Reform Calculus: Part I

Marcel B. Finan
Arkansas Tech University
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PREFACE

This supplement consists of my lectures of a freshmen-level mathematics class offered at Arkansas Tech University. The lectures are designed to accompany the textbook "*Calculus: Single and Multivariable*" written by the Harvard Consortium.

This book has been written in a way that can *be read by students*. That is, the text represents a serious effort to produce exposition that is accessible to a student at the freshmen or high school levels.

The lectures cover sections from Chapters 1, 2, 3, 4, 5, and 6. These chapters are basically well suited for a 4-hour one semester course in a first course Calculus.

Marcel B. Finan
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Contents

1 Functions and Change

Functions play a crucial role in mathematics. A function describes how one quantity depends on others. More precisely, when we say that a quantity y is a **function** of a quantity x we mean a rule that assigns to every possible value of x exactly one value of y. We call x the **input** and y the **output**. In **function notation** we write

$$y = f(x)$$

Since y depends on x it makes sense to call x the **independent variable** and y the **dependent variable**.

In applications of mathematics, functions are often representations of real world phenomena. Thus, the functions in this case are referred to as **mathematical models.** If the set of input values is a finite set then the models are known as **discrete** models. Otherwise, the models are known as **continuous** models. For example, if H represents the temperature after t hours for a specific day, then H is a discrete model. If A is the area of a circle of radius r then A is a continuous model.

There are four common ways in which functions are presented and used: By words, by tables, by graphs, and by formulas.

Example 1.1

The sales tax on an item is 6%. So if p denotes the price of the item and C the total cost of buying the item then if the item is sold at \$ 1 then the cost is $1 + (0.06)(1) = \$1.06$ or $C(1) = \$1.06$. If the item is sold at \$2 then the cost of buying the item is $2 + (0.06)(2) = \$2.12$, or $C(2) = \$2.12$, and so on. Thus we have a relationship between the quantities C and p such that each value of p determines exactly one value of C. In this case, we say that C is a function of p. Describes this function using words, a table, a graph, and a formula.

Solution.

•**Words:** To find the total cost, multiply the price of the item by 0.06 and add the result to the price.

•**Table:** The chart below gives the total cost of buying an item at price p as a function of p for $1 \leq p \leq 6$.

p	1	2	3	4	5	6
C	1.06	2.12	3.18	4.24	5.30	6.36

•**Graph:** The graph of the function C is obtained by plotting the data in the above table. See Figure 1.
•**Formula:** The formula that describes the relationship between C and p is given by

$$C(p) = 1.06p. \blacksquare$$

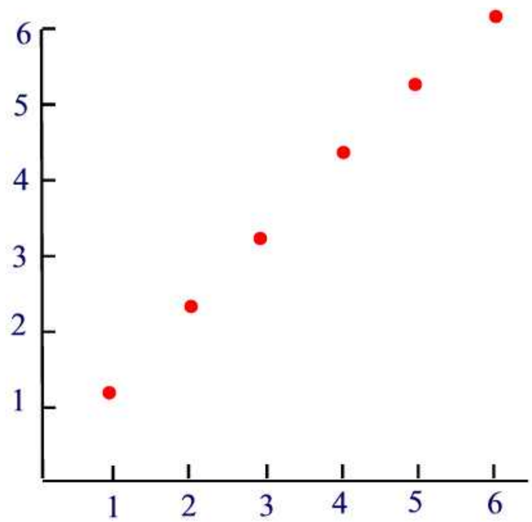


Figure 1

Emphasis of the Four Representations
A formula has the advantage of being both compact and precise. However, not much insight can be gained from a formula as from a table or a graph. A graph provides an overall view of a function and thus makes it easy to deduce important properties. Tables often clearly show trends that are not easily discerned from formulas, and in many cases tables of values are much easier to obtain than a formula.
Now, most of the functions that we will encounter in this course have formulas. For example, the area A of a circle is a function of its radius r. In function notation, we write $A(r) = \pi r^2$. However, there are functions that can not be represented by a formula. For example, the value of Dow Jones Industrial Average at the close of each business day. In this case the value depends on the date, but there is no known formula. Functions of this nature, are mostly represented by either a graph or a table of numerical data.

Example 1.2
The table below shows the daily low temperature for a one-week period in New York City during July.
(a) What was the low temperature on July 19?
(b) When was the low temperature $73°F$?
(c) Is the daily low temperature a function of the date?Explain.
(d) Can you express T as a formula?

D	17	18	19	20	21	22	23
T	73	77	69	73	75	75	70

Solution.
(a) The low temperature on July 19 was $69°F$.
(b) On July 17 and July 20 the low temperature was $73°F$.
(c) T is a function of D since each value of D determines exactly one value of T.
(d) T can not be expressed by an exact formula.■

So far, we have introduced rules between two quantities that define functions. Unfortunately, it is possible for two quantities to be related and yet for neither quantity to be a function of the other.

Example 1.3
Let x and y be two quantities related by the equation

$$x^2 + y^2 = 4.$$

(a) Is x a function of y? Explain.
(b) Is y a function of x? Explain.

Solution.
(a) For $y = 0$ we have two values of x, namely, $x = -2$ and $x = 2$. So x is not a function of y.
(b) For $x = 0$ we have two values of y, namely, $y = -2$ and $y = 2$. So y is not a function of x. ■

Next, suppose that the graph of a relationship between two quantities x and y is given. To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x-value and this violates the definition of a function.

Example 1.4
Which of the graphs (a), (b), (c) in Figure 2 represent y as a function of x?

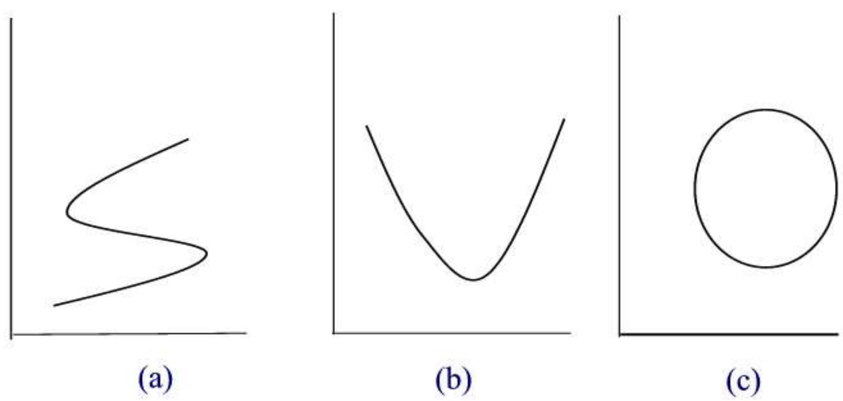


Figure 2

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

Domain and Range of a Function
If we try to find the possible input values that can be used in the function $y = \sqrt{x-2}$ we see that we must restrict x to the interval $[2, \infty)$, that is $x \geq 2$. Similarly, the function $y = \frac{1}{x^2}$ takes only certain values for the output, namely, $y > 0$. Thus, a function is often defined for certain values of x and the dependent variable often takes certain values.
The above discussion leads to the following definitions: By the **domain** of a function we mean all possible input values that yield an input value. Graphically, the domain is part of the horizontal axis. The **range** of a function is the collection of all possible output values. The range is part of the vertical axis.
The domain and range of a function can be found either algebraically or graphically.

• **Finding the Domain and the Range Algebraically**
When finding the domain of a function, ask yourself what values can't be used. Your domain is everything else. There are simple basic rules to consider:

• The domain of all polynomial functions, i.e. functions of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is nonnegative integer, is the Real numbers $\mathbb{R}$.
• Square root functions can not contain a negative underneath the radical. Set the expression under the radical greater than or equal to zero and solve for the variable. This will be your domain.
• Fractional functions, i.e. ratios of two functions, determine for which input values the numerator and denominator are not defined and the domain is everything else. For example, make sure not to divide by zero!

Example 1.5
Find, algebraically, the domain and the range of each of the following functions. Write your answers in interval notation:

(a) $y = \pi x^2$ (b) $y = \frac{1}{\sqrt{x-4}}$ (c) $y = 2 + \frac{1}{x}$.

Solution.
(a) Since the function is a polynomial then its domain is the interval $(-\infty, \infty)$. To find the range, solve the given equation for x in terms of y obtaining

$x = \pm\sqrt{\frac{y}{\pi}}$. Thus, x exists for $y \geq 0$. So the range is the interval $[0, \infty)$.

(b) The domain of $y = \frac{1}{\sqrt{x-4}}$ consists of all numbers x such that $x - 4 > 0$ or $x > 4$. That is, the interval $(4, \infty)$. To find the range, we solve for x in terms of $y > 0$ obtaining $x = 4 + \frac{1}{y^2}$. x exists for all $y > 0$. Thus, the range is the interval $(0, \infty)$.

(c) The domain of $y = 2 + \frac{1}{x}$ is the interval $(-\infty, 0) \cup (0, \infty)$. To find the range, write x in terms of y to obtain $x = \frac{1}{y-2}$. The values of y for which this later formula is defined is the range of the given function, that is, $(-\infty, 2) \cup (2, \infty)$.■

Remark 1.1
Note that the domain of the function $y = \pi x^2$ of the previous problem consists of all real numbers. If this function is used to model a real-world situation, that is, if the x stands for the radius of a circle and y is the corresponding area then the domain of y in this case consists of all numbers $x \geq 0$. In general, for a word problem the domain is the set of all x values such that the problem makes sense.

Finding the Domain and the Range Graphically
We often use a graphing calculator to find the domain and range of functions. In general, the domain will be the set of all x values that has corresponding points on the graph. We note that if there is an asymptote (shown as a vertical line on the TI series) we do not include that x value in the domain. To find the range, we seek the top and bottom of the graph. The range will be all points from the top to the bottom (minus the breaks in the graph).

Example 1.6
Use a graphing calculator to find the domain and the range of each of the following functions. Write your answers in interval notation:

(a) $y = \pi x^2$ (b) $y = \frac{1}{\sqrt{x-4}}$ (c) $y = 2 + \frac{1}{x}$.

Solution.
(a) The graph of $y = \pi x^2$ is given in Figure 3.

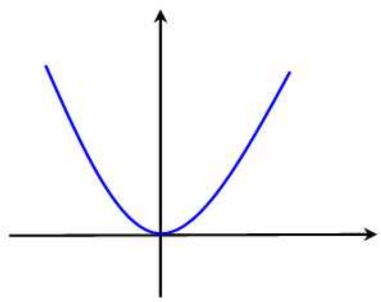


Figure 3

The domain is the set $(-\infty, \infty)$ and the range is $[0, \infty)$.
(b) The graph of $y = \frac{1}{\sqrt{x-4}}$ is given in Figure 4.

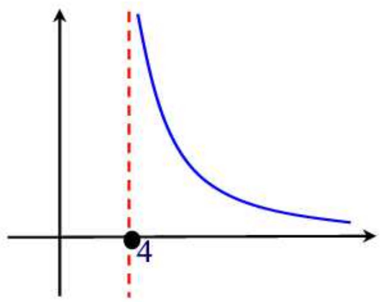


Figure 4

The domain is the set $(4, \infty)$ and the range is $(0, \infty)$.
(c) The graph of $y = 2 + \frac{1}{x}$ is given in Figure 5.

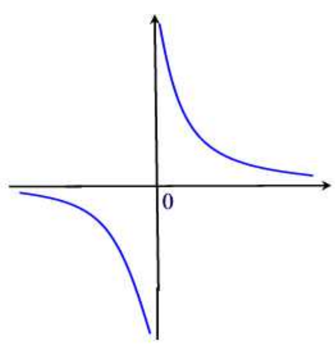


Figure 5

The domain is the set $(-\infty, 0) \cup (0, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$.■

Average Rate of Change

Functions given by tables of values have their limitations in that nearly always leave gaps. One way to fill these gaps is by using the **average rate of change**. For example, Table 1 below gives the population of the United States between the years 1950 - 1990.

d(year)	1950	1960	1970	1980	1990
N(in millions)	151.87	179.98	203.98	227.23	249.40

Table 1

This table does not give the population in 1972. One way to estimate $N(1972)$, is to find the average yearly rate of change of N from 1970 to 1980 given by

$$\frac{227.23 - 203.98}{10} = 2.325 \textit{ million people per year.}$$

Then,

$$N(1972) = N(1970) + 2(2.325) = 208.63 \textit{ million.}$$

Average rates of change can be calculated not only for functions given by tables but also for functions given by formulas. The **average rate of change** of a function $y = f(x)$ from $x = a$ to $x = b$ is given by the **difference quotient**

$$\frac{\Delta y}{\Delta x} = \frac{\text{Change in function value}}{\text{Change in x value}} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, this quantity represents the slope of the secant line going through the points $(a, f(a))$ and $(b, f(b))$ on the graph of $f(x)$. See Figure 6. The average rate of change of a function on an interval tells us how much the function changes, on average, per unit change of x within that interval. On some part of the interval, f may be changing rapidly, while on other parts f may be changing slowly. The average rate of change evens out these variations.

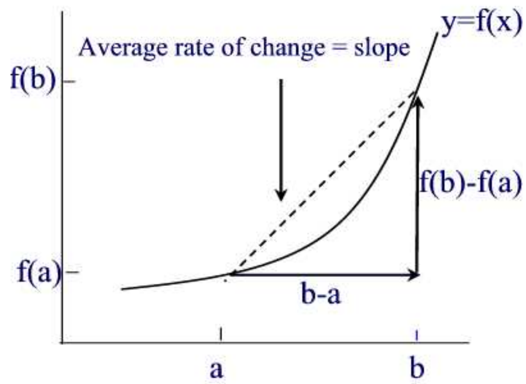


Figure 6

Example 1.7
Find the average value of the function $f(x) = x^2$ from $x = 3$ to $x = 5$.

Solution.
The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(3)}{5 - 3} = \frac{25 - 9}{2} = 8. \blacksquare$$

Average Rate of Change and Increasing/Decreasing Functions
Now, we would like to use the concept of the average rate of change to test whether a function is increasing or decreasing on a specific interval. First, we introduce the following definition: We say that a function is **increasing** if its graph climbs as x moves from left to right. That is, the function values increase as x increases. It is said to be **decreasing** if its graph falls as x moves from left to right. This means that the function values decrease as x increases.

As an application of the average rate of change, we can use such quantity to decide whether a function is increasing or decreasing. If a function f is increasing on an interval I then by taking any two points in the interval I, say $a < b$, we see that $f(a) < f(b)$ and in this case

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Going backward with this argument we see that if the average rate of change is positive in an interval then the function is increasing in that interval. Similarly, if the average rate of change is negative in an interval I then the function is decreasing there.

Some functions can be increasing on some intervals and decreasing on other intervals. These intervals can often be identified from the graph.

Example 1.8
Determine the intervals where the function is increasing and decreasing. See Figure 7.

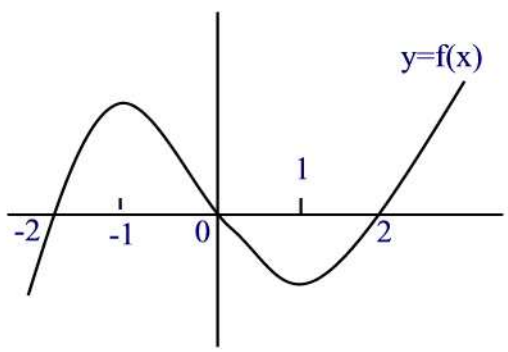


Figure 7

Solution.
The function is increasing on $(-\infty,-1)\cup(1,\infty)$ and decreasing on the interval $(-1,1)$.■

Linear Functions
In general, the average rate of change of a function is different on different intervals. For example, consider the function $f(x)=x^2$. The average rate of change of $f(x)$ on the interval $[0,1]$ is

$$\frac{f(1)-f(0)}{1-0}=1.$$

The average rate of change of $f(x)$ on $[1,2]$ is

$$\frac{f(2)-f(1)}{2-1}=3.$$

A **linear** function is a function with the property that the average rate of change on any interval is the same. We say that y is changing at a constant rate with respect to x. Thus, y changes by the some amount for every unit change in x. Geometrically, the graph is a straight line (and thus the term linear).

Now, suppose that $f(x)$ is a linear function of x. Then f changes at a constant rate m. That is, if we pick two points $(0, f(0))$ and $(x, f(x))$ then

$$m = \frac{f(x) - f(0)}{x - 0}.$$

That is, $f(x) = mx + f(0)$. This is the function notation of the linear function $f(x)$. Another notation is the equation notation, $y = mx + f(0)$. We will denote the number $f(0)$ by b. In this case, the linear function will be written as $f(x) = mx+b$ or $y = mx+b$. Since $b = f(0)$ then the point $(0, b)$ is the point where the line crosses the vertical line. We call it the **y-intercept**. So the y-intercept is the output corresponding to the input $x = 0$, sometimes known as the **initial value** of y.
If we pick any two points (x_1, y_1) and (x_2, y_2) on the graph of $f(x) = mx + b$ then we must have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We call m the **slope** of the line.

Example 1.9
The value of a new computer equipment is \$20,000 and the value drops at a constant rate so that it is worth \$ 0 after five years. Let $V(t)$ be the value of the computer equipment t years after the equipment is purchased.

(a) Find the slope m and the y-intercept b.
(b) Find a formula for $V(t)$.

Solution.
(a) Since $V(0) = 20,000$ and $V(5) = 0$ then the slope of $V(t)$ is

$$m = \frac{0 - 20,000}{5 - 0} = -4,000$$

and the vertical intercept is $V(0) = 20,000$.
(b) A formula of $V(t)$ is $V(t) = -4,000t + 20,000$. In financial terms, the

function $V(t)$ is known as the **straight-line depreciation** function.■

The Significance of the Parameters m and b

Next, we consider the significance of the parameters m and b in the **slope-intercept form** of a linear funtion $y = mx + b$.
We have seen that the graph of a linear function $f(x) = mx + b$ is a straight line. But a line can be horizontal, vertical, rising to the right or falling to the right. The slope is the parameter that provides information about the steepness of a straight line.

• If $m = 0$ then $f(x) = b$ is a constant function whose graph is a horizontal line at $(0, b)$.
• For a vertical line, the slope is undefined since any two points on the line have the same x-value and this leads to a division by zero in the formula for the slope. The equation of a vertical line has the form $x = a$.
• Suppose that the line is neither horizontal nor vertical. If $m > 0$ then by Section 3, $f(x)$ is increasing. That is, the line is rising to the right.
• If $m < 0$ then $f(x)$ is decreasing. That is, the line is falling to the right.
• The slope, m, tells us how fast the line is climbing or falling. The larger the value of m the more the line rises and the smaller the value of m the more the line falls.
The parameter b tells us where the line crosses the vertical axis.

Example 1.10
Arrange the slopes of the lines in Figure 8 from largest to smallest.

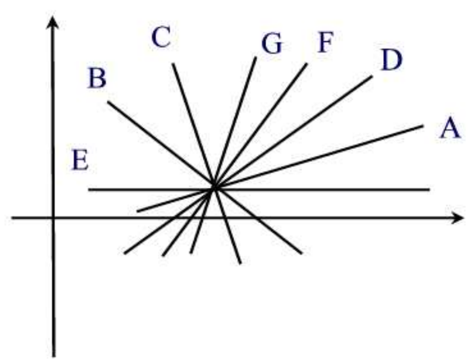


Figure 8

Solution.
According to Figure 8 we have

$$m_G > m_F > m_D > m_A > m_E > m_B > m_C.\blacksquare$$

Proportionality
Consider the linear function $y = mx + b$. If $b = 0$ then we say that the quantity y is **directly proportional** to x with **constant of proportionality** m. If $y = \frac{k}{x}$ then in this case we say that y is **inversely proportional** to x.

Recommended Problems (pp. 7 - 9): 3, 5, 8, 10, 12, 14, 17, 19, 20, 21, 23, 25, 27, 30, 32, 33, 35, 37, 38.

2 Exponential Functions

Exponential functions appear in many applications such as population growth, radioactive decay, and interest on bank loans.
Recall that linear functions are functions that change at a constant rate. For example, if $f(x) = mx + b$ then $f(x+1) = m(x+1) + b = f(x) + m$. So when x increases by 1, the y value increases by m. In contrast, an exponential function with base a is one that changes by constant multiples of a. That is, $f(x+1) = af(x)$. Writing $a = 1 + r$ we obtain $f(x+1) = f(x) + rf(x)$. Thus, an exponential function is a function that changes at a constant percent rate.
Exponential functions are used to model increasing quantities such as **population growth** problems.

Example 2.1
Suppose that you are observing the behavior of cell duplication in a lab. In one experiment, you started with one cell and the cells doubled every minute. That is, the population cell is increasing at the constant rate of 100%. Write an equation to determine the number (population) of cells after one hour.

Solution.
Table 2 below shows the number of cells for the first 5 minutes. Let $P(t)$ be the number of cells after t minutes.

t	0	1	2	3	4	5
P(t)	1	2	4	8	16	32

Table 2

At time 0, i.e t=0, the number of cells is 1 or $2^0 = 1$. After 1 minute, when $t = 1$, there are two cells or $2^1 = 2$. After 2 minutes, when $t = 2$, there are 4 cells or $2^2 = 4$.
Therefore, one formula to estimate the number of cells (size of population) after t minutes is the equation (model)

$$f(t) = 2^t.$$

It follows that $f(t)$ is an increasing function. Computing the rates of change to obtain

$$\begin{array}{rcl} \frac{f(1)-f(0)}{1-0} & = & 1 \\ \frac{f(2)-f(1)}{2-1} & = & 2 \\ \frac{f(3)-f(2)}{3-2} & = & 4 \\ \frac{f(4)-f(3)}{4-3} & = & 8 \\ \frac{f(5)-f(4)}{5-4} & = & 16. \end{array}$$

Thus, the rate of change is increasing. Geometrically, this means that the graph of $f(t)$ is concave up. See Figure 9.

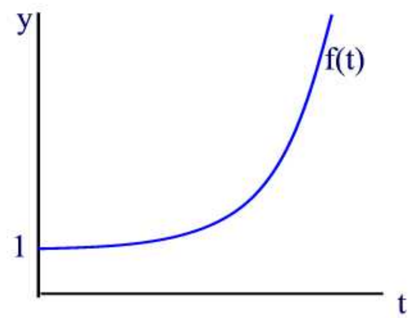


Figure 9

Now, to determine the number of cells after one hour we convert to minutes to obtain $t = 60$ minutes so that $f(60) = 2^{60} = 1.15 \times 10^{18}$ cells. ■

Exponential functions can also model decreasing quantities known as **decay models.**

Example 2.2
If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 50,000 cells?

Solution.
Let $P(t)$ be the number of cells after t minutes. Then $P(t+1) = P(t) - 45\%P(t)$ or $P(t+1) = 0.55P(t)$. By constructing a table of data we find

t	P(t)
0	5,000,000
1	2,750,000
2	1,512,500
3	831,875
4	457,531.25
5	251,642.19
6	138,403.20
7	76,121.76
8	41,866.97

So it takes 8 minutes for the population to reduce to less than 50,000 cells. A formula of $P(t)$ is $P(t) = 5,000,000(0.55)^t$. The graph of $P(t)$ is given in Figure 10.■

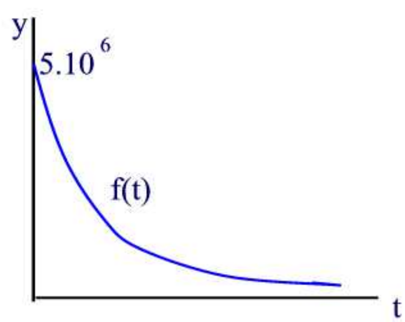


Figure 10

From the previous two examples, we see that an exponential function has the general form

$$P(t) = b \cdot a^t, a > 0\ a \neq 1.$$

Since $b = P(0)$ then we call b the **initial value.** We call a the base of $P(t)$. If $a > 1$, then $P(t)$ shows exponential growth with **growth factor** a. The graph of P will be similar in shape to that in Figure 27.
If $0 < a < 1$, then P shows exponential decay with **decay factor** a. The graph of P will be similar in shape to that in Figure 28.
Since $P(t+1) = aP(t)$ then $P(t+1) = P(t) + rP(t)$ where $r = a - 1$. We call r the **percent growth rate**.

Remark 2.1
Why a is restricted to $a > 0$ and $a \neq 1$? Since t is allowed to have any value then a negative a will create meaningless expressions such as $\sqrt{a}$ (if $t = \frac{1}{2}$). Also, for $a = 1$ the function $P(t) = b$ is called a **constant function** and its graph is a horizontal line.

Doubling Time
In some exponential models one is interested in finding the time for an exponential growing quantity to double. We call this time the **doubling time**. To find it, we start with the equation $b \cdot a^t = 2b$ or $a^t = 2$. Solving for t we find $t = \frac{\log 2}{\log a}$.

Example 2.3
Find the doubling time of a population growing according to $P = P_0 e^{0.2t}$.

Solution.
Setting the equation $P_0 e^{0.2t} = 2P_0$ and dividing both sides by P_0 to obtain $e^{0.2t} = 2$. Take ln of both sides to obtain $0.2t = \ln 2$. Thus, $t = \frac{\ln 2}{0.2} \approx 3.47.\blacksquare$

Half-Life
On the other hand, if a quantity is decaying exponentially then the time required for the initial quantity to reduce into half is called the **half-life**. To find it, we start with the equation $ba^t = \frac{b}{2}$ and we divide both sides by b to obtain $a^t = 0.5$. Take the log of both sides to obtain $t \log a = \log (0.5)$. Solving for t we find $t = \frac{\log (0.5)}{\log a}$.

Example 2.4
The half-life of Iodine-123 is about 13 hours. You begin with 50 grams of this substance. What is a formula for the amount of Iodine-123 remaining after t hours?

Solution.
Since the problem involves exponential decay then if $Q(t)$ is the quantity remaining after t hours then $Q(t) = 50a^t$ with $0 < a < 1$. But $Q(13) = 25$. That is, $50a^{13} = 25$ or $a^{13} = 0.5$. Thus $a = (0.5)^{\frac{1}{13}} \approx 0.95$ and $Q(t) = 50(0.95)^t.\blacksquare$

The Effect of the Parameters a and b

Recall that an exponential function with base a and initial value b is a function of the form $f(x) = b \cdot a^x$. In this section, we assume that $b > 0$. Since $b = f(0)$ then $(0, b)$ is the vertical intercept of $f(x)$. In this section we consider graphs of exponential functions.

Let's see the effect of the parameter b on the graph of $f(x) = ba^x$.

Example 2.5

Graph, on the same axes, the exponential functions $f_1(x) = 2 \cdot (1.1)^x$, $f_2(x) = (1.1)^x$, and $f_3(x) = 0.75(1.1)^x$.

Solution.

The three functions as shown in Figure 11.

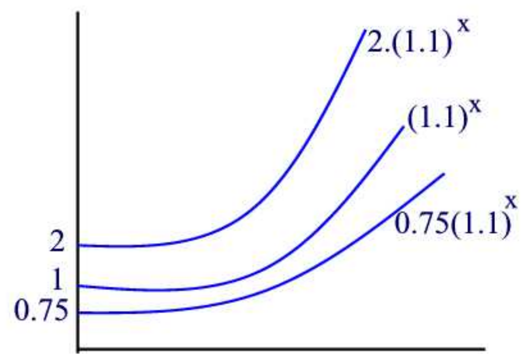


Figure 11

Note that these functions have the same growth factor but different b and therefore different vertical intercepts. ■

We know that the slope of a linear function measures the steepness of the graph. Similarly, the parameter a measures the steepness of the graph of an exponential function. First, we consider the effect of the growth factor on the graph.

Example 2.6

Graph, on the same axes, the exponential functions $f_1(x) = 4^x$, $f_2(x) = 3^x$, and $f_3(x) = 2^x$.

Solution.
Using a graphing calculator we find

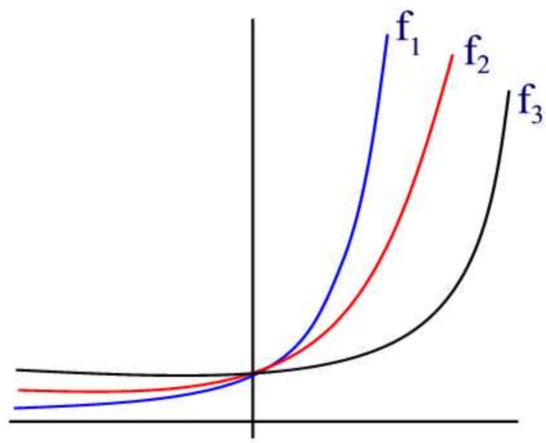


Figure 12

It follows that the greater the value of a, the more rapidly the graph rises. That is, the growth factor a affects the steepness of an exponential function. Also note that as x decreases, the function values approach the x-axis. Symbolically, as $x \to -\infty, y \to 0.$■

Next, we study the effect of the decay factor on the graph.

Example 2.7
Graph, on the same axes, the exponential functions $f_1(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$, $f_2(x) = 3^{-x}$, and $f_3(x) = 4^{-x}$.

Solution.
Using a graphing calculator we find

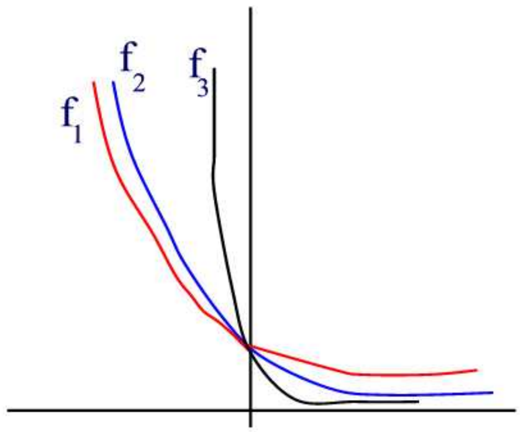


Figure 13

It follows that the smaller the value of a, the more rapidly the graph falls. Also as x increases, the function values approach the x-axis. Symbolically, as $x \to \infty, y \to 0$.

• **General Observations**
(i) For $a > 1$, as x decreases, the function values get closer and closer to 0. Symbolically, as $x \to -\infty, y \to 0$. For $0 < a < 1$, as x increases, the function values gets closer and closer to the x-axis. That is, as $x \to \infty, y \to 0$. We call the x-axis, a **horizontal asymptote**.
(ii) The domain of an exponential function consists of the set of all real numbers whereas the range consists of the set of all positive real numbers.
(iii) The graph of $f(x) = ba^x$ with $b > 0$ is always concave up.

Exponential Functions with Base e
When writing $y = be^t$ then we say that y is an exponential function with base e. Look at your calculator and locate the key **ln**. (This is called the natural logarithm function which will be discussed in the next section) Pick any positive number of your choice, say c, and compute $e^{\ln c}$. What do you notice? For any positive number c, you notice that $e^{\ln c} = c$. Thus, any positive number a can be written in the form $a = e^k$ where $k = \ln a$.

Now, suppose that $Q(t) = ba^t$. Then by the above paragraph we can write $a = e^k$. Thus,

$$Q(t) = b(e^k)^t = be^{kt}.$$

Note that if $k > 0$ then $e^k > 1$ so that $Q(t)$ represents an exponential growth and if $k < 0$ then $e^k < 1$ so that $Q(t)$ is an exponential decay.
We call the constant k the **continuous growth rate**.

Example 2.8
If $f(t) = 3(1.072)^t$ is rewritten as $f(t) = 3e^{kt}$, find k.

Solution.
By comparison of the two functions we find $e^k = 1.072$. Solving this equation graphically (e.g. using a calculator) we find $k \approx 0.695$.$\blacksquare$

Example 2.9
A population increases from its initial level of 7.3 million at the continuous rate of 2.2% per year. Find a formula for the population $P(t)$ as a function of the year t. When does the population reach 10 million?

Solution.
We are given the initial value 7.3 million and the continuous growth rate $k = 0.022$. Therefore, $P(t) = 7.3e^{0.022t}$. Next,we want to find the time when $P(t) = 10$. That is , $7.3e^{0.022t} = 10$. Divide both sides by 7.3 to obtain $e^{0.022t} \approx 1.37$. Solving this equation graphically to obtain $t \approx 14.3$.$\blacksquare$

Next, in order to convert from $Q(t) = be^{kt}$ to $Q(t) = ba^t$ we let $a = e^k$. For example, to convert the formula $Q(t) = 7e^{0.3t}$ to the form $Q(t) = ba^t$ we let $b = 7$ and $a = e^{0.3} \approx 1.35$. Thus, $Q(t) = 7(1.35)^t$.

Example 2.10
Find the annual percent rate and the continuous percent growth rate of $Q(t) = 200(0.886)^t$.

Solution.
The annual percent of decrease is $r = a-1 = 0.886-1 = -0.114 = -11.4\%$. To find the continuous percent growth rate we let $e^k = 0.886$ and solve for k graphically to obtain $k \approx -0.121 = -12.1\%$.$\blacksquare$

Recommended Problems:
pp. 14 - 17: 1, 2, 3, 7, 11, 14, 15, 19, 21, 23, 25, 27, 30, 33, 38.

3 Building New Functions from Old Ones

In this section we will discuss some procedures for building new functions from old ones. The first one is known as the composition of functions.
We start with an example of a real-life situation where composite functions are applied.

Example 3.1
You have two money machines, both of which increase any money inserted into them. The first machine doubles your money. The second adds five dollars. The money that comes out is described by $f(x) = 2x$ in the first case, and $g(x) = x + 5$ in the second case, where x is the number of dollars inserted. The machines can be hooked up so that the money coming out of one machine goes into the other. Find formulas for each of the two possible composition machines.

Solution.
Suppose first that x dollars enters the first machine. Then the amount of money that comes out is $f(x) = 2x$. This amount enters the second machine. The final amount coming out is given by $g(f(x)) = f(x) + 5 = 2x + 5$.
Now, if x dollars enters the second machine first, then the amount that comes out is $g(x) = x + 5$. If this amount enters the second machine then the final amount coming out is $f(g(x)) = 2(x + 5) = 2x + 10$. The function $f(g(x))$ is called the composition of f with g.■

In general, suppose we are given two functions f and g such that the range of g is contained in the domain of f so that the output of g can be used as input for f. We define a new function, called the **composition** of f with g, by the formula

$$(f \circ g)(x) = f(g(x)).$$

Using a Venn diagram (See Figure 14) we have

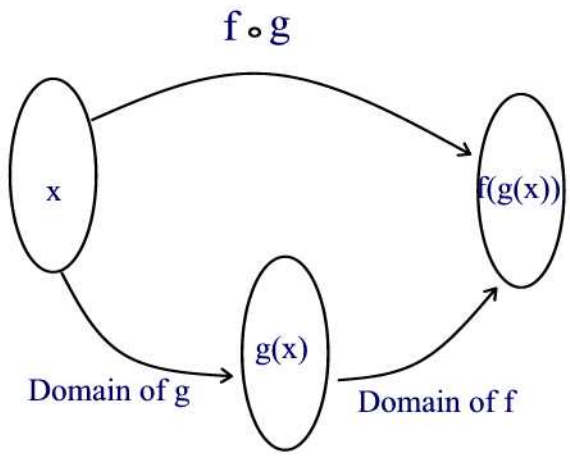


Figure 14

Example 3.2
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $f \circ g$ and $g \circ f$.
(b) Calculate $(f \circ g)(5)$ and $(g \circ f)(-3)$.
(c) Are $f \circ g$ and $g \circ f$ equal?

Solution.
(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5$. Similarly, $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2$.
(b) $(f \circ g)(5) = 2(5)^2 - 5 = 45$ and $(g \circ f)(-3) = 4(-3)^2 + 4(-3) - 2 = 22$.
(c) $f \circ g \neq g \circ f$.■

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

Example 3.3
Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $(f \circ f)(x)$.
(b) Find $[f \circ (f \circ g)](x)$.

Solution.
(a) $(f \circ f)(x) = f(f(x)) = f(2x+1) = 2(2x+1)+1 = 4x+3.$
(b) $[f \circ (f \circ g)](x) = f(f(g(x))) = f(f(x^2-3)) = f(2x^2-5) = 2(2x^2-5)+1 = 4x^2-9.$ ■

Decomposition of Functions
If a formula for $(f \circ g)(x)$ is given then the process of finding the formulas for f and g is called **decomposition.**

Example 3.4
Decompose $(f \circ g)(x) = \sqrt{1-4x^2}$.

Solution.
One possible answer is $f(x) = \sqrt{x}$ and $g(x) = 1-4x^2$. Another possible answer is $f(x) = \sqrt{1-x^2}$ and $g(x) = 2x$. Thus, decomposition of functions in not unique. ■

Inverse Functions
We have seen that when every vertical line crosses a curve at most once then the curve is the graph of a function f. We called this procedure the **vertical line test.** Now, if every horizontal line crosses the graph at most once then the function can be used to build a new function, called the **inverse function** and is denoted by f^{-1}, such that if f takes an input x to an output y then f^{-1} takes y as its input and x as its output. That is

$$f(x) = y \; if \; and \; only \; if \; f^{-1}(y) = x.$$

When a function has an inverse then we say that the function is **invertible.**

Remark 3.1
The test used to identify invertible functions which we discussed above is referred to as the **horizontal line test**.

Example 3.5
Use a graphing calculator to decide whether or not the function is invertible, that is, has an inverse function:
(a) $f(x) = x^3+7$ (b) $g(x) = |x|$.

Solution.
(a) Using a graphing calculator, the graph of $f(x)$ is given in Figure 15.

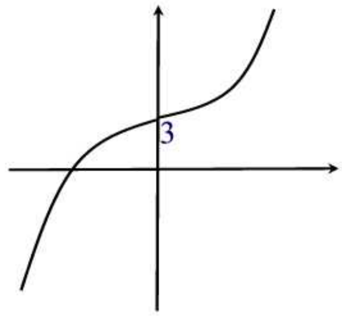


Figure 15

We see that every horizontal line crosses the graph once so the function is invertible.
(b) The graph of $g(x) = |x|$ (See Figure 16) shows that there are horizontal lines that cross the graph twice so that g is not invertible.$\blacksquare$

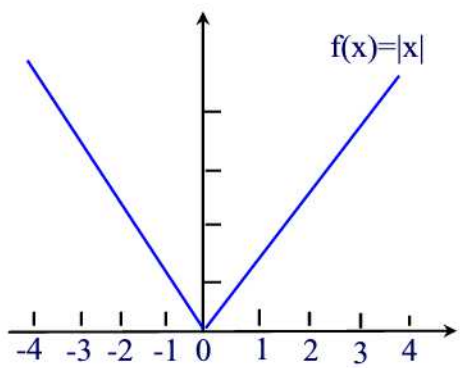


Figure 16

Remark 3.2
It is important not to confuse between $f^{-1}(x)$ and $(f(x))^{-1}$. The later is just the reciprocal of $f(x)$, that is, $(f(x))^{-1} = \frac{1}{f(x)}$ whereas the former is how the inverse function is represented.

Domain and Range of an Inverse Function
Figure 17 shows the relationship between f and f^{-1}.

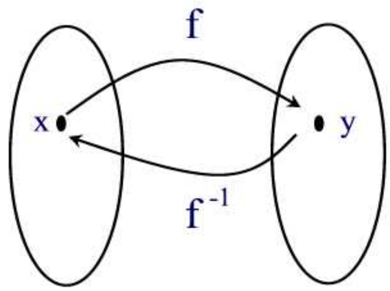


Figure 17

This figure shows that we get the inverse of a function by simply reversing the direction of the arrows. That is, the outputs of f are the inputs of f^{-1} and the outputs of f^{-1} are the inputs of f. It follows that

$$\text{Domain of } f^{-1} = \text{Range of f} \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of f}.$$

Example 3.6
Consider the function $f(x) = \sqrt{x-4}$.

(a) Find the domain and the range of $f(x)$.
(b) Use the horizontal line test to show that $f(x)$ has an inverse.
(c) What are the domain and range of f^{-1}?

Solution.
(a) The function $f(x)$ is defined for all $x \geq 4$. The range is the interval $[0, \infty)$.
(b) Graphing $f(x)$ we see that $f(x)$ satisfies the horizontal line test and so f has an inverse. See Figure 18.
(c) The domain of f^{-1} is the range of f, i.e. the interval $[0, \infty)$. The range of f^{-1} is the domain of f, that is, the interval $[4, \infty)$.■

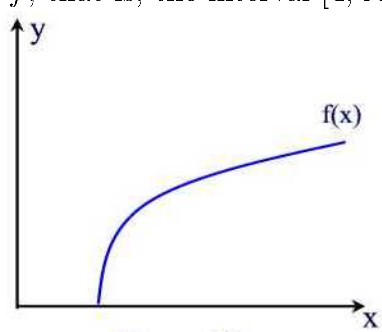


Figure 18

Finding a Formula for the Inverse Function
How do you find the formula for f^{-1} from the formula of f? The procedure consists of the following steps:

1. Replace $f(x)$ with y.
2. Interchange the letters x and y.
3. Solve for y in terms of x.
4. Replace y with $f^{-1}(x)$.

Example 3.7
Find the formula for the inverse function of $f(x) = x^3 + 7$.

Solution.
As seen in Example 3.5, $f(x)$ is invertible. We find its inverse as follows:

1. Replace $f(x)$ with y to obtain $y = x^3 + 7$.
2. Interchange x and y to obtain $x = y^3 + 7$.
3. Solve for y to obtain $y^3 = x - 7$ or $y = \sqrt[3]{x-7}$.
4. Replace y with $f^{-1}(x)$ to obtain $f^{-1}(x) = \sqrt[3]{x-7}$.■

Compositions of f and its Inverse
Suppose that f is an invertible function. Then the expressions $y = f(x)$ and $x = f^{-1}(y)$ are equivalent. So if x is in the domain of f then

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

and for y in the domain of f^{-1} we have

$$f(f^{-1}(y)) = f(x) = y$$

It follows that for two functions f and g to be inverses of each other we must have $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for x in the domain of f.

Example 3.8
Check that the pair of functions $f(x) = \frac{x}{4} - \frac{3}{2}$ and $g(x) = 4(x + \frac{3}{2})$ are inverses of each other.

Solution.
The domain and range of both functions consist of the set of all real numbers. Thus, for any real number x we have

$$f(g(x)) = f(4(x + \frac{3}{2})) = f(4x + 6) = \frac{4x + 6}{4} - \frac{3}{2} = x.$$

and

$$g(f(x)) = g(\frac{x}{4} - \frac{3}{2}) = 4(\frac{x}{4} - \frac{3}{2} + \frac{3}{2}) = x.$$

So f and g are inverses of each other.■

Combinations of Functions
Next, we are going to construct new functions from old ones using the operations of addition, subtraction, multiplication, and division.
Let $f(x)$ and $g(x)$ be two given functions. Then for all x in the common domain of these two functions we define new functions as follows:

• **Sum**: $(f + g)(x) = f(x) + g(x)$.
• **Difference**: $(f - g)(x) = f(x) - g(x)$.
• **Product**: $(f \cdot g)(x) = f(x) \cdot g(x)$.
• **Division**: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$.

In the following example we see how to construct the four functions discussed above when the individual functions are defined by formulas.

Example 3.9
Let $f(x) = x + 1$ and $g(x) = \sqrt{x + 3}$. Find the common domain and then find a formula for each of the functions $f + g, f - g, f \cdot g, \frac{f}{g}$.

Solution.
The domain of $f(x)$ consists of all real numbers whereas the domain of $g(x)$ consists of all numbers $x \geq 3$. Thus, the common domain is the interval $[-3, \infty)$. For any x in this domain we have

$$\begin{array}{rcl}
(f + g)(x) & = & x + 1 + \sqrt{x + 3} \\
(f - g)(x) & = & x + 1 - \sqrt{x + 3} \\
(f \cdot g)(x) & = & x\sqrt{x + 3} + \sqrt{x + 3} \\
\left(\frac{f}{g}\right)(x) & = & \frac{x+1}{\sqrt{x+3}} \textit{ provided } x > -3. \blacksquare
\end{array}$$

Transformations of Functions
We close this section by giving a summary of the various transformations obtained when either the input or the output of a function is altered.

Vertical Shifts: The graph of $f(x) + k$ with $k > 0$ is a vertical translation of the graph of $f(x)$, k units upward, whereas for $k < 0$ it is a shift by k units downward.
Horizontal Shifts: The graph of $f(x+k)$ with $k > 0$ is a horizontal translation of the graph of $f(x)$, k units to the left, whereas for $k < 0$ it is a shift by k units to the right.
Reflections about the x-axis: For a given function $f(x)$, the graph of $-f(x)$ is a reflection of the graph of $f(x)$ about the x-axis.
Reflections about the y-axis: For a given function $f(x)$, the graph of $f(-x)$ is a reflection of the graph of $f(x)$ about the y-axis.
Vertical Stretches and Compressions: If a function $f(x)$ is given, then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of k for $k > 1$, and a vertical compression for $0 < k < 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $kf(x)$ is a vertical stretch of the graph of $f(x)$ followed by a reflection about the x-axis. If $0 < |k| < 1$ then the graph of $kf(x)$ is a vertical compression of the graph of $f(x)$ by a factor of k followed by a reflection about the x-axis.
Horizontal Stretches and Compressions: If a function $f(x)$ is given, then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ for $0 < k < 1$, and a horizontal compression for $k > 1$.
What about $k < 0$? If $|k| > 1$ then the graph of $f(kx)$ is a horizontal compression of the graph of $f(x)$ followed by a reflection about the y-axis. If $0 < |k| < 1$ then the graph of $f(kx)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\frac{1}{k}$ followed by a reflection about the y-axis.

Odd and Even Functions
For a given function $f(x)$, the graph of the new function $-f(x)$ is the reflection of the graph of $f(x)$ about the x-axis.

Example 3.10
Graph the functions $f(x) = 2^x$ and $-f(x) = -2^x$ on the same axes.

Solution.
The graph of both $f(x) = 2^x$ and $-f(x)$ are shown in Figure 19.■

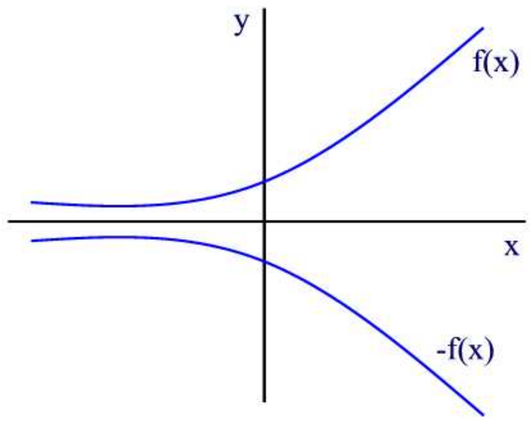


Figure 19

We know that the points x and $-x$ are on opposite sides of the x-axis. So the graph of the new function $f(-x)$ is the reflection of the graph of $f(x)$ about the y-axis.
If the reflection of the graph of $f(x)$ about the y-axis is the same as the graph of $f(x)$,i.e, $f(-x) = f(x)$, then we say that the graph of $f(x)$ is symmetric about the y-axis. We call such a function an **even** function.

Example 3.11
(a) Using a graphing calculator show that the function $f(x) = (x - x^3)^2$ is even.
(b) Now show that $f(x)$ is even algebraically.

Solution.
(a) The graph of $f(x)$ is symmetric about the y-axis so that $f(x)$ is even. See Figure 20.

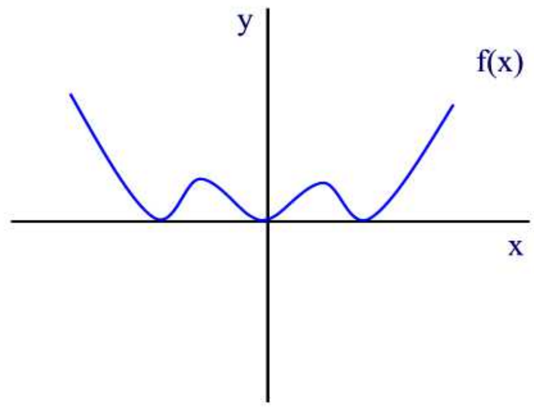


Figure 20

(b) Since $f(-x) = (-x-(-x)^3)^2 = (-x+x^3)^2 = [-(x-x^3)]^2 = (x-x^3)^2 = f(x)$ then $f(x)$ is even.■

Now, if the images $f(x)$ and $f(-x)$ are of opposite signs i.e, $f(-x) = -f(x)$, then the graph of $f(x)$ is symmetric about the origin. In this case, we say that $f(x)$ is **odd.** Alternatively, since $f(x) = -f(-x)$, if the graph of a function is reflected first across the y-axis and then across the x-axis and you get the graph of $f(x)$ again then the function is odd.

Example 3.12
(a) Using a graphing calculator show that the function $f(x) = \frac{1+x^2}{x-x^3}$ is odd.
(b) Now show that $f(x)$ is odd algebraically.

Solution.
(a) The graph of $f(x)$ is symmetric about the origin so that $f(x)$ is odd. See Figure 21.

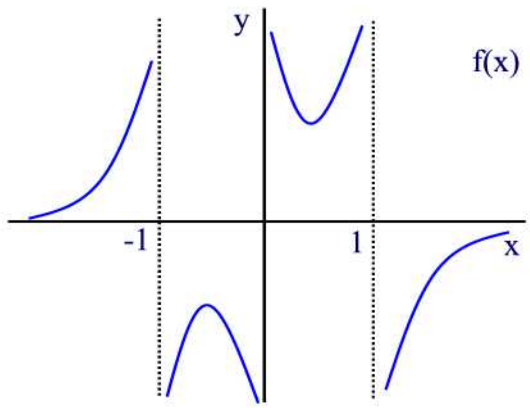


Figure 21

(b) Since $f(-x) = \frac{1+(-x)^2}{(-x)-(-x)^3} = \frac{1+x^2}{-x+x^3} = \frac{1+x^2}{-(x-x^3)} = -f(x)$ then $f(x)$ is odd.$\blacksquare$

A function can be either even, odd, or neither.

Example 3.13
(a) Show that the function $f(x) = x^2$ is even but not odd.
(b) Show that the function $f(x) = x^3$ is odd but not even.
(c) Show that the function $f(x) = x + x^2$ is neither odd nor even.
(d) Is there a function that is both even and odd? Explain.

Solution.
(a) Since $f(-x) = f(x)$ and $f(-x) \neq -f(x)$ then $f(x)$ is even but not odd.
(b) Since $f(-x) = -f(x)$ and $f(-x) \neq f(x)$ then $f(x)$ is odd but not even.
(c) Since $f(-x) = -x + x^2 \neq \pm f(x)$ then $f(x)$ is neither even nor odd.
(d) We are looking for a function such that $f(-x) = f(x)$ and $f(-x) = -f(x)$. This implies that $f(x) = -f(x)$ or $2f(x) = 0$. Dividing by 2 to obtain $f(x) = 0$. This function is both even and odd.$\blacksquare$

Recommended Problems (pp. 21 - 23): 2, 3, 5, 7, 11, 12, 13, 14, 17, 18, 25, 29, 31, 36.

4 Logarithmic Functions

An equation of the form $a^x = b$ can be solved graphically. That is, using a calculator we graph the horizontal line $y = b$ and the exponential function $y = a^x$ and then find the point of intersection.
In this section we discuss an algebraic way to solve equations of the form $a^x = b$ where a and b are positive constants. For this, we introduce two functions that are found in today's calculators, namely, the functions $\log x$ and $\ln x$.

If $x > 0$ then we define $\log x$ to be a number y that satisfies the equality $10^y = x$. For example, $\log 100 = 2$ since $10^2 = 100$. Similarly, $\log 0.01 = -2$ since $10^{-2} = 0.01$. We call $\log x$ the **common logarithm of x**. Thus,

$$y = \log x \ \ if\ and\ only\ if \ \ 10^y = x.$$

Similarly, we have

$$y = \ln x \ \ if\ and\ only\ if \ \ e^y = x.$$

We call $\ln x$ the **natural logarithm** of x.

Example 4.1
(a) Rewrite $\log 30 = 1.477$ using exponents instead of logarithms.
(b) Rewrite $10^{0.8} = 6.3096$ using logarithms instead of exponents.

Solution.
(a) $\log 30 = 1.477$ is equivalent to $10^{1.477} = 30$.
(b) $10^{0.8} = 6.3096$ is equivalent to $\log 6.3096 = 0.8.\blacksquare$

Example 4.2
Without a calculator evaluate the following expressions:

(a) $\log 1$ (b) $\log 10^0$ (c) $\log\left(\frac{1}{\sqrt{10}}\right)$ (d) $10^{\log 100}$ (e) $10^{\log(0.01)}$

Solution.
(a) $\log 1 = 0$ since $10^0 = 1$.
(b) $\log 10^0 = \log 1 = 0$ by (a).
(c) $\log\left(\frac{1}{\sqrt{10}}\right) = \log 10^{-\frac{1}{2}} = -\frac{1}{2}$.

(d) $10^{\log 100} = 10^2 = 100.$
(e) $10^{\log (0.01)} = 10^{-2} = 0.01.$■

Properties of Logarithms

(i) Since $10^x = 10^x$ we can write

$$\log 10^x = x$$

(ii) Since $\log x = \log x$ then

$$10^{\log x} = x$$

(iii) $\log 1 = 0$ since $10^0 = 1$.
(iv) $\log 10 = 1$ since $10^1 = 10$.
(v) Suppose that $m = \log a$ and $n = \log b$. Then $a = 10^m$ and $b = 10^n$. Thus, $a \cdot b = 10^m \cdot 10^n = 10^{m+n}$. Rewriting this using logs instead of exponents, we see that

$$\log (a \cdot b) = m + n = \log a + \log b.$$

(vi) If, in (v), instead of multiplying we divide, that is $\frac{a}{b} = \frac{10^m}{10^n} = 10^{m-n}$ then using logs again we find

$$\log \left(\frac{a}{b}\right) = \log a - \log b.$$

(vii) It follows from (vi) that if $a = b$ then $\log a - \log b = \log 1 = 0$ that is $\log a = \log b$.
(viii) Now, if $n = \log b$ then $b = 10^n$. Taking both sides to the power k we find $b^k = (10^n)^k = 10^{nk}$. Using logs instead of exponents we see that $\log b^k = nk = k \log b$ that is

$$\log b^k = k \log b.$$

Example 4.3
Solve the equation: $4(1.171)^x = 7(1.088)^x$.

Solution.
Rewriting the equation into the form $\left(\frac{1.171}{1.088}\right)^x = \frac{7}{4}$ and then using properties (vii) and (viii) to obtain

$$x \log \left(\frac{1.171}{1.088}\right) = \log \frac{7}{4}.$$

Thus,

$$x = \frac{\log \frac{7}{4}}{\log\left(\frac{1.171}{1.088}\right)}. \blacksquare$$

Example 4.4
Solve the equation $\log(2x+1)+3=0$.

Solution.
Subtract 3 from both sides to obtain $\log(2x+1) = -3$. Switch to exponential form to get $2x+1 = 10^{-3} = 0.001$. Subtract 1 and then divide by 2 to obtain $x = -0.4995$. $\blacksquare$

Remark 4.1
• All of the above arguments are valid for the function $\ln x$ for which we replace the number 10 by the number $e = 2.718\cdots$. That is, $\ln(a\cdot b) = \ln a + \ln b, \ln\frac{a}{b} = \ln a - \ln b$ etc.
• Keep in mind the following:
$\log(a+b) \neq \log a + \log b$. For example, $\log 2 \neq \log 1 + \log 1 = 0$.
$\log(a-b) \neq \log a - \log b$. For example, $\log(2-1) = \log 1 = 0$ whereas $\log 2 - \log 1 = \log 2 \neq 0$.
$\log(ab) \neq \log a \cdot \log b$. For example, $\log 1 = \log(2\cdot\frac{1}{2}) = 0$ whereas $\log 2 \cdot \log\frac{1}{2} = -\log^2 2 \neq 0$.
$\log\left(\frac{a}{b}\right) \neq \frac{\log a}{\log b}$. For example, letting $a = b = 2$ we find that $\log\frac{a}{b} = \log 1 = 0$ whereas $\frac{\log a}{\log b} = 1$.
$\log\left(\frac{1}{a}\right) \neq \frac{1}{\log a}$. For example, $\log\frac{1}{\frac{1}{2}} = \log 2$ whereas $\frac{1}{\log\frac{1}{2}} = -\frac{1}{\log 2}$.

Logarithmic Functions and Their Graphs
Next, we will graph logarithmic functions and determine a number of their general features.
We have seen that the notation $y = \log x$ is equivalent to $10^y = x$. Since 10 raised to any power is always positive then the domain of the function $\log x$ consists of all positive numbers. That is, $\log x$ cannot be used with negative numbers.
Now, let us sketch the graph of this function by first constructing the following chart:

x	$\log x$	Average Rate of Change
0	undefined	-
0.001	-3	-
0.01	-2	111.11
0.1	-1	11.11
1	0	1.11
10	1	0.11
100	2	0.011
1000	3	0.0011

From the chart we see that the graph is always increasing. Since the average rate of change is decreasing then the graph is always concave down. Now plotting these points and connecting them with a smooth curve to obtain

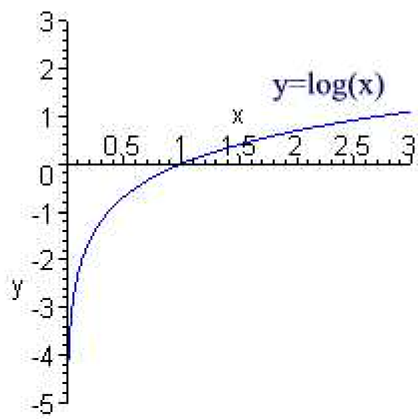


Figure 22

From the graph we observe the following properties:
(a) The range of $\log x$ consists of all real numbers.
(b) The graph never crosses the y-axis since a positive number raised to any power is always positive.
(c) The graph crosses the x-axis at $x = 1$.
(d) As x gets closer and closer to 0 from the right the function $\log x$ decreases without bound. That is, as $x \to 0^+, x \to -\infty$. We call the y-axis a **vertical asymptote**. In general, if a function increases or decreases without bound as x gets closer to a number a then we say that the line $x = a$ is a **vertical asymptote**.

Next, let's graph the function $y = 10^x$ by using the above process:

x	10^x	Average Rate of Change
-3	0.001	-
-2	0.01	0.009
-1	0.1	0.09
0	1	0.9
1	10	9
2	100	90
3	1000	900

Note that this chart can be obtained from the chart of $\log x$ discussed above by interchanging the variables x and y. This means, that the graph of $y = 10^x$ is a reflection of the graph of $y = \log x$ about the line $y = x$ as seen in Figure 23.

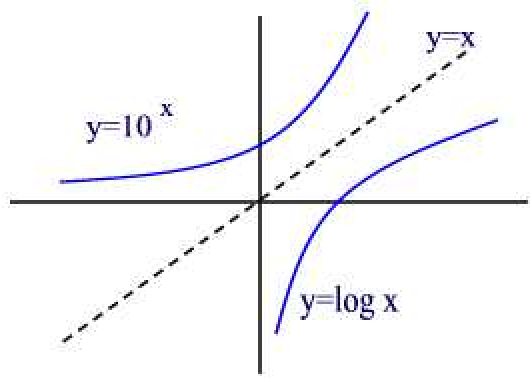


Figure 23

Example 4.5
Sketch the graphs of the functions $y = \ln x$ and $y = e^x$ on the same axes.

Solution.
The functions $y = \ln x$ and $y = e^x$ are inverses of each like the functions $y = \log x$ and $y = 10^x$. So their graphs are reflections of one another across the line $y = x$ as shown in Figure 24.■

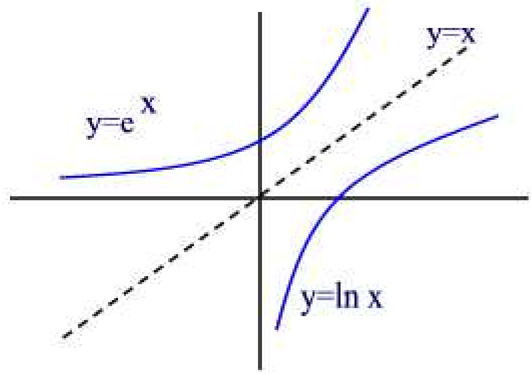


Figure 24

Recommended Problems (pp. 27 - 28): 2, 8, 14, 15, 19, 25, 30, 33, 35, 36, 37, 39, 40, 42, 44.

5 Trigonometric Functions

In this section, we review some of the results of trigonometry that we need in the sequel.

An **angle** is determined by rotating a ray (or a half-line) from one position, called the **initial side**, to a terminal position, called the **terminal side**, as shown in Figure 25. The point V is called the **vertex** of the angle.

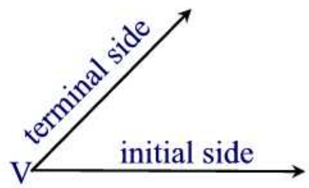


Figure 25

If the initial side is the positive x-axis then we say that the angle is in **standard position**. See Figure 26.

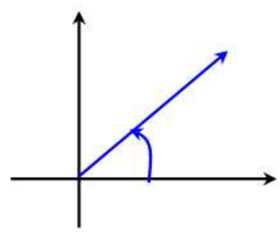


Figure 26

Angles that are obtained by a counterclockwise rotation are considered **positive** and those that are obtained clockwise are **negative** angles. See Figure 27.

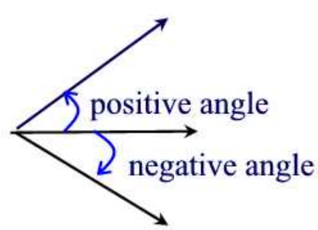


Figure 27

Angle Measure
To be able to compare angles, one needs to assign a unit of measurement for angles. The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side, this is how much the angle "opens". There are two commonly used measures of angles: **degrees** and **radians**

• **Degree Measure:**
If we rotate counterclockwise a ray about a fixed vertex and then return back to its initial position then we say that we have a one complete **revolution.** The angle in this case is said to have measure of 360 degrees, in symbol $360°$. Thus, $1°$ is $\frac{1}{360}$th of a revolution. See Figure 28.

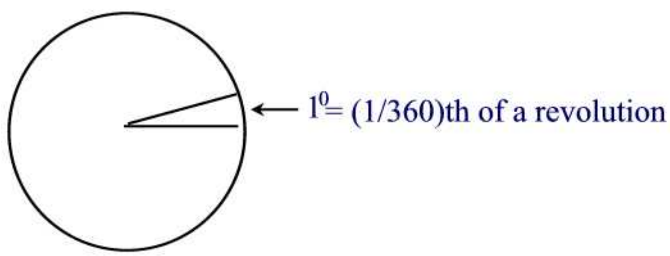


Figure 28

• **Radian Measure:**
A more natural method of measuring angles used in calculus and other branches of mathematics is the **radian** measure. The amount an angle opens is measured along the arc of the unit circle with its center at the vertex of the angle.(An angle whose vertex is the center of a circle is called a **central angle**.) One **radian**, abbreviated **rad**, is defined to be the measure of a central angle that intercepts an arc s of length one unit. See Figure 29.

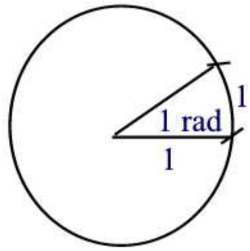


Figure 29

Since one complete revolution measured in radians is 2π radians and measured in degrees is $360°$ then we have the following conversion formulas:

$$1° = \frac{\pi}{180}\ rad \approx 0.01745\ rad \quad and \quad 1\ rad = (\frac{180}{\pi})° \approx 57.296°.$$

Example 5.1
Complete the following chart.

degree	30°	45°	60°	90°	180°	270°
radian						

Solution.

degree	30°	45°	60°	90°	180°	270°
radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

By the conversion formulas, we have, for example $30° = 30(1°) = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$. In a similar way we convert the remaining angles. ■

Example 5.2
Convert each angle in degrees to radians: (a) $150°$ (b) $-45°$.

Solution.
(a) $150° = 150(1°) = 150(\frac{\pi}{180}) = \frac{5\pi}{6} rad.$
(b) $-45° = -45(1°) = -45(\frac{\pi}{180}) = -\frac{\pi}{4} rad.$ ■

Example 5.3
Convert each angle in radians to degrees: (a) $-\frac{3\pi}{4}$ (b) $\frac{7\pi}{3}$.

Solution.
(a) $-\frac{3\pi}{4} = -\frac{3\pi}{4}(1\ rad) = -\frac{3\pi}{4}(\frac{180}{\pi})^\circ = -135^\circ$.
(b) $\frac{7\pi}{3} = \frac{7\pi}{3}(\frac{180}{\pi})^\circ = 420^\circ$ ■

Remark 5.1
When we write a trigonometric function, such as $\sin t$, then it is assumed that t is in radians. If we want to evaluate the trigonometric function of an angle measured in degrees we will use the degree notation such as $\cos 30^\circ$.

We next discuss a relationship between a central angle θ, measured in radians, and the length of the arc s that it intercepts.

Theorem 5.1
For a circle of radius r, a central angle of θ radians subtends an arc whose length s is given by the formula:

$$s = r\theta$$

Proof.
Suppose that $r > 1$. (A similar argument holds for $0 < r < 1$.) Draw the unit circle with the same center C (See Figure 30).

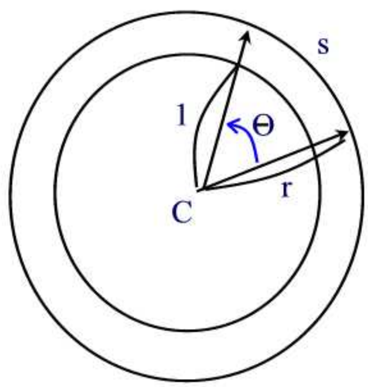


Figure 30

By definition of radian measure, the length of the arc determined by θ on the unit circle is also θ. From elementary geometry, we know that the ratio of the measures of the arc lengths are the same as the ratio of the corresponding radii. That is,

$$\frac{r}{1} = \frac{s}{\theta}.$$

Now the formula follows by cross-multiplying. ■

The above formula allows us to define the radian measure using a circle of any radius r. (See Figure 31).

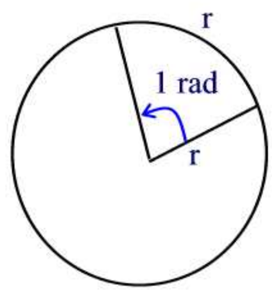


Figure 31

Example 5.4
Find the length of the arc of a circle of radius 2 meters subtended by a central angle of measure 0.25 radians.

Solution.
We are given that $r = 2\ m$ and $\theta = 0.25\ rad$. By the previous theorem we have:

$$s = r\theta = 2(0.25) = 0.5\ m \blacksquare$$

Example 5.5
Suppose that a central angle of measure $30°$ is subtended by an arc of length $\frac{\pi}{2}$ feet. Find the radius r of the circle.

Solution.
Substituting in the formula $s = r\theta$ we find $\frac{\pi}{2} = r\frac{\pi}{6}$. Solving for r to obtain $r = 3 feet$. ■

The Trigonometric Functions
By the **unit circle** we mean the circle with center at the point $O(0,0)$ and radius 1. Such a circle has the equation $x^2 + y^2 = 1$.
Now, let t be any real number. From the point $A(1,0)$, walk on the unit circle a distance t arriving at some point $P(a,b)$. Then the arc $\widehat{AP}$ subtends

a central angle θ. See Figure 32.

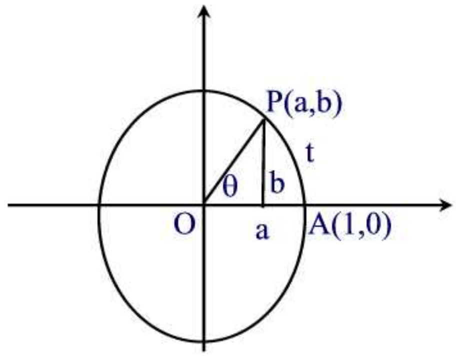


Figure 32

We define the following trigonometric functions:

$$\begin{array}{rclcrclcrcl} \sin t & = & b & \quad & \cos t & = & a & \quad & \tan t & = & \frac{b}{a} \\ \csc t & = & \frac{1}{b} & & \sec t & = & \frac{1}{a} & & \cot t & = & \frac{a}{b} \end{array}$$

where $a \neq 0$ and $b \neq 0$. If $a = 0$ then the functions $\sec t$ and $\tan t$ are undefined. If $b = 0$ then the functions $csct$ and $\cot t$ are undefined.
The above trigonometric functions are referred to as **circular functions**.
It follows from the above definition that

$$\sin^2 x + \cos^2 x = 1.$$

The table below lists the domain, range, and the periodicity of each of the triginometric functions discussed above.

$f(x)$	Domain	Range	Period
$\sin x$	$(-\infty, \infty)$	[-1,1]	2π
$\cos x$	$(-\infty, \infty)$	[-1,1]	2π
$\tan x$	$x \neq (2n+1)\frac{\pi}{2}$	$(-\infty, \infty)$	π
$\cot x$	$x \neq n\pi$	$(-\infty, \infty)$	π
$\sec x$	$x \neq (2n+1)\frac{\pi}{2}$	$(-\infty, -1] \cup [1, \infty)$	2π
$\csc x$	$x \neq n\pi$	$(-\infty, -1] \cup [1, \infty)$	2π

Example 5.6
Referring to Figure 33, answer the following questions.
(a) Express the area of ΔOBC in terms of $\sin\theta$ and $\cos\theta$.
(b) Express the area of ΔOBD in terms of $\tan\theta$.
(c) Use parts (a) and (b) to show

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

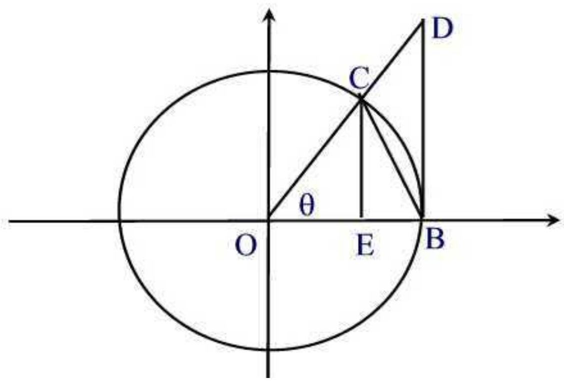


Figure 33

Solution.
(a) $Area\ \Delta OBC = \frac{1}{2}|EC||OB| = \frac{1}{2}|EC| = \frac{1}{2}\sin\theta$ since $\sin\theta = \frac{|EC|}{|OC|} = |EC|$.
(b) $Area\ \Delta OBD = \frac{1}{2}|BD||OB| = \frac{1}{2}|BD| = \frac{1}{2}\tan\theta$ since $\tan\theta = \frac{|DB|}{|OB|} = |DB|$.
(c) Using Figure 33 we see that

$$Area\ \Delta OBC < Area\ circular\ sector\ OBC < Area\ \Delta OBD$$

But the area of the circular sector OBC is $\frac{1}{2}\theta$. Hence,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

Multiplying through by $\frac{2}{\sin\theta}$ to obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.\blacksquare$$

Graphs of Sine, Cosine, and Tangent Functions

Figure 34 shows the graph of $y = sinx$.

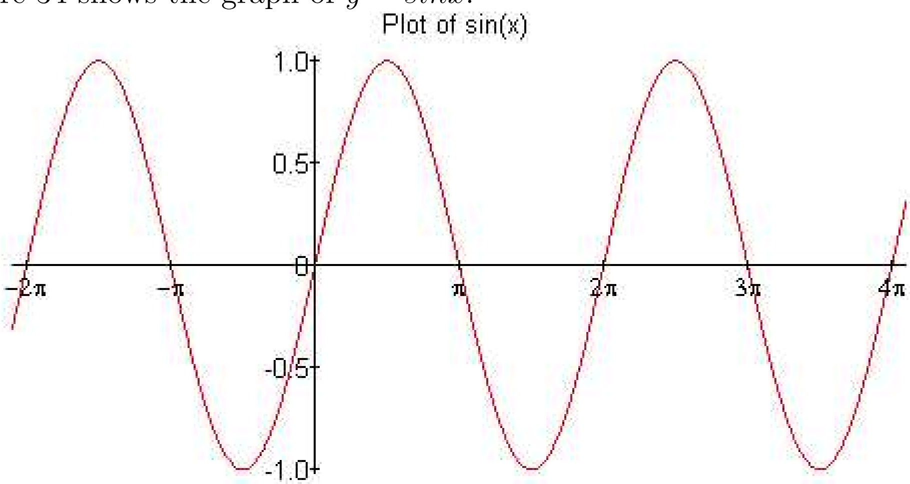


Figure 34

Figure 35 shows the graph of $y = cosx$.

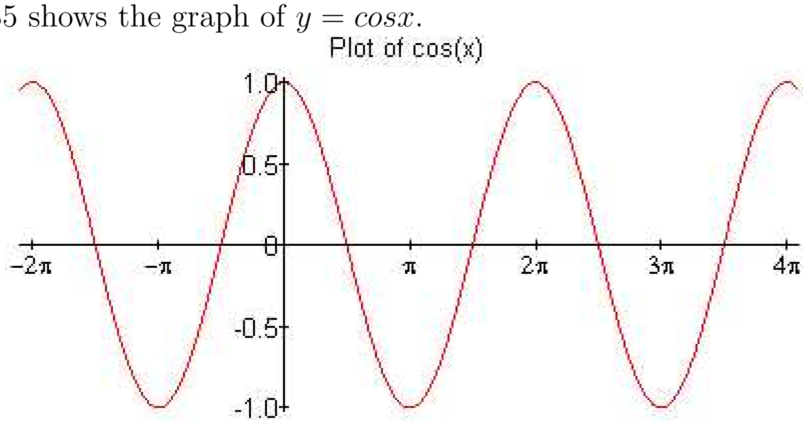


Figure 35

Figure 36 shows the graph of $y = tanx$.

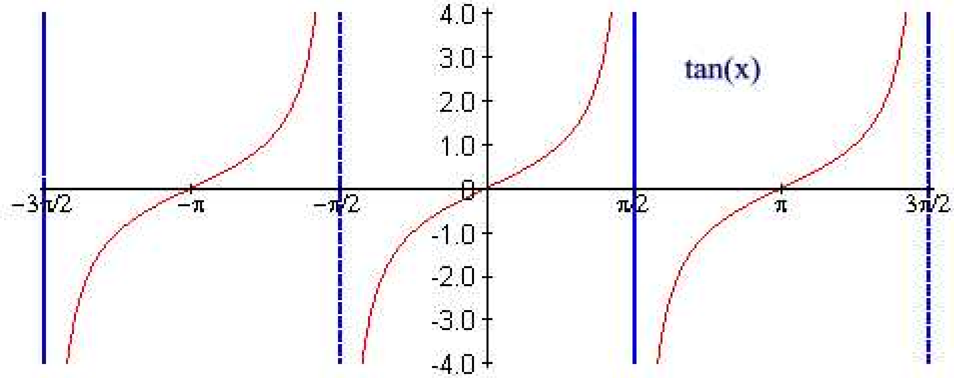


Figure 36

Amplitude and period of $y = a\sin(bx), y = a\cos(bx), b > 0$
We now consider graphs of functions that are transformations of the sine and cosine functions.

• **The parameter a:** This is outside the function and so deals with the output (i.e. the y values). Since $-1 \leq \sin(bx) \leq 1$ and $-1 \leq \cos(bx) \leq 1$ then $-a \leq a\sin(bx) \leq a$ and $-a \leq a\cos(bx) \leq a$. So, the range of the function $y = a\sin(bx)$ or the function $y = a\cos(bx)$ is the closed interval $[-a, a]$. The number $|a|$ is called the **amplitude**. Graphically, this number describes how tall the graph is. The amplitude is half the distance from the top of the curve to the bottom of the curve. If $b = 1$, the amplitude $|a|$ indicates a vertical stretch of the basic sine or cosine curve if $a > 1$, and a vertical compression if $0 < a < 1$. If $a < 0$ then a reflection about the x-axis is required.
Figure 37 shows the graph of $y = 2\sin x$ and the graph of $y = 3\sin x$.

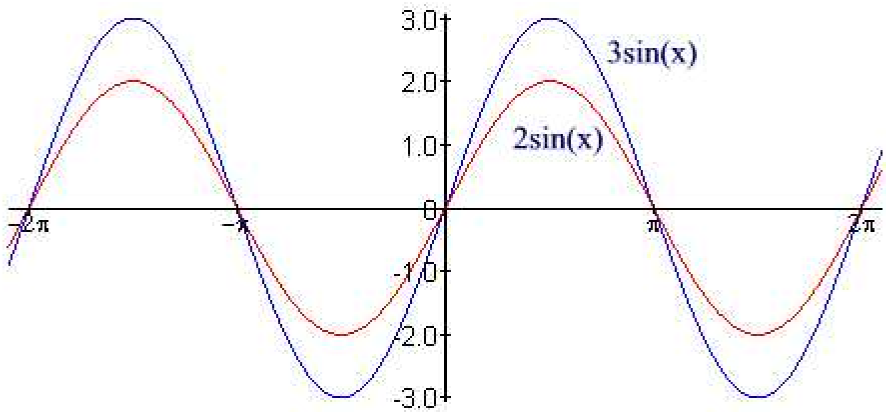


Figure 37

• **The parameter b:** This is inside the function and so effects the input (i.e. x values). Now, the graph of either $y = a\sin(bx)$ or $y = a\cos(bx)$ completes one period from $bx = 0$ to $bx = 2\pi$. By solving for x we find the interval of one period to be $[0, \frac{2\pi}{b}]$. Thus, the above mentioned functions have a period of $\frac{2\pi}{b}$. The number b tells you the number of cylces in the interval $[0, 2\pi]$. Graphically,b either stretches (if $b < 1$) or compresses (if $b > 1$) the graph horizontally.
Figure 38 shows the function $y = \sin x$ with period 2π and the function $y = \sin(2x)$ with period π.

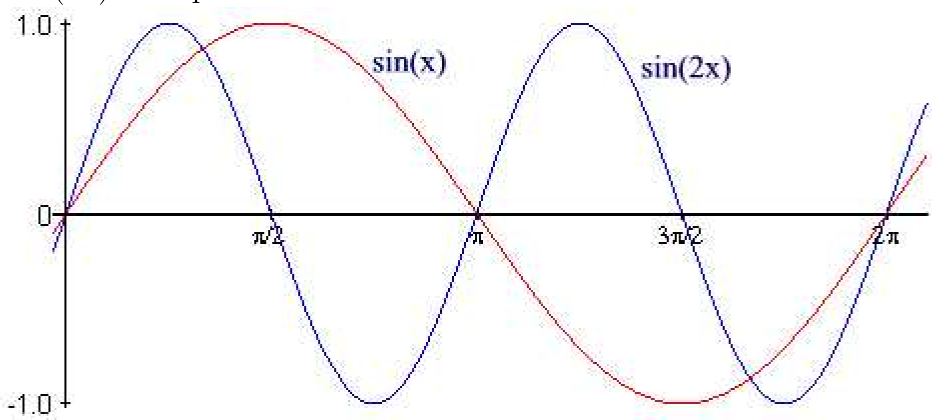


Figure 38

The Inverse Sine Function
The function $f(x) = \sin x$ is increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. See Figure 39. Thus, $f(x)$ is one-to-one and consequently it has an inverse denoted by

$$f^{-1}(x) = \sin^{-1} x.$$

We call this new function the **inverse sine function.**

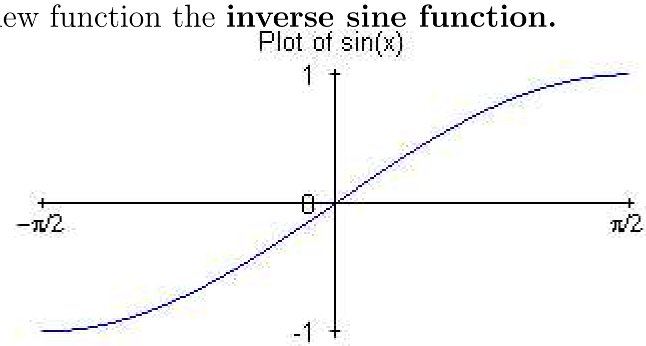


Figure 39

From the definition of inverse functions discussed in Section 4, we have the following properties of $f^{-1}(x)$:

(i) $Dom(\sin^{-1} x) = Range(\sin x) = [-1, 1]$.
(ii) $Range(\sin^{-1} x) = Dom(\sin x) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.
(iii) $\sin(\sin^{-1} x) = x$ for all $-1 \leq x \leq 1$.
(iv) $\sin^{-1}(\sin x) = x$ for all $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
(v) $y = \sin^{-1} x$ if and only if $\sin y = x$. Using words, the notation $y = \sin^{-1} x$ gives the angle y whose sine value is x.

Remark 5.2
If x is outside the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ then we look for the angle y in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin x = \sin y$. In this case, $\sin^{-1}(\sin x) = y$. For example, $\sin^{-1}(\sin \frac{5\pi}{6}) = \sin^{-1}(\sin \frac{\pi}{6}) = \frac{\pi}{6}$.

The graph of $y = \sin^{-1} x$ is the reflection of the graph of $y = \sin x$ about the line $y = x$ as shown in Figure 40.

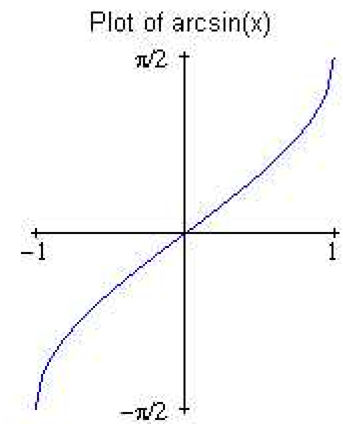


Figure 40

Example 5.7
Find the exact value of:
(a) $\sin^{-1} 1$ (b) $\sin^{-1} \frac{\sqrt{3}}{2}$ (c) $\sin^{-1}(-\frac{1}{2})$

Solution.
(a) Since $\sin \frac{\pi}{2} = 1$ then $\sin^{-1} 1 = \frac{\pi}{2}$.
(b) Since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ then $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$.
(c) Since $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ then $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$. ■

Example 5.8
Find the exact value of:
(a) $\sin(\sin^{-1} 2)$ (b) $\sin^{-1}(\sin \frac{\pi}{3})$.

Solution.
(a) $\sin(\sin^{-1} 2)$ is undefined since 2 is not in the domain of $\sin^{-1} x$.
(b) $\sin(\sin^{-1} \frac{\pi}{3}) = \frac{\Pi}{3}$.■

The Inverse Cosine function
In order to define the inverse cosine function, we will restrict the function $f(x) = \cos x$ over the interval $[0, \pi]$. There the function is always decreasing. See Figure 41. Therefore $f(x)$ is one-to-one function. Hence, its inverse will be denoted by

$$f^{-1}(x) = \cos^{-1} x.$$

We call $\cos^{-1} x$ the **inverse cosine function.**

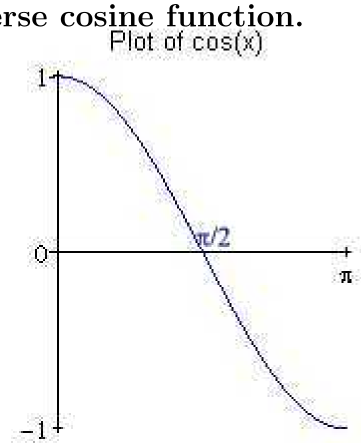


Figure 41

The following are consequences of the definition of inverse functions:
(i) $Dom(\cos^{-1} x) = Range(\cos x) = [-1, 1]$.
(ii) $Range(\cos^{-1} x) = Dom(\cos x) = [0, \pi]$.
(iii) $\cos(\cos^{-1} x) = x$ for all $-1 \leq x \leq 1$.
(iv) $\cos^{-1}(\cos x) = x$ for all $0 \leq x \leq \pi$.
(v) $y = \cos^{-1} x$ if and only if $\cos y = x$. Using words, the notation $y = \cos^{-1} x$ gives the angle y whose cosine value is x.

Remark 5.3
If x is outside the interval $[0, \pi]$ then we look for the angle y in the interval $[0, \pi]$ such that $\cos x = \cos y$. In this case, $\cos^{-1}(\cos x) = y$. For example, $\cos^{-1}(\cos \frac{7\pi}{6}) = \cos^{-1}(\cos \frac{5\pi}{6}) = \frac{5\pi}{6}$.

The graph of $y = \cos^{-1} x$ is the reflection of the graph of $y = \cos x$ about the line $y = x$ as shown in Figure 42.

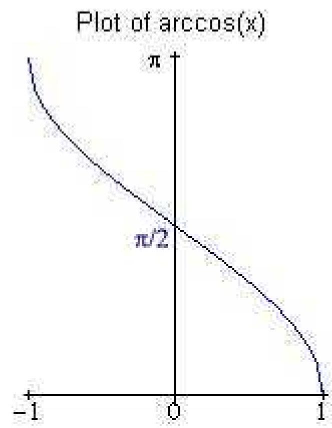


Figure 42

Example 5.9
Let $\theta = \cos^{-1} x$. Find the six trigonometric functions of θ.

Solution.
Let $u = \cos^{-1} x$. Then $\cos u = x$. Since $\sin^2 u + \cos^2 u = 1$ then $\sin u = \pm\sqrt{1-x^2}$. Since $0 \le u \le \pi$ then $\sin u \ge 0$ so that $\sin u = \sqrt{1-x^2}$. Thus,

$$\begin{aligned}
\sin(\cos^{-1} x) &= \sqrt{1-x^2}\\
\cos(\cos^{-1} x) &= x\\
\csc(\cos^{-1} x) &= \frac{1}{\sin(\cos^{-1} x)} = \frac{1}{\sqrt{1-x^2}}\\
\sec(\cos^{-1} x) &= \frac{1}{\cos(\cos^{-1} x)} = \frac{1}{x}\\
\tan(\cos^{-1} x) &= \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x)} = \frac{\sqrt{1-x^2}}{x}\\
\cot(\cos^{-1} x) &= \frac{1}{\tan(cos^{-1} x)} = \frac{x}{\sqrt{1-x^2}}. \blacksquare
\end{aligned}$$

Example 5.10
Find the exact value of:
(a) $\cos^{-1} \frac{\sqrt{2}}{2}$ (b) $\cos^{-1}(-\frac{1}{2})$.

Solution.
(a) $\cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$ since $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
(b) $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$. $\blacksquare$

The Inverse Tangent Function
Although not one-to-one on its full domain, the tangent function is one-to-one when restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ since it is increasing there (See

Figure 43).

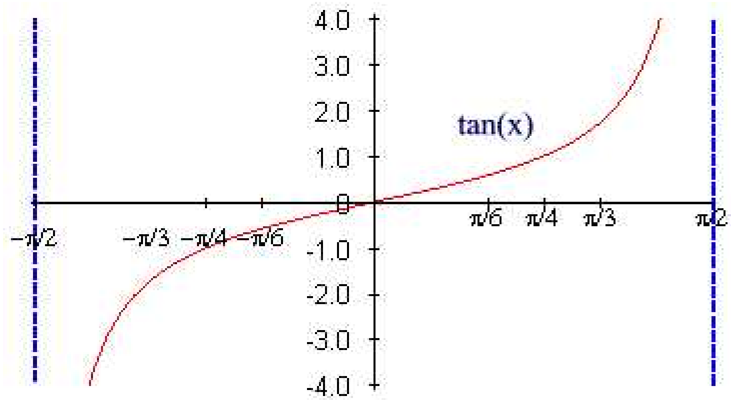


Figure 43

Thus, the inverse function exists and is denoted by

$$f^{-1}(x) = \tan^{-1} x.$$

We call this function the **inverse tangent function.**
As before, we have the following properties:
(i) $Dom(\tan^{-1} x) = Range(\tan x) = (-\infty, \infty)$.
(ii) $Range(\tan^{-1} x) = Dom(\tan x) = (-\frac{\pi}{2}, \frac{\pi}{2})$.
(iii) $\tan(\tan^{-1} x) = x$ for all x.
(iv) $\tan^{-1}(\tan x) = x$ for all $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(v) $y = \tan^{-1} x$ if and only if $\tan y = x$. In words, the notation $y = \tan^{-1} x$ means that y is the angle whose tangent value is x.

Remark 5.4
If x is outside the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $x \neq (2n+1)\frac{\pi}{2}$, where n is an integer, then we look for the angle y in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan x = \tan y$. In this case, $\tan^{-1}(\tan x) = y$. For example, $\tan^{-1}(\tan \frac{5\pi}{6}) = \tan^{-1}(\tan(-\frac{\pi}{6})) = -\frac{\pi}{6}$.

The graph of $y = \tan^{-1} x$ is the reflection of $y = \tan x$ about the line $y = x$ as shown in Figure 44.

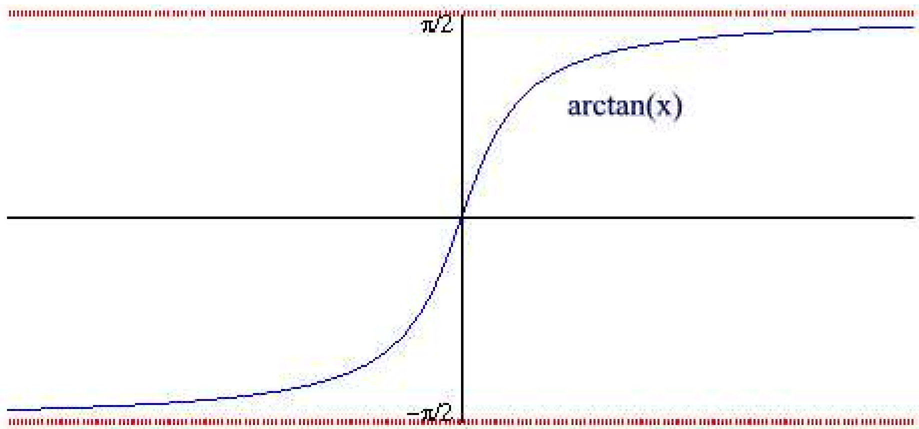


Figure 44

Example 5.11
Find the exact value of:
(a) $\tan^{-1}(\tan\frac{\pi}{4})$ (b) $\tan^{-1}(\tan\frac{7\pi}{5})$.

Solution.
(a) $\tan^{-1}(\tan\frac{\pi}{4}) = \frac{\pi}{4}$.
(b) $\tan^{-1}(\tan\frac{7\pi}{5}) = \tan^{-1}(\tan(\frac{2\pi}{5})) = \frac{2\pi}{5}$. ■

Example 5.12
Let $u = \tan^{-1} x$. Find the six trigonometric functions of u.

Solution.
Since $1 + \tan^2 u = \sec^2 u$ then $\sec u = \pm\sqrt{1+x^2}$. But $-\frac{\pi}{2} < u < \frac{\pi}{2}$ then $\sec u > 0$ so that $\sec u = \sqrt{1+x^2}$. Also, $\cot u = \frac{1}{\tan u} = \frac{1}{x}$. In summary,

$$\begin{aligned}
\sin(\tan^{-1} x) &= \frac{1}{\csc(\tan^{-1} x)} = \frac{x}{\sqrt{1+x^2}}\\
\cos(\tan^{-1} x) &= \frac{1}{\sec(\tan^{-1} x)} = \frac{1}{\sqrt{1+x^2}}\\
\csc(\tan^{-1} x) &= \frac{\sqrt{1+x^2}}{x}\\
\sec(\tan^{-1} x) &= \sqrt{1+x^2}\\
\tan(\tan^{-1} x) &= x\\
\cot(\tan^{-1} x) &= \frac{1}{x} \ \blacksquare
\end{aligned}$$

Recommended Problems (pp. 35 - 6): 1, 5, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19, 20, 23, 25, 27, 28, 33, 35.

6 Powers, Polynomial, and Rational Functions

In this section we discuss three important families of functions: powers, polynomials, and rational functions.

Power Functions
A function $f(x)$ is a **power function** of x if there is a nonzero constant k such that

$$f(x) = kx^n$$

The number n is called the **power** of x.

Remark 6.1
Recall that an exponential function has the form $f(x) = ba^x$, where the base a is fixed ans the exponent x varies. For a power function these properties are reversed- the base varies and the exponent remains constant.

Domains of Power Functions
If n is a non-negative integer then the domain of $f(x) = kx^n$ consists of all real numbers. If n is a negative integer then the domain of f consists of all nonzero real numbers.
If $n = \frac{r}{s}$, where r and s have no common factors, then the domain of $f(x)$ is all real numbers for s odd and $n > 0$ (all non zero real numbers for s odd and $n < 0$.) If s is even and $n > 0$ then the domain consists of all non-negative real numbers(all positive real numbers if $n < 0$.)

The Effect of n on the Graph of x^n
We assume that $k = 1$ and we will compare the graphs of $f(x) = x^n$ for various values of n. We will use graphing calculator to illustrate how power functions work and the role of n.
When $n = 0$ then the graph is a horizontal line at $(0, 1)$. When $n = 1$ then the graph is a straight line through the origin with slope equals to 1. See Figure .

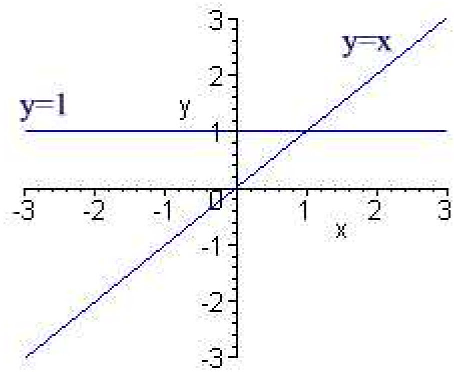


Figure 45

The graphs of all power functions with $n = 2, 4, 6, \cdots$ have the same characteristic $\cup -$ shape and they satisfy the following properties:

1. Pass through the points $(0,0), (1,1)$, and $(-1,1)$.
2. Decrease for negative values of x and increase for positive values of x.
3. Are symmetric about the y-axis because the functions are even.
4. Are concave up.
5. The graph of $y = x^4$ is flatter near the origin and steeper away from the origin than the graph of $y = x^2$. See Figure 46.

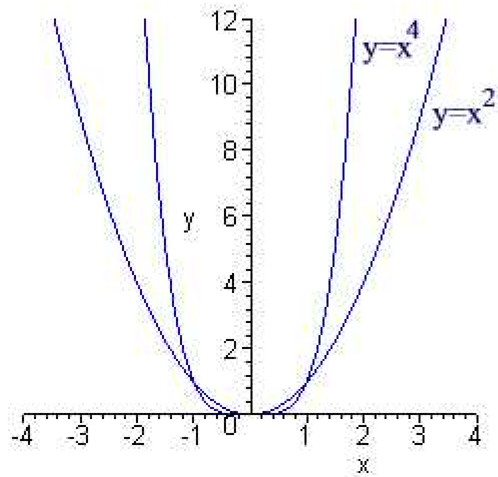


Figure 46

The graphs of power functions with $n = 1, 3, 5, \cdots$ resemble the side view of a chair and satisfy the following properties:

1. Pass through $(0, 0)$ and $(1, 1)$ and $(-1, -1)$.
2. Increase on every interval.
3. Are symmetric about the origin because the functions are odd.
4. Are concave down for negative values of x and concave up for positive values of x.
5. The graph of $y = x^5$ is flatter near the origin and steeper far from the origin than the graph of $y = x^3$. See Figure 47.

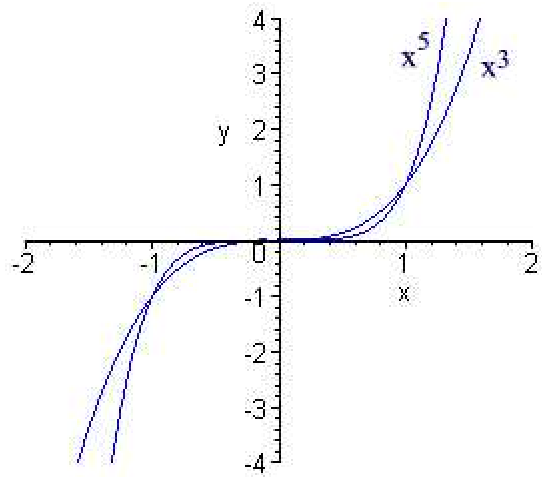


Figure 47

Graphs of $y = x^n$ with $n = -1, -3, \cdots$.
1. Passes through $(1, 1)$ and $(-1, -1)$ and does not have a y-intercept.
2. Is decreasing everywhere that it is defined.
3. Is symmetric about the origin because the function is odd.
4. Is concave down for negative values of x and concave up for positive values of x.
5. Has the x-axis as a horizontal asymptote and they y-axis as a vertical asymptote.
6. For $-1 < x < 1$, the graph of $y = \frac{1}{x}$ approaches the y-axis more rapidly then the graph of $y = \frac{1}{x^3}$. For $x < -1$ or $x > 1$ the graph of $y = \frac{1}{x^3}$ approach the x-axis more rapidly then the graph of $y = \frac{1}{x}$. See Figure 48.

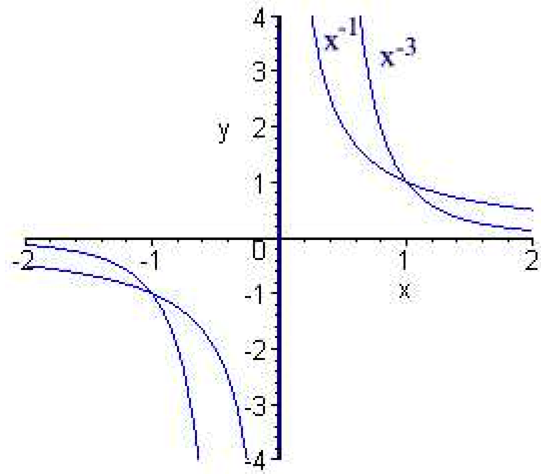


Figure 48

Graphs of $y = x^n$ with $n = -2, -4, \cdots$

1. Passes through $(1,1)$ and $(-1,1)$ and does not have a y- or x-intercept.
2. Is increasing for negative values of x and decreasing for positive values of x.
3. Is symmetric about the y-axis because the function is even.
4. Is concave up everywhere that it is defined.
5. Has the x-axis as a horizontal asymptote and the y-axis as vertical asymptote.
6. For $-1 < x < 1$, the graph of $y = \frac{1}{x^2}$ approaches the y-axis more rapidly then the graph of $y = \frac{1}{x^4}$. For $x < -1$ or $x > 1$ the graph of $y = \frac{1}{x^4}$ approach the x-axis more rapidly then the graph of $y = \frac{1}{x^2}$. See Figure 49.

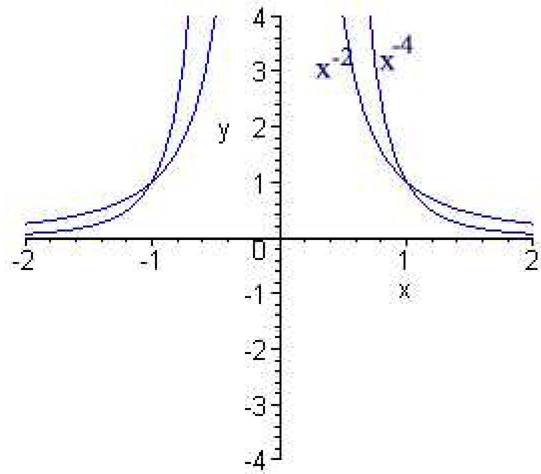


Figure 49

Graphs of $y = x^{\frac{1}{r}}$ where $r = 2, 4, \cdots$ has the following properties:
1. Domain consists of all non-negative real numbers.
2. Pass through $(0,0)$ and $(1,1)$.
3. Are increasing for $x > 0$.
4. Are concave down for $x > 0$.
5. The graph of $y = x^{\frac{1}{4}}$ is steeper near the origin and flatter away from the origin then the graph of $y = x^{\frac{1}{2}}$ See Figure 50.

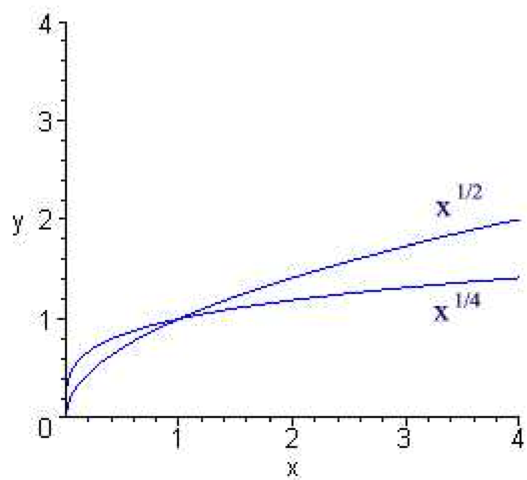


Figure 50

Graphs of $y = x^{\frac{1}{r}}$ where $r = 3, 5, \cdots$ has the following properties:
1. Domain consists for all real numbers.
2. Pass through $(0, 0)$, $(1, 1)$ and $(-1, -1)$.
3. Are increasing.
4. Are concave down for $x > 0$ and concave up for $x < 0$.
5. The graph of $y = x^{\frac{1}{5}}$ is steeper near the origin and flatter away from the origin then the graph of $y = x^{\frac{1}{3}}$. See Figure 51.

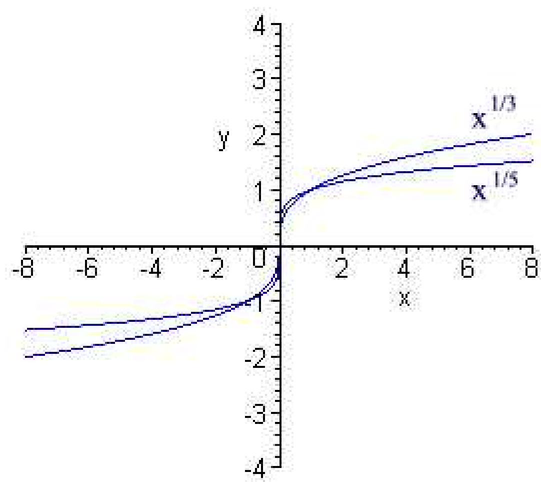


Figure 51

Polynomial Functions
Polynomial functions are among the simplest, most important, and most commonly used mathematical functions. These functions consist of one or more terms of variables with whole number exponents. (Whole numbers are positive integers and zero.) All such functions in one variable (usually x) can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are all real numbers, called the **coefficients** of $f(x)$. The number n is a non-negative integer. It is called the **degree** of the polynomial. A polynomial of degree zero is just a constant function. A polynomial of degree one is a linear function, of degree two a quadratic function, etc. The number a_n is called the **leading coefficient** and a_0 is

called the **constant term**.
Note that the terms in a polynomial are written in descending order of the exponents. Polynomials are defined for all values of x.

Example 6.1
Find the leading coefficient, the constant term and the degreee of the polynomial $f(x) = 4x^5 - x^3 + 3x^2 + x + 1$.

Solution.
The given polynomial is of degree 5, leading coefficient 4, and constant term 1. ■

A polynomial function will never involve terms where the variable occurs in a denominator, underneath a radical, as an input of either an exponential, logarithmic or trigonometric function.

Example 6.2
Determine whether the function is a polynomial function or not:

(a) $f(x) = 3x^4 - 4x^2 + 5x - 10$
(b) $g(x) = x^3 - e^x + 3$
(c) $h(x) = x^2 - 3x + \frac{1}{x} + 4$
(d) $i(x) = x^2 - \sqrt{x} - 5$
(e) $j(x) = x^3 - 3x^2 + 2x - 5\ln x - 3$.

Solution.
(a) $f(x)$ is a polynomial function of degree 4.
(b) $g(x)$ is not a ploynomial because one of the terms is an exponential function.
(c) $h(x)$ is not a polynomial because x is in the denominator of a fraction.
(d) $i(x)$ is not a polynomial because it contains a radical sign.
(e) $j(x)$ is not a polynomial because one of the terms is a logarithm of x.■

Long-Run Behavior of a Polynomial Function
If $f(x)$ and $g(x)$ are two functions such that $f(x) - g(x) \approx 0$ as x increases without bound then we say that $f(x)$ resembles $g(x)$ in the **long run**. For example, if n is any positive integer then $\frac{1}{x^n} \approx 0$ in the long run.
Now, if $f(x) = a_nx^n + a_{n-1}x^n + \cdots + a_1x + a_0$ then

$$f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$$

Since $\frac{1}{x^k} \approx 0$ in the long run, for each $1 \leq k \leq n$ then

$$f(x) \approx a_n x^n$$

in the long run.

Example 6.3
The polynomial function $f(x) = 1 - 2x^4 + x^3$ resembles the function $g(x) = -2x^4$ in the long run.

Zeros of a Polynomial Function
If f is a polynomial function in one variable, then the following statements are equivalent:

• $x = a$ is a **zero** or **root** of the function f.
• $x = a$ is a solution of the equation $f(x) = 0$.
• $(a, 0)$ is an x-intercept of the graph of f. That is, the point where the graph crosses the x-axis.

Example 6.4
Find the x-intercepts of the polynomial $f(x) = x^3 - x^2 - 6x$.

Solution.
Factoring the given function to obtain

$$\begin{aligned} f(x) &= x(x^2 - x - 6) \\ &= x(x-3)(x+2) \end{aligned}$$

Thus, the x-intercepts are the zeros of the equation

$$x(x-3)(x+2) = 0$$

That is, $x = 0, x = 3$, or $x = -2$.■

Graphs of a Polynomial Function
Polynomials are continuous and smooth everywhere:

• A continuous function means that it can be drawn without picking up your pencil. There are no jumps or holes in the graph of a polynomial function.

• A smooth curve means that there are no sharp turns (like an absolute value) in the graph of the function.
• The y-intercept of the polynomial is the constant term a_0.

The shape of a polynomial depends on the degree and leading coefficient:

• If the leading coefficient, a_n, of a polynomial is positive, then the right hand side of the graph will rise towards $+\infty$.
• If the leading coefficient, a_n, of a polynomial is negative, then the right hand side of the graph will fall towards $-\infty$.
• If the degree, n, of a polynomial is even, the left hand side will do the same as the right hand side.
• If the degree, n, of a polynomial is odd, the left hand side will do the opposite of the right hand side.

Example 6.5
According to the graphs given below, indicate the sign of a_n and the parity of n for each curve.

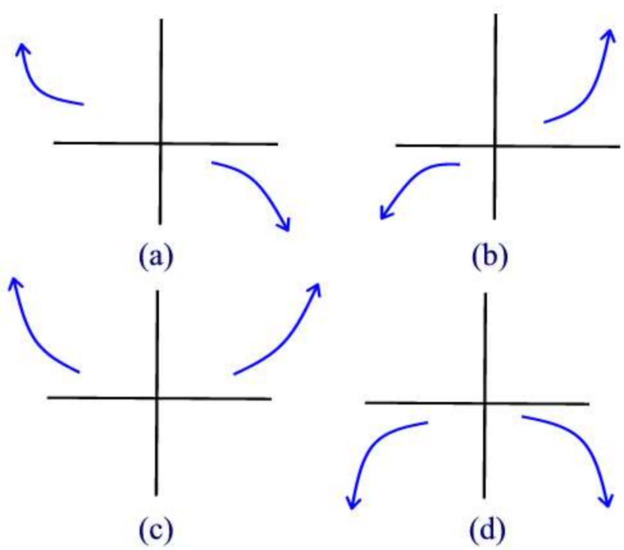


Figure 52

Solution.
(a) $a_n < 0$ and n is odd.

(b) $a_n > 0$ and n is odd.
(c) $a_n > 0$ and n is even.
(d) $a_n < 0$ and n is even.■

Short-Run Behavior of Polynomial Functions
We have seen in the discussion above that the two functions $f(x) = x^3 + 3x^2 + 3x + 1$ and $g(x) = x^3 - x^2 - 6x$ resemble the function $h(x) = x^3$ at the long-term behavior. See Figure 53.

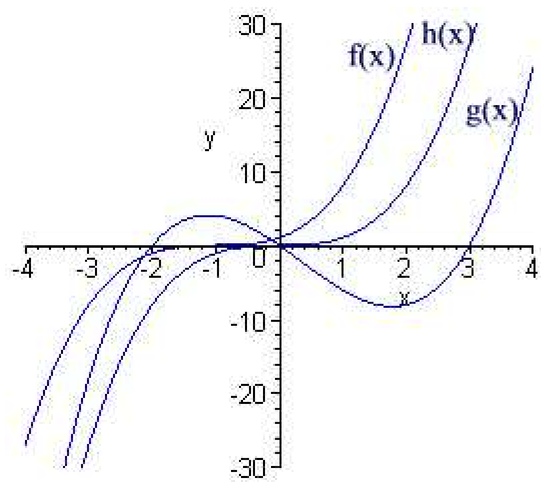


Figure 53

Although f and g have similar long-run behavior, they are not identical functions. This can be seen by studying the **short-run** (or local) behavior of these functions.
The short-run behavior of the graph of a function concerns graphical feature of the graph such as its intercepts or the number of bumps on the graph. For example, the function $f(x)$ has an x-intercept at $x = -1$ and y-intercept at $y = 1$ and no bumps. On the other hand, the function $g(x)$ has x-intercepts at $x = -2, 0, 3$, y-intercept at $y = 0$, and two bumps.
As you have noticed, the zeros (or roots) of a polynomial function is one of the important part of the short-run behavior. To find the zeros of a polynomial function, we can write it in factored form and then use the zero product rule which states that if $a \cdot b = 0$ then either $a = 0$ or $b = 0$. To illustrate, let us find the zeros of the function $g(x) = x^3 - x^2 - 6x$.

Factoring, we find

$$g(x) = x(x^2 - x - 6) = x(x-3)(x+2).$$

Setting $g(x) = 0$ and solving we find $x = 0, x = -2$, and $x = 3$. The number of zeros determines the number of bumps that a graph has since between any two consecutive zeros, there is a bump because the graph changes direction. Thus, as we see from the graph of $g(x)$ that $g(x)$ has two bumps.
Now, the function $f(x)$ has only one zero at $x = -1$. We call $x = -1$ a zero of **multiplicity three.**
It is easy to see that when a polynomial function has a zero of even multiplicity than the graph does not cross the x-axis at that point; on the contrary, if the zero is of odd multiplicity than the graph crosses the x-axis.

Example 6.6
Sketch the graph of $f(x) = (x+1)^3$ and $g(x) = (x+1)^2$.

Solution.
The graphs of $f(x)$ and $g(x)$ are shown in Figure 54.■

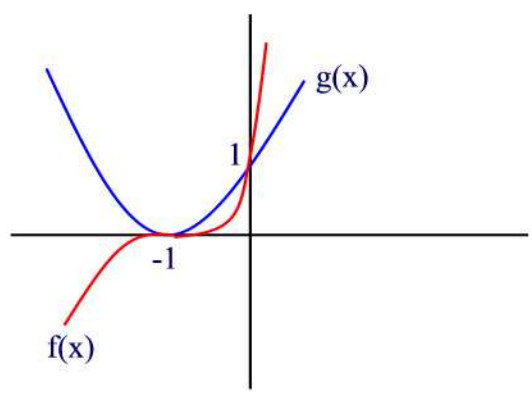


Figure 54

Rational Functions
A **rational function** is a function that is the ratio of two polynomial functions $\frac{f(x)}{g(x)}$. The domain consists of all numbers such that $g(x) \neq 0$.

Example 6.7
Find the domain of the function $f(x) = \frac{x-2}{x^2-x-6}$.

Solution.
The domain consists of all numbers x such that $x^2 - x - 6 \neq 0$. But this last quadratic expression is 0 when $x = -2$ or $x = 3$. Thus, the domain is the set $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.■

The Long-Run Behavior of Rational Functions
Given a rational function

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}.$$

We know that the top polynomial resembles $a_m x^m$ and the bottom polynomial resembles $b_n x^n$ in the long run. It follows that, in the long run, $f(x) \approx \frac{a_m x^m}{b_n x^n}$.

Example 6.8
Discuss the long run behavior of each of the following functions:

(a) $f(x) = \frac{3x^2+2x-4}{2x^2-x+1}$.
(b) $f(x) = \frac{2x+3}{x^3-2x^2+4}$.
(c) $f(x) = \frac{2x^2-3x-1}{x-2}$.

Solution.
(a) $f(x) = \frac{3x^2+2x-4}{2x^2-x+1} \approx \frac{3x^2}{2x^2} = \frac{3}{2}$.
(b) $f(x) = \frac{2x+3}{x^3-2x^2+4} \approx \frac{2x}{x^3} = \frac{2}{x^2}$.
(c) $f(x) = \frac{2x^2-3x-1}{x-2} \approx \frac{2x^2}{x} = 2x$.■

Horizontal Asymptote
Contrary to polynomial functions, it is possible for a rational function to level at $x \to \infty$ or $x \to -\infty$. That is, $f(x)$ approaches a value b as $x \to \infty$ or $x \to -\infty$. We call $y = b$ a **horizontal asymptote**. The graph of a rational function may cross its horizontal asymptote.

Example 6.9
Find the horizontal asymptote, if it exists, for each of the following functions:

(a) $f(x) = \frac{3x^2+2x-4}{2x^2-x+1}$.
(b) $f(x) = \frac{2x+3}{x^3-2x^2+4}$.
(c) $f(x) = \frac{2x^2-3x-1}{x-2}$.

Solution.
(a) As $x \to \pm\infty$, we have

$$\begin{aligned} f(x) &= \frac{3x^2+2x-4}{2x^2-x+1} \\ &= \frac{x^2}{x^2} \cdot \frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}} \to \frac{3}{2} \end{aligned}$$

so the line $y = \frac{3}{2}$ is the horizontal asymptote.
(b) As $x \to \pm\infty$, we have

$$\begin{aligned} f(x) &= \frac{2x+3}{x^3-2x^2+4} \\ &= \frac{x}{x^3} \cdot \frac{2+\frac{3}{x}}{1-\frac{2}{x}+\frac{4}{x^3}} \to 0 \end{aligned}$$

so the x-axis is the horizontal asymptote.
(c) As $x \to \pm\infty$, we have

$$\begin{aligned} f(x) &= \frac{2x^2-3x-1}{x-2} \\ &= \frac{x^2}{x} \cdot \frac{2-\frac{3}{x}-\frac{1}{x^2}}{1-\frac{2}{x}} \to \infty \end{aligned}$$

so the function has no horizontal asymptote.■

The Short-Run Behavior of Rational Functions
We close this section by studying the local behavior of rational functions which includes the zeros and the vertical asymptotes.

The Zeros of a Rational Function
The zeros of a rational function or its x-intercepts. They are those numbers that make the numerator zero and the denominator nonzero.

Example 6.10
Find the zeros of each of the following functions:

(a) $f(x) = \frac{x^2+x-2}{x-3}$ (b) $g(x) = \frac{x^2+x-2}{x-1}$.

Solution.
(a) Factoring the numerator we find $x^2 + x - 2 = (x-1)(x+2)$. Thus, the zeros of the numerator are 1 and -2. Since the denominator is different from zeros at these values then the zeros of $f(x)$ are 1 and -2.

(b) The zeros of the numerator are 1 and -2. Since 1 is also a zero of the denominator then $g(x)$ has -2 as the only zero.■

Vertical Asymptote
When the graph of a function either grows without bounds or decay without bounds as $x \to a$ then we say that $x = a$ is a **vertical asymptote**. For rational functions, the vertical asymptotes are the zeros of the denominator. Thus, if $x = a$ is a vertical asymptote then as x approaches a from either sides the function becomes either positively large or negatively large. The graph of a function never crosses its vertical asymptotes.

Example 6.11
Find the vertical asymptotes of the function $f(x) = \frac{2x-11}{x^2+2x-8}$

Solution.
Factoring $x^2 + 2x - 8 = 0$ we find $(x - 2)(x + 4) = 0$. Thus, the vertical asymptotes are the lines $x = 2$ and $x = -4$.■

Graphing Rational Functions
To graph a rational function $h(x) = \frac{f(x)}{g(x)}$:

1. Find the domain of $h(x)$ and therefore sketch the vertical asymptotes of $h(x)$.
2. Sketch the horizontal or the oblique asymptote if they exist.
3. Find the $x-$ *intercepts* of $h(x)$ by solving the equation $f(x) = 0$.
4. Find the y-intercept: $h(0)$
5. Draw the graph

Example 6.12
Sketch the graph of the function $f(x) = \frac{x(4-x)}{x^2-6x+5}$

Solution.

1. *Domain* $= (-\infty, 1) \cup (1, 5) \cup (5, \infty)$. The vertical asymptotes are $x = 1$ and $x = 5$.
2. As $x \to \pm\infty$, $f(x) \approx -1$ so the line $y = 1$ is the horizontal asymptote.
3. The x-intercepts are at $x = 0$ and $x = 4$.
4. The y-intercept is $y = 0$.

5. The graph is given in Figure 55.■

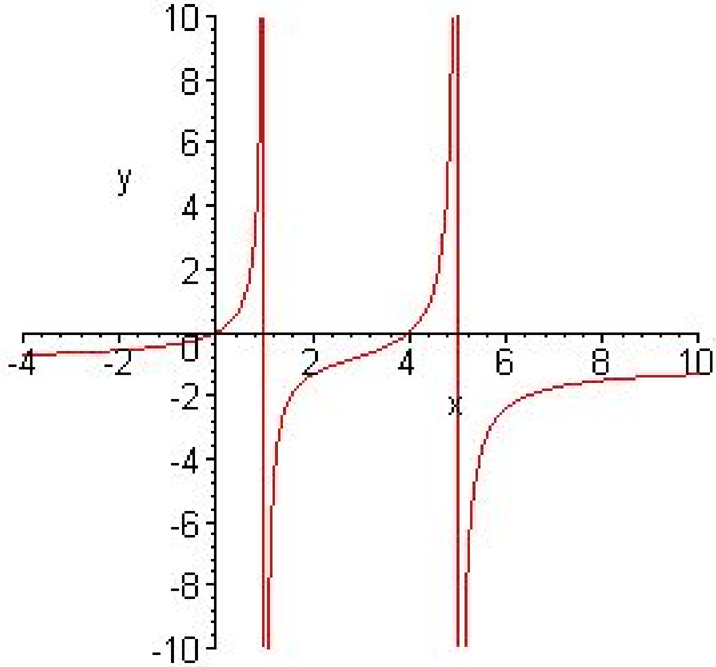


Figure 55

When Numerator and Denominator Have Common Zeros

We have seen in Example **??**, that the function $g(x) = \frac{x^2+x-2}{x-1}$ has a common zero a4 $x = 1$. You might wonder What the graph looks like. For $x \neq 1$, the function reduces to $g(x) = x + 2$. Thus, the graph of $g(x)$ is a straight line with a hole at $x = 1$ as shown in Figure 56.

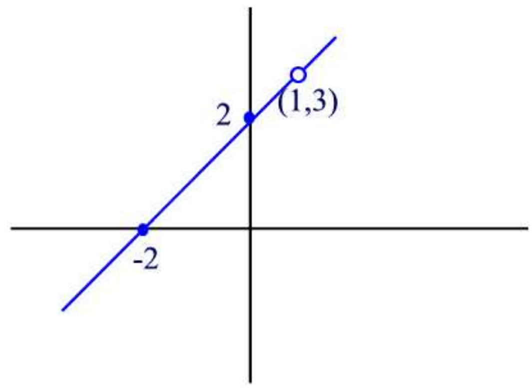


Figure 56

Recommended Problems (pp. 42 - 5): 3, 4, 6, 7, 9, 11, 13, 14, 23, 25, 26, 27, 32, 34.

7 Velocity as a Rate of Change

In this section, we discuss the concepts of the average rate of change and the instantaneous rate of change of a given function. As an application, we use the velocity of a moving object.

The motion of an object along a line at a particular instant is very difficult to define precisely. The modern approach consists of computing the average velocity over smaller and smaller time intervals containing the instant. To be more precise, let $s(t)$ be the **position** function of a moving object at time t. We would like to compute the velocity of the object at the instant $t = t_0$:

Average Velocity
We start by finding the **average velocity** of the object over the time interval $t_0 \leq t \leq t_0 + \Delta t$ given by the expression

$$\overline{v} = \frac{Distance\ Traveled}{Elapsed\ Time} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}.$$

Geometrically, the average velocity over the time interval $[t_0, t_0 + \Delta t]$ is just the slope of the line joining the points $(t_0, s(t_0))$ and $(t_0 + \Delta t, s(t_0 + \Delta t))$ on the graph of $s(t)$.(See Figure 57)

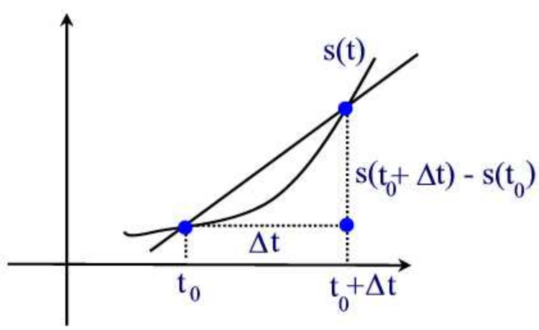


Figure 57

Example 7.1
An object moves a distance $s(t) = 16t^2$ feet from its starting point in t seconds. Complete the following table

time interval	[1.8,2]	[1.9,2]	[1.99,2]	[1.999,2]	[2,2.0001]	[2,2.001]	[2,2.01]
Average velocity							

Solution.

time interval	[1.8,2]	[1.9,2]	[1.99,2]	[1.999,2]	[2,2.0001]	[2,2.001]	[2,2.01]
Average velocity	60.8	62.4	63.84	63.98	64.0016	64.016	64.16

In general, the **average rate of change** of a function $y = f(x)$ over the interval $[a, b]$ is the difference quotient

$$\frac{f(b) - f(a)}{b - a}.$$

Instantaneous Velocity and Speed
The next step is to calculate the average velocity on smaller and smaller time intervals containing t_0(that is, make Δt close to zero). The average velocity in this case approaches what we would intuitively call the **instantaneous velocity** at time $t = t_0$ which is defined using the **limit** notation by

$$v(t_0) = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$

Geometrically, the instantaneous velocity at t_0 is the slope of the tangent line to the graph of $s(t)$ at the point $(t_0, s(t_0))$.(See Figure 58)

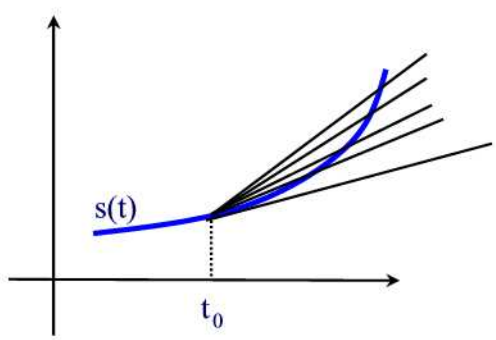


Figure 58

Example 7.2
For the distance function in Example 7.1, find the instantaneous velocity at $t = 2$.

Solution.
Examining the bottom row of the table in Example 7.1, we see that the average velocity seems to be approaching the value 64 as we shrink the time intervals. Thus, it is reasonable to expect the velocity to be $v(2) = 64\ ft/sec.$■

Speed of a Moving Object
We define the **speed** of a moving object to be the absolute value of the velocity function. Sometimes there is confusion between the words "speed" and "velocity". Speed is a nonnegative number that indicates how fast an object is moving, whereas velocity indicates both speed and direction(relative to a coordinate system). For example, if the object is moving along a vertical line we define a positive velocity when the object is going upward and a negative velocity when the object is going downward.

The Informal Definition of Limit
The notation of limit was introduced in defining the instantaneous velocity $v(t)$. Here is a general, informal, definition of the limit of a function.
The notation
$$\lim_{x\to a} f(x) = L$$
is read "the limit of f(x) as x approaches a is L" and means that the values of $f(x)$ approach a unique number L when the variable x is chosen sufficiently close to a, from either side, but not equal to a.
Sometimes we will find it convenient to represent the notation
$$\lim_{x\to a} f(x) = L$$
by writing $f(x) \to L$ as $x \to a$.

Example 7.3 *(Graphical evaluation of a limit)*
Evaluate graphically, $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$.

Solution.
The graph of $f(\theta) = \frac{\sin\theta}{\theta}$ is given by Figure 59. From this, we see that
$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.■$$

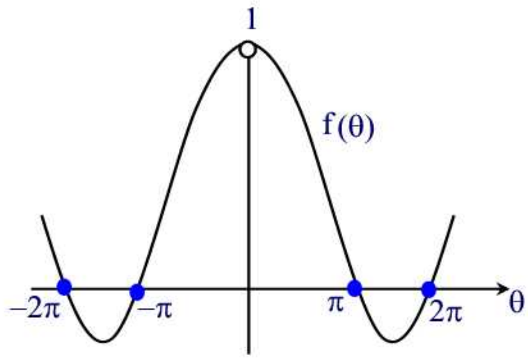


Figure 59

Example 7.4 *(Evaluating a limit numerically)*
Evaluate numerically $\lim_{t\to 0}\frac{16(2+t)^2-16(2)^2}{t}$.

Solution.

t	-.2	-.1	-.01	-0.001	0	.0001	.001	.01
$\frac{16(2+t)^2-16(2)^2}{t}$	60.8	62.4	63.84	63.98	undefined	64.0016	64.016	64.16

From the table we conclude that

$$\lim_{t\to 0}\frac{16(2+t)^2-16(2)^2}{t} = 64.\blacksquare$$

Example 7.5 *Evaluating a limit algebraically*
Use algebra to find $\lim_{t\to 0}\frac{16(2+t)^2-16(2)^2}{t}$.

Solution.
Expanding the numerator gives

$$\frac{16(2+t)^2-16(2)^2}{t} = \frac{16t^2+64t}{t}.$$

Thus,

$$\lim_{t\to 0}\frac{16(2+t)^2-16(2)^2}{t} = \lim_{t\to 0}(16t+64) = 64.\blacksquare$$

Generalizing the idea of instantaneous velocity, we define the **instantaneous rate of change** of a function $y = f(x)$ at $x = a$ to be

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or alternatively, by letting $h = x - a$

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Example 7.6
Evaluate

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

when $f(x) = x^2$.

Solution.
Since $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$ and $f(x) = x^2$ then

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0}(2x + h) = 2x. \blacksquare$$

Recommended Problems (pp. 61 - 62): 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15, 16, 18, 20.

8 The Formal Definition of Limit

In this section we will explore the concept of the limit of a function and then develop algebraic techniques for computing limits in a systematic way.

Definition and Properties of Limit
The limit of a function f is a tool for investigating the behavior of $f(x)$ as x gets closer and closer to a particular number a. For example, the function $s(t) = 16t^2$ of Example 7.1 approaches the value 64 as t approaches 2.
The notation

$$\lim_{x \to a} f(x) = L \tag{1}$$

is read "the limit of f(x) as x approaches a is L" and means that the values of $f(x)$ approach a unique number L when the variable x is chosen sufficiently close to a, from either side, but not equal to a.
Sometimes we will find it convenient to represent the notation

$$\lim_{x \to a} f(x) = L$$

by writing $f(x) \to L$ as $x \to a$.

Geometrically, the notation (1) means that for any tiny interval around L that you choose, you can find a tiny interval around a such that the function f takes any point in the tiny interval around a inside the tiny interval around L. This means that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if the distance between a point x and a is less than δ (i.e. the point x is inside the tiny interval around a) then the distance between $f(x)$ and L is less than ϵ (i.e, the point $f(x)$ is inside the tiny interval around L.). See Figure 60.

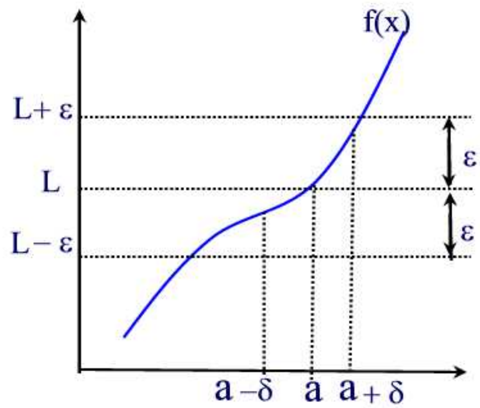


Figure 60

Symbolically, (1) is equivalent to

For every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |\mathrm{x} - \mathrm{a}| < \delta$ then $|\mathrm{f(x)} - \mathrm{L}| < \epsilon$. (2)

(2) is known as the formal definition of the limit of a function.

Remark 8.1
Realize that in the above definition the ϵ is given and the δ is to be found. In practice, one finds an explicit expression for δ in terms of ϵ.

Example 8.1
Show, by using the formal definition of limit, that $\lim_{x\to 2} x^2 = 4$.

Solution.
Let $\epsilon > 0$ be a small given positive number. We must find a $\delta > 0$ such that if $0 < |x-2| < \delta$ then $|x^2 - 4| < \epsilon$. Note that $|x^2-4| = |x-2||x+2|$. Since $0 < |x-2| < \delta$ then $|x+2| = |(x-2)+4| \le |x-2|+4 < \delta+4$. Since δ is supposed to be small so we can assume that $\delta < 1$ so that $|x+2| < 4+\delta < 5$. Thus, $|x^2-4| < 5\delta$. So choose $0 < \delta < 1$ such that $\epsilon = 5\delta$, i.e., $\delta = \frac{\epsilon}{5}$.■

Using the $\epsilon - \delta$ definition in finding the limit of a function is sometimes cumbersome. Instead, one can use the following basic properties listed in the next theorem.

Theorem 8.1
Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L'$. Then

(a) $\lim_{x\to a}(f(x) + g(x)) = L + L'$.
(b) $\lim_{x\to a}(f(x)g(x)) = L \cdot L'$.
(c) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{L'}$, *provided that* $L' \neq 0$.
(d) $\lim_{x\to a} k = k$, *for any constant* k.
(e) $\lim_{x\to a} x = a$.
(f) If $f(x)$ is a polynomial then $\lim_{x\to a} f(x) = f(a)$.

Proof.
(a) Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \ \ implies \ \ |f(x) - L| < \frac{\epsilon}{2}$$

and

$$0 < |x - a| < \delta_2 \ \ implies \ \ |g(x) - L'| < \frac{\epsilon}{2}.$$

Let δ be the smallest of the two numbers δ_1 and δ_2. Assume that $0 < |x-a| < \delta$. Then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$. Moreover,

$$|(f(x)+g(x))-(L+L')| = |(f(x)-L)+(g(x)-L')| \leq |f(x)-L|+|g(x)-L'| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

This establishes the first property.
(b) First, note the following

$$\begin{array}{rcl} |f(x)g(x) - LL'| & = & |(f(x) - L)g(x) + (g(x) - L')L| \\ & \leq & |f(x) - L||g(x)| + |g(x) - L'||L| \\ & = & |f(x) - L||g(x) - L' + L'| + |g(x) - L'||L| \\ & \leq & |f(x) - L|(|g(x) - L'| + |L'|) + |g(x) - L'||L|. \end{array}$$

Let $\epsilon > 0$ be given. Then there is $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $|g(x) - L'| < \frac{\epsilon}{2L}$. Also, there is a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|f(x) - L| < \frac{\epsilon(\frac{\epsilon}{2|L|}+|L'|)}{2}$. Now, choose δ to be the smallest of δ_1 and δ_2. Then for $0 < |x - a| < \delta$ we have

$$|f(x)g(x) - LL'| < \frac{\epsilon}{2}(\frac{\epsilon}{2|L|} + |L'|)\frac{1}{\frac{\epsilon}{2|L|} + |L'|} + \frac{\epsilon}{2|L|}|L| = \epsilon.$$

This proves (b).
(c) We will show that $\lim_{x\to a}\frac{1}{g(x)}=\frac{1}{L'}$. Let $\delta_1>0$ be such that whenever $0<|x-a|<\delta$ then $|g(x)-L'|<\frac{|L'|}{2}$. But $|L'|-|g(x)|\leq|g(x)-L'|<\frac{|L'|}{2}$. This implies that $|g(x)|>\frac{|L'|}{2}$ for $0<|x-a|<\delta_1$. On the other hand, there is a $\delta_2>0$ such that whenever $0<|x-a|<\delta_2$ then $|g(x)-L'|<\frac{|L'|^2}{2}\epsilon$. Let δ be the smallest of the δ_1 and δ_2. Then

$$\left|\frac{1}{g(x)}-\frac{1}{L'}\right| \;=\; \left|\frac{g(x)-L'}{g(x)L'}\right| \;<\; \frac{2}{|L'|^2}|g(x)-L'|<\epsilon.$$

This result and (b) yield a proof for (c).
(d) Let $\epsilon>0$ be given. Choose $\delta=\epsilon$. Then, if $0<|x-a|<\delta$ then $|k-k|=0<\epsilon$.
(e) Let $\epsilon>0$ be given. Choose $\delta=\epsilon$. Then $0<|x-a|<\delta$ implies $|x-a|<\epsilon$.
(f) This follows from (a) - (e).■

Example 8.2
Evaluate the following limits:
(i) $\lim_{x\to 2}(2x^5-9x^3+3x^2-11)$.
(ii) $\lim_{x\to -1}\frac{x^3-3x+7}{5x^2+9x+6}$.

Solution.
(i) $\lim_{x\to 2}(2x^5-9x^3+3x^2-11)=2(2)^5-9(2)^3+3(2)^2-11=-7$
(ii) $\lim_{x\to -1}\frac{x^3-3x+7}{5x^2+9x+6}=\frac{(-1)^3-3(-1)+7}{5(-1)^2+9(-1)+6}=\frac{9}{2}$.■

Sided Limits
By graphing the function $f(x)=\frac{|x|}{x}$, $x\neq 0$, (See Figure 61) one notices that $\lim_{x\to 0}f(x)$ does not exist. However, there is a limit if we approach zero from either the right (in this case the limit is 1) or the left of zero(the limit is -1).

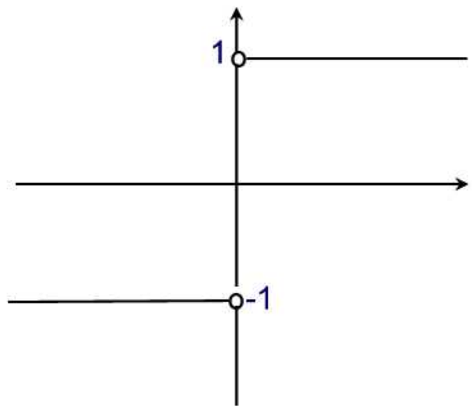


Figure 61

This leads to the following definitions:

Right-Hand Limit
We write $\lim_{x\to a^+} f(x) = L$ if we can make the values of $f(x)$ as close to L as we please by choosing x sufficiently close to a from the right.
Left-Hand Limit
We write $\lim_{x\to a^-} f(x) = L$ if we can make the values of $f(x)$ as close to L as we please by choosing x sufficiently close to a from the left.

Remark 8.2
When we write $\lim_{x\to a} f(x) = L$ we mean that $f(x)$ approaches L from both sides of a. That is, $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.

Limits That Do Not Exist
Whenever there is no number L such that $\lim_{x\to a} f(x) = L$, we say that $\lim_{x\to a} f(x)$ does not exist. Below we exhibit three examples in which limits fail to exist.

Example 8.3
Explain why $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution.
According to Figure 61, we have that $\lim_{x\to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x\to 0^-} \frac{|x|}{x} = -1$. Thus, by the above remark, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.■

Example 8.4
Use a calculator to graph the function $f(x) = \sin\left(\frac{1}{x}\right)$ and then conclude that $\lim_{x\to 0} f(x)$ does not exist.

Solution.
The graph of $f(x) = \sin\left(\frac{1}{x}\right)$ is given by Figure 62.

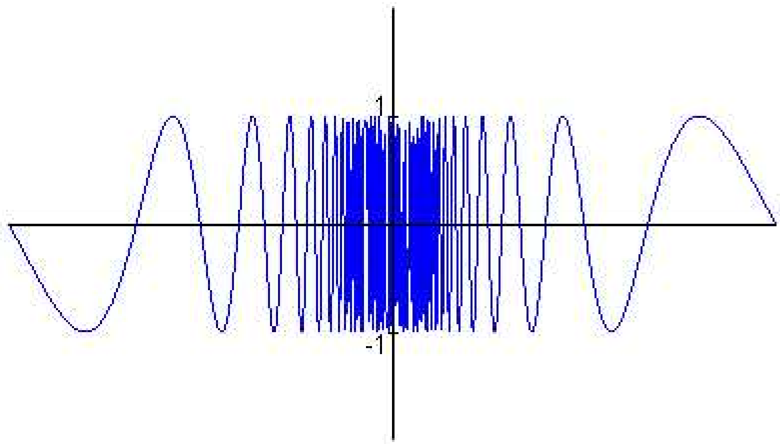


Figure 62

It follows that the graph oscillates between -1 and 1 as x approaches 0. Hence, $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.■

Example 8.5
(i) Use a calculator to graph $f(x) = \frac{1}{x^2}$.
(ii) Use part (i) to show that $\lim_{x\to 0} f(x) = \infty$.

Solution.
According to Figure 63, we see that the function increases without bound as x approaches 0. We write this using the notation

$$\lim_{x\to 0} f(x) = \infty.■$$

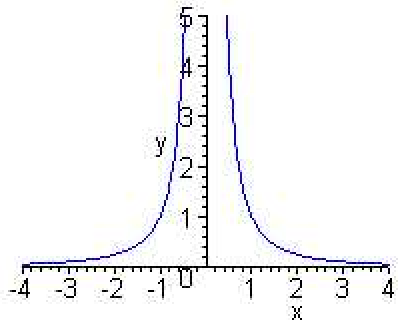


Figure 63

Infinite Limits and Vertical Asymptotes
A function f that decreases or increases without bound as x approaches a is said to **tend to infinity** at a. Symbolically, we write

$$\lim_{x \to a} f(x) = \infty$$

if f increases without bound and

$$\lim_{x \to a} f(x) = -\infty$$

if f decreases without bound.
In this case, we say that the graph of f has a **vertical asymptote** at $x = a$.

Warning. It is important to keep in mind that ∞ is not a number, but is merely a symbol denoting unrestricted growth in the magnitude of the function.

Example 8.6
Investigate $\lim_{x \to 0^+} \frac{1}{x}$ and $\lim_{x \to 0^-} \frac{1}{x}$.

Solution.
According to Figure 64, we see that $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. Thus, the y-axis is a vertical asymptote.■

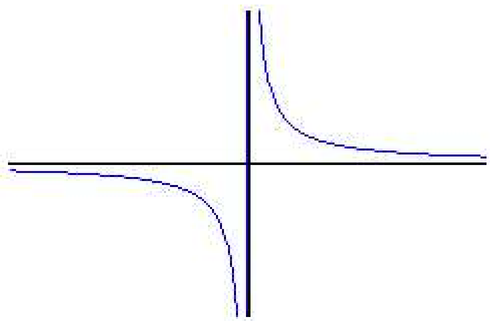


Figure 64

Limits at Infinity and Horizontal Asymptotes
Sometimes we want to know what happens to $f(x)$ as x increases without bound (i.e. $x \to \infty$) or x decreases without bound (i.e., $x \to -\infty$). If $f(x)$ gets as close to a number L as we please when x gets sufficiently large then we write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if $f(x)$ gets as close to a number L as we please when x decreases without bound then we write

$$\lim_{x\to-\infty} f(x) = L.$$

In this case, the graph of $f(x)$ has a **horizontal asymptote** at $y = L$.

Example 8.7
Investigate $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

Solution.
According to Figure 64, we see that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$. Thus, the x-axis is a horizontal asymptote.■

Recommended Problems (pp. 68 - 9): 1, 2, 7, 9, 12, 13, 15, 18, 20, 22, 24, 31, 33, 35, 37, 39.

9 The Concept of Continuity

The goal of this section is to introduce the concept of continuity. Graphically, a function is said to be continuous if its graph has no holes or jumps. Stated differently, a continuous function has a graph which can be drawn without lifting the pencil from the paper.

Continuity at a Point
We say that a function $f(x)$ is **continuous** at $x = a$ if and only if the functional values $f(x)$ get closer and closer to the value $f(a)$ as x is sufficiently close to a. We write

$$\lim_{x \to a} f(x) = f(a).$$

This means, that for any given $\epsilon > 0$ we can find a $\delta > 0$ such that

$$|x - a| < \delta \;\; implies \;\; |f(x) - f(a)| < \epsilon.$$

In words, we say "f is continuous at a" if, for each open interval J containing $f(a)$, we can find an open interval I containing a so that for each point x in I, $f(x)$ lies in the interval J. That is, given $\epsilon > 0$ we can find $\delta > 0$ such that

$$|x - a| < \delta \;\; implies \;\; |f(x) - f(a)| < \epsilon.$$

Example 9.1
(i) Theorem 8.1(f) shows that polynomials $P(x)$ are continuous everywhere.
(ii) Similarly, if $h(x) = \frac{f(x)}{g(x)}$ is a rational function and a is a number such that $g(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$$

That is, $h(x)$ is continuous at numbers that are not zeroes of the function $g(x)$. (iii) Trigonometric functions are continuous at values where they are defined.■

Discontinuity
A function $f(x)$ that is not continuous at $x = a$ is said to be **discontinuous** there. We exhibit three examples of discontinuous functions.

Example 9.2 *(Removable Discontinuity)*
Show that the function $f(x) = \frac{x^2+x-2}{x-1}$ is discontinuous at $x = 1$.

Solution.
Graphing the given function (see Figure 65) er find

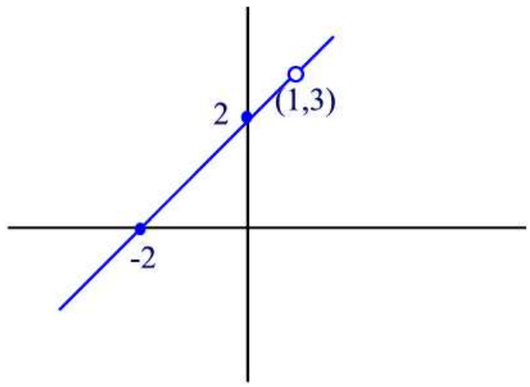


Figure 65

The dot indicates an excluded point on the graph. Thus, we see that $f(1)$ is undefined and therefore $f(x)$ is discontinuous at $x = 1$. Note that

$$\lim_{x \to 1} f(x) = 3.$$

Thus, if we redefine $f(x)$ in such a way that $f(1) = 3$ then we create a continuous function at $x = 1$. That is, the discontinuity is removable.■

Example 9.3 *(Infinite Discontinuity)*
Show that $f(x) = \frac{1}{x^2}$ is discontinuous at $x = 0$.

Solution.
According to Figure 63, we have that $\lim_{x \to 0} \frac{1}{x^2}$ does not exist. Thus, $f(x)$ is discontinuous at $x = 0$. Since $\lim_{x \to 0} f(x) = \infty$ then we call $x = 0$ an infinite discontinuity.■

Example 9.4 *(Jump Discontinuity)*
Show that $f(x) = \frac{|x|}{x}$ is discontinuous at $x = 0$.

Solution.
The fact that $f(x)$ is discontinuous at $x = 0$ follows from Figure 61. ■

The limit properties of Theorem 8.1 can be used to prove the following properties of continuous functions.

Theorem 9.1
If f and g are two continuous functions at $x = a$ and k is a constant then all of the following functions are continuous at $x = a$.

∘ **Scalar Multiple** kf
∘ **Sum and Difference** $f \pm g$
∘ **Product** $f \cdot g$
∘ **Quotient** $\frac{f}{g}$, provided that $g(a) \neq 0$.

Theorem 9.2 *(Continuity of Composite Functions)*
The composition of two continuous functions is continuous.

Proof.
Let g be a function continuous at a and f a function continuous at $g(a)$. we will show that $f \circ g$ is continuous at a. Let $\epsilon > 0$ be given. Since f is continuous at $g(a)$ then there is a $\delta' > 0$ such that

$$|x - g(a)| < \delta' \ \ implies \ \ |f(x) - f(g(a))| < \epsilon.$$

Now, since g is continuous at a then there is a $\delta' > 0$ such that

$$|x - a| < \delta \ \ implies \ \ |g(x) - g(a)| < \delta'.$$

Thus,

$$|x - a| < \delta \ \ implies \ \ |f(g(x)) - f(g(a))| < \epsilon.$$

This completes a proof of the theorem.■

Continuity on an Interval
We say that a function f is **continuous on the open interval** (a, b) if it is continuous at each number in this interval. If in addition, the function is continuous from the right of a, i.e. $\lim_{x \to a^+} f(x) = f(a)$, then we say that f is continuous on the interval $[a, b)$. If f is continuous from the left of b,i.e. $\lim_{x \to b^-} f(x) = f(b)$ then we say that f is continuous on the interval $(a, b]$. Finally, if f is continuous on the open interval (a, b), from the right at a and from the left at b then we say that f is continuous in the interval $[a, b]$.

Example 9.5
Find the interval(s) on which each of the given functions is continuous.

(i) $f_1(x) = \frac{x^2-1}{x^2-4}$.
(ii) $f_2(x) = \csc x$
(iii) $f_3(x) = \sin\left(\frac{1}{x}\right)$.
(iv) $f_5(x) = \begin{cases} 3-x, & \text{if } -5 \leq x < 2 \\ x-2, & \text{if } 2 \leq x < 5 \end{cases}$

Solution.
(i) $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$.
(ii) $x \neq n\pi$ where n is an integer.
(iii) $(-\infty,0) \cup (0,\infty)$ (See Example 8.4).
(iv) Since $\lim_{x\to 2^-} f_5(x) = \lim_{x\to 2^-}(3-x) = 1$ and $\lim_{x\to 2^+}(x-2) = 0$ then f_5 is continuous on the interval $[-5,2) \cup (2,5)$.■

The Intermediate Value Theorem
Continuity can be a very useful tool in solving equations. So if a function is continuous on an interval and changes sign then definitely it has to cross the x-axis. This shows that the function possesses a zero in that interval. This is a special case of the following theorem.

Theorem 9.3 *(Bolzano Theorem)*
Suppose that f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs. Then there is a number $a \leq c \leq b$ such that $f(c) = 0$.

Proof.
Without loss of generality we assume that $f(a) < 0 < f(b)$. The proof uses a result from advanced calculus known as the "Least Upper Bound Axiom":

"If S is a nonempty set of real numbers such that $|x| \leq M$ for all $x \in S$ then there is a number c such that $x \leq c$ for all $x \in S$. Moreover, if d is any number such that $x \leq d$ for all $x \in S$ then $c \leq d$. We call c a least upper bound of S."

Now, let $s = \{a \leq x \leq b : f(x) < 0\}$. Since $f(a) < 0$ then a belongs to S so that S is nonempty. Also, $|x| \leq b$ for all $x \in S$. By the Least upper Bound Axiom there is a least upper bound number $a \leq c \leq b$. We will show that $f(c) = 0$.
Suppose that $f(c) < 0$. Since f is continuous at c and $(2f(c), 0)$ is an open interval around $f(c)$ then there is an open interval $I = (s,t)$ around c such

that $2f(c) < f(x) < 0$ for all x in I. This shows that I is contained in S. Thus, $t \in S$ and $c < t$. This contradicts the definition of c. Hence, $f(c) \geq 0$. A similar argument shows that $f(c)$ cannot be positive. Hence, $f(c) = 0$. This conclude s a proof of the theorem.■

Theorem 9.4 *(Intermediate Value Theorem)*
Let f be a continuous function on $[a, b]$ with $f(a) < f(b)$. If $f(a) < d < f(b)$ then there is $a \leq c \leq b$ such that $f(c) = d$.

Proof.
Define $F(x) = f(x) - d$. Then, clearly, $F(x)$ is continuous on $[a, b]$. Moreover, $F(a) = f(a) - d < 0$ and $F(b) = f(b) - d > 0$. By the Bolzano Theorem there is $a \leq c \leq b$ such that $F(c) = 0$. That is, $f(c) = d$.■

Example 9.6
Show that $\cos x = x^3 - x$ has at least one zero on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.

Solution.
Let $f(x) = \cos x - x^3 + x$. Since $f(\frac{\pi}{4}) \approx 1.008 > 0$ and $f(\frac{\pi}{2}) \approx -2.305 < 0$ then by the IVT there is at least one number c in the interval $(\frac{\pi}{4}, \frac{\pi}{2})$ such that $f(c) = 0$.■

Recommended Problems (pp. 47 - 8): 1, 2, 5, 7, 9, 11, 14, 15, 16, 18, 19.

10 The Derivative at a Point

In this section we introduce the definition of the derivative and its geometrical significance.
In Section 7, we defined the instantaneous rate of change of a function $f(x)$ at a point $x = a$ to be the value that the difference quotient or the average rate of change

$$\frac{f(a+h) - f(a)}{h}$$

approaches over smaller and smaller intervals(i.e. when $h \to 0$). This instantaneous rate of change is called **the derivative of** $f(x)$ **with respect to x at** $x = a$ and will be denoted by $f'(a)$. Thus,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists then we say that f is **differentiable at a**. To **differentiate** a function $f(x)$ at $x = a$ means to find its derivative at the point $(a, f(a))$. The process of finding the derivative of a function is known as **differentiation.**

Example 10.1
Use the definition of the derivative to find $f'(x)$ where $f(x) = \sqrt{x}, x > 0$.

Solution.

$$\begin{array}{rcl} f'(x) & = & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ & = & \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ & = & \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\ & = & \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ & = & \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ & = & \frac{1}{2\sqrt{x}} \blacksquare \end{array}$$

Going back to Figure 58 of Section 7, we conclude that the number $f'(a)$ is the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$. The equation of the tangent line to the graph of $f(x)$ at $x = a$ is then given by the formula

$$y - f(a) = f'(a)(x - a).$$

The equation of the normal line to the graph of $f(x)$ at $x=a$ is given by

$$y - f(a) = -\frac{1}{f'(a)}(x-a),$$

assuming that $f'(a) \neq 0$.

Example 10.2
(i) Find the derivative of the function $f(x) = x^2$ at $x = 1$.
(ii) Write the equation of the tangent line to the graph of f at the point $(1, f(1))$.

Solution.
(i)

$$\begin{array}{rcl} f'(1) & = & \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ & = & \lim_{h\to 0} \frac{(1+h)^2-1}{h} \\ & = & \lim_{h\to 0} \frac{1+2h+h^2-1}{h} \\ & = & \lim_{h\to 0} \frac{h(2+h)}{h} \\ & = & \lim_{h\to 0}(2+h) = 2. \end{array}$$

(ii) The equation of the tangent line is given by

$$y - f(1) = f'(1)(x-1)$$

or in slope-intercept form

$$y = 2x - 1.\blacksquare$$

Example 10.3
Find the equation of the line that is perpendicular to the tangent line to $f(x) = x^2$ at $x = 1$.

Solution.
The equation of the line is given by

$$y = mx + b.$$

Since $m \times f'(1) = -1$ and $f'(1) = 2$ then $m = -\frac{1}{2}$. Thus, $y = -\frac{1}{2}x + b$. Since the line crosses the point $(1,1)$ then $1 = -\frac{1}{2} + b$ or $b = \frac{3}{2}$. Hence, the equation of the normal line is

$$y = -\frac{1}{2}x + \frac{3}{2}.\blacksquare$$

Remark 10.1
By letting $x = a + h$ in the definition of $f'(a)$ we obtain an alternative form of $f'(a)$ which is useful in computations and is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 10.4
Show that the derivative of $f(x) = \sin x$ at $x = 0$ is 1.

Solution.
Using the above remark and Example 7.3 of Section 7 we find

$$\begin{array}{rcl} f'(0) & = & \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\ & = & \lim_{x\to 0} \frac{\sin x - \sin 0}{x} \\ & = & \lim_{x\to 0} \frac{\sin x}{x} = 1 \blacksquare \end{array}$$

Recommended Problems (pp. 76 - 8): 1, 4, 7, 8, 11, 13, 15, 17, 19, 20, 22, 24, 25, 29, 30, 33.

11 The Derivative Function

Recall that a function f is differentiable at x if the following limit exists

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3}$$

Thus, we associate with the function f, a new function f' whose domain is the set of points x at which the limit (31.1) exists. We call the function f' the **derivative function** of f.

The Derivative Function Graphically
Since the derivative at a point represents the slope of the tangent line then one can obtain the graph of the derivative function from the graph of the original function. It is important to keep in mind the relationship between the graphs of f and f'. If $f'(x) > 0$ then the tangent line must be tilted upward and the graph of f is rising or increasing. Similarly, if $f'(x) < 0$ then the tangent line is tilted downward and the graph of f is falling or decreasing. If $f'(a) = 0$ then the tangent line is horizontal at $x = a$.

Example 11.1
Sketch the graph of the derivative of the function shown in Figure 66.

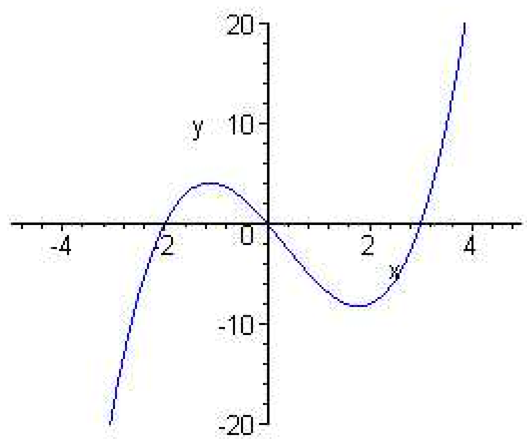


Figure 66

Solution.
Note that for $x < -1.12$ the derivative is positive and getting less and less positive. At $x \approx -1.12$ we have $f'(-1.12) = 0$. For $-1.12 < x < 0$ the

derivative is negative and getting more and more negative till reaching $x = 0$. For $0 < x < 1.79$ the derivative is less and less negative and at $x = 1.79$ we have $f'(1.79) = 0$. Finally, for $x > 1.79$ the derivative is getting more and more positive. Thus, a possible graph of f' is given in Figure 67. ■

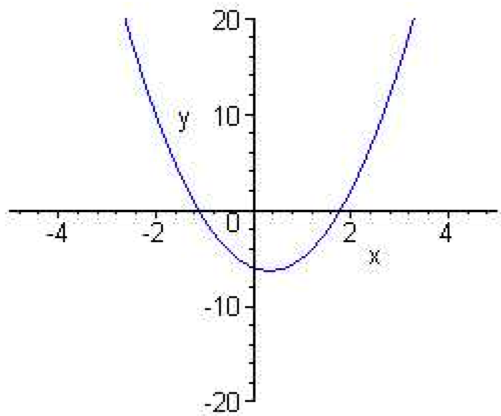


Figure 67

The Derivative Function Numerically
Here, we want to estimate the derivative of a function defined by a table. The derivative can be estimated by using the average rate of change or the difference quotient
$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$
If a is a left-endpoint then $f'(a)$ is estimated by
$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$
where $b > a$. If a is a right-endpoint then $f'(a)$ is estimated by
$$f'(a) \approx \frac{f(a) - f(b)}{a - b}$$
where $b < a$. If a is an interior point then $f'(a)$ is estimated by
$$f'(a) \approx \frac{1}{2}\left(\frac{f(a) - f(b)}{a - b} + \frac{f(c) - f(a)}{c - a}\right)$$
where $b < a < c$.

Example 11.2
Find approximate values for $f'(x)$ at each of the x-values given in the following table

x	0	5	10	15	20
f(x)	100	70	55	46	40

Solution.

$$\begin{array}{rcccr} f'(0) & \approx & \frac{f(5)-f(0)}{5} & = & -6 \\ f'(5) & \approx & \frac{1}{2}\left(\frac{f(10)-f(5)}{5}+\frac{f(5)-f(0)}{5}\right) & = & -4.5 \\ f'(10) & \approx & \frac{1}{2}\left(\frac{f(15)-f(10)}{5}+\frac{f(10)-f(5)}{5}\right) & = & -2.4 \\ f'(15) & \approx & \frac{1}{2}\left(\frac{f(20)-f(15)}{5}+\frac{f(15)-f(10)}{5}\right) & = & -1.5 \\ f'(20) & \approx & \frac{f(20)-f(15)}{5} & = & -1.2 \blacksquare \end{array}$$

The Derivative Function From a Formula
Now, if a formula for f is given then by applying the definition of $f'(x)$ as the limit of the difference quotient we can find a formula of f' as shown in the following two problems.

Example 11.3 *(Derivative of a Constant Function)*
Suppose that $f(x) = k$ for all x. Find a formula for $f'(x)$.

Solution.

$$\begin{array}{rcl} f'(x) & = & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ & = & \lim_{h\to 0} \frac{k-k}{h} = 0. \end{array}$$

Thus, $f'(x) = 0.\blacksquare$

Example 11.4 *(Derivative of a Linear Function)*
Find the derivative of the linear function $f(x) = mx + b$.

Solution.

$$\begin{array}{rcl} f'(x) & = & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ & = & \lim_{h\to 0} \frac{m(x+h)+b-(mx+b)}{h} \\ & = & \lim_{h\to 0} \frac{mh}{h} = m. \end{array}$$

Thus, $f'(x) = m.\blacksquare$

Recommended Problems (pp. 83 - 5): 1, 3, 5, 8, 11, 15, 16, 19, 21, 27, 28, 29, 33, 35, 40.

12 Leibniz Notation for The Derivative

When dealing with mathematical models that involve derivatives it is convenient to denote the prime (or Newton) notation of the derivative of a function $y = f(x)$ by $\frac{dy}{dx}$. That is,

$$\frac{dy}{dx} = f'(x).$$

This notation is called **Leibniz notation** (due to W.G. Leibniz). For example, we can write $\frac{dy}{dx} = 2x$ for $y' = 2x$.
When using Leibniz notation to denote the value of the derivative at a point a we will write

$$\left.\frac{dy}{dx}\right|_{x=a}$$

Thus, to evaluate $\frac{dy}{dx} = 2x$ at $x = 2$ we would write

$$\left.\frac{dy}{dx}\right|_{x=2} = 2x|_{x=2} = 2(2) = 4.$$

Remark 12.1
When you think about it, the Leibniz notation better indicates what is going on when you take a derivative than does the Newton notation. For one thing, it clearly shows that a derivative of a function is taken with respect to a particular independent variable. This will prove to be handy when we deal with the applications of the derivative in Section 29.

One of the advantages of Leibniz notation is the recognition of the units of the derivative. For example, if the position function $s(t)$ is expressed in meters and the time t in seconds then the units of the velocity function $\frac{ds}{dt}$ are meters/sec.
In general, the units of the derivative are the units of the dependent variable divided by the units of the independent variable.

Example 12.1
The cost, C (in dollars) to produce x gallons of ice cream can be expressed as $C = f(x)$. What are the units of measurements and the meaning of the statement $\left.\frac{dC}{dx}\right|_{x=200} = 1.4$?

Solution.
$\frac{dC}{dx}$ is measured in dollars per gallon. The notation

$$\left.\frac{dC}{dx}\right|_{x=200} = 1.4$$

means that if 200 gallons of ice cream have already been produced then the cost of producing the next gallon will be roughly 1.4 dollars.■

Recommended Problems (pp. 88 - 9): 1, 3, 5, 7, 9, 11, 13, 15, 16, 19.

13 The Second Derivative

Let $f(x)$ be a differentiable function. If the limit

$$\lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

exists then we say that the function $f'(x)$ is differentiable and we denote its derivative by $f''(x)$ or using Leibniz notation

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

We call $f''(x)$ the **second derivative** of $f(x)$.

Now, recall that if $f'(x) > 0$ (resp. $f'(x) < 0$) over an interval I then the function $f(x)$ is increasing (resp. decreasing on I). So if $f''(x) > 0$ on I then $f'(x)$ is increasing on I. This means that the slope of the tangent line to the graph of $f(x)$ is either getting more and more positive or less and less negative. This occurs only when the graph of f is bending up on the interval I. In this case, we say that f is **concave up** on I. Similarly, if $f''(x) < 0$ on I then $f'(x)$ is decreasing. This means that the slope of the tangent line to the graph of f is either getting less and less positive or more and more negative. This occurs when the graph of f is bending down. In this case, we say that f is **concave down.**

Remark 13.1
Note that when a curve is concave up then the tangent lines lie below the curve whereas when it is concave down then the tangent lines lie above the curve.

Example 13.1
Give the signs of f' and f'' for the function whose graph is given in Figure 68.

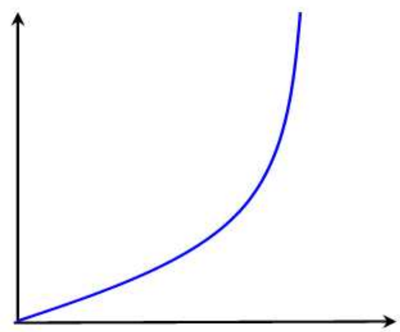


Figure 68

Solution.
Since f is always increasing then f' is always positive. Since the graph is concave up then f'' is always positive.■

Example 13.2
Find where the graph of $f(x) = x^3 + 3x + 1$ is concave up and where it is concave down.

Solution.
Finding the first and second derivatives of f we obtain $f'(x) = 3x^2 + 3$ and $f''(x) = 6x$. Thus, the graph of f is concave up for $x > 0$ and concave down for $x < 0$.■

As an application to the second derivative, we consider the motion of an object determined by the position function $s(t)$. Recall that the velocity of the object is defined to be the first derivative of $s(t)$, i.e.

$$v(t) = s'(t) = \frac{ds}{dt}$$

and the absolute value of $v(t)$ is the speed. When the object speeds up we say that he/she accelerates and when the object slows down we say that he/she decelerates. We define the **acceleration** of an object as the derivative of the velocity function and consequently as the second derivative of the position function

$$a(t) = \frac{d^2s}{dt^2} = \frac{ds}{dt}.$$

Example 13.3
A particle is moving along a straight line. If its distance, s, to the right of a fixed point is given by Figure 69, estimate:

(a) When the particle is moving to the right and when it is moving to the left.
(b) When the particle has positive acceleration and when it has negative acceleration.

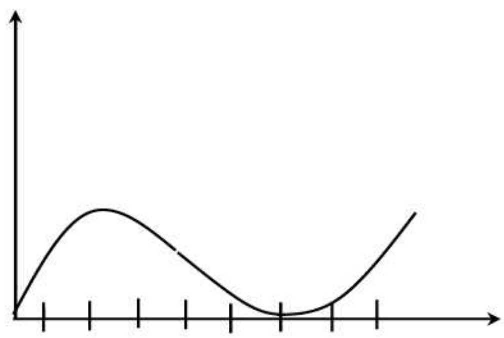


Figure 69

Solution.
(a) When s is increasing (i.e. $v > 0$) then the particle moves to the right. This occurs when $0 < t < \frac{2}{3}$ and for $t > 2$. On the other hand, the particle moves to the left when s is decreasing (i.e. $v < 0$). This happens when $\frac{2}{3} < t < 2$.
(b) Positive acceleration occurs when the graph is concave up. This occurs when $t > \frac{4}{3}$. The particle has negative acceleration when the curve is concave down, i.e for $t < \frac{4}{3}$.■

Recommended Problems (pp. 93 - 4): 1, 2, 3, 4, 5, 9, 10, 11, 13, 15, 20, 22, 23.

14 Continuity and Differentiability

So far we have been discussing functions that are differentiable. What about nondifferentiable functions? How do we know when a function is not differentiable?
A function fails to be differentiable at a point if:

• the point is a sharp corner point. In this case, the left-hand derivative and the right-hand derivative are different and therefore the limit of the difference quotient does not exist.
• The tangent line is vertical at the point since vertical lines have no slopes.
• The function is discontinuous at a point. (See Theorem below.)

Example 14.1
Show that the function $f(x) = |x|$ is not differentiable at $x = 0$. This is an example of a nondifferentiable function at a corner point.

Solution.
$f'(0)$ would exist if the following limit exists and is equal to $f'(0)$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

According to Figure 61 of Section 8 of these notes, $\lim_{h\to 0^-} \frac{|h|}{h} = -1$ whereas $\lim_{h\to 0^+} \frac{|h|}{h} = 1$. Thus, $\lim_{h\to 0} \frac{|h|}{h}$ does not exist. This shows that $f(x)$ is not differentiable at $x = 0$.■

Example 14.2
Show that $f(x) = x^{\frac{1}{3}}$ is not differentiable at $x = 0$. This is an example of a nondifferentiable function at a point where the tangent line is vertical.

Solution.
Figure 70 shows the graph of $f(x)$.

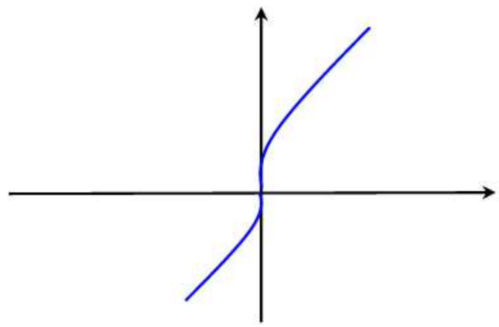


Figure 70

Notice that at $x = 0$ the tangent line is vertical. Looking at the difference quotient at $x = 0$ we find

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}} = \infty.$$

Thus, $f(x)$ has a vertical tangent at $x = 0$ and $f'(0)$ does not exist.■

Example 14.3
Show that the piecewise defined function

$$f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 3x-1 & \text{if } x > 1 \end{cases}$$

is not differentiable at $x = 1$. This shows an example of a nondifferentiable function at a sharp corner point.

Solution.
Finding the limit of the difference quotient from both the left of 1 and the right of 1 we obtain

$$\lim_{h\to 0^-}\frac{1+h+1-2}{h} = \lim_{h\to 0^-}\frac{h}{h} = 1$$

and

$$\lim_{h\to 0^+}\frac{3+3h-1-2}{h} = \lim_{h\to 0^+}\frac{3h}{h} = 3.$$

Thus,

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

does not exist and consequently, $f'(1)$ does not exist.■

Example 14.4

$$f(x)=\begin{cases} -x^2+5 & \text{if } x\le 2\\ x-2 & \text{if } x>2\end{cases}$$

is not differentiable at $x=2$. This shows an example of a nondifferentiable function at a point of discontinuity.

Solution.
Finding the left hand derivative we obtain

$$\begin{aligned}\lim_{h\to 0^-}\tfrac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-}\tfrac{5-(2+h)^2-(5-4)}{h}\\ &= \lim_{h\to 0^-}(-h-4)=-4.\end{aligned}$$

Similarly, the right hand derivative is

$$\begin{aligned}\lim_{h\to 0^+}\tfrac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+}\tfrac{(2+h)-2-1}{h}\\ &= \lim_{h\to 0^+}\tfrac{h-1}{h}=-\infty.\end{aligned}$$

It follows that $f'(2)$ does not exist.■

Now, if the graph of a function has a tangent line at a point, then we would expect to draw the graph continuously (without the pencil leaving the paper) near the point. This suggests that a differentiable function at a point is continuous at that point. We state this result as a theorem and then we provide a proof.

Theorem 14.1
If a function $f(x)$ is differentiable at $x=a$ then it is continuous there.

Proof.
Since $f'(a)$ exists then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a).$$

Thus,

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left(\tfrac{f(x)-f(a)}{x-a}\right)(x-a)=f'(a)\cdot 0=0$$

That is, $\lim_{x\to a}f(x)=f(a)$ and this shows that f is continuous at $x=a$.■

Remark 14.1
According to Problem 14.3, a continuous function need not be differentiable. That is, the converse of the above theorem is not true in general. So be careful not to consider all continuous functions to be differentiable.

Recommended Problems (pp. 98 - 9): 1, 3, 5, 7, 9, 10.

15 Derivatives of Power and Polynomial Functions

Finding the derivative function by using the limit of the difference quotient is sometimes difficult for functions with complicated expressions. Fortunately, there is an indirect way for computing derivatives that does not compute limits but instead uses formulas which we will derive in this section and in the coming sections.
We first derive a couple of formulas of differentiation.

Theorem 15.1
If f is differentiable and k is a constant then the new function $kf(x)$ is differentiable with derivative given by

$$[kf(x)]' = kf'(x).$$

Proof.

$$\begin{array}{rcl} [kf'(x) & = & \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h} = \lim_{h\to 0}\frac{k(f(x+h)-f(x))}{h} \\ & = & k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = kf'(x) \end{array}$$

where we used the fact that a constant can be taking across the limit sign by the properties of limits.■

Theorem 15.2
If $f(x)$ and $g(x)$ are two differentiable functions then the functions $f+g$ and $f-g$ are also differentiable with derivatives

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

Proof.
Again by using the definition of the derivative and the fact that the limit of a sum/difference is the sum/difference of limits we find

$$\begin{array}{rcl} [f(x)+g(x)]' & = & \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \\ & = & \lim_{h\to 0}\frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h} \\ & = & \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x). \end{array}$$

The same proof is valid for the difference formula.■

Next, we state and give a partial proof of a rule for finding the derivative of a power function of the form $f(x) = x^n$.

Theorem 15.3 *(Power Rule)*
For any real number n, the derivative of the function $y = x^n$ is given by the formula

$$\frac{dy}{dx} = nx^{n-1}$$

Proof.
We prove the result when n is a positive integer. We start by writing the definition of the derivative of any function $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Letting $h = ax - x$ we can rewrite the previous definition in the form

$$f'(x) = \lim_{a \to 1} \frac{f(ax) - f(x)}{ax - x}.$$

Thus,

$$f'(x) = \lim_{a \to 1} \frac{(ax)^n - x^n}{ax - x} = x^{n-1} \lim_{a \to 1} \frac{a^n - 1}{a - 1}.$$

Dividing $a^n - 1$ by $a - 1$ by the method of synthetic division we find

$$a^n - 1 = (a-1)(1 + a + a^2 + a^3 + \cdots + a^{n-1}).$$

Thus,

$$f'(x) = x^{n-1} \lim_{a \to 1} (1 + a + a^2 + \cdots + a^{n-1}) = nx^{n-1}. \blacksquare$$

Example 15.1
Use the power rule to differentiate the following:

(a) $y = x^{\frac{4}{3}}$ (b) $y = \frac{1}{\sqrt[3]{x}}$ (c) $y = x^{\pi}$.

Solution.
(a) Using the power rule with $n = \frac{4}{3}$ to obtain $y' = \frac{4}{3}x^{\frac{1}{3}}$.
(b) Since $y = x^{-\frac{1}{3}}$ then using the power rule with $n = -\frac{1}{3}$ to obtain $y' = -\frac{1}{3}x^{-\frac{4}{3}}$.
(c) Using the power rule with $n = \pi$ to obtain $y' = \pi x^{\pi - 1}$.$\blacksquare$

Remark 15.1
The derivative of a function of the form $y = 2^x$ is not $y' = x2^{x-1}$ because $y = 2^x$ is an exponential function and not a power function. A formula for finding the derivative of an exponential function will be discussed in the next section.

Now, combining the results discussed above, we can find the derivative of functions that are combinations of power functions of the form ax^n. In particular, the derivative of a polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is given by the formula

$$f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

Example 15.2
Find the derivative of the function $y = \sqrt{3}x^7 - \frac{x^5}{5} + \pi$.

Solution.
The derivative is $f'(x) = 7\sqrt{3}x^6 - x^4$.■

Example 15.3
Find the second derivative of $y = 5\sqrt[3]{x} - \frac{10}{x^4} + \frac{1}{2\sqrt{x}}$.

Solution.
Note that the given function can be written in the form $y = 5x^{\frac{1}{3}} - 10x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}$. Thus, the first derivative is

$$y' = \frac{5}{3}x^{-\frac{2}{3}} + 40x^{-5} - \frac{1}{4}x^{-\frac{3}{2}}.$$

The second derivative is

$$y'' = -\frac{10}{9}x^{-\frac{5}{3}} - 200x^{-6} + \frac{3}{8}x^{-\frac{5}{2}}.■$$

Recommended Problems (pp. 111 - 3): 1, 2, 5, 9, 10, 12, 13, 14, 22, 24, 26, 29, 35, 43, 45, 47, 49, 53, 57.

16 Derivatives of Exponential Functions

We start this section by looking at the limit

$$\lim_{h\to 0}\frac{e^h-1}{h}.$$

The chart below suggests that the limit is 1.

h	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$\frac{e^h-1}{h}$	0.995	0.9995	0.99995	undefined	1.0000	1.0005	1.005

Now, let's try and find the derivative of the function $f(x) = e^x$ at any number x. By the definition of the derivative and the limit above we see that

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x. \end{aligned}$$

This means that e^x is its own derivative:

$$\frac{d}{dx}(e^x) = e^x.$$

Now, suppose that the x in e^x is replaced by a differentiable function of x, say $u(x)$. We would like to find the derivative of e^u with respect to x, i.e., what is $\frac{d}{dx}(e^u)$?

Theorem 16.1

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}.$$

Proof.
By the definition of the derivative we have

$$\frac{d}{dx}(e^u) = \lim_{h\to 0}\frac{e^{u(x+h)}-e^{u(x)}}{h}.$$

Since u is differentiable at x then by letting

$$v = \frac{u(x+h)-u(x)}{h} - u'(x)$$

we find

$$u(x+h) = u(x) + (v + u'(x))h$$

with $\lim_{h\to 0} v = 0$. Similarly, we can write

$$e^{y+k} = e^y + (w + e^y)k$$

with $\lim_{k\to 0} w = 0$. In particular, letting $y = u(x)$ and $k = (v + u'(x))h$ we find

$$e^{u(x)+(v+u'(x))h} = e^{u(x)} + (w + e^{u(x)})(v + u'(x))h.$$

Hence,

$$\begin{array}{rcl} e^{u(x+h)} - e^{u(x)} & = & e^{u(x)+(v+u'(x))h} - e^{u(x)} \\ & = & e^{u(x)} + (w + e^{u(x)})(v + u'(x))h - e^{u(x)} \\ & = & (w + e^u)(v + u'(x))h \end{array}$$

Thus,

$$\begin{array}{rcl} \frac{d}{dx}(e^u) & = & \lim_{h\to 0} \frac{e^{u(x+h)} - e^{u(x)}}{h} \\ & = & \lim_{h\to 0}(w + e^u)(v + u'(x)) \\ & = & e^u u' \ \blacksquare \end{array}$$

We end this section, by finding the derivative of the function $f(x) = a^x$, where $a > 0$ and $a \neq 1$. First, note that $f(x) = a^x = (e^{\ln a})^x = e^{x\ln a}$. Thus, by Theorem 16.1 we see that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a}\frac{d}{dx}(x\ln a) = a^x \ln a.$$

Example 16.1
Find the derivative of each of the following functions:

(a) $f(x) = 3^x$ (b) $y = 2 \cdot 3^x + 5 \cdot e^{3x-4}$.

Solution.

(a) $f'(x) = 3^x \ln 3$.
(b) $y' = 2(3^x)' + 5(e^{3x-4})' = 2 \cdot 3^x \ln 3 + 5(3)e^{3x-4} = 2 \cdot 3^x \ln 3 + 15 \cdot e^{3x-4}$.$\blacksquare$

Recommended Problems (pp. 116 - 7): 1, 4, 8, 10, 12, 16, 21, 24, 26, 27, 29, 30, 32, 36, 40, 43, 45.

17 The Product and Quotient Rules

At this point we don't have the tools to find the derivative of either the function $f(x) = x^3e^{x^2}$ or the function $g(x) = \frac{x^2}{e^x}$. Looking closely at the function $f(x)$ we notice that this function is the product of two functions, namely, x^3 and e^{x^2}. On the other hand, the function $g(x)$ is the ratio of two functions. Thus, we hope to have a rule for differentiating a product of two functions and one for differentiating the ratio of two functions.
We start by finding the derivative of the product $u(x)v(x)$, where u and v are differentiable functions:

$$\begin{array}{rcl}(u(x)v(x))' &=& \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\ &=& \lim_{h\to 0}\frac{u(x+h)(v(x+h)-v(x))+v(x)(u(x+h)-u(x))}{h}\\ &=& \lim_{h\to 0}u(x+h)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\\ &=& u(x)v'(x)+u'(x)v(x).\end{array}$$

Note that since u is differentiable so it is continuous and therefore

$$\lim_{h\to 0}u(x+h) = u(x).$$

The formula

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{d}{dx}(v(x)) + \frac{d}{dx}(u(x))v(x). \tag{4}$$

is called the **product rule**.

Example 17.1
Find the derivative of $f(x) = x^3e^{x^2}$.

Solution.
Let $u(x) = x^3$ and $v(x) = e^{x^2}$. Then $u'(x) = 3x^2$ and $v'(x) = 2xe^{x^2}$. Thus, by the product rule we have

$$f'(x) = x^3(2x)e^{x^2} + 3x^2e^{x^2} = 2x^4e^{x^2} + 3x^2e^{x^2}.\blacksquare$$

The **quotient rule** is obtained from the product rule as follows: Let $f(x) = \frac{u(x)}{v(x)}$. Then $u(x) = f(x)v(x)$. Using the product rule, we find $u'(x) = f(x)v'(x) + f'(x)v(x)$. Solving for $f'(x)$ to obtain

$$f'(x) = \frac{u'(x) - f(x)v'(x)}{v(x)}.$$

Now replace $f(x)$ by $\frac{u(x)}{v(x)}$ to obtain

$$\begin{aligned}\left(\frac{u(x)}{v(x)}\right)' &= \frac{u'(x)-\frac{u(x)}{v(x)}v'(x)}{v(x)}\\ &= \frac{\frac{u'(x)v(x)-u(x)v'(x)}{v(x)}}{v(x)}\\ &= \frac{u'(x)v(x)-u(x)v'(x)}{(v(x))^2}.\end{aligned}$$

Example 17.2
Find the derivative of $g(x) = \frac{x^2}{e^x}$.

Solution.
Let $u(x) = x^2$ and $v(x) = e^x$. Then by the quotient rule we have

$$\begin{aligned}f'(x) &= \frac{(x^2)'e^x - x^2(e^x)'}{(e^x)^2}\\ &= \frac{2xe^x - x^2e^x}{e^{2x}} \blacksquare\end{aligned}$$

Example 17.3
Prove the power rule for integer exponents.

Solution.
In Section 15, we proved the result for positive integers. The result is trivially true when the exponent is zero. So suppose that $y = x^n$ with n a negative integer. Then $y = \frac{1}{x^{-n}}$ where $-n$ is a positive integer. Applying both the quotient rule and the power rule we find

$$y' = \frac{(0)(x^{-n}) - (-nx^{-n-1})}{x^{-2n}} = nx^{n-1}.\blacksquare$$

Recommended Problems (pp. 121 - 2): 1, 5, 9, 11, 22, 25, 29, 31, 33, 38, 41, 43, 44.

18 The Chain Rule

In this section we want to find the derivative of a composite function $f(g(x))$ where $f(x)$ and $g(x)$ are two differentiable functions.

Theorem 18.1
If f and g are differentiable then $f(g(x))$ is differentiable with derivative given by the formula

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

This result is known as the **chain rule**. Thus, the derivative of $f(g(x))$ is the derivative of $f(x)$ evaluated at $g(x)$ times the derivative of $g(x)$.

Proof.
By the definition of the derivative we have

$$\frac{d}{dx}f(g(x)) = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}.$$

Since g is differentiable at x then by letting

$$v = \frac{g(x+h) - g(x)}{h} - g'(x)$$

we find

$$g(x+h) = g(x) + (v + g'(x))h$$

with $\lim_{h\to 0} v = 0$. Similarly, we can write

$$f(y+k) = f(y) + (w + f'(y))k$$

with $\lim_{k\to 0} w = 0$. In particular, letting $y = g(x)$ and $k = (v + g'(x))h$ we find

$$f(g(x) + (v + g'(x))h) = f(g(x)) + (w + f'(g(x)))(v + g'(x))h.$$

Hence,

$$\begin{array}{rcc} f(g(x+h)) - f(g(x)) & = & f(g(x) + (v + g'(x))h) - f(g(x)) \\ & = & f(g(x)) + (w + f'(g(x)))(v + g'(x))h - f(g(x)) \\ & = & (w + f'(g(x)))(v + g'(x))h \end{array}$$

Thus,

$$\begin{array}{rcl} \frac{d}{dx}f(g(x)) & = & \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} \\ & = & \lim_{h\to 0}(w+f'(g(x)))(v+g'(x)) \\ & = & f'(g(x))g'(x). \end{array}$$

This completes a proof of the theorem. ■

Example 18.1
Find the derivative of $y = (4x^2 + 1)^7$.

Solution.
First note that $y = f(g(x))$ where $f(x) = x^7$ and $g(x) = 4x^2 + 1$. Thus, $f'(x) = 7x^6$, $f'(g(x)) = 7(4x^2+1)^6$ and $g'(x) = 8x$. So according to the chain rule, $y' = 7(4x^2 + 1)^6(8x) = 56x(4x^2 + 1)^6$.■

Example 18.2
Find the derivative of $f(x) = \frac{x}{x^2+1}$.

Solution.
We already know one way to find the derivative of this function which is the use of the quotient rule. Another way, is to use the product rule combined with the chain rule since $f(x) = x(x^2 + 1)^{-1}$.

$$\begin{array}{rcl} f'(x) & = & (x)'(x^2+1)^{-1} + x[(x^2+1)^{-1}]' \\ & = & (x^2+1)^{-1} - x(x^2+1)^{-2}(2x) \\ & = & \frac{1}{x^2+1} - \frac{2x^2}{(x^2+1)^2} \end{array}$$ ■

Example 18.3
Prove the power rule for rational exponents.

Solution.
Suppose that $y = x^{\frac{p}{q}}$, where p and q are integers with $q > 0$. Take the qth power of both sides to obtain $y^q = x^p$. Differentiate both sides with respect to x to obtain $qy^{q-1}y' = px^{p-1}$. Thus,

$$y' = \frac{p}{q}\frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

Note that we are assuming that x is chosen in such a way that $x^{\frac{p}{q}}$ is defined.■

Example 18.4
Show that $\frac{d}{dx}x^n = nx^{n-1}$ for $x > 0$ and n is any real number.

Solution.
Since $x^n = e^{n\ln x}$ then

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = e^{n\ln x} \cdot \frac{n}{x} = nx^{n-1}.\blacksquare$$

Recommended Problems (pp. 126 - 7): 1, 8, 11, 13, 18, 27, 31, 32, 38, 42, 49, 51, 53, 54, 55, 57, 64, 70.

19 Derivatives of Trigonometric Functions

The goal of this section is to find the derivatives of the six trigonometric functions.
We start by finding the derivative of $y = \sin x$. For this purpose, we remind the reader of the following trigonometric identity:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Using the definition of derivative, the above identity and the fact that $\lim_{h\to 0} \frac{\sin h}{h} = 1, \lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ we find

$$\begin{array}{rcl} y' &=& \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\ &=& \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &=& \lim_{h\to 0} \frac{\sin x(\cos h - 1)+\cos x \sin h}{h} \\ &=& \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = \cos x. \end{array}$$

Now, if $y = \sin u$ where u is a function of x then by the chain rule we have

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}.$$

As a result of this rule and the fact that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ and $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$ we can obtain the derivative of $\cos x$:

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$

If u is a function of x then by the chain rule

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}.$$

Example 19.1
Differentiate: (a) $2\sin(3x)$ (b) $\cos(x^2)$ (c) $e^{\sin x}$.

Solution.
(a) $\frac{d}{dx}(2\sin(3x)) = 2\cos(3x)(3x)' = 6\cos(3x)$.
(b) $\frac{d}{dx}(\cos(x^2)) = -\sin(x^2)(x^2)' = -2x\sin(x^2)$.
(c) $\frac{d}{dx}(e^{\sin x}) = e^{\sin x}(\sin x)' = \cos x e^{\sin x}$.$\blacksquare$

Example 19.2
(a) Use the quotient rule to find the derivative of the function $\sec x = \frac{1}{\cos x}$. Then use the chain rule to find $\frac{d}{dx}(\sec u)$.
(b) Use the quotient rule to find the derivative of the function $\csc x = \frac{1}{\sin x}$. Then use the chain rule to find $\frac{d}{dx}(\csc u)$.
(c) Use the quotient rule to find the derivative of the function $\tan x = \frac{\sin x}{\cos x}$. Then use the chain rule to find $\frac{d}{dx}(\tan u)$.
(d) Use the quotient rule to find the derivative of the function $\cot x = \frac{1}{\tan x}$. Then use the chain rule to find $\frac{d}{dx}(\cot u)$.

Solution.
(a) Since $\sec x = \frac{1}{\cos x}$ then by the quotient rule we have

$$(\sec x)' = \frac{(1)'\cos x - (1)(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

By the chain rule

$$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}.$$

(b) Since $\csc x = \frac{1}{\sin x}$ then by the quotient rule we have

$$(\csc x)' = \frac{(1)'\sin x - (1)(\sin x)'}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\csc x \cot x.$$

By the chain rule

$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}.$$

(c) Since $\tan x = \frac{\sin x}{\cos x}$ then by the quotient rule we have

$$(\tan x)' = \frac{(\sin x)'\cos x - (\sin x)(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

By the chain rule

$$\frac{d}{dx}(\tan u) = \frac{1}{\cos^2 u}\frac{du}{dx}.$$

(d) Since $\cot x = \frac{1}{\tan x}$ then by the quotient rule we have

$$(\cot x)' = \frac{(1)'\tan x - (1)(\tan x)'}{\tan^2 x} = \frac{-\frac{1}{\cos^2 x}}{\tan^2 x} = -\frac{1}{\sin^2 x}.$$

By the chain rule

$$\frac{d}{dx}(\cot u) = -\frac{1}{\sin^2 u}\frac{du}{dx}. \blacksquare$$

Remark 19.1
When differentiating trigonometric functions, the domain must be given in radians. Otherwise, conversion and possibly an application of the chain rule will be required. For example, if $f(x) = \sin x$, and x is in degrees, then $f'(x) = \frac{\pi}{180} \cos x$.

Recommended Problems (pp. 131 - 3): 6, 9, 15, 17, 27, 28, 33, 37, 39, 41, 44, 45, 51.

20 Applications of the Chain Rule

In this section we will use the chain rule to find the derivatives of logarithmic and inverse trigonometric functions.
To find the derivative of $\ln x$ we recall the result

$$\frac{d}{dx}(e^u) = e^u \frac{d}{dx}(u).$$

As a consequence of this rule, we can find the derivative of the function $\ln x$. Indeed, since $e^{\ln x} = x$ then differentiating both sides to obtain

$$e^{\ln x} \cdot \frac{d}{dx}(\ln x) = 1.$$

Solving for $\frac{d}{dx}(\ln x)$ we find

$$\frac{d}{dx}(\ln x) = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Example 20.1
(a) Use the chain rule to find the derivative of $\ln u$ where u is a function of x.
(b) What is the derivative of $\log_a x$? $\log_a u$? where u is a function of x.

Solution.
(a) By the chain rule

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}.$$

(b) Since $\log_a x = \frac{\ln x}{\ln a}$ then $(\log_a x)' = \frac{1}{x \ln a}$. By the chain rule

$$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}\frac{du}{dx}.\blacksquare$$

Example 20.2
Differentiate: (a) $\ln(x^2+1)$ (b) $\sqrt{1+\ln(1-y)}$.

Solution.
(a) By the chain rule

$$(\ln(x^2+1))' = \frac{(x^2+1)'}{x^2+1} = \frac{2x}{x^2+1}.$$

(b) Since $\sqrt{1+\ln(1-y)} = [1+\ln(1-y)]^{\frac{1}{2}}$ then by the chain rule

$$\sqrt{1+\ln(1-y)} = \frac{1}{2}[1+\ln(1-y)]^{-\frac{1}{2}}(1+\ln(1-y))' = \frac{1}{2}[1+\ln(1-y)]^{-\frac{1}{2}}\frac{-y'}{1-y}. \blacksquare$$

Next, we find the derivatives of the inverse trigonometric functions.
Recall that $y = \arcsin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, if and only if $\sin y = x$. Differentiating both sides of this equality with respect to x to obtain

$$\cos y \cdot y' = 1$$

or

$$y' = \frac{1}{\cos y}.$$

Using the trigonometric identity $\cos(\arcsin x) = \sqrt{1-x^2}$ we see that

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

Example 20.3
(a) Find the derivative of $\arccos x, 0 \le x \le \pi$.
(b) Find the derivative of $\arccos u$.
(c) Find the derivative of $\arctan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$.
(d) Find the derivative of $\arctan u$, where u is a function of x.

Solution.
(a) $y = \arccos x, 0 \le x \le \pi$, if and only if $\cos y = x$. Differentiating both sides of this equality with respect to x to obtain

$$-\sin y \cdot y' = 1$$

or

$$y' = -\frac{1}{\sin y}.$$

Using the trigonometric identity $\sin(\arccos x) = \sqrt{1-x^2}$ we see that

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}.$$

(b) By (a) and the chain rule we have

$$\frac{d}{dx}(\arccos u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

(c) $y = \arctan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$, if and only if $\tan y = x$. Differentiating both sides of this equality with respect to x to obtain

$$\frac{1}{\cos^2 y} \cdot y' = 1$$

or

$$y' = \cos^2 y.$$

Using the trigonometric identity $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ we see that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

(d) By (c) and the chain rule we have

$$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx}.\blacksquare$$

Example 20.4
Differentiate: (a) $\arctan(x^2)$ (b) $\arcsin(\tan x)$.

Solution.
(a) By (d) of the previous exercise we have

$$(\arctan(x^2))' = \frac{(x^2)'}{1+(x^2)^2} = \frac{2x}{1+x^4}.$$

(b)

$$(\arcsin(\tan x))' = \frac{1}{\sqrt{1-\tan^2 x}}(\tan x)' = \frac{1}{\sqrt{1-\tan^2 x}}\frac{1}{\cos^2 x}.\blacksquare$$

Recommended Problems (pp. 136 - 8): 1, 4, 7, 8, 10, 15, 19, 23, 26, 29, 36, 39, 43, 44, 46, 49.

21 Implicit Differentiation

So far functions have been defined **explicitly**, that is, they can be written in the form $y = f(x)$ such as $y = \sqrt{1-x^2}, -1 \leq x \leq 1$. This same function can be defined **implicitely** by the equation $x^2 + y^2 = 1$ where $0 \leq y \leq 1$. To find the derivative of the explicit form we use the chain rule to obtain

$$\frac{d}{dx}(1-x^2)^{\frac{1}{2}} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

To find the derivative of the implicit form we start by differentiating both sides of the equation with respect to x to obtain

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving this equation for $\frac{dy}{dx}$ we find

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}.$$

The purpose of this section is to find the derivatives of implicit functions. The process is known as **implicit differentiation** and consists of the following two steps:

Step 1. Differentiate both sides of the equation with respect to x. Remember that y is a function of x for part of the curve and use the chain rule when differentiating terms containing y.

Step 2. Solve the differentiated equation in Step 1 algebraically for $\frac{dy}{dx}$.

Example 21.1
Suppose that y is a differentiable function of x such that

$$x^2y + 2y^3 = 3x + 2y.$$

Find the equation of the tangent line to the graph at the point $(3, 1)$.

Solution.
Differentiating both sides to obtain

$$2xy + x^2y' + 6y^2y' = 3 + 2y'$$

Replacing $x = 3$ and $y = 1$ we find

$$6 + 9y' + 6y' = 3 + 2y'$$

Solving for y' to obtain $y' = -\frac{3}{13}$. Thus, the equation of the tangent line is given by

$$y - 1 = -\frac{3}{13}(x - 3)$$

or in standard form

$$3x + 13y - 22 = 0.\blacksquare$$

Example 21.2
Consider the equation $y^3 - xy = -6$.

(a) Find $\frac{dy}{dx}$ by implicit differentiation.
(b) Give a table of approximate y-values near $(7, 2)$ for $x = 6.8, 6.9, 7.0, 7.1, 7.2$.
(c) Find the y-value for $x = 6.8$ by substituting 6.8 in the equation and solving for y using a calculator. Compare your answer in part (b).

Solution.
(a) Using implicit differentiation we find

$$\begin{array}{ccc}
\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) & = & \frac{d}{dx}(-6) \\
3y^2\frac{dy}{dx} - y - x\frac{dy}{dx} & = & 0 \\
(3y^2 - x)\frac{dy}{dx} & = & y \\
\frac{dy}{dx} & = & \frac{y}{3y^2 - x}
\end{array}$$

When $x = 6$ and $y = 2$ we have

$$\frac{dy}{dx} = \frac{2}{12 - 7} = \frac{2}{5}.$$

(b)Using the tangent line approximation we find

$$y = 2 + \frac{2}{5}(x - 7) = 0.4x - 0.8.$$

We use this equation to calculate the following approximate y-values

x	6.8	6.9	7.0	7.1	7.2
Approximate y	1.92	1.96	2.00	2.04	2.08

(c) Letting $x = 6.8$ in the original equation we find

$$y^3 - 6.8y + 6 = 0.$$

Using a graphing calculator we find that $y \approx 1.915.\blacksquare$

Remark 21.1
It is important to make sure that y is a differentiable function of x before applying the implicit differentiation process. For example, there is no real valued function satisfying the equation $x^2 + y^2 = -1$ so y' does not exist. A careless application of the above technique might lead you to thinking that y' exists and is equal to

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Recommended Problems (pp. 140 - 1): 2, 9, 10, 15, 17, 21, 24, 26, 27.

22 Parametric Equations

In this section we define parametric equations in the plane and we illustrate how to draw their graphs.
Parametric equations in the plane is a pair of functions

$$x = f(t) \quad and \quad y = g(t)$$

which describe the coordinates of the points of some curve in the plane. The variable t is referred to as a **parameter.**

Example 22.1
Write the equation of a circle of radius r in both the Cartesian coordinates form and the parametric form.

Solution.
The equation of the circle in the Cartesian coordinates system is given by

$$x^2 + y^2 = r^2.$$

The parametric equations of this circle are given by

$$x = \cos t \quad and \quad y = \sin t \quad 0 \leq t \leq 2\pi. \blacksquare$$

Example 22.2
Figure 71 shows the graphs of two functions, $f(t)$ and $g(t)$. Describe the motion of the particle whose coordinates at time t are $x = f(t)$ and $y = g(t)$.

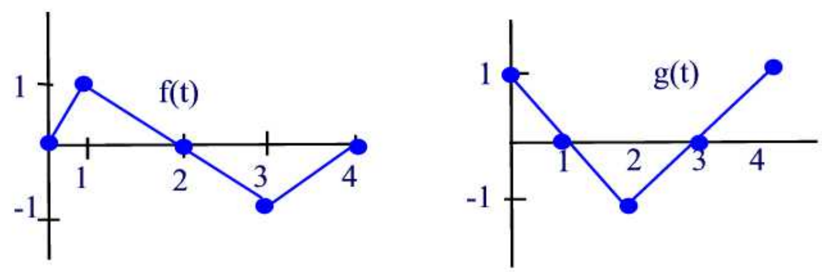


Figure 71

Solution.
For $0 \leq t \leq 1$, x moves at a constant rate from 0 to 1 and y moves at a

constant rate from 1 to 0. So the particle moves in a straight line from $(0,1)$ to $(1,0)$. For $1 \le t \le 2$, the particle moves on a straight line from $(1,0)$ to $(0,-1)$. For $2 \le t \le 3$ the particle moves on a straight line from $(0,-1)$ to $(-1,0)$. Finally, for $3 \le t \le 4$, the particle moves on the straight line from $(-1,0)$ to $(0,1)$. Thus, the particle traces the diamond shown in Figure 72.■

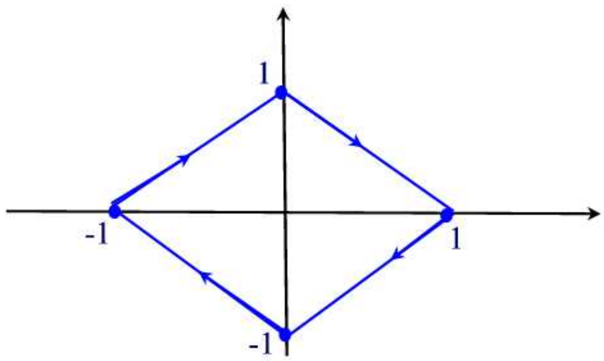


Figure 72

Remark 22.1
Parametric representations, in general, are not unique. That is, the same curve can be expressed by a number of different parametrizations. For example, the representations $(x = \cos t, y = \sin t)$ and $(x = \cos(3t), y = \sin(3t))$ both represent the unit circle. These various representations describe various motions of a moving particle on the unit circle. Looking at the tables below we see that the particle on the second curve is moving three times faster than the one moving on the curve of the previous example.

t	$\cos t$	$\sin t$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

t	$\cos(3t)$	$\sin(3t)$
0	1	0
$\frac{\pi}{6}$	0	1
$\frac{2\pi}{6}$	-1	0
$\frac{3\pi}{6}$	0	-1
$\frac{4\pi}{6}$	1	0

Parametric Equations of a Straight Line

Consider a straight line passing through two points (x_0, y_0) and (x_1, y_1). Then the equation of the line is given by

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0).$$

This is equivalent to

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0}.$$

Denote the common value by the parameter t to obtain

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0} = t.$$

Equating each fraction seperately to t and solving for x and y we find

$$x = x_0 + (x_1 - x_0)t \quad and \quad y = y_0 + (y_1 - y_0)t.$$

These equations are referred to as the parametric equations of the straight line.

Example 22.3
Find the parametric equations of the line passing through the points $(2, -1)$ and $(-1, 5)$.

Solution.
The parametric equations are given by

$$x = 2 - 3t \quad and \quad y = -1 + 6t. \blacksquare$$

Remark 22.2
Note that by restricting the parameter t in the previous problem to the interval $[0, 1]$ we obtain the parametric equations of the line segment from $(2, -1)$ to $(-1, 5)$.

Speed and Velocity
Suppose that the motion of a moving particle is given by a curve with parametric equations

$$x = f(t) \quad and \quad y = g(t).$$

We define the **instantaneous speed** by the formula

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

The quantity $v_x = \frac{dx}{dt}$ is the **instantaneous velocity** in the x-direction and $v_y = \frac{dy}{dt}$ is the **instantaneous velocity** in the y-direction.

Example 22.4
The motion of a particle is given by the parametric equations

$$x = t^2 - 3t, \quad y = t^2 - 2t.$$

(a) Does the particle ever come to a stop? If so, when and where?
(b) Is the particle ever moving straight up or down? If so, when and where?
(c) Is the particle ever moving straight horizontally right or left? If so, when and where?

Solution.
(a) Saying that the particle comes to a stop is equivalent to saying that $v = 0$, i.e. $v_x = v_y = 0$. Solving the equations

$$\begin{array}{lllllllll} v_x & = & \frac{dx}{dt} & = & 3t^2 - 3 & = & 3(t-1)(t+1) & = & 0 \\ v_y & = & \frac{dy}{dt} & = & 2t - 2 & = & 2(t-1) & = & 0 \end{array}$$

give $t = 1$. Thus, the particle stops when $t = 1$ at the point $(t^3 - 3t, t^2 - 2t)\,|_{t=1} = (-2, -1)$.
(b) Saying that the particle is traveling straight up or down is equivalent to saying that $v_x = 0$ and $v_y \neq 0$. Solving the equation $v_x = \frac{dx}{dt} = 3(t^2 - 1) = 0$ we find $t = \pm 1$. Since $v_y = 0$ at $t = 1$ then the only value of t is $t = -1$. The position of the particle at that time is $(t^3 - 3t, t^2 - 2t)\,|_{t=-1} = (2, 3)$.
(c) Saying that the particle is traveling straight left or right is equivalent to saying that $v_x \neq 0$ and $v_y = 0$. Solving the equation $v_y = \frac{dy}{dt} = 2(t-1) = 0$ we find $t = 1$. Since $v_x = 0$ at $t = 1$ then the particle is not moving at all.■

Slopes of Parametric Curves

Suppose that a curve is defined parametrically by the equations $x = f(t)$ and $y = g(t)$. The curve can be defined in Cartesian coordinates form either explicitely or implicitely. In the first case, we can write $y = h(x)$ and find $\frac{dy}{dx}$ using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Thus, we obtain the slope of the curve as a function of t :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Similar procedure applies if the curve is defined by an implicit function. In this case we use implicit differentiation.

Example 22.5
Find the slope of the tangent line to the curve defined parametrically by $x = t^3 - t$ and $y = t^2$ at $t = 2$. Find the equation of the tangent line.

Solution.
The point corresponding to $t = 2$ is $(6, 4)$. We have

$$\frac{dy}{dx}\Big|_{(6,4)} = \frac{dy/dt}{dx/dt}\Big|_{(6,4)} = \frac{2t}{3t^2-1}\Big|_{t=2} = \frac{4}{11}$$

The equation of the tangent line is given by

$$y - 4 = \frac{4}{11}(x - 6)$$

or n parametric form by

$$x = 6 + 11t, \text{``}y = 4 + 4t. \blacksquare$$

Parametrizing $y = f(x)$
The graph of any function defined in Cartesian coordinates form $y = f(x)$ can be parametrized by the equations:

$$x = t \;\; and \;\; y = f(t).$$

Example 22.6
Give parametric equations for the vertical line passing through the point $(-2, -3)$.

Solution.
A possible answer is: $x = -2, \; y = t. \blacksquare$

Recommended Problems (pp. 148 - 9): 1, 2, 5, 7, 9, 11, 15, 18, 19, 21, 22, 26, 30.

23 Linear Approximations and Differentials

Consider a function f and a point a. We want to find the best linear function passing through $(a, f(a))$ that approximate the values of $f(x)$ near a. If we let $g(x)$ be such a function and $E(x) = f(x) - g(x)$ be the error in the approximation then by best approximation we mean

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0.$$

Since $g(x)$ is linear and goes through $(a, f(a))$ then $g(x)$ has the form

$$g(x) = g(a) + L(x - a).$$

Suppose that f is differentiable at a then we choose $L = f'(a)$. In this case,

$$\begin{array}{rcl} \lim_{x\to a} \frac{E(x)}{x-a} & = & \lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a} \\ & = & \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a) \\ & = & f'(a) - f'(a) = 0 \end{array}$$

Thus, $g(x)$ is the best linear approximation of f near a. We call $g(x)$ the **tangent line approximation** of $f(x)$ at $x = a$ and we write

$$f(x) \approx f'(a)(x - a) + f(a)$$

It follows that the graph of $f(x)$ near a can be thought of as a straight line.

Remark 23.1
The **error** of the approximation is defined to be E(x)= Exact - Approximate and is given by

$$E(x) = f(x) - f'(a)(x - a) - f(a).$$

If $E(x) < 0$ then the approximation is an overestimate whereas if $E(x) > 0$ then the approximation is an underestimate. Figure 73 shows the tangent line approximation and its error.

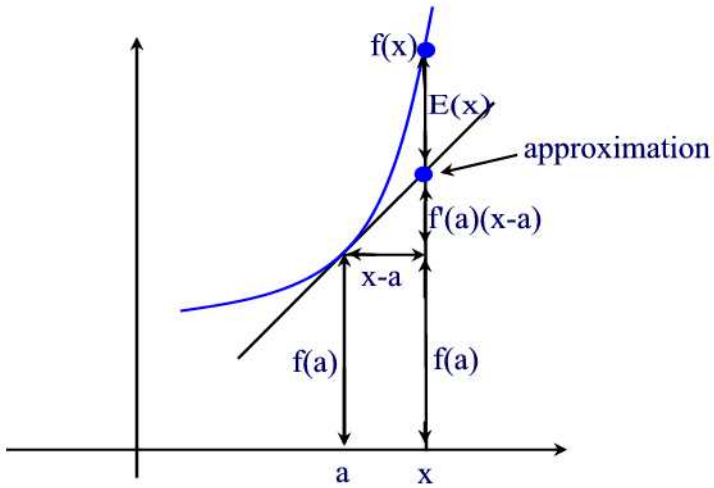


Figure 73

Example 23.1
Find the tangent line approximation of $f(x) = e^{kx}$ near $x = 0$.

Solution.
According to the formula above, we have

$$f(h) \approx f(0) + f'(0)h.$$

But $f(0) = 1$ and $f'(0) = k$ since $f'(x) = ke^{kx}$. Hence, for small h we have

$$f(h) \approx 1 + kh$$

That is, $e^{kx} \approx 1 + kx$ for x close to 0.■

Differentials
Now, suppose f is a differentiable function at x. Then we define the exact change of x along the curve from x to $x+h$ to be the quantity $\Delta x = h$ and the exact change in y along the curve to be the quantity $\Delta y = f(x) - f(x+h)$. We define the **differential** of x along the tangent line to the graph of $f(x)$ to be the quantity dx. Clearly, $dx = \Delta x$. Also, we define the **differential** of y along the tangent line to be the product

$$dy = f'(x)dx.$$

That is, dy is the **estimated** change in f or $dy \approx \Delta y$.
The Figure 74 exhibits all four quantities $\Delta x, \Delta y, dx$, and dy.

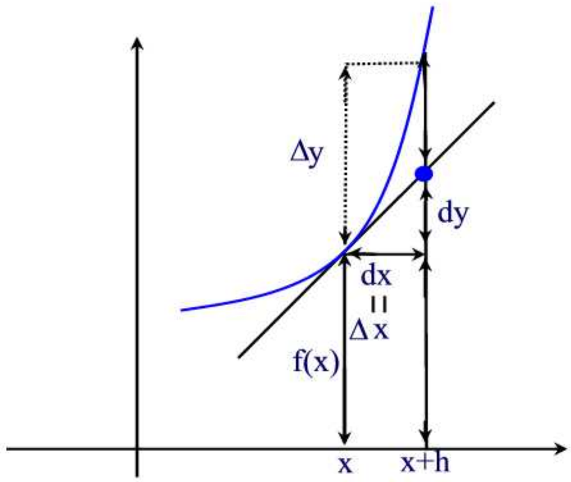


Figure 74

Example 23.2
Complete the following:

(a) $d(\sin x) =$
(b) $d(\ln x) =$
(c) $d(e^x) =$
(d) $d(x^n) =$

Solution.
(a) $d(\sin x) = \cos x dx$
(b) $d(\ln x) = \frac{dx}{x}$
(c) $d(e^x) = e^x dx$
(d) $d(x^n) = nx^{n-1}dx$ ■

Note that the linear approximation can be written now in terms of differentials

$$f(a + dx) \approx f(a) + dy$$

where $dy = f'(a)dx$.

Example 23.3
Approximate $\frac{1}{3.98}$ using differentials. Use a calculator to compute the error. Is the approxmation an overestimate or an underestimate?

Solution.
Let $f(x) = \frac{1}{x}$ then $f'(x) = -\frac{1}{x^2}$. Let $a = 4$ and $dx = -0.02$. Then

$$\begin{aligned} \tfrac{1}{3.98} &= f(a+dx) \approx f(4) + f'(4)dx \\ &= \tfrac{1}{4} - \tfrac{1}{16}(-0.02) = 0.25125 \end{aligned}$$

Using a calculator, we find

$$Error = \frac{1}{3.98} - 0.25125 \approx 6.28 \times 10^{-6}.$$

Hence the approximation is an underestimate.■

Example 23.4
Estimate the change in the area of a square if its edge length is decreased from 10 inches to 9.8 inches.

Solution.
Let x be the edge length, so the area is $A = x^2 = f(x)$. Since x changes from 10 to 9.8, the change in x is $dx = -0.2$. We have

$$dA = f'(x)dx = 2xdx,$$

so the estimated change in A is

$$dA = 2 \cdot 10 \cdot (-0.2) = -4.$$

The area will decrease by about 4 square inches.■

Recommended Problems (pp. 153 - 4): 1, 2, 3, 4, 5, 6, 7, 9, 15.

24 Indeterminate Forms and L'Hôpital's Rule

In this section we want to study limits of ratios of two functions

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

We have seen that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L' \neq 0$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{L'}.$$

Now, if $L \neq 0$ but $L' = 0$ then we say that the limit does not exist. What if both L and L' are zero? Sometimes simple algebraic manipulations can be used to find the limit. For example, if $f(x) = x + 3$ and $g(x) = x^2 + x - 6$ then

$$\lim_{x\to -3}\frac{x+3}{x^2+x-6} = \lim_{x\to -3}\frac{x+3}{(x+3)(x-2)} = \lim_{x\to -3}\frac{1}{x-2} = -\frac{1}{5}.$$

Another way of evaluating limits of the form $\frac{0}{0}$ is by applying the following result.

Theorem 24.1 *(L'Hôpital's Rule)*
Suppose that f and g are differentiable functions on (a,b) and x_0 is a point inside the interval such that $f(x_0) = g(x_0) = 0$. If $g'(x) \neq 0$ in (a,b) except possibly at x_0 then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

Proof.
We first show that

$$\lim_{x\to x_0^+}\frac{f(x)}{g(x)} = \lim_{x\to x_0^+}\frac{f'(x)}{g'(x)}.$$

The left limit case is similar, and the combination of these two cases establishes the result.
So fix $x > x_0$. Then the function $g(t)$ is continuous on $[x_0, x]$ and differentiable on (x_0, x) so that we can apply the Mean Value Theorem (See Section 31) and obtain a number c_x in (x_0, x) such that

$$\frac{g(x) - g(x_0)}{x - x_0} = g'(c_x) \neq 0.$$

This shows that $g(x) \neq 0$. Next, consider the function

$$F(t) = f(t) - \frac{f(x)}{g(x)} g(t).$$

This function is continuous in $[x_0, x]$ and differentiable on (x_0, x) with $F(x) = F(x_0) = 0$. So by the Mean Value Theorem there is a c in (x_0, x) such that $F'(c) = 0$. That is,

$$0 = f'(c) - \frac{f(x)}{g(x)} g'(c)$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

Thus,

$$\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = \lim_{c \to x_0^+} \frac{f'(c)}{g'(c)} = \lim_{x \to x_0^+} \frac{f'(x)}{g'(x)}. \blacksquare$$

Remark 24.1
1. In the case that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ then rewrite the limit as $\lim_{x \to x_0} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$ and use the above rule.
2. The x_0 in the theorem can be replaced by one of the symbols $-\infty$ or ∞.

Example 24.1
Compute $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$.

Solution.
Using L'Hôpital's rule we find

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2. \blacksquare$$

Example 24.2
Use L'Hôpital's rule to find $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(sinx)}{\frac{d}{dx}(x)} \\ &= \lim_{x \to 0} \cos x = \cos 0 = 1 \ \blacksquare \end{aligned}$$

Example 24.3
Calculate $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$.

Solution. Using L'Hôpital's rule twice we find

$$\begin{array}{rcl}\lim_{x\to 0}\frac{e^x-1-x}{x^2} & = & \lim_{x\to 0}\frac{e^x-1}{2x}\\ & = & \lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}\ \blacksquare\end{array}$$

Example 24.4
Calculate $\lim_{x\to\infty} xe^{-x}$.

Solution.
Since $\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty}\frac{x}{e^x}$ is of the form $\frac{\infty}{\infty}$ then we can apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0.\blacksquare$$

Example 24.5
Calculate $\lim_{x\to\infty}\frac{5x+e^{-x}}{7x}$.

Solution.
The given limit is of the form $\frac{\infty}{\infty}$ so we can apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty}\frac{5x+e^{-x}}{7x}=\lim_{x\to\infty}\frac{5-e^{-x}}{7}=\frac{5}{7}.\blacksquare$$

Example 24.6
We say that a function $g(x)$ **dominates** a function $f(x)$ if and only $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$, or equivalently, $\lim_{x\to\infty}\frac{g(x)}{f(x)}=\infty$. Use L'Hôpital's rule to show that the function $\sqrt{x}$ dominates the function $\ln x$.

Solution.
The limit $\frac{\ln x}{\sqrt{x}}$ as $x\to\infty$ is of the form $\frac{\infty}{\infty}$. So we can apply L'Hôpital's rule to find

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

Thus, $\sqrt{x}$ dominates $\ln x$.$\blacksquare$

Recommended Problems (p. 158): 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 18.

25 Using First and Second Derivatives

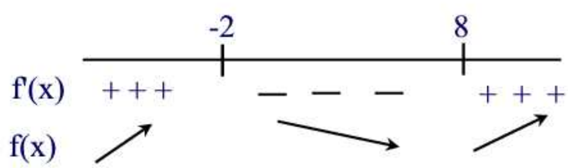
We start this section by reviewing what the first and second derivatives of a function tell us about its graph:

- If $f'(x) > 0$ on an open interval I then $f(x)$ is increasing on I.
- If $f'(x) < 0$ on an open interval I then $f(x)$ is decreasing on I.
- If $f''(x) > 0$ on an open interval I then $f(x)$ is concave up on I.
- If $f''(x) < 0$ on an open interval I then $f(x)$ is concave down on I.

Example 25.1
Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

(a) Find the intervals where the function is increasing/decreasing.
(b) Find the intervals where the function is concave up/down.

Solution.
(a) Finding the first derivative we obtain $f'(x) = 3x^2 - 18x - 48 = 3(x-8)(x+2)$. Constructing the chart of signs below



we see that $f(x)$ is increasing on $(-\infty, -2) \cup (8, \infty)$ and decreasing on $(-2, 8)$.
(b) Finding the second derivative, we obtain $f''(x) = 6x - 18 = 6(x-3)$. So, $f(x)$ is concave up on $(3, \infty)$ and concave down on $(-\infty, 3)$.■

<u>**Local Maxima and Minima**</u>
Points of interest on the graph of a function are those points that are the highest on the curve, or the lowest, in a specific interval. Such points are called **local extrema.** The highest point, say $f(a)$, is called a **local maximum** and satisfies $f(x) \leq f(a)$ for all x in an interval I. A **local minimum** is a point $f(a)$ such that $f(a) \leq f(x)$ for all x in an interval I containing a.

Example 25.2
Find the local maxima and the local minima of the function given in Figure 75.

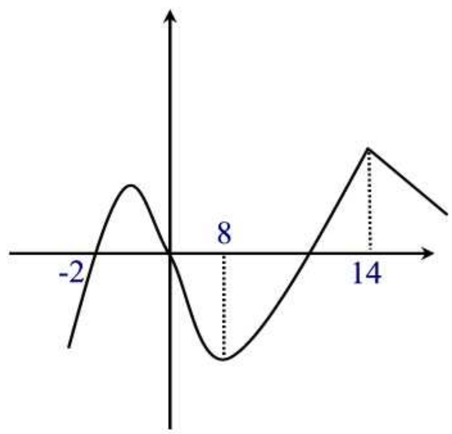


Figure 75

Solution.
The local maxima occur at $x = -2$ and $x = 14$ whereas the local minimum occurs at $x = 8$.■

Next we will discuss two procedures for finding local extrema. We notice from the previous example that local extrema occur at points p where the derivative is either zero or undefined. We call p a **critical number**, $f(p)$ a **critical value**, and $(p, f(p))$ a **critical point**. Most of the critical numbers that we will encounter in this book are of the form $f'(p) = 0$ type. The following theorem asserts that local extrema occur at the critical points. This theorem is proved in Section 31.

Theorem 25.1
Suppose that f is defined on an interval I and has a local maximum or minimum at an interior point a. If f is differentiable at a then $f'(a) = 0$.

Remark 25.1
By Theorem 25.1, local extrema are always critical points. The converse of this statement is not true in general. That is, there are critical points that are not local extrema of a function. An example of this situation is given next.

Example 25.3
Show that $f(x) = x^3$ has a critical point at $x = 0$ but 0 is neither a local maximum nor a local minimum.

Solution.
Finding the derivative to obtain $f'(x) = 3x^2$. Setting this to 0 we find the critical point $x = 0$. Since $f'(x)$ does not change sign at 0 then 0 is neither a local maximum nor a local minimum.■

The graph in Example 25.2 suggests two tests for finding local extrema. The first is known as the **first derivative test** and the second as the **second derivative test.**

<u>**First-Derivative Test**</u>
Suppose that a continuous function f has a critical point at p.
• If f' changes sign from negative to positive at p, then f has a local minimum at p.
• If f' changes sign from positive to negative at p, then f has a local maximum at p. See Figure 76.

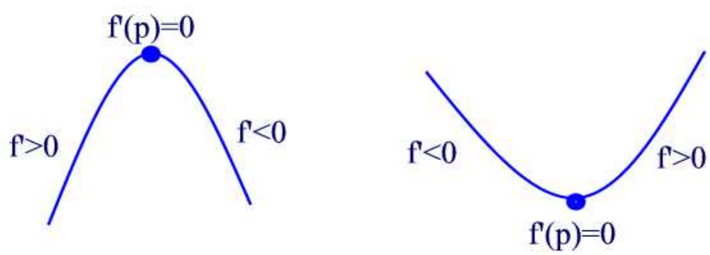


Figure 76

Example 25.4
(a) Find the local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$.
(b) Find the local extrema of the function $f(x) = \sin x + e^x, x \geq 0$.

Solution.
(a) Using the chart of signs of f' discussed in Example 25.1, we find that $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 8$.
(b) Finding the derivative to obtain $f'(x) = \cos x + e^x$. But for $x \geq 0$,

$1 \leq e^x$. Since $-1 \leq \cos x \leq 1$ then adding the two inequalities we see that $0 \leq \cos x + e^x$. This implies that f' does not change sign for $x \geq 0$. Therefore, there are no local maxima. The only local minimum occurs at $(0, 1)$.■

<u>**Second-Derivative Test**</u>

Let f be a continuous function such that $f'(p) = 0$.
- if $f''(p) > 0$ then f has a local minimum at p.
- if $f''(p) < 0$ then f has a local maximum at p.
- if $f''(p) = 0$ then the test fails. In this case, it is recommended that you use the first derivative test.

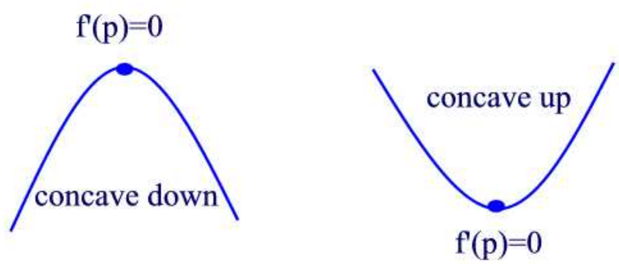


Figure 77

Example 25.5
Use the second derivative test to find the local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Solution.
The second derivative of $f(x)$ is given by $f''(x) = 6(x - 3)$. The critical numbers are -2 and 8. Since $f''(-2) = -30 < 0$ then $x = -2$ is a local maximum. Since $f''(8) = 30 > 0$ then $x = 8$ is a local minimum.■

Example 25.6
Find the local extrema of the function $f(x) = x^4$.

Solution.
Let's try and find the local extrema by using the second derivative test. Since $f'(x) = 4x^3$ then $x = 0$ is the only critical number. Since $f''(x) = 12x^2$ then $f''(0) = 0$. So the second derivative test is inconclusive. Now, using the first derivative test, we see that $f'(x)$ changes sign from negative to positive at

$x = 0$. Thus, $x = 0$ is a local minimum.■

Concavity and Points of Inflection

We have seen that a local extremum is a point where the first derivative changes sign. In this section we will discuss points where the second derivative changes sign. That is, the points where the graph of the function changes concavity. We call such points **points of inflection**.

How do you find the points of inflection? Well, since f'' changes sign on the two sides of an inflection point then it makes sense to say that points of inflection occur at points where either the second derivative is 0 or undefined.

Example 25.7

Find the point(s) of inflection of the function $f(x) = xe^{-x}$.

Solution.

Using the product rule to obtain $f'(x) = e^{-x} - xe^{-x}$. Using the product rule for the second time we find $f''(x) = e^{-x}(x-2)$. Thus, a candidate for a point of inflection is $x = 2$. Since $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$ then $x = 2$ is a point of inflection.■

Remark 25.2

We have seen that not every value of x where the derivative is zero or undefined is a local maximum or minimum. The same thing applies for points of inflection. That is, it is not always true that if the second derivative is 0 or undefined then automatically you have a point of inflection. It is critical that f'' changes sign at such a point in order to have a point of inflection.

Example 25.8

Consider the function $f(x) = x^4$. Show that $f''(0) = 0$ but 0 is not a point of inflection.

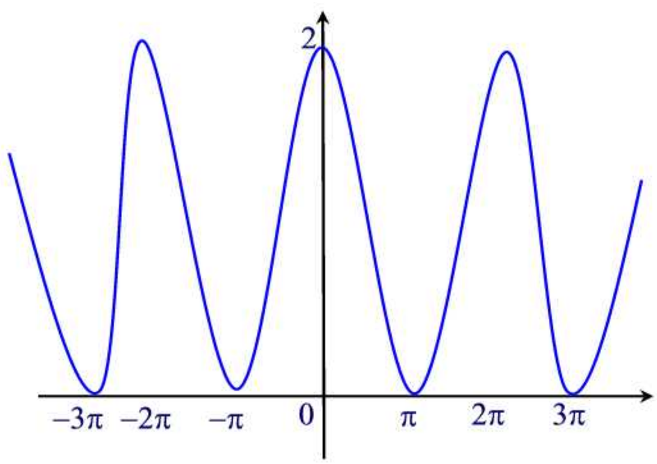
Solution.

The second derivative is given by the formula $f''(x) = 12x^2$. Clearly, $f''(0) = 0$. Since $f''(x) \geq 0$, that is, $f''(x)$ does not change sign then 0 is not a point of inflection.■

Example 25.9

Graph the derivative of the function $f(x) = x + \sin x$. Determine where f is increasing most rapidly, and least rapidly.

Solution.
The derivative of $f(x)$ is given by the expression $f'(x) = \cos x + 1 \geq 0$ so that $f(x)$ is always increasing. Now, $f(x)$ increases most rapidly at the maximum values of $f'(x)$ and increases least rapidly at the minimum values of $f'(x)$. Graphing the function $f'(x)$ we find



Thus, f increases most rapidly at $x = 2n\pi$ and least rapidly at $x = (2n+1)\pi$ where n is an integer.■

Example 25.10
Graph a function with the following properties:

- f has a critical point at $x = 4$ and an inflection point at $x = 8$.
- $f' < 0$ for $x < 4$ and $f' > 0$ for $x > 4$.
- $f'' > 0$ for $x < 8$ and $f'' < 0$ for $x > 8$.

Solution.
The graph is given in Figure 78.

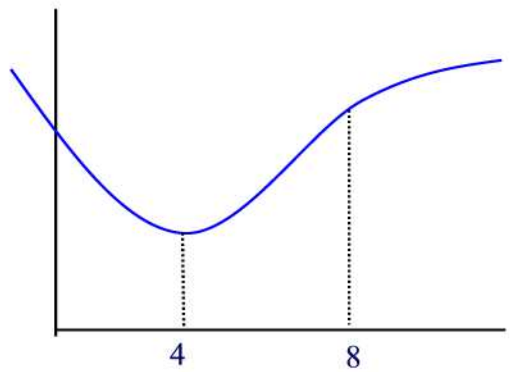


Figure 78

Recommended Problems (pp. 172 - 5): 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 18, 20, 22, 24, 26, 32, 40, 41, 42, 43.

26 Special Families of Curves

Consider the line $y = x$. The collection of lines $y = x + c$ is an example of a one-parameter family of curves. We call c the **parameter**. In this section, we will discuss certain family of functions that are of importance to us in later discussions.

Motion Under Gravity

The motion of a particle along a horizontal or a vertical line is given by

$$s(t) = -4.9t^2 + v_0 t + s_0$$

where the parameters v_0 and s_0 represent the initial velocity and the initial distance respectively.

Harmonic Motion

A simple harmonic motion is modeled by either the function $f(t) = a\sin(bt)$ or the function $f(t) = a\cos(bt)$. But the period of either function is known to be $p = \frac{2\pi}{|b|}$. The number $|a|$ is called the **amplitude**.

Remark 26.1

1. If maximum displacement occurs at $t = 0$ then the motion is modeled by the cosine function.
2. If zero displacement occurs at $t = 0$ then the motion is modeled by the sine function.

Example 26.1

Find the amplitude and the period of the simple harmoninc motion described by the equation

$$y = 3\cos\frac{2}{3}t.$$

Solution.

The amplitude is $|a| = |3| = 3$. The period is $= \frac{2\pi}{b} = \frac{2\pi}{\frac{2}{3}} = 3\pi.$■

Example 26.2

Find an equation of a simple harmoninc motion with period $p = \frac{2}{3}$ and amplitude 4. Assume that maximum displacement occurs at $t = 0$.

Solution.
Since maximum displacement occurs at $t = 0$ then $y = a\cos bt$. But $p = 2/3$ and $a = 4$ so that $y = 4\cos 3\pi t$.■

Bell Shaped-Curves

Bell-shaped curves occur in many common situations and have been studied extensively by statisticians. More specifically, we look at the two-parameters family of curves

$$y = e^{-\frac{(x-a)^2}{b}}$$

where $b > 0$. This family is related to the **normal density** function, studied in probability and statistics.
Let's first consider the case $b = 1$. In this case, $y = e^{-(x-a)^2}$, and the graph of this function is a horizontal shift of the graph of $y = e^{-x^2}$, by a units. The Figure 79 shows the cases, $a = -2, 0, 2$.

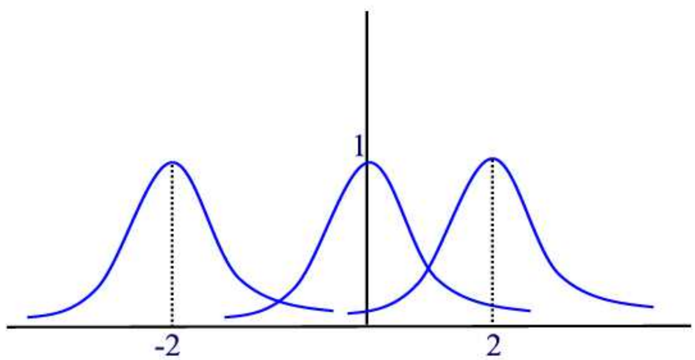


Figure 79

Next, consider the case where $a = 0$ and b being arbitrary. Using the chain rule one finds

$$y' = -\frac{2x}{b}e^{-\frac{x^2}{b}}.$$

Thus, $x = 0$ is a critical number. Using the product rule together with the chain rule one finds

$$y'' = \frac{2}{b}\left(\frac{2x^2}{b} - 1\right)e^{-\frac{x^2}{b}}$$

Thus, $y''(0) < 0$ so that by the second derivative test, $(0, 1)$ is a local maximum. Also, notice that at 0 the graph is concave down. But $e^{-\frac{x^2}{b}} \to 0$ as

$x \to \pm\infty$. Thus, the function must have points of inflection. The x-values of these points are the zeros of $y'' = 0$. That is,

$$\frac{2}{b}\left(\frac{2x^2}{b} - 1\right)e^{-\frac{x^2}{b}} = 0.$$

Solving this equation for x we find

$$x = \pm\sqrt{\frac{b}{2}}.$$

Summarizing, we see that the function $y = e^{-\frac{(x-a)^2}{b}}$ has a maximum at $x = a$ and inflection points at $x = a \pm \sqrt{\frac{b}{2}}$. Also, the parameter a gives the center of the bell whereas the parameter b determines how narrow or wide the bell is. The smaller b is, the closer the points of inflection are to the center a and so the bell is sharply peaked near a; the larger b is, the farther the points of inflection are from a,i.e. the bell is spread out.

Recommended Problems (pp. 179 - 180): 1, 4, 5, 7, 8, 13, 15, 16, 21, 23, 28.

27 Global Maxima and Minima

In this section we will look for the largest or the smallest values of a function on its domain. Such points are called **global extrema**. If $f(a)$ is the largest value then it satisfies the inequality $f(x) \leq f(a)$ for all x in the domain of f. We call $f(a)$ the **global or absolute maximum value** of f and the point $(a, f(a))$ the global maximum point. Similarly, if $f(a)$ is the smallest value of $f(x)$ then $f(a) \leq f(x)$ for all x in the domain of f. We call $f(a)$ the **absolute or global minimum value** of f and the point $(a, f(a))$ the global minimum.
The process of finding the global extrema is called **optimization**. Problems that involve finding the global extrema are called **optimization problems**.

How do we find the global extrema?
• If the function is continuous on a closed interval then the global extrema occur at either the critical points or the endpoints of the interval.

Problem 27.1
Find the global extrema of the function $f(x) = x^3 - 9x^2 - 48x + 52$ *on the closed interval* $[-5, 12]$.

Solution.
Finding the derivative of $f(x)$ we get $f'(x) = 3x^2 - 18x - 48$. Solving the equation $f'(x) = 0$ that is, $x^2 - 6x - 16 = 0$ we find the critical points at $x = 8$ and $x = -2$. Now, evaluating the function at these points and at the endpoints we find

$$\begin{array}{rcl} f(-5) & = & -58 \\ f(-2) & = & 104 \\ f(8) & = & -396 \\ f(12) & = & -92 \end{array}$$

It follows that $(-2, 104)$ is the global maximum point and $(8, -396)$ is the global minimum point.■
• If a function is continuous on an open interval or on all real numbers then it is recommended to find the global extrema by graphing the function.

Problem 27.2
Find the global extrema of the function $f(x) = 100(e^{-0.02x} - e^{-0.1x})$ *for* $x \geq 0$.

Solution.
Let's sketch the graph of this function. The standard process of graphing consists of the following steps:

Step 1. Find the critical numbers. Setting $f'(x) = 0$ to obtain

$$\begin{array}{rcccc}
100(-0.02e^{-0.02x} + 0.1e^{-0.1x}) & = & 0 & & \\
0.02e^{-0.02x} & = & 0.1e^{-0.1x} & & \\
\frac{e^{-0.02x}}{e^{-0.1x}} & = & \frac{0.1}{0.02} & & \\
e^{0.08x} & = & 5 & & \\
0.08x & = & \ln 5 & & \\
x & = & \frac{\ln 5}{0.08} & = & 20.12
\end{array}$$

Step 2. We construct the following chart:

x		20.12	
f'(x)	+	0	-
f(x)	$\nearrow$	53.50	$\searrow$

Step 3. Find the second derivative to obtain $f''(x) = 100(0.0004e^{-0.02x} - 0.01e^{-0.1x})$. Setting this to zero and solving for x as in Step 1 we find $x \approx 40.25$. Now we construct the table

x		40.25	
f"(x)	-	0	+
f(x)	$\cap$	f(40.25)	$\cup$

Step 4.Graph

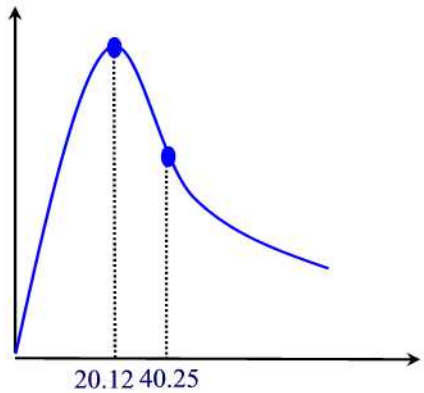


Figure 80

Thus, from the graph we see that $(20.12, 53.50)$ is a global maximum. The function has a global minimum at $x = 0$.■

Recommended Problems (pp. 185 - 8): 2, 4, 7, 8, 9, 13, 19, 33, 34.

28 Applications to Economics

Management of most businesses always aim to maximizing profit. In this section we will use the derivative to optimizie profit and revenue functions. We begin by introducing the cost, revenue, and profit functions.

The **cost function** C gives the cost $C(q)$ of manufacturing a quantity q of some good. A **linear cost function** has the form

$$C(q) = mq + b,$$

where the y-intercept b is called the **fixed cost**, i.e. the costs incurred even if nothing is produced, and the slope m is called the **variable costs per unit**.
The function $\overline{C(x)} = \frac{C(x)}{x}$ is called the **average cost function.**

Example 28.1
What is the cost function of manufacturing a product with fixed cost of \$400 and variable costs of \$40 per item, assuming the function is linear?

Solution.
The cost function is

$$C(q) = 40q + 400.\blacksquare$$

Example 28.2
Values of a linear cost function are shown below. What are the fixed costs and the variable costs per units? Find a formula for the cost function.

q	0	5	10	15	20
C(q)	5000	5020	5040	5060	5080

Solution.
The fixed costs are $b = C(0) = \$5,000$, the variable costs are

$$m = \frac{5020 - 5000}{5 - 0} = 4$$

The cost function is

$$C(q) = 4q + 5,000\blacksquare$$

A **revenue function** R gives the total revenue $R(q)$ from the sale of a quantity q at a unit price p dollars. Thus, $R(q) = pq$.

Example 28.3
A company that makes a certain brand of chairs has fixed costs of \$5,000 and marginal cost of \$30 per chair. The company sells the chairs for \$50 each. Find formulas for the cost and revenue functions.

Solution.
The cost function is $C(q) = 30q + 5000$. The revenue function is $R(q) = pq = 30q$.■

Profit is defined to be revenue minus cost. That is

$$P(q) = R(q) - C(q).$$

The **break-even point** is the point where the profit is zero, i.e. $R(q) = C(q)$.

Example 28.4
A company has cost and revenue functions, in dollars, given by $C(q) = 6,000 + 10q$ and $R(q) = 12q$.

(a) Graph the functions $C(q)$ and $R(q)$ on the same coordinate axes.
(b) Find the break-even point and illustrate it graphically.
(c) When does the company make a profit? Loses money?

Solution.
(a) The graph is given in Figure 81.

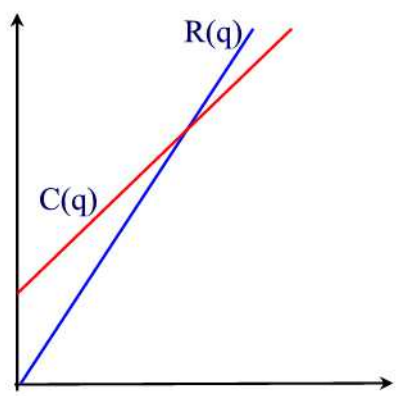


Figure 81

(b) The break-even point is the point of intersection of the two lines. To find the point, set $12q = 10q + 6000$ and solve for q to find $q = 3000$. Thus, the break-even point is the point $(3000, 36000)$.
(c) The company makes profit for $q > 3000$ and loses money for $q < 3000$.■

Marginal Analysis

Marginal analysis is an area of economics concerned with estimating the effect on quantities such as cost, revenue, and profit when the level of production is changed by a unit amount. For example, if $C(q)$ is the cost of producing q units of a certain commodity, then the **marginal cost**, MC(q), is the additional cost of producing one more unit and is given by the difference $MC(q) = C(q+1) - C(q)$. Using the estimation

$$C'(q) \approx \frac{C(q+1) - C(q)}{(q+1) - q} = C(q+1) - C(q)$$

we find that

$$MC(q) \approx C'(q)$$

and for this reason, we will compute the marginal cost by the derivative $C'(q)$.
Similarly, if $R(q)$ is the revenue obtained from producing q units of a commodity, then the **marginal revenue**, MR(q), is the additional revenue obtained from producing one more unit, and we compute $MR(q)$ by the derivative $R'(q)$.

Example 28.5

Let $C(q)$ represent the cost, $R(q)$ the revenue, and $P(q)$ the total profit, in dollars, of producing q units.

(a) If $C'(50) = 75$ and $R'(50) = 84$, approximately how much profit is earned by the 51st item?
(b) If $C'(90) = 71$ and $R'(90) = 68$, approximately how much profit is earned by the 91st item?

Solution.

(a) $P'(50) = R'(50) - C'(50) = 84 - 75 = 9$.
(b) $P'(90) = R'(90) - C'(90) = 68 - 71 = -3$. A loss by 3 dollars.■

Remark 28.1
Marginal cost and average cost can differ greatly. For example, suppose it costs \$1000 to produce 100 units and \$1020 to produce 101 units. The average cost per unit is \$10, but the marginal cost of the 101st unit is \$20. Similar remarks apply for the marginal revenue and the marginal profit.

Supply and Demand Curves

The quantity q manufactured and sold depends on the unit price p. In general, when the price goes up then manufacturers are willing to supply more of the product whereas consumers are going to reduce their buyings. Since consumers and manufacturers react differently to changes in price, there are two curves relating p and q.

The **supply curve** is the quantity that producers are willing to make at a given price. Thus, increasing price will increase quantity.
The **demand curve** is the amount that will be bought by consumers at a given price. Thus, decreasing price will increase quantity.
Even though quantity is a function of price, it is the tradition to use the vertical y-axis for the variable p and the horizontal x-axis for the variable q. The supply and demand curves intersect at point (q^*, p^*) called the **point of equilibrium**. We call p^* the **equilibrium price** and q^* the **equilibrium quantity**.

Example 28.6
Find the equilibrium point for the supply function $S(p) = 3p - 50$ and the demand function $D(p) = 100 - 2p$.

Solution.
Setting the equation $S(p^*) = D(p^*)$ to obtain $3p^* - 50 = 100 - 2p^*$. By adding $2p^* + 50$ to both sides we obtain $5p^* = 150$. Solving for p^* we find $p^* = 30$. Substituting this value in $S(p)$ we find $q^* = 3(30) - 50 = 40.$ ■

Optimizing Profit

Recall that the profit resulting from producing and selling q items is defined by
$$P(q) = R(q) - C(q)$$
where $C(q)$ is the total cost of producing a quantity q and $R(q)$ is the total revenue from selling a quantity q of some good.

To maximize or minimize profit over a closed interval, we optimize the profit function P. We know that global extrema occur at the critical numbers of P or at the endpoints of the interval. Thus, the process of optimization requires finding the critical numbers which are the zeros of the marginal profit function

$$P'(q) = R'(q) - C'(q) = 0$$

where $R'(q)$ is the marginal revenue function and $C'(q)$ is the marginal cost function. Thus, the global maximum or the global minimum of P occurs when

$$MR(q) = MC(q)$$

or at the endpoints of the interval.

Example 28.7
Find the quantity q which maximizes profit given the total revenue and cost functions

$$\begin{aligned} R(q) &= 5q - 0.003q^2 \\ C(q) &= 300 + 1.1q. \end{aligned}$$

where $0 \leq q \leq 800$ units. What production level gives the minimum profit?

Solution.
The profit function is given by

$$P(q) = R(q) - C(q) = -0.003q^2 + 3.9q - 300.$$

The critical numbers of P are the solutions to the equation $P'(q) = 0$. That is,

$$3.9 - 0.006q = 0$$

or $q = 650$ units. Since $P(0) = -\$300, P(800) = \$900$ and $P(650) = \$967.50$ then the maximum profit occurs when $q = 800$ units and the minimum profit(or loss) occurs when $q = 0$, i.e. when there is no production.■

Example 28.8
The total revenue and total cost curves for a product are given in Figure 82.

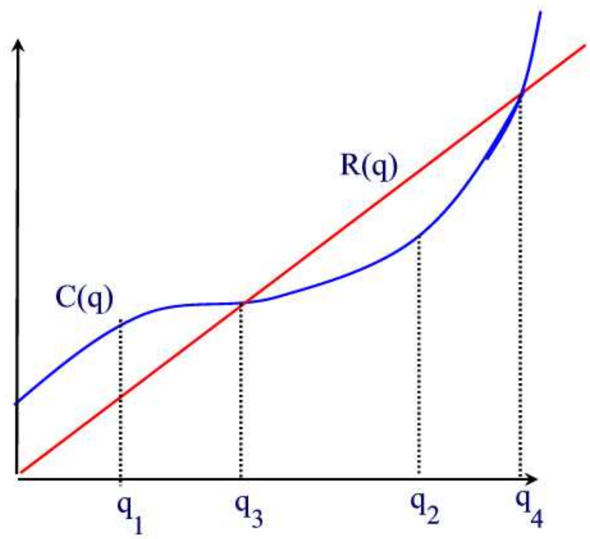


Figure 82

(a) Sketch the curves for the marginal revenue and marginal cost on the same axes. Show on this graph the quantities where marginal revenue equals marginal cost. What is the significance of these two quantities? At which quantity is profit maximum?
(b) Graph the profit function $P(q)$.

Solution.
(a) Since R is a straight line with positive slope then its derivative is a positive constant. That is, the graph of the marginal revenue is a horizontal line at some value $a > 0$. Since C is always increasing then its derivative MC is always positive. For $0 < q < q_3$ the curve is concave down so that MC is decreasing. For $q > q_3$ the graph of C is concave up and so MC is increasing. Thus, the graphs of C and R are shown in Figure 83.

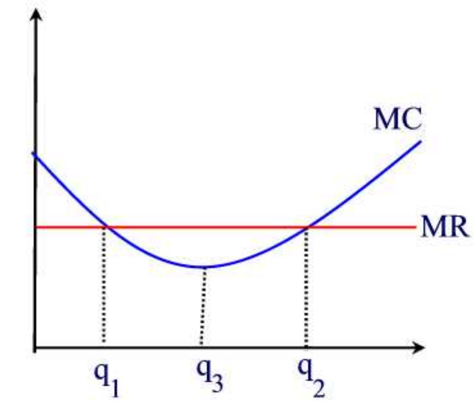


Figure 83

According to the graph, marginal revenue equals marginal cost at the values $q = q_1$ and $q = q_2$. So maximum profit occurs either at q_1, q_2 or at the endpoints. Notice that the production levels q_1 and q_2 correspond to the two points where the tangent line to C is parallel to the tangent line to R. Now, for $0 < q < q_1$ we have $MR < MC$ so that $P' = MR - MC < 0$ and this shows that P is decreasing. For $q_1 < q < q_2$, $MR > MC$ so that $P' > 0$ and hence P is increasing. So P changes from decreasing to increasing at q_1 which means that P has a minimum at q_1. Now, for $q > q_2$ we have that $MR < MC$ so that $P' < 0$ and P is decreasing. Thus, P changes from increasing to decreasing at q_2 so q_2 is a local maximum for P. So maximum profit occurs either at the endpoints or at q_2. Since profit is negative for $q < q_3$ and $q > q_4$ then the profit is maximum for $q = q_2$.

(b)

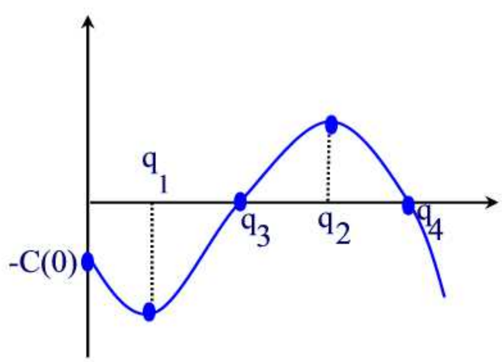


Figure 84

Recommended Problems (pp. 194 - 6): 1, 3, 5, 6, 7, 9, 10, 15.

29 Mathematical Modeling and Optimization

The branch of mathematics that consists of developing a mathematical framework that is used to solve a real world problem is referred to as **mathematical modeling**.
With the calculus skills that we have developed so far, we will use mathematical modeling and calculus to solve optimization problems. By **optimization** we mean the process of finding either the maximum or the minimum of a function.
We outline a general optimization procedure:

<u>Step 1.</u> Draw a figure (if appropriate) and label all quantities relevant to the problem.
<u>Step 2.</u> Name the quantity to be optimized. Find a formula for the quantity to be optimized.
<u>Step 3.</u> Use conditions in the problem to eliminate variables in order to express the quantity to be optimized in terms of a single variable.
<u>Step 4.</u> Find the practical domain of the variable in Step 3.
<u>Step 5.</u> If possible, use the methods you have learned so far to obtain the optimum value.

Example 29.1
A carpenter wants to make an open-topped box out of a rectangular sheet of tin 24 inches wide and 45 inches long. The carpenter plans to cut congruent squares out of each corner of the sheet and then bend and solder the edges of the sheet upward to form the sides of the box, as shown in Figure 85.

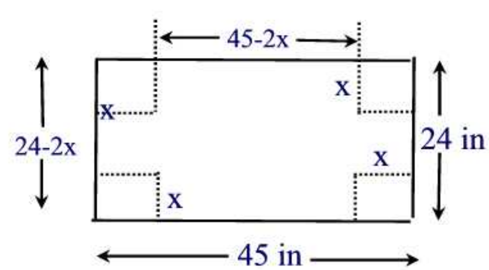


Figure 85

For what dimensions does the box have the greatest possible volume?

Solution.
Let x be the side of each square. When constructing the box, the length of the base is $45 - 2x$, the width is $24 - 2x$ and the height of the box is x. Thus, the volume is given by the formula

$$V = x(24 - 2x)(45 - 2x) = 4x^3 - 138x^2 + 1080x.$$

This quantity is to be maximized. The domain of this function consists of those numbers that satisfy $x \geq 0, 24 - 2x \geq 0$, and $45 - 2x \geq 0$. Solving this compound inequality we find that the domain is the interval $[0, 12]$.
We next find the critical numbers of the function $V(x)$ which are the zeros of $V'(x)$. That is, we solve the equation

$$12x^2 - 276x + 1080 = 0$$

or after dividing by 12 and factoring

$$(x - 18)(x - 5) = 0.$$

Thus, the critical numbers are $x = 18$ and $x = 5$. The value $x = 18$ is to be discarded since it is not in the interval $[0, 12]$. Evaluating $V(x)$ at $x = 0, x = 5$, and $x = 12$ we find $V(0) = 1080, V(5) = 2450$, and $V(12) = 0$. Thus, $V(x)$ is maximized at $x = 5$. So the box with the largest volume has dimensions $5\ in \times 14\ in \times 35\ in$.■

Example 29.2
You need to fence a rectangular play zone for children. What is the maximum area for this play zone if it is to fit into a right-triangular plot with sides measuring 4 m and 12 m?

Solution.
We start by drawing a picture of the play zone as shown in Figure 86.

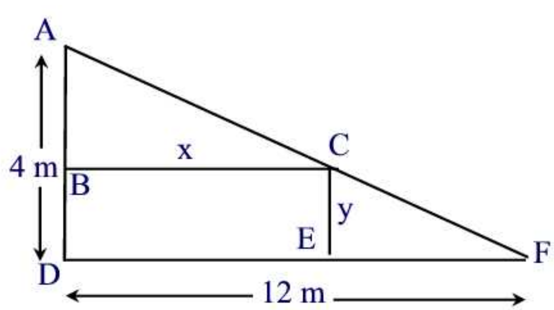


Figure 86

Let x and y be the dimensions of the inscribed rectangle. Then the quantity to be maximized is the area of the rectangle given by $A = xy$. First, we must express A as a function of a single variable. To do this, note that, the triangles ABC and ADF are similar triangles so that the corresponding sides are proportional and therefore we can write

$$\frac{4-y}{4} = \frac{x}{12}$$

or $y = 4 - \frac{1}{3}x$. Substituting back into A to obtain $A = x(4 - \frac{1}{3}x) = 4x - \frac{1}{3}x^2$ with $0 \leq x \leq 12$. Next, we find the critical numbers that are the zeros of the derivative function $A'(x) = 4 - \frac{2}{3}x$. Solving for x we find $x = 6$. Evaluating A at the endpoints and at the critical point we find $A(0) = A(12) = 0$ and $A(6) = 12$. Thus, the maximum area occurs when $x = 6$. In this case, $y = 4 - \frac{1}{3}(6) = 2$. Thus, the largest rectangular play zone that can be built in the triangular plot is a rectangle 6 m long and 2 m wide.■

Recommended Problems (pp. 201 - 3): 1, 3, 4, 6, 7, 11, 14, 16, 24.

30 Hyperbolic Functions

The hyperbolic functions possess many parallels with the trigonometric functions, both in naming and in their properties and inter-relationships.
The function e^x and its reciprocal e^{-x} can be combined to create three new functions:

Hyperbolic Cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Hyperbolic Sine

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Hyperbolic Tangent

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Example 30.1
Find $\cosh 0$ and $\sinh 0$.

Solution.
Substituting 0 for x in the formulas for $\cosh x$ and $\sinh x$ to obtain

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

and

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0.\blacksquare$$

Hyperbolic Functions Identities
Hyperbolic functions have similar identities then that of trigonometric functions.
Substituting $-x$ for x in the formula of $\cosh x$ to obtain

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x.$$

This says that $\cosh x$ is an even function.
Similarly, substituting $-x$ for x in the formula of $\cosh x$ to obtain

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh x.$$

That is, $\sinh x$ is an odd function.
Next, we derive an identity for the hyperbolic functions similar to the Pythagorean identity for the trigonometric functions.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\tfrac{e^x+e^{-x}}{2}\right)^2 - \left(\tfrac{e^x-e^{-x}}{2}\right)^2\\ &= \tfrac{e^{2x}+2+e^{-2x}}{4} - \tfrac{e^{2x}-2+e^{-2x}}{4} = 1.\end{aligned}$$

Thus, $(\cosh x, \sinh x)$ is a point on the hyperbola $x^2 - y^2 = 1$, thus, the name hyperbolic. (Recall that the functions $\sin x$ and $\cos x$ are called circular functions since $(\cos x, \sin x)$ is a point on the unit circle!)

Derivatives of Hyperbolic Functions
Sine $(e^x)' = e^x$ and $(e^{-x})' = -e^{-x}$ then we have

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Similarly,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

To find the derivative of $\tanh x$, we use the quotient rule

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Graphs of Hyperbolic Functions
The graphs of $\cosh x$ and $\sinh x$ are given in Figure 87.

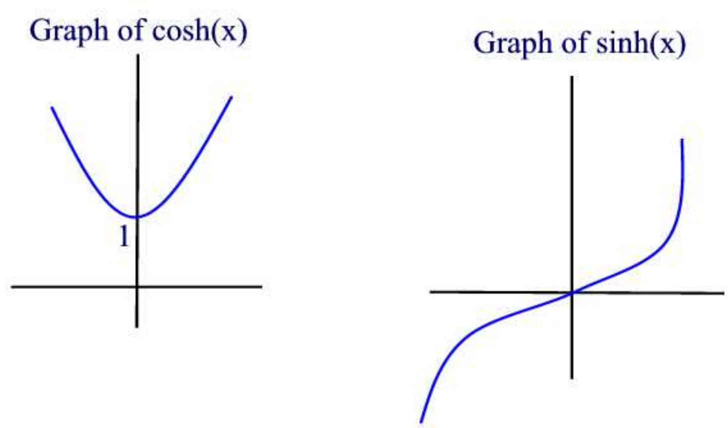


Figure 87

Recommended Problems (p. 206): 1, 3, 5, 7, 12, 13, 14, 15, 20.

31 The Mean Value Theorem

We start this section by showing that the derivative of a function evaluated at a local extremum is always zero.

Theorem 31.1
If a differentiable function f has a local extremum a inside an interval I then $f'(a) = 0$.

Proof.
We will prove the theorem for a local maximum. The proof for a local minimum is similar. So suppose that f has a local maximum at a on an open interval I. Then $f(x) \leq f(a)$ for all x in I. In particular, for h close to zero the point $a + h$ is in the interval I and therefore $f(a + h) \leq f(a)$ or $f(a + h) - f(a) \leq 0$. If h is close to zero from the left (i.e. $h < 0$) then Dividing both sides by h to obtain $\frac{f(a+h)-f(a)}{h} \geq 0$.(Note the inequality sign is reversed since we are dividing by a negative h.) Now, take the limit of both sides as $h \to 0^-$ to obtain

$$\lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h} \geq 0$$

That is, $f'(a) \geq 0$. If h is close to zero from the right, i.e. $h > 0$ then by dividing both sides of the inquality $f(a+h)-f(a) \leq 0$ by h and then letting $h \to 0^+$ we obtain

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \leq 0$$

That is, $f'(a) \leq 0$. We have shown that $0 \leq f'(a) \leq 0$. This is only true when $f'(a) = 0$.■

We have seen in Section 27 that a continuous function on a closed interval has a maximum and a minimum at either the critical numbers of the function or at the endpoints of the interval. This is a consequence of the following theorem

Theorem 31.2 *(The Extreme Value Theorem)*
A continuous function on a closed interval $[a, b]$ has both a global maximum and a global minimum on that interval.

The proof of this theorem is beyond the scope of this course and is therefore omitted.
Now consider the following problem: Let $f(x)$ be a continuous function on $[a, b]$ and differentiable in (a, b). Is there a point on the graph of f where the tangent line at that point is parallel to the line passing through the points $(a, f(a))$ and $(b, f(b))$? The answer is yes according to the following theorem

Theorem 31.3 *(Mean Value Theorem)*
If f is continuous on $[a, b]$ and differentiable in (a, b) then there is a number $a < c < b$ such that the tangent line to the graph at $(c, f(c))$ is parallel to the line passing through the points $(a, f(a))$ and $(b, f(b))$. That is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof.
We first consider the case when $f(a) = f(b)$. If f is a constant function then $f'(x) = 0$ in (a, b) and therefore letting $c = \frac{a+b}{2}$ we have $f'(c) = 0 = \frac{f(b)-f(a)}{b-a}$. So suppose that f is not a constant function. By Theorem 31.2, f has a global maximum and a global minimum and they are both different since f is not constant. So one of them, called it c, must be in (a, b). By Theorem 31.1, $f'(c) = 0 = \frac{f(b)-f(a)}{b-a}$.
Now, suppose that $f(a) \neq f(b)$. Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that g is differentiable in $[a, b]$ and continuous in (a,b). Moreover, $g(a) = g(b) = 0$. So by the first case, there is a $a < c < b$ such that $g'(c) = 0$. That is,

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

or

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \blacksquare$$

Example 31.1
Use the Mean Value Theorem to show the following: Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ on (a, b) then f is a constant function. That is, $f(x) = c$ for all x in $[a, b]$ where c is a constant.

Solution.
We just need to show that any two values x_1 and x_2 in $[a,b]$ have the same y-value, that is, $f(x_1) = f(x_2)$. By the MVT, there is $x_1 < c < x_2$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But $f'(c) = 0$ so that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$. This implies that $f(x_2) - f(x_1) = 0$ or $f(x_1) = f(x_2)$.$\blacksquare$

Theorem 31.4 *(The Increasing Function Theorem)*
Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable in (a,b).

(a) If $f'(x) > 0$ on $[a,b]$ then f is increasing on [a,b](that is if $x_1 < x_2$ then $f(x_1) < f(x_2)$, where $a \leq x_1, x_2 \leq b$).
(b) If $f'(x) \geq 0$ on (a,b) then f is nondecreasing on [a,b](that is if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$, where $a \leq x_1, x_2 \leq b$).

Proof.
Let x_1 and x_2 be two numbers in $[a,b]$ such that $x_1 < x_2$. By the Mean Value Theorem, there is a number c inside the interval (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

(a) Since $f'(x) > 0$ in (a,b) then the right hand side is positive. That is, $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$.
(b) Since $f'(x) \geq 0$ then $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0$ or $f(x_1) \leq f(x_2)$.$\blacksquare$

Theorem 31.5 *(The Racetrack Principle)*
Suppose that f and g are continuous on $[a,b]$ and differentiable in (a,b). Furthemore, suppose that $f'(x) \leq g'(x)$ in (a,b).
(a) If $f(a) = g(a)$ then $f(x) \leq g(x)$ for all $a \leq x \leq b$.
(b) If $f(b) = g(b)$ then $f(x) \geq g(x)$ for all $a \leq x \leq b$.

Proof.
(a) Let $h(x) = g(x) - f(x)$. Since $f'(x) \leq g'(x)$ for all $a < x < b$ then $h'(x) \geq 0$ for all $a < x < b$. By part (b) of the previous exercise, $h(x)$ is nondecreasing. So, for any $a \leq x \leq b$ we have $h(a) \leq h(x)$. But $h(a) = f(a) - g(a) = 0$. Thus, $h(x) \geq 0$ or $f(x) \leq g(x)$.
(b) Now, for $a \leq x \leq b$ we have $h(x) \leq h(b)$. But $h(b) = f(b) - g(b) = 0$. Thus, $h(x) \leq 0$ or $f(x) \geq g(x)$.$\blacksquare$

Recommended Problems (pp. 210 - 1): 1, 3, 7, 9, 11, 12, 13, 15, 19, 24, 26, 27.

32 Measuring The Distance Traveled

We have seen that the velocity of an object moving along the curve $s(t)$ is obtained by taking the average rate of change on smaller and samller intervals , that is finding the derivative of s, i.e. $v(t) = s'(t)$. In this and the following sections we want to go the opposite direction. That is, given the velocity function $v(t)$ we want to find the position function $s(t)$.
To be more precise, suppose that we want to estimate the distance s traveled by a car after 10 seconds of departure. Assume for example, that we are given the velocity of the car every two seconds as shown in the table below

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

Since we don't know the instantaneous velocity of the car at every momentm then we can not calculate the distance exactly. What we can do is to estimate the distance traveled. For the first two seconds, the velocity is at least 20 miles per second so that the distance traveled is at least $20 \times 2 = 40$ feet. Likewise, at least $30 \times 2 = 60$ feet has been traveled the next two seconds and so on. Thus, we obtain a lower estimate to the exact distance traveled

$$20 \times 2 + 30 \times 2 + 38 \times 2 + 44 \times 2 + 48 \times 2 = 360\ feet.$$

However, we can reason differently and get an overestimate to the total distance traveled as follows: For the first two seconds the car's velocity is at most 30 feet so that the car travels at most $30 \times 2 = 60$ feet. In the next two seconds, it travels $38 \times 2 = 76$ feet and so on. So an upper estimate of the total distance traveled is

$$30 \times 2 + 38 \times 2 + 44 \times 2 + 48 \times 2 + 50 \times 2 = 420\ feet$$

Hence,

$$360\ feet \leq\ Total\ distance\ traveled \leq 420\ feet.$$

Notice that the difference between the upper and lower estimates is 60 feet. Figure 88 shows both the lower estimate and the upper estimate. The graph of the velocity is obtained by plotting the points given in the above table and then connect them with a smooth curve. The area of the lower rectangles represent the lower estimate and the larger rectangles represent the upper estimate.

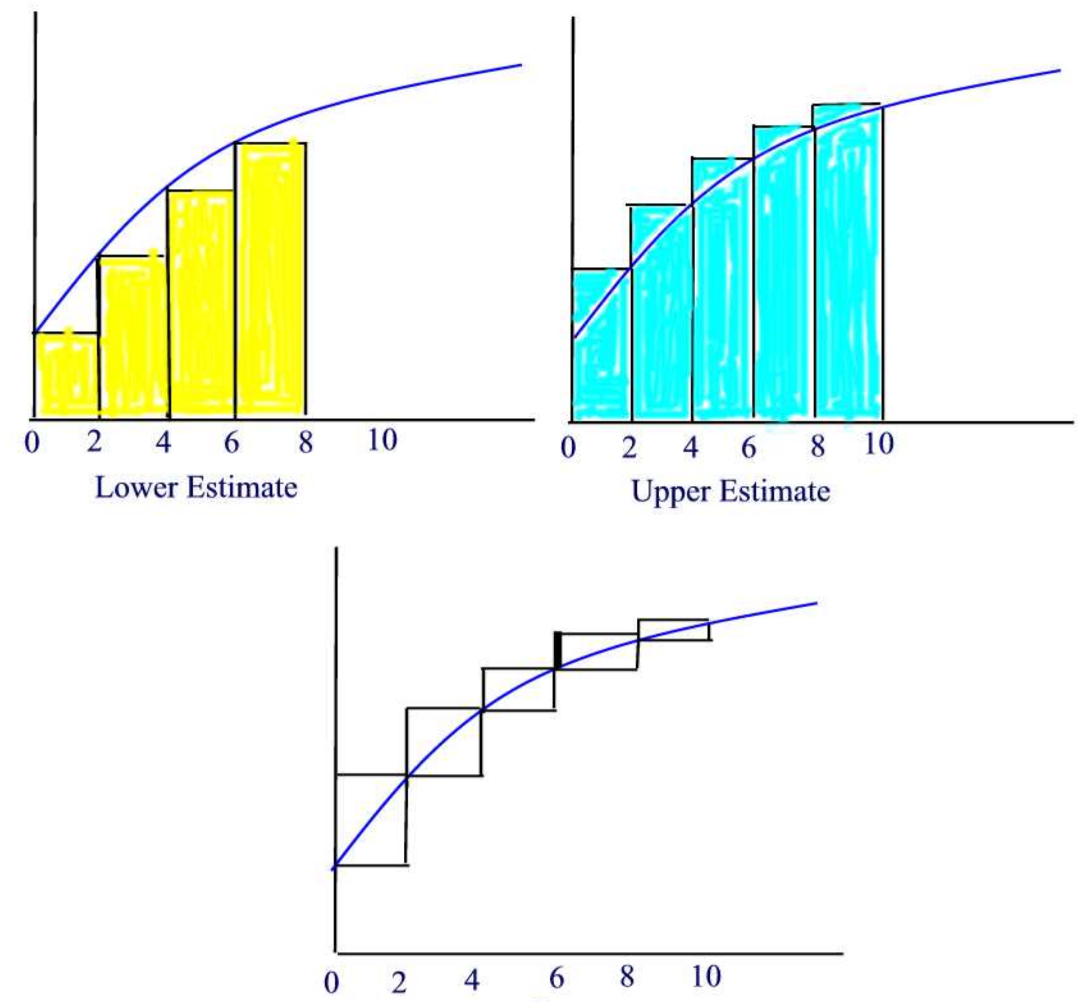


Figure 88

To visualize the difference between the upper and lower estimates, look at the above figure, and imagine that all the unshaded rectangles are pushed to the right and stacked on top of each other. This gives a rectangle of width 2 and height 30 so its area is the difference between the estimates.

Example 32.1
Suppose that the velocity of the car is given every second instead as shown in the table below. Find the lower and upper estimates of the total distance

traveled. What is the difference between the lower and upper estimates? Do you think that knowing the velocity at every second is a better estimate than knowing the velocity every two seconds?

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	35	38	42	44	46	48	49	50

Solution.
The lower estimate is

$$(20)(1) + (26)(1) + \cdots + (48)(1) + (49)(1) = 378 \; feet$$

and the upper estimate is

$$(26)(1) + (30)(1) + \cdots + (49)(1) + (50)(1) = 408 \; feet$$

Hence,

$$378 \; feet \leq \; Total \; distance \; traveled \leq 408 \; feet.$$

So the difference between the upper and lower estimates is $408 - 378 = 30 \; feet$. This shows that by increasing the partition points we get better and better estimate.■

Remark 32.1
Once the upper estimate and the lower estimate are found then one can get an even better estimate by taking the average of the two estimates.

The use of the average rate of change of the distance leads to finding the total distance traveled. This same method can be used to find the total change from the rate of change of other quantities.

Example 32.2
The following table gives world oil consumptions, in billions of barrels per year. Estimate the total oil consumption during this 20-year period.

Year	1980	1985	1990	1995	2000
Oil (barrels/yr)	22.3	23.0	23.9	24.9	27.0

Solution.
We underestimate the total oil consumption as follows:

$$22.3 \times 5 + 23.0 \times 5 + 23.9 \times 5 + 24.9 \times 5 = 470.5 \; billion\ barrels.$$

The overestimate is

$$23.0 \times 5 + 23.9 \times 5 + 24.9 \times 5 + 27.0 \times 5 = 494 \; billion\ barrels.$$

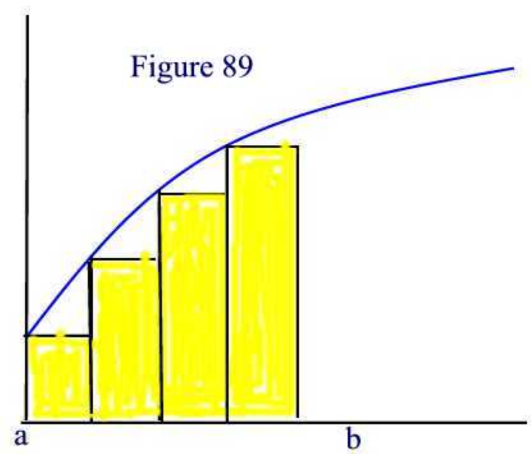
A good single estimate of the total oil consumption is the average of the above estimates. That is

$$Total\ oil\ consumption \approx \frac{470.5 + 494}{2} = 482.25 \; billion\ barrels.\blacksquare$$

• **Finding the Exact Distance Traveled**
Suppose that we want to find the total distance traveled over the time interval $a \leq t \leq b$. We take measurements of the velocity $v(t)$ at equally spaced times, $a = t_0, t_1, t_2, \cdots, t_n = b$. This means that we devide the interval $[a, b]$ into n equal pieces each of length $\Delta t = \frac{b-a}{n}$. We first use the left-end point of each interval $[t_{i-1}, t_i]$ and construct the **left-hand sum**

$$L(v, n) = v(t_0)\Delta t + v(t_1)\Delta t + \cdots + v(t_{n-1})\Delta t.$$

Geometrically, this sum represents the sum of areas of rectangles constructed by taking the height to be the value of the function at the left-endpoint of each subinterval. See Figure 89.



Secondly, we use the right-end point of each interval $[t_{i-1}, t_i]$ and construct the **right-hand sum**

$$R(v, n) = v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t.$$

Geometrically, this sum represents the sum of areas of rectangles constructed by taking the height to be the value of the function at the right-endpoint of each subinterval. See Figure 90.

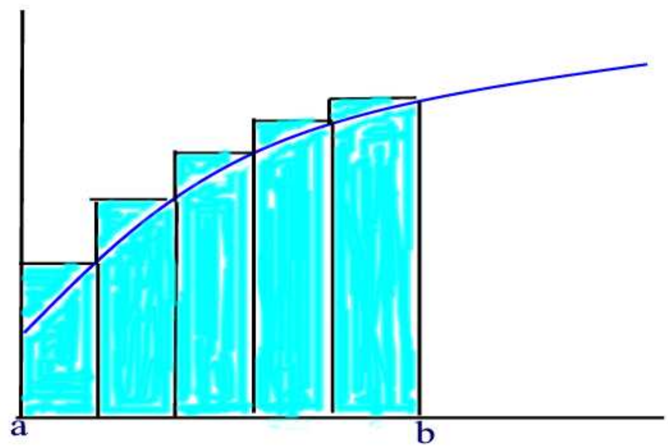


Figure 90

Now, the exact distance traveled lies between the two estimates. As we have seen earlier, by making the time interval smaller and smaller we can make the difference between the two estimates as small as we like. This is equivalent to letting $n \to \infty$. If the function $v(t)$ is continuous then the following two limits are equal to the exact distance traveled from $t = a$ to $t = b$.

$$\textit{Total distance traveled} = \lim_{n\to\infty} L(v, n) = \lim_{n\to\infty} R(v, n).$$

Geometrically, each of the above limit represents the area under the graph of $v(t)$ bounded by the lines $t = a, t = b$ and the horizontal axis. See Figure 91.

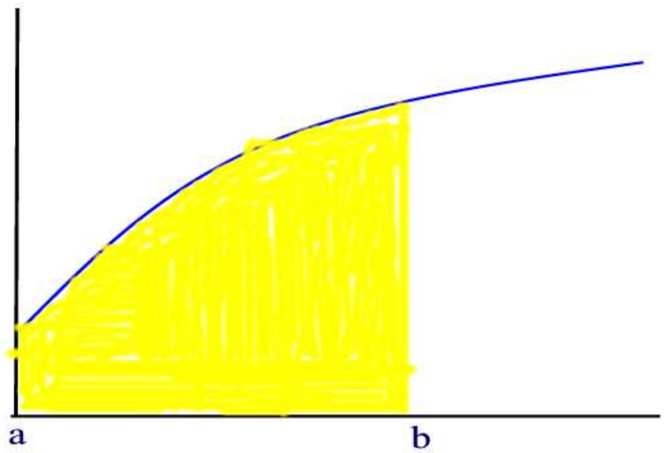


Figure 91

Remark 32.2
Notice that for an increasing function the left-hand sum is an underestimate whereas the right-hand sum is an overestimate. This role is reversed for a decreasing function.

Recommended Problems (pp. 227 - 8): 2, 3, 4, 5, 8, 10, 12, 13.

33 The Definite Integral

We discussed in the previous section how to find the exact distance traveled by taking the limit of both the left-hand sum and the right-hand sum as the number of subintervals increases without bound. In this section, we will construct these sums for any continuous function on a closed interval $[a, b]$. We start by dividing the interval [a,b] into n subintervals each of length

$$\Delta x = \frac{b-a}{n}.$$

Let $a = x_0, x_1, \cdots, x_{n-1}, x_n = b$ be the endpoints of the subdivisions. We construct the **left-hand sum** or the **left Riemann sum**

$$L(f, n) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

and the **right-hand sum** or the **right Riemann sum**

$$R(f, n) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

It is shown in advanced calculus that for a continuous function on a closed interval $[a, b]$ that as $n \to \infty$ both the left-hand sum and the right-hand sum exist and are equal. We denote the common value by the notation $\int_a^b f(x)dx$. Thus,

$$\begin{aligned} \textstyle\int_a^b f(x)dx &= \lim_{n\to\infty} L(f, n) \\ &= \lim_{n\to\infty} R(f, n). \end{aligned}$$

We call $\int_a^b f(x)dx$ the **definite integral** of f from $x = a$ to $x = b$. We call a the **lower limit** and b the **upper limit**. The function f is called the **integrand**.

Example 33.1

(a) On a sketch of $y = \ln x$, represent the left Riemann sum with $n = 2$ approximating $\int_1^2 \ln x dx$. Write out the terms in the sum, but do not evaluate.

(b) On a another sketch of $y = \ln x$, represent the right Riemann sum with $n = 2$ approximating $\int_1^2 \ln x dx$. Write out the terms in the sum, but do not evaluate.

(c) Which sum is an underestimate? Which sum is an overestimate?

Solution.
(a) The left Riemann sum is the sum

$$L(\ln x, 2) = \ln 1(0.5) + \ln(1.5)(0.5) = 0.5\ln(1.5).$$

The Left Riemann sum is shown in Figure 92.

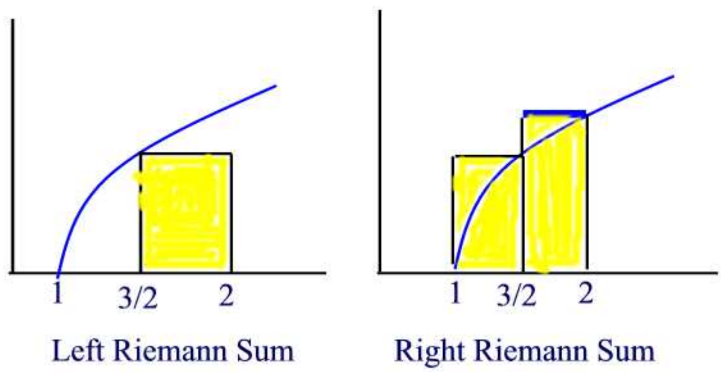


Figure 92

(b) The right Riemann sum is the sum

$$R(\ln x, 2) = \ln(1.5)(0.5) + \ln 2(0.5) = 0.5\ln(2)(1.5) = 0.5\ln 3.$$

The Right Riemann sum is shown in Figure 92.
(c) $L(\ln x, 2) < \int_1^2 \ln x dx < R(\ln x, 2)$.■

Geometrical Significance of $\int_a^b f(x)dx$
Case 1: $f(x) \geq 0$
Looking closely to either the left Riemann sum or the right Riemann sum we see that if $f(x) \geq 0$ then a term of the form $f(x)\Delta x$ represents the area of a rectangle. As n increases without bound, that is, the width Δx of the rectangles approaches zero, the rectangles fit the curve of the graph more exactly, and the sum of their areas gets closer and closer to the area under the graph, bounded by the vertical lines $x = a$ and $x = b$ and the x-axis. Thus,

$$\int_a^b f(x)dx = \textit{Area under graph of } f \textit{ between } a \textit{ and } b.$$

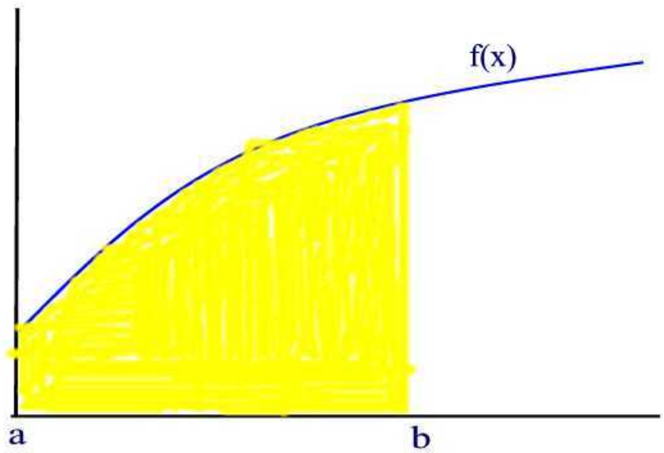


Figure 93

Example 33.2
Consider the integral $\int_{-1}^{1} \sqrt{1-x^2}dx$.

(a) Interpret the integral as an area, and find its exact value.
(b) Estimate the integral using a calculator.

Solution.
(a) Note that the equation of a circle centered at the origin and with radius 1 is given by $x^2 + y^2 = 1$. Solving for y we find $y = \pm\sqrt{1-x^2}$. The function $y = \sqrt{1-x^2}$ corresponds to the upper semicircle and the function $y = -\sqrt{1-x^2}$ corresponds to the lower semicircle. See Figure 94.

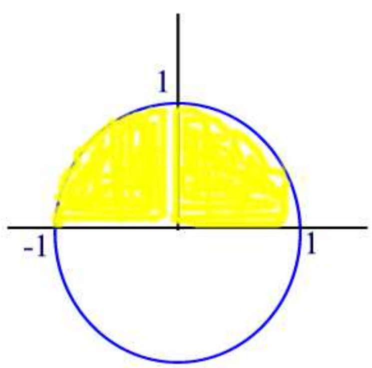


Figure 94

It follows that the given integral represents the area of the upper semicircle and therefore is equal to $\frac{\pi}{2}$. That is,

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{\pi}{2}.$$

(b) Using a TI-83 calculator we find

$$fnInt(\sqrt{1-x^2}, x, -1, 1) \approx 1.571.\blacksquare$$

Case 2:$f(x) \leq 0$

In this case, since each product of the form $f(x)\Delta x$ is less than or equal to zero then the area gets counted negatively. That is, the absolute value of the integral gives the area above the curve between $x = a$ and $x = b$.

Example 33.3

Find the area above the graph of $y = x^2 - 1$ from $x = -1$ to $x = 1$.

Solution.

The graph of $y = x^2 - 1$ is shown in Figure 95.

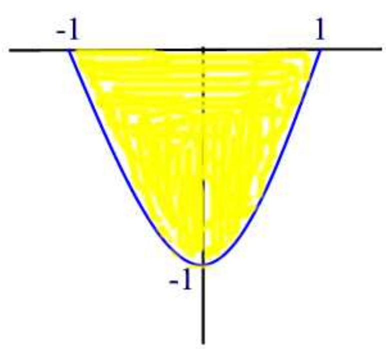


Figure 95

The area is given by $|\int_{-1}^{1}(x^2-1)dx| \approx |-1.33| = 1.33.\blacksquare$

Case 3:f changes sign

In this case, the integral is the sum of the areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively. If the integral is positive then the region above the x-axis has larger area than the region below the x-axis. If the integral is negative then the region below the x-axis has a larger area then the region above the x-axis.

Example 33.4
Find the area between the graph of $y = x^3$ and the x-axis from $x = -1$ to $x = 1$.

Solution.
The area is shown in Figure 96.

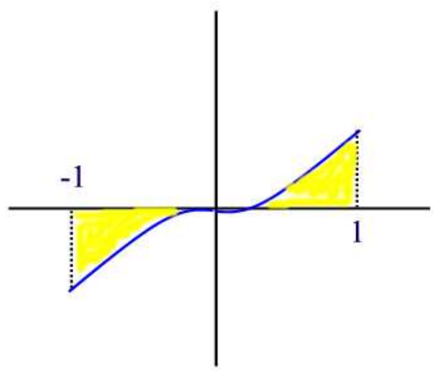


Figure 96

It follows that the area is given by

$$\left|\int_{-1}^{0} x^3 dx\right| + \int_0^1 x^3 dx = 0.5. \blacksquare$$

Recommended Problems (pp. 234 - 6): 2, 3, 7, 8, 9, 11, 15, 17, 20, 22, 24, 27, 30, 31, 32.

34 Interpretations of the Definite Integral

We start this section by showing that the definite integral of a rate of change gives the total change of the function. We define the total change of a function $F(t)$ from $t = a$ to $t = b$ to be the difference $F(b) - F(a)$. Suppose that $F(t)$ is continuous in [a,b] and differentiable in (a, b). Divide the interval $[a, b]$ into n equal subintervals each of length $\Delta t = \frac{b-a}{n}$. Let $a = t_0, t_1, \cdots, t_n = b$ be the partition points of the subdivision. Then on the interval $[t_0, t_1]$ the change in F can be estimated by the formula

$$F'(t_0) \approx \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}$$

or

$$F(t_0 + \Delta t) - F(t_0) \approx F'(t_0)\Delta t$$

that is

$$F(t_1) - F(t_0) \approx F'(t_0)\Delta t$$

On the interval $[t_1, t_2]$ we get the estimation

$$F(t_2) - F(t_1) = F'(t_1)\Delta t$$

Continuing in this fashion we find that on the interval $[t_{n-1}, t_n]$ we have

$$F(t_{n-1}) - F(t_n) \approx F'(t_{n-1})\Delta t.$$

Adding all these approximations we find that

$$F(t_n) - F(t_0) \approx \sum_{i=0}^{n-1} F'(t_i)\Delta t$$

Letting $n \to \infty$ we see that

$$F(b) - F(a) = \int_a^b F'(t)dt.$$

Example 34.1
The amount of waste a company produces, W, in metric tons per week, is approximated by $W = 3.75e^{-0.008t}$, where t is in weeks since January 1, 2000. Waste removal for the company costs \$15/ton. How much does the company pay for waste removal during the year 2000?

Solution.
The amount of tons produced during the year 2000 is just the definite integral $\int_0^{52} W(t)dt$. Using a calculator we find that

$$Total\ waste\ during\ the\ year = \int_0^{52} 3.75e^{-0.008t}dt \approx 159\ tons.$$

The cost to remove this quantity is $159 \times 15 = \$2385.$■

Remark 34.1
When using $\int_a^b f(x)dx$ in applications then its units is the product of the units of $f(x)$ with the units of x.

The Definite Integral as an Average
We know that the average of n given numbers is just the sum divided by n. What is the average in the continuous case. That is, what is the average of a continuous function on a closed interval $[a,b]$? Partition the interval into n equal subintervals each of length $\Delta t = \frac{b-a}{n}$ and let $a = t_0, t_1, t_2, \cdots, t_n$ be the division points. Then

$$Average\ of\ f(t)\ on\ [a,b] \approx \frac{f(t_0) + f(t_1) + \cdots + f(t_{n-1})}{n}$$

But $n = \frac{b-a}{\Delta t}$ so that

$$\begin{array}{rcl} Average\ of\ f(t)\ on\ [a,b] & \approx & \frac{1}{b-a}(f(t_0) + f(t_1) + \cdots + f(t_{n-1}))\Delta t \\ & = & \frac{1}{b-a}\sum_{i=0}^{n-1} f(t_i)\Delta t \end{array}$$

Letting $n \to \infty$ we see that

$$Average\ of\ f(t)\ on\ [a,b] = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 34.2
A bar of metal is cooling from $1000°C$ to room temperature, $20°C$. The temperature, H, of the bar t minutes after it starts cooling is given by

$$H = 20 + 980e^{-0.1t}.$$

Find the average temperature over the first hour.

Solution.
The average temperature is given by

$$Average\,temperature\,for\,the\,first\,hour = \frac{1}{60}\int_0^{60}(20+980e^{-0.1t})dt \approx 183°C.\blacksquare$$

Remark 34.2
From the definition of the average value we can write

$$(average\ value\ of\ f) \times (b-a) = \int_a^b f(x)dx$$

Geometrically, this says that the area of the rectangle of dimensions $(average\ value\ of\ f)\times(b-a)$ is equal to the area under the graph of $f(x)$ from $x=a$ to $x=b$. See Figure 97.

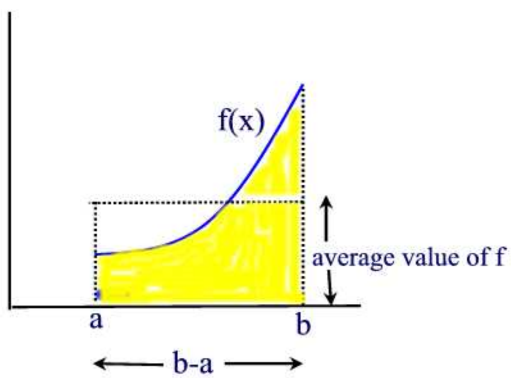


Figure 97

Recommended Problems (pp. 242 - 4): 3, 4, 6, 10, 12, 13, 14, 20, 21, 22, 23, 25, 31.

35 Theorems About Definite Integrals

The following result is considered among the most important result in calculus.

Theorem 35.1 *(The Fundamental Theorem of Calculus)*
If $f(x)$ is a continuous function on [a,b] and $F'(x) = f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

We call the function $F(x)$ an **antiderivative** of $f(x)$.

Proof.
Partition the interval [a,b] into n subintervals each of length $\Delta x = \frac{b-a}{n}$ and let $\{a = x_0, x_1, \cdots, x_n = b\}$ be the partition points. Applying the Mean Value Theorem on the interval $[x_0, x_1]$ we can find a number $x_0 < c_1 < x_1$ such that

$$F(x_1) - F(x_0) = F'(c_1)\Delta x.$$

Continuing this process on the remaining intervals we find

$$\begin{array}{ccc} F(x_2) - F(x_1) & = & F'(c_1)\Delta x \\ & \vdots & \\ F(x_n) - F(x_{n-1}) & = & F'(c_n)\Delta x \end{array}$$

Adding these equalities we find

$$F(x_n) - F(x_0) = \sum_{i=1}^{n} f(c_i)\Delta x$$

Letting $n \to \infty$ to obtain

$$F(b) - F(a) = \int_a^b f(x)dx \blacksquare$$

Example 35.1
Use FTC to compute $\int_1^2 2xdx$. Use a calculator to find the answer to the integral and compare.

Solution.
Since the derivative of x^2 is 2x then $F(x) = x^2$. Thus, by the FTC we have

$$\int_1^2 2xdx = F(2) - F(1) = 4 - 1 = 3.$$

Using a calculator we find $\int_1^2 2xdx = 3.$■

When a function is symmetric about the y-axis, i.e. the function is even, then the y-axis divide the region under consideration into two equal subregions. This leads to

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

If the function is symmetric about the origin, i.e. the function is odd, then one region in the reflection of the other about the x-axis followed by a reflection about the y-axis so that

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_{-a}^{0} f(x)dx - \int_{-a}^{0} f(x)dx = 0.$$

Example 35.2
Without using a calculator, find the value of the integral $\int_{-1}^{1}(x^3 - x)^{1001}dx$.

Solution.
Since the integrand is odd then $\int_{-1}^{1}(x^3 - x)^{1001}dx = 0.$■

Next, we discuss several properties of definite integrals. In what follows, the functions f and g are continuous on [a,b] and c is a constant. We partition the interval [a,b] into n subintervals and we let $\{a = x_0, x_1, \cdots, x_n = b\}$. Then

$$\int_a^b cf(x) = \lim_{n\to\infty}\sum_{i=0}^{n-1} cf(x_i)\Delta x = c\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x = c\int_a^b f(x)dx.$$

$$\begin{array}{lcc}
\int_a^b[f(x) \pm g(x)]dx & = & \lim_{n\to\infty}\sum_{i=0}^{n-1}[f(x_i) \pm g(x_i)]\Delta x \\
 & = & \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x \pm \lim_{n\to\infty}\sum_{i=0}^{n-1} g(x_i)\Delta x \\
 & = & \int_a^b f(x)dx \pm \int_a^b g(x)dx \\
\int_a^b f(x)dx & = & \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\frac{b-a}{n} \\
 & = & -\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\frac{a-b}{n} = -\int_b^a f(x)dx
\end{array}$$

It follows from the last equality that if we replace b by a then $\int_a^a f(x)dx = -\int_a^a f(x)dx$ or $2\int_a^a f(x)dx = 0$ and this implies that

$$\int_a^a f(x)dx = 0.$$

Now, if $f(x) \leq g(x)$ in $[a, b]$ then

$$\begin{array}{rcl} \int_a^b f(x)dx & = & \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x \\ & \leq & \lim_{n\to\infty} \sum_{i=0}^{n-1} g(x_i)\Delta x = \int_a^b g(x)dx \end{array}$$

If $f(x)$ satisfies $m \leq f(x) \leq M$ on [a,b] then applying the last result and the fact that $\int_a^b dx = b - a$ to get

$$m(b-a) \leq \int_a^b f(x)dx \leq (b-a)M.$$

Example 35.3
Without using a calculator, show that $\int_0^{\sqrt{\pi}} \sin(x^2)dx \leq \sqrt{\pi}$.

Solution.
Since $\sin(x^2) \leq 1$ for all x then by integrating both sides from 0 to $\sqrt{\pi}$ to obtain

$$\int_0^{\sqrt{\pi}} \sin(x^2)dx \leq \int_0^{\sqrt{\pi}} dx$$

But the right-hand integral is the area of the rectangle of length 1 and height $\sqrt{\pi}$ so that $\int_0^{\sqrt{\pi}} dx = \sqrt{\pi}$.■

Example 35.4
Show the following

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Solution.
Let $F(x)$ be a function such that $F'(x) = f(x)$. Then by the Fundamental Theorem of Calculus we have

$$\begin{array}{rcl} \int_a^c f(x)dx + \int_c^b f(x)dx & = & (F(c) - F(a)) + (F(b) - F(c)) \\ & = & F(b) - F(a) \\ & = & \int_a^b f(x)dx \blacksquare \end{array}$$

Recommended Problems (pp. 250 - 2): 1, 3, 6, 8, 10, 13, 15, 17, 18, 19, 21, 22.

36 Finding Antiderivatives Graphically and Numerically

Recall that if a function $f(x)$ is given then the new function $f'(x)$ is called the **derivative** of $f(x)$. For example, if $f(x) = x^2$ then $f'(x) = 2x$. This process is referred to as **differentiation**.
Now, if instead $f'(x)$ is known then the process of finding $f(x)$ is called **integration** or **antidifferentiation**. In this case, we call $f(x)$ an **antiderivative** of $f'(x)$. In general, if f and F are two functions such that $F' = f$ then we say that $F(x)$ is an antiderivative of $f(x)$. For example, x^2 is an antiderivative of $2x$ since $(x^2)' = 2x$. Note that, there are infinitely many antiderivatives of $2x$, namely, $x^2 + C$ where C is a constant. We call $x^2 + C$ the **general antiderivative** (or the **indefinite integral**) of x^2 and we represent this symbolically by

$$\int 2xdx = x^2 + C.$$

Next, we see how to reconstruct the graph of f given the graph of its derivative f'.

Example 36.1
The graph of $f'(x)$ is given in Figure 98.

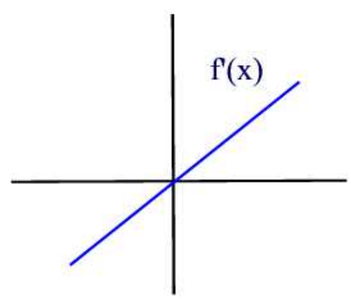


Figure 98

Sketch a graph of the function $f(x)$ satisfying $f(0) = 1$.

Solution.
Note that since $f'(x)$ is always increasing then $f''(x) > 0$ so that the graph of

$f(x)$ is always concave up. Since $f'(x) < 0$ for $x < 0$ then $f(x)$ is decreasing there. Similarly, since $f'(x) > 0$ for $x > 0$ then $f(x)$ is increasing there. Since $f'(0) = 0$ and $f(x)$ is decreasing and then increasing we conclude that $x = 0$ is a minimum. A graph of $f(x)$ is given in Figure 99.■

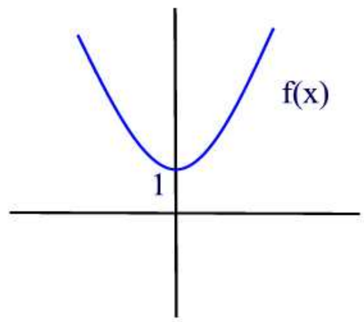


Figure 99

Example 36.2
The graph of $f'(x) = e^{-x^2}$ is given in Figure 100.

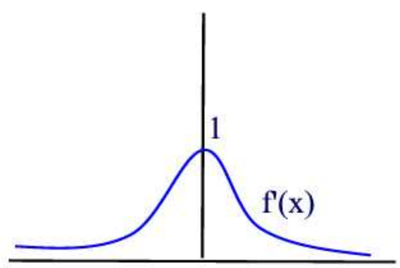


Figure 100

Sketch the graph of $f(x)$ satisfying $f(0) = 0$.

Solution.
Since $f'(x)$ is always positive then the graph of $f(x)$ is always increasing. Now, for $x < 0$, $f'(x)$ is increasing so that $f''(x) > 0$ and therefore $f(x)$ is concave up. For $x > 0$ the function $f'(x)$ is decreasing and so $f''(x) < 0$. That is, $f(x)$ is concave down there. Thus, $x = 0$ is a point of inflection. Finally, since $\lim_{x\to\pm\infty} f'(x) = 0$ then the graph of $f(x)$ levels off at both ends. See Figure 101.■

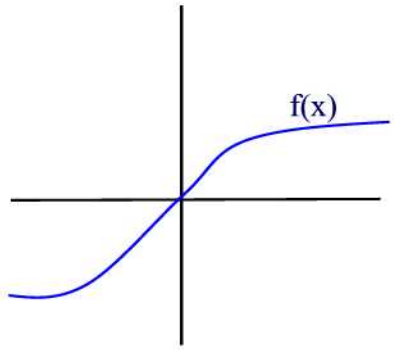


Figure 101

Next, we will reconstruct numerically the antiderivative f by using the Fundamental Theorem of Calculus: If $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$. In particular, we have

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Example 36.3
Suppose that $f'(t) = t\cos t$ and $f(0) = 2$. Find $f(0.3)$.

Solution.
Let $a = 0$ in the FTC to obtain

$$f(b) - f(0) = \int_0^b t\cos t dt.$$

But $f(0) = 2$ so the previous equation becomes

$$f(b) = 2 + \int_0^b t\cos t dt.$$

Thus,

$$f(0.3) = 2 + \int_0^{0.3} t\cos t dt.$$

Using the TI83 command $fnInt(x*cosx, x, 0, 0.3)$ we find that $\int_0^{0.3} t\cos t dt \approx 0.044$ so that $f(0.3) \approx 2.044.\blacksquare$

Now, recall that for $f(x) \geq 0$ the definite integral $\int_a^b f(x)dx$ represents the area under the graph of $f(x)$ between the lines $x = a$ and $x = b$. If the region is below the x-axis then $\int_a^b f(x)dx$ is the negative of the area of that region.

Example 36.4
Figure 102 shows the graph of $f'(x)$. Suppose that $f(-1) = -2$. Find $f(0), f(1)$, and $f(3)$.

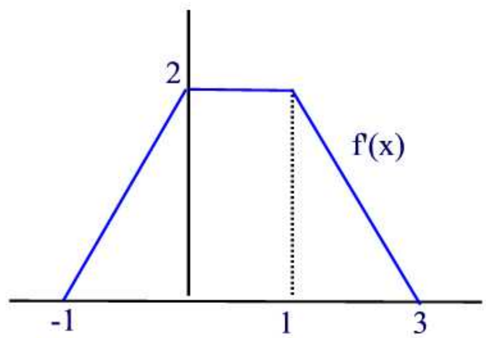


Figure 102

Solution.
By the FTC we have

$$f(b) = f(-1) + \int_{-1}^{b} f'(x)dx.$$

Thus,

$$\begin{aligned} f(0) &= f(-1) + \int_{-1}^0 f'(x)dx \\ &= -2 + \tfrac{1}{2}(1 \cdot 2) = -1 \\ f(1) &= f(0) + \int_0^1 f'(x)dx \\ &= -1 + 1 \cdot 2 = 1 \\ f(3) &= f(1) + \int_1^3 f'(x)dx = 1 + \tfrac{1}{2}(2 \cdot 2) = 3 \end{aligned}$$

where we compute $\int_a^t f'(x)dx$ by determining the area between f' and the horizontal axis for $a \leq x \leq t$.■

Recommended Problems (pp. 265 - 7): 1, 3, 5, 7, 9, 11, 13, 17, 18.

37 Analytical Construction of Antiderivatives

In this section we will find analytical expressions of antiderivatives. As discussed in the previous section, there are infinitely many antiderivatives of a given function $f(x)$. They all differ by a constant and the family of antiderivatives is represented by $F(x) + C$. The notation of the general antiderivative is called an **indefinite integral** and is written

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is the symbol of integration, $f(x)$ is called the **integrand** and C is called the **constant of integration**. Keep in mind the relationship between $f(x)$ and $F(x)$ which is given by $F'(x) = f(x)$.

Warning: The indefinite integral is a short-hand notation for a family of functions $F(x) + C$ with the property $F'(x) = f(x)$ for all x. It is not to be confused with the definite integral $\int_a^b f(x)dx$ which is a real number.

Example 37.1
Show that $\int 0dx = C$.

Solution.
Since the derivative of a constant function is always zero then

$$\int 0dx = C. \blacksquare$$

Example 37.2
Show that $\int kdx = kx + C$ where k is a constant.

Solution.
Since the derivative of kx is just k then

$$\int kdx = kx + C. \blacksquare$$

Example 37.3
Show that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$.

Solution.
By the power rule, if $F(x) = \frac{x^{n+1}}{n+1}$ then $F'(x) = x^n$. Thus,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C. \blacksquare$$

Note that this formula is valid only if $n \neq -1$ for if $n = -1$ we would have $\frac{x^0}{0}$ which doesn't make sense. The case $n = -1$ is treated in the next problem.

Example 37.4
Show that

$$\int \frac{dx}{x} = \ln|x| + C.$$

Solution.
Suppose first that $x > 0$ so that $\ln|x| = \ln x$. Then $(\ln|x|)' = (\ln x)' = \frac{1}{x}$. Now, if $x < 0$ then $\ln|x| = \ln(-x)$ and by the chain rule $(\ln|x|)' = (\ln(-x))' = \frac{-1}{-x} = \frac{1}{x}$. Thus, in both cases $(\ln|x|)' = \frac{1}{x}$. $\blacksquare$

Example 37.5
Show that for $a \neq 0$, $\int e^{ax}dx = \frac{e^{ax}}{a} + C$.

Solution.
If a is a nonzero constant and $F(x) = \frac{e^{ax}}{a}$ then $F'(x) = e^{ax}$. This shows that

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C. \blacksquare$$

Example 37.6
Show that $\int \cos(ax)dx = \frac{\sin(ax)}{a} + C$ where $a \neq 0$.

Solution.
If a is a nonzero constant and $F(x) = \frac{\sin(ax)}{a}$ then $F'(x) = \cos(ax)$. This shows that

$$\int \cos(ax)dx = \frac{\sin(ax)}{a} + C. \blacksquare$$

Example 37.7
Show that $\int \sin(ax)dx = -\frac{\cos(ax)}{a} + C$ where $a \neq 0$.

Solution.
If a is a nonzero constant and $F(x) = -\frac{\cos(ax)}{a}$ then $F'(x) = \sin(ax)$. This shows that

$$\int \sin(ax)dx = -\frac{\cos(ax)}{a} + C. \blacksquare$$

Example 37.8
Show that $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$.

Solution.
Let $F(x) = \arcsin x$. Then $F'(x) = \frac{1}{\sqrt{1-x^2}}$. Thus,

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C. \blacksquare$$

Example 37.9
Show that $\int \frac{1}{1+x^2}dx = \arctan x + C$.

Solution.
Let $F(x) = \arctan x$. Then $F'(x) = \frac{1}{1+x^2}$. Thus,

$$\int \frac{1}{1+x^2}dx = \arctan x + C. \blacksquare$$

Properties of Indefinite Integrals

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$$

To see why this property is true, let $F(x)$ be an antiderivative of $f(x)$ and $G(x)$ be an antiderivative of $g(x)$. The result follows from the fact that $\frac{d}{dx}[F(x) \pm G(x)] = f(x) \pm g(x)$.

$$\int cf(x)dx = c\int f(x)dx.$$

To see this, suppose that $F(x)$ is an antiderivative of $f(x)$. Then $\int f(x)dx = F(x) + C$. But $\frac{d}{dx}(cF(x)) = cf(x)$ so that $cF(x)$ is an antiderivative of $cf(x)$, that is, $\int cf(x)dx = cF(x) + C'$. This implies

$$\int cf(x)dx = cF(x)+C' = c(\int f(x)dx-C)+C' = c\int f(x)dx-cC+C' = c\int f(x)dx.$$

Note that the constant $-cC + C'$ is ignored since a constant of integration will result from $\int f(x)dx$.

Example 37.10
Find
$$\int (\sin(2x) - e^{-3x} + \frac{3}{x} - \frac{5}{x^3})dx.$$

Solution.
Using the linearity property of indefinite integrals together with the formulas of integration obtained above we have

$$\begin{array}{lll}\int (\sin(2x) - e^{-3x} + \frac{3}{x} - \frac{5}{x^3})dx &=& \int \sin(2x)dx - \int e^{-3x}dx + 3\int \frac{dx}{x} - 5\int x^{-3}dx\\ &=& -\frac{\cos(2x)}{2} + \frac{e^{-3x}}{3} + 3\ln|x| + \frac{5}{2x^2} + C\blacksquare\end{array}$$

Once we have found an antiderivative of $f(x)$, computing definite integrals is easy by the Fundamental Theorem of Calculus.

Example 37.11
Compute $\int_1^2 3x^2 dx$.

Solution.
Since $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, then by FTC

$$\int_1^2 3x^2 dx = x^3|_1^2 = 2^3 - 1^3 = 7.\blacksquare$$

Finally, we end this section by pointing out that not every funtion has a simple analytical antiderivative. For example, the function

$$Si(x) = \int_0^x \frac{\sin t}{t} dt$$

is an antiderivative of the function $\frac{\sin x}{x}$. There is no way of expressing $Si(x)$ in terms of simple analytic formula.

Recommended Problems (pp. 271 - 3): 1, 5, 9, 15, 16, 18, 20, 22, 27, 29, 36, 39, 46, 53, 56, 58, 67, 69, 73, 75, 77, 81.

38 Applications of Antiderivatives

In this section we will discuss four basic applications of antidifferentiation.

• **Antiderivatives and Differential Equations**
Antidifferentiation can be used in finding the **general solution** of the **differential equation**

$$\frac{dy}{dx} = f(x).$$

Note that the general solution is basically the general antiderivative of $f(x)$. A particular solution is one that satisfies an **initial condition** such as $y(x_0) = y_0$. A differential equation together with an initial condition is called an **initial value problem.**

Example 38.1
Find the general solution of the differential equation

$$\frac{dy}{dx} = \sin x + 2$$

and then graph the particular solution corresponding to the inital value problem $y(3) = 5$.

Solution.
The general solution is the general antiderivative of the function $f(x) = \sin x + 2$ which is given by $F(x) = -\cos x + 2x + C$. The solution corresponding to $F(3) = 5$ is given by $F(x) = -\cos x + 2x + \cos 3 - 1$. The graph is given in Figure 103

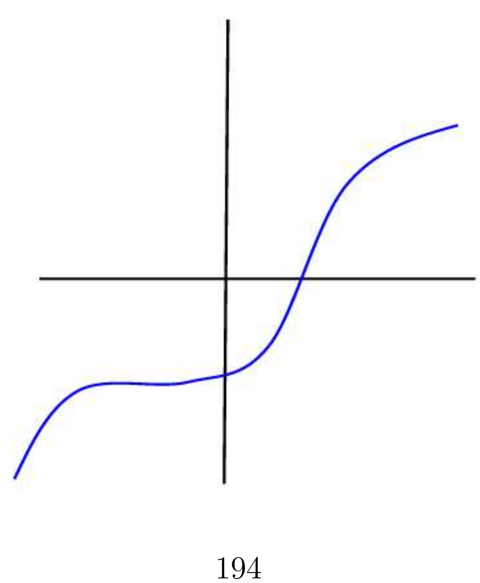


Figure 103

Motion along a Straight Line

Antidifferentiation can be used to find specific antiderivatives using initial conditions, including applications to motion along a line.

Example 38.2

An egg is dropped from an appartment window 20 meters above the ground. Assuming that the acceleration of the egg is $-16m/sec^2$ find
(a) the velocity of the egg as a function of time t;
(b) the position function $s(t)$;
(c) the time when the egg hits the ground and the corresponding velocity.

Solution

(a) The motion is described by the equation

$$\frac{dv}{dt} = -16m/sec^2.$$

The general solution of this equation is given by

$$v(t) = -16t + C.$$

Since the initial velocity is $v(0) = 0$ then $C = 0$ and therefore

$$v(t) = -16t.$$

(b) The solution to the equation

$$\frac{ds}{dt} = -16t$$

is the antiderivative

$$s(t) = -8t^2 + C.$$

But $s(0)$ is the initial distance of the egg from the ground, i.e $s(0) = 20$. Thus, $20 = C$ and therefore $s(t) = -8t^2 + 20$.
(c) The egg hits the ground when $s(t) = 0$ that is, $-8t^2 + 20 = 0$. Solving for $t > 0$ we find $t \approx 1.6sec$. At that time, $v = -16(1.6) = -25.6m/sec$. (The velocity is negative because we are considering up as positive and down as negative.)■

Example 38.3
An object is thrown vertically upward with a speed of $10m/sec$ from a height of 2 meters. Assuming that the acceleration of the object is $-9.8m/sec^2$ find
(a) the velocity of the object as a function of time t;
(b) the position function $s(t)$;
(c) the highest point the object reaches.

Solution.
(a) The motion of the object is described by the equation

$$\frac{dv}{dt} = -9.8.$$

Finding the general solution of this differential equation we have

$$v(t) = -9.8t + C.$$

Since $v(0) = 10$ then $C = 10$ so that $v(t) = -9.8t + 10$.
(b) Finding the general solution of the differential equation

$$\frac{ds}{dt} = -9.8t + 10$$

we have

$$s(t) = -4.9t^2 + 10t + C.$$

But $s(0) = 2$ so that $C = 2$. Hence, $s(t) = -4.9t^2 + 10t + 2$.
(c) The object reaches its highest point when the velocity is zero.Solving $-9.8t + 10 = 0$ we find $t \approx 1.02sec.$■

Antidifferentiation and Rate of Change
Antidifferentiation can also be used when dealing with rates of change.

Example 38.4
If the temperature is constant, then the rate of change of barometric pressure p with respect to altitude h is proportional to p. If $p = 40\ in$ at sea level and $p = 32\ in$ when $h = 1500\ ft$, find the pressure at an altitude of $6000\ ft$.

Solution.
Since the rate of change of p with respect to h is proportional to p then

$$\frac{dp}{dh} = kp,$$

where k is the constant of proportionality. To find the general solution of this differential equation we seperate the variables to obtain

$$\frac{dp}{p} = kdh.$$

Integrate both sides to obtain $\ln|p| = kh + C$ or $|p| = e^{kh+C}$. Thus, $p(h) = Ce^{kh}$. To find C we use the fact that $P(0) = 40$ so that $C = 40$. Hence, $P(h) = 40e^{kh}$. Now, to find the constant k we use the fact that $P(1500) = 32$. That is, $32 = 40e^{1500k}$. Solving this equation for k we find $k = \frac{\ln\left(\frac{32}{40}\right)}{1500} \approx -0.000149$. Hence,

$$P(h) = 40e^{-0.000149h}$$

and in particular $P(6000) = 40e^{-0.000149(6000)} \approx 16.36\ in.$■

Modeling Exponential Growth and Decay

Antidifferentiation can be used also to model exponential growth and decay of the form $\frac{dy}{dx} = ky$. This differential equation says that the rate of change of y with respect to x is proportional to y. The constant k is called the **constant of proportionality.** Solving the equation by the method of seperation of variables discussed earlier we find

$$y(x) = Ce^{kx}. \tag{5}$$

Equation (5) represents an exponential growth for $k > 0$, and exponential decay for $k < 0$. The constant C is the value of y when $x = 0$.

Example 38.5

A bank account earns interest continuously at a rate of 5% of the current balance per year. Assume that the initial deposit is \$1,000 and that no other deposits or withdrawels are made.

(a) Write the differential equation satisfied by the balance in the account.
(b) Solve the differential equation and graph the solution.

Solution.

(a) If A denotes the balance at time t then the differential equation that describes the model is given by

$$\frac{dA}{dt} = 0.05A.$$

(b) Since $A(0) = \$1,000$ then $A(t) = 1,000e^{0.05t}$. This function is shown in Figure 104.■

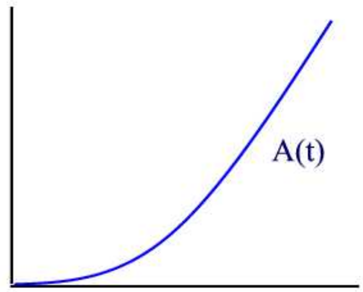


Figure 104

Recommended Problems (pp. 277 - 8): 2, 4, 7, 10, 11, 12, 13, 17.

39 The Second Fundamental Theorem of Calculus

We have seen so far that most of the functions that we considered have elementary antiderivatives, that is, antiderivatives that can be expressed as a linear combination of constants, powers of x, $\sin x, \cos x, e^x$ and $\ln x$. However, not all functions have antiderivatives that can be expressed in simple analytic formula and we already encountered an example of such functions, i.e. $\frac{\sin x}{x}$. In this section, we will present a method for constructing antiderivatives.

Theorem 39.1 *(Second Fundamental Theorem of Calculus)*
Suppose that f is continuous on an interval and a is any point of that interval. Then the funtion

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$. That is, $F'(x) = f(x)$.

Proof.
What we must show here is that $F'(x) = f(x)$, i.e.

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

From the definition of F we have that

$$\begin{array}{rcl} F(x+h) - F(x) & = & \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ & = & \int_a^{x+h} f(t)dt + \int_x^a f(t)dt \\ & = & \int_x^a f(t)dt + \int_a^{x+h} f(t)dt \\ & = & \int_x^{x+h} f(t)dt \end{array}$$

This is illustrated in Figure 105.

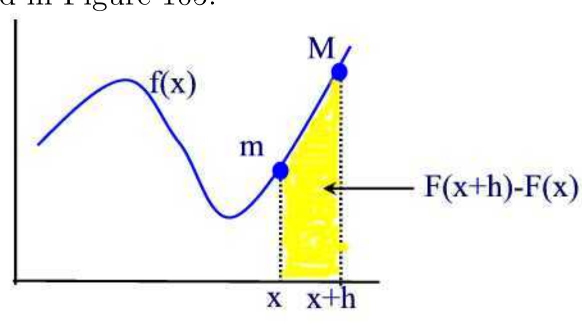


Figure 105

Since $f(x)$ is continuous on the interval $[x, x+h]$ then it is bounded there. This means that there exist two constants M and m such that

$$m \leq f(t) \leq M$$

for all t in $[x, x+h]$. See Figure 105.
Integrate this inequality from x to $x+h$ we find

$$m\int_x^{x+h} dt \leq \int_x^{x+h} f(t)dt \leq M\int_x^{x+h} dt$$

or

$$mh \leq F(x+h) - F(x) \leq Mh.$$

Assume first that $h > 0$. (The case $h < 0$ is similar.) Dividing through by h to obtain

$$m \leq \frac{F(x+h) - F(x)}{h} \leq M.$$

Since f is continuous at x then $\lim_{h\to 0} m = \lim_{h\to 0} M = f(x)$. Hence, by the squeeze law we have $F'(x) = f(x)$. ■

Example 39.1
According to the above theorem an antiderivative of the function $f(x) = \frac{\sin x}{x}$ is given by

$$Si(x) = \int_0^x \frac{\sin t}{t} dt.$$

(a) Find the values of $Si(x)$ for $x = 1, 2, 3$.
(b) Find the derivative of $xSi(x)$.

Solution.
(a) Using a TI 83 calculator with the command $fnInt(\frac{\sin x}{x}, x, 0, 1)$ we find $Si(1) = 0.95$. Similarly, we find $Si(2) = 1.61$ and $Si(3) = 1.85$.
(b) Applying the product rule,

$$\begin{array}{rcl} \frac{d}{dx}(xSi(x)) & = & (x)'Si(x) + x(Si(x))' \\ & = & Si(x) + x\frac{\sin x}{x} \\ & = & Si(x) + \sin x \blacksquare \end{array}$$

Recommended Problems (p. 281): 1, 2, 4, 5, 7, 8, 12, 16, 18, 19, 22, 25, 26.