CHAPTER FIVE

Wave–Body Interactions

The subject of this chapter is, as with Chapter 3, a discussion on the interaction between waves and oscillating bodies. Now, however, the discussion is limited to the case of interaction with water waves, which we discussed in some detail in Chapter 4. We start by studying body oscillations and wave forces on bodies. Next, we consider the phenomenon of wave generation by oscillating bodies, and we discuss some general relationships between the wave forces on bodies when they are held fixed and the waves they generate when they oscillate in otherwise calm water. Some particular consideration is given to two-dimensional cases and to axisymmetric cases. While most of the analysis is carried out in the frequency domain, some studies in the time domain are also made.

5.1 Six Modes of Body Motion: Wave Forces and Moments

To describe the motion of a body immersed in water, we need a coordinate system, for which we choose to have the z-axis passing through the centre of gravity of the body. The plane $z = 0$ defines the mean free surface. Further, we choose a reference point $(0, 0, z_0)$ on the z-axis—for instance, the centre of gravity or the origin (as in Figure 5.1). The body has a wet surface S, which separates it from the water. Consider a surface element dS at position $\vec{s}$ (Figure 5.1). The vector $\vec{s}$ originates in the chosen reference point. Let $\vec{U}$ be the velocity of the reference point. The surface element dS has a velocity

$$\vec{u} = \vec{U} + \vec{\Omega} \times \vec{s}, \tag{5.1}$$

where $\vec{\Omega}$ is the angular velocity vector corresponding to rotation about the reference point. Thus, with no rotation, $\vec{u} = \vec{U}$, and with no translation, $\vec{u} = \vec{\Omega} \times \vec{s}$. For a given body, the motion of each point $\vec{s}$ (of the wet body surface S) is characterised by the time-dependent vectors $\vec{U}$ and $\vec{\Omega}$.

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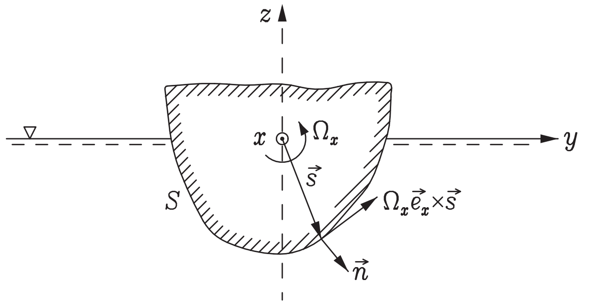

Figure 5.1: Body oscillating in water. Vector $\vec{s}$ gives the position of a point on the wet surface S, where the unit normal is $\vec{n}$.

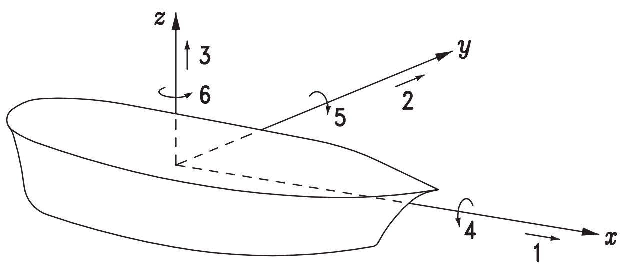

Figure 5.2: A rigid body has six modes of motion: surge (1), sway (2), heave (3), roll (4), pitch (5) and yaw (6).

The velocity potential ϕ must satisfy the boundary condition (4.15) everywhere on S:

$$\frac{\partial \phi}{\partial n} = \vec{n} \cdot \nabla \phi = u_n = \vec{U} \cdot \vec{n} + (\vec{\Omega} \times \vec{s}) \cdot \vec{n} = \vec{U} \cdot \vec{n} + \vec{\Omega} \cdot (\vec{s} \times \vec{n}). \tag{5.2}$$

Here, u_n is the normal component of the velocity $\vec{u}$ of the wet-surface element dS of the body.

5.1.1 Six Modes of Motion

The motion of the rigid body is characterised by six components, corresponding to six degrees of freedom or modes of (oscillatory) motion, as shown in Figure 5.2. The three translational modes are conventionally named surge, sway and heave, whereas the three rotational modes are named roll, pitch and yaw. Note that in the two-dimensional case (see Section 5.8)—that is, when there is no variation in the y-direction—there are only three modes of motion, namely surge, heave and pitch (two translations and one rotation).

We introduce six-dimensional generalised vectors: a velocity vector $\mathbf{u}$ with components

$$(u_1, u_2, u_3) \equiv (U_x, U_y, U_z) = \vec{U}, \tag{5.3}$$

$$(u_4, u_5, u_6) \equiv (\Omega_x, \Omega_y, \Omega_z) = \vec{\Omega}; \tag{5.4}$$

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and a normal vector $\mathbf{n}$ with components

$$(n_1, n_2, n_3) \equiv (n_x, n_y, n_z) = \vec{n}, \tag{5.5}$$

$$(n_4, n_5, n_6) \equiv \vec{s} \times \vec{n}. \tag{5.6}$$

Thus, $n_4 = (\vec{s}\times\vec{n})_x = s_y n_z - s_z n_y$, and so on. Note that the components numbered 1 to 3 have different dimensions than the remaining components numbered 4 to 6. Thus, u_1, u_2 and u_3 have SI units of m/s, whereas u_4, u_5 and u_6 have SI units of rad/s.

Inserting Eqs. (5.3)–(5.6) into the boundary condition (5.2), we can write it as

$$\frac{\partial \phi}{\partial n} = \sum_{j=1}^{6} u_j n_j = \mathbf{u}^{\mathrm{T}}\mathbf{n} \quad \text{on } S. \tag{5.7}$$

Here the superscript T denotes transpose. Thus, $\mathbf{u}^{\mathrm{T}} = (u_1, u_2, u_3, u_4, u_5, u_6)$ is a row vector, while $\mathbf{u}$ is the corresponding column vector. Moreover, $\mathbf{n} = (n_1, n_2, n_3, n_4, n_5, n_6)^{\mathrm{T}}$. These generalised vectors are six-dimensional.

Note that in the preceding equations, the variables $\vec{U}$, $\vec{\Omega}$, $u_j(t)$ and $\phi(x, y, z, t)$ may, in the case of harmonic oscillations and waves, be replaced by their complex amplitudes $\hat{\vec{U}}$, $\hat{\vec{\Omega}}$, $\hat{u}_j$ and $\hat{\phi}(x, y, z)$. Thus,

$$\frac{\partial \hat{\phi}}{\partial n} = \sum_{j=1}^{6} \hat{u}_j n_j = \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{n} \quad \text{on } S. \tag{5.8}$$

Let us, for a while, consider the radiation problem. When the body oscillates, a radiated wave ϕ_r is generated which is a superposition (linear combination) of radiated waves due to each of the six oscillation modes:

$$\hat{\phi}_r = \sum_{j=1}^{6} \varphi_j \hat{u}_j, \tag{5.9}$$

where $\varphi_j = \varphi_j(x, y, z)$ is a complex coefficient of proportionality. This frequency-dependent coefficient φ_j must satisfy the body-boundary condition

$$\frac{\partial \varphi_j}{\partial n} = n_j \quad \text{on } S, \tag{5.10}$$

since the complex amplitude $\hat{\phi}_r$ of the radiated velocity potential must satisfy the boundary condition (5.8). Whereas ϕ_r has SI unit m^2/s, φ_1, φ_2 and φ_3 have unit m, and φ_4, φ_5 and φ_6 have unit m^2 (or, more precisely, m^2/rad). The coefficient φ_j may be interpreted as the complex amplitude of the radiated velocity potential due to body oscillation in mode j with unit velocity amplitude ($\hat{u}_j = 1$). Coefficients φ_j are independent of the oscillation amplitude because boundary condition (5.2), as well as Laplace's equation (4.33), is linear. Finally, let us mention that, in the present linear theory, viscosity is neglected.

Coefficients φ_j must satisfy the same homogeneous equations as $\hat{\phi}_r$, namely the Laplace equation

$$\nabla^2 \varphi_j = 0, \tag{5.11}$$

the sea-bed boundary condition

$$\left[\frac{\partial \varphi_j}{\partial z}\right]_{z=-h} = 0 \tag{5.12}$$

and the free-surface boundary condition

$$\left[-\omega^2 \varphi_j + g\frac{\partial \varphi_j}{\partial z}\right]_{z=0} = 0. \tag{5.13}$$

[See Eqs. (4.33), (4.40) and (4.41).] Moreover, φ_j (and $\hat{\phi}_r$) must satisfy a radiation condition at infinite distance. This will be discussed later (also see Section 4.6).

Remember that in this linear theory it is consistent to take the boundary condition (5.13) at the mean free surface $z = 0$ instead of the instantaneous free surface $z = \eta(t)$. Similarly, when we use the boundary condition (5.10), we may consider S to be the mean wet surface of the body instead of the instantaneous wet surface.

5.1.2 Hydrodynamic Force Acting on a Body

Next, let us derive expressions for the force and the moment which act on the body, in terms of a given velocity potential ϕ.

Firstly, let us consider the vertical (or heave) force component F_z. The vertical force on the element dS (Figure 5.3) is $p(-n_z)dS = -pn_3dS$. Here p is the hydrodynamic pressure. Integrating over the wet surface S gives the total heave force

$$F_3 \equiv F_z = -\iint_S pn_3\, dS. \tag{5.14}$$

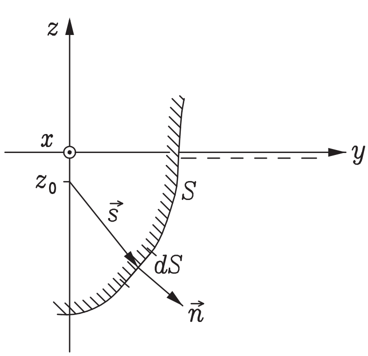

Figure 5.3: Surface element dS of the wet surface S of a rigid body with unit normal $\vec{n}$ and with position $\vec{s}$ relative to the chosen reference point $(x, y, z) = (0, 0, z_0)$ of the body.

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In terms of complex amplitudes, we have, using Eq. (4.37),

$$\hat{F}_3 = \hat{F}_z = i\omega\rho \iint_S \hat{\phi} n_3 \, dS. \tag{5.15}$$

Analogous expressions apply for the horizontal force components—that is, surge force F_1 and sway force F_2.

Next, let us consider the moment about the x-axis. The moment acting on surface element dS is

$$dM_x = s_y dF_z - s_z dF_y = (-pn_z s_y + pn_y s_z) dS. \tag{5.16}$$

Integrating over the whole wet surface S gives

$$M_x = -\iint_S p(s_y n_z - s_z n_y) dS = -\iint_S p(\vec{s} \times \vec{n})_x \, dS. \tag{5.17}$$

Thus, using Eq. (5.6)—that is, $n_4 = (\vec{s} \times \vec{n})_x$—we find that the roll moment is

$$M_x = -\iint_S p n_4 \, dS. \tag{5.18}$$

In terms of complex amplitudes,

$$\hat{M}_x = i\omega\rho \iint_S \hat{\phi} n_4 \, dS. \tag{5.19}$$

Similar expressions may be obtained for pitch moment M_y and yaw moment M_z.

We now define a generalised force vector having six components:

$$\mathbf{F} \equiv (F_1, F_2, F_3, F_4, F_5, F_6) \equiv (F_x, F_y, F_z, M_x, M_y, M_z) = (\vec{F}, \vec{M}). \tag{5.20}$$

The first three components have SI unit N (newton), and the three remaining ones have SI unit N m. For component j ($j = 1, 2, \ldots, 6$) we have

$$F_j = -\iint_S p n_j \, dS, \tag{5.21}$$

$$\hat{F}_j = i\omega\rho \iint_S \hat{\phi} n_j \, dS \tag{5.22}$$

for an arbitrary, given potential $\hat{\phi}$.

If the body is oscillating, surface element dS receives an instantaneous power

$$\begin{aligned} dP(t) &= -\vec{u} \cdot \vec{n} \, p \, dS \\ &= -(\vec{U} \cdot \vec{n} + \vec{\Omega} \times \vec{s} \cdot \vec{n}) p \, dS = -[\vec{U} \cdot \vec{n} + \vec{\Omega} \cdot (\vec{s} \times \vec{n})] p dS \\ &= -(u_1 n_1 + u_2 n_2 + u_3 n_3 + u_4 n_4 + u_5 n_5 + u_6 n_6) p \, dS. \end{aligned} \tag{5.23}$$

Integrating over the wet surface, we get the power received by the oscillating body:

$$P(t) = \vec{F}(t) \cdot \vec{U}(t) + \vec{M}(t) \cdot \vec{\Omega}(t) = \sum_{j=1}^{6} F_j u_j, \tag{5.24}$$

where we have used Eqs. (5.1), (5.3)–(5.6), (5.20) and (5.21).

5.1.3 Excitation Force

If the body is fixed, a non-vanishing potential $\hat{\phi}$ is the result of an incident wave only. In such a case, F_j is called the 'excitation force' (or, for $j = 4, 5, 6$, the 'excitation moment'). With no body motion, no radiated wave is generated. The generalised vector of the *excitation force* is written

$$\mathbf{F}_e \equiv (F_{e,1}, F_{e,2}, \ldots, F_{e,6}) = (\vec{F}_e, \vec{M}_e), \tag{5.25}$$

and using Eq. (5.22), we have

$$\hat{F}_{e,j} = i\omega\rho \iint_S (\hat{\phi}_0 + \hat{\phi}_d) n_j \, dS, \tag{5.26}$$

where $\hat{\phi}_0$ represents the undisturbed incident wave and $\hat{\phi}_d$ the diffracted wave. The latter is induced when the former does not satisfy the homogeneous boundary condition (4.16) on the fixed wet surface S. The boundary condition

$$-\frac{\partial \hat{\phi}_d}{\partial n} = \frac{\partial \hat{\phi}_0}{\partial n} \quad \text{on } S \tag{5.27}$$

has to be satisfied for the diffraction problem. Note that $\hat{\phi}_d$ and $\hat{\phi}_0$ satisfy the homogeneous boundary conditions (4.40) on the sea bed, $z = -h$, and (4.41) on the free water surface, $z = 0$.

If the diffraction term $\hat{\phi}_d$ is neglected in Eq. (5.26), the resulting force is termed the Froude–Krylov force, which is considered in more detail in Section 5.6. It may represent a reasonable approximation to the excitation force, in particular if the horizontal extension of the immersed body is very small compared to the wavelength. It may be computationally convenient to use such an approximation since it is not then required to solve the boundary-value problem for finding the diffraction potential $\hat{\phi}_d$.

Except for very simple geometries, it is not possible to find solutions to the diffraction problem in terms of elementary functions. However, for the case of a wave

$$\hat{\eta}_0 = Ae^{-ikx} \tag{5.28}$$

incident upon a vertical wall at $x = 0$, the diffracted wave is simply the totally reflected wave

$$\hat{\eta}_d = Ae^{ikx} \tag{5.29}$$

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(cf. Problem 4.5). Then, from Eq. (4.88), the velocity potential is given by

$$\hat{\phi}_0 + \hat{\phi}_d = -\frac{g}{i\omega} e(kz)(\hat{\eta}_0 + \hat{\eta}_d) = -\frac{2g}{i\omega} e(kz) A \cos(kx). \tag{5.30}$$

Using Eq. (5.26) and noting that $x = 0$ and $n_1 = -1$ on S, we now find the surge component of the excitation force (per unit width in the y-direction):

$$\hat{F}'_{e,1} = -i\omega\rho \int_{-a_2}^{-a_1} (\hat{\phi}_0 + \hat{\phi}_d)\, dz = 2\rho g A \int_{-a_2}^{-a_1} e(kz)\, dz \tag{5.31}$$

for the striplike piston shown in Figure 5.4. (The prime is used to denote a quantity per unit width. $F'_{e,1}$ has SI unit N/m.) Inserting the integrand from Eq. (4.52) and performing the integration give (cf. Problem 5.2)

$$\hat{F}'_{e,1} = 2\rho g A \frac{\sinh(kh - ka_1) - \sinh(kh - ka_2)}{k \cosh(kh)}. \tag{5.32}$$

Setting $a_2 = h$ and $a_1 = 0$, we find the surge excitation force per unit width of the entire vertical wall:

$$\hat{F}'_{e,1} = 2\rho g A \frac{\sinh(kh)}{k \cosh(kh)} = 2\rho A \left(\frac{\omega}{k}\right)^2, \tag{5.33}$$

where we have made use of the dispersion relationship (4.54).

If the incident wave is as given by Eq. (5.28), then, according to Eqs. (4.88) and (5.27), $\hat{\phi}_0$ and $\hat{\phi}_d$ are proportional to the complex elevation amplitude A at the origin $(x, y) = (0, 0)$. Hence, also all excitation force components $\hat{F}_{e,j}$ are proportional to A. It should be remembered that the excitation force on a body is the wave force when the body is not moving. Thus, for instance, the surge velocity u_1 (see Figure 5.4) was zero for the situation analysed earlier. In Section 5.2.3, however, we shall consider the case when $u_1 \neq 0$.

Although the excitation force in the preceding example is in phase with the incident wave elevation at the origin, there may, in the general case, be a

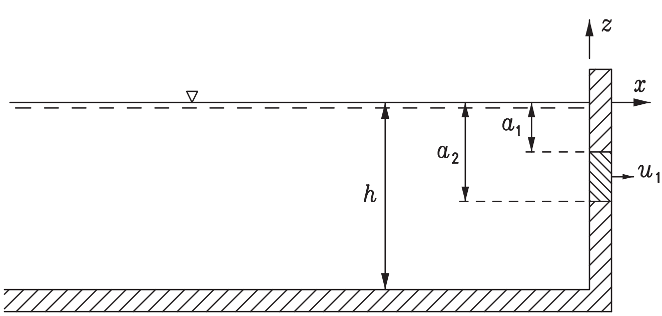

Figure 5.4: Piston in a vertical end wall of a basin with water of depth h. The piston, which is able to oscillate in surge (horizontal motion), occupies, in this two-dimensional problem, the strip $-a_2 < z < -a_1$.

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nonzero phase difference θ_j between these two variables such that the *excitation-force coefficient*

$$f_j = \frac{\hat{F}_{e,j}}{A} = \frac{|\hat{F}_{e,j}|}{|A|} e^{i\theta_j} \tag{5.34}$$

is, in general, a complex coefficient of proportionality. As we shall see later (in Section 5.6), for a floating body which is very much smaller than the wavelength, we have $\theta_1 \approx \pi$ and $\theta_3 \approx 0$ for the surge and heave modes, respectively. For a floating semisubmerged sphere on deep water, numerically computed values for θ_3 are given by the graph in Figure 5.5 (provided the sphere is located such that its vertical axis coincides with the z-axis). It can be seen from the graph that $\theta_3 < 0.1$ for $ka < 0.25$. Additional numerical information relating to a semisubmerged sphere is presented in Section 5.2.4.

5.2 Radiation from an Oscillating Body

5.2.1 The Radiation Impedance Matrix

In the following, let us consider a case when a body is oscillating in the absence of an incident wave. We shall study the forces acting on the body due to the wave which is radiated as a result of the body's oscillation. Let the body oscillate in a single mode j with complex velocity amplitude $\hat{u}_j$. The radiated wave is associated with a velocity potential ϕ_r given by

$$\hat{\phi}_r = \varphi_j \hat{u}_j, \tag{5.35}$$

where $\varphi_j = \varphi_j(x, y, z)$ is a coefficient of proportionality, as introduced in Eq. (5.9). The radiated wave reacts with a force on the body. Component j' of the force is

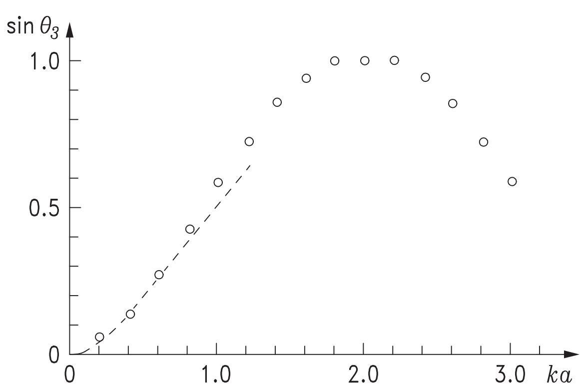

Figure 5.5: Phase angle θ_3 between the heave excitation force and the undisturbed incident wave elevation at the body centre for a semisubmerged floating sphere of radius a. The circle points are numerically computed by Greenhow [42]. The curve is computed by Kyllingstad [43] using a second-order scattering method.

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$$\hat{F}_{r,j'} = i\omega\rho \iint_S \varphi_j \hat{u}_j n_{j'} \, dS, \tag{5.36}$$

according to Eqs. (5.22) and (5.35).

Whether j denotes a translation mode ($j = 1, 2, 3$) or a rotation mode ($j = 4, 5, 6$), $\hat{u}_j$ is a constant under the integration. Hence, we may write

$$\hat{F}_{r,j'} = -Z_{j'j}\hat{u}_j, \tag{5.37}$$

where

$$Z_{j'j} = -i\omega\rho \iint_S \varphi_j n_{j'} \, dS \tag{5.38}$$

is an element of the so-called *radiation impedance* matrix. The SI units are N s/m = kg/s for $Z_{q'q}$ ($q', q = 1, 2, 3$), N s m (or N s m/rad) for $Z_{p'p}$ ($p', p = 4, 5, 6$) and N s for Z_{qp} and Z_{pq}.

Using the wet-surface boundary condition (5.10), we have

$$Z_{j'j} = -i\omega\rho \iint_S \varphi_j \frac{\partial \varphi_{j'}}{\partial n} \, dS. \tag{5.39}$$

Whereas φ_j is complex, $\partial\varphi_j/\partial n$ is real on S, since n_j is real [cf. Eq. (5.10)]. Hence, we may in the integrand replace $\partial\varphi_{j'}/\partial n$ by $\partial\varphi_{j'}^*/\partial n$ if we so wish. Thus, we have the alternative formula

$$Z_{j'j} = -i\omega\rho \iint_S \varphi_j \frac{\partial \varphi_{j'}^*}{\partial n} \, dS. \tag{5.40}$$

Note that φ_j has to satisfy boundary conditions elsewhere [see Eqs. (5.12)–(5.13)]. This means, for instance, that the radiation impedance matrix of a body in a wave channel does not have the same value as that of the same body in open sea [44, 45]. Note that a homogeneous boundary condition such as (4.16), namely $\partial\varphi_j/\partial n = 0$, has to be satisfied on the vertical walls of the wave channel.

We may interpret $-Z_{j'j}$ as the j' component of the reaction force due to the wave radiated by the body oscillating with unit amplitude in mode j ($\hat{u}_j = 1$). In fact, it is equal to the j component of the reaction force due to wave radiation from unit-amplitude oscillation in mode j' ($\hat{u}_{j'} = 1$). This is true because of the reciprocity relation

$$Z_{jj'} = Z_{j'j}, \tag{5.41}$$

which follows from Eqs. (5.39), (4.230) and (4.239), when we utilise the fact that φ_j and $\varphi_{j'}$ satisfy the same radiation condition. Thus, the radiation impedance matrix is symmetric.

For certain body geometries, some of the elements of the radiation impedance matrix vanish. If $y = 0$ is a plane of symmetry (which is typical for a ship hull), then n_2, n_4 and n_6 in Eq. (5.38) are odd functions of y, while φ_1, φ_3 and φ_5 are even functions. Hence, $Z_{21} = Z_{23} = Z_{25} = Z_{41} = Z_{43} = Z_{45} = Z_{61} = Z_{63} = Z_{65} = 0$. On the other hand, if $x = 0$ is a plane of symmetry, then n_1, n_5 and n_6 are odd functions of x, while φ_2, φ_3 and φ_4 are even functions. Hence, $Z_{12} = Z_{13} = Z_{14} = Z_{52} = Z_{53} = Z_{54} = Z_{62} = Z_{63} = Z_{64} = 0$. From this observation, while noting Eq. (5.41), it follows that if both $y = 0$ and $x = 0$ are planes of symmetry, then the only non-vanishing off-diagonal elements of the radiation impedance matrix are $Z_{15} = Z_{51}$ and $Z_{24} = Z_{42}$.

Since ω is real, it is convenient to split $Z_{j'j}$ into real and imaginary parts:

$$Z_{j'j} = R_{j'j} + iX_{j'j} = R_{j'j} + i\omega m_{j'j}, \tag{5.42}$$

where we call $\mathbf{R}$ the *radiation resistance* matrix, $\mathbf{X}$ the *radiation reactance* matrix and $\mathbf{m}$ the *added-mass* matrix. Note that some authors call $\mathbf{R}$ the 'added damping coefficient' matrix. Eq. (5.42) is a generalisation of the scalar equations (3.33) and (3.37).

5.2.2 Energy Interpretation of the Radiation Impedance

In Eq. (3.29), we have defined a radiation resistance by considering the radiated power. We shall later, in a rather general way, relate radiation resistance and reactance to power and energy associated with wave radiation. At present, however, let us just consider a diagonal element of the radiation impedance matrix. Using Eq. (5.40), we have

$$\frac{1}{2}Z_{jj}|\hat{u}_j|^2 = \frac{1}{2}Z_{jj}\hat{u}_j\hat{u}_j^* = \frac{1}{2}\iint_S(-i\omega\rho\varphi_j\hat{u}_j)\frac{\partial}{\partial n}\left(\varphi_j^*\hat{u}_j^*\right)\,dS. \tag{5.43}$$

Further, using Eqs. (5.35), (4.7) and (4.37), this becomes

$$\frac{1}{2}Z_{jj}|\hat{u}_j|^2 = \frac{1}{2}\iint_S(-i\omega\rho\hat{\phi}_r)\frac{\partial\hat{\phi}_r^*}{\partial n}\,dS = \frac{1}{2}\iint_S\hat{p}\hat{v}_n^*\,dS = \mathcal{P}_r. \tag{5.44}$$

Referring to Eqs. (3.17) and (3.20), we see that the real part of $\mathcal{P}_r$ is the (time-average) power P_r delivered to the fluid from the body oscillating in mode j:

$$P_r = \mathrm{Re}\{\mathcal{P}_r\} = \tfrac{1}{2}\mathrm{Re}\{Z_{jj}\}|\hat{u}_j|^2 = \tfrac{1}{2}R_{jj}|\hat{u}_j|^2. \tag{5.45}$$

Here R_{jj} is the diagonal element of the radiation resistance matrix. The real part P_r of the radiated 'complex power' $\mathcal{P}_r$ [see Eq. (2.90)] represents the radiated 'active power'. The imaginary part

$$\mathrm{Im}\{\mathcal{P}_r\} = \tfrac{1}{2}\mathrm{Im}\{Z_{jj}\}|\hat{u}_j|^2 = \tfrac{1}{2}X_{jj}|\hat{u}_j|^2 \tag{5.46}$$

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represents the radiated 'reactive power'. We may note that (see Problem 5.7 or Section 6.5)

$$\mathcal{P}_r = \sum_{j'=1}^{6} \sum_{j=1}^{6} Z_{j'j} \hat{u}_{j'} \hat{u}_j^* \tag{5.47}$$

is a generalisation of Eq. (5.45), which is valid when all except one of the oscillation modes have a vanishing amplitude.

5.2.3 Wavemaker in a Wave Channel

Let us consider an example referring to Figure 5.4, when the piston has a surge velocity with complex amplitude $\hat{u}_1 \neq 0$. To be slightly more general, we shall, for this example, write the inhomogeneous boundary condition (5.10) as

$$\frac{\partial \varphi_1}{\partial x} = c(z) \quad \text{for } x = 0, \tag{5.48}$$

where $c(z)$ is a given function. For the case shown in Figure 5.4, we must set $c(z) = 1$ for $-a_2 < z < -a_1$ and $c(z) = 0$ elsewhere. If we set $c(z) = 1$ for $-h < z < 0$, this corresponds to the entire vertical wall oscillating as a surging piston. If $c(z) = 1 + z/h$ for $-h < z < 0$, this corresponds to an oscillating flap, hinged at $z = -h$. Both are typical as wavemakers in wave channels.

The given oscillation at $x = 0$ generates a wave which propagates in the negative x-direction towards $x = -\infty$ (Figure 5.4). Hence, it is a requirement that our solution of the boundary-value problem is finite in the region $x < 0$ and that it satisfies the radiation condition at $x = -\infty$. Referring to Eq. (4.111), which is a solution to the boundary-value problem given by Eqs. (4.33), (4.40) and (4.41), we can immediately write down the following solution:

$$\varphi_1 = c_0 e^{ikx} Z_0(z) + \sum_{n=1}^{\infty} c_n e^{m_n x} Z_n(z) = \sum_{n=0}^{\infty} X_n(x) Z_n(z), \tag{5.49}$$

which (for $j = 1$) satisfies the Laplace equation (5.11), the homogeneous boundary conditions (5.12)–(5.13), the finiteness condition for $-\infty < x < 0$ and the radiation condition for $x = -\infty$. Concerning the orthogonal functions $Z_n(z)$ used in Eq. (5.49), we may refer to Eqs. (4.70)–(4.76). For convenience, we have introduced $m_0 = ik$ and

$$X_n(x) = c_n e^{m_n x}. \tag{5.50}$$

In Eq. (5.49), the terms corresponding to $n \geq 1$ represent evanescent waves which are non-negligible only near the wave generator. Thus, in the far-field region $-x \gg 1/m_1$, we have, asymptotically,

$$\varphi_1 \approx c_o e^{ikx} Z_0(z). \tag{5.51}$$

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In order to determine the unknown constants c_n in Eq. (5.49), we use the boundary condition (5.48) at $x = 0$:

$$c(z) = \left[\frac{\partial \varphi_1}{\partial x}\right]_{x=0} = \sum_{n=0}^{\infty} X_n'(0) Z_n(z). \tag{5.52}$$

We multiply by $Z_m^*(z)$, integrate from $z = -h$ to $z = 0$ and use the orthogonality and normalisation conditions (4.67) and (4.73). This gives

$$\int_{-h}^{0} c(z) Z_m^*(z)\, dz = \sum_{n=0}^{\infty} X_n'(0) \int_{-h}^{0} Z_m^*(z) Z_n(z)\, dz = X_m'(0) h. \tag{5.53}$$

That is,

$$X_n'(0) = \frac{1}{h} \int_{-h}^{0} c(z) Z_n^*(z)\, dz. \tag{5.54}$$

According to Eq. (5.50), we have

$$X_0'(0) = ikc_0, \quad X_n'(0) = m_n c_n, \tag{5.55}$$

and hence,

$$c_0 = \frac{1}{ikh} \int_{-h}^{0} c(z) Z_0^*(z)\, dz, \tag{5.56}$$

$$c_n = \frac{1}{m_n h} \int_{-h}^{0} c(z) Z_n^*(z)\, dz. \tag{5.57}$$

Thus, c_n is a kind of 'Fourier' coefficient for the expansion of the velocity-distribution function $c(z)$ in the function 'space' spanned by the complete function set $\{Z_n(z)\}$.

The velocity potential of the radiated wave is

$$\hat{\phi}_r = \varphi_1 \hat{u}_1, \tag{5.58}$$

where φ_1 is given by Eq. (5.49). The wave elevation of the radiated wave is [cf. Eq. (4.39)]

$$\hat{\eta}_r = -\frac{i\omega}{g} \left[\hat{\phi}_r\right]_{z=0} = -\frac{i\omega}{g} \hat{u}_1 \sum_{n=0}^{\infty} c_n Z_n(0) e^{m_n x}. \tag{5.59}$$

In the far-field region—that is, for $-x \gg 2h/\pi > 1/m_1$—the evanescent waves are negligible, and we have the asymptotic solution

$$\hat{\eta}_r \sim A_r^- e^{ikx} = a_1^- \hat{u}_1 e^{ikx}, \tag{5.60}$$

where

$$a_1^- = \frac{A_r^-}{\hat{u}_1} = -\frac{i\omega}{g} c_0 Z_0(0) = -\frac{i\omega}{g} c_0 \sqrt{\frac{2kh}{D(kh)}}. \tag{5.61}$$

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Eqs. (4.52) and (4.76) have been used in the last step. Note that a_1^- is real, since $c(z)$ and $Z_0(z)$ are real, and hence [cf. Eq. (5.56)], ic_0 is also real.

If Figure 5.4 represents a wave channel of width d, the radiation impedance of the wavemaker is, according to Eq. (5.40), with $\partial/\partial n = -\partial/\partial x$,

$$Z_{11} = i\omega\rho d \int_{-h}^{0} \left[\varphi_1 \frac{\partial \varphi_1^*}{\partial x}\right]_{x=0} dz. \tag{5.62}$$

Considering Figure 5.4 as representing a two-dimensional problem, we find that the radiation impedance per unit width is

$$\begin{aligned} Z'_{11} = \frac{Z_{11}}{d} &= i\omega\rho \int_{-h}^{0} \left[c_0 Z_0(z) + \sum_{n=1}^{\infty} c_n Z_n(z)\right] c^*(z)\, dz \\ &= i\omega\rho\left(-ikhc_0c_0^* + \sum_{n=1}^{\infty} m_n h c_n c_n^*\right), \end{aligned} \tag{5.63}$$

where we have used Eqs. (5.48), (5.49), (5.56) and (5.57) in Eq. (5.62). Hence,

$$Z'_{11} = \frac{Z_{11}}{d} = \omega k\rho h|c_0|^2 + i\omega\rho h \sum_{n=1}^{\infty} m_n |c_n|^2. \tag{5.64}$$

The radiation resistance is

$$R_{11} = \mathrm{Re}\{Z_{11}\} = \omega k\rho |c_0|^2 hd. \tag{5.65}$$

The added mass is

$$m_{11} = \frac{1}{\omega}\mathrm{Im}\{Z_{11}\} = \rho hd \sum_{n=1}^{\infty} m_n |c_n|^2 \tag{5.66}$$

[cf. Eq. (5.42)]. Because all m_n are positive, we see that all contributions to the radiation reactance ωm_{11} are positive. We also see that the radiation resistance is associated with the far field (corresponding to the propagating wave), whereas the radiation reactance and, hence, the added mass are associated with the near field (corresponding to the evanescent waves).

It might be instructive to consider the 'complex energy transport' $\mathcal{J}(x)$ associated with the radiated wave. Referring to Eqs. (2.90) and (4.126), we obtain the complex wave-energy transport (which, in this case, is in the negative x-direction):

$$\begin{aligned} \mathcal{J}(x) &= \frac{1}{2}\int_{-h}^{0} \hat{p}(-\hat{v}_x)^*\, dz = -\frac{1}{2} i\omega\rho \int_{-h}^{0} \hat{\phi}_r \left(-\frac{\partial \hat{\phi}_r^*}{\partial x}\right) dz \\ &= \frac{i\omega\rho}{2} \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} X_n(x) X_l'^*(x) \int_{-h}^{0} Z_n(z) Z_l^*(z)\, dz\, \hat{u}_1 \hat{u}_1^* \\ &= \frac{1}{2} i\omega\rho h \sum_{n=0}^{\infty} X_n(x) X_n'^*(x) |\hat{u}_1|^2, \end{aligned} \tag{5.67}$$

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where we have utilised Eqs. (5.49), (5.58) and (4.67). Using Eq. (5.50) gives

$$\mathcal{J}(x) = \left[\frac{1}{2}\omega kh\rho|c_0|^2 + \frac{1}{2}i\omega\rho h\sum_{n=1}^{\infty} m_n|c_n|^2 \exp(2m_n x)\right]|\hat{u}_1|^2. \tag{5.68}$$

We see that, because of orthogonality condition (4.67), there are no cross terms from different vertical eigenfunctions in the product.

The (active) energy transport per unit width of wave frontage is

$$J = \mathrm{Re}\{\mathcal{J}(x)\} = \tfrac{1}{2}\omega kh\rho|c_0|^2|\hat{u}_1|^2. \tag{5.69}$$

Moreover, we see that there is a reactive power $\mathrm{Im}\{\mathcal{J}(x)\}$ which exists only in the near field. This demonstrates that the radiation reactance and, hence, the added mass are related to reactive energy transport in the near-field region of the wavemaker. The active energy transport J in a plane wave in an ideal (loss-free) liquid is, of course, independent of x. We see that the radiated power, in accordance with Eq. (3.29), is

$$\tfrac{1}{2}R_{11}|\hat{u}_1|^2 = P_r = Jd = \mathrm{Re}\{\mathcal{J}(x)\}d = \mathcal{J}(-\infty)\,d. \tag{5.70}$$

Correspondingly, the radiation impedance Z_{11} may be expressed as

$$\tfrac{1}{2}Z_{11}|\hat{u}_1|^2 = \mathcal{J}(0)d, \tag{5.71}$$

in accordance with Eq. (5.44).

Imagine now that we have a slightly flexible wavemaker such that, for a particular value of k (and, hence, of ω), a horizontal oscillation can be realised where

$$c(z) = e(kz). \tag{5.72}$$

Using Eq. (4.76) in Eq. (5.56), we then find that

$$c_0 = -\frac{i}{k}\sqrt{\frac{D(kh)}{2kh}}, \tag{5.73}$$

and, from Eq. (4.67), that $c_n = 0$ for $n \geq 1$. From Eq. (5.65), we then find that

$$R_{11} = \frac{\omega\rho D(kh)d}{2k^2} = \frac{\rho d}{g}v_p^2 v_g \tag{5.74}$$

and, from Eq. (5.66), that $m_{11} = 0$. In such a case, we have wave generation without added mass and without any evanescent wave. The reason is that the far-field solution alone [cf. Eq. (5.51)] matches the wavemaker boundary condition (5.48) on the wavemaker's wet surface, when $c(z)$ is given as in Eq. (5.72).

In constrast, if we are able to realise

$$c(z) = \frac{\cos(m_1 z + m_1 h)}{\cos(m_1 h)}, \tag{5.75}$$

we generate only one evanescent mode and no progressive wave. (Note that, in this case, the lower and upper parts of the vertical wall have to oscillate in opposite phases.) For this example, the radiation resistance vanishes at the particular frequency for which Eq. (5.75) is satisfied. This shows that it is possible, at least in principle, to make oscillations in the water without producing a propagating wave. This knowledge may be of use if it is desired that a body oscillating in the sea with a particular frequency shall not generate undesirable waves.

Finally, let us consider the case of a stiff surging vertical plate. That is, we choose $c(z) = 1$ for $-h < z < 0$. Then, in general, $c_n \neq 0$ for all n. In particular, we have

$$c_0 = -\frac{i\omega^2}{gk^3h}\sqrt{\frac{2kh}{D(kh)}}, \tag{5.76}$$

as shown in Problem 5.3. From Eqs. (5.61) and (5.65), we then find that the far-field coefficient is

$$a_1^- = -\frac{2\omega^3}{g^2k^2D(kh)} = -\frac{v_p^2}{gv_g}, \tag{5.77}$$

and the radiation resistance is

$$R_{11} = \frac{2\omega^5\rho d}{g^2k^4D(kh)} = \frac{\rho d}{g}\frac{v_p^4}{v_g}. \tag{5.78}$$

Let us now, as a prelude to sections on reciprocity relations (Sections 5.4 and, in particular, 5.8), establish that the following relations hold in the present example, with respect to the interaction of a surging vertical plate with a wave on one of its sides (the left-hand side, as shown in Figure 5.4). The radiation resistance (5.78) and the excitation force (5.33) may be expressed in terms of the far-field coefficient (5.77) as

$$R_{11} = \frac{\rho g^2 D(kh)d}{2\omega}|a_1^-|^2 = \rho g d v_g |a_1^-|^2, \tag{5.79}$$

$$\hat{F}_{e,1} = -A\frac{\rho g^2 D(kh)d}{\omega}a_1^- = -2\rho g d v_g A a_1^-, \tag{5.80}$$

respectively. Using these relations in Eq. (3.45), we see that at optimum oscillation, the maximum power absorbed by the plate is

$$P_{\max} = \frac{|\hat{F}_{e,1}|^2}{8R_{11}} = \frac{\rho g^2 D(kh)d}{4\omega}|A|^2 = \frac{1}{2}\rho g d v_g |A|^2 = Jd, \tag{5.81}$$

which, according to Eq. (4.130), is the complete energy transport of the incident wave in the wave channel of width d. In this case, the absorbing plate generates a wave which, in the far-field region, cancels the reflected wave. Thus, $A_r^- = a_1^-\hat{u}_1 = -A$, or

$$\hat{u}_1 = \hat{u}_{1,\mathrm{opt}} = -A/a_1^- = -Aa_1^-/|a_1^-|^2 = \hat{F}_{e,1}/2R_{11}, \tag{5.82}$$

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in accordance with Eq. (3.46). Note that a_1^- is real in the present example. The results of Eqs. (5.81) and (5.82) are examples of matter discussed in a more general and systematic way in Section 8.3.

5.2.4 Other Body Geometries

For a general geometry, it is complicated to solve the radiation problem for an oscillating body; that is, it is complicated to solve the boundary-value problem represented by Eqs. (5.10)–(5.13). A simple example which is amenable to analytical solution was discussed in Section 5.2.3. For almost all other cases, a numerical solution is necessary. In the present subsection, some numerically obtained results will be presented for the radiation impedance and the excitation force of some particular bodies. These quantities are examples of the so-called hydrodynamic parameters.

Let us first consider a sphere of radius a semisubmerged on water of infinite depth. With this body geometry, it is obvious that rotary modes cannot generate any wave in an ideal fluid. This, together with the fact that both $x = 0$ and $y = 0$ are planes of symmetry, means that all off-diagonal elements of the radiation impedance vanish. Thus, the only non-vanishing elements are $Z_{11} = Z_{22}$ and Z_{33}, which we may write as

$$Z_{jj} = \tfrac{2}{3}\pi a^3 \rho\omega(\epsilon_{jj} + i\mu_{jj}), \tag{5.83}$$

where ϵ and μ are non-dimensionalised radiation resistance and added mass, respectively. These parameters are shown by curves in Figure 5.6, which is based on previous numerical results [46, 47]. We may note that the radiation resistance tends to zero as ka approaches infinity or zero. The added mass is finite in both of these limits. Observe that

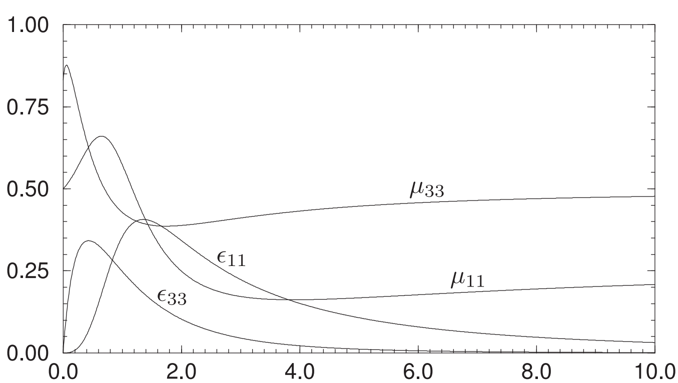

Figure 5.6: Non-dimensionalised surge and heave coefficients of radiation resistance and of added mass versus ka for a semisubmerged sphere of radius a on deep water. Here, $k = \omega^2/g$ is the angular repetency. The numerical values are from Havelock [46] and Hulme [47]. The radiation impedance for surge ($j = 1$) and heave ($j = 3$) is given by $Z_{jj} = \frac{2}{3}\pi a^3\rho\omega(\epsilon_{jj} + i\mu_{jj})$.

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$$\frac{R_{jj}}{\omega m_{jj}} = \frac{\epsilon_{jj}}{\mu_{jj}} = \mathcal{O}\{(ka)^n\} \quad \text{as } ka \to 0, \tag{5.84}$$

where $n = 2$ for the surge mode and $n = 1$ for the heave mode. It can be shown (cf. Problem 5.10) that

$$\epsilon_{33} \to (3\pi/4)ka \quad \text{as } ka \to 0. \tag{5.85}$$

When $ka \ll 1$, the diameter is much shorter than the wavelength. Also, for body geometries other than a sphere, it is generally true that the radiation reactance dominates over the radiation resistance when the extension of the body is small compared to the wavelength.

As the next example, we present in Figure 5.7 the excitation-force coefficient

$$f_3 \equiv F_{e,3}/A = \kappa_3 \rho g \pi a^2 \tag{5.86}$$

and the radiation impedance

$$Z_{33} = \tfrac{2}{3}\pi a^3 \rho\omega(\epsilon_{33} + i\mu_{33}) \tag{5.87}$$

for the heave mode of a floating truncated vertical cylinder of radius a and draft b. The cylinder axis coincides with the z-axis ($x = y = 0$). The numerical values for these hydrodynamic parameters have been computed using a method described by Eidsmoen [48].

We observe from Figure 5.7 that, for low frequencies ($ka \to 0$), the heave excitation force has a magnitude as we could expect from simply applying Archimedes' law, which neglects effects of wave interference and wave diffraction. For high frequencies ($ka \to \infty$), the force tends to zero, which is to be expected because of the mentioned effects, but also because of the decrease of hydrodynamic pressure with increasing submergence below the free water surface. From the lower-left-hand graph, we see that for large frequencies, the curve for the phase approaches a straight line which (in radians) has a steepness equal to 1. This means that the heave excitation force is, for $ka \to \infty$, in phase with the incident wave elevation at $x = -a$ (that is, where the wave first hits the cylinder), whereas for $ka \to 0$, it is in phase with the (undisturbed) incident wave elevation at $x = 0$.

The curves for ϵ_{33} and μ_{33} in Figure 5.7 are qualitatively similar to the corresponding curves in Figure 5.6 for the semisubmerged sphere. Note, however, that ϵ_{33} is, at larger frequencies, significantly smaller for the vertical cylinder. The limiting values for $ka \to 0$ are also different. It can be shown (cf. Problem 5.10) that for finite water depth h, $\epsilon_{33} \to 3\pi a/8h$, while for infinite water depth, $\epsilon_{33} \to 3\pi ka/4$, as $ka \to 0$. The reason for the different appearance of the two ϵ_{33} curves for $ka \to 0$ is that infinite water depth was assumed with the semisubmerged sphere results in Figure 5.6.

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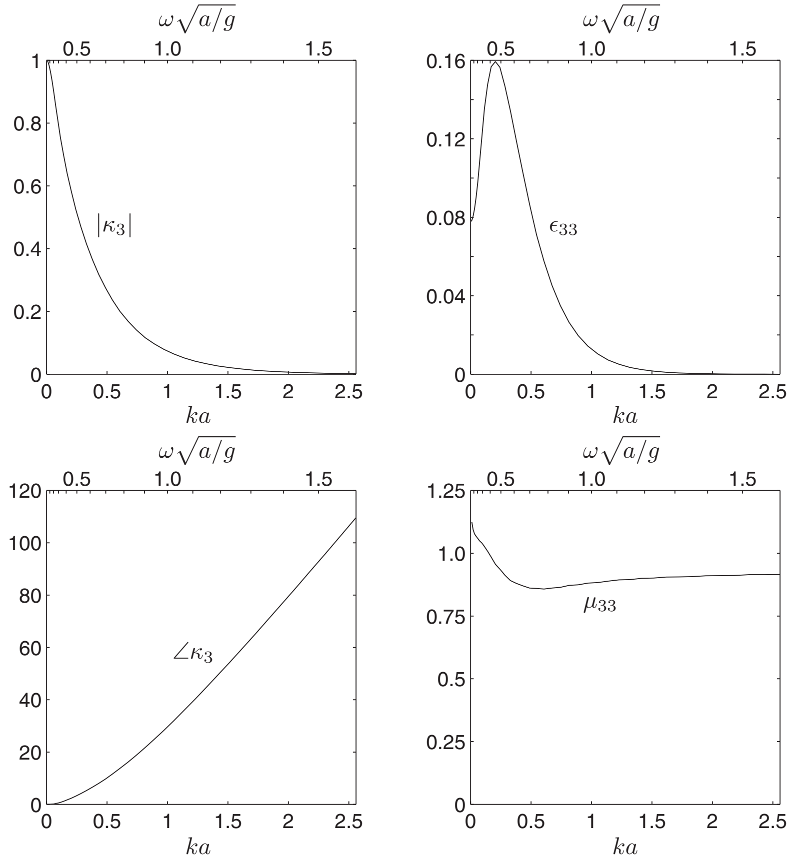

Figure 5.7: Non-dimensional hydrodynamic parameters for the heave mode of a floating truncated vertical cylinder of radius a and draft $b = 1.88a$ on water of depth $h = 15a$. In each graph, the horizontal scales are ka (lower scale) and $\omega\sqrt{a/g}$ (upper scale). The four graphs are, in dimensionless values, the amplitude $|\kappa_3|$ and phase $\angle\kappa_3$ (in degrees) of the excitation force, and the real and imaginary parts of the radiation impedance $\epsilon_{33} + i\mu_{33}$.

The final example for which numerical results are presented in this subsection is the International Ship and Offshore Structures Congress (ISSC) tension-leg platform (TLP) [49] in water of depth $h = 450$ m. The platform consists of four cylindrical columns (each with a radius of 8.435 m) and four pontoons of rectangular cross section (7.5 m wide and 10.5 m high) connecting the columns. The distance between the centres of adjacent columns is $2L = 86.25$ m. The draft of the TLP is 35.0 m, corresponding to the lowermost surface of the columns as well as of the pontoons. This is a large structure with a non-simple geometry. To numerically solve the boundary-value problems as given in Section 5.1.1 or in the first part of Section 4.2, we have applied the computer programme WAMIT [50]. The wet surface of the immersed TLP is approximated to 4,048 plane panels

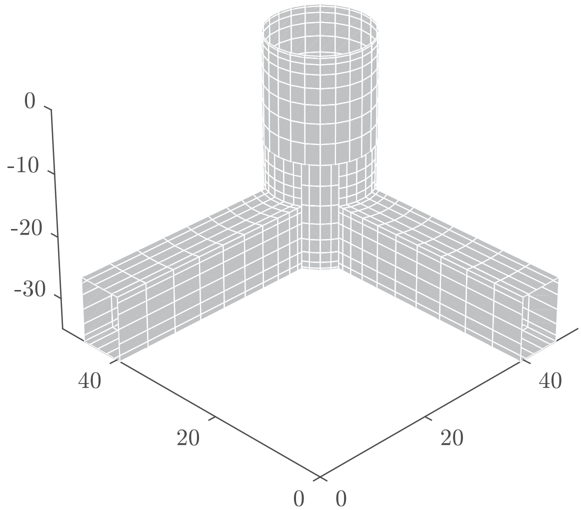

Figure 5.8: One quadrant of the ISSC TLP as approximated by 1,012 panels [50]. The scales on the indicated coordinate system are in metres.

(see Figure 5.8). The structure is symmetric with respect to the planes $x = 0$ and $y = 0$, and the width and length are equal. The computer programme WAMIT provides values for the fluid velocity, hydrodynamic pressure, body parameters (e.g., immersed volume, centre of gravity and hydrostatic coefficients), drift forces and motion response, in addition to the hydrodynamic parameters (i.e., excitation forces and moments, radiation resistance, added masses and inertia moments).

Dimensionless graphs based on the computed results for the hydrodynamic parameters of the TLP body are shown in Figure 5.9. The dimensionless hydrodynamic parameters $A_{jj'}$, $B_{jj'}$ and X_j are defined as follows from the radiation impedance and the excitation force:

$$Z_{jj'} = \omega\rho L^k(B_{jj'} + iA_{jj'}), \tag{5.88}$$

$$\hat{F}_{e,j} = f_j A = \rho g A L^m X_j, \tag{5.89}$$

where $m = 2$ for $j = 1,3$, while $m = 3$ for $j = 5$, and $k = 3$ for $(j,j') = (1,1)$ and $(j,j') = (3,3)$, while $k = 4$ for $(j,j') = (1,5)$ and $k = 5$ for $(j,j') = (5,5)$. The excitation force computed here applies for the case when the incident wave propagates in the x-direction, and A is the wave elevation at the origin $(x,y) = (0,0)$.

We observe that the curves in Figure 5.9 have a somewhat wavy nature. This is due to constructive and destructive wave interference associated with different parts of the TLP body. Let us consider, for instance, the surge mode. When the distance $2L$ between adjacent columns is an odd multiple of half wavelengths,

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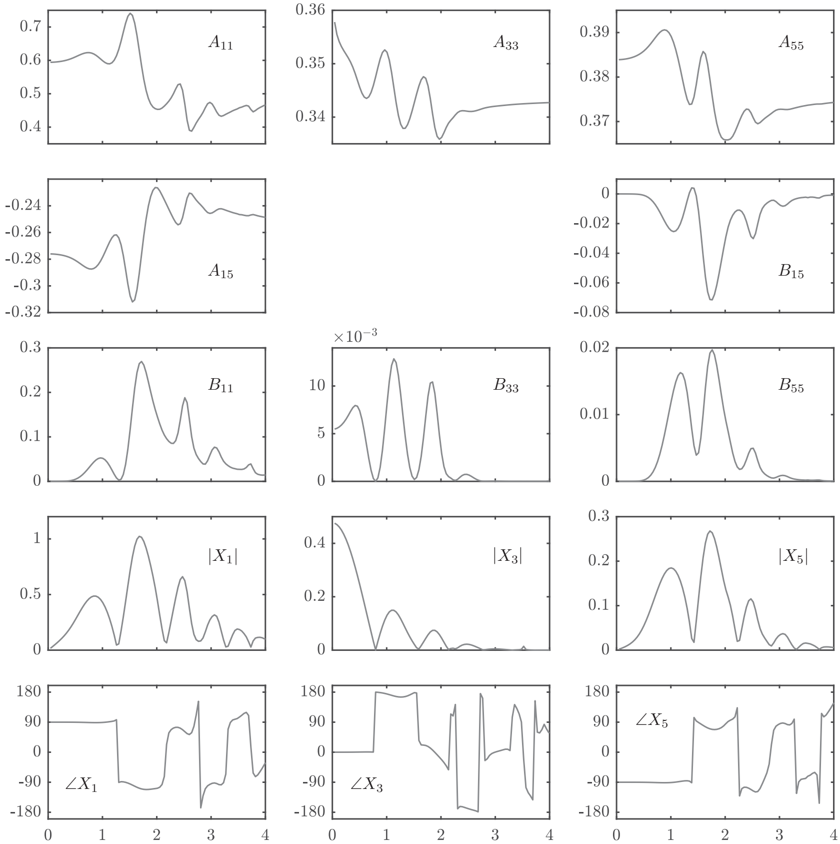

Figure 5.9: Hydrodynamic parameters for the ISSC TLP. The horizontal scale is the non-dimensional frequency $\Omega = \omega\sqrt{L/g}$, where $L = 43.125$ m is half of the distance between centres of adjacent TLP columns and $g = 9.81$ m/s^2. The eight graphs in the three upper rows give the radiation impedance [cf. Eq. (5.88)], and the six graphs in the two lower rows give the excitation force [cf. Eq. (5.89)] in dimensionless quantities for the modes surge ($j = 1$), heave ($j = 3$) and pitch ($j = 5$). In the lowest row, the phase is given in degrees.

corresponding to values of dimensionless frequency $\Omega = 1.25, 2.17, 2.80, \ldots,$ the surge excitation force contribution to one pair of columns is cancelled by the contribution from the remaining pair. Similarly, generated waves from the two pairs cancel each other. This explains why $|X_1|$ and B_{11} are so small at these frequencies. If, however, the distance $2L$ corresponds to a multiple of wavelengths, which happens for $\Omega = 1.77, 2.51, \ldots,$ then we expect constructive interference, which explains why $|X_1|$ and B_{11} are large at these frequencies. We see similar behaviour with the pitch mode. However, with the heave mode, maxima and minima for $|X_3|$ and B_{33} occur at other frequencies. Graphs similar to those shown in Figure 5.9 have been published for a TLP body with six columns [51]. They show, as we would expect, even more interference effects than disclosed by the wavy nature of the curves in Figure 5.9 for a four-column TLP.

A submerged body of the pontoon's cross section and of total length $8L$ displaces a water mass of 0.339 (in normalised units). It is interesting to note that this value is rather close to the added-mass value A_{33} for heave, as given in Figure 5.9. A deeply submerged horizontal cylinder of infinite length has an added mass for heave, as well as for surge, that equals the mass of the displaced water. If, however, the cylinder is only slightly below the water surface, the added mass may depend strongly on frequency, and in some cases, it may even be negative. Numerical results for the added mass and radiation resistance of a submerged horizontal cylinder, as well as a submerged vertical cylinder, have been published by McIver and Evans [23].

5.3 Impulse Response Functions in Hydrodynamics

We have discussed two kinds of wave–body interaction in the frequency domain. First (in Section 5.1), we introduced the excitation force—that is, the wave force on the body when the body is held fixed ($u_j = 0$ for $j = 1, \ldots, 6$); see Eqs. (5.26) and (5.27). Then, in Section 5.2.1, we introduced the hydrodynamic radiation parameters: the radiation impedance $Z_{jj'}$—see Eq. (5.39)—and its real and imaginary parts—that is, the radiation resistance $R_{jj'}$ and the radiation reactance $X_{jj'}$, respectively. Also, the added mass $m_{jj'} = X_{jj'}/\omega$ was introduced; see Eq. (5.42).

To the two kinds of interaction there correspond two kinds of boundary-value problems, which we call the excitation problem and the radiation problem. They have different inhomogeneous boundary conditions on the wet body surface S, namely as given by Eqs. (5.27) and (5.10), respectively.

Following the introduction of generalised six-dimensional vectors (in Section 5.1), we may define a six-dimensional column vector $\hat{\mathbf{F}}_e$ for the excitation force's complex amplitude. Similarly, we have a vector $\hat{\mathbf{u}}$, which represents the complex amplitudes of the six components of the velocity of the oscillating body. Correspondingly, we denote the radiation impedance matrix by $\mathbf{Z} = \mathbf{Z}(\omega)$, which is of dimension 6×6. The reaction force due to the oscillating body's radiation is given by

$$\hat{\mathbf{F}}_r = -\mathbf{Z}\hat{\mathbf{u}} = -\mathbf{Z}(\omega)\hat{\mathbf{u}}, \tag{5.90}$$

which is an alternative way of writing Eq. (5.37). We may interpret $-\mathbf{Z}$ as the transfer function of a linear system, where $\hat{\mathbf{u}}$ is the input and $\hat{\mathbf{F}}_r$ the output. Similar to this definition of a linear system for the radiation problem, we may define the following linear system for the excitation problem:

$$\hat{\mathbf{F}}_e = \mathbf{f}(\omega)A. \tag{5.91}$$

Here the system's input is $A = \hat{\eta}_0(0,0)$, which is the complex elevation amplitude of the (undisturbed) incident wave at the origin $(x, y) = (0, 0)$. Further, the transfer function is the six-dimensional column vector $\mathbf{f}$, which we shall call the excitation-force coefficient vector.

In the following, let us generalise this concept to situations in which the oscillation need not be sinusoidal. Then, referring to Eqs. (2.133), (2.158) and (2.163), we have the Fourier transforms of the reaction force $\mathbf{F}_{r,t}(t)$, due to radiation, and the excitation force $\mathbf{F}_{e,t}(t)$:

$$\mathbf{F}_r(\omega) = -\mathbf{Z}(\omega)\mathbf{u}(\omega), \tag{5.92}$$

$$\mathbf{F}_e(\omega) = \mathbf{f}(\omega)A(\omega). \tag{5.93}$$

Here $\mathbf{u}(\omega)$ is the Fourier transform of $\mathbf{u}_t(t)$, and $A(\omega)$ is the Fourier transform of $a(t) \equiv \eta_0(0,0,t)$, the wave elevation of the undisturbed incident wave at the origin $(x,y) = (0,0)$. (A subscript t is used to denote the inverse Fourier transforms, which are functions of time.)

The inverse Fourier transforms of the transfer functions $\mathbf{f}(\omega)$ and $\mathbf{Z}(\omega)$ correspond to time-domain impulse response functions, introduced into ship hydrodynamics by Cummins [52]. Let us first discuss the causal linear system given by Eq. (5.92). Afterwards, we shall discuss the system given by Eq. (5.93); we shall see that this system may be noncausal.

5.3.1 The Kramers–Kronig Relations in Hydrodynamic Radiation

We consider a body which oscillates and thereby generates a radiated wave on otherwise calm water. Let the body's oscillation velocity (in the time domain) be given by $\mathbf{u}_t(t)$. The reaction force $\mathbf{F}_{r,t}(t)$ is given by the convolution product

$$\mathbf{F}_{r,t}(t) = -\mathbf{z}(t) * \mathbf{u}_t(t) \tag{5.94}$$

in the time domain. [See Eq. (2.126), and remember that convolution is commutative. Note, however, that the matrix multiplication implied here is not commutative.] The impulse response matrix $-\mathbf{z}(t)$ is the negative of the inverse Fourier transform of the radiation impedance

$$\mathbf{z}(t) = \mathcal{F}^{-1}\{\mathbf{Z}(\omega)\}. \tag{5.95}$$

Note that this system is causal; that is,

$$\mathbf{z}(t) = 0 \quad \text{for } t < 0, \tag{5.96}$$

which means that there is no output $\mathbf{F}_{r,t}(t)$ before a non-vanishing $\mathbf{u}_t(t)$ is applied as input. For this reason, when we compute the convolution integral in Eq. (5.94), we may apply Eq. (2.172) instead of the more general Eq. (2.126), where the range of integration is from $t = -\infty$ to $t = +\infty$. The fact that $\mathbf{z}(t)$ is a causal function also has some consequence for its Fourier transform

$$\mathbf{Z}(\omega) = \mathbf{R}(\omega) + i\omega\mathbf{m}(\omega), \tag{5.97}$$

namely that the radiation-resistance matrix $\mathbf{R}(\omega)$ and the added-mass matrix $\mathbf{m}(\omega)$ are related by Kramers–Kronig relations. Apparently, Kotik and Mangulis

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[53] were the first to apply these relations to a hydrodynamic problem. Later, some consequences of the relations were discussed by, for example, Greenhow [54]. Here let us derive the relations in the following way. First we note that $\mathbf{m}(\omega)$ does not, in general, vanish in the limit $\omega \to \infty$. We wish to remove this singularity by considering the following related transfer function:

$$\mathbf{H}(\omega) = \mathbf{m}(\omega) - \mathbf{m}(\infty) + \mathbf{R}(\omega)/i\omega. \tag{5.98}$$

Note that $\mathbf{R}(\omega) \to 0$ as $\omega \to \infty$. We may assume that $\mathbf{R}(\omega)$ tends sufficiently fast to zero as $\omega \to 0$ to make $\mathbf{H}(\omega)$ non-singular at $\omega = 0$. The Kramers–Kronig relations (2.197) and (2.199) give for the real and imaginary parts of $\mathbf{H}(\omega)$

$$\mathbf{m}(\omega) - \mathbf{m}(\infty) = -\frac{2}{\pi}\int_0^\infty \frac{\mathbf{R}(y)}{\omega^2 - y^2}\,dy, \tag{5.99}$$

$$\mathbf{R}(\omega) = \frac{2\omega^2}{\pi}\int_0^\infty \frac{\mathbf{m}(y) - \mathbf{m}(\infty)}{\omega^2 - y^2}\,dy. \tag{5.100}$$

By using Eqs. (5.92), (5.97) and (5.98), we find that the reaction force due to radiation is given by

$$\mathbf{F}_r(\omega) = -\mathbf{Z}(\omega)\mathbf{u}(\omega) = \mathbf{F}'_r(\omega) - i\omega\mathbf{m}(\infty)\mathbf{u}(\omega), \tag{5.101}$$

where

$$\mathbf{F}'_r(\omega) = -i\omega\mathbf{H}(\omega)\mathbf{u}(\omega) = -\mathbf{K}(\omega)\mathbf{u}(\omega). \tag{5.102}$$

Here we have introduced an alternative transfer function

$$\begin{aligned}\mathbf{K}(\omega) = i\omega\mathbf{H}(\omega) &= \mathbf{Z}(\omega) - i\omega\mathbf{m}(\infty)\\ &= \mathbf{R}(\omega) + i\omega[\mathbf{m}(\omega) - \mathbf{m}(\infty)].\end{aligned} \tag{5.103}$$

The corresponding inverse Fourier transforms are

$$\mathbf{F}_{r,t}(t) = \mathbf{F}'_{r,t}(t) - \mathbf{m}(\infty)\dot{\mathbf{u}}_t(t), \tag{5.104}$$

with

$$\mathbf{F}'_{r,t}(t) = -\mathbf{h}(t) * \dot{\mathbf{u}}_t(t) = -\mathbf{k}(t) * \mathbf{u}_t(t), \tag{5.105}$$

where

$$\mathbf{h}(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{H}(\omega)e^{i\omega t}\,dt, \tag{5.106}$$

$$\mathbf{k}(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{K}(\omega)e^{i\omega t}\,dt. \tag{5.107}$$

Note that

$$\mathbf{k}(t) = \frac{d}{dt}\mathbf{h}(t). \tag{5.108}$$

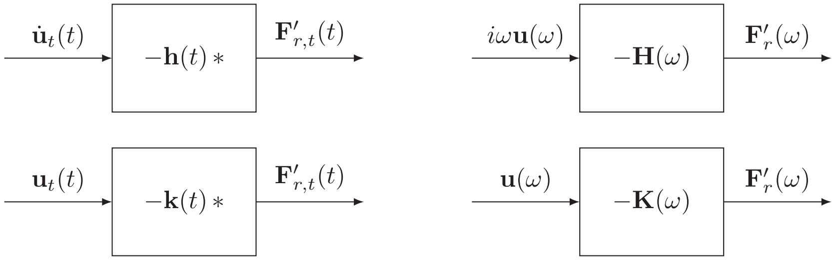

Figure 5.10: Block diagrams of linear systems in the time (left) and frequency (right) domains, where the acceleration (upper) or the velocity (lower) vectors of the oscillating body are considered as input and where a certain part [cf. Eqs. (5.98) and (5.102)] of the radiation reaction force vector is, in both cases, considered as the output.

The linear systems corresponding to the two alternative transfer functions $\mathbf{H}(\omega)$ and $\mathbf{K}(\omega)$—or, equivalently, to the convolutions in Eq. (5.105)—are represented by block diagrams in Figure 5.10.

Because of the condition of causality, we have $\mathbf{h}(t) = 0$ and $\mathbf{k}(t) = 0$ for $t < 0$. Moreover, using Eqs. (2.178), (2.179), (5.98) and (5.103), we find for $t > 0$

$$\begin{aligned}\mathbf{h}(t) &= \frac{2}{\pi}\int_0^\infty [\mathbf{m}(\omega) - \mathbf{m}(\infty)]\cos(\omega t)\,d\omega \\ &= \frac{2}{\pi}\int_0^\infty \frac{\mathbf{R}(\omega)}{\omega}\sin(\omega t)\,d\omega,\end{aligned} \tag{5.109}$$

$$\begin{aligned}\mathbf{k}(t) &= \frac{2}{\pi}\int_0^\infty \mathbf{R}(\omega)\cos(\omega t)\,d\omega = 2\mathcal{F}^{-1}\{\mathbf{R}(\omega)\} \\ &= -\frac{2}{\pi}\int_0^\infty \omega[\mathbf{m}(\omega) - \mathbf{m}(\infty)]\sin(\omega t)\,d\omega.\end{aligned} \tag{5.110}$$

Note that as a consequence of the principle of causality, all information contained in the two real matrix functions $\mathbf{R}(\omega)$ and $\mathbf{m}(\omega)$ is contained alternatively in the single real matrix function $\mathbf{k}(t)$ together with the constant matrix $\mathbf{m}(\infty)$. A function of the type $\mathbf{k}(t)$ has been applied in a time-domain analysis of a heaving body—for instance, by Count and Jefferys [55].

As an example, let us consider the heave mode of the floating truncated vertical cylinder for which the radiation resistance was given in Figure 5.7. The corresponding impulse response function $k_3(t)$ for radiation is given by the curve in the left-hand graph of Figure 5.11.

5.3.2 Noncausal Impulse Response for the Excitation Force

The impulse responses corresponding to the radiation impedance $\mathbf{Z}(\omega)$ and the related transfer functions $\mathbf{H}(\omega)$ and $\mathbf{K}(\omega)$, as described earlier, are causal because their inputs (velocity or acceleration of the oscillating body) are the actual causes of their responses (reaction forces from the radiated wave which is generated by the oscillating body).

In contrast, the impulse response functions related to the excitation forces are not necessarily causal, since the output excitation force, as well as the input

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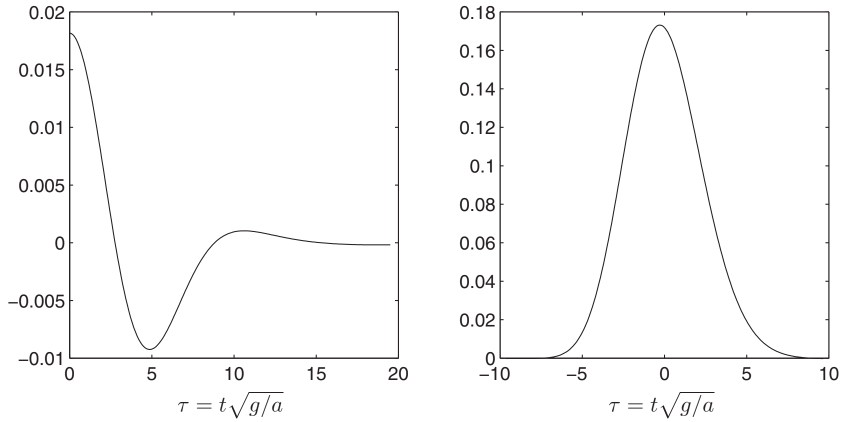

Figure 5.11: Impulse-response functions for the heave mode of a floating truncated vertical cylinder of radius a and draft $b = 1.88a$ on water of depth $h = 15a$. The scales are dimensionless. The dimensionless time on the horizontal scale is $t\sqrt{g/a}$. The curve in the left graph gives the dimensionless impulse-response function $k_3/(\pi\rho g a^2)$ for the radiation problem. The curve in the right graph gives the dimensionless impulse-response function $f_{t,3}/[\pi\rho(ga)^{3/2}]$ for the excitation force [cf. Eqs. (5.110) and (5.112)].

incident wave elevation at the origin (or at some other specified reference point), has a distant primary cause such as a storm or an oscillating wavemaker. The linear system, whose output is the excitation force, has the undisturbed incident wave elevation at the origin of the body as input. However, this wave may hit part of the body and exert a force before the arrival of the wave at the origin [56]. Note, however, that this is not sufficient [37] for a complete explanation of the noncausality of the impulse response function $\mathbf{f}_t(t)$ defined as follows.

The linear system associated with the excitation problem is represented by Eq. (5.93) or its inverse Fourier transform

$$\begin{aligned}\mathbf{F}_{e,t}(t) &= \mathbf{f}_t(t) * a(t)\\ &= \int_{-\infty}^{\infty} \mathbf{f}_t(t-\tau)a(\tau)\,d\tau = \int_{-\infty}^{\infty} \mathbf{f}_t(\tau)a(t-\tau)\,d\tau,\end{aligned} \tag{5.111}$$

where

$$\mathbf{f}_t(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{f}(\omega)e^{i\omega t}\,d\omega \tag{5.112}$$

is the inverse Fourier transform of the excitation-force-coefficient vector $\mathbf{f}(\omega)$, and where $a(t)$ is the wave elevation due to the (undisturbed) incident wave at the reference point $(x, y) = (x_0, y_0)$ of the body. We may choose this to be the origin—that is, $(x_0, y_0) = (0, 0)$. Note that, as a result of the noncausality of $\mathbf{f}_t(t)$, we could apply neither Eq. (2.178) nor Eq. (2.179) to compute the inverse Fourier transform in Eq. (5.112). Rather, we had to apply Eq. (2.167). When $\mathbf{f}_t(t) \neq 0$ for $t < 0$, it appears from Eq. (5.111) that in order to compute $\mathbf{F}_{e,t}(t)$, future information is required on the wave elevation $a(t)$. For this reason, it may

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be desirable to predict the incident wave a certain time length t_1 into the future, where t_1 is sufficiently large to make $\mathbf{f}_t(t)$ negligible for $t < -t_1$.

The statement that $\mathbf{f}_t(t)$ is not, in general, causal is easy to demonstrate by considering the example of Figure 5.4, where the surge excitation-force coefficient per unit width is

$$f_1'(\omega) = \hat{F}'_{e,1}/A = 2\rho(\omega/k)^2, \tag{5.113}$$

according to Eq. (5.33). Note that k is a function of ω as determined through the dispersion relationship (4.54). In this case, $f_1'(\omega)$ is real and an even function of ω. Hence, the inverse Fourier transform $f_{t,1}(t)$ is an even, and hence noncausal, function of time. Let us, for mathematical simplicity, consider the surge excitation force on an infinitesimal horizontal strip, corresponding to $z = -a_1$ and $z + \Delta z = -a_2$ in Figure 5.4, when the water is deep. Then with $k = \omega^2/g$ and $e(kz) = e^{kz}$ [see Eqs. (4.52) and (4.55)], we have, from Eq. (5.31),

$$\Delta f_1'(\omega) = \Delta F'_{e,1}/A = 2\rho g e^{\omega^2 z/g}\Delta z. \tag{5.114}$$

By taking the inverse Fourier transform [38, formula (1.4.11)], we have the corresponding impulse response

$$\Delta f'_{t,1}(t) = \rho g(-g/\pi z)^{1/2} \exp(gt^2/4z)\Delta z, \tag{5.115}$$

which is evidently noncausal, except for the case of $z = 0$, for which

$$\Delta f'_{t,1}(t) = 2\rho g\delta(t)\Delta z \tag{5.116}$$

(see Section 4.9.2 and Problem 5.8).

Taking the inverse Fourier transform of the heave excitation force given in Figure 5.7 for a floating truncated vertical cylinder, we find for this example the excitation impulse-response function $f_{3,t}$ as shown in the right-hand graph of Figure 5.11. Looking at the curves in this figure, we may say that the causal radiation impulse response 'remembers' roughly 15 to 20 $\sqrt{a/g}$ time units back into the past. In contrast, the noncausal excitation impulse 'remembers' only approximately eight $\sqrt{a/g}$ time units back into the past, but it also 'requires' information on the incident wave elevation approximately seven $\sqrt{a/g}$ time units into the future.

5.4 Reciprocity Relations

In this section, let us state and prove some relations (so-called reciprocity relations) which may be very useful when we study the interaction between a wave and an oscillating body.

We have already proven Eq. (5.41), which states that the radiation impedance is symmetric, $Z_{jj'} = Z_{j'j}$ or, in the matrix notation,

$$\mathbf{Z} = \mathbf{Z}^{\mathrm{T}}, \tag{5.117}$$

where $\mathbf{Z}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{Z}$. It follows that also the radiation resistance matrix and the added mass matrix are symmetric:

$$\mathbf{R} = \mathbf{R}^{\mathrm{T}}, \quad \mathbf{m} = \mathbf{m}^{\mathrm{T}}. \tag{5.118}$$

Reciprocity relations of this type are well known in several branches of physics. An analogy is, for instance, the symmetry of the electric impedance matrix for a linear electric network (see, e.g., section 2.13 in Goldman [57]).

Another analogy, from the subject of statics, is the relationship between a set of forces applied to a body of a linear elastic material and the resulting linear displacements of the points of force attack, where each displacement is in the same direction as the corresponding force. In this case, a reciprocity relation may be derived by considering a force F applied statically to two different points a and b. Applied to point a, the displacement is δ_{aa} at point a and δ_{ba} at point b, and the work done is $(F/2)\delta_{aa}$. If applied to point b, the displacement is δ_{ab} at point a and δ_{bb} at point b, and the work done is $(F/2)\delta_{bb}$. Further, if, in addition, an equally large force F is afterwards applied to point a, the displacements are $\delta_{ab} + \delta_{aa}$ at point a and $\delta_{bb} + \delta_{ba}$ at point b, since the principle of superposition is applicable for the linear system. The work done is

$$W_{ba} = (F/2)\delta_{bb} + (F/2)\delta_{aa} + F\delta_{ba}. \tag{5.119}$$

However, if a force F is first applied to point a and then a second force F applied to point b, the work would be

$$W_{ab} = (F/2)\delta_{aa} + (F/2)\delta_{bb} + F\delta_{ab}. \tag{5.120}$$

The work done is stored as elastic energy in the material, and this energy must be independent of the order of applying the two forces—that is, $W_{ba} = W_{ab}$. Hence,

$$\delta_{ab} = \delta_{ba}, \tag{5.121}$$

which proves the following reciprocity relation from the subject of statics: The displacement at point b due to a force at point a equals the displacement at point a if the same force is applied to point b (see, e.g., p. 423 in Timoshenko and Gere [58]).

Several kinds of reciprocity relations relating to hydrodynamics have been formulated and proven by Newman [34]. Some of these relations will be considered in the following.

5.4.1 Radiation Resistance in Terms of Far-Field Coefficients

The complex amplitude of the velocity potential of the wave radiated from an oscillating body may, according to Eq. (5.9), be written as

$$\hat{\phi}_r(x, y, z) = \boldsymbol{\varphi}^{\mathrm{T}}(x, y, z)\hat{\mathbf{u}} = \hat{\mathbf{u}}^{\mathrm{T}}\boldsymbol{\varphi}(x, y, z), \tag{5.122}$$

where we have introduced the column vectors $\boldsymbol{\varphi}(x,y,z)$ and $\hat{\mathbf{u}}$, which are six-dimensional if the wave is radiated from a single body oscillating in all its six degrees of freedom [see Eqs. (5.3)–(5.4)]. Note that, because $\hat{\phi}_r$ satisfies the radiation condition of an outgoing wave at infinity, we have [see Eq. (4.212) or Eq. (4.256)] the far-field approximations

$$\hat{\phi}_r \sim A_r(\theta)e(kz)(kr)^{-1/2}e^{-ikr}, \tag{5.123}$$

$$\boldsymbol{\varphi} \sim \mathbf{a}(\theta)e(kz)(kr)^{-1/2}e^{-ikr}, \tag{5.124}$$

where

$$A_r(\theta) = \mathbf{a}^{\mathrm{T}}(\theta)\hat{\mathbf{u}} = \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{a}(\theta) \tag{5.125}$$

is the far-field coefficient for the radiated wave. Note that each component φ_j of the column vector satisfies the radiation condition and that $a_j(\theta)$ is the far-field coefficient for this component. Using relation (4.272), we have the following Kochin functions for the radiated wave:

$$\begin{bmatrix} H_r(\theta) \\ \mathbf{h}(\theta) \end{bmatrix} = \sqrt{2\pi}\begin{bmatrix} A_r(\theta) \\ \mathbf{a}(\theta) \end{bmatrix} e^{i\pi/4}. \tag{5.126}$$

Here $\mathbf{h}(\theta)$ and $\mathbf{a}(\theta)$ are column vectors composed of the components $h_j(\theta)$ and $a_j(\theta)$, respectively.

As a further preparation for proof of reciprocity relations involving radiation parameters, let us introduce the following matrix version of Eq. (4.230):

$$\mathbf{I}(\boldsymbol{\varphi},\boldsymbol{\varphi}^{\mathrm{T}}) = \iint_S \left(\boldsymbol{\varphi}\frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial n} - \frac{\partial \boldsymbol{\varphi}}{\partial n}\boldsymbol{\varphi}^{\mathrm{T}}\right) dS. \tag{5.127}$$

Because $\boldsymbol{\varphi}$ satisfies the radiation condition, we may apply Eq. (4.239), which means that

$$\mathbf{I}(\boldsymbol{\varphi},\boldsymbol{\varphi}^{\mathrm{T}}) = 0. \tag{5.128}$$

However, the integral does not vanish when we take the complex conjugate of one of the two functions entering into the integral. Thus, the complex conjugate of Eqs. (4.244) and (4.245) may in matrix notation be rewritten as

$$\begin{aligned} \mathbf{I}(\boldsymbol{\varphi}^*,\boldsymbol{\varphi}^{\mathrm{T}}) &= -i\frac{D(kh)}{k}\int_0^{2\pi} \mathbf{a}^*(\theta)\mathbf{a}^{\mathrm{T}}(\theta)\, d\theta \\ &= -\lim_{r\to\infty} 2\iint_{S_\infty} \frac{\partial \boldsymbol{\varphi}^*}{\partial r}\boldsymbol{\varphi}^{\mathrm{T}}\, dS. \end{aligned} \tag{5.129}$$

Note that $\mathbf{I}(\boldsymbol{\varphi}^*,\boldsymbol{\varphi}^{\mathrm{T}})$, which should not be confused with the identity matrix $\mathbf{I}$, is a square matrix, which is of dimension 6×6 for the case of one (three-dimensional) single body oscillating in all its degrees of freedom.

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As Eq. (5.35) has been generalised to Eq. (5.122), it follows that expression (5.40) for the radiation impedance may be rewritten in matrix notation as

$$\mathbf{Z} = -i\omega\rho \iint_S \frac{\partial \boldsymbol{\varphi}^*}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}}\, dS. \tag{5.130}$$

The complex conjugate of this matrix is

$$\mathbf{Z}^* = i\omega\rho \iint_S \frac{\partial \boldsymbol{\varphi}}{\partial n} \boldsymbol{\varphi}^{\dagger}\, dS, \tag{5.131}$$

where $\boldsymbol{\varphi}^{\dagger} = (\boldsymbol{\varphi}^{\mathrm{T}})^*$ is the complex-conjugate transpose of $\boldsymbol{\varphi}$. Note, however, that $\mathbf{Z}^{\mathrm{T}} = \mathbf{Z}$ [see Eq. (5.117)], and hence,

$$\mathbf{Z}^* = \mathbf{Z}^{\dagger} = i\omega\rho \iint_S \boldsymbol{\varphi}^* \frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial n}\, dS. \tag{5.132}$$

The real part of $\mathbf{Z}$ is the radiation resistance matrix

$$\begin{aligned} \mathbf{R} &= \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^*) = \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^{\dagger}) \\ &= -\frac{i}{2}\omega\rho \iint_S \left(\frac{\partial \boldsymbol{\varphi}^*}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}} - \boldsymbol{\varphi}^* \frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial n} \right) dS = \frac{i}{2}\omega\rho\, \mathbf{I}(\boldsymbol{\varphi}^*, \boldsymbol{\varphi}^{\mathrm{T}}), \end{aligned} \tag{5.133}$$

where Eq. (5.127) has been used. (It has been assumed that ω is real.) Now, using Eq. (5.129) in Eq. (5.133), we have

$$\mathbf{R} = -i\omega\rho \lim_{r\to\infty} \iint_{S_\infty} \frac{\partial \boldsymbol{\varphi}^*}{\partial r} \boldsymbol{\varphi}^{\mathrm{T}}\, dS. \tag{5.134}$$

It is interesting to compare this with Eq. (5.130). If the integral is taken over S_∞ in the far field instead of over the wet surface S of the body, we arrive at the radiation resistance matrix instead of the radiation impedance matrix. This indicates that the radiation reactance and hence the added mass are somehow related to the near-field region of the wave-generating oscillating body.

The radiation resistance matrix may be expressed in terms of the far-field coefficient vector $\mathbf{a}(\theta)$ of the radiated wave. From Eqs. (5.126), (5.129) and (5.133), we obtain

$$\mathbf{R} = \frac{\omega\rho D(kh)}{2k} \int_0^{2\pi} \mathbf{a}^*(\theta)\, \mathbf{a}^{\mathrm{T}}(\theta)\, d\theta = \frac{\omega\rho D(kh)}{4\pi k} \int_0^{2\pi} \mathbf{h}^*(\beta)\, \mathbf{h}^{\mathrm{T}}(\beta)\, d\beta. \tag{5.135}$$

We have here been able to express the radiation resistance matrix in terms of far-field quantities of the radiated wave. The physical explanation of this is that the energy radiated from the oscillating body must be retrieved without loss in the far-field region of the assumed ideal fluid. It is possible to derive Eq. (5.135) simply by using this energy argument as a basis for the derivation (see Problem 5.7).

Whereas the radiation resistance is related to energy in the far-field region, the radiation reactance and, hence, the added mass are somehow related to the near-field region, as we remarked earlier when comparing Eqs. (5.130) and (5.134). We shall return to this matter in a more quantitative fashion in Section 5.5.4.

5.4.2 The Excitation Force: The Haskind Relation

Some important relations which express the excitation force in terms of radiation parameters are derived in the following. Using the wet-surface conditions (5.10) and (5.27), we may rewrite expression (5.26) for the excitation force as

$$\begin{aligned}\hat{F}_{e,j} &= i\omega\rho \iint_S \left[(\hat{\phi}_0 + \hat{\phi}_d)\frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial}{\partial n}(\hat{\phi}_0 + \hat{\phi}_d)\right] dS \\ &= i\omega\rho I(\hat{\phi}_0 + \hat{\phi}_d, \varphi_j) \\ &= i\omega\rho I(\hat{\phi}_0, \varphi_j) + i\omega\rho I(\hat{\phi}_d, \varphi_j),\end{aligned} \tag{5.136}$$

where integral (4.230) has been used. Further, since $\hat{\phi}_d$ and φ_j both satisfy the radiation condition, which means that $I(\hat{\phi}_d, \varphi_j) = 0$ according to Eq. (4.239), we obtain

$$\hat{F}_{e,j} = i\omega\rho I(\hat{\phi}_0, \varphi_j). \tag{5.137}$$

In order to calculate the excitation force from the formula (5.26), we need to know the diffraction potential at the wet surface S. Alternatively, the excitation force may be calculated from Eq. (5.137), where we need to know not the diffraction potential ϕ_d but the radiation potential φ_j, either at the wet surface S or at the control surface S_∞ in the far-field region [see Eq. (4.233)]. This latter method of calculating the excitation force was first pointed out by Haskind [59, 60] for the case of a single body. Equation (5.137) is one way of formulating the so-called Haskind relation.

Using the column vector $\boldsymbol{\varphi}$, which we introduced in Eq. (5.122), we may, as in Eq. (5.91), write the excitation force as a column vector

$$\hat{\mathbf{F}}_e = \mathbf{f}A = i\omega\rho \mathbf{I}(\hat{\phi}_0, \boldsymbol{\varphi}) \tag{5.138}$$

as an alternative to Eq. (5.137). We have also here introduced the excitation-force coefficient vector **f**.

In another version of the Haskind relation, the excitation force may also be expressed in terms of the radiated wave's far-field coefficients or its Kochin functions. We assume now that the incident plane wave is given as

$$\hat{\eta}_0 = A \exp\left[-ik(x\cos\beta + y\sin\beta)\right] = Ae^{-ikr(\beta)}. \tag{5.139}$$

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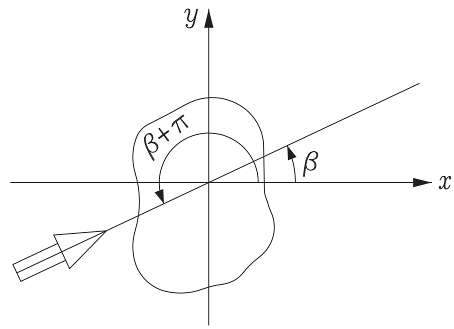

Figure 5.12: Excitation force experienced by an oscillating body due to a wave with angle of incidence β is related to the body's ability to radiate a wave in the opposite direction, $\theta = \beta + \pi$.

Here, β is the angle of wave incidence. See Eqs. (4.98) and (4.267). Note that the frequency-dependent excitation-force coefficient vector $\mathbf{f} = \hat{\mathbf{F}}_e/A$ is a function also of β—that is, $\mathbf{f} = \mathbf{f}(\beta)$. Applying now Eq. (4.271) with $\phi_i = \hat{\phi}_0$ (that is, $A_i = A$ and $\beta_i = \beta$) and with $\phi_j = \hat{\phi}_r$ (that is, $A_j = 0$), we have

$$I(\hat{\phi}_0, \hat{\phi}_r) = \frac{gD(kh)}{i\omega k} H_r(\beta \pm \pi) A. \tag{5.140}$$

Further, using this together with Eqs. (5.122), (5.125) and (5.126) in Eq. (5.138), we may write the excitation-force vector as

$$\hat{\mathbf{F}}_e = \mathbf{f}(\beta)A = i\omega\rho \mathbf{I}(\hat{\phi}_0, \boldsymbol{\varphi}) = \frac{\rho g D(kh)}{k} \mathbf{h}(\beta \pm \pi) A, \tag{5.141}$$

or for the excitation-force coefficient vector,

$$\mathbf{f}(\beta) = \frac{\rho g D(kh)}{k} \mathbf{h}(\beta \pm \pi) = \frac{\rho g D(kh)}{k} \sqrt{2\pi}\, \mathbf{a}(\beta \pm \pi) e^{i\pi/4}. \tag{5.142}$$

Reciprocity relation (5.142), which is another form of the Haskind relation, means that a body's ability to radiate a wave *into* a certain direction is related to the excitation force which the body experiences when a plane wave is incident *from* just that direction. See Figure 5.12. If we, in an experiment, measure the excitation-force coefficients for various directions of wave incidence β, we can also, by application of the reciprocity relation (5.142), obtain the corresponding far-field coefficients for waves radiated in the opposite directions.

5.4.3 Reciprocity Relation between Radiation Resistance and Excitation Force

By using the Haskind relation in the form (5.142) together with reciprocity relation (5.135) between the radiation resistance and the far-field coefficients, we find

$$\mathbf{R} = \frac{\omega k}{4\pi\rho g^2 D(kh)} \int_{-\pi}^{\pi} \mathbf{f}(\beta)\mathbf{f}^\dagger(\beta)\, d\beta = \frac{\omega k}{4\pi\rho g^2 D(kh)} \int_{-\pi}^{\pi} \mathbf{f}^*(\beta)\mathbf{f}^{\mathrm{T}}(\beta)\, d\beta. \tag{5.143}$$

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Here we have also utilised the fact that $\mathbf{R}$ is a symmetrical matrix; that is, it equals its own transpose. Remembering that $\mathbf{f}(\beta) = \hat{\mathbf{F}}_e(\beta)/A$, we may rewrite Eq. (5.143) as

$$\mathbf{R} = \frac{\omega k}{4\pi\rho g^2 D(kh)|A|^2} \int_{-\pi}^{\pi} \hat{\mathbf{F}}_e(\beta)\, \hat{\mathbf{F}}_e^{\dagger}(\beta)\, d\beta. \tag{5.144}$$

Introducing the wave-energy transport J per unit frontage [see Eq. (4.130)], we may also write the radiation resistance matrix as

$$\mathbf{R} = \frac{k}{16\pi J} \int_{-\pi}^{\pi} \hat{\mathbf{F}}_e(\beta)\, \hat{\mathbf{F}}_e^{\dagger}(\beta)\, d\beta. \tag{5.145}$$

If the body is axisymmetric, the heave component $F_{e,3}$ of the excitation force is independent of β (the angle of wave incidence), and we have

$$R_{33} = \frac{k}{8J} |\hat{F}_{e,3}|^2. \tag{5.146}$$

The coefficient in front of the integral in Eqs. (5.144) and (5.145) may also be written as $k/(8\pi\rho g v_g |A|^2)$ if Eq. (4.106) for the group velocity v_g is used.

5.4.4 Reciprocity Relation between Diffraction and Radiation Kochin Functions

A relation which may prove useful when considering wave interactions in the far field relates the diffraction and radiation Kochin functions. The starting point is the integral $I(\hat{\phi}_0 + \hat{\phi}_d, \varphi_j^*)$, which is a special case of integral (4.292):

$$I(\hat{\phi}_0 + \hat{\phi}_d, \varphi_j^*) = \iint_S \left[(\hat{\phi}_0 + \hat{\phi}_d) \frac{\partial \varphi_j^*}{\partial n} - \varphi_j^* \frac{\partial}{\partial n} (\hat{\phi}_0 + \hat{\phi}_d) \right] dS. \tag{5.147}$$

Before we proceed to evaluate this integral, observe that the body boundary condition (5.10) requires that $\partial\varphi_j/\partial n = n_j$ on S, where n_j is the j-component of $\mathbf{n}$. Since n_j is real, we have $\partial\varphi_j^*/\partial n = \partial\varphi_j/\partial n$ on S. Also, $\partial(\hat{\phi}_0 + \hat{\phi}_d)/\partial n = 0$ on S, due to boundary condition (5.27). Thus, we have the following identity:

$$I(\hat{\phi}_0 + \hat{\phi}_d, \varphi_j^*) = I(\hat{\phi}_0 + \hat{\phi}_d, \varphi_j) = I(\hat{\phi}_0, \varphi_j), \tag{5.148}$$

where the last equality follows from the fact that $I(\hat{\phi}_d, \varphi_j) = 0$ because $\hat{\phi}_d$ and φ_j satisfy the same radiation condition [cf. Eq. (4.239)].

We may express both sides of the identity in terms of Kochin functions. For the right-hand side, we have, from Eqs. (4.265)–(4.271),

$$I(\hat{\phi}_0, \varphi_j) = \frac{2 v_p v_g A}{i\omega} h_j(\beta \pm \pi). \tag{5.149}$$

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Likewise, for the left-hand side, we have, from Eqs. (4.288) and (4.292)–(4.295),

$$\begin{aligned} I(\hat{\phi}_0 + \hat{\phi}_d, \varphi_j^*) &= I(\hat{\phi}_0, \varphi_j^*) + I(\hat{\phi}_d, \varphi_j^*) \\ &= \frac{2v_p v_g A}{i\omega} h_j^*(\beta) + \frac{iv_p v_g}{\pi g} \int_0^{2\pi} H_d(\theta) h_j^*(\theta)\, d\theta. \end{aligned} \tag{5.150}$$

Equating the two sides yields

$$h_j(\beta \pm \pi) = h_j^*(\beta) - \frac{\omega}{2\pi g} \int_0^{2\pi} \frac{H_d(\theta)}{A} h_j^*(\theta)\, d\theta. \tag{5.151}$$

Multiplying by u_j^* and taking the sum, we have

$$\bar{H}_r(\beta \pm \pi) = H_r^*(\beta) - \frac{\omega}{2\pi g} \int_0^{2\pi} \frac{H_d(\theta)}{A} H_r^*(\theta)\, d\theta, \tag{5.152}$$

where

$$\bar{H}_r(\theta) \equiv \sum_j h_j(\theta) u_j^*. \tag{5.153}$$

5.5 Several Bodies Interacting with Waves

So far in this chapter, we have studied the interaction between waves and a body which may oscillate in six independent modes, as introduced in Section 5.1. There are two different kinds of interaction, as represented by two hydrodynamic parameters: the excitation force and the radiation impedance.

Let us now consider the interaction between waves and an arbitrary number of oscillating bodies, partly or totally submerged in water (see Figure 5.13). A systematic study of many hydrodynamically interacting bodies was made independently by Evans [61] and Falnes [25], following Budal's analysis [62] of wave-energy absorption by such a system of several bodies.

First, in the following subsection, we shall introduce the excitation force and the radiation impedance in a phenomenological way. Later, we shall relate these phenomenologically defined parameters with hydrodynamic theory.

In general, each oscillating body has six degrees of freedom: three translatory modes and three rotary modes. We chose a coordinate system with the z-axis pointing upwards and with the plane $z = 0$ coinciding with the mean free water surface. We consider each body as corresponding to six *oscillators*. Thus, if the number of oscillating bodies is N, there are $6N$ oscillators. Parameters pertaining to a particular oscillator are denoted by a subscript

$$i = 6(p - 1) + j. \tag{5.154}$$

Here, p is the number of the body, and $j = 1, 2, \ldots, 6$ is its mode number.

With an assumed angular frequency ω, the state of an oscillator is given by amplitude and phase, two real quantities which are conveniently incorporated

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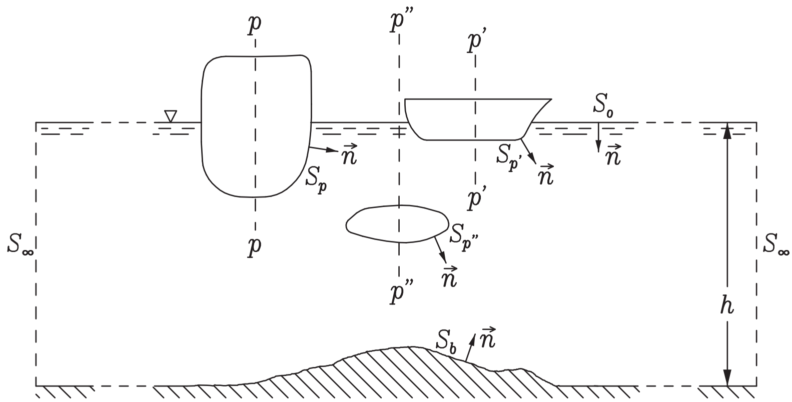

Figure 5.13: System of interacting oscillating bodies contained within an imaginary cylindrical control surface S_∞ at water depth h. Vertical lines through (the average position of) the centre of gravity of the bodies are indicated by p–p, p'–p' and p''–p''. Wet surfaces of oscillating bodies are indicated by S_p, $S_{p'}$ and $S_{p''}$. Fixed surfaces, including the sea bed, are given by S_b, while S_0 denotes the external free water surface. The arrows indicate unit normals pointing into the fluid region.

into a single complex quantity, the complex amplitude. We specify the state of oscillator number i by a complex velocity amplitude $\hat{u}_i$ or, alternatively, by $\hat{u}_{pj}$, which is a translatory velocity for the modes surge, sway and heave ($j = 1, 2, 3$), or an angular velocity for the modes roll, pitch and yaw ($j = 4, 5, 6$). See Section 5.1.1.

5.5.1 Phenomenological Discussion

When no oscillators are moving, an excitation force, represented by its complex amplitude $\hat{F}_{e,i}$, acts on oscillator number i. This excitation force is caused by an incoming wave and includes diffraction effects due to the fixed bodies. When oscillator number i' oscillates with a complex velocity amplitude $\hat{u}_{i'}$, it radiates a wave acting on oscillator i with an additional force, which has a complex amplitude $-Z_{ii'}\hat{u}_{i'}$. The complex coefficient $-Z_{ii'}$ is a factor of proportionality, and it depends on ω and on the geometry of the problem. Based on the principle of superposition, the total force acting on oscillator number i is given as

$$\hat{F}_{t,i} = \hat{F}_{e,i} - \sum_{i'} Z_{ii'}\hat{u}_{i'} = f_i A - \sum_{i'} Z_{ii'}\hat{u}_{i'}, \tag{5.155}$$

where the sum is taken over all oscillators, including oscillator number i.

The set of complex amplitudes of the total force, the excitation force and the oscillator velocity may be assembled into column vectors, $\hat{\mathbf{F}}_t$, $\hat{\mathbf{F}}_e$ and $\hat{\mathbf{u}}$, respectively. For all oscillators, the set of equations (5.155) may be written in matrix form as

$$\hat{\mathbf{F}}_t = \hat{\mathbf{F}}_e - \mathbf{Z}\hat{\mathbf{u}} = \mathbf{f}A - \mathbf{Z}\hat{\mathbf{u}}. \tag{5.156}$$

Here $\mathbf{Z}$ is a square matrix composed of elements $Z_{ii'}$.

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This complex matrix $\mathbf{Z}$ may be decomposed into its real part $\mathbf{R}$, the radiation resistance matrix, and its imaginary part $\mathbf{X}$, the radiation reactance matrix. Thus,

$$\mathbf{Z} = \mathbf{R} + i\mathbf{X} = \mathbf{R} + i\omega\mathbf{m}. \tag{5.157}$$

Here ω is the angular frequency, and $\mathbf{m}$ is the so-called hydrodynamic added-mass matrix.

In general, $\mathbf{Z}$ is a $6N{\times}6N$ matrix. It may be partitioned into $6{\times}6$ matrices $\mathbf{Z}_{pp'}$, defined as follows. Let us assume that body number p' is oscillating with complex velocity amplitudes, as given by the generalised six-dimensional vector $\hat{\mathbf{u}}_{p'}$. This results in a contribution $-\mathbf{Z}_{pp'}\hat{\mathbf{u}}_{p'}$ to the generalised six-dimensional vector, which represents the complex amplitudes of the radiation force on body number p. For the particular case of $p' = p$, $\mathbf{Z}_{pp'} = \mathbf{Z}_{pp}$ represents the ordinary radiation impedance matrix for body number p, but modified due to the presence of all the other bodies. The radiation impedance matrix for the whole system of the N oscillating bodies is, thus, a partitioned matrix of the type

$$\mathbf{Z} = \begin{bmatrix} \ddots & \vdots & & \vdots & \\ \cdots & \mathbf{Z}_{pp} & \cdots & \mathbf{Z}_{pp'} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & \mathbf{Z}_{p'p} & \cdots & \mathbf{Z}_{p'p'} & \cdots \\ & \vdots & & \vdots & \ddots \end{bmatrix}. \tag{5.158}$$

In this way, the matrix $\mathbf{Z}$ is composed of N^2 matrices.

If $p' \neq p$, and if the distance $d_{pp'}$ between the two bodies' vertical reference axes p–p and p'–p' (see Figure 5.13) is sufficiently large, we expect that matrix $\mathbf{Z}_{pp'}$ depends on the distance $d_{pp'}$ approximately as

$$\mathbf{Z}_{pp'} \sim \left(\mathbf{Z}_{pp'}\right)_0 \sqrt{d_0/d_{pp'}}\, \exp(-ikd_{pp'}), \tag{5.159}$$

where $\left(\mathbf{Z}_{pp'}\right)_0$ is independent of $d_{pp'}$. Compare this with approximation (4.212) or (4.236), and remember that in linear theory, the wave force contribution on a body is proportional to the corresponding wave hitting the body. We expect that approximation (5.159) might be useful when the spacing $d_{pp'}$ is large in comparison with the wavelength and with the largest horizontal extension (diameter) of bodies p and p'.

5.5.2 Hydrodynamic Formulation

In general, the velocity potential is composed of three main contributions:

$$\phi = \phi_0 + \phi_d + \phi_r. \tag{5.160}$$

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Here ϕ_0 represents the given incident wave, which results in a diffracted wave ϕ_d when all bodies are fixed. If the bodies are oscillating, then, additionally, a radiated wave ϕ_r is set up.

Note that, in general, ϕ_0 is the given velocity potential in case all the N bodies are absent. It could, for instance, represent a plane wave with complex amplitude, as given by Eqs. (4.88) and (4.98), or alternatively by Eq. (4.80). In the former case, it is a propagating incident wave. In the latter case, it is a plane wave which includes also a wave which is partly reflected from, for example, a straight coast line. If the coast topography is more irregular, the mathematical description of ϕ_0 would be more complicated. The term ϕ_d in Eq. (5.160) thus represents diffraction due to the immersed N bodies only.

The velocity potential associated with the radiated wave has a complex amplitude

$$\hat{\phi}_r = \sum_i \varphi_i \hat{u}_i, \tag{5.161}$$

where the sum is taken over all oscillators. The complex coefficient φ_i represents the velocity potential resulting from a unit velocity amplitude of oscillator number i, when all other oscillators are not moving. For one single body, we introduced the six-dimensional column vector $\boldsymbol{\varphi}$ and its transpose $\boldsymbol{\varphi}^{\mathrm{T}}$ in connection with Eqs. (5.122) and (5.127). If we extend the dimension to $6N$, the vector includes all φ_i as its components. Then we may write Eqs. (5.161) as

$$\hat{\phi}_r = \boldsymbol{\varphi}^{\mathrm{T}}\hat{\mathbf{u}} = \hat{\mathbf{u}}^{\mathrm{T}}\boldsymbol{\varphi}. \tag{5.162}$$

For each of the bodies, we may define corresponding six-dimensional column vectors $\boldsymbol{\varphi}_p$ and $\mathbf{u}_p$. Then we may write

$$\hat{\phi}_r = \sum_{p=1}^{N} \boldsymbol{\varphi}_p^{\mathrm{T}}\hat{\mathbf{u}}_p = \sum_{p=1}^{N} \hat{\mathbf{u}}_p^{\mathrm{T}}\boldsymbol{\varphi}_p. \tag{5.163}$$

At large horizontal distance r_p from the reference axis p–p of body number p, we have, according to Eq. (4.255), an asymptotic expression

$$\boldsymbol{\varphi}_p \sim \mathbf{b}_p(\theta_p)e(kz)(kr_p)^{-1/2}\exp\{-ikr_p\}. \tag{5.164}$$

Here $\mathbf{b}_p(\theta_p)$ is a six-dimensional vector composed of body number p's six far-field coefficients referred to the local origin (the point where the axis p–p crosses the plane of the mean water surface). See also Figure 4.9 (with subscript p instead of i) for a definition of the local coordinates r_p and θ_p.

All terms in Eq. (5.160) satisfy the Laplace equation (4.14) in the fluid domain and the usual homogeneous boundary conditions (4.27) and (4.16) or (4.17) on the free surface, $z = 0$, and on the sea bed. The radiation condition of outgoing waves at infinity has to be satisfied for the velocity potentials ϕ_d, ϕ_r

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and all φ_i. The inhomogeneous boundary condition (4.15) on the wet surfaces S_p of the oscillating bodies is satisfied, if

$$\frac{\partial}{\partial n}(\hat{\phi}_0 + \hat{\phi}_d) = 0 \quad \text{on all } S_p, \tag{5.165}$$

$$\frac{\partial \varphi_i}{\partial n} = \begin{cases} n_{pj} & \text{on } S_p \\ 0 & \text{on } S_{p'} (p' \neq p) \end{cases}, \tag{5.166}$$

where $\partial/\partial n$ is the normal derivative in the direction of the outward unit normal $\vec{n}$ to the surface of the body. In accordance with Eq. (5.154), n_{pj} is defined as the x, y or z component of $\vec{n}$ when $j = 1, 2$ or 3, respectively. Furthermore, n_{pj} is the x, y or z component of vector $\vec{s}_p \times \vec{n}$ when $j = 4, 5$ or 6, respectively. Here $\vec{s}_p$ is the position vector referred to a selected point on the reference axis p–p of body number p—for instance, the centre of gravity or the centre of buoyancy. Note that Eqs. (5.165) and (5.166) represent a generalisation of the corresponding boundary conditions (5.10) and (5.27) for the single body.

5.5.3 Radiation Impedance and Radiation Resistance Matrices

According to Eq. (5.21), the force on oscillator number i due to the potential $\varphi_{i'}$ (that is, due to a unit velocity amplitude of oscillator number i') is

$$-Z_{ii'} = -\iint_{S_p} p_{i'} n_{pj} \, dS_p = i\omega\rho \iint_S \varphi_{i'} \frac{\partial \varphi_i}{\partial n} dS, \tag{5.167}$$

where $p_{i'}$ represents the hydrodynamic pressure corresponding to $\varphi_{i'}$. The last integral, which represents a generalisation of Eq. (5.39), is, by virtue of Eq. (5.166), to be taken over S_p or over the totality of wet body surfaces

$$S = \sum_{p=1}^{N} S_p. \tag{5.168}$$

Since $\partial\varphi_i/\partial n$ is real everywhere on S according to boundary condition (5.166), we may use $\partial\varphi_i/\partial n$ or $\partial\varphi_i^*/\partial n$ as we wish in the preceding integrand. Note that Eq. (5.167) is valid for $i' = i$ as well as for $i' \neq i$. When $i' \neq i$, oscillator i' may pertain to any of the oscillating bodies, including body number p.

Matrix $\mathbf{Z}_{pp'}$, representing the radiation interaction between bodies p and p' [see Eq. (5.158)], may be written in terms of the (generally six-dimensional) column vectors $\boldsymbol{\varphi}_p$ and $\boldsymbol{\varphi}_{p'}$ as

$$\mathbf{Z}_{pp'} = -i\omega\rho \iint_S \frac{\partial \boldsymbol{\varphi}_p}{\partial n} \boldsymbol{\varphi}_{p'}^{\mathrm{T}} \, dS. \tag{5.169}$$

In the integrand here, we may replace $\boldsymbol{\varphi}_p$ by $\boldsymbol{\varphi}_p^*$.

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The radiation impedance matrix for the total system of N oscillating bodies may be written as

$$\mathbf{Z} = -i\omega\rho \iint_S \frac{\partial \boldsymbol{\varphi}}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}} \, dS. \tag{5.170}$$

If in the integrand we replace $\partial\boldsymbol{\varphi}/\partial n$ with $\partial\boldsymbol{\varphi}^*/\partial n$, we get the alternative expression

$$\mathbf{Z} = -i\omega\rho \iint_S \frac{\partial \boldsymbol{\varphi}^*}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}} \, dS, \tag{5.171}$$

which is a generalisation of Eq. (5.40).

Next, we show that the radiation impedance matrix $\mathbf{Z}$ is symmetric; that is,

$$\mathbf{Z} = \mathbf{Z}^{\mathrm{T}}. \tag{5.172}$$

From Eqs. (4.230) and (5.170) [see also Eq. (5.127)], it follows that

$$\mathbf{Z} - \mathbf{Z}^{\mathrm{T}} = -i\omega\rho \iint_S \left(\frac{\partial \boldsymbol{\varphi}}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}} - \boldsymbol{\varphi} \frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial n} \right) dS = i\omega\rho \, \mathbf{I}(\boldsymbol{\varphi}, \boldsymbol{\varphi}^{\mathrm{T}}) = 0, \tag{5.173}$$

where the last equality follows from Eq. (4.239) [see also Eq. (5.128)], since all components of vector $\boldsymbol{\varphi}$ satisfy the same radiation condition at infinity. We may note that reciprocity relation (5.172) is a generalisation of Eq. (5.41) or (5.117). Taking a look at Eq. (5.158), we see that Eq. (5.172) means

$$\mathbf{Z}_{p'p} = \mathbf{Z}_{pp'}^{\mathrm{T}}. \tag{5.174}$$

Note that matrix $\mathbf{Z}_{p'p}$ is not necessarily symmetric if $p' \neq p$. Physically, this means that the i component of the force on body p' due to body p oscillating with a unit amplitude in mode j is the same as the j component of the force on body p due to body p' oscillating with a unit amplitude in mode i but not the same as the j component of the force on body p' due to body p oscillating with a unit amplitude in mode i. Moreover, from Eq. (5.172), it follows that

$$\mathbf{Z}^* = (\mathbf{Z}^{\mathrm{T}})^* = \mathbf{Z}^{\dagger}. \tag{5.175}$$

Splitting the radiation impedance matrix into real and imaginary parts, as in Eq. (5.157), we may write the radiation resistance matrix as

$$\begin{aligned} \mathbf{R} &= \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^*) = \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^{\dagger}) = -\frac{1}{2} i\omega\rho \iint_S \left(\frac{\partial \boldsymbol{\varphi}^*}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}} - \boldsymbol{\varphi}^* \frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial n} \right) dS \\ &= \frac{i}{2} \omega\rho \, \mathbf{I}(\boldsymbol{\varphi}^*, \boldsymbol{\varphi}^{\mathrm{T}}) = -\frac{i}{2} \omega\rho \, \mathbf{I}(\boldsymbol{\varphi}, \boldsymbol{\varphi}^{\dagger}). \end{aligned} \tag{5.176}$$

Here we have used Eqs. (5.171), (5.172) and (5.175). Note that Eq. (5.176) is an extension of Eq. (5.133) from the case of 6×6 matrix to the case of $6N \times 6N$ matrix. In an analogous way, we may extend Eqs. (5.134) and (5.135). Thus, we may also write the $6N \times 6N$ radiation resistance matrix as

$$\mathbf{R} = -i\omega\rho \lim_{r\to\infty} \iint_{S_\infty} \frac{\partial \boldsymbol{\varphi}^*}{\partial r} \boldsymbol{\varphi}^{\mathrm{T}}\, dS \tag{5.177}$$

or as

$$\mathbf{R} = \frac{\omega\rho D(kh)}{2k} \int_0^{2\pi} \mathbf{a}^*(\theta)\mathbf{a}^{\mathrm{T}}(\theta)\, d\theta = \frac{\omega\rho D(kh)}{4\pi k} \int_0^{2\pi} \mathbf{h}^*(\theta)\mathbf{h}^{\mathrm{T}}(\theta)\, d\theta, \tag{5.178}$$

where $\mathbf{a}(\theta)$ and $\mathbf{h}(\theta)$ are $6N$-dimensional column vectors consisting of all far-field coefficients $a_i(\theta)$ for the radiated wave and of the corresponding Kochin functions $h_i(\theta)$, respectively. Note that the far-field coefficients $a_i(\theta)$ are referred to the common (global) origin. We may observe that since $\mathbf{R}$ is real, we may take the complex conjugate of the integrands without changing the resulting integral in Eq. (5.178).

We shall now prove the following relationship between the radiation resistance matrix $\mathbf{R}$ and the radiated power P_r:

$$P_r = \tfrac{1}{2}\hat{\mathbf{u}}^{\mathrm{T}}\mathbf{R}\hat{\mathbf{u}}^* = \tfrac{1}{2}\hat{\mathbf{u}}^{\dagger}\mathbf{R}\hat{\mathbf{u}}. \tag{5.179}$$

The last equality follows from the fact that P_r is real ($P_r^* = P_r$) and that $\mathbf{R}$ is a real matrix. Since $P_r \geq 0$, $\mathbf{R}$ is a positive semidefinite matrix (see Section 6.5).

In the far-field region ($r \to \infty$), the curvature of the wave front is negligible. The radiated power transport per unit width of the wave front is then

$$J(r,\theta) = \frac{\rho g^2 D(kh)}{4\omega} |\hat{\eta}_r(r,\theta)|^2 = \frac{\omega\rho D(kh)}{4} \left|\hat{\phi}_r(r,\theta,0)\right|^2, \tag{5.180}$$

where we have used Eqs. (4.39) and (4.130). In the far-field region ($r \to \infty$), we may use the asymptotic approximation (4.212)

$$\hat{\phi}_r \sim A_r(\theta)e(kz)(kr)^{-1/2}e^{-ikr}, \tag{5.181}$$

where

$$A_r(\theta) = \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{a}(\theta) = \mathbf{a}^{\mathrm{T}}(\theta)\hat{\mathbf{u}} \tag{5.182}$$

is the far-field coefficient for the radiated wave, in accordance with Eq. (5.162). Thus, the power radiated through the cylindrical control surface S_∞ of radius r, ($r \to \infty$) is

$$\begin{aligned} P_r &= \int_0^{2\pi} J_r(r,\theta) r\, d\theta = \frac{\omega\rho D(kh)}{4k} \int_0^{2\pi} A_r^*(\theta)A_r(\theta)\, d\theta \\ &= \frac{\omega\rho D(kh)}{4k} \int_0^{2\pi} \hat{\mathbf{u}}^{\dagger}\mathbf{a}^*(\theta)\mathbf{a}^{\mathrm{T}}(\theta)\hat{\mathbf{u}}\, d\theta. \end{aligned} \tag{5.183}$$

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[Also cf. Eq. (4.217).] Remembering that $\hat{\mathbf{u}}$ may be taken outside the last integral, we obtain Eq. (5.179) if we make use of Eq. (5.178).

Components $a_i(\theta)$ of the vector $\mathbf{a}(\theta)$ may be arranged in groups $\mathbf{a}_p(\theta)$ pertaining to body p, ($p = 1, 2, \ldots, N$). The real part of matrices $\mathbf{Z}_{pp'}$ on the right-hand side of Eq. (5.158) may, thus, according to Eqs. (5.169), (5.176) and (5.178), be written as

$$\mathbf{R}_{pp'} = \frac{i}{2}\omega\rho\mathbf{I}(\boldsymbol{\varphi}_p^*, \boldsymbol{\varphi}_{p'}^{\mathrm{T}}) = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{a}_p^*(\theta)\mathbf{a}_{p'}^{\mathrm{T}}(\theta)\, d\theta. \tag{5.184}$$

In analogy with Eqs. (4.255) and (4.256), we may use far-field coefficient vectors $\mathbf{b}_p(\theta_p)$ or $\mathbf{a}_p(\theta)$ referred to the local origin $(x_p, y_p, 0)$ or the global origin $(0, 0, 0)$, respectively. See Figure 4.9, which—with subscript i replaced by p—defines the angles θ_p and α_p and the distances x_p, y_p and d_p. The vertical axis p–p in Figure 5.13 is given by $(x, y) = (x_p, y_p)$. Note that vector $\mathbf{b}_p(\theta_p)$ appears also in Eq. (5.164). According to Eq. (4.261), we have

$$\mathbf{a}_p(\theta) = \mathbf{b}_p(\theta)\exp\left[ikd_p\cos\left(\alpha_p - \theta\right)\right] = \mathbf{b}_p(\theta)\exp\left[ik(x_p\cos\theta + y_p\sin\theta)\right]. \tag{5.185}$$

If we use this, or Eq. (4.289), in Eq. (5.184), we get

$$\mathbf{R}_{pp'} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{b}_p^*(\theta)\mathbf{b}_{p'}^{\mathrm{T}}(\theta)\exp\left[ik(r_p - r_{p'})\right] d\theta. \tag{5.186}$$

According to Eq. (4.290), the distance difference $r_p - r_{p'}$ entering in the exponent may be expressed in various ways, such as

$$\begin{aligned} r_p - r_{p'} &= (x_{p'} - x_p)\cos\theta + (y_{p'} - y_p)\sin\theta \\ &= d_{p'}\cos\left(\alpha_{p'} - \theta\right) - d_p\cos\left(\alpha_p - \theta\right) = d_{p'p}\cos\left(\alpha_{p'p} - \theta\right). \end{aligned} \tag{5.187}$$

Using the last expression, we have

$$\mathbf{R}_{pp'} = \frac{\omega\rho D(kh)}{2k}\int_0^{2\pi} \mathbf{b}_p^*(\theta)\mathbf{b}_{p'}^{\mathrm{T}}(\theta)\exp\left[ikd_{p'p}\cos(\alpha_{p'p} - \theta)\right] d\theta. \tag{5.188}$$

5.5.4 Radiation Reactance Relation to Near-Field Energy Mismatch

We note that, according to Eq. (5.157), the radiation reactance matrix may be written as

$$\mathbf{X} = \omega\mathbf{m} = -i(\mathbf{Z} - \mathbf{R}) = \omega\rho\left(\lim_{r\to\infty}\iint_{S_\infty}\frac{\partial\boldsymbol{\varphi}^*}{\partial r}\boldsymbol{\varphi}^{\mathrm{T}} dS - \iint_S \frac{\partial\boldsymbol{\varphi}^*}{\partial n}\boldsymbol{\varphi}^{\mathrm{T}} dS\right), \tag{5.189}$$

where we have used Eqs. (5.171) and (5.177). This indicates that the reactance matrix $\mathbf{X}$ and, hence, the added-mass matrix $\mathbf{m}$ are somehow related to the

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radiated wave in the near-field region. Equation (2.88) shows how the reactance of a simple mechanical oscillator is related to the difference between the kinetic and potential energy. In the following paragraphs, let us relate the radiation reactance matrix to the difference between kinetic and potential energy in the near-field region. We note that in the far-field region, as also in a plane wave, this difference vanishes [cf. Eqs. (4.117) and (4.123)]. To be explicit, we shall show that

$$\frac{1}{4}\hat{\mathbf{u}}^{\mathrm{T}}\mathbf{m}\hat{\mathbf{u}}^* = \frac{1}{4\omega}\hat{\mathbf{u}}^{\mathrm{T}}\mathbf{X}\hat{\mathbf{u}}^* = W_k - W_p, \tag{5.190}$$

where $W_k - W_p$ is the difference between the kinetic energy and the potential energy associated with the radiated wave $\hat{\phi}_r$, as given by Eq. (5.162). Negative added mass is possible if $W_p > W_k$ [23].

The (time-average) kinetic energy is [see Eq. (2.83)]

$$W_k = \frac{\rho}{4}\iiint_V \hat{\vec{v}}\cdot\hat{\vec{v}}^* dV = \frac{\rho}{4}\iiint_V \nabla\hat{\phi}_r^*\cdot\nabla\hat{\phi}_r\, dV, \tag{5.191}$$

where the integral is taken over the fluid volume shown in Figure 5.13. This volume is bounded by a closed surface consisting of the sea bed S_b, the control surface S_∞, the free water surface S_0 and the wave-generating surface S, defined by Eq. (5.168). Since $\hat{\phi}_r$ obeys Laplace's equation (4.33), we have

$$\nabla\cdot(\hat{\phi}_r^*\nabla\hat{\phi}_r) \equiv \hat{\phi}_r^*\nabla^2\hat{\phi}_r + \nabla\hat{\phi}_r^*\cdot\nabla\hat{\phi}_r = \nabla\hat{\phi}_r^*\cdot\nabla\hat{\phi}_r. \tag{5.192}$$

Hence, since the integrand is the divergence of the vector $\hat{\phi}_r^*\nabla\hat{\phi}_r$, we may, by using Gauss's divergence theorem, replace the volume integral (5.191) by the following closed-surface integral

$$W_k = \frac{\rho}{4}\oiint \hat{\phi}_r^*\nabla\hat{\phi}_r\cdot(-\vec{n})\, dS = -\frac{\rho}{4}\oiint \hat{\phi}_r^*\frac{\partial\hat{\phi}_r}{\partial n}\, dS, \tag{5.193}$$

where the unit normal $\vec{n}$ is pointing into the fluid (see Figure 5.13). Note that $\partial/\partial n = -\partial/\partial z$ on S_0 and $\partial/\partial n = -\partial/\partial r$ on S_∞. Further, we observe that the integrand vanishes on the sea bed S_b as a result of boundary condition (4.16).

By using Eqs. (4.39), (4.41) and (4.115), we obtain the (time-average) potential energy

$$\begin{aligned} W_p &= \iint_{S_0} E_p dS = \frac{\rho g}{4}\iint_{S_0}\hat{\eta}\hat{\eta}^* dS = \frac{\omega^2\rho}{4g}\iint_{S_0}\hat{\phi}_r^*\hat{\phi}_r dS \\ &= \frac{\rho}{4}\iint_{S_0}\hat{\phi}_r^*\frac{\partial\hat{\phi}_r}{\partial z}dS = -\frac{\rho}{4}\iint_{S_0}\hat{\phi}_r^*\frac{\partial\hat{\phi}_r}{\partial n}dS. \end{aligned} \tag{5.194}$$

Subtraction between Eqs. (5.193) and (5.194) gives

$$W_k - W_p = \frac{\rho}{4}\left(\iint_{S_\infty}\hat{\phi}_r^*\frac{\partial\hat{\phi}_r}{\partial r}dS - \iint_S \hat{\phi}_r^*\frac{\partial\hat{\phi}_r}{\partial n}dS\right). \tag{5.195}$$

Observing from Eq. (5.162) that we may write $\hat{\phi}_r^* = \hat{\mathbf{u}}^\dagger \boldsymbol{\varphi}^*$ and $\hat{\phi}_r = \boldsymbol{\varphi}^{\mathrm{T}} \hat{\mathbf{u}}$ and, further, that $\hat{\mathbf{u}}^\dagger$ and $\hat{\mathbf{u}}$ may be taken outside the integral, we find from Eqs. (5.189) and (5.195) that

$$W_k - W_p = \frac{1}{4\omega} \hat{\mathbf{u}}^\dagger \mathbf{X} \hat{\mathbf{u}} = \frac{1}{4} \hat{\mathbf{u}}^\dagger \mathbf{m} \hat{\mathbf{u}}. \tag{5.196}$$

Since $W_k - W_p$ is a real scalar and $\mathbf{X}$ (and $\mathbf{m}$) is a real symmetric matrix, the right-hand sides of Eqs. (5.189) and (5.196) must equal their own conjugates as well as their own transposes. Hence, we obtain Eq. (5.190) if we take the complex conjugate of Eq. (5.196). Note that the control cylinder S_∞ has to be sufficiently large in order to contain the complete near-field region inside S_∞.

An alternative expression for the added mass matrix is obtained as follows. Using Eqs. (5.157), (5.171) and (5.175) gives

$$\begin{aligned} \mathbf{m} &= \frac{\mathbf{X}}{\omega} = \frac{1}{2i\omega}(\mathbf{Z} - \mathbf{Z}^*) = \frac{1}{2i\omega}(\mathbf{Z} - \mathbf{Z}^\dagger) \\ &= -\frac{\rho}{2} \iint_S \left(\frac{\partial \boldsymbol{\varphi}^*}{\partial n} \boldsymbol{\varphi}^{\mathrm{T}} + \boldsymbol{\varphi}^* \frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial n} \right) dS = -\frac{\rho}{2} \iint_S \frac{\partial}{\partial n} (\boldsymbol{\varphi}^* \boldsymbol{\varphi}^{\mathrm{T}})\, dS. \end{aligned} \tag{5.197}$$

In this way, the added mass matrix is expressed as an integral over the wave-generating surface S (the totality of wet body surfaces) only.

It is emphasised that $W_k - W_p$ in identity (5.190) or (5.196) is the kinetic–potential energy difference when there is no other wave than a radiated wave resulting from a forced body motion on otherwise still water. Thus, only ϕ_r is accounted for in Eqs. (5.191)–(5.195). If, in addition, an incident wave is present, then there is an additional contribution to $W_k - W_p$. Let us now consider this more general case, where we shall follow the derivation given in [63, appendix B].

Since the total velocity potential $\phi = \phi_0 + \phi_d + \phi_r$ satisfies Laplace's equation and the homogeneous boundary condition on the sea bed, Eq. (5.195) applies also to ϕ as to ϕ_r, where it is now understood that $W_k - W_p$ in the equation is the kinetic–potential energy difference associated with ϕ. Denoting $\phi_s \equiv \phi_0 + \phi_d$, we thus have

$$W_k - W_p = \frac{\rho}{4} \left[\iint_{S_\infty} (\hat{\phi}_s + \hat{\phi}_r)^* \frac{\partial(\hat{\phi}_s + \hat{\phi}_r)}{\partial r} dS - \iint_S (\hat{\phi}_s + \hat{\phi}_r)^* \frac{\partial(\hat{\phi}_s + \hat{\phi}_r)}{\partial n} dS \right]. \tag{5.198}$$

We see that the integrands in this equation contain products of the type $(\hat{\phi}_s + \hat{\phi}_r)^*(\hat{\phi}_s + \hat{\phi}_r) = \hat{\phi}_s^*\hat{\phi}_s + \hat{\phi}_s^*\hat{\phi}_r + \hat{\phi}_r^*\hat{\phi}_s + \hat{\phi}_r^*\hat{\phi}_r$. The fourth term corresponds to Eq. (5.195). The first term corresponds to pure scattering—that is, the case in which the bodies are not oscillating. The remaining terms, $\hat{\phi}_s^*\hat{\phi}_r + \hat{\phi}_r^*\hat{\phi}_s$, represent the interaction between the radiated and scattered waves.

Let us first consider the pure scattering problem. In this case,

$$W_k - W_p = \frac{\rho}{4}\left(\iint_{S_\infty} \hat{\phi}_s^* \frac{\partial \hat{\phi}_s}{\partial r} dS - \iint_S \hat{\phi}_s^* \frac{\partial \hat{\phi}_s}{\partial n} dS\right). \tag{5.199}$$

Because of boundary condition (5.165), the second integral vanishes. The first integral may be split into

$$\iint_{S_\infty} \hat{\phi}_s^* \frac{\partial \hat{\phi}_s}{\partial r} dS \equiv I_{\infty,ss} = I_{\infty,00} + I_{\infty,0d} + I_{\infty,dd}, \tag{5.200}$$

where

$$I_{\infty,00} = \iint_{S_\infty} \hat{\phi}_0^* \frac{\partial \hat{\phi}_0}{\partial r} dS, \tag{5.201}$$

$$I_{\infty,0d} = \iint_{S_\infty} \left(\hat{\phi}_0^* \frac{\partial \hat{\phi}_d}{\partial r} + \hat{\phi}_d^* \frac{\partial \hat{\phi}_0}{\partial r}\right) dS, \tag{5.202}$$

$$I_{\infty,dd} = \iint_{S_\infty} \hat{\phi}_d^* \frac{\partial \hat{\phi}_d}{\partial r} dS. \tag{5.203}$$

Since $W_k - W_p$ is real, we may write

$$W_k - W_p = \frac{\rho}{4} I_{\infty,ss} = \frac{\rho}{8}(I_{\infty,ss} + I_{\infty,ss}^*). \tag{5.204}$$

The incident wave potential is given by [cf. Eq. (4.262)]

$$\hat{\phi}_0 = -\frac{g}{i\omega} A e(kz) \exp[-ikr\cos(\theta - \beta)], \tag{5.205}$$

while the diffracted wave potential in the far-field region has an asymptotic expression [cf. Eq. (4.212)]

$$\hat{\phi}_d \sim A_d(\theta) e(kz)(kr)^{-1/2} e^{-ikr}. \tag{5.206}$$

Inserting these expressions into Eqs. (5.201) and (5.203), we see that $I_{\infty,00} + I_{\infty,00}^*$ as well as $I_{\infty,dd} + I_{\infty,dd}^*$ vanish, and hence,

$$I_{\infty,ss} + I_{\infty,ss}^* = I_{\infty,0d} + I_{\infty,0d}^*. \tag{5.207}$$

Using the product rule, we find from Eq. (5.202) that

$$I_{\infty,0d} + I_{\infty,0d}^* = \iint_{S_\infty} \frac{\partial}{\partial r}(\hat{\phi}_0^* \hat{\phi}_d) dS + \text{c. c.} \equiv I'_{\infty,0d} + \text{c. c.} \tag{5.208}$$

Inserting Eqs. (5.205)–(5.206) and denoting $\varphi = \theta - \beta$, we then have

$$\begin{aligned} I'_{\infty,0d} &= -\iint_{S_\infty} ik(1-\cos\varphi)\hat{\phi}_0^*\hat{\phi}_d\,dS \\ &= -\int_{-h}^{0}\lim_{kr\to\infty}\int_0^{2\pi} ikr(1-\cos\varphi)\hat{\phi}_0^*\hat{\phi}_d\,d\varphi\,dz \\ &= -\frac{v_p v_g}{\omega}A^*\lim_{kr\to\infty}\int_0^{2\pi}\sqrt{kr}(1-\cos\varphi)A_d(\varphi+\beta)e^{-ikr(1-\cos\varphi)}d\varphi, \end{aligned} \tag{5.209}$$

where we have used Eqs. (4.107)–(4.108) to evaluate the integral over z. The integral over φ may be evaluated using the method of stationary phase, as described in Section 4.8. Application of Eq. (4.275) gives, after further manipulation,

$$I'_{\infty,0d} = \frac{v_p v_g}{\omega}|A|\sqrt{8\pi}\lim_{kr\to\infty}\exp\left[-i(2kr+3\pi/4+\alpha_0-\alpha_d)\right]|A_d(\pi+\beta)|, \tag{5.210}$$

where $\alpha_0 = \arg(A)$ and $\alpha_d = \arg\left[A_d(\pi+\beta)\right]$. Putting this back into Eq. (5.204), we finally arrive at an expression for the kinetic–potential energy difference in the pure scattering case:

$$W_k - W_p = \frac{\rho}{4}\frac{v_p v_g}{\omega}|A|\sqrt{8\pi}\lim_{kr\to\infty}\cos(2kr+3\pi/4+\alpha_0-\alpha_d)|A_d(\pi+\beta)|. \tag{5.211}$$

We may observe from this expression that, in the pure scattering case, $W_k - W_p = 0$ for discrete values of the distance r from origin:

$$kr = \frac{n\pi}{2} - \frac{\pi}{8} - \frac{\alpha_0}{2} + \frac{\alpha_d}{2}, \quad n = 0,1,2,\ldots. \tag{5.212}$$

Further, $W_k - W_p$ averages to zero over every half-wavelength increment of r.

Next, let us consider the kinetic–potential energy difference associated with the interaction between the radiated and scattered waves. According to Eq. (5.198),

$$W_k - W_p = \frac{\rho}{4}\left[\iint_{S_\infty}\left(\hat{\phi}_s^*\frac{\partial\hat{\phi}_r}{\partial r} + \hat{\phi}_r^*\frac{\partial\hat{\phi}_s}{\partial r}\right)dS - \iint_S\left(\hat{\phi}_s^*\frac{\partial\hat{\phi}_r}{\partial n} + \hat{\phi}_r^*\frac{\partial\hat{\phi}_s}{\partial n}\right)dS\right]. \tag{5.213}$$

Let us first discuss the integral over S_∞, which may be split into

$$\iint_{S_\infty}\left(\hat{\phi}_s^*\frac{\partial\hat{\phi}_r}{\partial r} + \hat{\phi}_r^*\frac{\partial\hat{\phi}_s}{\partial r}\right)dS \equiv I_{\infty,sr} = I_{\infty,0r} + I_{\infty,dr}, \tag{5.214}$$

where

$$I_{\infty,0r} = \iint_{S_\infty}\left(\hat{\phi}_0^*\frac{\partial\hat{\phi}_r}{\partial r} + \hat{\phi}_r^*\frac{\partial\hat{\phi}_0}{\partial r}\right)dS, \tag{5.215}$$

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$$I_{\infty,dr} = \iint_{S_\infty} \left(\hat{\phi}_d^* \frac{\partial \hat{\phi}_r}{\partial r} + \hat{\phi}_r^* \frac{\partial \hat{\phi}_d}{\partial r} \right) dS. \tag{5.216}$$

Since $\hat{\phi}_r$ has the same asymptotic form as $\hat{\phi}_d$ in the far field, the preceding discussion pertaining to $I_{\infty,0d}$ is also applicable to $I_{\infty,0r}$, namely that the integral $I_{\infty,0r}$ contributes essentially nothing to the kinetic–potential energy difference. As for the integral $I_{\infty,dr}$, we may write it as $(I_{\infty,dr} + I^*_{\infty,dr})/2$ because $W_k - W_p$ is real. Then, because Eqs. (4.237) and (4.241) are applicable to both $\hat{\phi}_r$ and $\hat{\phi}_d$, it follows that $I_{\infty,dr} + I^*_{\infty,dr} = 0$. What remains then is to evaluate the integral over S in Eq. (5.213). We observe that the second term of the integrand vanishes because of boundary condition (5.165), whereas

$$\iint_S \hat{\phi}_s^* \frac{\partial \hat{\phi}_r}{\partial n} dS = \iint_S \hat{\phi}_s^* \hat{\mathbf{u}}^{\mathrm{T}} \mathbf{n}\, dS = \hat{\mathbf{u}}^{\mathrm{T}} \iint_S \hat{\phi}_s^* \mathbf{n}\, dS = -\frac{1}{i\omega\rho} \hat{\mathbf{u}}^{\mathrm{T}} \hat{\mathbf{F}}_e^* \tag{5.217}$$

because of boundary condition (5.166) and Eq. (5.26) [also see Eq. (5.221)]. Here, $\hat{\mathbf{F}}_e$ is the excitation force vector [see Eq. (5.220)]. Therefore, the kinetic–potential energy difference due to the interaction between the radiated and scattered waves is

$$\begin{aligned} W_k - W_p &= -\frac{\rho}{8} \left(\iint_S \hat{\phi}_s^* \frac{\partial \hat{\phi}_r}{\partial n} dS + \text{c. c.} \right) = -\frac{\rho}{8} \left(-\frac{1}{i\omega\rho} \hat{\mathbf{u}}^{\mathrm{T}} \hat{\mathbf{F}}_e^* + \frac{1}{i\omega\rho} \hat{\mathbf{u}}^{\dagger} \hat{\mathbf{F}}_e \right) \\ &= -\frac{1}{4\omega} \mathrm{Im}\{\hat{\mathbf{u}}^{\dagger} \hat{\mathbf{F}}_e\} = -\frac{1}{4\omega} \mathrm{Im}\{\hat{\mathbf{F}}_e^{\mathrm{T}} \hat{\mathbf{u}}^*\}. \end{aligned} \tag{5.218}$$

Returning to the expression (5.198), we find that the kinetic–potential energy difference in the general case is given as

$$W_k - W_p = \frac{1}{4\omega} \left(\hat{\mathbf{u}}^{\mathrm{T}} \mathbf{X} \hat{\mathbf{u}}^* - \mathrm{Im}\{\hat{\mathbf{F}}_e^{\mathrm{T}} \hat{\mathbf{u}}^*\} \right). \tag{5.219}$$

5.5.5 Excitation Force Vector: The Haskind Relation

For each of the N bodies, we may define an excitation force vector $\mathbf{F}_{e,p}$, which has, in general, six components, as in Eq. (5.26) or (5.138). Note that boundary condition (5.27) or (5.165) for ϕ_d has to be satisfied on the wet surface, not only of body p but also of all other bodies. Thus, $\mathbf{F}_{e,p}$ represents the wave force on body p when *all* bodies are in a non-oscillating state.

For the total system of N bodies, we define a column vector $\mathbf{F}_e$ (in general, of dimension $6N$) for the excitation forces:

$$\mathbf{F}_e = (\mathbf{F}_{e,1}^{\mathrm{T}} \cdots \mathbf{F}_{e,p}^{\mathrm{T}} \cdots \mathbf{F}_{e,N}^{\mathrm{T}})^{\mathrm{T}}. \tag{5.220}$$

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Here $\mathbf{F}_{e,p}$ is the (generally six-dimensional) excitation force vector for body p. In accordance with Eq. (5.26), we may write

$$\hat{\mathbf{F}}_{e,p} = i\omega\rho \iint_{S_p} (\hat{\phi}_0 + \hat{\phi}_d)\mathbf{n}_p \, dS, \tag{5.221}$$

where $\mathbf{n}_p$ is the (six-dimensional) unit normal vector, as defined by Eqs. (5.5) and (5.6), but now for body p.

Since all components φ_i of vector $\boldsymbol{\varphi}$ satisfy boundary condition (5.166) and also the same radiation condition at infinity as $\hat{\phi}_d$, we may generalise Eqs. (5.136) and (5.138) to the case with N bodies:

$$\hat{\mathbf{F}}_e = i\omega\rho\mathbf{I}[(\hat{\phi}_0 + \hat{\phi}_d), \boldsymbol{\varphi}] = i\omega\rho\mathbf{I}(\hat{\phi}_0, \boldsymbol{\varphi}). \tag{5.222}$$

In particular, we have for body p

$$\hat{\mathbf{F}}_{e,p} = i\omega\rho\mathbf{I}(\hat{\phi}_0, \boldsymbol{\varphi}_p). \tag{5.223}$$

If the incident wave is plane in accordance with Eq. (5.139), we may also generalise Eqs. (5.141) and (5.142) to the case of N bodies. Thus, the excitation force vector is

$$\hat{\mathbf{F}}_e(\beta) = \mathbf{f}_g(\beta)A = \frac{\rho g D(kh)}{k}\mathbf{h}(\beta \pm \pi)A, \tag{5.224}$$

where

$$\mathbf{f}_g(\beta) = \frac{\rho g D(kh)}{k}\mathbf{h}(\beta \pm \pi) = \frac{\rho g D(kh)}{k}\sqrt{2\pi}\,\mathbf{a}(\beta \pm \pi)e^{i\pi/4} \tag{5.225}$$

is, in general, a $6N$-dimensional column vector consisting of the excitation force coefficients referred to the global origin $(x, y) = (0, 0)$. Note that the excitation force depends on the angle of incidence β of the incident wave.

Similarly, reciprocity relations (5.143)–(5.145), which express the radiation resistance matrix in terms of excitation force vectors, may be directly generalised to the $6N$-dimensional case.

Note that A in Eq. (5.224) is the elevation of the undisturbed incident wave at the (global) origin $(x, y) = (0, 0)$. Considering body p's excitation force vector $\mathbf{F}_{e,p}$, we have from Eqs. (5.224) and (5.225) that

$$\hat{\mathbf{F}}_{e,p}(\beta) = \mathbf{f}_{g,p}(\beta)A = \frac{\rho g D(kh)}{k}\sqrt{2\pi}\,\mathbf{a}_p(\beta \pm \pi)e^{i\pi/4}A. \tag{5.226}$$

Using relation (5.185) between the local and global far-field coefficients, we find

$$\hat{\mathbf{F}}_{e,p}(\beta) = \mathbf{f}_p(\beta)A_p = \frac{\rho g D(kh)}{k}\sqrt{2\pi}\,\mathbf{b}_p(\beta \pm \pi)e^{i\pi/4}A_p, \tag{5.227}$$

where

$$A_p = A \exp[-ik(x_p \cos\beta + y_p \sin\beta)] \tag{5.228}$$

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is [see Eq. (5.139)] the wave elevation corresponding to the undisturbed incident wave at body p's local origin $(x, y) = (x_p, y_p)$. In Eqs. (5.226) and (5.227), the excitation-force coefficients $\mathbf{f}_{g,p}(\beta)$ and $\mathbf{f}_p(\beta)$ for body p are referred to the global origin $(x, y) = (0, 0)$ and to the local origin $(x, y) = (x_p, y_p)$, respectively.

By using Eqs. (4.82) and (4.106) for the phase and group velocities, we may write the coefficient $\rho g D(kh)/h$, appearing in Eqs. (5.224)–(5.227), as $2v_g v_p \rho$.

5.5.6 Wide-Spacing Approximation

If the distance between two different bodies (p and p', for example) is sufficiently large, the diffraction and radiation interactions between the two bodies may be well represented by the wide-spacing approximation. The requirement is that one body is not influenced by the other body's near-field contribution to the diffracted or radiated waves. Here, let us illustrate the wide-spacing approximation for the radiation problem. We shall apply the asymptotic approximation (5.164) for both bodies. That means we have to assume equal water depth h for both bodies.

The interaction between the two bodies is then represented by the 6×6 matrix $\mathbf{Z}_{pp'}$. We consider the force

$$\Delta'\hat{\mathbf{F}}_p = -\mathbf{Z}_{pp'}\hat{\mathbf{u}}_{p'} \tag{5.229}$$

on body p due to the oscillation of body p' with velocity $\mathbf{u}_{p'}$. Using far-field approximation (5.164) with p' instead of p and also Eqs. (4.39) and (5.122), we find the corresponding wave elevation

$$\Delta'\hat{\eta}_r(r_{p'}, \theta_{p'}) = -\frac{i\omega}{g}\boldsymbol{\varphi}_{p'}^{\mathrm{T}}(r_{p'}, \theta_{p'}, 0)\hat{\mathbf{u}}_{p'} \tag{5.230}$$

$$\sim -\frac{i\omega}{g}\mathbf{b}_{p'}^{\mathrm{T}}(\theta_{p'})(kr_{p'})^{-1/2}\exp\{-ikr_{p'}\}\hat{\mathbf{u}}_{p'}. \tag{5.231}$$

If $kd_{pp'} \to \infty$, this wave may be considered as a plane wave when it arrives at body p, where $(r_{p'}, \theta_{p'}) = (d_{pp'}, \alpha_{pp'})$; see Figure 4.11. Using also Eq. (5.227) yields

$$\Delta'\hat{\mathbf{F}}_p \sim \frac{\rho g D(kh)}{k}\sqrt{2\pi}\,\mathbf{b}_p(\alpha_{pp'} \pm \pi)e^{i\pi/4}\Delta'\hat{\eta}_r(d_{pp'}, \alpha_{pp'}). \tag{5.232}$$

Inserting here from the approximation (5.231) and comparing with Eq. (5.229) result in the following asymptotic approximation for $\mathbf{Z}_{pp'}$ as $kd_{pp'} \to \infty$:

$$\mathbf{Z}_{pp'} \sim \frac{i\omega\rho D(kh)}{k}\sqrt{2\pi}\mathbf{b}_p(\alpha_{p'p})\mathbf{b}_{p'}^{\mathrm{T}}(\alpha_{pp'})(kd_{pp'})^{-1/2}\exp\left(-ikd_{pp'} + \frac{\pi}{4}\right). \tag{5.233}$$

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Note that $\mathbf{b}_p(\alpha_{p'p})$ represents the action on body p' from body p and vice versa for $\mathbf{b}_{p'}(\alpha_{pp'})$. The result agrees with the expected approximation (5.159). We also easily see that the general reciprocity relationship (5.172) is satisfied by approximation (5.233).

5.6 The Froude–Krylov Force and Small-Body Approximation

We may, in accordance with Eq. (5.221), decompose the excitation force vector $\mathbf{F}_{e,p}$ for body p into two parts:

$$\hat{\mathbf{F}}_{e,p} = \hat{\mathbf{F}}_{\mathrm{FK},p} + \hat{\mathbf{F}}_{d,p}, \tag{5.234}$$

where

$$\hat{\mathbf{F}}_{\mathrm{FK},p} = i\omega\rho \iint_{S_p} \hat{\phi}_0 \mathbf{n}_p dS \tag{5.235}$$

is the Froude–Krylov force vector, with $\mathbf{n}_p = (\vec{n}_p, \vec{s}_p \times \vec{n}_p)$ [see Eqs. (5.5)–(5.6)], and

$$\hat{\mathbf{F}}_{d,p} = i\omega\rho \iint_{S_p} \hat{\phi}_d \mathbf{n}_p dS \tag{5.236}$$

is the diffraction force vector. Alternatively, we have from the Haskind relation [cf. Eq. (5.223)]

$$\hat{\mathbf{F}}_{e,p} = i\omega\rho \mathbf{I}(\hat{\phi}_0, \boldsymbol{\varphi}_p) = i\omega\rho \iint_S \left(\hat{\phi}_0 \frac{\partial \boldsymbol{\varphi}_p}{\partial n} - \boldsymbol{\varphi}_p \frac{\partial \hat{\phi}_0}{\partial n} \right) dS. \tag{5.237}$$

The first term represents the Froude–Krylov force, since $\partial \boldsymbol{\varphi}_p / \partial n = 0$ on $S_{p'}$ when $p' \neq p$, and $\partial \boldsymbol{\varphi}_p / \partial n = \mathbf{n}_p$ on S_p; see Eq. (5.166). Hence, we may compute the diffraction force vector

$$\hat{\mathbf{F}}_{d,p} = -i\omega\rho \iint_S \boldsymbol{\varphi}_p \frac{\partial \hat{\phi}_0}{\partial n} dS \tag{5.238}$$

by solving the radiation problem instead of solving the scattering (or diffraction) problem.

With a given incident wave and a given geometry for body p, the Froude–Krylov force vector $\hat{\mathbf{F}}_{\mathrm{FK},p}$ is obtained by direct integration. To compute the diffraction force, it is necessary to solve a boundary-value problem corresponding to either scattering (diffraction) or radiation. The latter problem is frequently, but not always, the easier one to solve.

5.6.1 The Froude–Krylov Force and Moment

Next, we shall consider the translational components of the Froude–Krylov force. For

$$i = 6(p-1) + q \qquad (q = 1, 2, 3), \tag{5.239}$$

we write the three components of the Froude–Krylov force as

$$\hat{F}_{\mathrm{FK},i} = \hat{F}_{\mathrm{FK},pq} = i\omega\rho \iint_{S_p} \hat{\phi}_0 n_{pq} dS = i\omega\rho \iint_{S_p} \hat{\phi}_0 \vec{e}_q \cdot \vec{n}_p dS. \tag{5.240}$$

Here $\vec{e}_q$ $(q = 1, 2, 3)$ is the unit vector in the direction x, y or z, for surge, sway or heave, respectively. The unit normal components n_{pq} are as given by Eq. (5.5), but for body p. By making the union of the wet surface S_p and the water plane area S_{wp} of body p (see Figure 5.14), we may decompose this integral into one integral over the closed surface $S_p + S_{wp}$ and one integral over S_{wp}:

$$\hat{F}_{\mathrm{FK},pq} = i\omega\rho \oiint_{S_p+S_{wp}} \hat{\phi}_0 \vec{e}_q \cdot \vec{n}_p dS - i\omega\rho \iint_{S_{wp}} \hat{\phi}_0 \delta_{q3} dS. \tag{5.241}$$

Note that the last term vanishes except for the heave mode $(q = 3)$, because $n_q = \delta_{q3}$ on S_{wp}. We apply Gauss's theorem to the closed-surface integral and convert it to an integral over the volume V_p of the displaced water:

$$i\omega\rho \oiint_{S_p+S_{wp}} \hat{\phi}_0 \vec{e}_q \cdot \vec{n}_p dS = i\omega\rho \iiint_{V_p} (\nabla\hat{\phi}_0) \cdot \vec{e}_q dV = i\omega\rho \iiint_{V_p} \frac{\partial \hat{\phi}_0}{\partial x_q} dV \tag{5.242}$$

$(x_1 = x, x_2 = y, x_3 = z)$. This may be obtained directly from Gauss's divergence theorem applied to the divergence $\nabla \cdot (\hat{\phi}_0 \vec{e}_q)$ where $\vec{e}_q$ is a constant (unit) vector. Thus, we have

$$\hat{F}_{\mathrm{FK},pq} = i\omega\rho \iiint_{V_p} \frac{\partial \hat{\phi}_0}{\partial x_q} dV + \delta_{q3}\rho g \iint_{S_{wp}} \hat{\eta}_0 dS, \tag{5.243}$$

where Eq. (4.39) also has been used. Note that $\hat{\phi}_0$ and $\hat{\eta}_0$ are the velocity potential and the wave elevation of the undisturbed incident wave—that is, what the wave would have been in the absence of all the N bodies. Observing

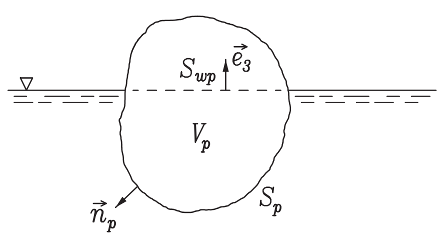

Figure 5.14: Floating body with water plane area S_{wp}, submerged volume V_p (volume of displaced water) and wet surface S_p with unit normal $\vec{n}_p$. The vertical unit vector is indicated by $\vec{e}_3$.

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that [see Eq. (4.36)] $i\omega\partial\hat{\phi}_0/\partial x_q = i\omega\hat{v}_{0q} = \hat{a}_{0q}$, we may alternatively write Eq. (5.243) as

$$\hat{F}_{\mathrm{FK},pq} = \rho \iiint_{V_p} \hat{a}_{0q} dV + \delta_{q3}\rho g \iint_{S_{wp}} \hat{\eta}_0 dS, \tag{5.244}$$

where $\hat{a}_{0q}$ represents the q component of the fluid acceleration due to the incident wave. In the particular case of a purely propagating incident plane wave, $\hat{\phi}_0$ is given by Eqs. (4.88) and (4.98). Then we have the vertical and horizontal components of the undisturbed fluid particle acceleration as [cf. Eqs. (4.90), (4.91) and (4.267)]

$$\begin{aligned} \hat{a}_{0z} = \hat{a}_{03} = i\omega\hat{v}_{03} &= -gkAe'(kz)\exp[-ikr(\beta)] \\ &= -\omega^2 A\frac{\sinh(kz+kh)}{\sinh(kh)}\exp[-ikr(\beta)], \end{aligned} \tag{5.245}$$

$$(\hat{a}_{0x}, \hat{a}_{0y}) = (\hat{a}_{01}, \hat{a}_{02}) = i\omega(\hat{v}_{01}, \hat{v}_{02}) = a_{0H}(\cos\beta, \sin\beta), \tag{5.246}$$

where

$$a_{0H} = ikgAe(kz)\exp[-ikr(\beta)]. \tag{5.247}$$

Finally, we consider the rotational modes of the Froude–Krylov 'force' (i.e., Froude–Krylov moments of roll, pitch and yaw) for body p. For

$$i = 6(p-1) + 3 + q \qquad (q = 1, 2, 3), \tag{5.248}$$

set $F_i = M_{pq}$. Thus, according to Eqs. (5.6) and (5.19), the Froude–Krylov moment is given by

$$\hat{M}_{\mathrm{FK},pq} = \hat{F}_{\mathrm{FK},i} = i\omega\rho \iint_{S_p} \hat{\phi}_0 n_{pq} dS = i\omega\rho \iint_{S_p} \hat{\phi}_0 \vec{e}_q \cdot (\vec{s}_p \times \vec{n}_p) dS, \tag{5.249}$$

where $\vec{s}_p$ is the vector from the body's reference point (x_p, y_p, z_p) to the surface element dS of the wet surface S_p (cf. Figures 5.3 and 5.13). On the water plane area S_{wp} (see Figure 5.14), we have $\vec{n}_p = \vec{e}_3$ and $\vec{e}_q \cdot (\vec{s}_p \times \vec{e}_3) = (y - y_p)\delta_{q1} - (x - x_p)\delta_{q2}$. Thus,

$$\begin{aligned} \hat{M}_{\mathrm{FK},pq} = i\omega\rho \oiint_{S_p + S_{wp}} &\hat{\phi}_0 \vec{e}_q \times \vec{s}_p \cdot \vec{n}_p dS \\ &- i\omega\rho \iint_{S_{wp}} \hat{\phi}_0 [(y - y_p)\delta_{q1} - (x - x_p)\delta_{q2}] dS. \end{aligned} \tag{5.250}$$

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Note that, for the yaw mode ($j = 6$ or $q = 3$), there is no contribution from the last term. Applying Gauss's theorem on the closed-surface integral, we have

$$\begin{aligned}\hat{M}_{\mathrm{FK},pq} &= i\omega\rho \iiint_{V_p} \nabla \cdot (\hat{\phi}_0 \vec{e}_q \times \vec{s}_p) dV \\ &\quad - i\omega\rho \iint_{S_{wp}} \hat{\phi}_0 [(y - y_p)\delta_{q1} - (x - x_p)\delta_{q2}] dS. \end{aligned} \tag{5.251}$$

In the volume integral here, the integrand may be written in various ways:

$$\begin{aligned}\nabla \cdot (\hat{\phi}_0 \vec{e}_q \times \vec{s}_p) &= \vec{s}_p \cdot [\nabla \times (\hat{\phi}_0 \vec{e}_q)] = -\hat{\phi}_0 \vec{e}_q \cdot (\nabla \times \vec{s}_p) \\ &= \vec{s}_p \cdot (\nabla \hat{\phi}_0 \times \vec{e}_q) = \vec{e}_q \cdot (\vec{s}_p \times \nabla \hat{\phi}_0) = (\vec{s}_p \times \nabla \hat{\phi}_0)_q \\ &= (\vec{s}_p \times \hat{\vec{v}}_0)_q = (\vec{s}_p \times \hat{\vec{a}}_0)_q \frac{1}{i\omega}. \end{aligned} \tag{5.252}$$

Here, $\nabla \hat{\phi}_0 = \hat{\vec{v}}$ represents the fluid velocity $\vec{v}_0$ corresponding to the undisturbed velocity potential ϕ_0, and $\vec{a}_0$ is the corresponding fluid acceleration.

5.6.2 The Diffraction Force

In the next subsection, we shall consider the Froude–Krylov force, the diffraction force and the excitation force for a small body. Then we shall make use of Eqs. (5.244), (5.251) and (5.252). We shall then also approximate the following equation for the diffraction force, which is still exact (within potential theory). Based on Eq. (5.238), we write the diffraction force as

$$\hat{\mathbf{F}}_{d,p} = -i\omega\rho \sum_{p'=1}^{N} \iint_{S_{p'}} \varphi_p \frac{\partial \hat{\phi}_0}{\partial n} dS. \tag{5.253}$$

On $S_{p'}$, we have

$$\frac{\partial \hat{\phi}_0}{\partial n} = \vec{n}_{p'} \cdot \nabla \hat{\phi}_0 = \vec{n}_{p'} \cdot \hat{\vec{v}}_0 = \frac{1}{i\omega} \vec{n}_{p'} \cdot \hat{\vec{a}}_0 = \frac{1}{i\omega} \sum_{q=1}^{3} n_{p'q} \hat{a}_{0q}, \tag{5.254}$$

and hence,

$$\hat{\mathbf{F}}_{d,p} = -\rho \sum_{q=1}^{3} \sum_{p'=1}^{N} \iint_{S_{p'}} \varphi_p n_{p'q} \hat{a}_{0q} dS. \tag{5.255}$$

Here the sum over p' shows that the diffracted waves, not only from body p but also from all the other bodies ($p \neq p'$), contribute to the diffraction force on body p.

5.6.3 Small-Body Approximation for a Group of Bodies

If the horizontal and vertical extensions of body p are very much shorter than one wavelength, we may in Eqs. (5.244) and (5.251), as an approximation, take incident-wave quantities—such as $\hat{\phi}_0$, $\hat{\vec{v}}_0$ and $\hat{\vec{a}}_0$—outside the integral. Then (for $q = 1, 2, 3$) the Froude–Krylov force and moment components are given by

$$\hat{F}_{\text{FK},pq} \approx \rho \hat{a}_{0pq} V_p + \delta_{q3} \rho g \hat{\eta}_{0p} S_{wp}, \tag{5.256}$$

$$\hat{M}_{\text{FK},pq} \approx \rho (\vec{\Gamma}_p \times \hat{\vec{a}}_{0p})_q + \rho g \hat{\eta}_{0p} (S_{yp}\delta_{1q} - S_{xp}\delta_{2q}), \tag{5.257}$$

where $\vec{a}_{0p}$ is the fluid acceleration due to the undisturbed incident wave at the chosen reference point $(x, y, z) = (x_p, y_p, z_p)$—for instance, the centre of mass or the centre of displaced volume—for body p. Furthermore, $\hat{\eta}_{0p}$ is the same wave's elevation at $(x, y) = (x_p, y_p)$. Also,

$$\vec{\Gamma}_p = \iiint_{V_p} \vec{s}_p dV_p, \tag{5.258}$$

$$S_{xp} = \iint_{S_{wp}} (x - x_p) dS, \qquad S_{yp} = \iint_{S_{wp}} (y - y_p) dS \tag{5.259}$$

are moments of the displaced volume V_p and of the water plane area S_{wp} of body p. Equations (4.39) and (5.252) have been used to obtain Eq. (5.257).

The small-body approximation corresponds to replacing the velocity potential $\hat{\phi}_0$ and its derivatives by the first (zero-order) term of their Taylor expansions about the reference point (x_p, y_p, z_p). Thus, assuming that $\hat{\phi}_0$ is as given by Eq. (4.83) or by Eqs. (4.88) and (4.98), we find it evident that the error of the approximation is small, of order $\mathcal{O}\{ka\}$ as $ka \to 0$, where

$$a = \max(|\vec{s}_p|). \tag{5.260}$$

Hence, we expect the approximation to be reasonably good if $ka \ll 1$—that is, if the linear extension of the body is very small compared to the wavelength. For this reason, we call it the small-body approximation or, alternatively, the long-wavelength approximation. The choice of the reference point is of little importance if $k \to 0$. However, in order for the approximation to be reasonably good also for cases when k is not close to zero, particular choices may be better than others. To assist in a good selection of the reference point, comparisons with experiments or with numerical computations may be useful.

Before approximating the diffraction force, we make the further assumption that not only body p but also each of the N bodies have horizontal and vertical extensions very small compared with one wavelength. For each body p', we may expand $\hat{a}_{0q}$ in the integrand of Eq. (5.255) as a Taylor series around the reference point $(x_{p'}, y_{p'}, z_{p'})$, and if we then neglect terms of order $\mathcal{O}\{ka\}$ as

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$ka \to 0$, the j component ($j = 1, 2, \ldots, 6$) of the diffraction force on body p may be approximated to

$$\hat{F}_{d,i} \approx \frac{1}{i\omega} \sum_{p'=1}^{N} \sum_{q=1}^{3} Z_{ii'} \hat{a}_{0p'q} = \sum_{p'=1}^{N} \sum_{q=1}^{3} Z_{ii'} \hat{v}_{0p'q}, \tag{5.261}$$

where $i = 6(p-1) + j$, $i' = 6(p'-1) + q$ and

$$Z_{ii'} = -i\omega\rho \iint_{S_{p'}} \varphi_i n_{p'q} dS \tag{5.262}$$

is an element of the radiation impedance matrix. See Eqs. (5.37), (5.154) and (5.167).

The excitation force is, in accordance with Eq. (5.234), given as the sum of the Froude–Krylov force and the diffraction force. Thus, the j component of the excitation force on body p is given by

$$\hat{F}_{e,i} = \hat{F}_{\mathrm{FK},i} + \hat{F}_{d,i}, \tag{5.263}$$

where $\hat{F}_{d,i}$ is given by Eq. (5.261) and $\hat{F}_{\mathrm{FK},i}$ by Eq. (5.256) for a translational mode ($j = q = 1, 2, 3$) or by Eq. (5.257) for a rotational mode ($j = q + 3 = 4, 5, 6$).

5.6.4 Small-Body Approximation for a Single Body

In the remaining part of this section, let us consider the case of only one body, and thus, we have $N = 1$, $p = p' = 1$, $i = j$ and $i' = q$. Hence, we may omit the subscript p on $\hat{a}_{0pq}$, $\hat{v}_{0pq}$ and $\hat{\eta}_{0p}$. Then the sum over p' in Eq. (5.261) has only one term, and, hence, the diffraction force is given by

$$\hat{F}_{d,j} \approx \frac{1}{i\omega} \sum_{q=1}^{3} Z_{jq} \hat{a}_{0q} = \sum_{q=1}^{3} Z_{jq} \hat{v}_{0q}, \tag{5.264}$$

which, in view of Eq. (5.42), may be written as

$$\hat{F}_{d,j} \approx \sum_{q=1}^{3} (m_{jq} \hat{a}_{0q} + R_{jq} \hat{v}_{0q}). \tag{5.265}$$

As we observed in Section 5.2.4, the radiation impedance for a small body is usually dominated by the radiation reactance; that is,

$$|\omega m_{jq}| = |X_{jq}| \gg R_{jq}. \tag{5.266}$$

On deep water, it can be shown that (see the following paragraphs)

$$|R_{33}/\omega m_{33}| = \mathcal{O}\{ka\} \tag{5.267}$$

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(and even smaller for other jq combinations) as $ka \to 0$. Then the diffraction force is

$$\hat{F}_{d,j} \approx \sum_{q=1}^{3} m_{jq}\hat{a}_{0q}. \tag{5.268}$$

This is a consistent approximation when we neglect terms of order $\mathcal{O}\{ka\}$ relative to terms of $\mathcal{O}\{1\}$. Kyllingstad [64] has given a more consistent higher-order approximation than Eq. (5.265).

The free-surface boundary condition is, in the case of deep water,

$$0 = \left(-\frac{\omega^2}{g} + \frac{\partial}{\partial z}\right)\varphi_j = \left(-k + \frac{\partial}{\partial z}\right)\varphi_j. \tag{5.269}$$

When $k \to 0$, this condition becomes

$$\frac{\partial \varphi_j}{\partial z} = 0 \quad \text{on } z = 0. \tag{5.270}$$

That is, it corresponds to a stiff plate (or ice) on the water. In this case, no surface wave can propagate on the surface as a result of the body's (slow!) oscillation. However, there is still a finite added mass in the limit $k \to 0$, because there is some finite amount of kinetic energy in the fluid; see Eq. (5.190). Small-body approximations (5.256)–(5.257) and reciprocity relation (5.144) show explicitly for a body floating on deep water (with $D = 1$) that when $ka \to 0$, R_{jq}/ω is of $\mathcal{O}\{k\}$ and of $\mathcal{O}\{a^4\}$ for $(j,q) = (3,3)$ and even smaller for other possible jq combinations. Moreover, in agreement with the text which follows Eq. (5.85) [also see Eq. (5.84)], it is reasonable to assume that m_{jq} is of $\mathcal{O}\{k^0\}$ and of $\mathcal{O}\{a^3\}$. Hence, $|R_{jq}/\omega m_{jq}|$ is of $\mathcal{O}\{ka\}$ or smaller.

Neglecting terms of order $\mathcal{O}\{ka\}$, we have, for the translational modes ($j = 1,2,3$), the following approximation for the excitation force on a single small body:

$$\hat{F}_{e,j} \approx \rho g \hat{\eta}_0 S_w \delta_{j3} + \rho V \hat{a}_{0j} + \sum_{q=1}^{3} m_{jq}\hat{a}_{0q}. \tag{5.271}$$

For the rotational modes ($j = 4,5,6$), an approximation to the excitation moment of the small body is obtained by the addition of the two approximations (5.257) and (5.268).

Let us consider the excitation force on a body having two mutually orthogonal vertical planes of symmetry (e.g., the planes $x = 0$ and $y = 0$). A special case of this is an axisymmetric body (the z-axis being the axis of symmetry). Because in this case φ_3 is symmetric, whereas n_1 and n_2 are antisymmetric, it follows from definition (5.38) of the radiation impedance that $Z_{13} = 0$ and $Z_{23} = 0$, and

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hence, $m_{13} = m_{23} = 0$. Based on a similar argument, $m_{12} = 0$. For $j = 1, 2, 3$, the preceding approximation for $F_{e,j}$ simplifies to

$$\hat{F}_{e,j} \approx \rho g \hat{\eta}_0 S_w \delta_{j3} + \rho V(1 + \mu_{jj})\hat{a}_{0j}, \tag{5.272}$$

where

$$\mu_{jj} = m_{jj}/\rho V \tag{5.273}$$

is the non-dimensionalised added-mass coefficient. If the undisturbed velocity potential represents a purely propagating wave, as given by Eqs. (4.88) and (4.98), and if the reference axis p–p of the single body (see Figure 5.13) is the vertical line $(x, y) = (0, 0)$, then $\hat{\eta}_0 = A$. Inserting this into Eq. (5.272), we obtain, for the heave excitation force,

$$F_{e,3} = F_{\mathrm{FK},3} + F_{d,3} \approx \left[\rho g S_w + (\rho V + m_{33})\frac{\hat{a}_{03}}{A}\right] A. \tag{5.274}$$

Using Eq. (5.245) gives

$$\hat{F}_{e,3} \approx \left[\rho g S_w - \omega^2(\rho V + m_{33})\frac{\sinh(kz_1 + kh)}{\sinh(kh)}\right] A, \tag{5.275}$$

where z_1 is the z-coordinate of the reference point for the body. On deep water, we now have

$$\hat{F}_{e,3} \approx [\rho g S_w - \omega^2 \rho V(1 + \mu_{33})\exp(kz_1)]A. \tag{5.276}$$

For a small floating body with $z_1 \approx 0$, this gives

$$\hat{F}_{e,3} \approx [\rho g S_w - \omega^2 \rho V(1 + \mu_{33})]A. \tag{5.277}$$

Note that as $\omega \to 0$, $\hat{F}_{e,3}/A \to \rho g S_w$, which is the hydrostatic (buoyancy) stiffness of the floating body (see Section 5.9.1).

For small values of ω, the first term $\rho g S_w$ dominates over the second term $-\omega^2 \rho V(1 + \mu_{33})$. On deep water, where $\omega^2 = gk$, the second term is approximately linear in k (forgetting the usually small frequency dependence of μ_{33}). This corresponds to the tangent at $ka = 0$ of the curve indicated in Figure 5.15.

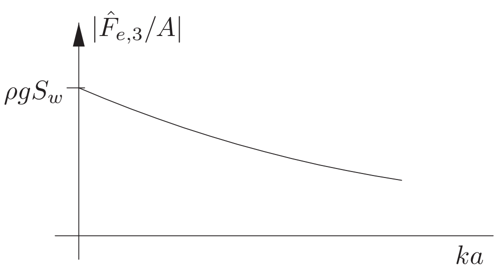

Figure 5.15: Typical variation of the amplitude of the heave force on a floating body for relatively small values of the angular repetency.

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For most body shapes, the heave excitation force $\hat{F}_{e,3}$ is closely in phase with the undisturbed wave amplitude A for those small values of ka, where the small-body approximation is applicable. An exception is a body shape with a relatively small water plane area, as indicated in Figure 5.16. Then the second term of the formula may become the dominating term even for a frequency interval within the range of validity of the small-body approximation. Then there is a phase shift of π between $\hat{F}_{e,3}$ and A within a narrow frequency interval near the angular frequency $\{gS_w/V(1+\mu_{33})\}^{1/2}$.

For a submerged body ($S_w = 0$), we have

$$\hat{F}_{e,3} \approx -\omega^2 \rho V(1+\mu_{33}) \exp(kz_1)A. \tag{5.278}$$

Now $\hat{F}_{e,3}$ and A are in antiphase everywhere within the region of validity of the approximation. This antiphase may be simply explained by the fact that the hydrodynamic pressure, which is in phase with the wave elevation [cf. Eq. (4.89)], is smaller below the submerged body than above it. It should be observed that even if the heave excitation force in the small-body approximation is in antiphase with the wave elevation, the body's vertical motion is essentially in phase with the wave elevation, provided no other external forces are applied to the submerged body. This follows from the fact that the body's mechanical impedance is dominated by inertia. Then the acceleration and excursion are essentially in phase and antiphase, respectively, with the excitation force. (See also Sections 2.2 and 5.9.) For small values of $ka = \omega^2 a/g$, the preceding

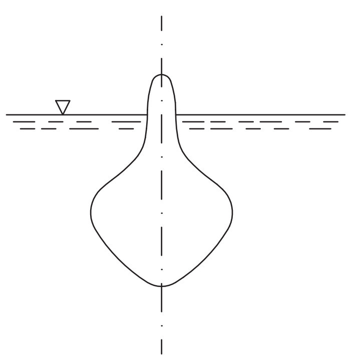

Figure 5.16: Floating body with a relatively small water plane area.

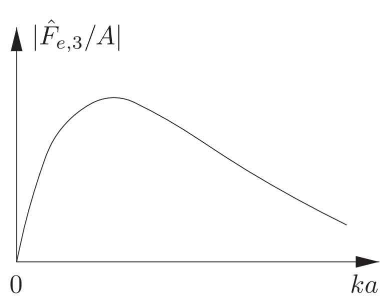

Figure 5.17: Typical variation of the amplitude of the heave force on a submerged body versus the angular repetency.

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approximate formula (5.278) for $\hat{F}_{e,3}$ corresponds to the first increasing part of the force amplitude curve shown in Figure 5.17. The decreasing part of the curve for larger values of ka is due to the exponential factor in formula (5.278).

For a horizontal cylinder whose centre is submerged at least one diameter below the free water surface, the added mass coefficients $\mu_{33} = \mu_{11}$ are close to 1, which is their exact value in an infinite fluid (without any free water surface) [23]. A similar approximation applies for the added-mass coefficients μ_{11}, μ_{22} and μ_{33} for a sufficiently submerged sphere where the added-mass coefficient is $\frac{1}{2}$ in an infinite fluid (cf. [28], p. 144).

5.7 Axisymmetric Oscillating System

A group of concentric axisymmetric bodies is an axisymmetric system. A case with three bodies is shown in Figure 5.18. Concerning the sea bed, we have to assume either deep water or a sea-bed structure which is axisymmetric and concentric with the bodies. (The latter is true if the water depth is constant and equal to h.) Because the reference axis (p–p in Figure 5.13) is the same for all bodies, we may choose it as the z-axis. Correspondingly, the global coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ coincide with the local coordinates $(x_p, y_p, z) = (r_p\cos\theta_p, r_p\sin\theta_p, z)$ associated with each body p (for all p).

For an axisymmetric body p, as shown in Figure 5.19, a wet-surface element dS in position has a unit normal

$$\vec{n}_p = (n_{px}, n_{py}, n_{pz}) = (n_{pr}\cos\theta, n_{pr}\sin\theta, n_{pz}) = (n_{p1}, n_{p2}, n_{p3}). \tag{5.279}$$

Further,

$$\vec{s}_p \times \vec{n}_p = (n_{p4}, n_{p5}, n_{p6}), \tag{5.280}$$

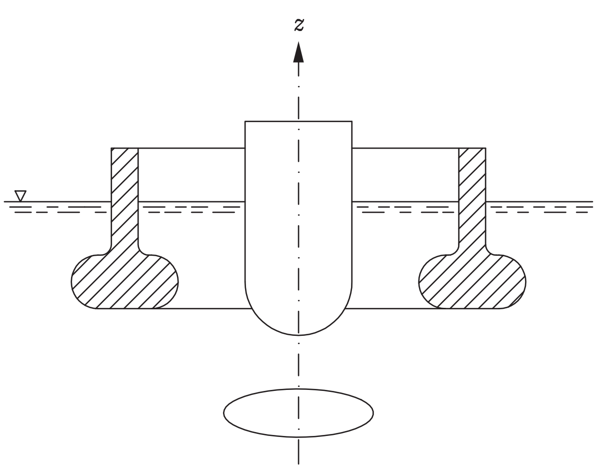

Figure 5.18: Axisymmetric system consisting of three concentric axisymmetric bodies: two floating and one submerged.

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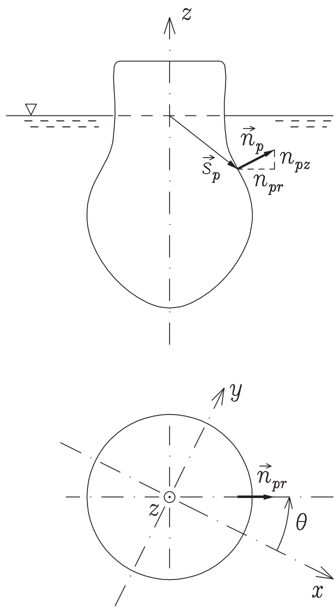

Figure 5.19: Side view and top view of a body symmetric with respect to the z-axis, with unit normal $\vec{n}_p$ on element dS of the wet surface in position $\vec{s}_p$ from the reference point $(0, 0, z_p) = (0, 0, 0)$.

where

$$\begin{aligned} n_{p4} &= yn_{pz} - zn_{py} = (rn_{pz} - zn_{pr})\sin\theta \equiv -n_{pM}\sin\theta \\ n_{p5} &= zn_{px} - xn_{pz} = (zn_{pr} - rn_{pz})\cos\theta = n_{pM}\cos\theta \\ n_{p6} &= xn_{py} - yn_{px} = rn_{pr}(\cos\theta\sin\theta - \sin\theta\cos\theta) = 0. \end{aligned} \tag{5.281}$$

To summarise, for $j = 1, \ldots, 6$, the components of the unit normal may be collected into the column vector

$$\mathbf{n}_p = (n_{pr}\cos\theta, n_{pr}\sin\theta, n_{pz}, -n_{pM}\sin\theta, n_{pM}\cos\theta, 0)^{\mathrm{T}}, \tag{5.282}$$

where

$$n_{pM} = zn_{pr} - rn_{pz}, \qquad n_{pr} = \sqrt{n_{px}^2 + n_{py}^2}. \tag{5.283}$$

Notice that for $j = 6$, boundary condition (5.166) is satisfied if $\varphi_{p6} = 0$, which was to be expected because an axisymmetric body yawing in an ideal fluid can generate no wave. The complex functions $\varphi_{pj} (j = 1, \ldots, 5)$ must satisfy the Laplace equation, and the usual homogeneous boundary conditions on the free surface $z = 0$ and on fixed surfaces such as the sea bed $z = -h$. Moreover, φ_{pj} must satisfy the radiation condition.

For an axisymmetric body, it is easy to see that a particular solution of the type

$$\varphi_{pj} = \varphi_{pj}(r, \theta, z) = \varphi_{pj0}(r, z)\Theta_j(\theta) \tag{5.284}$$

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satisfies Laplace's equation, provided $\Theta_j(\theta)$ is a function of the type given by Eq. (4.200). If we choose

$$\begin{aligned}\Theta_1(\theta) &= \Theta_5(\theta) = \cos\theta \\ \Theta_2(\theta) &= \Theta_4(\theta) = \sin\theta \\ \Theta_3(\theta) &= \Theta_6(\theta) = 1,\end{aligned} \tag{5.285}$$

then also the inhomogeneous boundary condition (5.166) on S_p can be satisfied in accordance with Eq. (5.282). (Note, however, that $\varphi_{p60} \equiv 0$.)

In the far-field region, we have, asymptotically, according to Eqs. (4.211) and (4.214),

$$\varphi_{pj} \sim (\text{constant}) \times \Theta_j(\theta)e(kz)H_n^{(2)}(kr), \tag{5.286}$$

with $n = 0$ for $j = 3$ and $n = 1$ for $j = 1, 2, 4$ and 5.

Since the solution for $\varphi_{pj}(r, \theta, z)$ is of the type of Eq. (5.284) also in the far-field region, we may write the Kochin functions and the far-field coefficients as follows:

$$\begin{aligned}h_{pj}(\theta) &= h_{pj0}\Theta_j(\theta) \\ a_{pj}(\theta) &= a_{pj0}\Theta_j(\theta).\end{aligned} \tag{5.287}$$

By comparing Eqs. (5.284) and (4.236), we see that in the far-field region, all the functions $\varphi_{pj0}(r, z)$ vary in the same way with r and z, since asymptotically we have

$$\varphi_{pj0}(r, z) \sim a_{pj0}e(kz)(kr)^{-1/2}e^{-ikr} \tag{5.288}$$

for $kr \to \infty$. Note that a_{pj0} is independent of r, θ and z.

Similarly, we may, according to Haskind relation (5.227), write the excitation force coefficients as

$$f_{pj}(\beta) = f_{pj0}\Theta_j(\beta \pm \pi), \tag{5.289}$$

where

$$f_{pj0} = \frac{\rho g D}{k} h_{pj0} = \frac{\rho g D}{k}\sqrt{2\pi}\, a_{pj0}\, e^{i\pi/4}. \tag{5.290}$$

Note from Eq. (5.285) that $\Theta_j(\beta \pm \pi) = -\Theta_j(\beta)$ for $j = 1, 2, 4, 5$.

5.7.1 The Radiation Impedance

The radiation impedance matrix $\mathbf{Z}$ is, according to Eq. (5.158), partitioned into 6×6 matrices $\mathbf{Z}_{pp'}$ of which each element is of the type [see Eqs. (5.166) and (5.169)]

$$Z_{pj,p'j'} = -i\omega\rho \iint_{S_p} \varphi_{p'j'} \frac{\partial \varphi_{pj}^*}{\partial n} dS. \tag{5.291}$$

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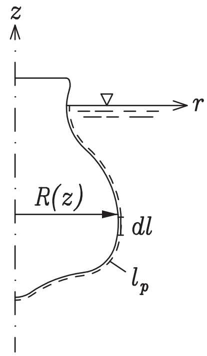

Figure 5.20: Wet surface S_p of an axisymmetric body is a surface of revolution generated by curve l_p.

Let us consider the wet body surface S_p as a surface of revolution generated by curve l_p, as shown in Figure 5.20. Then

$$\begin{aligned} Z_{pj,p'j'} &= -i\omega\rho \int_0^{2\pi} d\theta \int_{l_p} dl\, R(z)\varphi_{p'j'} \frac{\partial \varphi^*_{pj}}{\partial n} \\ &= -i\omega\rho \int_0^{2\pi} \Theta_{j'}(\theta)\Theta_j(\theta) d\theta \int_{l_p} \varphi_{p'j'0} \frac{\partial \varphi^*_{pj0}}{\partial n} R(z) dl. \end{aligned} \tag{5.292}$$

Note that $\varphi_{pj0} = \varphi_{pj0}(r, z)$. Now, since

$$\int_0^{2\pi} (\cos\theta, \sin\theta, \sin\theta\cos\theta) d\theta = (0, 0, 0), \tag{5.293}$$

$$\int_0^{2\pi} (\cos^2\theta, \sin^2\theta, 1^2) d\theta = (\pi, \pi, 2\pi), \tag{5.294}$$

the radiation impedance matrix $\mathbf{Z}_{pp'}$ is of the form

$$\mathbf{Z}_{pp'} = \begin{bmatrix} Z_{p1p'1} & 0 & 0 & 0 & Z_{p1p'5} & 0 \\ 0 & Z_{p1p'1} & 0 & Z_{p1p'5} & 0 & 0 \\ 0 & 0 & Z_{p3p'3} & 0 & 0 & 0 \\ 0 & Z_{p1p'5} & 0 & Z_{p5p'5} & 0 & 0 \\ Z_{p1p'5} & 0 & 0 & 0 & Z_{p5p'5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.295}$$

where we also utilised the fact that for the axisymmetric system, we have $Z_{p2p'2} = Z_{p1p'1}$, $Z_{p4p'4} = Z_{p5p'5}$, $Z_{p2p'4} = Z_{p1p'5}$ and $\varphi_{p6} = \varphi_{p60} \equiv 0$. Note that matrix $\mathbf{Z}_{pp'}$ is singular. Its rank is at most 5. [An $n \times n$ matrix is of rank k ($k \le n$) if there in the matrix exists a non-vanishing minor of dimension $k \times k$, but no larger.]

The *non-vanishing* elements of $\mathbf{Z}$ may be expressed as

$$Z_{pj,p'j'} = -i\omega\rho(1 + \delta_{3j}\delta_{3j'})\pi \int_{l_p} \varphi_{p'j'0} \frac{\partial \varphi^*_{pj0}}{\partial n} R(z) dl, \tag{5.296}$$

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where $\delta_{3j} = 1$ if $j = 3$ and $\delta_{3j} = 0$ otherwise. Thus, we have for the radiation impedance matrix,

$$Z_{pj,p'j'} = -i\omega\rho\sigma_{jj'}\pi \int_{l_p} \varphi_{p'j'0} \frac{\partial\varphi^*_{pj0}}{\partial n} R(z)dl, \tag{5.297}$$

where 27 of the 36 values of the numbers $\sigma_{jj'}$ vanish. There are just nine non-vanishing values, namely

$$\sigma_{33} = 2, \tag{5.298}$$

$$\sigma_{11} = \sigma_{22} = \sigma_{44} = \sigma_{55} = \sigma_{15} = \sigma_{51} = \sigma_{24} = \sigma_{42} = 1. \tag{5.299}$$

5.7.2 Radiation Resistance and Excitation Force

The radiation resistance matrix $\mathbf{R}_{pp'}$ is obtained by taking the real part of Eq. (5.295), which shows that at least 27 of the 36 elements vanish. To calculate the non-vanishing elements $R_{pj,p'j'}$, we take the integral along a generatrix r = constant ($r \to \infty$) of the control cylinder S_∞ in the far field (Figure 5.13) instead of along the curve l_p—that is, $r = R(z)$—as in Eq. (5.297) [cf. Eqs. (5.171) and (5.177)].

Thus, the radiation resistance matrix is

$$R_{pj,p'j'} = -i\omega\rho\sigma_{jj'}\pi \lim_{r\to\infty} \int_{-h}^{0} \varphi_{p'j'0} \frac{\partial\varphi^*_{pj0}}{\partial r} r dz. \tag{5.300}$$

From expression (5.288)—also see Eqs. (4.272), (5.284) and (5.287)—we have the asymptotic expression

$$\varphi_{pj0}(r,z) \sim h_{pj0}e(kz)(2\pi kr)^{-1/2}e^{-i(kr+\pi/4)}. \tag{5.301}$$

Thus, we have

$$\begin{aligned} R_{pj,p'j'} &= -i\omega\rho\sigma_{jj'}\pi h_{p'j'0}h^*_{pj0}\frac{ik}{2\pi k}\int_{-h}^{0} e^2(kz)dz \\ &= \frac{\omega\rho D(kh)}{4k}\sigma_{jj'}h_{p'j'0}h^*_{pj0} = \frac{\omega k}{4\rho g^2 D(kh)}\sigma_{jj'}f_{p'j'0}f^*_{pj0}, \end{aligned} \tag{5.302}$$

where we have used Eqs. (4.108) and (5.290). We note from Eq. (5.302) that the diagonal elements $R_{pj,pj}$ of the radiation resistance matrix are necessarily nonnegative. This is a result which was to be expected, because otherwise the principle of conservation of energy would be violated. If the only oscillation is mode j of body p, the radiated power in accordance with Eq. (5.179) is $\frac{1}{2}R_{pj,pj}|\hat{u}_j|^2$, which, of course, cannot be negative.

Since the radiation resistance matrix is real, it follows from Eq. (5.302) that for those combinations of j and j' for which $\sigma_{jj'}$ *does not vanish*, we have that

$$f_{pj0}f^*_{p'j'0} = f^*_{pj0}f_{p'j'0} \tag{5.303}$$

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is real. This is also in agreement with reciprocity relation (5.174), from which it follows that

$$R_{p'j',pj} = R_{pj,p'j'}. \tag{5.304}$$

Hence,

$$\frac{f_{p'j'0}}{f^*_{p'j'0}} = \frac{f_{pj0}}{f^*_{pj0}} \quad \text{or} \quad \frac{f^2_{p'j'0}}{|f_{p'j'0}|^2} = \frac{f^2_{pj0}}{|f_{pj0}|^2}. \tag{5.305}$$

Taking the square root gives

$$\frac{f_{p'j'0}}{|f_{p'j'0}|} = \pm\frac{f_{pj0}}{|f_{pj0}|}, \tag{5.306}$$

which shows that the excitation force coefficients $f_{p'j'0}$ and f_{pj0} have equal or opposite phases. Note that this applies only if $\sigma_{jj'} \neq 0$. Moreover, note from Eq. (5.302) that

$$R_{pj,p'j'} = \pm\sqrt{R_{pj,pj}R_{p'j',p'j'}} \tag{5.307}$$

is negative if and only if f_{pj0} and $f_{p'j'0}$ have opposite phases.

Considering just body p, we find that the excitation surge force and pitch moment are in equal or opposite phases, and the same can be said about the excitation sway force and roll moment. The only non-vanishing off-diagonal elements of $\mathbf{R}_{pp}$ are

$$R_{p1,p5} = R_{p5,p1} = \pm\sqrt{R_{p1,p1}R_{p5,p5}}, \tag{5.308}$$

$$R_{p2,p4} = R_{p4,p2} = \pm\sqrt{R_{p2,p2}R_{p4,p4}}, \tag{5.309}$$

which are equal due to the axial symmetry. From this, it follows that $\mathbf{R}_{pp}$ is a more singular matrix than $\mathbf{Z}_{pp}$. Their ranks are at most 3 and 5, respectively (see Problem 5.13).

Let us now consider a system of two bodies. Then, since $\sigma_{33} = 2 \neq 0$, Eq. (5.306) shows that the heave excitation forces for the two bodies have either equal or opposite phases. If both bodies are floating (as in the upper part of Figure 5.18), we know that they have the same phase if the frequency is sufficiently low for the small-body approximation (5.272) to be applicable. Then $R_{p3,p'3}$ is positive. However, if one body is floating while the other is submerged (as if the largest, annular body were removed from Figure 5.18), then, since the latter has a vanishing water plane area, the heave excitation forces are in opposite phases for low frequencies where the small-body approximation is applicable. In this case, $R_{p3,p'3}$ is negative. Moreover, for this system, the excitation surge forces and pitch moments for the two bodies have equal or opposite phases. The same may be said about the excitation sway forces and roll moments for the two bodies.

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For the two-body system, the radiation impedance matrix **Z**, a 12×12 matrix, has the structure

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{pp} & \mathbf{Z}_{pp'} \\ \mathbf{Z}_{p'p} & \mathbf{Z}_{p'p'} \end{bmatrix}. \tag{5.310}$$

We see from Eq. (5.295) that at least $4 \times 27 = 108$ of the $4 \times 36 = 144$ matrix elements vanish. Among the $4 \times 9 = 36$ remaining elements, no more than 23 different values are apparently possible because of the symmetry relation $\mathbf{Z}^{\mathrm{T}} = \mathbf{Z}$; see Eq. (5.173). However, there are in fact only 13 different nonzero values if we do not forget that even though the matrix (5.295) has only 9 non-vanishing elements, these 9 elements share only 4 or 5 different values (for $p' = p$ or $p' \neq p$, respectively). The matrix **Z** is singular, and its rank is at most 10. However, as a result of relation (5.302), the rank of the radiation resistance matrix $\mathbf{R} = \mathrm{Re}\{\mathbf{Z}\}$ cannot be more than 3. A physical explanation of this fact will be given in the next chapter (see Section 6.5.2). Let us here just mention that the maximum rank of 3 is related to the fact that there is only three different linearly independent functions $\Theta_j(\theta)$, as given in Eq. (5.285).

If a third body is included in our axisymmetric system, the rank of the 18×18 matrix **Z** cannot be larger than 15, but the rank of the 18×18 matrix **R** is still no more than 3.

If the radiation resistance matrix is singular, it is possible to force the bodies to oscillate in still water, without radiating a wave in the far-field region. Such a cancellation is possible in the far-field region [see asymptotic approximation (5.288)], but not in general in the near-field region, where fluid motion is associated with the added-mass matrix [see Eqs. (5.189)–(5.190)]. As an example, consider surge and pitch oscillations of a single body. Since $\Theta_1(\theta) = \Theta_5(\theta) = \cos\theta$ according to Eq. (5.285), it is possible to choose a combination of $\hat{u}_{11}$ and $\hat{u}_{15}$ such that the far-field waves radiated by the surge mode and the pitch mode cancel each other. From Eqs. (5.163), (5.284), (5.288) and (5.290), this is achieved if

$$\frac{\hat{u}_{11}}{\hat{u}_{15}} = -\frac{a_{150}}{a_{110}} = -\frac{h_{150}}{h_{110}} = -\frac{f_{150}}{f_{110}}. \tag{5.311}$$

A similar interrelationship holds for a combined sway and roll oscillation that does not produce any wave in the far field. Hence, a single immersed axisymmetric body which has a complex velocity amplitude represented, for instance, by the six-dimensional column vector

$$\hat{\mathbf{u}} = \left[-\frac{a_{150}}{a_{110}}\hat{u}_{15} \quad -\frac{a_{140}}{a_{120}}\hat{u}_{14} \quad 0 \quad \hat{u}_{14} \quad \hat{u}_{15} \quad \hat{u}_{16} \right]^{\mathrm{T}} \tag{5.312}$$

does not produce any far-field radiated wave. Observe that the number of mutually independent velocities is three.

Note that for a single axisymmetric body, it is not possible for the far-field radiated wave to vanish unless $\hat{u}_{13} = 0$. However, it is possible for an axisymmetric system composed of two or more bodies to have nonzero heave velocities and a vanishing far-field radiated wave. For example, with a system of two bodies, the isotropic part of the far-field radiated wave is cancelled, provided the heave motions of the two bodies are related through

$$\frac{\hat{u}_{23}}{\hat{u}_{13}} = -\frac{a_{130}}{a_{230}}. \tag{5.313}$$

The $\cos\theta$ part is cancelled, provided the pitch and surge motions of the two bodies are related through

$$\hat{u}_{25} = -(a_{110}u_{11} + a_{150}u_{15} + a_{210}u_{21})/a_{250}. \tag{5.314}$$

Similarly, the $\sin\theta$ part is cancelled, provided the roll and sway motions of the two bodies are related through

$$\hat{u}_{24} = -(a_{120}u_{12} + a_{140}u_{14} + a_{220}u_{22})/a_{240}. \tag{5.315}$$

Moreover, any yaw motions of the two bodies do not radiate any waves. Thus, we have nine mutually independent velocities. Observe that the number of mutually independent velocities equals $6N - r$, the difference between dimensionality $6N$ and rank r of the radiation resistance matrix $\mathbf{R}$.

5.7.3 Example: Two-Body System

Consider two bodies with a common vertical axis of symmetry. One body (number 1) is floating, whereas the other (number 2) is submerged, and we take only the heave mode into account. Then, according to the numbering scheme of Eq. (5.154), the force-excitation coefficients of interest are $f_3(\omega)$ and $f_9(\omega)$, both of which are independent of the angle β of wave incidence. Correspondingly, the radiation impedance matrix $\mathbf{Z}(\omega) = \mathbf{R}(\omega) + i\omega\mathbf{m}(\omega)$ has only the following elements of interest: Z_{33}, Z_{99} and $Z_{39} = Z_{93}$. It follows from Eqs. (4.130), (5.298) and (5.302) that

$$\frac{f_i f_{i'}^*}{8R_{ii'}} = \frac{J}{k|A|^2} = \frac{J\lambda}{2\pi|A|^2} = \frac{\rho g v_g \lambda}{4\pi}, \tag{5.316}$$

where the subscripts i and i' are 3 or 9. Furthermore, $\lambda = 2\pi/k$ is the wavelength, and v_g is the group velocity. (See also the last remark in Section 5.4.3.)

As an example, let us present some numerical results for the following geometry. The upper body is a floating cylinder with a hemisphere at the lower end. The radius is a, and the draft is $2.21a$ at equilibrium. The body displaces a water volume of $\pi a^3(1.21 + 2/3) \approx 5.89a^3$. The lower body is completely submerged, and it is shaped as an ellipsoid, with a radius of $2.27a$ and vertical extension $0.182a$. Its volume is $\pi a^3(2.27^2 \cdot 0.182)2/3 = 1.96a^3$. The centre of the ellipsoid is at a submergence $6.06a$, and the water depth is $h = 15a$.

For this geometry, hydrodynamic parameters have been computed using the computer code ©AQUADYN-2.1. The wet surfaces of the upper and

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lower bodies are approximated by 660 and 540 plane panels, respectively. The numerically computed hydrodynamic parameters are expected to have an accuracy of 2%–3% [65]. Excitation force coefficients $f_3(\omega)$ and $f_9(\omega)$ are shown in Figure 5.21. Observe that there is a frequency range, corresponding to $0.2 < ka < 0.4$, where the heave excitation forces on the two bodies are approximately equal but in opposite directions. We may refer to this as 'force compensation' [66, 67], and this has been utilised in some drilling platforms (semisubmersible floating vessels). Radiation resistances $R_{33}(\omega)$ and $R_{99}(\omega)$ are given by the curves in Figure 5.22. From Eq. (5.316), it follows that $R_{39}^2 = R_{93}^2 = R_{33}R_{99}$, and observing that $f_3/f_9 = -|f_3/f_9|$, we find that $R_{39} = R_{93}$ is negative:

$$R_{39} = R_{93} = -\sqrt{R_{33}R_{99}}. \tag{5.317}$$

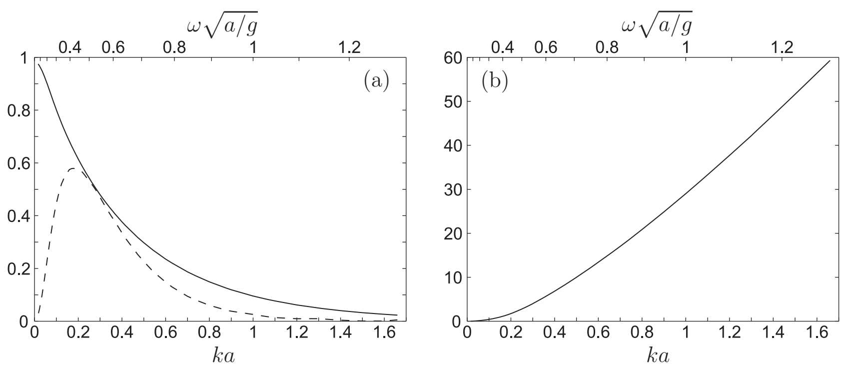

Figure 5.21: (a) Modulus of excitation force coefficients $|f_i(\omega)|/(\rho g\pi a^2)$ with $i = 3$ for the floating upper body (solid curve) and with $i = 9$ for the submerged lower body (dashed curve). (b) Phase angle $\angle f_3(\omega)$ (in degrees) for the heave excitation force coefficient of the floating upper body. It differs by π (180 degrees) from the corresponding phase angle $\angle f_9(\omega)$ for the submerged lower body.

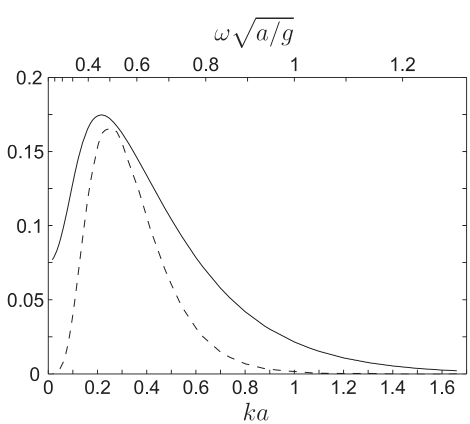

Figure 5.22: Normalised radiation resistances $R_{i,i'}(\omega)/(\omega\rho a^3 2\pi/3)$ for $(i, i') = (3, 3)$ (solid curve) and for $(i, i') = (9, 9)$ (dashed curve), versus normalised angular repetency ka (lower scale) and normalised frequency $\omega\sqrt{a/g}$ (upper scale).

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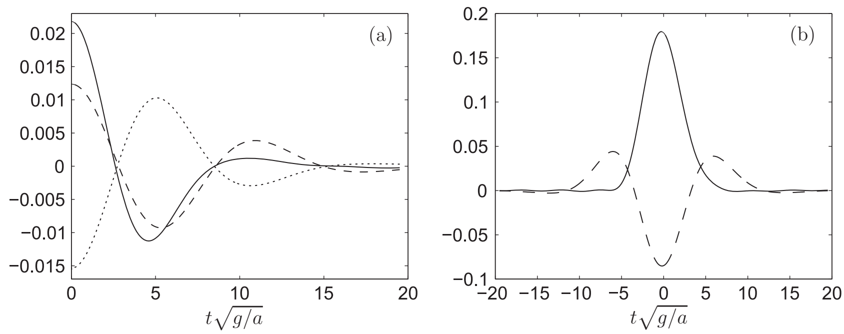

Figure 5.23: (a) Normalised radiation-problem impulse-response functions $k_{ii'}/(\pi\rho g a^2)$ versus normalised time $t\sqrt{g/a}$ for $(i,i') = (3,3)$ (solid curve), for $(i,i') = (9,9)$ (dashed curve) and for $(i,i') = (3,9)$ (dotted curve). (b) Normalised excitation-problem impulse-response function $f_{t,i}(t)(a/g)^{1/2}/(\pi\rho g a^2)$ versus normalised time $t(g/a)^{1/2}$ for $i = 3$ (solid curve) and for $i = 9$ (dashed curve).

The numerically computed values [65] of the added mass at infinite frequency are, in normalised quantity $m_{ii'}(\infty)/(\rho a^3 2\pi/3)$, given by 0.58, 15.0 and -0.156 for (i,i') equal (3,3), (9,9) and (3,9), respectively. These numerical values are needed if the dynamics of the two-body system is to be investigated in the time domain [cf. Section 5.3.1 and also Eq. (5.365)]. For such an investigation, the causal impulse-response functions $k_{33}(t)$, $k_{99}(t)$ and $k_{39}(t) = k_{93}(t)$ are also needed. For $t > 0$, they are, according to Eq. (5.110), equal to twice the inverse Fourier transform of $R_{33}(\omega)$, $R_{99}(\omega)$ and $R_{39}(\omega) = R_{93}(\omega)$, respectively. The curves in Figure 5.23(a) show computed numerical results for these impulse-response functions. Observe that they, together with the numerical values of $m_{ii'}(\infty)$ stated earlier, constitute, in principle, the same quantitative information as the radiation impedances $Z_{33}(\omega)$, $Z_{99}(\omega)$ and $Z_{39}(\omega)$.

Impulse-response functions $f_{t,3}(t)$ and $f_{t,9}(t)$ are inverse Fourier transforms of the excitation-force coefficients $f_3(\omega)$ and $f_9(\omega)$, respectively. The curves in Figure 5.23(b) show computed numerical values for the axisymmetric two-body example. Observe that these impulse-response functions do not vanish for $t < 0$, and hence, they are noncausal, as explained in Section 5.3.2.

5.8 Two-Dimensional System

For a two-dimensional system, there is no variation in the y-direction, and wave propagation is in the x-direction. Each body has three degrees of freedom (or modes of motion)—namely, surge, heave and pitch—corresponding to $j = 1$, $j = 3$ and $j = 5$, respectively.

Let us use a prime symbol on variables to denote physical quantities per unit width. Thus, F', m', Z' and R' denote force, mass, impedance and resistance, respectively, per unit width of the two-dimensional system. The total wave force on body p is per unit width given by

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$$\hat{\mathbf{F}}'_{pt} = \hat{\mathbf{F}}'_{pe} + \hat{\mathbf{F}}'_{pr} = \hat{\mathbf{F}}'_{pe} - \sum_{p'} \mathbf{Z}'_{pp'} \hat{\mathbf{u}}_{p'}, \tag{5.318}$$

which is a three-dimensional column vector. The two terms result from the incident wave and from the bodies' radiation, respectively. The 3×3 coefficient matrix $\mathbf{Z}'_{pp'}$ represents the radiative interaction from body p' on body p.

In analogy with Eq. (5.220), we may define $3N$-dimensional force vectors $\mathbf{F}'_t$ and $\mathbf{F}'_e$ and correspondingly $3N$-dimensional velocity vector $\mathbf{u}$, which are related by

$$\hat{\mathbf{F}}'_t = \hat{\mathbf{F}}'_e - \mathbf{Z}'\hat{\mathbf{u}}, \tag{5.319}$$

where the $3N \times 3N$ matrix $\mathbf{Z}'$ represents the radiation impedance per unit width.

The radiated wave has a velocity potential given by

$$\hat{\phi}_r = \sum_{p=1}^{N} \boldsymbol{\varphi}_p^{\mathrm{T}} \hat{\mathbf{u}}_p = \boldsymbol{\varphi}^{\mathrm{T}} \hat{\mathbf{u}}, \tag{5.320}$$

where $\boldsymbol{\varphi}_p$ and $\boldsymbol{\varphi}$ are coefficient column vectors of dimension 3 and $3N$, respectively. In accordance with Eq. (4.246), the far-field (asymptotic) approximation may be written as

$$\boldsymbol{\varphi} \sim -\frac{g}{i\omega} \mathbf{a}^{\pm} e(kz) e^{\mp ikx} \quad \text{as } x \to \pm\infty. \tag{5.321}$$

Moreover, taking the complex conjugate of Eq. (4.248), we find

$$\mathbf{I}'(\boldsymbol{\varphi}^*, \boldsymbol{\varphi}^{\mathrm{T}}) = -i \left(\frac{g}{\omega}\right)^2 D(kh)(\mathbf{a}^{+*}\mathbf{a}^{+T} + \mathbf{a}^{-*}\mathbf{a}^{-T}), \tag{5.322}$$

and Eq. (4.249) gives

$$\mathbf{I}'(\boldsymbol{\varphi}, \boldsymbol{\varphi}^{\mathrm{T}}) = 0. \tag{5.323}$$

The two-dimensional Kochin function for radiation is given by

$$\mathbf{h}'(0) = -\frac{k}{D(kh)} \mathbf{I}'[e(kz)e^{ikx}, \boldsymbol{\varphi}] = -\frac{gk}{\omega}\mathbf{a}^{+}, \tag{5.324}$$

$$\mathbf{h}'(\pi) = -\frac{k}{D(kh)} \mathbf{I}'[e(kz)e^{-ikx}, \boldsymbol{\varphi}] = -\frac{gk}{\omega}\mathbf{a}^{-}, \tag{5.325}$$

in agreement with Eqs. (4.268), (4.285) and (4.286). The Kochin function $H'_d(\beta)$ $(\beta = 0, \pi)$ for the diffracted wave is given by analogous expressions—with $\boldsymbol{\varphi}$, $\mathbf{a}^{\pm}$ and $\mathbf{h}'$ replaced by $\hat{\phi}_d$, $A_d^{\pm}$ and H'_d, respectively—in Eqs. (5.321)–(5.325). For the diffraction problem (that is, when the bodies are not moving), we may define the reflection coefficient

$$\Gamma_d = A_d^-/A \tag{5.326}$$

[see Eq. (4.84)] and the transmission coefficient

$$T_d = 1 + A_d^+/A. \tag{5.327}$$

These coefficients quantitatively describe how the incident wave of amplitude $|A|$ is divided into a reflected wave and a transmitted wave, which is propagating beyond the bodies. From energy conservation arguments, it is expected that

$$|\Gamma_d|^2 + |T_d|^2 = 1. \tag{5.328}$$

This can indeed be proved mathematically (see Problem 5.14).

Now let us consider some reciprocity relations involving the radiation resistance and the excitation force for a two-dimensional system. The radiation resistance per unit width is given by the $3N \times 3N$ matrix

$$\begin{aligned}\mathbf{R}' &= \frac{1}{2} i\omega\rho \mathbf{I}'(\boldsymbol{\varphi}^*, \boldsymbol{\varphi}^{\mathrm{T}}) = \frac{\rho g^2 D(kh)}{2\omega}(\mathbf{a}^{+*}\mathbf{a}^{+T} + \mathbf{a}^{-*}\mathbf{a}^{-T}) \\ &= \frac{\omega\rho D(kh)}{2k^2}[\mathbf{h}'^*(0)\mathbf{h}'^{\mathrm{T}}(0) + \mathbf{h}'^*(\pi)\mathbf{h}'^{\mathrm{T}}(\pi)] \\ &= \frac{\omega\rho D(kh)}{2k^2}[\mathbf{h}'(0)\mathbf{h}'^{\dagger}(0) + \mathbf{h}'(\pi)\mathbf{h}'^{\dagger}(\pi)].\end{aligned} \tag{5.329}$$

Compare Eqs. (5.176), (5.322), (5.324) and (5.325). Observe that because $\mathbf{R}'$ is real, we may, if we wish, take the complex conjugate of Eq. (5.329) without changing the result. The excitation force per unit width is

$$\mathbf{F}_e' = \mathbf{f}'(\beta)A = i\omega\rho\mathbf{I}'(\hat{\phi}_0, \boldsymbol{\varphi}), \tag{5.330}$$

where, for $\beta = 0$ or $\beta = \pi$,

$$\mathbf{f}'(\beta) = \frac{\rho g D(kh)}{k}\mathbf{h}'(\beta \pm \pi). \tag{5.331}$$

That is,

$$\mathbf{f}'(0) = \frac{\rho g D(kh)}{k}\mathbf{h}'(\pi) = -\frac{\rho g^2 D(kh)}{\omega}\mathbf{a}^-, \tag{5.332}$$

$$\mathbf{f}'(\pi) = \frac{\rho g D(kh)}{k}\mathbf{h}'(0) = -\frac{\rho g^2 D(kh)}{\omega}\mathbf{a}^+. \tag{5.333}$$

Compare Eqs. (5.222), (5.324) and (5.325). Also, the two-dimensional version of Eq. (5.224) has been used. Note that the excitation force on a surging piston in a wave channel, derived previously [see Eq. (5.80)], agrees with the general equation (5.330) combined with Eq. (5.332).

For a single symmetric body oscillating in heave, the wave radiation is symmetric, and hence,

$$\begin{bmatrix} a_3^- \\ h_3'(\pi) \\ f_3'(0) \end{bmatrix} = \begin{bmatrix} a_3^+ \\ h_3'(0) \\ f_3'(\pi) \end{bmatrix}. \tag{5.334}$$

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If it oscillates in surge and/or pitch, the radiation is antisymmetric, and hence,

$$\begin{bmatrix} a_j^- \\ h_j'(\pi) \\ f_j'(0) \end{bmatrix} = - \begin{bmatrix} a_j^+ \\ h_j'(0) \\ f_j'(\pi) \end{bmatrix} \quad \text{for } j = 1 \text{ and } j = 5. \tag{5.335}$$

Thus, according to Eq. (5.329), the radiation resistance matrix for the two-dimensional symmetric body is

$$\mathbf{R}' = \frac{\rho g^2 D(kh)}{\omega} \begin{bmatrix} |a_1^+|^2 & 0 & a_5^+ a_1^{+*} \\ 0 & |a_3^+|^2 & 0 \\ a_1^+ a_5^{+*} & 0 & |a_5^+|^2 \end{bmatrix}. \tag{5.336}$$

In this matrix, it is possible to find a non-vanishing minor of dimension 2×2, but not of dimension 3×3. Hence, $\mathbf{R}'$ is a singular matrix of rank no more than 2.

Since $\mathbf{R}'$ is a symmetrical and real matrix, we may use Eqs. (5.332) and (5.333) and then argue as we did in connection with Eqs. (5.303)–(5.308). We then conclude that the off-diagonal element of $\mathbf{R}'$ is positive or negative when the excitation force coefficients for surge and pitch for the symmetric two-dimensional body are in the same phase or in opposite phases, respectively.

It can be shown, in general, for a two-dimensional system of oscillating bodies, that the system's radiation resistance matrix has a rank which is at most 2. Let $\mathbf{x}$ and $\mathbf{y}$ be two arbitrary column vectors of order $3N$. Then the matrix $\mathbf{xy}^{\mathrm{T}}$ is of rank no more than 1 [16, p. 239]. The matrix $\mathbf{R}'$ in Eq. (5.329) is a sum of two such matrices. Hence, $\mathbf{R}'$ is a matrix of rank 2 or smaller. (The rank is smaller than 2 if the column vectors $\mathbf{a}^+$ and $\mathbf{a}^-$ are linearly dependent.) A physical explanation of why the rank is bounded is given in Section 8.3.

5.9 Motion Response

When an immersed body is not moving, it is subjected to an excitation force $\mathbf{F}_e$ if an incident wave exists. If the body is oscillating, it is subjected to two other forces—namely the radiation force $\mathbf{F}_r$, which we have already considered [see Eq. (5.36) or (5.104)], and a hydrostatic buoyancy force $\mathbf{F}_b$. The latter force originates from the static-pressure term $-\rho g z$ in Eq. (4.12), because the body's wet surface experiences varying hydrostatic pressure as a result of its oscillation. However, oscillation in surge, sway or yaw modes of motion does not produce any hydrostatic force. In the following paragraphs, let us limit ourselves to the study of heave motion in more detail. For a more general treatment of hydrostatic buoyancy forces and moments, refer to textbooks by Newman [28, chapter 6] or Mei et al. [1, chapter 8]. In linear theory, we shall assume that the hydrostatic buoyancy force $\mathbf{F}_b$ is proportional to the excursion $\mathbf{s}$ of the body from its equilibrium position. We write

$$\mathbf{F}_b = -\mathbf{S}_b \mathbf{s}, \tag{5.337}$$

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where the elements of the 'buoyancy stiffness' matrix $\mathbf{S}_b$ are coefficients of proportionality. Many of the 36 elements—for instance, all elements associated with surge or sway—vanish. Moreover, we note that the velocity is

$$\mathbf{u} = \frac{d}{dt}\mathbf{s}. \tag{5.338}$$

There may be additional forces acting on the oscillating body, such as a viscous force $\mathbf{F}_v$ and a control or load force $\mathbf{F}_u$ from a control mechanism or from a mechanism which delivers or absorbs energy. There may also be a friction force $\mathbf{F}_f$ due to this mechanism or other means for transmission of forces. Moreover, there may be mooring forces, which we shall not consider explicitly. We may assume that mooring forces are included in $\mathbf{F}_u$ (or in $\mathbf{F}_b$, if the mooring is flexible and linear theory applicable).

From Newton's law, the dynamic equation for the oscillating body may be written as

$$\mathbf{m}_m \frac{d^2}{dt^2}\mathbf{s} = \mathbf{F}_e + \mathbf{F}_r + \mathbf{F}_b + \mathbf{F}_v + \mathbf{F}_f + \mathbf{F}_u. \tag{5.339}$$

Matrix $\mathbf{m}_m$ represents the inertia of the oscillating body. The three first diagonal elements are $m_{m11} = m_{m22} = m_{m33} = m_m$, where m_m is the mass of the body. Some of the off-diagonal elements of $\mathbf{m}_m$ involving rotary modes of motion may be different from zero (see [28, chapter 6] or [1, chapter 8]). For j and j' equal to 4, 5 or 6, $m_{mjj'}$ is of dimension inertia moment.

The term $\mathbf{F}_u$ is introduced to serve some purpose. It may be a control force intended to reduce the oscillation of a ship or floating platform in rough seas, for instance, or it may represent a load force necessary for conversion of ocean-wave energy. The terms $\mathbf{F}_v$ and $\mathbf{F}_f$ represent unavoidable viscous and friction effects. Thus, although we in Section 4.1 assumed an ideal fluid, in this time-domain formulation, we may make a practical correction by introducing nonzero loss forces in Eq. (5.339). In general, such forces may depend explicitly on time as well as on $\mathbf{s}$ and $\mathbf{u}$. For mathematical convenience, however, we shall assume that $\mathbf{F}_f$ may be written as

$$\mathbf{F}_f = -\mathbf{R}_f\mathbf{u} = -\mathbf{R}_f\frac{d\mathbf{s}}{dt}, \tag{5.340}$$

where $\mathbf{R}_f$ is a constant friction resistance matrix. A viscosity resistance matrix $\mathbf{R}_v$ may be defined in an analogous manner.

5.9.1 Dynamics of a Floating Body in Heave

In this subsection, we assume that the body is constrained to oscillate in the heave mode only. Then we may consider the matrices in Eqs. (5.337), (5.339) and (5.340) as scalars S_b, m_m and R_f.

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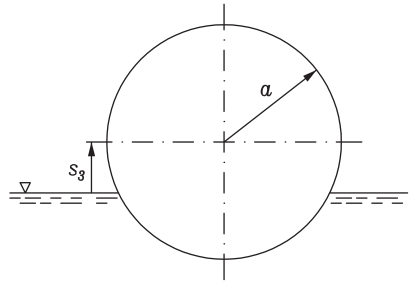

Figure 5.24: Floating sphere displaced a distance s_3 above its equilibrium semisubmerged position.

Let us first determine the buoyancy stiffness S_b. As an example, consider a sphere of radius a and mass $m_m = \rho 2\pi a^3/3$. Thus, in equilibrium, it is semisubmerged. When the sphere is displaced upwards a distance s_3, as shown in Figure 5.24, the volume of displaced water is $\pi(2a^3 - 3a^2 s_3 + s_3^3)/3$, and hence, there is a restoring force $F_b = -\rho g \pi(a^2 s_3 - s_3^3/3)$, from which we see that the buoyancy stiffness is

$$S_b = \pi \rho g a^2 (1 - s_3^2/3a^2) \tag{5.341}$$

for $|s_3| < a$. For small excursions, such that $|s_3| \ll a$, we have

$$S_b \approx \pi \rho g a^2, \tag{5.342}$$

which is independent of s_3. We note that it is ρg multiplied by the water plane area.

For a floating body, in general, the buoyancy stiffness (for small heave excursion s_3) is

$$S_b = \rho g S_w, \tag{5.343}$$

where S_w is the (equilibrium) water plane area of the body. For a floating vertical cylinder of radius a, we have $S_w = \pi a^2$. (For such a cylinder, the water plane area is independent of the heave position s_3. Then the hydrostatic stiffness S_b is constant even if s_3 is not very small.) For a freely floating body, the weight equals $\rho g V$, where V is the volume of displaced water at equilibrium.

For a body constrained to oscillate in the heave mode only, the dynamic equation (5.339) becomes

$$m_m \ddot{s}_3 = F_{e3} + F_{r3} + F_{b3} + F_{v3} + F_{f3} + F_{u3}. \tag{5.344}$$

In terms of complex amplitudes, this may be written as

$$\{-\omega^2[m_m + m_{33}(\omega)] + i\omega[R_v + R_f + R_{33}(\omega)] + S_b\}\hat{s}_3 = \hat{F}_{e3} + \hat{F}_{u3}, \tag{5.345}$$

when use has also been made of Eqs. (5.37), (5.42), (5.337), (5.338) and (5.340). Note that Eq. (5.345) corresponds to the Fourier transform of the time-domain

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equation (5.344) [cf. Eqs. (2.158) and (2.166)]. Introducing the ‘intrinsic’ transfer function

$$G_i(\omega) = S_b + i\omega R_l - \omega^2 m_m + i\omega Z_{33}(\omega), \tag{5.346}$$

where $R_l = R_v + R_f$ is the loss resistance, we write Eq. (5.345) simply as

$$G_i(\omega)\hat{s}_3 = \hat{F}_{e3} + \hat{F}_{u3} \equiv \hat{F}_{\text{ext}}. \tag{5.347}$$

We have adopted the adjective ‘intrinsic’ because all terms in G_i relate to the properties of the oscillating system, and Eq. (5.347) describes quantitatively how the system responds to an external force $F_{\text{ext}} = F_{e3} + F_{u3}$, of which the first term F_{e3} is given by an incident wave and the second term F_{u3} is determined by an operator or by a control device assisting the operation of the oscillating body. In particular cases, F_{e3} or F_{u3} may vanish. In the radiation problem, for instance, there is no incident wave, and $F_{e3} = 0$ ($F_{\text{ext}} = F_{u3}$).

We also introduce the ‘intrinsic mechanical impedance’

$$Z_i(\omega) = Z_{33}(\omega) + R_l + i\omega m_m + S_b[1/i\omega + \pi\delta(\omega)] \tag{5.348}$$

such that

$$G_i(\omega) = i\omega Z_i(\omega) \tag{5.349}$$

and

$$Z_i(\omega)\hat{u}_3 = \hat{F}_{e3} + \hat{F}_{u3} = \hat{F}_{\text{ext}}. \tag{5.350}$$

The last term with $\delta(\omega)$ in Eq. (5.348) secures that the inverse Fourier transform of $Z_i(\omega)$ is causal (see Problem 2.15). This term, which is of no significance if $\omega \neq 0$, may be of importance if the input and output functions in Eq. (5.350) were Fourier transforms rather than complex amplitudes. Note that this term does not explicitly show up in the expression (5.346) for $G_i(\omega)$ since $i\omega\delta(\omega) = 0$.

With given external force $\hat{F}_{e3} + \hat{F}_{u3}$, the solution of Eq. (5.350), for $\omega \neq 0$, is

$$\hat{u}_3 = \frac{\hat{F}_{\text{ext}}}{Z_i(\omega)} = \frac{\hat{F}_{\text{ext}}}{R_{33}(\omega) + R_l + i[\omega m_{33}(\omega) + \omega m_m - S_b/\omega]}, \tag{5.351}$$

where we have used Eq. (5.97) in Eq. (5.348). When the intrinsic reactance vanishes, we say that the system is in resonance. This happens for a frequency $\omega = \omega_0$ for which

$$\text{Im}\{Z_i(\omega_0)\} = 0 \quad \text{or} \quad \omega_0 = \sqrt{S_b/[m_m + m_{33}(\omega_0)]}. \tag{5.352}$$

Then the heave velocity is in phase with the external force. If $R_{33}(\omega)$ has a relatively slow variation with ω, we also have maximum velocity amplitude response $|\hat{u}_3/\hat{F}_{\text{ext}}|$ at a frequency close to ω_0.

Let us now for a moment consider a floating sphere of radius a, having a semisubmerged equilibrium position. When this sphere is heaving on deep water, the radiation impedance depends on the frequency, in accordance with Figure 5.6. The hydrostatic stiffness is $S_b = \rho g \pi a^2$, and the mass is $m_m = \rho(2\pi/3)a^3$. Furthermore, the added mass is $m_{33}(\omega) = m_m \mu_{33}$, where the dimensionless parameter $\mu_{33} = \mu_{33}(ka)$ is in the region $0.38 < \mu_{33} < 0.9$, as appears from Figure 5.6. Note that $\mu_{33} \to 0.5$ as $ka \to \infty$. Insertion into Eq. (5.352) gives the following condition for resonance of the heaving semisubmerged sphere:

$$\omega_0 = \sqrt{\frac{3g}{2a[1 + \mu_{33}(\omega_0)]}}. \tag{5.353}$$

We may solve for ω_0 by using iteration. If, as a first approximation, we assume a value $\mu_{33} \approx 0.5$, we have $\omega_0 \approx \sqrt{g/a}$. Next, we may improve the approximation. For $\omega_0 = \sqrt{g/a}$—that is, $k_0 a = \omega_0^2 a/g = 1$—we have $\mu_{33} = 0.43$. Then we have the following corrected value for the angular eigenfrequency:

$$\omega_0 \approx \sqrt{3g/2a(1 + 0.43)} = 1.025\sqrt{g/a}, \tag{5.354}$$

and, correspondingly, $k_0 a = \omega_0^2 a/g \approx 1.05$. For a sphere of 20 m diameter ($a = 10$ m), the resonance frequency is given by $\omega_0 \approx 1.01$ rad/s, $f_0 = \omega_0/2\pi \approx 0.161$ Hz. The resonance period is $T_0 = 1/f_0 \approx 6.2$ s.

As another example, let us consider a floating slender cylinder of radius a and depth of submergence l, as shown in Figure 5.25. We assume that $a \ll l$. The hydrostatic stiffness is $S_b = \rho g \pi a^2$, and the mass is $m_m = \rho \pi a^2 l$. Furthermore, the added mass is $m_{33} = m\mu_{33}$, where the dimensionless added-mass coefficient μ_{33} depends on ka, l/a and kh. From [68, p. 49], we take the value

$$m_{33} = 0.167\rho(2a)^3 = 0.64(2\pi/3)a^3\rho, \tag{5.355}$$

which is the high-frequency limit ($ka \gg 1$) on deep water ($kh \gg 1$) when the cylinder is relatively tall ($l/a \gg 1$). For the floating, truncated, vertical cylinder discussed in Section 5.2.4 (cf. Figure 5.7), condition $l/a \gg 1$ is not satisfied, and the added mass appears to exceed the value given by Eq. (5.355) by a

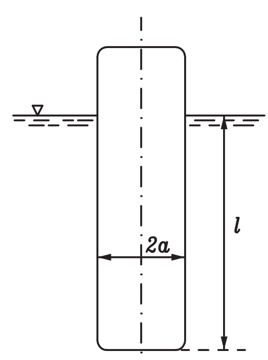

Figure 5.25: Floating vertical cylinder. The diameter is $2a$ and the depth of submergence is l.

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factor between 1.4 and 1.7. If $l \gg a$, we have $m_m \gg m_{33}$, and the angular eigenfrequency (natural frequency) is

$$\omega_0 = \sqrt{S_b/(m_m + m_{33})} \approx \sqrt{S_b/m_m} = \sqrt{g/l}. \tag{5.356}$$

Correspondingly, the resonance period is $T_0 \approx 2\pi\sqrt{l/g}$, which amounts to $T_0 =$ 10 s for a cylinder of draft $l = 25$ m.

Looking at Eqs. (5.347) and (5.350), we interpret $G_i(\omega)$ and $Z_i(\omega)$ as transfer functions of linear systems where the input functions are the heave excursion and heave velocity, respectively. For both systems, the output is the force, which we call external force. These linear systems 'produce' radiation force, friction force, restoring force, and inertial force, which are balanced by the external force.

However, if we consider the heave oscillation to be caused by the external force, it would seem more natural to consider linear systems in which input and output roles have been interchanged. Mathematically, these systems may be represented by

$$s_3(\omega) = H_i(\omega)F_{\text{ext}}(\omega), \tag{5.357}$$

$$u_3(\omega) = Y_i(\omega)F_{\text{ext}}(\omega), \tag{5.358}$$

where we have written Fourier transforms rather than complex amplitudes for the input and output quantities [cf. Eqs. (2.158) and (2.166)]. The transfer functions for these linear systems are

$$H_i(\omega) = 1/G_i(\omega), \tag{5.359}$$

$$Y_i(\omega) = i\omega H_i(\omega). \tag{5.360}$$

[Note that we could not define Y_i as the inverse of Z_i, since the inverse of $\delta(\omega)$ in Eq. (5.348) does not exist; see also Problem 2.15.]

By taking inverse Fourier transforms, we have the following time-domain representation of the four considered linear systems:

$$g_i(t) * s_3(t) = z_i(t) * u_3(t) = F_{\text{ext}}(t), \tag{5.361}$$

$$h_i(t) * F_{\text{ext}}(t) = s_3(t), \qquad y_i(t) * F_{\text{ext}}(t) = u_3(t). \tag{5.362}$$

The impulse response functions $g_i(t)$, $z_i(t)$, $h_i(t)$ and $y_i(t)$ entering into these convolutions are inverse Fourier transforms of $G_i(\omega)$, $Z_i(\omega)$, $H_i(\omega)$ and $Y_i(\omega)$, respectively. Using Table 2.2 and Eqs. (5.103) and (5.107) with Eqs. (5.348) and (5.349), we find

$$z_i(t) = k_{33}(t) + R_l\delta(t) + [m_{33}(\infty) + m_m]\dot{\delta}(t) + S_b U(t), \tag{5.363}$$

$$g_i(t) = \dot{z}_i(t) = \dot{k}_{33}(t) + R_l\dot{\delta}(t) + [m_{33}(\infty) + m_m]\ddot{\delta}(t) + S_b\delta(t). \tag{5.364}$$

As explained in Section 5.3, we know that the impulse response function $k_{33}(t)$ for radiation is causal. Then it is obvious that $z_i(t)$ and $g_i(t)$ are also causal.

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Note that without the term with $\delta(\omega)$ in Eq. (5.348), we would have had $\frac{1}{2}\mathrm{sgn}(t)$ instead of the causal unit step function $U(t)$ in Eq. (5.363), but Eq. (5.364) would have remained unchanged. Combining Eqs. (5.361) and (5.363), we may write the time-domain dynamic equation as

$$(m_{33}(\infty) + m_m)\ddot{s}_3(t) + R_l\dot{s}_3(t) + k_{33}(t) * \dot{s}_3(t) + S_b s_3(t) = F_{\mathrm{ext}}(t) \tag{5.365}$$

when we observe that $\dot{k}_{33}(t) * s_3(t) = k_{33}(t) * \dot{s}_3(t)$, since $k_{33}(t) \to 0$ as $t \to \infty$. [This may easily be shown by partial integration. See also Eq. (2.146).] Similarly, $\dot{\delta}(t) * s_3(t) = \delta(t) * \dot{s}_3(t) = \dot{s}_3(t)$ and $\ddot{\delta}(t) * s_3(t) = \dot{\delta}(t) * \dot{s}_3(t) = \ddot{s}_3(t)$.

It is reasonable to assume that impulse response functions $h_i(t)$ and $y_i(t)$ are also causal; this view may obtain support by studying Problem 2.15. A rigorous proof has, however, been delivered by Wehausen [69].

Problems

Problem 5.1: Surge and Pitch of a Hinged Plate

A rectangular plate is placed in a vertical position with its upper edge above a free water surface, while its lower edge is hinged below the water surface, at depth c. We choose a coordinate system such that the plate is in the vertical plane $x = 0$, and the hinge at the horizontal line $x = 0$, $z = -c$. Further, for this immersed plate, we choose a reference point $(x, y, z) = (0, 0, z_0)$; see Figure 5.3. The plate performs oscillatory sinusoidal rotary motion with respect to the hinge. The angular amplitude is β_5 (assumed to be small), and the angular frequency is ω. Determine the six-dimensional vectors $\mathbf{u}$ and $\mathbf{n}$, defined in Section 5.1.1, for the three cases:

(a) $z_0 = -c$ (reference point at the hinge)
(b) $z_0 = 0$ (reference point at the free water surface)
(c) $z_0 = -c/2$ (reference point midway between hinge and $z = 0$)

Problem 5.2: Excitation Force on a Surging Piston in a Wave Channel

A wave channel is assumed to extend from $x = 0$ to $x \to -\infty$. The water depth is h, and the channel width is d. Choose a coordinate system such that the wet surface of the end wall is $x = 0$, $0 < y < d$, $0 > z > -h$.

In the vertical end wall, there is a rectangular piston of width d_1 and height $a_2 - a_1$, corresponding to the surface

$$x = 0, \quad b < y < c, \quad -a_2 < z < -a_1,$$

where $c - b = d_1$ $(0 < b < c < d)$. In a situation in which the piston is not oscillating, the velocity potential is as given by Eq. (5.30). Derive an expression for the surge excitation force on the piston in terms of A, ρ, g, k, h, a_1, a_2 and d_1.

Problem 5.3: Radiation Resistance for a Vertical Plate

A wave channel of width d and water depth h has in one end ($x = 0$) a wave generator in the form of a stiff vertical plate (rectangular piston) which oscillates harmonically with velocity amplitude $\hat{u}_1$ (pure surge motion) and with angular frequency ω. In the opposite end of the wave channel, there is an ideal absorber. That is, we could consider the channel to be of infinite length ($0 < x < \infty$).

(a) Find an exact expression for the velocity potential of the generated wave in terms of an infinite series, where some parameters are defined implicitly through a transcendental equation.
(b) By means of elementary functions, express the radiation resistance R_{11} in terms of the angular repetency k, the fluid density ρ and the lengths h and d. Draw a curve for $(R_{11}\omega^3/\rho g^2 d)$ versus kh in a diagram.
(c) Also solve the same problem for the case when the plate is hinged at its lower edge (at $z = -h$), and let $\hat{u}_1$ now represent the horizontal component of the velocity amplitude at the mean water level.

Problem 5.4: Flap as a Wave Generator

An incident wave $\hat{\eta} = Ae^{-ikx}$ hits a vertical plate hinged with its lower end at water depth $z = -c$ at a horizontal position $x = 0$, where $0 < c < h$. At $x = 0$, $-h < z < -c$ (below the hinge), there is a fixed vertical wall. Consider the system as a wave channel of width $\Delta y = d$.

(a) Determine the excitation pitch moment $\hat{M}_y = \hat{F}_5$ expressed by ρ, g, k, A, h and d. Give simpler expressions for the two special cases $h \to \infty$ and $c = h$.
(b) Solve the radiation problem by obtaining an expression for $\varphi_5 = \hat{\phi}_r/\hat{u}_5$. Further derive a formula for the radiation impedance $Z_{55} = R_{55} + i\omega m_{55}$.
(c) Check that $R_{55} = |\hat{F}_5/A|^2\omega/(2\rho g^2 Dd)$. Check also that the excitation pitch moment $\hat{F}_5$ is recovered through application of the Haskind relation.

Problem 5.5: Pivoting Vertical Plate as a Wave Generator

This is the same as Problem 5.4, but with the plate extending down to $z = -h$, while the hinge is still at $z = -c$. Show for this case that, with proper choice of c/h ($0 < c < h/2$), it is possible to find a frequency where the radiated wave vanishes—that is, $R_{55} = 0$. [Hint: to find the repetency (wave number) corresponding to this frequency, a transcendental equation involving hyperbolic functions has to be solved. The solution may be graphically represented as the intersection of a straight line with the graph for $\tanh(kh/2)$.]

Problem 5.6: Circular Wave Generator

A vertical cylinder of radius a and height larger than the water depth stands on the sea bed, $z = -h$. The cylinder wall is pulsating in the radial direction

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with a complex velocity amplitude $\hat{u}_r(z) = \hat{u}_0 c(z)$, where the constant $\hat{u}_0 = \hat{u}_r(0)$ represents the pulsating velocity at the mean water level $z = 0$. Note that $c(0) = 1$. Assume that the system is axisymmetric; that is, there is no variation with the coordinate angle θ.

(a) Express the velocity potential $\hat{\phi} = \varphi \hat{u}_0$ for the generated wave as an infinite series in terms of the normalised vertical eigenfunctions $\{Z_n(z)\}$ and the corresponding eigenvalues $\lambda_0 = k^2$, $\lambda_1 = -m_1^2$, $\lambda_2 = -m_2^2, \ldots$.
(b) Express the radiation impedance Z_{00} as an infinite series, and find a simple expression for the radiation resistance R_{00}.

[Hint: for Bessel functions, we have the relations $(d/dx)J_0(x) = -J_1(x)$ and correspondingly for $Y_0(x)$, $H_0^{(2)}(x)$ and $K_0(x)$, whereas $(d/dx)I_0(x) = I_1(x)$. Here I_n and K_n are modified Bessel functions of order n and of the first and second kinds, respectively. Asymptotic expressions are $I_0(x) \to e^x/\sqrt{2\pi x}$ and $K_0(x) \to \pi e^{-x}/\sqrt{2\pi x}$ as $x \to +\infty$. Moreover, we have $J_1(x)Y_0(x) - J_0(x)Y_1(x) = 2/\pi x$.]

Problem 5.7: Radiation Resistance in Terms of Far-Field Coefficients

The time-average mechanical power radiated from an oscillating body may be expressed as

$$P_r = \sum_{j=1}^{6} \sum_{j'=1}^{6} \frac{1}{2} R_{jj'} \hat{u}_j \hat{u}_{j'}^*.$$

In an ideal fluid, this time-average power may be recovered in the radiated wave-power transport—for instance, in the far-field region where the formula for wave-power transport with a plane wave is applicable. In the far-field region, the radiated wave is

$$\hat{\phi}_r = \hat{\phi}_l(r, \theta, z) + A_r(\theta) e(kz) (kr)^{-1/2} e^{-ikr},$$

where the near-field part $\hat{\phi}_l$ is negligible, because $\hat{\phi}_l = \mathcal{O}\{r^{-1}\}$ as $r \to \infty$. Use the decomposition

$$A_r(\theta) = \sum_{j=1}^{6} a_j(\theta) \hat{u}_j$$

in deriving an expression for the radiation resistance matrix $R_{jj'}$ in terms of the far-field coefficient vector $a_j(\theta)$.

Problem 5.8: Excitation-Force Impulse Response

Consider the surge excitation force per unit width of a horizontal strip of a vertical wall normal to the wave incidence. The strip of height Δz covers the

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interval $(z - \Delta z, z)$ where $z < 0$. Show that for the excitation-force impulse-response function, we have

$$\Delta f'_{t,1}(t) = \rho g(g/\pi|z|)^{1/2} e^{gt^2/4z}\Delta z \to 2\rho g\delta(t)\Delta z$$

as $z \to 0$. Compare Sections 4.9.2 and 5.3.2.

Problem 5.9: Froude–Krylov Force for an Axisymmetric Body

The radius of the wet surface of an axisymmetric body with vertical axis $x = y = 0$ is given as the function $a = a(z)$. Show that the heave component of the Froude–Krylov force due to a plane incident wave $\hat{\eta} = Ae^{-ikx}$ is given by

$$f_{\mathrm{FK},3} = F_{\mathrm{FK},3}/A = \pi\rho g \int_{-b}^{-c} J_0(ka)\frac{da^2(z)}{dz} e(kz)\,dz,$$

where J_0 is the zero-order Bessel function of the first kind. Furthermore, b and c are the depth of submergence of the wet surface's bottom and top, respectively. If the body is floating, then $c = 0$. The lower end of the body is at $z = -b$.

For a floating vertical cylinder with hemispherical bottom, we have

$$a^2(z) = \begin{cases} a_0^2 & \text{for } -l < z < 0 \\ a_0^2 - (z+l)^2 & \text{for } -l - a_0 < z < -l, \end{cases}$$

where $l = b - a_0$ is the length of the cylindrical part of the wet surface. We may expand the above exact formula for $f_{\mathrm{FK},3}$ as a power series in ka_0 and kl. Assume deep water ($kh \gg 1$), and find the first few terms of this expansion. Compare with the Froude–Krylov force as obtained from the small-body approximation. [Hint: assume as known the following:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ix\cos\theta}\,d\theta = J_0(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - + \cdots .]$$

Problem 5.10: Radiation-Resistance Limit for Zero Frequency

For a floating axisymmetric body, we write the heave-mode radiation resistance as

$$R_{33} = \epsilon_{33}\omega\rho a^3 2\pi/3,$$

where a is the radius of the water plane area. Show that if the water depth is infinite, then

$$\epsilon_{33} \to ka3\pi/4 \quad \text{as } ka \to 0,$$

while for finite water depth h,

$$\epsilon_{33} \to 3\pi a/(8h) \quad \text{as } ka \to 0.$$

[Hint: make use of Eqs. (5.277) and (5.302).]

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Problem 5.11: Heave Excitation Force for a Semisubmerged Sphere

For a semisubmerged floating sphere of radius a, we write the heave excitation force as $F_3 = \kappa SA = \kappa\rho g\pi a^2 A$, where S is the buoyancy stiffness and A the complex amplitude of the undisturbed incident wave at the centre of the sphere. Further, κ is the non-dimensionalised excitation force. The radiation impedance is

$$Z_{33} = R_{33} + i\omega m_{33} = \omega\rho\tfrac{2}{3}\pi a^3(\epsilon + i\mu).$$

For deep water ($kh \gg 1$), values for $\epsilon = \epsilon(ka)$ and $\mu = \mu(ka)$ computed by Hulme (1982) are given by the following table:

ka	0	0.05	0.1	0.2	0.3	0.4
μ	0.8310	0.8764	0.8627	0.7938	0.7157	0.6452
ϵ	0	0.1036	0.1816	0.2793	0.3254	0.3410
ka	0.5	0.6	0.7	0.8	0.9	1
μ	0.5861	0.5381	0.4999	0.4698	0.4464	0.4284
ϵ	0.3391	0.3271	0.3098	0.2899	0.2691	0.2484

Compute numerical values of $|\kappa|$ from the values of ϵ by using the exact reciprocity relation between the radiation resistance and the excitation force. Also compute approximate values of κ from the values of μ by using the small-body approximation for the excitation force. Compare the values by drawing a curve for the exact $|\kappa|$ and the approximate κ in the same diagram, versus ka $(0 < ka < 1)$.

Problem 5.12: Radiation Resistance for a Cylindrical Body

A vertical cylinder of diameter $2a$ and of height $2l$ is (in order to reduce viscous losses) extended with a hemisphere in its lower end. When the cylinder is floating in equilibrium position, it has a draught of $l + a$ (and the cylinder's top surface is a distance of l above the free water surface). We shall assume deep water in the present problem.

If we assume, for simplicity, that the heave excitation force is

$$\hat{F}_3 \equiv F(l) \approx e^{-kl}F(0), \tag{1}$$

where $F(0)$ is the heave excitation force for a semisubmerged sphere of radius a, then it follows from the reciprocity relationship between the radiation resistance and the excitation force [cf. Eq. (4) in the following] that the radiation resistance for heave is

$$R_{33} \equiv R_0 \approx R_H e^{-2kl}, \tag{2}$$

where R_H is the radiation resistance for the semisubmerged sphere:

$$R_H = \omega\rho(2\pi/3)a^3\epsilon. \tag{3}$$

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Here, $\epsilon = \epsilon(ka)$ is Havelock's dimensionless damping coefficient as computed by Hulme [47]. Compare the following table or the table given in Problem 5.11. [It may be observed from the solution of Problem 5.9 that a relationship like (1) is valid exactly for the Froude–Krylov force. In the small-body case, approximation (1) is valid for the diffraction force if the variation of the added mass with l is neglected. Note, however, that this assumption is not exactly true, because a modification of the radiation resistance according to approximation (2) is associated with some modification of the added mass, according to the Kramers–Kronig relations. See Section 5.3.1.]

For an axisymmetric body on deep water, we have the reciprocity relation

$$R_0 = R_{33} = (\omega k/2\rho g^2)|f_3|^2, \tag{4}$$

where $f_3 = \hat{F}_3/A$ is the heave excitation force coefficient and A is the incident wave amplitude (complex amplitude at the origin). Express the dimensionless parameters $|f_3|/S$ and $\omega R_{33}/S$ in terms of ϵ, ka and kl, where $S = \rho g \pi a^2$ is the hydrostatic stiffness of the body. Compute numerical values for both these parameters with $l/a = 0, 4/3$ and $8/3$ for each of the ka values from the following table:

ka	0.01	0.05	0.1	0.15	0.2	0.3
ϵ	0.023	0.1036	0.1816	0.24	0.2793	0.3254

Problem 5.13: Eigenvalues of a Radiation Resistance Matrix

Given that the diagonal matrix elements are nonnegative, show that the 6×6 radiation resistance matrix for an axisymmetric body has at least three vanishing eigenvalues (or, expressed differently, one triple eigenvalue equal to zero), and hence, its rank is at most $6 - 3 = 3$. Also show that the three remaining eigenvalues are positive (or zero in exceptional cases) and that two of them have to be equal (or, expressed differently, there is a double eigenvalue which is not necessarily zero). [Hint: consider a real matrix of the form

$$\mathbf{r} = \begin{bmatrix} r_{11} & 0 & 0 & 0 & r_{15} & 0 \\ 0 & r_{22} & 0 & r_{24} & 0 & 0 \\ 0 & 0 & r_{33} & 0 & 0 & 0 \\ 0 & r_{24} & 0 & r_{44} & 0 & 0 \\ r_{15} & 0 & 0 & 0 & r_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $r_{33} \geq 0$, $r_{22} = r_{11} \geq 0$, $r_{44} = r_{55} \geq 0$, $r_{24} = r_{15}$ and $r_{15}^2 = r_{11} r_{55}$; cf. Eqs. (5.295), (5.302) and (5.309). The eigenvalues λ are the values which make the determinant of $\mathbf{r} - \lambda \mathbf{I}$ equal to zero, where $\mathbf{I}$ is the identity matrix. Factorising the determinant, which is a sixth-degree polynomial in λ, gives the eigenvalues in a straightforward manner.]

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Problem 5.14: Reciprocity Relations between Diffracted Waves

In agreement with Eq. (5.168), let S be the totality of wet surfaces of oscillating bodies, as indicated in Figure 5.13. It follows from boundary condition (5.165) and definition (4.230) that

$$I(\hat{\phi}_i, \hat{\phi}_j) = 0, \qquad I(\hat{\phi}_i^*, \hat{\phi}_j^*) = 0$$

if $\hat{\phi}_i$ and $\hat{\phi}_j$ are sums of an incident wave and a diffracted wave, namely

$$\hat{\phi}_j = (-g/i\omega)A_j e(kz)\exp\left[-ikr(\beta_j)\right] + \hat{\phi}_{d,j},$$

and similarly for $\hat{\phi}_i$ with subscript j replaced by i. (It is assumed that the bodies indicated in Figure 5.13 are held fixed.) Let us consider two-dimensional cases with two waves either from opposite directions ($\beta_i = \beta_1 = 0$ and $\beta_j = \beta_2 = \pi$, that is, $r(\beta_i) = x$ and $r(\beta_j) = -x$, respectively) or from the same direction ($\beta_i = \beta_j$).

(a) With Kochin functions $H'(\beta) = H'_d(\beta)$ for the diffracted waves, show that

$$A_i H_j'^*(\beta_i) + A_j^* H_i'(\beta_j) = \frac{\omega}{gk}\left[H_i'(0)H_j'^*(0) + H_i'(\pi)H_j'^*(\pi)\right].$$

[Hint: use Eq. (4.291) and the two-dimensional version of Eq. (4.295).]

(b) By considering the diffracted plane waves in the far-field region, let us define transmission coefficients T and reflection coefficients Γ as follows:

$$T_1 = 1 + A_1^+/A_1, \qquad \Gamma_1 = A_1^-/A_1,$$
$$T_2 = 1 + A_2^-/A_2, \qquad \Gamma_2 = A_2^+/A_2,$$

where the relationship between the far-field coefficients $A_j^\pm$ and the Kochin functions H_j' are given by Eqs. (4.285) and (4.286). Derive the following three relations:

$$\Gamma_j\Gamma_j^* + T_jT_j^* = 1 \quad \text{for } j = 1,2, \qquad T_1\Gamma_2^* + T_2^*\Gamma_1 = 0.$$

(The former of these relations represents energy conservation. For a geometrically symmetrical case, the latter relation reduces to a result derived in Problem 4.13.) [Hint: consider the three cases when (i,j) equals (1,1), (2,2) and (1,2).]

(c) Use the two-dimensional version of Eq. (4.271) to show that $T_2 = T_1$, as derived by Newman [34].

Problem 5.15: Diffracted and Radiated Two-Dimensional Waves

According to the boundary condition (5.166), $\partial\varphi/\partial n$ is real on S, the totality of wet surfaces of wave-generating bodies. Hence, we may replace φ by φ^* in

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Eq. (5.222) as well as in Eqs. (5.136)–(5.138). By combining Eqs. (5.222) and (5.224), this gives

$$\mathbf{I}'(\hat{\phi}_0 + \hat{\phi}_d, \boldsymbol{\varphi}^*) = \frac{gD(kh)}{i\omega k}\mathbf{h}'(\beta \pm \pi)A$$

for the two-dimensional case, where $\hat{\phi}_0 + \hat{\phi}_d = \hat{\phi}$ is as given in Problem 5.14 (when the subscript j is omitted).

(a) Use Eq. (4.291) and the two-dimensional version of Eq. (4.295) to show that (for $\beta = 0$)

$$\Gamma_d\,\mathbf{h}'^*(\pi) + T_d\,\mathbf{h}'^*(0) = \mathbf{h}'(\pi),$$

where $\Gamma_d = \Gamma_1$ and $T_d = T_1$. (Here the reflection coefficient Γ_1 and the transmission coefficient T_1 are defined in Problem 5.14.) [Hint: in Eqs. (4.291) and (4.295), set $H_i' = H_d'$, $H_j' = \mathbf{h}'$, $A_i = A$ and $A_j = 0$; cf. Newman [34].]

(b) Consider now a single (two-dimensional) body which has a symmetry plane $x = 0$ and which oscillates in the heave mode only. Show that in the far-field region beyond the body ($x \to \infty$), there is no propagating wave, provided the heave velocity is given as

$$\hat{u}_3 = T_d(gk/\omega h_3')A,$$

since the radiated wave then cancels the transmitted wave for the diffraction problem.

(c) Show that for maximum absorbed wave power, the heave velocity is given by

$$\hat{u}_3 = \hat{u}_{3,\mathrm{OPT}} = (gk/2\omega h_3'^*)A.$$

[Hint: use Eqs. (6.11) and (5.329)–(5.332).] What is the absorbed power if $\hat{u}_3 = 2\hat{u}_{3,\mathrm{OPT}}$?

(d) Find a possible value of $\hat{u}_3$ for which the resulting far-field wave propagating in the negative x-direction ($x \to -\infty$) is as large as the incident wave. Note that this value of $\hat{u}_3$ is not unique, since the phase of the wave propagating in the negative x-direction has not been specified.

(e) Make a comparative discussion of the various results for $\hat{u}_3$ obtained earlier, when we assume that the body is so small that $\Gamma_d \to 0$. Can you say something about the aforementioned unspecified phase for the two 'opposite' cases $\Gamma_d \to 0$ and $\Gamma_d \to 1$? (In the latter case, the body has to be large.)

Problem 5.16: Rectangular Piston in the End Wall of Wave Channel

A wave channel is assumed to extend from $x = 0$ to $x \to -\infty$. The water depth is h, and the channel width is d. Choose a coordinate system such that the wet surface of the end wall is $0 > z > -h$, $0 < y < d$.

In the vertical end wall, there is a rectangular piston of width d_1 and height $a_2 - a_1$, corresponding to the surface $x = 0$, $b < y < c$, $-a_2 < z < -a_1$, where $c - b = d_1$. The piston has a surging velocity of complex amplitude $\hat{u}_1$.

The velocity potential has a complex amplitude $\hat{\phi} = \varphi_1 \hat{u}_1$, where

$$\varphi_1 = \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} b_{nq} \exp(\gamma_{nq} x) \cos\left(\frac{q\pi y}{d}\right) Z_n(z).$$

Moreover,

$$\begin{aligned} \gamma_{nq} &= [m_n^2 + (q\pi/d)^2]^{1/2}, \\ Z_n(z) &= N_n^{-1/2} \cos[m_n(z+h)], \\ N_n &= \tfrac{1}{2} + \sin(2m_n h)/(4m_n h), \end{aligned}$$

and m_n is the $(n+1)$th solution of

$$\omega^2 = -gm \tan(mh)$$

such that, with $m_0 = ik$,

$$k^2 = -m_0^2 > -m_1^2 > -m_2^2 > \ldots > -m_n^2 > \ldots.$$

Show that $\hat{\phi}$ satisfies Laplace's equation in the fluid, the radiation condition at $x = -\infty$, and the homogeneous boundary conditions on the free surface $z = 0$, on the bottom $z = -h$ and on the walls $y = 0$ and $y = d$ of the wave channel. Further, show that the boundary conditions on the wall $x = 0$ are satisfied if the unknown coefficients are given by

$$b_{nq} = \sigma_q (d_1/d) t_q N_n^{-1/2} s_n / \gamma_{nq},$$

where $\sigma_0 = 1$ and $\sigma_q = 2$ for $q \geq 1$, and

$$\begin{aligned} t_q &= (d/q\pi d_1)[\sin(q\pi c/d) - \sin(q\pi b/d)], \\ s_n &= \{\sin[m_n(h-a_1)] - \sin[m_n(h-a_2)]\}/m_n h. \end{aligned}$$

Moreover, show that the radiation resistance is

$$R_{11} = \frac{\omega\rho s_0^2 d_1^2 h}{kN_0 d} \left\{ 1 + \sum_{q=1}^{q_0} \frac{2kt_q^2}{[k^2 - (q\pi/d)^2]^{1/2}} \right\},$$

where q_0 is the integer part of kd/π. Finally, determine the added mass m_{11} expressed as a double sum over n and q.

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