Master's degree in Computer Engineering for Robotics and Smart Industry

Advanced control systems

Report on the assignments given during the 2022/2023 a.y.

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1 Assignment 1 - DH table, direct/inverse kinematics, Jacobians

Given a 3 Degrees Of Freedom (DOF) manipulator with a Revolute-Prismatic-Revolute (RPR) configuration, the object of the assignment is to compute the Denavit-Hartenberg (DH) parameter table, compute the direct and inverse kinematics and compute the analytical and geometrical Jacobian matrices.

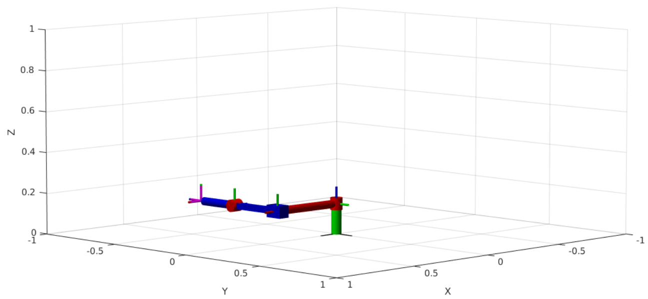

Figure 1: Visualization of the URDF of the PRP robot in the home configuration ($\theta_1 = 0, d_2 = 0, \theta_3 = 0$)

1.1 Denavit-Hartenberg parameters

First of all, reference frames were assigned to each joint according to the DH convention:

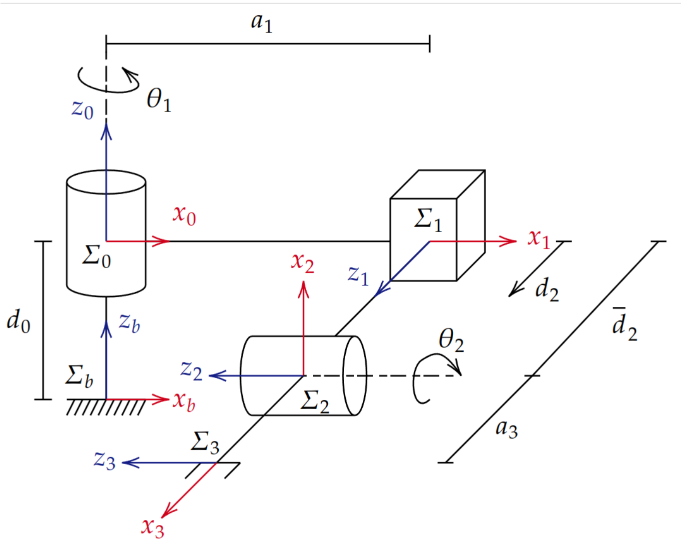

Figure 2: Reference frames assigned with the DH convention

Note that the DH frames do not correspond with the frames of the URDF model.
Then the DH parameter table was populated:

	a_i	α_i	d_i	θ_i
0	0	0	d_0	0
1	a_1	$\frac{\pi}{2}$	0	θ_1
2	0	$-\frac{\pi}{2}$	$d_2 + \bar{d}_2$	$\frac{\pi}{2}$
3	a_3	0	0	$\theta_3 - \frac{\pi}{2}$

Using the DH parameters, the transformation matrices between the frames were then computed. The transformation matrix between frame i and frame $i-1$ is in the form:

$$T_{i-1}^{i} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The computed transformation matrices are then:

$$T_b^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_0^1 = \begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_1^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & d_2+\bar{d}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} C\left(\theta_3-\frac{\pi}{2}\right) & -S\left(\theta_3-\frac{\pi}{2}\right) & 0 & a_3C\left(\theta_3-\frac{\pi}{2}\right) \\ S\left(\theta_3-\frac{\pi}{2}\right) & C\left(\theta_3-\frac{\pi}{2}\right) & 0 & a_3S\left(\theta_3-\frac{\pi}{2}\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_3 & C_3 & 0 & a_3S_3 \\ -C_3 & S_3 & 0 & -a_3C_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2 Direct kinematics

The direct kinematics of the manipulator are obtained by chaining the above transformation matrices, thus obtaining a global transformation matrix from the end-effector to the base of the manipulator:

$$T_b^3 = T_b^0T_0^1T_1^2T_2^3 = \underbrace{\begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^1} T_1^2T_2^3 = \underbrace{\begin{bmatrix} 0 & -S_1 & -C_1 & (d_2+\bar{d}_2)S_1 + a_1C_1 \\ 0 & C_1 & -S_1 & -(d_2+\bar{d}_2)C_1 + a_1S_1 \\ 1 & 0 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^2} T_2^3$$

$$= \begin{bmatrix} S_1C_3 & -S_1S_3 & -C_1 & (d_2+\bar{d}_2)S_1 + a_1C_1 + a_3S_1C_3 \\ -C_1C_3 & C_1S_3 & -S_1 & -(d_2+\bar{d}_2)C_1 + a_1S_1 - a_3C_1C_3 \\ S_3 & C_3 & 0 & d_0 + a_3S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_b^3 & p_3 \\ \bar{0} & 1 \end{bmatrix}$$

1.3 Inverse kinematics

From the direct kinematics, the position of the end-effector is given by:

$$p_3 = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} (d_2+\bar{d}_2)S_1 + a_1C_1 + a_3S_1C_3 \\ -(d_2+\bar{d}_2)C_1 + a_1S_1 - a_3C_1C_3 \\ d_0 + a_3S_3 \end{pmatrix}$$

Therefore an expression for θ_3 can immediately be derived:

$$p_z = d_0 + a_3S_3 \implies S_3 = \frac{p_z - d_0}{a_3}, C_3 = \pm\sqrt{1-S_3^2} = \pm\sqrt{\frac{a_3^2 - (p_z-d_0)^2}{a_3^2}} \implies \theta_3^{\pm} = \operatorname{atan2}(S_3, \pm C_3)$$

d_2 is determined by applying summing and squaring:

$$\begin{aligned} p_x^2 + p_y^2 = {} & S_1^2(d_2+\bar{d}_2)^2 + a_1^2C_1^2 + a_3^2S_1^2C_3^2 + \cancel{2a_1(d_2+\bar{d}_2)S_1C_1} - \cancel{2a_1a_3S_1C_1C_3} + 2a_3(d_2+\bar{d}_2)S_1^2C_3 + \\ & + C_1^2(d_2+\bar{d}_2)^2 + a_1^2S_1^2 + a_3^2C_1^2C_3^2 - \cancel{2a_1(d_2+\bar{d}_2)S_1C_1} + \cancel{2a_1a_3S_1C_1C_3} + 2a_3(d_2+\bar{d}_2)C_1^2C_3 = \\ & = (d_2+\bar{d}_2)^2 + 2a_3(d_2+\bar{d}_2)C_3 + a_1^2 + a_3^2C_3^2 \end{aligned}$$

Therefore d_2 is given by the solution of the quadratic equation:

$$(d_2+\bar{d}_2)^2 + 2a_3(d_2+\bar{d}_2)C_3 + a_1^2 + a_3^2C_3^2 - p_x^2 - p_y^2 = 0$$

$$(d_2+\bar{d}_2)_{12} = -\frac{\cancel{2}a_3C_3 \pm \sqrt{\cancel{4a_3^2C_3^2} - \cancel{4}(a_1^2 + \cancel{a_3^2C_3^2} - p_x^2 - p_y^2)}}{\cancel{2}} \implies d_{2,12} = -a_3C_3 - \bar{d}_2 \pm \sqrt{p_x^2 + p_y^2 - a_1^2}$$

An expression for θ_1 can be derived by considering the position p_1 of the origin of the frame Σ_1. From the transformation matrix T_b^1, p_1 is:

$$p_1 = \begin{pmatrix} a_1 C_1 \\ a_1 S_1 \\ d_0 \\ 1 \end{pmatrix}$$

p_1 is related to the position p_3 of the end-effector frame by the relation:

$$p_3 = T_1^3 p_1 \implies p_1 = {T_1^3}^{-1} p_3 \implies \begin{pmatrix} a_1 C_1 \\ a_1 S_1 \\ d_0 \\ 1 \end{pmatrix} = {T_1^3}^{-1} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

The transformation matrix T_1^3 is:

$$T_1^3 = T_1^2 T_2^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ S_3 & C_3 & 0 & a_3 S_3 \\ C_3 & -S_3 & 0 & (d_2 + \overline{d}_2) + a_3 C_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is always invertible, since its determinant is never null:

$$det(T_1^3) = -1 \begin{vmatrix} S_3 & C_3 & a_3 S_3 \\ C_3 & -S_3 & (d_2 + \overline{d}_2) + a_3 C_3 \\ 0 & 0 & 1 \end{vmatrix} = -(-S_3^2 - C_3^2) = 1$$

Therefore:

$$p_1 = {T_1^3}^{-1} p_3 = \begin{bmatrix} 0 & S_3 & C_3 & -(d_2 + \overline{d}_2) C_3 - a_3 \\ 0 & C_3 & -S_3 & (d_2 + \overline{d}_2) S_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_y S_3 + p_z C_3 - (d_2 \overline{d}_2) C_3 - a_3 \\ p_y C_3 - p_z S_3 + (d_2 + \overline{d}_2) S_3 \\ -p_x \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 C_1 \\ a_1 S_1 \\ d_0 \\ 1 \end{pmatrix}$$

Therefore:

$$C_1 = \frac{p_y S_3 + p_z C_3 - (d_2 \overline{d}_2) C_3 - a_3}{a_1}, S_1 = \frac{p_y C_3 - p_z S_3 + (d_2 + \overline{d}_2) S_3}{a_1} \implies \theta_1^{\pm} = \operatorname{atan2}(\pm S_1, C_1)$$

In summary:

$$\begin{aligned} \theta_3^{\pm} &= \operatorname{atan2}\left(\frac{p_z - d_0}{a_3}, \pm \sqrt{\frac{a_3^2 - (p_z - d_0)^2}{a_3^2}} \right) \\ d_{2,12} &= -a_3 C_3 - \overline{d}_2 \pm \sqrt{p_x^2 + p_y^2 - a_1^2} \\ \theta_1^{\pm} &= \operatorname{atan2}\left(\pm \frac{p_y C_3 - p_z S_3 + (d_2 + \overline{d}_2) S_3}{a_1}, \frac{p_y S_3 + p_z C_3 - (d_2 \overline{d}_2) C_3 - a_3}{a_1} \right) \end{aligned}$$

1.4 Analytical Jacobian

The analytical Jacobian is obtained by computing the gradient of the three components of the end-effector position with respect to the three joint variables:

$$\begin{aligned} \nabla p_x &= \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial d_2} & \frac{\partial p_x}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} (d_2 + \overline{d}_2) C_1 - a_1 S_1 + a_3 C_3 C_1 & S_1 & a_3 S_1 S_3 \end{bmatrix} \\ \nabla p_y &= \begin{bmatrix} \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial d_2} & \frac{\partial p_y}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} (d_2 + \overline{d}_2) S_1 + a_1 C_1 + a_3 S_1 C_3 & -C_1 & -a_3 S_1 S_3 \end{bmatrix} \\ \nabla p_z &= \begin{bmatrix} \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial d_2} & \frac{\partial p_z}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_3 C_3 \end{bmatrix} \end{aligned}$$

Therefore the analytical Jacobian is:

$$J_A = \begin{bmatrix} \nabla p_x \\ \nabla p_y \\ \nabla p_z \end{bmatrix} = \begin{bmatrix} (d_2+\overline{d}_2)C_1 - a_1S_1 + a_3C_3C_1 & S_1 & a_3S_1S_3 \\ (d_2+\overline{d}_2)S_1 + a_1C_1 + a_3S_1C_3 & -C_1 & -a_3C_1S_3 \\ 0 & 0 & a_3C_3 \end{bmatrix} \implies \dot{p}_3 = J_A\dot{q}$$

where $\dot{q} = \begin{bmatrix} \dot{\theta}_1 & \dot{d}_2 & \dot{\theta}_3 \end{bmatrix}^T$.

1.5 Geometric Jacobian

The i-th column of the geometric Jacobian ($i = 0, \dots, n-1$) is constructed as follows:

$$J_{Gi} = \begin{bmatrix} z_i \times (p_e - p_i) \\ z_i \end{bmatrix} \quad \text{(revolute joint)} \qquad J_{Gi} = \begin{bmatrix} z_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(revolute joint)}$$

where p_e is the position of the end-effector, p_i is the position of frame i and z_i is the direction of the z axis of frame i, all with respect to the base frame.

So:

$$p_2 = \begin{bmatrix} S_1(d_2+\overline{d}_2) + a_1C_1 \\ -C_1(d_2+\overline{d}_2) + a_1S_1 \\ d_0 \end{bmatrix}, z_2 = \begin{bmatrix} -C_1 \\ -S_1 \\ 0 \end{bmatrix} \implies J_{G2} = \begin{bmatrix} -a_3S_1S_3 & a_3C_1C_3 & -a_3C_3 & -C_1 & -S_1 & 0 \end{bmatrix}^T$$

$$p_1 = \begin{bmatrix} a_1C_1 \\ a_1S_1 \\ d_0 \end{bmatrix}, z_1 = \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \implies J_{G1} = \begin{bmatrix} S_1 & -C_1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ d_0 \end{bmatrix}, z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies J_{G0} = \begin{bmatrix} C_1(d_2+\overline{d}_2) - a_1S_1 - a_3C_1C_3 & S_1(d_2+\overline{d}_2) + a_1C_1 - a_3S_1C_3 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

Finally:

$$J_G = \begin{bmatrix} C_1(d_2+\overline{d}_2) - a_1S_1 - a_3C_1C_3 & S_1 & -a_3S_1S_3 \\ S_1(d_2+\overline{d}_2) + a_1C_1 - a_3S_1C_3 & -C_1 & a_3C_1C_3 \\ 0 & 0 & -a_3C_3 \\ 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} \dot{p}_3 \\ \dot{\omega}_3 \end{bmatrix} = J_G\dot{q}$$

2 Assignment 2 - Kinetic energy, potential energy

The object of the assignment is the computation of the kinetic energy and of the potential energy of the 3 DOF RPR manipulator.

To compute both the kinetic and potential energy, the positions of the centres of mass of each link are needed. With respect to each frame Σ_i attached to link i:

$$p_{l_1}^1 = \begin{bmatrix} \frac{h_1}{2} \\ 0 \\ 0 \end{bmatrix} \qquad p_{l_2}^2 = \begin{bmatrix} 0 \\ 0 \\ \frac{a_2}{2} \end{bmatrix} \qquad p_{l_3}^3 = \begin{bmatrix} 0 \\ -\frac{h_3}{2} \\ 0 \end{bmatrix}$$

Using the transformation matrices obtained with direct kinematics, the positions of the centres of mass with respect to the base frame Σ_b are:

$$p_{l_1} = R_b^1 p_{l_1}^1 + p_1 = \begin{bmatrix} \left(\frac{h}{2} + a_1\right) C_1 \\ \left(\frac{h}{2} + a_1\right) S_1 \\ d_0 \end{bmatrix} \qquad p_{l_2} = R_b^2 p_{l_2}^2 + p_2 = \begin{bmatrix} (q_2 + \overline{d}_2) S_1 + \left(a_1 - \frac{c_2}{2}\right) C_1 \\ -(q_2 + \overline{d}_2) C_1 + \left(a_1 - \frac{c_2}{2}\right) S_1 \\ d_0 \end{bmatrix}$$

$$p_{l_3} = R_b^3 p_{l_3}^3 + p_3 = \begin{bmatrix} (q_2 + \overline{d}_2) S_1 + a_1 C_1 + \frac{h_3}{2} S_1 S_3 + a_3 S_1 C_3 \\ -(q_2 + \overline{d}_2) C_1 + a_1 S_1 - \frac{h_3}{2} C_1 S_3 - a_3 C_1 C_3 \\ d_0 - \frac{h_3}{2} C_3 + a_3 S_3 \end{bmatrix}$$

2.1 Kinetic energy

The total kinetic energy for an open-chain manipulator is:

$$\mathcal{T}(q, \dot{q}) = \frac{1}{2} \dot{q}^T B(q) \dot{q} \qquad B(q) = \sum_{i=1}^{n} (m_{l_i} (J_P^{l_i})^T J_P^{l_i} + (J_O^{l_i})^T R_b^i I_{l_i}^i R_b^{i\,T} J_O^{l_i})$$

where $B(q)$ is the inertia matrix, $I_{l_i}^i$ are the inertia tensors with respect to Σ_i, $J_P^{l_i}$ and $J_O^{l_i}$ are the linear and angular partial Jacobian matrices and R_b^i are the rotation matrices that bring frame Σ_i to frame Σ_b.

The inertia tensors are obtained with Steiner's theorem:

$$\begin{aligned}
I_{l_1}^1 &= I_{l_1}^{C_1} + m_{l_1} S^T(p_{l_1}^1) S(p_{l_1}^1) \\
&= m_{l_1} \begin{bmatrix} \frac{1}{2}(r_1^2) & 0 & 0 \\ 0 & \frac{1}{2}(3r_1^2 + h_1^2) & 0 \\ 0 & 0 & \frac{1}{2}(3r_1^2 + h_1^2) \end{bmatrix} + m_{l_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{h_1^2}{4} & 0 \\ 0 & 0 & \frac{h_1^2}{4} \end{bmatrix} \\
&= \frac{1}{2} m_{l_1} \begin{bmatrix} r_1^2 & 0 & 0 \\ 0 & 3r_1^2 + \frac{3h_1^2}{2} & 0 \\ 0 & 0 & 3r_1^2 + \frac{3h_1^2}{2} \end{bmatrix} \\
I_{l_2}^2 &= I_{l_2}^{C_2} + m_{l_2} S^T(p_{l_2}^2) S(p_{l_2}^2) \\
&= m_{l_2} \begin{bmatrix} \frac{1}{12}(b_2^2 + c_2^2) & 0 & 0 \\ 0 & \frac{1}{12}(a_2^2 + c_2^2) & 0 \\ 0 & 0 & \frac{1}{12}(a_2^2 + b_2^2) \end{bmatrix} + m_{l_2} \begin{bmatrix} \frac{a_2^2}{4} & 0 & 0 \\ 0 & \frac{a_2^2}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \frac{1}{12} m_{l_2} \begin{bmatrix} 3a_2^2 + b_2^2 + c_2^2 & 0 & 0 \\ 0 & 4a_2^2 + c_2^2 & 0 \\ 0 & 0 & a_2^2 + b_2^2 \end{bmatrix} \\
I_{l_3}^3 &= I_{l_3}^{C_3} + m_{l_3} S^T(p_{l_3}^3) S(p_{l_3}^3) \\
&= m_{l_3} \begin{bmatrix} \frac{1}{2}(r_3^2) & 0 & 0 \\ 0 & \frac{1}{2}(3r_3^2 + h_3^2) & 0 \\ 0 & 0 & \frac{1}{2}(3r_3^2 + h_3^2) \end{bmatrix} + m_{l_3} \begin{bmatrix} \frac{h_3^2}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{h_3^2}{4} \end{bmatrix} \\
&= \frac{1}{2} m_{l_3} \begin{bmatrix} r_3^2 + \frac{h_3^2}{2} & 0 & 0 \\ 0 & 3r_3^2 + h_3^2 & 0 \\ 0 & 0 & 3r_3^2 + \frac{3}{2} h_3^2 \end{bmatrix}
\end{aligned}$$

The links are assumed to be made up of an homogeneous material, specifically aluminium. Therefore their masses are $m_{l_i} = \rho V_{l_i}$, where $\rho = 2710\ kg/m^3$ is the density of aluminium and V_{l_i} is the volume of link i.

The partial Jacobian matrices are constructed as follows:

$$J_P^{l_i} = \begin{bmatrix} j_{P1}^{l_i} & \dots & j_{Pj}^{l_i} & 0 & \dots & 0 \end{bmatrix} \text{ where } j_{Pj}^{l_i} = \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_{l_i} - p_{j-1}) & \text{revolute joint} \end{cases}$$

$$J_O^{l_i} = \begin{bmatrix} j_{O1}^{l_i} & \dots & j_{Oj}^{l_i} & 0 & \dots & 0 \end{bmatrix} \text{ where } j_{Oj}^{l_i} = \begin{cases} 0 & \text{prismatic joint} \\ z_{j-1} & \text{revolute joint} \end{cases}$$

where p_{j-1} is the position vector of the origin of frame Σ_{j-1} and z_{j-1} is the unit vector of axis z of frame Σ_{j-1}, all with respect of Σ_b.

$$J_P^{l_1} = \begin{bmatrix} j_{P1}^{l_1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -S_1\left(a_1 + \frac{h_1}{2}\right) & 0 & 0 \\ C_1\left(a_1 + \frac{h_1}{2}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_O^{l_1} = \begin{bmatrix} j_{O1}^{l_1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_1} = z_0 \times (p_{l_1} - p_0) = \begin{bmatrix} -S_1\left(a_1 + \frac{h_1}{2}\right) & C_1\left(a_1 + \frac{h_1}{2}\right) & 0 \end{bmatrix}^T \qquad j_{O1}^{l_1} = z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$J_P^{l_2} = \begin{bmatrix} j_{P1}^{l_2} & j_{P2}^{l_2} & 0 \end{bmatrix} = \begin{bmatrix} d_2C_1 + \frac{1}{2}(a_2S_1) + q_2C_1 & S_1 & 0 \\ d_2S_1 - \frac{1}{2}(a_2C_1) + q_2S_1 & -C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_O^{l_2} = \begin{bmatrix} j_{O1}^{l_2} & j_{O2}^{l_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_2} = z_0 \times (p_{l_2} - p_0) = \begin{bmatrix} d_2C_1 + \frac{1}{2}(a_2S_1) + q_2C_1 & d_2S_1 - \frac{1}{2}(a_2C_1) + q_2S_1 & 0 \end{bmatrix}^T \qquad j_{O1}^{l_2} = z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$j_{P2}^{l_2} = z_1 = \begin{bmatrix} S_1 & -C_1 & 0 \end{bmatrix}^T \qquad j_{O2}^{l_2} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

$$J_P^{l_3} = \begin{bmatrix} j_{P1}^{l_3} & j_{P2}^{l_3} & j_{P3}^{l_3} \end{bmatrix} = \begin{bmatrix} C_1\left(a_3C_3 + \frac{h_3}{2}S_3\right) & S_1 & S_1\left(\frac{h_3}{2}C_3 - a_3S_3\right) \\ S_1\left(a_3C_3 + \frac{h_3}{2}S_3\right) & -C_1 & -C_1\left(\frac{h_3}{2}C_3 - a_3S_3\right) \\ 0 & 0 & a_3C_3 + \frac{h_3}{2}S_3 \end{bmatrix} \qquad J_O^{l_3} = \begin{bmatrix} j_{O1}^{l_3} & j_{O2}^{l_3} & j_{O3}^{l_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_3} = z_0 \times (p_{l_3} - p_0) = \begin{bmatrix} C_1\left(a_3C_3 + \frac{h_3}{2}S_3\right) & S_1\left(a_3C_3 + \frac{h_3}{2}S_3\right) & 0 \end{bmatrix}^T \qquad j_{O1}^{l_3} = z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$j_{P2}^{l_3} = z_1 = \begin{bmatrix} S_1 & -C_1 & 0 \end{bmatrix}^T \qquad j_{O2}^{l_3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

$$j_{P3}^{l_3} = z_2 \times (p_{l_3} - p_2) = \begin{bmatrix} S_1\left(\frac{h_3}{2}C_3 - a_3S_3\right) & -C_1\left(\frac{h_3}{2}C_3 - a_3S_3\right) & a_3C_3 + \frac{h_3}{2}S_3 \end{bmatrix}^T \qquad j_{O3}^{l_3} = z_2 = \begin{bmatrix} -C_1 & -S_1 & 0 \end{bmatrix}^T$$

So the inertial matrices of each joint are:

$$\begin{aligned}
B_1(q) &= m_{l_1}(J_P^{l_1})^T J_P^{l_1} + (J_O^{l_1})^T R_b^1 I_{l_1}^1 {R_b^1}^T J_O^{l_1} \\
&= m_{l_1} \begin{bmatrix} a_1^2 + a_1h_1 + h_1^2 + \frac{3}{2}r_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
B_2(q) &= m_{l_2}(J_P^{l_2})^T J_P^{l_2} + (J_O^{l_2})^T R_b^2 I_{l_2}^2 {R_b^2}^T J_O^{l_2} \\
&= m_{l_2} \begin{bmatrix} \frac{1}{2}a_2^2 + \frac{1}{12}(b_2^2 + c_2^2) + d_2^2 + 2d_2q_2 + q_2^2 & \frac{1}{2}a_2^2 & 0 \\ \frac{1}{2}a_2^2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
B_3(q) &= m_{l_3}(J_P^{l_3})^T J_P^{l_3} + (J_O^{l_3})^T R_b^3 I_{l_3}^3 {R_b^3}^T J_O^{l_3} \\
&= m_{l_3} \begin{bmatrix} \frac{1}{2}(a_3^2 + r_3^2)(C_3^2 - S_3^2) + \frac{1}{2}a_3^2 + \frac{1}{2}r_3^2 + a_3h_3S_3C_3 & 0 & 0 \\ 0 & 1 & \frac{1}{2}h_3C_3 - a_3S_3 \\ 0 & \frac{1}{2}h_3C_3 - a_3S_3 & a_3^2 + h_3^2 + \frac{3}{2}r_3^2 \end{bmatrix}
\end{aligned}$$

So the overall inertia matrix is:

$$\begin{aligned}
B(q) &= B_1(q) + B_2(q) + B_3(q) \\
&= \begin{bmatrix} K & m_{l_2}\frac{1}{2}a_2^2 & 0 \\ m_{l_2}\frac{1}{2}a_2^2 & m_{l_2} + m_{l_3} & m_{l_3}(\frac{1}{2}h_3C_3 - a_3S_3) \\ 0 & m_{l_3}(\frac{1}{2}h_3C_3 - a_3S_3) & m_{l_3}(a_3^2 + h_3^2 + \frac{3}{2}r_3^2) \end{bmatrix}
\end{aligned}$$

with $K = m_{l_1}B_{1,11} + m_{l_2}B_{2,11} + m_{l_3}B_{3,11}$.

Finally, the kinetic energy is given by:

$$
\begin{aligned}
T &= \frac{1}{2}\dot{q}^T B(q)\dot{q} \\
&= \frac{1}{4}((2K + m_{l_2}a_2^2)\dot{q}_1^2 + (m_{l_2}(2 + a_2^2) + m_{l_3}(2 + h_3C_3 - 2a_3S_3))\dot{q}_2^2 + (m_{l_3}(h_3C_3 - 2a_3S_3 + 2a_3^2 + 2h_3^2 + 3r_3^2))\dot{q}_3^2)
\end{aligned}
$$

2.2 Potential energy

The potential energy is given by:

$$\mathcal{U}(q) = -\sum_{i=1}^{n} m_{l_i} g_0^T p_{l_i}$$

where $g_0 = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$ is the gravity acceleration vector in the base frame Σ_b.
So:

$$
\begin{aligned}
\mathcal{U}_1 &= -m_{l_1} g d_0 \\
\mathcal{U}_2 &= -m_{l_2} g d_0 \\
\mathcal{U}_3(q) &= -m_{l_3} g \left(d_0 - \frac{1}{2}h_3C_3 + a_3S_3\right)
\end{aligned}
$$

Finally:

$$\mathcal{U}(q) = -(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3(q)) = \left[(m_{l_1} + m_{l_2} + m_{l_3})d_0 + m_{l_3}\left(\frac{1}{2}h_3C_3 - a_3S_3\right)\right] g$$

3 Assignment 3 - Equations of motion

The equations of motion for an open chain robotic manipulator are:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s sign(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

Ignoring the contributions related to friction (F_v, F_s) and the contribution related to an external wrench (h_e), the equations reduce to:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where τ is the command torque, $C(q,\dot{q})$ is the matrix of Christoffel symbols of the first type and $g(q)$ is the vector of gravity terms.

The gravity term is given by:

$$g_i(q) = -\sum_{j=1}^{n} m_{l_i} g_0^T j_{Pi}^{l_j}(q) \rightarrow g(q) = \begin{bmatrix} 0 \\ 0 \\ m_{l_3} g\left(a_3C_3 + \frac{1}{2}h_3S_3\right) \end{bmatrix}$$

The c_{ij} elements of $C(q,\dot{q})$ are:

$$c_{ij} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k$$

where b_{ij}, b_{ik} and b_{jk} are the elements of the inertial matrix $B(q)$. The derivatives of the $B(q)$ matrix are:

$$\frac{\partial B}{\partial q_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial B}{\partial q_2} = \begin{bmatrix} 2m_{l_2}(d_2+q_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial B}{\partial q_3} = \begin{bmatrix} m_{l_3}(a_3h_3(C_3^2 - S_3^2) - 2(a_3^2 + r_3^2)C_3S_3) & 0 & 0 \\ 0 & 0 & -m_{l_3}\left(\frac{1}{2}h_3S_3 + a_3C_3\right) \\ 0 & -m_{l_3}\left(\frac{1}{2}h_3S_3 + a_3C_3\right) & 0 \end{bmatrix}$$

So the c_{ij} components are:

$$\begin{aligned}
c_{11} &= \frac{1}{2}\left(\frac{\partial b_{11}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_1} - \frac{\partial b_{11}}{\partial q_1}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{11}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_1} - \frac{\partial b_{12}}{\partial q_1}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{11}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_1} - \frac{\partial b_{13}}{\partial q_1}\right)\dot{q}_3 \\
&= m_{l_2}(d_2+q_2)\dot{q}_2 + \frac{1}{2}m_{l_3}(a_3h_3(C_3^2 - S_3^2) - 2(a_3^2 + r_3^2)C_3S_3)\dot{q}_3 \\
c_{12} &= \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_1}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_1}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_2} - \frac{\partial b_{23}}{\partial q_1}\right)\dot{q}_3 \\
&= m_{l_2}(d_2+q_2)\dot{q}_1 \\
c_{13} &= \frac{1}{2}\left(\frac{\partial b_{13}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_1}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{13}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_1}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{13}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_1}\right)\dot{q}_3 \\
&= \frac{1}{2}m_{l_3}(a_3h_3(C_3^2 - S_3^2) - 2(a_3^2 + r_3^2)C_3S_3)\dot{q}_1 \\
c_{22} &= \frac{1}{2}\left(\frac{\partial b_{22}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_2}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{22}}{\partial q_2} + \frac{\partial b_{22}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_2}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{22}}{\partial q_3} + \frac{\partial b_{23}}{\partial q_2} - \frac{\partial b_{23}}{\partial q_2}\right)\dot{q}_3 \\
&= 0 \\
c_{23} &= \frac{1}{2}\left(\frac{\partial b_{23}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_2}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{23}}{\partial q_2} + \frac{\partial b_{22}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_2}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{23}}{\partial q_3} + \frac{\partial b_{23}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_2}\right)\dot{q}_3 \\
&= -m_{l_3}\left(\frac{1}{2}h_3S_3 + a_3C_3\right)\dot{q}_3 \\
c_{33} &= \frac{1}{2}\left(\frac{\partial b_{33}}{\partial q_1} + \frac{\partial b_{31}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_3}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{33}}{\partial q_2} + \frac{\partial b_{32}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_3}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{33}}{\partial q_3} + \frac{\partial b_{33}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_3}\right)\dot{q}_3 \\
&= 0
\end{aligned}$$

So the equations of motion are:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where $q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T, \dot{q} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{bmatrix}^T, \ddot{q} = \begin{bmatrix} \ddot{q}_1 & \ddot{q}_2 & \ddot{q}_3 \end{bmatrix}^T$ and:

$$B(q) = \begin{bmatrix} K & m_{l_2}\frac{1}{2}a_2^2 & 0 \\ & m_{l_2} + m_{l_3} & m_{l_3}(\frac{1}{2}h_3C_3 - a_3S_3) \\ * & * & m_{l_3}(a_3^2 + h_3^2 + \frac{3}{2}r_3^2) \end{bmatrix}$$

$$\begin{aligned} K = {} & m_{l_1}\left(a_1^2 + a_1h_1 + h_1^2 + \frac{3}{2}r_1^2\right) + m_{l_2}\left(\frac{1}{2}a_2^2 + \frac{1}{12}(b_2^2 + c_2^2) + d_2^2 + 2d_2q_2 + q_2^2\right) \\ & + m_{l_3}\left(\frac{1}{2}(a_3^2 + r_3^2)(C_3^2 - S_3^2) + \frac{1}{2}a_3^2 + \frac{1}{2}r_3^2 + a_3h_3S_3C_3\right) \end{aligned}$$

$$C(q,\dot{q}) = \begin{bmatrix} m_{l_2}(d_2 + q_2)\dot{q}_2 + \frac{1}{2}m_{l_3}(a_3h_3(C_3^2 - S_3^2) - 2(a_3^2 + r_3^2)C_3S_3)\dot{q}_3 & m_{l_2}(d_2 + q_2)\dot{q}_1 & \frac{1}{2}m_{l_3}(a_3h_3(C_3^2 - S_3^2) - 2(a_3^2 + r_3^2)C_3S_3)\dot{q}_1 \\ * & 0 & -m_{l_3}\left(\frac{1}{2}h_3S_3 + a_3C_3\right)\dot{q}_3 \\ * & * & 0 \end{bmatrix}$$

$$g(q) = \begin{bmatrix} 0 \\ 0 \\ m_{l_3}g\left(a_3C_3 + \frac{1}{2}h_3S_3\right) \end{bmatrix}$$

4 Recursive Newton-Euler formulation