Fourier Series Solution of the Heat Equation

Physical Application; the Heat Equation In the early nineteenth century Joseph Fourier, a French scientist and mathematician who had accompanied Napoleon on his Egyptian campaign, introduced the idea of expanding functions in trigonometric series as a device to solve the heat conduction equation he had developed in his treatise *Theorie Analytique de la Chaleur* (Analytic Theory of Heat). The subject was greatly extended beyond Fourier's original contributions by later mathematicians but has continued to bear Fourier's name. Our treatment of Fourier series would clearly be incomplete without reference to the subject of heat conduction which led to Fourier's discovery.

We consider a long thin bar of heat conducting material; the length coordinate may be taken to be x in the interval $[0, L]$. We will suppose that the specific heat per unit length (a scalar indicating the capacity of a unit length of the material to hold heat) is σ and the heat conductivity is κ. We let N be a positive integer, $h = L/N$, and $x_k = kh,\ k = 0, 1, 2, \cdots, N$. For the present discussion we think of the bar formed into a ring. Thus $x_0 = 0$ is identified with $x_n = L$ and we use the wrap-around convention with respect to indices k lying outside the range 0 through N.

We suppose the *temperature* in the subinterval $I_k = [x_{k-1}, x_k]$ at a given time t can be adequately approximated by the scalar function $T_k(t)$. The *heat* contained in I_k is then $h\,\sigma\,T_k(t)$. The heat conductivity coefficient expresses the relationship between the rate of flow of heat and the temperature differential per unit length, i.e., $\frac{\partial T}{\partial x}$. Since our model is spatially discrete so far, we approximate $\frac{\partial T}{\partial x}(x_k)$ by the difference quotient $\frac{T_k - T_{k-1}}{h}$. The rate of heat flow into I_k is then $h\,\sigma\,\frac{dT_k}{dt}$ while the flow of heat into I_k from I_{k+1} and I_{k-1} is $\frac{\kappa}{h}\left(T_{k+1}(t) - T_k(t)\right)$

and $\frac{\kappa}{h}\,(T_{k-1}(t) \,-\, T_k(t))$, respectively. Assuming heat is conserved we obtain

$$h\,\sigma\,\frac{dT_k}{dt} \;=\; \frac{\kappa}{h}\,(T_{k+1}(t) \;-\; 2\,T_k(t) \;+\; T_{k-1}(t))\,.$$

Dividing by h we have

$$\sigma\,\frac{dT_k}{dt} \;=\; \kappa\,\frac{T_{k+1}(t) \;-\; 2\,T_k(t) \;+\; T_{k-1}(t)}{h^2}.$$

If we assume the actual heat distribution is a function $T(x,t)$ of both time, t, and space, x, the fraction on the right, a second difference divided by h^2, may be regarded as an approximation to $\frac{\partial^2 T}{\partial x^2}(x_k)$. Letting $N \,\to\, \infty$, $h \,=\, L/N$ tends to zero and, in the limit, we obtain for $T(x,t)$ the *partial differential equation*

$$\sigma\,\frac{\partial T}{\partial t}(x,t) \;=\; \kappa\,\frac{\partial^2 T}{\partial x^2}(x,t), \quad x \,\in\, [0,L], \;\; t \,>\, 0.$$

If there are external heat sources, or losses which we can represent by a function $f(x,t)$ the equation is augmented to the more general form

$$\sigma\,\frac{\partial T}{\partial t}(x,t) \;=\; \kappa\,\frac{\partial^2 T}{\partial x^2}(x,t) \;+\; f(x,t).$$

In addition to the partial differential equation it is necessary to give an *initial heat distribution*. We may suppose the initial instant corresponds to $t_0 \,=\, 0$ and then stipulate the initial data

$$T(x,0) \;=\; T_0(x)$$

with the initial heat distribution function $T_0(x)$ at least piecewise continuous as a function of x.

Sophisticated mathematical techniques are required to show that partial differential equations such as these actually have solutions, that the solutions are uniquely determined by the given initial data, etc. These questions are important but we do not pursue them here; our task is to describe a method for solution of equations of this type with the use of Fourier series techniques.

The Method of Separation of Variables Let us divide the partial differential equation shown earlier by the positive number σ, define $\kappa/\sigma \equiv \alpha$ and rename $\alpha\, f(x,t)$ as $f(x,t)$ again. Then we have

$$\frac{\partial T}{\partial t}(x,t) \,=\, \alpha\, \frac{\partial^2 T}{\partial x^2}(x,t) \,+\, f(x,t).$$

We begin with the homogeneous case $f(x,t) \equiv 0$. To implement the method of separation of variables we write $T(x,t) = z(t)\, y(x)$, thus expressing $T(x,t)$ as the product of a function of t and a function of x. Using $\dot{z}$ to denote $\frac{dz}{dt}$ and $y',\ y''$ to denote $\frac{dy}{dx},\ \frac{d^2y}{dx^2}$, respectively, we obtain

$$\dot{z}(t)\, y(x) \,=\, \alpha\, z(t)\, y''(x).$$

Assuming $z(t),\ y(x)$ are non-zero, we then have

$$\frac{\dot{z}(t)}{\alpha\, z(t)} \,=\, \frac{y''(x)}{y(x)}.$$

Since the left hand side is a constant with respect to x and the right hand side is a constant with respect to t, both sides must, in fact, be constant. It turns out that constant should be taken to be non-positive, so we indicate it as $-\,\omega^2$; thus

$$\frac{\dot{z}(t)}{\alpha\, z(t)} \,=\, \frac{y''(x)}{y(x)} \,=\, -\,\omega^2$$

and we then have two ordinary differential equations

$$\dot{z}(t) \,=\, -\,\alpha\,\omega^2\, z(t), \quad y''(x) \,=\, -\,\omega^2\, y(x).$$

We first deal with the second equation, writing it as

$$y''(x) \,+\, \omega^2\, y(x) \,=\, 0.$$

The general solution of this equation takes the form

$$y(x) = c \cos \omega x + d \sin \omega x.$$

Since we want $y(x)$ to be periodic with period L the choices for ω are

$$\omega = \frac{2\pi k}{L}, \; k = 0, 1, 2, \cdots.$$

The choice $k = 0$ is only useful for the cosine; $\cos 0 = 1$. Indexing the coefficients $c, \; d$ to correspond to the indicated choices of ω, we have solutions for the y equation in the forms

$$c_0 \text{ (a constant);}$$

$$c_k \cos \frac{2\pi k x}{L} + d_k \sin \frac{2\pi k x}{L}, \quad k = 1, 2, \cdots.$$

Now, for each indicated choice $\omega = \frac{2\pi k}{L}$ the z equation takes the form

$$\dot{z}(t) = -\alpha \frac{4\pi^2 k^2}{L^2} z(t)$$

which has the general solution

$$z(t) = c \exp \left(-\alpha \frac{4\pi^2 k^2}{L^2} t \right).$$

Absorbing the constant c appearing here into the earlier $c_k, \; d_k$ we have solutions of the homogeneous partial differential equation in the form

$$T(x,t) = c_0,$$

$$T(x,t) = \exp \left(-\alpha \frac{4\pi^2 k^2}{L^2} t \right) \left(c_k \cos \frac{2\pi k x}{L} + d_k \sin \frac{2\pi k x}{L} \right), k = 1, 2, \cdots.$$

Since we are working at this point with a linear homogeneous equation, any linear combination of these solutions will also be a solution.

This means we can represent a whole family of solutions, involving an infinite number of parameters, in the form

$$T(x,t) \;=\; c_0 \,+\, \sum_{k=1}^{\infty} \exp\left(-\,\alpha\,\frac{4\pi^2k^2}{L^2}\,t\right)\,\left(c_k\,\cos\frac{2\pi kx}{L} \,+\, d_k\,\sin\frac{2\pi kx}{L}\right).$$

It should be noted that this expression is a representation of $T(x,t)$ in the form of a Fourier series with coefficients depending on the time, t:

$$T(x,t) \;=\; c_0 \,+\, \sum_{k=1}^{\infty} \left(c_k(t)\,\cos\frac{2\pi kx}{L} \,+\, d_k(t)\,\sin\frac{2\pi kx}{L}\right),$$

where

$$c_k(t) \;=\; c_k\,\exp\left(-\,\alpha\,\frac{4\pi^2k^2}{L^2}\,t\right),\quad d_k(t) \;=\; d_k\,\exp\left(-\,\alpha\,\frac{4\pi^2k^2}{L^2}\,t\right).$$

The coefficients $c_k(t),\;\; d_k(t),\;\; k \;=\; 1,2,3,\cdots$ in the above representation of $T(x,t)$ remain undetermined, of course, to precisely the extent that the constants $c_k,\;\; d_k$ remain undetermined. In order to obtain definite values for these coefficients it is necessary to use the initial temperature distribution $T_0(x)$. This function has a Fourier series representation

$$T_0(x) \;=\; a_0 \,+\, \sum_{k=1}^{\infty} \left(a_k\,\cos\frac{2\pi kx}{L} \,+\, b_k\,\cos\frac{2\pi kx}{L}\right),$$

where

$$a_0 \;=\; \frac{1}{L}\int_0^L T_0(x)\,dx,\;\; a_k \;=\; \frac{2}{L}\int_0^L \cos\frac{2\pi kx}{L}\,T_0(x)\,dx,$$

$$b_k \;=\; \frac{2}{L}\int_0^L \sin\frac{2\pi kx}{L}\,T_0(x)\,dx.$$

To obtain agreement at $t \;=\; 0$ between our Fourier series representation of $T(x,0)$ and this Fourier series representation of $T_0(x)$ we require, since $\exp\left(-\,\alpha\,\frac{4\pi^2k^2}{L^2}\cdot 0\right) \;=\; 1$,

$$c_0 \;=\; a_0,\;\; c_k \;=\; a_k,\;\; d_k \;=\; b_k,\quad k \;=\; 1,2,3,\cdots.$$

Thus we have, in fact

$$T(x,t) \;=\; a_0 \;+\; \sum_{k=1}^{\infty} \exp\left(-\,\alpha\,\frac{4\pi^2 k^2}{L^2}\,t\right)\,\left(a_k\,\cos\frac{2\pi k x}{L} \;+\; b_k\,\sin\frac{2\pi k x}{L}\right),$$

where a_0, a_k, b_k, $k = 1, 2, 3, \cdots$ are the Fourier coefficients of the initial temperature distribution $T_0(x)$.

Example 1 In the section on **real Fourier series** we have computed the coefficients of the Fourier series for the function $f(x) = \pi - |x - \pi|$ on the interval $[0, 2\pi]$. Taking this function as the initial temperature distribution $T_0(x)$ for the spatially periodic heat conduction process discussed above, we have

$$T_0(x) \;=\; \frac{\pi}{2} \;-\; \sum_{\substack{k\,=\,1\\(k\text{ odd})}}^{\infty} \;\frac{4}{\pi k^2}\,\cos kx.$$

Accordingly, the solution of the periodic heat equation with this initial temperature distribution is

$$T(x,t) \;=\; \frac{\pi}{2} \;-\; \sum_{\substack{k\,=\,1\\(k\text{ odd})}}^{\infty} \;\frac{4}{\pi k^2}\,e^{-k^2 t}\,\cos kx.$$

Since the non-negative odd integers can be represented as $k = 2j + 1$, $j = 0, 1, 2, ...,$ we can write this without the qualifier "k odd" in the form

$$T(x,t) \;=\; \frac{\pi}{2} \;-\; \sum_{j=0}^{\infty} \frac{4}{\pi(2j+1)^2}\,e^{-(2j+1)^2 t}\,\cos(2j+1)x.$$