

Taylor's Formula

3.0. General Aspects

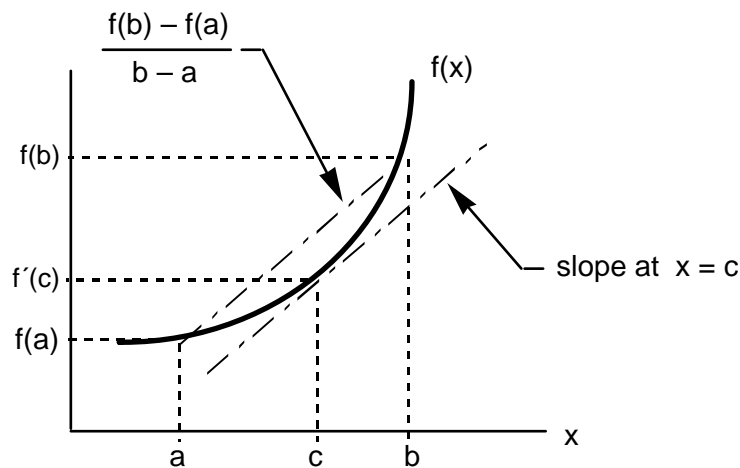
There are many methods for approximating a given function by polynomials. One of the most widely used is that involving Taylor's formula¹. Many numerical analysis techniques, and estimates of their associated errors are derived directly from Taylor series expansions. Before deriving Taylor's formula, recall the Mean Value Theorem.

Theorem (Mean Value Theorem)

If the function f is continuous on the closed interval $[a, b]$ and f is differentiable on the open interval (a, b) , then a point c , $a < c < b$ exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This is illustrated in Figure 3.1.



¹ Named in honor of the English mathematician Brook Taylor (1685 - 1731).

Figure 3.1. Schematic Illustration of the Mean Value Theorem

Taylor's formula can be viewed as being a generalization of the Mean Value Theorem. In particular, let f be a function such that f and its first n derivatives are continuous on $[a, b]$. Furthermore, let $f^{(n+1)}(x)$ exist for all x in (a, b) . Then there exists a number ξ in (a, b) such that

$$f(b) = f(a) + (b-a) \frac{f'(a)}{1!} + (b-a)^2 \frac{f''(a)}{2!} + \dots + (b-a)^n \frac{f^{(n)}(a)}{n!} + (b-a)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad (3.1)$$

Equation (3.1) also holds if $b < a$; in such a case, $[a, b]$ and (a, b) are replaced by $[b, a]$ and (b, a) , respectively. If, in the above equation b is replaced by x , Taylor's formula is obtained:

$$f(x) = f(a) + (x-a) \frac{f'(a)}{1!} + (x-a)^2 \frac{f''(a)}{2!} + \dots + (x-a)^n \frac{f^{(n)}(a)}{n!} + (x-a)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad (3.2)$$

where $x \leq \xi \leq a$. Equation (3.2) is commonly written in the form

$$f(x) = P_n(x; a) + R_n(x; a) \quad (3.3)$$

where

$$\begin{aligned} P_n(x; a) &= f(a) + (x-a) \frac{f'(a)}{1!} + (x-a)^2 \frac{f''(a)}{2!} + \dots + (x-a)^n \frac{f^{(n)}(a)}{n!} \\ &= \sum_{i=0}^n (x-a)^i \frac{f^{(i)}(a)}{i!} \end{aligned} \quad (3.4)$$

is called the n -th degree *Taylor polynomial* (or *Taylor series*) of the function f in the region near the point a . The quantity

$$R_n(x; a) = (x-a)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad (3.5)$$

is called the remainder term (or truncation error) associated with $P_n(x;a)$. As given above, $R_n(x;a)$ is referred to as the *derivative* or *Lagrange form* of the remainder ².

If $a = 0$, Taylor's formula simplifies to

$$f(x) = f(0) + (x) \frac{f'(0)}{1!} + (x)^2 \frac{f''(0)}{2!} + \dots + (x)^n \frac{f^{(n)}(0)}{n!} + (x)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad (3.6)$$

which is commonly referred to as the *Maclaurin series* ³.

If as $|x - a|$ is increased, a point is reached where the power series given by equation (3.2) is no longer convergent, then the point in question is outside the *radius of convergence*; i.e., it is no longer sufficiently close to $x = a$. If the series is convergent, then the value of $f(x)$ will be exact if an infinite number of terms are taken in the series.

The truncation error $R_n(x;a)$ also deserves further mention. In order to evaluate this error, the $(n + 1)$ st derivative of f must be known at every point ξ in the interval (x, a) . As such, it is clear that the error cannot be evaluated, for then the value of $f(x)$ on this interval would also be known, thus alleviating the need for the series expansion. However, since the extent of the interval is under the control of the analyst, it follows that the truncation error is referred to as being of the order of $(x - a)^{n+1}$ or $O(x - a)^{n+1}$. In general, for a Taylor's series for a given function, the following relationship holds between the error terms associated with convergent series truncated at n and $n + 1$ terms, respectively

$$O(x - a)^{n+1} < O(x - a)^n \quad (3.7)$$

Further insight into Taylor series expansions will be gained by considering the following two examples.

² Named in honor of the French mathematician Joseph L. Lagrange (1736 - 1813). Other (e.g. integral) forms of the remainder in Taylor's formula exist. Depending upon the function used, one form of the remainder may be more desirable to use than another.

³ Named in honor of the Scotch mathematician Colin Maclaurin (1698 - 1746). However, this formula was developed earlier by Taylor and by another English mathematician, James Stirling (1692 - 1770) [1].

3.1. Taylor's Series Expansions

The use and nature of Taylor's series expansions are better understood by studying the following examples.

Example 3.1. Taylor Series Expansion of $\exp(x)$.

Using a Taylor's series expansion, approximate the function $f(x) = e^x$ near the point $x=a=0$. Specializing equation (3.4) for the case of $f(x) = e^x$ and $a=0$, gives

$$P_n(x;0) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \quad (1)$$

Some insight into the associated approximation can be obtained through graphical means. If only one term is included in the series (i.e., $n=0$), the exponential function is approximated by a constant.

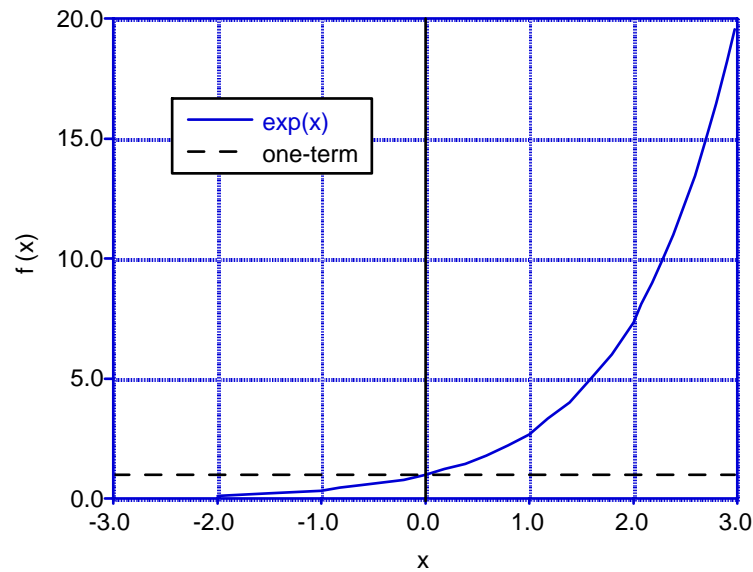


Figure E3.1.1. One Term Approximation for e^x

If two terms are retained in the series ($n=1$), the approximation becomes a straight line

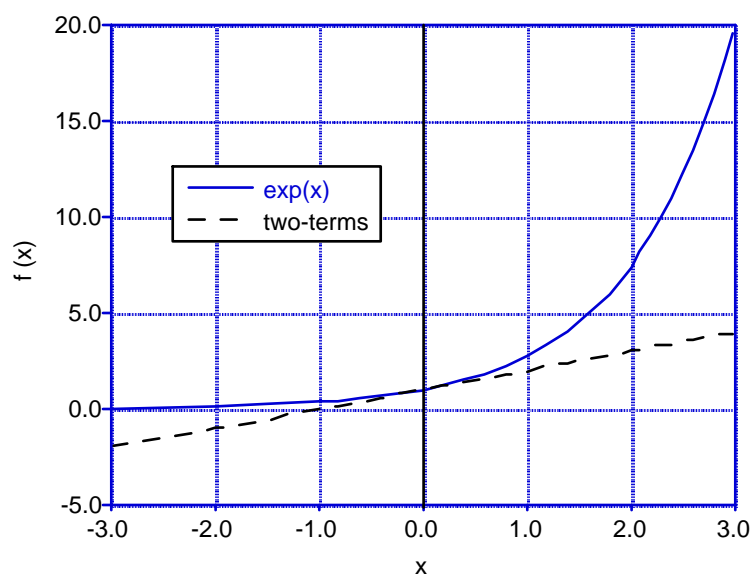


Figure E3.1.2. Two Term Approximation for e^x

Using three terms in the approximation (i.e., $n = 2$) gives a parabola.

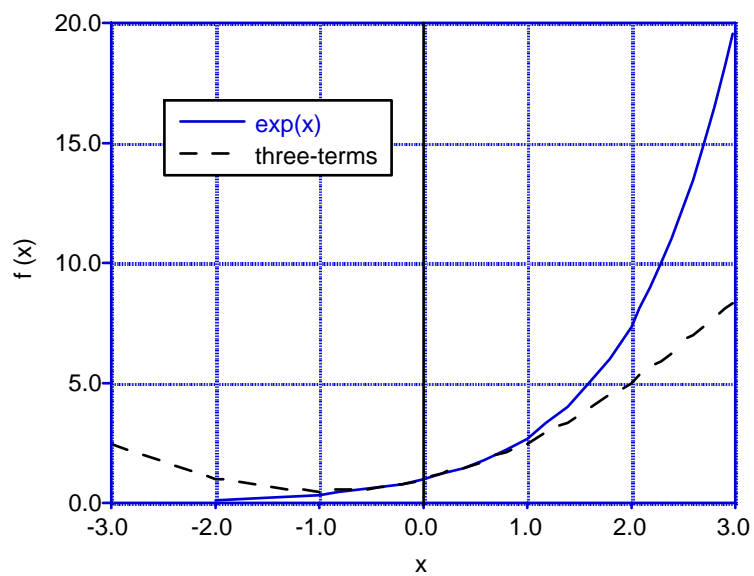


Figure E3.1.3. Three Term Approximation for e^x

Finally, an approximation using four terms ($n = 3$) gives a third order curve that approximates the exponential function more closely, but only in the vicinity of the point of expansion ($x = a = 0$).

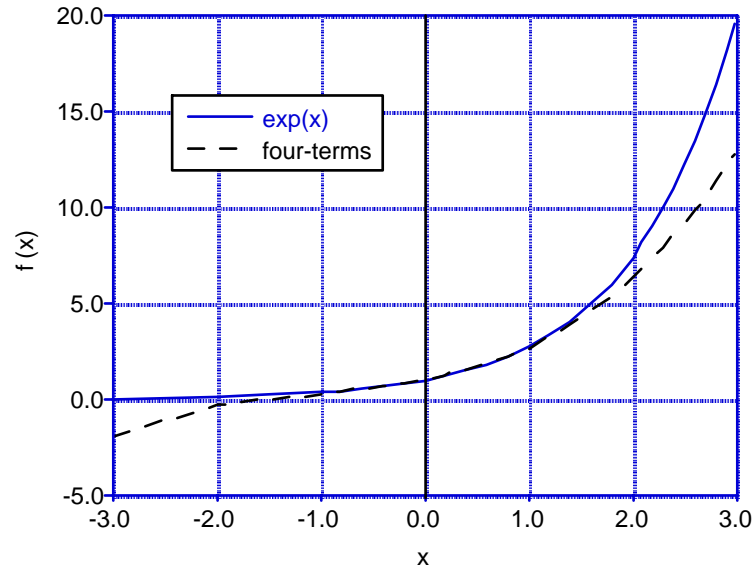


Figure E3.1.4. Four Term Approximation for e^x

In order to assess the quality of the approximation, choose $n = 3$ (i.e., four terms in the series). As such,

$$e^{(0.1)} \approx 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} = 1.10517 \quad (3.8\text{e} - 04\% \text{ relative error}) \quad (2)$$

$$e^{(1.0)} \approx 1 + 1.0 + \frac{1}{2} + \frac{1}{6} = 2.66667 \quad (1.9\% \text{ relative error}) \quad (3)$$

$$e^{(5.0)} \approx 1 + 5.0 + \frac{(5.0)^2}{2} + \frac{(5.0)^3}{6} = 39.33333 \quad (73.5\% \text{ relative error}) \quad (4)$$

These results clearly indicate that Taylor series expansions are useful *locally*; i.e., for x near a (the point of expansion).

Example 3.2. Taylor Series Expansion of $\cos(x)$.

Find the Taylor's series expansion of $\cos x$ about the point $x = \frac{\pi}{4}$. Since $f(x) = \cos x$, it follows from equation (3.3) that

$$\begin{aligned} \cos(x) = & \cos\left(\frac{\pi}{4}\right) - \left(x - \frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) - \frac{1}{2} \left(x - \frac{\pi}{4}\right)^2 \cos\left(\frac{\pi}{4}\right) + \cdots \\ & \cdots + \left(x - \frac{\pi}{4}\right)^n \frac{f^{(n)}(a)}{n!} + \left(x - \frac{\pi}{4}\right)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} \end{aligned} \quad (1)$$

To further investigate the performance of the above Taylor series, evaluate the accuracy of the approximation of $\cos \frac{\pi}{3}$. The exact value of $\cos \frac{\pi}{3}$ is 0.50. Beginning with the simplest approximation:

- One term ($n = 0$):

Clearly the approximation is a poor one; i.e., $\cos \frac{\pi}{3} \neq \cos \frac{\pi}{4}$. The associated relative error is 41.4 percent. Also note that the truncation error is $O\left(\frac{\pi}{3} - \frac{\pi}{4}\right)^1 = O(0.26180)$.

In a similar manner, the approximations generated using from one to four terms ($n = 0$ to $n = 3$) in the Taylor series are summarized in Table 3.2.1 below. Note that as n is increased, the terms present in the series for $(n - 1)$ require no re-computation.

n	approximation for $\cos \pi/3$	relative error (percent)	truncation error
0	0.70711	41.4	$O(0.26180)$
1	0.52199	4.4	$O(0.06854)$
2	0.49775	0.45	$O(0.01794)$
3	0.49987	0.03	$O(0.00470)$

Table E3.2.1. Taylor Series Approximations for $\cos \pi/3$

From the results presented in Table 3.2.1, it is clear that the Taylor series approximation for $\cos \frac{\pi}{3}$ is converging.

Next consider a point that is “farther” from $x = \frac{\pi}{4}$; for example $\cos \frac{9\pi}{16}$ (exact value is equal to -0.19509). Since an approximation involving one term is clearly unrealistic, the approximations generated using from two to five terms ($n = 1$ to $n = 4$) in the Taylor series are summarized in Table 3.2.2 below.

n	approximation for $\cos 9\pi/16$	relative error (percent)	truncation error
1	0.01291	107	$O(0.96383)$
2	-0.32786	68	$O(0.94624)$
3	-0.21634	10.9	$O(0.92897)$
4	-0.18897	3.1	$O(0.91201)$

Table E3.2.2. Taylor Series Approximations for $\cos \pi/3$

As n is increased, the truncation error is seen to be essentially unchanged. This observation suggests convergence, but at a very slow rate. Heuristically, this can be seen by noting that the quantity $\left(\frac{9\pi}{16} - \frac{\pi}{4}\right) = 0.98175$ is very close to 1.0; thus raising it to ever higher powers yields values very close to each other, and very close to 1.0. In conclusion, the Taylor series approximation for $\cos \frac{9\pi}{16}$ slowly converges to the exact result.

3.2. References

1. Leithold, L. The Calculus with Analytic Geometry. Harper and Row Publishers, New York, 1972.
2. Wylie, C. R. Advanced Engineering Mathematics, 4th ed. McGraw-Hill Book Co., 1975.