

# Master's degree in Computer Engineering for Robotics and Smart Industry

## Advanced control systems

Report on the assignments given during the 2022/2023 a.y.

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# 1 Assignment 1 - DH table, direct/inverse kinematics, Jacobians

Given a 3 Degrees Of Freedom (DOF) manipulator with a Revolute-Prismatic-Revolute (RPR) configuration, the object of the assignment is to compute the Denavit-Hartenberg (DH) parameter table, compute the direct and inverse kinematics and compute the analytical and geometrical Jacobian matrices.

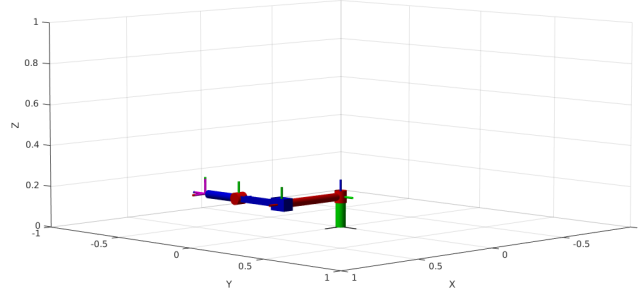


Figure 1: Visualization of the URDF of the PRP robot in the home configuration ( $q_1 = 0, q_2 = 0, \theta_3 = 0$ )

## 1.1 Denavit-Hartenberg parameters

First of all, reference frames were assigned to each joint according to the DH convention:

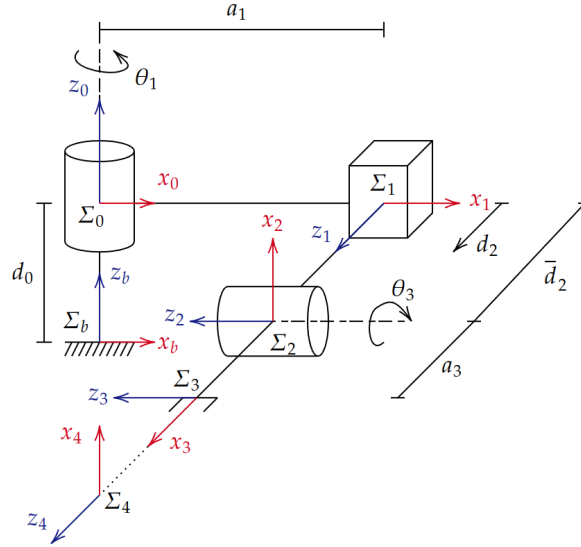


Figure 2: Reference frames assigned with the DH convention

Note that the DH frames do not correspond with the frames of the URDF model. Then the DH parameter table was populated:

	$a_i$	$\alpha_i$	$d_i$	$\theta_i$
0	0	0	$d_0$	0
1	$a_1$	$\frac{\pi}{2}$	0	$q_1 = \theta_1$
2	0	$-\frac{\pi}{2}$	$(q_2 = q_2) + \bar{d}_2$	$\frac{\pi}{2}$
3	$a_3$	0	0	$(q_3 = \theta_3) - \frac{\pi}{2}$
4	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$

The first row is the fixed offset in the  $z_b$  direction between world and frame  $\Sigma_0$ , while the last row is the fixed rotation that aligns the  $z$  axis of the end-effector to the approach direction.

Using the DH parameters, the transformation matrices between the frames were then computed. The transformation matrix between frame  $i$  and frame  $i - 1$  is in the form:

$$T_{i-1}^i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \cos(\alpha_i) & \sin(\theta_i) \sin(\alpha_i) & a_i \cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \cos(\alpha_i) & -\cos(\theta_i) \sin(\alpha_i) & a_i \sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The computed transformation matrices are then:

$$\begin{aligned} T_b^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & T_0^1 &= \begin{bmatrix} C_1 & 0 & S_1 & a_1 C_1 \\ S_1 & 0 & -C_1 & a_1 S_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & T_1^2 &= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & q_2 + \bar{d}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ T_2^3 &= \begin{bmatrix} C(q_3 - \frac{\pi}{2}) & -S(q_3 - \frac{\pi}{2}) & 0 & a_3 C(q_3 - \frac{\pi}{2}) \\ S(q_3 - \frac{\pi}{2}) & C(q_3 - \frac{\pi}{2}) & 0 & a_3 S(q_3 - \frac{\pi}{2}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_3 & C_3 & 0 & a_3 S_3 \\ -C_3 & S_3 & 0 & -a_3 C_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & T_3^4 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where  $C_i$  and  $S_i$  denote respectively  $\cos(q_i)$  and  $\sin(q_i)$ .

## 1.2 Direct kinematics

The direct kinematics of the manipulator are obtained by chaining the above transformation matrices, thus obtaining a global transformation matrix from the end-effector to the base of the manipulator:

$$\begin{aligned} T_b^4 &= T_b^0 T_0^1 T_1^2 T_2^3 T_3^4 = \underbrace{\begin{bmatrix} C_1 & 0 & S_1 & a_1 C_1 \\ S_1 & 0 & -C_1 & a_1 S_1 \\ 0 & 1 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^1} \underbrace{\begin{bmatrix} 0 & -S_1 & -C_1 & (q_2 + \bar{d}_2)S_1 + a_1 C_1 \\ 0 & C_1 & -S_1 & -(q_2 + \bar{d}_2)C_1 + a_1 S_1 \\ 1 & 0 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^2} T_2^3 T_3^4 \\ &= \underbrace{\begin{bmatrix} S_1 C_3 & -S_1 S_3 & -C_1 & (q_2 + \bar{d}_2)S_1 + a_1 C_1 + a_3 S_1 C_3 \\ -C_1 C_3 & C_1 S_3 & -S_1 & -(q_2 + \bar{d}_2)C_1 + a_1 S_1 - a_3 C_1 C_3 \\ S_3 & C_3 & 0 & d_0 + a_3 S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^3} T_3^4 \\ &= \begin{bmatrix} -S_1 S_3 & -C_1 & S_1 C_3 & (q_2 + \bar{d}_2)S_1 + a_1 C_1 + a_3 S_1 C_3 \\ C_1 S_3 & -S_1 & -C_1 C_3 & -(q_2 + \bar{d}_2)C_1 + a_1 S_1 - a_3 C_1 C_3 \\ C_3 & 0 & S_3 & d_0 + a_3 S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_b^4 & p_4 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

## 1.3 Inverse kinematics

From the direct kinematics, the position of the end-effector is given by:

$$p_4 = \begin{pmatrix} p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix} = \begin{pmatrix} (q_2 + \bar{d}_2)S_1 + a_1 C_1 + a_3 S_1 C_3 \\ -(q_2 + \bar{d}_2)C_1 + a_1 S_1 - a_3 C_1 C_3 \\ d_0 + a_3 S_3 \end{pmatrix}$$

Therefore an expression for  $q_3$  can immediately be derived:

$$p_{4z} = d_0 + a_3 S_3 \implies S_3 = \frac{p_{4z} - d_0}{a_3}, C_3 = \pm \sqrt{1 - S_3^2} = \pm \sqrt{\frac{a_3^2 - (p_{4z} - d_0)^2}{a_3^2}} \implies q_3^\pm = \text{atan2}(S_3, \pm C_3)$$

$q_2$  is determined by applying summing and squaring to the position of the origin of frame  $\Sigma_2$ :

$$p_2 = \begin{pmatrix} p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} = \begin{pmatrix} (q_2 + \bar{d}_2)S_1 + a_1 C_1 \\ -(q_2 + \bar{d}_2)C_1 + a_1 S_1 \\ d_0 \end{pmatrix}$$

$$\begin{aligned} p_{2x}^2 + p_{2y}^2 &= S_1^2(q_2 + \bar{d}_2)^2 + a_1^2 C_1^2 + \underline{2a_1(q_2 + \bar{d}_2)S_1 C_1} + C_1^2(q_2 + \bar{d}_2)^2 + a_1^2 S_1^2 - \underline{2a_1(q_2 + \bar{d}_2)S_1 C_1} = \\ &= (q_2 + \bar{d}_2)^2 + a_1^2 \end{aligned}$$

Therefore  $q_2$  is given by the solution of the quadratic equation:

$$q_2^2 + 2\bar{d}_2 q_2 + a_2 - p_{2x}^2 - p_{2y}^2 = 0$$

which is:

$$q_{2,12} = -\frac{2\bar{d}_2 \pm \sqrt{4\bar{d}_2^2 - 4(a_1^2 + \bar{d}_2^2 - p_{2x}^2 - p_{2y}^2)}}{2} = -\bar{d}_2 \pm \sqrt{p_{2x}^2 + p_{2y}^2 - a_1^2}$$

To choose the solution, both are computed and the correct one will be the one that satisfies joint limits.

To choose the correct sign for  $q_3$ ,  $q_2$  is recomputed by summing and squaring the  $x$  and  $y$  components of  $p_3$ :

$$\begin{aligned} p_{3x}^2 + p_{3y}^2 &= S_1^2(q_2 + \bar{d}_2)^2 + a_1^2 C_1^2 + a_3^2 S_1^2 C_3^2 + \underline{2a_1(q_2 + \bar{d}_2)S_1 C_1} + 2a_3(q_2 + \bar{d}_2)S_1^2 C_3 + \underline{2a_1 a_3 S_1 C_1 C_3} \\ &+ C_1^2(q_2 + \bar{d}_2)^2 + a_1^2 S_1^2 + a_3^2 C_1^2 C_3^2 - \underline{2a_1(q_2 + \bar{d}_2)S_1 C_1} + 2a_3(q_2 + \bar{d}_2)C_1^2 C_3 - \underline{2a_1 a_3 S_1 C_1 C_3} = \\ &= (q_2 + \bar{d}_2)^2 + a_1^2 + a_3^2 C_3^2 + 2a_3(q_2 + \bar{d}_2)C_3 \end{aligned}$$

and therefore:

$$q_{2,12} = -a_3 C_3 - \bar{d}_2 \pm \sqrt{p_{3x}^2 + p_{3y}^2 - a_1^2}$$

By computing the four possible solutions and checking which one is equal to the one obtained previously, it is possible to determine the correct sign for  $C_3$  and therefore the correct value for  $q_3$ .

Finally,  $q_1$  is determined by solving the system of equations:

$$\begin{cases} p_{2x} = (q_2 + \bar{d}_2)S_1 + a_1 C_1 \\ p_{2y} = -(q_2 + \bar{d}_2)C_1 + a_1 S_1 \end{cases}$$

in the unknowns  $C_1$  and  $S_1$ . The system yields:

$$C_1 = \frac{a_1 p_{2x} - (q_2 + \bar{d}_2) p_{2y}}{(q_2 + \bar{d}_2)^2 + a_1^2}, S_1 = \frac{p_{2x} - a_1 C_1}{(q_2 + \bar{d}_2)} \implies q_1 = \text{atan2}(S_1, C_1)$$

For the orientation  $o_4$  of the end-effector, the angles  $\alpha, \beta$  and  $\gamma$  can be derived by equating the rotation matrix  $R_b^3$  to the rotation matrix that expresses a  $ZXZ$  Euler angle rotation:

$$\begin{bmatrix} -S_1 S_3 & -C_1 & S_1 C_3 \\ C_1 S_3 & -S_1 & -C_1 C_3 \\ C_3 & 0 & S_3 \end{bmatrix} = \begin{bmatrix} C_\alpha C_\gamma - C_\beta S_\alpha S_\gamma & -C_\alpha S_\gamma - C_\beta C_\gamma S_\alpha & S_\alpha S_\beta \\ S_\alpha C_\gamma + C_\beta C_\alpha S_\gamma & -S_\alpha S_\gamma - C_\beta C_\gamma C_\alpha & -C_\alpha S_\beta \\ S_\beta S_\gamma & C_\gamma S_\beta & C_\beta \end{bmatrix}$$

$$\begin{cases} S_\alpha S_\beta = S_1 C_3, -C_\alpha S_\beta = -C_1 C_3 \implies S_\alpha = S_1, C_\alpha = C_1 & \implies \alpha = q_1 \\ C_\beta = S_3 & \implies \beta = \frac{\pi}{2} - q_3 \\ S_\beta S_\gamma = C_3 \implies S_\gamma = 1 & \implies \gamma = \frac{\pi}{2} \end{cases} \implies o_4 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} q_1 \\ \frac{\pi}{2} - q_3 \\ \frac{\pi}{2} \end{pmatrix}$$

## 1.4 Analytical Jacobian

The lines of the analytical Jacobian are the gradients of the pose (position and orientation) of the end-effector with respect to the joint variables:

$$\begin{aligned} \nabla p_x &= \begin{bmatrix} \frac{\partial p_x}{\partial q_1} & \frac{\partial p_x}{\partial q_2} & \frac{\partial p_x}{\partial q_3} \end{bmatrix} = \begin{bmatrix} (q_2 + \bar{d}_2)C_1 - a_1 S_1 + a_3 C_3 C_1 & S_1 & -a_3 S_1 S_3 \end{bmatrix} \\ \nabla p_y &= \begin{bmatrix} \frac{\partial p_y}{\partial q_1} & \frac{\partial p_y}{\partial q_2} & \frac{\partial p_y}{\partial q_3} \end{bmatrix} = \begin{bmatrix} (q_2 + \bar{d}_2)S_1 + a_1 C_1 + a_3 S_1 C_3 & -C_1 & a_3 S_1 S_3 \end{bmatrix} \\ \nabla p_z &= \begin{bmatrix} \frac{\partial p_z}{\partial q_1} & \frac{\partial p_z}{\partial q_2} & \frac{\partial p_z}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_3 C_3 \end{bmatrix} \\ \nabla \alpha &= \begin{bmatrix} \frac{\partial \alpha}{\partial q_1} & \frac{\partial \alpha}{\partial q_2} & \frac{\partial \alpha}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ \nabla \beta &= \begin{bmatrix} \frac{\partial \beta}{\partial q_1} & \frac{\partial \beta}{\partial q_2} & \frac{\partial \beta}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \\ \nabla \gamma &= \begin{bmatrix} \frac{\partial \gamma}{\partial q_1} & \frac{\partial \gamma}{\partial q_2} & \frac{\partial \gamma}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore the analytical Jacobian is:

$$J_A = \begin{bmatrix} \nabla p_x \\ \nabla p_y \\ \nabla p_z \\ \nabla \alpha \\ \nabla \beta \\ \nabla \gamma \end{bmatrix} = \begin{bmatrix} (q_2 + \bar{d}_2)C_1 - a_1S_1 + a_3C_3C_1 & S_1 & -a_3S_1S_3 \\ (q_2 + \bar{d}_2)S_1 + a_1C_1 + a_3S_1C_3 & -C_1 & a_3C_1S_3 \\ 0 & 0 & a_3C_3 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{p}_3 \\ \dot{o}_3 \end{bmatrix} = J_A \dot{q}$$

where  $\dot{q} = [\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3]^T$ .

## 1.5 Geometric Jacobian

The  $i$ -th column of the geometric Jacobian ( $i = 0, \dots, n-1$ ) is constructed as follows:

$$J_{Gi} = \begin{bmatrix} z_i \times (p_4 - p_i) \\ z_i \end{bmatrix} \quad (\text{revolute joint}) \quad J_{Gi} = \begin{bmatrix} z_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{prismatic joint})$$

where  $p_e$  is the position of the end-effector,  $p_i$  is the position of frame  $i$  and  $z_i$  is the direction of the  $z$  axis of frame  $i$ , all with respect to the base frame.

So:

$$p_2 = \begin{bmatrix} S_1(q_2 + \bar{d}_2) + a_1C_1 \\ -C_1(q_2 + \bar{d}_2) + a_1S_1 \\ d_0 \end{bmatrix}, z_2 = \begin{bmatrix} -C_1 \\ -S_1 \\ 0 \end{bmatrix} \Rightarrow J_{G2} = [-a_3S_1S_3 \quad a_3C_1C_3 \quad a_3C_3 \quad -C_1 \quad -S_1 \quad 0]^T$$

$$p_1 = \begin{bmatrix} a_1C_1 \\ a_1S_1 \\ d_0 \end{bmatrix}, z_1 = \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \Rightarrow J_{G1} = [S_1 \quad -C_1 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ d_0 \end{bmatrix}, z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow J_{G0} = [C_1(q_2 + \bar{d}_2) - a_1S_1 + a_3C_1C_3 \quad S_1(q_2 + \bar{d}_2) + a_1C_1 + a_3S_1C_3 \quad 0 \quad 0 \quad 0 \quad 1]^T$$

Finally:

$$J_G = \begin{bmatrix} C_1(q_2 + \bar{d}_2) - a_1S_1 + a_3C_1C_3 & S_1 & -a_3S_1S_3 \\ S_1(q_2 + \bar{d}_2) + a_1C_1 + a_3S_1C_3 & -C_1 & a_3C_1S_3 \\ 0 & 0 & a_3C_3 \\ 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{p}_3 \\ \omega_3 \end{bmatrix} = J_G \dot{q}$$

As expected, the rows of the geometric Jacobian related to linear velocity are the same ones found in the analytical Jacobian.



## 2 Assignment 2 - Kinetic energy, potential energy

The object of the assignment is the computation of the kinetic energy and of the potential energy of the 3 DOF RPR manipulator.

To compute both the kinetic and potential energy, the positions of the centres of mass of each link are needed. With respect to each frame  $\Sigma_i$  attached to link  $i$ :

$$p_{l_1}^1 = \begin{bmatrix} -\frac{h_1}{2} \\ 0 \\ 0 \end{bmatrix} \quad p_{l_2}^2 = \begin{bmatrix} 0 \\ \frac{a_2}{2} \\ 0 \end{bmatrix} \quad p_{l_3}^3 = \begin{bmatrix} -\frac{h_3}{2} \\ 0 \\ 0 \end{bmatrix}$$

Using the transformation matrices obtained with direct kinematics, the positions of the centres of mass with respect to the base frame  $\Sigma_b$  are:

$$\begin{aligned} p_{l_1} &= R_b^1 p_{l_1}^1 + p_1 = \begin{bmatrix} (a_1 - \frac{h_1}{2}) C_1 \\ (a_1 - \frac{h_1}{2}) S_1 \\ d_0 \end{bmatrix} & p_{l_2} &= R_b^2 p_{l_2}^2 + p_2 = \begin{bmatrix} (q_2 + \bar{d}_2 - \frac{a_2}{2}) S_1 + a_1 C_1 \\ -(q_2 + \bar{d}_2 - \frac{a_2}{2}) C_1 + a_1 S_1 \\ d_0 \end{bmatrix} \\ p_{l_3} &= R_b^3 p_{l_3}^3 + p_3 = \begin{bmatrix} (q_2 + \bar{d}_2) S_1 + a_1 C_1 + (a_3 - \frac{h_3}{2}) S_1 C_3 \\ -(q_2 + \bar{d}_2) C_1 + a_1 S_1 - (a_3 - \frac{h_3}{2}) C_1 C_3 \\ d_0 + (a_3 - \frac{h_3}{2}) S_3 \end{bmatrix} \end{aligned}$$

### 2.1 Kinetic energy

The total kinetic energy for an open-chain manipulator is:

$$\mathcal{T}(q, \dot{q}) = \frac{1}{2} \dot{q}^T B(q) \dot{q} \quad B(q) = \sum_{i=1}^n (m_{l_i} (J_P^{l_i})^T J_P^{l_i} + (J_O^{l_i})^T R_b^i I_{l_i}^i R_b^{iT} J_O^{l_i})$$

where  $B(q)$  is the inertia matrix,  $I_{l_i}^i$  are the inertia tensors with respect to  $\Sigma_i$ ,  $J_P^{l_i}$  and  $J_O^{l_i}$  are the linear and angular partial Jacobian matrices and  $R_b^i$  are the rotation matrices that bring frame  $\Sigma_i$  to frame  $\Sigma_b$ .

The inertia tensors are obtained with Steiner's theorem:

$$\begin{aligned} I_{l_1}^1 &= I_{l_1}^{C_1} + m_{l_1} S^T(p_{l_1}^1) S(p_{l_1}^1) \\ &= m_{l_1} \begin{bmatrix} \frac{1}{2}(r_1^2) & 0 & 0 \\ 0 & \frac{1}{2}(3r_1^2 + h_1^2) & 0 \\ 0 & 0 & \frac{1}{2}(3r_1^2 + h_1^2) \end{bmatrix} + m_{l_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{h_1^2}{4} & 0 \\ 0 & 0 & \frac{h_1^2}{4} \end{bmatrix} \\ &= \frac{1}{2} m_{l_1} \begin{bmatrix} r_1^2 & 0 & 0 \\ 0 & 3r_1^2 + \frac{3h_1^2}{2} & 0 \\ 0 & 0 & 3r_1^2 + \frac{3h_1^2}{2} \end{bmatrix} \\ I_{l_2}^2 &= I_{l_2}^{C_2} + m_{l_2} S^T(p_{l_2}^2) S(p_{l_2}^2) \\ &= m_{l_2} \begin{bmatrix} \frac{1}{12}(b_2^2 + c_2^2) & 0 & 0 \\ 0 & \frac{1}{12}(a_2^2 + c_2^2) & 0 \\ 0 & 0 & \frac{1}{12}(a_2^2 + b_2^2) \end{bmatrix} + m_{l_2} \begin{bmatrix} \frac{a_2^2}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a_2^2}{4} \end{bmatrix} \\ &= \frac{1}{12} m_{l_2} \begin{bmatrix} 3a_2^2 + b_2^2 + c_2^2 & 0 & 0 \\ 0 & a_2^2 + c_2^2 & 0 \\ 0 & 0 & 4a_2^2 + b_2^2 \end{bmatrix} \\ I_{l_3}^3 &= I_{l_3}^{C_3} + m_{l_3} S^T(p_{l_3}^3) S(p_{l_3}^3) \\ &= m_{l_3} \begin{bmatrix} \frac{1}{2}(r_3^2) & 0 & 0 \\ 0 & \frac{1}{2}(3r_3^2 + h_3^2) & 0 \\ 0 & 0 & \frac{1}{2}(3r_3^2 + h_3^2) \end{bmatrix} + m_{l_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{h_3^2}{4} & 0 \\ 0 & 0 & \frac{h_3^2}{4} \end{bmatrix} \\ &= \frac{1}{2} m_{l_3} \begin{bmatrix} r_3^2 & 0 & 0 \\ 0 & 3r_3^2 + \frac{3}{2}h_3^2 & 0 \\ 0 & 0 & 3r_3^2 + \frac{3}{2}h_3^2 \end{bmatrix} \end{aligned}$$

The links are assumed to be made up of an homogeneous material, specifically aluminium. Therefore the mass  $m_{l_i}$  of link  $i$  is  $m_{l_i} = \rho V_{l_i}$ , where  $\rho = 2710 \text{ kg/m}^3$  is the density of aluminium and  $V_{l_i}$  is the volume of link  $i$ .

The partial Jacobian matrices are constructed as follows:

$$J_P^{l_i} = \begin{bmatrix} j_{P1}^{l_i} & \dots & j_{Pj}^{l_i} & 0 & \dots & 0 \end{bmatrix} \text{ where } j_{Pj}^{l_i} = \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_{l_i} - p_{j-1}) & \text{revolute joint} \end{cases}$$

$$J_O^{l_i} = \begin{bmatrix} j_{O1}^{l_i} & \dots & j_{Oj}^{l_i} & 0 & \dots & 0 \end{bmatrix} \text{ where } j_{Oj}^{l_i} = \begin{cases} 0 & \text{prismatic joint} \\ z_{j-1} & \text{revolute joint} \end{cases}$$

where  $p_{j-1}$  is the position vector of the origin of frame  $\Sigma_{j-1}$  and  $z_{j-1}$  is the unit vector of axis  $z$  of frame  $\Sigma_{j-1}$ , all with respect of  $\Sigma_b$ .

$$J_P^{l_1} = [j_{P1}^{l_1} \ 0 \ 0] = \begin{bmatrix} -S_1(a_1 - \frac{h_1}{2}) & 0 & 0 \\ C_1(a_1 - \frac{h_1}{2}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_O^{l_1} = [j_{O1}^{l_1} \ 0 \ 0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_1} = z_0 \times (p_{l_1} - p_0) = [-S_1(a_1 - \frac{h_1}{2}) \ C_1(a_1 - \frac{h_1}{2}) \ 0]^T \quad j_{O1}^{l_1} = z_0 = [0 \ 0 \ 1]^T$$

$$J_P^{l_2} = [j_{P1}^{l_2} \ j_{P2}^{l_2} \ 0] = \begin{bmatrix} (d_2 + q_2 - \frac{a_2}{2})C_1 & S_1 & 0 \\ (d_2 + q_2 - \frac{a_2}{2})S_1 & -C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_O^{l_2} = [j_{O1}^{l_2} \ j_{O2}^{l_2} \ 0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_2} = z_0 \times (p_{l_2} - p_0) = [(d_2 + q_2 - \frac{a_2}{2})C_1 \ (d_2 + q_2 - \frac{a_2}{2})S_1 \ 0]^T \quad j_{O1}^{l_2} = z_0 = [0 \ 0 \ 1]^T$$

$$j_{P2}^{l_2} = z_1 = [S_1 \ -C_1 \ 0]^T \quad j_{O2}^{l_2} = [0 \ 0 \ 0]^T$$

$$J_P^{l_3} = [j_{P1}^{l_3} \ j_{P2}^{l_3} \ j_{P3}^{l_3}] = \begin{bmatrix} C_1C_3(a_3 - \frac{h_3}{2}) & S_1 & S_1S_3(a_3 - \frac{h_3}{2}) \\ S_1C_3(a_3 - \frac{h_3}{2}) & -C_1 & -C_1S_3(a_3 - \frac{h_3}{2}) \\ 0 & 0 & C_3(a_3 - \frac{h_3}{2}) \end{bmatrix} \quad J_O^{l_3} = [j_{O1}^{l_3} \ j_{O2}^{l_3} \ j_{O3}^{l_3}] = \begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_3} = z_0 \times (p_{l_3} - p_0) = [C_1C_3(a_3 - \frac{h_3}{2}) \ S_1C_3(a_3 - \frac{h_3}{2}) \ 0]^T \quad j_{O1}^{l_3} = z_0 = [0 \ 0 \ 1]^T$$

$$j_{P2}^{l_3} = z_1 = [S_1 \ -C_1 \ 0]^T \quad j_{O2}^{l_3} = [0 \ 0 \ 0]^T$$

$$j_{P3}^{l_3} = z_2 \times (p_{l_3} - p_2) = [S_1S_3(a_3 - \frac{h_3}{2}) \ -C_1S_3(a_3 - \frac{h_3}{2}) \ C_3(a_3 - \frac{h_3}{2})]^T \quad j_{O3}^{l_3} = z_2 = [-C_1 \ -S_1 \ 0]^T$$

So the inertial matrices of each joint are:

$$B_1(q) = m_{l_1}(J_P^{l_1})^T J_P^{l_1} + (J_O^{l_1})^T R_b^1 I_{l_1}^1 R_b^{1T} J_O^{l_1}$$

$$= m_{l_1} \begin{bmatrix} a_1^2 - a_1 h_1 + h_1^2 + \frac{3}{2}r_1^2 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$B_2(q) = m_{l_2}(J_P^{l_2})^T J_P^{l_2} + (J_O^{l_2})^T R_b^2 I_{l_2}^2 R_b^{2T} J_O^{l_2}$$

$$= m_{l_2} \begin{bmatrix} \frac{1}{2}a_2^2 - a_2(d_2 + q_2) + \frac{1}{12}(b_2^2 + c_2^2) + d_2^2 + 2d_2q_2 + q_2^2 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{bmatrix}$$

$$B_3(q) = m_{l_3}(J_P^{l_3})^T J_P^{l_3} + (J_O^{l_3})^T R_b^3 I_{l_3}^3 R_b^{3T} J_O^{l_3}$$

$$= m_{l_3} \begin{bmatrix} C_3^2(h_3^2 + a_3^2 - a_3 h_3 + \frac{3}{2}r_3^2) + \frac{1}{2}r_3^2 S_3^2 & 0 & 0 \\ * & 1 & -\frac{1}{2}S_3(2a_3 - h_3) \\ * & * & \frac{1}{2}(2a_3^2 - 2a_3 h_3 + 2h_3^2 + 3r_3^2) \end{bmatrix}$$

So the overall inertia matrix is:

$$B(q) = B_1(q) + B_2(q) + B_3(q)$$

$$= \begin{bmatrix} K & 0 & 0 \\ * & m_{l_2} + m_{l_3} & -\frac{1}{2}m_{l_3}S_3(2a_3 - h_3) \\ * & * & \frac{1}{2}(2a_3^2 - 2a_3 h_3 + 2h_3^2 + 3r_3^2) \end{bmatrix}$$

with  $K = m_{l_1}B_{1,11} + m_{l_2}B_{2,11} + m_{l_3}B_{3,11}$ .

Finally, the kinetic energy is given by:

$$\begin{aligned}
T &= \frac{1}{2} \dot{q}^T B(q) \dot{q} \\
&= \frac{1}{2} \left( m_{l_1} \left( a_1^2 + h_1^2 + \frac{3}{2} r_1^2 \right) + m_{l_2} \left( \frac{1}{2} a_2^2 + \frac{1}{12} (b_2^2 + c_2^2) + (d_2 + q_2)^2 \right) + m_{l_3} \left( a_3^2 + h_3^2 + \frac{3}{2} r_3^2 + a_3 h_3 S_3 \right) \right) \dot{q}_1^2 \\
&\quad + \frac{1}{2} (m_{l_2} + m_{l_3}) \dot{q}_2^2 + \frac{1}{2} \left( \frac{1}{2} m_{l_3} h_3 S_3 \right) \dot{q}_2 \dot{q}_3 \\
&\quad + \frac{1}{2} \left( \frac{1}{2} m_{l_3} (2a_3^2 + 2h_3 + 3h_3^2 + 3r_3^2 + h_3 S_3) \right) \dot{q}_3^2
\end{aligned}$$

## 2.2 Potential energy

The potential energy is given by:

$$\mathcal{U}(q) = - \sum_{i=1}^n m_{l_i} g_0^T p_{l_i}$$

where  $g_0 = [0 \quad 0 \quad -g]^T$  is the gravity acceleration vector in the base frame  $\Sigma_b$ .  
So:

$$\begin{aligned}
\mathcal{U}_1 &= -m_{l_1} g d_0 \\
\mathcal{U}_2 &= -m_{l_2} g d_0 \\
\mathcal{U}_3(q) &= -m_{l_3} g \left( d_0 + \left( a_3 - \frac{h_3}{2} \right) S_3 \right)
\end{aligned}$$

Finally:

$$\mathcal{U}(q) = -(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3(q)) = \left[ (m_{l_1} + m_{l_2} + m_{l_3}) d_0 + m_{l_3} \left( a_3 - \frac{h_3}{2} \right) S_3 \right] g$$

### 3 Assignment 3 - Equations of motion

The equations of motion for an open chain robotic manipulator are:

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + F_v\dot{q} + F_s \text{sign}(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

Ignoring the contributions related to friction ( $F_v, F_s$ ) and the contribution related to an external wrench ( $h_e$ ), the equations reduce to:

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

where  $\tau$  is the command torque,  $C(q, \dot{q})$  is the matrix of Christoffel symbols of the first type and  $g(q)$  is the vector of gravity terms.

The gravity term is given by:

$$g_i(q) = - \sum_{j=1}^n m_{l_i} g_0^T j_{P_i}^{l_j}(q) \rightarrow g(q) = \begin{bmatrix} 0 \\ 0 \\ m_{l_3} g (a_3 - \frac{1}{2} h_3) C_3 \end{bmatrix}$$

The  $c_{ij}$  elements of  $C(q, \dot{q})$  are:

$$c_{ij} = \sum_{k=1}^n \frac{1}{2} \left( \frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i} \right) \dot{q}_k$$

where  $b_{ij}$ ,  $b_{ik}$  and  $b_{jk}$  are the elements of the inertial matrix  $B(q)$ . The derivatives of the  $B(q)$  matrix are:

$$\begin{aligned} \frac{\partial B}{\partial q_1} &= \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \\ \frac{\partial B}{\partial q_2} &= \begin{bmatrix} m_{l_2}(2d_2 - a_2 + 2q_2) & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \\ \frac{\partial B}{\partial q_3} &= \begin{bmatrix} 2m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2) & 0 & 0 \\ * & 0 & -\frac{1}{2}m_{l_3}C_3(2a_3 - h_3) \\ * & * & 0 \end{bmatrix} \end{aligned}$$

So the  $c_{ij}$  components are:

$$\begin{aligned} c_{11} &= \frac{1}{2} \left( \frac{\partial b_{11}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_1} - \frac{\partial b_{11}}{\partial q_1} \right) \dot{q}_1 + \frac{1}{2} \left( \frac{\partial b_{11}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_1} - \frac{\partial b_{12}}{\partial q_1} \right) \dot{q}_2 + \frac{1}{2} \left( \frac{\partial b_{11}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_1} - \frac{\partial b_{13}}{\partial q_1} \right) \dot{q}_3 \\ &= \frac{1}{2} m_{l_2}(2d_2 + 2q_2 - a_2) \dot{q}_2 + m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2) \dot{q}_3 \\ c_{12} &= \frac{1}{2} \left( \frac{\partial b_{12}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_1} \right) \dot{q}_1 + \frac{1}{2} \left( \frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_1} \right) \dot{q}_2 + \frac{1}{2} \left( \frac{\partial b_{12}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_2} - \frac{\partial b_{23}}{\partial q_1} \right) \dot{q}_3 \\ &= \frac{1}{2} m_{l_2}(2d_2 + 2q_2 - a_2) \dot{q}_1 \\ c_{13} &= \frac{1}{2} \left( \frac{\partial b_{13}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_1} \right) \dot{q}_1 + \frac{1}{2} \left( \frac{\partial b_{13}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_1} \right) \dot{q}_2 + \frac{1}{2} \left( \frac{\partial b_{13}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_1} \right) \dot{q}_3 \\ &= m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2) \dot{q}_1 \\ c_{22} &= \frac{1}{2} \left( \frac{\partial b_{22}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_2} \right) \dot{q}_1 + \frac{1}{2} \left( \frac{\partial b_{22}}{\partial q_2} + \frac{\partial b_{22}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_2} \right) \dot{q}_2 + \frac{1}{2} \left( \frac{\partial b_{22}}{\partial q_3} + \frac{\partial b_{23}}{\partial q_2} - \frac{\partial b_{23}}{\partial q_2} \right) \dot{q}_3 \\ &= 0 \\ c_{23} &= \frac{1}{2} \left( \frac{\partial b_{23}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_2} \right) \dot{q}_1 + \frac{1}{2} \left( \frac{\partial b_{23}}{\partial q_2} + \frac{\partial b_{22}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_2} \right) \dot{q}_2 + \frac{1}{2} \left( \frac{\partial b_{23}}{\partial q_3} + \frac{\partial b_{23}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_2} \right) \dot{q}_3 \\ &= -\frac{1}{2} m_{l_3}C_3(2a_3 - h_3) \dot{q}_3 \\ c_{33} &= \frac{1}{2} \left( \frac{\partial b_{33}}{\partial q_1} + \frac{\partial b_{31}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_3} \right) \dot{q}_1 + \frac{1}{2} \left( \frac{\partial b_{33}}{\partial q_2} + \frac{\partial b_{32}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_3} \right) \dot{q}_2 + \frac{1}{2} \left( \frac{\partial b_{33}}{\partial q_3} + \frac{\partial b_{33}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_3} \right) \dot{q}_3 \\ &= 0 \end{aligned}$$

So the equations of motion are:

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

where  $q = [q_1 \quad q_2 \quad q_3]^T$ ,  $\dot{q} = [\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3]^T$ ,  $\ddot{q} = [\ddot{q}_1 \quad \ddot{q}_2 \quad \ddot{q}_3]^T$  and:

$$B(q) = \begin{bmatrix} K & 0 & 0 \\ * & m_{l_2} + m_{l_3} & -\frac{1}{2}m_{l_3}S_3(2a_3 - h_3) \\ * & * & \frac{1}{2}(2a_3^2 - 2a_3h_3 + 2h_3^2 + 3r_3^2) \end{bmatrix}$$

$$K = m_{l_1} \left( a_1^2 - a_1h_1 + h_1^2 + \frac{3}{2}r_1^2 \right) + m_{l_2} \left( \frac{1}{2}a_2^2 - a_2(d_2 + q_2) + \frac{1}{12}(b_2^2 + c_2^2) + d_2^2 + 2d_2q_2 + q_2^2 \right) \\ + m_{l_3} \left( C_3^2 \left( h_3^2 + a_3^2 - a_3h_3 + \frac{3}{2}r_3^2 \right) + \frac{1}{2}r_3^2S_3^2 \right)$$

$$C(q, \dot{q}) = \begin{bmatrix} \frac{1}{2}m_{l_2}(2d_2 + 2q_2 - a_2)\dot{q}_2 + m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2)\dot{q}_3 & \frac{1}{2}m_{l_2}(2d_2 + 2q_2 - a_2)\dot{q}_1 & m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2)\dot{q}_1 \\ * & 0 & -\frac{1}{2}m_{l_3}C_3(2a_3 - h_3)\dot{q}_3 \\ * & * & 0 \end{bmatrix}$$

$$g(q) = \begin{bmatrix} 0 \\ 0 \\ m_{l_3}g(a_3 - \frac{1}{2}h_3)C_3 \end{bmatrix}$$

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## 4 Assignment 4 - Recursive Newton-Euler formulation

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## 5 Assignment 5 - Operational space dynamic model