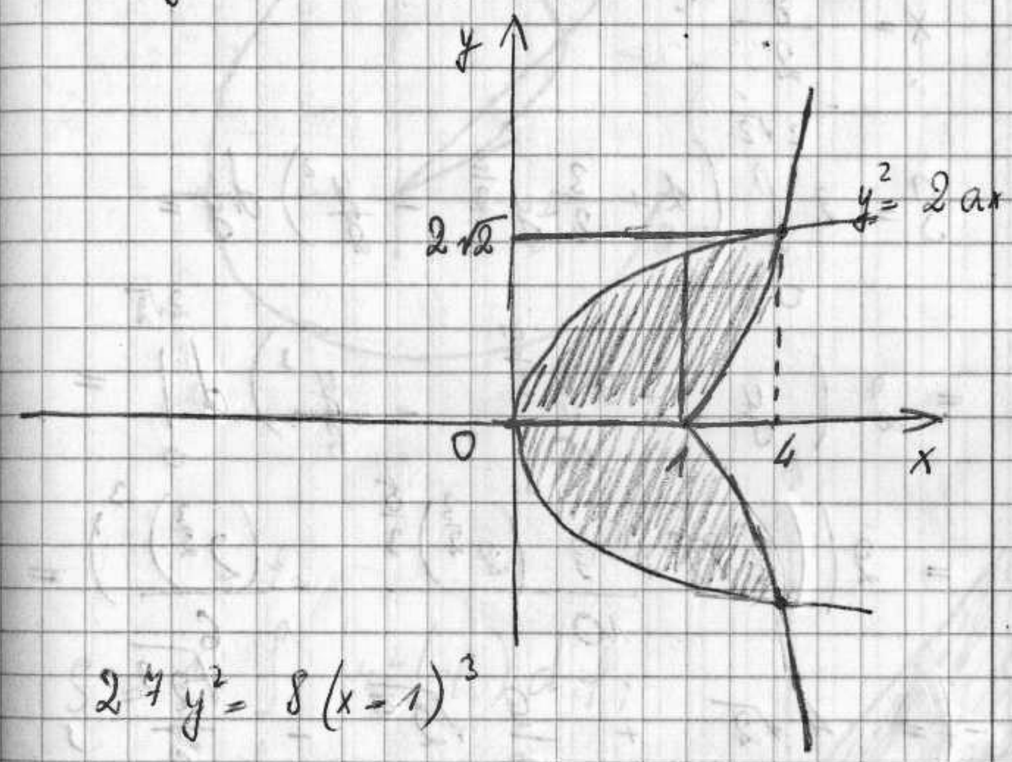
$$= -\frac{1}{\ln 10}\left(\left(\underset{0}{\ln 1} - 1\right) - \left(\frac{1}{10}\cdot\overset{-\ln 10}{\ln\frac{1}{10}} - \frac{1}{10}\right)\right) + \frac{1}{\ln 10}\left(\left(10\ln 10 - 10\right) - \left(\underset{0}{\ln 1} - 1\right)\right) = -\frac{1}{\ln 10}\left(-1 + \frac{1}{10} + \frac{1}{10}\right) + \frac{1}{\ln 10}\left(10 - 10 + 1\right) =$$

$$= \frac{1}{\ln 10}\left(0{,}9 - 9\right) + 9{,}9 - \frac{8{,}1}{\ln 10};$$

4) $y^2 = 2ax;\quad 27ay^2 = 8(x-a)^3;$

$a = 1;$

$y^2 = 2x;\quad 27y^2 = 8(x-1)^3;$



$27y^2 = 8(x-1)^3$

$y = \sqrt{\frac{8}{27}(x-1)^3};$

$27\cdot 2x = 8(x-1)^3$

$x = 4;$

$$x - 1 = \sqrt[3]{\frac{27y^2}{8}} = \frac{3}{2}\sqrt[3]{y^2};$$

$$x = 1 + \frac{3}{2}\, y^{\frac{2}{3}};$$

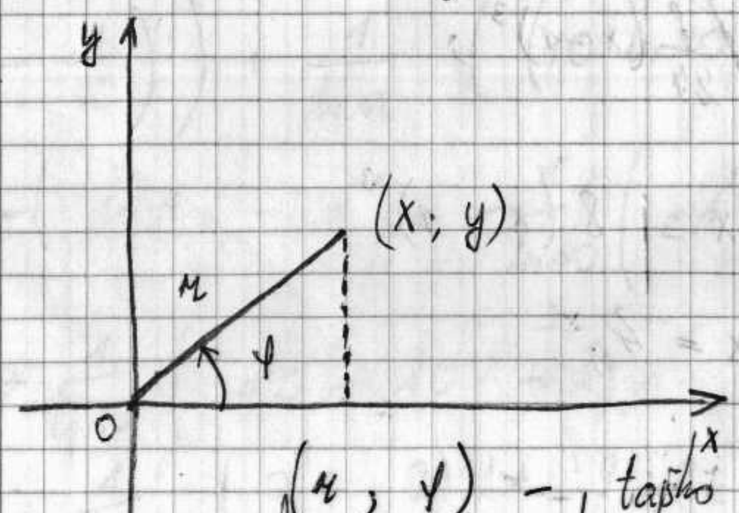
$$x = \frac{y^2}{2}$$

$$S = 2\int_0^{2\sqrt{2}} \left(1 + \frac{3}{2}\, y^{\frac{2}{3}} - \frac{y^2}{2}\right) dy =$$

$$= 2\left(y + \frac{9\, y^{\frac{5}{3}}}{10} - \frac{y^3}{6}\right)\Bigg|_0^{2\sqrt{2}} =$$

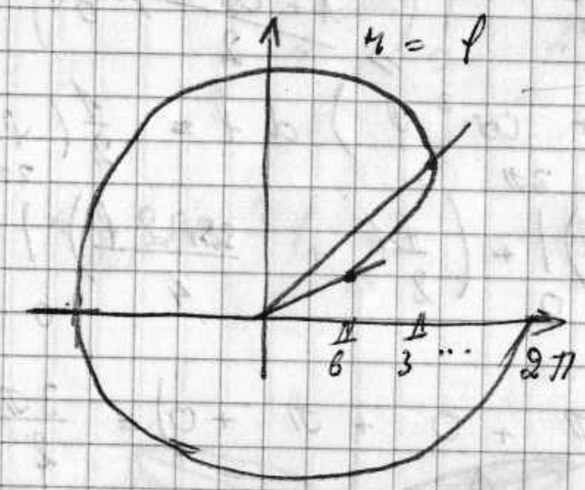
$$= 2\left(2\sqrt{2} + \frac{9\left(2^{\frac{3}{2}}\right)^{\frac{5}{3}}}{10} - \frac{\left(2^{\frac{3}{2}}\right)^3}{6}\right) =$$

$$= 4\sqrt{2} + \frac{9}{5}\sqrt{2^5} - \frac{\sqrt{2^9}}{3},$$

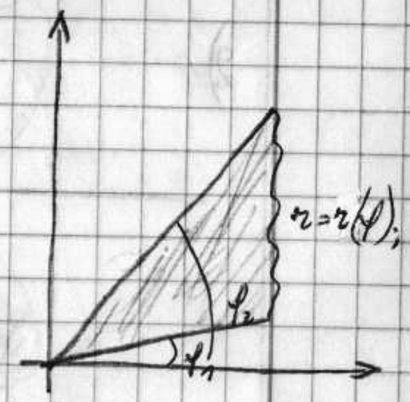


$(r;\ \varphi)$ – точка $(x;\ y)$

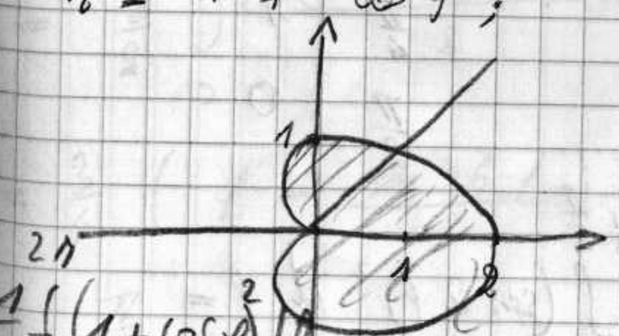
$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi \end{cases} \qquad r = r(\varphi);$$



$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)\, d\varphi;$$



$$r = 1 + \cos\varphi;$$



φ	r
0	2
$\frac{\pi}{4}$	$1 + \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	$1 - \frac{\sqrt{2}}{2}$

$$\frac{1}{2}\int^{2\pi} (1 + \cos\varphi)^2 d\varphi$$

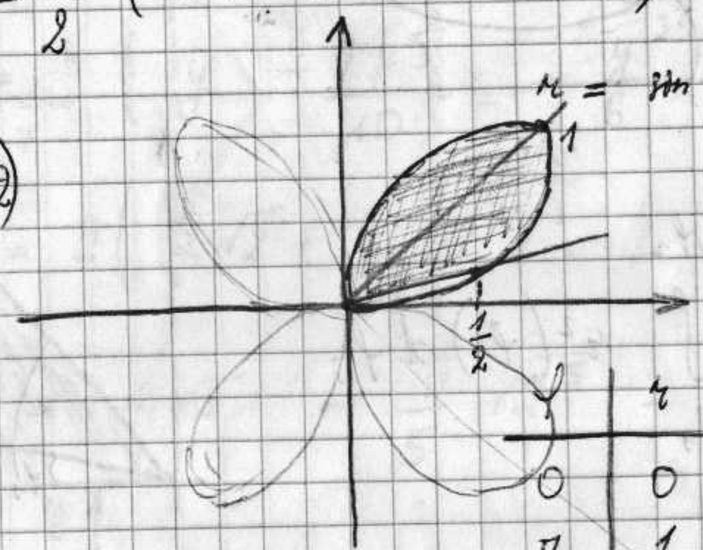
$r = \sin 2\varphi;$
$r = \cos 3\varphi;$

Namų darbai

1) $S = \frac{1}{2}\int_0^{2\pi}(1+\cos\varphi)^2\,d\varphi = \frac{1}{2}\int_0^{2\pi}(1 + \underbrace{\cos^2\varphi}_{=\frac{1+\cos 2\varphi}{2}} + 2\cos\varphi + \cos^2\varphi)\,d\varphi = \frac{1}{2}\left(\varphi + 2\sin\varphi\right)\Big|_0^{2\pi} + \left(\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right)\Big|_0^{2\pi} =$

$= \frac{1}{2}(2\pi + 0 + \pi + 0) = \frac{3\pi}{2};$

2)



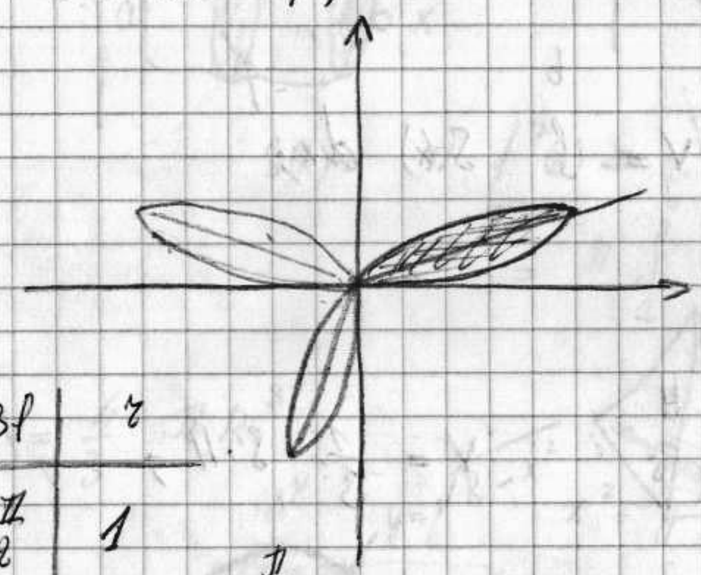
φ	r
0	0
$\frac{\pi}{12}$	$\frac{1}{2}$
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0

$S = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin 2\varphi)^2\,d\varphi =$

$= 2\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 4\varphi}{2}\right)^2 d\varphi =$

$= \left(\varphi - \frac{\sin 4\varphi}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2};$

3) $r = \sin 3\varphi;$



φ	3φ	r
$\frac{\pi}{6}$	$\frac{\pi}{2}$	1
$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{3}$	π	0

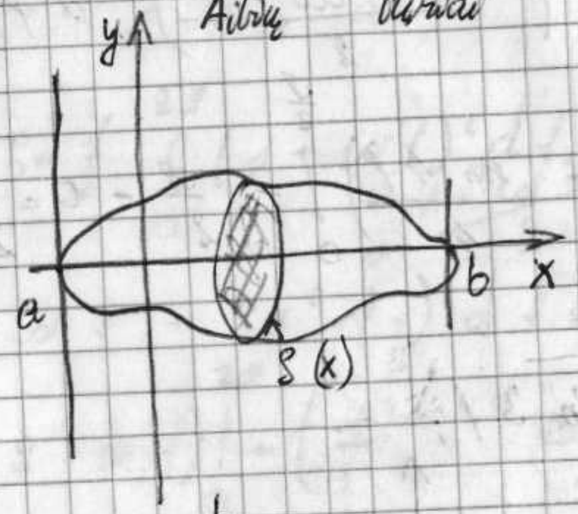
$S = \frac{3}{2}\int_0^{\frac{\pi}{3}}(\sin 3\varphi)^2\,d\varphi =$

$= \frac{3}{2}\int_0^{\frac{\pi}{3}}\frac{(1-\cos 6\varphi)}{2}\,d\varphi =$

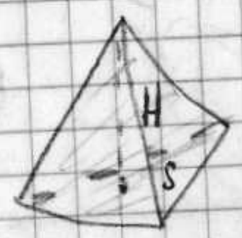
$= \frac{3}{4}\left(\varphi - \frac{\sin 6\varphi}{6}\right)\Big|_0^{\frac{\pi}{3}} = \frac{3}{4}\left(\frac{\pi}{3} - 0\right) =$

$= \frac{\pi}{4};$

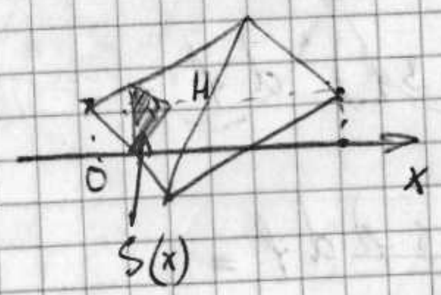
Aibių tūriai



$$V = \int_a^b S(x)\,dx;$$



$$V = \frac{1}{3}\cdot S\cdot H;$$

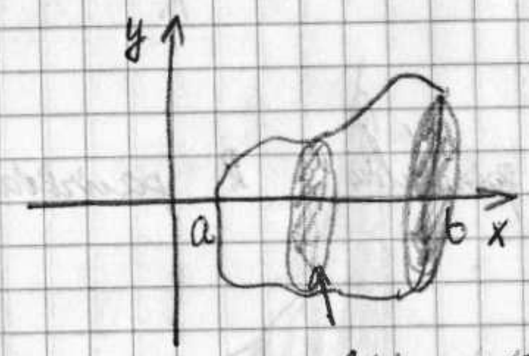


$$\frac{S(x)}{S} = \frac{x^2}{H^2} \Rightarrow$$

$$V = \int_0^H \frac{S}{H^2}x^2\,dx = \frac{S}{H^2}\frac{x^3}{3}\Big|_0^H =$$

$$= \frac{S}{3H^2};\qquad S(x) = \frac{S}{H^2}x^2;$$

$$H^3 = \frac{1}{3}SH;$$

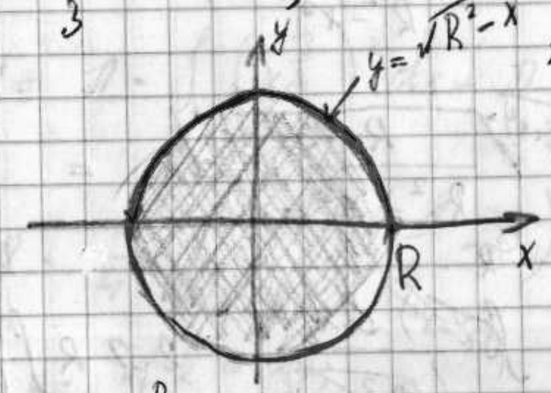


$$S(x) = \pi f^2(x) \Rightarrow V_{\text{sukimosi}} =$$

$$= \pi\int_a^b f^2(x)\,dx;$$

$$V = \frac{4}{3}\pi R^3;\qquad x\in[-R;R];$$

$$y = \sqrt{R^2-x^2};\qquad x^2+y^2=R^2;$$



$$V = \pi\int_{-R}^{R}\left(\sqrt{R^2-x^2}\right)^2dx = \pi\int_{-R}^{R}(R^2-x^2)\,dx =$$

$$= \pi\left(R^2x - \frac{x^3}{3}\right)\Big|_{-R}^{R} = \pi\left(R^3 - \frac{R^3}{3} -\right.$$

$$-\left(-R^3+\frac{R^3}{3}\right)\Big)\Big) = \frac{4}{3}\pi R^3;$$

$$S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx;$$

Sfero su spindulio R paviršiaus plotas;

$$S = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{1+\left(\frac{1\cdot(-2x)}{2\sqrt{R^2-x^2}}\right)^2}\,dx =$$

$$= 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx =$$

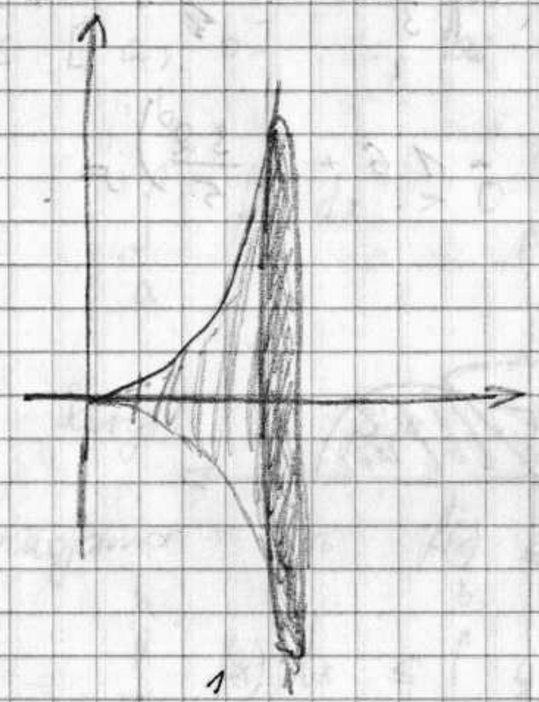
$$= 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{\frac{R^2}{R^2-x^2}}\,dx =$$

$$= 2\pi R\int_{-R}^{R}\frac{\sqrt{R^2-x^2}\,dx}{\sqrt{R^2-x^2}} = 2\pi R\int_{-R}^{R}dx =$$

$$= 4\pi R^2;$$

4) $y = x^{\frac{3}{2}}$, $x \in [0;1]$

Rasti ašies suksnos kūno tūrį;

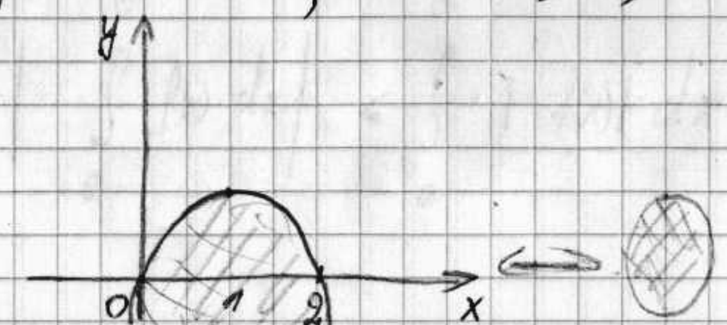


$$V = \pi\cdot\int_0^1\left(x^{\frac{3}{2}}\right)^2dx = \pi\int_0^1 x^3\,dx =$$

$$= \pi\cdot\frac{x^4}{4}\Big|_0^1 = \pi\left(\frac{1}{4}-0\right) = \frac{\pi}{4};$$

5) $y = x(2-x)$;

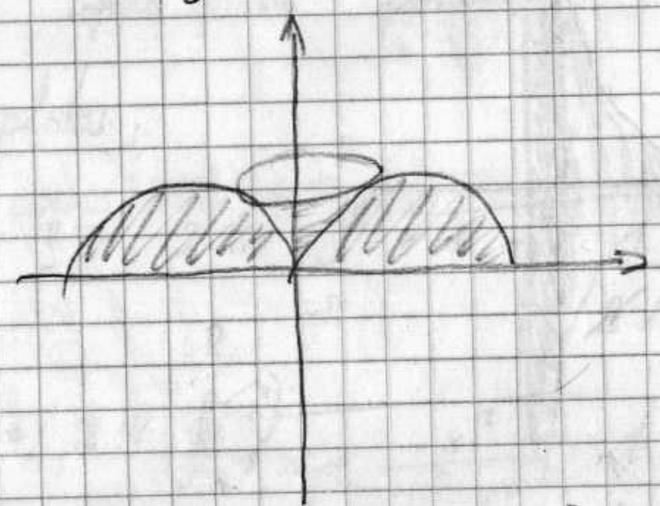
$y = 2x - x^2$, $x \in [0;2]$;



$$V = \pi \int_0^2 (2x - x^2)^2 dx = \pi \int_0^2 (4x^2 - 4x^3 +$$

$$+ x^4)\, dx = \pi \left(\frac{4}{3} x^3 - \frac{4}{4} x^4 + \frac{x^5}{5} \right) \Big|_0^2 =$$

$$= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) =$$



$y = \sin x$, $x \in [0, \pi]$

sukant apie x ir y ašį,

6. Savybė. Jeigu f integruojama intervale $[a; b]$ ir $f(x) \geq 0$ visiems $x \in [a; b]$, tai

$$\int_a^b f(x)\, dx > 0 ;$$

7. Savybė. Jeigu f ir g integruojamos ir $f(x) \leq g(x)$ $\forall x \in [a; b]$, tai $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$;

8. Savybė. Jei f integruojama intervale $[a; b]$, tai $|f|$ irgi integruojama intervale $[a; b]$ ir

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx ;$$

9. Savybė. Jei f integruojama intervale ~~...~~ $[a; b]$ ir $m \le f(x) \le M$ $\forall x \in [a; b]$, tai

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a);$$

10. Savybė. Jeigu f integruojama intervale $[a; b]$ ir $m \le f(x) \le M$, $\forall x \in [a; b]$, tai

$$\exists\ \mu \in [m; M],\ \text{kurioms}$$

$$\int_a^b f(x)\,dx = \mu(b-a);$$

11. Savybė. Jeigu f ~~integruojama~~ tolydi intervale $[a; b]$ ~~...~~, tai

$$\exists\ \xi \in (a; b) \text{ su savybe:}$$

$$\int_a^b f(x)\,dx = f(\xi)(b-a);$$

12. Savybė. Jei g integruojama intervale $[a; b]$, f tolydi intervale $[a; b]$ ir $g(x) \ge 0$ $\forall x \in [a; b]$, tai $\exists\ \xi \in (a; b)$:

$$\int_a^b f(x)\,g(x)\,dx = f(\xi) \cdot \int_a^b g(x)\,dx;$$

13. Savybė. Jei intervale $[a; b]$ funkcija $g(x)$ monotonė, o funkcija $f(x)$ integruojama, tai $\exists\ \xi \in (a; b)$ toks kad:

$$\int_a^b f(x)\, g(x)\, dx = g(x)\int_a^{\xi} f(x)\, dx + g(b)\int_{\xi}^{b} f(x)\, dx;$$

14. Savybė. Jei f ir g integruojamos $[a; b]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, tai

$$\left| \int_a^b f(x)\, g(x)\, dx \right| \leq \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \cdot \left(\int_a^b |g(x)|^q dx \right)^{\frac{1}{q}};$$

15. Savybė. Jei f, g integruojamos, $f(x) \geq 0$, $g(x) \geq 0$ $\forall x [a, b]$, $p > 1$, tai

$$\left(\int_a^b (f(x) \cdot g(x))^p dx \right)^{\frac{1}{p}} \leq \left(\int_a^b f^p(x)\, dx \right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\, dx \right)^{\frac{1}{p}};$$

▷ (1. Savybė.) Sakykime $\mathcal{P} = \{ a = x_0 < x_1 < x_2 < \ldots < x_n = b \}$; yra intervale $[a; b]$ padalinimas. Sudarykime integralinę sumą funkcijai $f(x) + g(x)$ intervaluose $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ pasirinkdami taškus ξ_i;

$$G_{f+g}(\mathcal{P}, \xi) = \sum_{i=1}^{n} \left(f(\xi_i) + g(\xi_i) \right) \cdot \underbrace{(x_i - x_{i-1})}_{\Delta x_i};$$

Aišku, kad

$$G_{f+g}(\mathcal{P}, \xi) = \sum_{i=1}^{n} f(\xi_i)\, \Delta x_i +$$

$$+ \sum_{i=1}^{n} g(\xi_i)\,\Delta x_i = \sigma_f(P;\xi) +$$

$$+ \sigma_g(P,\xi)$$

Kadangi f ir g yra integruojamos intervale $[a;b]$, tai $\exists$ riba:

$$\lim_{\Delta(P)\to 0} \sigma_f(P,\xi) = \int_a^b f(x)\,dx;$$

$$\lim_{\Delta(P)\to 0} \sigma_g(P,\xi) = \int_a^b g(x)\,dx;$$

čia $\Delta(P) = \max\limits_{1\le i\le n} \Delta x_i$;

Vadinasi $\exists$ ir riba:

$$\lim_{\Delta(P)\to 0} \sigma_{f+g}(P,\xi) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$

Taigi (pagal apibr.) funkcija $f(x)+g(x)$ integruojama intervale $[a;b]$ ir

$$\int_a^b (f(x)+g(x))\,dx = \lim_{\Delta(P)\to 0} \sigma(P,\xi) =$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx; \quad \triangleleft$$

$\triangleright$ (9. Savybė.) Sakykime $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ yra intervalo $[a,b]$ padalinimas. Sudarykime funkcijai $f(x)$ integralę sumą ant $[x_{i-1}, x_i]$ $i=1,2,\ldots,n$ pasirinkdami taškus ξ_i

Aišku, kad

$$\sigma_f(P,\xi) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i;$$

kadangi $m \leq f(x) \leq M \quad \forall x \in [a,b]$,

tai

$$\sigma_f(P, \xi) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i \geq$$

$$\geq m \sum_{i=1}^{n} \Delta x_i = m \cdot (x_1 - x_0 + x_2 - x_3 +$$

$$+ \ldots + x_n - x_{n-1}) = m(b-a);$$

$$\sigma_f(P, \xi) = \sum_{i=1}^{n} \underbrace{f(\xi_i)}_{<M}\,\Delta x_i \leq$$

$$\leq M \sum_{i=1}^{n} \Delta x_i = M(b-a);$$

Mokygytėje:

$$m(b-a) \leq \sigma_f(P, \xi) \leq M(b-a)$$

$$\ldots \quad , \text{ kai } \Delta(P) > 0.$$

gauname

$$m(b-a) \leq \lim_{\Delta(P) \to 0} \sigma(P, \xi) \leq M(b-a);$$

kadangi $f(x)$ integruojama, tai

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a);$$

§[1] Niutono - Leibnico formulė

Schyhime $f: [a, b] \to R$ integruojama funkcija. Pasirinkime $x \in [a, b]$.

Apibrėžkime naują funkciją:

$$F(x) = \int_a^x f(y)\,dy$$

Pagal §[1] 4.5. 2. Savybę gauname, kad funkcija $F(x)$ egzistuoja su visai $x \in [a, b]$.

Funkcja $F(x)$ zadana całką nazywamy całką zależną od górnej granicy.

Twierdzenie. Jeśli f jest całkowalna na przedziale $[a; b]$ i w każdym punkcie $x_0 \in (a; b)$, to

$$F'(x_0) = \left(\int_a^x f(y)\,dy \right)'_{x=x_0} = f(x_0);$$

▷ Załóżmy przedtem $x > x_0$

(2 sposoby) $$F(x) - F(x_0) = \int_a^x f(y)\,dy - \int_a^{x_0} f(y)\,dy =$$

$$= \int_a^{x_0} f(y)\,dy + \int_{x_0}^x f(y)\,dy - \int_a^{x_0} f(y)\,dy =$$

$$= \int_{x_0}^x f(y)\,dy;$$

Zatem,

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(y)\,dy \quad (1)$$

Załóżmy $\varepsilon > 0$ funkcja f w każdym punkcie x_0, tak

$$\lim_{y \to x_0} f(y) = f(x_0);$$

Zatem $\exists\ \delta > 0$, taki że

$$|f(y) - f(x_0)| < \varepsilon, \text{ tak } |y - x_0| < \delta;$$

Albo

$$f(x_0) - \varepsilon < f(y) < f(x_0) + \varepsilon, \text{ tak}$$

$$|y - x_0| < \delta;$$

Załóżmy $x_0 < x < x_0 + \delta$;

Tada wiemy $y \in (x_0, x)$

$|y - x_0| < \delta$. Todel wiemy $y \in (x_0, x)$

$$f(x_0) - \varepsilon < f(y) < f(x_0) + \varepsilon,$$

Pagal § 4.5. 9 twierdzenie:

$$(f(x_0) - \varepsilon)\cdot(x - x_0) \leq \int_{x_0}^{x} f(y)\,dy \leq$$

$$\leq (f(x_0) + \varepsilon)\cdot(x - x_0), \quad \text{kai}$$

$$x_0 < x < x_0 + \delta$$

Pasmandoję (1) gaunamę

$$f(x_0) - \varepsilon < \frac{F(x) - F(x_0)}{x - x_0} < f(x_0) + \varepsilon$$

Arba

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon$$

kai $x_0 < x < x_0 + \delta$.

Gauname, kad bet kuriem

$\varepsilon > 0$ $\exists$ $\delta > 0$, $x_0 < x < x_0 + \delta$:

$$\left| \frac{F(x) - f(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon;$$

Pagal ribo apibrėžimą

$$\lim_{x \to x_0 + 0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

Analogiškai gauname, kai $x < x_0$

$$\lim_{x \to x_0 - 0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

w drugą stronę wzoru, kad

$$F'(x_0) = f(x_0) \quad \triangle$$

$$\left(\int_0^x \sin y \, dy\right)' = \sin x \; ;$$

$$\left(\int_1^{x^2} y^2 \, dy\right)' = x^4 \cdot (x^2)' = 2x^5 \; ;$$

$$\left(\int_1^{x^2} y^2 \, dy\right)' = \left(\left(\frac{y^3}{3}\right)\Big|_1^{x^2}\right)' = \left(\frac{x^6}{3} - \frac{1}{3}\right)' =$$

$$= \frac{\cancel{6}^{2} x^5}{\cancel{3}_{1}} = 2x^5 \; ;$$

Wnioski:

1) Ciągłość w tych przedziale $[a;b]$ funkcja $f(x)$, zawsze przedziale ma pierwotną funkcja;

▷ Pokazuje mi $F(x) = \int_a^x f(y)\,dy$ ◁

2) Jeżeli $f(x)$ w tych przedziale $[a;b]$, to $d\left(\int_a^x f(y)\,dy\right) = f(x)\,dx$, $x \in (a;b)$;

3) N-L. Jeżeli w tych przedziale $[a;b]$, a ϕ jest jakąś pierwotną funkcją, to

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \; ;$$

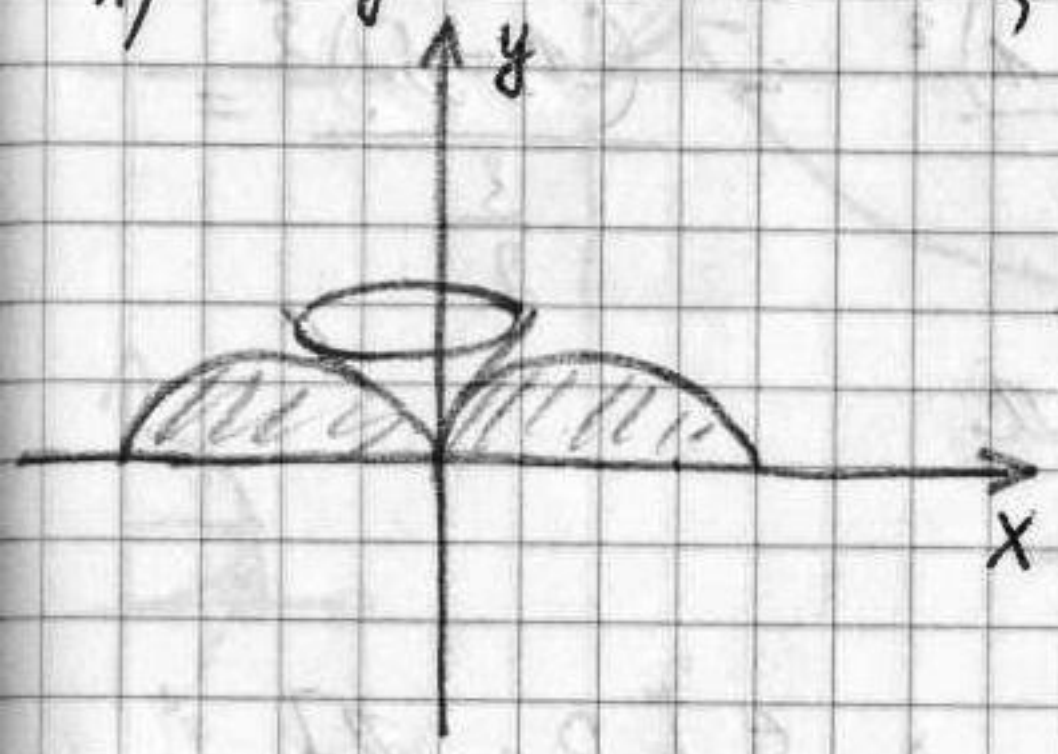
▷ Wybierzmy

$$F(x) = \int_a^x f(y)\,dy \; ;$$

Według twierdzenia $F(x)$

Namų darbai

1) $y = 2x - x^2$, $x \in [0; 2]$



$y = 2x - x^2$

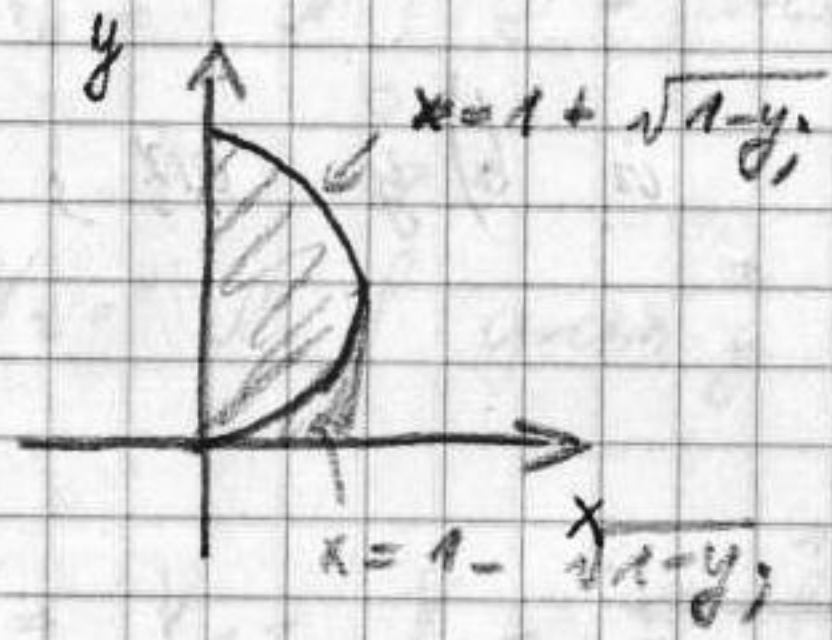
$x^2 - 2x + y = 0$

$x^2 - 2x + 1 - 1 + y = 0$

$(x-1)^2 = 1 - y$;

$x - 1 = \pm\sqrt{1-y}$;

$x = 1 \pm \sqrt{1-y}$;



$$V_1 = \pi \int_0^1 \left(1 + \sqrt{1-y}\right)^2 dy =$$

$$V_2 = \pi \int_0^1 \left(1 - \sqrt{1-y}\right)^2 dy;$$

$$V = V_1 - V_2 = \pi \int_0^1 \left(4\sqrt{1-y}\right) dy =$$

$$= -4\pi \int_0^1 \sqrt{1-y}\, d(1-y) =$$

$$= -4\pi \cdot \frac{(1-y)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 =$$

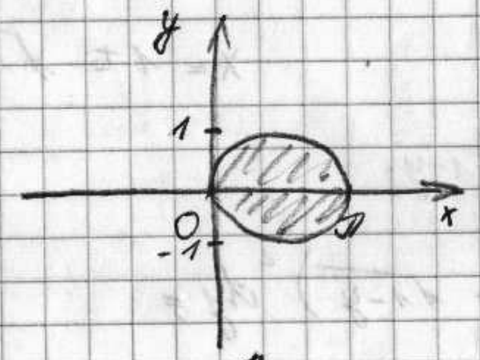
$$= -4\pi \cdot \frac{(1-1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(1-0)^{\frac{3}{2}}}{\frac{3}{2}} =$$

$$= 4\pi \cdot \frac{2}{3} = \frac{8}{3}\pi;$$

2) $y = \sin x, \quad x \in [0; \pi]$

apie a) x ašį ir b) y ašį;

a)

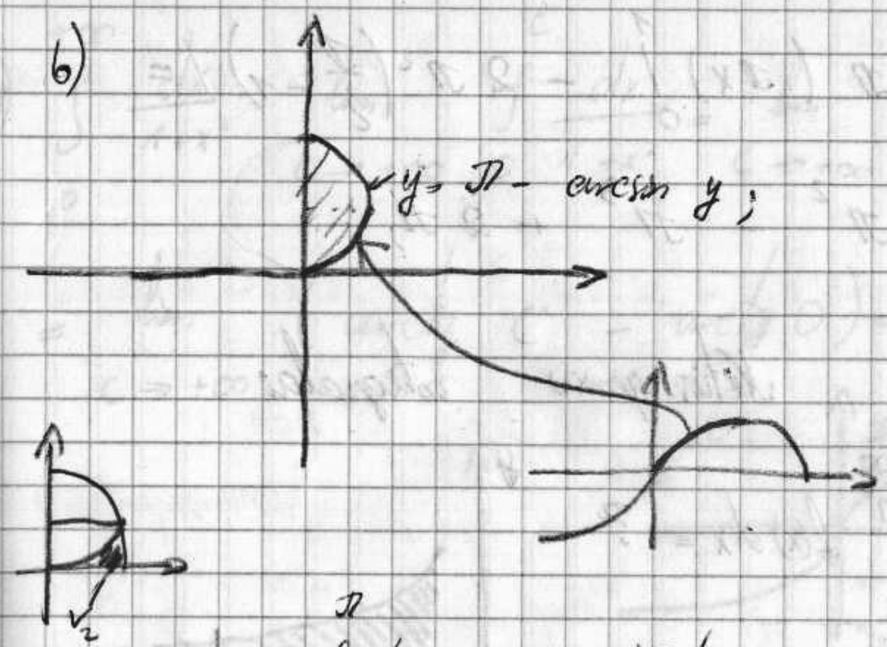


$$V_1 = \pi \int_0^{\pi} (\sin x)^2 dx = \pi \int_0^{\pi} \left(\frac{1-\cos(2x)}{2}\right) dx =$$

$$= \frac{1}{2}\pi \left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \frac{1}{2}\pi\left(\pi - \frac{\sin 2\pi}{2}\right)$$

$$= \frac{1}{2}\pi^2;$$

b)



$$V_1 = \pi \int_0^{\pi} (\pi - \arcsin y)^2 dy;$$

$$V_2 = \pi \int_0^{1} \arcsin^2 y \, dy;$$

$$V = V_1 - V_2 = \pi \int_0^1 (\pi^2 - 2\pi \arcsin y +$$

$$+ \arcsin^2 y - \arcsin^2 y)\, dy = \pi \int_0^1 (\pi - 2\arcsin y) dy =$$

$$= \int_0^{\frac{\pi}{2}} x \, d(\sin x) = x \cdot \sin x \Big|_0^{\frac{\pi}{2}} - \int_2^{\pi} \sin x \, dx =$$

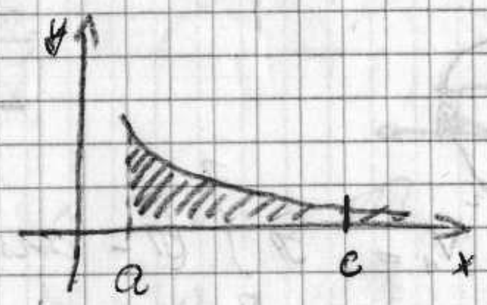
$$= x \cdot \sin x \Big|_0^{\frac{\pi}{2}} - \cos x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1;$$

$$= \pi (\pi x)\Big|_0^1 - 2\pi^2\left(\frac{\pi}{2} - 1\right) =$$

$$= \pi^2 - \pi^3 + 2\pi,$$

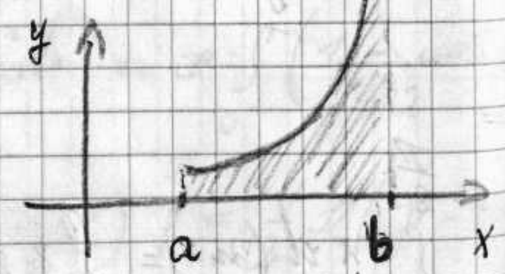
Netiesioginės integralai

$$\int_a^{+\infty} f(x)\,dx = ?$$



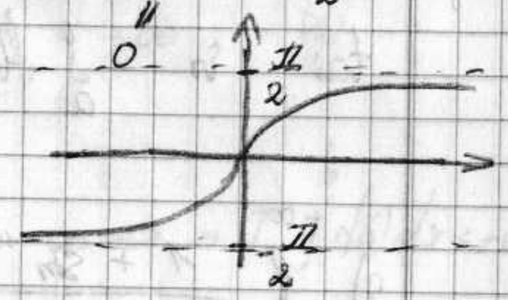
1) $\displaystyle\int_a^{+\infty} f(x)\,dx = \lim_{c\to\infty} \int_a^c f(x)\,dx = \lim_{x\to\infty} \int_a^x f(x)\,dx;$

2) $\displaystyle\int_a^b f(x)\,dx := \lim_{c\uparrow b} \int_a^c f(x)\,dx,$



1) $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{c\to+\infty} \int_0^c \frac{dx}{1+x^2} = \lim_{c\to+\infty} \operatorname{arctg} x\Big|_0^c =$

$\displaystyle = \lim_{c\to+\infty} (\operatorname{arctg} c - \operatorname{arctg} 0) = \frac{\pi}{2};$



$\displaystyle\int_a^{+\infty} \frac{dx}{x^p} < +\infty \iff p > 1 \quad (a>0);$

$\displaystyle\int_0^1 \frac{dx}{x^p} < +\infty \iff p < 1;$

$\displaystyle\int_a^b \frac{dx}{(b-x)^p} < +\infty \iff p < 1;$

$$0 \le f \le g \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx,$$

Teoda. Jeśli n.ił. $\int_a^b g(x)\,dx < +\infty$,

tai is $\int_a^b f(x)\,dx < +\infty$,

(konvergujo)

2) $$\int_1^{+\infty} \frac{1+\sin^{100} x}{x\sqrt{x}}\,dx \le \int_1^{+\infty} \frac{2}{x\sqrt{x}}\,dx =$$

$$= 2\int_1^{+\infty} \frac{dx}{x^{\frac{3}{2}}} < +\infty \quad \left(\text{nes } \frac{3}{2} > 1\right),$$

konvergujo;

3) $$\int_0^1 \frac{1+\sin^{100} x}{x\sqrt{x}}\,dx \ge \int_0^1 \frac{dx}{x^{\frac{3}{2}}} = +\infty;$$

$\left(\text{nes } \frac{3}{2} > 1\right)$;

divergujo;

$f, g \ge 0$; $\exists \int_a^c f(x)\,dx$ ir $\int_a^c g(x)\,dx$,

$c < b \le +\infty$

Jei $\exists\ A := \lim_{x \uparrow b} \frac{f(x)}{g(x)} < +\infty$, tai

n. I. $\int_a^b g(x)\,dx < +\infty \Rightarrow$ n. I. $\int_a^b f(x)\,dx < +\infty$;

Jei, beto, $A > 0$, tuo $\int_a^b g(x)\,dx < +\infty \Leftrightarrow$

$\Leftrightarrow \int_a^b f(x)\,dx < +\infty$;

4) $$\int_1^{+\infty} \frac{dx}{\sqrt[3]{x(1+x^5)}} \le \int_1^{+\infty} \frac{dx}{x^2} < +\infty$$

konvergujo;

$$\int_1^{+\infty} \frac{dx}{x^2}$$

$$I_1 \int_1^{+\infty} \frac{dx}{\sqrt[3]{x\left(x^5-\frac{1}{2}\right)}};$$

$$I_2 \int_1^{+\infty} \frac{dx}{x^2};$$

$$A := \lim_{x\to+\infty} \frac{\frac{1}{\sqrt[3]{x\left(x^5-\frac{1}{2}\right)}}}{\frac{1}{x^2}} =$$

$$= \lim_{x\to+\infty} \frac{x^2}{\sqrt[3]{x\left(x^5-\frac{1}{2}\right)}} = \lim_{x\to+\infty} \frac{1}{\sqrt[3]{\frac{x\left(x^5-\frac{1}{2}\right)}{x^6}}} = \lim_{x\to+\infty} \frac{1}{\sqrt[3]{1-\frac{1}{2x^5}}} =$$

Ale: $I_2 < +\infty \Rightarrow I_1 < +\infty$;

konwergują;

5) $I_1 \int_0^1 \frac{dx}{\sqrt[5]{x(x^5+1)}}$;

prze 0

„błędna": $I_1 < \int \frac{dx}{x^{\frac{2}{5}}} = +\infty \not\Rightarrow I_1 = +\infty$;

$$I_1 \le \int_0^1 \frac{dx}{\sqrt[3]{x}} = \int_0^1 \frac{dx}{x^{\frac{1}{3}}} < +\infty:$$

$\left(\text{nos } \frac{1}{3} < 1\right)$

konwergują;

6) $I_1 \int_0^1 \frac{dx}{\sqrt[3]{x(2-x^5)}}$;

$$I_2 = \int_0^1 \frac{dx}{\sqrt[3]{x}} < +\infty$$

1;

$$A := \lim_{x\to 0} \frac{\sqrt[3]{x(2-x^5)}}{\frac{1}{\sqrt[3]{x}}} = \lim_{x\to 0} \frac{1}{\sqrt[3]{2-x^5}} =$$

$$= \frac{1}{\sqrt[3]{2}} < +\infty; > 0$$

Zatem $I_2 < +\infty \Rightarrow I_1 < +\infty$

$\int_1^{+\infty} \frac{dx}{x\sqrt[3]{x^2+1}}$; $\int_0^1 \frac{dx}{x\sqrt[3]{x^2+1}}$; konwergują;

$\int_1^{+\infty} \frac{x\,dx}{1+x^3}$; $\int_1^{+\infty} \frac{x^2\,dx}{1+x^3}$; $\int_1^{+\infty} \frac{\operatorname{arctg} x}{x\sqrt{x}}\,dx$; „-"; „-";

§ Kintamųjų keitimas apibrėžtinėje integrale

Teorema. Sakykime patenkinto sąlygos:

1) Funkcija $f: [a; b] \to R$ tolydi intervale $[a; b]$;

2) Funkcija $g: [\alpha; \beta] \to [a; b]$ turi tolydžią išvestinę intervale $[\alpha; \beta]$; ir yra bijektyvi

3) $g(\alpha) = a$; $g(\beta) = b$;

Tada teisinga formulė:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\, g'(t)\,dt;$$

▷ Sakykime $F(x)$ $f(x)$ pirmykštė funkcija. Kadangi $f(x)$ tolydi, tai tokia pradinė f-ja egzistuoja. Pagal N-L formulę

$$\int_a^b f(x)\,dx = F(b) - F(a) =$$

$$= F(g(\beta)) - F(g(\alpha));$$

Kadangi $F(x)$ yra pirmykštė funkcija, tai $(F(g(t)))' =$

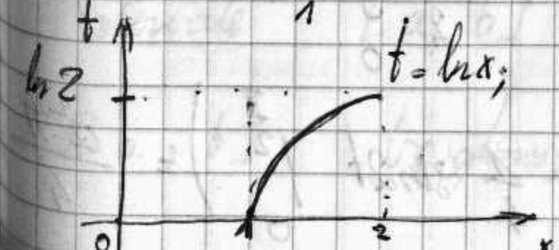
$$= F'(g(t)) \cdot g'(t) = f(g(t))\, g'(t);$$

Vadinasi $F(g(t))$ yra $f(g(t))\, g'(t)$ pirmykštė funkcija. Pagal N-L formulę

$$\int_\alpha^\beta f(g(t))\, g'(t)\,dt = F(g(\beta)) - F(g(\alpha)) \quad (2);$$

◁

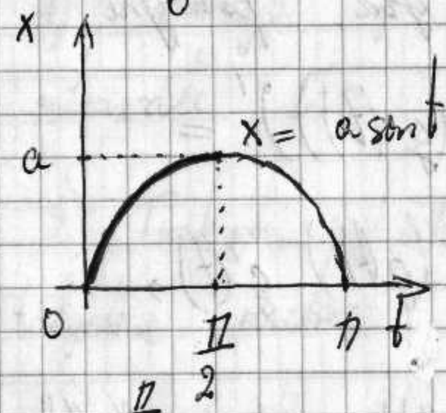
Pvz. 1) $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx = \left[\begin{array}{l} \ln x = t \\ d(\ln x) = dt \\ \frac{dx}{x} = dt \end{array}\right] =$



bijektyvus pakeitimas;

$$= \int_0^{\ln 2} t\,dt = \frac{t^2}{2}\Big|_0^{\ln 2} = \frac{1}{2}\ln^2 2;$$

2) $$\int_0^a \sqrt{a^2-x^2}\,dx = \left[\begin{array}{l} x = a\sin t \\ dx = a\cos t\,dt; \\ t = \arcsin\frac{x}{a} \end{array}\right]$$



$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\,\arccos t\,dt =$$

$$= a^2\int_0^{\frac{\pi}{2}} |\cos t|\cos t\,dt =$$

$$= a^2\int_0^{\frac{\pi}{2}} \cos^2 t\,dt = a^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt =$$

$$= a^2\left(\int_0^{\frac{\pi}{2}} \frac{dt}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2t\,dt\right) =$$

$$= a^2\left(\frac{t}{2}\Big|_0^{\frac{\pi}{2}} - \frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{2}}\right) = \frac{a^2\pi}{4};$$

§ Apibrėžtinio integralo integravimas dalimis

Teorema. Sakykime funkcijos $u(x)$ ir $v(x)$ turi tolydžias išvestines intervale $[a, b]$. Tada teisinga formulė:

$$\int_a^b u(x)\,d\,(x) = u\,(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)\,d\,u(x);$$

čia $$u(x)\cdot v(x)\Big|_a^b = u(b)\cdot v(b) - u(a)\cdot v(a);$$

§ Netikrosios integralai

Apibrėždami integralą $\int_a^b f(x)\,dx$ laikėme, kad funkcija $f(x)$ yra apriboto intervale $[a, b]$, o pats intervalas $[a, b]$ baigtinis;

Kaip integruoti intervale $[a, +\infty)$?

1. Apib. Sakykime $f(x)$ apibrėžta intervale nuo $[a, \infty)$.

Jeigu $\exists$ baigtinė riba:

$$\lim_{t\to\infty} \int_a^t f(x)\,dx;$$

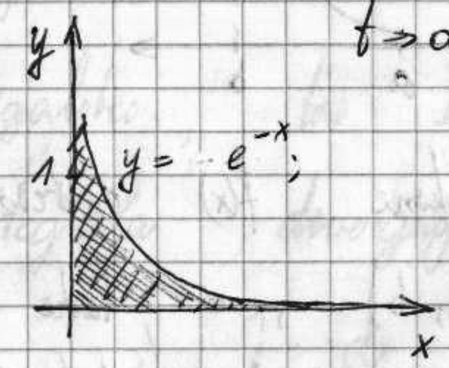
tai ši riba vadinama funkcijos $f(x)$ netikrosiomis integralu intervale $[a, \infty)$ ir žymima

$$\int_a^\infty f(x)\,dx;$$

Priešingu atveju $\int_a^\infty f(x)dx$ - diverguoja;

Pvz. $$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}dx =$$

$$= \lim_{t\to\infty} -e^{-x}\Big|_0^t = \lim_{t\to\infty}\left(-e^{-t} - \left(-e^{-0}\right)\right) = \lim_{t\to\infty}\left(1 - \frac{1}{e^t}\right) = 1;$$

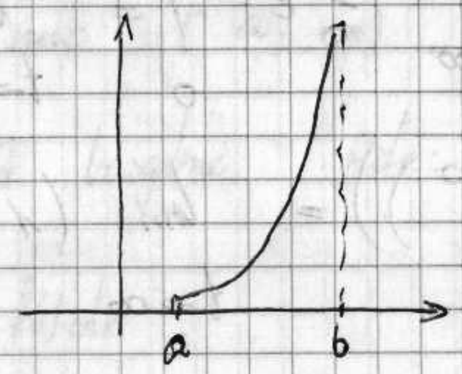


$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln x\Big|_1^t =$$

$$= \lim_{t\to\infty}(\ln t - \ln 1) = \lim_{t\to\infty} \ln t = \infty;$$

integralas diverguoja;

Kaip rasti $\int_a^b f(x)\,dx$ jeigu $f(x)$ taške b turi antro tipo trūkio tašką?



2. Apib. Sakykime $f(x)$ apibrėžta intervale nuo $[a, b)$, o taške b turi antro tipo trūkio tašką, (t.y. $\lim_{t\to b-0} f(x) = \pm\infty$ arba $\lim_{t\to b-0} f(x)\ \nexists$)

Jeigu $\exists$ riba:

$$\lim_{t\to b-0} \int_a^t f(x)\,dx,$$

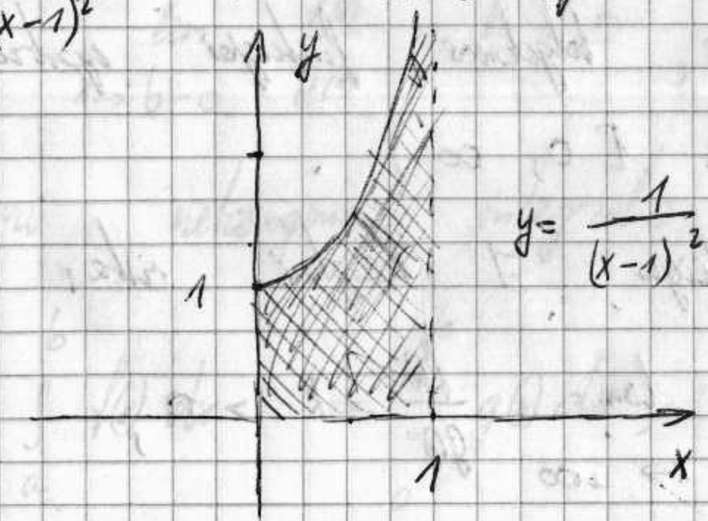
tai ši riba vadinama netikrojo funkcijos integralu intervale $[a; b)$ ir žymima

$$\int_a^b f(x)\,dx;$$

Jei minėta riba $\nexists$ arba lygi begalybei, tai sakoma netikrojo integrala diverguoja;

Pvz. 1) $\int_0^1 \frac{dx}{(x-1)^2} = \lim_{t\to 1-0} \int_0^t \frac{dx}{(x-1)^2} =$

$\frac{1}{(x-1)^2}$ turi antro tipo trūkį taške 1;



$y = \frac{1}{(x-1)^2}$;

$$= \lim_{t \to 1-0} \int_0^t (x-1)^{-2}\, d(x-1) =$$

$$= \lim_{t \to 1-0} \frac{(x-1)^{-1}}{-1} \Big|_0^t = \lim_{t \to 1-0} \left(\frac{-1}{t-1} + 1 \right)$$

$= +\infty$; integralas dverguoja;

Kartas nereikalaujame rasti nehistorinio integralo, o tik nustatyti ar jis $\exists$; Taikomi požymiai, tarp jų palyginimo;

1. Požymis. Sakykime $f(x)$ ir $g(x)$ teigiamos tolydžios funkcijos apibrėžtos intervale $[0; \infty)$.

Jeigu $\exists$ baigtinė riba:

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = R > 0,$$

tai nehistorinės integralai

$$\int_a^\infty f(x)\, dx \quad \text{ir} \quad \int_a^\infty g(x)\, dx;$$

kartu arba konverguoja arba dvergiuoja;

2. Pož. Sakykime $f(x)$ ir $g(x)$ teigiamos tolydžios funkcijos apibrėžtos intervale $[a; b)$ taškuose taške b $2-0$ tipo trūkio taškas.

Jeigu $\exists$ baigtinė riba

$$\lim_{x \to b-0} \frac{f(x)}{g(x)} = k > 0,$$

tai nehistorinės integralai

$$\int_a^b f(x)\, dx \quad \text{ir} \quad \int_a^b g(x)\, dx;$$

Prz. 1) Nustatysime ar integralas

$\int_0^1 \frac{2+\sin x}{(x-1)^2}\,dx$ yra baigtinis;

Lyginame integralus:

$\int_0^1 \frac{2+\sin x}{(x-1)^2}$ ir $\int_0^1 \frac{dx}{(x-1)^2}$;

Kadangi $\lim\limits_{x\to 1-0} \frac{\frac{2+\sin x}{(x-1)^2}}{\frac{1}{(x-1)^2}} =$

$= \lim\limits_{x\to 1-0} (2+\sin x) = 2+\sin 1 > 0$, tai

abu užrašyti integralai elgiasi

vienodai;

Buvo gauta:

$\int_0^1 \frac{dx}{(x-1)^2}$ diverguoja,

$\Downarrow$

$\int_0^1 \frac{2+\sin x}{(x-1)^2}\,dx$ irgi diverguoja;

Apibrėžtinių integralų

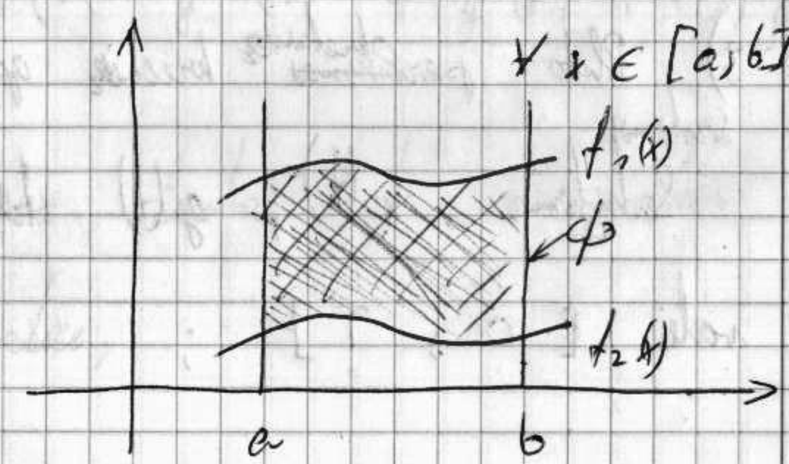
taikymai geometrijoje

1) Plotas ortogonaliose koordinatėse,

Tegul $f_1(x)$ ir $f_2(x)$ tolydžios

intervale $[a;b]$ ir $f_1 \geq f_2(x)$

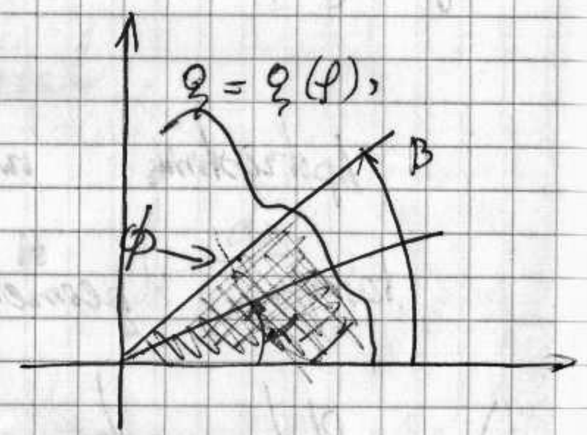
$\forall x \in [a,b]$.



$$S_\phi = \int_a^b \left(f_1(x) - f_2(x) \right) dx,$$

2) Płoszę polarne koordinatowe.

Tegub $\varrho = \varrho(\varphi)$ łożyło ukrocek $[\alpha, \beta]$



Toda

$$S_\phi = \frac{1}{2} \int_\alpha^\beta \varrho^2(\varphi)\, d\varphi;$$

3) Płoto parametrycznie krzywej zamkniętej

Sobytowej

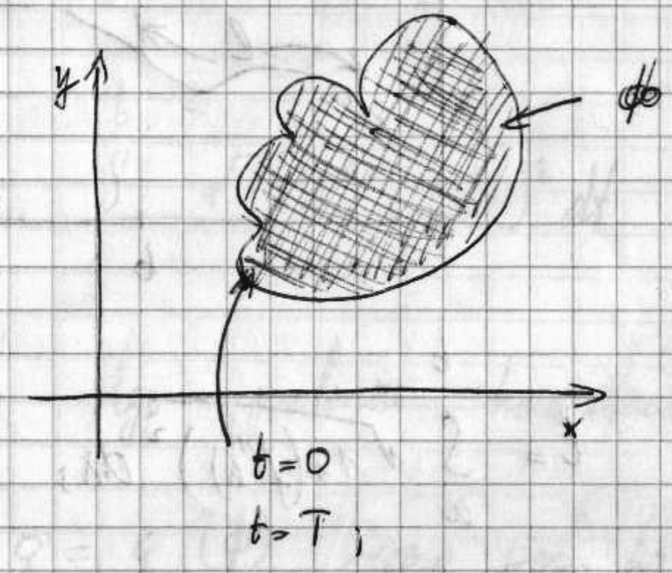
Sołyłowne $x(t)$, $y(t)$ łożyłożio unter-

vale $[0, T]$;

Sołyłowne krzywe $\begin{cases} x = x(t), \\ y = y(t) \end{cases}$ $t \in [0, T]$

yra wzdarza su tergrema ąpęłomo

horyzłomi



Toda

$$S_\phi = \int_0^T x(t)\, y(t)\, dt = -\int_0^T y(t)\, x'(t)\, dt =$$

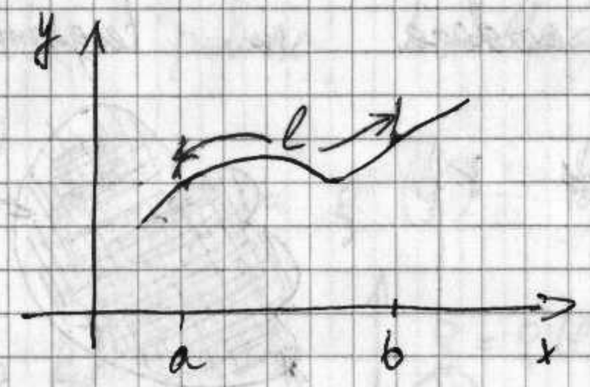
$$= \frac{1}{2} \int_0^T \left(x(t)\, y'(t) - y(t) \cdot x'(t) \right) dt$$

4) Krzywa łanko łyglo ortogonalnie

koordinatowe,

Załóżmy $f(x)$ tożsamo różniczkowalną

przedziale $[a; b]$



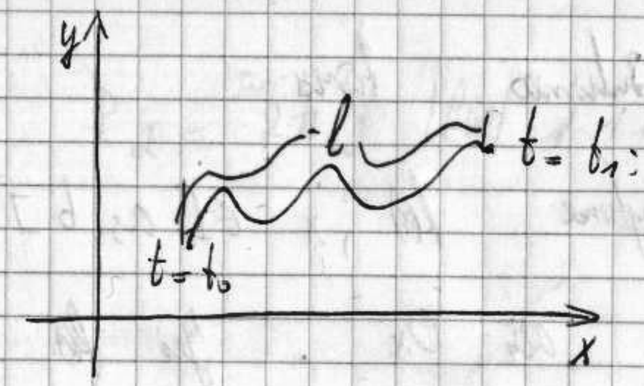
$$l = \int_a^b \sqrt{1+(f'(x))^2}\,dx;$$

5). Parametryzacja krzywej płaskiej;

Załóżmy krzywą możliwą łączyć:

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \quad t \in [t_0, t_1];$$

w $x(t)$, $y(t)$ są tożsamo różniczkowalnymi;

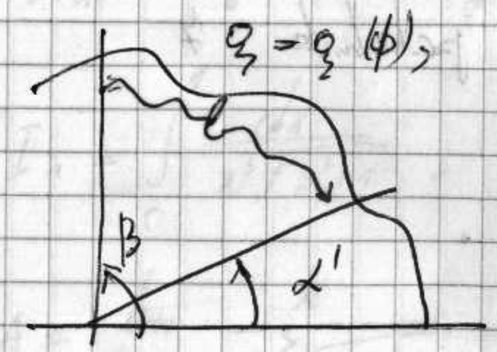


$$l = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

6) Krzywa dana w biegunowym współrzędnych.

Tego $\varrho = \varrho(\varphi)$ krzywa pol. współrz.

Tego $\varrho(\varphi)$ jest tożsamo różniczkowalną funk. przedz. $[\alpha'; \beta]$

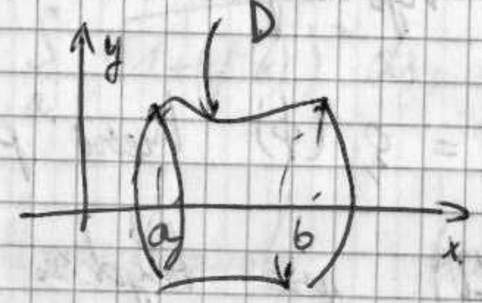


$$l = \int_{\alpha'}^{\beta} \sqrt{\varrho^2(\varphi) + (\varrho'(\varphi))^2}\,d\varphi;$$

7) Sukrinto tūris

Sakykime $f(x)$, $x \in [a, b]$ sukama aplink ašį Ox. Jei $f(x)$ tolydi interv. $[a, b]$, tai grafiko, kurio D tūris:

$$V_D = \pi \int_a^b f^2(x)\,dx,$$



8) Sukinio paviršiaus plotas;

Sakykime pateikiama γ ,

ir to

$$P_D = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2}\,dx,$$

Namų darbai

1) $I_1 = \int_1^{+\infty} \frac{dx}{x^3\sqrt{x^2+1}}$;

$$I_2 = \int_1^{+\infty} \frac{dx}{x^3\sqrt{x^2}} = \int_1^{+\infty} \frac{dx}{x^{\frac{5}{3}}}, \quad p = \frac{5}{3} > 1$$

$\Downarrow$

$I_1 \le I_2$ $\qquad$ I_2 konverguoja;

I_1 irgi konverguoja;

2) $I_1 = \int_0^1 \frac{dx}{x^3\sqrt{x^2+1}}$; $\qquad I_2 = \int_0^1 \frac{dx}{x}$;

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = A = \lim_{x\to 0} \frac{x}{x^3\sqrt{x^2+1}} = 1;$$

divergoja;

3) $I_1 = \int_1^{+\infty} \frac{x\,dx}{1+x^3} \leq \int_1^{+\infty} \frac{x\,dx}{x^3} = \int_1^{+\infty} \frac{dx}{x^2} < +\infty$ $\quad p = 2 > 1;$

$\Downarrow$

konwergencja;

4) $I_1 = \int_1^{+\infty} \frac{x\sqrt{x}\,dx}{1+x^3} < \int_1^{+\infty} \frac{x\sqrt{x}\,dx}{x^3} =$

$= \int_1^{+\infty} \frac{dx}{x^{\frac{3}{2}}} < +\infty \Rightarrow$ konwergencja;

$p = \frac{3}{2} > 1$

5) $I_1 = \int_1^{+\infty} \frac{x^2\sqrt{x}}{1+x^3}\,dx = \int_1^{+\infty} \frac{x^{\frac{5}{2}}\,dx}{1+x^3} \leq$

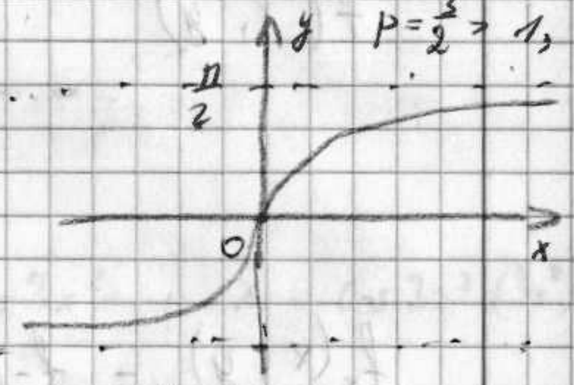
$\leq \int_1^{+\infty} \frac{x^{\frac{5}{2}}\,dx}{x^3} = \underbrace{\int_1^{+\infty} \frac{dx}{x^{\frac{1}{2}}}}_{I_2} = +\infty$ – „nieko";

$\frac{I_1}{I_2} \lim_{x\to+\infty} \frac{x^{\frac{1}{2}} \cdot x^{\frac{5}{2}}}{1+x^3} = \lim_{x\to+\infty} \frac{x^3}{1+x^3} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^3}+1} =$

$= 1;$ kadangi I_2 divergencja, tai ir I_1 divergencja;

6) $I_1 = \int_1^{+\infty} \frac{\operatorname{arctg} x}{x\sqrt{x}}\,dx \leq \int_1^{+\infty} \frac{\frac{\pi}{2}}{x\sqrt{x}}\,dx = \frac{\pi}{2}\int_1^{+\infty} \frac{dx}{x^{\frac{3}{2}}} < +\infty$

$\operatorname{arctg} x \leq \frac{\pi}{2}$



$p = \frac{3}{2} > 1,$

konwergencja;

7) $I_1 = \int_0^1 \frac{\operatorname{arctg} x}{x\sqrt{x}}\,dx \leq \frac{\pi}{2}\int_0^1 \frac{dx}{x^{\frac{3}{2}}} = +\infty$

kaip „nieko"

$I_2 = \int_0^1 \frac{x}{x\sqrt{x}}\,dx = \int_0^1 \frac{dx}{\sqrt{x}} = +\infty;$

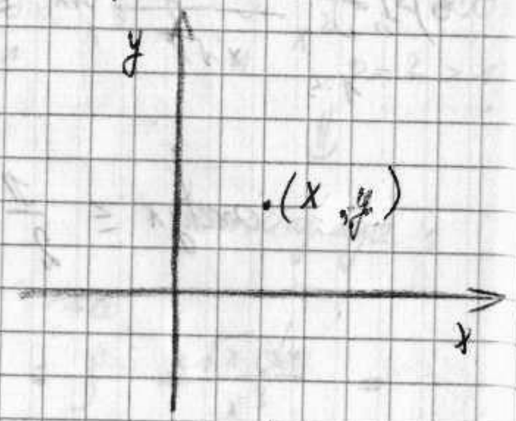
$A = \lim_{x\to 0} \frac{I_1}{I_2} = \lim_{x\to 0} \frac{\frac{\operatorname{arctg} x}{x\sqrt{x}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0} \frac{\operatorname{arctg} x}{x} =$

$= \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = \lim_{x\to 0} \frac{1}{1+x^2} = 1;$

$I_2 = +\infty \Rightarrow I_1 = +\infty$ $\quad I_1$ divergencja

Kintamųjų integralusse

$f(x, y)$



$f'_x(x, y)$ – f-jos f išvestinė pagal x, kai y laikomas fiksuotu (konstanta);

Analogiškai $f'_y(x, y)$;

$$\frac{\partial f(x,y)}{\partial x} = f'_x(x,y);$$

$$\frac{df(x,y)}{dx};$$

1) $$\frac{\partial (x^2, y^5)}{\partial x} = 2x y^5;$$

2) $$\frac{\partial (x^2 y^5)}{\partial y} = 5x^2 y^2;$$

3) $$\frac{\partial (x \cdot \sin x^2 \cdot y^3)}{\partial x} =$$

$$(x \cdot \sin 3x^2)' = \sin 3x^2 + x \cdot \cos 3x^2 (3x^2)' =$$

$$= \sin 3x^2 + x \cos 3x^2 \cdot 6x;$$

$$= \sin x^2 y^3 + x \cdot \cos x^2 y^3 \cdot 2xy^3;$$

4) $$\frac{\partial (x \cdot \sin x^2 \cdot y^3)}{\partial y} = x \cdot \cos x^2 \cdot y^3 \cdot 3x^2 y^2 =$$

$$= 3x^3 y^2 \cdot \cos x^2 y^3;$$

5) $$\frac{\partial (x^y)}{\partial x} = y x^{y-1};$$

6) $$\frac{\partial (x^y)}{\partial y} = x^y \ln x;$$

$$(a^x)' = a^x \ln a;$$

$$\frac{\partial^2 f(x,y)}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) =$$

$$= f''_{xy}(x,y);$$

$$\frac{\partial^2 f(x,y)}{\partial x\,\partial x} = \frac{\partial^2 f(x,y)}{\partial x^2} = f''_{xx}(x,y);$$

7) $f(x,y) = x^2 y^5;$

$$\frac{\partial f}{\partial x} = 2x\,y^5; \quad \frac{\partial f}{\partial y} = 5x^2 y^4;$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^5; \quad \frac{\partial^2 f}{\partial y^2} = 5x^2 \cdot 4y^3 =$$

$$= 20x^2y^3;$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = 2x \cdot 5y^4 = 10xy^4;$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = 5y^4 \cdot 2x = 10xy^4;$$

8) $f(x,y) = x \cdot y + \operatorname{arctg}(x^6 + \sin x^4);$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial}{\partial x}(x) = 1;$$

9) $f(x,y) = x^y;$

$$\frac{\partial f}{\partial x} = y\,x^{y-1}; \quad \frac{\partial f}{\partial y} = x^y \ln x;$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1) \cdot x^{y-2};$$

$$\frac{\partial^2 f}{\partial y^2} = x^y \ln x \ln x = x^y \ln^2 x;$$

$$(y\,x^{y-1})'_y = x^{y-1} + y\,x^{y-1}\ln x =$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \cancel{y\,x^{y-1}\ln x(y-1)'_y = y\,x^{y-1}\ln x};$$

$$= x^{y-1}(1 + y\ln x)$$

$$\frac{\partial^2 f}{\partial y \partial x} = (x^y \ln x)'_x = y x^{y-1} \ln x +$$

$$+ x^y \cdot \frac{1}{x} = y x^{y-1} \ln x + x^{y-1};$$

10) $f(x, y) = x y^2 \sin(x^2 \cdot y)$,

$$\frac{\partial f(x,y)}{\partial x} = y^2 (x \cdot \sin(x^2 \cdot y))' =$$

$$= y^2 (\sin(x^2 \cdot y) + x \sin'(x^2 \cdot y)) =$$

$$= y^2 (\sin(x^2 \cdot y) + x \cos(x^2 \cdot y) \cdot (x^2 \cdot y)') =$$

$$= y^2 (\sin(x^2 \cdot y) + 2x^2 y \cos(x^2 \cdot y)) =$$

$$= y^2 \cdot \sin(x^2 \cdot y) + 2x^2 y^3 \cos(x^2 \cdot y);$$

$$\frac{\partial f(x,y)}{\partial y} = x (y^2 \cdot \sin(x^2 \cdot y))' =$$

$$= x (2y \cdot \sin(x^2 \cdot y) + y^2 \cdot \cos(x^2 \cdot y) \cdot$$

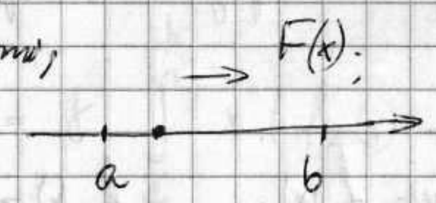
$$\cdot x^2) = 2xy \sin(x^2 \cdot y) + x^3 \cdot y^2 \cdot \cos(x^2 \cdot y);$$

2 eilės Apibrėžtinės integralų taikymai mechanikoje 2x ir 2y.

1) Kreivės jėgos darbas;

Sakykime tašką veikia jėga $F(x)$, judėdamas atkarpoje $[a, b]$;

Sakykime, be to, $F(x)$ yra tolydi, ir jėgos kryptis sutampa su taško judėjimo kryptimi;

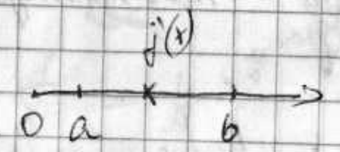


$$A = \int_a^b F(x)\,dx;$$

2) Strypo masė;

Sakykime intervale $[a; b]$ Ox ašyje guli strypas, kurio ~~masė~~ tankis taške x $(x \in [a, b])$ lygus $\rho(x)$.

Jeigu $j(x)$ tolydi, tai



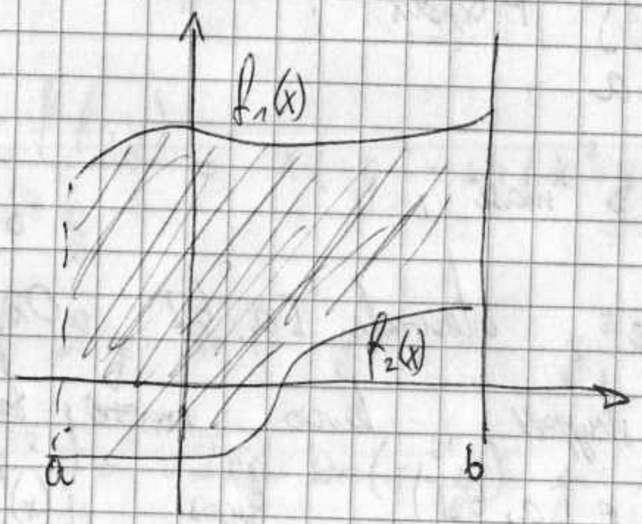
$$m = \int_a^b j(x)\,dx,$$

3) Plokščios figūros svorio centras;

Sakykime plokš. figūra aprėboja kreivėmis:

$y = f_1(x)$; $y = f_2(x)$; $f_1(x) > f_2(x)$,

$x \in [a; b]$



Jei $f_1(x)$ ir $f_2(x)$ yra tolydžios (intervale $[a; b]$); ...

tai svorio koordinatės randamos taip:

$$x_0 = \frac{m_y}{m}; \quad y_0 = \frac{m_x}{m};$$

$$m_x = \frac{1}{2}\gamma \int_a^b \left(f_2^2(x) - f_1^2(x)\right) dx;$$

$$m_y = \gamma \int_a^b x\left(f_2(x) - f_1(x)\right) dx;$$

$$m = \gamma \int_a^b \left(f_2(x) - f_1(x)\right) dx,$$

γ kelias kintamųjų funkcija;

n- matė erdvė ir jos taškų aibė

Aproksymacja. n - mate ordo R^n tashy $\bar{x}$ vadinane sutrashyta realy shavoshe rinhune $\bar{x} = (x_1, x_2, \dots, x_n)$, shavosha x_i $(i = 1, 2, \dots, n)$ vadinane tashe $\bar{x}$ i-qja koordinate,

Pra. $(1, 2, 3) \in R^3$;

$(-1, -4) \in R^2$;

$(1, 2, 0, -1, -3) \in R^5$,

Aproksimacja. Ashe qysh n - mates ordos tashy vadinane n - mate euklidine ordhe ir shpjome R^n;

kur $n = 1$ $R^n = R^1$ qya kise,

kur $n = 2$ $R^n = R^2$ qya plokshtina;

kur $n = 3$ $R^n = R^3$ qya prosta trimate ordhe;

Aproksimacja. Athemi tap dhjetë n-mate ordos tashy $\bar{x} = (x_1, x_2, \dots, x_n)$ ir $\bar{y} = (y_1, y_2, \dots, y_n)$ vadinane shasia:

$$\rho(\bar{x}, \bar{y}) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n-y_n)^2};$$

Parashtrim,

$$\rho((-1, 0), (-1, -3)) =$$

$$= \sqrt{(-1-(-1))^2+(0-(-3))^2} = 3;$$

$$\rho((1, 1, 2), (2, 2, 3)) =$$

$$= \sqrt{(1-2)^2+(1-2)^2+(2-3)^2} = \sqrt{3};$$

$$\rho((1, 1, 0, 0, 3), (0, 0, -1, -1, -5)) =$$

$$= \sqrt{1^2+1^2+1^2+1^2+2^2} = \sqrt{8} = 2\sqrt{2};$$

Athemi posh. aprox. qya athemi kosqe, plokshtnqje trimatqje ordhqje

apibrėžiamas.

Tenkina tokias savybes:

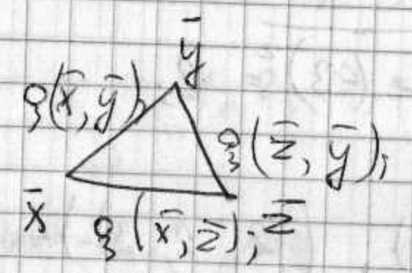
1) $\rho(\bar{x}, \bar{y}) \geq 0 \quad \forall \bar{x}, \bar{y} \in R^n$;

$\rho(\bar{x}, \bar{y}) = 0 \iff \bar{x} = \bar{y}$;

2) $\rho(\bar{x}, \bar{y}) = \rho(\bar{y}, \bar{x}), \quad \forall \bar{x}, \bar{y} \in R^n$;

3) $\rho(\bar{x}, \bar{y}) \leq \rho(\bar{x}, \bar{z}) + \rho(\bar{z}, \bar{y})$,

$\forall \bar{x}, \bar{y} \in R^n$,



Erdvės R^n atvirų rutulių

Sakykime $\bar{a} = (a_1, a_2, \dots, a_n) \in R^n$

fiksuotas taškas

Tada

$$U(\bar{a}, \varepsilon) = \{\bar{x} = (x_1, x_2, \dots, x_n) : \rho(\bar{x}, \bar{a}) < \varepsilon\}$$

n-mačiu rutuliu su centru taške $\bar{a}$ ir spindulio ε,

kai $n = 3$ $U(\bar{a}, \varepsilon)$ yra

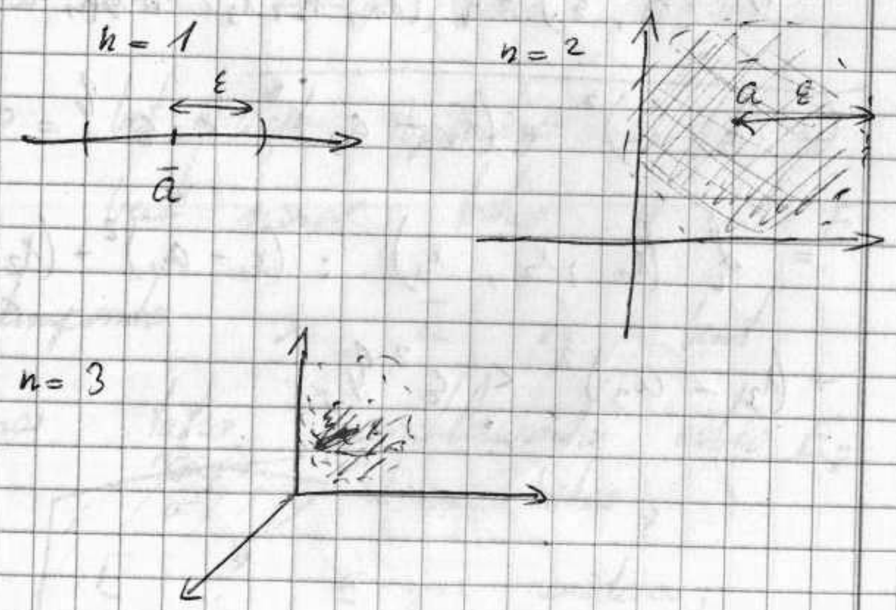
įprastas rutulys (rutulio vidus);

kai $n = 2$ $U(\bar{a}, \varepsilon)$ yra

skritulys (skritulio vidus);

kai $n = 1$ $U(\bar{a}, \varepsilon)$ yra

atvirasis intervalas $(\bar{a} - \varepsilon, \bar{a} + \varepsilon)$;



Iš čia, kai $n = 1$ turime

$$U(\bar{a}, \varepsilon) = \{x : \sqrt{(x-a)^2} < \varepsilon\} =$$

$$= \{x : |x-a| < \varepsilon\};$$

kai $n = 2$

$$U(\bar{a}, \varepsilon) = \{(x_1, x_2) : \sqrt{(x_1-a_1)^2+(x_2-a_2)^2} < \varepsilon\} =$$

$$= \{(x_1, x_2) : (x_1-a_2)^2 + (x_2-a_2)^2 < \varepsilon^2\};$$

kai $n = 3$

$$U(\bar{a}, \varepsilon) = \{(x_1, x_2, x_3) : \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2} < \varepsilon\} =$$

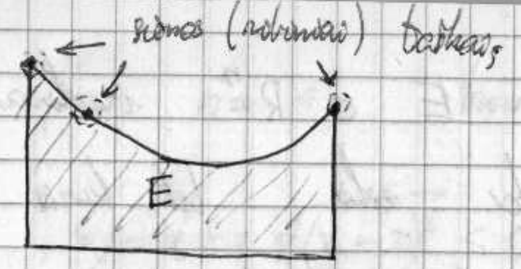
$$= \{(x_1, x_2, x_3) : (x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2 < \varepsilon^2\};$$

Aibę $E \subset R^n$ vadinsime atvirąja, jeigu tos aibės bet kurio taško $\bar{a}$ priklausos aibei E taškai su kokiu nors rutuliu $U(\bar{a}; \varepsilon)$;

Aibė $E \subset R^n$ vadinsime uždaroja, jeigu šitos aibės priklausos visi vidiniai taškai (šiame taškas). Aibės E šiame taškas, tai taškas $\bar{a}$, kurio bet kuriuo apliniuoju $U(\bar{a}, \varepsilon)$ yra bent vienas taškas iš aibės E nesutampantis su $\bar{a}$ ir bent vienas taškas nepriklausantis aibei E;

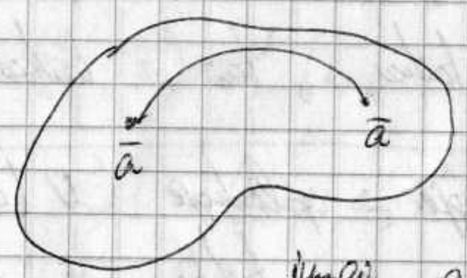
vidinis taškas;

E yra uždara;



E yra uždara,

Aibę $E \subset R^n$ vadinsime jungia, jeigu bet kuriems du, šios aibės taškus, galime surasti kelyklos kreivę, kuri šiuos taškus prijungia tai aibei;



jungi aibė;

nejungi aibė;

Kelias kintamųjų funkcija

Apibrėžimas. Jeigu n-mate euklidine erdvė R^n taškų aibė E kiekvienam taškui $\bar{a}$ pagal tam tikrą taisyklę priskiriamas vienintelis realus skaičius u, tai sakoma, kad aibėje E yra apibrėžta funkcija:

$$u = f(\bar{x}) = f(x_1, x_2, \dots, x_n);$$

Aibė E esantis apibrėžimo vadiname funkcijos f apibrėžimo sritimi;

Jei f. ap. sritis, nusakant f nenurodoma, tai laikoma f apib. sritimi laikoma aibė, kuriuje pakeitus f kuo prasmę;

Przy. funkcji:

(1) $z = x^2 + y^2, \quad (x, y) \in [0, 1]^2,$

ona jest funkcją $z : [0, 1]^2 \to R;$

(2) $u = xyz + x^2y^2 + z;$

ona jest funkcją $u : R^3 \to R;$

(3) $\xi = x_1^2 + x_2^2 + x_3^2 + x_4^2;$

ona jest funkcją $\xi : R^4 \to R;$

(4) $z = \ln(y - x^2);$

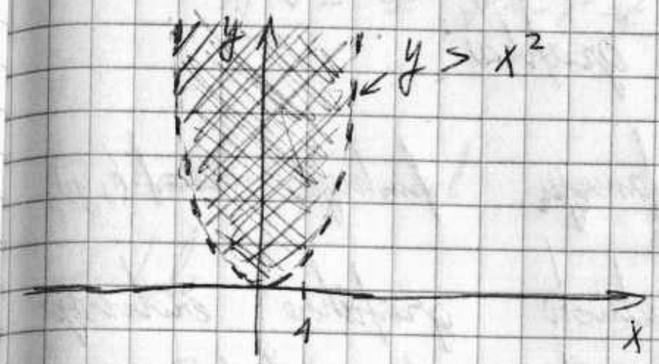
ona jest funkcją $z : E \to R,$

ho $E \subset R^2,$

Rysunek E, Arkusz, had

$E = \{(x, y) : y > x^2\}$

E rozwiązać;



(5) $z = \sqrt{x^2 + y^2 - 4};$

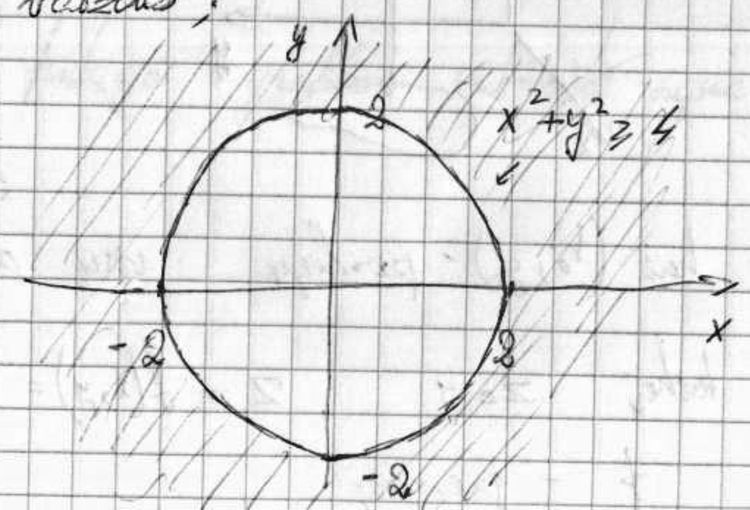
ona jest funkcją $z : E \to R$

są $E \subset R^2;$

rysunek E

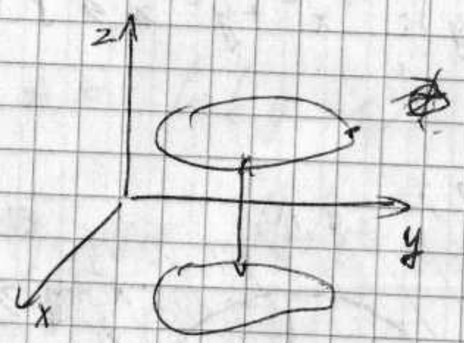
$E = \{(x, y) : x^2 + y^2 \geq 4\}$

E rozwiązać;



Dwóch kontynuów funkcji grafika

Dwóch kontynuów funkcja $z=f(x,y)$, gdzie powierzchnia grafiku ortogonalnego kwadratowy systemowe. Sekwencja E gra $f(x,y)$ ograniczona zwrot. kiedy-nam teshi $(x_0;y_0) \in E$ brzeżnem kontynent Oxy płaskimi iz same skończenie atiedame tesh $z_0=f(x_0;y_0)$

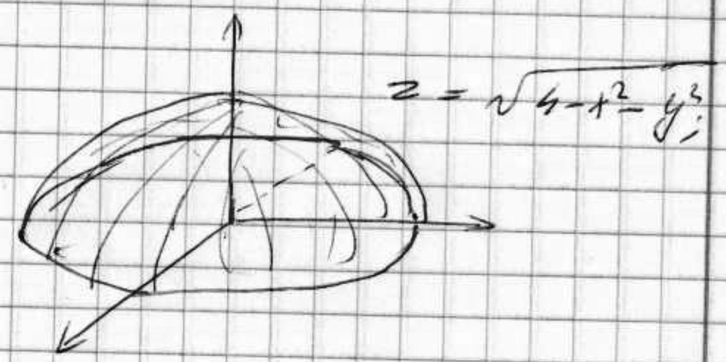


kai (x_0,y_0) pozostaje wszy ade E tesh z_0, $z=f(x,y)$ = grafiku f - powierzchni

Prz. 1) $z=\sqrt{4-x^2-y^2}$;

pasebum, kad z opit. sufo gra skritulo;

$E=\{(x,y): x^2+y^2 \le 4\}$



Jeigu $y=f(x_1, x_2, \dots, x_n)$ gra $n \ge 3$ kontynuoji funkcja, tai toks funkcijos grafiko mokslinis nejmanoma;