Transformation Methods
Penalty and Barrier methods

Study of Engineering Optimization

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Example6_1

$$\text{Minimize} \quad f(x) = (x_1 - 4)^2 + (x_2 - 4)^2$$

$$\text{Subject to} \quad h(x) = x_1 + x_2 - 5 = 0$$

Penalty Parameter

$$P(x, R) = (x_1 - 4)^2 + (x_2 - 4)^2 + \underbrace{R(x_1 + x_2 - 5)^2}_{\text{Penalty Term}}$$

Stationary Values

Table 6.1 Stationary Values of Parabolic Penalty

R	$x_1^{(t)} = x_2^{(t)}$	$f(x^{(t)})$	$h(x^{(t)})$	$P(x^{(t)}, R)$
0	4.0000	0.0000	3.0000	0.0000
0.1	3.7500	0.1250	2.5000	0.7500
1	3.0000	2.0000	1.0000	3.0000
10	2.5714	4.0818	0.1428	4.2857
100	2.5075	4.4551	0.0150	4.4776
∞	2.5000	4.5000	0.0000	4.5000

Example6_2 Bracket operator

$$\text{Minimize} \quad f(x) = (x_1 - 4)^2 + (x_2 - 4)^2$$

$$\text{Subject to} \quad g(x) = 5 - x_1 - x_2 \geqslant 0$$

$$P(x, R) = (x_1 - 4)^2 + (x_2 - 4)^2 + R\langle 5 - x_1 - x_2 \rangle^2$$

Log penalty(Barrier)

$$P(x, R) = f(x) - R\ \ln(g(x))$$

Table 6.2 Stationary Values of Log Penalty

R	$x_1 = x_2$	$f(x)$	$g(x)$	$-R\ \ln(g(x))$	$P(x, R)$
100	−1.8059	67.4170	8.6118	−215.3133	−147.8963
10	1.5000	12.5000	2.0000	−6.9315	5.5685
1	2.3486	5.4542	0.3028	+1.1947	6.6489
0.1	2.4835	4.5995	0.0034	+0.3411	4.9406
0.01	2.4983	4.5100	0.0034	+0.0568	4.5668
0	2.5000	4.5000	0.0000	0.0000	4.5000

Penalty and Barrier Methods

General classical constrained minimization problem

minimize $f(\mathbf{x})$

subject to

$\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$

$\mathbf{h}(\mathbf{x}) = \mathbf{0}$

Penalty methods are motivated by the desire to use unconstrained optimization techniques to solve constrained problems.

This is achieved by either

- adding a **penalty** for infeasibility and forcing the solution to feasibility and subsequent optimum, or
- adding a **barrier** to ensure that a feasible solution never becomes infeasible.

Penalty Methods

Penalty methods use a mathematical function that will increase the objective for any given constrained violation.

General transformation of constrained problem into an unconstrained problem:

$$\min P(\mathbf{x}, \mathbf{R}) = f(\mathbf{x}) + Q(R, h(\mathbf{x}), g(x))$$

where

- $f(\mathbf{x})$ is the objective function of the constrained problem
- **R** is a scalar denoted as the penalty or controlling parameter
- $Q(\mathbf{x},\mathbf{R})$ is a function which imposes penalities for infeasibility
- $P(\mathbf{x},\mathbf{R})$ is the (pseudo) transformed objective

Sequential Penalty Transformations

Sequential Penalty Transformations are the oldest penalty methods.

Also known as Sequential Unconstrained Minimization Techniques (SUMT) based upon the work of Fiacco and McCormick, 1968.

Two major classes exist in sequential methods:
1) First class uses a sequence of infeasible points and feasibility is obtained only at the optimum. These are referred to as **penalty function** or **exterior-point penalty function** methods.
2) Second class is characterized by the property of preserving feasibility at all times. These are referred to as **barrier function** methods or **interior-point penalty function** methods.

What to Choose?

- Some prefer barrier methods because even if they do not converge, you will still have a feasible solution.

- Others prefer penalty function methods because
  - You are less likely to be stuck in a feasible pocket with a local minimum.
  - Penalty methods are more robust because in practice you may often have an infeasible starting point.

- However, penalty functions typically require more function evaluations.

- Choice becomes simple if you have equality constraints.

Matlab Example

- To Matlab

Closing Remarks

Typically, you will encounter sequential approaches.

Various penalty functions Q(x) exist in the literature.

Various approaches to selecting the penalty parameter sequence exist. Simplest is to keep it constant during all iterations.

Thanks!