Introduction to NP-Completeness

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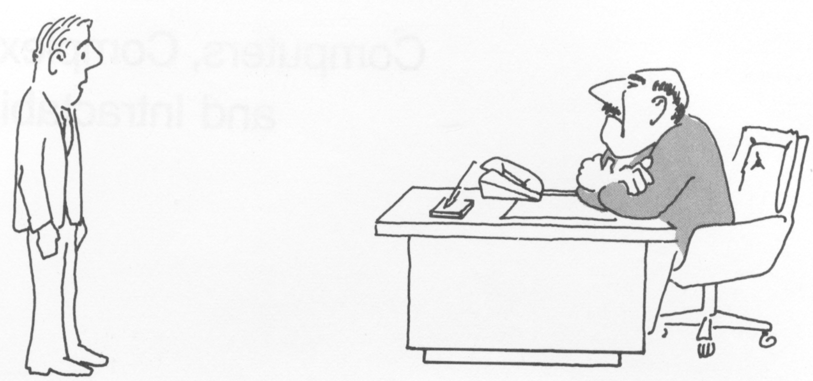
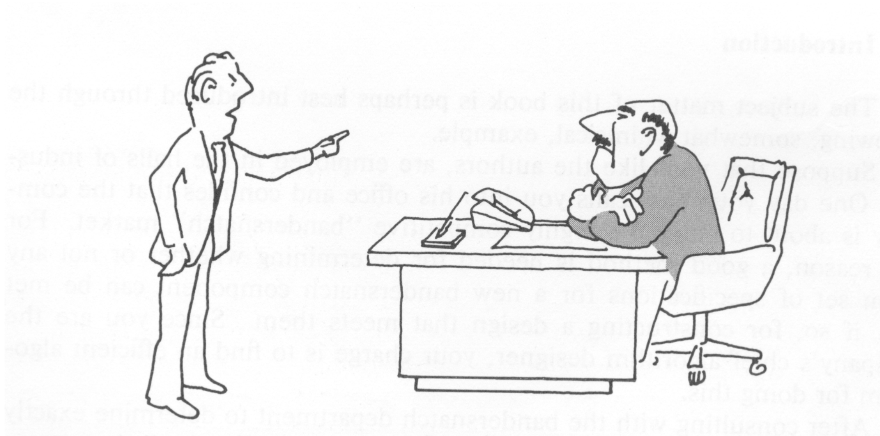
1 Introduction

The ‘NP’ stands for ‘nondeterministic polynomial time’, which stands for the fact that a solution can be checked (not: found) in polynomial time. This class of algorithms is informally characterized by the fact there is polymomial time for checking their solution. However, it is also true that there is no polynomial algorithm known for solving them.



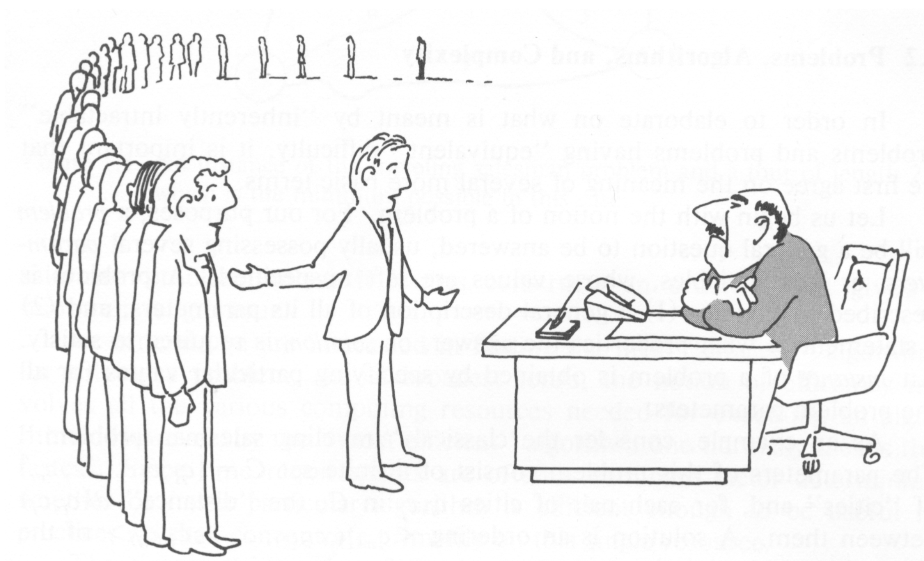
“I can’t find an efficient algorithm, I guess I’m just too dumb.”

The fact that there is no efficient algorithms *known* would not be bad if it could be proved that no efficient algorithm *exists.*



"I can't find an efficient algorithm, because no such algorithm is possible!"

However, also there exists no non-polynomial lower bound on the solution time. Thus, the question whether they can be solved in polynomial time is still open. Since methods in this class can all be translated into each other, having a solution method for one implies that methods exist for all of them. This also means that none of the problems in this class have been solved so far.



"I can't find an efficient algorithm, but neither can all these famous people."

2 Basics

2.1 Optimization versus decision problems

Many problems that are said to be NP-complete are optimization problems. For instance, in the traveling salesman problem the shortest route through a set of cities is asked. However, it is more convenient to look at decision problems, that is, problems that have a yes or no answer for each input.

It is not hard to transform an optimization problem into a decision problem. Rather than asking for an optimal solution, we determine a bound B, and ask whether there is a solution that is within that bound.

> **Exercise 1.** Finish this argument. Show that, if we can solve the optimization problem, we can solve the decision problem. Now, supposing we can solve the decision problem, how does that give a solution of the optimization problem? Assume that the outcome of the optimization problem is an integer quantity. Do you have to make other assumptions; discuss? What is the complexity of the one solution method given a certain complexity for the other?

2.2 Language of a problem

For each optimization or decision problem we can defined 'instances', which are ways of setting all the free variables of the problem. Since these variables are in sets of types that depend on the problem, we can not be more precise here. A problem can then be phrased as a question over this set of instances: which instances optimize the cost function, or which give a yes answer. That last set we will denote Y_Π.

Again depending on the problem, we can encode instances of a problem. For instance, in the traveling salesman problem, an instance would be encoded as the ordered list of cities to pass through.

With this, we can define the language of a problem:

$$L[\Pi, e] = \{\text{the instances in } Y_\Pi \text{ encoded under } e\}$$

2.3 Turing machines

A Turing machine, given some input, can halt with the yes state q_Y, the no state q_N, or can never halt. We say that a string is accepted if it leads the Turing machine to halt with q_Y. The language L_M of a Turing machine M is the set of strings that are accepted.

A deterministic Turing machine (DTM) M is said to solve a problem Π (under some encoding e), or equivalently to recognize $L[\Pi, e]$, if

- it halts for all strings over its input alphabet, and
- its language L_M is $L[\Pi, e]$.

Note that 'solving a problem' here actually means 'recognizing a solution of the problem'. This DTM is a solution checker, not a solution generator.

As an example, consider the recast the traveling salesman problem 'does a route, shorter than B, exist?'. The set of purported solutions are then lists of cities, and the DTM gives for each list a verdict ‘yes, this route is shorter than B’ or ‘no, this route is not shorter than B’.

3 Complexity classes

3.1 Class P

This allows us to define class P:

$$P = \{L : \text{there is DTM that recognizes } L \text{ in polynomial time}\}$$

and with this

$$\begin{aligned} \Pi \in P \quad &\equiv \quad L[\Pi, e] \in P \quad \text{for some encoding } e \\ &\equiv \quad \text{there is a polynomial time DTM that recognizes } L[\Pi, e] \end{aligned}$$

What this means is that for problems in P there is a polynomial time Turing machine that recognizes strings in Y_Π as valid, and that on strings not in Y_Π it both halts, and gives a negative verdict.

3.2 Class NP

Solving the traveling salesman problem may be hard, but if we have a network and some bound, and someone gives us an itinerary with the claim that it satisfies that bound, we can check this with a DTM in polynomial time. We can now build a non-deterministic Turing machine (NDTM) which ‘solves’ such problems: given a decision problem it ‘guesses’ some purported solution, and then checks (with a DTM) whether that solves the decision problem. The guessed solution is called a ‘certificate’.

Clearly, only if the decision problem has an answer of ‘true’ can the NDTM guess a solution, so the Y_Π of this problem is precisely the set of decision problem instances with a yes answer.

For instance, for the traveling salesman problem, the instances in Y_Π are a combination of a cities network plus a feasible bound on the travel time. The non-deterministic Turing machine would then guess an itinerary, and it can be checked in polynomial time that that is indeed a valid route, and that is satisfies the bound.

We can write this whole story compactly: a problem Π is in NP if there is a polynomial time function $A(\cdot, \cdot)$ such that

$$w \in Y_\Pi \Leftrightarrow \exists_C : A(w, C) = \mathrm{true}$$

and C itself can be polynomially generated.

The final condition on the generation of the certificate is needed for a total polynomial runtime, but in practice it is not much of a limitation. For instance, in the traveling salesman problem, a list of cities can be guessed in linear time.

> **Exercise 2.** Prove that NP is closed under union and intersection. What difficulty is there in showing that it is closed under complement taking?

3.3 Examples

As was indicated above, while finding a solution to a problem is often hard, checking that something is a solution is often fairly simply and doable in polynomial time. A nontrivial example of a polynomial time problem is checking whether a number is prime. This question was only settled in 2002. Before that there were polynomial time probabilistic testers, which would test a number and return a verdict with high reliability, or with a high probability of polynomial running time.

> **Exercise 3.** Why is the following algorithm not a linear time solution to the PRIME problem?
> for $i = 0 \ldots \sqrt{n}$:
>   if $\mathrm{mod}(n, i) \equiv 0$
>   return true

Other algorithms have provably an exponential running time. Examples here are finding the best move in chess, or checking statements in Pressburger arithmetic.

It is possible to find levels in between polynomial and exponential. The problem of factoring an integer (note that this is more powerful than primality testing) has a runtime of $O(\exp((n \cdot 64/9)^{1/3})(\log n)^{2/3})$. Interestingly, on a quantum computer, a polymial algorithm is known; see `http://en.wikipedia.org/wiki/Shors_algorithm`.

In the next section we will go further into the middle ground, of algorithms for which no polymomial time algorithm is known, but for which no exponential bound is known either.

4 NP-completeness

4.1 Transformations

Often it is possible to transform one problem into another. This has the advantage that, if you prove that one problem is in a certain class, the other one is also in that class. Of course this depends on the nature of the transformation.

We will here consider 'polynomial transformations'. Let L_1 and L_2 be the languages of two problems over alphabets $\sum_1^*$ and $\sum_2^*$ respectively, then f is a polymomial transformation of problem 1 into problem 2 if

- There is a DTM that computes $f(x)$ in time $T_f(x) \leq p_f(|x|)$ for some polynomial p_f, and
- For all $x \in \sum_1^*$, $x \in L_1$ iff $f(x_1) \in L_2$.

The transformation does not have to be a one-to-one mapping, so it is sometimes explicitly terms a 'many-to-one polynomial transformation'.

Lemma 1 *Suppose f is a polynomial transformation from L_1 to L_2, then*

$$L_2 \in P \Rightarrow L_1 \in P$$

Proof: assume that $M_2 : L_2 \to \{0, 1\}$ is a DTM that recognizes L_2, then $M_2 \circ f$ is a DTM that recognizes L_1, and this composition still runs in polynomial time $T_{M_2 \circ f}(x) \leq p_{T_2}(|p_f(|x|)|)$.

If L_1 transforms to L_2 (in polynomial time), we notate that as $L_1 \leq L_2$. This notation also suggests the idea that L_1 is easier than L_2.

It is easy to see that

$$L_1 \leq L_2 \wedge L_2 \leq L_3 \Rightarrow L_1 \leq L_3,$$

that is, the ‘transforms into’ relation is transitive.

4.2 NP-complete

A language L is said to be NP-complete if

- $L \in NP$, and
- for all $L' \in NP$: Ł$' \leq L$

(Languages that satisfy the second clause but not the first are called ‘NP-hard’. One example is the halting problem, which is known not to be decidable. In other words, the DTM that should recogize the language does not always halt with yes or no.)

Informally, the class NP-complete is that of the problems where a solution can be verified quickly (meaning, in polynomial time). On the other hand, P is the class of problems where the solution can be *computed* quickly. The question whether these classes are disjoint is open. In fact, you can win a million dollars by settling it one way or another.

Lemma 2 *If $L_1, L_2 \in NP$, L_1 is NP-complete, and $L_1 \leq L_2$, then L_2 is NP-complete.*

Proof: the only thing we need to check is that every $L' \leq L_2$ for all $L_2 \in NP$. Since L_1 is NP-complete, $L' \leq L_1$. Now use the transitivity of the transform relation.

4.3 Proving NP-completeness

Above we saw that, given one NP-complete problem, others can easily be proved NP-complete by constructing a mapping between the one problem and the other. This raises a bootstrapping question.

Stephen Cook was the first to prove the NP-completeness of any problem (1971), in his case the satisfiability problem. This is the problem of, given boolean variables $x_1 \ldots x_n$ and a logical formula $F(x_1, \ldots, x_n)$, deciding whether there is a way of specifying the variables such that the result is true.

Examples: the formula $x_1 \vee \neq x_1$ is always true; $x_1 \wedge \neq x_1$ is always false, and $x_1 \wedge \neq x_2$ is only true for the pair $(x_1 = T, x_2 = F)$. For the first and third examples, there are values of x_1, x_2 for which the formula is true, so these are satisfiable. The second example is not satisfiable.

The Boolean satisfiability problem is in NP because a non-deterministic Turing machine can guess an assignment of truth values to the variables, determine the value of the expression under that assignment, and accept if the assignment makes the entire expression true.

Now, to prove that the satisfiability problem is NP-complete, it remains to show that any language $L \in NP$ can polynomially be transformed into it. This is done by assuming a NDPT Turing machine for L, and transforming it into a logical formula, so that there is a correspondence between successful computation in the NDTM, and a satisfied logical formula.

Let the Turing machine be

$$M = \langle Q, s, \Sigma, F, \delta \rangle$$

where

Q is the set of states, and $s \in Q$ the initial state,
Σ the alphabet of tape symbols,
$F \subset Q$ the set of accepting states, and
$\delta \subset Q \times \Sigma \times Q \times \Sigma \times \{-1, +1\}$ the set of transitions,

and that M accepts or rejects an instance of the problem in time $p(n)$ where n is the size of the instance and $p(\cdot)$ is a polynomial function.

We describe for each instance I a Boolean expression which is satisfiable if and only if the machine M accepts I.

The Boolean expression uses the variables set out in the following table, where $q \in Q$, $-p(n) \leq i \leq p(n)$, $j \in \Sigma$, and $0 \leq k \leq p(n)$:

Variables	Intended interpretation	How many
T_{ijk}	True iff tape cell i contains symbol j at step k of the computation	$O(p(n)^2)$
H_{ik}	True iff the M's read/write head is at tape cell i at step k of the computation.	$O(p(n)^2)$
Q_{qk}	True iff M is in state q at step k of the computation.	$O(p(n))$

Define the Boolean expression B to be the conjunction of the clauses in table **??**, for all $-p(n) \leq i \leq p(n)$, $j \in \Sigma$, and $0 \leq k \leq p(n)$.

This table describes how to construct a logical formula in the variables T_{ijk}, H_{ik}, Q_{qk} (describing tape contents, head positions, and states, respectively) that corresponds to the Turing machine. If there is an accepting computation for M on input I, then B is satisfiable, by assigning T_{ijk}, H_{ik} and Q_{ik} their intended interpretations. On the other hand, if B is satisfiable, then there is an accepting computation for M on input I that follows the steps indicated by the assignments to the variables.

How large is B? There are $O(p(n)^2)$ Boolean variables, each of which may be encoded in space $O(\log p(n))$. The number of clauses is $O(p(n)^2)$. So the size of B is $O((\log p(n))p(n)^2)$. This is polynomial in n, the size of the input, so the transformation is certainly a polynomial-time reduction, as required.

For all:	Add the clauses	Interpretation	How many clauses?
initial conditions			
Tape cell i of the input I contains symbol j.	T_{ij0}	Initial contents of the tape.	$O(p(n))$
	Q_{s0}	Initial state of M	$O(1)$
	H_{00}	Initial position of read/write head.	$O(1)$
physical constraints			
symbols $j \neq j'$	$T_{ijk} \rightarrow \neg T_{ij'k}$	One symbol per tape cell.	$O(p(n)^2)$
states $q \neq q'$	$Q_{qk} \rightarrow \neg Q_{q'k}$	Only one state at a time.	$O(p(n))$
cells $i \neq i'$	$H_{ik} \rightarrow \neg H_{i'k}$	Only one head position at a time.	$O(p(n))$
Turing machine basics			
i, j, k	$T_{ijk} = T_{ij(k+1)} \vee H_{ik}$	Tape remains unchanged unless written.	$O(p(n)^2)$
$f \in F$	The disjunction of the clauses $Q_{f,p(n)}$	Must finish in an accepting state.	$O(1)$
transition table			
$(q, \sigma, q', \sigma', d) \in \delta$	The disjunction of the clauses $(H_{ik} \wedge Q_{qk} \wedge T_{i\sigma k}) \rightarrow (H_{(i+d)(k+1)} \wedge Q_{q'(k+1)} \wedge T_{i\sigma'(k+1)})$	Possible transitions at computation step k when head is at position i.	$O(p(n)^2)$

Table 1: Translation table from a NDPT Turing machine to a logic formula

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