ORF 522
Linear Optimization

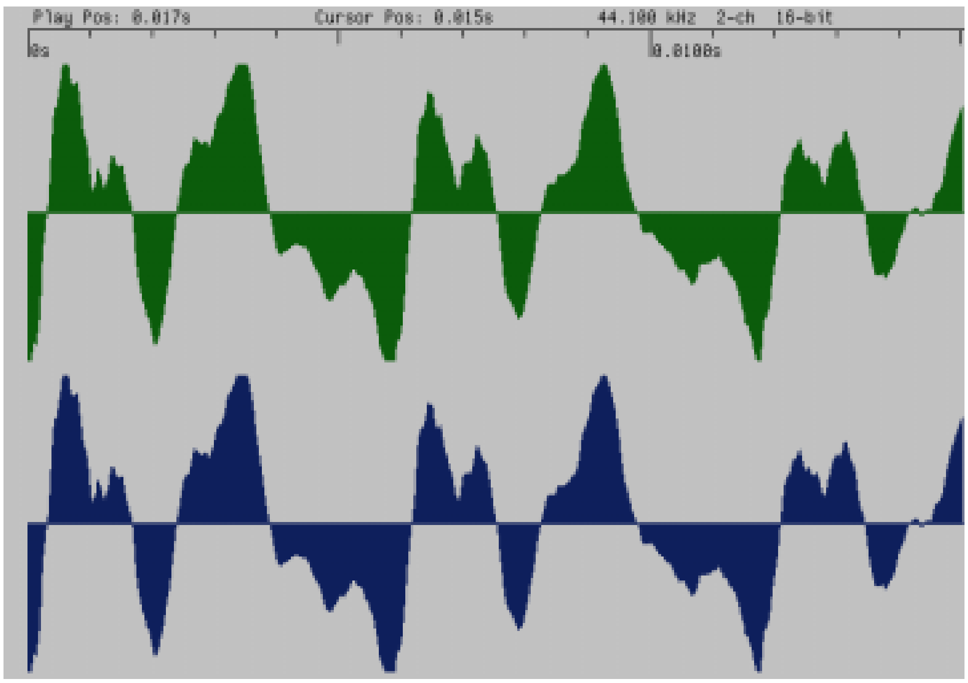
Lecture 19

Applications in Electrical Engineering

Digital Signal Processing[1]

[1]J.O. Coleman, A Systematic Approach to the Constrained Quadratic Optimization of Embedded FIR Filters

An Audio Signal



Digitization

44,100 samples/sec-channel

Each sample is a 2 byte (16 bit) integer
(i.e., between -32768 and 32767).

Stereo needs 176 kBytes/sec = 634 MBytes/hour.

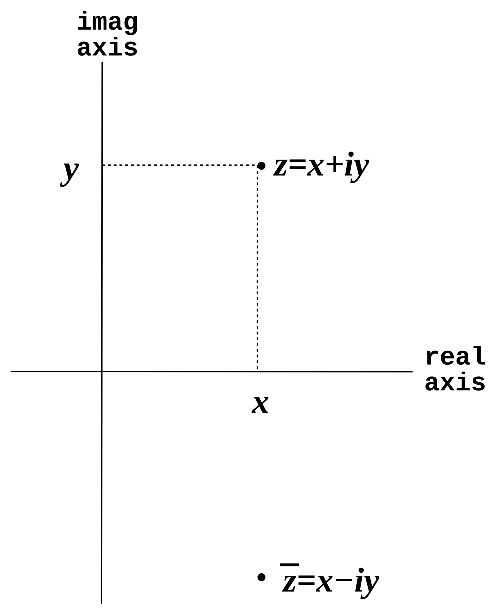
A Glimpse at the Data

0	-32768	20	23519	40	16975	60	12719
1	-32768	21	25247	41	16191	61	12223
2	-32768	22	27535	42	14799	62	11055
3	-30753	23	29471	43	12367	63	10479
4	-28865	24	31919	44	8607	64	10367
5	-29105	25	32767	45	5599	65	9951
6	-29201	26	32767	46	4927	66	10399
7	-26513	27	32767	47	5599	67	10255
8	-23681	28	32767	48	6799	68	8591
9	-18449	29	32031	49	9135	69	6671
10	-11025	30	29759	50	9839	70	5199
11	-6913	31	28399	51	8575	71	3823
12	-4337	32	28095	52	7615	72	2783
13	-1329	33	28399	53	5983	73	1567
14	1743	34	28751	54	5343	74	-785
15	6223	35	28751	55	6287	75	-4433
16	12111	36	26911	56	7183	76	-8737
17	17311	37	24063	57	8687	77	-12001
18	21311	38	21247	58	10927	78	-14369
19	23055	39	18415	59	12479	79	-16305

Complex Variables—A Brief Tutorial

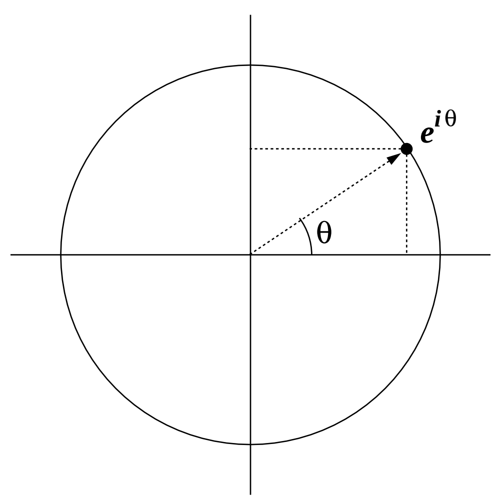
Complex Number: $z = x + iy$, where $i = \sqrt{-1}$

Complex Conjugate: $\bar{z} = x - iy$



Magnitude: $|z|^2 = z\bar{z} = x^2 + y^2$

The Unit Circle: $e^{i\theta} = \cos\theta + i\,\sin\theta$, $|e^{i\theta}|^2 = \cos^2\theta + \sin^2\theta = 1$



Input

Complex random sequence:

$$\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$$

Assume:

$$\mathbf{E}u_k = 0$$

Autocorrelation:

$$s_k = \mathbf{E}u_j \bar{u}_{j+k}$$

Assumption: independent of j (random sequence is called stationarity).

Spectral Density = Fourier Transform:

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} s_t e^{i\omega t} dt \\ &= \sum_k s_k e^{ik\omega} \end{aligned}$$

Power:

$$\mathbf{E}|u_0|^2 = s_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) d\omega$$

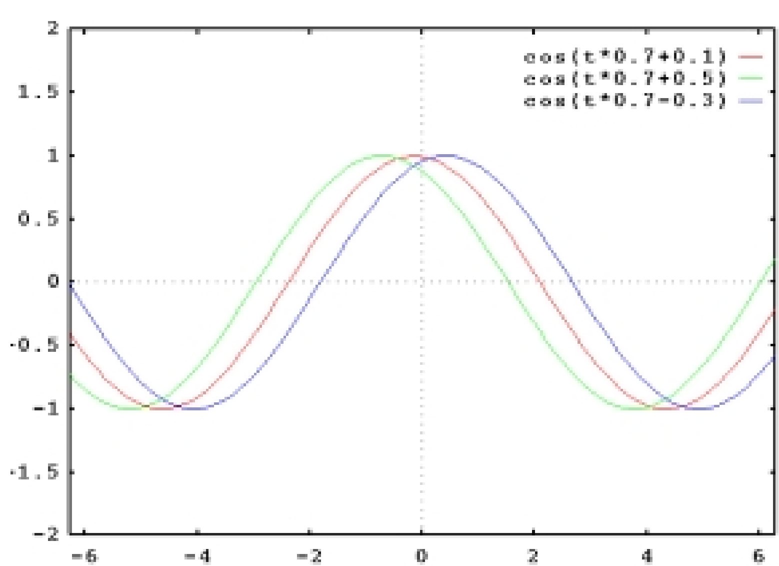
Example

A sine/cosine wave with frequency ω_0 but undetermined phase:

$$\begin{aligned} u_k &= e^{i(k\omega_0+\theta)} \\ &= \cos(k\omega_0+\theta) + i\sin(k\omega_0+\theta) \end{aligned}$$

where

$$\theta = \text{Uniform}(-\pi, \pi)$$



$$\mathbf{E}u_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k\omega_0+\theta)}d\theta = \frac{1}{2\pi}e^{ik\omega_0}\int_{-\pi}^{\pi} e^{i\theta}d\theta = 0$$

$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(j\omega_0+\theta)}e^{-i((j+k)\omega_0+\theta)}d\theta = e^{-ik\omega_0}$$

The spectral density is a delta function at ω_0:

$$S(\omega) = \cdots = 2\pi\delta_{\omega_0}(\omega)$$

Power:

$$P = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)d\omega = 1$$

Finite Impulse Response (FIR) Filter

Finite Impulse Response

$$h_{-n+1}, \ldots, h_{-1}, h_0, h_1, \ldots, h_{n-1}$$

Assumption: $h_{-k} = h_k$ (called linear phase filter)

Autocorrelation

$$g_k = \sum_j h_j h_{j+k}$$

Fourier Transform of $\{h_k\}$

$$\begin{aligned} H(\omega) &= \sum_k h_k e^{ik\omega} \\ &= h_0 + 2\sum_{k=1}^{n-1} h_k \cos(k\omega) \end{aligned}$$

Fourier Transform of $\{g_k\}$

$$\begin{aligned} G(\omega) &= \sum_k g_k e^{ik\omega} \\ &\vdots \\ &= |H(\omega)|^2 \end{aligned}$$

Output

At time k, the output includes the input from time $k - j$ multiplied by the impulse response h_j:

$$y_k = \sum_j h_j u_{k-j}.$$

The output is a convolution of the input and the filter.

Autocorrelation:

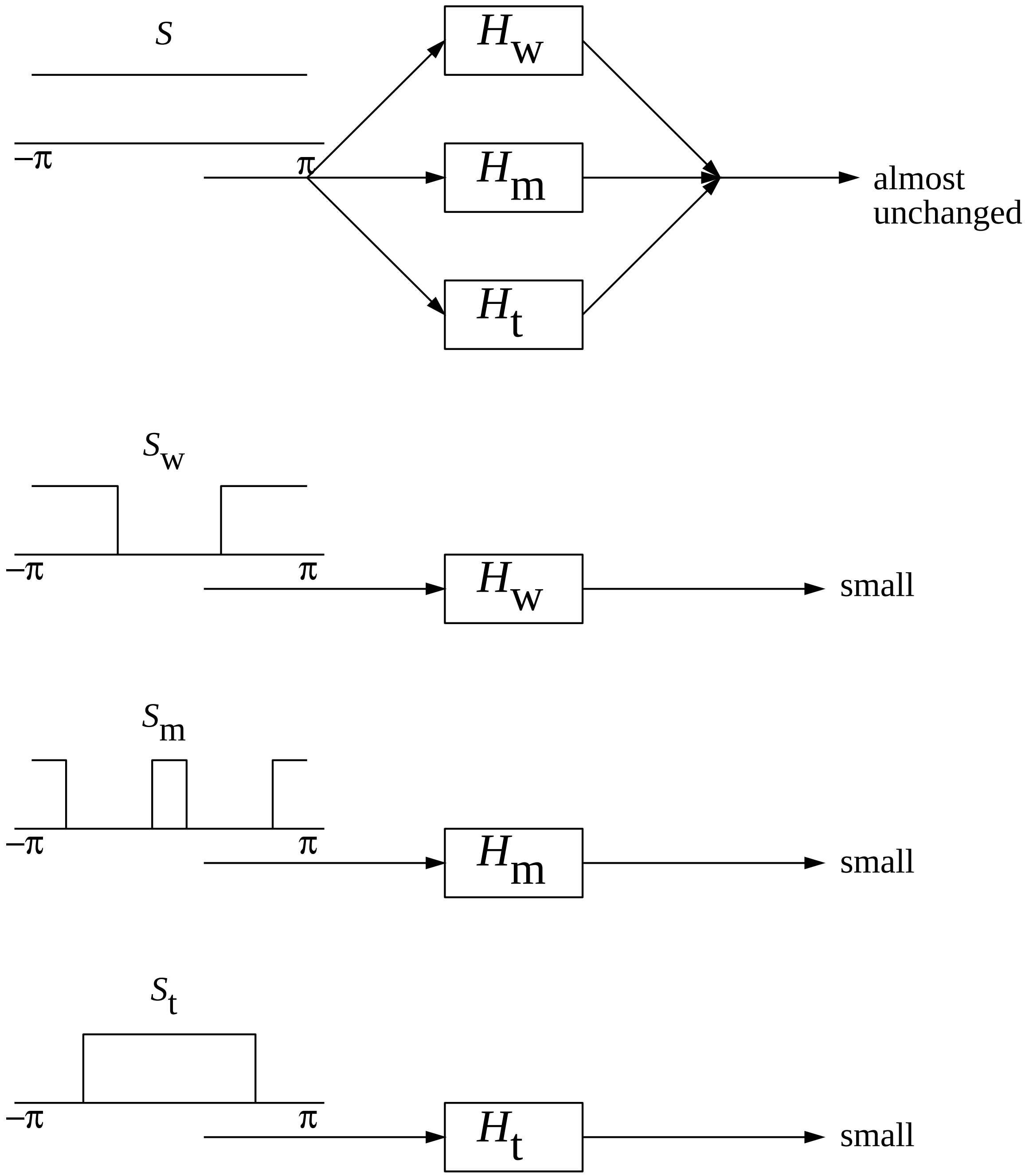
$$\begin{aligned} r_k &= \mathbf{E} y_j \bar{y}_{j+k} \\ &\vdots \\ &= \sum_j g_j s_{k-j}. \end{aligned}$$

Again, just a convolution.

Spectral Density:

$$\begin{aligned} R(\omega) &= \sum_k r_k e^{ik\omega} \\ &\vdots \\ &= G(\omega) S(\omega) \end{aligned}$$

A Design Specification



A Convex Optimization Problem

minimize:

$$\rho$$

subject to:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}(H_{\mathrm{w}}(\omega)+H_{\mathrm{m}}(\omega)+H_{\mathrm{t}}(\omega)-1)^2 d\omega &\le \epsilon \\
\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{w}}^2(\omega)S_{\mathrm{w}}(\omega)d\omega\right)^{1/2} &\le \rho \\
\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{m}}^2(\omega)S_{\mathrm{m}}(\omega)d\omega\right)^{1/2} &\le \rho \\
\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{t}}^2(\omega)S_{\mathrm{t}}(\omega)d\omega\right)^{1/2} &\le \rho
\end{aligned}$$

where

$$H_{\mathrm{s}}(\omega) = h_0^{\mathrm{s}} + 2\sum_{k=1}^{n-1} h_k^{\mathrm{s}} \cos(k\omega), \qquad \mathrm{s}=\mathrm{w},\mathrm{m},\mathrm{t}$$

Note: The integrals can be approximated by sums. They can also be integrated explicitly.

The ampl Model

```
# length 27 (-13,...,13) linear phase FIR filter

param n := 14;
param N := 1000;
param pi := 4*atan(1);
param eps := 1.0e-4;

set OMEGA := setof {j in -N..N} pi*j/N;

param Sw {omega in OMEGA} := (if abs(omega) > 0.4*pi then 1 else 0)
param Sm {omega in OMEGA} := (if abs(omega) < 0.2*pi
      || abs(omega) > 0.8*pi then 1 else 0);
param St {omega in OMEGA} := (if abs(omega) < 0.6*pi then 1 else 0)

var rho >= 0;
var hw {0..n-1};
var hm {0..n-1};
var ht {0..n-1};

var Hw {omega in OMEGA} =
    hw[0] + 2* sum {k in 1..n-1} (hw[k]*cos(k*omega));

var Hm {omega in OMEGA} =
    hm[0] + 2* sum {k in 1..n-1} (hm[k]*cos(k*omega));

var Ht {omega in OMEGA} =
    ht[0] + 2* sum {k in 1..n-1} (ht[k]*cos(k*omega));

minimize power_bnd: rho;

subject to passband:
    sqrt(sum {omega in OMEGA} (Hw[omega]+Hm[omega]+Ht[omega]-1)ˆ2)
    <= sqrt(N)*sqrt(2*pi*eps);

subject to wooferband:
    sqrt(sum {omega in OMEGA} Hw[omega]ˆ2 * Sw[omega])
    <= sqrt(N)*rho;

subject to midrangeband:
    sqrt(sum {omega in OMEGA} Hm[omega]ˆ2 * Sm[omega])
```

```
    <= sqrt(N)*rho;

subject to tweeterband:
    sqrt(sum {omega in OMEGA} Ht[omega]^2 * St[omega])
    <= sqrt(N)*rho;

let hw[0] := 2;
let hm[0] := 2;
let ht[0] := 2;

solve;

printf {omega in OMEGA}: "%7.4f %10.3e \n",
    omega, 20*log10(abs(Hw[omega])) > 3w.out;

printf {omega in OMEGA}: "%7.4f %10.3e \n",
    omega, 20*log10(abs(Hm[omega])) > 3m.out;

printf {omega in OMEGA}: "%7.4f %10.3e \n",
    omega, 20*log10(abs(Ht[omega])) > 3t.out;
```

The Iteration Log

```
LOQO 4.01: verbose=2
timing=1
variables: non-neg      1,  free       42,  bdd          0,  total       43
constraints: eq         0,  ineq        4,  ranged       0,  total        4
nonzeros:    A         87,  Q       1764
------------------------------------------------------------------------------
     |          Primal           |           Dual            | Sig
Iter |  Obj Value     Infeas     |   Obj Value     Infeas    | Fig  Status  P   M
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  1   0.0000000e+00  5.47e+00   -4.5048766e+01  8.20e+01
nonzeros:    L        990,  arith_ops              31041
  2   8.1566802e-02  2.67e+00   -2.2246490e+01  3.11e+01
  3   4.2789358e-02  9.91e-01   -8.2537485e+00  1.14e+01
  4   1.3387496e-01  2.06e-01   -1.6740002e+00  2.32e+00
  5   1.6865099e-02  1.16e-01    1.0434433e-01  1.86e-01    1
  6   1.3181610e-03  1.40e-01   -8.0805850e-03  1.50e-02    2
  7   7.5265116e-05  1.44e-01    4.5425492e-04  9.05e-04    3
  8   5.9279466e-06  1.39e-01   -3.6288050e-05  6.97e-05    4
  9   3.1343416e-06  7.84e-02    1.8886582e-05  3.31e-05    5                1
 10   2.6317200e-07  6.24e-02   -1.6970215e-06  1.62e-05    6
 11   6.9794369e-08  6.42e-02    3.1953216e-07  5.72e-06    7
 12   1.5662519e-08  1.32e+00   -4.2603108e-10  2.60e-06    8
 13   1.0162506e-08  2.59e-01    3.9640273e-08  7.34e-07    8      DF
 14   7.3901217e-09  2.07e-01    4.5719497e-08  4.30e-07    7      DF
 15   6.8298003e-09  1.83e-01    8.6045607e-08  2.07e-07    7      DF
    .
    .
    .
 28   2.3814666e-07  1.78e-01    1.6399637e-03  4.01e-07    3      DF
 29   9.4878775e-04  2.17e-01    3.3674241e-04  9.48e-03    3
 30   1.4551883e-03  1.01e-01    9.6862590e-04  6.88e-03    3
 31   1.7563445e-03  4.66e-02    1.3613840e-03  4.52e-03    3
 32   2.0179127e-03  1.25e-02    1.6853065e-03  2.00e-03    3
 33   2.1373955e-03  1.72e-03    1.9841897e-03  3.99e-04    4
 34   2.1589812e-03  1.17e-04    2.1435636e-03  2.86e-05    5
 35   2.1603608e-03  6.05e-06    2.1595563e-03  1.48e-06    6
 36   2.1604314e-03  3.03e-07    2.1603910e-03  7.43e-08    7   PF DF
 37   2.1604349e-03  1.52e-08    2.1604329e-03  3.72e-09    9   PF DF
----------------------
OPTIMAL SOLUTION FOUND
```

The Output