# Reform Calculus: Part II

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# PREFACE

This supplement consists of my lectures of a sophomore-level mathematics class offered at Arkansas Tech University. The lectures are designed to accompany the textbook "*Calculus: Single and Multivariable*" written by the Harvard Consortium.
This book has been written in a way that can *be read by students*. That is, the text represents a serious effort to produce exposition that is accessible to a student at freshmen or sophomore levels.
This supplement is a continuation of the previous calculus book. The lectures cover Chapters 7, 8, 9, 10, and 11. These chapters are well suited for a 4-hour one semester course of a second course Calculus.

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# Contents

## 40 Integration by Substitution

The purpose of this section is to evaluate the integral

$$\int f'(g(x))g'(x)dx. \tag{1}$$

This is done by letting $u = g(x)$. Then $du = g'(x)dx$. (See Section 23). Hence, we have the following

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C \tag{2}$$

The above procedure is referred to as the **method of integration by substitution**. Thus, when the integrand looks like a compound function times the derivative of the inside then try using substitution to integrate. Note also that this method of antidifferentiation reverses the chain rule of differentiation.
The following examples illustrate the use of this method.

**Example 40.1**
Find $\int 3x^2 \cos x^3 dx$.

**Solution.**
Let $u(x) = x^3$. Then $du = 3x^2 dx$ and therefore

$$\int 3x^2 \cos x^3 dx = \int \cos u du = \sin u + C = \sin x^3 + C. \blacksquare$$

The method of substitution works even when the derivative of the inside is missing a constant factor as shown in the next couple of examples.

**Example 40.2**
Find $\int xe^{x^2+1}dx$.

**Solution.**
Letting $u(x) = x^2 + 1$ then $du = 2xdx$. Thus,

$$\begin{array}{rcl} \int xe^{x^2+1}dx & = & \frac{1}{2}\int 2xe^{x^2+1}dx \\ & = & \frac{1}{2}\int e^u du = \frac{e^u}{2} + C \\ & = & \frac{1}{2}e^{x^2+1} + C. \end{array}$$

You may wonder why $\frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$ and not $\frac{1}{2}\int e^u du = \frac{1}{2}(e^u + C) = \frac{e^u}{2} + \frac{C}{2}$. The convention is always to add $C$ to whatever antiderivative we have calculated.$\blacksquare$

**Example 40.3**
Find $\int x^3\sqrt{x^4+5}dx$.

**Solution.**
Let $u = x^4 + 5$. Then $du = 4x^3dx$. Thus,

$$\begin{array}{rcl}\int x^3\sqrt{x^4+5}dx &=& \frac{1}{4}\int 4x^3\sqrt{x^4+5}dx\\ &=& \frac{1}{4}\int \sqrt{u}du = \frac{1}{4}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C\\ &=& \frac{1}{6}(x^4+5)^{\frac{3}{2}} + C \blacksquare\end{array}$$

In some situations, the integrand consists of a fraction having a function in the denominator and its derivative in the numerator. This leads to a natural logarithm as seen in the next two examples.

**Example 40.4**
Find $\int \frac{e^x}{e^x+1}dx$.

**Solution.**
Let $u = e^x + 1$. Then $du = e^xdx$. Thus,

$$\begin{array}{rcl}\int \frac{e^x}{e^x+1}dx &=& \int \frac{du}{u} = \ln|u| + C\\ &=& \ln|e^x+1| + C. \blacksquare\end{array}$$

**Example 40.5**
Find $\int \tan xdx$.

**Solution.**
Since $\tan x = \frac{\sin x}{\cos x}$ then letting $u = \cos x$ we find that $du = -\sin xdx$ and therefore

$$\int \tan xdx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C. \blacksquare$$

Next, we discuss the evaluation of a definite integral using the technique of substitution. From ( 2) we have that $f(g(x))$ is an antiderivative of the function $f'(g(x))g'(x)$. Applying the Fundamental Theorem of Calculus we can write

$$\int_a^b f'(g(x))g'(x)dx = f(g(x))|_a^b = f(g(b)) - f(g(a)).$$

If we let $u = g(x)$ then the previous formula reduces to

$$\int_a^b f'(g(x))g'(x)dx = f(g(b)) - f(g(a)) = \int_{g(a)}^{g(b)} f'(u)du.$$

**Warning**: When evaluating definite integrals, there is no constant of integration in the final answer.

**Example 40.6**
Compute $\int_0^2 xe^{x^2}dx$.

**Solution.**
Let $u(x) = x^2$. Then $du = 2xdx, u(0) = 0$, and $u(2) = 4$. Thus,

$$\int_0^2 xe^{x^2}dx = \frac{1}{2}\int_0^4 e^u du = \left.\frac{e^u}{2}\right|_0^4 = \frac{1}{2}(e^4 - 1). \blacksquare$$

**Example 40.7**
Compute $\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{\cos^2 x}dx$.

**Solution.**
Let $u = \tan x$. Then $du = \frac{dx}{\cos^2 x}, u(0) = 0$, and $u(\frac{\pi}{4}) = 1$. Thus,

$$\int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{\cos^2 x}dx = \int_0^1 u^3 du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}. \blacksquare$$

**Example 40.8**
Find $\int \sqrt{1 + \sqrt{x}}dx$.

**Solution.**
Let $u = 1 + \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2(u-1)}$ or $dx = 2(u-1)du$. Thus,

$$\begin{array}{rcl}\int \sqrt{1+\sqrt{x}}dx & = & \int 2\sqrt{u}(u-1)du = \int(2u\sqrt{u} - 2\sqrt{u})du \\ & = & \int(2u^{\frac{3}{2}} - 2u^{\frac{1}{2}})du \\ & = & 2\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\ & = & \frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C. \blacksquare\end{array}$$

## Practice Problems

**Exercise 40.1**
*Find* $\int x^2(1+2x^3)^2dx.$

**Exercise 40.2**
*Find* $\int x(x^2-4)^{\frac{7}{2}}dx.$

**Exercise 40.3**
*Find* $\int \frac{1}{\sqrt{4-x}}dx.$

**Exercise 40.4**
*Find* $\int \sin x(\cos x+5)^7dx.$

**Exercise 40.5**
*Find* $\int x^2e^{x^3+1}dx.$

**Exercise 40.6**
*Find* $\int \frac{(\ln x)^2}{x}dx.$

**Exercise 40.7**
*Find* $\int \frac{e^x+1}{e^x+x}dx.$

**Exercise 40.8**
*Find* $\int \frac{1+e^x}{\sqrt{x+e^x}}dx.$

**Exercise 40.9**
*Find* $\int \frac{x+1}{x^2+2x+19}dx.$

**Exercise 40.10**
*Find* $\int \frac{x\cos{(x^2)}}{\sqrt{\sin{(x^2)}}}dx.$

**Exercise 40.11**
*Compute* $\int_0^{\frac{\pi}{2}} e^{-\cos x}dx.$

**Exercise 40.12**
*Compute* $\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx.$

**Exercise 40.13**
*Compute* $\int_1^4 \frac{\cos\sqrt{x}}{\sqrt{x}}dx.$

**Exercise 40.14**
*Compute* $\int_{-1}^{1} \frac{1}{1+x^2} dx$.

**Exercise 40.15**
*Compute* $\int_1^3 \frac{dx}{(x+7)^2}$.

**Exercise 40.16**
*Compute* $\int_1^2 \frac{\sin x}{x} dx$.

**Exercise 40.17**
*Find the exact area under the graph of* $f(x) = xe^{x^2}$ *between* $x = 0$ *and* $x = 2$.

**Exercise 40.18**
*Find the exact area enclosed by the graph of* $y = \frac{4}{x}$, *the* $x$*-axis and the lines* $x = 2$ *and* $x = 4$.

**Exercise 40.19**
*Suppose* $\int_0^1 f(x)dx = 3$. *Calculate the following:*
*(a)* $\int_0^{0.5} f(2x)dx$ *(b)* $\int_0^1 f(x-1)dx$ *(c)* $\int_1^{1.5} f(3-2x)dx$.

**Exercise 40.20**
*If* $t$ *is in years since 1990, the population* $P$, *of the world in billions can be modeled by* $P(t) = 5.3e^{0.014t}$.
*(a) What does this model give the world population in 1990? in 2000?*
*(b) Use the Fundamental Theorem of Calculus to find the average population of the world during the 1990s.*

**Exercise 40.21**
*Decide whether the following statements are true or false. Give an explanation for your answer.*
*(a)* $\int f'(x) \cos(f(x))dx = sin(f(x)) + C$.
*(b)* $\int \frac{1}{f(x)} dx = \ln |f(x)| + C$.
*(c)* $\int x \sin(5 - x^2)dx$ *can be evaluated using substitution.*
*(d)* $\int \sin^7 \theta \cos^6 \theta d\theta$ *can be written as a polynomial with* $\cos\theta$ *as the variable.*

# 41 The Method of Integration by Parts

The integration by parts formula is an antidifferentiation method which reverses the product rule of differentiation. To see this, let $u$ and $v$ be two differentiable functions. Then the product rule asserts that the product function $uv$ is also differentiable and its derivative is given by

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

This says that an antiderivative of $u\frac{dv}{dx} + v\frac{du}{dx}$ is the function $uv$. In terms of indefinite integrals we have

$$\int (u\frac{dv}{dx} + \frac{du}{dx}v)dx = uv + C. \tag{3}$$

But the differential of $u$ is $du = \frac{du}{dx}dx$ and that of $v$ is $dv = \frac{dv}{dx}dx$. Hence, (3) becomes

$$\int udv + \int vdu = uv$$

or

$$\int udv = uv - \int vdu \tag{4}$$

Formula (4) is known as the **integration by parts formula.**

**Example 41.1**
Find $\int xe^x dx$.

**Solution.**
Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = \int e^x dx = e^x$. Note that in finding $v$ we did not include the constant of integration. We will write the constant C in the answer of $\int udv$. Now, substituting in formula (4) to obtain

$$\begin{array}{rcl}\int xe^x dx &=& xe^x - \int e^x dx\\ &=& xe^x - e^x + C. \blacksquare\end{array}$$

**Remark 41.1**
If we chose $u = e^x$ and $dv = xdx$ then we would have $du = e^x dx$ and $v = \frac{x^2}{2}$. In this case, formula (4) yields

$$\int xe^x dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x dx$$

and the second integral is definitely worse than the one to the left of the formula we had in the previous example, i.e. $\int xe^x dx$. It follows that for the method to be useful it is important to choose $u$ and $dv$ in such a way to make the integral on the right easier to find then the integral on the left.

**Example 41.2**
Find $\int x \sin x dx$.

**Solution.**
Let $u = x$ and $dv = \sin x$. Then $du = dx$ and $v = -\cos x$. Substituting in formula (4) to obtain

$$\begin{aligned}\int x \sin x dx &= -x\cos x - \int(-\cos x)dx\\ &= -x\cos x + \int \cos x dx\\ &= -x\cos x + \sin x + C. \blacksquare\end{aligned}$$

There are examples which don't look like good candidates for integration by parts because they don't appear to involve products, but for which the method works well. Such examples often involves $\ln x$ or the inverse trigonometric functions.

**Example 41.3**
Calculate $\int_1^5 \ln x dx$.

**Solution.**
Let $u = \ln x$ and $dv = dx$. Then $du = \frac{dx}{x}$ and $v = x$. Substituting in formula (4) we obtain

$$\begin{aligned}\int_1^5 \ln x dx &= x\ln x|_1^5 - \int_1^5 x\frac{1}{x}dx\\ &= x\ln x|_1^5 - \int_1^5 dx\\ &= x\ln x|_1^5 - x|_1^5\\ &= 5\ln 5 - \ln 1 - (5-1)\\ &= 5\ln 5 - 4. \blacksquare\end{aligned}$$

**Example 41.4**
Find $\int x^3 \ln x dx$.

**Solution.**
Let $u = \ln x$ and $dv = x^3 dx$. Then $du = \frac{dx}{x}$ and $v = \frac{x^4}{4}$. Substituting in

formula (4) to obtain

$$\begin{aligned}
\int x^3 \ln x dx &= \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}dx\\
&= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx\\
&= \frac{x^4}{4}\ln x - \frac{1}{4}\frac{x^4}{4} + C\\
&= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C. \blacksquare
\end{aligned}$$

Sometimes evaluating an integral might require integration by parts more than once as the following example shows.

**Example 41.5**
Find $\int x^2 \cos x dx$.

**Solution.**
Let $u = x^2$ and $dv = \cos x dx$. Then $du = 2xdx$ and $v = \sin x$. Substituting in formula (4) to obtain

$$\begin{aligned}
\int x^2 \cos x dx &= x^2 \sin x - \int 2x \sin x dx\\
&= x^2 \sin x - 2\int x \sin x dx\\
&= x^2 \sin x - 2(-x\cos x + \sin x) + C\\
&= x^2 \sin x + 2x\cos x - 2\sin x + C
\end{aligned}$$

where we have used Example 41.2 to evaluate $\int x \sin x dx$. $\blacksquare$

The following example illustrates a very useful technique: Use integration by parts to transform the integral into an expression containing another copy of the same integral, possibly multiplied by a coefficient, then solve for the original integral.

**Example 41.6**
Find $\int \sin^2 x dx$.

**Solution.**
Method I:
Using a half-angle formula to write $\sin^2 x = \frac{1-\cos 2x}{2}$. In this case the problem reduces to integrating $\frac{1-\cos 2x}{2}$. That is,

$$\begin{aligned}
\int \sin^2 x dx &= \frac{1}{2}\int (1-\cos 2x)dx\\
&= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx\\
&= \frac{1}{2}x - \frac{1}{4}\int \cos u du\\
&= \frac{1}{2}x - \frac{1}{4}\sin u + C\\
&= \frac{1}{2}x - \frac{1}{4}\sin 2x + C
\end{aligned}$$

where the substitution $u = 2x$ is used to evaluate the integral $\int \cos 2x dx$.

Method II:
We now use integration by parts formula to evaluate the given integral. Let $u = \sin x$ and $dv = \sin x dx$. Then $du = \cos x$ and $v = -\cos x$. Substituting in formula (4) to obtain

$$\begin{aligned} \int \sin^2 x dx &= -\sin x \cos x - \int -\cos^2 x dx \\ &= -\sin x \cos x + \int \cos^2 x dx. \end{aligned}$$

Using the trigonometric identities $\sin 2x = 2 \sin x \cos x$ and $\cos^2 x + \sin^2 x = 1$ we can rewrite the right-hand side of the above integral as

$$\begin{aligned} \int \sin^2 x dx &= -\tfrac{1}{2} \sin 2x + \int (1 - \sin^2 x) dx \\ &= -\tfrac{1}{2} \sin 2x + \int dx - \int \sin^2 x dx \\ &= -\tfrac{1}{2} \sin 2x + x - \int \sin^2 x dx \end{aligned}$$

Moving the right integral to the left side to obtain

$$2 \int \sin^2 x dx = -\frac{1}{2} \sin 2x + x$$

and finally dividing both sides by 2 to obtain

$$\int \sin^2 x dx = -\frac{1}{4} \sin 2x + \frac{x}{2} + C. \blacksquare$$

## Practice Problems

**Exercise 41.1**
*Find* $\int_0^1 \arctan x dx$.

**Exercise 41.2**
*Find* $\int x^2 e^{5x} dx$.

**Exercise 41.3**
*Find* $\int x^3 \ln x dx$.

**Exercise 41.4**
*Find* $\int x^2 \sin x dx$.

**Exercise 41.5**
*Find* $\int \cos^2 x dx$.

**Exercise 41.6**
*Find* $\int \frac{\ln x}{x^2} dx$.

**Exercise 41.7**
*Find* $\int x^5 \ln 5x dx$.

**Exercise 41.8**
*Find* $\int \frac{x}{\sqrt{5-x}} dx$.

**Exercise 41.9**
*Find* $\int (\ln x)^2 dx$.

**Exercise 41.10**
*Find* $\int x \arctan x^2 dx$.

**Exercise 41.11**
*Find* $\int x^5 \cos x^3 dx$.

**Exercise 41.12**
*Evaluate* $\int_0^{10} xe^{-x} dx$.

**Exercise 41.13**
*Evaluate* $\int_0^5 \ln(1+x) dx$.

**Exercise 41.14**
*Evaluate* $\int_0^1 \arcsin x dx$.

**Exercise 41.15**
*Find* $\int e^x \sin x dx$.

**Exercise 41.16**
*Compute* $\int \cos^2 \theta d\theta$ *in two different ways.*

**Exercise 41.17**
*Find the exact value of the area under the first arch of* $f(x) = x \sin x$.

**Exercise 41.18**
*Derive the following formula*

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

**Exercise 41.19**
*Let* $f$ *be twice differentiable function such that* $f(0) = 6, f(1) = 5$, *and* $f'(1) = 2$. *Evaluate the integral* $\int_0^1 x f''(x) dx$.

**Exercise 41.20**
*Suppose that* $F(a)$ *represents the area under the graph of* $y = x^2 e^{-x}$ *between* $x = 0$ *and* $x = a > 0$.
*(a) Find a formula for* $F(a)$.
*(b) Is* $F$ *increasing or decreasing function?*
*(c) Is* $F$ *concave up or down for* $0 < a < 2$?

**Exercise 41.21**
*In describing the behavior of an electron, we use wave functions* $\Psi_1, \Psi_2, \Psi_3, \cdots$ *of the form*

$$\Psi_n(x) = C_n \sin(n\pi x), \quad n = 1, 2, 3, \cdots$$

*where* $x$ *is the distance from a fixed point and* $C_n$ *is a positive constant.*
*(a) Find* $C_1$ *so that* $\Psi_1$ *satisfies*

$$\int_0^1 (\Psi_1(x))^2 dx = 1.$$

*(b) For any integer* $n$, *find* $C_n$ *so that*

$$\int_0^1 (\Psi_n(x))^2 dx = 1.$$

# 42 Tables of Integrals

If the standard integration techniques presented previously fail to yield an antiderivative, the last measure of despair are integral tables. These tables basically consist of collections of functions together with their antiderivatives. In order to use them, you may have to re-write the integrand function first in a standard form listed in the table. A short table of common formulas is given in the back cover of your textbook. The first example shows how to apply an integral with no manipulations of any kind.

**Example 42.1**
Find $\int \cos(2x)\cos(7x)dx$.

**Solution.**
Using formula II-11 with $a = 2$ and $b = 7$ we have

$$\int \cos(2x)\cos(7x)dx = \frac{1}{45}(7\cos(2x)\sin(7x) - 2\sin(2x)\cos(7x)) + C.\blacksquare$$

The following two examples use *reduction formulas.*

**Example 42.2**
*rm Find* $\int (x^2 - 3x + 2)e^{-4x}dx.$

**Solution.**
Using formula III-14 with $p(x) = x^2 - 3x + 2$ and $a = -4$ to obtain

$$\int (x^2-3x+2)e^{-4x}dx = -\frac{1}{4}(x^2-3x+2)e^{-4x} - \frac{1}{16}(2x-3)e^{-4x} - \frac{1}{32}e^{-4x} + C.\blacksquare$$

**Example 42.3**
Find $\int \sin^4 x dx$.

**Solution.**
Using the reduction formula IV-17 we find

$$\begin{array}{rcl}\int \sin^4 x dx & = & -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x dx\\ & = & -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C.\ \blacksquare\end{array}$$

Remember that some integrals can be evaluated without the use of an integral table as shown in the next example.

**Example 42.4**
Find $\int \cos^3 x \sin^4 x dx$.

**Solution.**
This integral can be evaluated without the use of a table of integral. Indeed, let $u = \sin x$ then $du = \cos x dx$ and

$$\begin{array}{rcl}\int \cos^3 x \sin^4 x dx & = & \int (1 - \sin^2 x)\cos x \sin^4 x dx \\ & = & \int (1-u^2)u^4 du = \int (u^4 - u^6) du \\ & = & \frac{u^5}{5} - \frac{u^7}{7} + C \\ & = & \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C. \blacksquare\end{array}$$

**Example 42.5**
Find $\int \sin^2 x \cos^2 x dx$.

**Solution.**
Using the trigonometric identity $\cos^2 x + \sin^2 x = 1$ and Formula IV-17 we have

$$\begin{array}{rcl}\int \sin^2 x \cos^2 x dx & = & \int \sin^2 x(1-\sin^2 x)dx \\ & = & \int \sin^2 x dx - \int \sin^4 x dx \\ & = & (-\frac{1}{2}\sin x \cos x + \frac{1}{2}x) - (-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x dx) \\ & = & -\frac{1}{2}\sin x \cos x + \frac{1}{2}x + \frac{1}{4}\sin^3 x \cos x - \frac{3}{4}(-\frac{1}{2}\sin x\cos x + \frac{1}{2}x) + C \\ & = & -\frac{1}{2}\sin x \cos x + \frac{1}{2}x + \frac{1}{4}\sin^3 x \cos x + \frac{3}{8}\sin x \cos x - \frac{3}{8}x + C \\ & = & -\frac{1}{8}\sin x \cos x + \frac{1}{4}\sin^3 x \cos x + \frac{1}{8}x + C. \blacksquare\end{array}$$

The integrands in the following require some manipulations to fit the entries in the table.

**Example 42.6** *(Factoring)*
Find $\int \frac{1}{x^2+4x+3}dx$.

**Solution.**
Factoring the denominator and then using Formula V-26 we obtain

$$\int \frac{1}{x^2+4x+3}dx = \int \frac{1}{(x+1)(x+3)}dx = \frac{1}{2}(\ln|x+1| - \ln|x+3|) + C. \blacksquare$$

**Example 42.7** *(Long division)*
Find $\int \frac{x^2+1}{x^2-1}dx$.

**Solution.**
Using the long division of polynomials we find

$$\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1} = 1 + \frac{2}{(x-1)(x+1)}.$$

This with Formula V-26 yield

$$\int \frac{x^2+1}{x^2-1}dx = x + 2[\frac{1}{2}(\ln|x-1| - \ln|x+1|) + C = x + \ln\left|\frac{x-1}{x+1}\right| + C.\blacksquare$$

**Example 42.8** *(Completing the square)*
Find $\int \frac{1}{x^2+4x+5}dx$.

**Solution.**
Completing the square we find $x^2+4x+5 = (x+2)^2+1$. Now, letting $u = x+2$ we can write

$$\int \frac{1}{x^2+4x+5}dx = \int \frac{1}{u^2+1}du.$$

Finally, using Formula V-24 we find

$$\int \frac{1}{x^2+4x+5}dx = \int \frac{1}{u^2+1}du = \arctan u + C = \arctan(x+2) + C.\blacksquare$$

**Example 42.9** *(Using substitution)*
Find $\int xe^{2x^2}\cos(2x^2)dx$.

**Solution.**
Let $u = 2x^2$. Then $du = 4xdx$ so that

$$\int xe^{2x^2}\cos(2x^2)dx = \frac{1}{4}\int e^u \cos u du.$$

Now, applying Formula II-9 to the last integral we find

$$\int xe^{2x^2}\cos(2x^2)dx = \frac{1}{8}e^{2x^2}(\cos(2x^2) + \sin(2x^2)) + C.\blacksquare$$

## Practice Problems.

**Exercise 42.1**
*Find* $\int x^5 \ln x dx$.

**Exercise 42.2**
*Find* $\int \sin x \cos^4 x dx$.

**Exercise 42.3**
*Find* $\int \sin{(3x)} \sin{(5x)} dx$.

**Exercise 42.4**
*Find* $\int \sin^3{(3x)} \cos^3{(3x)} dx$.

**Exercise 42.5**
*Find* $\int x^4 e^{3x} dx$.

**Exercise 42.6**
*Find* $\int \frac{x^2}{x^2+4} dx$.

**Exercise 42.7**
*Find* $\int \sin^3 x dx$.

**Exercise 42.8**
*Find* $\int \frac{dx}{4-x^2}$.

**Exercise 42.9**
*Find* $\int \frac{1}{x^2+4x+4} dx$.

**Exercise 42.10**
*Suppose* $n$ *is a positive integer and* $\Psi_n = C_n \sin{(n\pi x)}$ *is the wave function used in describing the behavior of an electron. If* $n$ *and* $m$ *are different integers, find*

$$\int_0^1 \Psi_n(x) \cdot \Psi_m(x) dx.$$

**Exercise 42.11**
*Decide whether the following statements are true or false. Give an explanation for your answer.*
*(a)* $\int \frac{1}{x^2+4x+5} dx$ *involves a natural logarithm.*
*(b)* $\int \frac{1}{x^2+4x-5} dx$ *involves an arctangent.*

# 43 Integration by Partial Fractions. Trigonometric Substitutions

**Method of Integration by Partial Fractions**

The method of integration by partial fractions is a technique for integrating rational functions, i.e. functions of the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.
The idea consists of writing the rational function as a sum of simpler fractions called **partial fractions**. This can be done in the following way:

*Step 1.* Use long division to find two polynomials $r(x)$ and $q(x)$ such that

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}.$$

Note that if the degree of $P(x)$ is smaller than that of $Q(x)$ then $q(x) = 0$ and $r(x) = P(x)$.

*Step 2.* Write $Q(x)$ as a product of factors of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$ where $ax^2+bx+c$ is irreducible, i.e. $ax^2+bx+c=0$ has no real zeros.

*Step 3.* Decompose $\frac{r(x)}{Q(x)}$ into a sum of partial fractions in the following way:
(1) For each factor of the form $(x-\alpha)^k$ write

$$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_k}{(x-\alpha)^k},$$

where the numbers $A_1, A_2, \cdots, A_k$ are to be determined.
(2) For each factor of the form $(ax^2+bx+c)^k$ write

$$\frac{B_1x + C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx+C_k}{(ax^2+bx+c)^k},$$

where the numbers $B_1, B_2, \cdots, B_k$ and $C_1, C_2, \cdots, C_k$ are to be determined.

*Step 4.* Multiply both sides by $Q(x)$ and simplify. This leads to an expression of the form

$$r(x) = \text{a polynomial whose coefficients are combinations of } \mathrm{A_i}, \mathrm{B_i}, \text{ and } \mathrm{C_i}.$$

Finally, we find the constants, $A_i, B_i$, and $C_i$ by equating the coefficients of like powers of $x$ on both sides of the last equation.

**Example 43.1**
Decompose into partial fractions $R(x) = \frac{x^3+x^2+2}{x^2-1}$.

**Solution.**
Step 1. $\frac{x^3+x^2+2}{x^2-1} = x + 1 + \frac{x+3}{x^2-1}$.
Step 2. $x^2 - 1 = (x-1)(x+1)$.
Step 3. $\frac{x+3}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$.
Step 4. Multiply both sides of the last equation by $(x-1)(x+1)$ to obtain

$$x + 3 = A(x-1) + B(x+1).$$

Expand the right hand side, collect terms with the same power of $x$, and indentify coefficients of the polynomials obtained on both sides:

$$x + 3 = (A+B)x + (B-A).$$

Hence, $A + B = 1$ and $B - A = 3$. Adding these two equations gives $B = 2$. Thus, $A = -1$ and so

$$\frac{x^3+x^2+2}{x^2-1} = x + 1 - \frac{1}{x+1} + \frac{2}{x-1}. \blacksquare$$

Now, after decomposing the rational function into a sum of partial fractions all we need to do is to integrate expressions of the form $\frac{A}{(x-\alpha)^n}$ or $\frac{Bx+C}{(ax^2+bx+c)^n}$.

**Example 43.2**
Find $\int \frac{1}{x(x-3)}dx$.

**Solution.**
We write

$$\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}.$$

Multiply both sides by $x(x-3)$ and simplify to obtain

$$1 = A(x-3) + Bx$$

or

$$1 = (A+B)x - 3A.$$

Now equating the coefficients of like powers of $x$ to obtain $-3A = 1$ and $A + B = 0$. Solving for $A$ and $B$ we find $A = -\frac{1}{3}$ and $B = \frac{1}{3}$. Thus,

$$\begin{aligned}
\int \frac{1}{x(x-3)}dx &= -\frac{1}{3}\int \frac{dx}{x} + \frac{1}{3}\int \frac{dx}{x-3} \\
&= -\frac{1}{3}\ln|x| + \frac{1}{3}\ln|x-3| + C \\
&= \frac{1}{3}\ln\left|\frac{x-3}{x}\right| + C. \blacksquare
\end{aligned}$$

**Example 43.3**
Find $\int \frac{3x+6}{x^2+3x}dx$.

**Solution.**
We factor the denominator and split the integrand into partial fractions:

$$\frac{3x+6}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}.$$

Multiplying both sides by $x(x+3)$ to obtain

$$\begin{aligned}
3x+6 &= A(x+3) + Bx \\
&= (A+B)x + 3A
\end{aligned}$$

Equating the coefficients of like powers of $x$ to obtain $3A = 6$ and $A+B = 3$. Thus, $A = 2$ and $B = 1$. Finally,

$$\int \frac{3x+6}{x^2+3x}dx = 2\int \frac{dx}{x} + \int \frac{dx}{x+3} = 2\ln|x| + \ln|x+3| + C. \blacksquare$$

**Example 43.4**
Find $\int \frac{x^2+1}{x(x+1)^2}dx$.

**Solution.**
We factor the denominator and split the integrand into partial fractions:

$$\frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying both sides by $x(x+1)^2$ and simplifying to obtain

$$\begin{aligned} x^2+1 &= A(x+1)^2+Bx(x+1)+Cx \\ &= (A+B)x^2+(2A+B+C)x+A. \end{aligned}$$

Equating coefficients of like powers of $x$ we find $A = 1, 2A + B + C = 0$ and $A + B = 1$. Thus, $B = 0$ and $C = -2$. Now integrating to obtain

$$\int \frac{x^2+1}{x(x+1)^2}dx = \int \frac{dx}{x} - 2\int \frac{dx}{(x+1)^2} = \ln|x| + \frac{2}{x+1} + C. \blacksquare$$

**Example 43.5**
Find $\int \frac{2x^2-x-1}{(x^2+1)(x-2)}dx$.

**Solution.**
We first write

$$\frac{2x^2-x-1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}.$$

Multiply both sides by $(x^2+1)(x-2)$ and simplify

$$\begin{aligned} 2x^2-x-1 &= (Ax+B)(x-2)+C(x^2+1) \\ &= (A+C)x^2+(-2A+B)x-2B+C. \end{aligned}$$

Equating coefficients of like powers of $x$

$$A+C=2, \quad -2A+B=-1, \quad -2B+C=-1.$$

Solving for $A, B$, and $C$ we find $A = B = C = 1$. Thus

$$\begin{aligned} \int \frac{2x^2-x-1}{(x^2+1)(x-2)}dx &= \int \frac{x+1}{x^2+1}dx + \int \frac{dx}{x-2} \\ &= \int \frac{x}{x^2+1}dx + \int \frac{dx}{x^2+1} + \int \frac{dx}{x-2} \\ &= \frac{1}{2}\ln(x^2+1) + \arctan x + \ln|x-2| + C. \blacksquare \end{aligned}$$

**Trigonometric Substitutions**

This section deals with integrands involving terms like $\sqrt{x^2-a^2}, \sqrt{x^2+a^2}$, and $\sqrt{a^2-x^2}$.

- **Integrands involving** $\sqrt{a^2-x^2}, -a \le x \le a$.

For each $x$ in the interval $[-a, a]$ there is a $\theta$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $x = a\sin\theta$. Thus, using the substitution $x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ to obtain

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2(1-\sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta} = a\cos\theta\end{aligned}$$

where we have used the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. Note that $\cos\theta \geq 0$ since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is important to point out here that by constructing a right triangle with one of the angle being $\theta$ then the hypotenuse of the triangle has length $a$, the opposite side has length $x$ and the adjacent side has length $\sqrt{a^2 - x^2}$. It follows that $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$. See Figure 106.

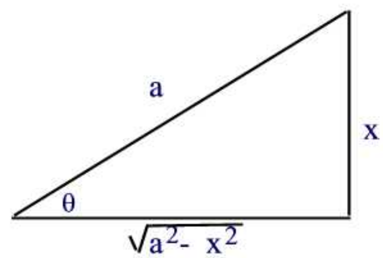



Figure 106

**Example 43.6**
Find $\int \frac{1}{\sqrt{4-x^2}}dx$.

**Solution.**
Let $x = 2\sin\theta, -\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$. Then

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta.$$

Moreover, $dx = 2\cos\theta d\theta$. It follows that

$$\int \frac{1}{\sqrt{4-x^2}}dx = \int \frac{2\cos\theta}{2\cos\theta}d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C. \blacksquare$$

- **Integrands involving $\sqrt{a^2 + x^2}$**

In this case, we let $x = a\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Such a substitution leads to

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = a\sec\theta$$

since $1 + \tan^2\theta = \sec^2\theta$ and $\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

**Remark 43.1**
Letting $\theta$ be the angle of a right triangle with opposite side $x$, adjacent side $a$, and hypotenuse $\sqrt{a^2+x^2}$ we find $\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$. See Figure 107.

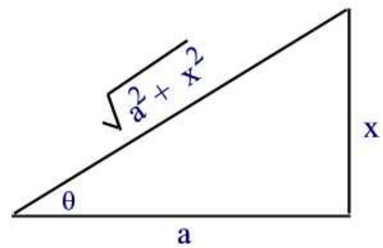



Figure 107

**Example 43.7**
Find $\int \frac{1}{\sqrt{x^2+9}}dx$.

**Solution.**
Let $x = 3\tan\theta$ with $-\frac{\pi}{2}\theta < \frac{\pi}{2}$. Then

$$\sqrt{x^2+9} = \sqrt{9+9\tan^2\theta} = \sqrt{9(1+\tan^2\theta)} = 3\sec\theta.$$

Moreover, $dx = 3\sec^2\theta d\theta$. Thus,

$$\int \frac{1}{\sqrt{x^2+9}}dx = \int \frac{3\sec^2\theta}{3\sec\theta}d\theta = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Now, considering a triangle with acute angle $\theta$, opposite side $x$, and adjacent side 3 we see that $\sec\theta = \frac{\sqrt{9+x^2}}{3}$ and $\tan\theta = \frac{x}{3}$. Thus,

$$\int \frac{1}{\sqrt{x^2+9}}dx = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C. \blacksquare$$

• **Integrands Involving** $\sqrt{x^2-a^2}, x \geq a$ *or* $x \leq -a$.
Here, we let $x = a\sec\theta$ with $0 \leq \theta < \pi, \theta \neq \frac{\pi}{2}$ so that

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\left|\tan\theta\right|.$$

**Remark 43.2**
Letting $\theta$ be the angle of a right triangle with opposite side $\sqrt{x^2-a^2}$, adjacent side $a$, and hypotenuse $x$ we find $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$. See Figure 108.

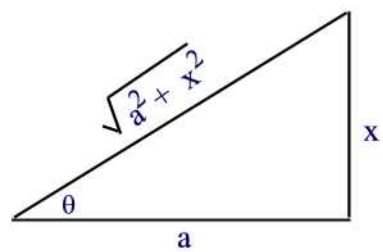



Figure 108

**Example 43.8**
Find $\int \frac{1}{x\sqrt{x^2-1}}dx$.

**Solution.**
Let $x = \sec\theta, 0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$. Then $dx = \sec\theta\tan\theta d\theta$ and

$$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta.$$

Thus,

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta}{\sec\theta|\tan\theta|}d\theta = \pm\int d\theta = \pm\theta + C = \left|sec^{-1}x\right| + C. \blacksquare$$

Completing the Square to Use Trigonometric Substitution

**Example 43.9**
Find $\int \frac{1}{\sqrt{x^2+6x+25}}dx$.

**Solution.**
Completing the square we find $x^2+6x+25 = (x+3)^2+16$. So let $x+3 = 4\tan\theta, -\frac{\pi}{2}\theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta d\theta$ and $\sqrt{x^2+6x+25} = \sqrt{16\sec^2\theta} = 4\sec\theta$. Thus,

$$\begin{array}{rcl}\int \frac{1}{\sqrt{x^2+6x+25}}dx &=& \int \frac{4\sec^2\theta}{4\sec\theta}d\theta = \int \sec\theta d\theta \\ &=& \ln|\sec\theta + \tan\theta| + C \\ &=& \ln|\frac{\sqrt{x^2+6x+25}}{4} + \frac{x+3}{4}| + C. \blacksquare\end{array}$$

We can summarize the above substitutions in the following table

| expression | substitution | identity |
|---|---|---|
| $\sqrt{a^2 - u^2}$ | $u = a\sin\theta$ | $1 - \sin^2\theta = \cos^2\theta$ |
| $\sqrt{a^2 + u^2}$ | $u = a\tan\theta$ | $1 + \tan^2\theta = \sec^2\theta$ |
| $\sqrt{u^2 - a^2}$ | $u = a\sec\theta$ | $\sec^2\theta - 1 = \tan^2\theta$ |

## Practice Problems

**Exercise 43.1**
*Find* $\int \frac{1}{x^2+4x+3}dx.$

**Exercise 43.2**
*Find* $\int \frac{1}{4-x^2}dx.$

**Exercise 43.3**
*Find* $\int \frac{1}{1+(x+2)^2}dx.$

**Exercise 43.4**
*Find* $\int \frac{1}{x^2+4x+5}dx.$

**Exercise 43.5**
*Find* $\int \frac{1}{x^2+4x+4}dx.$

**Exercise 43.6**
*Find* $\int \frac{1}{x^2-1}dx.$

**Exercise 43.7**
*Find* $\int \frac{1}{x^2-x}dx.$

**Exercise 43.8**
*Find* $\int \frac{3x+1}{x^2-3x+2}dx.$

**Exercise 43.9**
*Evaluate* $\int \frac{1}{3x-x^2}dx.$

**Exercise 43.10**
*Evaluate* $\int \frac{x^2+1}{x^2-3x+2}dx.$

**Exercise 43.11**
*Evaluate* $\int \frac{x^3}{x^2+3x+2}dx.$

**Exercise 43.12**
*Find* $\int \frac{1}{x^2+5x+4}dx.$

**Exercise 43.13**
*Find numbers* $A, B$ *and* $C$ *such that*

$$\frac{1}{(x^2+6x+14)(x-1)} = \frac{Ax+B}{x^2+6x+14} + \frac{C}{x-1}.$$

**Exercise 43.14**
*Write the fraction* $\frac{1}{e^{2x}-4e^x+3}$ *as the sum of two fractions.*

# 44 Numerical Approximation of Definite Integrals

Sometimes the integral of a function cannot be expressed with **elementary functions,** i.e. polynomial, trigonometric, exponential, logarithmic, or a suitable combination of these. For example, the functions $\sin(x^2)$ and $\frac{\sin x}{x}$ don't have simple antiderivatives, i.e. it is not possible to write down a simple analytic formula in terms of elementary fucntions. In cases like these we still can find an approximate value for the integral of such functions on an interval.

**Left- and Right-hand Riemann Sums**
We already know that we can approximate a definite integral $\int_a^b f(x)dx$ by a left- or right-hand Riemann sum for some finite n:

$$LEFT(f,n) = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

and

$$RIGHT(f,n) = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $x_0, x_1, \cdots, x_n$ are $n+1$ equally spaced points in $[a,b]$ with $x_0 = a, x_n = b$, and $\Delta x = \frac{b-a}{n}$.

**Example 44.1**
Approximate $\int_1^5 \frac{1}{x}dx$ using a left- and right-hand Riemann sum with 4 intervals, i.e. $n = 4$.

**Solution.**
We first construct the following chart.

| $x_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x_i)$ | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ |

Thus,

$$\begin{array}{rcl} LEFT(\frac{1}{x},4) &=& (f(1)+f(2)+f(3)+f(4))\Delta x \\ &=& (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})\cdot 1 \\ &=& \frac{25}{12} \approx 2.0833 \end{array}$$

and

$$\begin{aligned} RIGHT(\tfrac{1}{x}, 4) &= (f(2) + f(3) + f(4) + f(5))\Delta x \\ &= (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}) \cdot 1 \\ &= \tfrac{77}{60} \approx 1.2833. \end{aligned}$$

For comparison, we know that the exact value of the integral we are seeking to approximate is

$$\int_1^5 \frac{1}{x} dx = \ln 5 \approx 1.6094.$$

Thus, for $n = 4$ the left- and right-hand Riemann sums give poor approximations. Figures 109(a) and 109(b) illustrate why the left and right rules are so inaccurate. ■

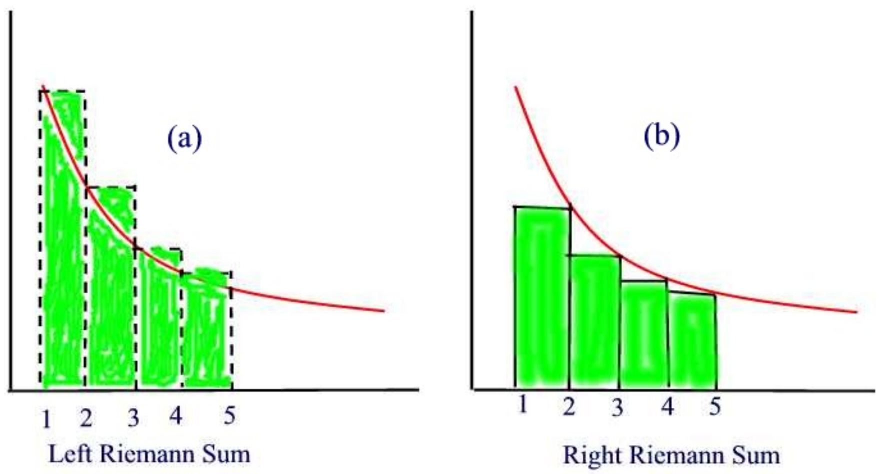



Figure 109

By drawing pictures of the geometric regions involved one can see easily that

$$LEFT(f,n) \le \int_a^b f(x)dx \le RIGHT(f,n), \quad \textit{when } f(x) \textit{ is increasing}$$

and

$$RIGHT(f,n) \le \int_a^b f(x)dx \le LEFT(f,n), \quad \textit{when } f(x) \textit{ is decreasing.}$$

So if $f(x)$ is increasing then $LEFT(f,n)$ is an underestimate while $RIGHT(f,n)$ is an overestimate. Similarly, when $f$ is decreasing $RIGHT(f,n)$ is an underestimate and $LEFT(f,n)$ is an overestimate.

**Midpoint Approximation**

Instead of using the left- or right-endpoints we use the midpoint. Recall that the midpoint of an interval $[x, y]$ is given by the midpoint formula $\frac{x+y}{2}$. The midpoint approximation is given by

$$MID(f, n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x.$$

**Example 44.2**

Approximate $\int_1^5 \frac{1}{x}dx$ with the midpoint rule using 4 intervals.

**Solution.**

Let $m_i$ denote the midpoint of the interval $[x_{i-1}, x_i]$. Then

| $m_i$ | 1.5 | 2.5 | 3.5 | 4.5 |
|---|---|---|---|---|
| $f(m_i)$ | $\frac{2}{3}$ | $\frac{2}{5}$ | $\frac{2}{7}$ | $\frac{2}{9}$ |

Thus,

$$\begin{array}{lcl} MID(\frac{1}{x}, 4) & = & (f(1.5) + f(2.5) + f(3.5) + f(4.5))\Delta x \\ & = & \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} \\ & \approx & 1.5746. \end{array}$$

This answer is fairly close to the exact answer. The midpoint rule approximates with rectangles on each subdivision that are partly above and partly below the graph, so the errors tend to balance out. See Figure 110.■

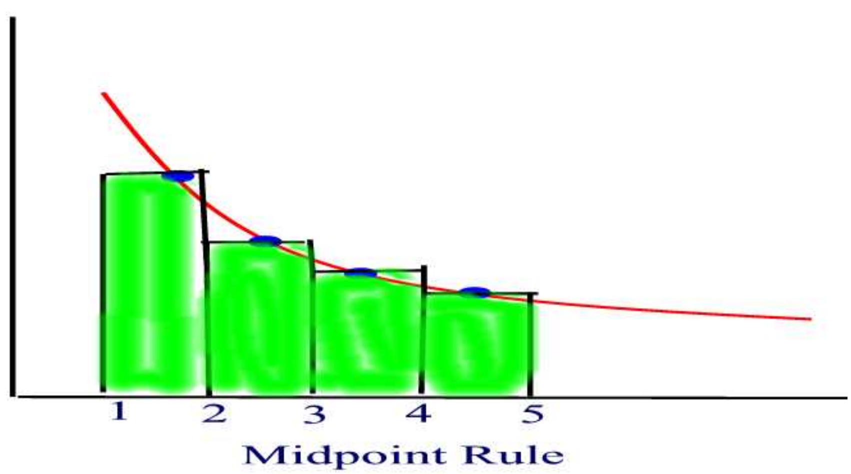


Figure 110

Note that if the graph of $f(x)$ is concave down on $[a, b]$ then $MID(f, n)$ is an overestimate of $\int_a^b f(x)dx$ since the midpoint rectangle and the trapezoid constracted by drawing the tangent line to the graph at the midpoint have the same area since the triangles ABC and CEF are equal. (See Figure 111)

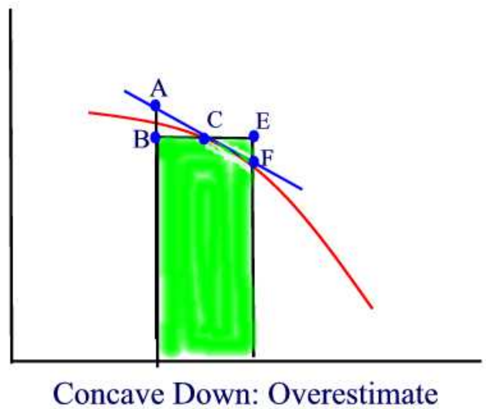



Figure 111

On the other hand, if $f(x)$ is concave up on $[a, b]$ then $MID(f, n)$ is an underestimate of $\int_a^b f(x)dx$. See Figure 112.

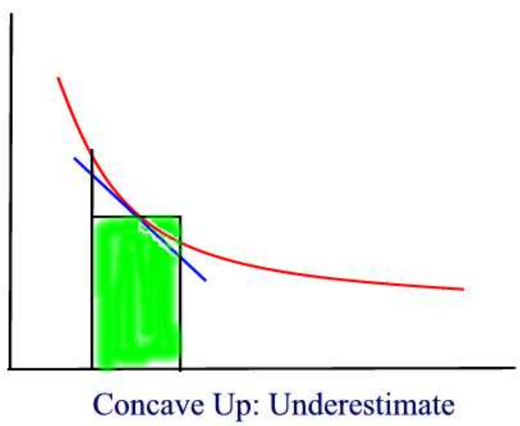



Figure 112

**Trapezoid Rule**

There is no reason why we should necessarily use rectangles to approximate $\int_a^b f(x)dx$. We could use the function values of both endpoints of the interval and approximate the interval by trapezoids instead. Recall that the area of a trapezoid with base $[x_{i-1}, x_i]$ and sides $f(x_{i-1})$ and $f(x_i)$ is given by

$$\frac{f(x_{i-1}) + f(x_i)}{2} \cdot \Delta x.$$

Thus, the trapezoid approximation is given by

$$TRAP(f, n) = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \cdot \Delta x = \frac{LEFT(f, n) + RIGHT(f, n)}{2}.$$

**Example 44.3**

Approximate $\int_1^5 \frac{1}{x}dx$ with the trapezoid rule using 4 intervals.

**Solution.**

Using the following chart

| $x_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $f(x_i)$ | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ |

we can write

$$\begin{array}{rcl} TRAP(\frac{1}{x}, 4) & = & (\frac{f(1)+f(2)}{2} + \frac{f(2)+f(3)}{2} + \frac{f(3)+f(4)}{2} + \frac{f(4)+f(5)}{2})\Delta x \\ & = & \frac{3}{4} + \frac{5}{12} + \frac{7}{24} + \frac{9}{40} \\ & \approx & 1.6833 \end{array}$$

This answer is fairly close to the exact answer.■

Note that if the graph of $f(x)$ is concave up then the area of each trapezoid is larger than the area under the graph so that $TRAP(f, n)$ is an overestimate of $\int_a^b f(x)dx$. See Figure 113(a). On the other hand, if the graph of $f(x)$ is concave down then the area fo each trapezoid is smaller than the area under the graph so that $TRAP(f, n)$ is an underestimate of the definite integral.

See Figure 113(b).

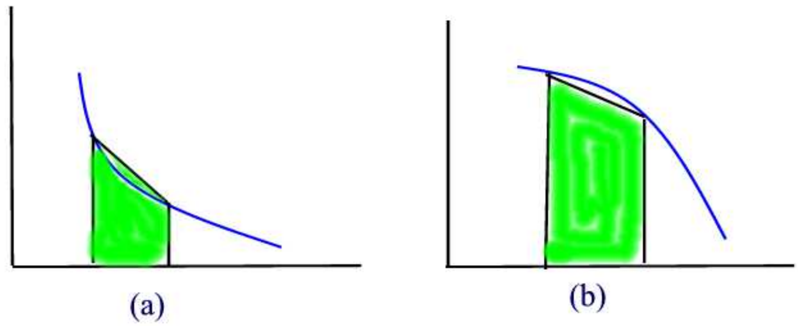



Figure 113

It follows from the above discussions that:

(i) If $f(x)$ is concave up and increasing on [a,b] then

$$LEFT(f,n) \leq MID(f,n) \leq \int_a^b f(x)dx \leq TRAP(f,n) \leq RIGHT(f,n).$$

(ii) If $f(x)$ is concave up and decreasing on [a,b] then

$$RIGHT(f,n) \leq MID(f,n) \leq \int_a^b f(x)dx \leq TRAP(f,n) \leq LEFT(f,n).$$

(iii) If $f(x)$ is concave down and increasing on [a,b] then

$$LEFT(f,n) \leq TRAP(f,n) \leq \int_a^b f(x)dx \leq MID(f,n) \leq RIGHT(f,n).$$

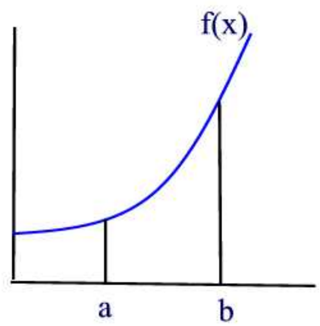
(iv) If $f(x)$ is concave down and decreasing on [a,b] then

$$RIGHT(f,n) \leq TRAP(f,n) \leq \int_a^b f(x)dx \leq MID(f,n) \leq LEFT(f,n).$$

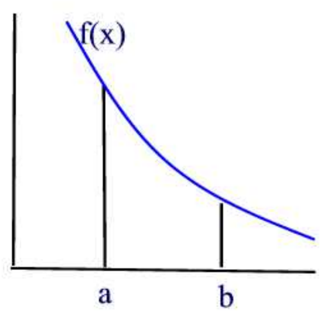
## Practice Problems

In Exercises 1 - 4 sketch the area given by the following approximations to $\int_a^b f(x)dx$. Identify each approximation as an overestimate or an underestimate.
(a) $LEFT(f,2)$ (b) $RIGHT(f,2)$ (c) $TRAP(f,2)$ (d) $MID(f,2)$

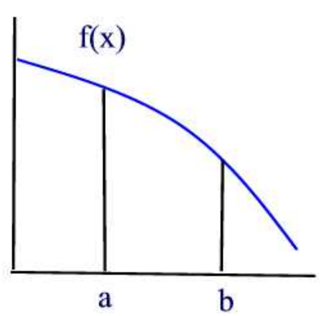
**Exercise 44.1**



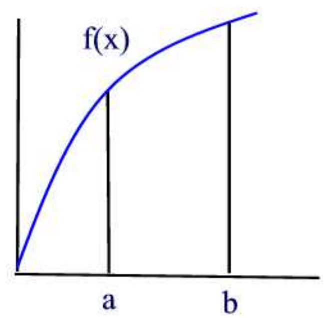

**Exercise 44.2**




**Exercise 44.3**



**Exercise 44.4**




**Exercise 44.5**
*(a) Estimate* $\int_0^1 \frac{1}{1+x^2}dx$ *by subdividing the interval* $[0,1]$ *into eight parts using:*
*(i) The left Riemann sum.*
*(ii) The right Riemann sum.*
*(iii) the trapeziod rule.*
*(b) Since the exact value of the integral is* $\frac{\pi}{4}$*, you can estimate the value of* $\pi$ *using part (a). Explain why your first estimate is too large and your second estimate is too small.*

**Exercise 44.6**
*(a) Find* $LEFT(x^2+1,2)$ *and* $RIGHT(x^2+1,2)$ *for* $\int_0^4 (x^2+1)dx$.
*(b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or an overestimate?*

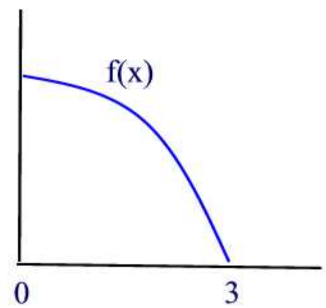
**Exercise 44.7**
*(a) Find* $MID(x^2+1,2)$ *and* $TRAP(x^2+1,2)$ *for* $\int_0^4 (x^2+1)dx$.
*(b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or an overestimate?*

**Exercise 44.8**
*Using the table, estimate the total distance traveled from time* $t=0$ *to time* $t=6$ *using LEFT, RIGHT, and TRAP.*

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $v$ | 3 | 4 | 5 | 4 | 7 | 8 | 11 |

**Exercise 44.9**
*Using the figure below, order the following approximations to the integral*

*$\int_0^3 f(x)dx$ and its exact value from smallest to largest: $LEFT(f,n), RIGHT(f,n), MID(f,n)$ and $TRAP(f,n)$.*




**Exercise 44.10**
*(a) Values for $f(x)$ are in the table. Which of the four approximation methods in this section is most likely to give the best estimate of $\int_0^{12} f(x)dx$? Estimate the integral using this method.*

| $x$ | 0 | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| $f(x)$ | 100 | 97 | 90 | 78 | 55 |

*(b) Assume $f(x)$ si continuous with no critical points or points of inflection on the interval $0 \leq x \leq 12$. Is the estimate found in part (a) an overestimate or an underestimate?Explain.*

**Exercise 44.11**
*Decide whether the following statements are true or false. Give an explanation for your answer. $f$ is assumed to be continuous on $[2,6]$.*
*(a) If $n = 10$, then the subdivision size is $\Delta x = \frac{1}{10}$.*
*(b) If we double the value of $n$, we make $\Delta x$ half as large.*
*(c)* $\mathrm{LEFT(f,10) \leq RIGHT(f,10)}$.
*(d) As $n$ approaches infinity, LEFT(f,n) approaches zero.*
*(e)* $\mathrm{LEFT(f,n) - RIGHT(f,n) = \frac{f(6)-f(2)}{n}}$.
*(f) Doubling $n$ decreases the difference $LEFT(f,n) - RIGHT(f,n)$ by exactly the factor $\frac{1}{2}$.*
*(g) If $LEFT(f,n) = RIGHT(f,n)$ for all $n$ then $f$ is a constant function.*

# 45 Simpson's Rule and Error Estimates

The trapezoid rule discussed in the previous section uses line segments to approximate the graph of the integrand, i.e. we approximate the area under the graph by using trapezoids. In this section, we will introduce an approximation technique, known as **simpson's rule**, that approximates the graph of the integrand by using parabolas instead (i.e. functions with equations $y = Ax^2 + Bx + C$.).

The process starts by dividing the interval $[a, b]$ into $n$ equal subintervals each of length $\Delta x = \frac{b-a}{n}$ using the partition points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. On the interval $[x_{i-1}, x_i]$ we want to approximate $f(x)$ by a quadratic function,i.e.

$$f(x) \approx Ax^2 + Bx + C$$

such that

$$\begin{array}{rcl} f(x_{i-1}) & = & Ax_{i-1}^2 + Bx_{i-1} + C \\ f(x_i) & = & Ax_i^2 + Bx_i + C \\ f(m_i) & = & Am_i^2 + Bm_i + C \end{array}$$

where $m_i$ is the midpoint of $[x_{i-1}, x_i]$. Thus,

$$\begin{array}{rcl} \int_{x_{i-1}}^{x_i} f(x)dx & \approx & \int_{x_{i-1}}^{x_i} (Ax^2 + Bx + C)dx \\ & = & \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_{x_{i-1}}^{x_i} \\ & = & \frac{A}{3}(x_i^3 - x_{i-1}^3) + \frac{B}{2}(x_i^2 - x_{i-1}^2) + C(x_i - x_{i-1}) \\ & = & \frac{A}{3}(x_i - x_{i-1})(x_i^2 + x_i x_{i-1} + x_{i-1}^2) + \frac{B}{2}(x_i - x_{i-1})(x_i + x_{i-1}) + C(x_i - x_{i-1}) \\ & = & \frac{\Delta x}{3}\left[A(x_i^2 + x_i x_{i-1} + x_{i-1}^2) + \frac{3}{2}B(x_i + x_{i-1}) + 3C\right]. \end{array}$$

But

$$\begin{array}{rcl} f(x_{i-1}) + 4f(m_i) + f(x_i) & = & Ax_{i-1}^2 + Bx_{i-1} + C + 4A\left(\frac{x_{i-1}+x_i}{2}\right)^2 + 4B\left(\frac{x_{i-1}+x_i}{2}\right) \\ & + & 4C + Ax_i^2 + Bx_i + C \\ & = & 2\left[A(x_i^2 + x_i x_{i-1} + x_{i-1}^2) + \frac{3}{2}B(x_{i-1} + x_i) + 3C\right]. \end{array}$$

It follows that

$$\int_{x_{i-1}}^{x_i} f(x)dx \approx \frac{\Delta x}{3}\left(\frac{f(x_{i-1})}{2} + 2f(m_i) + \frac{f(x_i)}{2}\right).$$

Hence,

$$\begin{array}{rcl} \int_a^b f(x)dx & \approx & \frac{1}{3}\sum_{i=1}^n \left(\frac{f(x_{i-1})+f(x_i)}{2}\Delta x + 2f(m_i)\Delta x\right) \\ & = & \frac{2\cdot MID(f,n) + TRAP(f,n)}{3}. \end{array}$$

We denote the expression on the right-hand side by $SIMP(f,n)$.

**Example 45.1**
Use Simpson's rule to approximate the value of $\pi$.

**Solution.**
First recall that $\int_0^1 \frac{1}{1+x^2}dx = \arctan x|_0^1 = \frac{\pi}{4}$. Let $n=2$ in Simpson's rule so that $\Delta x = \frac{1}{2}, x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, m_1 = \frac{1}{4}$, and $m_2 = \frac{3}{4}$. Thus,

$$\begin{array}{rcl} \int_0^1 \frac{1}{1+x^2}dx & \approx & SIMP(f,2) \\ & = & \frac{\Delta x}{3}\left(\frac{f(x_0)}{2} + 2f(m_1) + \frac{f(x_1)}{2} + \frac{f(x_1)}{2} + 2f(m_2) + \frac{f(x_2)}{2}\right) \\ & = & \frac{1}{6}\left(\frac{1}{2} + 2\cdot\frac{16}{17} + \frac{2}{5} + \frac{2}{5} + 2\cdot\frac{16}{25} + \frac{1}{4}\right) \\ & \approx & 0.7854. \end{array}$$

This produces the approximation $\pi = 4\int_0^1 \frac{1}{1+x^2}dx \approx 3.141568.$ ■

**Error Estimates**
We next discuss error estimates of the five numerical methods discussed so far. We define

$$Error = Exact\ Value - Approximate\ Value.$$

If the error is negative then the method produces an overestimate while if the error is positive the error produces an underestimate.
In general, one does not know the exact error, for otherwise one can find the exact value. Often, we work on finding an upper bound on the error and get an idea of how much work is involved in making the error smaller and thus obtaining better estimation.

**• Error in Left and Right Rules**
With a bit of work, it can be shown that

$$\left|\int_a^b f(x)dx - LEFT(f,n)\right| \leq \frac{M(b-a)^2}{2n}$$

and

$$\left|\int_a^b f(x)dx - RIGHT(f,n)\right| \leq \frac{M(b-a)^2}{2n}$$

where $M$ is the largest value of $|f'(x)|$ on the interval $[a,b]$. In words, the absolute value of the error in either the left-hand rule or the right-hand rule is bounded by a constant multiplied by $\frac{1}{n}$. Thus, doubling the number of intervals will decrease the error by a factor of $\frac{1}{2}$. Also, since $M = max\{|f'(x)| : a \leq x \leq b\}$ then the error depends on how steeply the graph of $f$ rises or falls. See Figure 114.

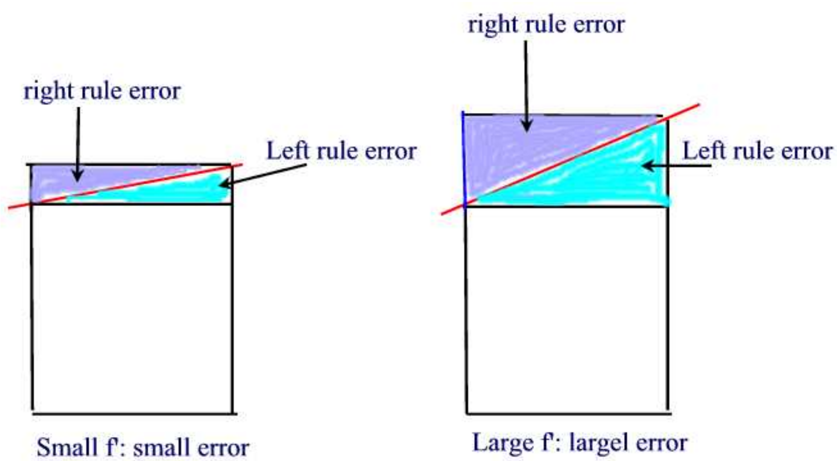



Figure 114

**Example 45.2**
Let $A = \int_1^{10} \frac{1}{x}dx \approx 2.302585$. Then by the definition of LEFT(f,n)and RIGHT(f,n) we have

$$\begin{array}{rcl} LEFT(f,n) & = & \sum_{i=0}^{n-1} f(x_i)\Delta x \\ & = & \sum_{i=0}^{n-1} \frac{1}{x_i}\frac{9}{n} \\ & = & \sum_{i=0}^{n-1} \frac{1}{1+\frac{9}{n}\cdot i} \cdot \frac{9}{n} \\ & = & \sum_{i=0}^{n-1} \frac{9}{n+9i}. \end{array}$$

Using a TI-83, one can find LEFT(f,n) for any $n$. For example, $LEFT(f, 160)$ can be found by typing

$$sum(seq(9/(160 + 9N), N, 0, 159, 1))$$

and then hit enter.
Similarly,

$$RIGHT(f,n) = \sum_{i=1}^{n} \frac{9}{n+9i}.$$

Thus, we have the following table

| $n$ | $RIGHT(\frac{1}{x},n)$ | $LEFT(\frac{1}{x},n)$ | $A-RIGHT(\frac{1}{x},n)$ | $A-LEFT(\frac{1}{x},n)$ |
|---|---|---|---|---|
| 10 | 1.960214 | 2.770214 | 0.342371 | −0.467629 |
| 20 | 2.116477 | 2.521477 | 0.186108 | −0.218892 |
| 40 | 2.205491 | 2.407991 | 0.097094 | −0.105406 |
| 80 | 2.253003 | 2.354253 | 0.049582 | −0.051668 |
| 160 | 2.277534 | 2.328159 | 0.025052 | −0.025574 |
| 320 | 2.289994 | 2.315307 | 0.012591 | −0.012722 |

In agreement with the result above, the errors in the right-hand rule or left-hand rule approximations decrease by a factor of, roughly, $\frac{1}{2}$ when we double the number of intervals. Also, note that the left errors are all negative since the function $\frac{1}{x}$ is decreasing so that LEFT is an overestimate. Similar remark for the right errors. ■

• **Error in the Trapezoid Rule**
It can be shown that the absolute value of the error in the trapezoid rule is bounded by

$$\left|\int_a^b f(x)dx - TRAP(f,n)\right| \le \frac{K(b-a)^3}{n^2}$$

where $K$ is the largest value of $|f''(x)|$ in the interval $[a,b]$. Thus, if we double the number of intervals then we should expect the error to decrease by a factor of $\frac{1}{4}$.

**Example 45.3**
Let $A = \int_1^{10} \frac{1}{x}dx$. Then by the definition of TRAP(f,n) we have

$$\begin{array}{rcl} TRAP(f,n) & = & \sum_{i=1}^n \frac{f(x_{i-1})+f(x_i)}{2}\cdot\Delta x \\ & = & \sum_{i=1}^n \frac{1}{2}\left(\frac{1}{x_{i-1}}+\frac{1}{x_i}\right)\cdot\frac{9}{n} \\ & = & \sum_{i=1}^n \frac{1}{2}\left(\frac{n}{n+9(i-1)}+\frac{n}{n+9i}\right)\cdot\frac{9}{n}. \end{array}$$

Using a TI-83 we have the table

| $n$ | $TRAP(\frac{1}{x},n)$ | $A-TRAP(\frac{1}{x},n)$ |
|---|---|---|
| 10 | 2.365214 | −0.062629 |
| 20 | 2.318977 | −0.016392 |
| 40 | 2.306741 | −0.004156 |
| 80 | 2.303628 | −0.001043 |
| 160 | 2.302846 | −0.000265 |
| 320 | 2.302650 | −0.000065 |

We see that the errors in the trapezoid rule approximations are significally smaller than the corresponding errors for the left-hand and right-hand rule approximations. Moreover, in agreement with the result above, the errors in the trapezoid rule approximations decrease by a factor of, roughly, $\frac{1}{4}$ when we double the number of intervals. Note that the errors are all negative due to the fact that the function $\frac{1}{x}$ is concave up so the trapezoid rule is an overestimate. Note also, that the trapezoid rule converges to the value of the definite integral at a significantly faster rate than do the left-hand rule or the right-hand rule.■

• **Error in Midpoint Rule**
An anlaysis of error similar to ones discussed before shows that

$$\left|\int_a^b f(x)dx - MID(f,n)\right| \leq C \cdot \frac{1}{n^2},$$

where $C$ depends on $|f''(x)|$. Hence, doubling the number of intervals will decrease the error by, roughly, a factor of $\frac{1}{4}$. Moreover, a more careful examination of the error would show that there is a sense in which it is typically on the order of $\frac{1}{2}$ the size of the error of the trapezoid rule.

**Example 45.4**
Let $A = \int_1^{10} \frac{1}{x}dx$. Then by the definition of MID(f,n) we have

$$\begin{array}{rcl} MID(f,n) & = & \sum_{i=1}^n f(m_i)\Delta x \\ & = & \sum_{i=1}^n \frac{1}{m_i} \cdot \frac{9}{n} \\ & = & \sum_{i=1}^n \frac{2}{x_{i-1}+x_i} \cdot \frac{9}{n} \\ & = & \sum_{i=1}^n \frac{18}{2n+18i-9}. \end{array}$$

Using a TI-83 we have the following table

| $n$ | $MID(\frac{1}{x},n)$ | $A - MID(\frac{1}{x},n)$ |
|---|---|---|
| 10 | 2.272740 | 0.029845 |
| 20 | 2.294504 | 0.008081 |
| 40 | 2.300515 | 0.002070 |
| 80 | 2.302064 | 0.000521 |
| 160 | 2.302455 | 0.000130 |
| 320 | 2.302552 | 0.000033 |

Notice that errors in the midpoint rule approximations decrease by a factor of, roughly, $\frac{1}{4}$ when we double the number of intervals. Note that the errors are all positive since the function $\frac{1}{x}$ is concave up so that the midpoint rule is an underestimate. Note also, that the error in each approximation is approximately $\frac{1}{2}$ of the corresponding error for the trapezoid rule.■

**Remark 45.1**
The errors in either the midpoint rule of the trapezoid rule not only depend on $n$ but also on the size of $f''$ and hence on the concavity of $f$. See Figure 115.

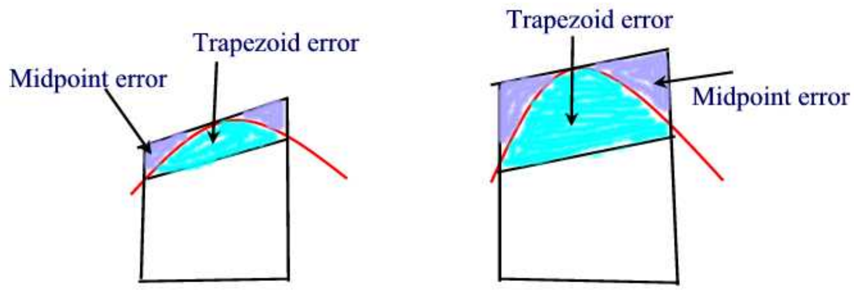



Figure 115

**• Error of Simpson's Rule**
It may be shown that the absolute value of the error using Simpson's rule is bounded by a constant multiple of $\frac{1}{n^4}$, resulting in a dramatic improvement over both the trapezoid and midpoint rules. For Simpson's rule, doubling the number of intervals decreases the error by, roughly $\frac{1}{16}$, a general fact that we can see some evidence in the following example. The constant depends on the size of the fourth derivative of $f$.

**Example 45.5**
Let $A = \int_1^{10} \frac{1}{x}dx$. Then by the definition of the Simpson's rule we have

$$SIMP(f,n) = \frac{2MID(f,n) + TRAP(f,n)}{3}.$$

Using Examples 45.3 and 45.4 we have the following table

| $n$ | $SIMP(\frac{1}{x},n)$ | $A - SIMP(\frac{1}{x},n)$ |
|---|---|---|
| 10 | 2.303565 | −0.000980 |
| 20 | 2.302662 | −0.000077 |
| 40 | 2.302590 | −0.000005 |
| 80 | 2.302585 | 0.000000 |
| 160 | 2.302455 | 0.000130 |
| 320 | 2.302552 | 0.000033■ |

## Practice Problems

**Exercise 45.1**
*(a) Compute SIMP(2) for* $\int_0^4 (x^2+1)dx$.
*(b) Use the Fundamental Theorem of Calculus to find* $\int_0^4 (x^2+1)dx$ *exactly.*
*(c) What is the error in* $SIMP(x^2+1,2)$ *for this integral?*

**Exercise 45.2**
*In this problem you will investigate the behavior of the errors in the approximation of the integral*

$$\int_1^2 \frac{1}{x}dx \approx 0.6931471806\cdots$$

*(a) For* $n = 2, 4, 8, 16, 32, 64, 128$ *intervals, find the left and right approximations and the absolute values of the errors in each. How do the errors change if* $n$ *is doubled?*
*(b) For the values of* $n$ *in part (a), compute the midpoint and trapezoid approximations and the absolute value of the errors in each. How do the errors change if* $n$ *is doubles?*
*(c) For* $n = 2, 4, 8, 16, 32$ *intervals, compute Simpson's error approximations and the absolute values of the errors in each. How do the errors change if* $n$ *is doubled?*

**Exercise 45.3**
*(a) What is the exact value of* $\int_0^2 (x^3+3x^2)dx$?
*(b) Find* $SIMP(x^3+3x^2,n)$ *for* $n = 2, 4, 100$. *What do you notice?*

**Exercise 45.4**
*(a) What is the exact value of* $\int_0^4 e^x dx$?
*(b) Find* $LEFT(e^x,2), RIGHT(e^x,2), TRAP(e^x,2), MID(e^x,2),$ *and* $SIMP(e^x,2)$. *Compute the error for each.*
*(c) Repeat part (b) with* $n = 4$.
*(d) For each rule in part (b), as* $n$ *goes from* $n = 2$ *to* $n = 4$, *does the error go down approximately as you would expect?Explain.*

# 46 Improper Integrals

A very common mistake among students is when evaluating the integral $\int_{-1}^{1}\frac{1}{x}dx$. A non careful student will just argue as follows

$$\int_{-1}^{1}\frac{1}{x}dx = [\ln|x|]_{-1}^{1} = 0.$$

Unfortunately, that's not the right answer as we will see below. The important fact ignored here is that the integrand is not continuous at $x = 0$.
Recall that the definite integral $\int_a^b f(x)dx$ was defined as the limit of a left- or right Riemann sum. We noted that the definite integral is always well-defined if:
(a) $f(x)$ is continuous on $[a, b]$,
(b) and if the domain of integration $[a, b]$ is finite.
**Improper integrals** are integrals in which one or both of these conditions are not met, i.e.,

(1) The interval of integration is infinite:

$$[a, \infty), (-\infty, b], (-\infty, \infty),$$

e.g.:

$$\int_1^{\infty}\frac{1}{x}dx.$$

(2) The integrand has an infinite discontinuity at some point $c$ in the interval $[a, b]$, i.e. the integrand is unbounded near $c$ :

$$\lim_{x\to c} f(x) = \pm\infty.$$

e.g.:

$$\int_0^1 \frac{1}{x}dx.$$

An improper integral may not be well defined or may have infinite value. In this case we say that the integral is **divergent**. In case an improper integral has a finite value then we say that it is **convergent.**
We will consider only improper integrals with positive integrands since they are the most common.

• **Unbounded Intervals of Integration**

The first type of improper integrals arises when the domain of integration is infinite. In case one of the limits of integration is infinite, we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

or

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$$

If both limits are infinite, then we choose any number $c$ in the domain of $f$ and define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$$

In this case, the integral is convergent if and only if both integrals on the right converge.

**Example 46.1**
Does the integral $\int_1^\infty \frac{1}{x^2}dx$ converge or diverge?

**Solution.**
We have

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}[-\frac{1}{x}]_1^b = \lim_{b\to\infty}(-\frac{1}{b}+1) = 1.$$

In terms of area, the given integral represents the area under the graph of $f(x) = \frac{1}{x^2}$ from $x = 1$ and extending infinitely to the right. The above improper integral says the following. Let $b > 1$ and obtain the area shown in Figure 116.

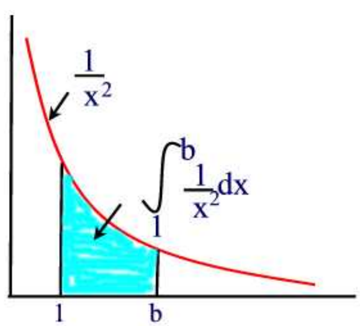



Figure 116

Then $\int_1^b \frac{1}{x^2}dx$ is the area under the graph of $f(x)$ from $x = 1$ to $x = b$. As $b$ gets larger and larger this area is close to 1.■

**Example 46.2**
Does the improper integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ converge or diverge?

**Solution.**
We have

$$\int_1^\infty \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty}[2\sqrt{x}]_1^b = \lim_{b\to\infty}(2\sqrt{b} - 2) = \infty.$$

So the improper integral is divergent.■

**Remark 46.1**
In general, some unbounded regions have finite areas while others have infinite areas. This is true whether a region extends to infinity along the x-axis or along the y-axis or both, and whether it extends to infinity on one or both sides of an axis. For example the area under the graph of $\frac{1}{x^2}$ is finite whereas that under the graph of $\frac{1}{\sqrt{x}}$ is infinite. This has to do with how fast each function approaches 0 as $x \to \infty$. The function $\frac{1}{x^2}$ approaches 0 more rapidly than that of $\frac{1}{\sqrt{x}}$.

The following example generalizes the results of the previous two examples.

**Example 46.3**
Determine for which values of $p$ the improper integral $\int_1^\infty \frac{1}{x^p}dx$ diverges.

**Solution.**
Suppose first that $p = 1$. Then

$$\begin{array}{rcl}\int_1^\infty \frac{1}{x}dx & = & \lim_{b\to\infty}\int_1^b \frac{1}{x}dx \\ & = & \lim_{b\to\infty}[\ln|x|]_1^b = \lim_{b\to\infty}\ln b = \infty\end{array}$$

so the improper integral is divergent.
Now, suppose that $p \neq 1$. Then

$$\begin{array}{rcl}\int_1^\infty \frac{1}{x^p}dx & = & \lim_{b\to\infty}\int_1^b x^{-p}dx \\ & = & \lim_{b\to\infty}[\frac{x^{-p+1}}{-p+1}]_1^b \\ & = & \lim_{b\to\infty}(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}).\end{array}$$

If $p < 1$ then $-p+1 > 0$ so that $\lim_{b\to\infty} b^{-p+1} = \infty$ and therefore the improper integral is divergent. If $p > 1$ then $-p+1 < 0$ so that $\lim_{b\to\infty} b^{-p+1} = 0$ and hence the improper integral converges:

$$\int_1^\infty \frac{1}{x^p}dx = \frac{-1}{-p+1}.\blacksquare$$

**Example 46.4**
For what values of $c$ is the improper integral $\int_0^\infty e^{cx}dx$ convergent?

**Solution.**
We have

$$\begin{array}{rcl}\int_0^\infty e^{cx}dx & = & \lim_{b\to\infty}\int_0^b e^{cx}dx = \lim_{b\to\infty}\frac{1}{c}e^{cx}|_0^b\\ & = & \lim_{b\to\infty}\frac{1}{c}(e^{cb}-1) = -1\end{array}$$

provided that $c < 0$. Otherwise, i.e. if $c \geq 0$, then the improper integral is divergent.$\blacksquare$

**Remark 46.2**
The previous two results are very useful and you may want to memorize them.

**Example 46.5**
Show that the improper integral $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$ converges.

**Solution.**
Splitting the integral into two as follows:

$$\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^\infty \frac{1}{1+x^2}dx.$$

Now,

$$\begin{array}{rcl}\int_{-\infty}^0 \frac{1}{1+x^2}dx & = & \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}dx = \lim_{a\to-\infty}\arctan x|_a^0\\ & = & \lim_{a\to-\infty}(\arctan 0 - \arctan a) = -(-\frac{\pi}{2}) = \frac{\pi}{2}.\end{array}$$

Similarly, we find that $\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}$ so that $\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi.\blacksquare$

**• Unbounded Integrands**
Suppose $f(x)$ is unbounded at $x = a$, that is $\lim_{x\to a^+} f(x)dx = \infty$. Then we define

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx.$$

Similarly, if $f(x)$ is unbounded at $x = b$, that is $\lim_{x\to b^-} f(x)dx = \infty$. Then we define

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx.$$

Now, if $f(x)$ is unbounded at an interior point $a < c < b$ then we define

$$\int_a^b f(x)dx = \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx.$$

If both limits exist then the integral on the right-hand side converges. If one of the limits does not exist then we say that the improper integral is divergent.

**Example 46.6**
Show that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges.

**Solution.**
Indeed,

$$\begin{array}{rcl}\int_0^1 \frac{1}{\sqrt{x}}dx & = & \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+} 2\sqrt{x}|_t^1 \\ & = & \lim_{t\to 0^+}(2-2\sqrt{t}) = 2.\end{array}$$

In terms of area, we pick an $a > 0$ as shown in Figure 117:

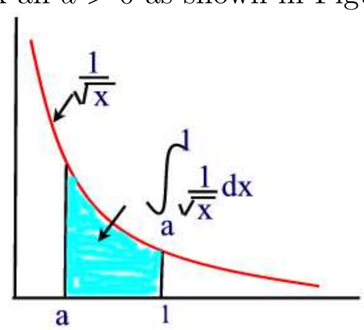



Figure 117

Then the shaded area is $\int_a^1 \frac{1}{\sqrt{x}}dx$. As $a$ approaches 0 from the right, the area approaches the value 2.■

**Example 46.7**
Investigate the convergence of $\int_0^2 \frac{1}{(x-2)^2}dx$.

**Solution.**
We deal with this improper integral as follows

$$\begin{array}{rcl}\int_0^2 \frac{1}{(x-2)^2}dx &=& \lim_{t\to 2^-}\int_0^t \frac{1}{(x-2)^2}dx = \lim_{t\to 2^-} -\frac{1}{(x-2)}\Big|_0^t \\ &=& \lim_{t\to 2^-}(-\frac{1}{t-2}-\frac{1}{2}) = \infty.\end{array}$$

So that the given improper integral is divergent.■

**Example 46.8**
Investigate the improper integral $\int_{-1}^1 \frac{1}{x}dx$.

**Solution.**
We first write

$$\int_{-1}^1 \frac{1}{x}dx = \int_{-1}^0 \frac{1}{x}dx + \int_0^1 \frac{1}{x}dx.$$

On one hand we have,

$$\begin{array}{rcl}\int_{-1}^0 \frac{1}{x}dx &=& \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x}dx = \lim_{t\to 0^-} \ln|x||_{-1}^t \\ &=& \lim_{t\to 0^-} \ln|t| = \infty.\end{array}$$

This shows that the improper integral $\int_{-1}^0 \frac{1}{x}dx$ is divergent and therefore the improper integral $\int_{-1}^1 \frac{1}{x}dx$ is divergent.■

**• Improper Integrals of Mixed Types**
Now, if the interval of integration is unbounded and the integrand is unbounded at one or more points inside the interval of integration we can evaluate the improper integral by decomposing it into a sum of improper integral with finite interval but where the integrand is unbounded and an improper integral with an infinite interval. If each component integrals converges, then we say that the original integral converges to the sum of the values of the component integrals. If one of the component integrals diverges, we say that the entire integral diverges.

**Example 46.9**
Investigate the convergence of $\int_0^\infty \frac{1}{x^2}dx$.

**Solution.**
Note that the interval of integration is infinite and the function is undefined at $x = 0$. So we write

$$\int_0^\infty \frac{1}{x^2}dx = \int_0^1 \frac{1}{x^2}dx + \int_1^\infty \frac{1}{x^2}dx.$$

But

$$\int_0^1 \frac{1}{x^2}dx = \lim_{t\to 0^-}\int_t^1 \frac{1}{x^2}dx = \lim_{t\to 0^-} -\frac{1}{x}|_t^1 = \lim_{t\to 0^-}(\frac{1}{t}-1) = \infty.$$

Thus, $\int_0^1 \frac{1}{x^2}dx$ diverges and consequently the improper integral $\int_0^\infty \frac{1}{x^2}dx$ diverges.■

## Practice Problems

**Exercise 46.1**
*Investigate the convergence of* $\int_0^\infty \frac{x}{e^x}dx$.

**Exercise 46.2**
*Investigate the convergence of* $\int_{-\infty}^0 \frac{e^x}{e^x+1}dx$.

**Exercise 46.3**
*Investigate the convergence of* $\int_{-\infty}^\infty \frac{1}{x^2+25}dx$.

**Exercise 46.4**
*Investigate the convergence of* $\int_0^4 \frac{1}{\sqrt{16-x^2}}dx$.

**Exercise 46.5**
*Investigate the convergence of* $\int_0^1 \frac{x^4+1}{x}dx$.

**Exercise 46.6**
*Investigate the convergence of* $\int_3^6 \frac{dx}{(4-x)^2}$.

**Exercise 46.7**
*Find the area under the curve* $y = xe^{-x}$ *for* $x \geq 0$.

**Exercise 46.8**
*Given that* $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$, *calculate the exact value of*

$$\int_{-\infty}^\infty e^{\frac{-(x-a)^2}{b}}dx.$$

**Exercise 46.9**
*Consider the improper integral*

$$\int_e^\infty x^p \ln x dx.$$

*For what values of p does the integral converge or diverge? What is the value of the integral when it converges?*

**Exercise 46.10**
*Consider the improper integral*

$$\int_0^e x^p \ln x dx.$$

*For what values of p does the integral converge or diverge? What is the value of the integral when it converges?*

**Exercise 46.11**

*The gamma function is defined for all $x > 0$ by the rule*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

*(a) Find $\Gamma(1)$ and $\Gamma(2)$.*
*(b) Integrate by parts with respect to $t$ to show that, for positive $n$,*

$$\Gamma(n+1) = n\Gamma(n).$$

*(c) Find a simple expression for $\Gamma(n)$ for positive integers.*

# 47 Comparison Tests for Improper Integrals

Sometimes it is difficult to find the exact value of an improper integral by antidifferentiation, for instance the integral $\int_0^\infty e^{-x^2}dx$. However, it is still possible to determine whether an improper integral converges or diverges. The idea is to compare the integral to one whose behavior we already know, such us

- the p-integral $\int_1^\infty \frac{1}{x^p}dx$ which converges for $p > 1$ and diverges otherwise;
- the integral $\int_0^\infty e^{cx}dx$ which converges for $c < 0$;
- the integral $\int_0^1 \frac{1}{x^p}dx$ which converges for $p < 1$ and diverges otherwise.

The comparison method consists of the following:

**Theorem 47.1**
Suppose that $f$ and $g$ are continuous and $0 \le g(x) \le f(x)$ for all $x \ge a$. Then

(a) if $\int_a^\infty f(x)dx$ is convergent, so is $\int_a^\infty g(x)dx$
(b) if $\int_a^\infty g(x)dx$ is divergent, so is $\int_a^\infty f(x)dx$.

This is only common sense: if the curve $y = g(x)$ lies below the curve $y = f(x)$, and the area of the region under the graph of $f(x)$ is finite, then of course so is the area of the region under the graph of $g(x)$. For a proof of this theorem see Exercise 47.11.
Similar results hold for the other types of improper integrals.

**Example 47.1**
Determine whether $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ converges.

**Solution.**
For $x \ge 1$ we have that $x^3 + 5 \ge x^3$ so that $\sqrt{x^3+5} \ge \sqrt{x^3}$. Thus, $\frac{1}{\sqrt{x^3+5}} \le \frac{1}{\sqrt{x^3}}$. Letting $f(x) = \frac{1}{\sqrt{x^3}}$ and $g(x) = \frac{1}{\sqrt{x^3+5}}$ then we have that $0 \le g(x) \le f(x)$. From the previous section we know that $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}dx$ is convergent, a p-integral with $p = \frac{3}{2} > 1$. By the comparison test, $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ is convergent.■

The next question is to estimate such a convergent improper integral.

**Example 47.2**
Estimate the value of $\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx$ with an error of less than 0.01.

**Solution.**
We want to find $b$ such that

$$\left|\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx - \int_1^b \frac{1}{\sqrt{x^3+5}}dx\right| < 0.01.$$

But

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx = \int_1^b \frac{1}{\sqrt{x^3+5}}dx + \int_b^\infty \frac{1}{\sqrt{x^3+5}}dx.$$

Thus, the problem is to find $b$ such that

$$\left|\int_b^\infty \frac{1}{\sqrt{x^3+5}}dx\right| < 0.01.$$

From the example above, we have

$$\int_b^\infty \frac{1}{\sqrt{x^3+5}}dx < \int_b^\infty \frac{1}{\sqrt{x^3}}dx = \frac{2}{\sqrt{b}}.$$

So it suffices to choose $b$ such that $\frac{2}{\sqrt{b}} < 0.01$ or $b > 40,000$, say for example $b = 45000$. In this case,

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}dx \approx \int_1^{45,000} \frac{1}{\sqrt{x^3+5}}dx = 1.69824. \blacksquare$$

**Example 47.3**
Investigate the convergence of $\int_4^\infty \frac{dx}{\ln x - 1}$.

**Solution.**
For $x \geq 4$ we know that $\ln x - 1 < \ln x < x$. Thus, $\frac{1}{\ln x - 1} > \frac{1}{x}$. Let $g(x) = \frac{1}{x}$ and $f(x) = \frac{1}{\ln x - 1}$. Thus, $0 \leq g(x) \leq f(x)$. Since $\int_4^\infty \frac{1}{x}dx = \int_1^\infty \frac{1}{x}dx - \int_1^4 \frac{1}{x}dx$ which is divergent since $\int_1^\infty \frac{1}{x}dx$ is divergent being a p-integral with $p = 1$. By the comparison test $\int_4^\infty \frac{dx}{\ln x - 1}$ is divergent. $\blacksquare$

**Example 47.4**
Investigate the convergence of the improper integral $\int_1^\infty \frac{\sin x + 3}{\sqrt{x}}dx$.

**Solution.**
We know that $-1 \leq \sin x \leq 1$. Thus $2 \leq \sin x + 3 \leq 4$. Since $x \geq 1$ then $\frac{2}{\sqrt{x}} \leq \frac{\sin x+3}{\sqrt{x}} \leq \frac{4}{\sqrt{x}}$. Note that the two integrals $\int_1^\infty \frac{2}{\sqrt{x}}dx$ and $\int_1^\infty \frac{4}{\sqrt{x}}dx$ are both divergent being a multiple of a p-integral with $p = \frac{1}{2} < 1$. If we let $g(x) = \frac{\sin x+3}{\sqrt{x}}$ and $f(x) = \frac{4}{\sqrt{x}}$ then we have no conclusion since $\int_1^\infty g(x)dx$ may or may not converge and still $\int_1^\infty g(x)dx \leq \int_1^\infty f(x)dx$. Now if we let $g(x) = \frac{2}{\sqrt{x}}$ and $f(x) = \frac{\sin x+3}{\sqrt{x}}$ then by the comparison test $\int_1^\infty \frac{\sin x+3}{\sqrt{x}}$ is divergent since $\int_1^\infty f(x)dx \geq \int_1^\infty g(x)dx$ and $\int_1^\infty g(x)dx$ is divergent.■

**Example 47.5**
Investigate the convergence of $\int_1^\infty e^{-\frac{1}{2}x^2}dx$.

**Solution.**
If $x \geq 2$ then $\frac{x}{2} \geq 1$. Multiply both sides of this inequality by $x \geq 2$ to obtain $\frac{1}{2}x^2 \geq x$. Now, multiply both sides of this last inequality by $-1$ to obtain $-\frac{1}{2}x^2 \leq -x$ and therefore $e^{-\frac{1}{2}x^2} \leq e^{-x}$ since the function $e^x$ is an increasing function. Thus,

$$\int_1^\infty e^{-\frac{1}{2}x^2}dx = \int_1^2 e^{-\frac{1}{2}x^2}dx + \int_2^\infty e^{-\frac{1}{2}x^2}dx.$$

But

$$\int_1^2 e^{-\frac{1}{2}x^2}dx \approx 0.34$$

and

$$\int_2^\infty e^{-\frac{1}{2}x^2}dx \leq \int_2^\infty e^{-x}dx \leq \int_0^\infty e^{-x}dx$$

so since $\int_0^\infty e^{-x}dx$ is convergent then $\int_2^\infty e^{-\frac{1}{2}x^2}dx$ is convergent. In conclusion, $\int_1^\infty e^{-\frac{1}{2}x^2}dx$ is convergent.■

Sometimes it is laborious to find a convenient function $f(x)$ with $g(x) \leq f(x)$, but we may know that $g(x)$ is no larger than a constant multiple of $f(x)$ for large enough x, and this is good enough. The most powerful test of this form in the course is this version of the **limit comparison test:**

**Theorem 47.2**
Let $f(x)$ and $g(x)$ be two positive and continuous functions on $[a, \infty)$.

(a)$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, or
(b)$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where $L$ is a finite positive number, or
(c)$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then

(a) If $\int_a^\infty g(x)dx$ converges, then so does $\int_a^\infty f(x)dx$.
(b) $\int_a^\infty g(x)dx$ converges if and only if $\int_a^\infty f(x)dx$ does.
(c) If $\int_a^\infty g(x)dx$ diverges, then so does $\int_a^\infty f(x)dx$.

**Proof.**
(a) Suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. Let $\epsilon > 0$ be given. Then there is a $b > a$ such that $\frac{f(x)}{g(x)} < \epsilon$ for all $x \geq b$. Thus, $f(x) < \epsilon g(x)$ for all $x \geq b$. By the comparison test, if $\int_a^\infty g(x)dx$ is convergent so does $\int_a^\infty f(x)dx$.
(b) Now, suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where $L$ is a finite positive constant. Let $\epsilon < L$. Then there is a constant $b > a$ such that for all $x \geq b$ we have

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon.$$

That is,

$$L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon.$$

Thus, for $x \geq b$ we have $(L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$. Now the result follows from the comparison test.
(c) Finally, suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. Then given there is $b > a$ such that $\frac{f(x)}{g(x)} \geq 1$ for all $x \geq b$. That is, $g(x) \leq f(x)$ for all $x \geq b$. Therefore, if $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.$\blacksquare$

**Remark 47.1**
The Comparison Test and Limit Comparison Test also apply, modified as appropriate, to other types of improper integrals.

**Example 47.6**
Show that the improper integral $\int_1^\infty \frac{1}{1+x^2}dx$ is convergent.

**Solution.**
Since the integral $\int_1^\infty \frac{dx}{x^2}$ is convergent (p-integral with $p > 1$) and since $\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2+1} = 1$ then by the limit comparison test we have that $\int_1^\infty \frac{dx}{x^2+1}$ is also convergent.$\blacksquare$

## Practice Problems.

**Exercise 47.1**
*Investigate the convergence of* $\int_1^\infty \frac{x^2}{x^4+1}dx$.

**Exercise 47.2**
*Investigate the convergence of* $\int_1^\infty \frac{x^2+1}{x^3+3x+2}dx$.

**Exercise 47.3**
*Investigate the convergence of* $\int_1^\infty \frac{1}{e^{5x}+2}dx$.

**Exercise 47.4**
*Investigate the convergence of* $\int_1^\infty \frac{dx}{x^2+x}$.

**Exercise 47.5**
*Investigate the convergence of* $\int_0^1 \frac{dx}{\sqrt{x^3+x}}$.

**Exercise 47.6**
*Investigate the convergence of* $\int_1^\infty \frac{2+\cos x}{x^2}dx$.

**Exercise 47.7**
*For what values of* $p$ *does the integral* $\int_2^\infty \frac{dx}{x(\ln x)^p}$ *converge or diverge?*

**Exercise 47.8**
*For what values of* $p$ *does the integral* $\int_1^2 \frac{dx}{x(\ln x)^p}$ *converge or diverge?*

**Exercise 47.9**
*Find the value of* $a$ *(to three decimal places) that makes*

$$\int_{-\infty}^{\infty} ae^{-\frac{x^2}{2}}dx = 1.$$

**Exercise 47.10**
*In Planck's Radiation Law, we encounter the integral*

$$\int_1^\infty \frac{dx}{x^5(e^{\frac{1}{x}}-1)}.$$

*(a) Graph the functions* $y = x+1$ *and* $y = e^x$. *Conclude from the graph that* $1+x \le e^x$ *for all* $x$.
*(b) Replacing* $x$ *by* $\frac{1}{x}$ *in (a), show that for all* $x$

$$e^{\frac{1}{x}} - 1 > \frac{1}{x}.$$

*(c) Use the comparison test to show that the original integral converges.*

**Exercise 47.11**
*Let $f(x) \geq 0$ for all $x \geq a$.*
*(a) Show that the sequence $a_n = \int_a^n f(x)dx$ is increasing.*
*(b) Suppose that $\int_a^b f(x)dx \leq M$ for all $b \geq a$, where $M > 0$. Show that $\int_a^\infty f(x)dx$ is convergent. Hint: You need to use the following result from Section 56: If $a_n$ is an increasing sequence of positive numbers such that $a_n \leq M$ for all $n \geq 1$ then $\lim_{a_n}$ exists.*
*(c) Proof the comparison test of improper integrals.*

# 48 Areas and Volumes

In this section we illustrate how the definite integral can be used to compute the the area of a region and the volume of certain solids. For example, the approach to finding the area of a region will be to think of the region as approximated by small elements, each of which is so geometrically simple that its area can be calculated directly, for example the slices might be rectangles, circles or triangles. Next, the areas of each of these elements are added to obtain a Riemann sum. The limit of such Riemann sum gives the desired area. The same idea applies in finding the volume of a solid. We refer to this process as the **method of slicing**.

**Finding Areas by The Method of Slicing**
When calculating the area of a region using Riemann sums, slice the region into thin pieces in which the geometry is so simple that the area can be estimated.

**Example 48.1**
Use horizontal slices to find the area of an isosceless triangle with vertices at $(0,0),(5,5)$ and $(10,0)$.

**Solution.**
We slice the triangle into $n$ horizontal slices, each slice being approximately a rectangle of width $\Delta y = y_i - y_{i-1}$ and length $w_i$. See Figure 118.

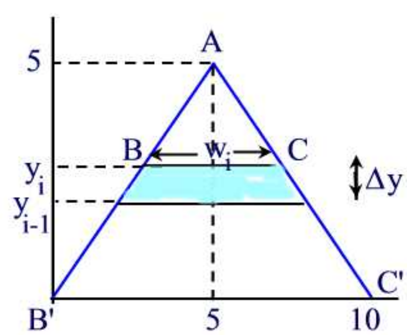



Figure 118

To find $w_i$, note that the triangles ABC and AB'C' are similar triangles so that

$$\frac{w_i}{10} = \frac{5 - y_i}{5}.$$

Solving for $w_i$, we find $w_i = 10 - 2y_i$. Thus, the area of each piece is approximately $(10 - 2y_i)\Delta y$ so that

$$Total\ Area \approx \sum_{i=1}^{n}(10 - 2y_i)\Delta y.$$

Letting $n \to \infty$ we get

$$Total\ Area = \int_0^5 (10 - 2y)dy = 10y - y^2\Big|_0^5 = 25\ square\ units.\blacksquare$$

**Example 48.2**
Use horizontal slices to set up a definite integral representing the area of a semicircle of radius 7cm.

**Solution.**
For simplicity, we assume that the circle is centered at $(0, 0)$. We slice the semicircle into $n$ thin slices of width $\Delta y = y_i - y_{i-1}$ and length $2\sqrt{49 - y_i^2}$. See Figure 119.

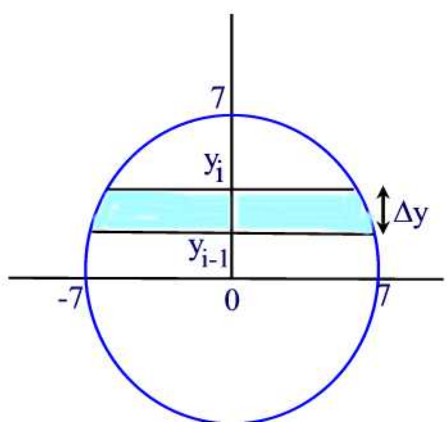



Figure 119

Thus, the area of a slice is approximately $2\sqrt{49 - y_i^2}\Delta y$ so that

$$Total\ Area \approx \sum_{i=1}^{n} 2\sqrt{49 - y_i^2}\Delta y.$$

Taking $n \to \infty$ we obtain

$$\begin{aligned} Total\ Area &= \int_0^7 2\sqrt{49-y^2}dy \\ &= 2 \cdot \tfrac{1}{2}\left[y\sqrt{49-y^2} + 40\arcsin\left(\tfrac{y}{7}\right)\right]_0^7 \\ &= \tfrac{49}{2}\pi\ cm^2 \ \blacksquare \end{aligned}$$

**• Finding Volumes by Slicing**

When calculating the volume of a solid using Riemann sums, slice the solid into thin pieces in which the geometry is so simple that the volume can be estimated.

**Example 48.3**

Use vertical slicing to find the volume of a cone with height 5 cm and radius of base 5 cm.

**Solution.**

We consider a cone centered at the origin and with axis the x-axis.

We divide the cone into $n$ thin disks each of thickness $\Delta x = x_i - x_{i-1}$ and radius $y_i = x_i$. See Figure 120.

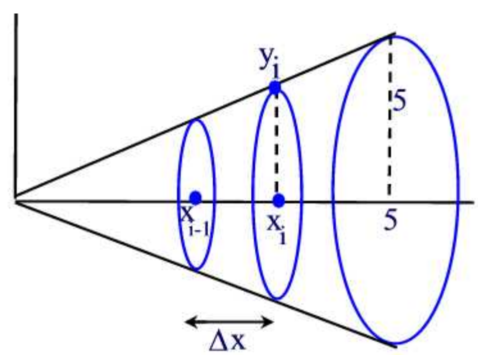



Figure 120

Thus, the volume of a slice is approximately $\pi x_i^2 \Delta x$ so that

$$Total\ Volume \approx \sum_{i=1}^{n} \pi x_i^2 \Delta x.$$

Letting $n \to \infty$ to obtain

$$Total\ Volume = \int_0^5 \pi x^2 dx = \pi \frac{x^3}{3}\Big|_0^5 = \frac{125}{3}\pi\ cm^3. \blacksquare$$

**Example 48.4**
The Great Pyramid of Egypt has a square base with side 755 feet long and height 410 feet. Find the volume of the Great Pyramid in cubic feet.

**Solution.**
The pyramid may be thought of as being made up of layers parallel to the base. Each layer is a thin rectangular box with square base and with thickness $\Delta z$. Figure 121 illustrates a slice of the pyramid.

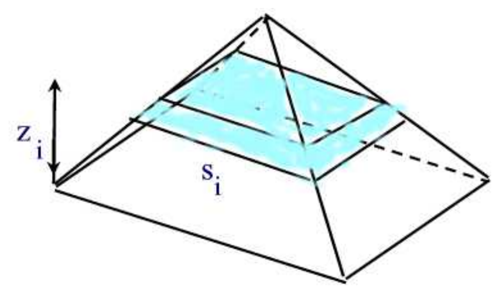



Figure 121

The following figure illustrates a triangular cross-section of a typical layer with a plane perpendicular at the center of the layer.

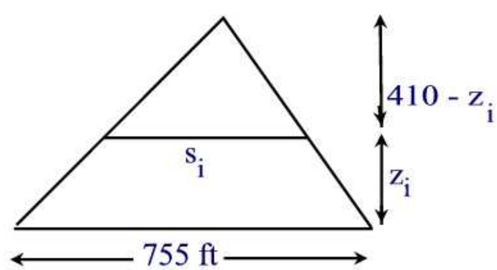



Figure 122

Let $s_i$ denote the length of the base of a typical layer, then the similar triangles shown above imply that

$$\frac{s_i}{755} = \frac{410 - z_i}{410},$$

where $z_i$ denotes the height above the horizontal that the center of the layer lies. Solving for $s_i$ one sees that the length of the rectangular box is given by the formula

$$s_i = 755 - \frac{755}{410}z_i.$$

The total volume is approximated by adding the volumes of the n layers

$$V \approx \sum_{i=1}^{n}[755 - \frac{755}{410}z_i]^2\Delta z.$$

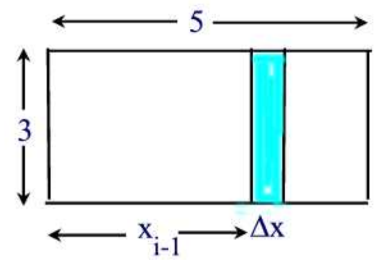
Letting $n \to \infty$ we obtain

$$\begin{array}{rcl} V & = & \int_0^{410}[755 - \frac{755}{410}z]^2dz = (\frac{755}{410})^2\int_0^{410}(410 - z)^2dz \\ & = & (\frac{755}{410})^2[-\frac{(410-z)^3}{3}]_0^{410} \\ & = & \frac{410}{3}(\frac{755}{410})^2 \approx 78 \ million \ ft^3 \ \blacksquare \end{array}$$

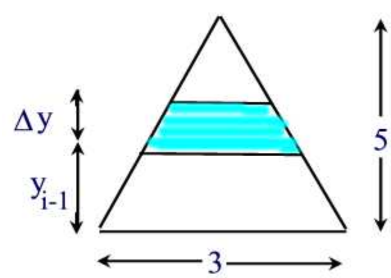
## Practice problems

In Exercises 1 - 5, write a Riemann sum and then use a definite integral representing the area of the region, using the strip shown. Evaluate the integral exactly.

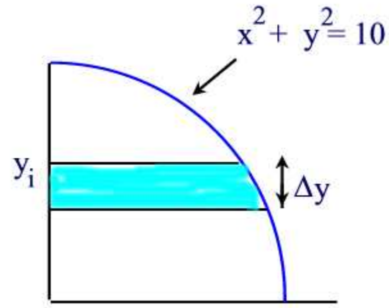
**Exercise 48.1**



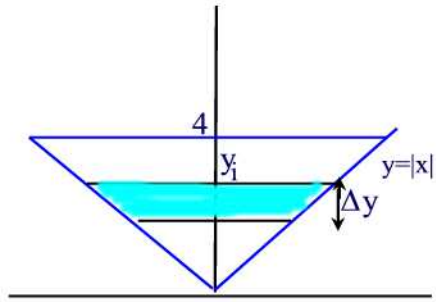

**Exercise 48.2**



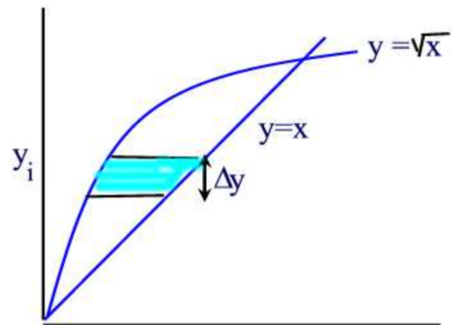

**Exercise 48.3**




**Exercise 48.4**

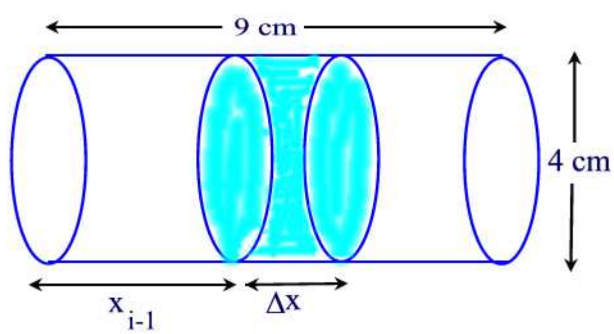



**Exercise 48.5**

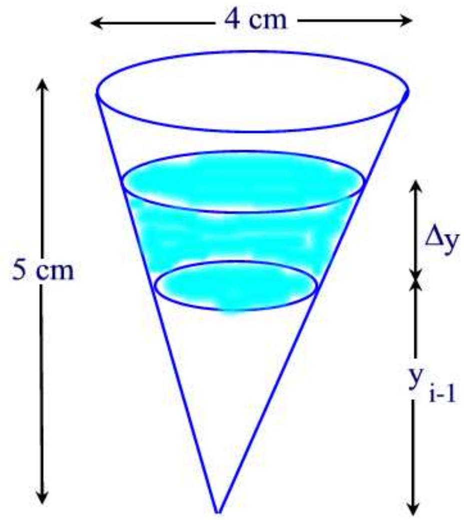



In Exercises 6 - 9, write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown. Evaluate the integral exactly.

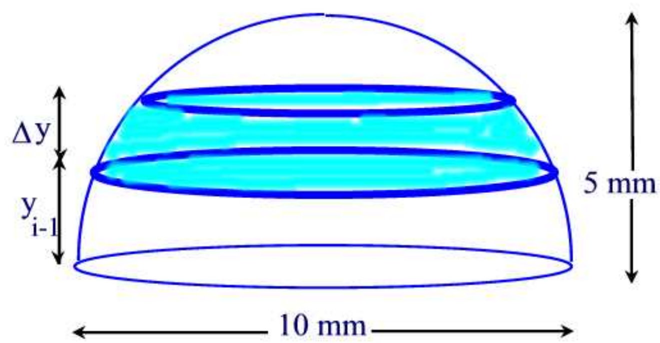
**Exercise 48.6**



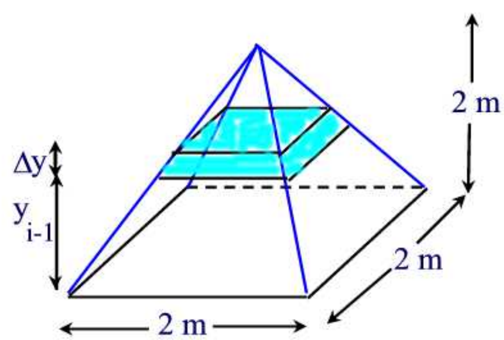

**Exercise 48.7**




**Exercise 48.8**




**Exercise 48.9**




**Exercise 48.10**

*Find the volume of a sphere of radius $r$ by slicing.*

**Exercise 48.11**

*A rectangular lake is 150 km long and 3 km wide. The vertical cross-section through the lake in Figure 123 shows that the lake is 0.2 km deep at the center. Set up and evaluate a definite integral giving the total volume of the lake.*

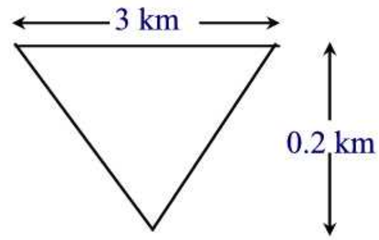



*Figure 123*

# 49 Solids of Revolution- Arc Length

In this section we will discuss the use of definite integrals in solving problems in geometry such as finding the

• volume of a solid of known cross-section,
• volume of a solid of revolution,
• arc length of a curve in the plane.

Recall the process of slicing which consists of the following steps:

◦ Divide the solid, region, or curve into small pieces whose volume, area, or length can be easily approximated.

◦ Add the volumes, areas, or lengths of all the pieces. (Thus obtaining a Riemann sum) that approximates the total volume, total area, or total length.

◦ Take the limit of the Riemann sum from the previous step as $n \to \infty$. This gives us a definite integral that gives the total volume, total area, or total length.

• **Volume of a Solid with Known Cross Section**
When we take a plane perpendicualr to a given solid then the common region between the plane and the solid is knwon as a **cross section.** By known cross sections we mean cross sections such as circles, squares, rectangles, or rings
Let $R$ be a solid lying alongside some interval $[a, b]$ of the x-axis. See Figure 124.

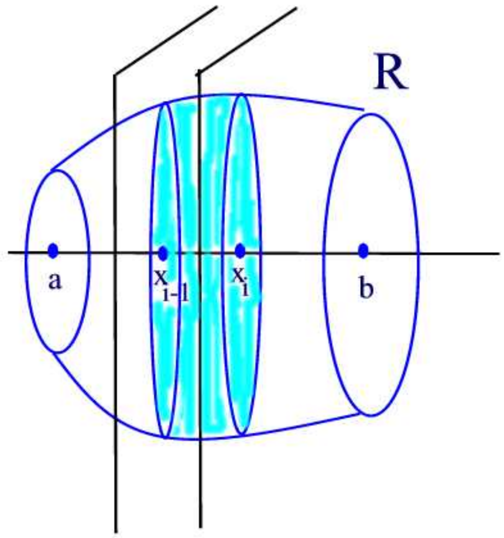



Figure 124

Divide the interval into $n$ equal subintervals with mesh points $x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. The planes that are perpendicular to the x-axis at the points $x_0, x_1, x_2 \cdots, x_n$ divide the solid into $n$ slices. Since the cross section of $R$ changes little along a subinterval $[x_{i-1}, x_i]$, the slab positioned alongside that subinterval can be considered a cylinder of height $\Delta x$ and whose base has area $A(x_i)$. So the volume of the slice is

$$\Delta V_i \approx A(x_i)\Delta x.$$

The approximate total volume of the solid is

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i)\Delta x.$$

Once again we recognize a Riemann sum at the right. Letting $n \to \infty$ we obtain the so-called *Cavalieri's principle:*

$$V = \int_a^b A(x)dx.$$

Of course, the formula can be applied to any axis. For instance if a solid lies alongside some interval $[a, b]$ on the y-axis, the formula becomes

$$V = \int_a^b A(y)dy.$$

**Example 49.1**
Find the volume of a cone of radius $r$ and height $h$.

**Solution.**
Assume that the cone is placed with its vertex in the origin and its axis on the x-axis as shown in Figure 125.

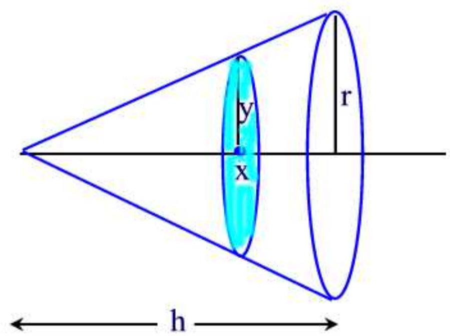



Figure 125

The cross section of the cone at each point $x$ is a circular disk of radius $y$. Using similar triangles, we find $y = \frac{xr}{h}$. Hence its area is $A(x) = \pi(\frac{xr}{h})^2$. The volume of the cone can now be computed by Cavalieri's formula:

$$V = \int_0^h \frac{\pi r^2}{h^2} x^2 dx = \frac{\pi r^2}{h^2} [\frac{x^3}{3}]_0^h = \frac{1}{3}\pi r^2 h.\blacksquare$$

**Example 49.2**
There is a solid whose bottom face is the disk $x^2 + y^2 \leq 1$ and every cross-section of the solid perpendicular to x-axis is a square. Find the volume of the solid.

**Solution.**
We view the solid a cardboard model shown in Figure 126.

Figure 126

A typical cross-section is a square of length side $s$ as shown in Figure 127.

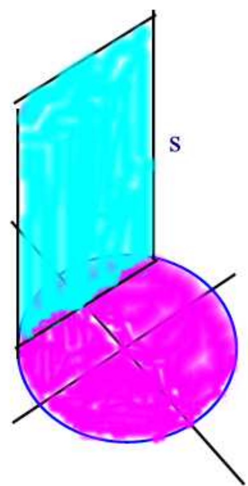



Figure 127

The length $s$ is given by the expression $s = 2\sqrt{1-y^2}$. Thus, the area of a cross section is $A(y) = s^2 = 4(1-y)^2$. By Cavalieri's formula the volume is

$$V = \int_{-1}^{1} 4(1-y^2)dy = 4\ y - \frac{1}{3}y^3\Big|_{-1}^{1} = \frac{4}{3}.\blacksquare$$

• **Volume of Solids of Revolution**

By a **solid of revolution** we mean a solid obtained by revolving a region around a line. Consider the solid of revolution obtained by revolving a plane region under the graph of $f(x)$ around the x-axis. See Figure 128.

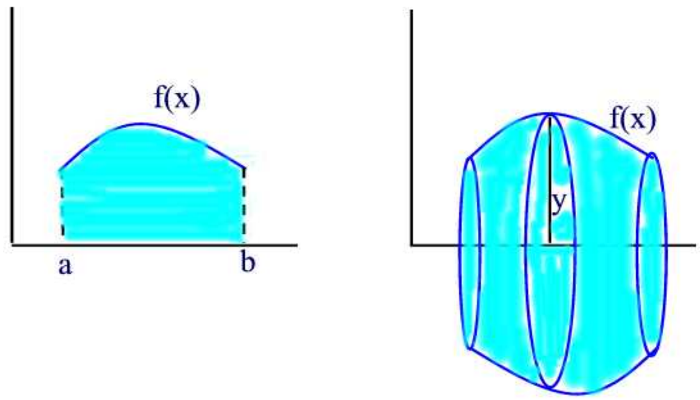



Figure 128

Each cross section is a circular disk of radius $y$, so its area is $A(x) = \pi y^2 = \pi[f(x)]^2$. Hence, by Cavalieri's principle, the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2 dx.$$

**Example 49.3**
The region bounded by the curve $y = \sqrt{x} + 1$ and the x-axis between $x = 0$ and $x = 9$ is revolved around the x-axis. Find the volume of this solid of revolution.

**Solution.**

The solid of revolution is given in Figure 129.

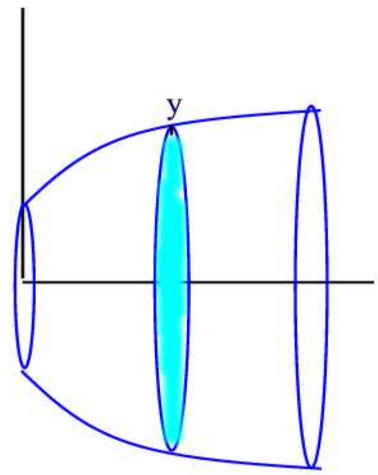



Figure 129

A cross-section is a disk of area $A(x) = \pi(\sqrt{x}+1)^2$. Thus, the total volume is given by

$$\begin{array}{rcl} V & = & \int_0^9 \pi(\sqrt{x}+1)^2 dx = \int_0^9 \pi(x+2\sqrt{x}+1)dx \\ & = & \pi[\frac{x^2}{2}+\frac{4}{3}x^{\frac{3}{2}}+x]_0^9 \\ & = & \pi\frac{171}{2} \approx 268.61 \textit{ cubic units} \blacksquare \end{array}$$

If the revolution is performed around the y-axis, the roles of $x$ and $y$ are interchanged so in that case the formula is

$$\int_a^b \pi x^2 dy,$$

where $x$ must be written as a function of $y$, i.e. $x = f^{-1}(y)$.

**Example 49.4**
The curve $y = x^2, 0 \leq x \leq 1$ is rotated about the y-axis. Find the volume of the resulting solid of revolution.

**Solution.**
The solid of revolution is shown in Figure 130.

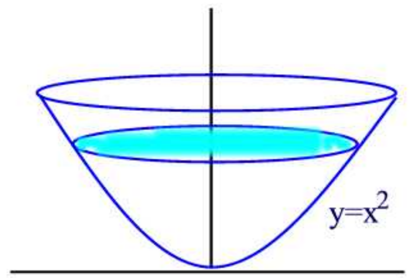



Figure 130

A cross-section is a disk of area $A(y) = \pi y$. Thus, by Vavalieri's principle the volume is

$$V = \int_0^1 \pi y dy = \pi \frac{y^2}{2}\Big|_0^1 \approx 1.571.\ \textit{cubic units}\blacksquare$$

If the region being revolved is the area between two curves $y = f(x)$ and $y = g(x)$, then each cross section is an annular ring (or washer) with outer radius $f(x)$ and inner radius $g(x)$ (assuming $f(x) \geq g(x) \geq 0$.) See Figure 131.

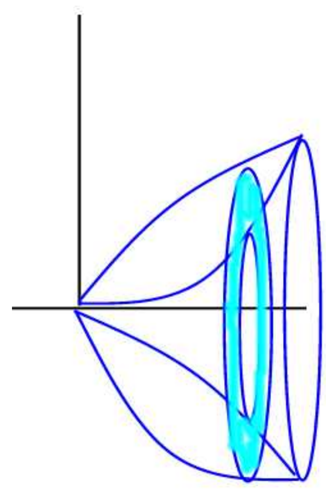


Figure 131

The area of the annular ring is $A(x) = \pi[(f(x))^2 - (g(x))^2]$, hence the volume of the solid will be

$$V = \int_a^b \pi[(y_{top})^2 - (y_{bottom})^2]dx = \int_a^b \pi[f(x)^2 - g(x)^2]dx.$$

If the revolution is performed around the y-axis, then

$$V = \int_a^b \pi[(x_{right})^2 - (x_{left})^2]dy.$$

**Example 49.5**
Find the volume of the solid obtained by revolving the area between $y = x^2$ and $y = \sqrt{x}$ around the x-axis.

**Solution.**
First we need to find the intersection points of these curves in order to find the interval of integration. Setting $x^2 = \sqrt{x}$ and solving for $x$ we find $(0,0)$ and $(1,1)$. Hence, we must integrate from $x = 0$ to $x = 1$.

$$\begin{array}{rcl} V &=& \int_0^1 \pi[(\sqrt{x})^2 - (x^2)^2]dx = \int_0^1 \pi(x - x^4)dx \\ &=& \pi\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{3\pi}{10}.\blacksquare \end{array}$$

**• Arc Length**
The definite integral can also be used to compute the length of a smooth curve (i.e. a curve with no corner points). Recall that when using the integral to find the area of a region one approximates the region by rectangles the sum of whose areas approximates the area of the region. In finding the length of an arc one approximates the arc by a finite set of straight line segments. An approximation of the length of the arc is made by using the well known formula for the length of a line segment and taking a sum . A limiting process then yields the definite integral which is equal to the length of the arc.
To elaborate the above statement, if an arc is just a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$ then its length can be found by the Pythagorean theorem:

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\Delta x + \Delta y}.$$

Now, if the arc is the graph of a function $f(x)$ defined on an interval $[a, b]$, then we divide the interval into $n$ equal subintervals. See Figure 132. The corresponding points in the arc have coordinates $(x_i, f(x_i))$, so two consecutive points are seperated by a distance equal to

$$s_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

But by the Mean Value Theorem there is a point $x_i^*$ in the interval $[x_{i-1}, x_i]$ such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x.$$

Hence,

$$s_i = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

The total length of the arc is

$$s \approx \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Again, we recognize the sum on the right-hand side as a Riemann sum which converges to the following integral

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx.$$

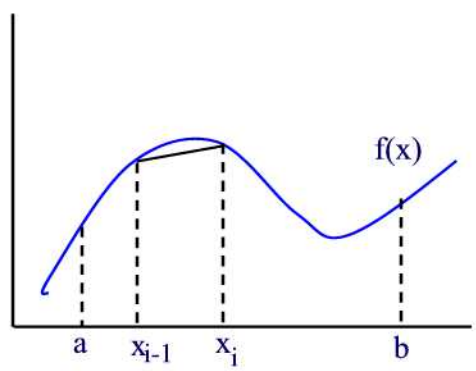



Figure 132

**Example 49.6**
Find the length of the arc defined by the curve $y = x^{\frac{3}{2}}$ between the points $(0, 0)$ and $(1, 1)$.

**Solution.**
Using the arc length formula we have

$$\begin{aligned} s &= \int_0^1 \sqrt{1+(\tfrac{dy}{dx})^2}dx = \int_0^1 \sqrt{1+[(x^{\frac{3}{2}})']^2}dx \\ &= \int_0^1 \sqrt{1+(\tfrac{3}{2}x^{\frac{1}{2}})^2}dx = \int_0^1 \sqrt{1+\tfrac{9x}{4}}dx \\ &= [\tfrac{1}{27}(4+9x)^{\frac{3}{2}}]_0^1 = \tfrac{1}{27}(13^{\frac{3}{2}}-8) \textit{ unit length} \blacksquare \end{aligned}$$

**Large Practice Problems**

In Exercises 1 - 3, the region is rotated around the x-axis. Find the volume.

**Exercise 49.1**
*Bounded by* $y = e^x, y = 0, x = -1, x = 1.$

**Exercise 49.2**
*Bounded by* $y = 4 - x^2, y = 0, x = -2, x = 1.$

**Exercise 49.3**
*Bounded by* $y = \cos x, y = 0, x = 0, x = \frac{\pi}{2}.$

In Exercises 4 - 5, find the arc length of the given function from $x = 0$ to $x = 2$.

**Exercise 49.4**
$f(x) = \sqrt{4 - x^2}.$

**Exercise 49.5**
$f(x) = \sqrt{x^3}.$

**Exercise 49.6**
*Find the length of the parametric curve* $x = \cos(e^t), y = \sin(e^t)$ *for* $0 \leq t \leq 1.$

**Exercise 49.7**
*Consider the hyperbola* $x^2 - y^2 = 1$ *in Figure 133.*

*(a) The shaded region* $2 \leq x \leq 3$ *is rotated around the x-axis. What is the volume generated?*
*(b) What is the arc length with* $y \geq 0$ *from* $x = 2$ *to* $x = 3$*?*

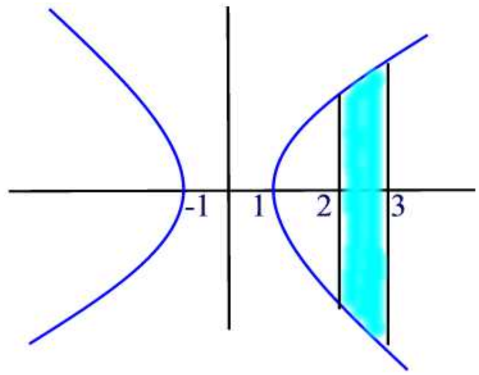



*Figure 133*

In Exercises 8 - 9, sketch the solid obtained by rotating each region around the indicated axis. Using the sketch, show how to approximate the volume of the solid by a Riemann sum, and hence find the volume.

**Exercise 49.8**
*Bounded by* $y = x^3, x = 1, y = -1$. *Axis:* $y = -1$.

**Exercise 49.9**
*Bounded by the first arch of* $y = \sin x, y = 0$. *Axis: x-axis.*

In Exercises 10 - 13 consider the region bounded by $y = e^x$, teh x-axis, and the lines $x = 0$ and $x = 1$. Find the volume of the following solids.

**Exercise 49.10**
*The solid obtained by rotating the region about the x-axis.*

**Exercise 49.11**
*The solid obtained by rotating the region about the horizontal line* $y = -3$.

**Exercise 49.12**
*The solid whose base is the given region and whose cross-sections perpendicular to the x-axis are squares.*

**Exercise 49.13**
*(a) Write an integral which represents the circumference of a circle of radius* $r$.
*(b) Evaluate the integral, and show that you get the answer you expect.*

# 50 Density and Center of Mass

In this section we discuss two of the applications of definite integrals, namely , the concepts of **density** and the **center of mass.**

**Density**

Density is used in different situations to describe similar concepts. We present some of the situations.

• There is the density of a substance, which indicates how much mass per volume unit (i.e. grams per $cm^3$)the substance has.
• There's population density (i.e. people per mile).
• Density of typed words on a page.
• There's density of fog, referring to the amount of water vapor in a volume of unit of air ( i.e. $kg/m^3$).

In all these cases we can use density to compute total mass, people, words, amount of water vapor, etc. If density is uniform, simply multiply the entire area/volume/etc by the density. If density is not uniform, then we divide the region /solid/etc. into small pieces so that the density is approximately uniform on each piece, then add all the pieces together to obtain a Riemann sum. Making the pieces smaller and smaller, i.e. letting $n \to \infty$ we obtain a definite integral.

**Example 50.1**
Find the total mass of a rod of length $l$ and (line-)density $\delta(x)$ where $x$ is the distance a length element from the left end.

**Solution.**
If the density $\delta$ was uniform then the mass $\Delta m$ of a length element $\Delta x$ would (by definition of $\delta$) be simply

$$\Delta m = \delta \Delta x.$$

However, the density is not uniform. So we can approximate the total mass of the rod by slicing it into thin segments $x_i \leq x \leq x_{i+1}$ each of length $\Delta x$ where the density is constant there, say $\delta(x) \approx \delta(x_i)$ for all $x$ in the interval $[x_i, x_{i+1}]$. See Figure 134.

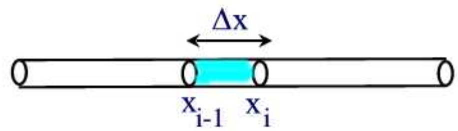



Figure 134

Then the mass of the ith piece is

$$\Delta m_i \approx \delta(x_i)\Delta x$$

and

$$M = \sum_{i=0}^{n-1} \Delta m_i \approx \sum_{i=0}^{n-1} \delta(x_i)\Delta x.$$

Taking the limit $n \to \infty$ (infinitely many thin segments) we obtain

$$M = \int_0^l \delta(x)dx. \blacksquare$$

**Example 50.2**
Find the total mass of a circular object of radius $R$ and (area-)density $\delta(r)$ where $r$ is the radius of an area element from the center.

**Solution.**
If the density $\delta$ was uniform then the mass $\Delta m$ of an area element $\Delta A$ would (by definition of $\delta$) be simply

$$\Delta m = \delta \Delta A.$$

Since the density is not uniform, we approximate the total mass of the circle by slicing it into thin concentric, circular rings, $r_i \leq r \leq r_{i+1}$ each of area $\Delta A(r_i)$ where the density is constant there, say $\delta(r) \approx \delta(r_i)$ for all $r$ in the interval $[r_i, r_{i+1}]$. See Figure 135.

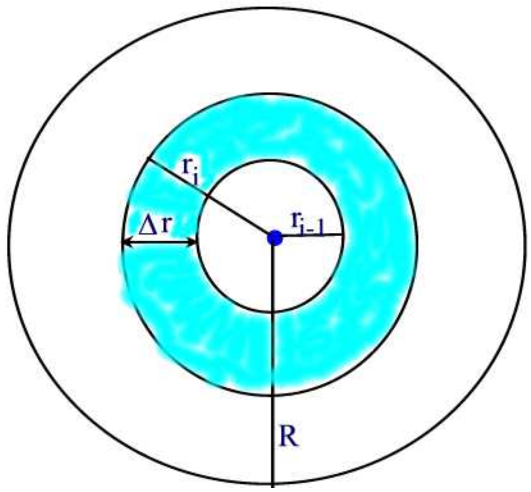



Figure 135

Then the mass of the ith ring is

$$\Delta m_i \approx \delta(r_i)\Delta A(r_i).$$

But

$$\begin{aligned}\Delta A(r_i) &= \pi(r_i+\Delta r)^2 - \pi r_i^2\\ &= 2\pi r_i\Delta r + \pi(\Delta r)^2.\end{aligned}$$

Since each slice is assumed to be very thin then we can ignore $(\Delta r)^2$. Thus, obtaining

$$\Delta m_i 2\pi r_i\delta(r_i)\Delta r.$$

It follows that the total mass is approximated by the sum

$$M = \sum_{i=0}^{n-1}\Delta m_i \approx \sum_{i=0}^{n-1} 2\pi r_i\delta(r_i)\Delta r.$$

Taking the limit $n \to \infty$ (infinitely many thin rings) we obtain

$$M = \int_0^R 2\pi r\delta(r)dr.\blacksquare$$

**Example 50.3**
Suppose we know the (volume-)density $\delta(r)$, i.e. the mass per volume element, of a spherical object of radius $R$ as a function of the radius $r$ from the center. Estimate the total mass of the object.

**Solution.**
If the density $\delta$ was uniform then the mass $\Delta m$ of a volume element $\Delta V$ would (by definition of $\delta$) be simply

$$\Delta m = \delta \Delta V.$$

Since the density is not uniform, then we approximate the total mass of the sphere by slicing it into thin concentric, spherical shells, $r_i \leq r \leq r_{i+1}$ each of volume $\Delta V(r_i)$ where the density is constant there, say $\delta(r) \approx \delta(r_i)$ for all $r$ in the interval $[r_i, r_{i+1}]$.

Then the mass of the ith spherical shell is

$$\Delta m_i \approx \delta(r_i)\Delta V(r_i).$$

But the volume of a thin spherical shell with inner radius $r_i$ and thickness $\Delta r$ is

$$\begin{array}{rcl}\Delta V_i(r_i) & = & \frac{4}{3}\pi(r_i + \Delta r)^3 - \frac{4}{3}\pi r_i^3 \\ & = & \frac{4}{3}\pi[r_i^3 + 3r_i^2\Delta r + 3r_i(\Delta r)^2 + (\Delta r)^3 - r_i^3] \\ & \approx & \frac{4}{3}\pi 3r_i^2\Delta r = 4\pi r_i^2 \Delta r\end{array}$$

since $(\Delta r)^2 \approx 0$ and $(\Delta r)^3 \approx 0$. Thus,

$$\Delta m_i \approx 4\pi r_i^2 \delta(r_i)\Delta r$$

and

$$M = \sum_{i=0}^{n-1} \Delta m_i \approx \sum_{i=0}^{n-1} 4\pi r_i^2 \delta(r_i)\Delta r.$$

Taking the limit $n \to \infty$ (infinitely many thin shells) we obtain

$$M = \int_0^R 4\pi r^2 \delta(r) dr. \blacksquare$$

<u>**Center of Mass**</u>
The **center of mass** is the so-called "balancing point" of an object (or system.) For example, when two children are sitting on a seesaw, the point at which the seesaw balances, i.e. becomes horizontal is the center of mass of the seesaw.

**Discrete Point Masses: One Dimensional Case**

Consider again the example of two children of mass $m_1$ and $m_2$ sitting on each side of a seesaw. It can be shown experimentally that the center of mass is a point $P$ on the seesaw such that

$$m_1 d_1 = m_2 d_2$$

where $d_1$ and $d_2$ are the distances from $m_1$ and $m_2$ to $P$ respectively.
In order to generalize this concept, we introduce an x-axis with points $m_1$ and $m_2$ located at points with coordinates $x_1$ and $x_2$. See Figure 136.

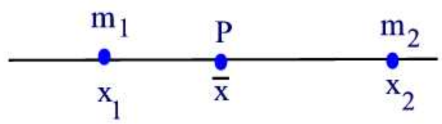



Figure 136

Since $P$ is the balancing point then we must have

$$m_1(\overline{x} - x_1) = m_2(x_2 - \overline{x}).$$

Solving for $\overline{x}$ we find

$$\overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The product $m_1 x_1$ is called the **moment of $m_1$ about the origin.**
The above result can be extended to a system with many points as follows:

The center of mass of a system of $n$ point-masses $m_1, m_2, \cdots, m_n$ located at $x_1, x_2, \cdots, x_n$ along the x-axis is given by the formula

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

**Continuous System:One Dimensional Case**
Next we consider a continuous system. Suppose that we have an object lying on the x-axis between $x = a$ and $x = b$. At point $x$, suppose that the object has mass density (mass per unit length) of $\delta(x)$. To calculate the center of

mass, we divide the object into $n$ pieces, each of length $\Delta x$. On each piece, the density is nearly constant, so the mass of the ith piece is

$$m_i \approx \delta(x_i)\Delta x.$$

The center of mass is then

$$\overline{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} \approx \frac{\sum_{i=1}^n x_i \delta(x_i)\Delta x}{\sum_{i=1}^n \delta(x_i)\Delta x}.$$

Letting $n \to \infty$ we obtain

$$\overline{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}.$$

**Example 50.4**
Find the center of mass of a 2-meter rod lying on the x-axis with its left end at the origin if its density is $\delta(x) = 15x^2\ kg/m.$

**Solution.**
The total mass is

$$M = \int_0^2 15x^2 dx = 5x^3\Big|_0^2 = 40\ kg.$$

The center of mass is

$$\overline{x} = \frac{\int_0^2 15x^3 dx}{40} = \frac{1}{40}\left[\frac{x^4}{4}\right]_0^2 = 1.5\ m. \blacksquare$$

**Two Dimensional System**

The concept of center of mass can be applied to two dimensional objects as well.
The determination of the center of mass in two dimensions is done in a similar manner. If a mass $m$ is located at a point $(x, y)$ then we define the **moment of $m$ about the x-axis** to be the product $my$ and the **monent of $m$ about the y-axis** to be the product $mx$. Let $(\overline{x}, \overline{y})$ be the center of mass. The procedure of finding formulas for $\overline{x}$ and $\overline{y}$ is the same as the one dimensional case. Add up the masses times their x-locations then divide by total mass to get $\overline{x}$. Next, add up the masses times their y-locations then divide by total mass to get $\overline{y}$. Hence the two formulas:

$$\overline{x} = \frac{\sum_{i=1}^n x_i m_i}{\sum_{i=1}^n m_i}, \quad \overline{y} = \frac{\sum_{i=1}^n y_i m_i}{\sum_{i=1}^n m_i}.$$

In the continuous case with uniform density $\delta$ we have

$$\overline{x} = \frac{\int x\delta A(x)dx}{M}, \quad \overline{y} = \frac{\int y\delta A(y)dy}{M}$$

where $A(x)$ and $A(y)$ are the lengths of strips perpendicular to the $x-$ and $y-$axes, respectively. Note that, for variable density, finding the center of mass requires the use of double and multiple integrals, topics that are discussed in Calculus III.

**Example 50.5**
Point-masses of 4, 8, 3, and 2 kilograms are located at $(-2,3), (2,-6), (7,-3)$, and $(5,1)$ respectively. Find the coordinates of the center of mass.

**Solution**
Applying the formula above we find

$$\overline{x} = \frac{(4)(3)+(8)(-6)+(3)(-3)+(2)(1)}{4+8+3+2} = \frac{39}{17}$$

and

$$\overline{y} = \frac{(4)(-2)+(8)(2)+(3)(7)+(2)(5)}{4+8+3+2} = -\frac{43}{17}.\blacksquare$$

**Example 50.6**
Suppose an isosceles triangle with uniform density, altitude $a$, and base $b$ is placed in the xy-plane as shown in Figure 137.

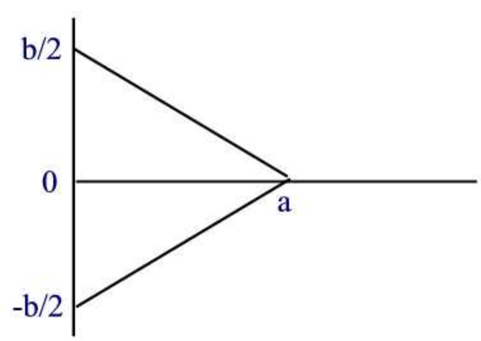



Figure 137

Show that the center of mass is at $\overline{x} = \frac{a}{3}, \overline{y} = 0$. Hence, show that the center of mass is independent of the triangle's base.

**Solution.**

Because the mass of the triangle is symmetrically distributed with respect to the x-axis, then $\overline{y} = 0$. If $\delta$ is the density of the triangle and $m$ is its mass then

$$\delta = \frac{m}{A} = \frac{m}{\frac{ab}{2}} = \frac{2m}{ab}.$$

We partition the triangle into vertical strips as shown in Figure 138.

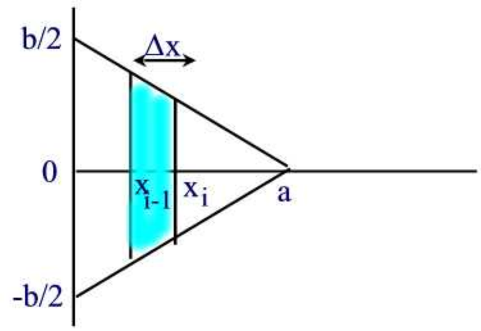



Figure 138

Let $h_i$ be the length of the base of the triangle with vertex at $(a, 0)$ and passing through $(x_i, 0)$. Using similar triangles we find that

$$\frac{h_i}{b} = \frac{a - x_i}{a}.$$

That is, $h_i = \frac{b}{a}(a - x_i)$. Hence, the area of the ith strip is $h_i \Delta x = \frac{b}{a}(a - x_i)\Delta x$ and the approximate mass of the ith strip is

$$m_i \approx \frac{b(a - x_i)}{a}\frac{2m}{ab}\Delta x = \frac{2m(a - x_i)}{a^2}\Delta x.$$

The approximate moment is

$$\sum_{i=1}^{n} x_i \frac{2m(a - x_i)}{a^2}\Delta x$$

and the exact moment is

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} x_i \tfrac{2m(a-x_i)}{a^2}\Delta x &= \int_0^a \tfrac{2mx(a-x)}{a^2}dx \\ &= \tfrac{2m}{a^2}\int_0^a (ax - x^2)dx = \tfrac{2m}{a^2}\left[\tfrac{ax^2}{2} - \tfrac{x^3}{3}\right]_0^a = \tfrac{ma}{3}. \end{aligned}$$

Hence,

$$\overline{x} = \frac{\frac{ma}{3}}{m} = \frac{a}{3}.$$

Finally, he center of mass is the point $(\frac{a}{3}, 0)$. ■

**Remark 50.1**

The center of mass of a body need not be within the body itself; the center of mass of a ring or a hollow cylinder of unifrom density is located in the enclosed space, not in the object itself.

## Practice Problems

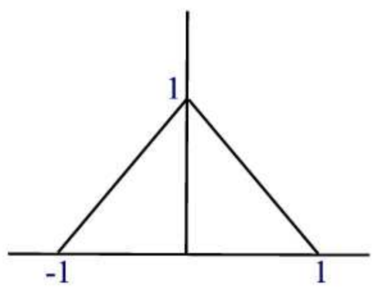
**Exercise 50.1**
*Find the mass of a rod of length 10 cm with density $\delta(x) = e^{-x}$ gm/cm at a distance of $x$ cm from the left end.*

**Exercise 50.2**
*A rod has length 2 meters. At a distance $x$ meters from its left end, the density of the rod is given by*

$$\delta(x) = 2 + 6x \ gm/m.$$

*(a) Write a Riemann sum approximating the total mass.*
*(b) Find the exact mass by converting the sum into a definite integral.*

**Exercise 50.3**
*Find the total mass of the triangular region shown below which has density $\delta(x) = 1 + x$ grams/cm$^2$.*




**Exercise 50.4**
*The density of oil in a circular oil slick on the surface of the ocean at a distance $r$ meters from the center of the slick is given by $\delta(r) = \frac{50}{1+r}$ kg/m$^2$.*

*(a) If the slick extends from $r = 0$ to $r = 10,000$ m, find a Riemann sum approximating the total mass of oil in the slick.*
*(b) Find the exact value of the mass of oil in the slick by turning your sum into an integral an evaluating it.*
*(c) Within what distance $r$ is half the oil in the slick contained?*

**Exercise 50.5**

*An exponential model for the density of the earth's atmosphere says that if the temperature of the atmosphere were constant, then the density of the atmosphere as a function of height h (in meters), above the surface of the earth would be given by*

$$\delta(h) = 1.28e^{-0.000124h}\ kg/m^3.$$

*(a) Write (but do not evaluate) a sum that approximates the mass of the portion of the atmosphere from $h = 0$ to $h = 100$ m (i.e. the first 100 meters above sea level). Assume the radius of the earth is 6400 km.*
*(b) Find the exact answer by turning your sum in part (a) into an integral. Evaluate the integral.*

**Exercise 50.6**

*Three point masses of 4 gm each are placed at $x = -6, 1, 4$ and 3. Where should a fourth point of mass 4 gm be placed to make the center of mass at the origin?*

**Exercise 50.7**

*A rod of length 3 meters with density $\delta(x) = 1 + x^2$ gm/m is positioned along the positive x-axis, with its left end at the origin. Find the total mass and the center of mass of the rod.*

**Exercise 50.8**

*A rod with density $\delta(x) = 2 + \sin x$ lies on the x-axis between $x = 0$ and $x = \pi$. Find the center of mass of the rod.*

# 51 Applications to Physics

It has been shown how calculus can be applied to find solutions to geometric problems such as problems concerned with computing area, volume, and arc length. In this section calculus is used to solve problems that arise from Physics.

• **The Concept of Work**

The work done by a constant force, $F$, in moving an object a distance , $d$, is equal to the product of the force and the distance moved. That is,

$$W = F \cdot d.$$

The SI (international) unit of work is the joule (J), which is the work done by a force of one Newton (N) pushing a body along one meter (m). Thus, 1 joule = 1 N-m. In the British system, a unit work is the foot-pound. Since $1N = 0.224809\ lb(1\ lb = 4.45\ N)$ and $1m = 3.28084\ ft\ (1\ ft = 0.305\ m)$ then $1J = 0.737561\ ft - lb(1\ ft - lb = 1.36\ J)$.

Now, in most cases the applied force is not constant, but varies over the straight line of motion. For example, suppose that the force, $F(x)$, acting on a particle as it moves along the straight line from a to b varies continuously. In order to find the total work done by the force we divide the interval $[a, b]$ into $n$ small equal subintervals $[x_{i-1}, x_i]$ so that the change in $F$ is small along each subinterval, i.e approximately constant. Then the work done by the force in moving the body from $x_{i-1}$ to $x_i$ is approximately:

$$\Delta W_i \approx F(x_i)\Delta x.$$

So the total work is

$$W = \sum_{i=1}^{n} \Delta W_i \approx \sum_{i=1}^{n} F(x_i)\Delta x.$$

As $n \to \infty$ the Riemann sum at the right converges to the following integral:

$$W = \int_a^b F(x)dx.$$

**Example 51.1**

Consider a spring on the x-axis so that its right end is at $x = 0$ when the spring is at its rest position. According to Hooke's Law, the force needed to stretch the spring from 0 to $x$ is proportional to $x$, i.e. $F(x) = kx$ where $k$ is called the **spring constant.** Find the work done in stretching the spring a length of $a$.

**Solution.**

The work needed to stretch the spring from 0 to a is then the integral

$$W = \int_0^a kx dx = \frac{ka^2}{2}. \blacksquare$$

• **Work Done Against Gravity**
Mass and weight are often confused. Weight is the force of gravity on an object. The mass of an object is the quantity of matter it comprises. An object's weight will vary depending on a given gravity. For example an object that weighs 10 pounds on earth is weightless in interstellar space. On the contrary, an object will have the exact same mass. Gravity causes objects to free fall with a constant acceleration ($9.8 meters/second^2$ on earth).
In the SI system, the unit of mass is the kilograms. Thus, 1 Kg of iron stands for the mass of iron present. Its weight is the amount of force exerted on it by gravity. Since $force = mass \times acceleration$ then 1 Kg of iron weighs $1Kg \cdot 9.8m/sec^2 = 9.8\frac{kg \cdot m}{sec^2} = 9.8N$. (A newton is a unit of force or weight). The unit of mass in the British system is slug.
Now, according to Newton's Law, the force of gravity at a distance $r$ from the center of the earth is

$$F(r) = \frac{k}{r^2}$$

where $k$ is some positive constant.
The work needed to lift a body from a point at distance $R_1$ from the center of the Earth to another point at distance $R_2$ is given by the integral

$$W = \int_{R_1}^{R_2} \frac{k}{r^2} dr = -\frac{k}{r}\Big|_{R_1}^{R_2} = k\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

**Example 51.2**
Find the work needed to lift a body of weight $1N$, 1000 km from the surface of the Earth. The Earth radius is 6378 Km.

**Solution.**
First we will find the value of $k$. Since $F(6378) = 1\ N$ then $\frac{k}{6378^2} = 1$, so $k = 6378^2$. Next we have $R_1 = 6378, R_2 = 6378 + 1000 = 7378km$, hence

$$W = 6378^2\left(\frac{1}{6378} - \frac{1}{7378}\right) = 864.462N - Km.$$

Since $1Km = 1000m$, the result in joule is

$$864.462N - Km = 864.462N \times 1000m = 864462J.\blacksquare$$

• **Work Done Filling (or Emptying) a Tank**

**Example 51.3**
A tank in the shape of a right circular cone of height 12 ft and radius 3 ft is inserted into the ground with its vertex pointing down and its top at ground level as shown in Figure 139. If the tank is filled with water (density $\rho = 62.4\ lb/ft^3$) to a depth of 6 ft, how much work is performed in pumping all the water in the tank to ground level? What changes if the water is pumped to a height of 3 ft above ground level?

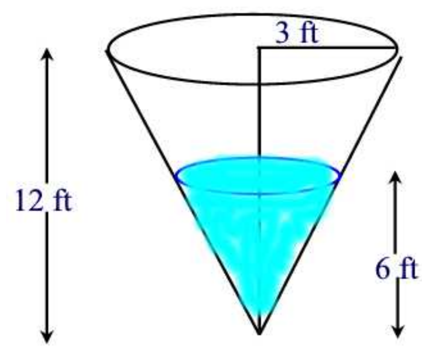



Figure 139

**Solution.**
Set up a coordinate system with the origin at the vertex of the cone and the y-axis as the axis of symmetry as shown in Figure 140.

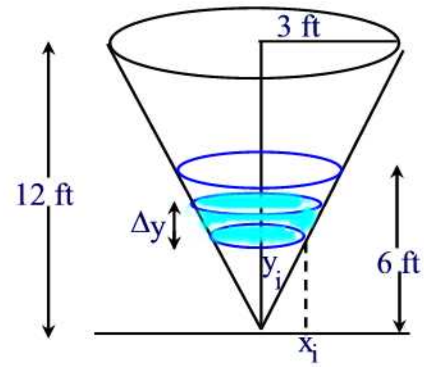



Figure 140

Consider a layer of distance $y_i$ from the vertex of the cone and with thickness $\Delta y$. The volume of such a circular layer is

$$V_i = \pi x_i^2 \Delta y.$$

Using similar triangles we find that

$$\frac{x_i}{3} = \frac{y_i}{12}$$

and consequently $x_i = \frac{y_i}{4}$. Thus,

$$V_i = \frac{\pi}{16} y_i^2 \Delta y.$$

Hence its weight is $m_i = 62.4 \frac{\pi}{16} y_i^2 \Delta y$. The work done to raise it to the top of the tank is

$$W_i = \frac{62.4\pi}{16} y_i^2 (12 - y_i) \Delta y.$$

Adding the works done to raise these slices we obtain the total work done to empty the tank:

$$\begin{array}{rcl} W & = & \int_0^6 \frac{62.4\pi}{16} y^2 (12 - y) dy = 3.9\pi \int_0^6 (12y^2 - y^3) dy \\ & = & 3.9\pi \left[4y^3 - \frac{1}{4}y^4\right]_0^6 = 2106\pi \ ft - lb. \end{array}$$

Now if the water is pumped to a height of 3 ft above ground level, all that changes is the distance moved by the layer of water. It becomes $12 + 3 - y_i = 15 - y_i$ and the work is given by

$$W = 62.4\pi \int_0^6 \frac{y^2}{16} (15 - y) dy \approx 9263 ft - lb. \blacksquare$$

• **Force and Pressure**
Pressure is the force per unit area acting on an object. You measure air, steam, gas pressure, and the fluid pressure in hydraulic systems in pounds per square inch (psi). However, you measure water pressure in pounds per square foot. So the pressure in a liquid is the force per unit area exerted by the liquid. The pressure is exerted equally in all directions and it increases in depth.
The pressure $p$ of a liquid at a given depth $h$ is given by the formula

$$p = \delta \cdot g \cdot h$$

where $g$ is the acceleration due to gravity and $\delta$ is the density of the liquid.
For a constant pressure over a given area, the force on the surface is given by

$$Force = Pressure \cdot Area.$$

If the pressure is variable then the force is found by dividing the surface into small pieces in such a way that the pressure is nearly constant and then write a Riemann sum that yields a definite integral giving the total force. Since the pressure varies with depth, we divide the surface into horizontal strips, each of which is at an approximately constant depth. The following example illustrates these concepts.

**Example 51.4**
Set up and calculate a definite integral giving the total force on the dam shown in Figure 141. The density of water is $\delta = 1000kg/m^3$.

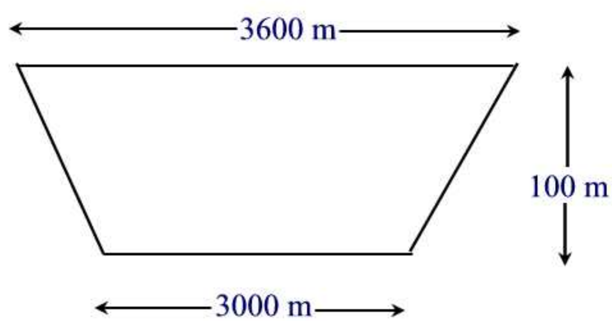



Figure 141

**Solution.**
We divide the dam into horizontal strips in which the pressure is almost constant. (See Figure 142). Let's find the area of a strip. The equation of the line going through the points $(1500, 0)$ and $(1800, 100)$ is $y = \frac{x}{3} - 500$. Thus,

$$A_i = 2x_i\Delta y$$

where $x_i = 3y_i + 1500$. Hence

$$A_i = (6y_i + 3000)\Delta y.$$

The pressure is given by

$$p_i = \delta g y_i = 1000 \cdot 9.8 y_i = 9800 y_i.$$

Thus, the total force is

$$F = \int_0^{100} 9800y(6y + 3000)dy = 9800 \left[2y^3 + 1500y^2\right]_0^{100} = 1.666 \cdot 10^{11}. \blacksquare$$

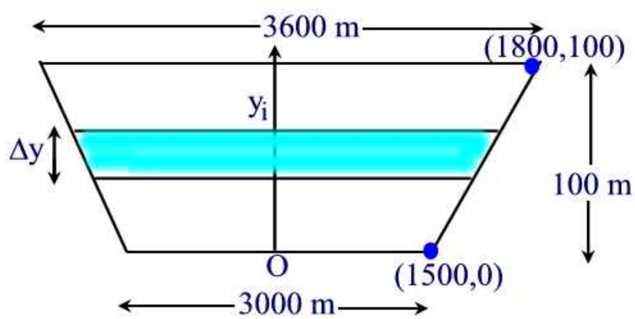



Figure 142

## Practice Problems.

In Exercises 1 - 2, the force, $F$, required to compress a spring by a distance $x$ meters is given by $F(x) = 3x$ newtons.

**Exercise 51.1**
*Find the work done in compressing the spring from $x = 1$ to $x = 2$.*

**Exercise 51.2**
*Find the work done to compress the spring to $x = 3$ starting at the equilibrium position ,$x = 0$.*

**Exercise 51.3**
*The gravitational force on a 1 kg object at a distance r meters from the center of the earth is $F(r) = \frac{4 \cdot 10^{14}}{r^2}$ newtons. Find the work done in moving the object from the surface of the earth to a height of $10^6$ meters above the surface. The radius of the earth is $6.4 \cdot 10^6$ meters.*

**Exercise 51.4**
*A rectangular water tank has length 20 ft, width 10 ft, and depth 15 ft. If the tank is full, how much work does it take to pump all the water out?*

**Exercise 51.5**
*A water tank is in the form of a right circular cylinder with height 20 ft and radius 6 ft. If the tank is half full of water, find the work required to pump all of it over the top rim.*

**Exercise 51.6**
*Suppose the tank in the previous problem is full of water. Find the work required to pump all of it to a point 10 ft above the top of the tank.*

# 52 Applications to Economics

In this section we consider some applications of definite integrals used in economics such as the present and future values of a continuous income stream and the consumers' and producers' surplus.

**Discrete Present and Future Value**
Many business deals involve payments in the future. For example, when a car or a home is bought on credits, payments are made over a period of time. The **future value**, FV, of a payment $P$ is the amount to which $P$ would have grown if deposited today in an interest bearing bank account. The **present value**, PV, of a future payment FV, is the amount that would have to be deposited in a bank account today to produce exactly FV in the account at the relevant time future.
If interest is compounded $n$ times a year at a rate $r$ for $t$ years, then the relationship between $FV$ and $PV$ is given by the formula

$$FV = PV(1 + \frac{r}{n})^{nt}.$$

In the case of continuous compound interest, the forumla is given by

$$FV = PVe^{rt}.$$

**Example 52.1**
You need \$10,000 in your account 3 years from now and the interest rate is 8% per year, compounded continuously. How much should you deposit now?

**Solution.**
We have $FV = \$10,000, r = 0.08, t = 3$ and we want to find $PV$. Solving the formula $FV = PVe^{rt}$ for $PV$ we find $PV = FVe^{-rt}$. Substituting to obtain, $PV = 10,000e^{-0.24} \approx \$7,866.28.$■

**Present and Future Value of a Continuous Income Stream**
When an income stream flows into an investment, the investment grows because of the continuous flows of money and the interest compounded on the money invested. Thus, two functions are required: a function defining the flow of money, and a function defining a function multiplier. In this section, we will find the present value and the future value of a continuous income stream.

Let $S(t)$ be the flow rate in dollars per year. To find the present value of a continuous income stream over a period of $M$ years we divide the interval $[0, M]$ into $n$ equal subintervals each of length $\Delta t = \frac{M}{n}$ and with division points $0 = t_0 < t_1 < \cdots < t_n = M$. That is, over each time interval we are assuming a single payment is made. Assuming interest $r$ is compounded continuously, the present value of the total money deposited is approximated by the following Riemann sum:

$$PV \approx S(t_1)e^{-rt_1}\Delta t + S(t_2)e^{-rt_2}\Delta t + \cdots S(t_n)e^{-rt_n}\Delta t = \sum_{i=1}^{n} S(t_i)e^{-rt_i}\Delta t.$$

Letting $\Delta t \to 0$, i.e. $n \to \infty$, we obtain

$$PV = \int_0^M S(t)e^{-rt}dt.$$

The future value is given by

$$FV = e^{rM} \int_0^M S(t)e^{-rt}dt.$$

**Example 52.2**
An investor is investing \$3.3 million a year in an account returning 9.4% APR. Assuming a continuous income stream and continuous compounding of interest, how much will these investments be worth 10 years from now?

**Solution.**
Using the formula for the future value defined above we find

$$FV = e^{.94} \int_0^{10} 3.3e^{-0.094t}dt \approx \$54.8 million. \blacksquare$$

**Example 52.3**
At what canstant, continuous rate must money be deposited into an account if the account contain \$20,000 in 5 years? The account earns 6% interest compounded continuously.

**Solution.**
Given $FV = \$20,000, M = 5, r = 0.06$. Since $S$ is assumed to be constant then we have

$$20,000 = S \int_0^5 e^{-0.06t}dt.$$

Solving for $S$ we find

$$S = \frac{20,000}{\int_0^5 e^{-0.06t}dt} \approx \$4,630 \text{ per year.}\blacksquare$$

**Supply and Demand Curves**

The quantity $q$ manufactured and sold depends on the unit price $p$. In general, when the price goes up then manufacturers are willing to supply more of the product whereas consumers are going to reduce their buyings. Since consumers and manufacturers react differently to changes in price, there are two curves relating $p$ and $q$.

The **supply curve** is the quantity that producers are willing to make at a given price. Thus, increasing price will increase quantity.
The **demand curve** is the amount that will be bought by consumers at a given price. Thus, decreasing price will increase quantity.
Even though quantity is a function of price, it is the tradition to use the vertical y-axis for the variable $p$ and the horizontal x-axis for the variable $q$. The supply and demand curves intersect at point $(q^*, p^*)$ called the **point of equilibrium**. We call $p^*$ the **equilibrium price** and $q^*$ the **equilibrium quantity**.

**Example 52.4**

Find the equilibrium point for the supply function $S(p) = 3p - 50$ and the demand function $D(p) = 100 - 2p$.

**Solution.**

Setting the equation $S(p^*) = D(p^*)$ to obtain $3p^* - 50 = 100 - 2p^*$. By adding $2p^* + 50$ to both sides we obtain $5p^* = 150$. Solving for $p^*$ we find $p^* = 30$. Substituting this value in $S(p)$ we find $q^* = 3(30) - 50 = 40.\blacksquare$

**Consumer and Producer Surplus**

The definitions of demand and supply must be remembered:

Demand tells us the price that consumers would be willing to pay for each different quantity. According to the law of demand, when the price increases the demand decreases and when the price decreases the demand increases. The graphical representation of the relationship between the quantity demanded

of a good and the price of the good is known as the **demand curve.**
Supply tells us the price that producers would be willing to charge in order to sell the different quantities. The law of supply asserts that as the price of a good rises, teh quantity supplied rises, and as the price of a good falls the quantity supplied falls. The graphical representation of the relationship between the quantity supplied of a good and the price of the good is known as the **supply curve.**
The demand and supply curve intersects at the **point of equilibrium** $(q^*, p^*)$. We call $p^*$ the **equilibrium price** and the $q^*$ the **equilibrium quantity.** See Figure 143.

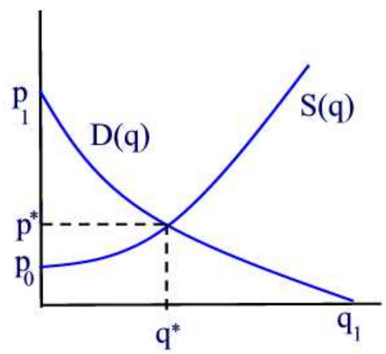



Figure 143

**Consumers' Surplus**
At the equilibrium level, the **consumers' surplus** is the difference between what consumers are willing to pay and their actual expenditure: It therefore represents the total amount saved by consumers who were willing to pay more than $p^*$ per unit.
To calculate the consumers' surplus, we first calculate the consumers' total expenditure. Divide the interval $[0, q^*]$ into $n$ equal pieces each of length $\Delta q$. According to Figure 144, the consumers' total expenditure is given by the sum

$$D(q_1)\Delta q + D(q_2)\Delta q + \cdots + D(q_n)\Delta q = \sum_{i=1}^{n} D(q_i)\Delta q.$$

Letting $\Delta q \to 0$ to obtain (See Figure 144)

$$Total\ Expenditure = \int_0^{q^*} D(q)dq.$$

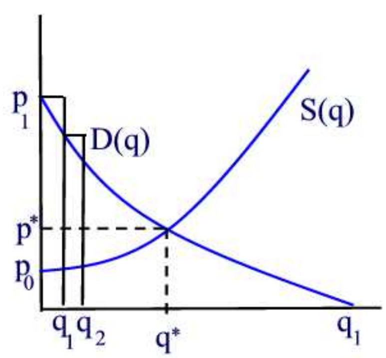



Figure 144

Thus,

$$Consumers'\ Surplus = \int_0^{q^*} (D(q) - p^*)dq.$$

Graphically, it is the area between demand curve and the horizontal line at $p^*$. See Figure 145.

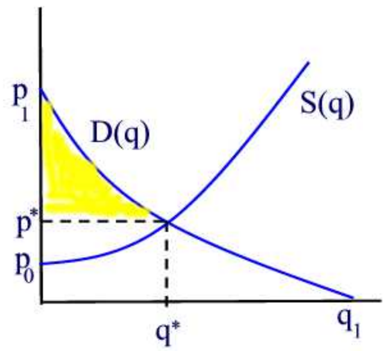



Figure 145

**Producers' Surplus**
The producers' surplus is the extra amount earned by producers who were willing to charge less than the selling price of $p^*$ per unit, and is given by

$$Producers'\ Surplus = \int_0^{q^*} (p^* - S(q))dq.$$

Graphically, it is the area between suuply curve and the horizontal line at $p^*$. See Figure 146.

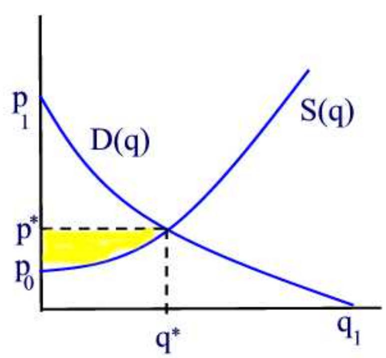



Figure 146

**Example 52.5**
The demand and supply equations are given by $D(q) = 60 - \frac{q^2}{10}$ and $S(q) = 30 + \frac{q^2}{5}$. Find the consumers' and producers' surplus at the equilibrium price.

**Solution.**
To find the Consumers and Producers surplus under equilibrium we first need to find the equilibrium point by setting supply=demand and solving for q:

$$30 + \frac{q^2}{5} = 60 - \frac{q^2}{10} \;\; implies \;\; q^* = 10.$$

Substituting this into the supply (or demand) equation we find the equilibrium price $p^* = 50$. Now we use formulas of the Consumers and Producers surplus:

$$\begin{array}{rcl} Consumers'\ Surplus & : & \int_0^{10} \left[\left(60 - \frac{q^2}{10}\right) - 50\right] dq \;\; = \;\; 10q - \frac{q^3}{30}\Big|_0^{10} \approx 66.67 \\ Producers'\ Surplus & : & \int_0^{10} \left[50 - \left(30 + \frac{q^2}{5}\right)\right] dq \;\; = \;\; 20q - \frac{q^3}{15}\Big|_0^{10} \approx 133.33 \;\blacksquare \end{array}$$

## Practice Problems

**Exercise 52.1**
*Find the future value of an income stream of \$1,000 per year, deposited into an account paying 8% interest, compounded continuously, over a 10-year period.*

**Exercise 52.2**
*Find the present and future values of an income stream of \$2,000 a year, for a period of 5 years, if the continuous interest rate is 8%.*

**Exercise 52.3**
*A person deposits money into a retirement account, which pays 7% interest compounded continuously, at a rate of \$1,000 per year for 20 years. Calculate:*

*(a) The balance in the account at the end of the 20 years.*
*(b) The amount of money actually deposited into the account.*
*(c) The interest earned during the 20 years.*

**Exercise 52.4**
*(a) A bank account earns 10% interest compounded continuously. At what (constant, continuous) rate must a parent deposit money into such an account in order to save \$100,000 in 10 years for a child's college expenses?*
*(b) If the parents decide instead to deposit a lump sum now in order to attain the goal of \$100,000 in 10 years, how much must be deposited now?*

**Exercise 52.5**
*Sales of Version 6.0 of a computer software package start out high and decrease exponentially. At time $t$, in years, the sales are $s(t) = 50e^{-t}$ thousands of dollars per year. After two years, Version 7.0 of the software is released and replaces Version 6.0. Assume that all income from software sales is immediately invested in government bonds which pay interest at a 6% rate compounded continuously, calculate the total value of sales of Version 6.0 over the two year period.*

**Exercise 52.6**
*An oil company discovered an oil reserve of 100 million barrels. For time*

$t > 0$, *in years, the company's extraction plan is a linear declining function of time as follows:*

$$q(t) = a - bt,$$

*where* $q(t)$ *is the rate of extraction of oil in millions of barrels per year at time* $t$ *and* $b = 0.1$ *and* $a = 10$.

*(a) How long does it take to exhaust the entire reserve?*
*(b) The oil price is a constant \$20 a barrel, the extraction caost per barrel is a constant \$10, and the market interest rate is 10% per year, compounded continuously. What is the present value of company's profit?*

**Exercise 52.7**
*The dairy industry is an example of acartel pricing: the government has set milk prices artificially high. On a supply and demand graph, label* $p^+$, *a price above the equilibrium price. Using the graph, describe the effect of forcing the price up to* $p^+$ *on:*

*(a) The consumer surplus.*
*(b) The producer surplus.*
*(c) The total gains from trade(Consumer surplus + producer surplus).*

# 53 Continuous Random Variables: Distribution Function and Density Function

Statistics is one of the major topics of mathematics. However, the study of statistics requires the study of probability theory.
What is probability? Before answering this question we start with some basic definitions.

An **experiment** is any operation whose outcome cannot predicted with certainty. The **sample space** $S$ of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be $S = \{1, 2, 3, 4, 5, 6\}$ where each digit represents a face of the die. An **event** is any subset of the sample space.

**Probability** is the measure of occurrence of an event. It is a number between 0 and 1. If the event is impossible to occur then its probability is 0. If the occurrence is certain then the probability is 1. The closer to 1 the probability is, the more likely the event is to occur.

A **random variable** $\alpha$ is a numerical valued function defined on a sample space. For example, in rolling two dice $\alpha$ might represent the sum of the points on the two dice. Similarly, in taking samples of college students $\alpha$ might represent the number of hours per week a student studies, or a student's GPA.
Random variables may be divided into two types: **discrete** random variables and **continuous** random variables. A discrete random variable is one that can assume only a countable number of values. It is usually the result of counting. A continuous random variable can assume any value in one or more intervals on a line. It is usually the result of a measurement.

**Example 53.1**
State whether the random variables are discrete or continuous:

(a) The height of a student in your class.
(b) The number of left-handed students in your class.

**Solution.**
(a) The randon variable in this case is a result of measurement and so it is a continuous random variable.
(b) The random variable takes whole positive integers as values and so is a discrete random variable.■

In this section, we limit our discussion to continuous random variables.

**Cumulative Distribution Function**
Let $S$ be a sample space and $\alpha : S \to \mathbb{R}$ be a continuous random variable. Then the **cumulative distribution function** (cdf) $F(t)$ of the variable $\alpha$ is defined as follows

$$F(t) = P(\alpha \leq t) = P(\{s \in S : \alpha(s) \leq t\})$$

i.e., $F(t)$ is equal to the probability that the variable $\alpha$ assumes values, which are less than or equal to $t$.

**Example 53.2**
If we think of an electron as a particle, the function

$$P(r) = 1 - (2r^2 + 2r + 1)e^{-2r}$$

is the cumulative distribution function of the distance, $r$, of the electron in a hydrogen atom from the center of the atom. The distance is measured in Bohr radii. (1 Bohr radius $= 5.29 \times 10^{-11}$m.) Interpret the meaning of $P(1)$.

**Solution.**
$P(1) = 1 - 5e^{-2} \approx 0.32$. This numbers says that the electron is within 1 Bohr radius from the center of the atom 32% of the time.■

Next, we discuss the properties of the cumulative distribution function $F(t)$ for a continuous random variable $\alpha$.

**Theorem 53.1**
The cumulative distribution function of a continuous random variable $\alpha$ satisfies the following properties:

(a) $0 \leq F(t) \leq 1$.

(b) $P(a \leq \alpha \leq b) = F(b) - F(a)$.
(c) $F(t)$ is a non-decreasing function, i.e. if $a \leq b$ then $F(a) \leq F(b)$.
(d) $F(t) \to 0$ as $t \to -\infty$ and $F(t) \to 1$ as $t \to \infty$.

**Proof.**
We will prove (a) - (c).
(a) Since $F$ is defined in terms of a probability then $0 \leq F(t) \leq 1$.
(b) Let $a$ and $b$ be two real numbers with $a < b$. Then

$$\begin{aligned} P(a \leq \alpha \leq b) &= P(\alpha \leq b) - P(\alpha \leq a) \\ &= F(b) - F(a). \end{aligned}$$

(c) This is a result of (b).■

Figure 147 illustrates a representative cdf.

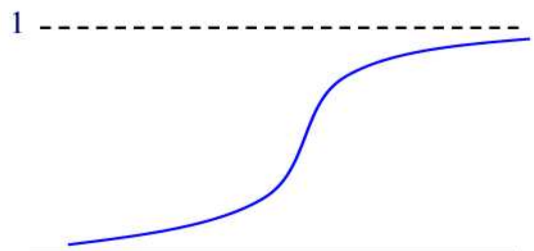



Figure 147

**Probability Density Function**
If $F(t)$ is the cumulative distribution function for a continuous random variable $\alpha$ then the **probability density function** (pdf) $f(t)$ for $\alpha$ satisfies

$$f(t) = F(t),$$

i.e., $f(t)$ is the derivative of the cumulative distribution function $F(t)$.
It follows from the definition of density function and the Fundamental Theorem of Calculus that

$$F(t) = \int_{-\infty}^{t} f(x)dx,$$

i.e. for every real number $t$, $F(t)$ is the area under the graph of $f$ to the left of $t$. Moreover,

$$\begin{array}{rcl} P(a \leq \alpha \leq b) & = & P(\alpha \leq b) - P(\alpha \leq a) \\ & = & F(b) - F(a) \\ & = & \int_{-\infty}^{b} f(t)dt - \int_{-\infty}^{a} f(t)dt \\ & = & \int_{a}^{b} f(t)dt \end{array}$$

**Theorem 53.2** *(More Properties of $f(t)$)*
(a) $f(t) \geq 0$ for all $t$.
(b) $\int -\infty^{\infty} f(t)dt = 1$.
(c) $\lim_{t\to-\infty} f(t) = 0 = \lim_{t\to\infty} f(t)$.

**Proof.**
(a) Since $f(t) = F'(t)$ and $F(t)$ is nondecreasing then $F'(t) \geq 0$ and so $f(t) \geq 0$.
(b) Since $\lim_{t\to\infty} F(t) = 1$ then $\int_{-\infty}^{\infty} f(t)dt = 1$.
(c) This follows from the fact that the graph of $F(t)$ levels off when $t \to \pm\infty$.■

The density function for a continuous random variable , the model for some real-life population of data, will usually be a smooth curve as shown in Figure 148.

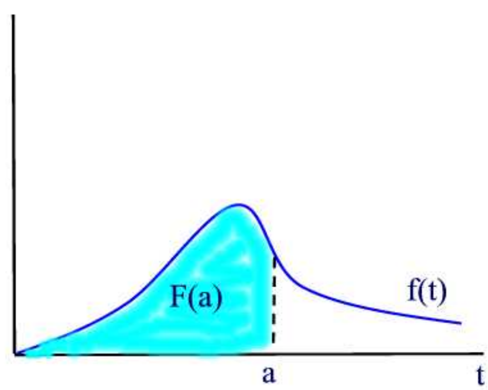



Figure 148

**Example 53.3**
Suppose that the function $f(t)$ defined below is the density function of some

random variable $\alpha$.

$$f(t) = \begin{cases} e^{-t} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Compute $P(-10 \leq \alpha \leq 10)$.

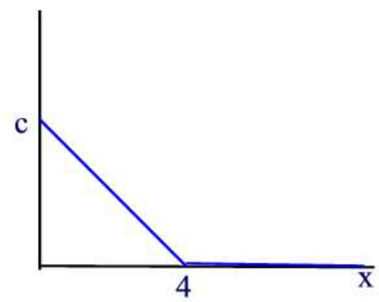
**Solution.**

$$\begin{array}{rcl} P(-10 \leq \alpha \leq 10) & = & \int_{-10}^{10} f(t)dt \\ & = & \int_{-10}^{0} f(t)dt + \int_{0}^{10} f(t)dt \\ & = & \int_{0}^{10} e^{-t}dt \\ & = & -e^{-t}|_{0}^{10} = 1 - e^{-10}. \blacksquare \end{array}$$

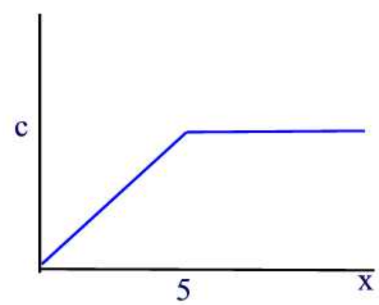
## Practice Problems

Decide if the function graphed in Exercises 1 - 4 is a probability density function or a cumulative distribution function. Give reasons. Find the value of $c$. Sketch and label the other function. That is, sketch and label the pdf if the problem shows a cdf, and the cdf if the problem shows a pdf.

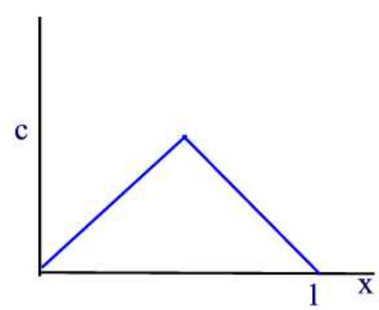
**Exercise 53.1**



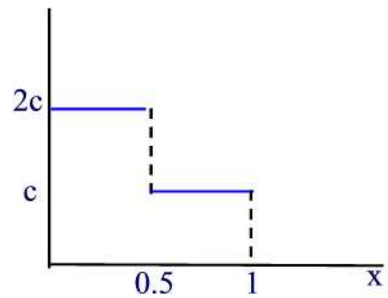

**Exercise 53.2**




**Exercise 53.3**



**Exercise 53.4**




**Exercise 53.5**
*A large number of people take a standardized test, receiving scores described by the density function p graphed in Figure 149. Does the density function imply that most people receive a score near 50? Explain why or why not?*

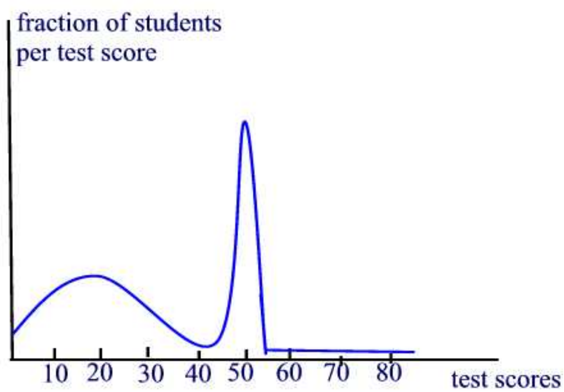



*Figure 149*

**Exercise 53.6**
*Suppose $F(x)$ is the cumulative distribution function for heights (in meters) of trees in a forest.*

*(a) Explain in terms of trees the meaning of the statement $F(7) = 0.6$.*
*(b) Which is greater, $F(6)$ or $F(7)$? Justify your answer in terms of trees.*

**Exercise 53.7**
*An experiment is done to determine the effect of two new fertilizers A and B on the growth of a species of peas. The cumulative distribution functions*

*of the heights of the mature peas without treatment and treated with each of A and A are graphed in Figure 150.*

*(a) About what height are most of the unfertilized plants?*
*(b) Explain in words the effect of the fertilizerd A and B on the mature height of the plants.*

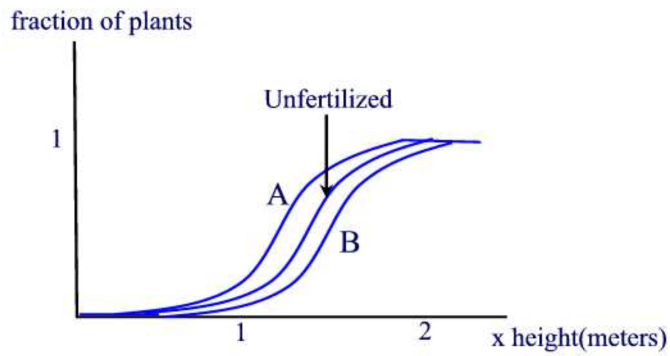



*Figure 150*

**Exercise 53.8**
*Figure 151 shows a density function and the corresponding cumulative distribution function. Which curve represents the density function and which represents the cumulative distribution function? Give a reason for your choice.*

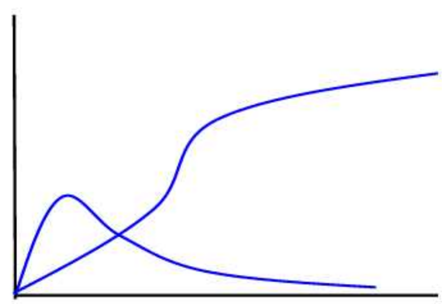


*Figure 151*

**Exercise 53.9**
*After measuring the duration of many telephone calls, the telephone company found their data was well approximated by the density function* $p(x) =$

$0.4e^{-0.4x}$, *where $x$ is the duration of call, in minutes.*

*(a) What percentage of calls last between 1 and 2 minutes?*
*(b) What percentage of calls last 1 minute or less?*
*(c) What percentage of calls last 3 minutes or more?*
*(d) Find the cumulative distribution function.*

**Exercise 53.10**
*Students at the University of California were surveyed and asked their grade point average.(The GPA ranges from 0 to 4, where 2 is just passing.) The distribution of GPAs is shown in Figure 152.*

*(a) Roughly what fraction of students are passing?*
*(b) Roughly what fraction of students have honor grades (i.e. GPA above 3)?*
*(c) Why do you think there is a peak around 2?*
*(d) Sketch the cumulative distribution function.*

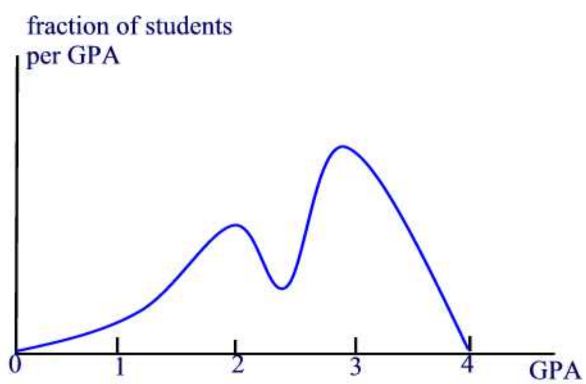



*Figure 152*

# 54 The Median and Mean

In this section we discuss two ways of measuring the "average" value for a distribution function, namely, the *median* and the *mean.*

**The Median**
The median income in the U.S. is the income $M$ such that half the population earn incomes $\leq M$ (so the other half earn incomes $\geq M$). In terms of probability, we can think of income as a random variable $\alpha$. Then the probability that $\alpha \leq M$ is $1/2$, and the probability that $\alpha \geq M$ is also $1/2$.
In general, if $\alpha$ is a continuous random variable then the **median** of $\alpha$ is a number $M$ such that

$$P(\alpha \leq M) = \frac{1}{2}.$$

If $p(x)$ is the probability density function of $\alpha$ then we can calculate the median by solving the equation

$$\int_{-\infty}^{M} p(x)dx = \frac{1}{2}.$$

Graphically, half the area under the graph of $p(x)$ is to the left of $M$. See Figure 153.

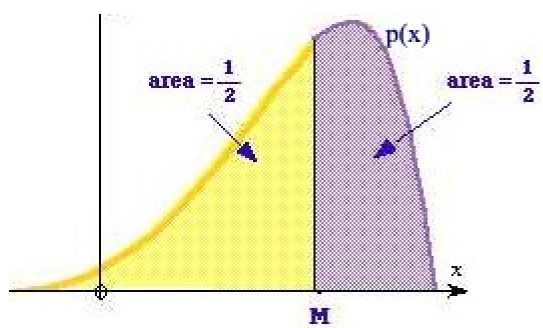



Figure 153

**Example 54.1**
The time in minutes between individuals joining the line at an Ottawa Post Office is a random variable with the density function

$$p(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Find the median time between individuals joining the line and interpret your answer.

**Solution.**
To find the median, we must solve

$$\int_{-\infty}^{M} p(x)dx = \frac{1}{2}$$

or

$$\int_{0}^{M} 2e^{-2x}dx = \frac{1}{2}.$$

Thus,

$$\begin{array}{ccc} -e^{-2x}|_0^M & = & \frac{1}{2} \\ 1-e^{-2M} & = & \frac{1}{2} \\ e^{-2M} & = & \frac{1}{2} \\ 2M & = & \ln 2 \\ M & = & \frac{\ln 2}{2} \approx 0.3466 \end{array}$$

This means that half the people get in line less than 0.3466 minutes (about 21 seconds) after the previous person, while half arrive more than 0.3466 minutes later.■

Sometimes we cannot solve the equation $\int_{-\infty}^{M} p(x)dx = 1/2$ for $M$ analytically, as the next example shows.

**Example 54.2**
Find the median for the random variable with density function

$$p(x) = \begin{cases} 30x^4(1-x), & x \ge 0 \\ 0, & x < 0 \end{cases}$$

**Solution.**

$$\begin{array}{ccc} \int_{-\infty}^{M} p(x)dx & = & \frac{1}{2} \\ \int_0^M 30x^4(1-x)dx & = & \frac{1}{2} \\ \int_0^M (30x^4-30x^5)dx & = & \frac{1}{2} \\ 6x^5-5x^6|_0^M & = & \frac{1}{2} \\ 12M^5-10M^6-1 & = & 0. \end{array}$$

There is no general analytical method for obtaining the solution, the only method we can use is numerical. Using a graphing calculator we find $M \approx 0.735.$■

**The Mean**

Let $S$ be a sample space and $\alpha : S \longrightarrow \mathbb{R}$ be a continuous random variable. The mean of $\alpha$ is the average value of all the values of $\alpha(s)$ where $s$ is an element of $S$. That is,

$$E(\alpha) = \frac{\sum_{s \in S} \alpha(s)}{|S|}$$

where $|S|$ denotes the number of elements of $S$. Finding the above sum when $S$ has a large number of elements is tedious. Instead, we will try to replace the sum by a definite integral.

Let $p(x)$ be the density function (for a continuous random variable $\alpha$) defined on a finite interval $[a, b]$. We will find the mean by the method of slicing. Break up the interval into $n$ subintervals $[x_{i-1}, x_i]$, each of length $\Delta x$, as we did for Riemann sums. From the previous section, we have

$$P(x_{i-1} \leq \alpha \leq x_i) \approx p(x_i)\Delta x,$$

i.e., the approximate area under the graph of $p$ over $[x_{i-1}, x_i]$). If we let $S_i$ be the set of elements in $S$ such that $x_{i-1} \leq \alpha(s) \leq x_i$ then

$$P(x_{i-1} \leq \alpha \leq x_i) = \frac{|S_i|}{|S|}.$$

(This a result from probability theory, if $A$ is a subset of $S$ then $P(A) = \frac{|A|}{|S|}$.) Thus, $|S_i| = |S|p(x_i)\Delta x$. Since for each $s$ in $S_i$ we have $\alpha(s) \approx x_i$ then $x_i p(x_i)|S|\Delta x$ represents the sum of all values satisfying $x_{i-1} \leq \alpha(s) \leq x_i$. Adding together all of the values of $\alpha(s)$ and averaging we find

$$E(X) \approx \sum_{i=1}^{n} x_i p(x_i)\Delta x.$$

Now these approximations get better as $n \to \infty$, and we notice that the sum above is a Riemann sum converging to

$$E(\alpha) = \int_a^b xp(x)dx.$$

The above argument applies if either $a$ or $b$ are infinite. In this case, one has to make sure that all improper integrals in question converge.

**Example 54.3**
Let $\alpha$ be a continuous random variable with density function

$$p(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Find the mean of $\alpha$.

**Solution.**
Using the integral formula for the mean and integration by parts we find

$$\begin{array}{rcl} E(\alpha) & = & \int_0^\infty 2xe^{-2x}dx \\ & = & \lim_{b\to\infty} \int_0^b 2xe^{-2x}dx \\ & = & \lim_{b\to\infty} \left[-xe^{-2x} - \frac{1}{2}e^{-2x}\right]_0^b \\ & = & \lim_{b\to\infty} \left(\frac{1}{2} - \frac{1}{2}e^{-2b} - be^{-2b}\right) = \frac{1}{2} \blacksquare \end{array}$$

**Normal Distribution**
When the graph of the density function $p(x)$ of a random variable is bell-shaped and symmetric about the line going through its central peak then we say that $\alpha$ has a **normal distribution.** Its density function has the formula

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu > 0$ and $\sigma > 0$ called the **standard deviation.**

**Example 54.4**
Consider the normal distribution, $p(x)$.

(a) Show that $p(x)$ has maximum when $x = \mu$.
(b) Show that $p(x)$ has points of inflection at $x = \mu + \sigma$ and $x = \mu - \sigma$.
(c) Describe in your own words what $\mu$ and $\sigma$ tell you about the distribuion.
(d) Represents $P(\mu - \sigma \leq \alpha \leq \mu + \sigma)$ graphically.

**Solution** (a) Finding the derivative $p'(x)$ we obtain

$$p'(x) = -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Thus, $x = \mu$ is the only critical number. Finding the second derivative we obtain

$$p''(x) = \frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left(\frac{(x-\mu)^2}{\sigma^2} - 1\right).$$

Hence, $p''(\mu) = -1 < 0$ so that $x = \mu$ is a maximum.
(b) Solving the equation $p''(x) = 0$ we find $x = \mu \pm \sigma$. If $\mu - \sigma < x < \mu + \sigma$ then $|x - \mu| < \sigma$ and therefore $\frac{(x-\mu)^2}{\sigma^2} < 1$ so that $p''(x) < 0$ and the graph of $p$ is concave down. Similarly, we see that $p''(x) >$ for $x < \mu - \sigma$ or $x > \mu + \sigma$ so that the graph of $p(x)$ is concave up. It follows that $p(x)$ has points of inflection at $x = \mu \pm \sigma$.
(c) It can be shown that $E(x) = \mu$ so that $\mu$ represents the mean of the distribution. Statisticians use the standard deviation of a continuous random variable $\alpha$ as a way of measuring its dispersion, or the degree to which is it "scattered." The larger $\sigma$ is, the wider the bell.
(c) The shaded region in Figure 154 represents $P(\mu - \sigma \leq \alpha \leq \mu + \sigma)$.■

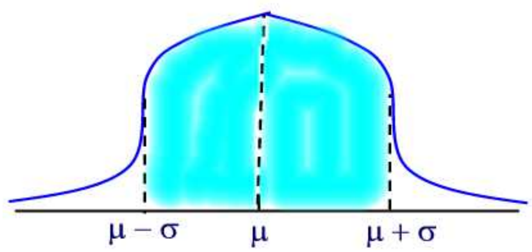



Figure 154

## Practice Problems

**Exercise 54.1**

*(a) Using a calculator or a computer, sketch graphs of the density function of the normal distribution*

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

*(i) For* $\mu = 5$ *and* $\sigma = 1, 2, 3$.
*(ii) For* $\sigma = 1$ *and* $\mu = 4, 5, 6$.
*(b) Explain how the graphs confirm that* $\mu$ *is the mean of the distribution and that* $\sigma$ *is a measure of how close;y the data is clustered around the mean.*

**Exercise 54.2**

*Suppose that* $x$ *measures the time (in hours) it takes for a student to complete an exam. Assume that all students are done within two hours and the density function for* $x$ *is given by*

$$p(x) = \begin{cases} \frac{1}{4}x^3, & \text{if } 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

*(a) What proportion of students take between 1.5 and 2.0 hours to finish the exam.*
*(b) What is the mean time for students to complete the exam?*
*(c) Compute the median of this distribution.*

**Exercise 54.3**

*In 1950 an experiment was done observing the time gaps between successive cars on the Arroyo Seco Freeway. The data show that the density function of these time gaps was given approximately by*

$$p(x) = ae^{-0.122x}$$

*where* $x$ *is the time in seconds and* $a$ *is a constant.*

*(a) Find the value of* $a$.
*(b) Find* $P$, *the cumulative distribution function.*
*(c) Find the median and mean time gap.*
*(d) Sketch rough graphs of* $p$ *and* $P$.

**Exercise 54.4**

*The distribution of IQ scores is often modeled by the normal distribution with mean 100 and standard deviation 15.*

*(a) Write a formula for the density distribution if IQ scores.*
*(b) Estimate the fraction of the population with IQ between 115 and 120.*

**Exercise 54.5**

*The speeds of cars on a road are approximately normally distributed with mean 58 km/hr and standard deviation 4 km/hr.*

*(a) What is the probability that a randomly selected car is going between 60 and 65 km/hr.*
*(b) What fraction of all cars are going slower than 52 km/hr?*

**Exercise 54.6**

*Let $P(x)$ be the cumulative distribution function for the income distribution in the US in 1973 (income is measured in thousands of dollars). Some values of $P(x)$ are int the following table:*

| Income x (thousands) | 1 | 4.4 | 7.8 | 12.6 | 20 | 50 |
|---|---|---|---|---|---|---|
| P(x)(%) | 1 | 10 | 25 | 50 | 75 | 99 |

*(a) What fraction of the population made between \$20,000 and \$50,000.*
*(b) What was the median income?*
*(c) Sketch a density function for this distribution. Where, approximately, does your density function have a maximum? What is the significance of this point, in terms of income distribution? How can you recognize this point on the graph of the density function and on the graph of the cumulative distribution?*

## 55 Geometric Series

Geometric series are frequently used in mathematics. They provide a good introduction of infinite series which we will discuss in the next section.
A **finite geometric series** is a finite sum that starts with an initial value $a$ and then obtain each new term of the series by multiplying by a **common ratio** $r$. Thus, an example of such a series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}.$$

**Example 55.1**
A defense contractor doubles its production every year after producing 1000 units in the first year. Let $S_{10}$ be the production after 10 years. Write $S_{10}$ as a geometric series.

**Solution.**
The sum is given by the finite geometric series

$$S_{10} = 1000 + 1000(2) + 1000(2^2) + \cdots + 1000(2^9). \blacksquare$$

Note that the above sum becomes extremely difficult if the number of years is very large. Thus, one might asks for a simple formula to find the sum for any number of years. We will derive next a formula for any finite geometric series.
Let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiply both sides of the above equation by $r$ to obtain

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Now subtracting this equation from the above equation, we get

$$(1 - r)S_n = a - ar^n = a(1 - r^n).$$

Finally, dividing both sides of the previous equation by $(1 - r)$ to obtain

$$S_n = a\frac{1 - r^n}{1 - r}.$$

Provided that $r \neq 1$. If $r = 1$ then $S_n = na$.

**Example 55.2**
Use the above sum to find the number of units the contractor produces after 10 years.

**Solution.**
Using the above formula with $n = 10, r = 2$, and $a = 1000$ to obtain

$$S_{10} = 1000\frac{1-2^{10}}{1-2} = 1,023,000.\blacksquare$$

A **finite arithmetic series** is a series that starts with initial value $a$ and such that the new term is obtained by adding a constant $r$. Thus, an example of a finite arithmetic series is the series

$$S_n = a + (a+r) + (a+2r) + \cdots + (a+(n-1)r).$$

**Example 55.3**
Find a formula for the series $S_n$. Hint: Write $S_n$ in reverse order and then add this to the initially given form of $S_n$.

**Solution.**
Write the sum in reverse order, i.e.

$$S_n = (a+(n-1)r) + (a+(n-2)r) + (a+(n-3)r) + \cdots + a.$$

Adding this expression of $S_n$ with the initially given expression to obtain

$$2S_n = (2a+(n-1)r) + (2a+(n-1)r) + \cdots + (2a+(n-1)r).$$

That is,

$$2S_n = n[2a+(n-1)r]$$

or

$$S_n = \frac{n}{2}[2a+(n-1)r].\blacksquare$$

**Example 55.4**
A military unit purchases 10 spare parts during the first month of a contract, 15 spare parts during the second month, 20 spare parts during the third month, 25 spare parts during the fourth month and so on. The acquisition officer wants to know the total number of spare parts the unit will have acquired after 50 months.

**Solution**
Let $S_{50}$ be the total number of spare parts after 50 months. Then $S_{50}$ is a finite arithmetic sum with $a = 10, n = 50$, and $r = 5$. Using the formula for an arithmetic finite sum we find

$$S_{50} = \frac{50}{2}[2(10) + (50-1)2] = 6,625 \textit{ spare parts.}\blacksquare$$

**Example 55.5**
Let $S_n$ be the sum of the first $n$ positive integers, i.e.

$$S_n = 1 + 2 + 3 + \cdots + n.$$

Find a formula for $S_n$.

**Solution.**
According to the formula for a finite arithmetic sum with $a = 1$ and $r = 1$ we have

$$S_n = \frac{n}{2}(n+1).\blacksquare$$

**Example 55.6**
Find the sum of the series

$$3 + \frac{3}{2} + \frac{3}{4} + \cdots + \frac{3}{2^{10}}.$$

**Solution.**
We have

$$\begin{aligned}
3 + \tfrac{3}{2} + \tfrac{3}{4} + \cdots + \tfrac{3}{2^{10}} &= 3(1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^{10}}) \\
&= 3\tfrac{1-\frac{1}{2^{11}}}{1-\frac{1}{2}} = 6(1 - 2^{-11}).\blacksquare
\end{aligned}$$

Now, an **infinite geometric series** with initial value $a$ and ratio $r$ is any infinite sum of the form

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

We define the following finite sums

$$\begin{aligned}
S_1 &= a \\
S_2 &= a + ar \\
S_3 &= a + ar + ar^2 \\
&\vdots \\
S_n &= a + ar + ar^2 + \cdots + ar^{n-1}.
\end{aligned}$$

We call the above sums, **partial sums**. Note that $S_n$ is a finite geometric series so that

$$S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1.$$

**Theorem 55.1**
(a) If $r = 1$ then $S_n = na$ and $S_n$ becomes large when $n$ is large if $a > 0$ and negatively large if $a < 0$.
(b) If $r = -1$ then $r^n$ alternates between $-1$ and 1 so that $r^n$ has no limit when $n$ is large.
(c) If $|r| > 1$ then the geometric series is divergent. (d) If $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$ and therefore $\lim_{n\to\infty} S_n = \frac{a}{1-r}$. In this case we say that the infinite geometric series **converges** with sum equals to $\frac{a}{1-r}$ and we write

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$

If a geometric series does not converge then we say that it **diverges.**

**Proof.**
The prove is given in Theorem 56.4 of the next section.■

**Example 55.7**
Determine whether the geometric series given below is convergent or divergent.

$$1000 + 1000(1.05) + 1000(1.05)^2 + \cdots$$

**Solution.**
Finding the nth partial sum we have

$$S_n = \frac{1000(1-(1.05)^n}{1-1.05}.$$

But $(1.05)^n \longrightarrow \infty$ as $n \longrightarrow \infty$ so that the given series is divergent.■

**Example 55.8**
Determine whether the geometric series given below is convergent or divergent.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

**Solution.**
Finding the nth partial sum we have

$$S_n = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}.$$

But $\left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \longrightarrow \infty$ so that the series converges to 2. ■

## Practice Problems

In Exercises 1 - 7, decide which of the following are geometric series. For those which are, give the first term and the ratio and the value of the sum. For those which are not, explain why not.

**Exercise 55.1**
$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

**Exercise 55.2**
$1 + \frac{1}{2} + \frac{1}{3} + \cdots$

**Exercise 55.3**
$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

**Exercise 55.4**
$y^2 + y^3 + y^4 + y^5 + \cdots$

**Exercise 55.5**
$1 - x + x^2 - x^3 + x^4 - \cdots$

**Exercise 55.6**
$1 - y^2 + y^4 - y^6 + \cdots$

**Exercise 55.7**
$1 + (2z) + (2z)^2 + (2z)^3 + \cdots$

**Exercise 55.8**
*Find the sum of the series* $\sum_{n=4}^{\infty}(\frac{1}{3})^n$.

**Exercise 55.9**
*A ball is dropped from a height of 10 feet and bounces. Each bounce is* $\frac{3}{4}$ *of the height of the bounce before. Thus after the ball hits the floor the first time, the ball rises to a height of* $10(\frac{3}{4}) = 7.5$ *feet, and after it hits the floor for the second time, it rises to a height of* $7.5(\frac{3}{4}) = 10(\frac{3}{4})^2 = 5.625$ *feet.*
*(a) Find an expression for the height to which the ball rises after it hits the floor for the nth time.*
*(b) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the first, second, third, and fourth times.*
*(c) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the nth time. Express your answer in closed-form.*

# 56 Convergence of Sequences and Series

We start this section by introducing the concept of a sequence and study its convergence.

**Convergence of Sequences**.
An **infinite sequence** is a list of numbers $a_1, a_2, a_3, \cdots$.

**Example 56.1**
(1) $1, \frac{1}{2}, \frac{1}{3}, \cdots$.
(2) $0, 2, 0, 2, 0, \cdots$. ■

Formally, a sequence is a function $f$ with domain the set of nonnegative integers $\mathbb{N}$. The image of $n$ is denoted by $f(n) = a_n$. We represent a sequence by the notation $\{a_n\}_{n=1}^{\infty}$ and we call $a_n$ the **nth term** of the sequence. For example, the nth term of the sequence (1) above is $a_n = \frac{1}{n}, n \geq 1$ while the nth term of (2) is $a_n = 1 + (-1)^n$.

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to a number $L$ if and only if $a_n$ approaches $L$ as $n$ gets larger and larger, i.e if the difference $|a_n - L|$ can be made as small as we wish by taking $n$ large enough. Using the $\epsilon - \delta$ argument, if $\epsilon > 0$ is given then we can find a positive integer $N$ such that

$$|a_n - L| < \epsilon \quad for\ n \geq N.$$

We write

$$\lim_{n \to \infty} a_n = L.$$

If a sequence does not converge then we say that it **diverges**.

**Example 56.2**
(1) Since $\lim_{n\to\infty} \frac{3n^2+5}{n^2+n+1} = 3$ then the sequence $\{\frac{3n^2+5}{n^2+n+1}\}_{n=1}^{\infty}$ converges to 3.
(2) Since $(-1)^n$ alternates between -1 and 1 then the sequence $\{1+(-1)^n\}_{n=1}^{\infty}$ alternates between 0 and 2 and so it diverges.■

**Theorem 56.1** *(Convergence Properties of Sequences)*
(a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
(b) $\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n$, where $k$ is a constant.
(c) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$.
(d) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided that $\lim_{n\to\infty} b_n \neq 0$.
(e) Squeeze Principle: If $a_n \leq b_n \leq c_n$ for all $n$ and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

**Proof.**
We will prove (a) and leave the rest as an exercise. Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L'$. We will show that $lim_{n\to\infty}(a_n+b_n) = L+L'$. Indeed, let $\epsilon > 0$ be given. There there exist poisitve integers $N_1$ and $N_2$ such that

$$|a_n - L| < \frac{\epsilon}{2} \quad for\ n \geq N_1$$

and

$$|b_n - L'| < \frac{\epsilon}{2} \quad for\ n \geq N_2.$$

Let $N = N_1 + N_2$. Then for $n \geq N$ we have

$$|(a_n + b_n) - (L + L')| \leq |a_n - L| + |b_n - L'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon. \blacksquare$$

Next, we discuss a very useful theorem that establishes the convergence of a given sequence (without, however, revealing the limit of the sequence). But first we introduce a couple of definitions:
We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if $a_n \leq a_m$ whenever $n \leq m$.
A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded from above** if there is a positive constant $C$ such that $a_n \leq C$ for all $n \geq 1$.

**Theorem 56.2** *(Monotone Convergence Theorem)*
An increasing sequence that is bounded from above is always convergent.

**Proof.**
Since $\{a_n\}_{n=1}^{\infty}$ is bounded from above then there exists a $C > 0$ such that $a_n \leq C$ for all $n \geq 1$. Let $c$ be the smallest positive constant such that $a_n \leq c$ for all $n \geq 1$. We will show that $\lim_{n\to\infty} a_n = c$.
Let $\epsilon > 0$. Then there is an integer $N$ such that (See Figure 155).

$$a_N > c - \epsilon.$$

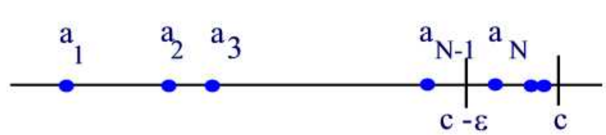



Figure 155

Since the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing then

$$a_n > c - \epsilon$$

for all $n \geq N$. Thus,

$$|a_n - c| = |c - a_n| < \epsilon$$

for all $n \geq N$. This shows that the sequence is convergent.■

The following theorem shows the connection between the convergence of a function and the convergence of a sequence.(See Figure 156.) This basically allows us to replace limits of sequences with limits of functions. In particular, this is useful for using L'Hôpital's rule in computing limits of sequences.

**Theorem 56.3**
If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ then $\lim_{n\to\infty} a_n = L$.

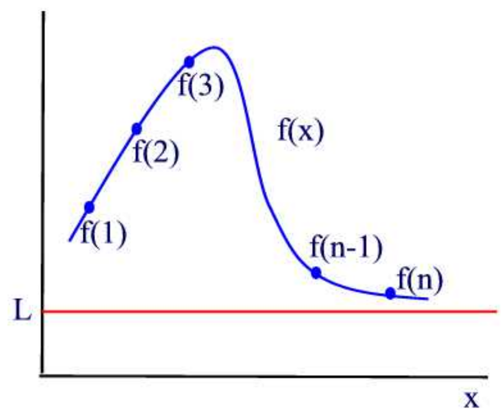



Figure 156

**Proof.**
Let $\epsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = L$ then we can find a $\delta > 0$ such that

$$|f(x) - L < \epsilon \quad if \quad |x| \geq \delta.$$

Let $N = (integer\ part\ of\ \delta) + 1$. Then $N$ is a positive integer greater than or equal to $\delta$. Thus, if $n \geq N$ then

$$|a_n - L| = |f(n) - L| < \epsilon.$$

This shows that $\lim_{n\to\infty} a_n = L$.■

**Example 56.3**
(a) Find $\lim_{n\to\infty}\frac{\ln(n+1)}{n+1}$.
(b) Find $\lim_{n\to\infty}\frac{\sin n}{\sqrt{n+1}}$.

**Solution.**
(a) Applying L'Hôpital's rule we have

$$\begin{array}{rcl}\lim_{n\to\infty}\frac{\ln(n+1)}{n+1} & = & \lim_{x\to\infty}\frac{\ln(x+1)}{x+1}\\ & = & \lim_{x\to\infty}\frac{\frac{1}{x+1}}{1}=0.\end{array}$$

(b) Since $-1\le\sin n\le 1$ then $-\frac{1}{\sqrt{n+1}}\le\frac{\sin n}{\sqrt{n+1}}\le\frac{1}{\sqrt{n+1}}$. But $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}=\lim_{x\to\infty}\frac{1}{\sqrt{x+1}}=0$. By the squeeze rule we have

$$\lim_{n\to\infty}\frac{\sin n}{\sqrt{n+1}}=0.\blacksquare$$

The following is a list of useful sequences.

**Theorem 56.4**
(a) If $r=1$ then $\lim_{n\to\infty}r^n=1$.
(b) If $r=-1$ then the sequence $\{r^n\}$ is divergent.
(c) If $|r|>1$ then the sequence $\{r^n\}$ is divergent.
(d) If $|r|<1$ then $\lim_{n\to\infty}r^n=0$.
(e) For $r>0$, $\lim_{n\to\infty}r^{\frac{1}{n}}=1$.

**Proof.**
(a) Trivial.
(b) If $r=-1$ then the sequence $\{r^n\}$ alternates between the two values -1 and 1 and so the sequence is divergent.
(c) $|r|>1$ implies $r>1$ or $r<-1$. Suppose first that $r>1$. Let $\epsilon>0$. Let $N$ be a positive integer greater than $\frac{\epsilon}{r-1}$. Then for $n\ge N$ we have

$$\begin{array}{rcl}r^n & = & (1+(r-1))^n\\ & \ge & 1+n(r-1)(by\ the\ binomial\ formula)\\ & > & 1+N(r-1)\\ & > & \epsilon.\end{array}$$

This shows that for any given positive number we can find a term in the sequence $\{r^n\}$ which is greater than the number. This means that $\{r^n\}\to\infty$

as $n \to \infty$.
If $r < -1$ then $r^n = (-1)^n(-r)^n$ with $-r > 1$. Thus, as $n$ becomes large, $r^n$ alternates between large positive numbers and large negative numbers so that again the limit $r^n$ does not exist.
(d) If $0 < r < 1$ then $r^n = \frac{1}{(r^{-1})^n}$ with $(r^{-1})^n \to \infty$ as $n \to \infty$. (See (c)). Hence, $r^n \to 0$ as $n \to \infty$.
If $-1 < r < 0$ then $0 < -r < 1$. In this case, $r^n = (-1)^n(-r)^n \to 0$ as $n \to \infty$.
(e) Let $d = r^{\frac{1}{n}}$. Then $\ln d = \frac{\ln c}{n} \to \infty$ as $n \to \infty$. Hence, $d = e^{\ln d} \to e^0 = 1$ as $n \to \infty$.■

**Convergence of Series**
The greek letter $\Sigma$ is used to denote summations such as

$$\Sigma_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

or

$$\Sigma_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n.$$

The sum of a sequence $\{a_n\}_{n=1}^{\infty}$ is called a **series**, denoted by

$$\Sigma_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

To determine whether this series converges or not we consider the sequence of **partial sums** defined as follows:

$$\begin{array}{rcl} S_1 & = & a_1 \\ S_2 & = & a_1 + a_2 \\ \vdots & & \\ S_n & = & a_1 + a_2 + \cdots + a_n. \end{array}$$

We say that a series $\Sigma_{n=1}^{\infty} a_n$ **converges** to a number $L$ if and only if the sequence $\{S_n\}_{n=1}^{\infty}$ converges to $L$ and we write

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

A series which is not convergent is said to **diverge.**

**Example 56.4**
(a) The geometric series $\Sigma_{n=1}^{\infty} x^n$ converges for $|x| < 1$ with sum equals to $\frac{x}{1-x}$.
(b) The series $\Sigma_{n=1}^{\infty}(-1)^n$ diverges since the sequence of partial sums alternates between the values -1 and 0. ■

**Example 56.5**
Show that the series $\Sigma_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges to 1.

**Solution.**
Using partial fractions we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus,

$$\begin{array}{rcc} S_1 & = & 1-\frac{1}{2} \\ S_2 & = & (1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3}) = 1-\frac{1}{3} \\ S_3 & = & S_2+(\frac{1}{3}-\frac{1}{4}) = (1-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4}) = 1-\frac{1}{4} \\ \vdots & & \\ S_n & = & 1-\frac{1}{n+1}. \end{array}$$

It follows that $\lim_{n\to\infty} S_n = 1.$■

Using Theorem 56.1 (a) and (b) we have the following properties of convergent series.

**Theorem 56.5**
If $\Sigma_{n=1}^{\infty}a_n$ and $\Sigma_{n=1}^{\infty}b_n$ are two convergent series and $k$ is a constant then
(a) $\Sigma_{n=1}^{\infty}(a_n \pm b_n) = \Sigma_{n=1}^{\infty}a_n \pm \Sigma_{n=1}^{\infty}b_n$
(b) $\Sigma_{n=1}^{\infty}ka_n = k\Sigma_{n=1}^{\infty}a_n$.

**Theorem 56.6**
Let $N$ be a positive integer. Suppose that $a_n = b_n$ for all $n \geq N$. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. Thus, changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.

**Proof.**
The proof follows from the equality $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N}(a_n - b_n) + \sum_{n=1}^{\infty}$ .■

Improper integrals can be used to determine the convergence or divergence of some series as shown by the following theorem.

**Theorem 56.7** *(The Integral Test)*
Suppose that $f(x)$ is a positive and decreasing function. Assume that $a_n =$

$f(n)$.
(a) If $\int_1^\infty f(x)dx$ converges, then the series $\Sigma_{n=1}^\infty a_n$ converges.
(b) If $\int_1^\infty f(x)dx$ diverges, then the series $\Sigma_{n=1}^\infty a_n$ diverges.

**Proof.**
Since $f$ is decreasing then for any $i \geq 1$

$$f(i+1) \leq \int_i^{i+1} f(x)dx \leq f(i).$$

See Figure157.

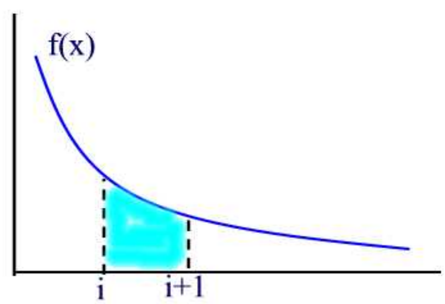



Figure 157

But $a_i = f(i)$ so that

$$a_{i+1} \leq \int_i^{i+1} f(x)dx \leq a_i.$$

Adding these inequalities for $i = 1$ to $i = n$ to obtain

$$a_2 + a_3 + \cdot a_{n+1} \leq \int_1^{n+1} f(x)dx \leq a_1 + a_2 + \cdot a_n.$$

If $\{S_n\}$ is the sequence of partial sums associated to the series $\sum_{n=1}^\infty a_n$ , then we have

$$S_{n+1} - a_1 \leq \int_1^{n+1} f(x)dx \leq S_n.$$

or equivalently

$$\int_1^{n+1} f(x)dx \leq S_n \leq a_1 + \int_1^n f(x)dx.$$

(a) If $\int_1^\infty f(x)dx$ converges then the se quence $\{S_n\}$ converges since

$$S_n \leq a_1 + \int_1^n f(x)dx.$$

Therefore $\sum_{n=1}^\infty a_n$ converges.
(b) If $\int_1^n f(x)dx$ is divergent then $\{S_n\}$ is divergent since

$$\int_1^{n+1} f(x)dx \leq S_n.$$

Thus, $\sum_{n=1}^\infty a_n$ is divergent.■

**Remark 56.1**
Note that if $c \geq 0$ then

$$\int_c^\infty f(x)dx = \int_c^1 f(x)dx + \int_1^\infty f(x)dx.$$

**Example 56.6**
Show that the series $\Sigma_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$. This series is referred to as **p-series**.

**Solution.**
We already know that the improper integral $\int_1^\infty \frac{1}{x^p}dx$ converges for $p > 1$ and diverges for $p \leq 1$. By the Integral Test the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.■

The following result provides a procedure for testing the divergence of a series. This is known as the the **nth term** test for convergence.

**Theorem 56.8**
If the series $\Sigma_{n=1}^\infty a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

**Solution.**
We know that $S_n = a_1 + a_2 + \cdots + a_n$ and $S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1} = S_n + a_n$ so it follows that $S_{n+1} - S_n = a_n$. Suppose that the series converges to a number $L$. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+1} = L$. Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_{n+1} - S_n) = L - L = 0$. ■

**Remark 56.2**
The theorem states that if we know that the series is convergent then $\lim_{n\to\infty} a_n = 0$. The converse is not true in general. That is, the condition $\lim_{n\to\infty} a_n = 0$ does not necessarily imply that the series $\sum_{n=1}^\infty a_n$ is convergent. By Example 56.6, the series $\sum_{n=1}^\infty \frac{1}{n}$ is divergent even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

## Practice Problems

Exercises 1 - 6 give expressions for the nth term $a_n$ of a sequence. Find the limit of each sequence, if it exists.

**Exercise 56.1**
$a_n = (0.2)^n$.

**Exercise 56.2**
$a_n = 2^n$.

**Exercise 56.3**
$a_n = 3 + e^{-2n}$.

**Exercise 56.4**
$a_n = \cos(\pi n)$.

**Exercise 56.5**
$a_n = \frac{3+4n}{5+7n}$.

**Exercise 56.6**
$a_n = \frac{2n+(-1)^n 5}{4n+(-1)^n 3}$.

**Exercise 56.7**
*Use the integral test to decide whether the series* $\Sigma_{n=1}^{\infty} \frac{n}{n^2+1}$ *converges or diverges.*

**Exercise 56.8**
*Use the integral test to decide whether the series* $\Sigma_{n=1}^{\infty} ne^{-n^2}$ *converges or diverges.*

Do the series in Exercises 9 - 12 converge or diverge?

**Exercise 56.9**
$\Sigma_{n=1}^{\infty}((\frac{3}{4})^n + \frac{1}{n})$.

**Exercise 56.10**
$\Sigma_{n=1}^{\infty} \frac{3}{n+2}$

**Exercise 56.11**
$\sum_{n=1}^{\infty} \frac{n}{n+1}$.

**Exercise 56.12**
$\sum_{n=1}^{\infty} \frac{3}{(2n-1)^2}$.

**Exercise 56.13**
*Find the sum* $\sum_{n=1}^{\infty} \frac{3^n+5}{4^n}$.

# 57 Tests for Convergence

### The Comparison Test

Going back to our discussion of improper integrals, we recall the comparison theorem which tells that an improper integral is convergent if it is smaller than a known convergent improper integral and is divergent if it is larger than a known divergent improper integral. A similar result holds for series.

**Theorem 57.1** *(Comparison test)*
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of nonnegative terms.
(i) If $a_n \leq b_n$ for $n \geq N$ and the series $\sum_{n=1}^{\infty} b_n$ is convergent then the series $\sum_{n=1}^{\infty} a_n$ is convergent as well.
(ii) If $a_n \leq b_n$ for $n \geq N$ and the series $\sum_{n=1}^{\infty} a_n$ is divergent then the series $\sum_{n=1}^{\infty} b_n$ is also divergent.

**Proof.**
Let $a'_n = a_{N+n-1}$ and $b'_n = b_{N+n-1}$. Since the convergence or divergence of a series is not affected by deleting a finite number of terms then either the series $\sum_{n=1}^{\infty} a'_n$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge. The same applies fo the series with terms $b_n$ and $b'_n$.
(a) Let $S_n$ and $T_n$ be the nth partial sums of $\sum_{n=1}^{\infty} a'_n$ and $\sum_{n=1}^{\infty} b'_n$. Then $S_n \leq T_n$. If $\sum_{n=1}^{\infty} b_n$ converges, say to $T$, then $T_n \to T$ as $n \to \infty$ and $T_n < T$ for all $n$ since $T_n$ is increasing. Thus, $S_n \leq T_n < T$. So we have that $S_n$ is increasing and bounded from above. By Theorem 56.2, $\{S_n\}$ is convergent and therefore $\sum_{n=1}^{\infty} a'_n$ is convergent. Hence, $\sum_{n=1}^{\infty} a_n$ converges.
(b) Now suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Then $\lim_{n\to\infty} S_n = \infty$. But $T_n \geq S_n$ so that $\lim_{n\to\infty} T_n = \infty$. This shows that $\sum_{n=1}^{\infty} b'_n$ is divergent and consequently the series $\sum_{n=1}^{\infty} b_n$ is divergent.$\blacksquare$

Keep in mind that the major purpose of this comparison test is that some series are very difficult to test for convergence directly. Instead we compare them to well known series such as the p-series.

**Example 57.1**
Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-n+1}}$ is divergent.

**Solution.**
Indeed, for $n \geq 1$ we have $n^2+(1-n) \leq n^2$ so that $\sqrt{n^2-n+1} \leq \sqrt{n^2} = n$. This implies that $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n^2-n+1}}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series) then the comparison test asserts that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-n+1}}$ is divergent.■

**Example 57.2**
Show that the series $\sum_{n=1}^{\infty} \frac{n-1}{n^3+n}$ is convergent.

**Solution.**
To see this, note that $n^3+n \geq n^3$ so that $\frac{1}{n^3+n} \leq \frac{1}{n^3}$. Thus, $\frac{n-1}{n^3+n} \leq \frac{n-1}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then by the comparison test the series $\sum_{n=1}^{\infty} \frac{n-1}{n^3+n}$ is also convergent.■

The difficulty with the comparison test is that when the nth term of a series $\sum_{n=1}^{\infty} a_n$ is complicated then it might be difficult to figure out the series $\sum_{n=1}^{\infty} b_n$ that need to be compared with. The following comparison test is often easier to apply, because after deciding on $\sum_{n=1}^{\infty} b_n$ we need only take a limit of the quotient $\frac{a_n}{b_n}$ as $n \to \infty$.

**Theorem 57.2**
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

then either both series converge or both diverge.

**Proof.**
Since $L > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ then there exists a positive integer $N$ such that for $n \geq N$ we have

$$|\frac{a_n}{b_n} - L| < \frac{L}{2}$$

i.e.

$$L - \frac{L}{2} < \frac{a_n}{b_n} < c + \frac{c}{2}$$

or

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3}{2}L.$$

This is equivalent to

$$\frac{L}{2}b_n < a_n < \frac{3}{2}Lb_n, \quad n \ge N.$$

If the series $\sum_{n=1}^{\infty} a_n$ converges then by the comparison test the series $\sum_{n=1}^{\infty} \frac{L}{2}b_n$ is convergent. By Theorem 56.5(b), the series $\sum_{n=1}^{\infty} b_n$ is also convergent. Conversely, if $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} \frac{3}{2}Lb_n$ is convergent and by the comparison test $\sum_{n=1}^{\infty} a_n$ is convergent. Similarly, $\sum_{n=1}^{\infty} a_n$ is divergent if and only if $\sum_{n=1}^{\infty} b_n$ is divergent.$\blacksquare$

**Example 57.3**
Determine whether the series $\sum_{n=1}^{\infty} \frac{3n+1}{4n^3+n^2-2}$ converges or diverges.

**Solution.**
For large $n$ we have $a_n = \frac{3n+1}{4n^3+n^2-2} \approx \frac{3n}{4n^3} = \frac{3}{4n^2}$. So let $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^3+n^2}{4n^3+n^2-2} = \frac{3}{4}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then so does the series $\sum_{n=1}^{\infty} \frac{3n+1}{4n^3+n^2-2}$. $\blacksquare$

**Absolute Convergence**

Consider a series $\sum_{n=1}^{\infty} a_n$ which has both positive and negative terms. We say that this series is **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent. The following theorem provides a test of convergence for series of the above type.

**Theorem 57.3**
If $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

**Proof.**
The proof of this result is quite simple. Let $b_n = |a_n| \ge 0$. By assumption, the series $\sum_{n=1}^{\infty} b_n = \sum_{n}^{\infty} |a_n|$ is convergent. But $|a_n| \le b_n$ for all $n$. Now, part (i) of the comparison test asserts that the series $\sum_{n=1}^{\infty} a_n$ must be convergent.$\blacksquare$

**Example 57.4**
Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is convergent.

**Solution.**
Indeed, the series of absolute values $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (p-series with $p=2$) so by the above theorem, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is also convergent.■

**Example 57.5**
Show that the series $1-x+x^2-x^3+\cdots$ is absolutely convergent for $|x|<1$.

**Solution.**
Since the geometric series $1+x+x^2+x^3+\cdots$ converges for $|x|<1$ then the given series is absolutely convergent.■

**Remark 57.1**
It is very important to be very careful with the statement of the above theorem. The theorem says that if we know that the series $\sum_{n=1}^{\infty}|a_n|$ is convergent then the series $\sum_{n=1}^{\infty}a_n$ is definitely convergent. However, it is possible that $\sum_{n=1}^{\infty}|a_n|$ is divergent and still $\sum_{n=1}^{\infty}a_n$ is convergent. The following example illustrates this situation.

**Example 57.6**
Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent but the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

**Solution.**
The alternating series test (see discussion below) asserts that the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ is convergent. However, the series $\sum_{n=1}^{\infty}|\frac{(-1)^{n-1}}{n}| = \sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (harmonic series)■

When a series is such that $\sum_{n=1}^{\infty}|a_n|$ is divergent but $\sum_{n=1}^{\infty}a_n$ is convergent then we say that the series $\sum_{n=1}^{\infty}a_n$ is **conditionally convergent**.

**The Ratio Test**

The integral test is hard to apply when the integrand involves factorials or complicated expressions. We shall now introduce a test that can be used to help determine convergence or divergence of series when other tests are not applicable.

**Theorem 57.4**
Let $\sum_{n=1}^{\infty}a_n$ be a series and suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$.

(a) If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges.
(b) If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.
(c) If $L = 1$ then the test fails, that is the test does not tell us anything about the convergence of the series.

**Proof.**
(a) Suppose that $0 \le L < 1$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ then we can find a positive integer $N$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| - L \le \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \frac{1-L}{2}, \ for\ n \ge N.$$

This is equivalent to

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{L+1}{2}, \ for\ n \ge N.$$

Let $r = \frac{L+1}{2}$. Clearly, $L < r < 1$. Thus,

$$|a_{n+1|} < r|a_n|, \ for\ n \ge N.$$

Hence,

$$\begin{array}{rcl} |a_{N+1}| & < & r|a_N| \\ |a_{N+2}| & < & r|a_{N+1}| < r^2|a_N| \\ |a_{N+3}| & < & r|a_{N+2}| < r^3|a_N| \\ \vdots & & \end{array}$$

Since the series $\sum_{n=1}^{\infty} r^n|a_N|$ is a geometric series with $r < 1$ then it is convergent. By the comparison test the series $\sum_{n=1}^{\infty} |a_{N+n}|$ is also convergent. By Theorem 57.3, the series $\sum_{n=1}^{\infty} a_{N+n}$ is convergent. Since the convergence or divergence is unaffected by deleting a finite number of terms then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
(b) Suppose now that $L > 1$. Then there is a positive integer $N$ such that for $n \ge N$

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \frac{L-1}{2}$$

or

$$-\frac{L-1}{2} < \left|\frac{a_{n+1}}{a_n}\right| - L < \frac{L-1}{2}$$

i.e.,

$$\left|\frac{a_{n+1}}{a_n}\right| > \frac{L+1}{2}, \ for\ n \ge N.$$

Let $r = \frac{L+1}{2}$. Then $L > r > 1$. Thus, for $n \geq N$ we have $\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $|a_{n+1}| > |a_n|$. This implies that $\lim_{n\to\infty} a_n \neq 0$ and by Theorem 56.8, the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(c) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. Then this series is divergent and $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$. On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent with $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = 1$. Thus, when $L = 1$ the test is inconclusive.■

**Example 57.7**
1. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent by the alternating series test. Note that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{n}{n+1} = 1$ i.e., $L = 1$ in the previous theorem.
2. The series $\sum_{n=1}^{\infty}(-1)^{n-1}$ is divergent. Also, note that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, i.e. $L = 1$.
3. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ is convergent since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$ so $L = 0 < 1$ in the above theorem.
4. The series $\sum_{n=1}^{\infty}(-2)^{n-1}$ is divergent since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 2 > 1$.■

**Remark 57.2**
When testing a series for convergence, normally concentrate on the nth term test and the ratio test. Use the comparison test only when both tests fail.

**Alternating Series Test**
By an **alternating series** we mean a series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where $a_n > 0$. For instance, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. Here $a_n = \frac{1}{n}$. The following theorem provides a way for testing alternating series for convergence.

**Theorem 57.5** *(Alternating Series Test)*
An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent if and only if:
(i) The sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing, i.e. $a_{n+1} < a_n$ for all $n$;
(ii) $\lim_{n\to\infty} a_n = 0$.

**Proof.**
Let $\{S_n\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$. Notice the following

$$
\begin{aligned}
S_4 - S_2 &= a_3 - a_4 \geq 0 \\
S_6 - S_4 &= a_5 - a_6 \geq 0 \\
S_8 - S_6 &= a_7 - a_8 \geq 0 \\
&\vdots
\end{aligned}
$$

Thus,

$$S_2 \leq S_4 \leq S_6 \leq S_8 \leq \cdots;$$

It follows that the sequence $\{S_{2n}\}$ is increasing. Moreover, for all $n \geq 1$ we have

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1.$$

Hence, the sequence $\{S_{2n}\}$ is bounded from above. By Theorem 56.2, there is an $S > 0$ such that $\lim_{n\to\infty} S_{2n} = S$.
Next, we consider the terms of $\{S_n\}$ with odd subscripts:

$$\begin{array}{rcl} S_1 - S_3 &=& a_2 - a_3 \geq 0 \\ S_3 - S_5 &=& a_4 - a_5 \geq 0 \\ S_5 - S_7 &=& a_6 - a_7 \geq 0 \vdots \end{array}$$

Thus,

$$S_1 \geq S_3 \geq S_5 \geq S_7 \geq \cdots;$$

It follows that the sequence $\{S_{2n+1}\}$ is decreasing. Moreover, $S_{2n+1} = S_{2n} + a_{2n+1}$. Thus, $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = S + 0 = S$. It follows that

$$\lim_{n\to\infty} S_n = S.$$

Figure 158 shows how the terms of $\{S_n\}$ and $S$ are ordered on a line.■

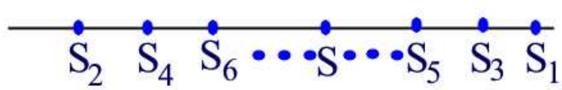



Figure 158

**Example 57.8**
Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

**Solution.**
To see this, let $a_n = \frac{1}{n}$. Then $n < n+1$ implies that $\frac{1}{n+1} < \frac{1}{n}$ that is $a_{n+1} < a_n$. Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. Hence, by the previous theorem the given series is convergent.■

**Remark 57.3**
(a)It follows from the above theorem that if $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n \leq a_1$.
(b) From Figure 158, for each $n \geq 1$, we have $S$ is between $S_n$ and $S_{n+1}$ so that the distance between $S$ and $S_n$ is less than the distance between $S_n$ and $S_{n+1}$. Thus, we have an upper bound for the error:

$$|S_n - S| < |S_{n+1} - S_n| = a_{n+1}.$$

(c) Keep in mind that the tests used in this section are basically used to test for convergence. However, when a series is convergent these tests do not provide a value for the sum.

## Practice Problems

Use the comparison test to determine whether the series in Exercises 1-7 converge.

**Exercise 57.1**
$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$.

**Exercise 57.2**
$\sum_{n=1}^{\infty} \frac{1}{\ln{(n+1)}}$.

**Exercise 57.3**
$\sum_{n=1}^{\infty} \frac{1}{3^n+1}$.

**Exercise 57.4**
$\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$.

**Exercise 57.5**
$\sum_{n=1}^{\infty} 2^{-n}\frac{n+1}{n+2}$.

**Exercise 57.6**
$\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$.

**Exercise 57.7**
$\sum_{n=1}^{\infty} \frac{n\sin{n}}{n^3+1}$.

**Exercise 57.8**
*Use the ratio test to show that the series* $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ *is convergent.*

**Exercise 57.9**
*Use the ratio test to show that the series* $\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$ *is divergent.*

**Exercise 57.10**
*Use the alternating series test to show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ *is convergent.*

Determine which of the series in Exercises 11 - 14 converge.

**Exercise 57.11**
$\sum_{n=1}^{\infty} e^{-n}$.

**Exercise 57.12**
$\sum_{n=1}^{\infty} e^n$.

**Exercise 57.13**
$\sum_{n=1}^{\infty} \frac{(n-1)!}{n^2}$.

**Exercise 57.14**
$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$.

**Exercise 57.15**
*Show that if* $\sum_{n=1}^{\infty} |a_n|$ *converges then* $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *converges.*

# 58 Power Series

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of numbers. Then a **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Note that a power series about $x = a$ always converges at $x = a$ with sum equals to $a_0$.

**Example 58.1**
1. A polynomial of degree $m$ is a power series about $x = 0$ since

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m.$$

Note that $a_n = 0$ for $n \geq m+1$.
2. The geometric series $1 + x + x^2 + \cdots$ is a power series about $x = 0$ with $a_n = 1$ for all n.
3. The series $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$ is not a power series since it has negative powers of $x$.
4. The series $1 + x + (x-1)^2 + (x-2)^3 + (x-3)^4 + \cdots$ is not a power series since each term is a power of a different quantity.■

**Convergence of Power Series**
To study the convergence of a power series about $x = a$ one starts by fixing $x$ and then constructing the partial sums

$$\begin{array}{rcc} S_0(x) & = & a_0, \\ S_1(x) & = & a_0 + a_1(x-a), \\ S_2(x) & = & a_0 + a_1(x-a) + a_2(x-a)^2, \\ \vdots & & \\ S_n(x) & = & a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n. \end{array}$$

Thus obtaining the sequence $\{S_n(x)\}_{n=0}^{\infty}$. If this sequence converges to a number $L$, i.e. $\lim_{n\to\infty} S_n(x) = L$, then we say that the power series **converges** to $L$ for the specific value of $x$. Otherwise, we say that the power series **diverges**.
Power series may converge for some values of $x$ and diverge for other values. The following theorems provides a tool for determining the values of $x$ for which the power series converges and those for which it diverges.

**Theorem 58.1**
Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series that converges for $x=c+a$ and diverges for $x=d+a$. Then

(i) $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for $|x-a|<|c|$; and
(ii) $\sum_{n=0}^{\infty} a_n(x-a)^n$ diverges for $|x-a|>|d|$.

**Proof.**
(i) There is nothing to show if $x=a$, i.e $c=0$. So we assume that $\sum_{n=0}^{\infty} a_n(c)^n$ converges with $c\neq 0$. By the nth term test, $\lim_{n\to\infty} a_nc^n=0$. Thus, there exists a positive integer $N$ such that $|a_nc^n|<1$ for all $n\geq N$. Let $M=\sum_{n=0}^{N-1}|a_nc^n|+1$. Then, $|a_nc^n|\leq M$ for all $n\geq 0$. Now, for $n\geq 0$ we have

$$|a_n(x-a)^n|=|a_nc^n|\cdot\left|\frac{x-a}{c}\right|^n\leq M\left|\frac{x-a}{c}\right|^n.$$

But the series $\sum_{n=0}^{\infty} M\left|\frac{x-a}{c}\right|^n$ is a convergent geometric series since $\left|\frac{x-a}{c}\right|<1$. Hence, by the comparison test $\sum_{n=0}^{\infty}|a_n(x-a)^n|$ converges.
(ii) Suppose that $|x-a|>|d|$. If $\sum_{n=0}^{\infty} a_n(x-a)^n$ is convergent then by (i), $\sum_{n=0}^{\infty} a_nd^n$ is absolutely convergent. This contradicts the fact that $\sum_{n=0}^{\infty} a_nd^n$ is divergent. Hence, $\sum_{n=0}^{\infty} a_n(x-a)^n$ diverges. ■

**Theorem 58.2**
Given a power series

$$\sum_{n=0}^{\infty} a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-a)^2+\cdots$$

Then one of following is true:
(i) The series converges only at $x=a$;
(ii) the series converges for all $x$;
(iii) There is some positive number $R$ such that the series converges absolutely for $|x-a|<R$ and diverges for $|x-a|>R$. The series may or may not converge for $|x-a|=R$. That is for the values $x=a-R$ and $x=a+R$.

**Proof.**
Let $C$ be the collection of all real numbers at which the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges. If $C=\{a\}$ the series converges only at $x=a$ and diverges for all $x\neq a$. This establishes (i). If $C=(-\infty,\infty)$ then the series converges for all

values of $x$. This establishes (ii). To prove (iii), we assume that $C \neq \{a\}$ and $C \neq (-\infty, \infty)$. The condition $C \neq (-\infty, \infty)$ guarantees the existence of a real number $d$ such that $\sum_{n=0}^{\infty} a_n d^n$ diverges. Hence, by applying the previous theorem, $|x-a| < |d|$ whenever $x$ is in $C$. So $C$ is bounded from above. Let $R > 0$ be the smallest upper bound of $C$. Thus, $R < |d|$. If $|x-a| > R$ then $x$ is not in $C$ and so $\sum_{n=0}^{\infty} a_n(x-a)^n$ divverges. If $|x-a| < R$ then we can find an $x_0$ is $C$ such that $|x-a| < x_0 \leq R$, by the way $R$ is defined. Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges then by the previous theorem $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges. Since $C \neq \{a\}$ then there is an $x$ in $C$ such that $x \neq a$; hence, $0 < |x-a| \leq R$. We cannot asserts what happens at either $x = a - R$ or $x = a + R$.■

The largest interval for which the power series converges is called the **interval of convergence**. If a power series converges at only the point $x = a$ then we define $R = 0$. If a power series converges for all values of $x$ then we define $R = \infty$. We call $R$ the **radius of convergence.**

**Finding the radius of convergence**
The next theorem gives a method for computing the radius of convergence of many series.

**Theorem 58.3**
Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series with $a_n \neq 0$ for all $n \geq 0$.

(a) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$ then $R = \infty$ and this means that the series converges for all $x$.
(b) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 0$ then $R = \frac{1}{L}$.
(c) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$ then $R = 0$ and this means that the series diverges for all $x \neq a$.

**Proof.**
Consider the ratio

$$\left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x-a|.$$

Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$. Then the sequence $\left\{\left|\frac{a_{n+1}}{a_n}\right| |x-a|\right\}_{n=0}^{\infty}$ converges to $R \cdot |x-a|$.

(a) If $L = 0$ then $|x-a| \cdot L = 0 < 1$ and the series converges for all real numbers $x$. Thus, $R = \infty$.
(b) Suppose that $L > 0$ and finite. If $R \cdot |x-a| < 1$ or $|x-a| < \frac{1}{L}$ then by the ratio test, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is convergent. If $|x-a| > \frac{1}{L}$ then by the ratio test, the series is divergent. Hence, $R = \frac{1}{L}$.
(c) If $L = \infty$ then the series diverges for all $x \neq a$. Thus, $R = 0$.■

**Example 58.2**
The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is $R = \infty$ since $\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$. Thus, the series converges everywhere.■

**Example 58.3**
Consider the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$. Then by the previous theorem $\frac{1}{R} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ so that $R = 1$. Hence, the series converges for $|x-1| < 1$ and diverges for $|x-1| > 1$. So the interval of convergence is the interval $0 < x < 2$. What about the endpoints $x = 0$ and $x = 2$? If we replace $x$ by 0 we obtain the series $-\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent (harmonic series). If we replace $x$ by 2 we obtain the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ which converges by the alternating series test. Thus, the interval of convergence is $0 < x \leq 2$.■

## Practice Problems

**Exercise 58.1**
*Find the radius of convergence of the power series* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$.

**Exercise 58.2**
*Find the radius and the interval of convergence of the series* $\sum_{n=0}^{\infty}2^{2n}x^{2n}$.

Write each of the series in Exercises 3 - 5 using sigma notation, i.e. $\sum c_n(x-a)^n$.

**Exercise 58.3**
$1-\frac{(x-1)^2}{2!}+\frac{(x-1)^4}{4!}-\frac{(x-1)^6}{6!}+\cdots$

**Exercise 58.4**
$(x-1)^3-\frac{(x-1)^5}{2!}+\frac{(x-1)^7}{4!}-\frac{(x-1)^9}{6!}+\cdots$

**Exercise 58.5**
$2(x+5)^3+3(x+5)^5+\frac{4(x+5)^7}{2!}+\frac{5(x+5)^9}{3!}+\cdots$

Use Theorem 58.3 to find the radius of convergence of the power series in Exercises 6 - 10.

**Exercise 58.6**
$\sum_{n=0}^{\infty}(5x)^n$.

**Exercise 58.7**
$\sum_{n=0}^{\infty}\frac{(n+1)}{2^n+n}x^n$.

**Exercise 58.8**
$x+4x^2+9x^3+16x^4+\cdots$

**Exercise 58.9**
$1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\cdots$

**Exercise 58.10**
$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

**Exercise 58.11**
*(a) Determine the radius of convergence of the series*

$$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

*What does this tell us about the interval of convergence of this series?*
*(b) Investigate convergence at the endpoints of the interval of convergence of this series.*

**Exercise 58.12**
*Show that the series* $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$ *converges for* $|x| < \frac{1}{2}$. *Investigate whether the series converges for* $x = \frac{1}{2}$ *and* $x = -\frac{1}{2}$.

## 59 Approximations by Taylor's Polynomials

In this and the next section we are interested in approximating function values by using polynomials which are easy to compute. The polynomials used in the process are referred to as **Taylor's polynomials.**

Let $f(x)$ be a function which has derivatives $f'(a), f''(a), f'''(a), \cdots, f^{(n)}(a)$. We want to approximate $f(x)$ by a polynomial of degree $n$ as follows.

$$f(x) \approx c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n.$$

Note that $f(a) \approx c_0$. Now if we take the first derivative of $f(x)$ we find

$$f'(x) \approx c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^n.$$

From this we see that $f'(a) \approx c_1$. Next, take the second derivative of $f(x)$ to obtain

$$f''(x) \approx 2 \cdot 1c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \cdots + n \cdot (n-1)c_n(x-a)^{n-2}$$

This implies that $f''(a) \approx 2 \cdot 1c_2 = 2!c_2$ or $c_2 \approx \frac{f''(a)}{2!}$. Again, taking the third order derivative of $f(x)$ we find

$$f'''(x) \approx 3 \cdot 2 \cdot 1c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + \cdots + n \cdot (n-1) \cdot (n-2)c_n(x-a)^{n-3}.$$

It follows that $f'''(a) \approx 3 \cdot 2 \cdot 1c_3 = 3!c_3$ or $c_3 \approx \frac{f'''(a)}{3!}$. Continuing this process of taking successive derivatives we find $c_4 \approx \frac{f^{(4)}(a)}{4!}, \cdots, c_n \approx \frac{f^{(n)}(a)}{n!}$. Hence, $f(x)$ can be approximated by the polynomial

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We define the **nth Taylor polynomial** of $f(x)$ at $x = a$ to be the polynomial

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

**Example 59.1**
Use the process of successive differentiation to write the polynomial

$$p(x) = x^2 + x + 2$$

in the form

$$p(x) = c_0 + c_1(x-2) + c_2(x-2)^2.$$

**Solution.**
We have

$$\begin{array}{rclcrclcl} p(x) & = & x^2+x+2, & c_0 & = & p(2) & = & 8 \\ p'(x) & = & 2x+1, & c_1 & = & p'(2) & = & 5 \\ p''(x) & = & 2, & c_2 & = & \frac{p''(2)}{2!} & = & 1 \end{array}$$

Thus,

$$p(x) = 8 + 5(x-2) + (x-2)^2 \blacksquare$$

**Example 59.2**
Find the 4th degree Taylor polynomial approximating

$$f(x) = \frac{1}{1+x}$$

near $a = 0$.

**Solution.**
We have

$$\begin{array}{rclcrclcl} f(x) & = & \frac{1}{x+1}, & c_0 & = & f(0) & = & 1 \\ f'(x) & = & -(x+1)^{-2}, & c_1 & = & f'(0) & = & -1 \\ f''(x) & = & 2(x+1)^{-3}, & c_2 & = & \frac{f''(0)}{2!} & = & \frac{2}{2}=1 \\ f'''(x) & = & -6(x+1)^{-4}, & c_3 & = & \frac{f'''(0)}{3!} & = & -1 \\ f^{(4)}(x) & = & 24(x+1)^{-5}, & c_4 & = & \frac{f^{(4)}(0)}{4!} = 1 & & \end{array}$$

Thus,

$$P_4(x) = 1 - x + x^2 - x^3 + x^4. \blacksquare$$

**Example 59.3**
Find the third degree Taylor polynomial approximating

$$f(x) = \arctan x,$$

near $a = 0$.

**Solution.**
We have

$$\begin{array}{rclcrclcl} f(x) & = & \arctan x, & c_0 & = & f(0) & = & 0 \\ f'(x) & = & \frac{1}{1+x^2}, & c_1 & = & f'(0) & = & 1 \\ f''(x) & = & -(1+x^2)^{-2}(2x), & c_2 & = & \frac{f''(0)}{2!} & = & 0 \\ f'''(x) & = & -2(1+x^2)^{-3}(4x^2) - 2(1+x^2)^{-2}, & c_3 & = & \frac{f'''(0)}{3!} = -\frac{1}{3} & & \end{array}$$

Thus,

$$P_3(x) = x - \frac{1}{3}x^3.\blacksquare$$

**Example 59.4**
Find the fifth degree Taylor polynomial approximating

$$f(x) = \ln(1+x),$$

near $a = 0$.

**Solution.**
We have

$$\begin{array}{rclcrclcr}
f(x) &=& \ln(1+x), & c_0 &=& f(0) &=& 0\\
f'(x) &=& \frac{1}{1+x}, & c_1 &=& f'(0) &=& 1\\
f''(x) &=& -\frac{1}{(1+x)^2}, & c_2 &=& \frac{f''(0)}{2!} &=& -\frac{1}{2}\\
f'''(x) &=& \frac{2}{(1+x)^3}, & c_3 &=& \frac{f'''(0)}{3!} &=& \frac{1}{3}\\
f^{(4)}(x) &=& -\frac{6}{(1+x)^4}, & c_4 &=& \frac{f^{(4)}(0)}{4!} &=& -\frac{1}{4}\\
f^{(5)}(x) &=& \frac{24}{(1+x)^5}, & c_5 &=& \frac{f^{(5)}(0)}{5!} &=& \frac{1}{5}
\end{array}$$

Thus,

$$P_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5.\blacksquare$$

**Example 59.5**
Find the fourth degree Taylor polynomial approximating

$$f(x) = \sin x,$$

near $a = \frac{\pi}{2}$.

**Solution.**
We have

$$\begin{array}{rclcrclcr}
f(x) &=& \sin x, & c_0 &=& f(\frac{\pi}{2}) &=& 1\\
f'(x) &=& \cos x, & c_1 &=& f'(\frac{\pi}{2}) &=& 0\\
f''(x) &=& -\sin x, & c_2 &=& \frac{f''(\frac{\pi}{2})}{2!} &=& -\frac{1}{2}\\
f'''(x) &=& -\cos x, & c_3 &=& \frac{f'''(\frac{\pi}{2})}{3!} &=& 0\\
f^{(4)}(x) &=& \sin x, & c_4 &=& \frac{f^{(4)}(\frac{\pi}{2})}{4!} &=& \frac{1}{24}
\end{array}$$

Thus,

$$P_4(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{24}(x - \frac{\pi}{2})^4.\blacksquare$$

**Example 59.6**
Suppose that the function $f(x)$ is approximated near $x = 0$ by a sixth degree Taylor polynomial

$$P_6(x) = 3x - 4x^3 + 5x^6.$$

Find the value of the following:
(a) $f(0)$ (b) $f'(0)$ (c) $f'''(0)$ (d) $f^{(5)}(0)$ (e) $f^{(6)}(0)$

**Solution.**
If

$$P_6(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6$$

then $c_0 = 0, c_1 = 3, c_2 = 0, c_3 = -4, c_4 = c_5 = 0$, and $c_6 = 5$.

(a) $f(0) = c_0 = 0$, (b) $f'(0) = c_1 = 3$, (c) $f'''(0) = 3!c_3 = -24$, (d) $f^{(5)}(0) = 5!c_5 = 0$, (e) $f^{(6)}(0) = 6!c_6 = 3600.$■

**Example 59.7**
Let $g(x)$ be a function such that $g(5) = 3, g'(5) = -1, g''(5) = 1$ and $g'''(5) = -3$.
(a) What is the Taylor polynomial of degree 3 for $g(x)$ near 5?
(b) Use (a) to approximate $g(4.9)$.

**Solution**
(a) We have: $c_0 = g(5) = 3, c_1 = g'(5) = -1, c_2 = \frac{g''(5)}{2!} = \frac{1}{2}$, and $c_3 = \frac{g'''(5)}{3!} = -\frac{1}{2}$. Thus, $P_3(x) = 3 - (x-5) + \frac{1}{2}(x-5)^2 - \frac{1}{2}(x-5)^3$.
(b) $g(4.9) = 3 - (4.9-5) + \frac{1}{2}(4.9-5)^2 - \frac{1}{2}(4.9-5)^3 = 3.1675.$■

## Practice Problems

**Exercise 59.1**
*Use the process of successive differentiation to write the polynomial*

$$p(x) = x^2 + x + 2$$

*in the form*

$$p(x) = c_0 + c_1(x-2) + c_2(x-2)^2.$$

**Exercise 59.2**
*Write the polynomial* $p(x) = x^3 - x^2 + 1$ *in powers of* $x - \frac{1}{2}$.

**Exercise 59.3**
*Let* $p(x) = x^3 - x^2 + 1$. *Compute* $p(0.50028)$ *to five decimal places.*

In Exercises 4 - 9, compute $P_n(x)$ for the given $f, n$ and $a$.

**Exercise 59.4**
$f(x) = \frac{1}{1+x}, n = 4, a = 0.$

**Exercise 59.5**
$f(x) = \arctan x, n = 3, a = 0.$

**Exercise 59.6**
$f(x) = \sqrt[3]{1-x}, n = 3, a = 0.$

**Exercise 59.7**
$f(x) = \ln(1+x), n = 5, a = 0.$

**Exercise 59.8**
$f(x) = \sin x, n = 4, a = \frac{\pi}{2}.$

**Exercise 59.9**
$f(x) = e^x, n = 4, a = 1.$

**Exercise 59.10**
*Suppose that the function* $f(x)$ *is approximated near* $x = 0$ *by a sixth degree Taylor polynomial*

$$P_6(x) = 3x - 4x^3 + 5x^6.$$

*Find the value of the following:*
*(a)* $f(0)$ *(b)* $f'(0)$ *(c)* $f'''(0)$ *(d)* $f^{(5)}(0)$ *(e)* $f^{(6)}(0)$

**Exercise 59.11**
*Let $g(x)$ be a function such that $g(5) = 3, g'(5) = -1, g''(5) = 1$ and $g'''(5) = -3$.*
*(a) What is the Taylor polynomial of degree 3 for $g(x)$ near 5?*
*(b) Use (a) to approximate $g(4.9)$.*

**Exercise 59.12**
*Find the second-degree Taylor polynomial for $f(x) = 4x^2-7x+2$ about $x = 0$. What do you notice?*

## 60 The Error in Taylor Polynomial Approximations: Taylor's Theorem

In this section we would like to see how good the Taylor polynomials approximation discussed in the previous section is.
To this end, let $E_n(x)$ be the error between the true value of $f(x)$ and the approximated value $P_n(x)$. That is, $E_n(x) = f(x) - P_n(x)$. The following theorem provides an explicit formula for $E_n(x)$.

**Theorem 60.1** *(Taylor's Theorem)*
Suppose that $f(x)$ has derivatives $f'(x), f''(x), \cdots, f^{(n)}(x), f^{(n+1)}(x)$ near $x = a$. Then there is a $c$ between $x$ and $a$ such that

$$f(x) = P_n(x) + E_n(x)$$

where

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

**Proof.**
The theorem is trivially true if $x = a$. In this case, we let $c = a$ and $E_n(x) = 0$. So suppose that $x \neq a$.
Let $E_n(x) = f(x) - P_n(x)$. Define the function

$$F(y) = \sum_{i=0}^{n} \frac{f^{(i)}(y)}{i!}(x-y)^i + \frac{E_n(x)}{(x-a)^{n+1}}(x-y)^{n+1}$$

where $y \neq a$. Then $F$ is continuous on $[a, b]$ and differentiable in $(a, b)$ with derivative

$$F'(y) = \frac{f^{(n+1)}(y)}{n!}(x-y)^n - \frac{(n+1)E_n(x)}{(x-a)^{n+1}}(x-y)^n.$$

By the Mean Value Theorem, there is $a < c < b$ such that

$$F'(c) = \frac{F(b) - F(a)}{b-a}.$$

But $F(a) = F(x) = f(x)$ so that $F'(c) = 0$. This implies that

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n = \frac{(n+1)E_n(x)}{(x-a)^{n+1}}(x-c)^n,$$

which reduces to $E_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1}$. ■

**Example 60.1**

We will find $E_n(x)$ for $f(x) = \sin x$ about $x = 0$. Using the idea of successive differentiation we find the following

$$\begin{array}{rclrcr}
f'(x) & = & \cos x, & f'(c) & = & \cos c \\
f''(x) & = & -\sin x, & f''(c) & = & -\sin c \\
f'''(x) & = & -\cos x, & f'''(c) & = & -\cos c \\
f^{(4)}(x) & = & \sin x, & f^{(4)}(c) & = & \sin c \\
f^{(5)}(x) & = & \cos x, & f^{(5)}(c) & = & \cos c \\
f^{(6)}(x) & = & -\sin x, & f^{(6)}(c) & = & -\sin c \\
f^{(7)}(x) & = & -\cos x, & f^{(7)}(c) & = & -\cos c \\
f^{(8)}(x) & = & \sin x, & f^{(8)}(c) & = & \sin c \\
f^{(9)}(x) & = & \cos x, & f^{(9)}(c) & = & \cos c \\
\vdots & & & & &
\end{array}$$

It follows that

$$f^{(n+1)}(c) = \begin{cases} \cos c & if\ n\ is\ a\ multiple\ of\ 4 \\ -\sin c & if\ n-1\ is\ a\ multiple\ of\ 4 \\ -\cos c & if\ n-2\ is\ a\ multiple\ of\ 4 \\ \sin c & if\ n-3\ is\ a\ multiple\ of\ 4. \end{cases}$$

Hence, using the above theorem we see that

$$E_n(x) = \begin{cases} \frac{\cos c}{(n+1)!}x^{n+1} & if\ n\ is\ a\ multiple\ of\ 4 \\ \frac{-\sin c}{(n+1)!}x^{n+1} & if\ n-1\ is\ a\ multiple\ of\ 4 \\ \frac{-\cos c}{(n+1)!}x^{n+1} & if\ n-2\ is\ a\ multiple\ of\ 4 \\ \frac{\sin c}{(n+1)!}x^{n+1} & if\ n-3\ is\ a\ multiple\ of\ 4. \end{cases}$$

Note that for all $n$ we have $|E_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$. But we know that $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ ( since the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent by the ratio test so its nth term converges to zero). Hence, $\lim_{n\to\infty} E_n(x) = 0$.

**Remark 60.1**
Note that from the above theorem, we have an upper bound for the error:

$$|f(x) - P_n(x)| = |E_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},$$

where $M$ is the maximum value of $f^{(n+1)}(x)$ on the interval between $a$ and $x$.

**Example 60.2**
Use Taylor's theorem to approximate $\sin 3^\circ$ to four decimal places accuracy; that is, the magnitude of the error is less than $0.5 \times 10^{-4}$.

**Solution.**
Since $^\circ$ is close to 0 then we approximate $\sin x$ by a Taylor polynomial near $x = 0$. Thus, by Taylor Theorem we have $\sin x = P_n(x) + E_n(x)$. Replace $x$ by $\frac{\pi}{60} = 3^\circ$ to obtain

$$\left|E_n\left(\frac{\pi}{60}\right)\right| \leq \left(\frac{\pi}{60}\right)^{n+1} \cdot \frac{1}{(n+1)!} < 0.5 \times 10^{-4}.$$

Solving for $n$ we find $n = 3$. Thus,

$$\sin 3^\circ = \sin\left(\frac{\pi}{60}\right) \approx \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} \approx 0.0523. \blacksquare$$

## Practice Problems

**Exercise 60.1**
*Find* $E_n(x)$ *for the function* $f(x) = \cos x$ *about* $x = 0$. *Show that* $\lim_{n\to\infty} E_n(x) = 0$.

**Exercise 60.2**
*Find* $E_n(x)$ *for the function* $f(x) = e^x$ *about* $x = 0$. *Show that* $\lim_{n\to\infty} E_n(x) = 0$.

**Exercise 60.3**
*Give a bound on the error* $E_4$, *when* $e^x$ *is approximated by its fourth-degree Taylor polynomial for* $-0.5 \leq x \leq 0.5$.

# 61 Taylor Series

Let $f(x)$ be a function with derivatives of any order at $x = a$, that is, $f$ is an **infinitely differentiable** function. Fix a value of $x$ near $a$ and consider the sequence of Taylor ploynomials $\{P_n(x)\}_{n=0}^{\infty}$ where

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $\lim_{n\to\infty} P_n(x)$ exists and is equal to $f(x)$ then we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The right-hand series is called the **Taylor series expansion of $f(x)$ about** $x = a$. We call $\frac{f^{(n)}(a)}{n!}(x-a)^n$ the **general term** of the series. It is a formmula that gives any term in the series.

**Example 61.1**
Let $f(x) = \frac{1}{1-x}$. Finding successive derivatives we have

$$\begin{array}{rclrcr}
f'(x) & = & (1-x)^{-2}, & f'(0) & = & 1 \\
f''(x) & = & 2(1-x)^{-3}, & f''(0) & = & 2 = 2! \\
f'''(x) & = & 3\cdot 2(1-x)^{-4}, & f'''(0) & = & 3! \\
\vdots & & & & & \\
f^{(n)}(x) & = & n\cdot(n-1)\cdots 3\cdot 2(1-x)^{-(n+1)}, & f^{(n)}(0) & = & n!
\end{array}$$

Thus,

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

This is the nth partial sum of a geometric series that converges for $|x| < 1$. Moreover,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

This shows that

$$f(x) = 1 + x + x^2 + \cdots$$

for all $-1 < x < 1$.

**Remark 61.1**
For a given function $f$ at a given $x$, it is possible that the Taylor series converges to a value different from $f(x)$. However, the Taylor series of most of the functions discussed in this section do converge to the original function.

**Example 61.2**
Show that the Taylor polynomials sequence $\{P_n(x)\}_{n=0}^{\infty}$ of the function $f(x)$ defined below about $x = 0$ converges to 0 for all $x$. Thus, if $x \neq 0$ then $\lim_{n\to\infty} P_n(x) \neq f(x)$.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-\frac{2}{x^2}} & \text{if } x \neq 0 \end{cases}$$

**Solution.**
Using a graphing calculator, one can see that the function $f$ and all its derivatives are flat at 0. That is, the derivatives of $f(x)$ of all orders are zero. Hence, $P_n(x) = 0$ for all $x$ and for all $n$. This shows that if $x \neq 0$ then $\lim_{n\to\infty} P_n(x) = 0 \neq f(x)$.■

**Taylor Series Expansion of $f(x) = \cos x$ About $x = 0$**
Let $f(x) = \cos x$. We will find the Taylor series expansion of $f(x)$ about $x = 0$. Indeed,

$$\begin{array}{rcl} f'(x) & = & -\sin x \\ f''(x) & = & -\cos x \\ f'''(x) & = & \sin x \\ f^{(4)}(x) & = & \cos x \\ \vdots & & \end{array}$$

We see that the derivatives go through a cycle of length 4 and then repeat that cycle forever. It follows that

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (-1)^{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

Hence,

$$P_{2n}(x) = P_{2n+1} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

. Now, consider the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and let $a_n = \frac{(-1)^n}{(2n)!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1.$$

This shows that the series is convergent for all values of $x$. It remains to show that the series converges to $\cos x$. For this purpose, we need to apply Taylor's Theorem discussed in the previous section. Write

$$\cos x = P_n(x) + E_n(x).$$

Then $\cos x - P_n(x) = E_n(x)$. By the previous section we showed that $\lim_{n\to\infty} E_n(x) = 0$. This implies that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$

**Taylor Series Expansion of $f(x) = \sin x$ About $x = 0$**

Let $f(x) = \sin x$. We will find the Taylor series expansion of $f(x)$ about $x = 0$. Indeed,

$$\begin{array}{rcl} f'(x) & = & \cos x \\ f''(x) & = & -\sin x \\ f'''(x) & = & -\cos x \\ f^{(4)}(x) & = & \sin x \\ \vdots & & \end{array}$$

We see that the derivatives go through a cycle of length 4 and then repeat that cycle forever. It follows that

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is even} \\ (-1)^{\frac{k-1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

Hence, for $n \geq 1$,

$$P_{2n}(x) = P_{2n1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

. Now, let $a_n = \frac{(-1)^n}{(2n+1)!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} = 0 < 1.$$

This shows that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ is convergent for all values of $x$. It remains to show that the series converges to $\sin x$. For this purpose, we need to apply Taylor's Theorem discussed in the previous section. Write

$$\sin x = P_n(x) + E_n(x).$$

From the previous section we see that $E_n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

**Taylor Series Expansion of $f(x) = e^x$ About $x = 0$**
For all nonnegative integer $k$ we have $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = 1$. Thus, the nth Taylor polynomial is given by the expression

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Let $a_n = \frac{1}{n!}$. Then by the ratio test we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1.$$

Thus, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all values of $x$. It remains to show that the series converges to $e^x$. For that, we need to use Taylor Theorem. Write $f(x) = P_n(x) + E_n(x)$ where

$$|E_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| = \frac{e^c}{(n+1)!}|x|^{n+1} \to 0, as\ n \to \infty,$$

since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. Hence,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

**Taylor Series Expansion of $f(x) = \ln(1+x)$ About $x = 0$**
Taking derivatives:

$$\begin{array}{rclrcr}
f'(x) & = & (1+x)^{-1}, & f'(0) & = & 1 = 0! \\
f''(x) & = & -(1+x)^{-2}, & f''(0) & = & -1! \\
f'''(x) & = & 2(1+x)^{-3}, & f'''(0) & = & 2! \\
f^{(4)}(x) & = & -6(1+x)^{-4}, & f^{(4)}(0) & = & -3!\vdots \\
f^{(n)}(x) & = & (-1)^{n-1}(n-1)!(1+x)^{-n}, & f^{(n)}(0) & = & (-1)^{n-1}(n-1)!.
\end{array}$$

Hence,

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k!}.$$

Letting $a_n = (-1)^{n-1}\frac{1}{n}$ and applying the ratio test we find that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

Hence, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for all $-1 < x < 1$. By the alternating series test we know that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent so the interval of convergence of the previous series is $-1 < x \le 1$. It remains to show that the series converges to $\ln(1+x)$.
Using Taylor theorem, we can write $f(x) = P_n(x) + E_n(x)$, where

$$|E_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \frac{1}{n!|1+c|^{n+1}} \cdot \frac{|x|^{n+1}}{(n+1)!} \to 0, \ as\ n \to \infty.$$

Hence,

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad -1 < x \le 1.$$

**Taylor Series Expansion of $f(x) = (1+x)^p$ About $x = 0$**
Finding successive derivatives:

$$\begin{array}{rclrcr}
f(x) & = & (1+x)^p, & f(0) & = & 1 \\
f'(x) & = & p(1+x)^{p-1}, & f'(0) & = & p \\
f''(x) & = & p(p-1)(1+x)^{p-2}, & f''(0) & = & p(p-1) \\
f'''(x) & = & p(p-1)(p-2)(1+x)^{p-3}, & f'''(0) & = & p(p-1)(p-2) \\
\vdots & & & & & \\
f^{(n)}(x) & = & p(p-1)\cdots(p-n+1)(1+x)^{p-n}, & f^{(n)}(0) & = & p(p-1)\cdots(p-n+1).
\end{array}$$

Hence,

$$P_n(x) = 1 + \sum_{k=1}^{n}\frac{p(p-1)\cdots(p-k+1)}{k!}x^k.$$

Consider the series $1 + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-n+1)}{k!}x^n$. Letting $a_n = \frac{p(p-1)\cdots(p-n+1)}{n!}$ and applying the ratio test we find

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{p(p-1)\cdots(p-n)n!}{(n+1)!p(p-1)\cdots(p-n+1)}\right| = \lim_{n\to\infty}\left|\frac{p-n}{n+1}\right| = 1.$$

Hence, the radius of convergence is 1 and therefore the series $1+\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-n+1)}{n!}x^n$ converges for all $-1 < x < 1$. It remains to show that the series converges to $(1+x)^p$.

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-n+1)}{n!}x^n, \quad -1 < x < 1.$$

According to Section 63, the function $f(x)$ is differentiable and has power series expansion

$$f'(x) = \sum_{n=1}^{\infty}\frac{np(p-1)\cdots(p-n+1)}{n!}x^{n-1}, \quad -1 < x < 1.$$

Thus,

$$\begin{array}{rcl} xf'(x)+f'(x) & = & \sum_{n=1}^{\infty}\frac{np(p-1)\cdots(p-n+1)}{n!}x^n + \sum_{n=1}^{\infty}\frac{np(p-1)\cdots(p-n+1)}{n!}x^{n-1} \\ & = & \sum_{n=1}^{\infty}\frac{np(p-1)\cdots(p-n+1)}{n!}x^n + p + \sum_{n=1}^{\infty}\frac{(n+1)p(p-1)\cdots(p-n)}{(n+1)!}x^n \\ & = & p + \sum_{n=1}^{\infty}\left[\frac{(n+1)p(p-1)\cdots(p-n)}{(n+1)!} + \frac{np(p-1)\cdots(p-n+1)}{n!}\right]x^n \\ & = & p + p\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-n+1)}{n!}x^n \\ & = & pf(x), \end{array}$$

where we have used the fact that

$$\frac{(n+1)p(p-1)\cdots(p-n)}{(n+1)!} + \frac{np(p-1)\cdots(p-n+1)}{n!} = p\cdot\frac{p(p-1)\cdots(p-n+1)}{n!}.$$

It follows that $f(x)$ satisfies the differential equation

$$xf'(x) + f(x) = pf(x)$$

or equivalently,

$$(1+x)f'(x) - pf(x) = 0.$$

Now, define the function $g(x) = \frac{f(x)}{(1+x)^p}, -1 < x < 1$. Then $g'$ is differentiable with derivative

$$\begin{array}{rcl} g'(x) & = & \frac{(1+x)^p f'(x) - f(x)p(1+x)^{p-1}}{(1+x)^{2p}} \\ & = & \frac{(1+x)f'(x)-pf(x)}{(1+x)^{p+1}} = 0. \end{array}$$

It follows that $g(x) = C$ for all $-1 < x < 1$. But $g(0) = f(0) = 1$ so that $C = 1$. This implies that $f(x) = (1+x)^p$.

**Example 61.3**
Use the Binomial series expansion to find the Taylor series expansion of $\frac{1}{1+x}$ about $x = 0$.

**Solution.**
Use the binomial series with $p = -1$ to obtain

$$\begin{array}{rcl}\frac{1}{1+x} & = & (1+x)^{-1} \\ & = & 1 - x + x^2 - x^3 + \cdots\end{array}$$

By the ratio test, this series converges for all $-1 < x < 1$.■

## Practice Problems

**Exercise 61.1**
*Consider the function* $f(x) = \sin x$.
*(a) Show that* $f^{(k)}(0) = 0$ *for* $k$ *even.*
*(b) Find the Taylor polynomials* $P_{2n}(x)$ *and* $P_{2n+1}(x)$.
*(c) Show that the series*

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

*converges for all* $x$.
*(d)Use Taylor's Theorem to show that* $\lim_{n\to\infty} E_n(x) = 0$.

**Exercise 61.2**
*Consider the function* $f(x) = e^x$.
*(a) Show that* $f^{(k)}(0) = 1$ *for all values of* $k$.
*(b) Find the Taylor polynomial* $P_n(x)$.
*(c) Show that the series*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

*converges for all* $x$.
*(d)Use Taylor's Theorem to show that* $\lim_{n\to\infty} E_n(x) = 0$.

**Exercise 61.3**
*Find the Taylor series of* $f(x) = \ln(1 + x)$ *about* $x = 0$, *and calculate its interval of convergence.*

**Exercise 61.4**
*Find the Taylor series of the function* $f(x) = (1 + x)^p$ *about* $x = 0$, *and calculate its interval of convergence.*

**Exercise 61.5**
*Find the Taylor series about* $x = 0$ *for* $\frac{1}{1+x}$.

In Exercises 6 - 9 find the general term of the Taylor series.

**Exercise 61.6**

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

**Exercise 61.7**

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

**Exercise 61.8**

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

**Exercise 61.9**

$$x^2 \cos(x^2) = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{12}}{6!} + \cdots$$

**Exercise 61.10**
*(a) Find the terms up to degree 6 of the Taylor series for* $f(x) = \sin(x^2)$ *about* $x = 0$ *by taking derivatives.*
*(b) Compare your result in part (a) to the series for* $\sin x$*. How could you have obtained your answer to part (a) from the series for* $\sin x$*?*

**Exercise 61.11**
*Find the radius of convergence of the Taylor series around* $x = 0$ *for* $\ln(1-x)$*.*

**Exercise 61.12**
*Find* $x$ *such that*

$$1 + x + x^2 + x^3 + \cdots = 5.$$

**Exercise 61.13**
*Find the sum of the following converging series*

$$1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots + \frac{2^n}{n!} + \cdots$$

**Exercise 61.14**
*Find the sum of the following converging series*

$$1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$$

**Exercise 61.15**

*Find the sum of the following converging series*

$$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots + \frac{(-1)^n}{(2n)!} + \cdots$$

# 62 Constructing New Taylor Series from Known Ones

Recall that the question of finding Taylor series of a function $f(x)$ about a point $x = a$ amounts to finding the coefficients of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!}$.
For some functions finding $f^{(n)}(a)$ is very tedious. Instead, we use other ways which are considered easier.
We first list five series that we discussed in the previous section:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \tag{5}$$

This series converges for all values of $x$.

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots \tag{6}$$

This series converges for $0 < x \le 2$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \tag{7}$$

This series converges for all values of $x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \tag{8}$$

This series converges for all values of $x$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \tag{9}$$

This series converges for $-1 < x < 1$.

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots \tag{10}$$

This series converges for $-1 < x < 1$.

Next, we discuss ways of constructing new series from the above listed series.

**New Series by Substitution**
One can derive new series from known series as suggested by the following examples.

**Example 62.1**
Find the Taylor series of $\frac{x}{e^x}$ about $x = 0$.

**Solution.**
Replacing $x$ by $-x$ in Formula 5 we find

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots$$

Now, multiplying both sides of this equality by $x$ to obtain

$$\frac{x}{e^x} = xe^{-x} = x - x^2 + \frac{x^3}{2!} + \cdots + (-1)^n \frac{x^{n+1}}{n!} + \cdots \blacksquare$$

**Example 62.2**
Find the Taylor series of $f(x) = \frac{1}{1+x^2}$ about $x = 0$.

**Solution.**
Replacing $x$ by $-x^2$ in Formula 9 we can write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

This series converges for $-1 < x < 1$. $\blacksquare$

**New Series by Differentiation and Integration**
The following result allows us to construct new series from old ones using the processes of differentiation and integration.

**Theorem 62.1**
Suppose that the power series

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

converges for all $x$ such that $|x-a| < R$. Then this series defines a function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

with domain the set of all $x$ such that $|x-a| < R$.

(a) The power series

$$c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

obtained by differentiating the original series term by term also has radius of convergence R and sums to $f'(x)$, i.e.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

(b) The power series

$$C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

obtained by integrating the original series term by term also has radius of convergence R and sums to $\int f(x)dx$, i.e.

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

The proof of this theorem is beyond the scope of calculus and therefore is omitted.

**Example 62.3**
Find the Taylor series about $x = 0$ of $\cos x$ from the series of $\sin x$.

**Solution.**
We know that $\frac{d}{dx}(\sin x) = \cos x$, so we start with Formula 7 and differentiate this series term by term we get the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \blacksquare$$

**Example 62.4**
Find the Taylor's series about $x = 0$ for arctanx from the series for $\frac{1}{1+x^2}$.

**Solution**
Integrating term by term of the series obtained in Example 62.2 we find

$$\arctan x = \int \frac{dx}{1+x^2} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

where $-1 < x < 1$. Since $\arctan 0 = 0$ then $C = 0$ and therefore

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \blacksquare$$

**Applications Of Taylor Series**

**Example 62.5**
Use the series of $\arctan x$ to estimate the numerical value of $\pi$.

**Solution.**
Substituting $x = 1$ in to the series for $\arctan x$ gives

$$\pi = 4 \arctan 1 = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots)$$

By using 500 terms of this series one finds

$$\pi \approx 3.140. \blacksquare$$

Remember what we've said that some functions have no antiderivative which can be expressed in terms of familiar functions. This makes evaluating definite integrals of these functions difficult because the Fundamental Theorem of Calculus cannot be used. However, if we have a series representation of a function, we can oftentimes use that to evaluate a definite integral.

**Example 62.6**
Estimate the value of $\int_0^1 \sin(x^2)dx$.

**Solution.**
The integrand has no antiderivative expressible in terms of familiar functions. However, we know how to find its Taylor series: we know that

$$\sin t = x - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$$

Now if we substitute $t = x^2$, we have

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

In spite of the fact that we cannot antidifferentiate the function, we can antidifferentiate the Taylor series:

$$\begin{array}{rcl}\int_0^1 \sin(x^2)dx & = & \int_0^1 (x^2 - \frac{x^6}{3!} + \frac{x^5}{5!} - \cdots)dx \\ & = & (\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \cdots)|_0^1 \\ & = & \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \cdots \\ & \approx & 0.31026 \blacksquare \end{array}$$

## Practice Problems

**Exercise 62.1**
*Find the first four nonzero terms of the Taylor series about $x = 0$ of the function $f(x) = \ln(1 - 2x)$.*

**Exercise 62.2**
*Find the first four nonzero terms of the Taylor series about $x = 0$ of the function $f(x) = \arcsin x$.*

**Exercise 62.3**
*Find the first four nonzero terms of the Taylor series about $x = 0$ of the function $f(x) = \frac{1}{\sqrt{1-x^2}}$.*

**Exercise 62.4**
*Find the first four nonzero terms of the Taylor series about $x = 0$ of the function $f(x) = x^3 \cos(x^2)$.*

**Exercise 62.5**
*Find the first four nonzero terms of the Taylor series about $x = 0$ of the function $f(x) = e^x \cos x$.*

**Exercise 62.6**
*Find the Taylor series about 0 for the function $f(x) = x\sin(x^2) - x^3$, including the general term.*

**Exercise 62.7**
*Expand the function $f(x) = \frac{1}{2+x}$ about $x = 0$ in terms of $\frac{x}{2}$.*

**Exercise 62.8**

*Consider the two functions $f(x) = e^{-x^2}$ and $g(x) = \frac{1}{1+x^2}$.*
*(a) Write the Taylor series for the two functions about $x = 0$. What is similar about the two series? What is different?*
*(b) Looking at the series, which function do you predict will be larger over the interval $(-1, 1)$? Graph both and see.*
*(c) Are the functions even, odd or neither?How might you see this by looking at the series expansions?*

# 63 Introduction to Fourier Series

The study of Fourier series is of fundamental importance in both theoretical and applied mathematics, and they are used for modelling phenomena in engineering and physics as well as in areas such as computer science and medical research.
Unlike the approximation by Taylor ploynomials, which requires the approximated function to be continuous and the approximation is near the value where the Taylor polynomial is centered, Fourier polynomials can be used to represent functions, both continuous and discontinuous, and approximation is done over larger intervals. Thus, Fourier approximations are rather global whereas approximations by Taylor polynomials are local.

To start with, suppose that $f$ is a function of period $2\pi$, that is, $f(x+2\pi) = f(x)$ for all $x$. Now, recall the trigonometric identities

$$\cos(mx)\sin(nx) = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x] \tag{11}$$

$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x] \tag{12}$$

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \tag{13}$$

For $n > 0$ we have

$$\int_{-\pi}^{\pi} \sin(nx)dx = \left[-\frac{1}{n}\cos(nx)\right]_{-\pi}^{\pi} = -\frac{1}{n}(\cos(n\pi) - \cos(-n\pi)) = 0 \tag{14}$$

and

$$\int_{-\pi}^{\pi} \cos(nx)dx = \left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{\pi} = -\frac{1}{n}(\sin(n\pi) - \sin(-n\pi)) = 0. \tag{15}$$

Next, for any $m \neq n$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx &= -\frac{1}{2}\left[\frac{1}{m+n}\cos((m+n)x) + \frac{1}{m-n}\cos((m-n)x)\right]_{-\pi}^{\pi} \\ &= 0\end{aligned} \tag{16}$$

and if $n = m \neq 0$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\cos(mx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2mx)dx = -\frac{1}{4m}\left[\cos(2mx)\right]_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

If $m = 0$ then $\sin(mx) = 0$ and the above integral is zero. If $n = 0$ then

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = \int_{-\pi}^{\pi} \sin(mx)dx = -\frac{1}{2}\left[\frac{1}{m}\cos(mx)\right]_{-\pi}^{\pi} = 0.$$

Next, for $n \neq m$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x)) + \frac{1}{m-n}\sin((m-n)x)\right]_{-\pi}^{\pi} \\ &= 0 \qquad (17)\end{aligned}$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx &= -\frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x)) - \frac{1}{m-n}\sin((m-n)x)\right]_{-\pi}^{\pi} \\ &= 0. \qquad (18)\end{aligned}$$

Finally, for $n \geq 1$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2(nx)dx &= \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2}dx \\ &= \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi \qquad (19)\end{aligned}$$

and similarly,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(nx)dx &= \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2}dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi. \qquad (20)\end{aligned}$$

We want to approximate $f(x)$ with a sum of trigonometric functions of the form

$$f(x) \approx F_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx). \qquad (21)$$

We call $F_n(x)$ the **Fourier polynomial of degree n.** The constants $a_0, a_1, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are called the **Fourier coefficents.**
The main question is to select the Fourier coefficients in such a way that $F_n(x)$ is a good approximation to $f(x)$.
Integrating both sides of (21) and using (14)-(20) we find that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Now, multiplying both sides of (21) by $\cos x$ and then integrating over the interval $[-\pi, \pi]$ we find that

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx.$$

Similarly, for $k = 2, 3, \cdots, n$ we have

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx.$$

In a similar way we show that, for $1 \leq k \leq n$,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx.$$

**Example 63.1**
Find the trigonometric polynomial of degree two of the function

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

**Solution.**
Indeed, we have

$$\begin{array}{rcl}
a_0 & = & \frac{1}{2\pi}\int_{-\pi}^0 0dx + \frac{1}{2\pi}\int_0^\pi 1dx = 0 + \frac{1}{2\pi}(\pi) \\
 & = & \frac{1}{2} \\
a_1 & = & \frac{1}{\pi}\int_0^\pi \cos x dx = \frac{1}{\pi}[\sin x]_0^\pi \\
 & = & 0 \\
b_1 & = & \frac{1}{\pi}\int_0^\pi \sin x dx = -\frac{1}{\pi}[\cos x]_0^\pi \\
 & = & \frac{2}{\pi} \\
a_2 & = & \frac{1}{\pi}\int_0^\pi \cos 2x dx = \frac{1}{2\pi}[\sin 2x]_0^\pi \\
 & = & 0 \\
b_2 & = & \frac{1}{\pi}\int_0^\pi \sin 2x dx = -\frac{1}{2\pi}[\cos 2x]_0^\pi \\
 & = & 0.
\end{array}$$

Thus, the trigonometric polynomial of degree 2 is

$$F_2(x) = F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x.$$

Figure 159, shows the graphs of the function $f$ together with $F_2(x)$.

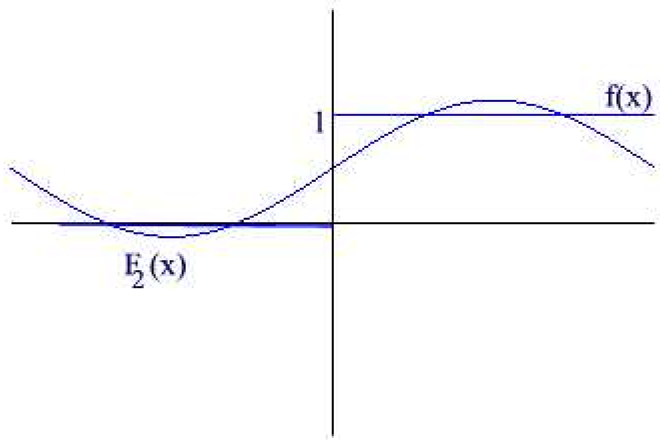



Figure 159

Without going through the details, we calculate the coefficients for the third degree Fourier polynomial to obtain

$$F_3(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x).$$

Graphing $f(x)$ and $F_3(x)$ we see

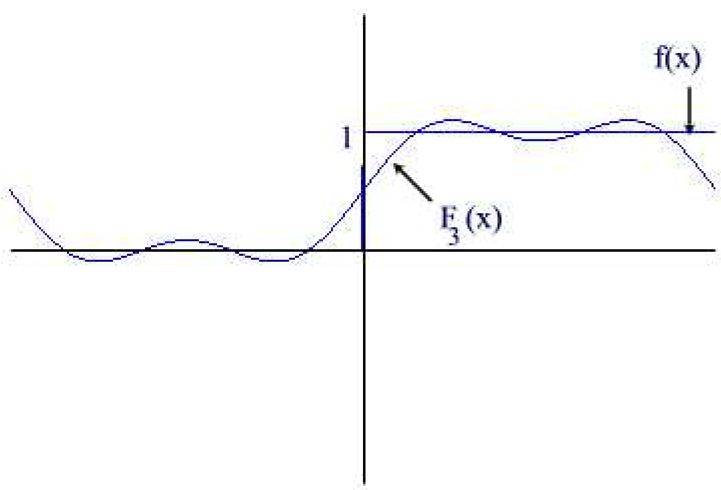


Figure 160

We conclude that a better approximation is found by taking Fourier polynomials of high degree. This leads to the introduction of the so called **Fourier series.**

Fix $x$ and construct the sequence of trigonometric polynomials $F_n(x)$. If $\lim_{n\to\infty} F_n(x)$ exists and is equal to $f(x)$ then we write

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

We call the right hand side series the **Fourier series** of $f(x)$.

**Example 63.2**

Find the Fourier series of the function

$$f(x) = x, \ -\pi \le x \le \pi.$$

**Solution.**

Since the function is odd then $a_n = 0$ for all $n \ge 0$. Thus, we look for $b_n$ for $n \ge 1$. Indeed,

$$\begin{array}{rcl} b_n &=& \frac{1}{\pi}\int_{-pi}^{pi} x \sin(nx)dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi} \\ &=& -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^{n+1}. \end{array}$$

Thus, the Fourier series is given by

$$f(x) = 2\left(\sin x - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{2} - \cdots\right).$$

So far we have been assuming that the underlying function $f(x)$ is $2\pi - periodic$. We consider next the case of a function that is $b - periodic$, i.e. $f(x+b) = f(x)$ for all $x$ in $[-\pi, \pi]$. Letting $x = \frac{bt}{\pi}$ then for $x$ in the interval $[-\pi, \pi]$ the variable $t$ is in the interval $[-\frac{b}{2}, \frac{b}{2}]$. Moreover, the new function

$$g(t) = f\left(\frac{bt}{2\pi}\right) = f(x)$$

is defined on the interval $[-\pi, \pi]$ and is $2\pi - \textit{periodic}$ since

$$g(t+2\pi) = f\left(\frac{b(t+2\pi)}{2\pi}\right) = f(x+b) = f(x) = g(t).$$

In this case, we can find the Fourier series of $g(t)$ as before obtaining

$$g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos{(nt)} + b_n \sin{(nt)}).$$

Substituting $t = \frac{2\pi x}{b}$ we get

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n x}{b}\right) + b_n \sin\left(\frac{2\pi n x}{b}\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)dx; \\ a_k &= \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\cos{(\tfrac{2\pi n x}{b})}dx; \\ b_k &= \frac{2}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\sin{(\tfrac{2\pi n x}{b})}dx. \end{aligned}$$

## Practice Problems

Which of the series in Exercises 1 - 4 are Fourier series?

**Exercise 63.1**
$1 + \cos x + \cos^2 x + \cos^3 x + \cdots$

**Exercise 63.2**
$\sin x + \sin(x+1) + \sin(x+2) + \cdots$

**Exercise 63.3**
$\frac{\cos x}{2} + \sin x - \frac{\cos(2x)}{4} - \frac{\sin(2x)}{2} + \frac{\cos(3x)}{8} + \frac{\sin(3x)}{8} + \cdots$

**Exercise 63.4**
$\frac{1}{2} - \frac{1}{3}\sin(2x) + \frac{1}{4}\sin(2x) - \frac{1}{5}\sin(3x) + \cdots$

**Exercise 63.5**
*Repeat Exercise 5 with the function*

$$f(x) = \begin{cases} -x, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

*Use a graphing calculator to draw the graph of each approximation.*

In Exercises 6 - 8, find the *nth* Fourier polynomial for the given functions, assuming them to be periodic with period $2\pi$. Graph the first three approximations with the original function.

**Exercise 63.6**
$f(x) = x^2, \quad -\pi < x \le \pi.$

**Exercise 63.7**

$$h(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 < x \le \pi \end{cases}$$

**Exercise 63.8**
$h(x) = x, \quad -\pi < x \le \pi.$

# 64 The Definition of a Differential Equation

The purpose of this and the coming sections is to introduce the concept of differential equation, explain what a solution is, and discuss graphical (slope fields) numerical (Euler's method) and analytical (seperation of variables) methods for finding the solutions.

• **Basic Definitions**
A **differential equation** ( or DE) is an equation involving an unknown function and its derivatives. The **order** of the differential equation is the order of the highest derivative of the unknown function involved in the equation. We say that a differential equation is **linear** if and only if the powers of the unknown function and all its derivatives are either 0 or 1. Thus, the equation

$$a(x)y' + b(x)y = c(x)$$

is a **first order linear differential equation** while the equation

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

is a **second order linear** differential equation.
If the righthand side of the above equations is zero then we say that the equation is **homogeneous** otherwise the equation is said to be **nonhomogeneous.**
By a **solution** to a differential equation we mean a function that satisfies the differential equation.

**Example 64.1**
Show that the function $y = 100 + e^{-t}$ is a solution to the DE

$$y' = 100 - y.$$

**Solution.**
Indeed, finding the first order derivative of $y$ we have $y' = -e^{-t}$. Also, $100 - y = 100 - (100 + e^{-t}) = -e^{-t}$. Thus, $y' = 100 - y$ so that $y = 100 + e^{-t}$ is a solution to the given DE.■

The process of the above example applies also with the function $y = 100 + Ce^{-t}$ where $C$ is any constant. Such a solution is referred to as the **general**

**solution** of the DE. The solution in the above example is obtained from the general solution by replacing $C$ by 1. We call this solution a **particular solution.** Also note that this particular solution satisfies the two equations

$$\left\{\begin{array}{rcl} y' & = & 100 - y \\ y(0) & = & 101. \end{array}\right.$$

A differential equation together with a set of conditions, known as **initial conditions**, is called an **initial value problem** (in short IVP). To **solve** a differential equation means to find its general solution.

**Example 64.2**
Show that $y = \sin(2t)$ is a particular solution to the equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

**Solution.**
Differentiating the function $y$ twice we obtain $y'' = -4\cos(2t)$. Thus $y'' + 4y = -4\cos(2t) + 4\cos(2t) = 0.$ ■

**Remark 64.1**
The general solution of a first order differential equation has only one constant since the solution is derived from integrating once. On the other hand, a second order differential equation contains two constants since the solution involves antidifferentiation twice.

## Practice Problems

**Exercise 64.1**
*Match the graphs in Figure 161 with the following descriptions.*

*(a) The population of a new species introduced onto a tropical island.*
*(b) The temperature of a metal ingot placed in a furnace and then removed.*
*(c) The speed of a car traveling at uniform speed and then breaking uniformly.*
*(d) The mass of carbon-14 in a historical specimen.*
*(e) The concetration of tree pollen in the air over the course of a year.*

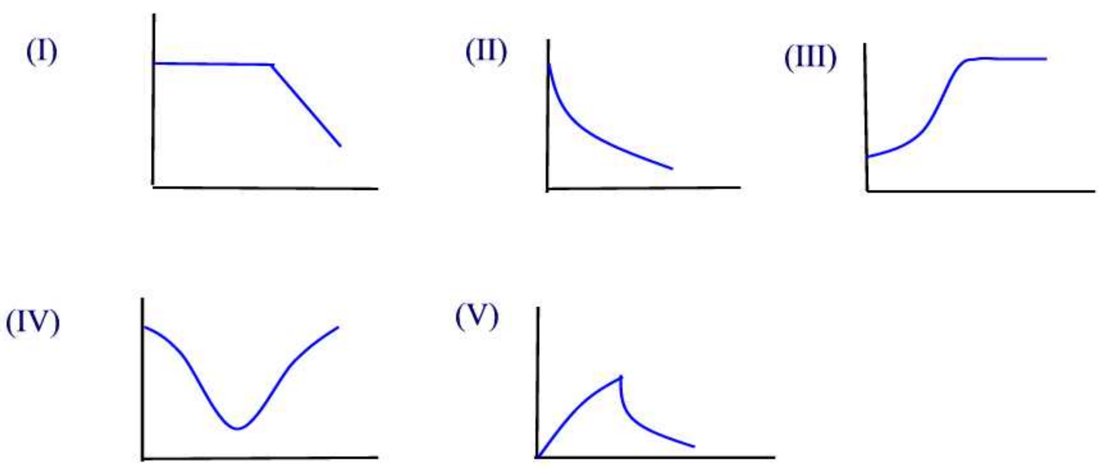



*Figure 161*

**Exercise 64.2**
*Fill in the missing values in the table given if you know that* $\frac{dy}{dx} = 0.5y$. *Assume the rate of growth given by* $\frac{dy}{dx}$ *is approximately constant over each unit time interval and that the initial value of* $y$ *is 8.*

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 8 | | | | |

**Exercise 64.3**
*Is* $y = x^3$ *a solution to the differential equation*

$$xy' - 3y = 0?$$

**Exercise 64.4**
*Find the values of $k$ for which $y = x^2 + k$ is a solution to the differential equation*

$$2y - xy' = 10.$$

**Exercise 64.5**
*For what values of $k$ (if any) does $y = 5+3e^{kx}$ satisfy the differential equation:*

$$\frac{dy}{dx} = 10 - 2y?$$

**Exercise 64.6**
*Find the value(s) of $\omega$ for which $y = \cos\omega t$ satisfies*

$$\frac{d^2y}{dx^2} + 9y = 0.$$

**Exercise 64.7**
*(a) Show that $P = \frac{1}{(1+e^{-t}}$ satisfies the logistic equation*

$$\frac{dP}{dt} = P(1 - P).$$

*(b) What is the limiting value of $P$ as $t \to \infty$?*

**Exercise 64.8**
*Match the following differential equations and possible solutions. (Note the given functions may satisfy more tan one equation or none, and some equations may have more than one solution.)*

| | | | |
|---|---|---|---|
| $(a)$ | $y'' = y$ | $(I)$ | $y = \cos x$ |
| $(b)$ | $y' = -y$ | $(II)$ | $y = \cos(-x)$ |
| $(c)$ | $y' = \frac{1}{y}$ | $(III)$ | $y = x^2$ |
| $(d)$ | $y'' = -y$ | $(IV)$ | $y = e^x + e^{-x}$ |
| $(e)$ | $x^2y'' - 2y = 0$ | $(V)$ | $y = \sqrt{2x}$ |

# 65 Slope Fields

In this section we describe a graphical method for solving initial value problems known as the **slope fields** or **direction fields**. This method is often practical when analytical methods for solving a differential equation are not possible. An array of short line segments in the plane having the property that the line plotted at a point $(x, y)$ has slope $f(x, y)$ is called a **slope field** for the differential equation $\frac{dy}{dx} = f(x, y)$. Slope fields are basically used to visualize the family of solutions of a given differential equation.

**Example 65.1**
Find the slope of the particular solution to the differential equation

$$\frac{dy}{dx} = 2x$$

passing through $(0, -1)$. What is the general solution?

**Solution.**
Figure 162 shows the slope field of the particular solution to the given DE passing through the point $(0, -1)$. The figure was plotted using the following MAPLE commands:
>with(plots):
>with(DEtools):
>slopeplot:= DEplot(diff(y(x),x)=2*x,y(x),x=-3..3,y=-3..3):
>g:=plot($x^2 - 1$, x=-3..3, y=-3..3, color=black):
>display([slopeplot,g]);

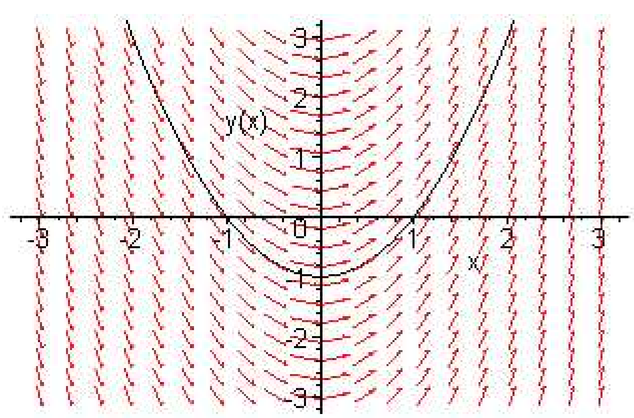


Figure 162

The solution curves look like parabolas. Thus, the general solution is given by the equation $y = x^2 + C$.■

**Example 65.2**
Using the slope fields, guess the form of the solution curves of the differential equation
$$\frac{dy}{dx} = -\frac{x}{y}.$$

**Solution.**
The slope fields (See Figure 163) is obtained by executing the following Maple commands

> with(plots):
> with (DEtools):
> slopeplot := DEplot($diff(y(x), x) = -x/y, y(x), x = -2..2, y = -2..2$):
> display(slopeplot);

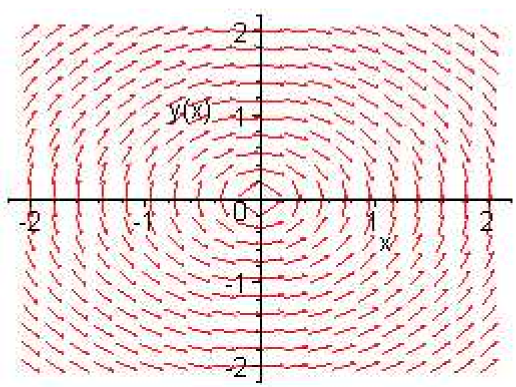



Figure 163

The solution curves look like circles centered at the origin. Thus, the general solution is given implicitly by the equation $x^2 + y^2 = C$ where $C$ is a positive constant.■

**Example 65.3**
Find the slope field of the differential equation $y' = 2 - y$ and then sketch the solution curves with initial conditions $y(0) = 1, y(1) = 0$ and $y(0) = 3$.

**Solution.**
The slope field and the three curves are shown in Figure 164:

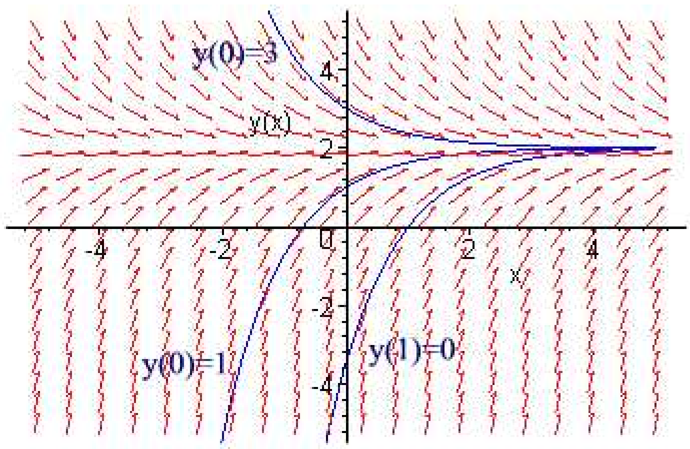



Figure 164

Note that for each solution curve $\lim_{x\to\infty} y(x) = 2.$■

**Remark 65.1**
Finally, we point out here that even though one can draw solution curves, some do not have simple formula.

## Practice Problems

**Exercise 65.1**
*The slope field for the equation* $y' = x + y$ *is shown in Figure 165.*

*(a) Carefully sketch the solutions that pass through the points*
*(i)* $(0,0)$ *(ii)* $(-3,1)$ *(iii)* $(-1,0)$
*(b) From your sketch, write the equation of the solution passing through* $(-1,0)$.
*(c) Check your solution to part (b) by substituting it into the differential equation.*

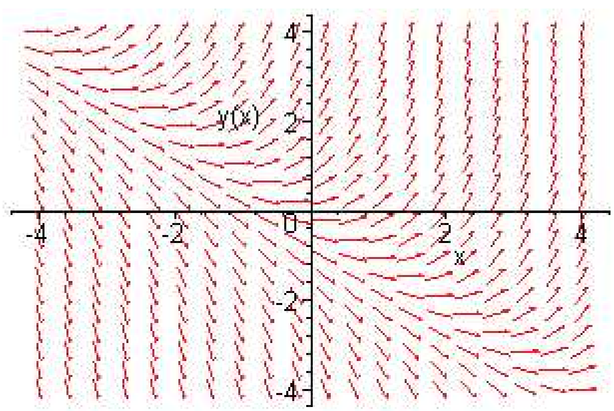



*Figure 165*

**Exercise 65.2**
*The slope field for the equation* $\frac{dP}{dt} = 0.1P(10 - P)$, *for* $P \geq 0$, *is in Figure 166.*

*(a) Plot the solutions through the points*
*(i)* $(0,0)$ *(ii)* $(1,4)$ *(iii)* $(4,1)$
*(iv)* $(-5,1)$ *(v)* $-2,12)$ *(vi)* $(-2,10)$.
*(b) For which positive values of* $P$ *are the solutions increasing?decreasing? What is the limiting value of* $P$ *as* $t$ *approaches infinity?*

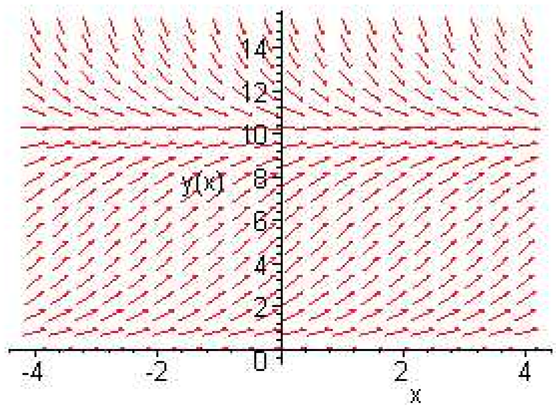



*Figure 166*

**Exercise 65.3**
*The slope field for the equation* $\frac{dy}{dx} = \sin x \sin y$, *is in Figure 167.*

*(a) Plot the solutions through the points*
*(i)* $(0, -2)$ *(ii)* $(0, \pi)$.
*(b) What is the equation of the solution that passes through* $(0, n\pi)$, *where* $n$ *is an integer?*

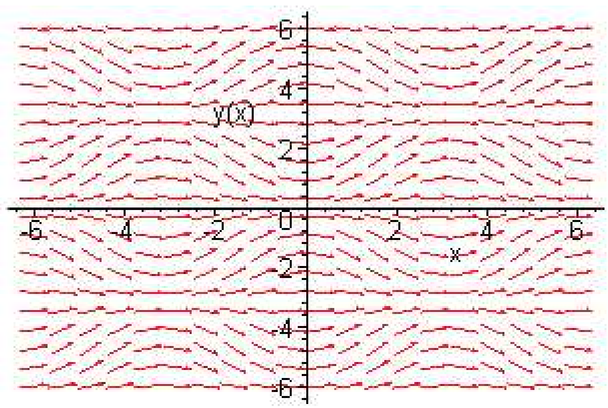



*Figure 167*

**Exercise 65.4**
*One of the slope fields in Figure 168 has the equation* $y' = \frac{x+y}{x-y}$. *Which one?*

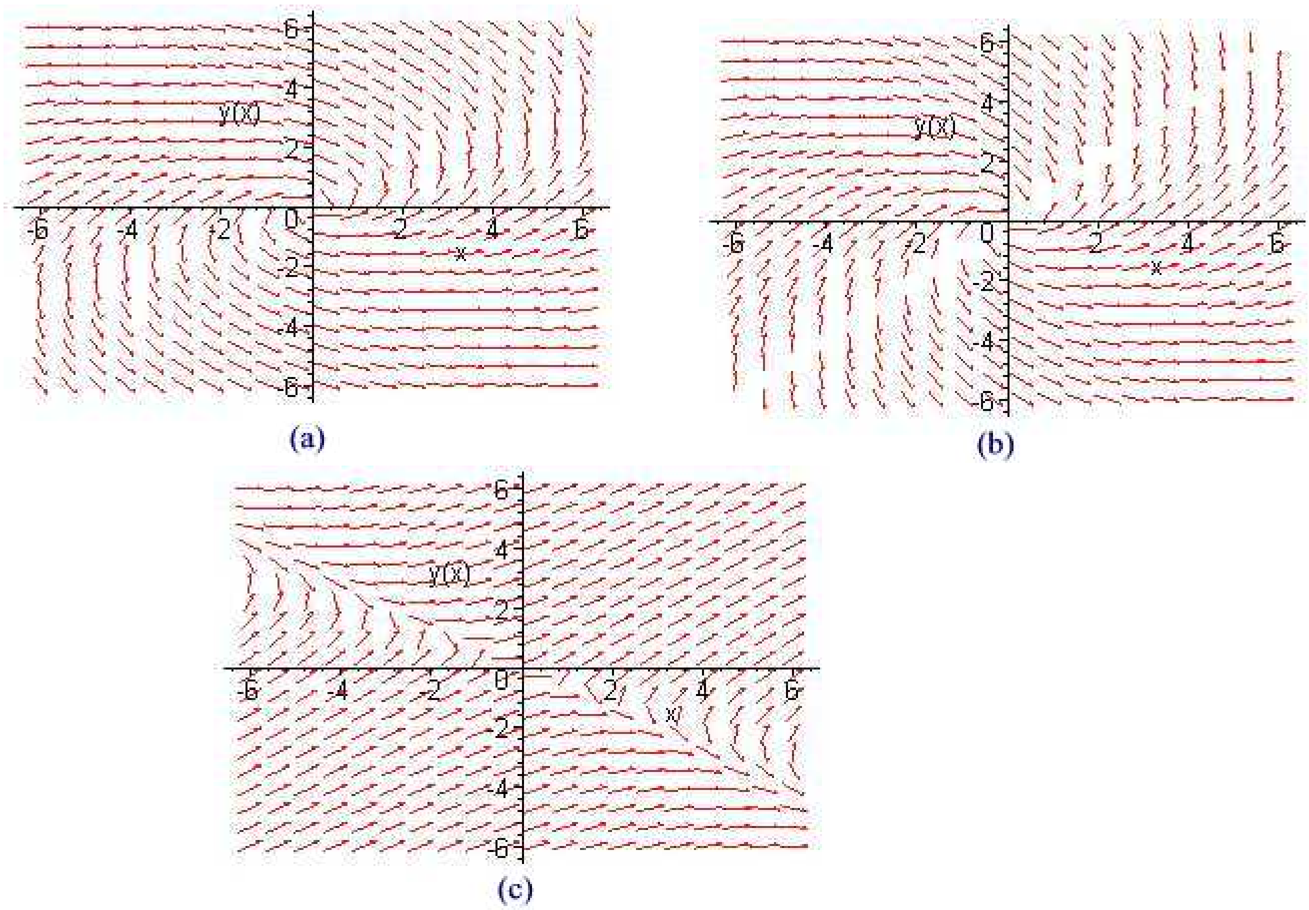


*Figure 168*

**Exercise 65.5**

**Exercise 65.6**

*Match the slope fields in Figure 169 with their differential equations:*

$$\begin{array}{llllll} (a) & y' = 1 + y^2 & (b) & y' = x & (c) & y' = \sin x \\ (d) & y' = y & (e) & y' = x - y & (f) & y' = 4 - y. \end{array}$$

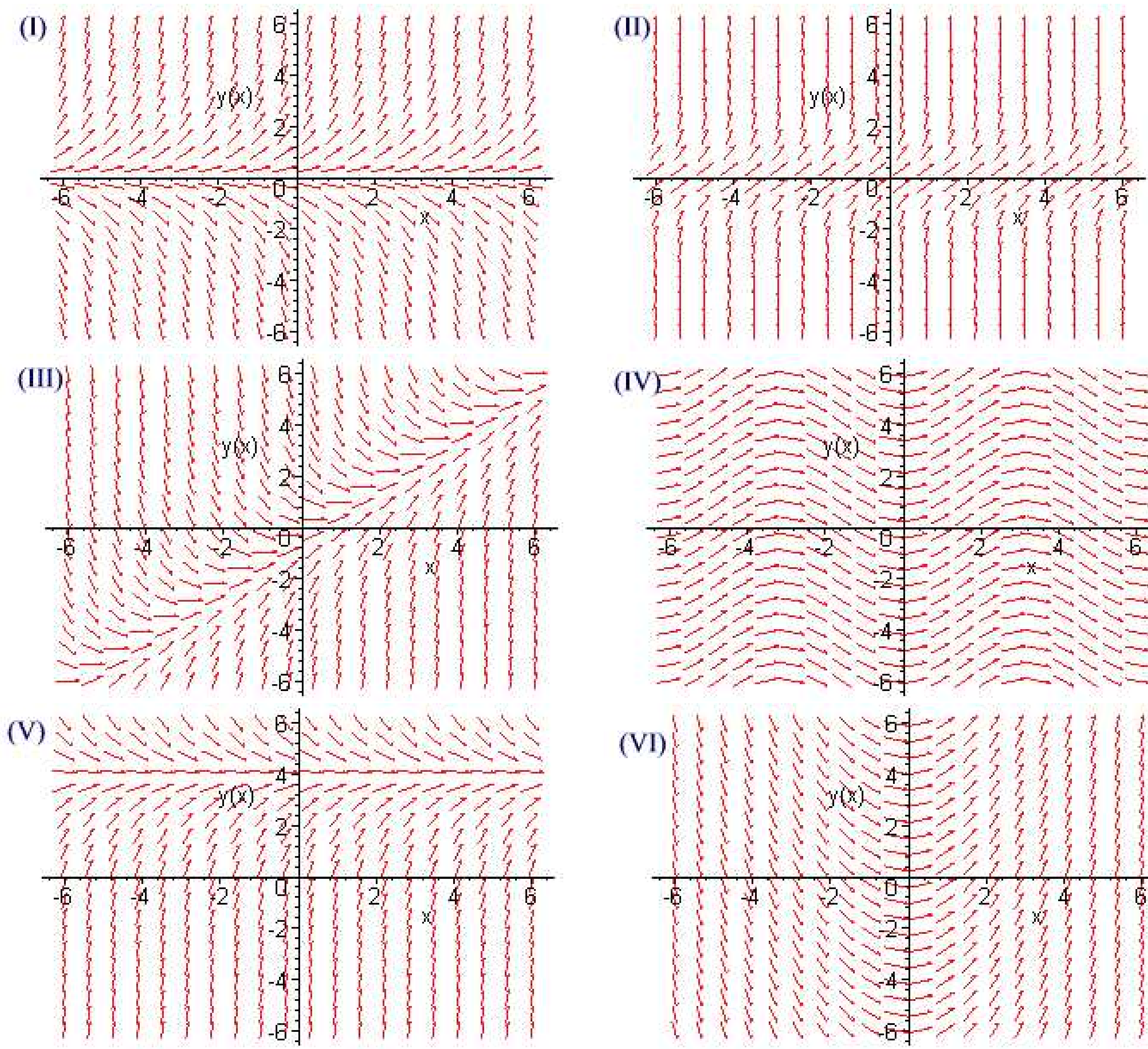



*Figure 169*

## 66 Euler's Method

In the previous section, we introduced direction fields as means for obtaining a picture of various solutions to a differential equation, but sometimes we need more than a rough graph of a solution. **Euler's method** is a numerical method for estimating the value of a solution of the initial value problem:

$$\begin{cases} \frac{dy}{dx} & = & f(x,y) \\ y(x_0) & = & y_0 \end{cases}$$

at a given point.

We now describe the way the method works. Recall the definition of the derivative of a function $y(x)$ at a point $x_0$ given by

$$y'(x_0) = \lim_{x \to x_0} \frac{y - y_0}{x - x_0}.$$

For $x = x_0 + h$ with $h$ small, that is $x$ is close to $x_0$, we can write

$$y'(x_0) \approx \frac{y - y_0}{x - x_0}$$

or

$$y - y_0 \approx y'(x_0)(x - x_0)$$

i.e.,

$$y \approx y_0 + hf(x_0, y_0).$$

Thus, we are approximating the solution curve $y = y(x)$ near $(x_0, y_0)$ by the tangent line to the curve at this point.

Next, we let $x_1 = x_0 + h$. We estimate $y(x_1) = y_1$ using the equation

$$y_1 \approx y_0 + hf(x_0, y_0)$$

Next, we let $x_2 = x_1 + h$ and estimate $y(x_2) = y_2$ using the equation

$$y_2 \approx y_1 + hf(x_1, y_1).$$

After $n$ steps, we let $x_{n+1} = x_n + h$ and we estimate $y(x_{n+1}) = y_{n+1}$ using the equation

$$y_{n+1} \approx y_n + hf(x_n, y_n).$$

Now, if we want to use this method to estimate $y(b)$ starting from $y(a)$ and using $n$ steps then $h = \frac{b-a}{n}$.

Geometrically, we obtain a sequence of line segments (tangent lines) that approximates the shape of the solution curve. (See Figure 170).

Here is an example to help you understand this process.

**Example 66.1**
Suppose that $y(0) = 1$ and $\frac{dy}{dx} = y$. Estimate $y(0.5)$ in 5 steps using Euler's method.

**Solution.**
We have $a = 0, b = 0.5, y(0) = 1$ and $n = 5$. Therefore, $h = \frac{b-a}{n} = 0.1$. The following chart lists the steps needed:

| $k$ | $x_k$ | $y_k$ | $f(x_k, y_k)h$ |
|---|---|---|---|
| 0 | 0 | 1 | 0.1 |
| 1 | 0.1 | 1.1 | 0.11 |
| 2 | 0.2 | 1.21 | 0.121 |
| 3 | 0.3 | 1.331 | 0.1331 |
| 4 | 0.4 | 1.4641 | 0.14641 |
| 5 | 0.5 | 1.61051 | |

Thus, $y(0.5) \approx 1.61051$.■

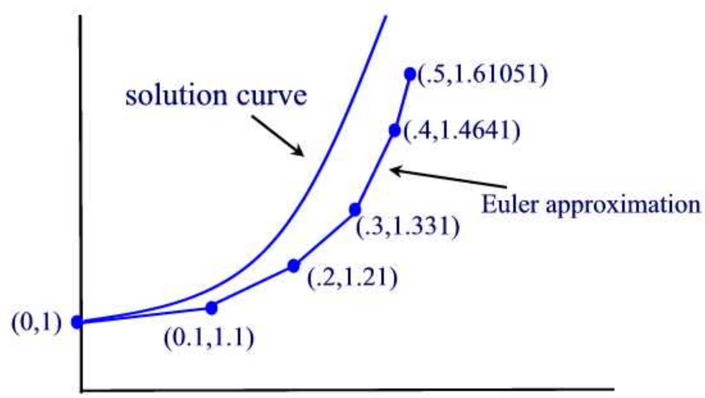



Figure 170

**Remark 66.1**
1. Euler's method approximates the value of the solution at a given point; it does not give an explicit formula of the solution.

2. It can be shown that the error in Euler's method is proportional to $\frac{1}{n}$. Thus, doubling the number of mesh points will decrease the error by $\frac{1}{2}$.

## Practice Problems

**Exercise 66.1**
*Consider the differential equation $y' = x+y$. Use Euler's method with $h = 0.1$ to estimate $y$ when $x = 0.4$ for the solution curves satisfying*

$$(a)\ y(0) = 1 \quad (b)\ y(-1) = 0.$$

**Exercise 66.2**
*Consider the solution to the differential equation $y' = y$ passing through $y(0) = 1$.*

*(a) Sketch the slope field for this differential equation, and sketch the solution through the point $(0,1)$.*
*(b) Use Euler's method with step size $h = 0.1$ to estimate the solution at $x = 0.1, 0.2, \cdots, 1$.*
*(c) Plot the estimated solution on the slope field; compare the solution and the slope field.*
*(d) Check that $y = e^x$ is the solution to $y' = y$ with $y(0) = 1$.*

**Exercise 66.3**
*(a) Use ten steps of Euler's method to determine an approximate solution for the differential equation $y' = x^3, y(0) = 0$, using $h = 0.1$.*
*(b) What is the exact solution? Compare it to the computed approximation.*
*(c) Use a sketch of the slope field for this equation to explain the results of part (b).*

**Exercise 66.4**
*(a) Use Euler's method to approximate the value of $y$ at $x = 1$ on the solution curve to the differential equation*

$$\frac{dy}{dx} = x^3 - y^3$$

*that passes through $(0,0)$. Use $h = \frac{1}{5}$(i.e., 5 steps).*
*(b) Using Figure 171, sketch the solution that passes through $(0,0)$. Show the approximation you made in part (a).*
*(c) Using the slope field, say whether your answer to part (a) is an overestimate or an underestimate.*

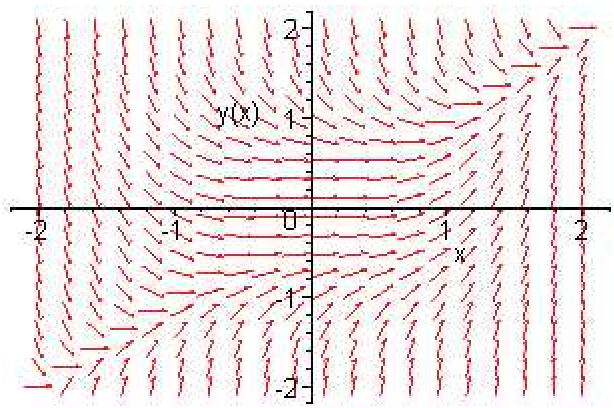



*Figure 171*

**Exercise 66.5**
*Consider the differential equation* $y' = (\sin x)(\sin y)$.

*(a) Calculate approximate y-values using Euler's method with three steps and* $h = 0.1$, *starting at each of the following points:*
*(i)* $(0, 2)$ *(ii)* $(0, \pi)$.
*(b) Find the slope field and explain your solution to part(a) (ii).*

**Exercise 66.6**
*Consider the differential equation*

$$\frac{dy}{dx} = 2x, \quad y(0) = 1.$$

*(a) Use Euler's method with two steps to estimate* $y$ *when* $x = 1$. *Now use four steps.*
*(b) What is the formula for the exact value of* $y$?
*(c) Does the error in Euler's approximation behave as predicted in Remark 66.1 (2)?*

## 67 Seperation of Variables

In this section, we remind the reader of the method of seperation of variables discussed in Section *Applications of Antiderivatives*. This method provides an analytical expression for the solution of a differential equation.

**Example 67.1**
Use the method of seperation of variables to find the solution to the differential equation

$$\frac{dy}{dx} = 2x.$$

**Solution.**
The given equation leads to $dy = 2xdx$. Now integrate both sides

$$\int dy = \int 2xdx$$

and this leads to $y = x^2 + C$.■

**Example 67.2**
Solve by means of the method of seperation of variables the DE

$$\frac{dy}{dx} = ky$$

where $k$ is any constant.

**Solution.**
The given equation leads to $\frac{dy}{y} = kdx$. Integrating both sides of this equation leads to $\ln|y| = kx + C$. Thus, $y = \pm e^{kx+C} = \pm e^C \cdot e^{kx}$ or $y = Ce^{kx}$. A model where $k > 0$ is said to represent an exponential growth whereas the model represents an exponential decay for $k < 0$.■

In general, consider the differential equation

$$\frac{dy}{dx} = f(x)g(y).$$

Multiply both sides by $\frac{1}{g(y)}$ to obtain

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Integrate both sides with respect to x to obtain

$$\int \frac{1}{g(y)} y' dx = \int f(x) dx$$

But $dy = y'dx$ so that

$$\int \frac{1}{g(y)} dy = \int f(x) dx.$$

**Example 67.3**
Solve the differential equation

$$\frac{dy}{dx} = \frac{y \ln y}{x}$$

**Solution.**
Seperating the variables and integrating we find

$$\int \frac{dy}{y \ln y} = \int \frac{dx}{x}.$$

Letting $u = \ln y$ we find

$$\int \frac{du}{u} = \int \frac{dx}{x}.$$

Thus,

$$\ln |u| = \ln |x| + C$$

or

$$\ln \left| \frac{u}{x} \right| = C$$

so that

$$\ln y = u = Cx$$

and this implies that

$$y = e^{Cx}. \blacksquare$$

## Practice Problems

Find the solutions to the differential equations in Exercises 1 - 8, subject to the given initial conditions.

**Exercise 67.1**
$y\frac{dy}{dx} = 1, \quad y(0) = 1.$

**Exercise 67.2**
$\frac{dy}{dx} = \frac{y}{2}, \quad y(0) = 100.$

**Exercise 67.3**
$\frac{1}{y}\frac{dy}{dx} = 5, \quad y(1) = 5.$

**Exercise 67.4**
$\frac{dy}{dx} = xy, \quad y(0) = 1.$

**Exercise 67.5**
$\frac{dy}{dx} = 2y - 4, \quad y(2) = 5.$

**Exercise 67.6**
$\frac{dy}{dx} + y = 1, \quad y(1) = 0.1.$

**Exercise 67.7**
$\frac{dy}{dx} = xe^y, \quad y(0) = 0.$

**Exercise 67.8**
$x(x+1)\frac{dy}{dx} = y^2, \quad y(1) = 1.$

Solve the differential equations in Exercises 9 - 11. Assume $a, b$, and $k$ are constants.

**Exercise 67.9**

$$\frac{dy}{dx} - \frac{y}{k} = 0.$$

**Exercise 67.10**

$$\frac{dy}{dx} - ay = b.$$

**Exercise 67.11**

$$\frac{dy}{dx} = ky^2(1+x^2).$$

Solve the differential equations in Exercises 12 - 13. Assume $x \geq 0, y \geq 0$.

**Exercise 67.12**

$$x\frac{dy}{dx} = (1+\ln x)\tan y.$$

**Exercise 67.13**

$$\frac{dy}{dx} = -y\ln\left(\frac{y}{2}\right), \quad y(0) = 1.$$

**Exercise 67.14**
*Find the general solution to the differential equation modeling how a person learns:*

$$\frac{dy}{dx} = 100 - y.$$

*(b) Plot the slope field of this differential equation and sketch solutions with* $y(0) = 5$ *and* $y(0) = 110$.
*(c) For each of the initial conditions in part (b), find the particular solution and add to your sketch.*

# 68 Differential Equations: Growth and Decay Models

Differential equations whose solutions involve exponential growth or decay are discussed. Everyday real-world problems involving these models are also introduced.
Consider the differential equation

$$\frac{dP}{dt} = kP.$$

Using the method of seperation of variables we find $\frac{dP}{P} = kdt$. Integrating both sides to obtain $\ln|P| = kt+C$ or $P(t) = Ce^{kt}$. Note that $C = P(0) = P_0$. Thus, $P(t) = P_0e^{kt}$. We say that the solution represents a **growth** model when $k > 0$ and a **decay** model when $k < 0$.

**Applications for Growth/Decay Models**

**Example 68.1** *(Doubling Time)*
A certain population grows exponentially. The population grows from 3500 people to 6245 people in 8 years. How long will it take for the original population to double ?This time is called the **doubling time.**

**Solution.**
We want to find the value of "t" that will yield a population of 7000 people. So if $7000 = 3500e^{rt}$ then $e^{rt} = 2$. To find $r$ we use the equality $6245 = 3500e^{8r}$. Taking the natual logarithm of both sides we find $8r = \ln\left(\frac{6245}{3500}\right)$ or $r = \frac{1}{8}\ln\left(\frac{6245}{3500}\right) \approx 0.0724$. Thus, $t = \frac{\ln 2}{0.0724} \approx 9.58\ years$. ■

**Example 68.2** *(Half-Life)*
A team of archaeologists thinks they may have discovered Fred Flintsone's fossilized bowling ball. But they want to determine whether the fossil is authentic before they report their discover to "ABC's Nightline." (Otherwise they run the risk of showing up on "Hard Copy" instead.) Fortunately, one of the scientists is a graduate of ATU's Math 2924, so he calls upon his experience as follows:
The radioactive substance (Carbon 14) has a half-life of 5730 years. By measuring the amount of present in a fossil, scientists can estimate how old the

fossil is.
Analysis of the "Flinstone bowling ball" determines that 15% of the radioactive substance has already decayed. How old is the fossil ?

**Solution.**
"Radioactive decay" means that we have a function of the form $A(t) = A_0e^{rt}$. Using the given information we can find $r$. Indeed, $0.5A_0 = A_0e^{5730r}$. Solving for $r$ we find $r \approx -0.000120968094$. Next, we want to find the desired t. Since $A(t) = .85A_0$ then $A_0e^{-0.000120968094t} = 0.85A_0$. Thus, $t = -\frac{1}{0.000120968094}\ln(0.85) \approx 1343.5$ *years*.■

**Continuous Compound Interest**
Continuously compounded interest is interest that is compounded an infinite number of times per year on a particular investment for a specific number of years. That is, we want $n$ in the compound interest formula

$$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt} = P_0\left[\left(1 + \frac{r}{n}\right)^n\right]^t$$

to be "very large". But it can be shown that

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r.$$

Thus, the formula for the continuous compound interest is given by

$$P(t) = P_0e^{rt}.$$

We call $r$ the **continuous growth rate.** The **annual growth rate** is found by solving the equation $e^r = k + 1$ for $k$. That is, $k = e^r - 1$.

**Example 68.3**
A certain investment grows from a balance of \$2500 to \$4132 in 12 years. The investment account offers continuously compounded interest. What is the annual interest rate ?

**Solution.**
Since $P(t) = 2500e^{rt}$ then $4132 = 2500e^{12r}$. Thus, $e^r = \left(\frac{4132}{2500}\right)^{\frac{1}{12}}$ so that the annual interest rate is

$$e^r - 1 = \left(\frac{4132}{2500}\right)^{\frac{1}{12}} - 1 \approx 4.19\%.■$$

**Newton's Law of Heating or Cooling**
Imagine that you are really hungry and in one minute the pizza that you are cooking in the oven will be finished and ready to eat. But it is going to be very hot coming out of the oven. How long will it take for the pizza, which is in an oven heated to 450 degrees Fahrenheit, to cool down to a temperature comfortable enough to eat and enjoy without burning your mouth?

Have you ever wondered how forensic examiners can provide detectives with a time of death (or at least an approximation of the time of death) based on the temperature of the body when it was first discovered?

All of these situations have answers because of Newton's Law of Heating or Cooling. The general idea is that *over time an object will heat up or cool down to the temperature of its surroundings.* In terms of differential equations, the temperature of an object changes, over time, at a rate proportional to the difference between its temperature and that of its surroundings:

$$\frac{dH}{dt} = k(H - S)$$

where $S$ is the temperature of the surroundings. The object is being heated when $k > 0$ and is cooling down when $k < 0$.

**Example 68.4**
The temperature of a cup of coffee is initially $150°F$. After two minutes its temperature cools to $130°F$. If the surrounding temperature of the room remains constant at $70°F$, how much longer must I wait until the coffee cools to $110°F$?

**Solution.**
The differential equation that to be solved is

$$\frac{dH}{dt} = -k(H - 70), \quad k > 0.$$

Using the seperation of variables technique, we have

$$\begin{array}{rcl} \frac{dH}{H-70} & = & -kdt \\ \int \frac{dH}{H-70} & = & \int -kdt \\ H - 70 & = & Ce^{-kt} \end{array}$$

Since $H(0) = 150°F$ then $150 - 70 = C$ so that $H(t) = 70 + 80e^{-kt}$. To find $k$ we use the fact that $T(2) = 130$. In this case, $130 = 70 + 80e^{-2k}$ or $e^{-2k} = \frac{3}{4}$. Hence, $k = -\frac{1}{2}\ln\frac{3}{4} = \ln\sqrt{\frac{4}{3}}$.
To finish the problem we must solve for t in the equation

$$110 = 70 + 80e^{-kt}.$$

From this equation, we find $e^{-kt} = 0.5$ or $t = -\frac{1}{k}\ln 0.5 = -\frac{1}{\ln\sqrt{\frac{4}{3}}}\ln 0.5 \approx 4.81$ *minutes*. Thus, I need to wait an additional 2.81 minutes.■

**Equilibrium Solutions**
If $y' = f(y)$ then a solution $y$ to the equation $y' = 0$ is called an **equilibrium solution.** The graph of such a solution is a horizontal line. An equilibrium solution is said to be **stable** if a small change in the initial conditions results in a solution that approaches the equilibrium solution as $t \to \infty$. If the solution veers away from the equilibrium solution as $t \to \infty$ then the solution is said to be **unstable**. See Figure 172.

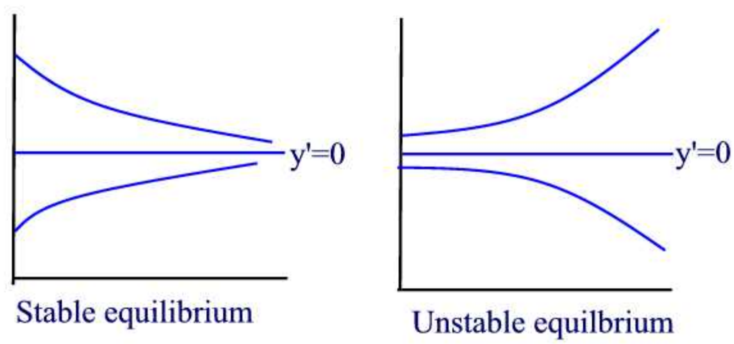



Figure 172

**Example 68.5**
The equilibrium solution of the previous example is $H(t) = 70$. Now, if $t \to \infty$ then $H(t) = 70 + 80e^{-kt} \longrightarrow 70$. That is, the equilibrium solution is a stable one.■

## Practice Problems

**Exercise 68.1**
*Each of the curves in Figure 173 represents the balance in a bank account into which a single deposit was made at time zero. Assuming continuously compounded interest, find:*

*(a) The curve representing the largest initial deposit.*
*(b) The curve representing the largest interest rate.*
*(c) Two curves representing the same initial deposit.*
*(d) Two curves representing the same interest rate.*

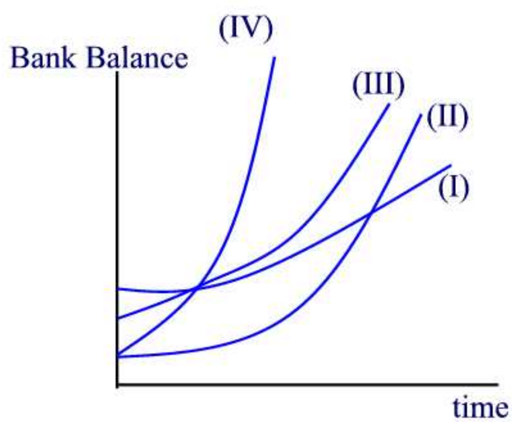



*Figure 173*

**Exercise 68.2**
*(a) What are the equilibrium solutions to the differential equation*

$$\frac{dy}{dt} = 0.2(y-3)(y+2)?$$

*(b) Use a slope field to determine whether each equilibrium solution is stable or unstable.*

**Exercise 68.3**
*A yam is put in a* $200^\circ C$ *oven and heats up according to the differential equation*

$$\frac{dH}{dt} = -k(H-200), \quad k \textit{ is a positive constant.}$$

*(a) If the yam is at $20^\circ C$ when it is put in the oven, solve the differential equation.*
*(b) Find k using the fact that after 30 minutes the temperature of the yam is $120^\circ C$.*

**Exercise 68.4**
*The rate of growth of a tumor is proportional to the size of the tumor.*

*(a) Write a differential equation satisfied by $S$, the size of the tumor, in mm, as a function of time, t.*
*(b) Find the general solution to the differential equation.*
*(c) If the tumor is 5 mm across at time $t = 0$, what does that tell you about the solution?*
*(d) If, in addition, the tumor is 8 mm across at time $t = 3$, what does that tell you about the solution?*

**Exercise 68.5**
*Hydrocodone bitartrate is used as a cough suppressant. After the drug is fully absorbes, the quantity of drug in the body decreases at a rate proportional to the amount left in the body. The hal-life of hydrocodone bitartrate in the body is 3.8 hours, and the usual dose is 10 mg.*

*(a) Write a differential equation for the quantity, $Q$, of hydrocodone bitartrate in the body at time t, inhourd, since the drug was fully absorbed.*
*(b) Solve the differential equation given in part (a).*
*(c) Use the half-life to find the constant of proportionality.*
*(d) How much of the 10 mg dose is still in the body after 12 hours?*

**Exercise 68.6**
*(a) If $B = f(t)$ is the balance at time t of a bank account that earns interest at a rate r%, compounded continuously, what is the differential equation describing the rate at which the balance changes? What is the constant of proportionality, in terms of r?*
*(b) What is the solution to the differential equation?*
*(c) Sketch the graph of $B = f(t)$ for an account that starts with $1,000 and earns interest at the following rates:*

*)i) 4%   (ii) 10%   (iii) 15%*

**Exercise 68.7**

*Morphine is often used as a pain-relieving drug. The half-life of morphine in the body is 2 hours. Suppose morphine is administered to a patient intravenously at a rate of 2.5 mg per hour, and the rate at which the morphine is eliminated is proportional to the amount present.*

*(a) Show that the constant of proportionality for the rate at which morphine leaves the body (in mg/hour) is* $k = -0.347$.
*(b) Write a differential equation for the quantity,* $Q$, *of morphine in the blood after* $t$ *hours.*
*(c) Use the differential equation to find the equilibrium solution.( This is the long-term amount of morphine in the body, once the system is stabilized.)*

**Exercise 68.8**

*A detective finds a murder victim at 9 am. The temperature of the body is measured at* $90.3°F$. *One hour later, the temperature of the body is* $89.0°F$. *The temperature of the room has been maintained at a constant* $68°F$.

*(a) Assuming the temperature,* $T$, *of the body obeys Newton's Law of cooling, write a differential equation for* $T$.
*(b) Solve the differential equation to estimate the time the murder occurred.*

# 69 First Order Differential Equations Models

Since most physical situations involve something changing, derivatives come into play resulting in a differential equation. In this section, We will investigate examples of how differential equations can model such situations.

**Example 69.1** *(Compartmental Analysis)*
Consider a tank with volume 100 liters containing a salt solution. Suppose a solution with 2kg/liter of salt flows into the tank at a rate of 5 liters/min. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of 5 liters/min. If initially there is 20 kg of salt in the tank, how much salt will be in the tank as a function of time?

**Solution.**
Let $y(t)$ denote the amount of salt in kg in the tank after $t$ minutes. We use a fundament property of rates:

$$\text{TotalRate} = \text{ Rate in } - \text{ Rate out.}$$

To find the rate in we use

$$\frac{kg}{min} = \frac{liters}{min} \cdot \frac{kg}{liter} = (5)(2) = 10kg/min.$$

The rate at which salt leaves the tank is equal to the rate of flow of solution out of the tank times the concentration of salt in the solution. Thus, the rate out is

$$\frac{kg}{min} = \frac{liters}{min} \cdot \frac{kg}{liter} = (5)(\frac{y}{100}) = \frac{y}{20}kg/min.$$

The differential equation for the amount of salt is

$$\left\{ \begin{array}{lcl} y' & = & 10 - \frac{y}{20} \\ y(0) & = & 20. \end{array} \right.$$

Using the method of seperation of variable we find

$$\begin{array}{ccc} y' & = & 10 - 0.05y \\ \frac{dy}{10-0.05y} & = & dt \\ \int \frac{dy}{10-0.05y} & = & \int dt \\ -20\ln|10 - 0.05y| & = & t + C \\ 10 - 0.05y & = & Ce^{-0.05t} \\ y & = & 200 - Ce^{-0.05t}. \end{array}$$

But $y(0) = 20$ so that $C = 180$. Hence, the amount of salt in the tank after t minutes is given by the formula

$$y(t) = 200 - 180e^{-0.05t}.$$

**Example 69.2** *(Falling Body)*
A 50 kg mass is shot from a cannon straight up with an initial velocity of 10 m/sec off a bridge that is 100 feet above the ground. If air resistance is given by $5v$, determine the velocity of the mass when it hits the ground.

**Solution.**
Figure 174 shows a sketch of the situation.

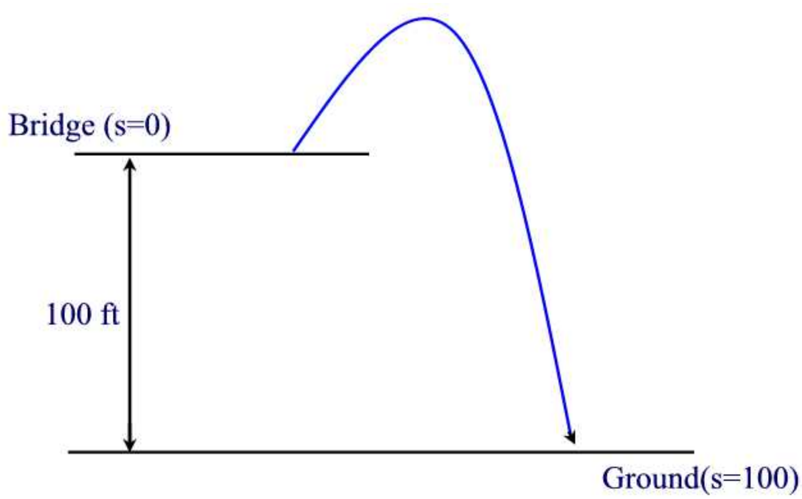



Figure 174

Since the vast majority of the motion will be in the downward direction we will assume that everything acting in the downward direction should be positive.
The motion of the mass consists of two phases. The initial phase in which the mass is rising in the air and the second phase when the mass is on its way down. Figure 175 shows the forces that are acting on the object on the way up and on the way down.

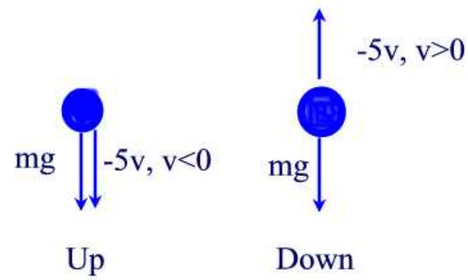



Figure 175

By Newton's Second Law of Motion

$$Force = Mass \times Acceleration.$$

The initial value problem for the first phase is

$$mg - 5v = m\frac{dv}{dt}, \quad v(0) = -10$$

and that for the second phase is

$$mg - 5v = m\frac{dv}{dt}, \quad v(t_0) = 0$$

where $t_0$ is the time when the object is at the highest point and is ready to start on the way down. Note that at this time the velocity would be zero. Note that the differential equation for both of the situations is identical. This wont always happen, but in those cases where it does, we can ignore the second initial value condition and just let the first govern the whole process. Thus, we will consider solving the linear first order differential equation

$$mg - 5v = m\frac{dv}{dt}, \quad v(0) = -10.$$

This is done as follows:

$$\begin{array}{ccc}
m\frac{dv}{dt} & = & mg - 5v \\
\frac{dv}{dt} & = & -\frac{5}{m}\left(v - \frac{mg}{5}\right) \\
\int \frac{dv}{v - mg/5} & = & -\frac{5}{m}\int dt \\
\ln|v - mg/5| & = & -\frac{5}{m}t + C \\
v - mg/5 & = & Ce^{-\frac{5}{m}t} \\
v & = & \frac{mg}{5} + Ce^{-\frac{5}{m}t}.
\end{array}$$

But, $m = 50, g = 9.8m/sec^2, v(0) = -10$ so that $C = -108$. Thus,

$$v(t) = 98 - 108e^{-0.1t}.$$

Now, to find the velocity of the mass when it hits the ground we must first the time when it hits the ground. Let $s(t)$ be the position of the function with respect to the bridge. Then

$$s(t) = \int (98 - 108e^{-0.1t})dt = 98t + 1080e^{-0.1t} + C.$$

Since $s(0) = 0$ then $C = -1080$. Thus,

$$s(t) = 98t + 1080e^{-0.1t} - 1080.$$

Using a calculator, solve the equation $s(t) = 100$ to find $t \approx 5.98147$. Thus, the mass hits the ground with a velocity of

$$v(5.98147) \approx 38.61841. \blacksquare$$

**Example 69.3** *(Lottery Winnings)*
You just won the lottery. You put your \$5,000,000 in winnings into a fund that has a rate of return of 4%. Each year you use \$300,000. How much money will you have twenty years from now?

**Solution**
Again, using a propert of rates we have

$$\text{TotalRate} = \text{ Rate in } - \text{ Rate out.}$$

where the rate out is \$300,000 and the rate in is $0.04x$ where $x(t)$ denotes the balance in the account after $t$ years. Thus,

$$\frac{dx}{dt} = 0.4x - 300,000.$$

This is both linear first order and separable. We separate and integrate to obtain

$$\begin{array}{ccc}
\int \frac{dx}{0.4x-300,000} & = & \int dt \\
25 \ln |0.4x - 300,000| & = & t + C \\
0.4x - 300,000 & = & Ce^{\frac{t}{25}} \\
x & = & 7,500,00 + Ce^{\frac{t}{25}}.
\end{array}$$

Since $x(0) = 5,000,000$ we find $C = -2,500,000$. Finally,

$$x(20) = 7,500,000 - 2,500,000e^{\frac{20}{25}} = \$1,936,148. \blacksquare$$

## Practice Problems

**Exercise 69.1**
*The velocity, $v$, of a dust particle of mass $m$ and acceleration $a$ satisfies the equation*

$$m\frac{dv}{dt} = mg - kv, \quad \text{where g and k are constants..}$$

*By differentiating this equation, find a differential equation satisfied by $a$. (Your may contain $m, g, k$, but not $v$.) Solve for $a$, given that $a(0) = g$.*

**Exercise 69.2**
*A deposit is made to a bank account paying 8% interest compounded continuously. Payments totaling \$2,000 per year are made from this account.*

*(a) Write a differential equation for the balance, $B$, in the account after $t$ years.*
*(b) Write the solution to the differential equation.*
*(c) How much is in the account after 5 years is the initial deposit is:*
*(i) \$20,000?   (ii) \$30,000?*

**Exercise 69.3**
*At time $t = 0$, a bottle of juice at $90°F$ is stood in a mountain stream whose temperature is $50°F$. After 5 minutes, its temperature is $80°F$. Let $H(t)$ denote the temperature of the juice at time $t$ in minutes.*

*(a) Write a differential equation for $H(t)$, using Newton's Law of Cooling.*
*(b) Solve the differential equation.*
*(c) When will the temperature of the juice have dropped to $60°F$?*

**Exercise 69.4**
*According to a simple physiological model, an athletic adult needs 20 calories per day per pound of body weight to maintain his weight. If he consumes more or fewer calories than those required to maitain his weight, his weight changes at a rate proportional to difference between the number of calories consumed and the number needed to maitain his current weight; the constant of proportionality is 1/3500 pounds per calorie. Suppose that a particular person has a constant calorie intake of $I$ calories per day. Let $W(t)$ be the person's weight in pounds at time $t$ (mesured in days).*

*(a) What differential equation has solution $W(t)$?*
*(b) Solve this differential equation.*
*(c) Graph $W(t)$ if the person starts out weighing 160 pounds and consumes 3000 calories a day.*

**Exercise 69.5**
*Water leaks out of a barrel at a rate proportional to the square root of the depth of the water at the time. If the water level starts at 36 inches and drops to 35 inches in 1 hour, how long will it take for all of the water to leak out of the barrel?*

**Exercise 69.6**
*A spherical snowball melts at a rate proportional to its surface area.*

*(a) Write a differential equation for its volume, $V$.*
*(b) If the initial value is $V_0$, solve the differential equation and graph the solution.*
*(c) When does the snowball disappear?*

**Exercise 69.7**
*When a course ends, students start to forget the material they have learned. One model assumes that the rate at which a student forgets material is proportional to the difference between the material currently remembered and some positive constant, $a$.*

*(a) Let $y = f(t)$ be the fraction of the original material remembered $t$ weeks after the course has ended. Set up a differential equation for $y$. Your equation will contain two constants; the constant $a$ is less than $y$ for all $t$.*
*(b) Solve the differential equation.*
*(c) Describe the practical meaning (in terms of the amount remembered) of the constants in the solution $y = f(t)$.*

**Exercise 69.8**
*An aquarium pool has volume $2 \cdot 10^6$ liters. The pool initially contains pure fresh water. At $t = 0$ minutes, water containing 10 grams/liter of salt is poured into the pool at a rate of 60 liters/minute. The salt water instantly mixes with the fresh water, and the excess mixture is drained out of the pool*

*at eh same rate (60 liters/minute).*

*(a) Write a differential equation for $S(t)$, the mass of salt in the pool at time $t$.*
*(b) Solve the differential equation to find $S(t)$.*
*(c) What happens to $S(t)$ as $t \to \infty$.*

# 70 The Logistic Model

One of the consequences of exponential growth is that the output $f(t)$ increases indefinitely in the long run. However, in some situations there is a limit $L$ to how large $f(t)$ can get. For example, the population of bacteria in a laboratory culture, where the food supply is limited. In such situations, the rate of growth slows as the population reaches the carrying capacity. One useful model is the **logistic growth model.**
Thus, logistic functions model *resource-limited* exponential growth.
A **logistic function** involves three positive parameters $L, C, k$ and has the from

$$f(t) = \frac{L}{1 + Ce^{-kt}}.$$

We next investigate the meaning of these parameters. From our knowledge of the graph of $e^{-x}$ we can easily see that $e^{-kt} \to 0$ as $t \to \infty$. Thus, $f(t) \to L$ as $t \to \infty$. It follows that the parameter $L$ represents the limiting value of the output past which the output cannot grow. We call $L$ the **carrying capacity.**
Now, to interpret the meaning of $C$, we let $t = 0$ in the formula for $f(t)$ and obtain $(1 + C)f(0) = L$. This shows that $C$ is the number of times that the initial output must grow to reach $L$. Finally, the parameter $k$ affects the steepness of the curve, that is, as $k$ increases, the curve approaches the asymptote $y = L$ more rapidly.

**Example 70.1**
Show that a logistic function is approximately exponential function with continuous growth rate $k$ for small values of $t$.

**Solution.**
Rewriting a logistic function in the form

$$f(t) = \frac{Le^{kt}}{e^{kt} + C}$$

we see that $f(t) \approx \frac{L}{1+C}e^{kt}$ for small values of $t$.■

**Graphs of Logistic Functions**
Graphing the logistic function $f(t) = \frac{185}{1+48e^{-0.032t}}$ (See Figure 175) we find

Figure 175

As is clear from the graph above, a logistic function shows that initial exponential growth is followed by a period in which growth slows and then levels off, approaching (but never attaining) a maximum upper limit.
Notice the characteristic S-shape which is typical of logistic functions.

**Point of Diminishing Returns**
Another important feature of any logistic curve is related to its shape: *every logistic curve has a single inflection point which separates the curve into two equal regions of opposite concavity.* This inflection point is called the **point of diminishing returns.**

**Finding the Coordinates of the Point of Diminishing Returns**
To find the point of inflection of a logistic function of the form $P = f(t) = \frac{L}{1+Ce^{-kt}}$, we notice that $P$ satisfies the equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

Using the product rule we find

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}\left(1 - \frac{2P}{L}\right).$$

Since $\frac{dP}{dt} > 0$ we conclude that $\frac{d^2P}{dt^2} = 0$ at $P = \frac{L}{2}$.
To find $y$, we set $y = \frac{L}{2}$ and solve for $t$ :

$$\begin{array}{rcl}
\frac{L}{2} & = & \frac{L}{1+Ce^{-kt}} \\
\frac{2}{L} & = & \frac{1+Ce^{-kt}}{L} \\
2 & = & 1 + Ce^{-kt} \\
1 & = & Ce^{-kt} \\
e^{kt} & = & C \\
kt & = & \ln C \\
t & = & \frac{\ln C}{k}
\end{array}$$

Thus, the coordinates of the diminishing point of returns are $\left(\frac{\ln C}{k}, \frac{L}{2}\right)$.

Logistic functions are good models of biological population growth in species which have grown so large that they are near to saturating their ecosystems, or of the spread of information within societies. They are also common in marketing, where they chart the sales of new products over time.

**Example 70.2**
The following table shows that results of a study by the United Nations (New York Times, November 17, 1995) which has found that world population growth is slowing. It indicates the year in which world population has reached a given value:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Year | 1927 | 1960 | 1974 | 1987 | 1999 | 2011 | 2025 | 2041 | 2071 |
| Billion | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

(a) Construct a scatterplot of the data, using the input variable $t$ is the number of years since 1900 and output variable $P =$ worldpopulation (in billions).
(b) Using a logistic regression, fit a logistic function to this data.
(c) Find the point of diminishing returns. Interpret its meaning.

**Solution.**
(a) For this part, we recommand the reader to use a TI for the plot.
(b) Using a TI with the logisitc regression we find $L = 11.5, C = 12.8, k = 0.0266$. Thus,

$$P = \frac{11.5}{1 + 12.8e^{-0.0266t}}.$$

(c) The inflection point on the world population curve occurs when $t = \frac{\ln C}{k} = \frac{\ln 12.8}{0.0266} \approx 95.8$. In other words, according to the model, in 1995 world population attained 5.75 billion, half its limiting value of 11.5 billion. From this year on, population will continue to increase but at a slower and slower rate. ■

## Practice Problems

**Exercise 70.1**
*The Tojolobal Mayan Indian community in Southern Mexico has available a fixed amount of land. The proportion, $P$, of land in use for farming $t$ years after 1935 is modeled with the logistic function*

$$P(t) = \frac{1}{1 + 2.968e^{-0.0275t}}.$$

*(a) What proportion of the land was in use for farming in 1935?*
*(b) What is the long run prediction of this model? (c) When was half the land in use for farming?*
*(d) When is the proportion of land used for farming increasing most rapidly?*

**Exercise 70.2**
*The total number of people infected with a virus often grows like a logistic curve. Suppose that 10 people originally have the virus, and that in the early stages of the virus (with time, $t$, measured in weeks), the number of people infected is increasing exponentially with $k1.78$. It is estimated that, in the long run, approximately 5000 people become infected.*

*(a) Use this information to find a logistic function to model this situation.*
*(b) Sketch a graph of your answer to part (a).*
*(c) Use your graph to estimate the length of time until the rate at which people are becoming infected starts to decrease. What is the vertical coordinate at this point?*

**Exercise 70.3**
*The table below gives the percentage, $P$, of households with a VCR, as a function of year.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *Year* | *'78* | *'79* | *'80* | *'81* | *'82* | *'83* | *'84* |
| *P%* | *0.3* | *0.5* | *1.1* | *1.8* | *3.1* | *5.5* | *10.6* |
| *Year* | *'85* | *'86* | *'87* | *'88* | *'89* | *'90* | *'91* |
| *P%* | *20.8* | *36.0* | *48.7* | *58.0* | *64.6* | *71.9* | *71.9* |

*(a) Explain why a logistic model is reasonable one to use for this data.*
*(b) Use the data to estimate the point of inflection of $P$. What limiting value $L$ does the point of inflection predict? Does this limiting value appear to be*

*accurate given the percentages for 1990 and 1991?*
*(c) The best logistic equation for this data turns out to be the following. What limiting value does this model predict?*

$$P = \frac{75}{1 + 316.75e^{-0.699t}}.$$

**Exercise 70.4**
*An alternative method of finding the analytic solution to the logistic equation*

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

*uses the substitution* $P = 1/u$.

*(a) Show that*

$$\frac{dP}{dt} = -\frac{1}{u^2}\frac{du}{dt}.$$

*(b) Rewrite the logistic equation in terms of u and t, and solve for u in terms of t.*
*(c) Using your answer to part (b), find P as a function of t.*

**Exercise 70.5**
*Any population, P, for which we can ignore immigration, satisfies*

$$\frac{dP}{dt} = Birth\ rate - death\ rate.$$

*For organisms which need a partner for reproduction but rely on a chance encounter for meeting a mate, the birth rate is proportional to the square of the population. Thus, the population of such a type of organism satisfies a differential equation of the form*

$$\frac{dP}{dt} = aP^2 - bP,\ with\ a, b > 0.$$

*Suppose that* $a = 0.02$ *and* $b = 0.08$.

*(a) Sketch the slope field for this differential equation for* $0 \leq t \leq 50, 0 \leq P \leq 8$.
*(b) Use your slope field to sketch the general shape of the solutions to the differential equation satisfying the following initial conditions:*
*(i)* $P(0) = 1$ *(ii)* $P(0) = 3$ *(iii)* $P(0) = 4$ *(iv)* $P(0) = 5$.
*(c) Are there any equilibrium values of the population? If so, are they stable?*

# 71 Second-Order Differential Equations Models

In this section we consider solving a second order differential equation of the form

$$y'' + \omega^2 y = 0. \tag{22}$$

We begin with the observation that if $y$ satisfies this equation, then $y''$ is equal to a constant multiple of $y$. Hence it would be reasonable to begin with $y = e^{\lambda x}$, for some constant $\lambda$, as an initial guess. In that case, $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ so $y$ will be a solution to (22) if and only if

$$\lambda^2 e^{\lambda x} + \omega^2 e^{\lambda x} = 0.$$

Dividing through by $e^{\lambda x}$ to obtain

$$\lambda^2 + \omega^2 = 0$$

This equation has complex solutions and can be factored as

$$(\lambda - i\omega)(\lambda + i\omega) = 0, \ \textit{where}\ i = \sqrt{-1}.$$

Hence, $\lambda = \pm i\omega$. But, by definition

$$e^{\pm i\omega x} = \cos\omega x \pm i \sin\omega x.$$

Since $e^{i\omega x}$ and $e^{-i\omega x}$ are solutions then by the *principle of superposition* so do

$$\frac{1}{2}(e^{i\omega x} + e^{-i\omega x}) = \cos\omega x$$

and

$$\frac{1}{2i}(e^{i\omega x} - e^{-i\omega x}) = \sin\omega x.$$

Hence, a general solution to (22) is

$$y = C_1 \cos(\omega x) + C_2 \sin(\omega x). \tag{23}$$

**Oscillations of a Mass on a Spring**

Consider a mass $m$ attached to the end of a spring hanging from a ceiling. We assume that the mass of the spring is negligible in comparison with the

mass $m$. When the system is left undisturbed, the force of gravity is balanced by the force the spring exerts on the mass, and the spring is in **equilibrium position.** If we pull down on the mass, we feel a force pulling upwards and if we push the mass upward we feel a force pushing the mass down. If we push the mass up and then releases it then the mass oscillates up and down around the equilibrium position.
By Hooke's Law, the net force, $F$, exerted on the mass is proportional to the displacement, $s$ :

$$F = -ks$$

where $k > 0$ is the spring constant and depends on the physical properties of the spring. The negative sign is to indicate that the net force is in the opposite direction to the displacement.
Now, by Newton's Second Law of Motion, we have

$$Net\ Force = Mass \times Acceleration.$$

Thus, the motion of a mass on a spring is described by the second order differential equation

$$-ks = m\frac{d^2s}{dt^2}$$

or

$$\frac{d^2s}{dt^2} + \frac{k}{m}s = 0$$

By (23), the general solution to this equation is given by

$$s(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

**Example 71.1**
Find the solution to the equation

$$\frac{d^2s}{dt^2} + 4s = 0$$

satisfying the boundary conditions $s(0) = 0$ and $s(\frac{\pi}{4}) = 20$.

**Solution**
Since $\omega^2 = 4$ then $\omega = 2$ so that the general solution is given by

$$s(t) = C_1 \cos 2t + C_2 \sin 2t.$$

The condition $s(0) = 0$ leads to $C_1 = 0$ and $s(\frac{\pi}{4}) = 20$ leads to $C_2 = 20$. Hence, $s(t) = 20\sin 2t$.$\blacksquare$

Now, let $A = \sqrt{C_1^2 + C_2^2}$, $\cos\phi = \frac{C_2}{A}$, and $\sin\phi = \frac{C_1}{A}$. Hence,

$$\begin{array}{rcl} C_1\cos(\omega x) + C_2\sin(\omega x) & = & A\sin\phi\cos(\omega x) + A\cos\phi\sin(\omega x) \\ & = & A\sin(\omega x + \phi) \end{array}$$

Since $A$ is half the distance between the maximum and the minimum points on the graph of $A\sin(\omega x + \phi)$ then it is called the **amplitude.** The angle $\phi$ is called the **phase shift**. It is assumed that $-\pi < \phi \le \pi$. The **period** of the motion is given by $T = \frac{2\pi}{\omega}$.

**Example 71.2**
Find the amplitude, period, and the phase shift of the solution to the equation

$$y'' + 16y = 0,\ y(0) = 5, y'(0) = 0.$$

**Solution.**
Since $\omega = 4$ then the general solution is

$$y(x) = C_1\cos(4x) + C_2\sin(4x).$$

Since $y(0) = 5$ then $C_1 = 5$. Since $y'(0) = 0$ then $4C_2 = 0$ so that $C_2 = 0$. Hence, $y = 5\cos(4x)$. Thus, $y = 5\sin(4x + \phi)$ where $\cos\phi = 0$ and $\sin\phi = 1$. The amplitude of this motion is $A = 5$, the period is $T = \frac{\pi}{2}$, and the phase shift is $\phi = \frac{\pi}{2}$.$\blacksquare$

## Practice Problems

**Exercise 71.1**
*Check by differentiation that* $y(t) = 2\cos t + 3\sin t$ *is a solution to* $y'' + y = 0$.

**Exercise 71.2**
*Check by differentiation that* $y(t) = A\cos t + B\sin t$ *is a solution to* $y'' + y = 0$ *for any constants* $A$ *and* $B$.

**Exercise 71.3**
*Check by differentiation that* $y(t) = A\cos\omega t + B\sin\omega t$ *is a solution to* $y'' + \omega^2 y = 0$ *for all values of* $A$ *and* $B$.

**Exercise 71.4**
*What values of* $\alpha$ *and* $A$ *make* $y(t) = A\cos\alpha t$ *a solution to* $y'' + 5y = 0$ *such that* $y'(1) = 3$?

**Exercise 71.5**
*Write the function* $s(t) = \cos t - \sin t$ *as a single sine function. Draw its graph.*

**Exercise 71.6**
*Find the amplitude of the oscillation* $y(t) = 3\sin 2t + 4\cos 2t$.

**Exercise 71.7**
*Write the function* $y(t) = 5\sin(2t) + 12\cos(2t)$ *in the form* $y(t) = A\sin(\omega t + \alpha)$.

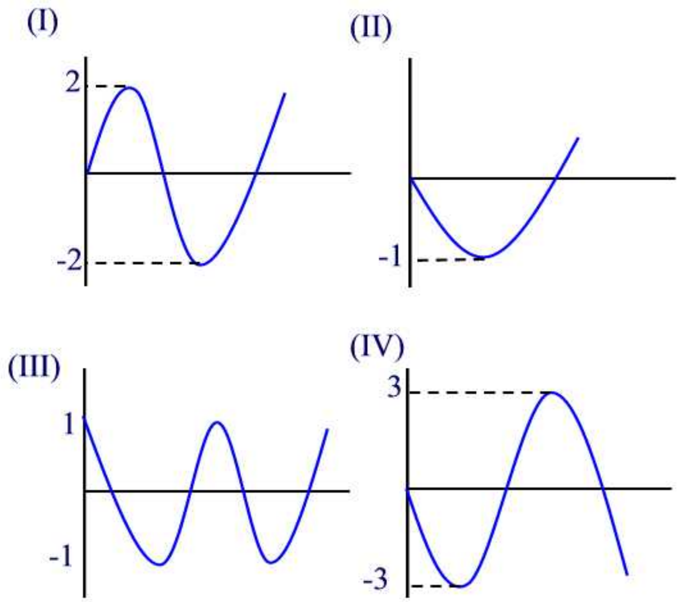
**Exercise 71.8**
*(a) Find the general solution of the differential equation*

$$y'' + 9y = 0.$$

*(b) For each of the following initial conditions, find a particular solution.*

*(i)* $y(0) = 0, y'(0) = 1$ *(ii)* $y(0) = 1, y(1) = 0$ *(iii)* $y(0) = 0, y(1) = 1$.
*(c) Sketch a graph of the solutions found in part (b).*

**Exercise 71.9**
*Each graph in the figure below reprents a solution to one of the differential equations:*
*(a)* $x'' + x = 0$, *(b)* $x'' + 4x = 0$ *(c)* $x'' + 16x = 0$.
*Assuming the t-scales on the four graphs are the same, which graph represents a solution to which equation? Find an equation for each graph.*




**Exercise 71.10**
*The following differential equations represent oscillating springs.*

$$\begin{array}{rrcrrcrrcr} (i) & s''+4s & = & 0 & s(0) & = & 5, & s'(0) & = & 0 \\ (ii) & 4s''+s & = & 0 & s(0) & = & 10, & s'(0) & = & 0 \\ (iii) & s''+6s & = & 0 & s(0) & = & 4, & s'(0) & = & 0 \\ (iv) & 6s''+s & = & 0 & s(0) & = & 20, & s'(0) & = & 0 \end{array}$$

*Which differential equation represents:*

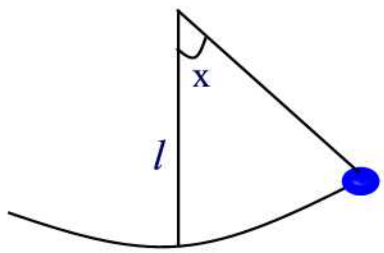
*(a) The spring oscillating most quickly (i.e. with the shortest period)?*
*(b) The spring oscillating with the largest amplitude?*
*(c) The spring oscillating most most slowly (i.e. with the longest period)?*
*(d) The spring with largest maximum velocity?*

**Exercise 71.11**

*A pendulum of length $l$ makes an angle of $x$ radians with the vertical (see figure below). When $x$ is small, it can be shown that, approximately:*

$$\frac{d^2x}{dt^2} = -\frac{g}{l}x,$$

*where $g$ is the acceleration due to gravity.*




*(a) Solve this differential equation assuming that $x(0) = 0$ and $x'(0) = v_0$.*
*(b) Solve this equation assuming that the pendulum is let go from the position where $x = x_0$, i.e., the velocity of the pendulum is zero. Measure $t$ from the moment when the pendulum is let go.*

**Exercise 71.12**

*A brick of mass 3 kg hangs from the end of a spring. When the brick is at rest, the spring is stretched by 2 cm. The spring is then stretched an additional 5 cm and released. Assume there is no air resistance.*

*(a) Set up a differential equation with initial conditions describing the motion.*
*(b) Solve the differential equation.*

**Exercise 71.13**

*Consider the motion described by the differential equations:*

$$\begin{array}{llllllll} (a) & x'' + 16x & = & 0, & x(0) & = & 5, & x'(0) = 0. \\ (b) & 25x'' + x & = & 0, & x(0) & = & -1, & x'(0) = 2. \end{array}$$

*In each case, find a formula for $x(t)$ and calculate the amplitude and period of the motion.*

# 72 Second-Order Homogeneous Linear Differential Equations

By a **homogeneous linear second order differential equation with constant coefficients** we mean an equation of the form

$$ay'' + by' + cy = 0 \tag{24}$$

We begin with the observation that if $y$ satisfies this equation, then $y''$ is equal to a sum of constant multiples of $y$ and $y'$. Hence it would be reasonable to begin with $y = e^{\lambda x}$, for some constant $\lambda$, as an initial guess. In that case, $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ so $y$ will be a solution to (24) if and only if

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

Dividing by $e^{\lambda x}$ gives the **characteristic equation**

$$a\lambda^2 + b\lambda + c = 0 \tag{25}$$

To solve this equation for $\lambda$ we need to consider the following three cases:

• If $b^2 - 4ac > 0$ then (25) has two distinct solutions $C_1 e^{\lambda_1 x}$ and $C_2 e^{\lambda_2 x}$, where $C_1$ and $C_2$ are arbitrary constants. By the *principle of superposition*, the general solution to (24) is given by

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

If $\lambda_1 < 0$ and $\lambda_2 < 0$ then the equations are called **overdamped**(i.e. a lot of friction in the system).

• If $b^2 - 4ac = 0$ then (25) has only one solution, $\lambda = -\frac{b}{2a}$. By substitution, we can check that both $y = C_1 e^{\lambda x}$ and $y = C_2 x e^{\lambda x}$ are solutions to (24) so that the general solution is given by

$$y = (C_1 + C_2 x)e^{\lambda x}$$

If $\lambda < 0$ then the equation is said to be **critically damped**.

• If $b^2 - 4ac < 0$ then (25) has complex conjugate roots $\lambda_1 = \frac{-b-i\sqrt{4ac-b^2}}{2a}$ and $\lambda_2 = \frac{-b+i\sqrt{4ac-b^2}}{2a}$. Writing $\lambda_1 = p - iq$ and $\lambda_2 = p + iq$ and using the fact that

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

we see that

$$\frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x}) = e^{px}\cos{(qx)}$$

is a solution to (24) and

$$\frac{1}{2i}(e^{\lambda_1 x} - e^{\lambda_2 x}) = e^{px}\sin{(qx)}$$

so the general solution is given by

$$y = C_1 e^{-\frac{b}{2a}x}\cos{(\frac{\sqrt{4ac-b^2}}{2a})}x + C_2 e^{-\frac{b}{2a}x}\sin{(\frac{\sqrt{4ac-b^2}}{2a})}x.$$

If $-\frac{b}{2a} < 0$ then the above oscillations are called **underdamped**

**Example 72.1**
Find the solution of $y'' - 3y' + 2y = 0$ satisfying $y(0) = 1$ and $y'(0) = 0$.

**Solution.**
Comparing the given equation with (24) we find $a = 1, b = -3$, and $c = 2$. Thus, $b^2 - 4ac = 1 > 0$ and therefore

$$y = C_1 e^x + C_2 e^{2x}$$

The conditions $y(0) = 1$ and $y'(0) = 0$ yield the linear system of equations

$$\begin{cases} C_1 + C_2 & = & 1 \\ C_1 + 2C_2 & = & 0 \end{cases}$$

Solving this system we find, $C_1 = -1$ and $C_2 = 2$. Thus,

$$y = -e^x + 2e^{2x}.\blacksquare$$

**Example 72.2**
Find the solution of $y'' - 2y' + y = 0$ satisfying $y(0) = 2$ and $y(1) = 0$.

**Solution.**
Comparing the given equation with (24) we find $a = 1, b = -2$, and $c = 1$. Thus, $b^2 - 4ac = 0$ and therefore

$$y = (C_1 + C_2 x)e^x$$

The condition $y(0) = 2$ yields $C_1 = 2$ and $y(1) = 0$ yields $C_2 = -2$. Thus,

$$y = 2(1 - x)e^x.\blacksquare$$

**Example 72.3**
Find the solution of $y'' + y = 0$ satisfying $y(0) = 1$ and $y'(0) = 0$.

**Solution.**
Comparing the given equation with (24) we find $a = 1, b = 0$, and $c = 1$. Thus, $b^2 - 4ac = -4 < 0$ and therefore

$$y = C_1 \cos x + C_2 \sin x$$

The condition $y(0) = 1$ yields $C_1 = 1$ and $y'(0) = 0$ yields $C_2 = 0$. Thus,

$$y = \cos x. \blacksquare$$

## Practice Problems

**Exercise 72.1**
*Find the general solution to the given differential equation.*

1. $y'' + 4y' + 3y = 0.$
2. $s'' + 7s = 0.$
3. $\frac{d^2P}{dt^2} + \frac{dP}{dt} + P = 0.$

**Exercise 72.2**
*Solve the initial value problem.*

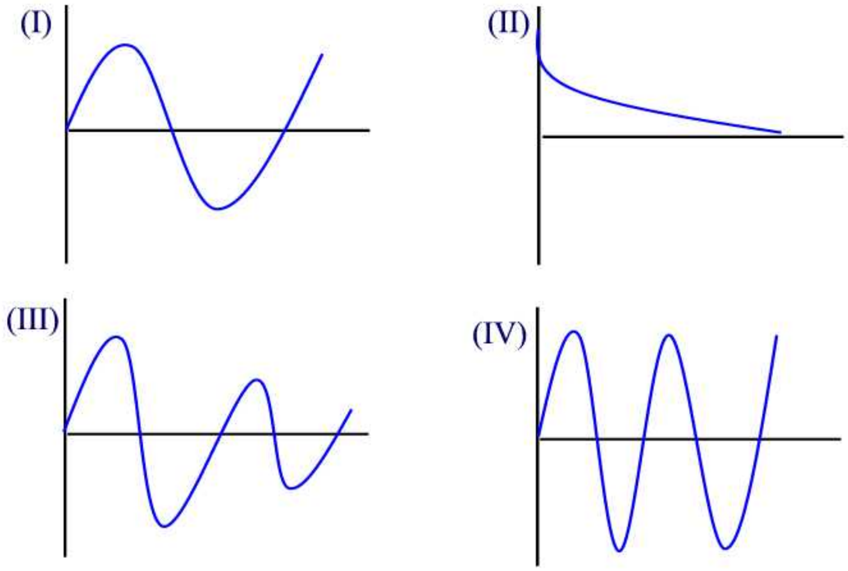
1. $y'' + 5y' + 5y = 0, y(0) = 1, y'(0) = 0.$
2. $y'' - 3y' - 4y = 0, y(0) = 1, y'(0) = 0.$
3. $y'' + 6y' + 5y = 0, y(0) = 1, y'(0) = 0.$
4. $y'' + 6y' + 10y = 0, y(0) = 0, y'(0) = 2.$

**Exercise 72.3**
*Solve the boundary value problem.*

1. $y'' + 5y' + 6y = 0, y(0) = 1, y(1) = 0.$
2. $p'' + 2p' + 2p = 0, p(0) = 0, p(\frac{\pi}{2}) = 20.$

**Exercise 72.4**
*Match the graphs of solutions in the figure with the differential equations below.*

*(a)* $x'' + 4x = 0$
*(b)* $x'' - 4x = 0$
*(c)* $x'' - 0.2x' + 1.01x = 0$
*(d)* $x'' + 0.2x' + 1.01x = 0.$




**Exercise 72.5**
*If* $y = 2^{2t}$ *is a solution to the differential equation*

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + ky = 0,$$

*find the value of the constant* $k$ *and the general solution to this equation.*

**Exercise 72.6**
*For each of the differential equations given below, find the values of* $c$ *that make the general solution:*
*(a) overdamped (b) underdamped (c) critically damped.*

1. $s'' + 4s' + cs = 0$
2. $s'' + 2\sqrt{2}s' + cs = 0$
3. $s'' + 6s' + cs = 0$

**Exercise 72.7**
*Find a solution to the following differential equation which satisfies* $z(0) = 3$ *and does not tend to infinity as* $t \to \infty$ :

$$\frac{d^2z}{dt^2} + \frac{dz}{dt} - 2z = 0.$$