Hilber-type calculus

Definition 1.3.7. Hilbert-type calculus for propositional logic (denoted *HPC*) consists of axioms:

- *1.1.* $F \supset (G \supset F)$;
- *1.2.* $\big(F \supset (G \supset H)\big) \supset \big((F \supset G) \supset (F \supset H)\big)$;
- *2.1.* $(F \wedge G) \supset F$;
- *2.2.* $(F \wedge G) \supset G$;
- *2.3.* $(F \supset G) \supset \Big((F \supset H) \supset \big(F \supset (G \wedge H)\big)\Big)$;
- *3.1.* $F \supset (F \vee G)$;
- *3.2.* $G \supset (F \vee G)$;
- *3.3.* $(F \supset H) \supset \Big((G \supset H) \supset \big((F \vee G) \supset H\big)\Big)$;
- *4.1.* $(F \supset G) \supset (\neg G \supset \neg F)$;
- *4.2.* $F \supset \neg\neg F$;
- *4.3.* $\neg\neg F \supset F$;

Definition 2.4.3. Hilbert-type calculus for predicate logic (denoted *HPR*) consists of axioms of *HPC* and:

H

- *5.1* $\forall x F(x) \supset F(t)$,
- *5.2* $F(t) \supset \exists x F(x)$.

Here, t is a term, which is free with respect to variable x in formula $F(x)$.

The rules of *HPR* are:

$$\frac{F \quad F \supset G}{G}\,MP \qquad \frac{G \supset F(y)}{G \supset \forall x F(x)}\,\forall \qquad \frac{F(y) \supset G}{\exists x F(x) \supset G}\,\exists$$

Example 1.3.9. A derivation of $p \supset p$ in *HPC* is as follows:

1.	$\big(p \supset ((p \supset p) \supset p)\big) \supset \big((p \supset (p \supset p)) \supset (p \supset p)\big)$	Axiom *1.2*, $\{p/F, p\supset p/G, p/H\}$.
2.	$p \supset \big((p \supset p) \supset p\big)$	Axiom *1.1*, $\{p/F, p\supset p/G\}$.
3.	$\big(p \supset (p \supset p)\big) \supset (p \supset p)$	*MP* rule from $\boxed{2}$ and $\boxed{1}$.
4.	$p \supset (p \supset p)$	Axiom *1.1*, $\{p/F, p/G\}$.
5.	$p \supset p$	*MP* rule from $\boxed{4}$ and $\boxed{3}$.

Example 2.4.4. Let's show, that from formula $\forall x \forall y P(x, y)$ another formula $\forall y \forall x P(x, y)$ is derivable. Let's construct a derivation:

1.	$\forall x \forall y P(x, y)$	An assumption.
2.	$\forall x \forall y P(x, y) \supset \forall y P(a, y)$	Axiom *5.1*.
3.	$\forall y P(a, y)$	*MP* rule from $\boxed{1}$ and $\boxed{2}$.
4.	$\forall y P(a, y) \supset P(a, b)$	Axiom *5.1*.
5.	$P(a, b)$	*MP* rule from $\boxed{3}$ and $\boxed{4}$.
6.	$\forall x P(x, b)$	$\forall$ rule from $\boxed{5}$.
7.	$\forall y \forall x P(x, y)$	$\forall$ rule from $\boxed{6}$.

Sequential calculus

Definition 1.3.13. The original Gentzen-type calculus for propositional logic (GPC_o) consists of an axiom $F \to F$, structural rules:

Weakening:

$$\frac{\Gamma \to \Delta}{F, \Gamma \to \Delta}\,(w \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, F}\,(\to w)$$

Contraction:

$$\frac{F, F, \Gamma \to \Delta}{F, \Gamma \to \Delta}\,(c \to) \qquad \frac{\Gamma \to \Delta, F, F}{\Gamma \to \Delta, F}\,(\to c)$$

Exchange:

$$\frac{\Gamma_1, G, F, \Gamma_2 \to \Delta}{\Gamma_1, F, G, \Gamma_2 \to \Delta}\,(e \to) \qquad \frac{\Gamma \to \Delta_1, G, F, \Delta_2}{\Gamma \to \Delta_1, F, G, \Delta_2}\,(\to e)$$

logical rules:

Negation:

$$\frac{\Gamma \to \Delta, F}{\neg F, \Gamma \to \Delta}\,(\neg \to) \qquad \frac{F, \Gamma \to \Delta}{\Gamma \to \Delta, \neg F}\,(\to \neg)$$

Conjunction:

$$\frac{F, \Gamma \to \Delta}{F \wedge G, \Gamma \to \Delta}\,(\wedge \to)_1 \qquad \frac{G, \Gamma \to \Delta}{F \wedge G, \Gamma \to \Delta}\,(\wedge \to)_2$$

$$\frac{\Gamma \to \Delta, F \qquad \Gamma \to \Delta, G}{\Gamma \to \Delta, F \wedge G}\,(\to \wedge)$$

Disjunction:

$$\frac{F, \Gamma \to \Delta \qquad G, \Gamma \to \Delta}{F \vee G, \Gamma \to \Delta}\,(\vee \to)$$

$$\frac{\Gamma \to \Delta, F}{\Gamma \to \Delta, F \vee G}\,(\to \vee)_1 \qquad \frac{\Gamma \to \Delta, G}{\Gamma \to \Delta, F \vee G}\,(\to \vee)_2$$

Implication:

$$\frac{\Gamma \to \Delta, F \qquad G, \Gamma \to \Delta}{F \supset G, \Gamma \to \Delta}\,(\supset \to) \qquad \frac{F, \Gamma \to \Delta, G}{\Gamma \to \Delta, F \supset G}\,(\to \supset)$$

and the cut rule:

$$\frac{\Gamma_1 \to \Delta_1, F \qquad F, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}\,(\text{cut } F)$$

$$\frac{F(z), \Gamma \to \Delta}{\exists x F(x), \Gamma \to \Delta}\,(\exists \to) \qquad \frac{\Gamma \to \Delta, \exists x F(x), F(t)}{\Gamma \to \Delta, \exists x F(x)}\,(\to \exists)$$

$$\frac{F(t), \forall x F(x), \Gamma \to \Delta}{\forall x F(x), \Gamma \to \Delta}\,(\forall \to) \qquad \frac{\Gamma \to \Delta, F(z)}{\Gamma \to \Delta, \forall x F(x)}\,(\to \forall)$$

Here z is a new variable, which doesn't belong to Γ, Δ, $\exists x F(x)$ nor $\forall x F(x)$ and t is a term, which is free with respect to x in formula $F(x)$.

Example 1.3.14. A derivation in *GPC* of the formula used in Example 1.3.9 is obvious, so a derivation tree of axiom *2.3* of *HPC* is provided instead.

$$
\begin{prooftree}
\AxiomC{$F \to F$}
\RightLabel{$(\to w)$}
\UnaryInfC{$F \to F, G \wedge H$}
\RightLabel{$(\to e)$}
\UnaryInfC{$F \to G \wedge H, F$}
\RightLabel{$(w \to)$}
\UnaryInfC{$F \supset G, F \to G \wedge H, F$}
\RightLabel{$(e \to)$}
\UnaryInfC{$F, F \supset G \to G \wedge H, F$}
\AxiomC{$F \to F$}
\RightLabel{$(\to w)$}
\UnaryInfC{$F \to F, G \wedge H$}
\RightLabel{$(\to e)$}
\UnaryInfC{$F \to G \wedge H, F$}
\RightLabel{$(w \to)$}
\UnaryInfC{$H, F \to G \wedge H, F$}
\AxiomC{$G \to G$}
\RightLabel{$(w \to)$}
\UnaryInfC{$F, G \to G$}
\RightLabel{$(e \to)$}
\UnaryInfC{$G, F \to G$}
\RightLabel{$(w \to)$}
\UnaryInfC{$H, G, F \to G$}
\RightLabel{$(e \to)$}
\UnaryInfC{$G, H, F \to G$}
\AxiomC{$H \to H$}
\RightLabel{$(w \to)$}
\UnaryInfC{$F, H \to H$}
\RightLabel{$(e \to)$}
\UnaryInfC{$H, F \to H$}
\RightLabel{$(w \to)$}
\UnaryInfC{$G, H, F \to H$}
\RightLabel{$(\to \wedge)$}
\BinaryInfC{$G, H, F \to G \wedge H$}
\RightLabel{$(\supset \to)$}
\BinaryInfC{$F \supset G, H, F \to G \wedge H$}
\RightLabel{$(e \to)$}
\UnaryInfC{$H, F \supset G, F \to G \wedge H$}
\RightLabel{$(e \to)$}
\UnaryInfC{$H, F, F \supset G \to G \wedge H$}
\RightLabel{$(\supset \to)$}
\BinaryInfC{$F \supset H, F, F \supset G \to G \wedge H$}
\RightLabel{$(e \to)$}
\UnaryInfC{$F, F \supset H, F \supset G \to G \wedge H$}
\RightLabel{$(\to \supset)$}
\UnaryInfC{$F \supset H, F \supset G \to F \supset (G \wedge H)$}
\RightLabel{$(\to \supset)$}
\UnaryInfC{$F \supset G \to (F \supset H) \supset \big(F \supset (G \wedge H)\big)$}
\RightLabel{$(\to \supset)$}
\UnaryInfC{$\to (F \supset G) \supset \Big((F \supset H) \supset \big(F \supset (G \wedge H)\big)\Big)$}
\end{prooftree}
$$

However such calculus has some drawbacks too. First of all, it is not obvious how to chose the main formula in the cut rule.

Example 1.3.15. The choice of main formula in the cut rule is not obvious:

$$
\begin{prooftree}
\AxiomC{$p \to p$}
\RightLabel{$(\to w)$}
\UnaryInfC{$p \to p, q$}
\RightLabel{$(\to e)$}
\UnaryInfC{$p \to q, p$}
\AxiomC{$q \to q$}
\RightLabel{$(w \to)$}
\UnaryInfC{$p, q \to q$}
\RightLabel{$(e \to)$}
\UnaryInfC{$q, p \to q$}
\RightLabel{$(\supset \to)$}
\BinaryInfC{$p \supset q, p \to q$}
\RightLabel{$(e \to)$}
\UnaryInfC{$p, p \supset q \to q$}
\AxiomC{$q \to q$}
\RightLabel{$(\to w)$}
\UnaryInfC{$q \to q, r$}
\RightLabel{$(\to e)$}
\UnaryInfC{$q \to r, q$}
\AxiomC{$r \to r$}
\RightLabel{$(w \to)$}
\UnaryInfC{$q, r \to r$}
\RightLabel{$(e \to)$}
\UnaryInfC{$r, q \to r$}
\RightLabel{$(\supset \to)$}
\BinaryInfC{$q \supset r, q \to r$}
\RightLabel{$(e \to)$}
\UnaryInfC{$q, q \supset r \to r$}
\RightLabel{(cut q)}
\BinaryInfC{$p, p \supset q, q \supset r \to r$}
\end{prooftree}
$$

Next, there are a lot of applications of different structural rules:

Example 1.3.16. Consider the following derivation:

$$
\begin{prooftree}
\AxiomC{$p \to p$}
\RightLabel{$(w \to)$}
\UnaryInfC{$q, p \to p$}
\AxiomC{$q \to q$}
\RightLabel{$(w \to)$}
\UnaryInfC{$p, q \to q$}
\RightLabel{$(e \to)$}
\UnaryInfC{$q, p \to q$}
\RightLabel{$(\to \wedge)$}
\BinaryInfC{$q, p \to p \wedge q$}
\RightLabel{$(\to \vee)_1$}
\UnaryInfC{$q, p \to (p \wedge q) \vee (p \wedge r)$}
\AxiomC{$p \to p$}
\RightLabel{$(w \to)$}
\UnaryInfC{$r, p \to p$}
\AxiomC{$r \to r$}
\RightLabel{$(w \to)$}
\UnaryInfC{$p, r \to r$}
\RightLabel{$(e \to)$}
\UnaryInfC{$r, p \to r$}
\RightLabel{$(\to \wedge)$}
\BinaryInfC{$r, p \to p \wedge r$}
\RightLabel{$(\to \vee)_2$}
\UnaryInfC{$r, p \to (p \wedge q) \vee (p \wedge r)$}
\RightLabel{$(\vee \to)$}
\BinaryInfC{$q \vee r, p \to (p \wedge q) \vee (p \wedge r)$}
\RightLabel{$(\wedge \to)_2$}
\UnaryInfC{$p \wedge (q \vee r), p \to (p \wedge q) \vee (p \wedge r)$}
\RightLabel{$(e \to)$}
\UnaryInfC{$p, p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$}
\RightLabel{$(\wedge \to)_1$}
\UnaryInfC{$p \wedge (q \vee r), p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$}
\RightLabel{$(c \to)$}
\UnaryInfC{$p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$}
\end{prooftree}
$$

Gentzen-type calculus

Aksiomos: $F \vdash F$.

Struktūrinės taisyklės:

(silpninimas) $$\frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}, \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F},$$

(prastinimas) $$\frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}, \qquad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F},$$

(perstatymas) $$\frac{\Gamma_1, F, G, \Gamma_2 \vdash \Delta}{\Gamma_1, G, F, \Gamma_2 \vdash \Delta}, \qquad \frac{\Gamma \vdash \Delta_1, F, G, \Delta_2}{\Gamma \vdash \Delta_1, G, F, \Delta_2}.$$

Loginių operacijų taisyklės:

$$(\neg \vdash) \quad \frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta}, \qquad (\vdash \neg) \quad \frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F},$$

$$(\& \vdash) \quad \frac{F, G, \Gamma \vdash \Delta}{F \& G, \Gamma \vdash \Delta}, \qquad (\vdash \&) \quad \frac{\Gamma \vdash \Delta, F \qquad \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \& G},$$

$$(\vee \vdash) \quad \frac{F, \Gamma \vdash \Delta \qquad G, \Gamma \vdash \Delta}{F \vee G, \Gamma \vdash \Delta}, \qquad (\vdash \vee) \quad \frac{\Gamma \vdash \Delta, F, G}{\Gamma \vdash \Delta, F \vee G},$$

$$(\rightarrow \vdash) \quad \frac{\Gamma \vdash \Delta, F \qquad G, \Gamma \vdash \Delta}{F \rightarrow G, \Gamma \vdash \Delta}, \qquad (\vdash \rightarrow) \quad \frac{F, \Gamma \vdash \Delta, G}{\Gamma \vdash \Delta, F \rightarrow G}.$$

Pjūvio taisyklė:

$$\frac{\Gamma_1 \vdash \Delta_1, F \qquad F, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}.$$

Kvantorinės taisyklės:

$$(\exists \vdash) \quad \frac{F(z), \Gamma \vdash \Delta}{\exists x F(x), \Gamma \vdash \Delta}, \qquad (\vdash \exists) \quad \frac{\Gamma \vdash \Delta, F(t), \exists x F(x)}{\Gamma \vdash \Delta \exists x F(x)},$$

$$(\forall \vdash) \quad \frac{F(t), \forall x F(x), \Gamma \vdash \Delta}{\forall x F(x), \Gamma \vdash \Delta}, \qquad (\vdash \forall) \quad \frac{\Gamma \vdash \Delta, F(z)}{\Gamma \vdash \Delta \forall x F(x)}.$$

Čia z yra naujas kintamasis, neįeinantis į Γ, Δ, $\exists x F(x)$ arba $\forall x F(x)$, t – termas, laisvas kintamojo x atžvilgiu formulėje $F(x)$.

Primename, kad sekvencijoje $\Gamma \vdash \Delta$ seka Γ vadinama ***antecedentu***, o Δ – ***sukcedentu***. Sakoma: formulė F priklauso sekvencijos antecedentui, jei ji yra sekoje Γ; formulė priklauso sukcedentui, jei ji yra sekoje Δ.

Pavyzdžiai:

1. Parodykime, kad sekvencija $F\&G, \neg H \vdash (\neg F \vee \neg G) \to H$ išvedama nagrinėjamajame skaičiavime:

$$
\begin{array}{c}
\begin{array}{c}
F \vdash F \\ \hline
F \vdash F, H \\ \hline
F \vdash H, F \\ \hline
\neg H, F \vdash H, F \\ \hline
F, \neg H \vdash H, F \\ \hline
G, F, \neg H \vdash H, F \\ \hline
F, G, \neg H \vdash H, F \\ \hline
\neg F, F, G, \neg H \vdash H
\end{array}
\qquad
\begin{array}{c}
G \vdash G \\ \hline
G \vdash G, H \\ \hline
G \vdash H, G \\ \hline
\neg H, G \vdash H, G \\ \hline
G, \neg H \vdash H, G \\ \hline
F, G, \neg H \vdash H, G \\ \hline
\neg G, F, G, \neg H \vdash H
\end{array} \\ \hline
\neg F \vee \neg G, F, G, \neg H \vdash H \\ \hline
F, G, \neg H \vdash (\neg F \vee \neg G) \to H \\ \hline
F\&G, \neg H \vdash (\neg F \vee \neg G) \to H
\end{array}.
$$

2. Sekvencija $\vdash \exists x \forall y A(x, y) \to \forall y \exists x A(x, y)$ taip pat išvedama:

$$
\begin{array}{c}
A(a, b) \vdash A(a, b) \\ \hline
A(a, b) \vdash A(a, b), \exists x A(x, b) \\ \hline
\forall y A(a, y), A(a, b) \vdash A(a, b), \exists x A(x, b) \\ \hline
A(a, b), \forall y A(a, y) \vdash A(a, b), \exists x A(x, b) \\ \hline
A(a, b), \forall y A(a, y) \vdash \exists x A(x, b) \\ \hline
\forall y A(a, y) \vdash \exists x A(x, b) \\ \hline
\forall y A(a, y) \vdash \forall y \exists x A(x, y) \\ \hline
\exists x \forall y A(x, y) \vdash \forall y \exists x A(x, y) \\ \hline
\vdash \exists x \forall y A(x, y) \to \forall y \exists x A(x, y)
\end{array}.
$$

Example 2.5.4. Let's show that sequent $\to \forall x \forall y P(x, y) \supset \forall y \forall x P(x, y)$ is derivable in calculus GPR. Recall, that this sequent is similar to the case analysed in Example 2.4.4.

$$
\begin{array}{cl}
P(x_1, y_1), \forall y P(x_1, y), \forall x \forall y P(x, y) \to P(x_1, y_1) & \\ \hline
\forall y P(x_1, y), \forall x \forall y P(x, y) \to P(x_1, y_1) & (\forall \to) \\ \hline
\forall x \forall y P(x, y) \to P(x_1, y_1) & (\forall \to) \\ \hline
\forall x \forall y P(x, y) \to \forall x P(x, y_1) & (\to \forall) \\ \hline
\forall x \forall y P(x, y) \to \forall y \forall x P(x, y) & (\to \forall) \\ \hline
\to \forall x \forall y P(x, y) \supset \forall y \forall x P(x, y) & (\to \supset)
\end{array}
$$

Example 2.5.5. Let's derive sequent: $\to \exists x \forall y P(x, y) \supset \forall y \exists x P(x, y)$ in calculus GPR.

$$
\begin{array}{cl}
P(x_1, y_1), \forall y P(x_1, y) \to \exists x P(x, y_1), P(x_1, y_1) & \\ \hline
P(x_1, y_1), \forall y P(x_1, y) \to \exists x P(x, y_1) & (\to \exists) \\ \hline
\forall y P(x_1, y) \to \exists x P(x, y_1) & (\forall \to) \\ \hline
\forall y P(x_1, y) \to \forall y \exists x P(x, y) & (\to \forall) \\ \hline
\exists x \forall y P(x, y) \to \forall y \exists x P(x, y) & (\exists \to) \\ \hline
\to \exists x \forall y P(x, y) \supset \forall y \exists x P(x, y) & (\to \supset)
\end{array}
$$

Example 2.5.6. Now a derivation of a well known tautology of predicate logic is presented: $\to \big(\neg \forall x P(x) \supset \exists x \neg P(x)\big) \wedge \big(\exists x \neg P(x) \supset \neg \forall x P(x)\big)$.

$$
\begin{array}{c}
\begin{array}{cl}
P(z) \to \exists x \neg P(x), P(z) & \\ \hline
\to \exists x \neg P(x), \neg P(z), P(z) & (\to \neg) \\ \hline
\to \exists x \neg P(x), P(z) & (\to \exists) \\ \hline
\to \exists x \neg P(x), \forall x P(x) & (\to \forall) \\ \hline
\neg \forall x P(x) \to \exists x \neg P(x) & (\neg \to) \\ \hline
\to \neg \forall x P(x) \supset \exists x \neg P(x) & (\to \supset)
\end{array}
\qquad
\begin{array}{cl}
P(z), \forall x P(x) \to P(z) & \\ \hline
\forall x P(x) \to P(z) & (\forall \to) \\ \hline
\forall x P(x), \neg P(z) \to & (\neg \to) \\ \hline
\forall x P(x), \exists x \neg P(x) \to & (\exists \to) \\ \hline
\exists x \neg P(x) \to \neg \forall x P(x) & (\to \neg) \\ \hline
\to \exists x \neg P(x) \supset \neg \forall x P(x) & (\to \supset)
\end{array} \\ \hline
\to \big(\neg \forall x P(x) \supset \exists x \neg P(x)\big) \wedge \big(\exists x \neg P(x) \supset \neg \forall x P(x)\big) \quad (\to \wedge)
\end{array}
$$

Gentzen-type calculus
For intuitionistic logic

Definition 2.10.6. Gentzen-type calculus for intuitionistic logic (denoted *GIN*) is composed of axiom $F \to F$, structural rules:

Weakening:

$$\frac{\Gamma \to \Delta}{F, \Gamma \to \Delta}\,(w \to) \qquad \frac{\Gamma \to}{\Gamma \to F}\,(\to w)$$

Contraction:

$$\frac{F, F, \Gamma \to \Delta}{F, \Gamma \to \Delta}\,(c \to)$$

Exchange:

$$\frac{\Gamma_1, G, F, \Gamma_2 \to \Delta}{\Gamma_1, F, G, \Gamma_2 \to \Delta}\,(e \to)$$

Logical rules:

Negation:

$$\frac{\Gamma \to F}{\neg F, \Gamma \to \Delta}\,(\neg \to) \qquad \frac{F, \Gamma \to}{\Gamma \to \neg F}\,(\to \neg)$$

Conjunction:

$$\frac{F, G, \Gamma \to \Delta}{F \wedge G, \Gamma \to \Delta}\,(\wedge \to) \qquad \frac{\Gamma \to F \qquad \Gamma \to G}{\Gamma \to F \wedge G}\,(\to \wedge)$$

Disjunction:

$$\frac{F, \Gamma \to \Delta \qquad G, \Gamma \to \Delta}{F \vee G, \Gamma \to \Delta}\,(\vee \to)$$

$$\frac{\Gamma \to F}{\Gamma \to F \vee G}\,(\to \vee)_1 \qquad \frac{\Gamma \to G}{\Gamma \to F \vee G}\,(\to \vee)_2$$

Implication:

$$\frac{\Gamma \to F \qquad G, \Gamma \to \Delta}{F \supset G, \Gamma \to \Delta}\,(\supset \to) \qquad \frac{F, \Gamma \to G}{\Gamma \to F \supset G}\,(\to \supset)$$

Quantifier rules:

Existence:

$$\frac{F(z), \Gamma \to \Delta}{\exists x F(x), \Gamma \to \Delta}\,(\exists \to) \qquad \frac{\Gamma \to F(t)}{\Gamma \to \Delta, \exists x F(x)}\,(\to \exists)$$

Universality:

$$\frac{F(t), \forall x F(x), \Gamma \to \Delta}{\forall x F(x), \Gamma \to \Delta}\,(\forall \to) \qquad \frac{\Gamma \to F(z)}{\Gamma \to \forall x F(x)}\,(\to \forall)$$

Variable z and term t must satisfy the same requirements as in the case of GPR_o.

And the cut rule:

$$\frac{\Gamma_1 \to F \qquad F, \Gamma_2 \to \Delta}{\Gamma_1, \Gamma_2 \to \Delta}\,(\text{cut } F)$$

Note that set Δ in this calculus represents one formula or an empty set. Succedent is not necessary, however it must contain no more than one formula.

Equivalency

Lemma 1.3.29. *These formulas are equivalent in propositional logic:*

- $\neg\neg p \equiv p$,
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$,
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$,
- $p \supset q \equiv \neg p \vee q$,
- $p \leftrightarrow q \equiv (p \supset q) \wedge (q \supset p)$.

Lemma 2.1.19. *These formulas are equivalent in predicate logic:*

1. $\forall x \forall y F \equiv \forall y \forall x F$,
2. $\exists x \exists y F \equiv \exists y \exists x F$,
3. $\forall x F(x) \equiv \forall y F(y)$, *here* y *is not part of* $F(x)$ *and* x *is not part of* $F(y)$,
4. $\exists x F(x) \equiv \exists y F(y)$, *here* y *is not part of* $F(x)$ *and* x *is not part of* $F(y)$,
5. $\neg\forall x F \equiv \exists x \neg F$,
6. $\neg\exists x F \equiv \forall x \neg H$.

1. $\neg\forall x F(x) \equiv \exists x \neg F(x)$,
2. $\neg\exists x F(x) \equiv \forall x \neg F(x)$,
3. $\exists x F(x) \equiv \exists y F(y)$, where y is a new variable, not part of $F(x)$,
4. $\forall x F(x) \equiv \forall y F(y)$, where y is a new variable, not part of $F(x)$,
5. $\forall x F(x) \wedge G \equiv \forall x \big(F(x) \wedge G\big)$, where x is not part of G,
6. $\exists x F(x) \wedge G \equiv \exists x \big(F(x) \wedge G\big)$, where x is not part of G,
7. $\forall x F(x) \vee G \equiv \forall x \big(F(x) \vee G\big)$, where x is not part of G,
8. $\exists x F(x) \vee G \equiv \exists x \big(F(x) \vee G\big)$, where x is not part of G,

   1. $\forall x F(x) \wedge \forall x G(x) \equiv \forall x \big(F(x) \wedge G(x)\big)$,
   2. $\exists x F(x) \vee \exists x G(x) \equiv \exists x \big(F(x) \vee G(x)\big)$,

Function symbols / Compactness

Definition 2.3.1. A *term* is defined as follows:

- Individual constant is a term.
- Individual variable is a term.
- If f^n is function symbol and $t_1, t_2, \ldots, t_n$ are terms, then $f(t_1, t_2, \ldots, t_n)$ is also a term.

Example 2.3.2. Let $\mathcal{M} = \{a, b, c\}$, $f(x)$ is one-place function symbol. Then the following expressions are terms: a, b, c, x, y, z, $f(x)$, $f(a)$, $f(c)$, $f\big(f(x)\big)$, $f\big(f(a)\big)$, $f\Big(f\big(f(y)\big)\Big)$,....

Definition 2.3.3. *Formula* is defined recursively as follows:

1. If P is n-place predicate variable and $t_1, t_2, \ldots, t_n$ are terms, then $P(t_1, t_2, \ldots, t_n)$ is a formula.
2. If p is a propositional variable, then p is a formula.
3. If F is a formula, then $\neg F$ is also a formula.
4. If F and G are formulas, then $(F \wedge G)$, $(F \vee G)$, $(F \supset G)$ and $(F \leftrightarrow G)$ are formulas.
5. if F is a formula and x is individual variable, then $\exists x F$ and $\forall x F$ are also formulas.

2.7 Compactness

In this section only propositional formulas are analysed. However, the results presented here are needed for the next section.

Definition 2.7.1. A set of formulas is *finite satisfiable*, if every finite subset of the set is satisfiable. A set of formulas $\{F_1, F_2, \ldots\}$ is *satisfiable* if there exists an interpretation ν such that $\nu \vDash F_i$ for every $i \in [1, \infty)$.

If a set of formulas is not satisfiable, then it is called *contradictory*. I.e. a set of propositional formulas $\mathcal{F}$ is contradictory if for any interpretation ν there is a formula $F \in \mathcal{F}$ such that $\nu \nvDash F$.

Definition 2.7.2. A set of formulas $\mathcal{F}$ is *maximal*, if:

- $\mathcal{F}$ is finite satisfiable and
- for any formula F either $F \in \mathcal{F}$ or $\neg F \in \mathcal{F}$.

2.6 Semantic tableaux method

Suppose, sequent $F_1, F_2, \ldots, F_n \to G_1, G_2, \ldots, G_m$ is derivable in sequent calculus for predicate logic GPR and the derivation is $\mathcal{D}$. Then sequent $F_1, F_2, \ldots, F_n, \neg G_1, \neg G_2, \ldots, \neg G_m \to$ is also derivable and the derivation is:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{F_1, F_2, \ldots, F_n \to G_1, G_2, \ldots, G_m}}{F_1, F_2, \ldots, F_n, \neg G_1 \to G_2, G_3, \ldots, G_m}(\neg \to)}{F_1, F_2, \ldots, F_n, \neg G_1, \neg G_2 \to G_3, G_4, \ldots, G_m}(\neg \to)$$

$$\ldots$$

$$F_1, F_2, \ldots, F_n, \neg G_1, \neg G_2, \ldots, \neg G_m \to$$

$$\frac{F, \Gamma \to}{\neg\neg F, \Gamma \to} \qquad \frac{\neg F, \Gamma \to \qquad \neg G, \Gamma \to}{\neg(F \wedge G), \Gamma \to} \qquad \frac{\neg F, \neg G, \Gamma \to}{\neg(F \vee G), \Gamma \to}$$

$$\frac{F, \neg G, \Gamma \to}{\neg(F \supset G), \Gamma \to} \qquad \frac{\neg F(t), \neg\exists x F(x), \Gamma \to}{\neg\exists x F(x), \Gamma \to} \qquad \frac{\neg F(z), \Gamma \to}{\neg\forall x F(x), \Gamma \to}$$

The change to rule $(\supset \to)$ is obvious:

$$\frac{\neg F, \Gamma \to \qquad G, \Gamma \to}{F \supset G, \Gamma \to}(\supset \to)$$

Conjunction rules:

$$\frac{F \wedge G}{\begin{matrix}F\\G\end{matrix}} \qquad \frac{\neg(F \vee G)}{\begin{matrix}\neg F\\\neg G\end{matrix}} \qquad \frac{\neg(F \supset G)}{\begin{matrix}F\\\neg G\end{matrix}}$$

Disjunction rules:

$$\frac{\neg(F \wedge G)}{\neg F \mid \neg G} \qquad \frac{F \vee G}{F \mid G} \qquad \frac{F \supset G}{\neg F \mid G}$$

Double negation rule:

$$\frac{\neg\neg F}{F}$$

Quantifier rules:

$$\frac{\forall x F(x)}{F(t)} \qquad \frac{\neg\forall x F(x)}{\neg F(z)} \qquad \frac{\exists x F(x)}{F(z)} \qquad \frac{\neg\exists x F(x)}{\neg F(t)}$$

Here z is a new free variable, which is not part of any formula of the branch, and t is some main term.

2.8 Semantic trees

Now let's return to predicate logic. In this section predicate formulas with function symbols but without free variables are analysed. Moreover, formulas must be transformed into normal prenex form and skolemized.

Definition 2.8.1. A *Herbrand universe* $\mathcal{H}$ of formula F is defined as follows:

- Every constant of formula F is part of $\mathcal{H}$. If F does not contain any constants, then $a \in \mathcal{H}$.
- If f^n is n-place function symbol in formula F and $t_1, t_2, \dots, t_n \in \mathcal{H}$, then $f(t_1, t_2, \dots, t_n) \in \mathcal{H}$

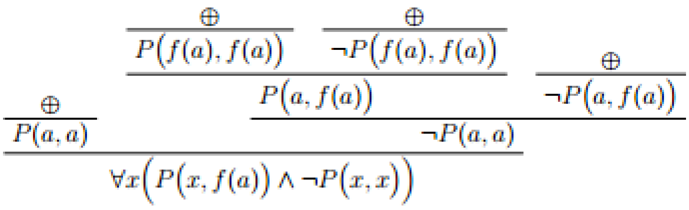
It must be noted that according to the definition Herbrand universe of any formula is never empty. It is finite, if the formula does not contain function symbols, or infinite but countable.

Definition 2.8.2. A *Herbrand base* $\mathcal{B}$ of formula F is the set of all the atomic formulas $P(t_1, t_2, \dots, t_n)$, where P^n is n-place predicate variable of F and $t_i, i \in [1, n]$ are some elements of Herbrand universe of F.

Definition 2.8.3. *H-interpretation* (or *Herbrand interpretation*) of formula F is a set $\{\alpha_1 P_1, \alpha_2 P_2, \dots\}$, where $\alpha_i \in \{\neg, \varnothing\}$ and $P_i \in \mathcal{B}, i \in [1, \infty)$. If $\alpha_i = \neg$, then it is understood that P_i is false with the H-interpretation, otherwise P_i is true.

Example 2.8.5. Let's start by finding a finite semantic tree of predicate formula $\forall x\Big(P\big(x, f(a)\big) \wedge \neg P\big(x, x\big)\Big)$.

$$\mathcal{H} = \{a, f(a), f\big(f(a)\big), \dots\}, \mathcal{B} = \{P\big(a, a\big), P\big(a, f(a)\big), P\big(f(a), f(a)\big), \dots\}$$



In this case set $\mathcal{T}$ is equal to:

$$\{F_1 = P\big(a, f(a)\big) \wedge \neg P\big(a, a\big),$$

$$F_2 = P\big(f(a), f(a)\big) \wedge \neg P\big(f(a), f(a)\big),$$

$$F_3 = P\Big(f\big(f(a)\big), f(a)\Big) \wedge \neg P\Big(f\big(f(a)\big), f\big(f(a)\big)\Big), \dots\}$$

Let's analyse branches from left to right. With interpretation $\{P(a, a)\}$ formula F_1 is false. With interpretations $\{\neg P\big(a, a\big), P\big(a, f(a)\big), P\big(f(a), f(a)\big)\}$ and $\{\neg P\big(a, a\big), P\big(a, f(a)\big), \neg P\big(f(a), f(a)\big)\}$ formula F_2 is false. With interpretation $\{\neg P\big(a, a\big), \neg P\big(a, f(a)\big)\}$ once again formula F_1 is false. Thus the contradictory subset consists of two formulas: F_1 and F_2.

2.9 Resolution method

2.9.1 The description of resolution calculus

Once again, in this section only skolemized normal prenex form predicate formulas are analysed. In order to apply resolution method a set of disjuncts must be obtained. As mentioned in the previous section, all the analysed formulas are of the form $\forall x_1 \forall x_2 \dots \forall x_n G(x_1, x_2, \dots, x_n)$. This formula is not satisfiable iff a set $\mathcal{T} = \{G(t_1, t_2 \dots, t_n) : t_i \in \mathcal{H}, i \in [1, n]\}$, where $\mathcal{H}$ is Herbrand universe of the formula, is contradictory. From the fact that $\mathcal{H}$ is either finite or countable and there are only finite number (n) of individual variables in the formula it follows that $\mathcal{T}$ is either finite or countable.

Now let's transform formula $G(x_1, x_2, \dots, x_n)$ into NCF. Suppose that NCF of $G(x_1, x_2, \dots, x_n)$ is $\bigwedge\limits_{j=1}^{m} H_j(x_1, x_2, \dots, x_n)$, where $H_j, j \in [1, m]$ are disjuncts. Let's analyse a set of disjuncts

Definition 2.9.1. A substitution α is called a *unifier* of formulas (terms) F_1 and F_2, if $F_1\alpha = F_2\alpha$.

Definition 2.9.2. A unifier α is the *most general* one for formulas (terms) F_1 and F_2, if for any unifier β of F_1 and F_2 there is a substitution γ, such that $\beta = \alpha \circ \gamma$ (i.e. β is a composition of α and γ).

Substitution α is a *unifier* of set $\{F_1, F_2, \dots, F_n\}$, if $F_1\alpha = F_2\alpha = \dots = F_n\alpha$. Substitution α is the *most general unifier* of the set, if it is a unifier of the set and for any unifier of the set β there is a substitution γ such that $\beta = \alpha \circ \gamma$.

Example 2.9.4. Atomic formulas $P\Big(x, f\big(f(y)\big)\Big)$ and $P\Big(f(z), f\Big(f\big(g(a)\big)\Big)\Big)$ are unifiable. The unifiers are, for example, $\big\{f(a)/x, g(a)/y, a/z\big\}$ and $\Big\{f\big(f(a)\big)/x, g(a)/y, f(a)/z\Big\}$. The most general unifier is $\big\{f(z)/x, g(a)/y\big\}$.

Of course, not all the expressions are unifiable. E.g. formulas $F(t_1)$ and $F(t_2)$ are not unifiable, if terms t_1 and t_2 are:

- two different constants,
- a variable x and term t which contains x, $t \neq x$,
- a constant and function symbol,
- terms starting with different function symbols.

Now a rule of resolution for predicate formulas is:

$$\frac{F \vee l_1 \qquad l_2 \vee G}{(F \vee G)\alpha}\,\alpha$$

Here l_1 and l_2 are literals, one of which is $P(t_1^1, t_2^1, \dots, t_n^1)$ and another one is $\neg P(t_1^2, t_2^2, \dots, t_n^2)$. Substitution α is a unifier of $P(t_1^1, t_2^1, \dots, t_n^1)$ and $P(t_1^2, t_2^2, \dots, t_n^2)$.

In order to derive formula in resolution calculus the following steps must be taken:

1. it must be transformed into normal prenex form;
2. it must be skolemized, $\forall$ quantifiers must be removed;
3. NCF of skolemized formula must be found;
4. a set composed of the disjuncts of the NCF must be created;
5. the set of disjuncts must be derived in resolution calculus using resolution rule for predicate logic.

Example 2.9.5. Using resolution method let's prove that if some binary relation is irreflexive and transitive, then it is asymmetric.

Let's denote a binary relation between x and y as predicate $P(x,y)$. Irreflexivity is defined as $F_1 = \forall x \neg P(x,x)$ and transitivity is defined as $F_2 = \forall x \forall y \forall z \big((P(x,y) \wedge P(y,z)) \supset P(x,z) \big)$. Asymmetry is defined as $F_3 = \forall x \forall y \big(P(x,y) \supset \neg P(y,x) \big)$.

The aim is to prove that from F_1 and F_2 follows F_3. This can be done by showing that set $\{F_1, F_2, \neg F_3\}$ is contradictory.

Let's transform the set into set of disjuncts. After removing $\forall$ quantifier from F_1, a disjunct $\neg P(x,x)$ is obtained. Formula F_2 is also skolemized, thus let's remove all the $\forall$ quantifiers and transform it into NCF:

$$\big(P(x,y) \wedge P(y,z)\big) \supset P(x,z) = \neg\big(P(x,y) \wedge P(y,z)\big) \vee P(x,z) =$$

$$= \neg P(x,y) \vee \neg P(y,z) \vee P(x,z)$$

Thus formula F_2 results in one disjunct: $\neg P(x,y) \vee \neg P(y,z) \vee P(x,z)$. Finally with F_3 every step of the algorithm must be performed. First it must be transformed into normal prenex form:

$$\neg \forall x \forall y \big(P(x,y) \supset \neg P(y,x) \big) = \exists x \exists y \neg \big(P(x,y) \supset \neg P(y,x) \big) =$$

$$= \exists x \exists y \neg \big(\neg P(x,y) \vee \neg P(y,x) \big) = \exists x \exists y \big(P(x,y) \wedge P(y,x) \big)$$

After skolemization formula $P(a,b) \wedge P(b,a)$ is obtained and this results in two disjuncts $P(a,b)$ and $P(b,a)$.

Therefore, $\mathcal{S} = \{\neg P(x,x), \neg P(x,y) \vee \neg P(y,z) \vee P(x,z), P(a,b), P(b,a)\}$. Now the following derivation can be obtained:

$$\dfrac{\neg P(x,x) \qquad \dfrac{P(b,a) \qquad \dfrac{P(a,b) \qquad \neg P(x,y) \vee \neg P(y,z) \vee P(x,z)}{\neg P(b,z) \vee P(a,z)}\{a/x, b/y\}}{P(a,a)}\{a/z\}}{\square}\{a/x\}$$

2.9.2 Linear tactics

Suppose formula F is derivable from set $\mathcal{S}$ and $T_1, T_2, \ldots T_n$ is a sequence of applications of resolution rule in the derivation. If a conclusion of T_n is F and for every $i \in [2,n]$ one of the premisses of T_i is a conclusion of T_{i-1}, then it is said that the derivation follows the linear tactics.

Example 2.9.6. Let's transform the following derivation into the one that follows the linear tactics:

$$\dfrac{\dfrac{\mathcal{D}}{F_3 \vee p} \qquad \dfrac{F_1 \vee \neg p \vee q \qquad F_2 \vee \neg q}{F_1 \vee F_2 \vee \neg p}}{F_1 \vee F_2 \vee F_3}$$

Here $\mathcal{D}$ denotes a derivation of $F_3 \vee p$, which follows linear tactics. The solution is as follows:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{F_3 \vee p} \qquad F_1 \vee \neg p \vee q}{F_1 \vee F_3 \vee q} \qquad F_2 \vee \neg q}{F_1 \vee F_2 \vee F_3}$$

Example 2.9.7. Let's derive the set used in Example 1.3.25 in resolution calculus by following the linear tactics:

$$\dfrac{F \qquad \dfrac{\neg F \vee H \qquad \dfrac{\neg F \vee G \qquad \neg G \vee \neg H}{\neg F \vee \neg H}}{\neg F}}{\square}$$

Resolution method

2.9.3 Tactics of semantic resolution

Suppose, $\mathcal{S}$ is a set of disjuncts and $\mathcal{I}$ is some interpretation, which divides $\mathcal{S}$ into two subsets $\mathcal{S}_+$ and $\mathcal{S}_-$. Here $\mathcal{S}_+$ consists of formulas from $\mathcal{S}$ that are true with interpretation $\mathcal{I}$ and $\mathcal{S}_-$ consists of formulas from $\mathcal{S}$ that are false with interpretation $\mathcal{I}$.

According to the tactics, premisses of the application of resolution rule must be part of different subsets. After the application, the conclusion is also assigned to $\mathcal{S}_+$ or $\mathcal{S}_-$ depending on if it is true with $\mathcal{I}$ or not.

Example 2.9.8. Let $\mathcal{S} = \{p \vee q \vee \neg r, \neg p \vee q, \neg q \vee \neg r, r\}$ and interpretation $\mathcal{I}$ defined as follows: $\mathcal{I}(p) = \top$, $\mathcal{I}(q) = \top$ and $\mathcal{I}(r) = \bot$.

In this case $\mathcal{S}_+ = \{p \vee q \vee \neg r, \neg p \vee q, \neg q \vee \neg r\}$ and $\mathcal{S}_- = \{r\}$.

The derivation is as follows:

$$\dfrac{\dfrac{\dfrac{p \vee q \vee \neg r \qquad r}{p \vee q} \qquad \dfrac{\neg q \vee \neg r \qquad r}{\neg q}}{p} \qquad \dfrac{\neg p \vee q \qquad \dfrac{\neg q \vee \neg r \qquad r}{\neg q}}{\neg p}}{\square}$$

It should be noted that the conclusions are included into subsets as soon as they are obtained. At the end $\mathcal{S}_+ = \{p \vee q \vee \neg r, \neg p \vee q, \neg q \vee \neg r, p \vee q, p\}$ and $\mathcal{S}_- = \{r, \neg q, \neg p\}$.

Example 2.9.9. Let's find the derivation of

$$\mathcal{S} = \{p_1 \vee p_2, p_1 \vee \neg p_2, \neg p_1 \vee p_2, \neg p_1 \vee \neg p_2\}$$

using semantic resolution tactics, when the interpretation $\mathcal{I}$ is $\mathcal{I}(p_1) = \top$, $\mathcal{I}(p_2) = \bot$.

Based on the given information $\mathcal{S}_+ = \{p_1 \vee p_2, p_1 \vee \neg p_2, \neg p_1 \vee \neg p_2\}$ and $\mathcal{S}_- = \{\neg p_1 \vee p_2\}$. Thus the derivation would be as follows:

$$\dfrac{\dfrac{p_1 \vee \neg p_2 \qquad \dfrac{p_1 \vee p_2 \qquad \neg p_1 \vee p_2}{p_2}}{p_1} \qquad \dfrac{\neg p_1 \vee \neg p_2 \qquad \neg p_1 \vee p_2}{\neg p_1}}{\square}$$

The subsets are:

$$\mathcal{S}_+ = \{p_1 \vee p_2, p_1 \vee \neg p_2, \neg p_1 \vee \neg p_2, p_1\}, \mathcal{S}_- = \{\neg p_1 \vee p_2, p_2, \neg p_1\}$$

2.9.4 Absorption tactics

It is said that disjunct $F_1 = l \vee G$ is absorbed by another disjunct F_2, if $F_2 = \neg l \vee G \vee G'$. Here l is a literal, $\neg\neg l = l$, G and G' are disjuncts and G' may be empty.

According to absorption tactics, a resolution rule can be applied only if one of the disjuncts absorbs the other one. More precisely, the rule can be applied to F_1 and F_2 with unifier α is either $F_1\alpha$ absorbs or is absorbed by $F_2\alpha$.

Example 2.9.12. Suppose, it is possible to transform any derivation tree to the one which follows the absorption tactics, if there are no more than n applications of resolution rule (induction hypothesis). A derivation tree of $F \vee G$ with $n+1$ application of resolution rule is as follows:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{F \vee p \vee q} \qquad \dfrac{\mathcal{D}_2}{F \vee p \vee \neg q}}{F \vee p} \qquad \dfrac{\mathcal{D}_3}{G \vee \neg p}}{F \vee G}\,(*)$$

The $(*)$ denotes an application of resolution rule which doesn't follow the absorption tactics. Let's prove, that it is possible to derive disjunct $F \vee G$ in resolution method using absorption tactics.

Let's analyse the two derivations:

$$\dfrac{\dfrac{\mathcal{D}_3}{G \vee \neg p} \qquad \dfrac{\mathcal{D}_1}{F \vee p \vee q}}{F \vee G \vee q} \quad \text{and} \quad \dfrac{\dfrac{\mathcal{D}_3}{G \vee \neg p} \qquad \dfrac{\mathcal{D}_2}{F \vee p \vee \neg q}}{F \vee G \vee \neg q}$$

Both of them have no more than n applications of resolution rule. Therefore, according to the induction hypothesis, it is possible to derive both disjuncts $F \vee G \vee q$ and $F \vee G \vee \neg q$ using absorption tactics. Let's denote the resulting derivations $\mathcal{D}_1'$ and $\mathcal{D}_2'$. Then the derivation of $F \vee G$ that follows the absorption tactics is as follows:

$$\dfrac{\dfrac{\mathcal{D}_1'}{F \vee G \vee q} \qquad \dfrac{\mathcal{D}_2'}{F \vee G \vee \neg q}}{F \vee G}$$