Linear Programming: Chapter 7 Sensitivity and Parametric Analysis

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Restarting

Consider an optimal dictionary:

$$\zeta = \zeta^* - z_{\mathcal{N}}^* T x_{\mathcal{N}}
x_{\mathcal{B}} = x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}}.$$

Recall definitions of $x_{\mathcal{B}}^*$, $z_{\mathcal{N}}^*$, and ζ^* :

$$x_{\mathcal{B}}^* = B^{-1}b$$

$$z_{\mathcal{N}}^* = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$$

$$\zeta^* = c_{\mathcal{B}}^T B^{-1}b.$$

Now, suppose objective coefficients change from c to \tilde{c} .

To adjust current dictionary,

- ullet recompute $z_{\mathcal{N}}^*$, and
- recompute ζ^* .

Note that $x_{\mathcal{B}}^*$ remains unchanged. Therefore,

- Adjusted dictionary is primal feasible.
- Apply primal simplex method.
- Likely to reach optimality quickly.

Had it been the right-hand sides b that changed, then

- Adjusted dictionary would be dual feasible.
- Could apply dual simplex method.

Ranging

Given an optimal dictionary:

$$\zeta = \zeta^* - z_{\mathcal{N}}^* T x_{\mathcal{N}}
x_{\mathcal{B}} = x_{\mathcal{B}}^* - B^{-1} N x_{\mathcal{N}}.$$

Question: If c were to change to

$$\tilde{c} = c + \mu \Delta c$$

for what range of μ 's does the current basis remain optimal?

Recall that:

$$z_{\mathcal{N}}^* = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}$$

Therefore, dual variables change as follows by $\mu \Delta z_N$ where

$$\Delta z_{\mathcal{N}} = (B^{-1}N)^T \Delta c_{\mathcal{B}} - \Delta c_{\mathcal{N}}$$

We want:

$$z_{\mathcal{N}}^* + \mu \Delta z_{\mathcal{N}} \ge 0$$

From familiar ratio tests, we get

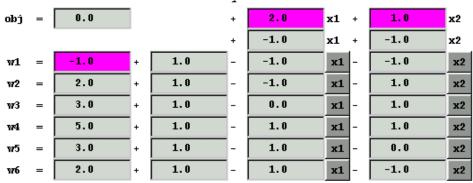
$$\left(\min_{j\in\mathcal{N}} - \frac{\Delta z_j}{z_j^*}\right)^{-1} \le \mu \le \left(\max_{j\in\mathcal{N}} - \frac{\Delta z_j}{z_j^*}\right)^{-1}.$$

Comments:

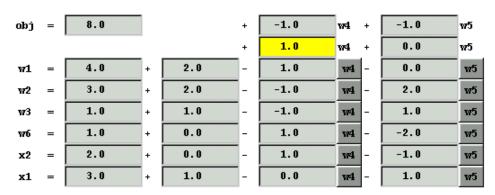
- A similar analysis works for changes to the right-hand side.
- An example is worked out in the text.

Ranging with the Pivot Tool.

An initial dictionary:



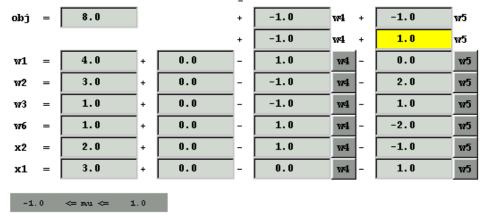
The optimal dictionary:



Question: If the coefficient on x_2 in original problem were changed to $1 + \mu$ (and everything remains unchanged), for what range of μ 's does the current basis remain optimal?

Ranging with the Pivot Tool-Continued.

Set artificial rhs column to zeros. Set artificial objective row to " x_2 ":



The range of μ values is shown at the bottom of the pivot tool.

The Primal-Dual Simplex Method.

An Example

Initial Dictionary:

Note: neither primal nor dual feasible.

Perturb

Introduce a parameter μ and perturb:

For μ large, dictionary is **optimal**.

Question: For which μ values is dictionary optimal? Answer:

Note: only those marked with (*) give inequalities that omit $\mu=0$. Tightest:

$$\mu \geq 11$$

Achieved by: objective row perturbation on x_2 . Let x_2 enter.

Who Leaves?

Do ratio test using current lowest μ value, i.e. $\mu = 11$:

Tightest:

$$4 + 11 - 3x_2 \ge 0.$$

Achieved by: constraint containing basic variable w_2 .

Let w_2 leave.

After the pivot:

Second Pivot

Using the advanced pivot tool, the current dictionary is:

obj	=	14.6667			+	-14.0	x1 +	-3.6667	w2 +	2.0	x 3
					+	0.0	x1 +	0.3333	w2 +	-1.0	x 3
w1	=	1.0	+	0.0]-	-4.0	x1 -	-1.0	w2 -	0.0	х3
x 2	=	1.3333	+	0.3333]-	1.0	x1 -	0.3333	w2 -	0.0	х3
w3	=	2.0	+	0.0]-	-3.0	x1 -	-1.0	w2 -	2.0	х3
w4	=	-4.0	+	1.0	-	-3.0	x1 -	0.0	w2 -	-5.0	x 3

Note: the parameter μ is not shown. But it is there!

Question: For which μ values is dictionary optimal? Answer:

Tightest lower bound:

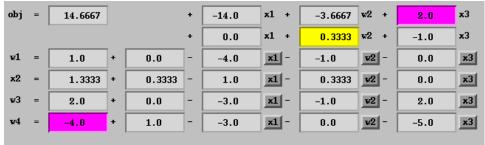
$$\mu > 4$$

Achieved by: constraint containing basic variable w_4 . Let w_4 leave.

Second Pivot-Continued

Who shall enter?

Recall the current dictionary:



Do *dual-type* ratio test using current lowest μ value, i.e. $\mu = 4$:

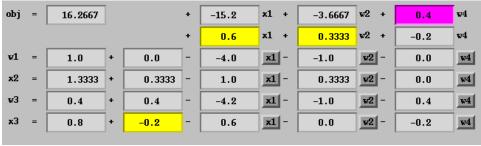
Tightest:

$$-2 + 1 * 4 - 5y_4 \ge 0.$$

Achieved by: objective term containing nonbasic variable x_3 . Let x_3 enter.

Third Pivot

The current dictionary is:



Question: For which μ values is dictionary optimal? Answer:

Tightest lower bound:

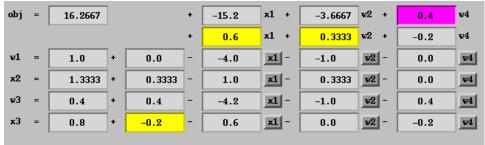
$$\mu \geq 2$$

Achieved by: objective term containing nonbasic variable w_4 . Let w_4 enter.

Third Pivot-Continued

Who shall leave?

Recall the current dictionary:



Do primal-type ratio test using current lowest μ value, i.e. $\mu=2$:

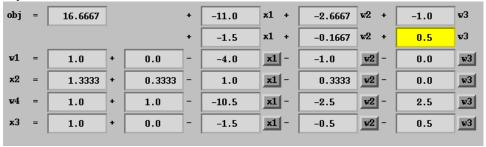
Tightest:

$$0.4 + 0.4 * 2 - 0.4w_4 > 0$$

Achieved by: constraint containing basic variable w_3 . Let w_3 leave.

Fourth Pivot

The current dictionary is:



It's optimal!

Also, the range of μ values includes $\mu=0$:

That is,

$$-1 \le \mu \le 2$$

Range of μ values is shown at bottom of pivot tool. Invalid ranges are highlighted in yellow.