ORF 522
Linear Optimization

Lecture 22

More Applications

Facility Location

Euclidean Multi-Facility Location Problem

Given m existing facilities with known locations: $a_1, a_2, \ldots, a_m$.

Wish to create n new facilities at locations to be determined: $x_1, x_2, \ldots, x_n$.

Anticipated level of activity between existing location a_i and to-be-created location x_j is w_{ij}.

Anticipated level of activity between a pair of to-be-created locations, say x_{j_1} and x_{j_2} is $v_{j_1 j_2}$.

Assumption is that total cost per time period is proportional to the sum over all pairs of the activity level times the distance separating the facilities:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \|x_j - a_i\| + \sum_{1 \le j_1 < j_2 \le n} v_{j_1 j_2} \|x_{j_1} - x_{j_2}\|$$

Minimize cost.

AMPL Model

```
# Euclidean multiple facility location problem

# Objective Function:  convex
# Constraint Functions: none

param m := 200;         # number of existing facilities
param n1 := 5;
param n2 := 5;
param n := n1*n2;               # number of new facilities

param a {1..m, 1..2};   # coordinates of existing facility
param w {1..m, 1..n};   # weights associated with old-new connections
param v {1..n, 1..n};   # weights associated with new-new connections

var x {1..n, 1..2};

minimize sumEucl:
  sum {i in 1..m, j in 1..n}
      w[i,j]*sqrt( sum {k in 1..2} (x[j,k] - a[i,k])^2 )
  +
  sum {j in 1..n, jj in 1..n: j<jj}
      v[j,jj]*sqrt( sum {k in 1..2} (x[j,k] - x[jj,k])^2 ) ;

let {i in 1..m, k in 1..2}          a[i,k]  := Uniform01();
let {j in 1..n, jj in 1..n: j < jj} v[j,jj] := 0.2;

let {j1 in 1..n1, j2 in 1..n2} x[j1+n1*(j2-1),1] := (j1-0.5)/n1;
let {j1 in 1..n1, j2 in 1..n2} x[j1+n1*(j2-1),2] := (j2-0.5)/n2;

let {i in 1..m, j in 1..n}
    w[i,j] := (if abs(a[i,1]-x[j,1]) <= 1/(2*n1)
               && abs(a[i,2]-x[j,2]) <= 1/(2*n2)
               then 0.95 else
                   (if abs(a[i,1]-x[j,1]) <= 2/(2*n1)
                    && abs(a[i,2]-x[j,2]) <= 2/(2*n2)
                    then 0.05 else 0
                   )
                );

solve;
```

Solver Output

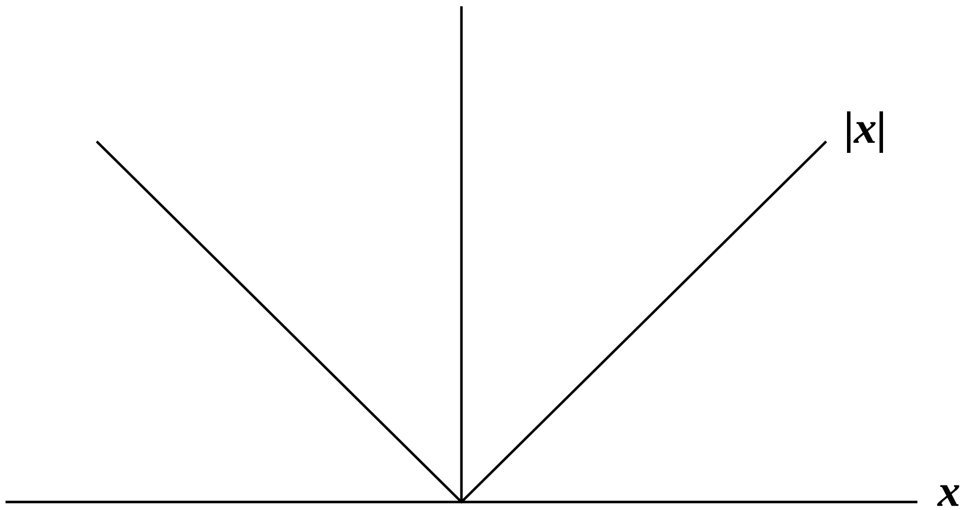
```
LOQO 4.01: verbose=2
timing=1
variables: non-neg      0,  free       50,  bdd          0,  total       50
constraints: eq         0,  ineq        0,  ranged       0,  total        0
nonzeros:    A          0,  Q       2500
-----------------------------------------------------------------------------
     |          Primal          |           Dual           | Sig
Iter |  Obj Value     Infeas    |   Obj Value     Infeas   | Fig  Status  P    M
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  1   5.0683483e+01  4.06e+00   1.5860298e+01  4.10e-02
nonzeros:    L       1225,  arith_ops             44150
  2   4.7652037e+01  1.11e+00   4.6195122e+01  3.20e-02    2
  3   4.7583289e+01  2.57e-01   4.5523472e+01  2.88e-02    1
  4   4.6828161e+01  3.44e-02   4.5203480e+01  2.01e-02    1
  5   4.6551449e+01  2.00e-02   4.5490988e+01  1.14e-02    2                1
  6   4.6491900e+01  1.31e-03   4.7230810e+01  8.28e-03    2
  7   4.6452894e+01  7.03e-04   4.6584414e+01  4.93e-03    3                1
  8   4.6430717e+01  3.77e-04   4.7474154e+01  3.61e-03    2                1
  9   4.6421124e+01  2.89e-04   4.6233922e+01  2.58e-03    2                2
 10   4.6420786e+01  2.55e-04   4.7740428e+01  1.54e-03    2                3
 11   4.6419206e+01  2.40e-04   4.7435290e+01  1.06e-03    2                4
 12   4.6418726e+01  2.26e-04   4.5684701e+01  8.04e-04    2                4
 13   4.6418086e+01  2.23e-04   4.7414950e+01  1.51e-04    2                6
 14   4.6418074e+01  2.21e-04   4.6617446e+01  9.36e-05    2                7
 15   4.6418052e+01  2.20e-04   4.7426454e+01  1.41e-04    2                8
   .
   .
   .
190   4.6417738e+01  4.96e-07   4.6125750e+01  2.10e-05    2   PF          10
191   4.6417720e+01  4.95e-07   4.6227208e+01  2.38e-05    2   PF          10
192   4.6417745e+01  4.95e-07   4.7632147e+01  6.49e-05    2   PF          10
193   4.6417716e+01  4.94e-07   4.6121593e+01  5.27e-06    2   PF          10
194   4.6417732e+01  4.94e-07   4.7737017e+01  5.33e-05    2   PF          10
195   4.6417730e+01  4.94e-07   4.6121402e+01  2.99e-05    2   PF          10
196   4.6417701e+01  4.93e-07   4.7717304e+01  2.02e-06    2   PF          10
197   4.6417759e+01  4.93e-07   4.5565235e+01  4.63e-05    2   PF          10
198   4.6417760e+01  4.92e-07   4.7716529e+01  3.55e-05    2   PF           9
199   4.6417750e+01  4.91e-07   4.6143010e+01  5.68e-05    2   PF          10
200   4.6417742e+01  4.91e-07   4.7611966e+01  1.60e-05    2   PF          10
```

Primal makes slow but steady progress.

Dual thrashes.

What's Wrong?

The Euclidean distance function is not differentiable at zero.



Could "smooth out" the point:

$$\|x\| \approx \sqrt{\epsilon + x_1^2 + x_2^2}$$

AMPL model gets changed in a trivial fashion:

```
param eps := 1.0e-8;

minimize sumEucl:
  sum {i in 1..m, j in 1..n}
      w[i,j]*sqrt( eps + sum {k in 1..2} (x[j,k] - a[i,k])^2 )
  +
  sum {j in 1..n, jj in 1..n: j<jj}
      v[j,jj]*sqrt( eps + sum {k in 1..2} (x[j,k] - x[jj,k])^2 ) ;
```

New Output

```
LOQO 4.01: verbose=2
timing=1
variables: non-neg      0,  free       50,  bdd          0,  total       50
constraints: eq         0,  ineq        0,  ranged       0,  total        0
nonzeros:    A          0,  Q        2500
----------------------------------------------------------------------------
     |         Primal           |           Dual           | Sig
Iter |  Obj Value     Infeas    |   Obj Value     Infeas   | Fig  Status  P   M
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  1   5.0683505e+01  4.06e+00    1.5860622e+01  4.10e-02
nonzeros:    L       1225,  arith_ops             44150
  2   4.7652492e+01  1.11e+00    4.6210165e+01  3.20e-02    2
  3   4.7583663e+01  2.57e-01    4.5491809e+01  2.89e-02    1
  4   4.6826624e+01  3.44e-02    4.5205843e+01  2.01e-02    1
  5   4.6551811e+01  2.00e-02    4.5479338e+01  1.14e-02    2               1
  6   4.6491604e+01  1.32e-03    4.7258596e+01  8.29e-03    2
  7   4.6452796e+01  7.10e-04    4.6562637e+01  5.00e-03    3               1
  8   4.6430035e+01  3.81e-04    4.7471052e+01  3.55e-03    2               1
  9   4.6421129e+01  2.92e-04    4.6278203e+01  2.66e-03    3               2
 10   4.6420602e+01  2.58e-04    4.7663293e+01  1.44e-03    2               3
 11   4.6419423e+01  2.28e-04    4.6181634e+01  5.46e-04    2               3
 12   4.6418436e+01  2.14e-04    4.7554731e+01  4.64e-04    2               4
 13   4.6418043e+01  2.02e-04    4.6418794e+01  2.11e-04    5               4
 14   4.6417990e+01  1.02e-05    4.6618888e+01  1.52e-04    2
 15   4.6417915e+01  5.33e-06    4.6590873e+01  5.49e-05    2               1
 16   4.6417907e+01  2.68e-07    4.6443550e+01  7.18e-06    3   PF
 17   4.6417906e+01  1.35e-08    4.6421050e+01  9.52e-07    4   PF DF
 18   4.6417906e+01  6.76e-10    4.6418113e+01  6.50e-08    5   PF DF
 19   4.6417906e+01  3.39e-11    4.6417917e+01  3.35e-09    7   PF DF
 20   4.6417906e+01  1.70e-12    4.6417907e+01  1.68e-10    8   PF DF
----------------------
OPTIMAL SOLUTION FOUND
```

Oh what a difference an ϵ can make!

The Answer