Linear Programming: Chapter 13
Network Flows: Algorithms

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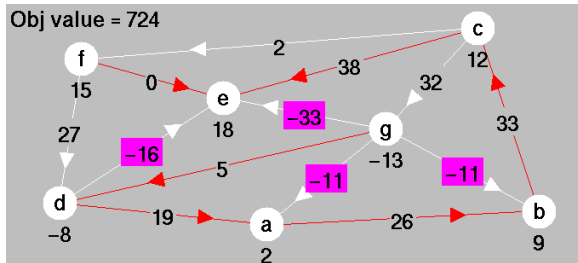http://www.princeton.edu/∼rvdb

Agenda

- Primal Network Simplex Method

- Dual Network Simplex Method

- Two-Phase Network Simplex Method

- One-Phase Primal-Dual Network Simplex Method

- Planar Graphs
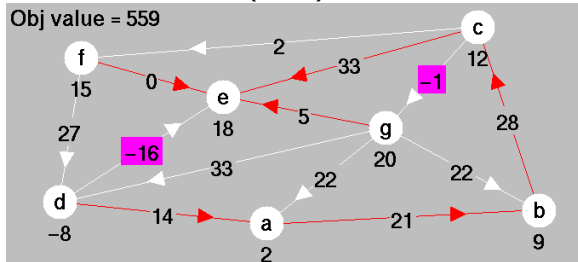
- Integrality Theorem

Primal Network Simplex Method

Used when all primal flows are nonnegative (i.e., primal feasible).

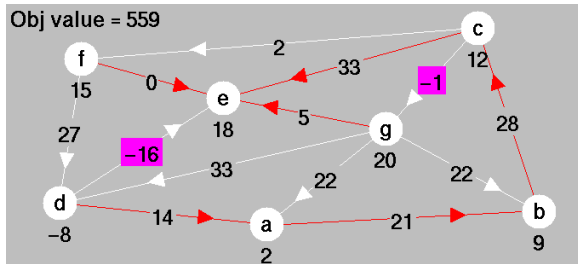Entering arc: (g,e)
Leaving arc: (g,d)

Pivot Rules:

Entering arc: Pick a nontree arc having a negative (i.e. infeasible) dual slack.

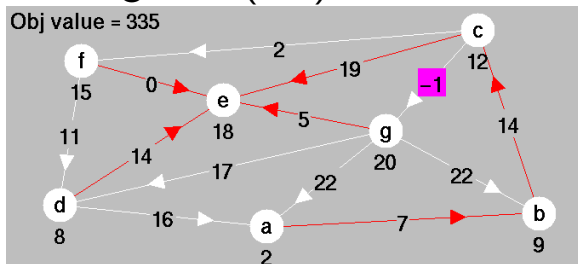*Leaving arc:* Add entering arc to make a *cycle*.
Leaving arc is an arc on the cycle, pointing in the *opposite* direction to the entering arc, and
of all such arcs, it is the one with the *smallest* primal flow.

Primal Method—Second Pivot
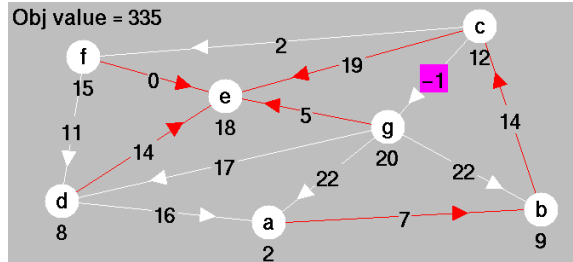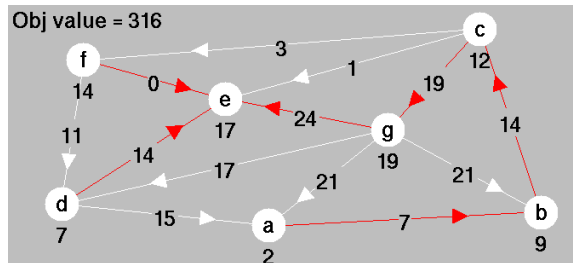

Entering arc: (d,e)

Leaving arc: (d,a)

Explanation of leaving arc rule:

- Increase flow on (d,e).

- Each unit increase produces a unit *increase* on arcs pointing in the *same* direction.

- Each unit increase produces a unit *decrease* on arcs pointing in the *opposite* direction.

- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.
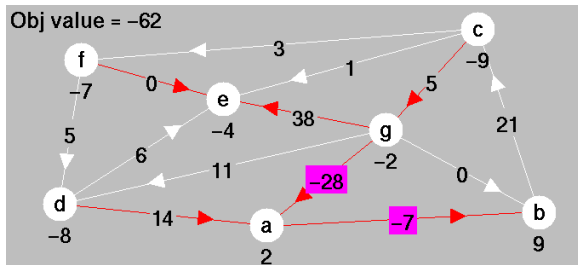
Primal Method—Third Pivot

Entering arc: (c,g)
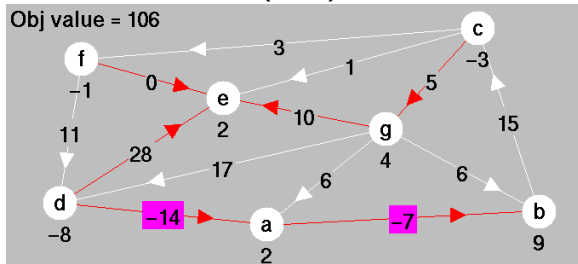Leaving arc: (c,e)

Optimal!

Dual Network Simplex Method

Used when all dual slacks are nonnegative (i.e., dual feasible).

Obj value = −62

Leaving arc: (g,a)
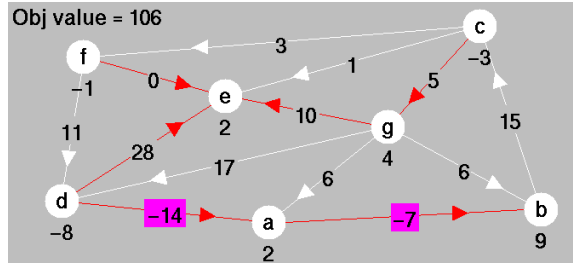Entering arc: (d,e)

Obj value = 106

Pivot Rules:

Leaving arc: Pick a tree arc having a negative (i.e. infeasible) primal flow.
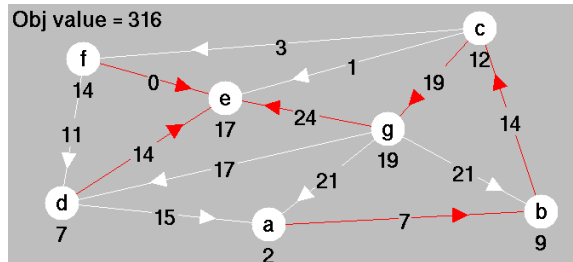
Entering arc: Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc re-connecting the spanning tree with an arc in the *opposite* direction, and, of all such arcs, is the one with the *smallest* dual slack.

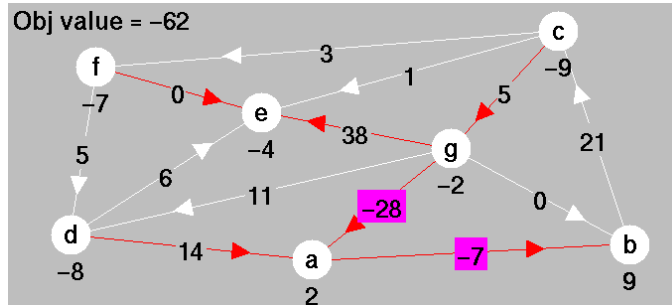# Dual Network Simplex Method

Second Pivot

Leaving arc: (d,a)
Entering arc: (b,c)

Optimal!

Explanation of Entering Arc Rule

Recall initial tree solution:

Leaving arc: (g,a)
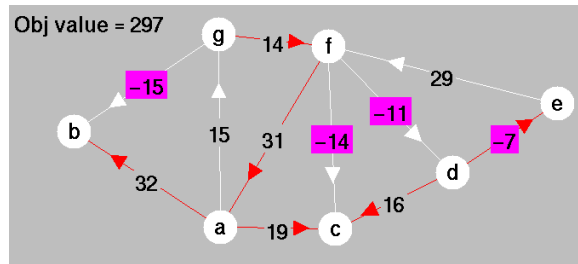Entering arc: (d,e)

- Remove leaving arc. Need to find a reconnecting arc.

- Consider some reconnecting arc. Add flow to it.

  - If it reconnects in the same direction as leaving arc, such as (f,d), then flow on leaving arc decreases.

  - Therefore, leaving arc's flow can't be *raised* to zero.

  - Therefore, leaving arc can't leave. No good.

- Consider a potential arc reconnecting in the opposite direction, say (b,c).

  - Its dual slack will drop to zero.

  - All other reconnecting arcs pointing in the same direction will drop by the same amount.

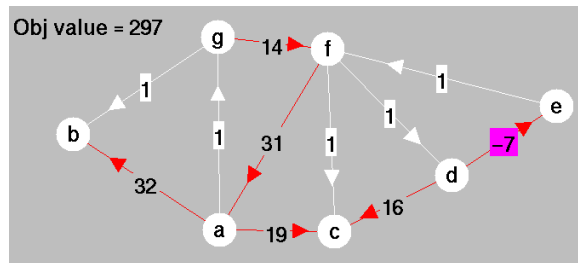 - To maintain nonnegativity of all the others, must pick the one that drops the least.

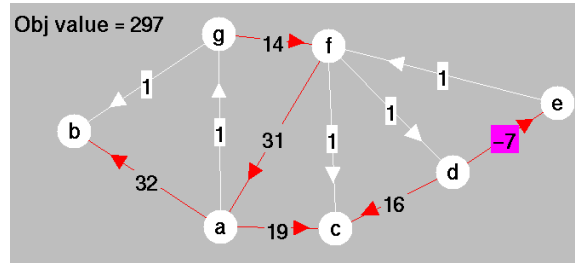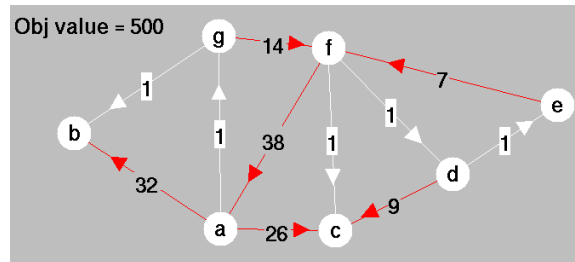# Two-Phase Network Simplex Method

Example.

- Turn off display of dual slacks.

- Turn on display of artificial dual slacks.

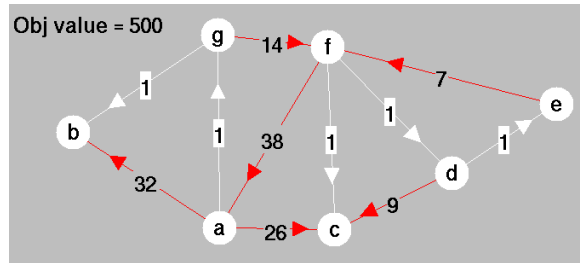# Two-Phase Method–First Pivot

Use dual network simplex method.
Leaving arc: (d,e)   Entering arc: (e,f)

Dual Feasible!

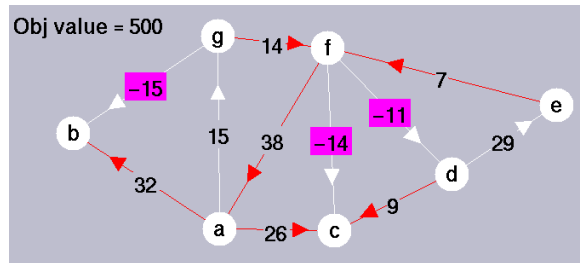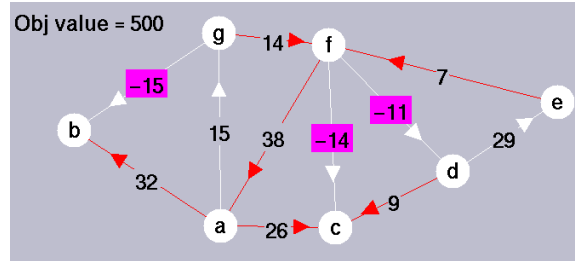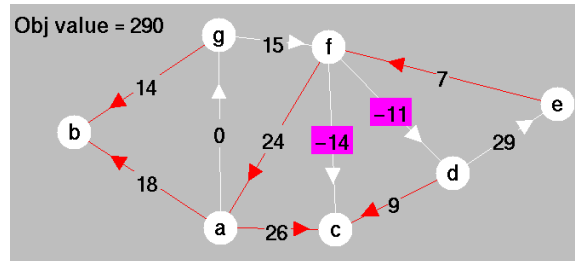# Two-Phase Method–Phase II

- Turn off display of artificial dual slacks.
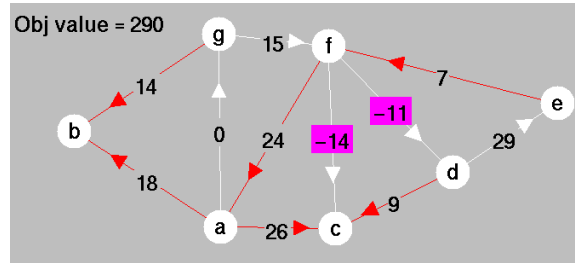
- Turn on display of dual slacks.

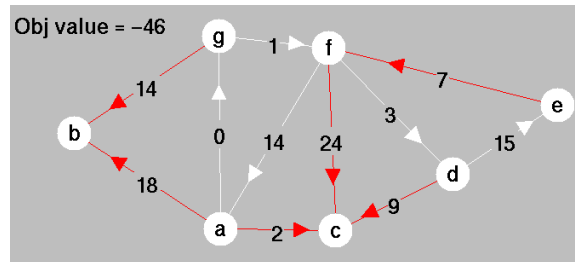# Two-Phase Method–Second Pivot

Entering arc: (g,b)
Leaving arc: (g,f)

Two-Phase Method–Third Pivot
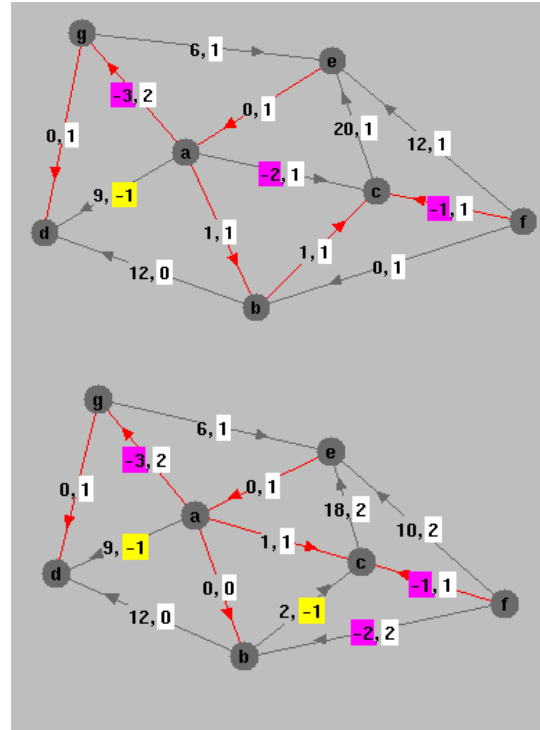

Entering arc: (f,c)
Leaving arc: (a,f)

Optimal!

Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

Parametric Self-Dual Method

- Artificial flows and slacks are multiplied by a parameter μ.

- In the Figure, $6, 1$ represents $6 + 1\mu$.

- *Question:* For which μ values is dictionary optimal?

- *Answer:*

$$
\begin{array}{rcccl}
1 & + & \mu & \geq & 0 \quad (a, b) \\
-2 & + & \mu & \geq & 0 \quad (a, c) \\
& & \mu & \geq & 0 \quad (a, d) \\
& & \mu & \geq & 0 \quad (e, a) \\
-3 & + & \mu & \geq & 0 \quad (a, g) \\
& & \mu & \geq & 0 \quad (b, c) \\
3 & + & \mu & \geq & 0 \quad (b, d)
\end{array}
$$

$$
\begin{array}{rcccl}
& & \mu & \geq & 0 \quad (f, b) \\
20 & + & \mu & \geq & 0 \quad (c, e) \\
-1 & + & \mu & \geq & 0 \quad (f, c) \\
-9 & + & \mu & \geq & 0 \quad (g, d) \\
12 & + & \mu & \geq & 0 \quad (f, e) \\
6 & + & \mu & \geq & 0 \quad (g, e)
\end{array}
$$

- That is, $9 \leq \mu < \infty$.

- Lower bound on μ is generated by arc (g,d).

- Therefore, (g,d) enters.

- Arc (a,d) leaves.

Second Iteration

- Range of μ values: $2 \le \mu \le 9$.
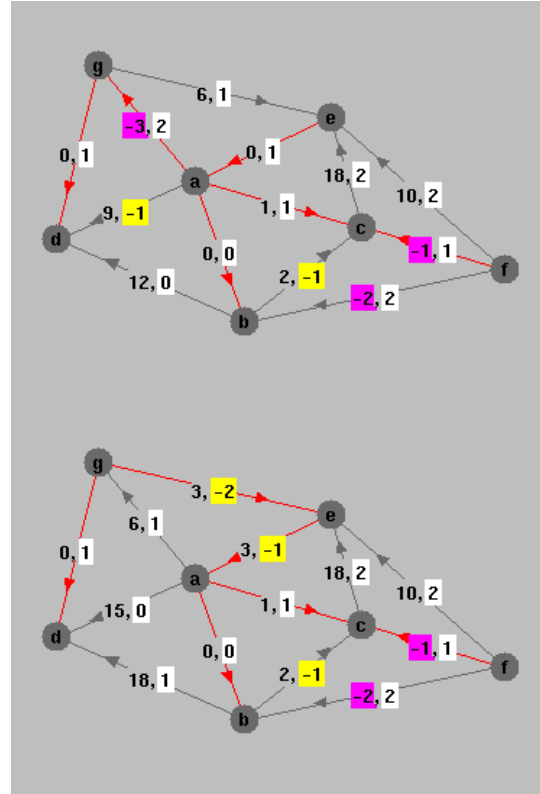
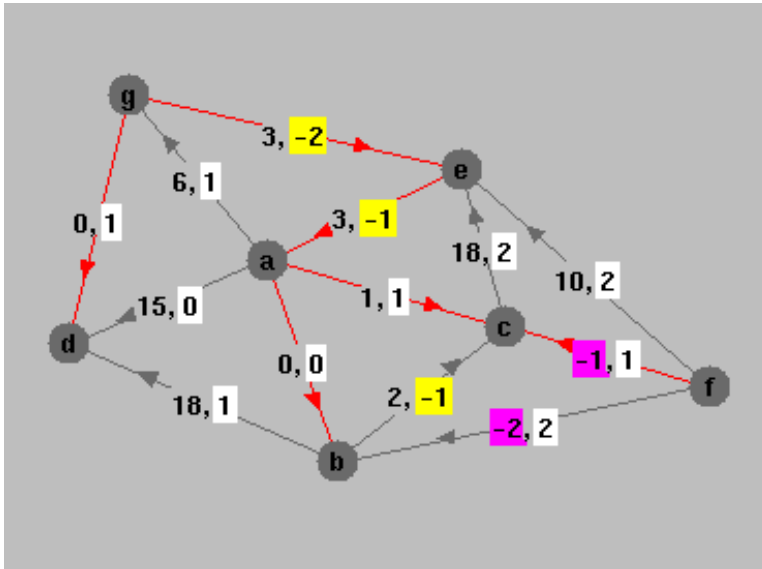- Entering arc: (a,c)

- Leaving arc: (b,c)

New tree:

Third Iteration

- Range of μ values: $1.5 \leq \mu \leq 2$.

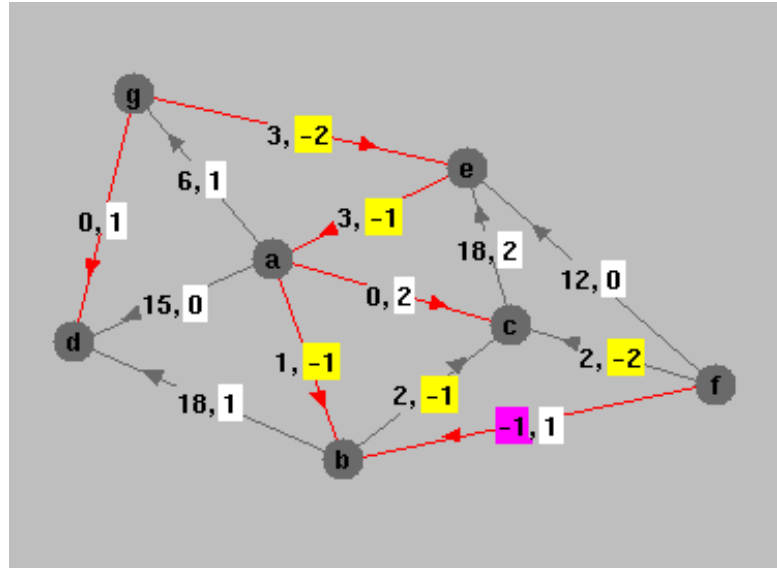- Leaving arc: (a,g)

- Entering arc: (g,e)

New tree:

Fourth Iteration

- Range of μ values: $1 \leq \mu \leq 1.5$.

- A tie:
  - Arc (f,b) enters, or
  - Arc (f,c) leaves.

- Decide arbitrarily:
  - Leaving arc: (f,c)
  - Entering arc: (f,b)

Fifth Iteration

- Range of μ values: $1 \le \mu \le 1$.
- Leaving arc: (f,b)
- Nothing to Enter.

Primal Infeasible!

Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

Planar Networks

Definition. *Network is called* planar *if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a dual—dual nodes are* faces *of primal network.*

Notes:

- Dual node A is "node at infinity".

- Primal spanning tree shown in red.

- Dual spanning tree shown in blue (don't forget node A).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*
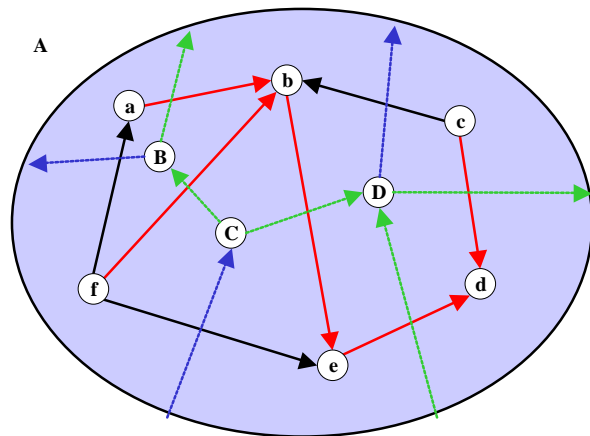
Integrality Theorem

Theorem. *Assuming integer data, every basic feasible solution assigns integer flow to every arc.*

Corollary. *Assuming integer data, every basic optimal solution assigns integer flow to every arc.*