I. VIENO KINTAMOJO FUNKCIJŲ INTEGRALINIS SKAIČIAVIMAS

1. Individualios užduotys

Funkcijos $f(x)$ pirmykðte vadinama tokia funkcija $F(x)$, kuriai teisinga lygybë $F'(x) = f(x)$.
Jei $F(x)$ yra funkcijos $f(x)$ pirmykðtë ir C – bet kuris realusis skaièius, tai $F(x)+ C$ irgi yra funkcijos $f(x)$ pirmykðtë funkcija.
Funkcijos $f(x)$ neapibrëþtiniu integralu vadinama ðios funkcijos visø pirmykðèiø funkcijø aibë $F(x)+ C$. Raðoma:

$$\int f(x)dx = F(x) + C.$$

Pagrindinës neapibrëþtinio integralo savybës

1. $\left(\int f(x)dx\right)' = \left(F(x) + C\right)' = f(x)$,
2. $d\left(\int f(x)dx\right) = \left(\int f(x)dx\right)' dx = f(x)dx$,
3. $\int df(x) = \int F'(x)dx = \int f(x)dx = F(x) + C$,
4. $\int af(x)dx = a\int f(x)dx$,
5. $\int \left(f(x) + g(x)\right)dx = \int f(x)dx + \int g(x)dx$.

Pagrindiniø integralø lentelë

1. $\int 0dx = C$,
2. $\int 1dx = \int dx = x + C$,
3. $\int x^{\alpha} dx = \dfrac{x^{\alpha+1}}{\alpha + 1} + C, \quad \alpha \neq -1$,
4. $\int \dfrac{dx}{x} = \ln|x| + C$,
5. $\int e^x dx = e^x + C$,
6. $\int a^x dx = \dfrac{a^x}{\ln a} + C$,
7. $\int \sin x dx = -\cos x + C$,
8. $\int \cos x dx = \sin x + C$,

9. $\int \frac{dx}{\sin^2 x} = -ctgx + C$, 10. $\int \frac{dx}{\cos^2 x} = tgx + C$,

11. $\int \frac{dx}{\sin x} = \ln\left| tg\left(\frac{x}{2}\right)\right| + C$,

12. $\int \frac{dx}{\cos x} = \ln\left| tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$,

13. $\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$,

14. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$,

15. $\int \frac{dx}{1 + x^2} = arctgx + C$, 16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} arctg\frac{x}{a} + C$,

17. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$,

18. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C$,

19. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left| x + \sqrt{x^2 \pm a^2}\right| + C$,

20. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left| x + \sqrt{x^2 \pm a^2}\right| + C$,

21. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$.

Integravimo metodai

1. Tiesioginio integravimo metodas
2. Ákëlimo uþ diferencialo þenklo metodas
3. Kintamojo keitimo metodas
4. Integravimo dalimis metodas.

Tiesioginio integravimo metodas

Ðis metodas pagrástas pagrindiniø integralø lentelës ir savybiø taikymu bei pointegralinës funkcijos tapaèiaisiais pertvarkiais.

Pavyzdþiai

1) $\int \frac{dx}{9+x^2} = \frac{1}{3} arctg \frac{x}{3} + C$,

2) $\int \frac{dx}{\sqrt{9+x^2}} = \ln\left| x + \sqrt{9+x^2} \right| + C$,

3) $\int \frac{dx}{9-x^2} = \frac{1}{6} \ln\left| \frac{3+x}{3-x} \right| + C$,

4) $\int \left(x + 3x^2 \right) dx = \frac{x^2}{2} + x^3 + C$,

5) $\int \frac{x^2+2}{x^2+1} dx = \int \frac{\left(x^2+1 \right) + 1}{x^2+1} dx = \int dx + \int \frac{dx}{x^2+1} =$

$= x + arctgx + C$,

6) $\int tg^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx =$

$= \int \frac{dx}{\cos^2 x} - \int dx = tgx - x + C$.

Ákëlimo uþ diferencialo þenklo metodas

Ðis metodas pagrástas trijø ákëlimo uþ diferencialo þenklo taisykliø ir vienos integralø savybës taikymu.

I taisyklë. Prie funkcijos, esanèios uþ diferencialo þenklo, galima pridëti bet kurá skaièiø:

$$du(x)= d(u(x)+a).$$

II taisyklë. Norint funkcijà, esanèià uþ diferencialo þenklo padauginti ið kurio nors nelygaus nuliui skaièiaus, reikia ið ðio skaièiaus padalinti diferencialà (integralà):

$$du(x)=\frac{1}{a}\, d(au(x)), \qquad \int f(x)du(x) = \frac{1}{a}\int f(x)d(au(x)).$$

III taisyklë (þr. antràjà savybæ). Norint funkcijà, esanèià prieð diferencialo þenklà, pakelti uþ diferencialo þenklo, reikia jà suintegruoti:

$$g(x)dx= d\left(\int g(x)dx\right).$$

Savybë. Jei $\int f(x)dx = F(x) + C$ ir $u=u(x)$, tai

$$\int f(u)du = F(u) + C.$$

Kintamojo keitimo metodas

Sakykime, kad reikia rasti integralà $\int f(x)dx$. Norëdami gauti paprastesná integralà, keièiame kintamàjá pagal lygybæ $t=u(x)$ arba $x=\varphi(t)$. Tuomet

$$\int f(x)dx = \int f(\varphi(t))d\varphi(t) = \int f(\varphi(t))\varphi'(t)dt =$$

$$= \int g(t)dt = G(t) + C.$$

Paprasèiausia nauju kintamuoju paþymëti uþ diferencialo þenklo esanèià funkcijà: $t=u(x)$.

Pavyzdþiai

1) $\int \frac{xdx}{x^2+1} = \int \frac{d\left(\frac{x^2}{2}\right)}{x^2+1} = \frac{1}{2}\int \frac{d\left(x^2+1\right)}{x^2+1} =$

$= \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln\left(x^2+1\right) + C$,

2) $\int e^{2\cos x}\sin xdx = \int e^{2\cos x}d(-\cos x) =$

$= -\frac{1}{2}\int e^{2\cos x}d(2\cos x) = -\frac{1}{2}\int e^t dt = -\frac{1}{2}e^t + C =$

$= -\frac{1}{2}e^{2\cos x} + C$,

3) $\int \frac{dx}{x\sqrt{1-\ln^2 x}} = \int \frac{d\ln x}{\sqrt{1-\ln^2 x}} = \int \frac{dt}{\sqrt{1-t^2}}$ = *arscint* + *C* = = *arcsin*(ln*x*) + *C*,

4) $\int \cos(3x+5)dx = \frac{1}{3}\int \cos(3x+5)d(3x+5) =$

$= \frac{1}{3}\int \cos tdt = \frac{1}{3}\sin t + C = \frac{1}{3}\sin(3x+5) + C$.

Integravimo dalimis metodas

Tai integralø apskaièiavimas taikant integravimo dalimis formulæ:

$$\int udv = uv - \int vdu.$$

Ðis metodas daþniausiai taikomas tuomet, kai reikia integruoti tokià dviejø funkcijø sandaugà: $P_n(x)\cdot f(x)$; èia $P_n(x)$ yra *n*-ojo laipsnio daugianaris ($n \geq 0$), o *f*(*x*) – rodiklinë, logaritminë, trigonometrinë arba atvirkðtinë trigonometrinë funkcija. Funkcijà *v* galima gauti keliant kurá nors dauginamàjá $P_n(x)$ ar *f*(*x*) uþ diferencialo þenklo. Kai yra galimybë kelti uþ diferencialo þenklo rodiklinæ, trigonometrinæ ir laipsninæ funkcijas, galima prisilaikyti nurodyto pirmumo.

Pavyzdþiai

1) $\int \ln x dx = x \cdot \ln x - \int x d(\ln x) = x \cdot \ln x - \int x \cdot \frac{1}{x} dx =$

$= x\ln x - x + C$,

2) $\int x \cdot \cos x dx = \int x d \sin x = x \cdot \sin x - \int \sin x dx =$

$= x \cdot \sin x + \cos x + C$,

Racionaliøjø funkcijø integravimas

Racionalià ja funkcija $R(x)$ vadinamas daugianariø

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$$

ir

$$Q_m(x) = b_0 x^m + b_1 x^{m-1} + ... + b_{m-1} x + b_m$$

santykis: $R(x) = \dfrac{P_n(x)}{Q_m(x)}$.

Racionaliosios funkcijos integruojamos keliais etapais.

1) Atkreipiame dëmesá á skaitiklio ir vardiklio daugianariø laipsnius n ir m. Jei $n \geq m$, racionaliojoje funkcijoje, kuri ðiuo atveju vadinama netaisyklingàja, iðskiriame sveikàjà dalá.

2) Atkreipiame dëmesá á vardiklio daugianario $Q_m(x)$ uþraðymo formà. Treèiojo ar aukðtesniojo laipsnio daugianaris turi bûti iðreikðtas tiesiniø ir kvadratiniø (su neigiamais diskriminantais) dauginamøjø sandauga. Jei $m = 2$, vardiklyje galima iðskirti dvinario kvadratà ir gautàjá dvinará paþymëti nauju kintamuoju.

3) Taisyklingàjà racionaliàjà funkcijà iðreiðkiame paprasèiausiø racionaliøjø funkcijø suma.

4) Integruojame racionaliosios funkcijos $R(x)$ sveikàjà dalá ir paprasèiausias racionaliàsias funkcijas.

Pavyzdþiai

1) Apskaièiuokime integralà $\int \frac{dx}{x(x+3)}$. Matome, kad racionalioji funkcija yra taisyklingoji. Ðià funkcijà iðreikðime dviejø paprasèiausiø racionaliøjø funkcijø suma:

$$\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3)+Bx}{x(x+3)} = \frac{(A+B)x+3A}{x(x+3)}.$$

Iš tapatybës $(A+B)x+3A \equiv 1$ rasime neapibrëþtuosius koeficientus *A* ir *B*, pavyzdþiui, sulyginæ koeficientus prie vienodø *x* laipsniø:

$$\begin{cases} A+B=0, \\ 3A=1. \end{cases}$$

Iš ðios sistemos gauname: $A=\frac{1}{3}, \quad B=-\frac{1}{3}.$

Tuomet

$$\int \frac{dx}{x(x+3)} = \int \left(\frac{\frac{1}{3}}{x} + \frac{-\frac{1}{3}}{x+3} \right) dx = \frac{1}{3}\ln|x| - \frac{1}{3}\ln|x+3| + C =$$

$$= \frac{1}{3}\ln\left|\frac{x}{x+3}\right| + C.$$

2) Apskaièiuokime integralà $\int \frac{x^2+x+12}{(x+1)(x^2-9)}dx$. Pointegralinæ funkcijà iðreikðime trijø paprasèiausiø racionaliøjø funkcijø suma ir jas suintegruosime.

$$\frac{x^2+x+12}{(x+1)(x^2-9)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{x+3}.$$

Neapibrëþtuosius koeficientus *A*, *B*, *C* rasime iš tapatybës:

$x^2 + x + 12 \equiv A(x-3)(x+3) + B(x+1)(x+3) + C(x+1)(x-3).$

Turëdami tapatybæ, sulyginame koeficientus prie vienodø x laipsniø arba parenkame tris x reikðmes, pavyzdþiui,

$x = 0$, $x = 3$, $x = -3$.

Gauname: $A = -\frac{3}{2}$, $B = 1$, $C = \frac{3}{2}$.

Tuomet

$$\int \frac{x^2 + x + 12}{(x+1)(x^2-9)} dx = -\frac{3}{2}\int \frac{dx}{x+1} + \int \frac{dx}{x-3} + \frac{3}{2}\int \frac{dx}{x+3} =$$

$$= -\frac{3}{2}\ln|x+1| + \ln|x-3| + \frac{3}{2}\ln|x+3| + C.$$

Iracionaliøjø reiðkiniø integravimas

Ðio tipo integraluose daþniausiai taikomas kintamojo keitimo metodas.

1) Jei pointegralinëje funkcijoje vien tik ðaknys ið x, arba ðaknys ið tiesinës funkcijos $ax+b$, arba ðaknys ið tiesiniø funkcijø dalmens $\frac{ax+b}{cx+d}$, tai naujas kintamasis ávedamas taip:

$$\sqrt[s]{x} = t, \quad \sqrt[s]{ax+b} = t, \quad \sqrt[s]{\frac{ax+b}{cx+d}} = t;$$

èia s – ðaknø rodikliø maþiausiasis bendrasis kartotinis. Pakeitæ integravimo kintamàjá, gauname racionaliosios funkcijos integralà.

2) Kai reikia apskaièiuoti integralus

$$\int \sqrt{ax^2 + bx + c}\, dx, \quad \int (Ax + B)\sqrt{ax^2 + bx + c}\, dx,$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \quad \int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx,$$

ið pradþiø kvadratiniame trinaryje iðskiriame dvinario kvadratà ir tà dvinará paþymime nauju kintamuoju.

3) Kai reikia apskaièiuoti integralà

$\int \frac{dx}{(x-\alpha)^n \cdot \sqrt{ax^2+bx+c}}$ ($n \in N$), integravimo kintamàjá

keièiame pagal lygybæ $t = \frac{1}{x-\alpha}$.

4) Kai pointegralinëje funkcijoje yra kvadratinës ðaknys ið kvadratiniø dvinariø , tai gali bûti taikomi ðie trigonometriniai keitiniai:

$$\sqrt{a^2 - x^2}, \quad x = a\sin t;$$

$$\sqrt{x^2 - a^2}, \quad x = \frac{a}{\sin t};$$

$$\sqrt{a^2 + x^2}, \quad x = a\,\mathrm{tg}\,t.$$

5) Diferencialiniu binomu vadinamas reiðkinys

$x^m \cdot \left(a+bx^n\right)^p$, èia m, n, p – racionalieji skaièiai. Ðie reiðkiniai suintegruojami tik trimis atvejais:

a) p – sveikasis skaièius; kai $p < 0$, taikomas keitinys $x = t^s$, èia s – trupmenø m ir n bendrasis vardiklis;

b) $\frac{m+1}{n}$ – sveikasis skaièius; ðiuo atveju taikomas keitinys $a+bx^n = t^s$, èia s – trupmenos p vardiklis;

c) $\frac{m+1}{n} + p$ – sveikasis skaièius; ðiuo atveju taikomas keitinys $a+bx^n = t^s \cdot x^n$, èia s – trupmenos p vardiklis.

Pavyzdþiai

1) Apskaièiuokime integralà $\int \frac{dx}{\left(1+\sqrt[4]{x}\right)\cdot\sqrt{x}} = I_1$.

Ávesime naujà kintamàjá $t=\sqrt[4]{x}$. Tuomet $x=t^4$, $dx=4t^3dt$,

$$I_1=\int\frac{4t^3dt}{(1+t)t^2} = 4\int\frac{t}{t+1}dt =$$

$$=4\int\frac{(t+1)-1}{t+1}dt = 4\left(t-\ln|t+1|\right)+C =$$

$$= 4\left(\sqrt[4]{x}-\ln\left|\sqrt[4]{x}+1\right|\right)+C.$$

2) Apskaièiuokime integralà $\int\frac{dx}{\sqrt{3+2x-x^2}} = I_2$.

Kvadratiniame trinaryje iðskirkime dvinario kvadratà ir tà dvinará paþymëkime nauju kintamuoju:

$$3 + 2x - x^2 = 4 - (x-1)^2 = 4 - t^2.$$

Tuomet

$$I_2=\int\frac{dt}{\sqrt{4-t^2}} = \arcsin\frac{t}{2}+C = \arcsin\frac{x-1}{2}+C.$$

3) Apskaièiuokime integralà $\int\frac{dx}{x\cdot\sqrt{5x^2-2x+1}} = I_3$.

Keitinio lygybës: $t=\frac{1}{x}$, $x=\frac{1}{t}$, $dx=-\frac{1}{t^2}dt$.

Tuomet imdami, kad $x > 0$, gauname:

$$I_3=\int\frac{-dt}{\sqrt{t^2-2t+5}} = -\int\frac{dt}{\sqrt{(t-1)^2+4}} = -\int\frac{dz}{\sqrt{z^2+4}} =$$

$$=-\ln\left|z+\sqrt{z^2+4}\right|+C = -\ln\left|t-1+\sqrt{t^2-2t+5}\right|+C =$$

$$= -\ln\left|\frac{1}{x}-1+\sqrt{\frac{1}{x^2}-\frac{2}{x}+5}\right|+C =$$

$$= \ln\frac{x}{1-x+\sqrt{5x^2-2x+1}} + C.$$

4) Apskaièiuokime integralà $\int \frac{\sqrt{1+x^3}}{x}\,dx$ = I_4.

Laikydami pointegralinæ funkcijà diferencialiniu binomu

$x^m \cdot \left(a+bx^n\right)^p$, matome, kad $\frac{m+1}{n}$ = 0.

Todël taikomas keitinys $1+x^3=t^2$. Tuomet $3x^2dx = 2tdt$,

$$I_4=\int \frac{\sqrt{1+x^3}\cdot x^2}{x^3}dx = \frac{2}{3}\int \frac{t^2}{t^2-1}dt = \frac{2}{3}\int \frac{(t^2-1)+1}{t^2-1}dt =$$

$$=\frac{2}{3}\int dt+\frac{2}{3}\int \frac{dt}{t^2-1} = \frac{2}{3}t+\frac{2}{3}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C =$$

$$= \frac{2}{3}\sqrt{1+x^3}+\frac{1}{3}\ln\left|\frac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}\right|+C.$$

Trigonometriniø reiðkiniø integravimas

Integruojant trigonometrinius reiðkinius taikomi visi keturi integravimo metodai: tiesioginio integravimo, ákëlimo uþ diferencialo þenklo, kintamojo keitimo ir integravimo dalimis.

Pertvarkant pointegralinæ funkcijà gali bûti taikomos ðios trigonometrinës formulës:

$$\sin^2\alpha+\cos^2\alpha=1,\quad tg\alpha=\frac{\sin\alpha}{\cos\alpha},\quad ctg\alpha=\frac{\cos\alpha}{\sin\alpha},$$

$$1+tg^2\alpha=\frac{1}{\cos^2\alpha},\quad 1+ctg^2\alpha=\frac{1}{\sin^2\alpha};$$

$$\sin^2\alpha=\frac{1}{2}(1-\cos2\alpha),\quad \cos^2\alpha=\frac{1}{2}(1+\cos2\alpha);$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha\ ,\ \sin 2\alpha = \frac{2tg\alpha}{1+tg^2\alpha},$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,\quad \cos 2\alpha = 2\cos^2\alpha - 1,$$

$$\cos 2\alpha = 1 - 2\sin^2\alpha,\quad \cos 2\alpha = \frac{1-tg^2\alpha}{1+tg^2\alpha},$$

$$tg2\alpha = \frac{2tg\alpha}{1-tg^2\alpha};$$

$$\sin\alpha\cdot\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right),$$

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right),$$

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right).$$

Kintamojo keitimo metodas taikomas tokio tipo integraluose (R – racionalioji funkcija):

$$\int R(\sin x)\cdot\cos x dx = \int R(\sin x)d\sin x,\qquad \sin x = t;$$

$$\int R(\cos x)\cdot\sin x dx = -\int R(\cos x)d\cos x,\qquad \cos x = t;$$

$$\int R(tgx)dx,\quad \int R(tgx)\cdot\frac{1}{\cos^2 x}dx = \int R(tgx)d(tgx),$$

$$\int\frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx,\quad \int\frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x},$$

$tgx = t$;

$$\int R(ctgx)dx,\quad \int R(ctgx)\cdot\frac{1}{\sin^2 x}dx = -\int R(ctgx)d(ctgx),$$

$ctgx = t$;

$\int R(tg\frac{x}{2})dx$, $\int \frac{dx}{a\sin x + b\cos x + c}$,

$tg\frac{x}{2} = t$ (universalusis keitinys)

Pavyzdþiai

1) $\int ctgx dx = \int \frac{\cos x}{\sin x}dx = \int \frac{d\sin x}{\sin x} = \ln|\sin x| + C,$

2) $\int \frac{1}{\sin^2 x \cdot \cos x}dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos x}dx =$

$= \int \frac{dx}{\cos x} + \int \frac{\cos x}{\sin^2 x}dx = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + \int \frac{d\sin x}{\sin^2 x} =$

$= \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \frac{1}{\sin x} + C.$

1 uþdavinys. Apskaièiuokite neapibrëþtinius integralus

1) $\int \frac{\arcsin x}{\sqrt{1 - x^2}}dx$, $\int \ln x dx$, $\int \frac{x + 4}{x + 1}dx$,

$\int \frac{1}{3x + \sqrt[3]{x^2}}dx$, $\int \frac{1}{\sin x + \cos x + 2}dx$

2) $\int \frac{1}{\sqrt{2 - x}}dx$, $\int \arcsin 3x dx$, $\int \frac{x^3 - x^2 - 6x + 5}{(x + 2)(x - 3)}dx$,

$\int x\sqrt{x + 2}dx$, $\int \frac{1}{\sin^2 x + \sin x \cos x}dx$

3) $\int \frac{2x}{\sqrt{1-4x^2}}dx$, $\int x\sin x dx$, $\int \frac{x^3-2x^2+3x+2}{x^3-3x^2+2x}dx$,

$\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx$, $\int ctg^2 x dx$

4) $\int e^{-\frac{x}{2}}dx$, $\int e^x \sin x dx$, $\int \frac{2x+10}{(x+2)(x-1)}dx$,

$\int \frac{1}{(1+\sqrt[4]{x})\sqrt[4]{x^3}}dx$, $\int \frac{1}{\cos^6 x}dx$

5) $\int \frac{e^{tgx}}{\cos^2 x}dx$, $\int e^{-x}(x+1)dx$, $\int \frac{2x+10}{(x+3)(x^2+x-2)}dx$,

$\int \frac{\sqrt{x+49}}{x}dx$, $\int \frac{1}{2\sin x+\cos x+2}dx$

6) $\int x5^{-x^2}dx$, $\int xe^{2x}dx$, $\int \frac{x^2+x+1}{x(x+1)^2}dx$, $\int x\sqrt{x+1}dx$,

$\int \frac{\cos^3 x}{\sin x}dx$

7) $\int \frac{3x^2}{4+x^6}dx$, $\int x\cos 5x dx$, $\int \frac{x^2+x+2}{x(x^2+1)}dx$,

$\int \frac{\sqrt{x}}{1+\sqrt{x}}dx$, $\int tg^3 x dx$

8) $\int \frac{\sqrt{1+\ln x}}{x}dx$, $\int x^2 \ln x dx$, $\int \frac{5x+7}{(x-1)(x^2+5x+6)}dx$,

$\int x\sqrt{x+3}dx$, $\int \frac{1}{2\sin x+2\cos x+3}dx$

9) $\int \frac{\sin x}{\cos^2 x}dx$, $\int xe^{3x}dx$, $\int \frac{x^2+4x+7}{(x+3)(x+1)}dx$,

$\int \frac{1}{x^3\sqrt{x^2-1}}dx$, $\int \sin^2 x\cos^3 xdx$

10) $\int \frac{e^x}{1+e^{2x}}dx$, $\int e^x\cos 2xdx$, $\int \frac{x^3-x^2-5x-2}{(x+2)(x-3)}dx$,

$\int \frac{1}{x^2\sqrt{x^2+1}}dx$, $\int \frac{\sin^2 x}{\cos^4 x}dx$

11) $\int \frac{\cos x}{\sqrt{\sin x+2}}dx$, $\int \sin(\ln x)dx$, $\int \frac{2x^2+x+1}{(x^2+x)(x+1)}dx$,

$\int \frac{dx}{(1+\sqrt[4]{x+1})\sqrt[4]{x+1}}$, $\int \sin^2 2x\cos^2 2xdx$

12) $\int x\sqrt[5]{5-x^2}dx$, $\int x\sin 3xdx$, $\int \frac{7x+5}{(x+2)(x^2+2x-3)}dx$,

$\int x\sqrt{x+4}dx$, $\int \frac{1}{2\sin x+3\cos x+4}dx$

13) $\int \frac{x}{\sqrt{x^4-3}}dx$, $\int \frac{\arcsin x}{x^2}dx$, $\int \frac{x^2+x+2}{x^2(x+1)}dx$,

$\int \frac{x^3}{\sqrt{4-x^2}}dx$, $\int \frac{1}{\sin^2 x\cos^4 x}dx$

14) $\int \frac{x^3}{5-x^4}dx$, $\int e^x \sin 3x dx$, $\int \frac{2x+13}{(x+2)(x-1)}dx$,

$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$, $\int \frac{1}{\sin^2 x - \sin x \cos x}dx$

15) $\int \frac{\ln^2 x}{x}dx$, $\int \frac{\ln x}{x^2}dx$, $\int \frac{x^2+3x+3}{x^3+x}dx$,

$\int x\sqrt{x+5}dx$, $\int \frac{1}{\sin x + 3\cos x + 4}dx$

16) $\int \frac{e^x}{1-e^{2x}}dx$, $\int x\cos 4x dx$, $\int \frac{x^3-x^2-4x-1}{x^2-x-6}dx$,

$\int \frac{x^4}{\sqrt{4-x^2}}dx$, $\int \frac{\cos^2 x}{\sin^6 x}dx$

17) $\int \frac{1}{\left(1-tg^2 x\right)\cos^2 x}dx$, $\int x \operatorname{arctg} 2x dx$,

$\int \frac{x^2+4x+9}{(x+1)(x+3)}dx$, $\int \frac{\sqrt{x+25}}{x}dx$, $\int \sin 3x \cos^3 3x dx$

18) $\int x\cos(x^2-4)dx$, $\int e^{2x}\cos x dx$, $\int \frac{x^3+3x+4}{x(x-1)(x-2)}dx$,

$\int \frac{1}{1+\sqrt{x}}dx$, $\int \frac{1}{\sin x + 2\cos x + 3}dx$

19) $\int \frac{\sin x dx}{\sqrt{\cos^2 x + 2}}$, $\int \arcsin^2 x dx$, $\int \frac{x^2+2x+4}{x^2(x+1)}dx$,

$\int \frac{dx}{\sqrt[3]{(3x+1)^2}+\sqrt{3x+1}}$, $\quad \int \frac{dx}{\sin^2 x + 25\cos^2 x}$

20) $\int \frac{4x^3}{\sqrt{x^4-1}}dx$, $\quad \int (2x+1)e^{-2x}dx$,

$\int \frac{x^2+2x+3}{x(x^2+1)}dx$, $\quad \int \frac{dx}{x\sqrt{4-x^2}}$, $\quad \int \frac{\cos^2 x}{\sin^4 x}dx$

21) $\int \frac{x^2}{x^6-9}dx$, $\quad \int (x^2+1)e^x dx$, $\quad \int \frac{2x^2+3x+1}{x^3+x^2}dx$,

$\int x\sqrt{x+7}dx$, $\quad \int \sin 2x \cos^2 2x dx$

22) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx$, $\quad \int \frac{\ln x}{\sqrt{x}}dx$, $\quad \int \frac{17x+7}{(x+2)(x^2+2x-3)}dx$,

$\int \frac{dx}{x^2\sqrt{4-x^2}}$, $\quad \int \frac{1}{\sin^2 x \cos^2 x}dx$

23) $\int \frac{\sin \ln x}{x}dx$, $\quad \int x^2 \arcsin x dx$, $\quad \int \frac{2x^2+2x+1}{(x^2+x)(x+1)}dx$,

$\int \frac{\sqrt{x+16}}{x}dx$, $\quad \int \frac{1}{\sin^2 x - 2\sin x \cos x}dx$

24) $\int \frac{2x}{1+x^4}dx$, $\quad \int e^{2x}\cos 3x dx$, $\quad \int \frac{x^3-x^2-4x+9}{(x-3)(x+2)}dx$,

$\int \frac{\sqrt{x+25}+5}{\sqrt{x+25}-5}dx$, $\quad \int \frac{\cos^3 x}{\sin^5 x}dx$

25) $\int \frac{1}{\sqrt{5-3x}}dx$, $\int \ln\left(x+\sqrt{1+x^2}\right)dx$, $\int \frac{x^2+x+2}{x(x^2+1)}dx$,

$\int \frac{\sqrt{x+36}-6}{\sqrt{x+36}-6}dx$, $\int \frac{dx}{3\cos^2 x+2\sin x\cos x}$

26) $\int \frac{3x^2}{\sqrt{x^6-1}}dx$, $\int x^2 arctgx dx$, $\int \frac{x^2+x+3}{x^2(x+1)}dx$,

$\int \frac{dx}{x^3\sqrt{9-x^2}}$, $\int \frac{\cos^2 x}{\sin^6 x}dx$

27) $\int \frac{8x^3}{\sqrt{1-x^8}}dx$, $\int (3x+1)e^{-x}dx$, $\int \frac{x^2+4x+11}{(x+3)(x+1)}dx$,

$\int \frac{1-2\sqrt{x}}{1+2\sqrt{x}}dx$, $\int \frac{\sin^3 x}{\cos^2 x}dx$

28) $\int \frac{e^x dx}{\sqrt{e^{2x}-2}}$, $\int x\cos 2x dx$, $\int \frac{x^2+2x+4}{x(x^2+1)}dx$,

$\int \frac{\sqrt{x+4}}{x}dx$, $\int \frac{\cos^2 x}{\sin x}dx$

29) $\int \sqrt{\frac{\arcsin x}{1-x^2}}dx$, $\int x^5 \ln x dx$, $\int \frac{x+14}{(x-1)(x+2)}dx$,

$\int \frac{dx}{\sqrt[3]{(x+1)^2}+\sqrt{x+1}}$, $\int \frac{dx}{3\sin x+3\cos x+4}$

30) $\int \frac{\sin 2x}{9+\cos^2 x}dx$, $\int x^2 arctg 2x dx$, $\int \frac{3x^2+2x+1}{x(x^2+2x+1)}dx$,

$\int \frac{3\sqrt{x}+1}{3\sqrt{x}-1}dx$, $\int \cos^3 x\sqrt{\sin x}dx$

2. Išspręstosios užduotys

1 Uždavinys. Apskaičiuokite neapibrėžtinius integralus.

1)

$$a)\int \frac{\arcsin x}{\sqrt{1-x^2}}dx = \int \arcsin x d(\arcsin x) = \int t dt = \frac{t^2}{2} + C =$$

$$= \frac{(\arcsin x)^2}{2} + C.$$

$$b)\int \ln x dx = x\ln x - \int x d(\ln x) = x\ln x - \int x \cdot \frac{1}{x}dx =$$

$$= x\ln x - \int dx = x\ln x - x + C.$$

$$c)\int \frac{x+4}{x+1}dx = \int \frac{x+1+3}{x+1}dx = \int \frac{x+1}{x+1}dx + \int \frac{3}{x+1}dx = \int dx + 3\int \frac{dx}{x+1} =$$

$$= x + 3\int \frac{d(x+1)}{x+1} = x + 3\ln|x+1| + C.$$

$$d)\int \frac{1}{3x+\sqrt[3]{x^2}}dx = I_1.$$

Įveskime naują kintamąjį $t = \sqrt[3]{x}$.

Tuomet $x = t^3, dx = 3t^2 dt,$

$$I_1 = \int \frac{1}{3t^3+t^2} \cdot 3t^2 dt = 3\int \frac{dt}{3t+1} = \int \frac{d(3t)}{3t+1} = \int \frac{d(3t+1)}{3t+1} =$$

$$= \ln|3t+1| + C = \ln\left|3\sqrt[3]{x}+1\right| + C.$$

Įveskime naują kintamąjį $t = tg\frac{x}{2}$.

$$e)\int \frac{1}{\sin x + \cos x + 2}dx = I_1.$$

Tuomet $\sin x = \frac{2t}{1+t^2}; \cos\ x = \frac{1-t^2}{1+t^2}; dx = \frac{2dt}{1+t^2}$,

$$I_1 = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2 \cdot \frac{1+t^2}{1+t^2}} = \int \frac{2dt}{2t+1-t^2+2+2t^2} =$$

$$= \int \frac{2dt}{t^2+2t+3} = \int \frac{2dt}{t^2+2t+1+2} = \int \frac{2dt}{(t+1)^2+2} =$$

$$= 2\int \frac{d(t+1)}{(t+1)^2+\left(\sqrt{2}\right)^2} = 2\frac{1}{\sqrt{2}} arctg\left(\frac{t+1}{\sqrt{2}}\right) + C =$$

$$= \sqrt{2} arctg\left(\frac{tg\frac{x}{2}+1}{\sqrt{2}}\right) + C.$$

2)

$$a)\int \frac{1}{\sqrt{2-x}}dx = -\int \frac{1}{\sqrt{2-x}}d(-x) = -\int \frac{1}{\sqrt{2-x}}d(2-x) =$$

$$= -\int \frac{dt}{\sqrt{t}} = -\int t^{\frac{1}{2}}dt = -2t^{\frac{1}{2}} + C = -2\sqrt{2-x} + C.$$

$$b)\int \arcsin 3x dx = \frac{1}{3}\int \arcsin 3x dx = \frac{1}{3}\int \arcsin t dt =$$

$$= \frac{1}{3} t \arcsin t - \frac{1}{3}\int t d \arcsin t = \frac{1}{3} t \arcsin t - \frac{1}{3}\int \frac{t}{\sqrt{1-t^2}}dt =$$

$$= \frac{1}{3} t \arcsin t - \frac{1}{6}\int \frac{d\left(t^2\right)}{\sqrt{1-t^2}} = \frac{1}{3} t \arcsin t + \frac{1}{6}\int \frac{d\left(-t^2\right)}{\sqrt{1-t^2}} =$$

$$= \frac{1}{3} t \arcsin t + \frac{1}{6}\int \frac{d\left(1-t^2\right)}{\sqrt{1-t^2}} = \frac{1}{3} t \arcsin t + \frac{1}{3}\sqrt{1-t^2} + C =$$

$$= x \arcsin 3x + \frac{1}{3}\sqrt{1-9x^2} + C.$$

$$c)\int\frac{x^3-x^2-6x+5}{(x+2)(x-3)}dx=\int\frac{x\left(x^2-x-6\right)+5}{(x+2)(x-3)}dx=$$
$$=\int\frac{x(x+2)(x-3)+5}{(x+2)(x-3)}dx=\int\frac{x(x+2)(x-3)}{(x+2)(x-3)}dx+$$
$$+5\int\frac{dx}{(x+2)(x-3)}=\int xdx+5\int\frac{A}{x+2}dx+5\int\frac{B}{x-3}dx=I_1.$$

Tuomet

$$\frac{1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}=\frac{Ax-3A+Bx-2B}{(x+2)(x-3)}=$$
$$=\frac{(A+B)x-(3A+2B)}{(x+2)(x-3)};$$

Iš tapatybės $(A+B)x-(3A+2B)\equiv 1$ rašome neapibrėžtus koeficientus A ir B, sulyginę koeficientus prie vienodų x laipsnių:

$$\begin{cases} A+B=0; \\ -3A-2B=1. \end{cases}$$

Iš šios sistemos gauname: $A=-\frac{1}{3}, B=\frac{1}{3}.$

Tuomet

$$I_1=\frac{x^2}{2}-\frac{5}{3}\int\frac{dx}{x+2}+\frac{5}{3}\int\frac{dx}{x-3}=$$
$$=\frac{x^2}{2}-\frac{5}{3}\ln|x+2|+\frac{5}{3}\ln|x-3|+C.$$

$d)\int x\sqrt{x+2}dx = I_1.$

Įvesime naują kintamąjį $t = \sqrt{x+2}$.

Tuomet $x + 2 = t^2; x = t^2 - 2; dx = 2tdt.$

$$I_1 = \int (t^2 - 2)t \cdot 2tdt = 2\int t^4 dt - 4\int t^2 dt = \frac{2}{5}t^5 - \frac{4}{3}t^3 + C =$$

$$= \frac{2}{5}\left(\sqrt{x+2}\right)^5 - \frac{4}{3}\left(\sqrt{x+2}\right)^3 + C.$$

$$e)\int \frac{1}{\sin^2 x + \sin x \cos x}dx = \int \frac{1}{\sin^2 x\left(1 + \frac{\cos x}{\sin x}\right)}dx =$$

$$= \int \frac{d(-ctgx)}{1+ctgx} = -\int \frac{d(ctgx)}{1+ctgx} = -\int \frac{d(1+ctgx)}{1+ctgx} = -\int \frac{dU}{U} =$$

$$= -\ln|U| + C = -\ln|1+ctgx| + C.$$

3)

$$a)\int \frac{2x}{\sqrt{1-4x^2}}dx = \int \frac{dx^2}{\sqrt{1-4x^2}} = -\frac{1}{4}\int \frac{d(-4x^2)}{\sqrt{1-4x^2}} =$$

$$= -\frac{1}{4}\int \frac{d(1-4x^2)}{\sqrt{1-4x^2}} = -\frac{1}{4}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\sqrt{t} + C =$$

$$= -\frac{1}{2}\sqrt{1-4x^2} + C.$$

$$b)\int x\sin xdx = -\int xd\cos x = -x\cos x + \int \cos xdx =$$

$$= -x\cos x + \sin x + C.$$

$$c)\int \frac{x^3-2x^2+3x+2}{x^3-3x^2+2x}dx=\int \frac{x^3-3x^2+2x+x^2+x+2}{x^3-3x^2+2x}dx=$$

$$=\int \frac{x^3-3x^2+2x}{x^3-3x^2+2x}dx+\int \frac{x^2+x+2}{x^3-3x^2+2x}dx=$$

$$=\int dx+\int \frac{x^2+x+2}{x(x-1)(x-2)}dx=I_1.$$

Tuomet

$$\frac{x^2+x+2}{x(x-1)(x-2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}=$$

$$=\frac{A(x-1)(x-2)+Bx(x-2)+Cx(x-1)}{x(x-1)(x-2)}=$$

$$=\frac{(A+B+C)x^2+(-3A-2B-C)x+2A}{x(x-1)(x-2)};$$

Iš tapatybės $(A+B+C)x^2+(-3A-2B-C)x+2A\equiv x^2+x+2$ rasime neapibrėžtus koeficientus *A*, *B* ir *C*, sulyginę koeficientus prie vienodų *x* laipsnių:

$$\begin{cases} A+B+C=1, \\ -3A-2B-C=1, \\ 2A=2. \end{cases}$$

Iš šios sistemos gauname: $A = 1;\ B = -4;\ C = 4$.

Tuomet

$$I_1=\int dx+\int \frac{dx}{x}-4\int \frac{dx}{x-1}+4\int \frac{dx}{x-2}=$$

$$=\frac{x^2}{2}+\ln|x|-4\ln|x-1|+4\ln|x-2|+C.$$

$$d)\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx = \int \frac{(\sqrt{x+1}+1)^2}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}dx =$$

$$= \int \frac{x+1+2\sqrt{x+1}+1}{x+1-1}dx = \int \frac{x}{x}dx + \int \frac{2\sqrt{x+1}}{x}dx + \frac{dx}{x} =$$

$$= x + \ln|x| + 2\int \frac{\sqrt{x+1}}{x}dx = I_1.$$

Įvesime naują kintamąjį $t = \sqrt{x+1}.$

Tuomet $x+1 = t^2, x = t^2 - 1, dx = 2tdt,$

$$I_1 = x + \ln|x| + 2\int \frac{t \cdot 2tdt}{t^2-1} = x + \ln|x| + 4\int \frac{t^2-1+1}{t^2-1}dt =$$

$$= x + \ln|x| + 4\int \frac{t^2-1}{t^2-1}dt + 4\int \frac{dt}{t^2-1} = x + \ln|x| + 4\int dt +$$

$$+ 4 \cdot \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = x + \ln|x| + 4\sqrt{x+1} + 2\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

$$e)\int ctg^2 x dx = I_1.$$

Įvesime naują kintamąjį $t = ctgx.$

Tuomet $x = arcctgt; dx = -\dfrac{1}{1+t^2}dt,$

$$I_1 = -\int \frac{t^2}{1+t^2}dt = -\int \frac{1-1+t^2}{1+t^2}dt = \int \frac{dt}{1+t^2} - \int \frac{1+t^2}{1+t^2}dt =$$

$$= arctgt - \int dt = arctg(ctgx) - ctgx + C.$$