CHAPTER EIGHT

Wave-Energy Converter Arrays

It is, in particular, the wavelengths of persisting wind-generated ocean waves that limit the power capacity of individual WEC units to the sub-megawatt region. Thus, a sizeable wave power plant needs to be an array of many individual WEC units. An example of such an array is shown in Figure 8.1. When a WEC unit has been successfully developed and sufficiently tested in the sea, the next development stage is to develop a future industry for mass production of WEC units. This is still far ahead. Strangely enough, however, for quite a long time, there has been a significant research interest in WEC arrays. Theoretical studies of WEC-body arrays were first made by Budal [62] and afterwards, more systematically, by Falnes [25] and Evans [61]. Budal also took into consideration absorbed-power limitation due to the finite hull volume of the array's WEC bodies [45]. WEC arrays consisting of WEC bodies and OWCs were considered by Falnes and McIver [109] and, independently, by Fernandes [110]. A more recent review is given by Falnes and Kurniawan [63]. Before proceeding to a systematic discussion, we first present a phenomenological description of the matter.

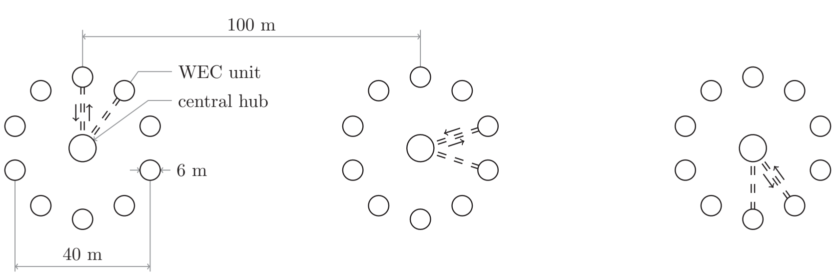

Figure 8.1: Groups of 10 WEC units, each of power capacity 200 kW, provide hydraulic power to a central hub containing gas accumulators for energy storage and a 2 MW electric generator (as proposed by Budal in 1979 [108]).

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8.1 WEC Array Consisting of Several Bodies

The analysis of an array of WEC bodies, each oscillating in up to six degrees of freedom, is similar to the analysis of a single WEC body oscillating in several degrees of freedom, discussed in Section 6.5.

Let us consider a system of N ($N \geq 1$) bodies, each of them oscillating in up to six degrees of freedom. As in Section 5.5, to each motion mode j ($j = 1, 2, \ldots 6$) of body p ($p = 1, 2, \ldots N$), we associate an oscillator number $i = 6(p-1) + j$. In accordance with Eq. (5.155), the total wave force on oscillator i has a complex amplitude

$$\hat{F}_{t,i} = \hat{F}_{e,i} + \hat{F}_{r,i} = \hat{F}_{e,i} - \sum_{i'=1}^{6N} Z_{ii'} \hat{u}_{i'}, \tag{8.1}$$

where $Z_{ii'}$ is an element of the radiation impedance matrix discussed in Sections 5.5.1 and 5.5.3. The only difference between this equation and Eq. (6.81), which pertains to a single WEC body oscillating in six degrees of freedom, is that the sum is here taken up to $6N$ instead of 6.

The (time-average) power absorbed by oscillator i is as given by Eq. (6.82), with index j replaced by i. Upon inserting the expression for $\hat{F}_{t,i}$ from Eq. (8.1) into Eq. (6.82), we find that Eqs. (6.82)–(6.86) are applicable to the array, provided we take the sums up to $6N$ instead of 6. Correspondingly, the velocity vector $\hat{\mathbf{u}}$ and excitation force vector $\hat{\mathbf{F}}_e$ in these equations are each a $6N$-dimensional column vector, whereas the radiation resistance matrix $\mathbf{R}$ is of dimension $6N \times 6N$. Thus, Eq. (6.95) for the maximum absorbed wave power is also applicable to the array of WEC bodies.

By extending Eqs. (6.82)–(6.86) to be applicable for an array of WEC bodies, we have also generalised the collective oscillation amplitude U and the collective excitation-power coefficient $E(\beta)$, which were introduced for a single WEC body in Section 6.5 (see also Section 6.2), to be valid for an array of WEC bodies:

$$|U|^2 = \tfrac{1}{2}\hat{\mathbf{u}}^\dagger \mathbf{R}\hat{\mathbf{u}} \tag{8.2}$$

$$E(\beta) = \tfrac{1}{4}\mathbf{f}^{\mathrm{T}}(\beta)\hat{\mathbf{u}}^*, \tag{8.3}$$

according to Eqs. (6.85)–(6.86). With these more generalised definitions of $E(\beta)$ and U, the simple equations (6.32)–(6.37), including the wave-power-island equation

$$P_{\mathrm{MAX}}(\beta) - P = |U_0(\beta) - U|^2, \tag{8.4}$$

is applicable even for a WEC-body array. The optimum condition (6.34) leads to the condition

$$\mathbf{R}\hat{\mathbf{u}}_{\mathrm{OPT}} = \hat{\mathbf{F}}_e/2 \tag{8.5}$$

for the optimum velocity vector $\hat{\mathbf{u}}_{\mathrm{OPT}}$, as in the case of a single WEC body; cf. Eq. (6.90). Furthermore, Eq. (6.35) gives

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$$P_{\mathrm{MAX}} = \tfrac{1}{4}\hat{\mathbf{F}}_e^{\mathrm{T}}\hat{\mathbf{u}}^*_{\mathrm{OPT}} = \tfrac{1}{2}\hat{\mathbf{u}}^{\dagger}_{\mathrm{OPT}}\mathbf{R}\hat{\mathbf{u}}_{\mathrm{OPT}}, \tag{8.6}$$

as in Eq. (6.95).

Note that $P_{\mathrm{MAX}} = |U_0(\beta)|^2$ is proportional to the wave amplitude squared and depends on the angle β of wave incidence. Assuming that the WEC array is provided with sufficient control devices, we may consider U to be an independent variable, on which the absorbed wave power P depends.

We shall say a few more words about the collective oscillation amplitude U and the collective excitation-power coefficient $E(\beta)$. When considering a single WEC body oscillating in one mode, we have U and $E(\beta)$ expressed in terms of the radiation Kochin function; see Eqs. (6.30)–(6.31). For the array, generalised expressions can be obtained from Eqs. (8.2)–(8.3) by making use of relations (5.224) and (5.178), which yield

$$|U|^2 = \frac{\omega\rho v_p v_g}{4\pi g}\int_0^{2\pi}\hat{\mathbf{u}}^{\dagger}\mathbf{h}^*(\theta)\mathbf{h}^{\mathrm{T}}(\theta)\hat{\mathbf{u}}\,d\theta = \frac{\omega\rho v_p v_g}{4\pi g}\int_0^{2\pi}|H_r(\theta)|^2\,d\theta, \tag{8.7}$$

$$E(\beta) = \frac{\rho v_p v_g}{2}\mathbf{h}^{\mathrm{T}}(\beta\pm\pi)\hat{\mathbf{u}}^* = \frac{\rho v_p v_g}{2}\bar{H}_r(\beta\pm\pi). \tag{8.8}$$

Here we have also made use of Eq. (4.107) and, as in Eqs. (5.152)–(5.153), introduced the 'adjoint companion'

$$\bar{H}_r(\theta) = \mathbf{h}^{\mathrm{T}}(\theta)\hat{\mathbf{u}}^* = \hat{\mathbf{u}}^{\dagger}\mathbf{h}(\theta) \tag{8.9}$$

of the radiation Kochin function

$$H_r(\theta) = \mathbf{h}^{\mathrm{T}}(\theta)\hat{\mathbf{u}} = \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{h}(\theta). \tag{8.10}$$

Alternative ways to arrive at the same expressions are discussed in [63]. Notice that Eqs. (8.7)–(8.8), when applied to the case of only one oscillator, lead to Eqs. (6.30)–(6.31).

Inserting expressions (8.7)–(8.8) into the optimum condition (6.34), which is applicable also for a WEC-body array, we have the condition

$$A\bar{H}_{r0}(\beta\pm\pi) = \frac{\omega}{2\pi g}\int_0^{2\pi}|H_{r0}(\theta)|^2 d\theta, \tag{8.11}$$

which the optimum radiated wave's Kochin function H_{r0} and its adjoint companion $\bar{H}_{r0}$ need to satisfy. Using relation (5.152), we can alternatively express this condition as

$$AH^*_{r0}(\beta) = \frac{\omega}{2\pi g}\int_0^{2\pi}[H_d(\theta) + H_{r0}(\theta)]H^*_{r0}(\theta)\,d\theta, \tag{8.12}$$

which involves the diffracted wave's Kochin function $H_d(\theta)$.

8.1.1 Optimum Gain Function for WEC-Body Arrays

In Eq. (6.35), we presented several different expressions for the maximum absorbed power. We shall find it convenient to add the following expressions also:

$$P_{\text{MAX}} = \frac{P^2_{\text{MAX}}}{P_{\text{MAX}}} = \frac{(P_{e,\text{OPT}}/2)^2}{P_{r,\text{OPT}}} = \frac{|AE_0(\beta)|^2}{|U_0|^2}. \tag{8.13}$$

Inserting Eqs. (8.7)–(8.8) gives

$$P_{\text{MAX}} = \frac{|AE_0(\beta)|^2}{|U_0|^2} = \frac{\rho g v_g |A|^2}{2k} G(\beta) = \frac{J}{k} G(\beta) = J d_{a,\text{MAX}}, \tag{8.14}$$

where J is the wave-energy transport or the wave-power level, as given by Eq. (4.130), and $d_a = P/J$ is the absorption width. Moreover, we have introduced the optimum gain function

$$G(\beta) = \frac{2\pi |\bar{H}_{r0}(\beta \pm \pi)|^2}{\int_0^{2\pi} |H_{r0}(\theta)|^2 d\theta}. \tag{8.15}$$

The optimum gain function for a single body oscillating in one mode is given earlier in Eq. (6.14). Observe that, with only one oscillator, we have $|\bar{H}_r(\theta)|^2 = |h_j(\theta)u_j^*|^2 = |h_j(\theta)u_j|^2 = |H_r(\theta)|^2$, and thus Eq. (8.15) reduces to Eq. (6.14). We must note, however, that in general $|\bar{H}_r(\theta)|^2 \neq |H_r(\theta)|^2$ for the multi-mode case.

Equation (8.15) tells us that in order to maximise power absorption, it is important for a WEC system to have the ability to radiate a wave in one predominant direction.

8.1.2 Similarity-Transformed WEC-Radiation Matrix

We have now generalised the optimum gain function for a single body oscillating in one mode, Eq. (6.14), to that for an array of bodies each oscillating in more than one mode, Eq. (8.15). When considering the one-body, one-mode case, we have also derived the directional average of the gain function, as given by Eq. (6.15), which follows immediately from Eq. (6.14). For an array of WEC bodies oscillating in multiple modes, it is less straightforward to obtain the directional average of the gain function from Eq. (8.15), and so another method needs to be employed, that is, by using similarity transformation; see, for example, [16, chapter V] and Section 2.4, where this method was briefly discussed. For this purpose, we need to diagonalise the radiation-resistance matrix $\mathbf{R}$, which is a matrix of dimension $6N \times 6N$.

This diagonalisation procedure may preferably be performed on a slightly more general radiation-damping matrix $\mathbf{D}$, which is complex and *Hermitian*—that is,

$$\mathbf{D}^\dagger \equiv (\mathbf{D}^\text{T})^* = \mathbf{D}. \tag{8.16}$$

(This radiation-damping matrix is applicable for a WEC array consisting of WEC bodies as well as OWCs). Thus, the symmetric real radiation resistance matrix $\mathbf{R}$ is a special case of such a Hermitian complex matrix $\mathbf{D}$, which we, for a general case, shall suppose to be an $M \times M$ matrix. For the particular case of an array consisting of N WEC bodies and no OWCs, $M = 6N$. Then, $\mathbf{D} = \mathbf{R}$. It can be shown (see Section 8.2.3) that just as the real radiation resistance matrix $\mathbf{R}$, this more general radiation-damping matrix $\mathbf{D}$ is also positive semidefinite. Thus, $\hat{\mathbf{u}}^\dagger \mathbf{D} \hat{\mathbf{u}} \geq 0$ for any M-dimensional complex column vector $\hat{\mathbf{u}}$.

Our first step is to determine the eigenvalues Λ_i and the corresponding normalised and mutually orthogonal eigenvectors $\mathbf{e}_i$ of the Hermitian matrix $\mathbf{D}$, for $i = 1, 2, \ldots, M-1, M$. They have to satisfy the following system of linear homogeneous equations:

$$\mathbf{D}\mathbf{e}_i = \Lambda_i \mathbf{e}_i. \tag{8.17}$$

Because matrix $\mathbf{D}$ is Hermitian, all its eigenvalues Λ_i are real [16, page 109], and since it is positive semidefinite, they are all nonnegative. They are the M solutions of the M-degree equation $|\mathbf{D} - \Lambda \mathbf{I}| = 0$, where $\mathbf{I}$ is the $M \times M$ identity matrix, for which all its diagonal entries equal 1, whereas all the other entries equal 0. Further, $|\mathbf{D} - \Lambda \mathbf{I}|$ is the determinant of matrix $(\mathbf{D} - \Lambda \mathbf{I})$. We may wish to arrange the M eigenvalues in a descending order—that is,

$$\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_M \geq 0. \tag{8.18}$$

Note that if $\mathbf{e}_i$ is a possible solution when $\Lambda = \Lambda_i$, then also $C_i \mathbf{e}_i$ is a solution, where C_i is an arbitrary complex scalar. It is convenient to normalise the eigenvectors $\mathbf{e}_i$ in such a way that the condition

$$\mathbf{e}_i^\dagger \mathbf{e}_{i'} = \delta_{ii'} = \begin{cases} 1 & \text{for } i = i' \\ 0 & \text{for } i \neq i' \end{cases} \tag{8.19}$$

applies. With this choice, we may consider this particular set of eigenvectors to be a complete set of mutually orthogonal unit vectors in our M-dimensional complex space [16, chapter V].

Let us now define the $M \times M$ matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \cdots & \mathbf{e}_M \end{bmatrix}. \tag{8.20}$$

It then follows from Eq. (8.19) that $\mathbf{S}^\dagger \mathbf{S} = \mathbf{I}$. Hence, $\mathbf{S}^{-1} = \mathbf{S}^\dagger$, or the inverse of matrix $\mathbf{S}$ is equal to its conjugate transpose.

We are now ready to carry out a similarity transformation of vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{F}}_e$. Let

$$\hat{\mathbf{u}}' = \mathbf{S}^{-1} \hat{\mathbf{u}} = \mathbf{S}^\dagger \hat{\mathbf{u}}, \qquad \hat{\mathbf{F}}_e' = \mathbf{S}^{-1} \hat{\mathbf{F}}_e = \mathbf{S}^\dagger \hat{\mathbf{F}}_e \tag{8.21}$$

and conversely, $\hat{\mathbf{u}} = \mathbf{S}\hat{\mathbf{u}}'$ and $\hat{\mathbf{F}}_e = \mathbf{S}\hat{\mathbf{F}}_e'$. Then, the collective oscillation amplitude U and the collective excitation-power coefficient $E(\beta)$ satisfy the two equations

$$|U|^2 = \hat{\mathbf{u}}^\dagger \mathbf{D} \hat{\mathbf{u}}/2 = \hat{\mathbf{u}}'^\dagger \mathbf{D}' \hat{\mathbf{u}}'/2, \tag{8.22}$$

$$E(\beta) = \hat{\mathbf{u}}^\dagger \hat{\mathbf{F}}_e/(4A) = \hat{\mathbf{u}}'^\dagger \hat{\mathbf{F}}'_e/(4A), \tag{8.23}$$

where

$$\mathbf{D}' = \mathbf{S}^\dagger \mathbf{D} \mathbf{S}. \tag{8.24}$$

Using Eq. (8.17), we find that, except in its main diagonal, this similarity-transformed radiation matrix

$$\mathbf{D}' = \mathbf{S}^{-1} \mathbf{D} \mathbf{S} = \mathbf{S}^{-1} \mathbf{S} \,\mathrm{diag}\,(\Lambda_1, \Lambda_2, \cdots, \Lambda_M) = \mathrm{diag}\,(\Lambda_1, \Lambda_2, \cdots, \Lambda_M) \tag{8.25}$$

has only zero elements. The diagonal elements are given by the eigenvalues. Note that the Hermitian matrix $\mathbf{D}$, in general, may be complex. However, the similarity-transformed, diagonalised, matrix $\mathbf{D}'$ is necessarily real since all the eigenvalues are real.

For some particular systems, matrix $\mathbf{D}$ is singular. Then $|\mathbf{D}| = 0$, and thus, at least one of its eigenvalues equals zero. If

$$\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_{r_\mathbf{D}} > \Lambda_{r_\mathbf{D}+1} = \Lambda_{r_\mathbf{D}+2} = \cdots = \Lambda_M = 0, \tag{8.26}$$

where the integer $r_\mathbf{D}$ satisfies $1 \leq r_\mathbf{D} < M$, then we say that the radiation-damping matrix $\mathbf{D}$ is singular and of rank $r_\mathbf{D}$. (The matrix is, however, non-singular if $r_\mathbf{D} = M$.) An example of such systems, an axisymmetric system of three concentric bodies, is discussed in Section 5.7.2. For this system, $M = 18$ possible oscillating modes, but the rank $r_\mathbf{R}$ of the radiation-resistance matrix $\mathbf{R}$ is no more than 3.

8.1.3 Alternative Expression for Maximum Absorbed Power of a WEC-Body Array

By now, it is not yet obvious how the diagonalisation procedure described earlier can be useful. This will become clear in a moment.

For our WEC array consisting of N immersed WEC bodies and M oscillation modes, where $M \leq 6N$, the $M \times M$ radiation resistance matrix may, according to Eq. (5.145), be written as

$$\mathbf{R} = \frac{k}{16\pi J} \int_{-\pi}^{\pi} \hat{\mathbf{F}}_e(\beta) \hat{\mathbf{F}}_e^\dagger(\beta)\, d\beta. \tag{8.27}$$

In agreement with Eqs. (8.21) and (8.25), the similarity-transformed radiation-resistance matrix is

$$\mathbf{R}' = \mathbf{S}^\dagger \mathbf{R} \mathbf{S} = \frac{k}{16\pi J} \int_{-\pi}^{\pi} \mathbf{S}^\dagger \hat{\mathbf{F}}_e(\beta) \hat{\mathbf{F}}_e^\dagger(\beta) \mathbf{S}\, d\beta = \frac{k}{16\pi J} \int_{-\pi}^{\pi} \hat{\mathbf{F}}'_e(\beta) \hat{\mathbf{F}}'^\dagger_e(\beta)\, d\beta. \tag{8.28}$$

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From Eqs. (8.25) and (8.28), we can conclude that the only nonzero elements of matrix $\mathbf{R}'$ are the following main diagonal elements:

$$R'_{ii} = \Lambda_i = \frac{k}{16\pi J} \int_{-\pi}^{\pi} |\hat{F}'_{e,i}(\beta)|^2 \, d\beta, \quad \text{for } i = 1, 2, \ldots, M-1, M. \tag{8.29}$$

By the use of Eqs. (8.21) and (8.24), the optimum condition (8.5) may be reformulated as

$$\mathbf{R}'\hat{\mathbf{u}}'_{\text{OPT}} = \hat{\mathbf{F}}'_e/2, \tag{8.30}$$

that is,

$$R'_{ii}\hat{u}'_{i,\text{OPT}}(\beta) = \Lambda_i \hat{u}'_{i,\text{OPT}}(\beta) = \tfrac{1}{2}\hat{F}'_{e,i}(\beta) \quad \text{for } i = 1, 2, \ldots, M, \tag{8.31}$$

and the maximum absorbed wave power, Eq. (8.6), may be rewritten as

$$P_{\text{MAX}}(\beta) = \frac{1}{2}\hat{\mathbf{u}}'^{\dagger}_{\text{OPT}}(\beta)\mathbf{R}'\hat{\mathbf{u}}'_{\text{OPT}}(\beta) = \frac{1}{2}\sum_{i=1}^{M} \Lambda_i |\hat{u}'_{i,\text{OPT}}(\beta)|^2. \tag{8.32}$$

Note that the use of Eqs. (8.22) and (8.23) in combination with Eqs. (6.34) and (6.35) would lead to the same results.

In certain cases, the radiation resistance matrix $\mathbf{R}$ may be singular and of rank $r_{\mathbf{R}}$, which is an integer that satisfies $1 \le r_{\mathbf{R}} < M$. Then, because $\Lambda_i = 0$ for $r_{\mathbf{R}} < i \le M$, we have [45, equation (135)]

$$P_{\text{MAX}}(\beta) = \frac{1}{2}\sum_{i=1}^{M} \Lambda_i |\hat{u}'_{i,\text{OPT}}(\beta)|^2 = \frac{1}{2}\sum_{i=1}^{r_{\mathbf{R}}} \Lambda_i |\hat{u}'_{i,\text{OPT}}(\beta)|^2 = \frac{1}{8}\sum_{i=1}^{r_{\mathbf{R}}} \frac{|\hat{F}'_{e,i}(\beta)|^2}{\Lambda_i}, \tag{8.33}$$

where we, in the last step, made use of Eq. (8.31). Observe that if the smallest positive eigenvalue $\Lambda_{r_{\mathbf{R}}}$ is very small, the required optimum velocity amplitude $|\hat{u}'_{i,\text{OPT}}(\beta)|$ may be impractically large! Note that for $r_{\mathbf{R}} < i \le M$, we have no particular optimum requirement for $\hat{u}'_{i,\text{OPT}}(\beta)$. Then the optimum value of $\hat{u}'_{i,\text{OPT}}(\beta)$ is ambiguous, although the maximum absorbed power is unambiguous, as is made clear by Eq. (8.33).

Notice that because of similarity transformation, matrix $\mathbf{R}'$ is a diagonal matrix with the main diagonal elements being the eigenvalues of matrix $\mathbf{R}$; see Eq. (8.29). The consequence of this is that the maximum absorbed power can be expressed as a simple sum, as given by Eq. (8.33); cf. Eq. (8.6).

With this alternative expression for $P_{\text{MAX}}(\beta)$, the WEC-body array's maximum absorbed power averaged over all directions is

$$P_{\text{MAX,average}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\text{MAX}}(\beta)\, d\beta = \frac{1}{16\pi}\sum_{i=1}^{r_{\mathbf{R}}} \frac{1}{\Lambda_i}\int_{-\pi}^{\pi} |\hat{F}'_{e,i}(\beta)|^2 \, d\beta. \tag{8.34}$$

If we here eliminate the integral by using Eq. (8.29), we obtain the simple relationship

$$P_{\text{MAX,average}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\text{MAX}}(\beta)\,d\beta = \sum_{i=1}^{r_{\mathbf{R}}}\frac{J}{k} = \frac{r_{\mathbf{R}}J}{k} = r_{\mathbf{R}}\frac{J\lambda}{2\pi}. \tag{8.35}$$

It is interesting to note that this is just a factor of $r_{\mathbf{R}}$ larger than the direction-averaged wave power which may be absorbed by one single, WEC body, oscillating in only one mode, as given by Eqs. (6.13) and (6.15). For the array, the optimum power-gain function $G(\beta)$ has a directional average that equals the rank $r_{\mathbf{R}}$ of the WEC array's radiation-resistance matrix $\mathbf{R}$.

8.1.4 Example: Two-Body Axisymmetric System

Let us consider two concentric axisymmetric buoys, $p = 1$ and $p = 2$, restricted to oscillate in the heave mode only. Then the relevant oscillating modes are $i = 6(p-1)+3$; that is, $i = 3$ and $i = 9$ [cf. Eq. (5.154)]. According to Eq. (5.302), the system's radiation resistance matrix is

$$\mathbf{R} = \begin{bmatrix} R_{33} & R_{39} \\ R_{93} & R_{99} \end{bmatrix} = \frac{\omega k}{2\rho g^2 D(kh)} \begin{bmatrix} |f_{30}|^2 & f_{30}f_{90}^* \\ f_{90}f_{30}^* & |f_{90}|^2 \end{bmatrix}. \tag{8.36}$$

This result is also in agreement with Eq. (5.316). It is easy to verify that the corresponding determinant vanishes, which means that $\mathbf{R}$ is a singular matrix. Choosing an arbitrary value U_9, we find that the indeterminate equation (6.90) for optimum gives

$$U_3 = \frac{f_{30}A/2 - R_{39}U_9}{R_{33}} = \frac{f_{90}A/2 - R_{99}U_9}{R_{93}}. \tag{8.37}$$

From Eq. (6.93), the maximum power is

$$P_{\text{MAX}} = \frac{(f_{30}U_3^* + f_{90}U_9^*)A}{4} = \frac{|f_{30}A|^2}{8R_{33}} = \frac{f_{30}f_{90}^*|A|^2}{8R_{93}} = \frac{|f_{90}A|^2}{8R_{99}}, \tag{8.38}$$

which is independent of the choice of U_9. Inserting from Eqs. (8.36) and (4.130), we get

$$P_{\text{MAX}} = \frac{\rho g^2 D(kh)}{4\omega k}|A|^2 = \frac{J}{k}. \tag{8.39}$$

Thus, the maximum absorption width of this two-body axisymmetric system is the same as that of a single axisymmetric body oscillating in heave, as given by Eq. (6.104). This observation gives a clue to understanding why radiation resistance matrices may turn out to be singular.

The maximum absorbed power as given by Eqs. (6.103) and (8.39) corresponds to optimum destructive interference between the incident plane wave and the isotropically radiated circular wave generated by the heaving oscillation

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of one body or two bodies [in the case of Eq. (6.103) or Eq. (8.39), respectively]; see Figure 6.2. As for this optimum interference, it does not matter from which of the two bodies the circular wave originates. It is not possible to increase the maximum absorbed power by adding another isotropically radiating body to the system. This is the physical explanation for the singularity of the radiation resistance matrix given by Eq. (8.36). If only isotropically radiating heave modes are involved, the rank of the matrix cannot be more than one.

8.1.5 Example: Two Axisymmetric Bodies a Distance Apart

As a second example, let us consider a system of two equal axisymmetric bodies with their vertical symmetry axes located at $(x, y) = (\mp d/2, 0)$. We shall assume that they are oscillating in the heave mode only. Then the excitation-force vector is of the form $\hat{\mathbf{F}}_e = \left[\hat{F}_{e,3}\hat{F}_{e,9}\right]^{\mathrm{T}}$. Further, the radiation resistance matrix may be written as

$$\mathbf{R} = \begin{bmatrix} R_d & R_c \\ R_c & R_d \end{bmatrix}. \tag{8.40}$$

Note that the diagonal entry R_d is positive, while the off-diagonal entry R_c may be positive or negative, depending on the distance d between the two bodies. Since waves radiated from the two distinct bodies cannot cancel each other in all directions in the far-field region—which may be proved by summing two asymptotic expressions (4.256) and (4.260)—the matrix $\mathbf{R}$ is non-singular, and hence, $R_c^2 < R_d^2$. (We assume that the body does not have such a peculiar shape that we may find R_d to vanish for some particular frequency.)

The eigenvalues $\Lambda = \Lambda_{1,2}$ satisfy the algebraic equation

$$\begin{aligned} 0 = |\mathbf{R} - \Lambda \mathbf{I}| = (R_d - \Lambda)^2 - R_c^2 &= (\Lambda - R_d - R_c)(\Lambda - R_d + R_c) \\ &\equiv (\Lambda - \Lambda_1)(\Lambda - \Lambda_2). \end{aligned} \tag{8.41}$$

Thus,

$$\Lambda_1 = R_d + R_c > 0, \qquad \Lambda_2 = R_d - R_c > 0. \tag{8.42}$$

It is easy to show that, for this case, the corresponding eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$, as defined by Eq. (8.17) and normalised according to Eq. (8.19), are given by

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \equiv \mathbf{S} = \mathbf{S}^{\mathrm{T}} = \mathbf{S}^{\dagger} = \mathbf{S}^{-1}, \tag{8.43}$$

where we have also used definition (8.20). In accordance with Eq. (8.21), the excitation force and the optimum oscillation velocity vectors transform as

$$\hat{\mathbf{F}}_e' = \begin{bmatrix} \hat{F}_{e,3}' \\ \hat{F}_{e,9}' \end{bmatrix} = \mathbf{S}^{-1}\hat{\mathbf{F}}_e = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{F}_{e,3} + \hat{F}_{e,9} \\ \hat{F}_{e,3} - \hat{F}_{e,9} \end{bmatrix}, \tag{8.44}$$

$$\hat{\mathbf{u}}'_{\mathrm{OPT}} = \begin{bmatrix} \hat{u}'_{3,\mathrm{OPT}} \\ \hat{u}'_{9,\mathrm{OPT}} \end{bmatrix} = \mathbf{S}^{-1}\hat{\mathbf{u}}_{\mathrm{OPT}} = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{u}_{3,\mathrm{OPT}} + \hat{u}_{9,\mathrm{OPT}} \\ \hat{u}_{3,\mathrm{OPT}} - \hat{u}_{9,\mathrm{OPT}}, \end{bmatrix}. \tag{8.45}$$

For this example, where the 2×2 matrix $\mathbf{R}$ is non-singular, Eqs. (8.33)–(8.35) are applicable, with $r_{\mathbf{R}} = M = 2$. The maximum power absorbed by the two heaving bodies is

$$\begin{aligned} P_{\mathrm{MAX}}(\beta) &= \frac{|\hat{F}_{e,3}(\beta) + \hat{F}_{e,9}(\beta)|^2}{16(R_d + R_c)} + \frac{|\hat{F}_{e,3}(\beta) - \hat{F}_{e,9}(\beta)|^2}{16(R_d - R_c)} \\ &= \frac{(R_d + R_c)|\hat{u}_{3,\mathrm{OPT}} + \hat{u}_{9,\mathrm{OPT}}|^2}{4} + \frac{(R_d - R_c)|\hat{u}_{3,\mathrm{OPT}} - \hat{u}_{9,\mathrm{OPT}}|^2}{4}. \end{aligned} \tag{8.46}$$

The optimum complex velocity amplitudes of the two bodies satisfy

$$\hat{u}_{3,\mathrm{OPT}} + \hat{u}_{9,\mathrm{OPT}} = \frac{\hat{F}_{e,3}(\beta) + \hat{F}_{e,9}(\beta)}{2(R_d + R_c)}, \quad \hat{u}_{3,\mathrm{OPT}} - \hat{u}_{9,\mathrm{OPT}} = \frac{\hat{F}_{e,3}(\beta) - \hat{F}_{e,9}(\beta)}{2(R_d - R_c)}, \tag{8.47}$$

according to Eq. (8.31).

From a wave-body interaction point of view, it is interesting to note that the first rhs term in Eq. (8.46) and the first equation of (8.47) correspond to a suboptimum situation when the two equal heaving bodies cooperate as a source-mode (monopole) radiator—that is, when the constraint $\hat{u}_9 = \hat{u}_3$ is applied. Then the two bodies are constrained to heave with equal amplitudes and equal phases. In contrast, the last rhs term in Eq. (8.46) and the last equation of (8.47) correspond to a suboptimum situation when the two bodies are constrained to cooperate as a dipole-mode radiator—that is, when the constraint $\hat{u}_9 = -\hat{u}_3$ is applied. In general, Eqs. (8.46)–(8.47) may be considered to quantify the optimum situation for this combined monopole-dipole wave-absorbing system.

If the maximum radius of each body is sufficiently small—say, less than 1/40 of a wavelength—it may be considered as a point absorber, for which the heave excitation force $\hat{F}_e$ is dominated by the Froude–Krylov force, and the diffraction force may be neglected. If, moreover, the centre-to-centre distance d between the two bodies is large in comparison with the maximum body radius, then

$$\hat{\mathbf{F}}_e = \begin{bmatrix} \hat{F}_{e,3} \\ \hat{F}_{e,9} \end{bmatrix} \approx \hat{F}_{e0} \begin{bmatrix} \exp\{ik(d/2)\cos\beta\} \\ \exp\{-ik(d/2)\cos\beta\} \end{bmatrix}, \tag{8.48}$$

where $\hat{F}_{e0} = \rho g \pi a^2 A$. Here a is the water plane radius of each body, and A is the complex amplitude of the incident wave elevation at the chosen origin $(x, y) = (0, 0)$. For this point-absorber case, the entries in the radiation-resistance matrix $\mathbf{R}$ in Eq. (8.40) are approximately given by [25, equations 43–44] [see also Eq. (8.27)]

$$R_d \approx R_0 = \frac{k|\hat{F}_{e0}|^2}{8J} = \frac{k|\hat{F}_{e0}/A|^2}{4\rho g v_g}, \qquad R_c \approx R_0 J_0(kd), \tag{8.49}$$

where J_0 denotes Bessel function of the first kind and order zero. We observe that matrix $\mathbf{R}$ is non-singular, and moreover,

$$R_d + R_c \approx R_0[1 + J_0(kd)], \qquad R_d - R_c \approx R_0[1 - J_0(kd)] \tag{8.50}$$

are positive, since $-1 < J_0(kd) < 1$ for $kd > 0$.

Using Eqs. (8.46)–(8.50), we find

$$\begin{bmatrix} \hat{F}_{e,3} + \hat{F}_{e,9} \\ \hat{F}_{e,3} - \hat{F}_{e,9} \end{bmatrix} \approx 2\hat{F}_{e0} \begin{bmatrix} \cos\{(kd/2)\cos\beta\} \\ i\sin\{(kd/2)\cos\beta\} \end{bmatrix} \tag{8.51}$$

and

$$\begin{aligned} P_{\mathrm{MAX}}(\beta) &\approx \frac{|F_{e,0}|^2}{4R_0} \left\{ \frac{\cos^2[k(d/2)\cos\beta]}{1 + J_0(kd)} + \frac{\sin^2[k(d/2)\cos\beta]}{1 - J_0(kd)} \right\} \\ &= \frac{|F_{e,0}|^2}{4R_0} \frac{1 - J_0(kd)\cos(kd\cos\beta)}{1 - J_0^2(kd)}. \end{aligned} \tag{8.52}$$

Note that, in general, this maximum absorbed power is not equally divided between the two bodies.

We may note from the point-absorber approximation (8.48) that, since $\hat{F}_{e0}/A$ is real, $f_i(\beta) = \hat{F}_{e,i}(\beta)/A = \hat{F}^*_{e,i}(\beta + \pi)/A^* = f_i^*(\beta + \pi)$ for $i = 3, 9$. Correspondingly, we then find from Eqs. (8.3), (8.8) and (8.9)–(8.10) that $\bar{H}_r(\beta) = h_3(\beta)\hat{u}_3^* + h_9(\beta)\hat{u}_9^* = h_3^*(\beta + \pi)\hat{u}_3^* + h_9^*(\beta + \pi)\hat{u}_9^* = H_r^*(\beta + \pi)$ and, similarly, $\bar{H}_r(\beta + \pi) = H_r^*(\beta)$. Thus, for the two heaving bodies, we have here explicitly demonstrated that in the point-absorber limit the term containing the integral on the rhs of Eq. (5.152) is, as expected, negligible because diffraction effects are in this case negligible.

8.1.6 Maximum Absorbed Power with Amplitude Constraints

In practice, there are certain limitations on excursion, velocity and acceleration of oscillating bodies. Thus, for all designed bodies of the system there are upper bounds to amplitudes. For sufficiently low waves, these limitations are not reached, and the optimisation without constraints in the previous subsections is applicable. For moderate and large wave heights, such limitations or amplitude constraints may be important.

Cases in which constraints come into play for only some of the bodies of a system are complicated to analyse. More complicated numerical optimisation has to be applied [43, 111]. It is simpler when the wave height is so large that no optimum oscillation amplitude is less than the bound due to the design specifications of the system [45, 62]. It is also relatively simple to analyse a case in which only one single, but global, constraint is involved [112, 113]. However, this case is not further pursued here.

8.2 WEC Array of Oscillating Bodies and OWCs

Let us consider a WEC array consisting of N_B WEC bodies and N_C OWC chambers. If an OWC chamber structure can oscillate, it belongs also to the set of oscillating bodies. The oscillating bodies may be partly or completely submerged. The equilibrium level of an OWC's internal water surface may differ from the mean level of the external water surface, provided the internal air pressure, at equilibrium, is correspondingly adjusted (see Figure 4.8).

We may assume that each WEC body is, in general, free to oscillate in all its six modes of motion. Then the WEC-array system has

$$M = 6N_B + N_C \tag{8.53}$$

independent oscillators. The states of the $6N_B$ WEC-body oscillators are characterised by velocity components u_{ij}, where the first subscript ($i = 1, 2, \ldots, N_B$) denotes the body number. The second subscript ($j = 1, 2, \ldots, 6$) denotes the mode of body motion. For the remaining oscillators, the states are given by dynamic pressures p_k ($k = 1, 2, \ldots, N_C$) of the air in the OWC chambers. For a given frequency, the state of each oscillator is then given by a single complex amplitude ($\hat{u}_{ij}$ or $\hat{p}_k$).

Let a plane incident wave

$$\eta_0(x, y) = Ae^{-ikr(\beta)} \tag{8.54}$$

be given as in Eq. (4.98). Here A is the complex elevation amplitude at the origin $r(\beta) = 0$, and

$$r(\beta) = x\cos\beta + y\sin\beta\,, \tag{8.55}$$

where (x, y) are horizontal Cartesian coordinates and β is the angle of incidence [cf. Eq. (4.267)].

Let us first consider the case in which the oscillation amplitude is zero for each oscillator. The incident wave produces a hydrodynamic force on the wet surface S_i of any immersed body i and a volume flow due to the induced motion of the internal water surface S_k of any OWC unit k. Provided the bodies are not moving ($u_{ij} = 0$ for all i and j) and provided there is no dynamic air pressure above all OWCs' internal water surface ($p_k = 0$ for all k), the force and volume flow are called the *excitation force* $F_{e,ij}$ and the *excitation volume flow* $Q_{e,k}$, respectively. With the assumption of linear theory, their complex amplitudes are proportional to the incident wave amplitude A—that is, $\hat{F}_{e,ij}(\beta) = f_{ij}(\beta)A$ and $\hat{Q}_{e,k}(\beta) = q_k(\beta)A$ or, in column-vector notation,

$$\hat{\mathbf{F}}_e(\beta) = \mathbf{f}(\beta)A \quad \text{and} \quad \hat{\mathbf{Q}}_e(\beta) = \mathbf{q}(\beta)A. \tag{8.56}$$

Here, the complex coefficients of proportionality $f_{ij}(\beta)$ and $q_k(\beta)$ or, in column-vector notation, $\mathbf{f}(\beta)$ and $\mathbf{q}(\beta)$ are functions of β and also of ω. They are termed excitation coefficients, the *excitation force coefficient* and the *excitation volume flow coefficient*, respectively.

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8.2.1 Wave-Interacting OWCs and Bodies

Next, consider the case in which the amplitudes of all oscillators may be different from zero; thus, $u_{ij} \neq 0$ and $p_k \neq 0$. Because of our assumption of linearity, we have proportionality between input and output. Let us introduce additional complex coefficients of proportionality ($Z_{ij,i'j'}$, $H_{ij,k}$, $Y_{k,k'}$ and $H_{k,ij}$). Moreover, we may use the principle of superposition. Then, in terms of its complex amplitude, we write the j component of the total force acting on body i as

$$\hat{F}_{t,ij} = f_{ij}A - \sum_{i'j'} Z_{ij,i'j'}\, \hat{u}_{i'j'} - \sum_{k} H_{ij,k}\, \hat{p}_k, \tag{8.57}$$

where the second sum runs from $k = 1$ to $k = N_C$. In the first sum, i' runs from 1 to N_B and j' from 1 to 6. [Instead of using a single index $l = 6(i' - 1) + j'$, we denote a body oscillation by an apparent double index $i'j'$ to distinguish it from a pressure oscillator, denoted by a single index k.] In order to write Eq. (8.57) in vectorial form, we introduce the two matrices

$$\mathbf{Z} = \{Z_{ij,i'j'}\} \quad \text{and} \quad \mathbf{H} \equiv \mathbf{H}_{up} = \{H_{ij,k}\}. \tag{8.58}$$

Then we may write Eq. (8.57) in matrix form as

$$\hat{\mathbf{F}}_t = \mathbf{f}A - \mathbf{Z}\hat{\mathbf{u}} - \mathbf{H}_{up}\hat{\mathbf{p}} = \hat{\mathbf{F}}_e - \mathbf{Z}\hat{\mathbf{u}} - \mathbf{H}\hat{\mathbf{p}}. \tag{8.59}$$

Furthermore, the total volume flow due to the oscillation of the internal water surface S_k is

$$\hat{Q}_{t,k} = q_k A - \sum_{k'} Y_{k,k'}\, \hat{p}_{k'} - \sum_{ij} H_{k,ij}\, \hat{u}_{ij}, \tag{8.60}$$

or, in matrix notation,

$$\hat{\mathbf{Q}}_t = \mathbf{q}A - \mathbf{Y}\hat{\mathbf{p}} - \mathbf{H}_{pu}\hat{\mathbf{u}} = \hat{\mathbf{Q}}_e - \mathbf{Y}\hat{\mathbf{p}} + \mathbf{H}^{\mathrm{T}}\hat{\mathbf{u}}, \tag{8.61}$$

where we utilised the fact that

$$H_{k,ij} = -H_{ij,k} \quad \text{and thus} \quad \mathbf{H}_{pu} = -\mathbf{H}^{\mathrm{T}}; \tag{8.62}$$

see Problem 8.2 and Section 8.2.5.

It can be shown (see Section 8.2.5) that matrices $\mathbf{Z}$ and $\mathbf{Y}$ are symmetric, which means that they do not change by transposition, or

$$\mathbf{Z}^{\mathrm{T}} = \mathbf{Z}, \qquad \mathbf{Y}^{\mathrm{T}} = \mathbf{Y}. \tag{8.63}$$

The three matrices $\mathbf{Z}$, $\mathbf{Y}$ and $\mathbf{H}$ have dimensions $6N_B \times 6N_B$, $N_C \times N_C$, and $6N_B \times N_C$, respectively.

Equations (8.59) and (8.61) represent a linear system, which may be illustrated in a block diagram as shown in Figure 8.2. The two sets of coefficients $Z_{ij,i'j'}$ and $H_{ij,k}$ in Eq. (8.57) compose the radiation impedance matrix $\mathbf{Z}$ (cf. Section 5.5) and the *OWC-to-body coupling matrix* $\mathbf{H}$ for the oscillating

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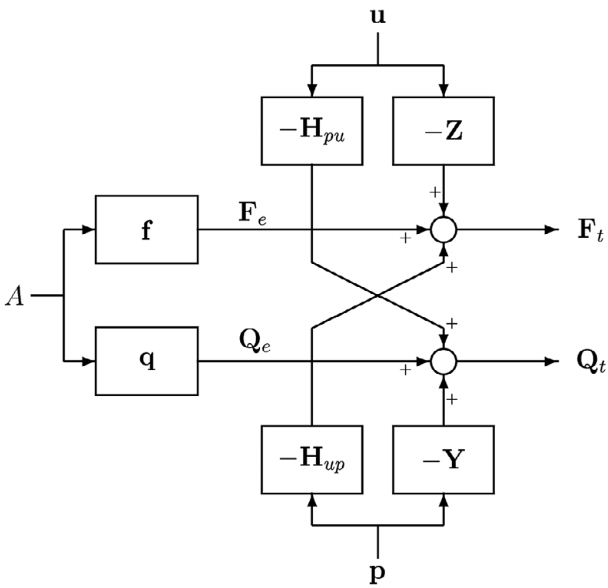

Figure 8.2: Block diagram of the system of oscillating bodies and OWCs. The incident wave, the oscillating bodies' velocities and the oscillating air-chamber pressures (represented by A, $\mathbf{u}$ and $\mathbf{p}$, respectively) are considered as inputs to the system. The body forces and the volume flows (represented by $\mathbf{F}_t$ and $\mathbf{Q}_t$) are considered as outputs here. The diagram illustrates Eqs. (8.59) and (8.61).

bodies. Similarly, the set of complex coefficients $Y_{k,k'}$ in Eq. (8.60) composes the radiation admittance matrix $\mathbf{Y}$ for the oscillating surface pressure distribution, as introduced first by Evans [98]. In the same equation we introduced also the body-to-OWC coupling coefficients $H_{k,ij}$ which compose the matrix $\mathbf{H}_{pu}$.

The two sets of complex coefficients $H_{ij,k}$ and $H_{k,ij}$ have the dimension of length squared for $j = 1,2,3$ and length cubed for $j = 4,5,6$. They represent hydrodynamical coupling between the WEC array's oscillating bodies and its OWCs.

Mechanical impedance, as introduced in Section 2.2.2, has the dimension of force divided by velocity. Contrary to common usage in electrical circuit theory, we have here, for notational convenience, defined an admittance $Y_{k,k'}$ which is not dimensionally inverse to impedance $Z_{ij,i'j'}$. Note that $Y'_{k,k'} \equiv Y_{k,k'}/S_k S_{k'}$ is dimensionally inverse to $Z_{ij,i'j'}$, but it is more convenient to write just $Y_{k,k'}$ instead of $S_k S_{k'} Y'_{k,k'}$ in Eq. (8.60).

Defining the three M dimensional column vectors

$$\hat{\mathbf{F}}_{FQ,t} \equiv \begin{bmatrix} \hat{\mathbf{F}}_t \\ -\hat{\mathbf{Q}}_t \end{bmatrix}, \quad \hat{\mathbf{F}}_{FQ,e} \equiv \begin{bmatrix} \hat{\mathbf{F}}_e \\ -\hat{\mathbf{Q}}_e \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{u}}_{FQ} \equiv \begin{bmatrix} \hat{\mathbf{u}} \\ -\hat{\mathbf{p}} \end{bmatrix}, \tag{8.64}$$

and defining the complex $M \times M$ *total-radiation-impedance matrix* as

$$\mathbf{Z}_{FQ} \equiv \begin{bmatrix} \mathbf{Z} & -\mathbf{H} \\ \mathbf{H}^{\mathrm{T}} & \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{R} + i\mathbf{X} & -(\mathbf{C} + i\mathbf{J}) \\ (\mathbf{C} + i\mathbf{J})^{\mathrm{T}} & \mathbf{G} + i\mathbf{B} \end{bmatrix}, \tag{8.65}$$

we can combine the two equations (8.59) and (8.61) into a single matrix equation

$$\hat{\mathbf{F}}_{FQ,t} = \hat{\mathbf{F}}_{FQ,e} + \hat{\mathbf{F}}_{FQ,r} = \hat{\mathbf{F}}_{FQ,e} - \mathbf{Z}_{FQ}\hat{\mathbf{u}}_{FQ}, \tag{8.66}$$

which specialises to Eq. (6.81) for the one-body case ($N_B = 1$ and $N_C = 0$).

For the purpose of further discussion to follow, we have also, in Eq. (8.65), split complex matrix entries into real and imaginary parts: $\mathbf{Z} = \mathbf{R}+i\mathbf{X}$, $\mathbf{Y} = \mathbf{G}+i\mathbf{B}$ and $\mathbf{H} = \mathbf{C} + i\mathbf{J}$, where the radiation resistance matrix $\mathbf{R}$, the radiation reactance matrix $\mathbf{X}$, the radiation conductance matrix $\mathbf{G}$, the radiation susceptance matrix $\mathbf{B}$, as well as the matrices $\mathbf{C}$ and $\mathbf{J}$, are real, whereas matrices $\mathbf{Z}$, $\mathbf{Y}$ and $\mathbf{H}$ are complex. All these nine matrices are frequency dependent.

8.2.2 Active Power and Reactive Power

For a simple oscillating system, we considered, in Section 2.3.1, active power and reactive power, which are related to produced or consumed power and to stored power, respectively. We also defined a ‘complex power’ $\mathcal{P} = R|\hat{u}|^2/2 + iX|\hat{u}|^2/2$, where the real and imaginary parts are the active power and the reactive power, respectively; see Eq. (2.90). Here, R and X denote resistance and reactance, respectively.

Furthermore, in Section 3.4, we introduced the concepts of radiation resistance, radiation reactance and radiation impedance, which have been a matter of discussion in most of the subsequent chapters. For instance, in Section 5.5.4, we discussed how the radiation reactance, and thus the ‘added mass’, is related to the—usually positive—difference between time-average kinetic and potential energies in the near-field region of oscillating bodies. As we shall discuss in some detail later, the three real matrices $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{J}$ are associated with active power, while the three remaining ones, matrices $\mathbf{X}$, $\mathbf{B}$ and $\mathbf{C}$, are associated with reactive power.

Let us split the total radiation matrix $\mathbf{Z}_{FQ}$ as given by Eq. (8.65), not into real and imaginary parts, but into active and reactive parts, as follows:

$$\mathbf{Z}_{FQ} = \mathbf{Z}_{FQ,\,\mathrm{active}} + \mathbf{Z}_{FQ,\,\mathrm{reactive}}, \tag{8.67}$$

where

$$\mathbf{Z}_{FQ,\,\mathrm{active}} = \begin{bmatrix} \mathbf{R} & -i\mathbf{J} \\ i\mathbf{J}^{\mathrm{T}} & \mathbf{G} \end{bmatrix} \equiv \mathbf{D}, \tag{8.68}$$

$$\mathbf{Z}_{FQ,\,\mathrm{reactive}} = i\begin{bmatrix} \mathbf{X} & i\mathbf{C} \\ -i\mathbf{C}^{\mathrm{T}} & \mathbf{B} \end{bmatrix}. \tag{8.69}$$

According to Eqs. (8.65)–(8.69), the total radiation force is

$$\hat{\mathbf{F}}_{FQ,r} = -\mathbf{Z}_{FQ}\,\hat{\mathbf{u}}_{FQ} = -\mathbf{D}\hat{\mathbf{u}}_{FQ} + \mathbf{Z}_{FQ,\,\mathrm{reactive}}\,\hat{\mathbf{u}}_{FQ}. \tag{8.70}$$

Thus, we have here, for the present WEC array, extended Eqs. (3.32)–(3.33), where we defined the radiation force $F_{r,i}$ and split it into active and reactive

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components. Observe that both matrices $\mathbf{D} = \mathbf{Z}_{FQ,\text{active}}$ and $(\mathbf{Z}_{FQ,\text{reactive}}/i)$ are not real and symmetric, but complex and Hermitian, since $\mathbf{D}^\dagger = \mathbf{D}$ and $(\mathbf{Z}_{FQ,\text{reactive}}/i) = (\mathbf{Z}_{FQ,\text{reactive}}/i)^\dagger$, due to relations (8.62)–(8.63). Matrices $\mathbf{D}$ and $(\mathbf{Z}_{FQ,\text{reactive}}/\mathrm{i})$ are real and symmetric if the WEC array contains no OWCs or no oscillating bodies, that is, in cases where $\mathbf{Z}_{FQ} = \mathbf{Z} = \mathbf{R}+\mathrm{i}\mathbf{X}$ or $\mathbf{Z}_{FQ} = \mathbf{Y} = \mathbf{G}+\mathrm{i}\mathbf{B}$, respectively.

As discussed in Section 8.1.2, all eigenvalues of Hermitian matrices are real [16, page 109]. Moreover, as we shall show later, matrix $\mathbf{D}$ is positive semidefinite; see Eq. (8.75). Therefore, the eigenvalues of $\mathbf{D}$ are necessarily nonnegative. The matrix $(\mathbf{Z}_{FQ,\text{reactive}}/i)$ may, however, have negative as well as positive eigenvalues.

Since matrix $\mathbf{D}$ is Hermitian, it is possible to diagonalise it as in Eq. (8.25). It follows that, for any M-dimensional complex column vector $\hat{\mathbf{u}}_{FQ}$, the scalar matrix product $\hat{\mathbf{u}}_{FQ}^\dagger \mathbf{D} \hat{\mathbf{u}}_{FQ} = \hat{\mathbf{u}}_{FQ}'^\dagger \mathbf{D}' \hat{\mathbf{u}}_{FQ}'$ is real. By the same argument, $\hat{\mathbf{u}}_{FQ}^\dagger \mathbf{Z}_{FQ,\text{reactive}} \hat{\mathbf{u}}_{FQ}$ is purely imaginary.

If we premultiply Eq. (8.70) by $-\hat{\mathbf{u}}_{FQ}^\dagger/2$, we obtain the 'complex radiated power'

$$\mathcal{P}_r = -\hat{\mathbf{u}}_{FQ}^\dagger \hat{\mathbf{F}}_{FQ,r}/2 = \hat{\mathbf{u}}_{FQ}^\dagger \mathbf{D} \hat{\mathbf{u}}_{FQ}/2 + \hat{\mathbf{u}}_{FQ}^\dagger \mathbf{Z}_{FQ,\text{reactive}} \hat{\mathbf{u}}_{FQ}/2, \tag{8.71}$$

where the last term, the reactive-power term $\hat{\mathbf{u}}_{FQ}^\dagger \mathbf{Z}_{FQ,\text{reactive}} \hat{\mathbf{u}}_{FQ}/2$, is purely imaginary, whereas the first term, the radiated-power term $\hat{\mathbf{u}}_{FQ}^\dagger \mathbf{D} \hat{\mathbf{u}}_{FQ}/2 \equiv P_r$, is real and nonnegative—see inequality (8.75). Moreover, if we premultiply by $\hat{\mathbf{u}}_{FQ}^\dagger/2$ the excitation vector $\hat{\mathbf{F}}_{FQ,e}$, which was introduced in Eq. (8.64), we get the 'complex excitation power'

$$\mathcal{P}_e = \hat{\mathbf{u}}_{FQ}^\dagger \hat{\mathbf{F}}_{FQ,e}/2. \tag{8.72}$$

Here the imaginary part, $\mathrm{Im}\{\mathcal{P}_e\}$, represents reactive power; see Eq. (5.218).

8.2.3 Wave Power Converted by the WEC Array

For any oscillation vector $\hat{\mathbf{u}}_{FQ} = \begin{bmatrix} \hat{\mathbf{u}} & -\hat{\mathbf{p}} \end{bmatrix}^{\mathrm{T}}$, the time-average wave power absorbed by the array is $P = \mathrm{Re}\{\mathcal{P}\} = \mathrm{Re}\{\mathcal{P}_e\} - \mathrm{Re}\{\mathcal{P}_r\} = P_e - P_r$, where the excitation power P_e and the radiated power P_r are given by

$$P_e = \mathrm{Re}\{\mathcal{P}_e\} = \frac{\hat{\mathbf{u}}_{FQ}^\dagger \hat{\mathbf{F}}_{FQ,e} + \hat{\mathbf{F}}_{FQ,e}^\dagger \hat{\mathbf{u}}_{FQ}}{4} \quad \text{and} \quad P_r = \mathrm{Re}\{\mathcal{P}_r\} = \frac{\hat{\mathbf{u}}_{FQ}^\dagger \mathbf{D} \hat{\mathbf{u}}_{FQ}}{2}; \tag{8.73}$$

cf. Eqs. (6.85)–(6.86) and (7.26)–(7.27). We may express this in the form of Eq. (6.32) or (6.84), provided we define the collective excitation-power coefficient $E(\beta)$ and the collective oscillation amplitude U by

$$E(\beta) = \frac{\hat{\mathbf{u}}_{FQ}^\dagger \hat{\mathbf{F}}_{FQ,e}}{4A} \quad \text{and} \quad |U|^2 = UU^* = \frac{\hat{\mathbf{u}}_{FQ}^\dagger \mathbf{D} \hat{\mathbf{u}}_{FQ}}{2}, \tag{8.74}$$

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where we still choose the otherwise arbitrary phase angle of U such as to make $A^*E^*(\beta)/U$ a real and positive quantity.

For a case with no incident wave, thus $\hat{\mathbf{F}}_{FQ,e} = \mathbf{0}$ (which means that $P_e = 0$), energy conservation requires that the absorbed wave power $P = P_e - P_r = -P_r = -\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ}/2$ cannot be positive. Hence, for all possible finite oscillation-state vectors $\hat{\mathbf{u}}_{FQ}$, we have

$$\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ} \geq 0. \tag{8.75}$$

Thus, in general, the radiation damping matrix $\mathbf{D}$ is positive semidefinite. It is singular in cases when its determinant vanishes, $|\mathbf{D}| = 0$. Otherwise, it is positive definite, $\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ} > 0$. This is a generalisation of inequality (6.88).

The maximum wave power absorbed by the array may be written in various ways, such as

$$P_{\mathrm{MAX}} = \frac{P_{e,\mathrm{OPT}}}{2} \equiv \frac{\hat{\mathbf{F}}_{FQ,e}^{\dagger}\hat{\mathbf{u}}_{FQ,0}}{4} = \frac{\hat{\mathbf{u}}_{FQ,0}^{\dagger}\hat{\mathbf{F}}_{FQ,e}}{4} = P_{r,\mathrm{OPT}} \equiv \frac{\hat{\mathbf{u}}_{FQ,0}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ,0}}{2}, \tag{8.76}$$

where $\hat{\mathbf{u}}_{FQ,0} \equiv \hat{\mathbf{u}}_{FQ,\mathrm{OPT}}(\beta)$ is an optimum value of the oscillation-state vector $\hat{\mathbf{u}}_{FQ}$ that has to satisfy the optimum condition

$$\mathbf{D}\hat{\mathbf{u}}_{FQ,0}(\beta) = \hat{\mathbf{F}}_{FQ,e}(\beta)/2. \tag{8.77}$$

Assuming that the WEC array is equipped with sufficient control devices, we may here consider the complex-velocity column vector $\hat{\mathbf{u}}_{FQ}$ to be an independent variable, whereas its optimum value $\hat{\mathbf{u}}_{FQ,0}(\beta)$ is proportional to the incident wave amplitude and also dependent on the angle of wave incidence β.

By manipulating Eqs. (8.16), (8.73), (8.76) and (8.77), we can show that

$$P_{\mathrm{MAX}}(\beta) - P = \tfrac{1}{2}[\hat{\mathbf{u}}_{FQ} - \hat{\mathbf{u}}_{FQ,0}(\beta)]^{\dagger}\mathbf{D}[\hat{\mathbf{u}}_{FQ} - \hat{\mathbf{u}}_{FQ,0}(\beta)]. \tag{8.78}$$

For a fixed value of the absorbed wave power P, where $P < P_{\mathrm{MAX}}$, Eq. (8.78) represents an 'ellipsoid' in the complex M-dimensional $\hat{\mathbf{u}}_{FQ}$ space, $\mathbb{C}^M$—but reduced to an $r_{\mathbf{D}}$-dimensional $\hat{\mathbf{u}}_{FQ}$ space, $\mathbb{C}^{r_{\mathbf{D}}}$, in cases where the radiation damping matrix $\mathbf{D}$ is singular and of rank $r_{\mathbf{D}} < M$. The centre of the 'ellipsoid' is at the optimum point $\hat{\mathbf{u}}_{FQ} = \hat{\mathbf{u}}_{FQ,0}$. The elliptical semiaxes are $\sqrt{2(P_{\mathrm{MAX}} - P)/\Lambda_i}$ for $i = 1, 2, \ldots, r_{\mathbf{D}}$, where Λ_i is any member of the positive definite (nonzero) eigenvalues of matrix $\mathbf{D}$—cf. Eq. (8.17). The 'ellipsoid' that corresponds to $P = 0$ runs through, for example, points $\hat{\mathbf{u}}_{FQ} = \mathbf{0}$ and $\hat{\mathbf{u}}_{FQ} = 2\,\hat{\mathbf{u}}_{FQ,0}$. The degenerate 'ellipsoid' that corresponds to $P = P_{\mathrm{MAX}}$ is just one point, which represents the (unconstrained) optimum situation. Choosing smaller P increases the size of the 'ellipsoid'. If $M = 1$, then the 'ellipsoid' simplifies to a circle in the complex $\hat{u}_1$ plane; cf. Problem 4.16. Considering how the absorbed power P varies with $\hat{\mathbf{u}}_{FQ}$, we may think of relationship (8.78) as a 'paraboloid' in the complex M-dimensional $\hat{\mathbf{u}}_{FQ}$ space, $\mathbb{C}^M$. The top point of this 'paraboloid' corresponds to the optimum, $(\hat{\mathbf{u}}_{FQ,0}, P_{\mathrm{MAX}})$. Here, M should be replaced by $r_{\mathbf{D}}$ if the radiation matrix $\mathbf{D}$ is singular.

The simple equation (6.37), which for a fixed absorbed-power value P represents a circle in the complex U plane, can be shown to be equivalent to Eq. (8.78) above, which represents an 'ellipsoid' in the complex $\hat{\mathbf{u}}_{FQ}$ space, by making use of Eqs. (6.36), (8.74) and (8.77). Starting from (6.37), we have

$$\begin{aligned} P_{\mathrm{MAX}} - P &= |U_0(\beta) - U|^2 = U_0 U_0^* - U_0 U^* - U_0^* U + U U^* \\ &= \frac{1}{2}\hat{\mathbf{u}}_{FQ,0}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ,0} - AE - A^* E^* + \frac{1}{2}\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ} \\ &= \frac{1}{2}\hat{\mathbf{u}}_{FQ,0}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ,0} - \frac{1}{4}\hat{\mathbf{u}}_{FQ}^{\dagger}\hat{\mathbf{F}}_{FQ,e} - \frac{1}{4}\hat{\mathbf{F}}_{FQ,e}^{\dagger}\hat{\mathbf{u}}_{FQ} + \frac{1}{2}\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ} \\ &= \frac{1}{2}\hat{\mathbf{u}}_{FQ,0}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ,0} - \frac{1}{2}\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ,0} - \frac{1}{2}\hat{\mathbf{u}}_{FQ,0}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ} + \frac{1}{2}\hat{\mathbf{u}}_{FQ}^{\dagger}\mathbf{D}\hat{\mathbf{u}}_{FQ} \\ &= \frac{1}{2}[\hat{\mathbf{u}}_{FQ} - \hat{\mathbf{u}}_{FQ,0}(\beta)]^{\dagger}\mathbf{D}[\hat{\mathbf{u}}_{FQ} - \hat{\mathbf{u}}_{FQ,0}(\beta)], \end{aligned} \tag{8.79}$$

noting that the collective excitation-power coefficient $E(\beta)$ is a scalar and, thus, equals its own transpose, and recalling the Hermitian property (8.16) of the radiation-damping matrix **D**.

Proof (8.79) also serves to demonstrate that, with the generalisations (8.74), Eqs. (6.32)–(6.37) are valid not only for a single, one-mode oscillating body but even for an array consisting of several WEC units—oscillating bodies as well as OWCs. In particular, Eq. (8.4), as illustrated in Figure 6.5, is applicable even to the general case of wave energy absorption by an array of oscillating bodies as well as OWCs.

The involved physical quantities, $\hat{\mathbf{F}}_{FQ,e}$ and $\hat{\mathbf{u}}_{FQ}$, pertain to the wave-interacting surfaces of the WEC array. Thus, if inverse Fourier transformation is applied to, for example, Eqs. (6.32) and (8.74), these equations also may be applied to analyse the WEC array's wave-power absorption even in the case of non-sinusoidal time variation. (However, this is not possible for expressions where the excitation forces and volume flows are expressed in terms of Kochin functions or other far-field coefficients.)

In terms of similarity-transformed excitation amplitudes $\hat{F}'_{FQ,e,i}(\beta)$ and corresponding optimum oscillation amplitudes $\hat{u}'_{FQ,i0}(\beta)$, the maximum absorbed power may be written as

$$P_{\mathrm{MAX}} = \sum_{i=1}^{M} \frac{|\hat{F}'_{FQ,e,i}(\beta)|^2}{8\Lambda_i} = \frac{1}{2}\sum_{i=1}^{M} \Lambda_i\, |\hat{u}'_{FQ,i0}(\beta)|^2, \tag{8.80}$$

corresponding to the optimum condition

$$\Lambda_i \hat{u}'_{FQ,i0}(\beta) = \tfrac{1}{2}\hat{F}'_{FQ,e,i}(\beta). \tag{8.81}$$

Likewise, Eq. (8.78) may be simplified to

$$\begin{aligned} 2(P_{\mathrm{MAX}} - P) &= [\hat{\mathbf{u}}'_{FQ} - \hat{\mathbf{u}}'_{FQ,0}(\beta)]^{\dagger}\mathbf{D}'[\hat{\mathbf{u}}'_{FQ} - \hat{\mathbf{u}}'_{FQ,0}(\beta)] \\ &= \sum_{i=1}^{M} \Lambda_i\, |\hat{u}'_{FQ,i} - \hat{u}'_{FQ,i0}(\beta)|^2. \end{aligned} \tag{8.82}$$

8.2.4 Example: WEC Body Containing an OWC

For simplicity, let us consider a WEC body that contains one OWC and oscillates in only one body mode. It could, for example, correspond to a backward-bent duct buoy (BBDB) device [114] constrained to oscillate in pitch only or an axisymmetric wave-powered navigation buoy [115] constrained to oscillate in heave only. Both devices were invented in Japan by Yoshio Masuda. During the early 1980s, sea tests of a heaving body containing an OWC were carried out also in Norway [116].

In this simple case, the two M-dimensional column vectors $\hat{\mathbf{u}}_{FQ}$ and $\hat{\mathbf{F}}_{FQ,e}$, introduced by Eq. (8.64), are simplified to two-dimensional column vectors,

$$\hat{\mathbf{u}}_{FQ} = \begin{bmatrix} \hat{u} \\ -\hat{p} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{F}}_{FQ,e}(\beta) = \begin{bmatrix} \hat{F}_e(\beta) \\ -\hat{Q}_e(\beta) \end{bmatrix}. \tag{8.83}$$

Correspondingly, the radiation-damping matrix introduced by Eq. (8.68) is, in this case,

$$\mathbf{D} = \begin{bmatrix} R & -\mathrm{i}J \\ \mathrm{i}J & G \end{bmatrix}, \tag{8.84}$$

which is a Hermitian matrix of dimension 2×2.

The two eigenvalues Λ_1 and Λ_2 of this radiation damping matrix (8.84) are solutions of the second-degree algebraic equation $|\mathbf{D} - \Lambda\mathbf{I}| = \Lambda^2 - (R+G)\Lambda + RG - J^2 = 0$. Thus, Λ_1 and Λ_2 are given by

$$\Lambda_i = \left[R + G - (-1)^i \sqrt{(R+G)^2 - 4(RG - J^2)} \right] /2 \quad \text{for } i = 1, 2. \tag{8.85}$$

The corresponding two eigenvectors, which satisfy Eqs. (8.17) and (8.19), are

$$\mathbf{e}_i = C_i \begin{bmatrix} \mathrm{i}J \\ R - \Lambda_i \end{bmatrix}, \quad \text{where} \quad C_i = 1/\sqrt{(R - \Lambda_i)^2 + J^2}. \tag{8.86}$$

If the system we are dealing with is a heaving axisymmetric body containing an axisymmetric OWC, we have

$$J^2 = RG, \tag{8.87}$$

and thus, from Eq. (8.85), we see that $\Lambda_1 = R + G$ and $\Lambda_2 = 0$, which means that in this case, matrix $\mathbf{D}$ is singular and of rank $r_{\mathbf{D}} = 1$. Therefore, there is only one term in the sum of the rhs of Eq. (8.82):

$$2(P_{\mathrm{MAX}} - P) = (R + G)|\hat{u}'_{FQ,1} - \hat{u}'_{FQ,10}(\beta)|^2, \tag{8.88}$$

which represents a circle in the complex $\hat{u}'_{FQ,1}$ plane. The centre of the circle is at $\hat{u}'_{FQ,1} = \hat{u}'_{FQ,10}(\beta)$ and the radius is $\sqrt{2(P_{\mathrm{MAX}} - P)/(R+G)}$. Because of the singularity of the radiation damping matrix, the similarity-transformed variable $\hat{u}'_{FQ,2}$ is irrelevant and may have any arbitrary value without influencing the maximum absorbed power.

It is furthermore possible to show from the optimum condition (8.77) and Eq. (8.87) that $iG\hat{F}_e = J\hat{Q}_e$ in this case, which leads to

$$P_{\text{MAX}} = \frac{|\hat{F}_e|^2}{8R} = \frac{i\hat{F}_e\hat{Q}_e^*}{8J} = \frac{|\hat{Q}_e|^2}{8G} \tag{8.89}$$

upon application of Eq. (8.76). Notice that because of the singularity of the radiation damping matrix, the maximum power that can be absorbed by the system is equal to that of a single heaving body without any OWC [cf. Eq. (6.12)] or of a single OWC contained in a fixed body [cf. Eq. (7.43)].

The physical reason for the singularity is that both modes, the heaving-body mode and the OWC mode, can radiate only isotropic outgoing waves. To obtain the maximum absorbed wave power, the optimum isotropically radiated wave may be realised by any optimum combined wave radiation from the axisymmetric OWC and the heaving axisymmetric body. The transformed oscillation $\hat{u}'_{FQ,2}$ corresponds to a situation where the heave mode and the OWC mode cancel each other's radiated waves in the far-field region.

8.2.5 Reciprocity Relations for Radiation Parameters

Our task in the following is to prove reciprocity relations (8.62)–(8.63), which underlie the results presented so far in this section. Note that these relations are important not only for arrays of many units of WEC bodies and OWCs but also for a single WEC unit in the form of an OWC housed in a floating structure, as we have just discussed. As in Section 5.5.2, we decompose the velocity potential into incident, diffracted and radiated components. The radiated wave potential can be further decomposed as

$$\hat{\phi}_r = \sum_{ij} \varphi_{ij}\hat{u}_{ij} + \sum_k \varphi_k \hat{p}_k = \boldsymbol{\varphi}_u^{\mathrm{T}}\hat{\mathbf{u}} + \boldsymbol{\varphi}_p^{\mathrm{T}}\hat{\mathbf{p}}, \tag{8.90}$$

where $\boldsymbol{\varphi}_u$ is a column vector composed of all the complex potential coefficients φ_{ij}, which depend on x, y, z and ω. Similarly, $\boldsymbol{\varphi}_p$ is the column vector composed of all the coefficients φ_k.

The j component of the total force on body i due to the hydrodynamic pressure $\hat{p} = -i\omega\rho\hat{\phi}$ is obtained by integration,

$$\hat{F}_{t,ij} = -\iint_{S_i} \hat{p} n_{ij}\, dS = i\omega\rho \iint_{S_i} n_{ij}\hat{\phi}\, dS, \tag{8.91}$$

in accordance with Eq. (5.22). The total volume flow through the mean water surface S_k is

$$\hat{Q}_{t,k} = \iint_{S_k} \hat{v}_z\, dS = \iint_{S_k} \frac{\partial\hat{\phi}}{\partial z}\, dS. \tag{8.92}$$

Using decompositions (5.160) and (8.90), we easily see that $\hat{F}_{t,ij}$ is as given by Eq. (8.57) with

$$Z_{ij,i'j'} = -i\omega\rho \iint_{S_i} n_{ij}\varphi_{i'j'}\, dS, \tag{8.93}$$

$$H_{ij,k} = -i\omega\rho \iint_{S_i} n_{ij}\varphi_k\, dS, \tag{8.94}$$

and $\hat{Q}_{t,k}$ is as given by Eq. (8.60) with

$$Y_{k,k'} = -\iint_{S_k} \frac{\partial}{\partial z}\varphi_{k'}\, dS, \tag{8.95}$$

$$H_{k,ij} = -\iint_{S_k} \frac{\partial}{\partial z}\varphi_{ij}\, dS. \tag{8.96}$$

Because of the boundary condition on the wet surfaces $S_i, i = 1, 2, \ldots, N_B$,

$$\frac{\partial\varphi_{i'j}}{\partial n} = n_{ij}\delta_{ii'} = \begin{cases} n_{ij}, & \text{for } i' = i \\ 0, & \text{for } i' \neq i, \end{cases} \tag{8.97}$$

we may rewrite expression (8.93) for the radiation impedance matrix as

$$Z_{ij,i'j'} = -i\omega\rho \iint_{S_i} \frac{\partial\varphi_{ij}}{\partial n}\varphi_{i'j'}\, dS \tag{8.98}$$

and extend the region of integration from S_i to include also the wet surfaces of all the other bodies. Further, from the boundary condition for φ_{ij} on S_k,

$$\left(\frac{\partial}{\partial z} - \frac{\omega^2}{g}\right)\varphi_{ij} = 0, \tag{8.99}$$

we observe that

$$0 = -i\omega\rho \iint_{S_k} \left(\frac{\partial}{\partial n} + \frac{\omega^2}{g}\right)\varphi_{ij}\,\varphi_{i'j'}\, dS, \tag{8.100}$$

since $\partial/\partial n = -\partial/\partial z$ on S_k. We are now in the position to extend the region of integration in Eq. (8.98) to the totality of wave-generating surfaces

$$S = \sum_{i=1}^{N_B} S_i + \sum_{k=1}^{N_C} S_k. \tag{8.101}$$

By summation, we obtain, after reverting to matrix notation,

$$\mathbf{Z} = -i\omega\rho \iint_S \frac{\partial\boldsymbol{\varphi}_u}{\partial n}\boldsymbol{\varphi}_u^{\mathrm{T}}\, dS - \sum_k i\omega\rho \iint_{S_k} \frac{\omega^2}{g}\boldsymbol{\varphi}_u\boldsymbol{\varphi}_u^{\mathrm{T}}\, dS. \tag{8.102}$$

Alternatively, since $\partial\varphi_{ij}/\partial n$ is real on S_i, we may replace $\partial\varphi_{ij}/\partial n$ by $\partial\varphi_{ij}^*/\partial n$ in Eq. (8.98). We may also replace φ_{ij} by φ_{ij}^* in Eq. (8.100). These give

$$\mathbf{Z} = -i\omega\rho \iint_S \frac{\partial\boldsymbol{\varphi}_u^*}{\partial n}\boldsymbol{\varphi}_u^{\mathrm{T}}\, dS - \sum_k i\omega\rho \iint_{S_k} \frac{\omega^2}{g}\boldsymbol{\varphi}_u^*\boldsymbol{\varphi}_u^{\mathrm{T}}\, dS. \tag{8.103}$$

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Note that these expressions for the radiation impedance matrix $\mathbf{Z}$ are extensions of Eq. (5.170)–(5.171) to the case where wave-generating OWCs are also included in the system.

Subtracting from Eq. (8.102) its own transpose, we find that the last term (the sum) does not contribute because of cancellation, and hence, $\mathbf{Z} - \mathbf{Z}^{\mathrm{T}}$ vanishes due to Eqs. (5.127)–(5.128). As a consequence, $\mathbf{Z}^{\mathrm{T}} = \mathbf{Z}$, which means that $\mathbf{Z}$ is a symmetric matrix, just as in the case of no OWCs; see Section 5.5.3.

Next, let us consider the radiation admittance matrix $\mathbf{Y}$, as given in component version by Eq. (8.95). The boundary condition for $\varphi_{k'}$ on S_k,

$$\left(\frac{\partial}{\partial z} - \frac{\omega^2}{g}\right)\varphi_{k'} = -\frac{i\omega}{\rho g}\delta_{kk'} = \begin{cases} i\omega/\rho g & \text{for } k' = k \\ 0 & \text{for } k' \neq k, \end{cases} \tag{8.104}$$

may be written as $\delta_{kk'} = -[i\omega\rho + (\rho g/i\omega)\partial/\partial z]\varphi_{k'}$. Inserting this into Eq. (8.95) gives

$$\begin{aligned} Y_{k,k'} &= \iint_{S_k} \frac{\partial \varphi_{k'}}{\partial z}\left(i\omega\rho + \frac{\rho g}{i\omega}\frac{\partial}{\partial z}\right)\varphi_k \, dS \\ &= \sum_{k''}\iint_{S_{k''}} \frac{\partial \varphi_{k'}}{\partial z}\left(i\omega\rho + \frac{\rho g}{i\omega}\frac{\partial}{\partial z}\right)\varphi_k \, dS. \end{aligned} \tag{8.105}$$

Since $\partial/\partial n = -\partial/\partial z$ on $S_{k''}$ and $\partial\varphi_{k'}/\partial n = 0$ on S_i, we have

$$\begin{aligned} Y_{k,k'} &= \iint_{S} \frac{\partial \varphi_{k'}}{\partial n}\left(-i\omega\rho + \frac{\rho g}{i\omega}\frac{\partial}{\partial n}\right)\varphi_k \, dS \\ &= -i\omega\rho\iint_S \varphi_k \frac{\partial\varphi_{k'}}{\partial n}\, dS + \frac{\rho g}{i\omega}\iint_S \frac{\partial\varphi_{k'}}{\partial n}\frac{\partial\varphi_k}{\partial n}\, dS. \end{aligned} \tag{8.106}$$

Since φ_k and $\varphi_{k'}$ satisfy the same radiation condition, $Y_{k,k'} - Y_{k',k}$ vanishes according to Eqs. (4.230) and (4.239). Hence, $\mathbf{Y}^{\mathrm{T}} = \mathbf{Y}$, which means that the radiation admittance matrix is symmetric. Note that because, for real ω, we have

$$1 = -\left(i\omega\rho + \frac{\rho g}{i\omega}\frac{\partial}{\partial z}\right)\varphi_k = \left(i\omega\rho + \frac{\rho g}{i\omega}\frac{\partial}{\partial z}\right)\varphi_k^* \quad \text{on } S_k, \tag{8.107}$$

we also have

$$Y_{k,k'} = i\omega\rho\iint_S \varphi_k^* \frac{\partial\varphi_{k'}}{\partial n}\, dS - \frac{\rho g}{i\omega}\iint_S \frac{\partial\varphi_{k'}}{\partial n}\frac{\partial\varphi_k^*}{\partial n}\, dS. \tag{8.108}$$

Next, let us present a proof of the reciprocity relation (8.62). As a first step, we use the inhomogeneous boundary condition (8.97) for φ_{ij} in Eq. (8.94), giving

$$H_{ij,k} = -i\omega\rho\sum_{i'}\iint_{S_{i'}} \varphi_k \frac{\partial\varphi_{ij}}{\partial n}\, dS. \tag{8.109}$$

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In view of boundary condition (8.99) and noting that $\partial/\partial n = -\partial/\partial z$ on $S_{k'}$, we can rewrite this as

$$H_{ij,k} = -i\omega\rho \iint_S \varphi_k \frac{\partial \varphi_{ij}}{\partial n}\, dS - \frac{i\omega^3 \rho}{g} \sum_{k'} \iint_{S_{k'}} \varphi_k \varphi_{ij}\, dS. \tag{8.110}$$

The integration surface S is defined by Eq. (8.101).

Moreover, using the inhomogeneous boundary condition (8.104) for φ_k in Eq. (8.96), we find

$$H_{k,ij} = \frac{\rho g}{i\omega} \sum_{k'} \iint_{S_{k'}} \frac{\partial \varphi_{ij}}{\partial z} \left(\frac{\partial \varphi_k}{\partial z} - \frac{\omega^2}{g} \varphi_k \right) dS. \tag{8.111}$$

Further, using the homogeneous boundary condition (8.99) for φ_{ij} and also noting that $\partial/\partial z = -\partial/\partial n$ on $S_{k'}$ and $\partial \varphi_k/\partial n = 0$ on S_i, we obtain

$$H_{k,ij} = i\omega\rho \iint_S \varphi_{ij} \frac{\partial \varphi_k}{\partial n}\, dS + \frac{i\omega^3 \rho}{g} \sum_{k'} \iint_{S_{k'}} \varphi_{ij} \varphi_k\, dS. \tag{8.112}$$

Adding this to Eq. (8.110), we find that $H_{ij,k} + H_{k,ij} = 0$ because of Eqs. (4.230) and (4.239). Thus, we have $H_{ij,k} = -H_{k,ij}$, in agreement with statement (8.62).

8.2.6 Reciprocity Relations for the Radiation Damping Matrix

The radiation damping matrix $\mathbf{D}$ is expressed in terms of matrices $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{J}$ according to definition (8.68). Here we shall derive reciprocity relations which express these matrices in terms of far-field coefficients, Kochin functions or excitation parameters.

First, let us express $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{J}$ in terms of the surface integral (4.230). From definition (8.65), adding Eq. (8.103) to its complex conjugate—or its conjugate transpose, because $\mathbf{Z}$ is symmetric—gives the radiation resistance matrix $\mathbf{R}$. Using also definition (5.127) gives

$$\mathbf{R} = \mathrm{Re}\{\mathbf{Z}\} = \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^\dagger) = \frac{i\omega\rho}{2} \mathbf{I}(\boldsymbol{\varphi}_u^*, \boldsymbol{\varphi}_u^{\mathrm{T}}) = -\frac{i\omega\rho}{2} \mathbf{I}(\boldsymbol{\varphi}_u, \boldsymbol{\varphi}_u^\dagger). \tag{8.113}$$

Because the last term (the sum) in Eq. (8.103) is cancelled out, Eq. (8.113) differs from Eq. (5.176) only by the subscript u.

Similarly, writing Eq. (8.108) in matrix notation and adding it to its conjugate transpose gives the radiation conductance matrix $\mathbf{G}$. Using also definition (5.127), we have

$$\mathbf{G} = \mathrm{Re}\{\mathbf{Y}\} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^\dagger) = \frac{i\omega\rho}{2} \mathbf{I}(\boldsymbol{\varphi}_p^*, \boldsymbol{\varphi}_p^{\mathrm{T}}) = -\frac{i\omega\rho}{2} \mathbf{I}(\boldsymbol{\varphi}_p, \boldsymbol{\varphi}_p^\dagger). \tag{8.114}$$

We have here assumed that ω is real, and we have observed the fact that $\mathbf{G}$ is a symmetric real matrix.

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For real ω, the right-hand side of Eq. (8.104) is purely imaginary. Hence, we may in the integrand of Eq. (8.111) replace φ_k with $-\varphi_k^*$. The same replacement may be made in Eq. (8.112). Further, since the right-hand side of Eq. (8.97) is real, we may replace φ_{ij} with φ_{ij}^* in Eq. (8.109) and, hence, also in Eq. (8.110). From definition (8.65), we have

$$J_{ij,k} = \mathrm{Im}\{H_{ij,k}\} = \frac{1}{2i}(H_{ij,k} - H_{ij,k}^*) = \frac{1}{2i}(H_{ij,k} + H_{k,ij}^*). \tag{8.115}$$

Taking the sum of Eq. (8.110) with φ_{ij} replaced by φ_{ij}^* and of the complex conjugate of Eq. (8.112) with φ_k replaced by $-\varphi_k^*$, we obtain

$$J_{ij,k} = -\frac{\omega\rho}{2}I(\varphi_k, \varphi_{ij}^*) = \frac{\omega\rho}{2}I(\varphi_{ij}^*, \varphi_k), \tag{8.116}$$

where we have used Eqs. (4.230) and (4.234). In matrix notation,

$$\mathbf{J} = \frac{\omega\rho}{2}\mathbf{I}(\boldsymbol{\varphi}_u^*, \boldsymbol{\varphi}_p^{\mathrm{T}}) = \frac{\omega\rho}{2}\mathbf{I}(\boldsymbol{\varphi}_u, \boldsymbol{\varphi}_p^{\dagger}). \tag{8.117}$$

By using the general relation (4.244)—as well as its complex conjugate—for waves $\psi_{i,j}$ satisfying the radiation condition, we find that Eqs. (8.113), (8.114) and (8.117) give

$$\mathbf{R} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_u(\theta)\mathbf{a}_u^{\dagger}(\theta)\, d\theta = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_u^*(\theta)\mathbf{a}_u^{\mathrm{T}}(\theta)\, d\theta, \tag{8.118}$$

$$\mathbf{G} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_p(\theta)\mathbf{a}_p^{\dagger}(\theta)\, d\theta = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_p^*(\theta)\mathbf{a}_p^{\mathrm{T}}(\theta)\, d\theta, \tag{8.119}$$

$$i\mathbf{J} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_u^*(\theta)\mathbf{a}_p^{\mathrm{T}}(\theta)\, d\theta = -\frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_u(\theta)\mathbf{a}_p^{\dagger}(\theta)\, d\theta, \tag{8.120}$$

respectively. Observe that we have assumed ω and k to be real and that $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{J}$, by definition, are real. Further, by transposing and conjugating, we get

$$i\mathbf{J}^{\mathrm{T}} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_p(\theta)\mathbf{a}_u^{\dagger}(\theta)\, d\theta = -\frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_p^*(\theta)\mathbf{a}_u^{\mathrm{T}}(\theta)\, d\theta. \tag{8.121}$$

By using the preceding expressions in the definition (8.65) of the radiation damping matrix, we have, similarly,

$$\mathbf{D} = \begin{bmatrix} \mathbf{R} & -i\mathbf{J} \\ i\mathbf{J}^{\mathrm{T}} & \mathbf{G} \end{bmatrix} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}(\theta)\,\mathbf{a}^{\dagger}(\theta)\, d\theta, \tag{8.122}$$

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where $\mathbf{a}^{\mathrm{T}}(\theta) \equiv [\mathbf{a}_u^{\mathrm{T}}(\theta), \mathbf{a}_p^{\mathrm{T}}(\theta)]$. We have now expressed the radiation parameters in terms of the far-field coefficients $a_{ij}(\theta)$ and $a_k(\theta)$, which make up the vectors $\mathbf{a}_u(\theta)$ and $\mathbf{a}_p(\theta)$, respectively.

To express the radiation parameters in terms of the excitation parameters, we proceed as follows. The excitation force and the excitation volume flow are defined as

$$\hat{F}_{e,ij} = f_{ij}A = i\omega\rho \iint_{S_i} n_{ij}(\hat{\phi}_0 + \hat{\phi}_d)\, dS, \tag{8.123}$$

$$\hat{Q}_{e,k} = q_k A = \iint_{S_k} \frac{\partial}{\partial z}(\hat{\phi}_0 + \hat{\phi}_d)\, dS \tag{8.124}$$

[cf. Eqs. (8.91) and (8.92)]. Making use of the boundary conditions on S_i and S_k and extending the region of integration as we did in Section 8.2.5, we can write

$$\hat{F}_{e,ij} = i\omega\rho I(\hat{\phi}_0, \varphi_{ij}), \tag{8.125}$$

$$\hat{Q}_{e,k} = -i\omega\rho I(\hat{\phi}_0, \varphi_k), \tag{8.126}$$

which is an extension of Haskind's formula (5.222). Using relation (5.140), we may further write

$$\hat{F}_{e,ij}(\beta) = \frac{\rho g D(kh)}{k} h_{ij}(\beta \pm \pi) A, \tag{8.127}$$

$$\hat{Q}_{e,k}(\beta) = -\frac{\rho g D(kh)}{k} h_k(\beta \pm \pi) A, \tag{8.128}$$

or, in vector notation,

$$\hat{\mathbf{F}}_{FQ,e}(\beta) \equiv \begin{bmatrix} \hat{\mathbf{F}}_e(\beta) \\ -\hat{\mathbf{Q}}_e(\beta) \end{bmatrix} = \frac{\rho g D(kh)}{k} A \begin{bmatrix} \mathbf{h}_u(\beta \pm \pi) \\ \mathbf{h}_p(\beta \pm \pi) \end{bmatrix} \equiv \frac{\rho g D(kh)}{k} A \mathbf{h}(\beta \pm \pi). \tag{8.129}$$

Hence,

$$\begin{aligned} \mathbf{D} &= \frac{\omega\rho D(kh)}{2k} \int_0^{2\pi} \mathbf{a}(\theta)\, \mathbf{a}^{\dagger}(\theta)\, d\theta = \frac{\omega\rho D(kh)}{4\pi k} \int_0^{2\pi} \mathbf{h}(\theta)\, \mathbf{h}^{\dagger}(\theta)\, d\theta \\ &= \frac{k}{16\pi J} \int_{-\pi}^{\pi} \hat{\mathbf{F}}_{FQ,e}(\beta)\, \hat{\mathbf{F}}^{\dagger}_{FQ,e}(\beta)\, d\beta, \end{aligned} \tag{8.130}$$

where we have used Eqs. (4.272) and (4.130). Here J is the wave energy transport.

For the case with only oscillating bodies and no OWCs, Eq. (8.130) specialises to Eqs. (5.135) and (5.145), and $\mathbf{D} = \mathbf{R}$ is then a real symmetrical matrix. Also, Eqs. (7.50) and (7.51) are special cases of Eq. (8.130) when the system contains just one OWC.

All diagonal elements of **D** are real, and they are also nonnegative. In particular,

$$R_{ij,ij} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi}|a_{ij}(\theta)|^2\,d\theta = \frac{k}{16\pi J}\int_{-\pi}^{\pi}|\hat{F}_{e,ij}(\beta)|^2\,d\beta \geq 0, \tag{8.131}$$

$$G_{kk} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi}|a_{k}(\theta)|^2\,d\theta = \frac{k}{16\pi J}\int_{-\pi}^{\pi}|\hat{Q}_{e,k}(\beta)|^2\,d\beta \geq 0. \tag{8.132}$$

Hence, the sum of all diagonal elements—that is, the so-called trace of the radiation damping matrix **D**—is necessarily real and nonnegative. The trace of the matrix equals also the sum of all eigenvalues of the matrix (cf., for example, [16, p. 86]). This is consistent with inequality (8.75), which means that the Hermitian matrix **D** is positive semidefinite. Such a matrix have only real nonnegative eigenvalues (cf., for example, [16, p. 109]).

8.3 Two-Dimensional WEC Body

A short descriptive discussion and explanation of wave-energy conversion is given in Section 6.1, with reference to Figure 6.1. This may serve to explain how a WEC terminator, which is a two-dimensional WEC unit, may absorb energy from a plane incident wave by radiating a wave that has certain amplitude and phase relationships relative to the incident wave. A famous example of such a two-dimensional WEC unit is the Salter Duck device, well-known from a 1974 publication [73]. It is a nonsymmetrical WEC unit operating in the pitch mode. Another well-known example is the Bristol Cylinder proposed by Evans [75]. It oscillates in two modes, surge and heave, with equally large amplitudes and with phases differing by a quarter wave period. Consequently, the submerged horizontal cylinder, including its axis, moves uniformly along a circle in the xz-plane.

The early theory for a two-dimensional WEC body was developed independently by Newman [34], Evans [75] and Mei [117]. Alternatively, this situation may be considered as a special case of an array of equal prism-body WEC units aligned closely together along the y-direction. Then theoretical results for infinite WEC-body arrays may be applicable for analysing a two-dimensional WEC-body system [45].

8.3.1 Maximum Power Absorbed by 2-D WEC Body

Two-dimensional parameters were considered in Sections 4.7–4.8 and particularly in Section 5.8. Denoting by a prime ($'$), we introduced per-unit width values for some quantities (for example, forces F', excitation force coefficients f'_e, radiation-resistance matrix $\mathbf{R}'$, Kochin-function coefficients h', power P') but

https://doi.org/10.1017/9781108674812.009 Published online by Cambridge University Press

not for some other quantities (for example, velocities u, far-field coefficients $a^{\pm}$). In the following, we discuss the theoretical maximum power absorbed by a two-dimensional WEC-body unit. For this case (see Section 5.8), Eqs. (6.95) for maximum power and Eq. (6.90) for optimum oscillation are modified into

$$P'_{\rm MAX} = \tfrac{1}{4}\hat{\mathbf{F}}_e'^{\rm T}\hat{\mathbf{u}}^*_{\rm OPT} = \tfrac{1}{2}\hat{\mathbf{u}}^{\rm T}_{\rm OPT}\mathbf{R}'\hat{\mathbf{u}}^*_{\rm OPT}, \qquad (8.133)$$

where the optimum complex velocity amplitude $\hat{\mathbf{u}}_{\rm OPT}$ has to satisfy the algebraic equation

$$\mathbf{R}'\hat{\mathbf{u}}_{\rm OPT} = \tfrac{1}{2}\hat{\mathbf{F}}'_e. \qquad (8.134)$$

We now express the excitation force and the radiation resistance (both per unit width) in terms of the Kochin functions by using Eqs. (5.329)–(5.333). We shall assume that the given incident wave is propagating in the positive x-direction ($\beta = 0$). Then the maximum absorbed power per unit width is rewritten as

$$P'_{\rm MAX} = \frac{1}{4}\hat{\mathbf{F}}_e'^{\rm T}\hat{\mathbf{u}}^*_{\rm OPT} = \frac{1}{4}\hat{\mathbf{F}}_e'^{\dagger}\hat{\mathbf{u}}_{\rm OPT} = \frac{\rho g D(kh)}{4k}A^*\mathbf{h}'^{\dagger}(\pi)\hat{\mathbf{u}}_{\rm OPT}, \qquad (8.135)$$

where the optimum complex amplitudes $\hat{\mathbf{u}}_{\rm OPT}$ are given by

$$[\mathbf{h}'(0)\mathbf{h}'^{\dagger}(0) + \mathbf{h}'(\pi)\mathbf{h}'^{\dagger}(\pi)]\hat{\mathbf{u}}_{\rm OPT} = \frac{gk}{\omega}\mathbf{h}'(\pi)A. \qquad (8.136)$$

Let us introduce the relative absorbed power

$$\epsilon = \frac{P'}{J} = \frac{4\omega P'}{\rho g^2 D(kh)|A|^2} \qquad (8.137)$$

and the relative optimum oscillation amplitude

$$\zeta = \frac{\omega}{gk}\frac{\hat{\mathbf{u}}_{\rm OPT}}{A}. \qquad (8.138)$$

Note that ϵ is just the fraction of the incident wave power transport (4.130) being absorbed. Its maximum value

$$\epsilon_{\rm MAX} = \frac{4\omega}{\rho g^2 D(kh)|A|^2}\frac{\rho g D(kh)}{4k}A^*\mathbf{h}'^{\dagger}(\pi)\frac{gk}{\omega}A\,\zeta = \mathbf{h}'^{\dagger}(\pi)\,\zeta \qquad (8.139)$$

is obtained when the vector of relative amplitudes ζ satisfies the equation

$$[\mathbf{h}'(0)\mathbf{h}'^{\dagger}(0) + \mathbf{h}'(\pi)\mathbf{h}'^{\dagger}(\pi)]\,\zeta = \mathbf{h}'(\pi). \qquad (8.140)$$

Note that the expression within the brackets here is a normalised version of the radiation resistance matrix, which is real and, hence, is equal to its own complex conjugate. [See the last expression of Eq. (5.329).] The optimum condition (8.140) may be written as

$$\mathbf{h}'(\pi)[\mathbf{h}'^{\dagger}(\pi)\zeta - 1] + \mathbf{h}'(0)\mathbf{h}'^{\dagger}(0)\zeta = 0. \qquad (8.141)$$

A matrix of the type $\mathbf{h}\mathbf{h}^\dagger$ is of rank 1 (or of rank zero in the trivial case when $\mathbf{h}^\dagger\mathbf{h} = 0$) [16, p. 239]. If the vectors $\mathbf{h}'(0)$ and $\mathbf{h}'(\pi)$ are linearly independent, the radiation resistance matrix is of rank 2. Otherwise, its rank is at most equal to 1. In general, the radiation resistance of an oscillating two-dimensional body is a 3×3 matrix. It follows that this matrix is necessarily singular, since its rank is 2 or less.

Let us assume that the two vectors $\mathbf{h}'(\pi)$ and $\mathbf{h}'(0)$ are linearly independent. Then the condition (8.141) for optimum cannot be satisfied unless

$$\mathbf{h}'^\dagger(\pi)\boldsymbol{\zeta} - 1 = 0 \tag{8.142}$$

$$\mathbf{h}'^\dagger(0)\boldsymbol{\zeta} = 0. \tag{8.143}$$

Thus, we have two scalar equations which the components of the optimum amplitude vector $\boldsymbol{\zeta}$ have to satisfy. Hence, the system of equations is indeterminate if $\boldsymbol{\zeta}$ has more than two components—that is, for a case where the radiation resistance matrix is singular, as stated previously.

Combining Eqs. (8.139) and (8.142), we obtain the unambiguous value

$$\epsilon_{\mathrm{MAX}} = \mathbf{h}'^\dagger(\pi)\boldsymbol{\zeta} = 1 \tag{8.144}$$

for the maximum relative absorbed wave power. This means that 100% of the incident wave energy is absorbed by the oscillating body. It can be shown [118, equations (53)–(59)] that when the conditions (8.142) and (8.143) are satisfied, then radiated waves cancel waves diffracted towards the left ($x \to -\infty$) and the sum of the incident wave and waves diffracted towards the right ($x \to +\infty$).

A necessary, but not sufficient, condition for the vectors $\mathbf{h}'(\pi)$ and $\mathbf{h}'(0)$ to be linearly independent is that they are of order two or more. That is, at least two modes of oscillation have to be involved.

If the body oscillates in one mode only, the aforementioned vectors simplify to scalars. Hence, Eqs. (8.139) and (8.140) in this case simplify to

$$\epsilon_{\mathrm{MAX}} = h'^*(\pi)\zeta \tag{8.145}$$

$$(|h'(0)|^2 + |h'(\pi)|^2)\zeta = h'(\pi). \tag{8.146}$$

This means that not more than a fraction

$$\epsilon_{\mathrm{MAX}} = \frac{1}{1 + |h'(0)/h'(\pi)|^2} \tag{8.147}$$

of the incident wave energy can be absorbed by the body. If the oscillating body is able to radiate a wave only in the negative direction, then $h'(0) = 0$, and hence, 100% absorption is possible. If equally large waves are radiated in opposite directions, we have $|h'(0)| = |h'(\pi)|$, which means that not more than 50% of the incident wave energy can be absorbed.

It is necessary that the body has a nonsymmetric radiation ability if it is to absorb more than 50% of the incident wave energy. An example of such a nonsymmetric body is the Salter Duck, intended to oscillate in the pitch mode. Already in 1974, Salter reported [73, 81] measured absorbed power corresponding to more than 80% of the incident wave power.

Next let us consider an oscillating body which (in its time-average position) is symmetric with respect to the $x = 0$ plane. It is obvious that the wave radiated by the heave motion is symmetric, while the waves radiated by the surge motion and by the pitch motion are antisymmetric. In terms of the Kochin functions, this means that

$$h_3'(\pi) = h_3'(0) \tag{8.148}$$

$$h_1'(\pi) = -h_1'(0), \quad h_5'(\pi) = -h_5'(0). \tag{8.149}$$

It may be remarked that if the body is a horizontal circular cylinder, and if the pitch rotation is about the cylinder axis, then $|h_5'| = 0$. Further, if the cylinder is completely submerged on deep water, we have

$$h_1'(0) = -ih_3'(0). \tag{8.150}$$

This means that if the cylinder axis is oscillating with equal amplitudes in heave and surge with phases differing by $\pi/2$—that is, if the centre of the cylinder is describing a circle in the xz-plane—then the waves generated by the cylinder motion travel away from the cylinder along the free surface, but in one direction only. This was first shown in a theoretical work by Ogilvie [119]. On this theoretical basis, Evans proposed the so-called Bristol cylinder WEC body [75].

Returning now to the case of a general symmetric body oscillating in the three modes, heave ($j = 3$), surge ($j = 1$) and pitch ($j = 5$), the vectors $\mathbf{h}'(\pi)$ and $\mathbf{h}'(0)$ are linearly independent; cf. Eqs. (8.148)–(8.149). If only two modes are involved and if the two modes are heave and surge or heave and pitch, the vectors are linearly independent. Thus, in all these cases, 100% absorption is possible if the optimum oscillation can be ideally fulfilled.

However, if surge and pitch are the only two modes involved in the body's oscillation, the vectors $\mathbf{h}'(\pi)$ and $\mathbf{h}'(0)$ are linearly dependent, i.e., $\mathbf{h}'(\pi) = -\mathbf{h}'(0)$; cf. (8.149). In this case, the optimum condition gives

$$[\mathbf{h}'(\pi) - \mathbf{h}'(0)]^\dagger \zeta = 1 \tag{8.151}$$

and, hence,

$$\epsilon_{\mathrm{MAX}} = \mathbf{h}'^\dagger(\pi)\zeta = \tfrac{1}{2}, \tag{8.152}$$

which means that not more than 50% power absorption is possible in this case.

We noted earlier that the 3×3 matrix for the radiation resistance is singular. There is a good reason for this. It is possible to absorb 100% of the incident wave power by optimum oscillation in two modes, one symmetric mode (heave) and one antisymmetric mode (surge or pitch). Hence, it is not possible to absorb more wave power by including a third mode. With all three modes involved in the optimisation problem, **R** is singular, and hence, the system of equations for determining the optimum values of U_1, U_3 and U_5 is indeterminate. The optimum complex amplitude U_3 is determined. If U_1 is arbitrarily chosen, then U_5 is determined, and vice versa. In this way, the antisymmetric wave, resulting from the combined surge-and-pitch oscillation, is optimum in the far-field region. If both the symmetric wave and the antisymmetric wave are optimum, all incident wave energy is absorbed by the oscillating body. (See also Problem 8.1.)

Problems

Problem 8.1: Maximum Absorbed Power by Symmetric 2-D Body

Show that if a plane wave is perpendicularly incident on a two-dimensional body which has a vertical symmetry plane parallel to the wave front, then, at optimum oscillation, the body absorbs

(a) half of the incident wave power if it oscillates in the surge and pitch modes only, and
(b) all incident wave power if it oscillates in surge and heave only or if it oscillates in surge, heave and pitch.

State in each case the conditions which the optimum complex velocity amplitudes have to satisfy.

Problem 8.2: Linear Electric 2-Port

Consider an electric '2-port' (also termed '4-pole'). The input voltage, input current, output voltage and output current have complex amplitudes U_l, I_1, U_2 and I_2, respectively. Assuming that the electric circuit is linear, we may express the voltages in terms of the currents as

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \mathbf{Z} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \text{where} \quad \mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

is the impedance matrix of the 2-port.

By solving linear algebraic equation, we find it possible to write this relation in some other ways, such as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \mathbf{Y} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \begin{bmatrix} U_2 \\ I_2 \end{bmatrix} = \mathbf{K} \begin{bmatrix} U_1 \\ I_1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} U_1 \\ I_2 \end{bmatrix} = \mathbf{H} \begin{bmatrix} I_1 \\ U_2 \end{bmatrix},$$

where **Y** is the 'admittance' matrix, **K** the 'chain' matrix and **H** the 'hybrid' matrix. Express the four matrix elements of **Y** and of **H** in terms of the four matrix elements of **Z**.

If the impedance matrix is symmetric—that is, if $Z_{21} = Z_{12}$—then the linear electric circuit is said to be 'reciprocal'. What is the corresponding condition on the hybrid matrix?

One possible electrical analogy of a mechanical system is to consider force and velocity as analogous to voltage and current, respectively. For a mechanical system consisting of one OWC and a single one-mode oscillating body, the preceding hybrid matrix may be considered as the electrical analogue of the radiation matrix in Eq. (8.65). Discuss this analogy.

Problem 8.3: Maximum Absorbed Power

For a system of OWCs and oscillating bodies, the absorbed power may be written as [cf. Eq. (8.73)]

$$P = P(\hat{\mathbf{u}}_{FQ}) = \tfrac{1}{4}(\hat{\mathbf{F}}^{\mathrm{T}}_{FQ,e}\hat{\mathbf{u}}^{*}_{FQ} + \hat{\mathbf{F}}^{\dagger}_{FQ,e}\hat{\mathbf{u}}_{FQ}) - \tfrac{1}{2}\hat{\mathbf{u}}^{\dagger}_{FQ}\mathbf{D}\hat{\mathbf{u}}_{FQ}. \tag{1}$$

(a) Assuming that the radiation damping matrix $\mathbf{D}$ is non-singular (that is, $\mathbf{D}^{-1}$ exists), prove that

$$P_{\mathrm{MAX}} = \tfrac{1}{8}\hat{\mathbf{F}}^{\dagger}_{FQ,e}\mathbf{D}^{-1}\hat{\mathbf{F}}_{FQ,e},$$

and derive an explicit expression for the optimum oscillation vector $\hat{\mathbf{u}}_{FQ,0}$ in terms of the excitation vector $\hat{\mathbf{F}}_{FQ,e}$. Observe that $\mathbf{D}$ is a complex, Hermitian and positive semidefinite matrix. [Hint: introduce the vector $\hat{\boldsymbol{\delta}} = \hat{\mathbf{u}}_{FQ} - (\mathbf{D}^{*})^{-1}\hat{\mathbf{F}}_{FQ,e}/2$, and consider the nonnegative quantity $\hat{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{D}\hat{\boldsymbol{\delta}}/2$ which would have been the radiated power if $\hat{\boldsymbol{\delta}}$ had been the complex velocity amplitude vector.]

(b) On the basis of Eq. (1), show that the maximum absorbed power is as given by Eq. (8.76) where the optimum oscillation corresponds to $\mathbf{U}$ satisfying Eq. (8.77). [Hint: make the necessary generalisation of the derivation of Eq. (6.92) from Eq. (6.91).]

(c) If the radiation damping matrix $\mathbf{D}$ is singular, Eq. (8.77) has an infinity of possible solutions for $\hat{\mathbf{u}}_{FQ,0}$. Assume that $\mathbf{U}_1$ and $\mathbf{U}_2$ are two different possible solutions. Show that $P(\mathbf{U}_1) = P(\mathbf{U}_2)$. Thus, in spite of the indeterminateness of Eq. (8.77), the maximum absorbed power P_{MAX} is unambiguous. [Hint: use Eqs. (8.16), (8.76) and (8.77).]