CHAPTER THREE

Interaction between Oscillations and Waves

There are many different types of waves in nature. Apart from the visible waves on the surface of oceans and lakes, there are, for instance, sound waves, light waves and other electromagnetic waves. This chapter gives a brief description of waves in general and compares surface waves on water with other types of waves. It also presents a simple generic discussion on the interaction between waves and oscillations. One phenomenon is generated waves radiated from an oscillator, and another phenomenon is oscillations excited by a wave incident upon the oscillating system. We shall define the radiation resistance in terms of the power associated with the wave generated by the oscillator. The 'added mass' is related to added energy associated with the wave-generating process, not to kinetic energy alone but to the difference between kinetic and potential energies.

3.1 Comparison of Waves on Water with Other Waves

Waves on water propagate along a surface. Acoustic waves in a fluid and electromagnetic waves in free space may propagate in any direction in a three-dimensional space. Waves on a stretched string propagate along a line (in a one-dimensional 'space'). The same may be said about waves on water in a canal and about guided acoustic waves or guided electromagnetic waves along cylindrical structures, although in these cases the physical quantities (pressure, velocity, electric field, magnetic field, etc.) may vary in directions transverse to the direction of wave propagation.

As was mentioned in Chapter 2, there is an exchange of kinetic energy and potential energy in a mechanical oscillator (or magnetic energy and electric energy in the electric analogue). In a propagating wave, too, there is interaction between different forms of energy—for instance, magnetic and electric energy with electromagnetic waves, and kinetic and potential energy with mechanical waves, such as acoustic waves and water waves. With an acoustic wave, the potential energy is associated with the elasticity of the medium in which

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the wave propagates. The potential energy with a water wave is due to gravity and surface tension. The contribution from the elasticity of water is negligible, because waves on water propagate rather slowly as compared with the velocity of sound in water. Gravity is responsible for the potential energy associated with the lifting of water from the wave troughs to the wave crests. As the waves increase the area of the interface between water and air, the work done against surface tension is converted to potential energy. For wavelengths in the range of 10^{-3} m to 10^{-1} m, both types of potential energy are important. For shorter waves, so-called capillary waves, the effect of gravity may be neglected (see Problems 3.2 and 4.1). For longer waves, so-called gravity waves, surface tension may be neglected, as we shall do in the following, since we restrict our study to water waves of wavelengths exceeding 0.25 m.

Oscillations are represented by physical quantities which vary with time. For waves, the quantities also vary with the spatial coordinates. In the present chapter, we shall consider waves which vary sinusoidally with time. Such waves are called 'harmonic' or 'monochromatic'. When daling with sea waves, they are also characterised as 'regular' if their time variation is sinusoidal.

Let

$$p = p(x, y, z, t) = \mathrm{Re}\left\{\hat{p}(x, y, z)e^{i\omega t}\right\} \tag{3.1}$$

represent a general harmonic wave, and let p denote the dynamic pressure in a fluid. (The total pressure is $p_{\mathrm{tot}} = p_{\mathrm{stat}} + p$, where the static pressure p_{stat} is independent of time.) The complex pressure amplitude $\hat{p}$ is a function of the spatial coordinates x, y and z.

For a plane acoustic wave propagating in a direction x, we have

$$p = p(x, t) = \mathrm{Re}\left\{Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}\right\}, \tag{3.2}$$

where $k = 2\pi/\lambda$ is the angular repetency (wave number) and λ is the wavelength. The first and second terms represent waves propagating in the positive and negative x direction, respectively. Assume that an observer moves with a velocity $v_p = \omega/k$ in the positive x direction. Then he or she will experience a constant phase $(\omega t - kx)$ of the first right-hand term in Eq. (3.2). If the observer moves with same speed in the opposite direction, he or she will experience a constant phase $(\omega t + kx)$ of the last term in Eq. (3.2). For this reason, $v_p = \omega/k$ is called the phase velocity. At a certain instant, the phase is constant on all planes perpendicular to the direction of wave propagation. For this reason, the wave is called plane. In contrast, an acoustic wave

$$p(r, t) = \mathrm{Re}\left\{(C/r)e^{i(\omega t - kr)}\right\} \tag{3.3}$$

radiated from a spherical loudspeaker in open air may be called a spherical wave, because the phase $(\omega t - kr)$ is the same everywhere on an envisaged sphere with a radius r from the centre of the loudspeaker. Note that in this geometrical case, the pressure amplitude $|C|/r$ decreases with the distance from the loudspeaker.

The pressure amplitudes $|A|$ and $|B|$ are constant for the two oppositely propagating waves corresponding to the two right-hand terms in Eq. (3.2). An acoustic wave in a fluid is a longitudinal wave, because the fluid oscillates only in the direction of wave propagation. We shall not discuss acoustic waves in detail here but just mention that if the sound pressure is given by Eq. (3.2), then the fluid velocity is

$$v = v_x = \frac{1}{\rho c}\mathrm{Re}\left\{Ae^{i(\omega t-kx)} - Be^{i(\omega t+kx)}\right\}, \tag{3.4}$$

where c is the sound velocity and ρ is the (static) mass density of the fluid. The readers who are interested in the derivation of Eqs. (3.2) and (3.4) and the wave equation for acoustic waves are referred to textbooks in acoustics [21].

For a plane harmonic wave that propagates along a water surface, an expression similar to Eq. (3.2) applies. However, then A and B cannot be constants; they depend on the vertical coordinate z (chosen to have positive direction upwards). On 'deep water', A and B are then proportional to e^{kz}, as will be shown later in Chapter 4. Then $p = p(x, z, t)$, and

$$\hat{p} = \hat{p}(x, z) = A(z)e^{-ikx} + B(z)e^{ikx}. \tag{3.5}$$

In the case of a plane and linearly polarised electromagnetic wave propagating in free space, the electric field is

$$E(x, t) = \mathrm{Re}\left\{Ae^{i(\omega t-kx)} + Be^{i(\omega t+kx)}\right\}, \tag{3.6}$$

where A and B are again complex constants.

3.2 Dispersion, Phase Velocity and Group Velocity

Both acoustic waves and electromagnetic waves are non-dispersive, which means that the phase velocity is independent of the frequency. The dispersion relation (the relationship between ω and k) is

$$\omega = ck, \tag{3.7}$$

where c is the constant speed of sound or light, respectively.

Gravity waves on water are, in general, dispersive. As will be shown later, in Chapter 4 (Section 4.2), the relationship for waves on deep water is

$$\omega^2 = gk, \tag{3.8}$$

where g is the acceleration of gravity. Using this dispersion relationship, we find that the phase velocity is

$$v_p \equiv \omega/k = g/\omega = \sqrt{g/k}. \tag{3.9}$$

Note that in the study of the propagation of dispersive waves (for which the phase velocity depends on frequency), we have to distinguish between phase

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velocity and group velocity. Let us assume that the dispersion relationship may be written as $F(\omega, k) = 0$, where F is a differentiable function of two variables. Then the group velocity is defined as

$$v_g = \frac{d\omega}{dk} = -\frac{\partial F/\partial k}{\partial F/\partial \omega}. \tag{3.10}$$

As shown in Problem 3.1, if several propagating waves with slightly different frequencies are superimposed on each other, the result may be interpreted as a group of waves, each of them moving with the phase velocity, whereas the amplitude of the group of individual waves are modulated by an envelope moving with the group velocity. Here let us just consider the following simpler example of two superimposed harmonic waves of angular frequency $\omega \pm \Delta\omega$ and angular repetency $k \pm \Delta k$, namely

$$\begin{aligned} p(x,t) &= D\cos\{(\omega - \Delta\omega)t - (k - \Delta k)x\} + D\cos\{(\omega + \Delta\omega)t - (k + \Delta k)x\} \\ &= 2D\cos\{(\Delta\omega)t - (\Delta k)x\}\cos(\omega t - kx), \end{aligned} \tag{3.11}$$

where an elementary trigonometric identity has been used in the last step. If $\Delta\omega \ll \omega$, this last expression for $p(x,t)$ is the product of a fast varying function, representing a wave with propagation speed ω/k (the phase velocity), and a slowly varying function, representing an amplitude envelope, moving in the positive x direction with a propagation speed $\Delta\omega/\Delta k$, which tends to the group velocity if $\Delta\omega$ (and, correspondingly, Δk) tend to zero.

Using Eq. (3.8), we find that for a wave on deep water, the group velocity is

$$v_g \equiv d\omega/dk = g/(2\omega) = v_p/2. \tag{3.12}$$

We shall see later, in Chapter 4, that this group velocity may be interpreted as the speed with which energy is transported by a deep-water wave. Although electromagnetic waves are non-dispersive in free space, such waves propagating along telephone lines or along optical fibres have, in general, some dispersion. In this case, it is of interest to know that not only the energy but also the information carried by the wave are usually propagated with a speed equal to the group velocity.

3.3 Wave Power and Energy Transport

Next let us consider the energy, power and intensity associated with waves. Intensity I is the time-average energy transport per unit time and per unit area in the direction of wave propagation. Whereas the dimension in SI units is J (joule) for energy and W (watt) for power, it is J/(s m^2) = W/m^2 for intensity. For surface waves on water and for acoustic waves, the intensity is

$$I = \overline{p_{\text{tot}} v} = \overline{(p_{\text{stat}} + p)v}, \tag{3.13}$$

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where the total pressure p_{tot} is the sum of the static pressure p_{stat} and the dynamic pressure p, and where $v = v_x$ is the fluid particle velocity component in the direction of wave propagation. (The overbar denotes time average.) Because the time-average particle velocity is zero, $\overline{v} = 0$, we have

$$I = p_{\mathrm{stat}}\overline{v} + \overline{pv} = \overline{pv}. \tag{3.14}$$

For a harmonic wave, we have

$$p = \mathrm{Re}\left\{\hat{p}e^{i\omega t}\right\}, \quad v_x = \mathrm{Re}\left\{\hat{v}_x e^{i\omega t}\right\}, \tag{3.15}$$

where $\hat{p} = \hat{p}(x, y, z)$ and $\hat{v}_x = \hat{v}_x(x, y, z)$ are complex amplitudes at (x, y, z) of the dynamic pressure p and of the x component of the fluid particle velocity, respectively. In analogy with the derivation of Eq. (2.76), we then have

$$I = I_x = I_x(x, y, z) = \overline{pv_x} = \tfrac{1}{2}\mathrm{Re}\{\hat{p}\hat{v}_x^*\}. \tag{3.16}$$

Strictly speaking, v and, hence, I are vectors:

$$\vec{I} = \vec{I}(x, y, z) = \overline{p\vec{v}} = \tfrac{1}{2}\mathrm{Re}\{\hat{p}\hat{\vec{v}}^*\}. \tag{3.17}$$

For an electromagnetic wave the intensity may be defined as the time-average of the so-called Poynting vector, which is well known in electromagnetics (see, e.g., Panofsky and Phillips [22]).

For a plane acoustic wave propagating in the positive x direction, the sound pressure is as given by Eq. (3.2), and the oscillating fluid velocity is as given by Eq. (3.4), with $B = 0$ and the constant $|A|$ being equal to the pressure amplitude. Then the sound intensity, as given by Eq. (3.16), is constant (independent of x, y and z).

For a plane gravity wave on water propagating in the positive x direction, the dynamic pressure is as given by Eq. (3.5), with $B(z) \equiv 0$. Refer to Chapter 4 for a discussion of the oscillating fluid velocity $\vec{v}$ associated with this wave. It is just mentioned here that if the water is 'deep', then both p and $\vec{v}$ are proportional to e^{kz}. Hence, the intensity is proportional to e^{2kz}, meaning that the intensity decreases exponentially with the distance downward from the water surface. Thus,

$$I_x = I_0 e^{2kz}, \tag{3.18}$$

where I_0 is the intensity at the (average) water surface, $z = 0$. By integrating $I_x = I_x(z)$ from $z = -\infty$ to $z = 0$, we arrive at the wave-energy transport

$$J = \int_{-\infty}^{0} I_x(z)\,dz = I_0 \int_{-\infty}^{0} e^{2kz}\,dz = I_0/2k, \tag{3.19}$$

which is the wave energy transported per unit time through an envisaged vertical strip of unit width parallel to the wave front—that is, parallel to the planes of constant phase of the propagating wave.

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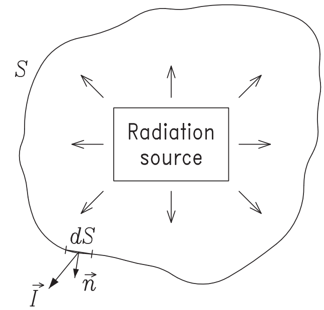

Figure 3.1: General source of radiation surrounded by an envisaged closed surface S.

Let us now consider an arbitrary, envisaged, closed surface with a radiation source (or wave generator) inside, as indicated in Figure 3.1. The radiation source could be an oscillating body immersed in water or another kind of wave generator. (In acoustics, the source could be a loudspeaker; in radio engineering, a transmitting antenna; and in optics, a light-emitting atom.)

The radiated power (energy per unit time) passing through the closed surface S may be expressed as an integral of the intensity over the surface,

$$P_r = \oiint \vec{I} \cdot \vec{n}\, dS = \oiint \vec{I} \cdot d\vec{S}, \tag{3.20}$$

where $\vec{n}$ is the unit normal and $d\vec{S} \equiv \vec{n}\, dS$. When a spherical loudspeaker radiates isotropically in open air, the sound intensity $I(r)$ is independent of direction, and then the radiated power through an envisaged spherical surface of radius r is

$$P_r = I(r)\, 4\pi r^2. \tag{3.21}$$

Here we have neglected reflection and absorption of the acoustic wave from ground and other obstacles. If we assume that the air does not absorb acoustic wave energy, P_r must be independent of r. Hence, $I(r)$ is inversely proportional to the square of the distance r:

$$I(r) = I(a)\, (a/r)^2, \tag{3.22}$$

where $I(a)$ is the intensity for $r = a$. With this result in mind, it is easier to accept the statement, as implied in Eq. (3.3) for the spherical wave, that the amplitude of the sound pressure is inversely proportional to the distance r.

Assume now that an axisymmetric body immersed in water of depth h is performing vertical oscillations. Then an axisymmetric wave generation will take place, and trains of circular waves will radiate along the water surface, outwards from the oscillating body. The power radiated through an envisaged vertical cylinder of large radius r may, according to Eq. (3.20), be written as

$$P_r = \int_{-h}^{0} I(r, z)\, 2\pi r\, dz = J_r 2\pi r, \tag{3.23}$$

where

$$J_r = \int_{-h}^{0} I(r,z)\,dz \tag{3.24}$$

is the radiated wave-energy transport (per unit width of the wave front). If there is no loss of wave energy in the water, P_r is independent of the distance r from the axis of the oscillating body. Consequently, J_r is inversely proportional to r:

$$J_r(r) = J_r(a)\,(a/r), \tag{3.25}$$

where $J_r(a)$ is the wave-energy transport at $r = a$. From this, we would expect that the dynamic pressure and other physical quantities associated with the radiated wave have amplitudes that are inverse to the square root of r. As we shall discuss in more detail later, in Chapters 4 and 5, this result is true, provided r is large enough. (There may be significant deviation from this result if the distance from the oscillating body is shorter than one wavelength.)

3.4 Radiation Resistance and Radiation Impedance

If the mass m indicated in Figure 2.1 is the membrane of a loudspeaker, an acoustic wave will be generated due to the oscillation of the system. Or, if the mass m_m is immersed in water, as indicated in Figure 3.2, a water wave will be generated. Let us now consider a wave-tank laboratory where such an immersed body of mass m_m is suspended through a spring S_m and a mechanical resistance R_m, as indicated in Figure 2.1. Alternatively, the body could be a membrane suspended in a frame inside a loudspeaker cabinet. Assume that an external force

$$F(t) = \mathrm{Re}\{\hat{F}e^{i\omega t}\} \tag{3.26}$$

is applied to the body, resulting in a forced oscillatory motion with velocity

$$u(t) = \mathrm{Re}\{\hat{u}e^{i\omega t}\}. \tag{3.27}$$

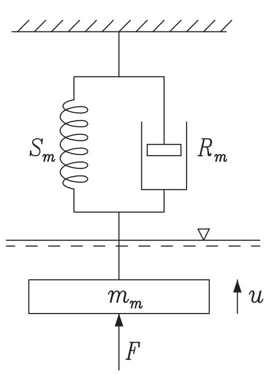

Figure 3.2: A body of mass m_m suspended in water through a spring S_m and a damper R_m.

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The power consumed by the mechanical damper is (in time average)

$$P_m = \tfrac{1}{2} R_m |\hat{u}|^2, \tag{3.28}$$

in agreement with Eq. (2.79). The oscillating body generates a wave which carries away a radiated power P_r [cf. Figure 3.1 and Eq. (3.20)]. In analogy with Eq. (3.28), we write

$$P_r = \tfrac{1}{2} R_r |\hat{u}|^2, \tag{3.29}$$

which defines the so-called *radiation resistance* R_r.

Due to the radiated wave, a reaction force F_r acts on the body in addition to the externally applied force F. With the assumption of linear theory, F_r is also varying as a harmonic oscillation—that is, $F_r = \mathrm{Re}\{\hat{F}_r e^{i\omega t}\}$. The dynamics of the system is then described by the following extension of Eq. (2.63) or Eq. (2.64):

$$i\omega m_m \hat{u} + R_m \hat{u} + (S_m / i\omega)\hat{u} = \hat{F} + \hat{F}_r, \tag{3.30}$$

or, in terms of the mechanical impedance Z_m:

$$Z_m \hat{u} = \hat{F} + \hat{F}_r. \tag{3.31}$$

Setting

$$\hat{F}_r = -Z_r \hat{u}, \tag{3.32}$$

we define an added impedance, or the so-called *radiation impedance* Z_r.

In general, Z_r is a complex function of ω:

$$Z_r = Z_r(\omega) = R_r(\omega) + iX_r(\omega), \tag{3.33}$$

which also depends on the geometry of the radiating system. Now we have from Eqs. (3.31) and (3.32) that

$$(Z_m + Z_r)\hat{u} = \hat{F}, \tag{3.34}$$

which gives the complex velocity amplitude

$$\hat{u} = \frac{\hat{F}}{Z_m + Z_r} = \frac{\hat{F}}{(R_m + R_r) + i(\omega m_m + X_r - S_m/\omega)}. \tag{3.35}$$

Comparing this result with Eq. (2.53), we observe that the oscillatory motion is modified because the oscillating mass has been immersed in water. The motion of the immersed body results in motion of the water surrounding the body. Some energy, represented by the radiated power (3.29), is carried away. Moreover, some energy is stored as kinetic energy, due to the velocity of the water, and as potential energy, due to gravity when the water surface is deformed and water is lifted from troughs to crests. The energy stored in the water is added to the energy stored in the mechanical system itself. Referring to Eq. (2.88), we may thus relate the *radiation reactance* $X_r(\omega)$ to the difference between the average

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values of the added kinetic energy and the added potential energy. The radiation reactance $X_r(\omega)$ is frequently written as ωm_r, where

$$m_r = m_r(\omega) = X_r(\omega)/\omega \tag{3.36}$$

is the so-called *added mass*, which is usually positive. There are, however, exceptional cases in which the added potential energy is larger than the added kinetic energy, and in such cases, the added mass becomes negative [23].

Combining Eqs. (3.33) and (3.36), we may write the radiation impedance as

$$Z_r(\omega) = R_r(\omega) + i\omega m_r(\omega). \tag{3.37}$$

Here, as well as in acoustics [21], radiation impedance has the dimension of force divided by velocity, and hence, the SI unit is $[Z_r] = [R_r] = \text{Ns/m} = \text{kg/s}$. Analogously, in theory for radio antennae, the (electric) radiation impedance has dimension voltage divided by current, and the corresponding SI unit is $\Omega = \text{V/A}$. The term ‘radiation impedance’ is commonly used in connection with microphones and loudspeakers in acoustics [21] and also in connection with receiving and transmitting antennae in electromagnetics. A few authors [24, 25] have also adopted the term ‘radiation impedance’ in hydrodynamics, in connection with the generation and absorption of gravity waves on water. This term will also be used in the subsequent text.

3.5 Resonance Absorption

In the preceding section, we assumed that an external force F was given [cf. Eq. (3.26) and Figure 3.2]. This external force could have been applied through a motor or some other mechanism, not shown in Figure 3.2. In addition to the external force, a reaction force F_r was taken into consideration in Eq. (3.30), and we assumed in Eq. (3.32) that this reaction force is linear in the velocity u. If the motion had been prevented (by choosing at least one of the parameters S_m, R_m and m_m sufficiently large), then only the external force F would remain.

Let us now assume that this force is applied through an incident wave. We shall adopt the term ‘excitation force’ for the wave force F_e which acts on the immersed body when it is not moving—that is, when $u = 0$. For this case, we replace $\hat{F}$ in Eqs. (3.30), (3.31), (3.34) and (3.35) by $\hat{F}_e$. According to Eq. (3.35), the body’s velocity is then given by

$$\hat{u} = \frac{\hat{F}_e}{R_m + R_r + i[\omega(m_m + m_r) - S_m/\omega]}. \tag{3.38}$$

The power absorbed in the mechanical damper resistance R_m is [cf. Eq. (3.28)]

$$P_a = \frac{R_m}{2}|\hat{u}|^2 = \frac{(R_m/2)|\hat{F}_e|^2}{(R_m + R_r)^2 + (\omega m_m + \omega m_r - S_m/\omega)^2}. \tag{3.39}$$

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Note that R_m could, in an ideal case, represent a load resistance and that P_a correspondingly represents useful power being consumed by the load resistance. We note that $P_a = 0$ for $R_m = 0$ and for $R_m = \infty$, and that $P_a > 0$ for $0 < R_m < \infty$. Thus, there is a maximum of absorbed power when $\partial P_a/\partial R_m = 0$, which occurs if

$$R_m = \left[R_r^2 + (\omega m_m + \omega m_r - S_m/\omega)^2\right]^{1/2} \equiv R_{m,\text{opt}}, \tag{3.40}$$

for which we have the maximum absorbed power

$$P_{a,\max} = \frac{|\hat{F}_e|^2/4}{R_r + \left[R_r^2 + (\omega m_m + \omega m_r - S_m/\omega)^2\right]^{1/2}}. \tag{3.41}$$

See Problems 3.7 and 3.8.

Furthermore, we see by inspection of Eq. (3.39) that if we, for arbitrary R_m, can choose m_m and S_m such that

$$\omega m_m + \omega m_r - S_m/\omega = 0, \tag{3.42}$$

then the absorbed power has the maximum value

$$P_a = \frac{R_m|\hat{F}_e|^2/2}{(R_m + R_r)^2}. \tag{3.43}$$

If we now choose R_m in accordance with condition (3.40), which now becomes

$$R_m = R_r \equiv R_{m,\text{OPT}}, \tag{3.44}$$

the maximum absorbed power is

$$P_{a,\text{MAX}} = |\hat{F}_e|^2/(8R_r), \tag{3.45}$$

and in this case, Eq. (3.38) simplifies to

$$\hat{u} = \hat{F}_e/(2R_r) \equiv \hat{u}_{\text{OPT}}. \tag{3.46}$$

When condition (3.42) is satisfied, we have resonance. We see from Eq. (3.38) that the oscillation velocity is in phase with the excitation force, since the ratio between the complex amplitudes $\hat{u}$ and $\hat{F}$ is then real. We may refer to Eq. (3.42) as the ‘resonance condition’ or the ‘optimum phase condition’. Note that this condition is independent of the chosen value of the mechanical damper resistance R_m, and the maximum absorbed power is as given by Eq. (3.43).

If the optimum phase condition cannot be satisfied, then the maximum absorbed power is as given by Eq. (3.41), provided the ‘optimum amplitude condition’ (3.40) is satisfied.

If the optimum phase condition and the optimum amplitude condition can be satisfied simultaneously, then the maximum absorbed power is as given by

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Eq. (3.45), and the optimum oscillation is as given by Eq. (3.46). In later chapters (Chapters 6–8), we encounter situations analogous to this optimum case of power absorption from a water wave. Note, however, that the discussion in the present section is also applicable to absorption of energy from an acoustic wave by a microphone or from an electromagnetic wave by a receiving antenna. (In this latter case, u is the electric current flowing in the electric terminal of the antenna, and F_e is the excitation voltage—that is, the voltage which is induced by the incident electromagnetic wave at the terminal when $u = 0$.)

Let us now, for simplicity, neglect the frequency dependence of the radiation resistance R_r and of the added mass m_r. Introducing the natural angular frequency (eigenfrequency)

$$\omega_0 = \sqrt{S_m/(m_m + m_r)} \tag{3.47}$$

into Eq. (3.39), we rewrite the absorbed power as

$$P_a(\omega) = \frac{R_m|\hat{F}_e(\omega)|^2}{2(R_m + R_r)^2} \frac{1}{1 + (\omega_0/2\delta)^2(\omega/\omega_0 - \omega_0/\omega)^2}\,, \tag{3.48}$$

where

$$\delta = \frac{R_m + R_r}{2(m_m + m_r)} = \frac{(R_m + R_r)\omega_0^2}{2S_m} \tag{3.49}$$

is the so-called damping coefficient of the oscillator. Note that Eqs. (3.47) and (3.49) are extensions of definitions given in Eq. (2.5).

Referring to Eq. (2.20), we see that the relative absorbed-power response

$$\frac{P_a(\omega)/|\hat{F}_e(\omega)|^2}{P_a(\omega_0)/|\hat{F}_e(\omega_0)|^2} = \frac{1}{1 + (\omega_0/2\delta)^2(\omega/\omega_0 - \omega_0/\omega)^2}\,, \tag{3.50}$$

which has its maximum value of 1 at resonance ($\omega = \omega_0$), exceeds $\frac{1}{2}$ in a frequency interval $\omega_l < \omega < \omega_u$, where

$$\omega_u - \omega_l = (\Delta\omega)_{\text{res}} = 2\delta = \frac{R_m + R_r}{m_m + m_r} = \frac{(R_m + R_r)\omega_0^2}{S_m}. \tag{3.51}$$

The relative absorbed-power response versus frequency is plotted in Figure 3.3 for two different values of the damping factor

$$\frac{\delta}{\omega_0} = \frac{R_m + R_r}{2\omega_0(m_m + m_r)} = \frac{R_m + R_r}{2\sqrt{S_m(m_m + m_r)}} = \frac{(R_m + R_r)\omega_0}{2S_m}. \tag{3.52}$$

The last equalities in Eqs. (3.49), (3.51) and (3.52) have been obtained by using Eq. (3.47) to eliminate $(m_m + m_r)$. Observe that increasing (decreasing) the spring stiffness S_m increases (decreases) the natural frequency ω_0 and decreases (increases) the relative bandwidth $(\Delta\omega)_{\text{res}}/\omega_0$. On the other hand, increasing (decreasing) the mass m_m decreases (increases) both ω_0 and $(\Delta\omega)_{\text{res}}/\omega_0$.

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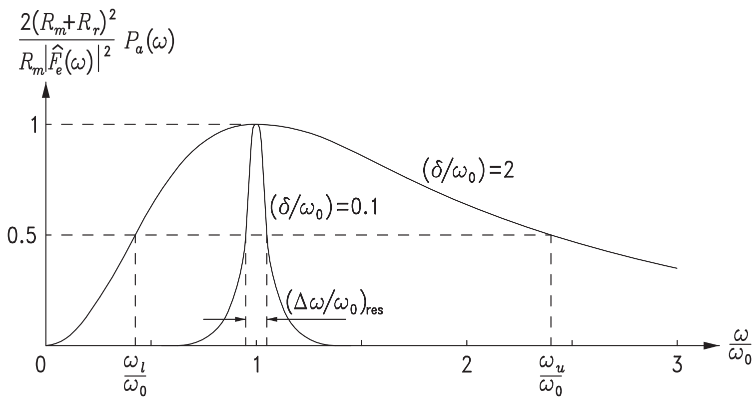

Figure 3.3: Frequency response of absorbed power for two different values of damping factor δ/ω_0.

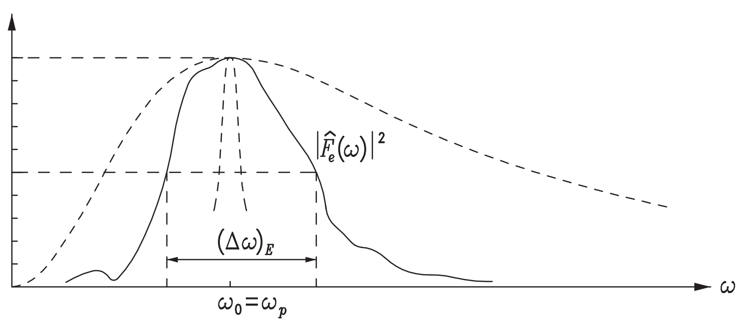

Figure 3.4: Representation of the wave spectrum (solid curve) compared with power absorption responses for the two cases given in Figure 3.3 (dashed curves).

Note that $|\hat{F}_e(\omega)|^2$ is a representation of the spectrum of the incident wave. An example is indicated in Figure 3.4, where $|\hat{F}_e(\omega)|^2$ is maximum at some angular frequency ω_p and exceeds half of its maximum in an interval of length $(\Delta\omega)_E$. Assume that an absorbing system has been chosen, for which $\omega_0 = \omega_p$. If a sufficiently large damping factor δ/ω_0 is chosen, we have that $(\Delta\omega)_{\text{res}} > (\Delta\omega)_E$. If we wish to absorb as much wave energy as possible, we should choose $R_m \geq R_r$ according to Eqs. (3.40) and (3.44). Then, from Eq. (3.51), we have

$$(\Delta\omega)_{\text{res}} \geq \frac{2R_r}{m_m + m_r}. \tag{3.53}$$

The two dashed curves in Figure 3.4 represent two different wave-absorbing oscillators, one with a narrow bandwidth, the other with a wide one. Evidently, the narrow-bandwidth oscillator can absorb efficiently from only a small part of the indicated wave spectrum.

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Problems

Problem 3.1: Group Velocity

(a) Show that the sum $s = s_1 + s_2$ of two waves of equal amplitudes and different frequencies, $s_j(x,t) = A\cos(\omega_j t - k_j x)$ (for $j = 1, 2$), may be written as

$$s(x,t) = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t - \frac{k_1 - k_2}{2}x\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t - \frac{k_1 + k_2}{2}x\right)$$

either by using a trigonometric formula or by applying the method of complex representation of harmonic waves. Set $\omega_1 = \omega + \Delta\omega/2$ and $\omega_2 = \omega - \Delta\omega/2$ and discuss the case $\Delta\omega \to 0$.

(b) Further, consider the more general case

$$s(x,t) = \tfrac{1}{2}\textstyle\sum_j A_j \exp\left[i(\omega_j t - k_j x)\right] + \text{c. c.}$$

Assume that $|A_j|$ has a maximum value $|A_m|$ for $j = m$ ($m \gg 1$) and that $|A_j|$ is negligible when ω_j deviates from ω_m by more than a relatively small amount $\Delta\omega$ ($\Delta\omega \ll \omega_m$, 'narrow spectrum'). Show that the 'signal' $s(x,t)$ may be interpreted as a 'carrier wave', of angular frequency ω_m and angular repetency k_m, modulated by an 'envelope wave' propagating with the group velocity $(d\omega/dk)_m$, provided the dispersion curve $\omega = \omega(k)$ or $k = k(\omega)$ may be approximated by its tangent at (k_m, ω_m) in the interval where $|A_j|$ is not negligible.

Problem 3.2: Capillary-Gravity Surface Wave

If the contribution to the wave's potential energy from capillary forces, in addition to the contribution from gravitational forces, is taken into consideration, then the phase velocity for waves on deep water is given by

$$v_p = \sqrt{g/k + \gamma k/\rho}$$

instead of by Eq. (3.9), where $\gamma = 0.07\,\text{N/m}$ is the surface tension on the air–water interface. Find the corresponding dispersion relationship which replaces Eq. (3.8). Derive also an expression for the group velocity. Find the numerical value of the phase velocity for a ripple of wavelength $2\pi/k = 10\,\text{mm}$ (assume $g = 9.8\,\text{m/s}^2$ and $\rho = 1.0 \times 10^3\,\text{kg/m}^3$). Finally, derive expressions and numerical values for the wavelength and the frequency of the ripple which has the minimum phase velocity.

Problem 3.3: Optical Dispersion in Gas

For a gas, the refraction index $n = c/v_p$ (the ratio between the speed of light in vacuum c and the phase velocity v_p) is to a good approximation

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$$n = 1 + p/(\omega_0^2 - \omega^2)$$

for frequencies where n is not significantly different from 1. Here, p is a positive constant and ω_0 is a resonance angular frequency for the gas. Verify that the group velocity $v_g = d\omega/dk$ for an electromagnetic wave (light wave) in the gas (contrary to the phase velocity $v_p = \omega/k$) is smaller than c, for $\omega \ll \omega_0$ as well as for $\omega \gg \omega_0$.

Problem 3.4: Radiation Impedance for Spherical Loudspeaker

An acoustic wave is radiated from (generated by) a pulsating sphere of radius a. The surface of the sphere oscillates with a radial velocity $u = \hat{u}e^{i\omega t}$. For $r > a$, the wave may be represented by the sound pressure

$$p = (A/r)\, e^{i(\omega t - kr)}$$

and the (radially directed) particle speed

$$v = v_r = [1 + 1/(ikr)]\,(p/\rho c),$$

where the constants ρ and $c = \omega/k$ are the fluid density and the speed of sound, respectively [21, pp. 114 and 163].

Use the boundary condition $u = [v]_{r=a}$ to determine the unknown coefficient A. Then derive expressions for the radiation impedance Z_r, radiation resistance R_r and added mass m_r as functions of the angular repetency k. (Base the derivation of Z_r on the reaction force from the wave on the surface of the sphere. Check the result for R_r by deriving an expression for the radiated power.)

Problem 3.5: Acoustic Point Absorber

Let us consider a pulsating sphere as a microphone. Let the radius a of the sphere be so small ($ka \ll 1$) that the microphone may be considered as an acoustic point absorber. When $ka \ll 1$, the radiation resistance is approximately

$$R_r \approx 4\pi a^4 k^2 \rho c.$$

The spherical shell of the microphone has a mass m and a stiffness S against radial displacements. Moreover, it has a mechanical resistance R representing conversion of absorbed acoustical power to electric power.

A plane harmonic wave

$$p_i = A_0\, e^{i(\omega t - kx)}$$

is incident upon the sphere. For a point absorber, we may disregard reflected or diffracted waves. At a certain frequency ($\omega = \omega_0$) we have resonance. Find the value of R for which the absorbed power P has its maximum $P_{\max}$ and express $P_{\max}$ in terms of A_0, a, k and ρc. Also show that the maximum absorption cross

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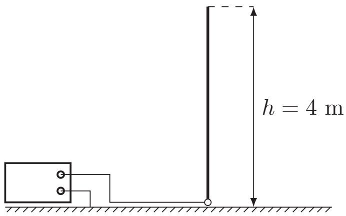

Figure 3.5: Vertical antenna connected to an electric circuit for transmission or reception of electromagnetic waves.

section of the spherical microphone is $\lambda^2/4\pi$. (This might be compared with the absorption cross section $A_a = \lambda^2/2\pi$ for a microphone of the type of a plane vibrating piston in a plane stiff wall, as shown, for instance, in [26]. The absorption cross section is defined as the absorbed power divided by the sound intensity $I_i = \frac{1}{2}|A_0|^2/\rho c$ of the incident wave.)

Problem 3.6: Short Dipole Antenna

A vertical grounded antenna of height $h = 4$ m has, at frequency $\nu = \omega/2\pi = 3$ MHz (or wavelength $\lambda = 100$ m), an effective height $h_{\text{eff}} = 2$ m. The input port for the antenna is at the ground plane and the antenna is coupled to an electric circuit as indicated in Figure 3.5. Show that the radiation resistance is $R_r = 0.63\ \Omega$. The antenna is being used for reception of a plane electromagnetic wave with vertically polarised electric field of amplitude $|E_i| = 10^{-3}$ V/m.

The antenna circuit is resistance matched as well as tuned to resonance. Calculate the power which is absorbed by the antenna circuit. Also calculate the absorption cross section for the antenna.

[Hint: for the grounded antenna, the effective dipole length is $l_{\text{eff}} = 2h_{\text{eff}}$. For a Hertz dipole, of infinitesimal length l_{eff}, the radiation resistance is $R_r = (kl_{\text{eff}})^2 Z_0/6\pi$, where $Z_0 = (\mu_0/\epsilon_0)^{1/2} = 377\ \Omega$. The intensity (average power per unit area) of the incident electromagnetic wave is $\frac{1}{2}|E_i|^2/Z_0$.]

Problem 3.7: Optimum Load Resistance

Derive Eqs. (3.40) and (3.41) for the optimum load resistance R_m and the corresponding absorbed power $P_{a,\max}$.

Problem 3.8: Maximum Absorbed Power

Show that Eq. (3.39) for the absorbed power may be reformulated as

$$\frac{8R_r}{|\hat{F}_e|^2}P_a = \frac{4R_mR_r}{(R_m + R_r)^2 + X^2} = 1 - \frac{(R_m - R_r)^2 + X^2}{(R_m + R_r)^2 + X^2} = 1 - \left|1 - \frac{2R_r\hat{u}}{\hat{F}_e}\right|^2,$$

where $X = \omega m_m + \omega m_r - S_m/\omega$. From this, Eqs. (3.42)–(3.46) may be obtained simply by inspection.

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Problem 3.9: Power Radiated from Oscillating Submerged Body

Assume that the mass m of Problem 2.12 is submerged in water, where it generates a wave. Further, assume that the radiation resistance $R_r = 1$ Ns/m is included as one-third of the total resistance $R = 3$ Ns/m. Similarly, assume that the added mass m_r is included in the total mass $m = 6$ kg. Here we neglect the fact that R_r and m_r vary with frequency. How large is the radiated power at frequencies f and f_0 defined in Problem 2.12?

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