The Theory of NP-Completeness

Tractable and intractable problems
NP-complete problems

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- NP-complete problems

Classifying problems

- Classify problems as tractable or intractable.
- Problem is *tractable* if there **exists at least one** polynomial bound algorithm that solves it.
- An algorithm is *polynomial bound* if its worst case growth rate can be bound by a polynomial $p(n)$ in the size n of the problem

$$p(n) = a_n n^k + \ldots + a_1 n + a_0 \text{ where } k \text{ is a constant}$$

Intractable problems

- Problem is *intractable* if it is not tractable.
- **All** algorithms that solve the problem are not polynomial bound.
- It has a worst case growth rate *f*(*n*) which cannot be bound by a polynomial p(n) in the size n of the problem.
- For intractable problems the bounds are:

$$f(n) = c^n \text{ , or } n^{\log n}\text{, etc.}$$

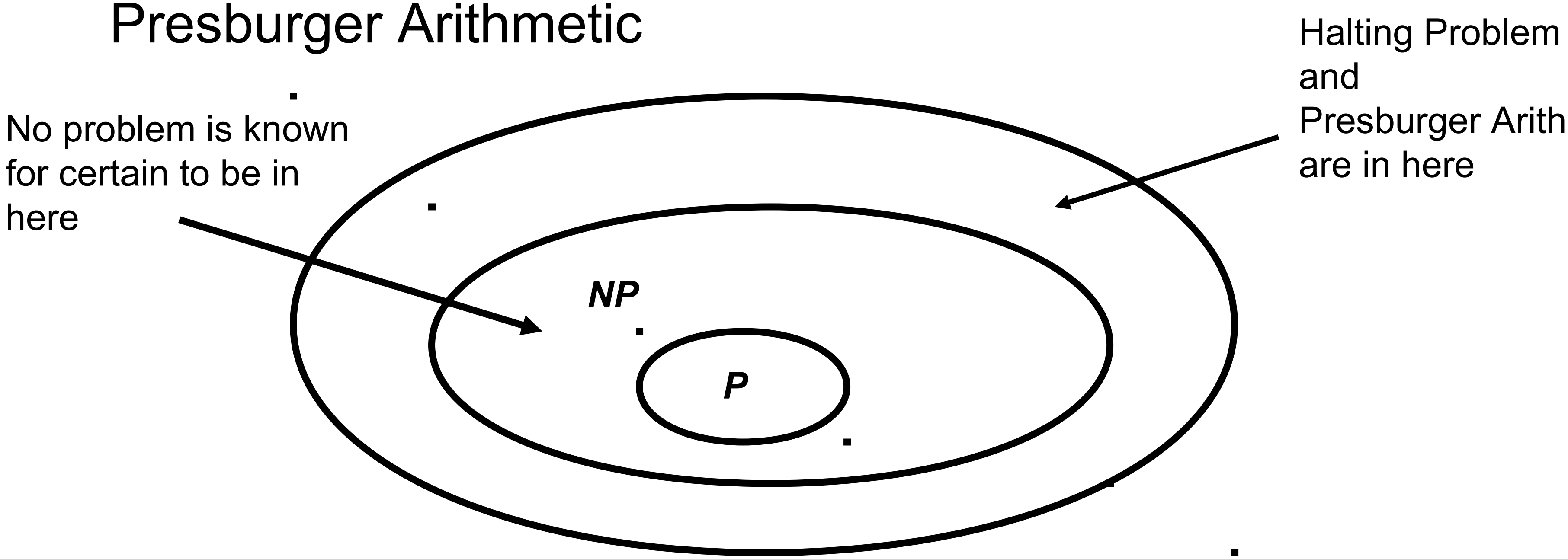
Why is this classification useful?

- If problem is intractable, no point in trying to find an efficient algorithm
- All algorithms will be too slow for **large inputs**.

Intractable problems

- Turing showed some problems are so hard that no algorithm can solve them (undecidable)

- Other researchers showed some decidable problems from automata, mathematical logic, etc. are intractable: Presburger Arithmetic



Hard practical problems

- There are many practical problems for which no one has yet found a polynomial bound algorithm.

- Examples: traveling salesperson, 0/1 knapsack, graph coloring, bin packing etc.

- Most design automation problems such as testing and routing.

- Many networks, database and graph problems.

How are they solved?

- A variety of algorithms based on backtracking, branch and bound, dynamic programming, etc.

- None can be shown to be polynomial bound

The theory of NP completeness

- The theory of NP-completeness enables showing that these problems are at least as hard as *NP-complete* problems

- Practical implication of knowing problem is NP-complete is that it is **probably** intractable (whether it is or not has not been proved yet)

- So any algorithm that solves it will probably be very slow for large inputs

We will need to discuss

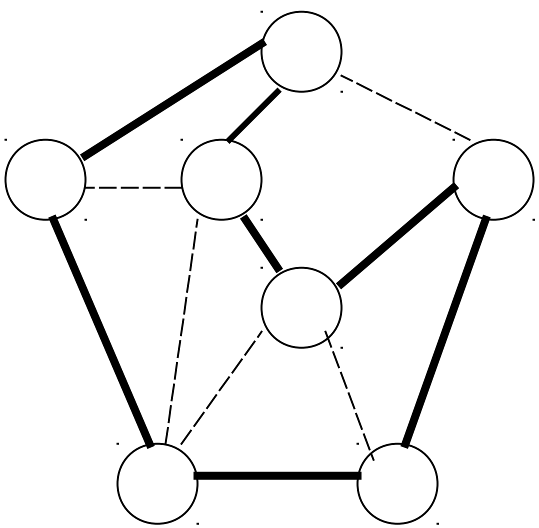
- Decision problems
- Converting optimization problems into decision problems
- The relationship between an optimization problem and its decision version
- The class P
- Verification algorithms
- The class NP
- The concept of polynomial transformations
- The class of NP-complete problems

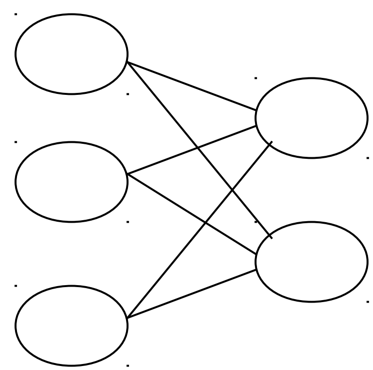
Decision Problems

- A *decision* problem answers *yes or no* for a given input
- Examples:
  - Given a graph *G* Is there a path from *s* to *t* of length at most *k*?
  - Does graph *G* contain a Hamiltonian cycle?
  - Given a graph *G* is it bipartite?

A decision problem: HAMILTONIAN-CYCLE

- A *Hamiltonian cycle* of a graph *G* is a cycle that includes each vertex of the graph exactly once.
- Problem: Given a graph G, does G have a Hamiltonian cycle?





Converting to decision problems

- Optimization problems can be converted to decision problems (typically) by adding a bound B on the value to optimize, and asking the question:
  - Is there a solution whose value is at most B? (for a minimization problem)
  - Is there a solution whose value is at least B? (for a maximization problem)

An optimization problem: traveling salesman

- Given:
  - A finite set $C=\{c_1,\ldots,c_m\}$ of cities,
  - A distance function $d(c_i, c_j)$ of nonnegative numbers.
- Find the length of the **minimum** distance tour which includes every city exactly once

A decision problem for traveling salesman (TS)

- Given a finite set $C=\{c_1,...,c_m\}$ of cities, a distance function $d(c_i, c_j)$ of nonnegative numbers and a bound B
- Is there a tour of all the cities (in which each city is visited exactly once) with total length **at most B**?
- There is no known polynomial bound algorithm for TS.

The relation between

- If we have a solution to the optimization problem we can compare the solution to the bound and answer “yes” or “no”.

- Therefore if the optimization problem is tractable so is the decision problem

- If the decision problem is “hard” the optimization problems are also “hard”
  - If the optimization was easy then the decision problem is easy.

The class P

- P is the class of decision problems that are polynomial bounded

- Is the following problem in P?
  - Given a weighted graph G, is there a spanning tree of weight at most B?

- The decision versions of problems such as shortest distance, and minimum spanning tree belong to P

The goal of verification algorithms

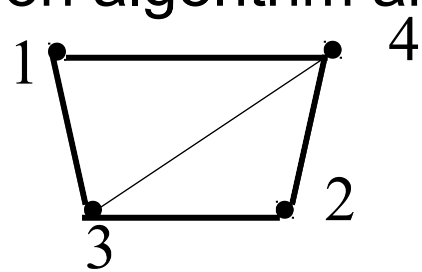
- The goal of a verification algorithm is to verify a “yes” answer to a decision problem’s input (i.e., if the answer is “yes” the verification algorithm verify this answer)

- The inputs to the verification algorithm are:
  - the original input (problem instance) and
  - a *certificate* (possible solution)

Verification Algorithms

- A *verification algorithm*, takes a problem instance x and *answers "yes"*, if there **exists** a certificate y such that the answer for x with certificate y is "yes"

- Consider HAMILTONIAN-CYCLE
- A problem *instance* x lists the vertices and edges of G: ({1,2,3,4}, {(3,2), (2,4), (3,4), (4,1), (1, 3)})
- There **exists** a certificate y = (3, 2, 4, 1, 3) for which the verification algorithm answers "yes"



Polynomial bound verification algorithms

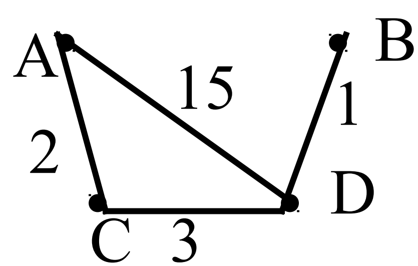
- Given a decision problem d.
- A verification algorithm for d is *polynomial bound* if given an input x to d, there exists a certificate y, such that $|y|=O(|x|^c)$ where c is a constant, and a polynomial bound algorithm A(x, y) that verifies an answer "yes" for d with input x

Note: |y| is the size of the certificate, |x| is the size of the input

The problem PATH

- PATH denotes the decision problem version of shortest path.
- PATH: Given a graph *G*, a start vertex *u*, and an end vertex *v*. Does there exist a path in *G*, from *u* to *v* of length at most *k?*

- The instance is: G=({A, B, C, D}, {(A, C,2), (A, D, 15), (C,D, 3), (D, B, 1)} k=6
- A certificate y=(A, C, D, B)



A verification algorithm for PATH

- Verification algorithm:
  - Given the problem instance x and a certificate *y*
    - Check that *y* is indeed a path from *u* to *v*.
    - Verify that the length of *y* is at most *k*

- Is the verification algorithm for PATH polynomial bound?
- Is the size of y polynomial in the size of x?
- Is the verification algorithm polynomial bound?

Example: A verification algorithm for TS

- Given a problem instance x for TS and a certificate *y*
  - Check that *y* is indeed a cycle that includes every vertex exactly once
  - Verify that the length of the cycle is at most B

- Is the size of y polynomial in the size of x?
- Is the verification algorithm polynomial?

The class NP

- NP is the class of decision problems for which there is a polynomial bounded verification algorithm

- It can be shown that:
  - all decision problems in P, and
  - decision problems such as traveling salesman, knapsack, bin pack, are also in NP

The relation between P and NP

- P $\subseteq$ NP

- If a problem is solvable in polynomial time, a polynomial time verification algorithm can easily be designed that *ignores the certificate* and answers "yes" for all inputs with the answer "yes".

The relation between P and NP

- It is not known whether P = NP.
- Problems in P can be *solved* “quickly”
- Problems in NP can be *verified* “quickly”.
- It is easier to verify a solution than to solve a problem.
- Some researchers believe that P and NP are not the same class.

Polynomial reductions

- **Motivation**: The definition of NP-completeness uses the notion of *polynomial reductions* of one problem A to another problem B, written as

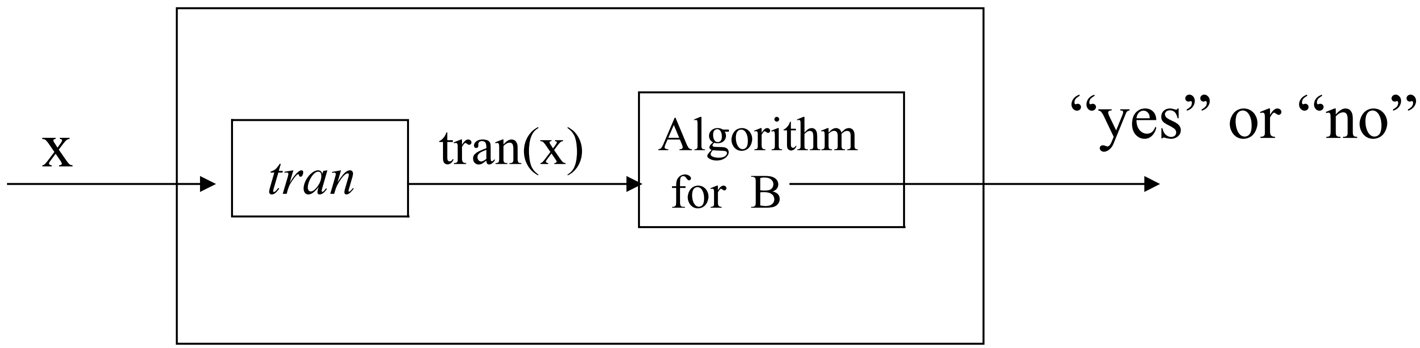
$$A \propto B$$

- Let *tran* be a function that converts any input x for decision problem A into input tran(x) for decision problem B

Polynomial reductions

tran is a *polynomial reduction* from A to B if:

1. *tran* can be computed in polynomial bounded time
2. The answer to A for input x is *yes* if and only if the answer to B for input *tran*(x) is *yes*.



Algorithm for A

Two simple problems

- A: Given n Boolean variables with values $x_1,\ldots,x_n$, does at least one variable have the value True?
- B: Given n integers $i_1,\ldots,i_n$ is $max\{i_1,\ldots,i_n\}>0$?

Algorithm for B :
Check the integers one after the other.
If one is positive stop and answer "*yes*",
otherwise (if none is positive) stop and answer "*no*".

Example:
n=4.
Given integers: -1, 0, 3, and 20.
Algorithm for B answers "*yes*".
Given integers: -1, 0, 0, and 0.
Algorithm for B answers "*no*".

Is there a transformation?

- Can we transform an instance of A into an instance of B?
- Yes.

```
            tran(x)
for ( j =1; j =< n; j ++)
        if (x_j == true)
                i_j=1
        else   // x_j=false
   i_j = 0
```

T(false, false, true, false)= 0,0,1,0

- Is this transformation polynomial bounded? yes

Does it satisfy all the requirements?

- Can we show that when the answer for an instance x_1, ...,x_n of A is "*yes*" the answer for the transformed instance *tran*($x_1,\ldots,x_n$)= $i_1,\ldots,i_n$ of B is also "*yes*"?

- If the answer for the given instance $x_1,\ldots,x_n$ of A is "*yes*", there is some x_j=true.

- The transformation assigns i_j=1.

- Therefore the answer for problem B is also "*yes*" (the maximum is positive)

The other direction

- Can we also show that when the answer for problem B with input *tran*($x_1,\ldots,x_n$)= $i_1,\ldots,i_n$ is "*yes*", the answer for the instance $x_1,\ldots,x_n$ of A is also "*yes*"?
- If the answer for problem B is "*yes*", it means that there is an $i_j>0$ in the transformed instance.
- i_j is either 0 or 1 in the transformed instance. So $i_j=1$, and therefore x_j=true.
- So the answer for A is also "*yes*"

Polynomial reductions

Theorem:

If $A \propto B$ and B is in P, then A is in P

If A is not in P then B is also not in P

NP-complete problems

- A problem A is *NP-complete* if
  1. It is in NP and
  2. For every other problem A' in NP, $A' \propto A$

- A problem A is *NP-hard* if
  For every other problem A' in NP, $A' \propto A$

Examples of NP-Complete problems

- Cook's theorem
  – Satisfiability is NP-complete
- This was the first problem shown to be NP-complete

- Other problems
  – the decision version of knapsack,
  – the decision version of traveling salesman

Coping with NP- Complete Problems

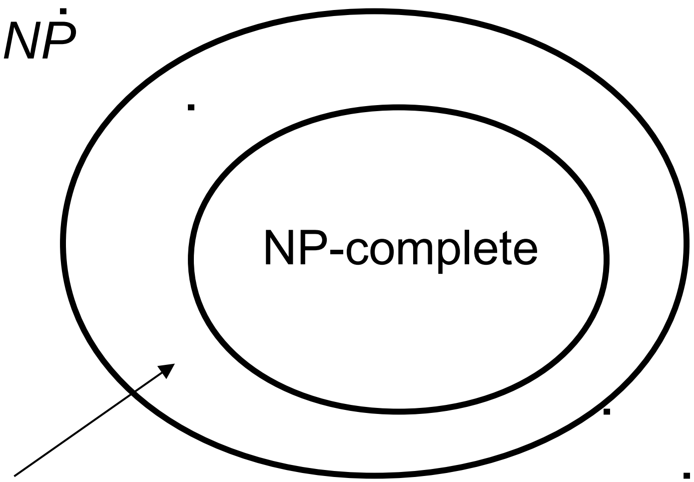
- To solve use approximations, heuristics, etc.
- Sometimes we need to solve only a restricted version of the problem.
- If restricted problem tractable design an algorithm for restricted problem

NP-complete problems: Theorem

If any NP-complete problem is in P, then P = NP.

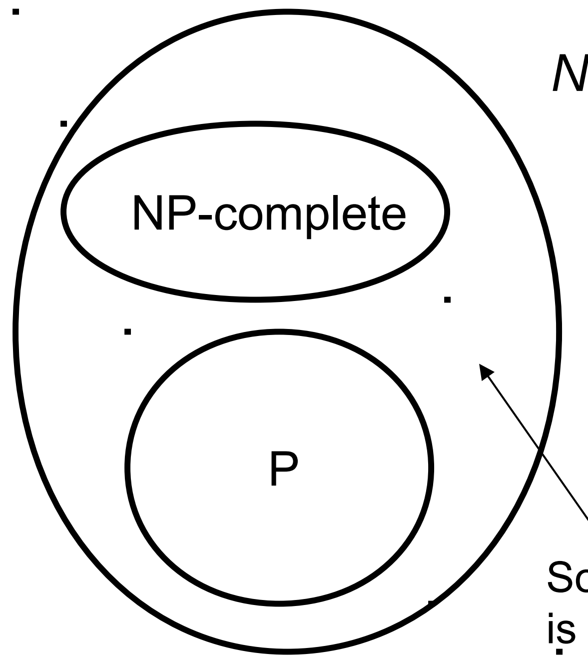
If any NP-complete problem is not polynomial bound, then all NP-Complete problems are not polynomial bound.

$P = NP$



The trival decision problem that always answers “yes” in here

$NP \neq P$



Some Problem is definitely in here

NP-completeness and Reducibility

- The existence of NP-complete problems is leads us suspect that $P \neq NP$.
- If HAMILTONIAN CYCLE could be solved in polynomial time, every problem in NP can be solved in polynomial time.
- If HAMILTONIAN CYCLE could not be solved in polynomial time, every NP-complete problem can not be solved in polynomial time.

The Satisfiability problem

- First, Conjunctive Normal Form (CNF) will be defined
- Then, the Satisfiability problem will be defined
- Finally, we will show a polynomial bounded verification algorithm for the problem

Conjunctive Normal Form (CNF)

- A *logical (Boolean) variable* is a variable that may be assigned the value *true* or *false* (p, q, r and s are Boolean variables)
- A *literal* is a logical variable or the negation of a logical variable (p and $\neg$q are literals)
- A *clause* is a disjunction of literals
  (($p \vee q \vee s$) and ($\neg q \vee r$) are clauses)

Conjunctive Normal Form (CNF)

- A logical (Boolean) expression is in Conjunctive Normal Form if it is a conjunction of *clauses.*
- The following expression is in conjunctive normal form:

$(p \vee q \vee s) \wedge (\neg q \vee r) \wedge (\neg p \vee r) \wedge (\neg r \vee s) \wedge (\neg p \vee \neg s \vee \neg q)$

The Satisfiability problem

- Is there a truth assignment to the n variables of a logical expression in Conjunctive Normal Form which makes the value of the expression true?
- For the answer to be “yes”, all clauses must evaluate to true
- Otherwise the answer is “no”

The Satisfiability problem

- $p=T$, $q=F$, $r=T$ and $s=T$ is a truth assignment for:

$(p \vee q \vee s) \wedge (\neg q \vee r) \wedge (\neg p \vee r) \wedge (\neg r \vee s) \wedge (\neg p \vee \neg s \vee \neg q)$

- Note that if $q=F$ then $\neg q=T$
- Each clause evaluates to true

A verification algorithm for Satisfiability

1. Check that the certificate s is a string of exactly n characters which are T or F.
2. **while** (there are unchecked clauses) {
   select next clause
   if (clause evaluates to false) **return**(“no”) }
3. **return** (“yes”)

- Is verification algorithm polynomial bound?
- Satisfiability is in NP since there exists a polynomial bound verification algorithm for it