CHAPTER TWO

Mathematical Description of Oscillations

In this chapter, which is a brief introduction to the theory of oscillations, a simple mechanical oscillation system is used to introduce concepts such as free and forced oscillations, state-space analysis and representation of sinusoidally varying physical quantities by their complex amplitudes. In order to be somewhat more general, causal and noncausal linear systems are also looked at and Fourier transform is used to relate the system's transfer function to its impulse response function. With an assumption of sinusoidal (or 'harmonic') oscillations, some important relations are derived which involve power and stored energy on one hand and the parameters of the oscillating system on the other hand. The concepts of resonance and bandwidth are also introduced.

2.1 Free and Forced Oscillations of a Simple Oscillator

Let us consider a simple mechanical oscillator in the form of a mass–spring–damper system. A mass m is suspended through a spring and a mechanical damper, as indicated in Figure 2.1. Because of the application of an external force F, the mass has a position displacement x from its equilibrium position.

Newton's law gives

$$m\ddot{x} = F + F_R + F_S, \tag{2.1}$$

where F_S is the spring force and F_R is the damper force.

If we assume that the spring and the damper have linear characteristics, then we can write $F_S = -Sx$ and $F_R = -R\dot{x}$, where the 'stiffness' S and the 'mechanical resistance' R are coefficients of proportionality, independent of the displacement x and the velocity $u = \dot{x}$. Thus, we have the following linear differential equation with constant coefficients:

$$m\ddot{x} + R\dot{x} + Sx = F, \tag{2.2}$$

where an overdot is used to denote differentiation with respect to time t.

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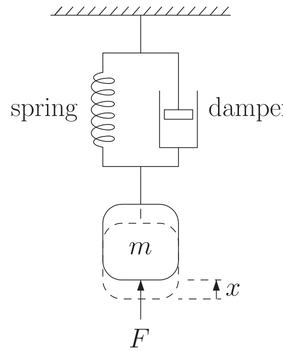

Figure 2.1: Mechanical oscillator in the form of a mass–spring–damper system.

2.1.1 Free Oscillation

If the external force is absent—that is, $F = 0$—we may have the so-called free oscillation if the system is released at a certain instant $t = 0$, with some initial energy

$$W_0 = W_{p0} + W_{k0} = Sx_0^2/2 + mu_0^2/2, \tag{2.3}$$

written here as a sum of potential and kinetic energy, where x_0 is the initial displacement and u_0 the initial velocity. It is easy to show (see Problem 2.1) that the general solution to Eq. (2.2), when $F = 0$, is

$$x = (C_1 \cos \omega_d t + C_2 \sin \omega_d t)\, e^{-\delta t}, \tag{2.4}$$

where

$$\delta = R/2m, \quad \omega_0 = \sqrt{S/m}, \quad \omega_d = \sqrt{\omega_0^2 - \delta^2} \tag{2.5}$$

are the damping coefficient, the undamped natural angular frequency and the damped angular frequency, respectively. The integration constants C_1 and C_2 may be determined from the initial conditions as (see Problem 2.1)

$$C_1 = x_0, \quad C_2 = (u_0 + x_0\delta)/\omega_d. \tag{2.6}$$

For the particular case of zero damping force, the oscillation is purely sinusoidal with a period $T_0 = 2\pi/\omega_0$, which is the so-called natural period of the oscillator. The free oscillation as given by Eq. (2.4) is an exponentially damped sinusoidal oscillation with 'period' $T_d = 2\pi/\omega_d$, during which a fraction $1 - \exp(-4\pi\delta/\omega_d)$ of the energy in the system is lost, as a result of power consumption in the damping resistance R.

The time histories of the normalised displacement x/x_0 and energy $W/W_0 = (Sx^2 + mu^2)/2W_0$ of an oscillator with an initial displacement x_0 when $F = 0$ and $u_0 = 0$ are shown in Figure 2.2. The amplitude of the oscillation decays

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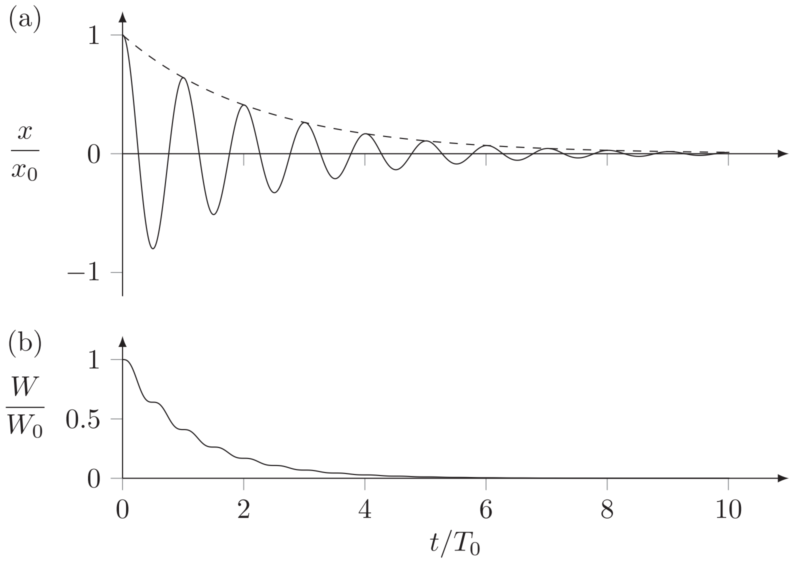

Figure 2.2: Time histories of normalised (a) displacement and (b) energy of a mechanical oscillator for a given initial displacement $x = x_0$, in the absence of external force, $F = 0$, and initial velocity, $u_0 = 0$.

exponentially. If the ratio between two successive peaks is known—for example, from physical measurements—the damping coefficient can be obtained as

$$\delta = \log\left(x(t)/x(t + T_d)\right)/T_d \tag{2.7}$$

according to Eq. (2.4). As expected, the rate of loss of energy in the system is largest when the instantaneous velocity is largest. The energy curve has a horizontal tangent and an inflection point at instants when the velocity is zero and changes sign.

We define the oscillator's *quality factor* Q as the ratio between the stored energy and the average energy loss during a time interval of length $1/\omega_d$:

$$Q = \left(1 - e^{-2\delta/\omega_d}\right)^{-1}. \tag{2.8}$$

If the damping coefficient δ is small, then Q is large. When $\delta/\omega_0 \ll 1$, the following expansions (see Problem 2.2) may be useful:

$$\begin{aligned} Q &= \frac{\omega_0}{2\delta}\left(1 + \frac{\delta}{\omega_0} - \frac{1}{6}\frac{\delta^2}{\omega_0^2} + \mathcal{O}\left\{\frac{\delta^3}{\omega_0^3}\right\}\right) \\ &\approx \frac{\omega_0}{2\delta} = \frac{\omega_0 m}{R} = \frac{S}{\omega_0 R} = \frac{(Sm)^{1/2}}{R}, \end{aligned} \tag{2.9}$$

$$\frac{\delta}{\omega_0} = \frac{1}{2Q}\left(1 + \frac{1}{2Q} + \frac{5}{24Q^2} + \mathcal{O}\left\{Q^{-3}\right\}\right) \approx \frac{1}{2Q}. \tag{2.10}$$

As a result of the energy loss, the freely oscillating system comes eventually to rest. The free oscillation is 'overdamped' if ω_d is imaginary—that is, if

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$\delta > \omega_0$ or $R > 2(Sm)^{1/2}$. [The quality factor Q, as defined by Eq. (2.8), is then complex, and it loses its physical significance.] Then the general solution of the differential equation (2.2) is a linear combination of two real, decaying exponential functions. The case of 'critical damping'—that is, when $R = 2(Sm)^{1/2}$ or $\omega_d = 0$—needs special consideration, which we omit here. (See, however, Problem 2.11.)

2.1.2 Forced Oscillation

When the differential equation (2.2) is inhomogeneous—that is, if $F = F(t) \neq 0$—the general solution may be written as a particular solution plus the general solution (2.4) of the corresponding homogeneous equation (corresponding to $F = 0$).

Let us now consider the case in which the driving external force $F(t)$ has a sinusoidal time variation with angular frequency $\omega = 2\pi/T$, where T is the period. Let

$$F(t) = F_0 \cos(\omega t + \varphi_F), \tag{2.11}$$

where F_0 is the amplitude and φ_F the phase constant for the force. It is convenient to choose a particular solution of the form where

$$x(t) = x_0 \cos(\omega t + \varphi_x) \tag{2.12}$$

is the position and

$$u(t) = \dot{x}(t) = u_0 \cos(\omega t + \varphi_u) \tag{2.13}$$

is the corresponding velocity of the mass m. Here the amplitudes are related by $u_0 = \omega x_0$ and the phase constants by $\varphi_u - \varphi_x = \pi/2$.

For Eq. (2.12) to be a particular solution of the differential equation (2.2), it is necessary (see Problem 2.3) that the excursion amplitude is

$$x_0 = \frac{u_0}{\omega} = \frac{F_0}{|Z|\omega} \tag{2.14}$$

and that the phase difference

$$\varphi = \varphi_F - \varphi_u = \varphi_F - \varphi_x - \pi/2 \tag{2.15}$$

is an angle which is in quadrant no. 1 or no. 4 and which satisfies

$$\tan\varphi = (\omega m - S/\omega)/R. \tag{2.16}$$

Here

$$|Z| = \sqrt{R^2 + (\omega m - S/\omega)^2} \tag{2.17}$$

is the absolute value (modulus) of the complex mechanical impedance, which is discussed later.

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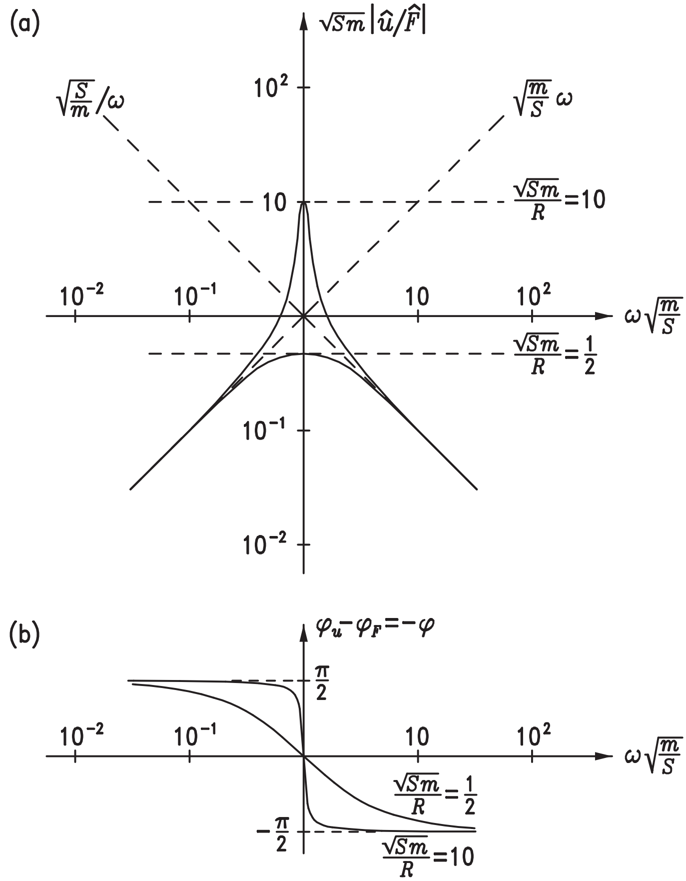

Figure 2.3: Frequency response of relation between velocity u and applied force F in normalised units, for two different values of the damping coefficient. (a) Amplitude (modulus) response with both scales logarithmic. (b) Phase response with linear scale for the phase difference.

The ‘forced oscillation’, Eq. (2.12) or (2.13), is a response to the driving force, Eq. (2.11). Let us now assume that F_0 is independent of ω and then discuss the responses $x_0(\omega)$ and $u_0(\omega)$, starting with $u_0(\omega) = F_0/|Z(\omega)|$. Noting that $|Z|_{\min} = R$ for $\omega = \omega_0 = \sqrt{S/m}$ and that $|Z| \to \infty$ for $\omega = 0$ as well as for $\omega \to \infty$, we see that $(u_0/F_0)_{\max} = 1/R$ for $\omega = \omega_0$ and that $u_0(0) = u_0(\infty) = 0$. We have resonance at $\omega = \omega_0$, where the ‘reactive’ contribution $\omega m - S/\omega$ to the mechanical impedance vanishes.

Graphs of the non-dimensionalised velocity response $\sqrt{Sm}\,u_0/F_0$ versus ω/ω_0 are shown in Figure 2.3 for $\sqrt{Sm}/R$ equal to 10 and 0.5. Note that the graphs are symmetric with respect to $\omega = \omega_0$ when the frequency scale is logarithmic. The phase difference φ as given by Eq. (2.16) is also shown in Figure 2.3. The graphs of Figure 2.3, where the amplitude response is presented in a double logarithmic diagram and the phase response in a semilogarithmic diagram, are usually called Bode plots or Bode diagrams [12].

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Next, we consider the resonance bandwidth, the frequency interval $(\Delta\omega)_{\rm res}$ where

$$\frac{u_0(\omega)}{F_0} > \frac{1}{\sqrt{2}}\left(\frac{u_0}{F_0}\right)_{\rm max} = \frac{1}{R\sqrt{2}}, \tag{2.18}$$

that is, where the kinetic energy exceeds half of the maximum value. At the upper and lower edges of the interval, ω_u and ω_l, the two terms of the radicand in Eq. (2.17) are equally large. Thus, we have

$$\omega_u m - S/\omega_u = R = S/\omega_l - \omega_l m. \tag{2.19}$$

Instead of solving these two equations, we note from the aforementioned symmetry that $\omega_u\omega_l = \omega_0^2 = S/m$—that is, $S/\omega_u = m\omega_l$ and $S/\omega_l = m\omega_u$. Evidently,

$$(\Delta\omega)_{\rm res} = \omega_u - \omega_l = R/m = 2\delta. \tag{2.20}$$

The relative bandwidth is

$$\frac{(\Delta\omega)_{\rm res}}{\omega_0} = \frac{2\delta}{\omega_0} = \frac{R}{\sqrt{Sm}}, \tag{2.21}$$

and it is seen that this is inverse to the maximum non-dimensionalised velocity response $\sqrt{Sm}u_0(\omega_0)/F_0 = \sqrt{Sm}/R$, as indicated on the graph in Figure 2.3. From Eq. (2.9), we see that this is approximately equal to the quality factor Q, when this is large ($Q \gg 1$). In the same case,

$$(\Delta\omega)_{\rm res}/\omega_0 \approx 1/Q. \tag{2.22}$$

Next we consider the excursion response—which, in non-dimensionalised form, may be written as $Sx_0(\omega)/F_0$. It equals unity for $\omega = 0$ and zero for $\omega = \infty$. At resonance, its value is

$$S\,x_0(\omega_0)/F_0 = S/\omega_0 R = \sqrt{Sm}/R = \omega_0/2\delta, \tag{2.23}$$

as obtained by using Eqs. (2.14) and (2.17). Note that $x_0(\omega)$ has its maximum at a frequency which is lower than the resonance frequency. It can be shown (see Problem 2.4) that, if $R < \sqrt{2Sm}$ or $\delta < \omega_0/\sqrt{2}$, then

$$\frac{Sx_{0,\rm max}}{F_0} = \frac{\omega_0}{2\delta}\left(1 - \frac{\delta^2}{\omega_0^2}\right)^{-1/2} \quad \text{at } \omega = \omega_0\left(1 - \frac{2\delta^2}{\omega_0^2}\right)^{1/2}, \tag{2.24}$$

and if $R > \sqrt{2Sm}$, then

$$\frac{Sx_{0,\rm max}}{F_0} = \frac{S\,x_0(0)}{F_0} = 1. \tag{2.25}$$

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For large values of Q, there is only a small difference between $x_0(\omega_0)/F_0$ and $x_{0,\max}/F_0$. Using Eq. (2.10), we find

$$\frac{Sx_0(\omega_0)}{F_0} = Q - \frac{1}{2} + \frac{1}{24Q} + \mathcal{O}\{Q^{-2}\} \tag{2.26}$$

and

$$\frac{Sx_{0,\max}}{F_0} = Q - \frac{1}{2} + \frac{1}{6Q} + \mathcal{O}\{Q^{-2}\} \tag{2.27}$$

for

$$\omega = \omega_0(1 - \frac{1}{4Q^2} + \mathcal{O}\{Q^{-3}\}). \tag{2.28}$$

2.1.3 Electric Analogue: Remarks on the Quality Factor

For readers with a background in electric circuits, it may be of interest to note that the mechanical system of Figure 2.1 is analogous to the electric circuit shown in Figure 2.4, where an inductance m, a capacitance $1/S$, and an electric resistance R are connected in series.

The force F is analogous to the driving voltage, the position x is analogous to the electric charge on the capacitance, and the velocity u is analogous to the electric current. If Kirchhoff's law is applied to the circuit, Eq. (2.2) results. The 'capacitive reactance' S/ω is related to the capacitance's ability to store electric energy (analogous to potential energy in the spring of Figure 2.1), and the 'inductive reactance' ωm is related to the inductance's ability to store magnetic energy (analogous to kinetic energy in the mass of Figure 2.1).

The electric (or potential) energy is zero when $x(t) = 0$, and the magnetic (or kinetic) energy is zero when $u(t) = \dot{x}(t) = 0$. The instants for $x(t) = 0$ and those for $u(t) = 0$ are displaced by a quarter of a period $\pi/2\omega_0$, and at resonance the maximum values for the electric and magnetic (or potential and kinetic) energies are equal, $mu_0^2/2 = m\omega_0^2 x_0^2/2 = Sx_0^2/2$, because $m\omega_0 = S/\omega_0$. Thus, at resonance, the stored energy is swinging back and forth between the two energy stores, twice every period of the system's forced oscillation.

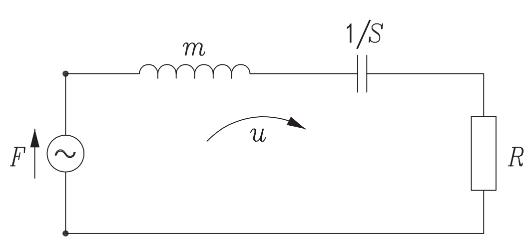

Figure 2.4: Electric analogue of the mechanical system shown in Figure 2.1.

By Eq. (2.8), we have defined the quality factor Q as the ratio between the stored energy and the average energy loss during a time interval $1/\omega_d$ of the free oscillation. An alternative definition would have resulted if instead the forced oscillation at resonance had been considered, for a time interval $1/\omega_0$ (and not $1/\omega_d$). The stored energy is $mu_0^2/2$ and the average lost energy is $Ru_0^2/2\omega_0$ during a time $1/\omega_0$ (as is shown in more detail later, in Section 2.3).

Such an alternative quality factor equals the right-hand side of the approximation (2.9) and would have been equal to the inverse of the relative bandwidth (2.21), to the non-dimensionalised excursion amplitude (2.23) at resonance (that is, the ratio between the excursion at resonance and the excursion at zero frequency), and to the ratio of the reactance parts $\omega_0 m$ and S/ω_0 to the damping resistance R. The term quality factor is usually used only when it is large, $Q \gg 1$. In that case, the relative difference between the two definitions is of little importance.

2.2 Complex Representation of Harmonic Oscillations

2.2.1 Complex Amplitudes and Phasors

When dealing with sinusoidal oscillations, it is mathematically convenient to apply the method of complex representation, involving complex amplitudes and phasors. A great advantage with the method is that differentiation with respect to time is simply represented by multiplying with $i\omega$, where i is the imaginary unit ($i = \sqrt{-1}$). We consider again the forced oscillations represented by the excursion response $x(t)$ or velocity response $u(t)$ due to an applied external sinusoidal force $F(t)$, as given by Eqs. (2.12), (2.13) and (2.11), respectively.

With the use of Euler's formula

$$e^{i\psi} = \cos\psi + i\sin\psi, \tag{2.29}$$

or, equivalently,

$$\cos\psi = (e^{i\psi} + e^{-i\psi})/2, \qquad \sin\psi = (e^{i\psi} - e^{-i\psi})/2i, \tag{2.30}$$

the oscillating quantity $x(t)$ may be rewritten as

$$\begin{aligned} x(t) &= x_0\cos(\omega t + \varphi_x) \\ &= \tfrac{1}{2}x_0[e^{i(\omega t+\varphi_x)} + e^{-i(\omega t+\varphi_x)}] \\ &= \tfrac{1}{2}x_0[e^{i\varphi_x}e^{i\omega t} + e^{-i\varphi_x}e^{-i\omega t}]. \end{aligned} \tag{2.31}$$

Introducing the *complex amplitude* (see Figure 2.5)

$$\hat{x} = x_0 e^{i\varphi_x} = x_0\cos\varphi_x + ix_0\sin\varphi_x \tag{2.32}$$

and the complex conjugate of $\hat{x}$

$$\hat{x}^* = x_0 e^{-i\varphi_x} = x_0\cos\varphi_x - ix_0\sin\varphi_x, \tag{2.33}$$

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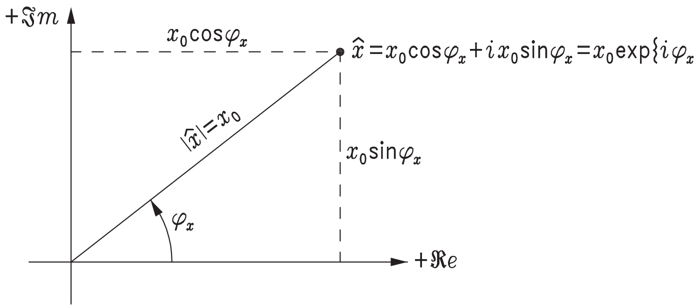

Figure 2.5: Complex-plane decomposition of the complex amplitude $\hat{x}$.

we have

$$2x(t) = \hat{x}e^{i\omega t} + \hat{x}^* e^{-i\omega t}. \tag{2.34}$$

Note that the sum is real, while the two terms are complex (and conjugate to each other). Another formalism is

$$2x(t) = \hat{x}e^{i\omega t} + \text{c. c.}, \tag{2.35}$$

where c. c. denotes complex conjugate of the preceding term.

The complex amplitude (2.32) contains information on two parameters:

1. the (absolute) amplitude $|\hat{x}| = x_0$, which is real and positive, and
2. the phase constant $\varphi_x = \arg \hat{x}$, which is an angle to be given in units of radians (rad) or degrees (°).

Multiplying Eq. (2.32) with $e^{i\omega t}$, we have

$$\hat{x}e^{i\omega t} = x_0 e^{i(\omega t + \varphi_x)} = x_0[\cos(\omega t + \varphi_x) + i \sin(\omega t + \varphi_x)], \tag{2.36}$$

from which we observe that

$$x(t) = x_0 \cos(\omega t + \varphi_x) = \mathrm{Re}\left\{\hat{x}e^{i\omega t}\right\} = \mathrm{Re}\left\{x_0 e^{i(\omega t + \varphi_x)}\right\}, \tag{2.37}$$

or that the instantaneous value equals the real part of the product of the complex amplitude $\hat{x}$ and $e^{i\omega t}$. On the complex plane, $x(t)$ equals the projection of the rotating vector $\hat{x}e^{i\omega t}$ on the real axis (see Figure 2.6). The value $x(t)$ is negative when $\omega t + \varphi_x$ is an angle in the second or third quadrant of the complex plane.

A summary is given in Table 2.1, where the real function $x(t)$ denotes a physical quantity in harmonic oscillation. [Note: frequently, one may see it written as $x(t) = \hat{x}e^{i\omega t}$. In this case, because $x(t)$ is complex, it must be implicitly understood that it is the real part of $x(t)$ that represents the physical quantity.]

Let us now consider position, velocity and acceleration and their interrelations. If the position of an oscillating mass point is given by Eq. (2.37), then the velocity is

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Table 2.1: Various mathematical expressions for a sinusoidal oscillation, where $\hat{x} = x_0 \exp(i\varphi_x)$ is the complex amplitude.

$x(t) = x_0 \cos(\omega t + \varphi_x)$
$x(t) = \frac{1}{2}(\hat{x}e^{i\omega t} + \hat{x}^* e^{-i\omega t})$
$x(t) = \frac{1}{2}(\hat{x}e^{i\omega t} + \text{c. c.})$
$x(t) = \text{Re}\left\{\hat{x}e^{i\omega t}\right\}$

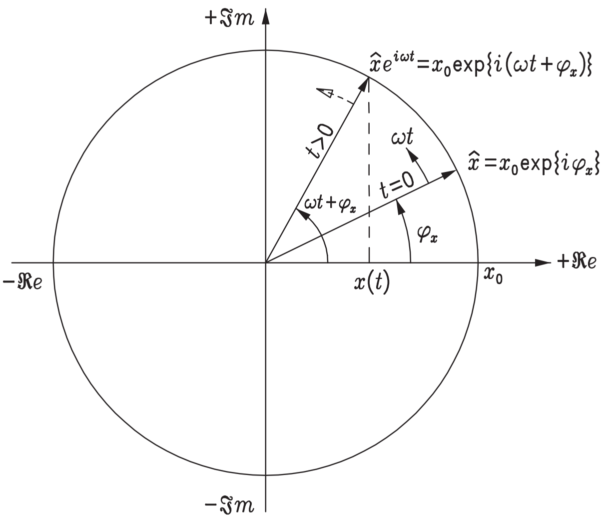

Figure 2.6: Phasor shown at times $t = 0$ and $t > 0$.

$$\begin{aligned} u = \dot{x} = \frac{dx}{dt} &= -\omega x_0 \sin(\omega t + \varphi_x) \\ &= \omega x_0 \cos(\omega t + \varphi_x + \pi/2) \\ &= \text{Re}\{\omega x_0 \exp[i(\omega t + \varphi_x + \pi/2)]\}. \end{aligned} \tag{2.38}$$

Because $e^{i\pi/2} = \cos(\pi/2) + i \sin(\pi/2) = i$, this gives

$$u(t) = \dot{x} = \text{Re}\{i\omega x_0 e^{i(\omega t+\varphi_x)}\} = \text{Re}\{i\omega\hat{x}e^{i\omega t}\}. \tag{2.39}$$

Differentiation of Eq. (2.34) gives the same result:

$$\begin{aligned} u = \dot{x} &= \tfrac{1}{2}(i\omega\hat{x}e^{i\omega t} - i\omega\hat{x}^* e^{-i\omega t}) \\ &= \tfrac{1}{2}(i\omega\hat{x}e^{i\omega t} + \text{c. c.}) \\ &= \text{Re}\{i\omega\hat{x}e^{i\omega t}\}. \end{aligned} \tag{2.40}$$

A more straightforward way to arrive at the same result, however, is by using the fact that $d(\text{Re}\{\hat{x}e^{i\omega t}\})/dt = \text{Re}\{d(\hat{x}e^{i\omega t})/dt\}$, which immediately gives the result.

Since $u(t) = \text{Re}\{\hat{u}e^{i\omega t}\}$, the complex velocity amplitude is therefore

$$\hat{u} = i\omega\hat{x}. \tag{2.41}$$

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Thus, to differentiate a physical quantity (in harmonic oscillation) with respect to time t means to multiply the corresponding complex amplitude by $i\omega$. Formally, $i\omega$ is a simpler operator than d/dt. This is one advantage of using the complex representation of harmonic oscillations.

Similarly, the acceleration is

$$a(t) = \dot{u} = \frac{du}{dt} = \mathrm{Re}\left\{i\omega\hat{u}e^{i\omega t}\right\}. \tag{2.42}$$

Thus, the complex acceleration amplitude is

$$\hat{a} = i\omega\hat{u} = i\omega\, i\omega\hat{x} = -\omega^2\hat{x}. \tag{2.43}$$

In analogy with Eq. (2.32), we may use Eqs. (2.11) and (2.13) to write the complex amplitudes for the external force and the velocity as

$$\hat{F} = F_0 e^{i\varphi_F}, \quad \hat{u} = u_0 e^{i\varphi_u}, \tag{2.44}$$

respectively. Because

$$\hat{u} = \hat{\dot{x}} = \omega x_0 \exp\left[i(\varphi_x + \pi/2)\right], \tag{2.45}$$

we have

$$u_0 = \omega x_0, \qquad \varphi_u = \varphi_x + \pi/2. \tag{2.46}$$

Similarly,

$$a_0 = \omega u_0 = \omega^2 x_0, \qquad \varphi_a = \varphi_u + \pi/2 = \varphi_x + \pi. \tag{2.47}$$

The complex amplitudes $\hat{x}$, $\hat{u}$ and $\hat{a}$ are illustrated in the phase diagram of Figure 2.7. Instead of letting the complex-plane vectors (the phasors) rotate in

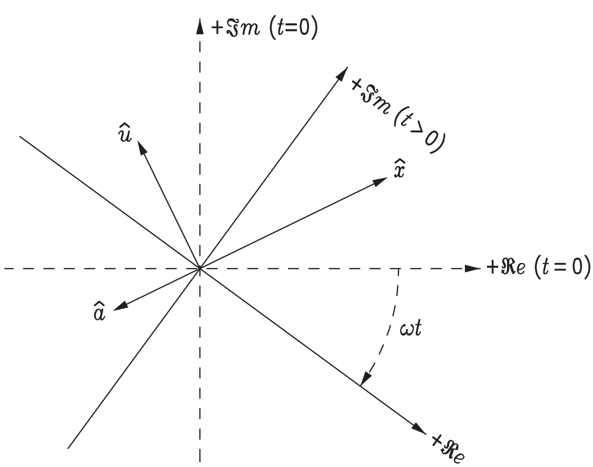

Figure 2.7: Phasor diagram for position x, velocity u and acceleration a.

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the counterclockwise direction, we may envisage the phasors being stationary, while the coordinate system (the real and imaginary axes) rotates in the clockwise direction. Projection of the phasors on the real axis equals the instantaneous values of position $x(t)$, velocity $u(t)$ and acceleration $a(t)$.

2.2.2 Mechanical Impedance

We shall now see how the dynamic equation (2.2) will be simplified, when we represent sinusoidally varying quantities by their complex amplitudes. Inserting

$$F(t) = \frac{1}{2}(\hat{F}e^{i\omega t} + \hat{F}^* e^{-i\omega t}), \tag{2.48}$$

$$x(t) = \frac{1}{2}(\hat{x}e^{i\omega t} + \hat{x}^* e^{-i\omega t}) \tag{2.49}$$

into Eq. (2.2) gives

$$\left[\hat{F} - \left(R + i\omega m + \frac{S}{i\omega}\right)\hat{u}\right]e^{i\omega t} + \left[\hat{F}^* - \left(R - i\omega m - \frac{S}{i\omega}\right)\hat{u}^*\right]e^{-i\omega t} = 0. \tag{2.50}$$

Introducing the complex *mechanical impedance*

$$Z = R + i\omega m + S/(i\omega) = R + i(\omega m - S/\omega) \tag{2.51}$$

gives

$$(\hat{F} - Z\hat{u})e^{i\omega t} + (\hat{F}^* - Z^*\hat{u}^*)e^{-i\omega t} = 0. \tag{2.52}$$

This equation is satisfied for arbitrary t if $\hat{F} - Z\hat{u} = 0$, giving

$$\hat{u} = \hat{F}/Z \tag{2.53}$$

or

$$\hat{x} = \hat{u}/(i\omega) = \hat{F}/(i\omega Z). \tag{2.54}$$

When the absolute value (modulus) is taken on both sides of this equation, Eq. (2.14) is obtained—if Eq. (2.44) is also observed.

An electric impedance is the ratio between complex amplitudes of voltage and current. Analogously, the mechanical impedance is the ratio between the complex force amplitude and the complex velocity amplitude. In SI units, mechanical impedance has the dimension [Z] = Ns/m = kg/s. The impedance $Z = R + iX$ is a complex function of ω. Here R is the *mechanical resistance*, and $X = \omega m - S/\omega$ is the *mechanical reactance*, which is indicated as a function of ω in Figure 2.8. Note that $X = 0$ for $\omega = \omega_0 = \sqrt{S/m}$. For this frequency, corresponding to resonance, the velocity-amplitude response is maximum:

$$|\hat{u}/\hat{F}|_{\max} = |u_0/F_0|_{\max} = 1/|Z|_{\min} = 1/R. \tag{2.55}$$

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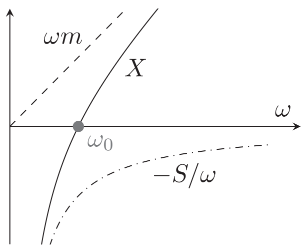

Figure 2.8: Mechanical reactance of the oscillating system versus frequency.

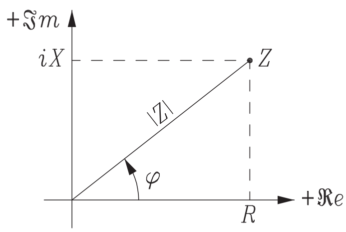

Figure 2.9: Complex-plane decomposition of the mechanical impedance Z.

The impedance Z, as shown in the complex plane of Figure 2.9, is

$$Z = R + iX = |Z|e^{i\varphi}, \tag{2.56}$$

where

$$|Z| = \sqrt{R^2 + X^2}, \tag{2.57}$$

in accordance with Eq. (2.17). Further, we have

$$\cos\varphi = R/|Z|, \quad \sin\varphi = X/|Z|, \tag{2.58}$$

from which Eq. (2.16) follows. Because $R > 0$, we have $|\varphi| < \pi/2$. At resonance, $\varphi = 0$ because $X = 0$.

Inserting Eq. (2.56) into Eq. (2.53), we have

$$\hat{u} = \frac{\hat{F}}{Z} = \frac{|\hat{F}|e^{i\varphi_F}}{|Z|e^{i\varphi}} = |\hat{F}/Z| \exp\left[i(\varphi_F - \varphi)\right] \tag{2.59}$$

or

$$|\hat{u}| = |\hat{F}|/|Z|, \quad \varphi_u = \varphi_F - \varphi. \tag{2.60}$$

We may choose the time origin $t = 0$ such that $\varphi_F = 0$; that is, $\hat{F}$ is real and positive. Then $\varphi_u = -\varphi$. A case with positive φ is illustrated with phasors in Figure 2.10 and with functions of time in Figure 2.11.

When $\varphi > 0$ (as in Figures 2.10 and 2.11), the velocity u is said to lag in phase, or to have a phase lag, relative to the force F. In this case, corresponding to $\omega > \omega_0$, the reactance is dominated by the inertia term ωm. In the opposite case, $\omega_0 > \omega$, $\varphi < 0$, the reactance is dominated by the stiffness, and the velocity leads in phase (or has a phase advance) relative to the force.

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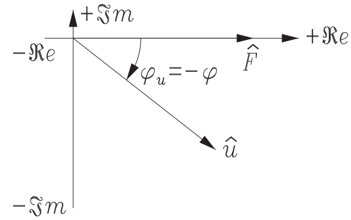

Figure 2.10: Complex amplitudes of applied force F and of resulting velocity u (when $\varphi > 0$).

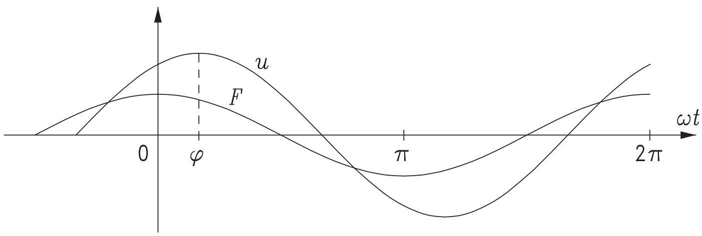

Figure 2.11: Applied force F and resulting velocity u versus time (when $\varphi > 0$). The velocity has a phase lag relative to the force.

Frequency-response diagrams for two different values of damping resistance R are shown in Figure 2.3. If $\omega_0 \ll \omega$, we have $Z \approx i\omega m$; that is, $\hat{u} \approx \hat{F}/(i\omega m)$ or $\hat{a} = i\omega\hat{u} \approx \hat{F}/m$. Thus,

$$ma(t) \approx F(t). \tag{2.61}$$

In contrast, if $\omega_0 \gg \omega$, we have $Z \approx -iS/\omega$; that is, $\hat{u} \approx i\omega\hat{F}/S$ or $\hat{x} = \hat{u}/i\omega \approx \hat{F}/S$. Hence,

$$Sx(t) \approx F(t). \tag{2.62}$$

The differential equation (2.2) expresses how the applied external force is balanced against inertia, damping and stiffness forces. The force balance may also be represented by complex amplitude relations:

$$m\hat{a} + R\hat{u} + S\hat{x} = \hat{F}, \tag{2.63}$$

where $\hat{a} = i\omega\hat{u}$ and $\hat{x} = \hat{u}/i\omega$, or

$$(R + iX)\hat{u} = Z\hat{u} = \hat{F}, \tag{2.64}$$

or, graphically, by a force-phasor diagram. Figure 2.12 shows a phasor diagram for the numerical example $R = 4$ kg/s, $\omega m = 5$ kg/s and $S/\omega = 2$ kg/s. For those values, we obtain $X = 3$ kg/s, $|Z| = 5$ kg/s and $\varphi = 0.64$ rad $= 37°$. The damping force $R\hat{u}$ is in phase with the velocity $\hat{u}$. The stiffness force $S\hat{x}$ and the inertia force $m\hat{a}$ are in anti-phase. The former has a phase lag of $\pi/2$, and the latter a phase lead of $\pi/2$, relative to the velocity.

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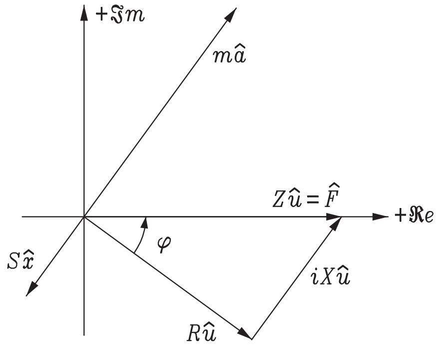

Figure 2.12: Phasor diagram showing balance of forces.

2.3 Power and Energy Relations

We shall now consider energy delivered by the external force to the simple mechanical oscillating system, shown in Figure 2.1, and see how this energy is exchanged with energy consumed in the damping resistance R and with stored potential and kinetic energy in the spring S and mass m. The mechanical power (rate of work) delivered by supplying the external force $F(t)$ is

$$P(t) = F(t)u(t) = F(t)\dot{x}(t), \tag{2.65}$$

where

$$\begin{aligned} F(t) &= F_m(t) - F_R(t) - F_S(t) \\ &= ma(t) + Ru(t) + Sx(t), \end{aligned} \tag{2.66}$$

in accordance with the dynamic equation (2.2). Thus, we have

$$P(t) = P_R(t) + \left[P_k(t) + P_p(t)\right], \tag{2.67}$$

where the powers delivered to R, m and S are

$$P_R(t) = -F_R(t)u(t) = Ru^2, \tag{2.68}$$

$$P_k(t) = F_m(t)u(t) = m\dot{u}u = \frac{d}{dt}W_k(t), \tag{2.69}$$

$$P_p(t) = -F_S(t)u(t) = S\dot{x}x = \frac{d}{dt}W_p(t), \tag{2.70}$$

respectively, where

$$W_k(t) = m[u(t)]^2/2 \tag{2.71}$$

is the kinetic energy and

$$W_p(t) = S[x(t)]^2/2 \tag{2.72}$$

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is the potential energy stored in spring S. The energy stored in the oscillating system is

$$W(t) = W_k(t) + W_p(t), \tag{2.73}$$

and the corresponding power or rate of change of energy is

$$P_k(t) + P_p(t) = \frac{d}{dt}W(t). \tag{2.74}$$

The delivered power $P(t)$ has two components:

1. $P_R(t)$, which is consumed in the damping resistance R (cf. instantaneous 'active power'), and
2. $P_k(t) + P_p(t)$ (cf. instantaneous 'reactive power'), which is exchanged with stored kinetic energy (in mass m) and potential energy (in spring S).

2.3.1 Harmonic Oscillation: Active Power and Reactive Power

With harmonic oscillation, we have

$$\begin{aligned} P(t) &= F(t)\,u(t) \\ &= \tfrac{1}{2}(\hat{F}e^{i\omega t} + \hat{F}^* e^{-i\omega t})\tfrac{1}{2}(\hat{u}e^{i\omega t} + \hat{u}^* e^{-i\omega t}) \\ &= \tfrac{1}{4}(\hat{F}\hat{u}^* + \hat{F}^*\hat{u} + \hat{F}\hat{u}e^{2i\omega t} + \hat{F}^*\hat{u}^* e^{-2i\omega t}). \end{aligned} \tag{2.75}$$

The sum of the two last terms, being complex conjugate to each other, is a harmonic oscillation with angular frequency 2ω, and hence, it has a time-average value equal to zero. The delivered time-average power is

$$P \equiv \overline{P(t)} = \frac{1}{4}\hat{F}\hat{u}^* + \text{c.c.} = \frac{1}{2}\text{Re}\{\hat{F}\hat{u}^*\} \tag{2.76}$$

$$\begin{aligned} &= \frac{1}{2}\text{Re}\{Z\hat{u}\hat{u}^*\} = \frac{1}{2}\text{Re}\{Z|\hat{u}|^2\} = \frac{R}{2}|\hat{u}|^2 \\ &= \frac{R}{2}\left|\frac{\hat{F}}{Z}\right|^2 = \frac{R}{2|Z|^2}|\hat{F}|^2 = \frac{R}{2(R^2+X^2)}|\hat{F}|^2. \end{aligned} \tag{2.77}$$

The consumed power is

$$\begin{aligned} P_R(t) &= Ru^2 = \frac{R}{4}\left(\hat{u}e^{i\omega t} + \hat{u}^* e^{-i\omega t}\right)^2 \\ &= \frac{R}{4}\left(2\hat{u}\hat{u}^* + \hat{u}^2 e^{i2\omega t} + \hat{u}^{*2}e^{-i2\omega t}\right). \end{aligned} \tag{2.78}$$

Here, the two last terms have a zero time-average sum. Thus,

$$P_R = \overline{P_R(t)} = \frac{R}{2}\,\hat{u}\hat{u}^* = \frac{R}{2}|\hat{u}|^2 = \overline{P(t)}. \tag{2.79}$$

Hence, the consumed power and the delivered power are equal in time average.

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The instantaneous values are (if we choose $\varphi_F = 0$)

$$\begin{aligned} P(t) &= \overline{P(t)} + (\tfrac{1}{4}\hat{F}\hat{u}e^{2i\omega t} + \text{c. c.}) \\ &= \overline{P(t)} + \tfrac{1}{2}|\hat{F}\hat{u}|\cos(2\omega t - \varphi) \end{aligned} \tag{2.80}$$

for the delivered power and

$$\begin{aligned} P_R(t) &= \overline{P(t)} + (\tfrac{1}{4}\hat{R}\hat{u}^2 e^{2i\omega t} + \text{c. c.}) \\ &= \overline{P(t)} + \tfrac{1}{2}\hat{R}|\hat{u}|^2\cos(2\omega t - 2\varphi) \end{aligned} \tag{2.81}$$

for the consumed power. Thus, in general, $P_R(t) \neq P(t)$. The difference $P(t) - P_R(t) = P_k(t) + P_p(t)$ is instantaneous reactive power exchanged with stored energy in the system.

In the special case of resonance ($\omega = \omega_0$, $\varphi = 0$, $Z = R$), we have $P_R(t) = P(t) = (R/2)|\hat{u}|^2(1 + \cos 2\omega t) = R|\hat{u}|^2\cos^2\omega t$. Hence, at resonance, there is no reactive power delivered. The stored energy is constant. It is alternating between kinetic energy and potential energy, which have equal maximum values.

In general, the stored instantaneous kinetic energy is

$$\begin{aligned} W_k(t) &= \frac{m}{2}[u(t)]^2 = \frac{m}{2}|\hat{u}|^2\cos^2(\omega t - \varphi) \\ &= \frac{m}{4}|\hat{u}|^2[1 + \cos(2\omega t - 2\varphi)], \end{aligned} \tag{2.82}$$

where the average kinetic energy is

$$W_k \equiv \overline{W_k(t)} = \frac{m}{4}|\hat{u}|^2 = \frac{m}{4}\hat{u}\hat{u}^*, \tag{2.83}$$

and the instantaneous potential energy is

$$\begin{aligned} W_p(t) &= \frac{S}{2}[x(t)]^2 = \frac{S}{2}|\hat{x}|^2\sin^2(\omega t - \varphi) \\ &= \frac{S}{4}|\hat{x}|^2[1 - \cos(2\omega t - 2\varphi)], \end{aligned} \tag{2.84}$$

where the average potential energy is

$$W_p \equiv \overline{W_p(t)} = \frac{S}{4}|\hat{x}|^2 = \frac{S}{4\omega^2}|\hat{u}|^2. \tag{2.85}$$

The instantaneous total stored energy is

$$\begin{aligned} W(t) &= W_k(t) + W_p(t) \\ &= W_k + W_p + (W_k - W_p)\cos(2\omega t - 2\varphi), \end{aligned} \tag{2.86}$$

which is of time average

$$\begin{aligned} W &\equiv \overline{W(t)} = W_k + W_p = \frac{1}{4}\left(m|\hat{u}|^2 + S|\hat{x}|^2\right) \\ &= \frac{m}{4}|\hat{u}|^2\left[1 + \left(\frac{\omega_0}{\omega}\right)^2\right]. \end{aligned} \tag{2.87}$$

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The amplitude of the oscillating part of the total stored energy is

$$W_k - W_p = \frac{1}{4\omega}\left(\omega m - \frac{S}{\omega}\right)|\hat{u}|^2 = \frac{X}{4\omega}|\hat{u}|^2 = \frac{X}{4\omega}\hat{u}\hat{u}^*. \tag{2.88}$$

The instantaneous reactive power is

$$\begin{aligned} P_k(t) + P_p(t) &= \frac{d}{dt}W(t) = -2\omega(W_k - W_p)\sin(2\omega t - 2\varphi) \\ &= -\frac{X}{2}|\hat{u}|^2 \sin(2\omega t - 2\varphi). \end{aligned} \tag{2.89}$$

We observe that the reactance and the reactive power may be related to the difference between the kinetic energy and the potential energy stored in the system. At resonance ($\omega = \omega_0$), the maximum kinetic energy $m|\hat{u}|^2/2$ equals the maximum potential energy $S|\hat{x}|^2/2 = S\omega_0^{-2}|\hat{u}|^2/2$, and the reactance vanishes. If $\omega < \omega_0$, the maximum potential energy is larger than the maximum kinetic energy, and the mechanical reactance differs from zero (and is negative) because some of the stored energy has to be exchanged with the external energy, back and forth twice every oscillation period, due to the imbalance between the two types of energy store within the system. If $\omega > \omega_0$, the maximum kinetic energy is larger than the maximum potential energy, and the mechanical reactance is positive. It may be noted that the average values W_k and W_p are just half of the maximum values $m|\hat{u}|^2/2$ and $S|\hat{x}|^2/2$ of the kinetic and potential energies, respectively.

If we define the delivered 'complex power' as

$$\mathcal{P} = \tfrac{1}{2}\hat{F}\hat{u}^* = \tfrac{1}{2}Z\hat{u}\hat{u}^* = \tfrac{1}{2}R|\hat{u}|^2 + i\tfrac{1}{2}X|\hat{u}|^2, \tag{2.90}$$

we see from Eqs. (2.76) and (2.81) that $\text{Re}\{\mathcal{P}\} = P$ equals the average delivered power, the average consumed power and the amplitude of the oscillating part of the consumed power. Moreover, from Eq. (2.89), we see that $\text{Im}\{\mathcal{P}\}$ is the amplitude of the instantaneous reactive power. Finally, we see from Eq. (2.80) that $|\mathcal{P}|$ is the amplitude of the oscillating part of the delivered power. Note that the time-independent quantities $\text{Re}\{\mathcal{P}\}$, $\text{Im}\{\mathcal{P}\}$ and $|\mathcal{P}|$ are sometimes called *active power*, *reactive power* and *apparent power*, respectively.

2.4 State-Space Analysis

The simple oscillator shown in Figure 2.1 is represented by Eq. (2.2), which is a second-order linear differential equation with constant coefficients. It is convenient to reformulate it as the following set of two simultaneous differential equations of first order:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{R}{m}x_2 - \frac{S}{m}x_1 + \frac{1}{m}u_1, \tag{2.91}$$

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where we have introduced the state variables

$$x_1(t) = x(t), \qquad x_2(t) = u(t) = \dot{x}(t) \tag{2.92}$$

and the input variable

$$u_1(t) = F(t). \tag{2.93}$$

Equations (2.91) may be written in matrix notation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -S/m & -R/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u_1. \tag{2.94}$$

For a more general case of a linear system which is represented by linear differential equations with constant coefficients, these may be represented in the state-variable form [13]:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \tag{2.95}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}. \tag{2.96}$$

If there are n state variables $x_1(t), \ldots, x_n(t)$, r input variables $u_1(t), \ldots, u_r(t)$ and m output variables $y_1(t), \ldots, y_m(t)$, then the system matrix (or state matrix) $\mathbf{A}$ is of dimension $n \times n$, the input matrix $\mathbf{B}$ of dimension $n \times r$ and the output matrix $\mathbf{C}$ of dimension $m \times n$. The $m \times r$ matrix $\mathbf{D}$, which is zero in many cases, represents direct coupling between the input and the output.

When the identity matrix is denoted by $\mathbf{I}$, the solution of the vectorial differential equation (2.95) may be written as (see Problem 2.9)

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau, \tag{2.97}$$

where the matrix exponential is defined as the series

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots, \tag{2.98}$$

which is a matrix of dimension $n \times n$ and which commutes with the matrix $\mathbf{A}$. Hence, we may perform algebraic manipulations (including integration and differentiation with respect to t) with $e^{\mathbf{A}t}$ in the same way as we do with e^{st} where s is a scalar constant. Inserting Eq. (2.97) into Eq. (2.96) gives the output vector

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau + \mathbf{Du}(t) \tag{2.99}$$

in terms of the input vector $\mathbf{u}(t)$ and the initial-value vector $\mathbf{x}(t_0)$. In many cases, we analyse situations where $\mathbf{u}(t) = 0$ for $t < t_0$ and $\mathbf{x}(t_0) = 0$. Then the first right-hand term in Eq. (2.99) vanishes. Note from the integral term in Eq. (2.99) that the output $\mathbf{y}$ at time t is influenced by the input $\mathbf{u}$ at an earlier time τ ($\tau < t$).

It may be difficult to make an accurate numerical computation of the matrix exponential, particularly if the matrix order n is large. Various methods have been proposed for computation and discussion of the matrix exponential [14, 15]. Which method is best depends on the particular problem.

In some cases (but far from always), the system matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{m}_i$ ($i = 1, 2, \ldots, n$), where

$$\mathbf{A}\mathbf{m}_i = \Lambda_i \mathbf{m}_i. \tag{2.100}$$

The eigenvalues Λ_i, which may or may not be distinct, are solutions of (roots in) the nth degree equation

$$\det(\mathbf{A} - \Lambda\mathbf{I}) \equiv |\mathbf{A} - \Lambda\mathbf{I}| = 0. \tag{2.101}$$

It may be mathematically convenient to transform the system matrix $\mathbf{A}$ to a particular (usually simpler) matrix $\mathbf{A}'$ by means of a similarity transformation [13, 16]

$$\mathbf{A}' = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \tag{2.102}$$

where $\mathbf{M}$ is a non-singular $n \times n$ matrix, which implies that its inverse $\mathbf{M}^{-1}$ exists. It is well-known that $\mathbf{A}'$ has the same eigenvalues as $\mathbf{A}$. If the matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{m}_i$, then $\mathbf{A}'$ is diagonal,

$$\mathbf{A}' = \mathrm{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n), \tag{2.103}$$

if $\mathbf{M}$ is chosen as

$$\mathbf{M} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n), \tag{2.104}$$

which is the $n \times n$ matrix whose ith column is the eigenvector $\mathbf{m}_i$. Because all the vectors $\mathbf{m}_i$ are linearly independent, this implies that the inverse matrix $\mathbf{M}^{-1}$ exists. In this case the matrix exponential is also simply a diagonal matrix,

$$e^{\mathbf{A}'t} = \mathrm{diag}\left(e^{\Lambda_1 t}, e^{\Lambda_2 t}, \ldots, e^{\Lambda_n t}\right). \tag{2.105}$$

Introducing also the transformed state vector

$$\mathbf{x}' = \mathbf{M}^{-1}\mathbf{x} \quad (\text{i.e., } \mathbf{x} = \mathbf{M}\mathbf{x}') \tag{2.106}$$

and the transformed matrices

$$\mathbf{B}' = \mathbf{M}^{-1}\mathbf{B}, \quad \mathbf{C}' = \mathbf{C}\mathbf{M}, \tag{2.107}$$

we rewrite Eqs. (2.95) and (2.96) as

$$\dot{\mathbf{x}}' = \mathbf{A}'\mathbf{x}' + \mathbf{B}'\mathbf{u}, \tag{2.108}$$

$$\mathbf{y} = \mathbf{C}'\mathbf{x}' + \mathbf{D}\mathbf{u}. \tag{2.109}$$

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In cases in which the matrix $\mathbf{A}'$ and hence also its exponential are diagonal, as in Eqs. (2.103) and (2.105), it is easy to compute the output $\mathbf{y}(t)$ from Eq. (2.99) with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{x}_0$ replaced by $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$ and $\mathbf{x}'_0$, respectively.

The use of state-space analysis is a well-known mathematical tool, for instance in control engineering [13]. Except when the state-space dimension n is rather low, such as $n = 2$ in the example below, the state-space analysis is performed numerically [14, 15].

As a simple example, we again use the oscillator shown in Figure 2.1, in which we shall consider the force $F(t)$ as input, $u_1(t) = F(t)$, and the displacement $x(t)$ as output, $y_1(t) = x(t) = x_1(t)$. The matrices to use with Eq. (2.96) are

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \tag{2.110}$$

and with Eq. (2.95),

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -S/m & -R/m \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}. \tag{2.111}$$

See Eq. (2.94). Thus, with $r = 1$ and $m = 1$, this is a SISO (single-input single-output) system. The state matrix is of dimension 2×2. With this example, the eigenvalue equation (2.101) has the two solutions

$$\Lambda_1 = -\delta + i\omega_d, \quad \Lambda_2 = -\delta - i\omega_d, \tag{2.112}$$

where δ and ω_d are given by Eq. (2.5). The corresponding eigenvectors are (see Problem 2.10)

$$\mathbf{m}_1 = c_1 \begin{bmatrix} 1 \\ \Lambda_1 \end{bmatrix}, \quad \mathbf{m}_2 = c_2 \begin{bmatrix} 1 \\ \Lambda_2 \end{bmatrix}, \tag{2.113}$$

where c_1 and c_2 are arbitrary constants, which may be chosen equal to 1. These eigenvectors are linearly independent if $\Lambda_2 \neq \Lambda_1$ (i.e., $\omega_d \neq 0$). The application of Eqs. (2.99) to (2.109) results in an output which agrees with Eqs. (2.4) and (2.6) for the case of free oscillations—that is, when $F(t) = 0$ (see Problem 2.10). For the case of 'critical damping' ($\omega_d = 0$ and $\Lambda_1 = \Lambda_2 = \Lambda = -\delta$), there are not two linearly independent eigenvectors. Then neither $\mathbf{A}'$ nor $e^{\mathbf{A}'t}$ is diagonal (see Problem 2.11).

2.5 Linear Systems

We have discussed in detail the simple mechanical system in Figure 2.1 (or the analogous electric system shown in Figure 2.4). It obeys the constant-coefficient linear differential equation (2.2). Further, we have just touched upon another linear system, represented by a more general set of linear differential equations with constant coefficients, Eq. (2.95). Next, let us discuss linear systems more generally.

A system may be defined as a collection of components, parts or units which influence each other mutually through relations between causes and resulting effects. Some of the interrelated physical variables denoted by $u_1(t), u_2(t), \ldots, u_r(t)$ may be considered as input to the system, whereas other variables, denoted by $y_1(t), y_2(t), \ldots, y_m(t)$, are considered as outputs. We may contract these variables into an r-dimensional input vector $\mathbf{u}(t)$ and an m-dimensional output vector $\mathbf{y}(t)$.

An example is the mechanical system shown in Figure 2.1, where the external force $F(t) = u_1(t)$ is the input variable, and the output variables are the excursion $x(t) = y_1(t)$, the velocity $u(t) = y_2(t)$ and the acceleration $a(t) = y_3(t)$ of the oscillating mass. In this case, the dimensions of the input and output vectors are $r = 1$ and $m = 3$, respectively. If the system parameters R, m and S are constant or, more precisely, independent of the oscillation amplitudes, then this particular system is linear. However, if the power $P_R(t) = R[u(t)]^2 = y_4(t)$ consumed by the resistance R had also been included as a fourth member of our chosen set of output variables, then our mathematically considered system would have been nonlinear. For instance, if the input $u_1(t) = F(t)$ had been doubled, then also $y_1(t)$, $y_2(t)$ and $y_3(t)$ would have been doubled, but $y_4(t)$ would have been increased by a factor of four.

The mathematical relationship between the variables may be expressed as

$$\mathbf{y}(t) = \mathbf{T}\{\mathbf{u}(t)\}, \tag{2.114}$$

where the operator symbol $\mathbf{T}$ designates the law for determining $\mathbf{y}(t)$ from $\mathbf{u}(t)$. If we in Eq. (2.99) set $\mathbf{x}(t_0) = \mathbf{0}$ and $\mathbf{D} = \mathbf{0}$, we have the following example of such a relationship:

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau. \tag{2.115}$$

With this example, the state variables obey a set of differential equations with constant coefficients and all the initial values at $t = t_0$ are zero. The output $\mathbf{y}$ at time t is a result of the input $\mathbf{u}(t)$, only, during the time interval (t_0, t).

A system is linear if, for any two input vectors $\mathbf{u}_a(t)$ and $\mathbf{u}_b(t)$ with corresponding output vectors $\mathbf{y}_a(t) = \mathbf{T}\{\mathbf{u}_a(t)\}$ and $\mathbf{y}_b(t) = \mathbf{T}\{\mathbf{u}_b(t)\}$, we have

$$\alpha_a\mathbf{y}_a(t) + \alpha_b\mathbf{y}_b(t) = \mathbf{T}\{\alpha_a\mathbf{u}_a(t) + \alpha_b\mathbf{u}_b(t)\} \tag{2.116}$$

for arbitrary constants α_a and α_b. The output and input in Eq. (2.116) are sums of two terms. It is straightforward to generalise this to a finite number of terms. Extension to infinite sums and integrals is an additional requirement which we shall include in our definition of a linear system. According to this definition, the superposition principle applies for linear systems. Since $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are constant matrices, the system represented by Eq. (2.115) is linear.

In some simple cases, there is an instantaneous relation between input and output—for instance, between the voltage across an electric resistance and the resulting electric current or between the acceleration of a body and the net force applied to it. In most cases, systems are dynamic. Then the instantaneous output variables depend both on previous and present values of the input variables. For instance, the velocity of a body depends on the force applied to the body at previous instants.

In some cases, it may be convenient to define a system in which both the input and the output are chosen to be variables caused by some other variable(s). Then it may happen that the present-time output variables may even be influenced by future values of the input variables. Such a system is said to be noncausal. Otherwise, for causal systems the output (the response) cannot exist before the input (the cause).

In the remainder of this chapter, let us consider a linear system with a single input ($r = 1$) and a single output ($m = 1$). The input function $u(t)$ and the output function $y(t)$ are then scalar functions. For this case, we write

$$y(t) = L\{u(t)\} \tag{2.117}$$

instead of Eq. (2.114). (Because the system is linear, we have written L instead of T to designate the law determining the system's output from its input.) In Sections 2.5.2 and 2.6.2, we introduce a mathematical function, the transfer function, which characterises the linear system.

2.5.1 The Delta Function and Related Distributions

As a preparation, let us first give a few mathematical definitions. Let $\varphi(\tau)$ be an arbitrary function which is continuous and infinitely many times differentiable at $t = \tau$. Then the impulse function $\delta(t)$, also called the Dirac delta function, delta function, or, more properly, delta distribution, is defined [17] by the property

$$\varphi(0) = \int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt \tag{2.118}$$

or, more generally,

$$\varphi(t) = \int_{-\infty}^{\infty} \delta(\tau - t)\varphi(\tau)\,d\tau. \tag{2.119}$$

In particular, if $\varphi(t) \equiv 1$ and $t = 0$, then this gives

$$\int_{-\infty}^{\infty} \delta(\tau)\,d\tau = 1. \tag{2.120}$$

Although the delta distribution is not a mathematical function in the ordinary sense, it is a meaningful statement to say that for $t \neq 0$, $\delta(t) = 0$. The derivative $\dot{\delta}(t) = d\delta(t)/dt$ of the impulse function is defined by

$$\dot{\varphi}(0) = -\int_{-\infty}^{\infty} \dot{\delta}(t)\,\varphi(t)\,dt, \tag{2.121}$$

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and the nth derivative $\delta^{(n)}(t) = d^n\delta(t)/dt^n$ by

$$\varphi^{(n)}(0) = (-1)^n \int_{-\infty}^{\infty} \delta^{(n)}(t)\,\varphi(t)\,dt. \tag{2.122}$$

The impulse function $\delta(t)$ is even and the derivative $\delta^{(n)}(t)$ is even if n is even, and odd if n is odd [17].

The function

$$\mathrm{sgn}(t) = 2U(t) - 1 = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases} \tag{2.123}$$

is an odd function, whose derivative is $2\delta(t)$. In Eq. (2.123), we have introduced the (Heaviside) unit step function $U(t)$, which equals 1 for $t > 0$ and 0 for $t < 0$. Note that $\dot{U}(t) = \delta(t)$.

2.5.2 Impulse Response: Time-Invariant System

Let us next assume that the input to a linear system is an impulse at time t_1. By this, we mean that the input function is $u(t) = \delta(t - t_1)$. The system's output $y(t) = h(t, t_1)$, which corresponds to the impulse input, is called the *impulse response*. By means of Eq. (2.119), we may write an arbitrary input as

$$u(t) = \int_{-\infty}^{\infty} u(t_1)\,\delta(t - t_1)\,dt_1, \tag{2.124}$$

which may be interpreted as a superposition of impulse inputs $u(t_1)\,\delta(t - t_1)\,dt_1$. Because the system is linear, we may superpose the corresponding outputs $u(t_1)\,h(t, t_1)\,dt_1$. Thus, the resulting output is

$$y(t) = \int_{-\infty}^{\infty} u(t_1)\,h(t, t_1)\,dt_1 \equiv L\{u(t)\}. \tag{2.125}$$

If we know the impulse response $h(t, t_1)$, we can use Eq. (2.125) to find the output corresponding to an arbitrary given input.

A linear system in which the impulse response $h(t, t_1)$ depends on t and t_1 only through the time difference $t - t_1$ is called a time-invariant linear system. Then the integral in Eq. (2.125) becomes a convolution integral

$$y(t) = \int_{-\infty}^{\infty} u(t_1)\,h(t - t_1)\,dt_1 \equiv u(t) * h(t). \tag{2.126}$$

Note that convolution is commutative, i.e., $h(t) * u(t) = u(t) * h(t)$. [See Eq. (2.145), which proves this statement of commutativity.]

In many cases, a system which is linear and time invariant may be represented by a set of simultaneous first-order differential equations with constant

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coefficients, as discussed in Section 2.4. Let us, as an example of such a case, consider a SISO system, for which Eq. (2.115) simplifies to

$$y(t) = \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau, \tag{2.127}$$

where the input matrix $\mathbf{B}$ is of dimension $n \times 1$ and the output matrix $\mathbf{C}$ of dimension $1 \times n$, and where we have chosen $t_0 = 0$. Thus, for this special case, the impulse response is given by

$$h(t) = \begin{cases} 0 & \text{for } t < 0 \\ \mathbf{C}e^{\mathbf{A}t}\mathbf{B} & \text{for } t > 0, \end{cases} \tag{2.128}$$

as is easily seen if we compare Eq. (2.127) with Eq. (2.126). For this example, the system is causal.

We say that a linear time-invariant system is causal if the impulse response vanishes for negative times—that is, if

$$h(t) = 0 \quad \text{for } t < 0. \tag{2.129}$$

Let us now return to the general case of a linear, time-invariant, SISO system. If the input is $u_e(t) = e^{i\omega t}$, then, with $y_e(t) = L\{e^{i\omega t}\}$, we have

$$\begin{aligned} y_e(t_1 + t) &= L\{e^{i\omega(t_1+t)}\} = L\{e^{i\omega t_1}e^{i\omega t}\} \\ &= e^{i\omega t_1}L\{e^{i\omega t}\} = u_e(t_1)\,y_e(t), \end{aligned} \tag{2.130}$$

where t_1 is an arbitrary constant (of dimension time). Here we utilise the fact that the system is time invariant and linear. Inserting $t = 0$ and $t_1 = t$ into Eq. (2.130), we obtain

$$y_e(t) = y_e(0)\,e^{i\omega t} \equiv H(\omega)e^{i\omega t}, \tag{2.131}$$

which shows that if the input is an exponential function of time, then the output equals the input multiplied by a time-independent coefficient, which we have denoted $H(\omega)$. If the input is

$$u(t) = \tfrac{1}{2}(\hat{u}e^{i\omega t} + \hat{u}^* e^{-i\omega t}), \tag{2.132}$$

then the output is

$$y(t) = \tfrac{1}{2}(\hat{y}e^{i\omega t} + \hat{y}^* e^{-i\omega t}), \tag{2.133}$$

with

$$\hat{y} = H(\omega)\hat{u}, \quad \hat{y}^* = H^*(\omega)\hat{u}^*. \tag{2.134}$$

Note that $H(\omega)$, which characterises the system, may be complex and dependent on ω (see Figure 2.13). We shall see in the next section that $H(\omega)$, which we call the system's *transfer function*, is the Fourier transform of the impulse response $h(t)$.

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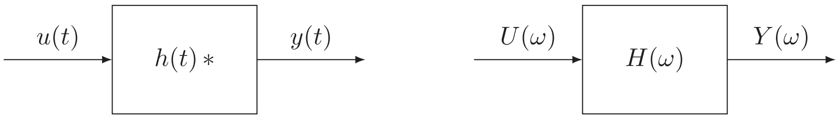

Figure 2.13: System with input signal $u(t)$ and output signal $y(t)$. The corresponding Fourier transforms are $U(\omega)$ and $Y(\omega)$ respectively. The linear system's impulse response $h(t)$ is the inverse Fourier transform of the transfer function $H(\omega)$.

Returning again to the example of forced oscillations in the system shown in Figure 2.1, where we consider the external force $F(t)$ as input and the velocity $u(t)$ as output, we see from Eq. (2.53) that the transfer function is $H(\omega) = 1/Z$, where $Z = Z(\omega)$ is the mechanical impedance, defined by Eq. (2.51).

2.6 Fourier Transform and Other Integral Transforms

We have previously studied forced sinusoidal oscillations (Section 2.1) and we have introduced complex amplitudes and phasors as convenient means to analyse sinusoidal oscillations (Section 2.2). In cases in which linear theory is applicable, the obtained results may also be useful with quantities which do not necessarily vary sinusoidally with time. This follows from the following two facts. Firstly, functions of a rather general class may be decomposed into harmonic components according to Fourier analysis. Secondly, the superposition principle is applicable when linear theory is valid. A condition for the success of this method of approach is that the physical systems considered are time invariant. This implies that the inherent characteristics of the system remain the same at any time (see Section 2.5).

Instead of studying how the complex amplitude of a physical quantity varies with frequency, we shall now study the physical quantity's variation with time. The main content of this section includes a short review of Fourier analysis and of the connection between causality and the Kramers–Kronig relations.

2.6.1 Fourier Transformation in Brief

It is assumed that the reader is familiar with Fourier analysis. For convenience, however, let us collect some of the main formulas related to the Fourier transformation. For a more rigorous treatment, the reader may consult many textbooks, such as those by Papoulis [17] or Bracewell [18].

If the function $f(t)$ belongs to a certain class of reasonably well-behaved functions, its Fourier transform $F(\omega)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \equiv \mathcal{F}\{f(t)\}, \tag{2.135}$$

and the inverse transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \equiv \mathcal{F}^{-1}\{F(\omega)\}. \tag{2.136}$$

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Table 2.2: Some relations between functions of time and their corresponding Fourier transforms.

Function of Time	**Fourier Transform**
$f(t)$	$F(\omega)$
$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
$F(t)$	$2\pi f(-\omega)$
$f(at)$ (a real)	$(1/\lvert a\rvert)F(\omega/a)$
$f(t - t_0)$	$F(\omega)\exp(-it_0\omega)$
$f(t)\exp(i\omega_0 t)$	$F(\omega - \omega_0)$
$df(t)/dt = \dot{f}(t)$	$i\omega F(\omega)$
$\int_{-\infty}^{t} f(\tau)\, d\tau$	$F(\omega)/i\omega + \pi F(0)\delta(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\hat{u}\exp(i\omega_0 t) + \hat{u}^* \exp(-i\omega_0 t)$	$\hat{u}2\pi\delta(\omega - \omega_0) + \hat{u}^* 2\pi\delta(\omega + \omega_0)$
$\mathrm{sgn}(t)$	$2/i\omega$
$U(t) = [\mathrm{sgn}(t) + 1]/2$	$1/i\omega + \pi\delta(\omega)$
$d\delta(t)/dt = \dot{\delta}(t)$	$i\omega$

Note that these integrals, as well as other Fourier integrals in this book, must be interpreted as Cauchy principal value integrals when necessary (Papoulis [17], p. 10).

Transformations (2.135) and (2.136) are linear. Other theorems concerning symmetry, time scaling, time shifting, frequency shifting, time differentiation and time integration are summarised in Table 2.2. The table also includes some transformation pairs relating to the delta function and its derivative (both of which belong to the class of generalised functions, also termed 'distributions'), the unit step function (Heaviside function) and the signum function.

If $f(t)$ is a real function, then

$$F^*(-\omega) = F(\omega). \tag{2.137}$$

The real and imaginary parts of the Fourier transform are even and odd functions, respectively:

$$R(\omega) = \mathrm{Re}\{F(\omega)\} = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt = R(-\omega), \tag{2.138}$$

$$X(\omega) = \mathrm{Im}\{F(\omega)\} = -\int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt = -X(-\omega). \tag{2.139}$$

Moreover, if the real $f(t)$ is an even function, $f(t) = f_e(t)$, say—that is, $f_e(-t) = f_e(t)$—its Fourier transform is real and even,

$$F(\omega) = R(\omega) = 2\int_{0}^{\infty} f_e(t)\cos(\omega t)\, dt, \tag{2.140}$$

and the inverse transform may be written as

$$f_e(t) = \frac{1}{\pi} \int_0^\infty R(\omega) \cos(\omega t)\, d\omega. \tag{2.141}$$

In contrast, if the real $f(t)$ is an odd function, $f(t) = f_o(t)$, say—that is, $f_o(-t) = -f_o(t)$—its Fourier transform is purely imaginary

$$F(\omega) = iX(\omega) = -2i \int_0^\infty f_o(t) \sin(\omega t)\, dt, \tag{2.142}$$

and its inverse transform may be written as

$$f_o(t) = -\frac{1}{\pi} \int_0^\infty X(\omega) \sin(\omega t)\, d\omega. \tag{2.143}$$

Frequently used in connection with Fourier analysis is the convolution theorem. The function

$$f(t) = \int_{-\infty}^\infty f_1(t-\tau) f_2(\tau)\, d\tau \equiv f_1(t) * f_2(t) \tag{2.144}$$

is termed the convolution (convolution product) of the two functions $f_1(t)$ and $f_2(t)$, as we have seen in Eq. (2.126). A very useful relation is given by the convolution theorem, which states that the Fourier transform of the convolution is

$$\mathcal{F}\{f_1(t) * f_2(t)\} = F_1(\omega)\, F_2(\omega) = \mathcal{F}\{f_1(t)\}\, \mathcal{F}\{f_2(t)\}. \tag{2.145}$$

Thus convolution in the time domain corresponds to ordinary multiplication in the frequency domain. It is implied that the convolution is commutative—that is, $f_1(t) * f_2(t) = f_2(t) * f_1(t)$. If we multiply Eq. (2.145) by $i\omega$, it is the Fourier transform of

$$f_1(t) * \frac{df_2(t)}{dt} = f_2(t) * \frac{df_1(t)}{dt}. \tag{2.146}$$

By using the symmetry theorem

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega) \tag{2.147}$$

(see Table 2.2), we can show that

$$\mathcal{F}\{f_1(t) f_2(t)\} = \frac{1}{2\pi} \int_{-\infty}^\infty F_1(\omega - y)\, F_2(y)\, dy = \frac{1}{2\pi} F_1(\omega) * F_2(\omega). \tag{2.148}$$

This is the frequency convolution theorem.

Let us next consider the relationship between Fourier integrals and Fourier series, as follows. Let $f_1(t) = 0$ for $|t| > T/2$. Then from Eq. (2.135), its Fourier transform is

$$F_1(\omega) = \int_{-T/2}^{T/2} f_1(t)\, e^{-i\omega t}\, dt. \tag{2.149}$$

Now we define a periodic function

$$f_T(t) = f_T(t+T) \tag{2.150}$$

for all t $(-\infty < t < \infty)$, where

$$f_T(t) = f_1(t), \quad -T/2 < t < T/2. \tag{2.151}$$

We may write the periodic function as a Fourier series,

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_0 t), \tag{2.152}$$

where

$$\omega_0 = 2\pi/T \tag{2.153}$$

and

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\ \exp(-in\omega_0 t)\, dt = \frac{1}{T} F_1(n\omega_0). \tag{2.154}$$

If f_T is real, then

$$c_{-n} = c_n^*, \tag{2.155}$$

which corresponds to Eq. (2.137). The Fourier transform of $f_T(t)$ is

$$\begin{aligned} F_T(\omega) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n \exp\left[i(n\omega_0 - \omega)t\right] dt \\ &= 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0) \\ &= \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} F_1(n\omega_0)\, \delta(\omega - n\omega_0). \end{aligned} \tag{2.156}$$

Thus, it follows that the Fourier transform of a periodic function is discontinuous and is made up of impulses weighted by the Fourier transform of the function $f_1(t)$ for $\omega = n\omega_0$, $(n = 1, 2, \dots)$. Note that $|F_1(n\omega_0)|/T$ may be interpreted as the amplitude of Fourier component number n of $f_T(t)$. See Eq. (2.154).

2.6.2 Time-Invariant Linear System

Fourier analysis may be applied to the study of linear systems. Let the input signal to a linear system be $u(t)$ and the corresponding output signal (response) be $y(t)$. We shall assume that the system is linear and time invariant; that is, it has the same characteristics now as in the past and in the future. We shall mostly consider systems which are causal, which means that there is no output response before a causing signal has been applied to the input. However, it is sometimes of practical interest to consider noncausal systems as well (see Sections 4.9 and 5.3.2).

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The Fourier transform of the impulse response $h(t)$ [see Eq. (2.126)] is

$$H(\omega) = R(\omega) + iX(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t} dt, \tag{2.157}$$

which we shall call the transfer function of the system. [If the system is not causal, that is, if the condition (2.129) is not satisfied, some authors use the term ‘frequency-response function’, instead of ‘transfer function’.] The real and imaginary parts of the transfer function are denoted by $R(\omega)$ and $X(\omega)$, respectively.

The Fourier transform of the output response $y(t)$, as given by the convolution product (2.126), is

$$Y(\omega) = H(\omega)\, U(\omega), \tag{2.158}$$

according to the convolution theorem (2.145), where $U(\omega)$ is the Fourier transform of the input signal $u(t)$. A system (block diagram) interpretation of Eq. (2.158) is given by the right-hand part of Figure 2.13.

For the simple case in which the transfer function is independent of frequency,

$$H(\omega) = H_0, \tag{2.159}$$

where H_0 is constant, the impulse response is

$$h(t) = H_0 \delta(t) \tag{2.160}$$

(see Table 2.2). For an arbitrary input $u(t)$, the response then becomes

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} H_0 \delta(t-\tau)u(\tau)d\tau = H_0 u(t), \tag{2.161}$$

which—apart for the constant H_0—is a distortion-free reproduction of the input signal.

As another example, let us consider a harmonic input function

$$u(t) = \tfrac{1}{2}\hat{u}e^{i\omega_0 t} + \tfrac{1}{2}\hat{u}^* e^{-i\omega_0 t}, \tag{2.162}$$

where $\hat{u}$ is the complex amplitude and $\hat{u}^*$ its conjugate. The Fourier transform is

$$U(\omega) = \pi\hat{u}\delta(\omega - \omega_0) + \pi\hat{u}^*\delta(\omega + \omega_0) \tag{2.163}$$

(see Table 2.2). The response is given by

$$\begin{aligned} Y(\omega) &= H(\omega)\, U(\omega) \\ &= \pi\hat{u}\delta(\omega - \omega_0)\, H(\omega) + \pi\hat{u}^*\delta(\omega + \omega_0)\, H(\omega) \\ &= \pi\hat{u}\delta(\omega - \omega_0)\, H(\omega_0) + \pi\hat{u}^*\delta(\omega + \omega_0)\, H(-\omega_0). \end{aligned} \tag{2.164}$$

Now, since $h(t)$ is real, we have $H(-\omega_0) = H^*(\omega_0)$ in accordance with Eq. (2.137). Hence, we may write

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$$y(t) = \tfrac{1}{2}\hat{y}e^{i\omega_0 t} + \tfrac{1}{2}\hat{y}^* e^{-i\omega_0 t}, \tag{2.165}$$

where

$$\hat{y} = H(\omega_0)\hat{u}, \quad \hat{y}^* = H^*(\omega_0)\hat{u}^*, \tag{2.166}$$

in agreement with Eq. (2.134). It follows that the complex amplitude of the response equals the product of the transfer function and the complex amplitude of the input signal. According to Eq. (2.166), we may consider $\hat{u}$ as input to a linear system in the frequency domain, where H is the transfer function and $\hat{y}$ the output response.

Assuming that the impulse response $h(t)$ is a real function of time, we next wish to express the impulse response in terms of the real and imaginary parts of the transfer function:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [R(\omega)\cos(\omega t) - X(\omega)\sin(\omega t)]d\omega \\ &\quad + \frac{i}{2\pi}\int_{-\infty}^{\infty} [R(\omega)\sin(\omega t) + X(\omega)\cos(\omega t)]d\omega. \end{aligned} \tag{2.167}$$

If $h(t)$ is real, Eqs. (2.137)–(2.139) hold. Thus, because $R(\omega)$ and $\cos(\omega t)$ are even functions of ω, whereas $X(\omega)$ and $\sin(\omega t)$ are odd functions of ω, the imaginary part in Eq. (2.167) vanishes. Moreover, we can rewrite the (real) impulse response as

$$h(t) = h_e(t) + h_o(t), \tag{2.168}$$

where

$$h_e(t) = \frac{1}{\pi}\int_0^{\infty} R(\omega)\cos(\omega t)\,d\omega, \tag{2.169}$$

$$h_o(t) = -\frac{1}{\pi}\int_0^{\infty} X(\omega)\sin(\omega t)\,d\omega. \tag{2.170}$$

Notice that $h_e(t)$ and $h_o(t)$ are even and odd functions of t, respectively. Thus, according to Eq. (2.168), the real impulse response $h(t)$ is split into even and odd parts. The corresponding transforms are even and odd functions of ω, respectively:

$$\mathcal{F}\{h_e(t)\} = R(\omega), \quad \mathcal{F}\{h_o(t)\} = iX(\omega). \tag{2.171}$$

2.6.3 Kramers–Kronig Relations and Hilbert Transform

The function $h(t)$ is the response upon application of an input impulse at $t = 0$. Hence, for a causal system, Eq. (2.129) holds because there can be no response before an input signal has been applied. Then Eq. (2.126) and the commutativity of the convolution product give

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$$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)\,d\tau = \int_{0}^{\infty} h(\tau)u(t-\tau)\,d\tau. \tag{2.172}$$

Further, if also the input is causal (i.e., if $u(t) = 0$ for $t < 0$), then

$$y(t) = h(t) * u(t) = \int_{0}^{t} h(t-\tau)u(\tau)\,d\tau = \int_{0}^{t} h(\tau)u(t-\tau)\,d\tau. \tag{2.173}$$

This is the version of the convolution theorem used in the theory of the (one-sided) Laplace transform, which may be applied to causal systems assumed to be dead (quiescent) previous to an initial instant, $t = 0$. Using the Fourier transform when $h(t) = 0$ for $t < 0$ and inserting s for $i\omega$, we obtain the following relation between the Laplace transform H_L and the transfer function H:

$$H_L(s) \equiv \int_{0}^{\infty} h(t)e^{-st}\,dt = H(s/i) = H(-is). \tag{2.174}$$

Using the convolution theorem (2.145), we obtain the Laplace transform of Eq. (2.173) as

$$Y_L(s) = H_L(s)\,U_L(s), \tag{2.175}$$

where $U_L(s)$ is the Laplace transform of $u(t)$, which is assumed to be vanishing for $t < 0$.

For a constant transfer function $H(\omega) = H_0$, the impulse response is given by Eq. (2.160) as $H_0\delta(t)$. Note that this is an example of a causal impulse response. In this case, the response to a causal input $u(t)$ is $y(t) = H_0\,u(t)$, which means that $y(t) = 0$ for $t < 0$ if $u(t) = 0$ for $t < 0$. This example of a causal impulse response, $H_0\delta(t)$, is an even function. Another example of a causal impulse response is $h(t) = H_{00}\dot{\delta}(t)$, for which the transfer function is $H(\omega) = i\omega H_{00}$ (cf. Table 2.2). This impulse response is an odd function.

In general, however, any causal impulse response $h(t)$ which 'has a memory of the past' is neither an even nor an odd function, because $h(t)$ vanishes for all negative t but not for all positive t. Next, let us make some observations for this more general situation. We shall first exclude cases for which $H(\infty) \neq 0$, such as in the preceding example, where $H(\infty) = H_0$.

For a causal function $h(t)$ decomposed into even and odd parts according to Eq. (2.168), we have

$$h(t) = \begin{cases} 2h_e(t) = 2h_o(t) & \text{for } t > 0 \\ h_e(0) & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{2.176}$$

and, in general (for $t \neq 0$),

$$h_o(t) = h_e(t)\,\mathrm{sgn}(t), \quad h_e(t) = h_o(t)\,\mathrm{sgn}(t), \tag{2.177}$$

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because the condition $h(t) = 0$ for $t < 0$ must be satisfied. Using Eqs. (2.169), (2.170), and (2.176) gives, for $t > 0$,

$$h(t) = 2h_e(t) = \frac{2}{\pi}\int_0^\infty R(\omega)\cos(\omega t)\,d\omega, \tag{2.178}$$

$$h(t) = 2h_o(t) = -\frac{2}{\pi}\int_0^\infty X(\omega)\sin(\omega t)\,d\omega. \tag{2.179}$$

It should be emphasised that these expressions apply for $t > 0$ only, whereas for $t < 0$, we have $h(t) = 0$. Note that the two alternative expressions (2.178) and (2.179) imply a relationship between $R(\omega)$ and $X(\omega)$, the real and imaginary parts of the Fourier transform of a causal, real function.

Using

$$\mathcal{F}\{\mathrm{sgn}(t)\} = 2/i\omega \tag{2.180}$$

(see Table 2.2) and the frequency convolution theorem (2.148), we obtain the Fourier transforms of Eqs. (2.177):

$$\begin{aligned}\mathcal{F}\{h_e(t)\} &= \mathcal{F}\{h_o(t)\mathrm{sgn}(t)\} = R(\omega)\\ &= \frac{1}{2\pi}iX(\omega) * \frac{2}{i\omega} = \frac{1}{\pi\omega} * X(\omega),\end{aligned} \tag{2.181}$$

$$\begin{aligned}\mathcal{F}\{h_o(t)\} &= \mathcal{F}\{h_e(t)\,\mathrm{sgn}(t)\} = iX(\omega)\\ &= \frac{1}{2\pi}R(\omega) * \frac{2}{i\omega} = -\frac{i}{\pi\omega} * R(\omega).\end{aligned} \tag{2.182}$$

Writing convolutions explicitly in terms of integrals, we have

$$R(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{X(y)}{\omega - y}\,dy, \tag{2.183}$$

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{\omega - y}\,dy. \tag{2.184}$$

Note that the integrand is singular for $y = \omega$ and that the integrals should be understood as principal value integrals (cf. footnote on p. 10 in Papoulis [17]). Relations (2.183) and (2.184) are called Kramers–Kronig relations [19, 20]. If for a causal function the real/imaginary part of the Fourier transform is known for all frequencies, then the remaining imaginary/real part of the transform is given by a principal-value integral. For the integrals to exist, it is necessary that

$$R(\omega) + iX(\omega) = H(\omega) \to 0 \quad \text{when } \omega \to \infty. \tag{2.185}$$

If this is not true, we may conveniently subtract the singular part of $H(\omega)$. Let us consider the case in which $h(t)$ has an impulse singularity. Note (from Table 2.2) that

$$\mathcal{F}\{\delta(t)\} = 1, \quad \mathcal{F}\{\delta(t - t_0)\} = \exp(-i\omega t_0). \tag{2.186}$$

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These Fourier transforms do not satisfy condition (2.185). If $H(\infty) \neq 0$, we define a modified transfer function

$$H'(\omega) = H(\omega) - H(\infty) \tag{2.187}$$

and a new corresponding causal impulse response function

$$h'(t) = h(t) - H(\infty)\,\delta(t). \tag{2.188}$$

For instance, if

$$H(\infty) = R(\infty) \neq 0, \quad X(\infty) = 0, \tag{2.189}$$

we have

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(y) - R(\infty)}{\omega - y}\,dy, \tag{2.190}$$

$$R(\omega) - R(\infty) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(y)}{\omega - y}\,dy. \tag{2.191}$$

Multiplying the former of the two equations by i and summing yield the Hilbert transform

$$H'(\omega) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{H'(y)}{\omega - y}\,dy. \tag{2.192}$$

An alternative formulation of the Kramers–Kronig relations is obtained by noting that because $h(t)$ is real, we have

$$R(-\omega) = R(\omega), \quad X(-\omega) = -X(\omega) \tag{2.193}$$

[cf. Eqs. (2.138)–(2.139)]. Then

$$-\pi X(\omega) = \int_{-\infty}^{\infty} \frac{R(y)}{\omega - y}\,dy = \int_{-\infty}^{0} \frac{R(z)}{\omega - z}\,dz + \int_{0}^{\infty} \frac{R(y)}{\omega - y}\,dy. \tag{2.194}$$

Because

$$\int_{-\infty}^{0} \frac{R(z)}{\omega - z}\,dz = \int_{0}^{\infty} \frac{R(-y)}{\omega + y}\,dy = \int_{0}^{\infty} \frac{R(y)}{\omega + y}\,dy \tag{2.195}$$

and

$$\frac{1}{\omega + y} + \frac{1}{\omega - y} = \frac{2\omega}{\omega^2 - y^2}, \tag{2.196}$$

we obtain

$$X(\omega) = -\frac{2\omega}{\pi}\int_{0}^{\infty} \frac{R(y)}{\omega^2 - y^2}\,dy. \tag{2.197}$$

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Similarly, we find

$$\begin{aligned}\pi R(\omega) &= \int_{-\infty}^{\infty} \frac{X(y)}{\omega - y}\,dy = \int_0^{\infty} \frac{X(-y)}{\omega + y}\,dy + \int_0^{\infty} \frac{X(y)}{\omega - y}\,dy \\ &= \int_0^{\infty} X(y)\left(\frac{1}{\omega - y} - \frac{1}{\omega + y}\right) dy,\end{aligned} \tag{2.198}$$

which gives

$$R(\omega) = \frac{2}{\pi}\int_0^{\infty} \frac{yX(y)}{\omega^2 - y^2}\,dy. \tag{2.199}$$

2.6.4 An Energy Relation for Non-sinusoidal Oscillation

In Section 2.3, we derived the formula (2.76) for the time-average power or rate of work associated with sinusoidal oscillations in a mechanical system, namely

$$P = \tfrac{1}{4}\hat{f}\hat{u}^* + \tfrac{1}{4}\hat{f}^*\hat{u} \tag{2.200}$$

(note that we here denote force by the variable f instead of F, as we did previously). For a more general oscillation which is not periodic in time, it is more convenient to consider the total work done than the time-average of the rate of work. The instantaneous rate of work is $P(t) = f(t)u(t)$—that is, the product of the instantaneous values of force f and velocity u. Hence, the total work done is

$$W = \int_{-\infty}^{\infty} f(t)u(t)\,dt. \tag{2.201}$$

By applying Parseval's theorem, or the frequency convolution theorem (2.148) with $\omega = 0$, we find that Eq. (2.201) gives

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)U(-\omega)\,d\omega, \tag{2.202}$$

where $F(\omega)$ and $U(\omega)$ are the Fourier transforms of the force $f(t)$ and the velocity $u(t)$, respectively. We shall assume that $F(\omega)$ and $U(\omega)$ are related through the transfer function

$$Y(\omega) = 1/Z(\omega) = U(\omega)/F(\omega), \tag{2.203}$$

where

$$Z(\omega) = R(\omega) + iX(\omega) \tag{2.204}$$

is a mechanical impedance. Its real and imaginary parts are the mechanical resistance $R(\omega)$ and the mechanical reactance $X(\omega)$, respectively. It may, for instance, be given by Eq. (2.51).

If $f(t)$ and $u(t)$ are real functions, then Eq. (2.137) is applicable to $F(\omega)$ and $U(\omega)$. Further, using also Eqs. (2.203) and (2.204), Eq. (2.202) becomes

$$\begin{aligned} W &= \frac{1}{2\pi}\int_0^\infty (F(\omega)U^*(\omega) + F^*(\omega)U(\omega))\,d\omega \\ &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\{F(\omega)U^*(\omega)\}\,d\omega \\ &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\{Z(\omega)\,U(\omega)U^*(\omega)\}\,d\omega \\ &= \frac{1}{\pi}\int_0^\infty R(\omega)|U(\omega)|^2\,d\omega. \end{aligned} \tag{2.205}$$

It may be interesting (see Problem 2.14) to compare this result with Eqs. (2.76) and (2.77), where the time-average mechanical power is expressed in terms of complex amplitudes.

Problems

Problem 2.1: Free Oscillation

Show that the equation $m\ddot{x} + R\dot{x} + Sx = 0$ has the general solution

$$x = (C_1 \cos\omega_d t + C_2 \sin\omega_d t)\, e^{-\delta t}.$$

Further, show that the integration constants C_1 and C_2, as given by

$$C_1 = x_0, \quad C_2 = (u_0 + x_0\delta)/\omega_d,$$

satisfy the initial conditions $x(0) = x_0$ and $\dot{x}(0) = u_0$.

Problem 2.2: Quality Factor or Q Value for Resonator

Using definition (2.8), derive the equations

$$\begin{aligned} Q &= \frac{\omega_0}{2\delta}\left(1 + \frac{\delta}{\omega_0} - \frac{1}{6}\frac{\delta^2}{\omega_0^2} + \mathcal{O}\left\{\frac{\delta^3}{\omega_0^3}\right\}\right) \\ &\approx \frac{\omega_0}{2\delta} = \frac{\omega_0 m}{R} = \frac{S}{\omega_0 R} = \frac{(Sm)^{1/2}}{R} \end{aligned}$$

and

$$\frac{\delta}{\omega_0} = \frac{1}{2Q}\left(1 + \frac{1}{2Q} + \frac{5}{24Q^2} + \mathcal{O}\left\{Q^{-3}\right\}\right) \approx \frac{1}{2Q}.$$

[Hint: use Taylor series for an exponential function and for a binomial, and/or use the method of successive approximations.]

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Problem 2.3: Forced Oscillation

Show that the equation

$$m\ddot{x} + R\dot{x} + Sx = F(t) = F_0 \cos(\omega t + \varphi_F)$$

has a particular solution

$$x(t) = x_0 \cos(\omega t + \varphi_x),$$

where the phase difference $\varphi = \varphi_F - \varphi_u = \varphi_F - \varphi_x - \pi/2$ is an angle in quadrant 1 or 4, and satisfies

$$\tan\varphi = (\omega m - S/\omega)/R.$$

Also show that

$$x_0 = \frac{u_0}{\omega} = \frac{F_0}{|Z|\omega},$$

where $|Z| = \sqrt{R^2 + (\omega m - S/\omega)^2}$.

Problem 2.4: Excursion Response Maximum

Using the results of Problem 2.3, discuss the nondimensionalised excursion ratio $|\xi| \equiv Sx_0/F_0$ versus the frequency ratio $\gamma \equiv \omega/\omega_0 = \omega\sqrt{m/S}$. Find the frequency at which $|\xi|$ is maximum. Determine also the inequality which $\delta = R/2m$ has to satisfy so that $1 < |\xi|_{\max} < \sqrt{2}$. Observe that $|\xi|_{\max} > |\xi(\gamma = 1)|$ and that $|\xi|_{\max} \to |\xi(\gamma = 1)|$, as $\delta/\omega_0 \to 0$.

Problem 2.5: Amplitude and Phase Constant

Determine the numerical values of the amplitude A and the phase constant φ for the oscillation

$$x = A\cos(62.8\,t + \varphi)$$

when the initial conditions at $t = 0$ are as follows: position $x(0) = 50$ mm, velocity $\dot{x}(0) = 0.8$ m/s. Determine the acceleration at $t = 0$. Draw a phasor diagram for the complex amplitudes of position (in scale 1:1), velocity (1 m/s $\widehat{=}$ 20 mm) and acceleration (1 m/s^2 $\widehat{=}$ 0.5 mm).

Problem 2.6: Complex Representations of Harmonic Oscillation

The following harmonic oscillation is given:

$$u(t) = 100\sin(\omega t) - 50\cos(\omega t).$$

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Rewrite $u(t)$ as

(a) a cosine function with phase constant,
(b) a sine function with phase constant,
(c) the real part of a complex quantity,
(d) the imaginary part of a complex quantity, and
(e) the sum of two complex conjugate quantities.

Problem 2.7: Superposed Oscillations of the Same Frequency

An oscillation is given as a superposition of individual oscillations:

$$\begin{aligned} x(t) = {} & 3\cos(\omega t + \pi/6) + 6\sin(\omega t + 3\pi/2) \\ & - 3.5\sin(\omega t + \pi/3) + 2.2\sin(\omega t - \pi/9). \end{aligned}$$

(a) Numerically determine the amplitude and the phase constant of the resultant oscillation.
(b) Rewrite $x(t)$ in complex form.
(c) Draw phasors for the four individual oscillations in the same diagram, and construct the resultant phasor. Compare it with the preceding numerical computation.

Problem 2.8: Resonance Bandwidth

According to Eq. (2.53), the inverse of the mechanical impedance Z may be interpreted as the transfer function of a system in which applied force $\hat{F}$ is the input and velocity $\hat{u}$ the output. The maximum modulus $|Y|_{\max}$ of this transfer function $Y = 1/Z$ is $1/R$, corresponding to the resonance frequency $\omega_0 = \sqrt{S/m}$. Derive an expression for the upper and lower frequencies, ω_u and ω_l, respectively, at which $|Y|/|Y|_{\max} = R|Y| = 1/\sqrt{2}$, in terms of ω_0 and $\delta = R/2m$. Also determine the resonance bandwidth $(\Delta\omega)_{\text{res}} = \omega_u - \omega_l$. Compare the relative bandwidth $(\omega_u - \omega_l)/\omega_0$ with the inverse of the quality factor Q defined in Section 2.1.1.

Problem 2.9: Solving System of Linear Differential Equations

Show that the solution

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

satisfies the vectorial differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$.

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Problem 2.10: State-Space Description of Oscillation

Let matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ be given by Eqs. (2.110)–(2.111). Determine the eigenvalues Λ_1 and Λ_2 in terms of the coefficients δ and ω_d given by Eq. (2.5). Assume that $\Lambda_2 \neq \Lambda_1$ and that $\Lambda_1 - \Lambda_2$ either has a positive imaginary part, or otherwise that $\Lambda_1 - \Lambda_2$ is real and positive. Show that $\mathbf{A}$ has the linearly independent eigenvectors $\mathbf{m}_1$ and $\mathbf{m}_2$ with transposed vectors $\mathbf{m}_i^{\mathrm{T}} = c_i \begin{bmatrix} 1 & \Lambda_i \end{bmatrix}$ $(i = 1, 2)$, where c_1 and c_2 are arbitrary constants. Determine the inverse of matrix $\mathbf{M} = \begin{bmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{bmatrix}$, and show that $\mathbf{A}'$ defined by the similarity transformation (2.102) is a diagonal matrix. Next perform the transformations (2.106) and (2.107); find $\mathbf{x}'$, $\mathbf{B}'$ and $\mathbf{C}'$; and use Eq. (2.99) to obtain the solution of Eqs. (2.108) and (2.109). In the final answer, set $t_0 = 0$, and replace y_1, x_1, x_2 and u_1 with x, x, u and F, respectively. Finally, write down the solution for free oscillations—i.e., for $F(t) = 0$. Compare the result with Eqs. (2.4) and (2.6).

Problem 2.11: Critically Damped Oscillation

Discuss the solution of Problem 2.10 for the case when $\Lambda_1 = \Lambda_2 = \Lambda$. In this case, $\mathbf{A}$ does not have two linearly independent eigenvectors, and the transformed matrix $\mathbf{A}'$ is not diagonal. For the similarity transformation, choose the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \Lambda & 1 \end{bmatrix},$$

and perform the various tasks as in Problem 2.10. A special challenge is to determine matrix exponential $e^{\mathbf{A}'t}$ by using definition (2.98).

Problem 2.12: Mechanical Impedance and Power

A mass of $m = 6$ kg is suspended by a spring of stiffness $S = 100$ N/m. The oscillating system has a mechanical resistance of $R = 3$ Ns/m. The system is excited by an alternating force $F(t) = |\hat{F}| \cos(\omega t)$, where the force amplitude is $|\hat{F}| = 1$ N and the frequency $f = \omega/2\pi = 1$ Hz.

(a) Determine the mechanical impedance $Z = R + iX = |Z|e^{i\varphi}$. State numerical values for R, X, $|Z|$ and φ.

(b) Determine the complex amplitudes $\hat{F}$ for the force, $\hat{s}$ for the position, $\hat{u}$ for the velocity and $\hat{a}$ for the acceleration. Draw the complex amplitudes as vectors in a phasor diagram.

(c) Find the frequency $f_0 = \omega_0/2\pi$ for which the mechanical reactance vanishes.

(d) Find the time-averaged mechanical power P which the force $|\hat{F}| = 1$ N supplies to the system, at the two frequencies f and f_0.

Problem 2.13: Convolution with Sinusoidal Oscillation

Show that if $f(t)$ varies sinusoidally with an angular frequency ω_0, then the convolution product $g(t) = h(t) * f(t)$ may be written as a linear combination of $f(t)$ and $\dot{f}(t)$. Assuming that the transfer function $H(\omega)$ is known, determine the coefficients of the linear combination.

Problem 2.14: Work in Terms of Complex Amplitude or Fourier Integral

Consider the particular case of harmonic oscillations. Discuss the mathematical relationship between Eq. (2.205) for the total work and the equation

$$P = \frac{1}{4}(\hat{F}\hat{u}^* + \hat{F}^*\hat{u}) = \frac{1}{2}\mathrm{Re}\{\hat{F}\hat{u}^*\}$$

for the power of a mechanical system performing harmonical oscillation [cf. Eq. (2.76)]. Consider also the dimensions (or SI units) of the various physical quantities.

Problem 2.15: Mechanical Impedance at Zero Frequency

When we, by Eq. (2.51), defined the mechanical impedance

$$Z(\omega) = R + i(\omega m - S/\omega),$$

we tacitly assumed that $\omega \neq 0$. We define the transfer functions $Y(\omega) = 1/Z(\omega)$, $G(\omega) = i\omega Z(\omega)$, and $H(\omega) = 1/G(\omega) = Y(\omega)/i\omega$. Show that the corresponding impulse response functions $y(t)$, $g(t)$ and $h(t)$ are causal. [Hint: for transfer functions with poles, apply the method of contour integration when considering the inverse Fourier transform.] If the variables concerned are $s(t)$, $u(t)$ and $F(t)$, choose input and output variables for the four linear systems concerned. Further, show that in order to make $z(t) = \mathcal{F}^{-1}\{Z(\omega)\}$ a causal impulse response function, it is necessary to add a term to the impedance such that

$$Z(\omega) = R + i\omega m + S[1/i\omega + \pi\delta(\omega)].$$

Explain the physical significance of the last term.

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