CHAPTER SEVEN

Oscillating Water Columns and Other Types of Wave-Energy Converters

7.1 Oscillating Water Column WECs

Many of the wave-energy converters which have been investigated in several countries are of the oscillating water column (OWC) type. By ‘oscillating water column’, we understand the water contained inside a hollow structure with a submerged opening, where the water inside the structure is communicating with the water of the open sea (see Figure 4.1). The power takeoff may be hydraulic, but pneumatic power takeoff using air turbines is more common. In the latter case, there is a dynamic air pressure above the water surface inside the OWC chamber. In such a case, the OWC may be referred to as a ‘periodic surface pressure’ [97] or an ‘oscillating surface-pressure distribution’ [98]. It is this kind of OWC which is the subject of study in the present section. As with an oscillating body (cf. Chapter 5), two kinds of interaction are considered: the radiation problem and the excitation problem. The radiation problem concerns the radiation of waves due to an oscillating dynamic air pressure above the interface. The excitation problem concerns the oscillation due to an incident wave when the dynamic air pressure is zero. Comparisons are made with wave–body interactions. Wave-energy extraction by OWCs is also discussed.

7.1.1 The Applied-Pressure Description for a Single OWC

In approximate theoretical studies of an OWC, one may think of the internal water surface S_k as an imaginary, weightless, rigid piston which is considered as an oscillating body. Such a theory does not correctly model the hydrodynamics because the boundary condition (4.38) is not exactly satisfied. It may, however, give good approximate results for low frequencies, when the wavelength is very long compared to the (characteristic) horizontal length of the internal water surface S_k. Then, far from resonances of this surface, it moves approximately as a horizontal plane piston.

https://doi.org/10.1017/9781108674812.008 Published online by Cambridge University Press

As opposed to this rigid-piston approximation, general and more correct theoretical results are presented here, based on the linearised hydrodynamical theory for an ideal irrotational fluid [97, 98].

The basic equations for the complex amplitude of the velocity potential are given by Eqs. (4.33)–(4.35) as

$$\nabla^2\hat{\phi} = 0 \quad \text{in the fluid region,} \tag{7.1}$$

$$\left[\frac{\partial\hat{\phi}}{\partial n}\right]_{S_b} = 0 \quad \text{on fixed solid surfaces } S_b, \tag{7.2}$$

$$\left[-\omega^2\hat{\phi} + g\frac{\partial\hat{\phi}}{\partial z}\right]_{S_0} = 0 \quad \text{on the free water surface } S_0, \tag{7.3}$$

$$\left[-\omega^2\hat{\phi} + g\frac{\partial\hat{\phi}}{\partial z}\right]_{S_k} = -\frac{i\omega}{\rho}\hat{p}_k \quad \text{on the internal water surface } S_k. \tag{7.4}$$

Of these four linear equations in $\hat{\phi}$, the first three are homogeneous, whereas Eq. (7.4) is inhomogeneous. The right-hand side of Eq. (7.4) is a driving term due to the dynamic air pressure or 'air-pressure fluctuation' p_k. This system of linear equations may be supplemented with a radiation condition at infinite distance, as discussed in Sections 4.1, 4.3 and 4.6. In the case of an incident wave, this may serve as the driving function.

Let there be an incident wave [cf. Eqs. (4.88) and (4.98)]

$$\hat{\phi}_0 = \frac{-g}{i\omega}e(kz)\hat{\eta}_0, \tag{7.5}$$

where

$$\hat{\eta}_0 = A\exp[-ik(x\cos\beta + y\sin\beta)]. \tag{7.6}$$

It is assumed that the sea bed is horizontal at a depth $z = -h$. The angle of incidence is β with respect to the x-axis. We decompose the resulting velocity potential as

$$\phi = \phi_0 + \phi_d + \phi_r, \tag{7.7}$$

where the two last terms represent a diffracted wave and a radiated wave, respectively. We require

$$\nabla^2\begin{bmatrix}\phi_0\\ \phi_d\\ \phi_r\end{bmatrix} = 0 \qquad \text{in the fluid region,} \tag{7.8}$$

$$\left(-\omega^2 + g\frac{\partial}{\partial z}\right)\begin{bmatrix}\hat{\phi}_0\\ \hat{\phi}_d\\ \hat{\phi}_r\end{bmatrix} = 0 \qquad \text{on } S_0, \tag{7.9}$$

and

$$\frac{\partial}{\partial n}\begin{bmatrix}\phi_0 + \phi_d\\ \phi_r\end{bmatrix} = 0 \qquad \text{on } S_b. \tag{7.10}$$

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(Note that $\partial\phi_0/\partial n = 0$ and, hence, also $\partial\phi_d/\partial n = 0$ on the plane sea bed $z = -h$, which is a part of S_b.) Furthermore,

$$\left(-\omega^2 + g\frac{\partial}{\partial z}\right)(\hat{\phi}_0 + \hat{\phi}_d) = 0 \qquad \text{on } S_k \tag{7.11}$$

$$\left(-\omega^2 + g\frac{\partial}{\partial z}\right)\hat{\phi}_r = -\frac{i\omega}{\rho}\hat{p}_k \qquad \text{on } S_k. \tag{7.12}$$

With the use of Eq. (7.7), it is easy to verify that if the homogeneous Eqs. (7.8)–(7.11) and the inhomogeneous boundary condition (7.12) are all satisfied, then the required Eqs. (7.1)–(7.4) are necessarily satisfied.

Writing

$$\hat{\phi}_r = \varphi_k \hat{p}_k, \tag{7.13}$$

we find that the proportionality coefficient φ_k must satisfy the Laplace equation, the preceding homogeneous boundary conditions for $\hat{\phi}_r$ on S_0 and S_b, and the inhomogeneous boundary condition

$$\left(-\omega^2 + g\frac{\partial}{\partial z}\right)\varphi_k = -\frac{i\omega}{\rho} \qquad \text{on } S_k. \tag{7.14}$$

The coefficient φ_k introduced in Eq. (7.13) is analogous to the coefficient φ_j in Eq. (5.9) or (5.35).

The volume flow produced by the oscillating internal water surface is given by

$$Q_{t,k} = \iint_{S_k} v_z \, dS = \iint_{S_k} \frac{\partial\phi}{\partial z} \, dS. \tag{7.15}$$

The SI unit for volume flow is m^3/s. It is convenient to decompose the total volume flow into two terms:

$$Q_{t,k} = Q_{e,k} + Q_{r,k}, \tag{7.16}$$

where we have introduced an *excitation volume flow*

$$Q_{e,k} = \iint_{S_k} \frac{\partial}{\partial z}(\phi_0 + \phi_d) \, dS \tag{7.17}$$

and a *radiation volume flow*

$$Q_{r,k} = \iint_{S_k} \frac{\partial\phi_r}{\partial z} \, dS. \tag{7.18}$$

Since ϕ_0 and ϕ_d are linear in A and ϕ_r is linear in p_k, we have in terms of complex amplitudes that

$$\hat{Q}_{t,k} = \hat{Q}_{e,k} + \hat{Q}_{r,k} = q_{e,k}A - Y_{kk}\hat{p}_k, \tag{7.19}$$

where we have introduced the *excitation volume-flow coefficient*

$$q_{e,k} = \hat{Q}_{e,k}/A = \iint_{S_k} \frac{\partial}{\partial z}(\hat{\phi}_0 + \hat{\phi}_d)\frac{1}{A}\, dS \tag{7.20}$$

and the *radiation admittance*

$$Y_{kk} = -\iint_{S_k} \frac{\partial \varphi_k}{\partial z}\, dS. \tag{7.21}$$

Note that the excitation volume flow $Q_{e,k}$ is the volume flow when the air-pressure fluctuation is zero ($p_k = 0$). This is a consequence of the decompositions (7.7) and (7.16) which we have chosen (following Evans [98]). Also observe that the larger $|Y_{kk}|$ is, the larger $|Q_{r,k}|$ is admitted for a given $|\hat{p}_k|$. The term 'admittance' for the ratio between the complex amplitudes of volume flow and air-pressure fluctuation is adopted from the theory of electric circuits, where electric admittance is the inverse of electric impedance. Their real/imaginary parts are called conductance/susceptance and resistance/reactance, respectively. Thus, we may decompose the radiation admittance into real and imaginary parts, $Y_{kk} = G_{kk} + iB_{kk}$, where

$$G_{kk} = \mathrm{Re}\{Y_{kk}\} = \tfrac{1}{2}(Y_{kk} + Y_{kk}^*) \tag{7.22}$$

is the *radiation conductance* and

$$B_{kk} = \mathrm{Im}\{Y_{kk}\} \tag{7.23}$$

is the *radiation susceptance*. The SI unit for Y_{kk}, G_{kk} and B_{kk} is $\mathrm{m}^3\,\mathrm{s}^{-1}/\mathrm{Pa} = \mathrm{m}^5\,\mathrm{s}^{-1}\,\mathrm{N}^{-1}$. Thus, in the present context, the product of mechanical impedance (cf. Chapters 2 and 5) and (pneumatic/hydraulic) admittance is not a dimensionless quantity. Its SI unit would be $(\mathrm{N\,s\,m}^{-1})(\mathrm{m}^5\,\mathrm{s}^{-1}\,\mathrm{N}^{-1}) = \mathrm{m}^4$.

7.1.2 Absorbed Power and Radiation Conductance

Whereas the (pneumatic/hydraulic) admittance represents the ratio between volume flow and pressure fluctuation, the product of these two quantities has the dimension of power, in SI units, $(\mathrm{m}^3\,\mathrm{s}^{-1})(\mathrm{Nm}^{-2}) = \mathrm{N\,m\,s}^{-1} = \mathrm{W}$. Proceeding in analogy with Eqs. (6.3)–(6.8), we have for the (time-average) power P absorbed from the wave,

$$P = \overline{p_k(t)Q_{t,k}(t)} = \tfrac{1}{2}\mathrm{Re}\{\hat{p}_k\hat{Q}_{t,k}^*\} = \tfrac{1}{2}\mathrm{Re}\{\hat{p}_k(\hat{Q}_{e,k} - Y_{kk}\hat{p}_k)^*\}, \tag{7.24}$$

where we have used Eq. (7.19) and an analogue of Eqs. (2.75)–(2.76). As in Eq. (6.4), we write the absorbed power as

$$P = P_e - P_r. \tag{7.25}$$

The excitation power is

$$P_e = \tfrac{1}{4}\hat{p}_k\hat{Q}_{e,k}^* + \tfrac{1}{4}\hat{p}_k^*\hat{Q}_{e,k} = \tfrac{1}{2}\mathrm{Re}\{\hat{p}_k\hat{Q}_{e,k}^*\} = \tfrac{1}{2}\mathrm{Re}\{\hat{p}_k q_{e,k}^* A^*\}, \tag{7.26}$$

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and the radiated power is

$$P_r = \tfrac{1}{4}\hat{p}_k\hat{Y}^*_{kk}\hat{p}^*_k + \tfrac{1}{4}\hat{p}^*_k\hat{Y}_{kk}\hat{p}_k = \tfrac{1}{2}G_{kk}|\hat{p}_k|^2, \tag{7.27}$$

where G_{kk} is given by Eq. (7.22). Since $P_r \geq 0$, the radiation conductance G_{kk} cannot be negative.

7.1.3 Reactive Power and Radiation Susceptance

Whereas the radiation conductance G_{kk} is related to radiated power, the radiation susceptance B_{kk} represents reactive power, and it may be related to the difference $W_k - W_p$ between kinetic energy and potential energy in the near-field region of the wave radiated as a result of the air-pressure fluctuation p_k. To be more explicit, it can be shown [63] that

$$\frac{1}{4\omega}B_{kk}|\hat{p}_k|^2 = W_k - W_p, \tag{7.28}$$

which is analogous to Eq. (5.190). Also compare Eqs. (2.85)–(2.88).

As the frequency goes to zero, the kinetic energy W_k tends to zero, whereas the potential energy W_p, in accordance with Eqs. (7.69)–(7.72), reduces to

$$W_{p0} = \rho g S_k|\hat{\eta}_{k0}|^2/4 - \rho g S_k|\eta_{k0}|^2/2 = -\rho g S_k|\hat{\eta}_{k0}|^2/4. \tag{7.29}$$

Note that $\hat{p}_k \to -\rho g\hat{\eta}_k$ as $\omega \to 0$. Thus, as $\omega \to 0$,

$$B_{kk} \to -\frac{4\omega W_{p0}}{|\hat{p}_k|^2} = \frac{\omega S_k}{\rho g}. \tag{7.30}$$

Note that the effect of hydrostatic stiffness is included in the radiation susceptance B_{kk}. This is in contrast to the case of a heaving body, where the effect is represented by a separate term—such as the term S_b in Eq. (5.346)—and not in the radiation reactance (or added mass). From the discussion in Section 5.9.1 on resonances for heaving bodies, it appears plausible that OWCs also have resonances. From physical arguments, we expect that Eq. (5.356) represents approximately the resonance frequency for an OWC contained in a surface-piercing vertical tube which is submerged to a depth l and which has a diameter that is small in comparison with l. Since the effect of hydrostatic stiffness is included in the radiation susceptance B_{kk}, resonance occurs for frequencies where $B_{kk} = 0$—that is, for frequencies where the radiation admittance Y_{kk} is real.

7.1.4 Example: Axisymmetric OWC

As an example, consider a circular vertical tube, the lower end of which just touches the water surface (Figure 7.1). This is an OWC chamber with a particularly simple geometry.

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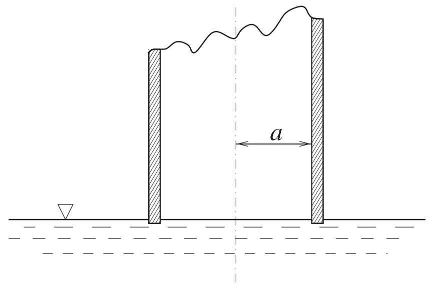

Figure 7.1: A vertical circular tube just penetrating the free water surface represents a simple form of an OWC chamber.

Since diffraction is negligible in this case, it is easy [cf. Problem 7.1 and Eq. (7.45)] to derive expressions for the excitation volume flow $Q_{e,k}$ and the radiation conductance G_{kk}, namely

$$\hat{Q}_{e,k} = q_{e,k} A \quad \text{with } q_{e,k} = \frac{i\omega}{k} 2\pi a J_1(ka), \tag{7.31}$$

$$G_{kk} = \frac{k}{8J} |\hat{Q}_{e,k}|^2 = \frac{\omega k}{2\rho g^2 D} |q_{e,k}|^2 = \frac{2\omega}{\rho g} \left[\pi a J_1(ka)\right]^2 . \tag{7.32}$$

Here J_1 is the first-order Bessel function of the first kind. Since

$$(2/ka) J_1(ka) = 1 + \mathcal{O}\{k^2 a^2\} \quad \text{as } ka \to 0, \tag{7.33}$$

we have

$$\hat{Q}_{e,k} \approx i\omega A \pi a^2 \tag{7.34}$$

in the long-wavelength limit—that is, for $ka \ll 1$. Since $i\omega A$ is the complex amplitude of the vertical component of the fluid velocity corresponding to the incident wave at the origin, this is a result to be expected in the long-wavelength case.

We know that G_{kk} can never be negative. However, G_{kk} (as well as $\hat{Q}_{e,k}$) has the same zeros as $J_1(ka)$, namely $ka = 3.832, 7.016, 10.173, \ldots$.

The derivation of the radiation susceptance

$$B_{kk} = \mathrm{Im}\{Y_{kk}\} \tag{7.35}$$

is more complicated. Results for deep water are given by Evans [98]. The graph in Figure 7.2 demonstrates that B_{kk} is positive for low frequencies, whereas negative values are most typically found for high frequencies. The zero crossings in the graph show the first seven resonances. The first of them is at $ka = 1.96$ corresponding to a disc of radius of approximately three-tenths of a wavelength ($a/\lambda = ka/2\pi = 1.96/2\pi = 0.31$). The corresponding angular eigenfrequency is

$$\omega_0 = \sqrt{k_0 g} = \sqrt{1.96\, g/a} = 1.4\sqrt{g/a}. \tag{7.36}$$

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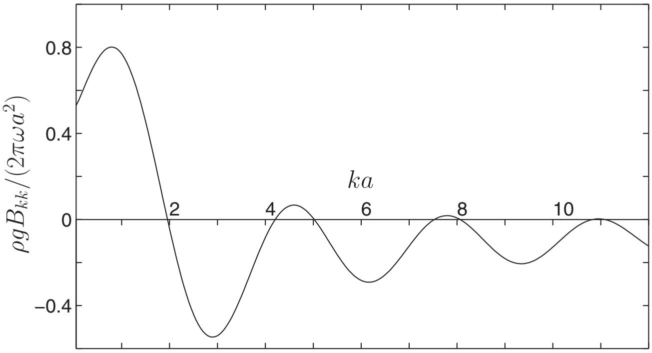

Figure 7.2: Radiation susceptance $B_{kk} = \mathrm{Im}\{Y_{kk}\}$ for a circular OWC with a negligible submergence [98]. Here $(\rho g/2\omega\pi a^2)B_{kk} \to 1/2$ as $ka \to 0$, in agreement with Eq. (7.30).

If we associate this lowest eigenfrequency with that of a heaving, rigid, weightless, circular disc of radius a and, hence, with hydrostatic stiffness $S_b = \rho g \pi a^2$, its added mass at ω_0 would be

$$m_{33} = \frac{S_b}{\omega_0^2} = \frac{\rho g \pi a^2}{1.96\, g/a} = 0.51\, \pi a^3 \rho = 0.76\, \frac{2\pi}{3} a^3 \rho. \tag{7.37}$$

This equals approximately three-quarters of the mass of water displaced by a hemisphere of radius a. Note that the effect of hydrostatic stiffness is included in the radiation susceptance B_{kk} (as opposed to the radiation reactance of a floating body such as, for instance, the envisaged rigid disk). In the long-wavelength limit ($ka \to 0$), this hydrostatic effect is the sole contribution to B_{kk}. Thus, in accordance with Eq. (7.30),

$$B_{kk} \to \omega \pi a^2/(\rho g) \quad \text{as } ka \to 0. \tag{7.38}$$

Note that the considered circular tube with negligible submergence can sustain only infinitesimal amplitudes. A more practical case would require a finite submergence of the tube.

7.1.5 Maximum Absorbed Power

In the present section, let us make a discussion analogous to that of Section 6.2.1 for the case of an oscillating body. The excitation power, as given by Eq. (7.26), may be written as

$$P_e = \frac{1}{2}|\hat{p}_k \hat{Q}_{e,k}| \cos(\gamma_k) = \frac{1}{2}|\hat{p}_k q_{e,k} A| \cos(\gamma_k), \tag{7.39}$$

where γ_k is the phase angle between the air-pressure fluctuation $\hat{p}_k$ and the excitation volume flow $\hat{Q}_{e,k}$. Note that, for a given incident wave and a given phase angle, P_e is linear in the air-pressure amplitude $|\hat{p}_k|$, whereas the radiated power P_r, as given by Eq. (7.27), is quadratic in $|\hat{p}_k|$. Hence, the absorbed power

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$P = P_e - P_r$ may be represented by a parabola, just as in the case of an oscillating body (cf. Figure 6.3). It is easy to show that P is maximum when $\hat{p}_k$ has the optimum value

$$|\hat{p}_k|_{\text{opt}} = \frac{|\hat{Q}_{e,k}|}{2G_{kk}} \cos(\gamma_k), \tag{7.40}$$

corresponding to the maximum absorbed power

$$P = P_{\max} \equiv \frac{|\hat{Q}_{e,k}|^2}{8G_{kk}} \cos^2(\gamma_k). \tag{7.41}$$

Also note that

$$P_{r,\text{opt}} = \tfrac{1}{2} P_{e,\text{opt}} = P_{\max} \tag{7.42}$$

and that $P = 0$ for $|\hat{p}_k| = 2|\hat{p}_k|_{\text{opt}}$.

Further, if it may be accomplished (by resonance or by phase control) that the air-chamber pressure fluctuation p_k is in phase with the excitation volume flow $Q_{e,k}$, then $\gamma_k = 0$, and the absorbed power takes the maximum value

$$P_{\text{MAX}} = \frac{|\hat{Q}_{e,k}|^2}{8G_{kk}} \tag{7.43}$$

when

$$\hat{p}_k = \hat{p}_{k,\text{OPT}} = \frac{\hat{Q}_{e,k}}{2G_{kk}}. \tag{7.44}$$

This maximum absorbed power must correspond to maximum destructive interference, as discussed in Section 6.1. Thus, we expect that Eqs. (6.103)–(6.104) also hold if the optimally absorbing system is an axisymmetric OWC. With this in mind, it follows from Eqs. (7.20), (7.43) and (6.103) that

$$G_{kk} = \frac{k}{8J} |\hat{Q}_{e,k}|^2 = \frac{\omega k}{2\rho g^2 D(kh)} |q_{e,k}|^2. \tag{7.45}$$

This reciprocity relation for the axisymmetric case, where $\hat{Q}_{e,k} = q_{e,k} A$ is independent of the angle β of wave incidence, is analogous to the reciprocity relation (5.146) for a heaving axisymmetric body.

If we define complex quantities U and $E(\beta)$ as

$$|U|^2 = P_r = \tfrac{1}{2} G_{kk} |p_k|^2, \tag{7.46}$$

$$E(\beta) = p_k^* q_e(\beta)/4, \tag{7.47}$$

we may see that the absorbed power P of an OWC as given by Eqs. (7.25)–(7.27) may be expressed in terms of U and $E(\beta)$ in the same way as Eq. (6.32). This means that Eqs. (6.34)–(6.37)—especially the simple formula (6.37), which represents the wave-power island in Figure 6.5—are also applicable to an OWC. We may see, for instance, that Eqs. (6.34) and (6.35) lead to the same expressions for P_{MAX} and $\hat{p}_{k,\text{OPT}}$ as given by Eqs. (7.43) and (7.44).

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7.1.6 Reciprocity Relations for an OWC

For the OWC, there are reciprocity relations similar to those of the oscillating body. At present, let us just state some of the relations and defer the proofs to Section 8.2.6. Relations for the three-dimensional case as well as for the two-dimensional case are given next.

In presenting the reciprocity relations, we need the far-field coefficients for radiated waves. In the three-dimensional case, this coefficient $a_k(\theta)$ is, in analogy with Eq. (5.123), given by the asymptotic expression for the radiated wave:

$$\hat{\phi}_r \sim \hat{p}_k a_k(\theta) e(kz)(kr)^{-\frac{1}{2}} e^{-ikr} \tag{7.48}$$

as $kr \to \infty$. The two-dimensional far-field coefficient $a_k^{\pm}$ is, in analogy with Eqs. (4.88) and (5.321), defined by the asymptotic expression

$$\hat{\eta}_r \sim \hat{p}_k a_k^{\pm} e^{-ik|x|}, \tag{7.49}$$

where the plus and minus signs refer to radiation in the positive and negative x-directions, respectively.

For the three-dimensional case, the relations analogous to Eqs. (5.135) and (5.145) are as follows. The radiation conductance $G_{kk} = \mathrm{Re}\{Y_{kk}\}$ is related to the far-field coefficient $a_k(\theta)$ by

$$G_{kk} = \frac{\omega\rho D(kh)}{2k} \int_0^{2\pi} |a_k(\theta)|^2 \, d\theta \tag{7.50}$$

and to the excitation volume flow $\hat{Q}_{e,k}(\beta)$ by

$$G_{kk} = \frac{k}{16\pi J} \int_{-\pi}^{\pi} |\hat{Q}_{e,k}(\beta)|^2 \, d\beta. \tag{7.51}$$

We may observe that Eq. (7.45) follows directly from Eq. (7.51) when $\hat{Q}_{e,k}$ is independent of β. The analogue of the Haskind relation is

$$q_{e,k}(\beta) = \hat{Q}_{e,k}(\beta)/A = -\rho g[D(kh)/k]\sqrt{2\pi} a_k(\beta \pm \pi) e^{i\pi/4}. \tag{7.52}$$

The minus sign is not a misprint, as the reader might suspect when comparing with Eq. (5.142). The minus sign is there because a positive air pressure (that is, an increase in air pressure) pushes the OWC downward instead of upward, resulting in a negative volume flow. On the other hand, a positive vertical force (that is, an upward force) on a floating body pushes it also upward, resulting in a positive body velocity.

For the case of propagation in a wave channel of width d under conditions of no cross waves—that is, for $kd < \pi$ (see Problem 4.6)—the reciprocity relations for the radiation conductance per unit width are

$$G'_{kk} = \frac{\rho g^2 D(kh)}{2\omega} (|a_k^+|^2 + |a_k^-|^2) \tag{7.53}$$

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$$G'_{kk} = \frac{1}{8J}\left(\left|\frac{\hat{Q}_{e,k}(0)}{d}\right|^2 + \left|\frac{\hat{Q}_{e,k}(\pi)}{d}\right|^2\right) = \frac{1}{8J}\left(|\hat{Q}'_{e,k}(0)|^2 + |\hat{Q}'_{e,k}(\pi)|^2\right), \tag{7.54}$$

whereas the excitation volume flow may be expressed as

$$\hat{Q}_{e,k}(0) = A\frac{\rho g^2 D(kh)d}{\omega}a_k^- \tag{7.55}$$

if the incident wave originates from $x = -\infty$ (that is, if $\beta = 0$) and

$$\hat{Q}_{e,k}(\pi) = A\frac{\rho g^2 D(kh)d}{\omega}a_k^+ \tag{7.56}$$

if it originates from $x = +\infty$ (i.e., $\beta = \pi$). These relations apply, of course, to the two-dimensional case, since it was assumed that no cross wave exists in the wave channel. For the two-dimensional case, we have introduced quantities per unit width (in the y-direction) denoted by $\hat{Q}'_{e,k}$ for the excitation volume flow and G'_{kk} for the radiation conductance. These reciprocity relations for a two-dimensional OWC are analogous to similar reciprocity relations for two-dimensional bodies. See Eqs. (5.329)–(5.333).

Let us, as an application, consider the two-dimensional situation with an incident wave (in the positive x-direction) in a wave channel of width $d < \pi/k$. A combination of Eqs. (7.54) and (7.43) gives, for this case,

$$P_{\mathrm{MAX}} = \frac{Jd\,|\hat{Q}_{e,k}(0)|^2}{|\hat{Q}_{e,k}(0)|^2 + |\hat{Q}_{e,k}(\pi)|^2} \tag{7.57}$$

or, alternatively,

$$P_{\mathrm{MAX}} = \frac{Jd}{1 + |a_k^+/a_k^-|^2}, \tag{7.58}$$

where Eqs. (7.55) and (7.56) have been used. Note that the OWC result, Eq. (7.58), is analogous to the oscillating-body result, Eq. (8.147). [To see this, one may find it helpful to use Eqs. (5.324) and (5.325) to express the Kochin functions in terms of far-field coefficients.] For an OWC which radiates symmetrically (equal waves in both directions, i.e., $a_k^+ = a_k^-$), we have $P_{\mathrm{MAX}} = Jd/2$, which corresponds to 50% absorption of the incident wave power. On the other hand, if the OWC is unable to radiate in the positive direction ($a_k^+ = 0$), then $P_{\mathrm{MAX}} = Jd$, which means 100% wave absorption. The case of $a_k^+ = 0$ may be realised, for instance, by an OWC spanning a wave channel at the downstream end where the absorbing beach has been removed. With an open air chamber ($\hat{p}_k = 0$), no power is absorbed ($P = 0$) and the incident wave will be totally reflected. With optimum air-chamber pressure, according to Eq. (7.44), we have 100% absorption, which means that the optimum radiated wave in the negative direction just cancels out the reflected one.

7.1.7 OWC with Pneumatic Power Takeoff

Next, let us discuss the practical possibility of achieving an optimum phase and optimum amplitude of the air-chamber pressure. We assume that the air chamber has a volume V_a and that an air turbine is placed in a duct between the chamber and the outer atmosphere. For simplicity, we represent the turbine by a pneumatic admittance Λ_t, which we assume to be a constant at our disposal. (To a reasonable approximation, a Wells turbine [99] is linear, and its Λ_t is real and, within certain limits, inversely proportional to the speed of rotation.) We assume that Λ_t is independent of the air-chamber pressure $\hat{p}_k$, which requires that

$$|\hat{p}_k| \ll p_a, \tag{7.59}$$

where p_a is the ambient absolute air pressure.

If we neglect air compressibility, the volume flow at the turbine equals that at the internal water surface, which means that the air pressure is given by

$$\hat{p}_k = \hat{Q}_{t,k}/\Lambda_t. \tag{7.60}$$

If air compressibility is taken into consideration, it can be shown (see Problem 7.2) that

$$\hat{p}_k = \hat{Q}_{t,k}/\Lambda, \tag{7.61}$$

where

$$\Lambda = \Lambda_t + i\omega(V_a/\kappa p_a) \tag{7.62}$$

and κ is the exponent in the gas law of adiabatic compression ($\kappa = 1.4$ for air). The imaginary part of Λ may be of some importance in a full-scale OWC, but it is usually negligible in downscaled laboratory model experiments.

Thus, the dynamics of the OWC determines the air-chamber pressure by the relation

$$\hat{p}_k = q_{e,k}A/(Y_{kk} + \Lambda), \tag{7.63}$$

which is obtained by combining Eqs. (7.19) and (7.61). Hence, the optimum phase condition (that $\hat{p}_k$ is in phase with the excitation volume flow $\hat{Q}_{e,k} = q_{e,k}A$ or that $\gamma_k = 0$) is fulfilled if

$$\mathrm{Im}\{Y_{kk} + \Lambda\} = B_{kk} + \frac{\omega V_a}{\kappa p_a} = 0. \tag{7.64}$$

Compare Eqs. (7.23) and (7.62). If air compressibility is negligible, this corresponds to the hydrodynamic resonance condition $B_{kk} = 0$. Moreover, comparison of Eqs. (7.63) and (7.44) shows that the amplitude $\hat{p}_k$ is optimum if the additional condition

$$\mathrm{Re}\{\Lambda\} = G_{kk} \equiv \mathrm{Re}\{Y_{kk}\} \tag{7.65}$$

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is satisfied. That is, we have to choose a (real) turbine admittance Λ_t which is equal to the radiation conductance G_{kk}.

As a numerical example, let us consider an axisymmetric OWC, as shown in Figure 7.1, and let the diameter be $2a = 8$ m. Thus, the internal water surface is $S_k = \pi a^2 = 50\,\mathrm{m}^2$. Further, assume that the average air-chamber volume is $V_a \approx 300$ m^3 and the ambient air pressure is $p_a = 10^5$ Pa. For a typical wave period $T = 2\pi/\omega = 9$ s, the (deep-water) wavelength is $\lambda = 2\pi/k = 126$ m—that is, $ka = 0.20$. Thus $J_1(ka) = 0.099$, and Eq. (7.32) gives $G_{kk} = 2.2 \times 10^{-4}$ m^5/(s N). From the curve in Figure 7.2, we see that $(\rho g/2\pi\omega a^2)B_{kk} \approx 0.5$—that is, $B_{kk} \approx 0.0018$ m^5/(s N), which is larger than G_{kk} by one order of magnitude. Taking into consideration the effect of air compressibility, we have $\mathrm{Im}\{\Lambda\} = \omega V_a/(\kappa p_a) = 0.0015$ m^5/(s N), which is also positive and of the same order of magnitude as B_{kk}. Hence, the optimum phase condition is far from being satisfied in this example. If, however, the phase could be optimised by some artificial means—for instance, by control of air valves in the system—then the optimum turbine admittance would be $\Lambda_t = G_{kk} \approx 0.0002$ m^5/(s N).

Finally, let us add some remarks of practical relevance. We have used linear analysis and neglected viscous effects. This is a reasonable approximation to reality only if the oscillation amplitude $\hat{s}_b$ of the water at the barrier (the inlet mouth) of the OWC does not exceed the radius of curvature ρ_b at the barrier. Then the so-called Keulegan–Carpenter number $N_{\mathrm{KC}} = \pi|\hat{v}_b|/(\omega\rho_b)$, where v_b is the water velocity at the barrier, is small and is less than π. See Eq. (6.80). In practice, the circular tube of the OWC has to have a finite wall thickness and a finite submergence, in contrast to the case discussed in connection with Figure 7.1, where both these dimensions were assumed to approach zero. If the wall thickness is, say, 1 m, the radius of curvature at the lower end could be 0.5 m. Then we may expect that the present linear theory is applicable if the volume flow does not exceed $0.5\,\mathrm{m} \times 50\,\mathrm{m}^2 \times 2\pi/(9\,\mathrm{s}) \approx 17\,\mathrm{m}^3/\mathrm{s} \sim 20\,\mathrm{m}^3/\mathrm{s}$, which, according to Eq. (7.43), corresponds to P_{MAX} not exceeding $(20\ \mathrm{m}^3/\mathrm{s})^2/(8 \times 0.0002\ \mathrm{m}^5\,\mathrm{s}^{-1}\,\mathrm{N}^{-1}) = 0.25 \times 10^6$ W $= 0.25$ MW. This limit for linear behaviour may be increased by increasing the radius of curvature at the lower barrier end. This may be achieved, for instance, by making the lower end of the vertical tube somewhat horn shaped.

7.1.8 Potential Energy of an Oscillating Water Column

The potential energy inside an OWC is associated with water being lifted against gravity and also against the air pressure. Per unit horizontal area of the mean air–water interface S_k of the OWC, the potential energy relative to the sea bed (cf. Figure 4.1 and Section 4.4.1) is equal to

$$(\rho g/2)(h + z_k + \eta_k)^2 + (p_{k0} + p_k)\eta_k. \tag{7.66}$$

The increase in relation to calm water is

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$$(\rho g/2)\eta_k^2 + \rho g h \eta_k + p_{\text{atm}}\eta_k + p_k\eta_k, \tag{7.67}$$

where we have used Eq. (4.13). The second and third terms have vanishing average values. Hence, the time-average potential energy per unit horizontal area is

$$E_p|_{S_k} = (\rho g/2)\overline{\eta_k^2(t)} + \overline{p_k(t)\eta_k(t)}, \tag{7.68}$$

which agrees with Eq. (4.114), applicable for the case in which the dynamic air pressure p_k is zero. Note that although $p_k(t)$ does not vary with the horizontal coordinates, $\eta_k(t)$ may depend on the horizontal coordinates (x, y). Thus, $E_p|_{S_k}$, which by definition is independent of time, may also depend on the horizontal coordinates (x, y).

For cases in which η_k and p_k vary sinusoidally with angular frequency ω, we may write $E_p|_{S_k}$ in terms of complex amplitudes:

$$E_p|_{S_k} = (\rho g/4)|\hat{\eta}_k|^2 + (1/2)\text{Re}\{\hat{p}_k\hat{\eta}_k^*\} = (\rho g\hat{\eta}_k\hat{\eta}_k^* + \hat{p}_k\hat{\eta}_k^* + \hat{p}_k^*\hat{\eta}_k)/4. \tag{7.69}$$

In the case of zero dynamic air pressure, only the first term remains, in agreement with Eq. (4.115).

Note that the first term in the preceding expressions for $E_p|_{S_k}$, if multiplied by infinitesimal horizontal area ΔS, may be interpreted as the potential energy of a vertical water column of cross section ΔS and stiffness $\rho g \Delta S$. This is easily seen by comparing with Eqs. (2.84) and (2.85). For a semisubmerged column-shaped rigid body, the corresponding stiffness is called buoyancy stiffness or hydrostatic stiffness; see Section 5.9.1.

Next, we wish to express $E_p|_{S_k}$ in terms of the velocity potential. Using Eqs. (4.38) and (7.4), we find

$$\hat{\eta}_k = \frac{1}{i\omega}\left[\frac{\partial\hat{\phi}}{\partial z}\right]_{S_k}, \tag{7.70}$$

$$\hat{p}_k = -i\omega\rho\left[\hat{\phi} - \frac{g}{\omega^2}\frac{\partial\hat{\phi}}{\partial z}\right]_{S_k}. \tag{7.71}$$

Thus, we may express the time-average potential-energy surface density (which is a real quantity) in terms of the (complex) velocity potential as

$$E_p|_{S_k} = -\frac{\rho g}{4\omega^2}\left[\frac{\partial\hat{\phi}}{\partial z}\frac{\partial\hat{\phi}^*}{\partial z}\right]_{S_k} + \frac{\rho}{4}\left[\hat{\phi}\frac{\partial\hat{\phi}^*}{\partial z} + \hat{\phi}^*\frac{\partial\hat{\phi}}{\partial z}\right]_{S_k}. \tag{7.72}$$

On the interface S_0 between the water and open air, where zero dynamic air pressure is assumed, the simpler boundary condition (7.3) replaces (7.4). There the surface density of potential energy may be expressed in various ways as

$$\begin{aligned} E_p|_{S_0} &= \frac{\rho g}{4}|\hat{\eta}|^2 = \frac{\rho g}{4\omega^2}\left[\frac{\partial\hat{\phi}}{\partial z}\frac{\partial\hat{\phi}^*}{\partial z}\right]_{S_0} = \frac{\rho\omega^2}{4g}\left[\hat{\phi}^*\hat{\phi}\right]_{S_0} \\ &= \frac{\rho}{4}\left[\hat{\phi}\frac{\partial\hat{\phi}^*}{\partial z}\right]_{S_0} = \frac{\rho}{4}\left[\hat{\phi}^*\frac{\partial\hat{\phi}}{\partial z}\right]_{S_0}. \end{aligned} \tag{7.73}$$

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This expression was applied in Eq. (5.194). We may note that expression (7.72) for $E_p|_{S_k}$ is valid for $E_p|_{S_0}$ if S_k is replaced by S_0. (However, the opposite procedure—to replace S_0 in Eq. (7.73) by S_k—will yield $E_p|_{S_k}$ only if $\hat{p}_k = 0$.)

7.2 WEC Bodies Oscillating in Unconventional Modes of Motion

There exists WEC bodies which oscillate in modes other than the six rigid-body modes we have described in Section 5.1. Examples include WEC bodies which oscillate in an incline such as the sloped IPS buoy [100], articulated WEC bodies such as the articulated raft [101, 102], WEC bodies that oscillate by expansion and contraction of their physical volumes such as the Archimedes Wave Swing (AWS) device [103] and WECs utilising flexible bodies [104–106]. The analysis of such WECs is facilitated by the use of generalised modes [107], which we shall describe next. As we shall see in a moment, the theory also encompasses the conventional rigid-body modes described earlier.

7.2.1 Generalised Modes of Motion

Following Newman [107], we can define a general mode of motion in terms of a 'shape function' $\vec{S}_j(\mathbf{x})$, which is a vector with Cartesian components $S_{jx}(\mathbf{x}), S_{jy}(\mathbf{x})$ and $S_{jz}(\mathbf{x})$. The displacement of an arbitrary point $\mathbf{x} = (x, y, z)$ within the body is given by

$$\vec{s}(\mathbf{x}) = \sum_j \hat{s}_j \vec{S}_j(\mathbf{x}), \tag{7.74}$$

where $\hat{s}_j$ is the complex displacement amplitude of the body in mode j. With this definition, the shape function $\vec{S}_j$ for conventional rigid-body translation ($j = 1, 2, 3$) is a unit vector in the corresponding direction, whereas for rigid-body rotation ($j = 4, 5, 6$) about a reference point $\mathbf{x}_0$, it is $\vec{S}_j = \vec{S}_{j-3} \times \vec{s}$, where $\vec{s}$ is the position of point $\mathbf{x}$ relative to the reference point. For example, the shape function for the surge mode ($j = 1$) is simply a unit vector in the x-direction—that is, $\vec{S}_1 = (1, 0, 0)^{\mathrm{T}}$—whereas the shape function for the roll mode ($j = 4$) is $\vec{S}_4 = \vec{S}_1 \times \vec{s} = (0, -s_z, s_y)^{\mathrm{T}}$. In general, however, the shape function $\vec{S}_j(\mathbf{x})$ can take any form corresponding to the general mode of motion.

The normal component of $\vec{S}_j(\mathbf{x})$ on the wet body surface S is expressed as

$$n_j = \vec{S}_j \cdot \vec{n} = S_{jx} n_x + S_{jy} n_y + S_{jz} n_z, \tag{7.75}$$

where the unit normal vector $\vec{n}$ points out of the body and into the fluid domain. It is easy to show that Eqs. (5.5)–(5.6), which are applicable to conventional rigid-body modes, agree with the preceding general expression, since they can be written as

$$n_j = \begin{cases} \vec{n} \cdot \vec{S}_j & \text{for } j = 1, 2, 3 \\ (\vec{s} \times \vec{n}) \cdot \vec{S}_{j-3} = (\vec{S}_{j-3} \times \vec{s}) \cdot \vec{n} & \text{for } j = 4, 5, 6, \end{cases} \tag{7.76}$$

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with $\vec{S}_j$ for $j = 1, 2, 3$ defined as a unit vector in the corresponding direction.

In accordance with linear theory, the generalised hydrodynamic force corresponding to mode j is defined as

$$\hat{F}_j = i\omega\rho \iint_S \hat{\phi} n_j \, dS; \tag{7.77}$$

cf. Eqs. (5.21)–(5.22). Here, $\hat{\phi}$ is the complex amplitude of the total velocity potential, which may be expressed as the sum of the incident wave potential $\hat{\phi}_0$, diffracted wave potential $\hat{\phi}_d$ and radiated wave potential $\hat{\phi}_r$.

The contribution from the incident and diffracted wave potentials is defined as the wave excitation force $\hat{F}_{e,j}$ [cf. Eq. (5.26)]:

$$\hat{F}_{e,j} = i\omega\rho \iint_S (\hat{\phi}_0 + \hat{\phi}_d) n_j \, dS = i\omega\rho \iint_S (\hat{\phi}_0 + \hat{\phi}_d) \frac{\partial \varphi_j}{\partial n} \, dS, \tag{7.78}$$

where φ_j is the unit-amplitude radiation potential, according to the definition

$$\hat{\phi}_r = \sum_j \hat{u}_j \varphi_j, \tag{7.79}$$

where $\hat{u}_j$ is the velocity amplitude corresponding to mode j. The second equality in Eq. (7.78) is the consequence of the body boundary condition $\partial \hat{\phi}_r / \partial n = \hat{u}_n = \sum_j \hat{u}_j n_j$; cf. Eq. (5.8).

The contribution from the radiation potential, as in Section 5.2.1, can be expressed in terms of the radiation impedance $Z_{jj'}$:

$$\hat{F}_{r,j} = -\sum_{j'} Z_{jj'} \hat{u}_{j'}, \tag{7.80}$$

where

$$Z_{jj'} = -i\omega\rho \iint_S \varphi_{j'} n_j \, dS = -i\omega\rho \iint_S \varphi_{j'} \frac{\partial \varphi_j}{\partial n} \, dS, \tag{7.81}$$

which in turn can be expressed in terms of the added inertia $m_{jj'}$ and radiation resistance $R_{jj'}$:

$$Z_{jj'} = R_{jj'} + i\omega m_{jj'}. \tag{7.82}$$

The hydrostatic restoring force coefficients—that is, the generalised hydrostatic force in mode j due to a unit displacement in mode j'—are given as [107]

$$K_{jj'} = -\rho g \iint_S n_{j'} (S_{jz} + z D_j) dS, \tag{7.83}$$

where D_j is the divergence of the shape function—that is, $D_j = \nabla \cdot \vec{S}_j(\mathbf{x})$. For certain modes, such as the conventional rigid-body modes, $D_j = 0$, and, thus, $K_{jj'}$ reduces to

$$K_{jj'} = -\rho g \iint_S n_{j'} S_{jz} dS. \tag{7.84}$$

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For example, the hydrostatic restoring force coefficient for rigid-body heave is $K_{33} = -\rho g \iint_S n_z dS = \rho g S_w$, where S_w is the mean water-plane area of the body, in agreement with Eq. (5.343).

It should be noted that Eq. (7.83) gives the restoring force coefficients due to the hydrostatic pressure alone. The gravitational force due to the body mass also contributes to the total restoring force, and this must be considered in the dynamics of the body. For example, from physical arguments, we know that a displacement in sway of a free floating body symmetric about the $y = 0$ plane should not result in any restoring moment in roll. However, Eq. (7.83) gives $K_{42} = -\rho g \iint_S n_y s_y dS = -\rho g V$, where V is the mean displaced volume of the body. This hydrostatic contribution is exactly balanced by the gravitational contribution, which is given by $m_m g$, where m_m is the mass of the body. Since $m_m = \rho V$ for a free floating body, the gravitational contribution exactly balances the hydrostatic contribution, resulting in zero total roll restoring moment due to displacement in sway.

The coefficients of the generalised mass matrix are given as

$$M_{jj'} = \iiint_V \rho_m(\mathbf{x}) \mathbf{S}_j^{\mathrm{T}} \mathbf{S}_{j'} \, dS, \tag{7.85}$$

where $\rho_m(\mathbf{x})$ is the mass density of the body, and the integral is taken over the total volume of the body (not to be confused with the displaced volume in the preceding paragraph). This expression follows from the definition of the kinetic energy of the body and from writing the velocity of any point $\mathbf{x}$ within the body as $\sum_j \hat{u}_j \vec{S}_j(\mathbf{x})$.

7.2.2 Absorbed Power

Notice that expressions (7.77)–(7.82) have the same form as the corresponding expressions for a rigid body oscillating in conventional modes of motion, as treated in Chapter 5. It follows that the power absorbed by WECs oscillating in general modes of motion is given by the same equations as found in Chapter 6, such as Eqs. (6.84)–(6.86) and (6.95). The wave-power island (cf. Figure 6.5) is also applicable for these WECs.

7.2.3 Example: Two-Body Axisymmetric System

To illustrate the theory just described, let us consider a simple example of a two-body axisymmetric system, constrained to oscillate in heave. In Sections 5.7.3 and 8.1.4, this two-body system is analysed by considering the heave motion of each body as two independent modes. Alternatively, we may treat the two-body system as one combined body and consider the heave motion of the combined body as the first mode and the heave motion of the second body relative to the combined body as the second mode. In this case, the first mode is a conventional rigid-body mode, whereas the second mode is a generalised mode.

Using the first approach—that is, considering the heave motion of each body as two independent modes—we may write the total hydrodynamic force acting on the bodies as

$$\begin{bmatrix}\hat{F}_3\\ \hat{F}_9\end{bmatrix} = \begin{bmatrix}\hat{F}_{e,3}\\ \hat{F}_{e,9}\end{bmatrix} - \begin{bmatrix}Z_{33} & Z_{39}\\ Z_{93} & Z_{99}\end{bmatrix}\begin{bmatrix}\hat{u}_3\\ \hat{u}_9\end{bmatrix}, \tag{7.86}$$

where the modes are numbered according to Eq. (5.154). Alternatively, with the second approach—that is, with a conventional heave mode and a generalised relative heave mode—we write

$$\begin{bmatrix}\hat{F}_0\\ \hat{F}_7\end{bmatrix} = \begin{bmatrix}\hat{F}_{e,0}\\ \hat{F}_{e,7}\end{bmatrix} - \begin{bmatrix}Z_{00} & Z_{07}\\ Z_{70} & Z_{77}\end{bmatrix}\begin{bmatrix}\hat{u}_0\\ \hat{u}_7\end{bmatrix}, \tag{7.87}$$

where $j = 7$ denotes the generalised mode and $j = 0$ denotes the heave mode of the combined body, to distinguish it from $j = 3$, which has been used to denote the heave mode of the first body. By definition,

$$\hat{u}_7 = \hat{u}_9 - \hat{u}_3, \tag{7.88}$$

$$\hat{u}_0 = \hat{u}_3, \tag{7.89}$$

which, according to Eq. (7.79), lead to

$$\varphi_0 = \varphi_3 + \varphi_9, \tag{7.90}$$

$$\varphi_7 = \varphi_9. \tag{7.91}$$

The shape function of the generalised mode can be written as

$$\vec{S}_7 = \begin{cases}[0, 0, 1]^{\mathrm{T}} & \text{for } \mathbf{x} \text{ within body 2}\\ 0 & \text{elsewhere.}\end{cases} \tag{7.92}$$

Thus,

$$n_7 = \vec{S}_7 \cdot \vec{n} = \begin{cases}n_z & \text{for } \mathbf{x} \text{ on body 2}\\ 0 & \text{elsewhere.}\end{cases} \tag{7.93}$$

It follows from definitions (7.77) and (7.78) that

$$\hat{F}_7 = \hat{F}_9, \tag{7.94}$$

$$\hat{F}_0 = \hat{F}_3 + \hat{F}_9, \tag{7.95}$$

$$\hat{F}_{e,7} = \hat{F}_{e,9}, \tag{7.96}$$

$$\hat{F}_{e,0} = \hat{F}_{e,3} + \hat{F}_{e,9}. \tag{7.97}$$

Substituting Eqs. (7.88)–(7.89) and (7.94)–(7.97) into Eqs. (7.86)–(7.87), we arrive at these identities:

$$Z_{00} = Z_{33} + Z_{99} + 2Z_{39}, \tag{7.98}$$

$$Z_{07} = Z_{39} + Z_{99}, \tag{7.99}$$

$$Z_{77} = Z_{99}, \tag{7.100}$$

which are in agreement with definition (7.81) for the radiation impedance if we make use of Eqs. (7.90)–(7.91). Hence, the two approaches, Eqs. (7.86) and (7.87), are equivalent.

If the two bodies are both floating and piercing the water surface as in the system shown in Figure 5.18 but without the submerged body, then the mass matrix and the restoring matrix of the system read as

$$\mathbf{M} = \begin{bmatrix} m_{m1} & 0 \\ 0 & m_{m2} \end{bmatrix}, \quad \mathbf{S}_b = \rho g \begin{bmatrix} S_{w1} & 0 \\ 0 & S_{w2} \end{bmatrix} \tag{7.101}$$

if the first approach is used, whereas if the second approach is used, they read as

$$\mathbf{M} = \begin{bmatrix} m_{m1} + m_{m2} & m_{m2} \\ m_{m2} & m_{m2} \end{bmatrix}, \quad \mathbf{S}_b = \rho g \begin{bmatrix} S_{w1} + S_{w2} & S_{w2} \\ S_{w2} & S_{w2} \end{bmatrix}. \tag{7.102}$$

Here, m_{m1} and m_{m2} are the masses, whereas S_{w1} and S_{w2} are the mean water-plane areas, of body 1 and body 2, respectively. These expressions can be obtained from Eqs. (7.85) and (7.84).

The mean power that is absorbed by the two-body axisymmetric system can be obtained from Eqs. (6.84)–(6.86). With the first approach,

$$\hat{\mathbf{F}}_e^{\mathrm{T}} \hat{\mathbf{u}}^* = \hat{F}_{e,3} \hat{u}_3^* + \hat{F}_{e,9} \hat{u}_9^*, \tag{7.103}$$

whereas with the second approach,

$$\hat{\mathbf{F}}_e^{\mathrm{T}} \hat{\mathbf{u}}^* = \hat{F}_{e,0} \hat{u}_0^* + \hat{F}_{e,7} \hat{u}_7^*. \tag{7.104}$$

Substituting Eqs. (7.96)–(7.97) and (7.88)–(7.89) into Eq. (7.104), we obtain the same expression as in Eq. (7.103). Thus, the excitation power P_e is the same for both approaches. By making similar substitutions, we can also show that the product $\hat{\mathbf{u}}^{\mathrm{T}} \mathbf{R} \hat{\mathbf{u}}^*$, and hence, the radiated power P_r, is the same for both approaches.

7.2.4 Example: Submerged Body with Movable Top

As a second example, let us consider a completely submerged body in the form of a vertical cylinder with a movable top but otherwise fixed. The AWS device [103] is a WEC that belongs to this category. The body then has only one mode, which we shall denote with the index $j = 7$. The shape function $\vec{S}_7$ is

$$\vec{S}_7(\mathbf{x}) = \begin{cases} [0, 0, 1]^{\mathrm{T}} & \text{for } \mathbf{x} \text{ within the movable top} \\ 0 & \text{elsewhere,} \end{cases} \tag{7.105}$$

and the normal component of $\vec{S}_7$ on the wet body surface is, therefore,

$$n_7 = \vec{S}_7 \cdot \vec{n} = \begin{cases} n_z & \text{for } \mathbf{x} \text{ on the movable top} \\ 0 & \text{elsewhere.} \end{cases} \tag{7.106}$$

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On the wet surface of the movable top, the unit normal vector is pointing upwards; thus, $n_z = 1$ on this surface. According to Eq. (7.84), the hydrostatic restoring coefficient is therefore negative and is given as

$$K_{77} = -\rho g S_{\text{top}}, \tag{7.107}$$

where S_{top} is the projected area of the top on the horizontal plane.

Problems

Problem 7.1: Circular Cylindrical OWC

The lower end of a vertical thin-walled tube of radius a just penetrates the free water surface. The tube constitutes the air chamber above an OWC. Assume the fluid to be ideal, and neglect diffraction. Derive an expression for the excitation volume flux $\hat{Q}_e$ when there is an incident plane wave

$$\hat{\phi}_0 = \frac{-g}{i\omega} e(kz)\hat{\eta}_0,$$
$$\hat{\eta}_0 = Ae^{-ikx}.$$

Let the z-axis coincide with the axis of the tube.

Furthermore, find an expression for the radiation conductance G_{kk} of the OWC. Simplify the expression for G_{kk} for the case of deep water ($kh \ll 1$). [Hint: use the reciprocity relation (7.51). We may also use the integrals

$$\int_0^x tJ_0(t)\,dt = xJ_1(x), \qquad \int_0^{2\pi} e^{ix\cos t}\,dt = 2\pi J_0(x),$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions of the first kind and of order zero and one, respectively.]

Problem 7.2: Air Compressibility

Use the expression

$$p_a V_a^\kappa = (p_a + \Delta p_a)(V_a + \Delta V_a)^\kappa$$

for adiabatic compression to derive the last term in Eq. (7.62). Assume that Δp_a and ΔV_a are so small that small terms of higher order may be neglected.

Problem 7.3: Two Identical Bodies a Distance Apart

Consider a system of two identical bodies, body A and body B, say, separated at a given horizontal distance from each other. Each of the bodies is constrained to

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oscillate in heave only but is otherwise free to oscillate independently of the other. The motion of the system is therefore described by two independent modes. We may choose the first mode to be the heave motion of body A and the second mode to be the heave motion of body B. Let us denote these modes by $j = 3$ and $j = 9$, respectively. Alternatively, we may choose the first mode to be both bodies moving the same distance upwards and the second mode to be the bodies moving in opposite directions. Let us denote these modes by $j = 1$ and $j = 2$, respectively.

(a) Write the shape functions $\vec{S}_j(\mathbf{x})$ corresponding to mode $j = 3, 9, 1, 2$, as well as their normal components.
(b) Based on the definitions of the total hydrodynamic force, Eq. (7.77), and of the wave excitation force, Eq. (7.78), express $\hat{F}_1$ and $\hat{F}_2$ in terms of $\hat{F}_3$ and $\hat{F}_9$, and also, $\hat{F}_{e,1}$ and $\hat{F}_{e,2}$ in terms of $\hat{F}_{e,3}$ and $\hat{F}_{e,9}$.
(c) From definition (7.74), express $\hat{u}_1$ and $\hat{u}_2$ in terms of $\hat{u}_3$ and $\hat{u}_9$.
(d) Then, using definition (7.79), express φ_1 and φ_2 in terms of φ_3 and φ_9.
(e) Making use of the relationship just derived, express Z_{11}, Z_{12} and Z_{22} in terms of Z_{33}, Z_{39} and Z_{99}. Also show that $Z_{12} = 0$ [Hint: make use of the fact that because the two bodies are identical, $Z_{33} = Z_{99}$].
(f) Finally, show that the expressions for the mean absorbed power obtained using the two different approaches are exactly the same.