

## **CHAPTER FOUR**

# **Gravity Waves on Water**

The subject of this chapter is the study of waves on an ideal fluid, namely a fluid which is incompressible and in which wave motion takes place without loss of mechanical energy. It is also assumed that the fluid motion is irrotational and that the wave amplitude is so small that linear theory is applicable. Starting from basic hydrodynamics, we shall derive the dispersion relationship for waves on water which is deep or otherwise has a constant depth. Plane and circular waves are discussed, and the transport of energy and momentum associated with wave propagation is considered. The final parts of the chapter introduce some concepts and derive some mathematical relations, which turn out to be very useful when, in subsequent chapters, interactions between waves and oscillating systems are discussed.

### **4.1 Basic Equations: Linearisation**

Let us start with two basic hydrodynamic equations which express conservation of mass and momentum, namely the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad (4.1)$$

and the Navier–Stokes equation

$$\frac{D\vec{v}}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p_{\text{tot}} + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{f}. \quad (4.2)$$

Here  $\rho$  is the mass density of the fluid,  $\vec{v}$  is the velocity of the flowing fluid element,  $p_{\text{tot}}$  is the pressure of the fluid and  $\nu = \eta/\rho$  is the kinematic viscosity coefficient, which we shall neglect by assuming the fluid to be ideal. Hence, we set  $\nu = 0$ . Finally,  $\vec{f}$  is external force per unit volume. Here we consider only gravitational force—that is,

$$\vec{f} = \rho \vec{g}, \quad (4.3)$$

where  $\vec{g}$  is the acceleration due to gravity. With the introduced assumptions, Eq. (4.2) becomes

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p_{\text{tot}} + \vec{g}. \quad (4.4)$$

For an incompressible fluid,  $\rho$  is constant, and Eq. (4.1) gives

$$\nabla \cdot \vec{v} = 0. \quad (4.5)$$

Further, we assume that the fluid is irrotational, which mathematically means that

$$\nabla \times \vec{v} = 0. \quad (4.6)$$

Because of the vector identity  $\nabla \times \nabla \phi \equiv 0$  for any scalar function  $\phi$ , we can then write

$$\vec{v} = \nabla \phi, \quad (4.7)$$

where  $\phi$  is the so-called *velocity potential*. Inserting into Eq. (4.4) and using the vector identity

$$\vec{v} \times (\nabla \times \vec{v}) \equiv \frac{1}{2} \nabla v^2 - (\vec{v} \cdot \nabla) \vec{v} \quad (4.8)$$

and the relation  $\vec{g} = -\nabla(gz)$  (the  $z$ -axis is pointing upwards) give

$$\nabla \left( \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \frac{p_{\text{tot}}}{\rho} + gz \right) = 0. \quad (4.9)$$

Integration gives

$$\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \frac{p_{\text{tot}}}{\rho} + gz = C, \quad (4.10)$$

where  $C$  is an integration constant or some function of time. This is (a non-stationary version of) the so-called Bernoulli equation.

Because  $v^2 = \nabla \phi \cdot \nabla \phi$  according to Eq. (4.7), the scalar equation (4.10) for the scalar quantity  $\phi$  replaces the vectorial equation (4.4) for the vectorial quantity  $\vec{v}$ . This is a mathematical convenience which is a benefit resulting from the assumption of irrotational flow.

For the static case, when the fluid is not in motion,  $\vec{v} = 0$  and  $\phi = \text{constant}$ . Then Eq. (4.10) gives

$$p_{\text{tot}} = p_{\text{stat}} = -\rho g z + \rho C. \quad (4.11)$$

At the free surface,  $z = 0$ , we have  $p_{\text{tot}} = p_{\text{atm}}$ , where  $p_{\text{atm}}$  is the atmospheric air pressure. Note that we here neglect surface tension on the air–fluid interface. This gives  $C = p_{\text{atm}}/\rho$  and

$$p_{\text{stat}} = -\rho g z + p_{\text{atm}}. \quad (4.12)$$

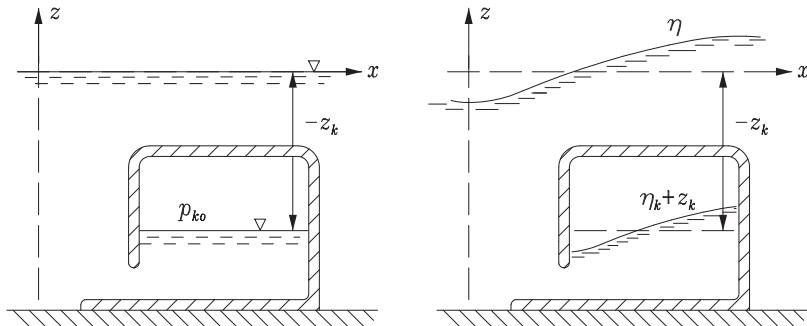


Figure 4.1: Submerged chamber for an OWC with an equilibrium water level below the mean sea level and containing entrapped air with elevated static pressure  $p_{k0}$ .

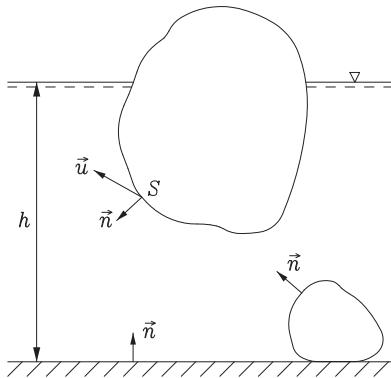


Figure 4.2: Unit normals  $\vec{n}$  on wet body surfaces  $S$  (interfaces between water and rigid bodies) directed into the fluid domain. Body velocity  $\vec{u}$  may be a function of the considered point of  $S$ .

The hydrostatic pressure increases linearly with the vertical displacement below the surface of the fluid.

The static air pressure in a submerged air chamber (Figure 4.1) is

$$p_{k0} = \rho g(-z_k) + p_{\text{atm}}, \quad (4.13)$$

where the water surface below the entrapped air is at depth  $-z_k$ .

The conditions of incompressibility (4.5) and of irrotational motion (4.7) require that Laplace's equation

$$\nabla^2 \phi = 0 \quad (4.14)$$

must be satisfied throughout the fluid. Solutions of this partial differential equation must satisfy certain boundary conditions, which are considered in the following paragraphs.

On a solid-body boundary moving with velocity  $\vec{u}$  (Figure 4.2), we have

$$u_n \equiv \vec{u} \cdot \vec{n} = v_n = \nabla \phi \cdot \vec{n} \equiv \partial \phi / \partial n, \quad (4.15)$$

as there is no fluid flow through the boundary. In an ideal fluid which is inviscid ( $\nu = 0$ ), no condition is required on the tangential component of  $\vec{v} = \nabla\phi$ . On a solid surface not in motion, we have

$$\partial\phi/\partial n = 0. \quad (4.16)$$

On a horizontal bottom of depth  $h$ , we have, in particular,

$$\partial\phi/\partial z = 0 \quad \text{at } z = -h. \quad (4.17)$$

Notice that the acceleration of gravity  $g$ , which is an important quantity for ocean waves, does not enter into Laplace's equation (4.14) or the body boundary condition (4.15). Moreover, these equations do not contain any derivative with respect to time. However, the fact that gravity waves can exist on water is associated with the presence of the quantity  $g$ , as well as of a time derivative, in the following free-surface boundary condition.

On the free surface  $z = \eta(x, y, t)$ , which is the interface between water and open air, the pressure in the fluid equals the air pressure (if we neglect capillary forces—which are, however, considered in Problem 4.1). Using  $[p_{\text{tot}}]_{z=\eta} = p_{\text{atm}}$  in the Bernoulli equation (4.10) gives

$$\left[ \frac{\partial\phi}{\partial t} + \frac{v^2}{2} \right]_{z=\eta} + g\eta = C - \frac{p_{\text{atm}}}{\rho}. \quad (4.18)$$

In the static case, the left-hand side vanishes. Hence,  $C = p_{\text{atm}}/\rho$ , and with  $\vec{v} = \nabla\phi$  [cf. Eq. (4.7)] we have the free-surface boundary condition

$$g\eta + \left[ \frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi \right]_{z=\eta} = 0. \quad (4.19)$$

In a wave-power converter of the oscillating-water-column (OWC) type with pneumatic power takeoff, the air pressure is not constant above the OWC. Let the dynamic part of the air pressure be  $p_k$ . The total air pressure is

$$p_{\text{air}} = p_{k0} + p_k = \rho g(-z_k) + p_{\text{atm}} + p_k, \quad (4.20)$$

where we have used Eq. (4.13). (Except for submerged OWCs, the equilibrium water level is at  $z_k = 0$ , which is also the case if air pressure fluctuation due to wind is considered.) Let  $\eta_k = \eta_k(x, y, t)$  denote the vertical deviation of the water surface from its equilibrium position below the entrapped air (see Figure 4.1). Thus,  $\eta = \eta_k$  when  $z_k = 0$ . From the Bernoulli equation (4.10), we now have

$$\begin{aligned} \left[ \frac{\partial\phi}{\partial t} + \frac{v^2}{2} \right]_{z=\eta_k+z_k} + g\eta_k + gz_k &= C - \left[ \frac{p_{\text{tot}}}{\rho} \right]_{z=\eta_k+z_k} \\ &= C - \frac{p_{\text{air}}}{\rho} = C - (-gz_k) - \frac{p_{\text{atm}} + p_k}{\rho}. \end{aligned} \quad (4.21)$$

Using  $C - p_{\text{atm}}/\rho = 0$  and Eq. (4.7) gives

$$g\eta_k + \frac{p_k}{\rho} + \left[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi \right]_{z=\eta_k+z_k} = 0. \quad (4.22)$$

This is the so-called *dynamic boundary condition*, which we linearise to

$$g\eta_k + \frac{p_k}{\rho} + \left[ \frac{\partial \phi}{\partial t} \right]_{z=z_k} = 0 \quad (4.23)$$

when we assume that the dynamic variables like  $\phi, \eta, \eta_k$  and all their derivatives are small, and we neglect small terms of second or higher order, such as  $v^2 = \nabla \phi \cdot \nabla \phi$ . Also the difference

$$\left[ \frac{\partial \phi}{\partial t} \right]_{z=\eta_k+z_k} - \left[ \frac{\partial \phi}{\partial t} \right]_{z=z_k} = \eta_k \left[ \frac{\partial^2 \phi}{\partial z \partial t} \right]_{z=z_k} + \dots \quad (4.24)$$

may then be neglected. [Note that the right-hand side of Eq. (4.24) is a Taylor expansion where only the first non-vanishing term is explicitly written. It is a product of two small quantities,  $\eta_k$  and a derivative of  $\phi$ .]

In addition to the dynamic boundary condition, there is also a *kinematic boundary condition* on the interface between water and air. Physically, it is the condition that a fluid particle on the interface stays on the interface as this undulates as a result of wave motion. For the linearised case, the kinematic boundary condition is simply

$$\left[ \frac{\partial \phi}{\partial z} \right]_{z=z_k} = [v_z]_{z=z_k} = \frac{\partial \eta_k}{\partial t}, \quad (4.25)$$

where  $\eta_k$  may be replaced by  $\eta$  for the open-air case. (Readers interested in learning about the kinematic boundary condition for the more general nonlinear case may consult, e.g., [1, chapter 1].) Taking the time derivative of Eq. (4.23) and inserting it into (4.25) give

$$\left[ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right]_{z=z_k} = -\frac{1}{\rho} \frac{\partial p_k}{\partial t}. \quad (4.26)$$

With zero dynamic air pressure (for  $z = 0$ ), we get, in particular,

$$\left[ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right]_{z=0} = 0. \quad (4.27)$$

Note that time  $t$  enters explicitly only into the free-surface boundary conditions. (It does not enter into the partial differential equation  $\nabla^2 \phi = 0$  and the remaining boundary conditions.) Thus, without a free surface, the solution  $\phi = \phi(x, y, z, t)$  could not represent a wave.

We may also note that the boundary condition (4.15) has to be satisfied on the wet surface of the moving body. However, if the body is oscillating with

a small amplitude, we may make the linear approximation that the boundary condition (4.15) is to be applied at the time-average (or equilibrium) position of the wet surface of the oscillating body.

Any solution  $\phi = \phi(x, y, z, t)$  for the velocity potential has to satisfy Laplace's equation (4.14) in the fluid domain and the inhomogeneous boundary conditions (4.15) and (4.26), which may sometimes or somewhere simplify to the homogeneous boundary conditions (4.16) and (4.27), respectively. For cases in which the fluid domain is of infinite extent, later sections (Sections 4.3 and 4.6) supplement the preceding boundary conditions with a 'radiation condition' at infinite distance.

From the velocity potential  $\phi(x, y, z, t)$ , which is an auxiliary mathematical function, we can derive the following physical quantities. Everywhere in the fluid domain, we can derive the fluid velocity from Eq. (4.7),

$$\vec{v} = \vec{v}(x, y, z, t) = \nabla\phi, \quad (4.28)$$

and the *hydrodynamic pressure* from the dynamic part of Eq. (4.10),

$$p = p(x, y, z, t) = -\rho \left( \frac{\partial \phi}{\partial t} + \frac{v^2}{2} \right) \approx -\rho \frac{\partial \phi}{\partial t}, \quad (4.29)$$

where in the last step we have neglected the small term of second order. Moreover, the elevation of the interface between water and entrapped air is given by the linearised dynamic boundary condition (4.23) as

$$\eta_k = \eta_k(x, y, t) = -\frac{1}{g} \left[ \frac{\partial \phi}{\partial t} \right]_{z=z_k} - \frac{1}{\rho g} p_k. \quad (4.30)$$

The *wave elevation* (the elevation of the interface between the water and the open, constant-pressure air) is

$$\eta = \eta(x, y, t) = -\frac{1}{g} \left[ \frac{\partial \phi}{\partial t} \right]_{z=0}. \quad (4.31)$$

## 4.2 Harmonic Waves on Water of Constant Depth

Except when otherwise stated, let us in the following consider the case of a plane horizontal sea bottom. If the water is sufficiently deep, the sea bottom does not influence the waves on the water surface. Then the shape of the bottom is of no concern for the waves. If the water is not sufficiently deep, and the sea bottom is not horizontal, an analysis differing from the following analysis is required.

With sinusoidal time variation, we write

$$\phi = \phi(x, y, z, t) = \operatorname{Re}\{\hat{\phi}(x, y, z)e^{i\omega t}\}, \quad (4.32)$$

where  $\hat{\phi}$  is the complex amplitude of the velocity potential at the space point  $(x, y, z)$ . Similarly, we define the complex amplitudes  $\hat{v} = \hat{v}(x, y, z)$ ,  $\hat{p} = \hat{p}(x, y, z)$ ,

$\hat{\eta}_k = \hat{\eta}_k(x, y)$  and  $\hat{\eta} = \hat{\eta}(x, y)$ . The linearised basic equations, the Laplace equation (4.14) and the boundary conditions (4.15) and (4.26), now become

$$\nabla^2 \hat{\phi} = 0 \quad (4.33)$$

everywhere in the water,

$$[\partial \hat{\phi} / \partial n]_S = \hat{u}_n \quad (4.34)$$

on the wet surface of solid bodies (Figure 4.2) and

$$\left[ -\omega^2 \hat{\phi} + g \frac{\partial \hat{\phi}}{\partial z} \right]_{z=z_k} = -\frac{i\omega}{\rho} \hat{p}_k \quad (4.35)$$

on the water-air surfaces (Figure 4.1). Here,  $\hat{u}_n$  and  $\hat{p}_k$  are also complex amplitudes. Note that the normal component  $u_n$  of the motion of the wet body surface (Figure 4.2) is a function of the point considered on that surface, whereas the dynamic pressure  $p_k$  is assumed to have the same value everywhere inside the volume of entrapped air (Figure 4.1). Equations (4.28)–(4.31) for determining the physical variables become, in terms of complex amplitudes,

$$\hat{\vec{v}} = \nabla \hat{\phi}, \quad (4.36)$$

$$\hat{p} = -i\omega \rho \hat{\phi}, \quad (4.37)$$

$$\hat{\eta}_k = -\frac{i\omega}{g} [\hat{\phi}]_{z=z_k} - \frac{1}{\rho g} \hat{p}_k, \quad (4.38)$$

$$\hat{\eta} = -\frac{i\omega}{g} [\hat{\phi}]_{z=0}. \quad (4.39)$$

The remaining part of this section (and also Sections 4.3–4.6) discusses some particular solutions which satisfy Laplace's equation and the homogeneous boundary conditions

$$\left[ \frac{\partial \hat{\phi}}{\partial z} \right]_{z=-h} = 0, \quad (4.40)$$

$$\left[ -\omega^2 \hat{\phi} + g \frac{\partial \hat{\phi}}{\partial z} \right]_{z=0} = 0. \quad (4.41)$$

These solutions will thus satisfy the boundary conditions on a (non-moving) horizontal bottom of a sea of depth  $h$  and the boundary condition at the free water surface (the interface between water and air), where the air pressure is constant. Additional (inhomogeneous) boundary conditions will be imposed later (e.g., in Section 4.7).

Using the method of separation of variables, we seek a particular solution of the form

$$\hat{\phi}(x, y, z) = H(x, y)Z(z). \quad (4.42)$$

Inserting this into the Laplace equation (4.33) and dividing by  $\hat{\phi}$ , we get

$$0 = \frac{\nabla^2 \hat{\phi}}{\hat{\phi}} = \frac{1}{H} \left[ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right] + \frac{1}{Z} \frac{d^2 Z}{dz^2} \quad (4.43)$$

or

$$-\frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{H} \left[ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right] \equiv \frac{1}{H} \nabla_H^2 H. \quad (4.44)$$

The left-hand side of Eq. (4.44) is a function of  $z$  only. The right-hand side is a function of  $x$  and  $y$ . This is impossible unless both sides equal a constant,  $-k^2$ , say. Thus, from Eq. (4.44), we get two equations:

$$\frac{d^2 Z(z)}{dz^2} = k^2 Z(z), \quad (4.45)$$

$$\nabla_H^2 H(x, y) = -k^2 H(x, y). \quad (4.46)$$

We note here that the separation constant  $k^2$  has the dimension of inverse length squared. Later, when discussing a solution of the two-dimensional Helmholtz equation (4.46), we shall see that  $k$  may be interpreted as the angular repetency of a propagating wave.

Let us, however, first discuss the following solution of Eq. (4.45):

$$Z(z) = c_+ e^{kz} + c_- e^{-kz}. \quad (4.47)$$

Here  $c_+$  and  $c_-$  are two integration constants. Because we have two new integration constants when solving Eq. (4.46) for  $H$ , we may choose  $Z(0) = 1$ , which means that  $c_+ + c_- = 1$ . Another relation to determine the integration constants is provided by the bottom boundary condition

$$\frac{\partial \hat{\phi}}{\partial z} = \frac{dZ(z)}{dz} H(x, y) = 0 \quad \text{for } z = -h. \quad (4.48)$$

It is then easy to show that (see Problem 4.2)

$$c_{\pm} = \frac{e^{\pm kh}}{e^{kh} + e^{-kh}}. \quad (4.49)$$

From Eq. (4.47) it now follows that

$$Z(z) = \frac{e^{k(z+h)} + e^{-k(z+h)}}{e^{kh} + e^{-kh}}. \quad (4.50)$$

Hence, we have the following particular solution of the Laplace equation:

$$\hat{\phi} = H(x, y)e(kz), \quad (4.51)$$

where  $H(x, y)$  has to satisfy the Helmholtz equation (4.46), and where

$$e(kz) = \frac{\cosh(kz + kh)}{\cosh(kh)} = \frac{e^{k(z+h)} + e^{-k(z+h)}}{e^{kh} + e^{-kh}} = \frac{1 + e^{-2k(z+h)}}{1 + e^{-2kh}} e^{kz}. \quad (4.52)$$

To be strict, since this is a function of two variables, we should perhaps have denoted the function by  $e(kz, kh)$ . We prefer, however, to use the simpler notation  $e(kz)$ , in particular because, for the deep-water case,  $kh \gg 1$ , it tends to the exponential function  $e(kz) \approx e^{kz}$  (although it then approaches  $2e^{kz}$  when  $z$  approaches  $-h$ , a  $z$ -coordinate which is usually of little practical interest in the deep-water case). We may note that the solution (4.51) is applicable for the deep-water case even if the water depth is not constant, provided  $kh_{\min} \gg 1$ .

In order to satisfy the free-surface boundary condition (4.41), we require

$$\omega^2 = \omega^2 e(0) = g \left[ \frac{de(kz)}{dz} \right]_{z=0} = gk \frac{\sinh(kh)}{\cosh(kh)} \quad (4.53)$$

or

$$\omega^2 = gk \tanh(kh), \quad (4.54)$$

which for deep water ( $kh \gg 1$ ) simplifies to

$$\omega^2 = gk, \quad (4.55)$$

a result which has already been presented in Eq. (3.8).

We have now derived the dispersion equation (4.54), which is a relation between the angular frequency  $\omega$  and the angular repetency  $k$ . Rewriting Eq. (4.54) as

$$\omega^2/(gk) = \tanh(kh), \quad (4.56)$$

we observe that, in the interval  $0 < k < +\infty$ , the right-hand side is monotonically increasing from 0 to 1, while the left-hand side, for a given  $\omega$ , is monotonically decreasing from  $+\infty$  to 0. Hence, there is one, and only one, positive  $k$  which satisfies (4.56), and there is correspondingly one, and only one, negative solution, since both sides of the equation are odd functions of  $k$ . Thus, there is only one possible positive value of the separation constant  $k^2$  in Eqs. (4.45)–(4.46). We may raise the question whether negative or even complex values are possible.

Replacing  $k^2$  by  $\lambda_n$  and  $\hat{\phi}(x, y, z) = Z(z)H(x, y)$  by

$$\hat{\phi}_n(x, y, z) = Z_n(z)H_n(x, y), \quad (4.57)$$

we rewrite Eqs. (4.45)–(4.46) as

$$\nabla_H^2 H_n(x, y) = -\lambda_n H_n(x, y), \quad (4.58)$$

$$Z_n''(z) = \lambda_n Z_n(z), \quad (4.59)$$

where the integer subscript  $n$  is used to label the various possible solutions. Since  $\hat{\phi}_n(x, y, z)$  has to satisfy the boundary conditions (4.40)–(4.41), the functions  $Z_n(z)$  are subject to the boundary conditions

$$Z'_n(-h) = 0, \quad (4.60)$$

$$Z'_n(0) = \frac{\omega^2}{g} Z_n(0). \quad (4.61)$$

In Eq. (4.59), the separation constant  $\lambda_n$  is an eigenvalue and  $Z_n(z)$  is the corresponding eigenfunction. We shall now show that all eigenvalues have to be real. Let us consider two possible eigenvalues  $\lambda_n$  and  $\lambda_m$  with corresponding eigenfunctions  $Z_n(z)$  and  $Z_m(z)$ . Replacing  $n$  by  $m$  in Eq. (4.59)—assuming  $\omega$  is real—and taking the complex conjugate give

$$Z_m^{*''}(z) = \lambda_m^* Z_m^*(z). \quad (4.62)$$

Now, let us multiply Eq. (4.59) by  $Z_m^*(z)$  and Eq. (4.62) by  $Z_n(z)$ . Subtracting and integrating then give

$$I \equiv \int_{-h}^0 [Z_m^*(z)Z_n''(z) - Z_n(z)Z_m^{*''}(z)] dz = (\lambda_n - \lambda_m^*) \int_{-h}^0 Z_m^*(z)Z_n(z) dz. \quad (4.63)$$

Noting that the integrand in the first integral of Eq. (4.63) may be written as

$$Z_m^*(z)Z_n''(z) - Z_n(z)Z_m^{*''}(z) = \frac{d}{dz} [Z_m^*(z)Z_n'(z) - Z_n(z)Z_m^{*'}(z)] \quad (4.64)$$

and that the boundary conditions (4.60)–(4.61) apply to  $Z_m^*$  as well, we find that the integral  $I$  vanishes. This is true because

$$\begin{aligned} I &= [Z_m^*(z)Z_n'(z) - Z_n(z)Z_m^{*'}(z)]_{-h}^0 \\ &= Z_m^*(0) \frac{\omega^2}{g} Z_n(0) - Z_n(0) \frac{\omega^2}{g} Z_m^*(0) - 0 - 0 \\ &= 0. \end{aligned} \quad (4.65)$$

We have here tacitly assumed (as we shall do throughout) that  $\omega$  is real—that is,  $\omega^* = \omega$ . Hence, we have shown that for all  $m$  and  $n$ ,

$$(\lambda_n - \lambda_m^*) \int_{-h}^0 Z_m^*(z)Z_n(z) dz = 0. \quad (4.66)$$

For  $m = n$ , we have for non-trivial solutions  $|Z_n(z)| \neq 0$  that  $(\lambda_n - \lambda_n^*) = 0$ . This means that  $\lambda_n$  is real for all  $n$ .

For  $\lambda_n \neq \lambda_m$ , Eq. (4.66) gives the following orthogonality condition for the eigenfunctions  $\{Z_n(z)\}$ :

$$\int_{-h}^0 Z_m^*(z)Z_n(z) dz = 0. \quad (4.67)$$

We have shown that the eigenvalues are real and that there is only one positive eigenvalue  $k^2$  satisfying the dispersion equation (4.54). The other eigenvalues have to be negative. We shall label the eigenvalues in decreasing order as

$$\lambda_0 = k^2 > \lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_n > \dots \quad (4.68)$$

Thus,  $\lambda_n$  is negative if  $n \geq 1$ . A negative eigenvalue may be conveniently written as

$$\lambda_n = -m_n^2, \quad (4.69)$$

where  $m_n$  is real. Let us now in Eq. (4.47) replace  $k$  with  $-im_n$  and  $Z(z)$  with  $Z_n(z)$ . One of the two integration constants may then be eliminated by using boundary condition (4.60). The resulting eigenfunction is (see Problem 4.3)

$$Z_n(z) = N_n^{-1/2} \cos(m_n z + m_n h), \quad (4.70)$$

where  $N_n^{-1/2}$  is an arbitrary integration constant. This result also follows from Eqs. (4.50) and (4.52) if we observe that  $\cosh[-im_n(z+h)] = \cos[-m_n(z+h)] = \cos[m_n(z+h)]$ . The free-surface boundary condition (4.61) is satisfied if (see Problem 4.3)

$$\omega^2/(gm_n) = -\tan(m_n h). \quad (4.71)$$

This equation also follows from Eq. (4.56) if  $k$  is replaced by  $-im_n$ . The left-hand side of Eq. (4.71) decreases monotonically from  $+\infty$  to 0 in the interval  $0 < m_n < +\infty$ . The right-hand side decreases monotonically from  $+\infty$  to 0 in the interval  $(n - 1/2)\pi/h < m_n < n\pi/h$ . Hence, Eq. (4.71) has a solution in this latter interval. Noting that  $n = 1, 2, 3, \dots$ , we find that there is an infinite, but numerable, number of solutions for  $m_n$ . For the solutions, we have (see Problem 4.3)

$$m_n \rightarrow \frac{n\pi}{h} - \frac{\omega^2}{n\pi g} \rightarrow \frac{n\pi}{h} \quad \text{as } n \rightarrow \infty. \quad (4.72)$$

Further (as is also shown in Problem 4.3), if

$$\frac{1}{h} \int_{-h}^0 |Z_n(z)|^2 dz = 1 \quad (4.73)$$

is chosen as a normalisation condition, then the integration constant in Eq. (4.70) is given by

$$N_n = \frac{1}{2} \left[ 1 + \frac{\sin(2m_n h)}{2m_n h} \right]. \quad (4.74)$$

For  $n = 0$ , we set  $m_0 = ik$  and then we have, in particular,

$$N_0 = \frac{1}{2} \left[ 1 + \frac{\sinh(2kh)}{2kh} \right]. \quad (4.75)$$

We may note (see Problem 4.2) that

$$Z_0(z) = \sqrt{\frac{2kh}{D(kh)}} e(kz), \quad (4.76)$$

where  $e(kz)$  is given by Eq. (4.52) and

$$D(kh) = \left[ 1 + \frac{2kh}{\sinh(2kh)} \right] \tanh(kh). \quad (4.77)$$

The orthogonal set of eigenfunctions  $\{Z_n(z)\}$  is complete if the function  $Z_0(z)$  is included in the set—that is, if  $n = 0, 1, 2, 3, \dots$ . The completeness follows from the fact that the eigenvalue problem (4.59)–(4.61) is a Sturm–Liouville problem [27].

We have now discussed possible solutions of Eq. (4.45) or (4.59) with the homogeneous boundary conditions (4.40) and (4.41), or (4.60) and (4.61). It remains for us to discuss solutions of the Helmholtz equation (4.46) or (4.58). In the discussion that follows, we start by considering the case with  $n = 0$ —that is,  $m_n = m_0 = ik$ . Then we take Eq. (4.46) as the starting point.

### 4.3 Plane Waves: Propagation Velocities

We consider a two-dimensional case with no variation in the  $y$ -direction. Setting  $\partial/\partial y = 0$ , we have from Eqs. (4.44) and (4.46)

$$d^2H(x)/dx^2 = -k^2H(x), \quad (4.78)$$

which has the general solution

$$H(x) = ae^{-ikx} + be^{ikx}, \quad (4.79)$$

where  $a$  and  $b$  are arbitrary integration constants. Hence, from (4.51),

$$\hat{\phi} = e(kz)(ae^{-ikx} + be^{ikx}) \quad (4.80)$$

and from (4.32),

$$\phi = \operatorname{Re}\{\hat{\phi}e^{i\omega t}\} = \operatorname{Re}\{ae^{i(\omega t-kx)} + be^{i(\omega t+kx)}\}e(kz), \quad (4.81)$$

which demonstrates that  $k$  is the angular repetency (wave number).

The first and second terms represent plane waves propagating in the positive and negative  $x$ -directions, respectively, with a phase velocity

$$v_p \equiv \frac{\omega}{k} = \frac{g}{\omega} \tanh(kh) = \sqrt{\frac{g}{k} \tanh(kh)}, \quad (4.82)$$

which is obtained from the dispersion relation (4.54). Note that  $v_p$  is the velocity by which the wave crest (or a line of constant phase) propagates (in a direction perpendicular to the wave crest). Later (in Section 4.4), we prove that the wave energy associated with a harmonic plane wave is transported with the group velocity  $v_g = d\omega/dk$  [cf. Eq. (3.10)], which, in general, differs from the phase velocity. The quantity  $v_g$  bears its name because it is the velocity by which a wave group, composed of harmonic waves with slightly different frequencies, propagates (see Problem 3.1). For instance, a swell wave, which originates from

a storm centre a distance  $L$  from a coast line, reaches this coast line a time  $L/v_g$  later (see Problem 4.4).

In Eq. (4.80), we might interpret the first term as an incident wave and the second term as a reflected wave. Introducing  $\Gamma = b/a$ , which is a complex reflection coefficient, and  $A = -i\omega a/g$ , which is the complex elevation amplitude at  $x = 0$  in the case of no reflection, we may rewrite Eq. (4.80) as

$$\hat{\phi} = -\frac{g}{i\omega} A e(kz) (e^{-ikx} + \Gamma e^{ikx}). \quad (4.83)$$

Correspondingly, the wave elevation  $\eta = \eta(x, t)$  has a complex amplitude [cf. Eqs. (4.39) and (4.52)]

$$\hat{\eta} = \hat{\eta}(x) = A (e^{-ikx} + \Gamma e^{ikx}). \quad (4.84)$$

Setting

$$\hat{\eta}_f = A e^{-ikx}, \quad (4.85)$$

$$\hat{\eta}_b = B e^{ikx} = A \Gamma e^{ikx}, \quad (4.86)$$

we have

$$\hat{\eta} = \hat{\eta}_f + \hat{\eta}_b, \quad (4.87)$$

$$\hat{\phi} = -\frac{g}{i\omega} \hat{\eta} e(kz), \quad (4.88)$$

$$\hat{p} = \rho g \hat{\eta} e(kz), \quad (4.89)$$

where we have also made use of Eq. (4.37). Note that the hydrodynamic pressure  $p$  decreases monotonically (exponentially for deep water) with the distance ( $-z$ ) below the mean free surface,  $z = 0$ . For the fluid velocity, we have, from Eqs. (4.36) and (4.52),

$$\hat{v}_x = \frac{\partial \hat{\phi}}{\partial x} = g \frac{k}{\omega} e(kz) (\hat{\eta}_f - \hat{\eta}_b) = \omega \frac{\cosh(kz + kh)}{\sinh(kh)} (\hat{\eta}_f - \hat{\eta}_b), \quad (4.90)$$

$$\hat{v}_z = \frac{\partial \hat{\phi}}{\partial z} = g \frac{ik}{\omega} e'(kz) (\hat{\eta}_f + \hat{\eta}_b) = i\omega \frac{\sinh(kz + kh)}{\sinh(kh)} (\hat{\eta}_f + \hat{\eta}_b), \quad (4.91)$$

where

$$e'(kz) = \frac{de(kz)}{d(kz)} = \frac{\sinh(kz + kh)}{\cosh(kh)} = e(kz) \tanh(kz + kh). \quad (4.92)$$

We have used the dispersion relation (4.54) to obtain the last expressions for  $\hat{v}_x$  and  $\hat{v}_z$ .

If we have a progressive wave in the forward (positive  $x$ ) direction ( $\hat{\eta}_b = 0$ ) on deep water ( $kh \gg 1$ ), the motion of fluid particles is circularly polarised

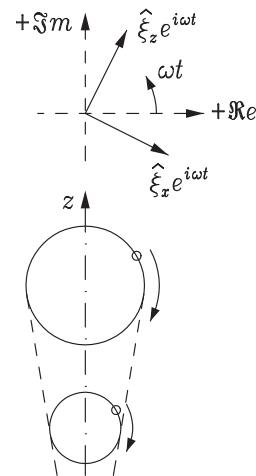


Figure 4.3: Phasor diagram of the  $x$ - and  $z$ -components of the fluid particle displacement (top) and fluid particle trajectories (bottom).

with a negative sense of rotation (in the clockwise direction in the  $xz$ -plane), notably

$$\hat{v}_x = \omega \hat{\eta}_f e^{kz}, \quad (4.93)$$

$$\hat{v}_z = i\omega \hat{\eta}_f e^{kz}. \quad (4.94)$$

The displacement of the fluid particles (as shown in Figure 4.3) are given by

$$\hat{\xi} = \hat{v}/i\omega, \quad (4.95)$$

$$\hat{\xi}_x = -i\hat{\eta}_f e^{kz}, \quad (4.96)$$

$$\hat{\xi}_z = \hat{\eta}_f e^{kz}. \quad (4.97)$$

For a progressive wave in the backwards direction ( $\hat{\eta}_f = 0$ ) on deep water, the motion is circularly polarised with a positive sense of rotation.

For a plane wave propagating in a direction making an angle  $\beta$  with the  $x$ -axis, the wave elevation is given by

$$\hat{\eta} = A \exp [-ik(x \cos \beta + y \sin \beta)], \quad (4.98)$$

which can be easily shown by considering transformation between two coordinate systems  $(x', y')$  and  $(x, y)$  which differ by a rotation angle  $\beta$ . The corresponding velocity potential  $\hat{\phi}$ , still given by Eq. (4.88) but now with the new  $\hat{\eta}$ , is a solution of the Laplace equation (4.33), satisfying the boundary conditions on the sea bed,  $z = -h$ , and on the free water surface,  $z = 0$  (see Problem 4.5).

For deep water ( $kh \gg 1$ ), the dispersion relation (4.54) is approximately  $\omega^2 = gk$ , in agreement with Eq. (4.55) and the previous statement, Eq. (3.8). Note that

$$\tanh(kh) > \begin{cases} 0.95 & \text{for } kh > 1.83 \text{ or } h > 0.3\lambda, \\ 0.99 & \text{for } kh > 2.6 \text{ or } h > 0.42\lambda. \end{cases} \quad (4.99)$$

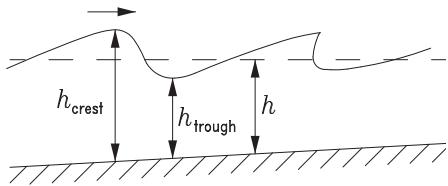


Figure 4.4: Increasing wave steepness as a wave propagates on shallow water.

Hence, depending on the desired accuracy, we may assume deep water when the depth is at least one third or one half of the wavelength, respectively. For deep water, the wavelength is

$$\lambda = 2\pi/k = 2\pi g/\omega^2 = (g/2\pi)T^2 = (1.56 \text{ m/s}^2)T^2. \quad (4.100)$$

As previously found [see Eqs. (3.9) and (3.12)], for deep water we have

$$2v_g = v_p = g/\omega = \sqrt{g/k}. \quad (4.101)$$

In contrast, for shallow water with horizontal bottom,  $kh \ll 1$ , a power series expansion of Eq. (4.54) gives

$$\omega^2 = gk(kh + \dots) \approx ghk^2. \quad (4.102)$$

(Here the error is less than 1% or 5% if the water depth is less than 1/36 or 1/16 of the wavelength, respectively.) In this approximation a wave is not dispersive, because

$$v_g = v_p = \sqrt{gh} \quad (4.103)$$

is independent of  $k$  and  $\omega$ . The group velocity  $v_g$  differs, however, from the phase velocity  $v_p$  in the general dispersive case, as is considered next. The present theory is linear, but let us nevertheless try to give a qualitative explanation of wave breaking on shallow water. The formula (4.103) indicates that the wave passes faster on the wave crest than on the wave trough because  $h_{\text{crest}} > h_{\text{trough}}$  (see Figure 4.4). Finally there will be a vertical edge and then, of course, linear theory does not apply since  $\partial\eta/\partial x \rightarrow \infty$ .

For constant water depth  $h$ , the phase velocity  $v_p$  is, in the general case, given by Eq. (4.82). In order to obtain the group velocity  $v_g$ , we differentiate the dispersion equation (4.54):

$$\begin{aligned} 2\omega d\omega &= gdk \tanh(kh) + \frac{gk}{\cosh^2(kh)} hdk \\ &= \frac{dk}{k} gk \tanh(kh) + \frac{gk \tanh(kh)}{\cosh(kh) \sinh(kh)} hdk \\ &= \frac{dk}{k} \omega^2 + \frac{2\omega^2 hdk}{\sinh(2kh)}. \end{aligned} \quad (4.104)$$

Hence,

$$v_g = \frac{d\omega}{dk} = \frac{\omega}{2k} \left[ 1 + \frac{2kh}{\sinh(2kh)} \right]. \quad (4.105)$$

Since  $v_p = \omega/k$ , we then obtain

$$v_g = \frac{D(kh)}{2 \tanh(kh)} v_p = \frac{g}{2\omega} D(kh), \quad (4.106)$$

where we have, for convenience, used the depth function  $D(kh)$  defined by Eq. (4.77). Because we shall refer to this function frequently later, here we present some alternative expressions:

$$\begin{aligned} D(kh) &= \left[ 1 + \frac{2kh}{\sinh(2kh)} \right] \tanh(kh) \\ &= \tanh(kh) + \frac{kh}{\cosh^2(kh)} \\ &= \tanh(kh) + kh - kh \tanh^2(kh) \\ &= \left[ 1 - (\omega^2/gk)^2 \right] kh + \omega^2/gk \\ &= (2\omega/g)v_g = (2k/g)v_p v_g. \end{aligned} \quad (4.107)$$

In the last equality, the somewhat complicated mathematical function  $D(kh)$  has been more simply expressed by the physical quantities  $v_g$  and  $v_p$ , the group and phase velocities. We shall frequently also need the relationship (see Problem 4.12)

$$2k \int_{-h}^0 e^2(kz) dz = D(kh), \quad (4.108)$$

where the vertical eigenfunction  $e(kz)$  is given by Eq. (4.52) and the depth function  $D(kh)$  by Eq. (4.107).

Note that on deep water,  $kh \gg 1$ , we have  $D(kh) \approx 1$  and  $\tanh(kh) \approx 1$ . For small values of  $kh$ , we have  $\tanh(kh) = kh + \dots$  and  $D(kh) = 2kh + \dots$ . Whereas  $\tanh(kh)$  is a monotonically increasing function, it can be shown that  $D(kh)$  has a maximum  $D_{\max} = x_0$  for  $kh = x_0$ , where  $x_0 = 1.1996786$  is a solution of the transcendental equation  $x_0 \tanh(x_0) = 1$  (see Problem 4.12). This maximum occurs for  $\omega^2 h/g = 1$ .

For arbitrary  $\omega$ , we can obtain  $k$  from the transcendental dispersion equation (4.54) for instance by applying some numerical iteration procedure, and then we obtain  $v_p$  and  $v_g$  from Eqs. (4.82) and (4.106). The relationships are shown graphically in the curves of Figure 4.5.

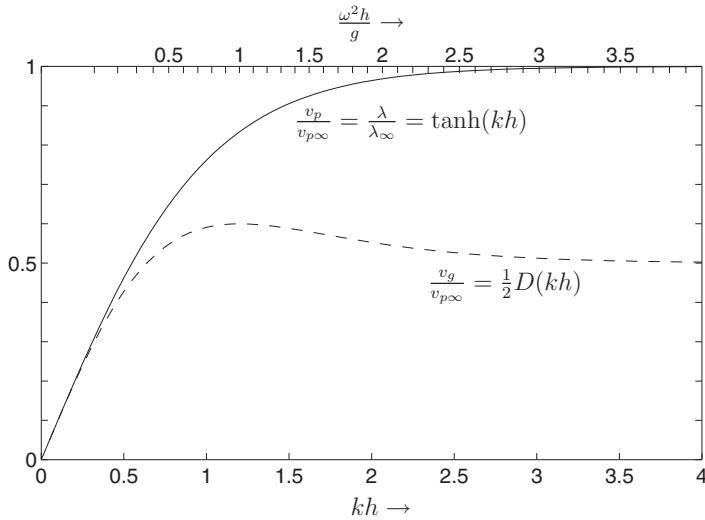


Figure 4.5: Phase velocity  $v_p$ , group velocity  $v_g$  and wavelength  $\lambda = 2\pi/k$  as functions of depth  $h$  for a given frequency, or as functions of frequency for a given water depth. The subscript  $\infty$  corresponds to (infinitely) deep water.

We have now discussed a propagating wave, which corresponds to the case with  $n = 0$  in Eqs. (4.70), (4.71) and (4.74). Let now  $n$  be an arbitrary non-negative integer. Setting again  $\partial/\partial y = 0$ , we find that Eq. (4.58), with Eq. (4.69) inserted, has a solution

$$H_n(x) = a_n e^{-m_n x} + b_n e^{m_n x}, \quad (4.109)$$

where  $a_n$  and  $b_n$  are integration constants. The corresponding complex amplitude of the velocity potential is, from Eq. (4.57),

$$\hat{\phi}_n(x, z) = (a_n e^{-m_n x} + b_n e^{m_n x}) Z_n(z). \quad (4.110)$$

Note that this is a particular solution, which satisfies the homogeneous Laplace equation (4.33) and the homogeneous boundary conditions (4.40) and (4.41). Hence, a superposition of such solutions as Eq. (4.110) is a solution:

$$\hat{\phi}(x, z) = \sum_{n=0}^{\infty} (a_n e^{-m_n x} + b_n e^{m_n x}) N_n^{-1/2} \cos(m_n z + m_n h), \quad (4.111)$$

where we have used Eq. (4.70).

Remember that for the term with  $n = 0$  we set  $m_n = ik$ . Here we choose  $k$  and  $m_n$  (for  $n \geq 1$ ) to be positive. Note that the term with  $a_n$  decays with increasing  $x$ , or, for  $n = 0$ , is a wave progressing in the positive  $x$ -direction. The term with  $b_n$  decays with decreasing  $x$  or, for  $n = 0$ , is a wave progressing in the negative  $x$ -direction. If our region of interest is  $0 < x < +\infty$ , we thus

have to set  $b_n = 0$  for all  $n$ , except for the possibility that  $b_0 \neq 0$  for a case where a wave is incident from infinite distance,  $x = +\infty$ . For  $n \geq 1$ , this is a consequence of avoiding infinite values. The *radiation condition* was briefly mentioned previously (in Section 4.1). We have the opportunity to apply such a condition here. When  $b_0 = 0$ , the radiation condition of an outgoing wave at infinite distance is satisfied. This means that there is no contribution of the form  $e^{ikx}$  to the potential  $\hat{\phi}$ . Similarly, if our region of interest is  $-\infty < x < 0$ , then  $a_n = 0$  for all  $n$  if the radiation condition is satisfied as  $x \rightarrow -\infty$ . However, if a wave is incident from  $x = -\infty$ , then  $a_0 \neq 0$ . If our region of interest is finite,  $x_1 < x < x_2$ , for example, then we may have  $a_n \neq 0$  and  $b_n \neq 0$  for all  $n$ .

For the two-dimensional case, Eq. (4.111) represents a general plane-wave solution for the velocity potential in a uniform fluid of constant depth. The terms with  $n = 0$  are propagating waves, whereas the terms with  $n \geq 1$  are ‘evanescent waves’. Propagation is along the  $x$ -axis. If  $x$  in Eq. (4.111) is replaced by  $(x \cos \beta + y \sin \beta)$ , then the plane-wave propagation is along a direction which makes an angle  $\beta$  with the  $x$ -axis [cf. Eq. (4.98)].

## 4.4 Wave Transport of Energy and Momentum

### 4.4.1 Potential Energy

Let us consider the potential energy associated with the elevation of water from the wave troughs to the wave crests. Per unit (horizontal) area, the potential energy relative to the sea bed equals the product of  $\rho g(h + \eta)$ , the water weight per unit area, and  $(h + \eta)/2$ , the height of the water mass centre above the sea bed:

$$(\rho g/2)(h + \eta)^2 = (\rho g/2)h^2 + \rho gh\eta + (\rho g/2)\eta^2. \quad (4.112)$$

The increase in relation to calm water is

$$\rho gh\eta + (\rho g/2)\eta^2, \quad (4.113)$$

where the first term has a vanishing average value.

Hence, the time-average potential energy per unit (horizontal) area is

$$E_p(x, y) = (\rho g/2)\overline{\eta^2(x, y, t)}. \quad (4.114)$$

In the case of a harmonic wave,

$$E_p(x, y) = (\rho g/4)|\hat{\eta}(x, y)|^2. \quad (4.115)$$

In particular, for the harmonic plane wave given by Eq. (4.87), we have

$$\begin{aligned} E_p(x) &= \frac{\rho g}{4}|\hat{\eta}_f + \hat{\eta}_b|^2 = \frac{\rho g}{4}\left(|\hat{\eta}_f|^2 + |\hat{\eta}_b|^2 + \hat{\eta}_f\hat{\eta}_b^* + \hat{\eta}_f^*\hat{\eta}_b\right) \\ &= \frac{\rho g}{4}\left(|A|^2 + |B|^2 + AB^*e^{-i2kx} + A^*Be^{i2kx}\right). \end{aligned} \quad (4.116)$$

If  $AB \neq 0$ , this expression for  $E_p(x)$  contains a term which varies sinusoidally with  $x$  with a ‘wavelength’  $\lambda/2 = 2\pi/2k = \pi/k$ . This sinusoidal variation does not contribute if we average over an  $x$  interval which is either very long or, alternatively, an integer multiple of  $\lambda/2$ . Denoting this average by  $\langle E_p \rangle$ , we have

$$\langle E_p \rangle = (\rho g/4) (|A|^2 + |B|^2), \quad (4.117)$$

and for a progressive plane wave ( $B = 0$ ),

$$\langle E_p \rangle = (\rho g/4)|A|^2. \quad (4.118)$$

In Section 7.1.8, we shall show that for an OWC the time-average potential energy per unit horizontal area includes an additional contribution from the dynamic air pressure  $p_k(t)$  in the chamber above the water column—that is,

$$E_p(x, y) = (\rho g/2)\overline{\eta_k^2(x, y, t)} + \overline{p_k(t)\eta_k(x, y, t)} \quad (4.119)$$

on the air–water interface of the OWC.

#### 4.4.2 Kinetic Energy

For simplicity, we consider a progressive, plane, harmonic wave on deep water. The fluid velocity is given by Eqs. (4.93)–(4.94). The average kinetic energy per unit volume is

$$\frac{1}{2}\rho\frac{1}{2}\text{Re}\left\{|\hat{v}_x|^2 + |\hat{v}_z|^2\right\} = \frac{\rho}{4}\left(|\hat{v}_x|^2 + |\hat{v}_z|^2\right) = \frac{\rho}{2}\omega^2|\hat{\eta}_f|^2e^{2kz} = \frac{\rho}{2}\omega^2|A|^2e^{2kz}. \quad (4.120)$$

By integrating from  $z = -\infty$  to  $z = 0$ , we obtain the average kinetic energy per unit (horizontal) area:

$$E_k = \frac{\rho}{2}\omega^2|A|^2 \int_{-\infty}^0 e^{2kz} dz = \frac{\rho}{2}\frac{\omega^2}{2k}|A|^2. \quad (4.121)$$

Using  $\omega^2 = gk$ , we obtain

$$E_k = (\rho g/4)|A|^2. \quad (4.122)$$

It can be shown (cf. Problem 4.7) that this expression for  $E_k$  is also valid for an arbitrary constant water depth  $h$ . Moreover, it can be shown (cf. Problem 4.8) that for a plane wave, as given by Eq. (4.87), the kinetic energy per unit horizontal surface, averaged both over time and over the horizontal plane, is

$$\langle E_k \rangle = (\rho g/4) (|A|^2 + |B|^2). \quad (4.123)$$

### 4.4.3 Total Stored Energy

The total energy is the sum of potential energy and kinetic energy. For a progressive plane harmonic wave, the time-average stored energy per unit (horizontal) area is

$$E = E_k + E_p = 2E_k = 2E_p = (\rho g/2)|A|^2. \quad (4.124)$$

Averaging also over the horizontal plane, we have for a plane wave as given by Eq. (4.87)

$$\langle E \rangle = 2\langle E_k \rangle = 2\langle E_p \rangle = (\rho g/2) (|A|^2 + |B|^2). \quad (4.125)$$

### 4.4.4 Wave-Energy Transport

Consider the energy transport of a plane harmonic wave propagating in the  $x$ -direction. Per unit (vertical) area, the time-average power propagating in the positive  $x$ -direction equals the intensity [cf. Eq. (3.16)]

$$I = \overline{p v_x} = \frac{1}{2} \operatorname{Re}\{\hat{p} \hat{v}_x^*\}. \quad (4.126)$$

Note that since  $\hat{p} \hat{v}_z^*$  is purely imaginary [cf. Eqs. (4.89) and (4.91)], the intensity has no  $z$ -component.

Inserting for  $\hat{p}$  and  $\hat{v}_x$  from Eqs. (4.89)–(4.90), we need the product

$$(\hat{\eta}_f + \hat{\eta}_b)(\hat{\eta}_f - \hat{\eta}_b)^* = |\hat{\eta}_f|^2 - |\hat{\eta}_b|^2 + (\hat{\eta}_f^* \hat{\eta}_b - \hat{\eta}_f \hat{\eta}_b^*). \quad (4.127)$$

Since the last term here is purely imaginary, we get

$$I = \frac{k \rho g^2}{2\omega} (|\hat{\eta}_f|^2 - |\hat{\eta}_b|^2) e^2(kz). \quad (4.128)$$

Integrating from  $z = -h$  to  $z = 0$  gives the transported wave power per unit width of the wave front:

$$\begin{aligned} J &= \int_{-h}^0 I dz = \frac{\rho g^2}{4\omega} (|\hat{\eta}_f|^2 - |\hat{\eta}_b|^2) 2k \int_{-h}^0 e^2(kz) dz \\ &= \frac{\rho g^2 D(kh)}{4\omega} (|\hat{\eta}_f|^2 - |\hat{\eta}_b|^2) \\ &= \frac{\rho g^2 D(kh)}{4\omega} (|A|^2 - |B|^2) = \frac{\rho g}{2} v_g (|A|^2 - |B|^2), \end{aligned} \quad (4.129)$$

where we have also used relations (4.107)–(4.108).

For a purely progressive wave ( $\hat{\eta}_b = 0$ ,  $\hat{\eta}_f = A e^{-ikx}$ ), this gives

$$J = \frac{\rho g^2 D(kh)}{4\omega} |\hat{\eta}_f|^2 = \frac{\rho g^2 D(kh)}{4\omega} |A|^2 = \frac{\rho g}{2} v_g |A|^2. \quad (4.130)$$

Introducing the period  $T = 2\pi/\omega$  and the wave height  $H = 2|A|$ , we have for deep water ( $kh \gg 1$ ,  $D(kh) \approx 1$ )

$$J = \frac{\rho g^2}{32\pi} TH^2 = (976 \text{ Ws}^{-1}\text{m}^{-3}) TH^2, \quad (4.131)$$

with  $\rho = 1020 \text{ kg/m}^3$  for sea water. For  $T = 10 \text{ s}$  and  $H = 2 \text{ m}$ , this gives

$$J = 3.9 \times 10^4 \text{ W/m} \approx 40 \text{ kW/m}. \quad (4.132)$$

In the case of a reflecting fixed wall, for instance at a plane  $x = 0$ , we have  $B = A$ —which, according to Eq. (4.129), means that  $J = 0$ , since the incident power is cancelled by the reflected power. Imagine that a device at  $x = 0$  extracts all the incident wave energy. Then it is necessary that the device radiates a wave which cancels the otherwise reflected wave. In this situation, a net energy transport as given by Eq. (4.130) would result, for  $x < 0$ .

We shall call the quantity  $J$  the *wave-energy transport*. (Note that some authors call this quantity ‘wave-energy flux’ or ‘wave-power flux’. This terminology will be avoided here, because of the confusion resulting from improper discrimination between ‘flux’ and ‘flux density’ in different branches of physics.) An alternative term for  $J$  could be *wave-power level* [7].

#### 4.4.5 Relation between Energy Transport and Stored Energy

For a progressive plane harmonic wave, the energy transport  $J$  (energy per unit time and unit width of wave frontage) is given by Eq. (4.130), and the stored energy per unit horizontal area is  $E$  as given by Eq. (4.124). We may use this to define an energy transport velocity  $v_E$  by

$$J = v_E E. \quad (4.133)$$

Thus,

$$v_E = J/E = gD(kh)/(2\omega). \quad (4.134)$$

Comparing with the expression (4.106) for the group velocity  $v_g$ , we see that  $v_E = v_g$ . Hence,

$$J = v_g E. \quad (4.135)$$

Note that this simple relationship is valid for a purely progressive plane harmonic wave.

For a wave given by Eq. (4.87), we have, from Eqs. (4.129) and (4.125),

$$J = v_g \langle E \rangle \frac{|A|^2 - |B|^2}{|A|^2 + |B|^2}. \quad (4.136)$$

Thus, in this case, measurement of

$$\langle E \rangle = \rho g \langle \eta^2(x, y, t) \rangle \quad (4.137)$$

does not alone determine wave-energy transport  $J$ . Indiscriminate use of Eq. (4.135) would then result in an overestimation of  $J$ .

#### 4.4.6 Momentum Transport and Momentum Density of a Wave

As we have seen, waves are associated with energy transport  $J$  and with stored energy density  $E$ , which is equally divided between kinetic energy density  $E_k$  and potential energy density  $E_p$ . With a plane progressive wave  $\hat{\eta} = Ae^{-ikx}$ , these relations are stated mathematically by Eqs. (4.124), (4.130) and (4.133)–(4.135). Moreover, the wave is associated with a momentum and, hence, a mean mass transport. A propagating wave induces a mean drift of water, the so-called Stokes drift (see [28], p. 251).

The  $x$ -component of the momentum (the rate of the mass transport) per unit volume is

$$\rho v_x = \rho \frac{\partial \phi}{\partial x}. \quad (4.138)$$

Per unit area of free water surface, the momentum is given by

$$M_x = \int_{-h}^{\eta} \rho v_x dz. \quad (4.139)$$

Integration over  $-h < z < 0$  (instead of  $-h < z < \eta$ ) would yield a quantity with zero time-average because  $v_x$  varies sinusoidally with time. Hence, in lowest-order approximation, the time-average momentum density (per unit area of the free surface) is

$$\overline{M}_x = \overline{\int_0^{\eta} \rho v_x dz} = \overline{\eta [\rho v_x]_{z=0}} + \dots \approx \overline{\eta [\rho v_x]_{z=0}}. \quad (4.140)$$

Thus, in linearised theory,  $J$ ,  $E$ ,  $E_k$ ,  $E_p$  and  $\overline{M}_x$  are quadratic in the wave amplitude. In analogy with Eq. (2.77), we have, with sinusoidal variation with time,

$$\overline{M}_x = \overline{\eta [\rho v_x]_{z=0}} = \frac{\rho}{2} \operatorname{Re}\{\hat{\eta}^*[\hat{v}_x]_{z=0}\}. \quad (4.141)$$

Let us now consider a plane wave with complex amplitudes of elevation, velocity potential and horizontal fluid velocity given by Eqs. (4.87), (4.88) and (4.90), respectively. Using these equations together with Eq. (4.141) gives

$$\begin{aligned} \overline{M}_x &= \frac{\rho g k}{2\omega} \operatorname{Re}\{(Ae^{-ikx} - Be^{ikx})(A^* e^{-ikx} - B^* e^{ikx})\} \\ &= \frac{\rho g k}{2\omega} \operatorname{Re}\{|A|^2 - |B|^2 + (AB^* e^{-i2kx} - A^* B e^{i2kx})\}. \end{aligned} \quad (4.142)$$

Noting that the last term is purely imaginary, we obtain

$$\overline{M}_x = \frac{\rho g k}{2\omega} (|A|^2 - |B|^2). \quad (4.143)$$

If we compare this result with Eq. (4.129) and use Eqs. (4.82) and (4.106), we find the following relationship between the average momentum density and the wave-energy transport:

$$J = v_p v_g \overline{M}_x. \quad (4.144)$$

For a progressive wave, Eq. (4.135) is applicable and then

$$E = J/v_g = v_p \overline{M}_x. \quad (4.145)$$

Let us next consider the momentum transport through a vertical plane normal to the direction of wave propagation. Per unit width of the wave front the momentum transport in the  $x$ -direction due to the wave is [29]

$$\mathcal{I}_x = \overline{\int_{-h}^{\eta} p_{\text{tot}} + \rho v_x^2 dz} - \left( - \int_{-h}^0 \rho g z dz \right), \quad (4.146)$$

where the second term is the momentum transport in the absence of waves. The integrals in Eq. (4.146) may be split into the following parts:

$$\mathcal{I}_x = \int_{-h}^0 \overline{p_{\text{tot}}} dz + \int_{-h}^0 \overline{\rho v_x^2} dz + \overline{\int_0^{\eta} p_{\text{tot}} dz} + \int_{-h}^0 \rho g z dz. \quad (4.147)$$

Because we keep only terms up to the second order, the second integral is taken to the mean free surface  $z = 0$  instead of the instantaneous free surface  $z = \eta$ . Note that for the first and second integrals, since the limits of integration are constants, we can transfer the time average on the integrands.

Now, inserting the expression for the total pressure  $p_{\text{tot}}$  as given by the Bernoulli equation (4.10):

$$p_{\text{tot}} = -\rho \frac{\partial \phi}{\partial t} - \frac{\rho}{2} (v_x^2 + v_z^2) - \rho g z, \quad (4.148)$$

we arrive at

$$\mathcal{I}_x = - \overline{\int_0^{\eta} \rho \left( g z + \frac{\partial \phi}{\partial t} \right) dz} + \int_{-h}^0 \overline{\frac{\rho}{2} (v_x^2 - v_z^2)} dz \equiv \mathcal{I}_{x1} + \mathcal{I}_{x2}. \quad (4.149)$$

Because the upper limit of the first integral is a first-order quantity and we are evaluating the integral correct to second order, only terms up to the first order are kept in the integrand [cf. (4.140)]. Thus,

$$\begin{aligned} \mathcal{I}_{x1} &= - \overline{\int_0^{\eta} \rho \left( g z + \left[ \frac{\partial \phi}{\partial t} \right]_{z=0} \right) dz} = - \overline{\int_0^{\eta} \rho (g z - g \eta) dz} \\ &= - \frac{\rho g}{2} \overline{[z^2]_0^{\eta}} + \rho g \overline{\eta^2} = \frac{\rho g}{2} \overline{\eta^2}. \end{aligned} \quad (4.150)$$

With the plane wave as given by Eqs. (4.87), (4.88), (4.90) and (4.91), we therefore have

$$\mathcal{I}_{x1} = \frac{\rho g}{4} \left( |A|^2 + |B|^2 + AB^* e^{-i2kx} + A^* B e^{i2kx} \right) \quad (4.151)$$

and

$$\begin{aligned} \mathcal{I}_{x2} &= \frac{\rho}{4} \left( \frac{gk}{\omega} \right)^2 \left( |A|^2 + |B|^2 \right) \int_{-h}^0 e^2(kz) [1 - \tanh^2(kz + kh)] dz \\ &\quad - \frac{\rho}{4} \left( \frac{gk}{\omega} \right)^2 (AB^* e^{-i2kx} + A^* B e^{i2kx}) \int_{-h}^0 e^2(kz) [1 + \tanh^2(kz + kh)] dz \\ &= \frac{\rho g}{4} \left( |A|^2 + |B|^2 \right) \frac{2kh}{\sinh(2kh)} - \frac{\rho g}{4} (AB^* e^{-i2kx} + A^* B e^{i2kx}). \end{aligned} \quad (4.152)$$

Taking the sum, we finally have

$$\mathcal{I}_x = \mathcal{I}_{x1} + \mathcal{I}_{x2} = \frac{\rho g}{4} \left( |A|^2 + |B|^2 \right) \left( 1 + \frac{2kh}{\sinh(2kh)} \right), \quad (4.153)$$

which may alternatively be expressed as

$$\mathcal{I}_x = \frac{\rho g^2 k D(kh)}{4\omega^2} (|A|^2 + |B|^2) \quad (4.154)$$

using Eq. (4.107). For a progressive wave ( $B = 0$ ), this simplifies to

$$\mathcal{I}_x = \frac{\rho g^2 k D(kh)}{4\omega^2} |A|^2. \quad (4.155)$$

Comparing this with Eqs. (4.143)–(4.145) gives

$$\mathcal{I}_x = \frac{g D(kh)}{2\omega} \overline{M}_x = v_g \overline{M}_x = \frac{J}{v_p} = \frac{E v_g}{v_p}. \quad (4.156)$$

Thus, for a purely progressive wave, the momentum transport equals the momentum density multiplied by the group velocity. This may be interpreted as follows: the momentum associated with the wave is propagated with a speed equal to the group velocity.

Defining

$$\overline{M}_{x,+} = \frac{\rho g k}{2\omega} |A|^2, \quad \overline{M}_{x,-} = \frac{\rho g k}{2\omega} |B|^2, \quad (4.157)$$

we may rewrite Eqs. (4.143) and (4.154) as

$$\overline{M}_x = \overline{M}_{x,+} - \overline{M}_{x,-}, \quad (4.158)$$

$$\mathcal{I}_x = v_g \overline{M}_{x,+} + v_g \overline{M}_{x,-}. \quad (4.159)$$

Note the minus sign associated with the wave propagating in the negative  $x$ -direction in the expression for  $\bar{M}_x$ . In the expression for  $\mathcal{I}_x$  this minus sign is cancelled by the negative group velocity for this wave.

#### 4.4.7 Drift Forces due to Absorption and Reflection of Wave Energy

If an incident wave  $\hat{\eta} = Ae^{-ikx}$  is completely absorbed at the plane  $x = 0$ , then  $\mathcal{I}_x = 0$  for  $x > 0$ , whereas  $\mathcal{I}_x > 0$  for  $x < 0$ . Thus, the momentum transport is stopped at  $x = 0$ . In a time  $\Delta t$ , a momentum  $\bar{M}_x \Delta x = F'_d \Delta t$  disappears. Here  $\Delta x = v_g \Delta t$  is the distance of momentum transport during the time  $\Delta t$ . A force  $F'_d$  per unit width of wave front is required to stop the momentum transport. This (time-average) drift force is given by

$$F'_d = \frac{\Delta x}{\Delta t} \bar{M}_x = v_g \bar{M}_x = \mathcal{I}_x = \frac{\rho g^2 k D(kh)}{4\omega^2} |A|^2. \quad (4.160)$$

Using the dispersion relationship (4.54), we find that this gives

$$F'_d = \frac{\rho g}{4} \frac{D(kh)}{\tanh(kh)} |A|^2. \quad (4.161)$$

Note that on deep water [see Eqs. (4.101) and (4.106)],

$$F'_d = (\rho g/4) |A|^2; \quad (4.162)$$

that is, the ratio between  $F'_d$  and  $|A|^2$  is independent of frequency.

The drift force must be taken up by the anchor or mooring system of the wave absorber. Note that the drift force is a time-average force usually much smaller than the amplitude of the oscillatory force. With complete absorption of a wave with amplitude  $|A| = 1$  m on deep water, the drift force is

$$F'_d = (1020 \times 9.81/4) 1^2 = 2500 \text{ N/m}. \quad (4.163)$$

The drift force is twice as large if the wave is completely reflected instead of completely absorbed, because of the equally large but oppositely directed momentum transport due to the reflected wave [cf. Eq. (4.154)].

A two-dimensional wave-energy absorber (or an infinitely long array of equally interspaced three-dimensional wave absorbers) may partly reflect and partly transmit the incident wave as indicated in Figure 4.6. In this case, the drift force is given by

$$F'_d = \mathcal{I}_{x \text{ lhs}} - \mathcal{I}_{x \text{ rhs}}, \quad (4.164)$$

where the two terms represent the momentum transport on the left-hand side (lhs) and right-hand side (rhs) of the absorbing system indicated in Figure 4.6. Thus,

$$F'_d = \frac{\rho g}{4} \frac{D(kh)}{\tanh(kh)} (|A|^2 + |B|^2 - |A_t|^2), \quad (4.165)$$

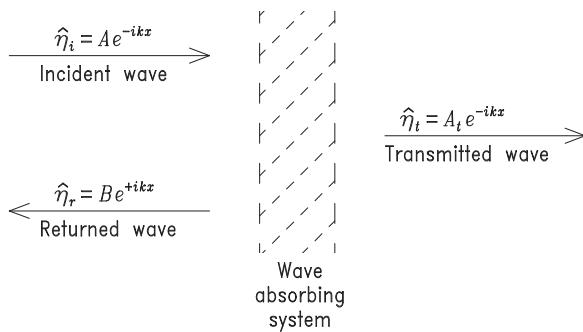


Figure 4.6: Wave which is incident upon a wave-absorbing system is partly absorbed, partly transmitted downstream and partly reflected and/or radiated upstream.

where  $A_t$  denotes the complex amplitude of the transmitted wave. Applying Eqs. (4.129) and (4.130) for the wave-energy transport and using the principle of conservation of energy, we obtain

$$\begin{aligned} F'_d &= \frac{\rho g^2 D(kh)}{4\omega} \frac{k}{\omega} (|A|^2 + |B|^2 - |A_t|^2) = \frac{k}{\omega} (J_i + J_r - J_t) \\ &= \frac{k}{\omega} (J_i - J_r - J_t + 2J_r) = \frac{k}{\omega} (P' + 2J_r), \end{aligned} \quad (4.166)$$

where  $P'$  is the absorbed power per unit width. Further,  $J_i$ ,  $J_r$  and  $J_t$  are the wave-energy transport for the incident, reflected and transmitted waves, respectively. Assuming that  $J_r$  is positive, we have  $F'_d > (k/\omega)P' = F'_{d\min}$ . As an example, let us consider absorption of 0.5 MW from a wave of period  $T = 10$  s in deep water ( $\omega/k = g/\omega$ ). This is associated with a drift force

$$F_d > F_{d\min} = \frac{2\pi}{10} \frac{1}{9.81} 5 \times 10^5 = 32 \times 10^3 \text{ N} (\approx 3.2 \text{ tons}). \quad (4.167)$$

## 4.5 Real Ocean Waves

Harmonic waves were the main subject of the previous section. Such waves are also called ‘regular waves’, as opposed to the real ‘irregular’ waves of the ocean. The swells—that is, travelling waves which have left their regions of generation by winds—are closer to harmonic waves than the more irregular, locally wind-generated sea waves.

Irregular waves are of a stochastic nature, and they may be considered, at least approximately, as a superposition of many different frequencies. Usually only statistical information is, at best, available for the amplitude, the phase and the direction of propagation for each individual harmonic wave. Most of the energy content of ocean waves is associated with waves of periods in the interval from 5 s to 15 s. According to Eq. (4.100), the corresponding interval of deep-water wavelengths is from 40 m to 350 m.

The present section relates wave spectra to superposition of plane waves on deep water, or otherwise water of finite but constant depth. For more thorough studies, readers may consult other literature [30, 31].

For a progressive harmonic plane wave, the stored energy per unit surface is

$$E = E_k + E_p = 2E_k = 2E_p = \frac{\rho g}{2} |\hat{\eta}_f|^2 = \rho g \overline{\eta^2(x, y, t)} \quad (4.168)$$

according to linear wave theory. See Eqs. (4.85), (4.115) and (4.125). Note that for a harmonic plane wave,  $\overline{\eta^2}$  is independent of  $x$  and  $y$ . Still assuming linear theory, the superposition principle is applicable, which means that a real sea state may be described in terms of components of harmonic waves. Correspondingly, we may generalise and rewrite Eq. (4.168) as

$$E = \rho g \overline{\eta^2(x, y, t)} = \rho g \int_0^\infty S(f) df, \quad (4.169)$$

where

$$\overline{\eta^2(x, y, t)} = \int_0^\infty S(f) df \equiv \frac{H_s^2}{16}. \quad (4.170)$$

Here  $S(f)$  is called the energy spectrum, or simply the spectrum. The so-called *significant wave height*  $H_s$  is defined as four times the square root of the integral of the spectrum. Sometimes the wave spectrum is defined in terms of the angular frequency  $\omega = 2\pi f$ . The corresponding spectrum may be written  $S_\omega(\omega)$ , where

$$\int_0^\infty S_\omega(\omega) d\omega = \int_0^\infty S(f) df. \quad (4.171)$$

Thus,

$$S(f) = S_f(f) = 2\pi S_\omega(2\pi f) = 2\pi S_\omega(\omega), \quad (4.172)$$

where the units of  $S$  and  $S_\omega$  are  $\text{m}^2/\text{Hz}$  and  $\text{m}^2\text{s}/\text{rad}$ , respectively.

In a more precise spectral description, the direction of propagation and the phase of each harmonic component should also be taken into account. A harmonic plane wave with direction of incidence given by  $\beta$  (the angle between the propagation direction and the  $x$ -axis) has an elevation given by

$$\eta(x, y, t) = \operatorname{Re}\{\eta_i \exp[i(\omega t - kx \cos \beta - ky \sin \beta)]\} \quad (4.173)$$

[cf. Eq. (4.98)]. Note that the complex amplitude  $\eta_i = \eta_i(\omega) = |\eta_i(\omega)|e^{i\psi}$  contains information on the phase  $\psi$  as well as the amplitude  $|\eta_i|$ . Consider a general sea state  $\eta(x, y, t)$  decomposed into harmonic components

$$\eta(x, y, t) = \sum_m \sum_n \operatorname{Re}\{\eta_i(\omega_m, \beta_n) \exp[i(\omega_m t - k_m x \cos \beta_n - k_n y \sin \beta_n)]\}, \quad (4.174)$$

where  $\omega_m$  and  $k_m$  are related through the dispersion relationship (4.54). Although any frequency  $\omega_m > 0$  and any angle of incidence  $-\pi < \beta_n \leq \pi$  are, in principle, possible in this summation, only certain finite intervals for  $\omega_m$  and  $\beta_n$  contribute significantly to the complete irregular wave in most cases. For convenience, we shall assume that in the sums over  $\omega_m$  and  $\beta_n$  we have  $\omega_{m+1} - \omega_m = \Delta\omega$  independent of  $m$ , and  $\beta_{n+1} - \beta_n = \Delta\beta$  independent of  $n$ .

Taking the square of the elevation (4.174) and then averaging with respect to time as well as the horizontal coordinates  $x$  and  $y$  results in

$$\langle \eta^2 \rangle = \sum_m \sum_n \frac{1}{2} |\eta_i(\omega_m, \beta_n)|^2. \quad (4.175)$$

Introducing the direction-resolved energy spectrum

$$s(f, \beta) = 2\pi s_\omega(\omega, \beta), \quad (4.176)$$

where

$$\Delta\omega \Delta\beta s_\omega(\omega_m, \beta_n) = \frac{1}{2} |\eta(\omega_m, \beta_n)|^2, \quad (4.177)$$

and replacing the sums by integrals, we rewrite the wave-elevation variance as

$$\langle \eta^2 \rangle = \int_0^\infty \int_{-\pi}^\pi s_\omega(\omega, \beta) d\beta d\omega = \int_0^\infty \int_{-\pi}^\pi s(f, \beta) d\beta df. \quad (4.178)$$

The (direction-integrated) energy spectrum is

$$S(f) = \int_{-\pi}^\pi s(f, \beta) d\beta. \quad (4.179)$$

In accordance with Eq. (4.168), the average stored energy per unit horizontal surface is

$$\langle E \rangle = \rho g \langle \eta^2 \rangle = \rho g \int_0^\infty \int_{-\pi}^\pi s(f, \beta) d\beta df = \rho g \int_0^\infty S(f) df. \quad (4.180)$$

From wave measurements, much statistical information has been obtained for the spectral functions  $S(f)$  and  $s(f, \beta)$  from various ocean regions. It has been found that wave spectra have some general characteristics which may be approximately described by semi-empirical mathematical relations. The most well-known functional relation is the Pierson–Moskowitz (PM) spectrum,

$$S(f) = (A/f^5) \exp(-B/f^4) \quad (4.181)$$

(for  $f > 0$ ). Various parameterisations have been proposed for  $A$  and  $B$ . One possible variant is

$$A = BH_s^2/4, \quad B = (5/4)f_p^4. \quad (4.182)$$

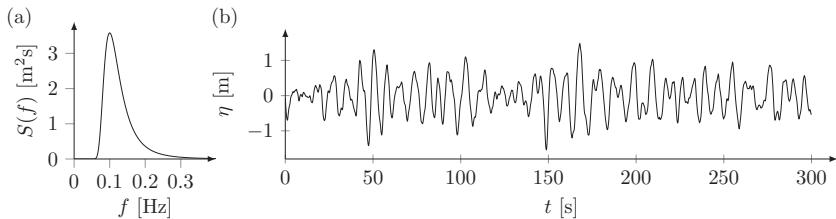


Figure 4.7: (a) Example of a Pierson–Moskowitz spectrum with  $H_s = 2$  m and  $T_p = 10$  s. (b) Single realisation of wave elevation at  $(x, y) = (0, 0)$  simulated from the given spectrum.

Here  $H_s$  is the significant wave height and  $f_p$  is the ‘peak frequency’ (the frequency for which  $S$  has its maximum). The corresponding wave period  $T_p = 1/f_p$  is called the ‘peak period’. Note that the energy period  $T_J$  as defined in Eq. (4.189) is different from the peak period  $T_p$ . In general,  $T_J < T_p$ .

For the direction-resolved spectrum, we may write

$$s(f, \beta) = D(\beta, f) S(f), \quad (4.183)$$

where it is required that

$$\int_{-\pi}^{\pi} D(\beta, f) d\beta = 1 \quad (4.184)$$

[cf. Eq. (4.179)]. One proposal for the directional distribution (neglecting its possible frequency dependence) is

$$D(\beta) = \begin{cases} (2/\pi) \cos^2(\beta - \beta_0) & \text{for } |\beta - \beta_0| < \pi/2 \\ 0 & \text{otherwise,} \end{cases} \quad (4.185)$$

where  $\beta_0$  is the predominant angle of incidence.

When a real irregular wave is simulated from a specified spectrum, a common procedure is to sum a finite number of wave components according to Eq. (4.174). Care must be exercised when selecting the complex amplitudes  $\eta(\omega_m, \beta_n)$  and the number of wave components to ensure that the resulting wave has the correct statistical properties. Figure 4.7 shows an example of a simulated wave elevation at  $(x, y) = (0, 0)$ , which we have generated from a Pierson–Moskowitz spectrum defined in Eq. (4.181), with  $H_s = 2$  m and  $T_p = 10$  s.

#### 4.5.1 Wave-Energy Transport of Irregular Waves

Because of the superposition principle, Eq. (4.135)—which is valid for a progressive plane harmonic wave—can be applied to a superposition of progressive plane harmonic waves. Thus,

$$J = \rho g \sum_m \frac{1}{2} |\eta_i(\omega_m)|^2 v_g(\omega_m) = \rho g \sum_m \Delta\omega S_\omega(\omega_m) v_g(\omega_m). \quad (4.186)$$

Replacing the sums by integrals, we have an expression of the energy transport for irregular plane waves:

$$J = \rho g \int_0^\infty S_\omega(\omega) v_g(\omega) d\omega = \rho g \int_0^\infty S(f) v_g(f) df. \quad (4.187)$$

On deep water, an expression corresponding to Eq. (4.131), which is applicable for regular waves, can be obtained for irregular waves by noting that  $v_g = g/2\omega$  for deep water, and thus,

$$J = \frac{\rho g^2}{2} \int_0^\infty S_\omega(\omega) \omega^{-1} d\omega = \frac{\rho g^2}{4\pi} \int_0^\infty S(f) f^{-1} df. \quad (4.188)$$

Defining the *energy period*  $T_J$  as

$$T_J = \frac{2\pi \int_0^\infty S_\omega(\omega) \omega^{-1} d\omega}{\int_0^\infty S_\omega(\omega) d\omega} = \frac{\int_0^\infty S(f) f^{-1} df}{\int_0^\infty S(f) df} \quad (4.189)$$

and using relationship (4.170), we have, for plane irregular waves on deep water,

$$J = \frac{\rho g^2}{64\pi} T_J H_s^2 = (488 \text{ Ws}^{-1}\text{m}^{-3}) T_J H_s^2. \quad (4.190)$$

As an example, for an irregular wave with energy period  $T_J = 10$  s and significant wave height  $H_s = 2$  m, the energy transport is  $J \approx 20$  kW/m.

## 4.6 Circular Waves

Using the method of separation of variables in a Cartesian  $(x, y, z)$  coordinate system we have studied plane-wave solutions of the Laplace equation (4.33) with the free-surface and sea-bed homogeneous boundary conditions (4.41) and (4.40), respectively. Explicit expressions for plane-wave solutions are given, for instance, by Eq. (4.80) or by Eq. (4.88) with Eq. (4.98). These are examples of plane waves propagating along the  $x$ -axis, or in a direction which differs by an angle  $\beta$  from the  $x$ -axis, respectively. Another plane-wave solution which includes evanescent plane waves is given by Eq. (4.111).

In the following paragraphs, let us discuss solutions of the problem by using the method of separation of variables in a cylindrical coordinate system  $(r, \theta, z)$ . The form of the Helmholtz equation (4.46) or (4.58) will be affected. The differential equation (4.45) or (4.59) with boundary conditions (4.60) and (4.61) remains, however, unchanged. Hence, we may still utilise the function  $e(kz)$  defined by Eq. (4.52) and the orthogonal set of function  $\{Z_n(z)\}$ , which is defined by Eq. (4.70) and which is a complete set if  $n = 0, 1, 2, \dots$ . In the following, we shall concentrate on the propagating wave (that is,  $n = 0$ ) and leave the evanescent waves aside.

The coordinates in the horizontal plane are related by  $(x, y) = (r \cos \theta, r \sin \theta)$ , and we replace  $H(x, y)$  by  $H(r, \theta)$ . The Helmholtz equation (4.46) now becomes

$$-k^2 H = \nabla_H^2 H = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2}. \quad (4.191)$$

Once more, we use the method of separation of variables to obtain a particular solution

$$H(r, \theta) = R(r) \Theta(\theta). \quad (4.192)$$

Inserting into Eq. (4.191) and dividing by  $H/r^2$  give

$$\frac{r^2}{R} \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + k^2 R \right) = -\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = m^2, \quad (4.193)$$

where  $m$  is a constant because the left-hand side is a function of  $r$  only, while the right-hand side is independent of  $r$ . Hence,

$$\frac{d^2 \Theta}{d\theta^2} = \Theta''(\theta) = -m^2 \Theta, \quad (4.194)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( k^2 - \frac{m^2}{r^2} \right) R = 0. \quad (4.195)$$

The general solution of Eq. (4.194) may be written as

$$\Theta = \Theta_m = c_c \cos(m\theta) + c_s \sin(m\theta), \quad (4.196)$$

where  $c_c$  and  $c_s$  are integration constants. We require an unambiguous solution; that is

$$\Theta_m(\theta + 2\pi) = \Theta_m(\theta). \quad (4.197)$$

This requires  $m$  to be an integer:  $m = 0, 1, 2, 3, \dots$ . Since there is also an integration constant in the solution for  $R(r)$ , we can set

$$\Theta_{\max} = 1, \quad (4.198)$$

such that

$$c_c^2 + c_s^2 = 1. \quad (4.199)$$

Then we may write

$$\begin{aligned} \Theta_m(\theta) &= \cos(m\theta + \psi_m) \\ &= \cos(\psi_m) \cos(m\theta) - \sin(\psi_m) \sin(m\theta), \end{aligned} \quad (4.200)$$

where  $\psi_m$  is an integration constant which is not currently specified.

Before considering the general solution of the Bessel differential equation (4.195), we note that in an ideal fluid which is non-viscous and, hence, loss-free, the power  $P_r$  which passes an envisaged cylinder with large radius  $r$  has to be independent of  $r$ ; that is

$$P_r = J 2\pi r \quad (4.201)$$

is independent of  $r$ . Hence, the energy transport  $J$  is inversely proportional to the distance  $r$ . For large  $r$ , the circular wave is approximately plane (the curvature of the wave front is of little importance). Then Eq. (4.130) for  $J$  applies. This shows that the wave-elevation amplitude  $|\hat{\eta}|$  is proportional to  $r^{-1/2}$ . Hence,  $\hat{\phi}$  is proportional to  $r^{-1/2}$  [because  $\hat{\phi} = (-g/i\omega)e(kz)\hat{\eta}$  according to Eq. (4.88)], and thus,  $|R(r)|$  is proportional to  $r^{-1/2}$ . However,  $R(r)$  also has, for an outgoing wave, a phase which varies with  $r$  as  $-kr$ . For large  $r$ , we then expect that

$$\hat{\phi} \propto r^{-1/2} e^{-ikr} e(kz), \quad (4.202)$$

where the coefficient of proportionality may be complex.

Bessel's differential equation (4.195) has two linearly independent solutions  $J_m(kr)$  and  $Y_m(kr)$ , which are Bessel functions of order  $m$ . The function  $J_m(kr)$  is of the first kind and  $Y_m(kr)$  is of the second kind. The function  $J_m(kr)$  is finite for  $r = 0$ , whereas  $Y_m(kr) \rightarrow \infty$  when  $r \rightarrow 0$ . Other singularities do not exist for  $r \neq \infty$ . For large  $r$ ,  $J_m(kr)$  and  $Y_m(kr)$  approximate cosine and sine functions multiplied by  $r^{-1/2}$ . In analogy with Euler's formulas

$$e^{ikr} = \cos(kr) + i \sin(kr), \quad (4.203)$$

$$e^{-ikr} = \cos(kr) - i \sin(kr), \quad (4.204)$$

the Hankel functions of order  $m$  and of the first kind and second kind are defined by

$$H_m^{(1)}(kr) = J_m(kr) + i Y_m(kr), \quad (4.205)$$

$$H_m^{(2)}(kr) = J_m(kr) - i Y_m(kr), \quad (4.206)$$

respectively. Note that

$$H_m^{(1)}(kr) = [H_m^{(2)}(kr)]^*. \quad (4.207)$$

Asymptotically, we have [32]

$$H_m^{(2)}(kr) \approx \sqrt{\frac{2}{\pi kr}} \exp \left[ -i \left( kr - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right] \left( 1 + \mathcal{O} \left\{ \frac{1}{kr} \right\} \right) \quad (4.208)$$

as  $kr \rightarrow \infty$ . Hence, the solution of the differential equation (4.195) can be written as

$$R(r) = a_m H_m^{(2)}(kr) + b_m H_m^{(1)}(kr), \quad (4.209)$$

where  $a_m$  and  $b_m$  are integration constants. From the asymptotic expression (4.208), we see that  $H_m^{(2)}(kr)$  represents an outgoing (divergent) wave and from Eq. (4.207) that  $H_m^{(1)}(kr)$  represents an incoming (convergent) wave. [It might here be compared with the two terms in each of Eqs. (4.80), (4.81) and (4.83) for the case of propagating plane waves.]

In Section 4.1, it was mentioned that the boundary conditions might have to be supplemented by a *radiation condition*. Let us now introduce the radiation

condition of outgoing wave at infinity, which means we have to set  $b_m = 0$ . Then we have from Eqs. (4.51), (4.192) and (4.209) the solution

$$\hat{\phi} = a_m H_m^{(2)}(kr) \Theta_m(\theta) e(kz) \quad (4.210)$$

for the complex amplitude of the velocity potential. A sum of similar solutions

$$\hat{\phi} = \sum_{m=0}^{\infty} a_m H_m^{(2)}(kr) \Theta_m(\theta) e(kz) \quad (4.211)$$

is also a solution which satisfies the radiation condition, the dispersion equation (4.54) and Laplace's equation (4.33). This solution satisfies the homogeneous boundary conditions (4.40) and (4.41) on the sea bed ( $z = -h$ ) and on the free water surface ( $z = 0$ ).

According to the asymptotic expression (4.208), the particular solution (4.211)—of the partial differential equation (4.191)—is, in the asymptotic limit  $kr \gg 1$ ,

$$\hat{\phi} \approx A(\theta) e(kz) (kr)^{-1/2} e^{-ikr}, \quad (4.212)$$

where

$$A(\theta) \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} a_m \Theta_m(\theta) \exp \left[ i \left( \frac{m\pi}{2} + \frac{\pi}{4} \right) \right] \quad (4.213)$$

is the so-called far-field coefficient of the outgoing wave. If we insert Eq. (4.200) for  $\Theta_m(\theta)$ , we may note that Eq. (4.213) is a Fourier-series representation of the far-field coefficient.

The outgoing wave could originate from a radiation source at or close to the origin  $r = 0$ . The  $z$ -variation according to factor  $e(kz)$ , as in the particular solution (4.211), can satisfy the inhomogeneous boundary conditions, such as Eq. (4.15) or (4.26), at the radiation source only in simple particular cases. A more general solution which satisfies Laplace's equation (4.33), the boundary conditions and the radiation condition as  $r \rightarrow \infty$  is

$$\hat{\phi} = \hat{\phi}_l(r, \theta, z) + A(\theta) e(kz) (kr)^{-1/2} e^{-ikr}. \quad (4.214)$$

Here the last term represents the so-called *far field* while the first term  $\hat{\phi}_l$  represents the *near field*, a local velocity potential which connects the far field to the radiation source in such a way that the inhomogeneous boundary conditions at the radiation source are satisfied. The term  $\hat{\phi}_l(r, \theta, z)$  may contain terms corresponding to evanescent waves. For large  $r$ , such terms decay exponentially as  $\exp(-m_1 r)$  (or faster) with increasing  $r$ , where  $m_1$  is the smallest positive solution of Eq. (4.71)—that is,  $\pi/(2h) < m_1 < \pi/h$ . (See Problems 4.3 and 5.6.) It can be shown that, in general, the local potential  $\hat{\phi}_l$  decreases with distance  $r$  at least as fast as  $1/r$  when  $r \rightarrow \infty$  (cf. Wehausen and Laitone [33], pp. 475–478).

The wave elevation corresponding to Eq. (4.214) is given by

$$\hat{\eta} = \hat{\eta}(r, \theta) = -\frac{i\omega}{g}\hat{\phi}_l(r, \theta, 0) - \frac{i\omega}{g}A(\theta)(kr)^{-1/2}e^{-ikr}, \quad (4.215)$$

which is obtained by using Eq. (4.39). For large values of  $kr$ , where  $\hat{\phi}_l$  is negligible,  $(kr)^{-1/2}$  varies relatively little over one wavelength, which means that the curvature of the wave front is of minor importance. Then we may use the energy transport formula (4.130) for plane waves and there replace the wave-elevation amplitude  $|A|$  by  $|(\omega/g)A(\theta)(kr)^{-1/2}|$ . The radiated energy transport per unit wave frontage then becomes

$$J(r, \theta) = \frac{1}{kr} \frac{D(kh)\rho g^2}{4\omega} \frac{\omega^2}{g^2} |A(\theta)|^2 = \frac{\omega\rho D(kh)}{4kr} |A(\theta)|^2. \quad (4.216)$$

The radiated power is

$$P_r = \int_0^{2\pi} J(r, \theta) r d\theta = \frac{\omega\rho D(kh)}{4k} \int_0^{2\pi} |A(\theta)|^2 d\theta. \quad (4.217)$$

For a circularly symmetric radiation source, the far-field coefficient  $A(\theta)$  must be independent of  $\theta$ . Hence, according to Eq. (4.213), we have

$$A(\theta) = \sqrt{2/\pi}a_0 e^{i\pi/4}. \quad (4.218)$$

Hence, the power associated with the outgoing wave is

$$P_r = \frac{\omega\rho D(kh)}{k} |a_0|^2 = \frac{\pi\omega\rho D(kh)}{2k} |A|^2, \quad (4.219)$$

where  $A$  is now the  $\theta$ -independent far-field coefficient. This is a relation we may use in combination with Eq. (3.29) to obtain an expression for the radiation resistance for an oscillating body which generates an axisymmetric (circularly symmetric) wave. See also Section 5.5.3 and, in particular, Eq. (5.183).

## 4.7 A Useful Integral Based on Green's Theorem

This section is a general mathematical preparation for further studies in the following sections and chapters. We start this section by stating that Green's theorem

$$\oint\oint \left( \varphi_i \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_i}{\partial n} \right) dS = 0 \quad (4.220)$$

applies to two arbitrary differentiable functions  $\varphi_i$  and  $\varphi_j$ , both of which satisfy the (three-dimensional) Helmholtz equation

$$\nabla^2 \varphi = \lambda \varphi \quad (4.221)$$

within the volume  $V$  contained inside the closed surface of integration. In the integrand,  $\partial/\partial n = \vec{n} \cdot \nabla$  is the normal component of the gradient on the surface of integration. The ‘eigenvalue’  $\lambda$  is a constant, whereas  $\varphi_i$  and  $\varphi_j$  depend on the spatial coordinates. In particular, Green’s theorem is applicable to two functions,  $\phi_i$  and  $\phi_j$ , which satisfy Laplace’s equation (corresponding to  $\lambda = 0$ ).

Green’s theorem follows from Gauss’s divergence theorem

$$\iiint_V \nabla \cdot \vec{A} dV = \oint S A_n dS = \oint S \vec{n} \cdot \vec{A} dS \quad (4.222)$$

if we consider

$$\vec{A} = \varphi_i \nabla \varphi_j - \varphi_j \nabla \varphi_i. \quad (4.223)$$

Then

$$A_n = \varphi_i \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_i}{\partial n} \quad (4.224)$$

and

$$\begin{aligned} \nabla \cdot \vec{A} &= \varphi_i \nabla^2 \varphi_j + \nabla \varphi_i \cdot \nabla \varphi_j - \varphi_j \nabla^2 \varphi_i - \nabla \varphi_j \cdot \nabla \varphi_i \\ &= \varphi_i \nabla^2 \varphi_j - \varphi_j \nabla^2 \varphi_i. \end{aligned} \quad (4.225)$$

In view of the Helmholtz equation (4.221), we have

$$\nabla \cdot \vec{A} = \varphi_i \lambda \varphi_j - \varphi_j \lambda \varphi_i = 0, \quad (4.226)$$

from which Green’s theorem follows as a corollary to Gauss’s theorem.

Let us consider a finite region of the sea containing one or more structures, fixed or oscillating, as indicated in Figure 4.8. It is assumed that outside this region, the sea is unbounded in all horizontal directions. Moreover, the water is deep there, or otherwise the water has a constant depth  $h$ , outside the

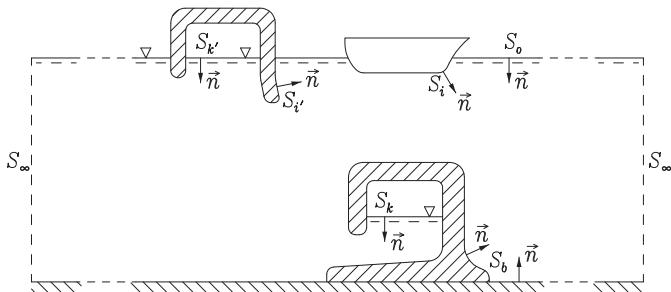


Figure 4.8: System of bodies and chambers for air-pressure distributions (OWCs) contained within an imaginary cylindrical control surface  $S_\infty$ . Wetted surfaces of oscillating bodies are indicated by  $S_r$  and  $S_i$ , whereas  $S_k$  and  $S_{k'}$  denote internal water surfaces. Fixed surfaces, including the sea bed, are given by  $S_b$ , and  $S_0$  denotes the external free water surface. The arrows indicate unit normals pointing into the fluid region.

mentioned finite region. The structures, some of which may contain air chambers above OWCs, diffract incoming waves. The phenomenon of diffraction occurs when the incident plane wave alone, such as given by Eq. (4.88), violates the boundary conditions (4.34) and (4.35) on the structures. Furthermore, waves may be generated by oscillating bodies or by oscillating air pressures above the water, such as in the entrapped air within the chambers indicated in the figure.

Let us next apply Green's theorem to the fluid region shown in Figure 4.8. The fluid region is contained inside a closed surface composed of the following surfaces:

$S$ , which is the sum (union) of all  $N_k$  internal water surfaces  $S_k$  above the OWCs and of the wet surface  $S_i$  of all  $N_i$  bodies, i.e.,  $S = \sum_{k=1}^{N_k} S_k + \sum_{i=1}^{N_i} S_i$   
 $S_0$ , which is the free water surface (at  $z = 0$ ) external to bodies and chamber structures for OWCs

$S_b$ , which is the sum (union) of all wet surfaces of fixed rigid structures, including fixed OWC chamber structures and the sea bed (which is the plane  $z = -h$  in case the sea bed is horizontal)

$S_\infty$ , which is a 'control' surface, an envisaged vertical cylinder. If this cylinder is circular, we may denote its radius by  $r$ . In many cases, we consider the limit  $r \rightarrow \infty$

If the extension of  $S_0$  and of the fluid region is not infinite—that is, if the fluid is contained in a finite basin—the surfaces  $S_0$  and  $S_b$  intersect along a closed curve, and  $S_\infty$  does not come into play. We may also consider a case in which an infinite coast line intersects the chosen control surface  $S_\infty$ .

Let  $\phi_i$  and  $\phi_j$  be two arbitrary functions which satisfy Laplace's equation

$$\nabla^2 \phi_{i,j} = 0 \quad (4.227)$$

and the homogeneous boundary conditions:

$$\frac{\partial}{\partial n} \phi_{i,j} = 0 \quad \text{on } S_b, \quad (4.228)$$

$$\left( \omega^2 - g \frac{\partial}{\partial z} \right) \phi_{i,j} = \left( \omega^2 + g \frac{\partial}{\partial n} \right) \phi_{i,j} = 0 \quad \text{on } S_0. \quad (4.229)$$

We define the very useful integral

$$I(\phi_i, \phi_j) \equiv \iint_S \left( \phi_i \frac{\partial \phi_j}{\partial n} - \frac{\partial \phi_i}{\partial n} \phi_j \right) dS. \quad (4.230)$$

Next, we shall show that, instead of integrating over  $S$ , the totality of wave-generating surfaces, we may integrate over  $S_\infty$ , a 'control' surface in the far-field region of waves radiated from  $S$ . Note that the integrand in Eq. (4.230) vanishes on  $S_b$  and on  $S_0$  because  $(\partial \phi_{i,j}/\partial n) = 0$  on  $S_b$  and

$$\phi_i \frac{\partial \phi_j}{\partial n} - \frac{\partial \phi_i}{\partial n} \phi_j = -\phi_i \frac{\omega^2}{g} \phi_j + \frac{\omega^2}{g} \phi_i \phi_j = 0 \quad \text{on } S_0. \quad (4.231)$$

Hence, it follows from Green's theorem (4.220) that

$$I(\phi_i, \phi_j) = - \iint_{S_\infty} \left( \phi_i \frac{\partial \phi_j}{\partial n} - \frac{\partial \phi_i}{\partial n} \phi_j \right) dS \quad (4.232)$$

or

$$I(\phi_i, \phi_j) = \iint_{S_\infty} \left( \phi_i \frac{\partial \phi_j}{\partial r} - \frac{\partial \phi_i}{\partial r} \phi_j \right) dS. \quad (4.233)$$

Note that if the control surface  $S_\infty$  is a cylinder which is not circular, this formula still applies, provided  $\partial/\partial r$  is interpreted as the normal component of the gradient on  $S_\infty$  pointing in the outwards direction. Note that Eq. (4.230) is a definition, while Eq. (4.233) is a theorem.

Let us make the following comments:

- (i)  $\phi_i$  and  $\phi_j$  may represent velocity potentials of waves generated by the oscillating rigid-body surfaces  $S_i$  or oscillating internal water surfaces  $S_k$ . Alternatively, they may represent other kinds of waves. The constraints on  $\phi_i$  and  $\phi_j$  are that they have to satisfy the homogeneous boundary conditions (4.228) and (4.229) on  $S_b$  and  $S_0$ .
- (ii)  $\phi_i$  and/or  $\phi_j$  may be matrices provided they (that is, all matrix elements) satisfy the mentioned homogeneous conditions. If both  $\phi_i$  and  $\phi_j$  are matrices which do not commute, their order of appearance in products is important.
- (iii) We may (for real  $\omega$ ) replace  $\phi_i$  and/or  $\phi_j$  by their complex conjugate, because  $\phi_i^*$  and  $\phi_j^*$  also satisfy Laplace's equation and the homogeneous boundary conditions on  $S_b$  and  $S_0$ .
- (iv) If at least one of  $\phi_i$  and  $\phi_j$  is a scalar function, or if they otherwise commute, we have

$$I(\phi_j, \phi_i) = -I(\phi_i, \phi_j). \quad (4.234)$$

- (v) Observe that we could include some finite part of surface  $S_b$  to belong to surface  $S$ . For instance, the surface of a fixed piece of rock on the sea bed (or of the fixed sea bed-mounted OWC structure shown in Figure 4.8) could be included in the surface  $S$  as a wet surface oscillating with zero amplitude.
- (vi) Definition (4.230) and theorem (4.233) are also valid if the boundary condition (4.228) on  $S_b$  is replaced by

$$\frac{\partial}{\partial n} \phi_{i,j} + C_n \phi_{i,j} = 0, \quad (4.235)$$

where  $C_n$  is a complex constant or a complex function given along the surface  $S_b$ . For this to be true when  $C_n$  is not real, however, the two functions  $\phi_i$  and  $\phi_j$  have to satisfy the same radiation condition; that means

it would not be allowable to replace just one of the two functions by its complex conjugate.

We next consider the integral (4.233) when  $\phi_i$  and  $\phi_j$ , in addition to satisfying the homogeneous boundary conditions (4.228) and (4.229), also satisfy a radiation condition as  $r \rightarrow \infty$ .

Waves satisfying the radiation condition of outgoing waves at infinite distance from the wave source, such as diffracted waves or radiated waves, have—in the three-dimensional case—an asymptotic expression of the type

$$\phi = \psi \sim A(\theta)e(kz)(kr)^{-1/2}e^{-ikr} \quad \text{as } kr \rightarrow \infty \quad (4.236)$$

[see Eqs. (4.212) and (4.214)]. In the remaining part of Section 4.7, and in Section 4.8, let us use the symbol  $\psi$  to denote the velocity potential of a general wave which satisfies the radiation condition. We have

$$\frac{\partial \psi}{\partial r} \sim -ik\psi \quad (4.237)$$

when we include only the dominating term of the asymptotic expansion in the far-field region. Thus, for the two waves  $\psi_i$  and  $\psi_j$ , we have

$$\lim_{r \rightarrow \infty} \iint_{S_\infty} \left( \psi_i \frac{\partial \psi_j}{\partial r} - \frac{\partial \psi_i}{\partial r} \psi_j \right) dS = 0. \quad (4.238)$$

Hence, according to Eq. (4.233),

$$I(\psi_i, \psi_j) = 0. \quad (4.239)$$

Note that also

$$I(\psi_i^*, \psi_j^*) = 0 \quad (4.240)$$

because

$$\frac{\partial \psi^*}{\partial r} \sim ik\psi^*. \quad (4.241)$$

However, since  $\psi_i$  and  $\psi_j^*$  satisfy opposite radiation conditions,

$$I(\psi_i, \psi_j^*) \neq 0. \quad (4.242)$$

We have

$$\begin{aligned} I(\psi_i, \psi_j^*) &= \lim_{r \rightarrow \infty} \iint_{S_\infty} A_i(\theta)A_j^*(\theta)e^2(kz)\frac{2ik}{kr} dS \\ &= 2i \int_{-h}^0 e^2(kz) dz \int_0^{2\pi} A_i(\theta)A_j^*(\theta) d\theta. \end{aligned} \quad (4.243)$$

Using Eq. (4.108) gives

$$I(\psi_i, \psi_j^*) = i \frac{D(kh)}{k} \int_0^{2\pi} A_i(\theta)A_j^*(\theta) d\theta. \quad (4.244)$$

Note that since the two terms of the integrand  $\psi_i(\partial\psi_j^*/\partial r) - (\partial\psi_i/\partial r)\psi_j^*$  have equal contributions in the far-field region, we also have

$$I(\psi_i, \psi_j^*) = \lim_{r \rightarrow \infty} 2 \iint_{S_\infty} \psi_i \frac{\partial\psi_j^*}{\partial r} dS. \quad (4.245)$$

For the two-dimensional case, possible evanescent waves—set up as a result of inhomogeneous boundary conditions in the finite part of the fluid region (see Figure 4.8)—are negligible in the far-field region. Thus, all terms corresponding to  $n \geq 1$  in Eq. (4.111) may be omitted there. The remaining terms for  $n = 0$  correspond to Eq. (4.83). Consequently, for waves satisfying the radiation condition, we have the far-field asymptotic expression

$$\phi = \psi \sim -\frac{g}{i\omega} A^\pm e(kz) e^{\mp ikx} = -\frac{g}{i\omega} A^\pm e(kz) e^{-ik|x|}. \quad (4.246)$$

The upper signs apply for  $x \rightarrow +\infty$  ( $\theta = 0$ ), and the lower signs for  $x \rightarrow -\infty$  ( $\theta = \pi$ ). The constants  $A^+$  and  $A^-$  represent wave elevations.

Consider now a vertical cylinder  $S_\infty$  with rectangular cross section of width  $d$  in the  $y$ -direction and of an arbitrarily large length in the  $x$ -direction. Then

$$\begin{aligned} I(\psi_i, \psi_j^*) &= \iint_{S_\infty} \left( \psi_i \frac{\partial\psi_j^*}{\partial r} - \frac{\partial\psi_i}{\partial r} \psi_j^* \right) dS \\ &= \left( \frac{-g}{i\omega} \right) \left( \frac{-g}{-i\omega} \right) (A_i^+ A_j^{+\ast} + A_i^- A_j^{-\ast}) 2ik \int_{-h}^0 e^2(kz) dz \int_{-d/2}^{d/2} dy. \end{aligned} \quad (4.247)$$

Per unit width in the  $y$ -direction, we have

$$I'(\psi_i, \psi_j^*) \equiv \frac{I(\psi_i, \psi_j^*)}{d} = i \left( \frac{g}{\omega} \right)^2 D(kh) (A_i^+ A_j^{+\ast} + A_i^- A_j^{-\ast}). \quad (4.248)$$

Also in the two-dimensional case it is easily seen that, for two waves satisfying the same radiation condition, we have

$$I'(\psi_i, \psi_j) = 0, \quad I'(\psi_i^*, \psi_j^*) = 0. \quad (4.249)$$

## 4.8 Far-Field Coefficients and Kochin Functions

In Section 4.6, we discussed circular waves and derived the asymptotic expression (4.212) for an outgoing wave. At very large distance ( $r \rightarrow \infty$ ), all the waves diffracted and radiated from the structures shown in Figure 4.8 seem to originate from a region near the origin ( $r = 0$ ). The resulting outgoing wave is, in linear theory, given by an asymptotic expression such as (4.212), where the far-field coefficient is a superposition of outgoing waves from the individual diffracting and radiating structures; that is,

$$A(\theta) = \sum_i A_i(\theta), \quad (4.250)$$

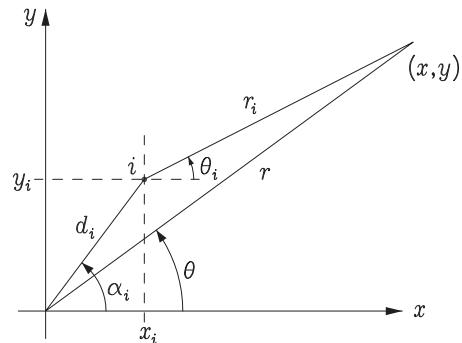


Figure 4.9: Horizontal coordinates, Cartesian  $(x, y)$  and polar  $(r, \theta)$ . Global coordinates with respect to the common origin  $O$ . Local coordinates with respect to the local origin  $O'$  of the (radiating and/or diffracting) structure number  $i$ .

where it is summed over all the sources for outgoing circular waves. Let

$$(x_i, y_i) = (d_i \cos \alpha_i, d_i \sin \alpha_i) \quad (4.251)$$

represent the local origin (or the vertical line through a reference point, e.g., centre of gravity) of source number  $i$  for the outgoing circular wave. See Figure 4.9 for definition of the lengths  $d_i$  and  $r_i$  and of the angles  $\alpha_i$ ,  $\theta_i$  and  $\theta$ . The horizontal coordinates of an arbitrary point may be written as

$$(x, y) = (r \cos \theta, r \sin \theta) = (x_i + r_i \cos \theta_i, y_i + r_i \sin \theta_i). \quad (4.252)$$

We shall assume that all sources for outgoing waves are located in a bounded region in the neighbourhood of the origin, that is,

$$r \gg \max(d_i) \quad \text{for all } i, \quad (4.253)$$

and that

$$kr \gg 1. \quad (4.254)$$

Then for very large distances, each term of the velocity potential  $\sum \phi_i = \sum \psi_i$  of the resulting outgoing wave is given by an asymptotic expression such as [cf. Eq. (4.212)]

$$\phi_i = \psi_i \sim B_i(\theta_i) e(kz) (kr_i)^{-1/2} \exp(-ikr_i) \quad (4.255)$$

or

$$\phi_i = \psi_i \sim A_i(\theta) e(kz) (kr)^{-1/2} \exp(-ikr). \quad (4.256)$$

Here, the complex function  $B_i(\theta)$  is the far-field coefficient referring to the source's local origin, and  $A_i(\theta)$  is the far-field coefficient, for the same wave source number  $i$ , referring to the common origin. For very distant field points  $(x, y)$ —that is,  $r \gg d_i$ —we have the following asymptotic approximations for

the relation between local and global coordinates (see Figure 4.9):

$$\theta_i \approx \theta, \quad (4.257)$$

$$r_i \approx r - d_i \cos(\alpha_i - \theta), \quad (4.258)$$

$$(kr_i)^{-1/2} \approx (kr)^{-1/2}. \quad (4.259)$$

For the outgoing wave from source number  $i$ , we have

$$\begin{aligned} \phi_i &\sim B_i(\theta_i) e(kz) (kr_i)^{-1/2} \exp(-ikr_i) \\ &\sim B_i(\theta) e(kz) (kr)^{-1/2} \exp[ikd_i \cos(\alpha_i - \theta)] \exp(-ikr). \end{aligned} \quad (4.260)$$

Note that we use a higher-order approximation for  $r_i$  in the phase than in the amplitude (modulus), as also usual in interference and diffraction theories, for instance, in optics.

Hence, we have the following relations between the far-field coefficients

$$\begin{aligned} A_i(\theta) &= B_i(\theta) \exp[ikd_i \cos(\alpha_i - \theta)] \\ &= B_i(\theta) \exp[ik(x_i \cos \theta + y_i \sin \theta)]. \end{aligned} \quad (4.261)$$

For a single axisymmetric body oscillating in heave only, the far-field coefficient  $B_0$ , say, is independent of  $\theta$ . If the vertical symmetry axis coincides with the  $z$ -axis, also the corresponding  $A_0$  is independent of  $\theta$ . However, if  $d_i \neq 0$ ,  $A_0$  varies with  $\theta$ , due to the exponential factor in Eq. (4.261).

Radiated waves and diffracted waves satisfy the radiation condition of outgoing waves at infinite distance. They are represented asymptotically by expressions such as (4.255) or (4.256), where the amplitude, the phase and the direction dependence are determined by the far-field coefficient.

An incident wave  $\Phi$ , for instance [see Eqs. (4.88) and (4.98)],

$$\Phi = \hat{\phi}_0 = -\frac{g}{i\omega} Ae(kz) \exp[-ik(x \cos \beta + y \sin \beta)], \quad (4.262)$$

does not satisfy the radiation condition. Let  $\phi_i$  and  $\phi_j$  be two arbitrary waves, where

$$\phi_{i,j} = \Phi_{i,j} + \psi_{i,j}. \quad (4.263)$$

Here  $\psi_i$  and  $\psi_j$  are two waves which satisfy the radiation condition (of outgoing waves at infinity), and  $\Phi_i$  and  $\Phi_j$  are two arbitrary incident waves. From the definition of the integral (4.230), we see that

$$I(\phi_i, \phi_j) = I(\Phi_i, \Phi_j) + I(\Phi_i, \psi_j) + I(\psi_i, \Phi_j) + I(\psi_i, \psi_j). \quad (4.264)$$

Since  $\psi_i$  and  $\psi_j$  satisfy the same radiation condition, we have from Eq. (4.239) that  $I(\psi_i, \psi_j) = 0$ . Further, since  $\Phi_i$  and  $\Phi_j$  satisfy the homogeneous boundary conditions on the planes  $z = 0$  and  $z = -h$ , and since they satisfy Laplace's equation everywhere in the volume region between these planes, *including* the

volume region occupied by the oscillator structures, it follows from Green's theorem (4.220) and Eq. (4.233) that also  $I(\Phi_i, \Phi_j) = 0$ . Hence,

$$\begin{aligned} I(\phi_i, \phi_j) &= I(\Phi_i, \psi_j) + I(\psi_i, \Phi_j) \\ &= I(\Phi_i, \psi_j) - I(\Phi_j, \psi_i), \end{aligned} \quad (4.265)$$

where we have in the last step also used Eq. (4.234), for which it is necessary to assume that  $\Phi_j$  commutes with  $\psi_i$ .

Let us now consider the two incident waves

$$\Phi_{i,j} = -\frac{g}{i\omega} A_{i,j} e(kz) \exp[-ikr(\beta_{i,j})], \quad (4.266)$$

where

$$r(\beta) \equiv x \cos \beta + y \sin \beta. \quad (4.267)$$

Note that the two waves have different complex elevation amplitudes  $A_i$  and  $A_j$ , and different angles of incidence  $\beta_i$  and  $\beta_j$ . They have, however, equal period  $T = 2\pi/\omega$  and, hence, equal wavelengths  $\lambda = 2\pi/k$ .

Following Newman [34], we define the so-called Kochin function

$$H_j(\beta) \equiv -\frac{k}{D(kh)} I\{e(kz)e^{ikr(\beta)}, \psi_j\}. \quad (4.268)$$

Note that  $H_j(\beta)$  depends on a chosen angle  $\beta$ . Otherwise,  $H_j(\beta)$  is a property of the wave  $\psi_j$  which satisfies the radiation condition. We shall show that  $H_j(\theta)$  is proportional to the far-field coefficient  $A_j(\theta)$  of the wave  $\psi_j$ . According to Eq. (4.267),

$$r(\beta \pm \pi) = -r(\beta). \quad (4.269)$$

Hence,

$$H_j(\beta \pm \pi) = -\frac{k}{D(kh)} I\{e(kz)e^{-ikr(\beta)}, \psi_j\}. \quad (4.270)$$

Using this in combination with the two chosen incident waves  $\Phi_i$  and  $\Phi_j$  given by Eq. (4.266), we have from Eq. (4.265),

$$I(\phi_i, \phi_j) = \frac{gD(kh)}{i\omega k} \{A_i H_j(\beta_i \pm \pi) - A_j H_i(\beta_j \pm \pi)\}. \quad (4.271)$$

Note that this result is based on Eq. (4.265) and hence on Eq. (4.234). Thus, observe that  $A_j$  has to commute with  $H_i$  when Eq. (4.271) is being applied. Also note that the given complex wave-elevation amplitude  $A_j$  here, and in Eq. (4.266), should not be confused with the function  $A_j(\theta)$ , which is the far-field coefficient.

Next, we shall show that

$$H_j(\theta) = \sqrt{2\pi} A_j(\theta) e^{i\pi/4}. \quad (4.272)$$

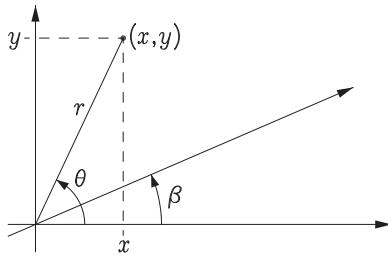


Figure 4.10: Cartesian and polar coordinates in the horizontal plane, with  $\beta$  as the angle of wave incidence.

Using Eqs. (4.233) and (4.256) in Eq. (4.268) gives

$$H_j(\beta) = -\frac{k}{D(kh)} I \left\{ e(kz) e^{ikr(\beta)}, A_j(\theta) e(kz) (kr)^{-1/2} e^{-ikr} \right\}, \quad (4.273)$$

where the integration surface is now  $S_\infty$  (cf. Figure 4.8). It should be observed here that  $r$  is the (horizontal) radial coordinate (see Figure 4.9), whereas  $r(\beta)$  is given by Eq. (4.267).

We shall assume that the control surface  $S_\infty$  is a circular cylinder of radius  $r$ . The integration variables are then  $\theta$  and  $z$  (cf. Figure 4.10), whereas  $r$  is a constant which tends to infinity. Noting that  $r(\beta) \equiv x \cos \beta + y \sin \beta = r \cos \theta \cos \beta + r \sin \theta \sin \beta = r \cos(\theta - \beta)$  and using Eq. (4.108), we have

$$H_j(\beta) = \lim_{kr \rightarrow \infty} \frac{i}{2} \int_0^{2\pi} \sqrt{kr} A_j(\theta) [1 + \cos(\theta - \beta)] \exp\{-ikr[1 - \cos(\theta - \beta)]\} d\theta. \quad (4.274)$$

We now substitute  $\varphi = \theta - \beta$  as the new integration variable. Further, using the asymptotic expression (for  $u \rightarrow \infty$ )

$$\lim_{u \rightarrow \infty} \int_{-\beta}^{2\pi - \beta} f(\varphi) \exp[-iu(1 - \cos \varphi)] d\varphi = \lim_{u \rightarrow \infty} \sqrt{\frac{\pi}{u}} \left[ (1 - i)f(0) + (1 + i)e^{-i2u}f(\pi) \right], \quad (4.275)$$

which is derived in the next paragraph, we obtain

$$H_j(\beta) = \frac{i}{2} (1 - i) \sqrt{\pi} A_j(\beta) (1 + \cos 0) = (1 + i) \sqrt{\pi} A_j(\beta) = \sqrt{2\pi} A_j(\beta) e^{i\pi/4}, \quad (4.276)$$

in accordance with the earlier statement (4.272) which was to be proven.

The previously mentioned mathematical relation (4.275) is derived as follows, by the ‘method of stationary phase’. To be slightly more general, we shall first consider the asymptotic limit (as  $u \rightarrow \infty$ ) of the integral

$$I_{ab} \equiv \int_a^b f(\varphi) e^{-iu\psi(\varphi)} d\varphi, \quad (4.277)$$

where  $f(\varphi)$  and  $\psi(\varphi)$  are analytic functions in the interval  $a < \varphi < b$ . Moreover,  $\psi(\varphi)$  is real and has one, and only one, extremum  $\psi(c)$  in the interval, such that  $\psi'(c) = 0$  and  $\psi''(c) \neq 0$ , where  $a < c < b$ . If we take a constant  $e^{-iu\psi(c)}$  outside

the integral, and if we then let  $u \rightarrow \infty$ , the integrand oscillates infinitely fast with  $\varphi$ , except near the ‘stationary’ point  $\varphi = c$ , where the imaginary exponent varies slowly with  $\varphi$ . Hence, the contribution to the integral is negligible outside the interval  $c - \epsilon < \varphi < c + \epsilon$ , where  $\epsilon$  is a small positive number. Using a Taylor expansion for  $\psi(\varphi)$  around  $\varphi = c$ , we have

$$\lim_{u \rightarrow \infty} I_{ab} = \lim_{u \rightarrow \infty} f(c) e^{-iu\psi(c)} \int_{c-\epsilon}^{c+\epsilon} \exp [-iu\psi''(c)(\varphi - c)^2/2] d\varphi. \quad (4.278)$$

We now take

$$\alpha = (\varphi - c) \sqrt{u\psi''(c) \operatorname{sgn}[\psi''(c)]/2} = (\varphi - c) \sqrt{u|\psi''(c)|/2} \quad (4.279)$$

as the new integration variable. Note that the new integration limits are given by

$$\alpha = \pm \epsilon \sqrt{u|\psi''(c)|/2} \rightarrow \pm \infty \quad \text{as } u \rightarrow \infty. \quad (4.280)$$

This gives

$$\begin{aligned} \lim_{u \rightarrow \infty} I_{ab} &= \lim_{u \rightarrow \infty} f(c) \sqrt{\frac{2}{u|\psi''(c)|}} e^{-iu\psi(c)} \int_{-\infty}^{\infty} \exp [-i\alpha^2 \operatorname{sgn}\{\psi''(c)\}] d\alpha \\ &= \lim_{u \rightarrow \infty} f(c) \sqrt{\frac{2}{u|\psi''(c)|}} e^{-iu\psi(c)} \left[ \int_{-\infty}^{\infty} \cos \alpha^2 d\alpha - i \operatorname{sgn}\{\psi''(c)\} \int_{-\infty}^{\infty} \sin \alpha^2 d\alpha \right] \\ &= \lim_{u \rightarrow \infty} f(c) \sqrt{\frac{\pi}{u|\psi''(c)|}} e^{-iu\psi(c)} [1 - i \operatorname{sgn}\{\psi''(c)\}], \end{aligned} \quad (4.281)$$

since each of the two last integrals is known to have the value  $\sqrt{\pi/2}$ . Note that for  $\psi(\varphi) = 1 - \cos \varphi$ , we have  $\psi'(\varphi) = 0$  for  $\varphi = 0$  and  $\varphi = \pi$ . Moreover, observing that  $\psi(0) = 0$ ,  $\psi'(0) = 1$ ,  $\psi(\pi) = 2$  and  $\psi'(\pi) = -1$ , we can easily see that Eq. (4.275) follows from Eqs. (4.277) and (4.281).

In the two-dimensional case ( $\partial/\partial y \equiv 0$ ), the bodies and air chambers shown as cross sections in Figure 4.8 are assumed to have infinite extension in the  $y$ -direction. To make the integral (4.232) finite, we integrate over that part of the surface  $S$  which is contained within the interval  $-d/2 < y < d/2$ , where we shall later let the width  $d$  tend to infinity. The corresponding integral per unit width is  $I' = I/d$ , as in Eq. (4.248). The surface  $S_\infty$  may be taken as the two planes  $x = r^\pm$ , where the constants  $r^\pm$  are sufficiently large ( $r^\pm \rightarrow \pm\infty$ ). The two-dimensional Kochin function  $H'_j(\beta)$  is defined as in Eq. (4.268), with  $I$  replaced

by  $I'$ . In accordance with Eq. (4.233), we take the integral over  $S_\infty$  instead of  $S$ , and we use the far-field approximation (4.246) to obtain

$$H_j(\beta) = -\frac{k}{D(kh)} \int_h^0 e^{2(kz)} dz \int_{-d/2}^{d/2} \exp(iky \sin \beta) dy (G^+ + G^-), \quad (4.282)$$

where

$$G^\pm = (1 \pm \cos \beta) \left( \frac{gk}{\omega} \right) A_j^\pm \exp(ikr^\pm \cos \beta) \exp\{\mp ikr^\pm\}. \quad (4.283)$$

The integral over  $z$  is  $D(kh)/2k$  in accordance with Eq. (4.108) and, moreover,

$$\lim_{d \rightarrow \infty} \frac{1}{d} \int_{-d/2}^{d/2} \exp(iky \sin \beta) dy = \begin{cases} 1 & \text{if } \sin \beta = 0 \\ 0 & \text{if } \sin \beta \neq 0. \end{cases} \quad (4.284)$$

Note that for a two-dimensional case, the angle of incidence is either  $\beta = 0$  or  $\beta = \pi$ . In both cases,  $\sin \beta = 0$ . Thus the two-dimensional Kochin function per unit width,  $H'_j(\beta) = H_j(\beta)/d$ , is given by

$$H'_j(0) = -\frac{gk}{\omega} A_j^+, \quad (4.285)$$

$$H'_j(\pi) = -\frac{gk}{\omega} A_j^-. \quad (4.286)$$

Otherwise, if  $\sin \beta \neq 0$ , we have  $H'_j(\beta) = 0$ . Here we have derived the two-dimensional version of the relation (4.272) between the Kochin function and the far-field coefficient.

Let us finally consider some relations between the Kochin functions, the far-field coefficients and the useful integral (4.233). For outgoing wave source number  $j$ , we have the Kochin function as given by Eqs. (4.268) and (4.272). For the outgoing wave, we have the asymptotic approximation

$$\begin{aligned} \psi_j &\sim A_j(\theta) e(kz) (kr)^{-1/2} e^{-ikr} \\ &= H_j(\theta) e(kz) (2\pi kr)^{-1/2} \exp(-ikr - i\pi/4). \end{aligned} \quad (4.287)$$

For two waves  $\psi_i$  and  $\psi_j$  satisfying the same radiation condition, we have, from Eq. (4.244),

$$I(\psi_i, \psi_j^*) = i \frac{D(kh)}{k} \int_0^{2\pi} A_i(\theta) A_j^*(\theta) d\theta = \frac{iD(kh)}{2\pi k} \int_0^{2\pi} H_i(\theta) H_j^*(\theta) d\theta. \quad (4.288)$$

Using the relation (4.261) between the far-field coefficients referred to the global origin and the local origin for wave source number  $j$ , we have

$$\begin{aligned} I(\psi_i, \psi_j^*) &= i \frac{D(kh)}{k} \int_0^{2\pi} A_i(\theta) A_j^*(\theta) d\theta \\ &= i \frac{D(kh)}{k} \int_0^{2\pi} B_i(\theta) B_j^*(\theta) \exp[ik(r_j - r_i)] d\theta. \end{aligned} \quad (4.289)$$

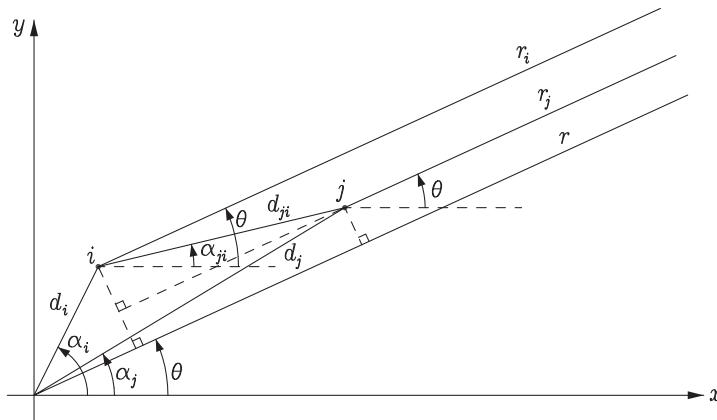


Figure 4.11: Definition sketch for horizontal distances and angles determining the position of the common origin and the local origins for outgoing wave sources number  $i$  and number  $j$ .

The exponent in the latter integrand may be rewritten by using the geometrical relations

$$\begin{aligned} r_j - r_i &= (x_i - x_j) \cos \theta + (y_i - y_j) \sin \theta \\ &= d_i \cos(\alpha_i - \theta) - d_j \cos(\alpha_j - \theta) \\ &= d_{ij} \cos(\alpha_{ij} - \theta), \end{aligned} \quad (4.290)$$

where distances and angles are defined in Figure 4.11 ( $d_{ij} = d_{ji}$ ,  $\alpha_{ij} = \alpha_{ji} + \pi$ ).

The two-dimensional version of Eq. (4.288) is

$$I'(\psi_i, \psi_j^*) = \frac{iD(kh)}{k^2} \left[ H'_i(0)H_j'^*(0) + H'_i(\pi)H_j'^*(\pi) \right], \quad (4.291)$$

which is obtained by combination of Eqs. (4.248), (4.285) and (4.286).

For waves not satisfying the radiation condition, such as  $\phi_i$  and  $\phi_j$  in Eq. (4.263), we shall now consider  $I(\phi_i, \phi_j^*)$ . Assuming that  $\Phi_j^*$  commutes with  $\psi_i$ , we obtain

$$I(\phi_i, \phi_j^*) = I(\Phi_i, \psi_j^*) - I(\Phi_j^*, \psi_i) + I(\psi_i, \psi_j^*) \quad (4.292)$$

in a similar way as we derived Eq. (4.265). Combining Eq. (4.292) with Eqs. (4.266) and (4.268), from which also follows

$$\Phi_j^* = \frac{g}{i\omega} A_j^* e(kz) \exp[ikr(\beta_j)], \quad (4.293)$$

$$H_j^*(\beta) = -\frac{k}{D(kh)} I \left\{ e(kz) e^{-ikr(\beta)}, \psi_j^* \right\}, \quad (4.294)$$

we find

$$I(\phi_i, \phi_j^*) = \frac{gD(kh)}{i\omega k} \left[ A_i H_j^*(\beta_i) + A_j^* H_i(\beta_j) \right] + I(\psi_i, \psi_j^*). \quad (4.295)$$

Here we may use Eq. (4.288) to express the last term in terms of Kochin functions. For the two-dimensional case, we instead use Eq. (4.291), and then we also replace  $I$  and  $H_{ij}$  by  $I'$  and  $H'_{ij}$ , respectively.

If we compare Eqs. (4.271) and (4.295), we notice two differences apart from the conjugation of  $\phi_j$ . Firstly, because of Eq. (4.239), Eq. (4.271) contains no term corresponding to the last term in Eq. (4.295). Secondly, in contrast to Eq. (4.271), there is not a shift of an angle of  $\pi$  in the argument of the Kochin functions in Eq. (4.295).

Equations (4.271) and (4.295) were used by Newman [34] as a base for deriving a set of reciprocity relations between parameters associated with various waves. In the next chapter, we shall derive and discuss some of these reciprocity relations.

## 4.9 Waves in the Time Domain

When analysing waves, we may choose between a frequency-domain or a time-domain approach. The frequency domain usually receives greater attention, as it also does here in this chapter, from Section 4.2 and onward. More recently, however, time-domain investigations have also been carried out [35, 36]. Such investigations may be based on a numerical solution of the time-domain Laplace equation (4.14) with boundary conditions (4.15), (4.16), (4.26) and (4.27). It is outside the scope of this chapter to discuss such investigations in detail, but let us here make a few comments based on applying inverse Fourier transform to some of the frequency-domain results. To be specific, let us discuss two linear systems: one which relates wave elevations at two locations and another which relates the hydrodynamic pressure to the wave elevation.

We start by assuming that an incident plane wave is propagating in the positive  $x$ -direction. In the frequency domain, the wave elevation may be expressed as

$$\eta(x, \omega) = A(\omega)e^{-ikx}. \quad (4.296)$$

Here  $A(\omega)$  is the Fourier transform of the incident wave elevation at  $x = 0$ :

$$A(\omega) = \eta(0, \omega) = \int_{-\infty}^{\infty} \eta(0, t)e^{-i\omega t} dt. \quad (4.297)$$

To ensure wave propagation in the positive  $x$ -direction, we have to choose a solution of the dispersion equation,  $k = k(\omega)$ , for which the angular repetency  $k$  has the same sign as the angular frequency  $\omega$ .

### 4.9.1 Relation between Wave Elevations at Two Locations

As real sea waves have a finite coherence time, it is possible to predict, with a certain probability, the wave elevation at a given point on the basis of wave

measurement at the same point. However, a more deterministic type of prediction of the incident wave elevation may be expected if the wave is measured at some distance  $l$  from the body's position  $x_B$ , in the ‘upwave’ direction (that is, opposite to the direction of wave propagation). We then assume that the measurement takes place at  $x = x_A$ , where

$$x_B - x_A = l > 0. \quad (4.298)$$

Choosing  $\eta(x_A, \omega)$  and  $\eta(x_B, \omega)$  as input and output, respectively, of a linear system, we find from Eq. (4.296) that the transfer function of the system is

$$H_l(\omega) = e^{-ik(\omega)l}. \quad (4.299)$$

The corresponding impulse response function is

$$h_l(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_l(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(\omega)l+i\omega t} d\omega. \quad (4.300)$$

It can be shown that this linear system is noncausal—that is,  $h_l(t) \neq 0$  for  $t < 0$  [cf. Eq. (2.129)]. We shall demonstrate this here for the case of waves on deep water ( $h \rightarrow \infty$ ). Further details and generalisation to finite water depth can be found in [37].

We start by rewriting the impulse response (4.300) as

$$h_l(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\cos(\omega t - kl) + i \sin(\omega t - kl)] d\omega. \quad (4.301)$$

Since  $k$  has the same sign as  $\omega$  and since the sine function is odd, the imaginary part of the integral vanishes. Moreover, the cosine function is even, and for deep water, we have  $k = \omega^2/g$  [cf. Eq. (4.55)]. Hence,

$$h_l(t) = \frac{1}{\pi} \int_0^{\infty} \cos(\omega t - \omega^2 l/g) d\omega, \quad (4.302)$$

which can further be decomposed into even and odd parts:

$$h_e(t) = \frac{1}{\pi} \int_0^{\infty} \cos \frac{\omega^2 l}{g} \cos(\omega t) d\omega, \quad (4.303)$$

$$h_o(t) = \frac{1}{\pi} \int_0^{\infty} \sin \frac{\omega^2 l}{g} \sin(\omega t) d\omega. \quad (4.304)$$

Using a table of Fourier transforms [38], we find, for  $t > 0$ ,

$$h_e(t) = \frac{1}{4} \left( \frac{2g}{\pi l} \right)^{1/2} \left( \cos \frac{gt^2}{4l} + \sin \frac{gt^2}{4l} \right), \quad (4.305)$$

$$h_o(t) = \frac{1}{2} \left( \frac{2g}{\pi l} \right)^{1/2} \left( C_2 \left\{ t \left( \frac{g}{2\pi l} \right)^{1/2} \right\} \cos \frac{gt^2}{4l} + S_2 \left\{ t \left( \frac{g}{2\pi l} \right)^{1/2} \right\} \sin \frac{gt^2}{4l} \right). \quad (4.306)$$

Here,  $C_2$  and  $S_2$  are Fresnel integrals, which are defined as

$$C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x t^{-1/2} \cos t dt \quad (4.307)$$

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x t^{-1/2} \sin t dt \quad (4.308)$$

for  $x > 0$ . As  $|h_e(t)| \neq |h_o(t)|$ , the impulse response  $h(t) = h_e(t) + h_o(t)$  is not causal [cf. Eq. (2.176)].

That this linear system is noncausal may be explained by the fact that the wave elevation at one location (input) is not the actual cause of the wave elevation at another location (output). The real cause of the output, as well as the input, may be a wavemaker (in the laboratory) or a distant storm (on the ocean).

Nevertheless, although the impulse response function  $h_l(t)$  is, strictly speaking, not causal [39], it is approximately causal and particularly so if  $l$  is large. With  $l = 400$  m, remarkably good prediction of computer-simulated sea swells may be obtained for at least half a minute into the future [40]. Previous experimental work [41] on real sea waves gave a less accurate prediction of the surface elevation but a more satisfactory prediction of the hydrodynamic pressure on the sea bottom at location  $x_B$ .

#### 4.9.2 Relation between Hydrodynamic Pressure and Wave Elevation

Let us now consider a linear system in which the input is the wave elevation at some position  $(x, y)$  on the average water surface and in which the output is the hydrodynamic pressure. Then according to Eq. (4.89), the transfer function is

$$H_p(\omega) = \frac{p(x, y, z; \omega)}{\eta(x, y; \omega)} = \rho g e(kz), \quad (4.309)$$

which for deep water becomes

$$H_p(\omega) = \rho g e^{\omega^2 z / g}. \quad (4.310)$$

(Note that  $z \leq 0$ .) Observe that  $H_p(\omega)$  is real, and hence its inverse Fourier transform is an even function of  $t$  (see Section 2.6.1), and for deep water is given by [38]

$$h_p(t) = h_p(-t) = \frac{1}{2} \rho g (-g/\pi z)^{1/2} e^{gt^2/4z}, \quad (4.311)$$

which is, evidently, not causal, except for the case of  $z = 0$ , which gives

$$h_p(t) = \rho g \delta(t). \quad (4.312)$$

Compared to the value at  $t = 0$ ,  $h_p(t)$  is reduced to 1% for  $t = \pm 4.3\sqrt{-z/g}$ , which increases with the submergence of the considered point. Thus a deeper point has a ‘more noncausal’ impulse response function for its hydrodynamic pressure.

Referring also to Eqs. (4.52) and (4.54), we see that  $H_p(\omega)$  is an even function of  $\omega$  also in the case of finite water depth. Moreover,  $H_p(\omega)$  is real when  $\omega$  is real. It follows that the corresponding inverse Fourier transform  $h_p(t)$  is an even function of  $t$ , and hence, it is a noncausal impulse response function.

Physically we may interpret this as follows. There is an effect on the hydrodynamic pressure also from the previous and the following wave troughs at the instant when a wave crest is passing. The deeper the considered location is, the more this non-instantaneous effect contributes.

## Problems

### Problem 4.1: Deriving Dispersion Relation Including Capillarity

Consider an infinitesimal surface element of length  $\Delta x$  in the  $x$ -direction and unit length in the  $y$ -direction. The net capillary attraction force is

$$p_k \Delta x = \gamma \sin(\theta) - \gamma \sin(\theta + \Delta\theta),$$

where  $\gamma$  is the surface tension ( $[\gamma] = \text{N/m}$ ). Further,  $\theta$  and  $\theta + \Delta\theta$  are the angles between the surface and the horizontal plane at the positions  $x$  and  $x + \Delta x$ , respectively.

Derive a modified dispersion relation including capillarity, by inserting a linearisation of the previously mentioned surface tension force into the Bernoulli equation (4.10). Recall that a discussion of this dispersion relation, for the deep-water case, is the subject of Problem 3.2. [Hint: first show that the free-surface boundary condition (4.41) has to be replaced by

$$\left[ -\omega^2 \hat{\phi} + g \frac{\partial \hat{\phi}}{\partial z} - \frac{\gamma}{\rho} \frac{\partial^3 \hat{\phi}}{\partial x^2 \partial z} \right]_{z=0} = 0.$$

Then use this condition in combination with Eq. (4.80)—which already satisfies the Laplace equation as well as the sea-bed boundary condition (4.40)—to determine a new dispersion relation to replace Eq. (4.54).]

### Problem 4.2: Vertical Functions from Bottom Boundary Condition

Use the boundary condition (4.40) for  $z = -h$  together with Eq. (4.42) in the determination of the integration constants  $c_+$  and  $c_-$  in Eq. (4.47) for  $Z(z)$ . One additional condition is required. Determine both integration constants, and write down an explicit expression for  $Z(z)$  for both of the following two cases of a second condition:

- (a)  $Z(0) = 1$ ,
- (b)  $\int_{-h}^0 |Z(z)|^2 dz = h$ .

### Problem 4.3: Vertical Functions for Evanescent Solutions

- (a) Show that the equation  $Z_n''(z) = -m_n^2 Z_n(z)$  with the boundary condition  $Z_n'(-h) = 0$  has a particular solution

$$Z_n(z) = C_n \cos(m_n z + m_n h).$$

- (b) Further, show that the boundary condition  $Z_n'(0) = \omega^2 Z_n(0)/g$  is satisfied if

$$\omega^2 = -g m_n \tan(m_n h). \quad (1)$$

- (c) Discuss possible solutions of Eq. (1) as well as normalisation of the corresponding functions  $Z_n(z)$ .

### Problem 4.4: Distance to Wave Origin

Assume that the ocean is calm and that suddenly a storm develops. Waves are generated, and swells propagate away from the storm region. Far away, at a distance  $l$  from the storm centre, swells of frequency  $f = 1/T = \omega/2\pi$  are recorded a certain time  $\tau$  after the start of the storm. We assume deep water between the storm centre and the place where the swells are recorded. Derive an expression for the ‘waiting time’  $\tau$  in terms of  $l, f$  and  $g$ , where  $g$  is the acceleration due to gravity. The starting point of the derivation should be the dispersion relationship  $\omega^2 = gk$  for waves on deep water.

In the South Pacific, at Tuvalu, a long swell was registered as follows:

Date	Time	$T$
16 Nov 1991	0300 GMT	22.2 s
	1400 GMT	20.0 s
17 Nov 1991	0100 GMT	18.2 s

Find the distance to the swell source. Approximately on which date did this storm occur? Assuming the swell was travelling southward, determine the latitude at which the swell was formed, given that Tuvalu is located at  $8^\circ 30' S$  ( $1'$  corresponds to a nautical mile = 1852 m. Correspondingly, 90 latitude degrees—the distance from the equator to the globe’s North or South Pole—is  $10^7$  m).

### Problem 4.5: Reflection of Plane Wave at Vertical Wall

- (a) Show that a gravity wave for which the elevation has a complex amplitude

$$\hat{\eta}_i = \eta_0 \exp(-ik_x x - ik_y y) \quad (2)$$

may propagate on an infinite lake of constant depth  $h$ . That is, show that this wave satisfies the partial differential equation

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right] \hat{\eta} = 0.$$

- (b) Moreover, it is assumed that the boundary conditions at the free water surface  $z = 0$  and at the bottom  $z = -h$  are satisfied (although a proof of this statement is not required in the present problem). Express  $k_x$  and  $k_y$  by the angular repetency  $k$  and the angle  $\beta$  between the direction of propagation and the  $x$ -axis.
- (c) Assume that the incident wave (2) in the region  $x < 0$  is reflected at a fixed vertical wall in the plane  $x = 0$ . Use the additional boundary condition at the wall to determine the resultant wave

$$\hat{\eta} = \hat{\eta}_i + \hat{\eta}_r,$$

where  $\hat{\eta}_r$  represents the reflected wave. Check that the reflection angle equals the angle of incidence.

#### Problem 4.6: Cross Waves in a Wave Channel

An infinitely long wave channel in the region  $-d/2 < x < d/2$  is bounded by fixed plane walls, the planes  $x = -d/2$  and  $x = d/2$ . The channel has constant water depth  $h$ . Show that a harmonic wave of the type

$$\hat{\eta} = (Ae^{-ik_xx} + Be^{ik_xx}) e^{-ik_yy}$$

may propagate in the  $y$ -direction. Use the boundary conditions to express possible values of  $k_x$  in terms of  $d$ . Find the limiting value of the angular frequency, below which no cross waves can propagate.

Derive expressions for the phase velocity  $v_p = \omega/k_y$  and the group velocity  $v_g = d\omega/dk_y$  for the cross waves which may propagate along the wave channel.

#### Problem 4.7: Kinetic Energy for Progressive Wave

Derive an expression for the time-average of the stored kinetic energy per unit volume associated with a progressive wave

$$\hat{\phi} = -(g/i\omega)A e(kz) e^{-ikx}$$

in terms of  $\rho$ ,  $g$ ,  $k$ ,  $\omega$ ,  $|A|$  and  $z$ .

#### Problem 4.8: Kinetic Energy Density

Prove that a plane wave

$$\hat{\eta} = Ae^{-ikx} + Be^{ikx}$$

on water of constant depth  $h$  is associated with a kinetic energy density per unit horizontal area equal to

$$\langle E_k \rangle = (\rho g / 4) (|A|^2 + |B|^2)$$

(averaged over time and over the horizontal plane). Also prove that the time-averaged kinetic energy per unit volume is

$$e_k = \frac{\rho}{2} \left[ \frac{e(kz) g k}{\omega} \right]^2 (|A|^2 + |B|^2) - \frac{\rho g^2 k^2}{4\omega^2 \cosh^2(kh)} |\hat{\eta}|^2.$$

### Problem 4.9: Propagation Velocities on Intermediate Water Depth

Find numerical values for the group velocity and phase velocity for a wave of period  $T = 9$  s on water of depth  $h = 8$  m, and also for the case when  $h = 6$  m. This problem involves numerical (or graphical) solution of the transcendental equation  $\omega^2 = gk \tanh(kh)$  (the dispersion relation). The acceleration of gravity is  $g = 9.81$  m/s<sup>2</sup>.

### Problem 4.10: Evanescent Waves

The complex amplitude  $\hat{\phi}$  of a velocity potential satisfying Laplace's equation may be written

$$\hat{\phi} = X(x)Z(z),$$

where  $X(x)$  and  $Z(z)$  have to satisfy the corresponding differential equations

$$X''(x) = -\lambda X(x),$$

$$Z''(z) = \lambda Z(z).$$

Further,  $Z(z)$  has to satisfy the boundary conditions

$$Z'(-h) = 0, \quad -\omega^2 Z(0) + gZ'(0) = 0.$$

Solutions for  $\hat{\phi}$  corresponding to negative values of  $\lambda$ —that is,  $\lambda = -m^2$ —represent the so-called evanescent waves.

Consider a propagating wave travelling in an infinite basin of water depth  $h$ , where some discontinuities at  $x = x_1$  and  $x = x_2$  ( $x_2 > x_1$ ) introduce evanescent waves. Show that an evanescent wave in the  $x$ -region  $-\infty < x < x_1$  or  $x_2 < x < \infty$  does not transport any time-average power. Is it possible that real energy transport is associated with an evanescent wave in the region  $x_1 < x < x_2$ ?

### Problem 4.11: Slowly Varying Water Depth and Channel Width

A regular wave  $\eta = \eta(x, t) = A_0 \cos(\omega t - k_0 x)$  is travelling in deep water. Its amplitude is  $A_0 = 1$  m, and its period is  $T = 9$  s.

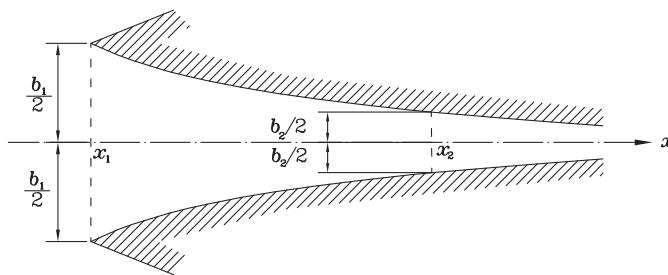


Figure 4.12: Tapered horizontal channel for wave concentration.

- (a) Give mathematical expressions and numerical values for the angular frequency  $\omega$  and the angular repetency (wave number)  $k_0$ . Also determine the wavelength  $\lambda_0$ . Give mathematical expressions and numerical values for the phase velocity  $v_{p0}$ , the group velocity  $v_{g0}$  and the maximum water particle velocity  $v_{0,\max}$ .

Now assume that this wave is moving toward a coast line which is normal to the  $x$ -axis. The water depth  $h = h(x)$  varies slowly with  $x$ , which means that we may neglect reflection (or partial reflection) of the wave. If we also neglect energy losses (e.g., due to friction at the sea bed), the wave-energy transport  $J$  (wave power per unit width of the wave front) stays constant as the wave travels into shallower water. (Note: for simplicity, we assume that the water depth does not vary with the  $y$ -coordinate.)

- (b) At some position  $x = x_1$ , the water depth is  $h = h(x_1) = 8$  m. Here the phase velocity is  $v_{p1} = 8.27$  m/s, and the group velocity is  $v_{g1} = 7.24$  m/s. Give relative numerical values of the wavelength  $\lambda_1$  and the wave amplitude  $A_1$  at this location; that is, state the numerical values of  $(\lambda_0/\lambda_1)$  and  $(A_1/A_0)$ .  
(c) Note that for a 9 s wave in 8 m depth, neither the deep-water nor the shallow-water approximations give accurate values for  $v_{g1}$  and  $v_{p1}$ . If you had used those approximations, how many percent too small or too large would the computed values be for  $v_{g1}$  and  $v_{p1}$ ?

A tapered horizontal wave channel with vertical walls has its entrance (mouth) at the position  $x_1$  (see Figure 4.12). The entrance width is  $b_1$  ( $= 30$  m). As an approximation, we assume that the wave continues as a plane wave into the channel, but the narrower width available to the wave results in an increased wave amplitude. If the channel width  $b = b(x)$  changes slowly with  $x$ , we may neglect partial reflections of the wave. Neglecting also other energy losses:

- (d) derive an expression for the wave amplitude  $A_2$  at  $x_2$  in terms of  $A_1$ ,  $b_1$  and  $b_2$ , where  $b_2$  is the channel width at  $x_2$  ( $x_2 > x_1$ ). The water depth is still 8 m at  $x_2$ . The bottom of the vertical channel wall is at the same depth, of course. The top of the vertical wall is 3.5 m above the mean water level. Find the numerical value of  $b_2$  when it is given that  $A_2 = 3.5$  m. (Remember that in the deep sea, the wave amplitude is  $A_0 = 1$  m.)

- (e) If the water depth at  $x_2$  had been 6 m instead of 8 m while the depth at the entrance remains 8 m, find for this case the wave amplitude  $A'_2$  at  $x'_2$  in terms of  $A_1$ ,  $b_1$ ,  $b_2$ ,  $v_{g1}$  and  $v_{g2}$ . In 6 m depth the 9 s wave has a group velocity  $v_{g2} = 6.60$  m/s. With this channel (where the depth decreases slowly with increasing  $x$ ), what would be the width  $b'_2$  at  $x'_2$  if  $A'_2 = 3.5$  m?
- (f) If 5% of the energy was lost (in friction and other processes) when the wave travelled from deep water to the position  $x_1$ , how would this influence the answers under point (b)? (Will the numerical values be increased, decreased or unchanged?) If changed, find the new values.
- (g) Moreover, if there is also energy loss when the wave travels from  $x_1$  to  $x_2$ , how would that influence the answers under point (d)? Assuming that this energy loss is 20% (and that 5% of the deep-water wave energy was lost before the entrance of the channel), obtain new numerical values for the answers under (d).

### Problem 4.12: Depth Function $D(kh)$

- (a) Show that the depth function  $D(kh)$  defined by Eq. (4.77) has a maximum  $x_0$  for  $kh = x_0$ , where  $x_0$  is a root of the transcendental equation

$$x_0 \tanh(x_0) = 1,$$

which corresponds to  $h = h_0 \equiv g/\omega^2$ .

- (b) Further, show that

$$2k \int_{-h}^0 e^2(kz) dz = D(kh),$$

where

$$e(kz) = \frac{\cosh(kh + kz)}{\cosh(kh)}.$$

(See also Problem 4.2.)

- (c) Show that the depth function may be written as

$$D(kh) = \left(1 + \frac{4khe^{-2kh}}{1 - e^{-4kh}}\right) \frac{1 - e^{-2kh}}{1 + e^{-2kh}}.$$

This expression is recommended for use in computer programmes when large values of  $kh$  are used.

### Problem 4.13: Transmission and Reflection at a Barrier

We consider the following two-dimensional problem. In a wave channel which is infinitely long in the  $x$ -direction and which has a constant water depth  $h$ , there is

at  $x = 0$  a barrier, a stiff plate of negligible thickness, occupying the region  $x = 0$ ,  $-h_1 > z > -h_2$ , where  $h \geq h_2 > h_1 \geq 0$ ). A wave propagating in the positive  $x$ -direction originates from  $x = -\infty$ , and it is partially reflected and partially transmitted at the barrier.

In the region  $x < 0$ , the complex amplitude of the velocity potential may be written as

$$\hat{\phi}(x, z) = A_0 Z_0(z) e^{-ikx} + \sum_{n=0}^{\infty} b_n Z_n(z) e^{m_n x},$$

where the first term on the right-hand side represents the incident wave. The orthogonal and complete set of vertical eigenfunctions  $\{Z_n(z)\}$  is defined by Eq. (4.70), and  $m_n$  ( $n \geq 1$ ) is a positive solution of Eq. (4.71), while  $m_0 = ik$ . See also Eqs. (4.74), (4.75) and (4.111). For the region  $x > 0$ , we correspondingly write

$$\hat{\phi}(x, z) = \sum_{n=0}^{\infty} a_n Z_n(z) \exp(-m_n x).$$

While the coefficients  $\{a_n\}$  and  $\{b_n\}$  are unknown, we consider  $A_0$  to be known. We define the complex reflection and transmission coefficients by

$$\Gamma = b_0/A_0, \quad T = a_0/A_0,$$

respectively. Obviously,  $\hat{\phi}(x, z)$  satisfies Laplace's equation and the homogeneous boundary conditions (4.40) at  $z = -h$  and (4.41) at  $z = 0$ .

Next we have to apply continuity and boundary conditions at the plane  $x = 0$ . Firstly,  $\partial\hat{\phi}/\partial x$ , representing the horizontal component of the fluid velocity, has to be continuous there (for  $0 > z > -h$ ), and moreover, it has to vanish on the barrier ( $-h_1 > z > -h_2$ ). Finally,  $\hat{\phi}$ , representing the hydrodynamic pressure [see Eq. (4.37)], has to be continuous above and below the barrier ( $0 > z > -h_1$  and  $-h_2 > z > h$ ). On this basis, show that  $a_n = -b_n$  for  $n \geq 1$ , whereas  $a_0 = A_0 - b_0$ . The latter result means that  $\Gamma + T = 1$ , and energy conservation means that  $|\Gamma|^2 + |T|^2 = 1$ . Using this, show that  $\Gamma T^* + \Gamma^* T = 0$ . Moreover, show that the set of coefficients  $\{b_n\}$  has to satisfy an equation of the type

$$\sum_{m=0}^{\infty} B_{mn} b_m = c_n A_0 \quad \text{for } n = 0, 1, 2, 3, \dots,$$

which may be solved approximately by truncating the infinite sum and then using a computer for a numerical solution. Derive an expression for  $B_{mn}$  and for  $c_n$ . [Hint: use the orthogonality condition (4.67) and (4.73)—that is,

$$\int_{-h}^0 Z_m(z) Z_m(z) dz = h \delta_{mn}.$$

If the integral is taken only over the interval  $-h_1 > z > -h_2$ , it takes another value, which we may denote by  $h D_{mn}$ . Likewise, we define  $h E_{mn}$  as the integral

taken over the remaining two parts of the interval  $0 > z > -h$ . Thus,  $D_{mn} + E_{mn} = \delta_{mn}$ . Observe that both  $D_{mn}$  and  $E_{mn}$  have to enter (explicitly or implicitly) into the expression for  $B_{mn}$ , in order for  $B_{mn}$  to be influenced by the boundary condition on the barrier as well as by the condition of continuous hydrodynamic pressure above and below the barrier.]

### Problem 4.14: Energy-Absorbing Wall

Assume that a wave

$$\hat{\eta}_i = A \exp[-ik(x \cos \beta + y \sin \beta)], \quad |\beta| < \pi/2$$

is incident on a vertical wall  $x = 0$  where there is a boundary condition

$$\hat{p} = Z_n \hat{v}_x = Z_n \frac{\partial \hat{\phi}}{\partial x} \quad \text{at } x = 0.$$

Here the constant parameter  $Z_n$  is a distributed surface impedance (of dimension Pa s/m = N s/m<sup>3</sup>). Show that the time-average absorbed power per unit surface area of the wall is  $\frac{1}{2} \operatorname{Re}\{Z_n\} |\hat{v}_x|^2$ . Further, show that a wave

$$\hat{\eta}_r = \Gamma A \exp[ik(x \cos \beta - y \sin \beta)]$$

is reflected from the wall. Assume infinite water depth, and derive an expression for  $\Gamma$  in terms of  $\omega$ ,  $\rho$ , and  $Z_n$ . Show that  $|\Gamma| = 1$  if  $\operatorname{Re}\{Z_n\} = 0$  and that  $|\Gamma| < 1$  if  $\operatorname{Re}\{Z_n\} > 0$ .

### Problem 4.15: Wave-Energy Transport with Two Waves

A harmonic wave on water of constant depth  $h$  is composed of two plane waves propagating in different directions. Let the wave elevation be given by

$$\hat{\eta} = Ae^{-ikx} + Be^{-ik(x \cos \beta + y \sin \beta)}.$$

For the superposition of these two plane waves, derive an expression for the intensity

$$\vec{I} = \overline{p(t)\vec{v}(t)},$$

which, by definition, is a time-independent vector. Further, referring to Section 4.4.4, show that the wave-power-level vector may be expressed as

$$\vec{J} = (\rho g/2)v_g \left[ \vec{e}_x (|A|^2 + |B|^2 \cos \beta) + \vec{e}_y |B|^2 \sin \beta \right] + \vec{s}(x, y),$$

where the spatially dependent vector  $\vec{s}(x, y)$  is solenoidal. Find an expression for  $\vec{s}(x, y)$ , and show explicitly that  $\nabla \cdot \vec{s} = 0$ , which has the physical significance that there is nowhere, in the water, accumulation of any permanent wave energy

(active energy—but possibly only of reactive energy). Discuss the cases  $\beta = 0$ ,  $\beta = \pi$  and  $\beta = \pi/2$ .

Derive expressions for the space-averaged wave-energy transport  $\langle J_x \rangle$  and  $\langle J_y \rangle$ . Pay special attention to the case of  $\cos \beta = 1$ .

### Problem 4.16: One Plane Wave and One Circular Wave

Let us assume that a harmonic wave on water of constant depth  $h$  is composed of one plane wave and one circular wave. The wave elevation  $\eta(r, \theta, t)$  has a complex amplitude  $\hat{\eta} = \hat{\eta}(r, \theta)$  given as

$$\begin{aligned}\hat{\eta} &= \hat{\eta}_A + \hat{\eta}_B = Ae^{-ikr \cos \theta} + \frac{B}{\sqrt{kr}}e^{-ikr} + \dots \\ &= A \left( e^{-ikr \cos \theta} + \frac{b}{\sqrt{kr}}e^{-ikr} + \dots \right),\end{aligned}$$

where  $b = B/A$  is dimensionless. Cartesian coordinates  $(x, y, z)$  have been replaced by cylindrical ones  $(r, \theta, z)$ ; thus,  $x = r \cos \theta$  and  $y = r \sin \theta$ . In relation to  $\hat{\eta}_B$ , additional terms of order  $(kr)^{-3/2}$ , which may be neglected for  $kr \gg 1$ , are denoted by three dots ( $\dots$ ). The wave's velocity potential is given by its complex amplitude

$$\hat{\phi}(r, \theta, z) = -\hat{\eta}(r, \theta) e(kz)g/i\omega,$$

where  $e(kz) = \cosh(kz + kh)/\cosh kh$ .

Our aim is to derive and discuss an expression for the wave power  $P_{\text{cyl}}$ , which is propagated outwards through an envisaged vertical, cylindrical, axisymmetric surface of radius  $r \gg 1/k$ .

- (a) As a first step, derive complex-amplitude expressions for the hydrodynamic pressure  $\hat{p} = \hat{p}_A + \hat{p}_B$  and for the radial component of the water-particle velocity  $\hat{v}_r = \hat{v}_{A,r} + \hat{v}_{B,r}$ . Here we may neglect terms that, for  $kr \gg 1$ , decrease faster towards zero than  $1/\sqrt{kr}$ . Consequently, we shall, in the following, neglect terms that decrease faster towards zero than  $1/kr$ .
- (b) Secondly, derive an expression for the resulting radial component of the intensity

$$I_r(r, \theta, z) = \frac{1}{2} \operatorname{Re}\{\hat{p}\hat{v}_r^*\} = \frac{1}{4}(\hat{p}\hat{v}_r^* + \hat{p}^*\hat{v}_r).$$

- (c) Further, show that the radial component  $J_r$  of the wave-power level (wave-energy transport) is

$$J_r(r, \theta) = J_A \left( \cos \theta + \frac{|b|^2}{kr} + \frac{\operatorname{Re}\{be^{-ikr(1-\cos \theta)} \cos \theta + b^*e^{-ikr(1-\cos \theta)}\}}{\sqrt{kr}} + \dots \right),$$

where

$$J_A = \frac{\rho g^2 D(kh)|A|^2}{4\omega} = v_g(\rho g/2)|A|^2 = v_g(\rho g/2)|\hat{\eta}_A|^2$$

is the wave-power level for the plane wave alone. Note that, in the preceding expression for  $J_r(r, \theta)$ , the first and second terms correspond to the direct contributions from the plane wave alone and the circular wave alone, respectively.

- (d) Next, assuming that  $kr \rightarrow \infty$ , show that the time-average wave-power

$$P_{\text{cyl}} = P_{\text{cyl, outwards}} = \int_0^{2\pi} J_r(r, \theta) r d\theta = r \int_{-\beta}^{2\pi-\beta} J_r(r, \theta) d\theta$$

which passes outwards through the envisaged cylindrical surface of radius  $r$  is given by

$$P_{\text{cyl}} = \frac{J_A}{k} (|1 + \zeta|^2 - 1), \quad \text{where } \zeta = b\sqrt{2\pi}e^{-i\pi/4} = (1 - i)b\sqrt{\pi}.$$

- (e) Is it reasonable that this power is independent of  $r$ , the radius of the envisaged cylindrical surface? Explain—for instance, by physical arguments—why this expression for  $P_{\text{cyl}}$  may be applicable/valid even for values of  $r$  that are too small to satisfy the condition  $kr \gg 1$ .
- (f) Let us assume that the complex plane-wave elevation amplitude  $A$  has a fixed value, and then discuss how the real-valued (positive or negative)  $P_{\text{cyl}}$  varies with the complex ratio  $B/A = b$ , or with the dimensionless complex variable  $\zeta$ . Which kind of a closed curve, in the complex  $B/A$  (or  $\zeta$ ) plane, separates regions for which  $P_{\text{cyl}}$  has opposite signs?
- (g) Show that  $P_{\text{cyl, min}} = -J_A/k = -(\lambda/2\pi)J_A$ .

[Hint: when solving this Problem 4.16, it may be helpful to apply, e.g., Eqs. (4.36)–(4.37), (4.52), (4.85)–(4.90), (4.126), (4.129)–(4.130), (4.214)–(4.219), and, finally, (4.274)–(4.276).]