Master's degree in Computer Engineering for Robotics and Smart Industry

Advanced control systems

Report on the assignments given during the 2022/2023 a.y.

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Contents

1 Assignment 1 - DH table, direct/inverse kinematics, Jacobians

Given a 3 Degrees Of Freedom (DOF) manipulator with a Revolute-Prismatic-Revolute (RPR) configuration, the object of the assignment is to compute the Denavit-Hartenberg (DH) parameter table, compute the direct and inverse kinematics and compute the analytical and geometrical Jacobian matrices.

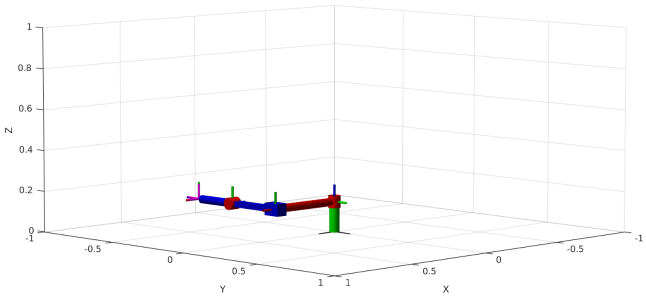

Figure 1: Visualization of the URDF of the PRP robot in the home configuration ($q_1 = 0, q_2 = 0, \theta_3 = 0$)

1.1 Denavit-Hartenberg parameters

First of all, reference frames were assigned to each joint according to the DH convention:

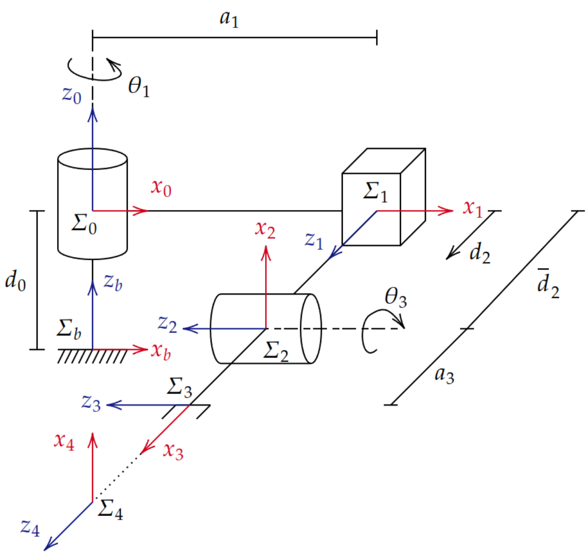

Figure 2: Reference frames assigned with the DH convention

Note that the DH frames do not correspond with the frames of the URDF model.
Then the DH parameter table was populated:

	a_i	α_i	d_i	θ_i
0	0	0	d_0	0
1	a_1	$\frac{\pi}{2}$	0	$q_1 = \theta_1$
2	0	$-\frac{\pi}{2}$	$(q_2 = q_2) + \bar{d}_2$	$\frac{\pi}{2}$
3	a_3	0	0	$(q_3 = \theta_3) - \frac{\pi}{2}$
4	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$

The first row is the fixed offset in the z_b direction between world and frame Σ_0, while the last row is the fixed rotation that aligns the z axis of the end-effector to the approach direction.

Using the DH parameters, the transformation matrices between the frames were then computed. The transformation matrix between frame i and frame $i-1$ is in the form:

$$T_{i-1}^{i} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The computed transformation matrices are then:

$$T_b^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_0^1 = \begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_1^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & q_2+\overline{d}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} C\left(q_3-\frac{\pi}{2}\right) & -S\left(q_3-\frac{\pi}{2}\right) & 0 & a_3C\left(q_3-\frac{\pi}{2}\right) \\ S\left(q_3-\frac{\pi}{2}\right) & C\left(q_3-\frac{\pi}{2}\right) & 0 & a_3S\left(q_3-\frac{\pi}{2}\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_3 & C_3 & 0 & a_3S_3 \\ -C_3 & S_3 & 0 & -a_3C_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_3^4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where C_i and S_i denote respectively $\cos(q_i)$ and $\sin(q_i)$.

1.2 Direct kinematics

The direct kinematics of the manipulator are obtained by chaining the above transformation matrices, thus obtaining a global transformation matrix from the end-effector to the base of the manipulator:

$$T_b^4 = T_b^0T_0^1T_1^2T_2^3T_3^4 = \underbrace{\begin{bmatrix} C_1 & 0 & S_1 & a_1C_1 \\ S_1 & 0 & -C_1 & a_1S_1 \\ 0 & 1 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^1} T_1^2T_2^3T_3^4 = \underbrace{\begin{bmatrix} 0 & -S_1 & -C_1 & (q_2+\overline{d}_2)S_1+a_1C_1 \\ 0 & C_1 & -S_1 & -(q_2+\overline{d}_2)C_1+a_1S_1 \\ 1 & 0 & 0 & d_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^2} T_2^3T_3^4$$

$$= \underbrace{\begin{bmatrix} S_1C_3 & -S_1S_3 & -C_1 & (q_2+\overline{d}_2)S_1+a_1C_1+a_3S_1C_3 \\ -C_1C_3 & C_1S_3 & -S_1 & -(q_2+\overline{d}_2)C_1+a_1S_1-a_3C_1C_3 \\ S_3 & C_3 & 0 & d_0+a_3S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T_b^3} T_3^4$$

$$= \begin{bmatrix} -S_1S_3 & -C_1 & S_1C_3 & (q_2+\overline{d}_2)S_1+a_1C_1+a_3S_1C_3 \\ C_1S_3 & -S_1 & -C_1C_3 & -(q_2+\overline{d}_2)C_1+a_1S_1-a_3C_1C_3 \\ C_3 & 0 & S_3 & d_0+a_3S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_b^4 & p_4 \\ \overline{0} & 1 \end{bmatrix}$$

1.3 Inverse kinematics

From the direct kinematics, the position of the end-effector is given by:

$$p_4 = \begin{pmatrix} p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix} = \begin{pmatrix} (q_2+\overline{d}_2)S_1+a_1C_1+a_3S_1C_3 \\ -(q_2+\overline{d}_2)C_1+a_1S_1-a_3C_1C_3 \\ d_0+a_3S_3 \end{pmatrix}$$

Therefore an expression for q_3 can immediately be derived:

$$p_{4z} = d_0+a_3S_3 \implies S_3 = \frac{p_{4z}-d_0}{a_3}, C_3 = \pm\sqrt{1-S_3^2} = \pm\sqrt{\frac{a_3^2-(p_{4z}-d_0)^2}{a_3^2}} \implies q_3^{\pm} = \operatorname{atan2}(S_3, \pm C_3)$$

q_2 is determined by applying summing and squaring to the position of the origin of frame Σ_2:

$$p_2 = \begin{pmatrix} p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} = \begin{pmatrix} (q_2+\overline{d}_2)S_1+a_1C_1 \\ -(q_2+\overline{d}_2)C_1+a_1S_1 \\ d_0 \end{pmatrix}$$

$$
\begin{aligned}
p_{2x}^2 + p_{2y}^2 &= S_1^2(q_2+\overline{d}_2)^2 + a_1^2C_1^2 + \cancel{2a_1(q_2+\overline{d}_2)S_1C_1} + C_1^2(q_2+\overline{d}_2)^2 + a_1^2S_1^2 - \cancel{2a_1(q_2+\overline{d}_2)S_1C_1} = \\
&= (q_2+\overline{d}_2)^2 + a_1^2
\end{aligned}
$$

Therefore q_2 is given by the solution of the quadratic equation:

$$q_2^2 + 2\overline{d}_2 q_2 + a_2 - p_{2x}^2 - p_{2y}^2 = 0$$

which is:

$$q_{2,12} = -\frac{\cancel{2}\overline{d}_2 \pm \sqrt{\cancel{4}\cancel{\overline{d}_2^2} - \cancel{4}(a_1^2 + \cancel{\overline{d}_2^2} - p_{2x}^2 - p_{2y}^2)}}{\cancel{2}} = -\overline{d}_2 \pm \sqrt{p_{2x}^2 + p_{2y}^2 - a_1^2}$$

To choose the solution, both are computed and the correct one will be the one that satisfies joint limits.
To choose the correct sign for q_3, q_2 is recomputed by summing and squaring the x and y components of p_3:

$$
\begin{aligned}
p_{3x}^2 + p_{3y}^2 &= S_1^2(q_2+\overline{d}_2)^2 + a_1^2C_1^2 + a_3^2S_1^2C_3^2 + \cancel{2a_1(q_2+\overline{d}_2)S_1C_1} + 2a_3(q_2+\overline{d}_2)S_1^2C_3 + \cancel{2a_1a_3S_1C_1C_3} \\
&+ C_1^2(q_2+\overline{d}_2)^2 + a_1^2S_1^2 + a_3^2C_1^2C_3^2 - \cancel{2a_1(q_2+\overline{d}_2)S_1C_1} + 2a_3(q_2+\overline{d}_2)C_1^2C_3 - \cancel{2a_1a_3S_1C_1C_3} = \\
&= (q_2+\overline{d}_2)^2 + a_1^2 + a_3^2C_3^2 + 2a_3(q_2+\overline{d}_2)C_3
\end{aligned}
$$

and therefore:

$$q_{2,12} = -a_3C_3 - \overline{d}_2 \pm \sqrt{p_{4x}^2 + p_{4y}^2 - a_1^2}$$

By computing the four possible solutions and checking which one is equal to the one obtained previously, it is possible to determine the correct sign for C_3 and therefore the correct value for q_3.

Finally, q_1 is determined by solving the system of equations:

$$
\begin{cases}
p_{2x} = (q_2+\overline{d}_2)S_1 + a_1C_1 \\
p_{2y} = -(q_2+\overline{d}_2)C_1 + a_1S_1
\end{cases}
$$

in the unknowns C_1 and S_1. The system yields:

$$C_1 = \frac{a_1p_{2x} - (q_2+\overline{d}_2)p_{2y}}{(q_2+\overline{d}_2)^2 + a_1^2}, S_1 = \frac{p_{2x} - a_1C_1}{(q_2+\overline{d}_2)} \implies q_1 = \operatorname{atan2}(S_1, C_1)$$

For the orientation o_4 of the end-effector, the angles α, β and γ can be derived by equating the rotation matrix R_b^3 to the rotation matrix that expresses a ZXZ Euler angle rotation:

$$
\begin{bmatrix}
-S_1S_3 & -C_1 & S_1C_3 \\
C_1S_3 & -S_1 & -C_1C_3 \\
C_3 & 0 & S_3
\end{bmatrix}
=
\begin{bmatrix}
C_\alpha C_\gamma - C_\beta S_\alpha S_\gamma & -C_\alpha S_\gamma - C_\beta C_\gamma S_\alpha & S_\alpha S_\beta \\
S_\alpha C_\gamma + C_\beta C_\alpha S_\gamma & -S_\alpha S_\gamma - C_\beta C_\gamma C_\alpha & -C_\alpha S_\beta \\
S_\beta S_\gamma & C_\gamma S_\beta & C_\beta
\end{bmatrix}
$$

$$
\begin{cases}
S_\alpha S_\beta = S_1C_3, -C_\alpha S_\beta = -C_1C_3 \implies S_\alpha = S_1, C_\alpha = C_1 & \implies \alpha = q_1 \\
C_\beta = S_3 & \implies \beta = \frac{\pi}{2} - q_3 \\
S_\beta S_\gamma = C_3 \implies S_\gamma = 1 & \implies \gamma = \frac{\pi}{2}
\end{cases}
\implies o_4 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} q_1 \\ \frac{\pi}{2} - q_3 \\ \frac{\pi}{2} \end{pmatrix}
$$

1.4 Analytical Jacobian

The lines of the analytical Jacobian are the gradients of the pose (position and orientation) of the end-effector with respect to the joint variables:

$$
\begin{aligned}
\nabla p_x &= \begin{bmatrix} \frac{\partial p_x}{\partial q_1} & \frac{\partial p_x}{\partial q_2} & \frac{\partial p_x}{\partial q_3} \end{bmatrix} = \begin{bmatrix} (q_2+\overline{d}_2)C_1 - a_1S_1 + a_3C_3C_1 & S_1 & -a_3S_1S_3 \end{bmatrix} \\
\nabla p_y &= \begin{bmatrix} \frac{\partial p_y}{\partial q_1} & \frac{\partial p_y}{\partial q_2} & \frac{\partial p_y}{\partial q_3} \end{bmatrix} = \begin{bmatrix} (q_2+\overline{d}_2)S_1 + a_1C_1 + a_3S_1C_3 & -C_1 & a_3S_1S_3 \end{bmatrix} \\
\nabla p_z &= \begin{bmatrix} \frac{\partial p_z}{\partial q_1} & \frac{\partial p_z}{\partial q_2} & \frac{\partial p_z}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_3C_3 \end{bmatrix} \\
\nabla \alpha &= \begin{bmatrix} \frac{\partial \alpha}{\partial q_1} & \frac{\partial \alpha}{\partial q_2} & \frac{\partial \alpha}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
\nabla \beta &= \begin{bmatrix} \frac{\partial \beta}{\partial q_1} & \frac{\partial \alpha}{\partial q_2} & \frac{\partial \beta}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \\
\nabla \gamma &= \begin{bmatrix} \frac{\partial \gamma}{\partial q_1} & \frac{\partial \gamma}{\partial q_2} & \frac{\partial \gamma}{\partial q_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Therefore the analytical Jacobian is:

$$J_A = \begin{bmatrix} \nabla p_x \\ \nabla p_y \\ \nabla p_z \\ \nabla \alpha \\ \nabla \beta \\ \nabla \gamma \end{bmatrix} = \begin{bmatrix} (q_2+\overline{d}_2)C_1 - a_1 S_1 + a_3 C_3 C_1 & S_1 & -a_3 S_1 S_3 \\ (q_2+\overline{d}_2)S_1 + a_1 C_1 + a_3 S_1 C_3 & -C_1 & a_3 C_1 S_3 \\ 0 & 0 & a_3 C_3 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} \dot{p}_3 \\ \dot{o}_3 \end{bmatrix} = J_A \dot{q}$$

where $\dot{q} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{bmatrix}^T$.

1.5 Geometric Jacobian

The i-th column of the geometric Jacobian ($i = 0, \ldots, n-1$) is constructed as follows:

$$J_{Gi} = \begin{bmatrix} z_i \times (p_4 - p_i) \\ z_i \end{bmatrix} \quad \text{(revolute joint)} \qquad J_{Gi} = \begin{bmatrix} z_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(prismatic joint)}$$

where p_e is the position of the end-effector, p_i is the position of frame i and z_i is the direction of the z axis of frame i, all with respect to the base frame.

So:

$$p_2 = \begin{bmatrix} S_1(q_2+\overline{d}_2) + a_1 C_1 \\ -C_1(q_2+\overline{d}_2) + a_1 S_1 \\ d_0 \end{bmatrix}, z_2 = \begin{bmatrix} -C_1 \\ -S_1 \\ 0 \end{bmatrix} \implies J_{G2} = \begin{bmatrix} -a_3 S_1 S_3 & a_3 C_1 C_3 & a_3 C_3 & -C_1 & -S_1 & 0 \end{bmatrix}^T$$

$$p_1 = \begin{bmatrix} a_1 C_1 \\ a_1 S_1 \\ d_0 \end{bmatrix}, z_1 = \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \implies J_{G1} = \begin{bmatrix} S_1 & -C_1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ d_0 \end{bmatrix}, z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies J_{G0} = \begin{bmatrix} C_1(q_2+\overline{d}_2) - a_1 S_1 + a_3 C_1 C_3 & S_1(q_2+\overline{d}_2) + a_1 C_1 + a_3 S_1 C_3 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

Finally:

$$J_G = \begin{bmatrix} C_1(q_2+\overline{d}_2) - a_1 S_1 + a_3 C_1 C_3 & S_1 & -a_3 S_1 S_3 \\ S_1(q_2+\overline{d}_2) + a_1 C_1 + a_3 S_1 C_3 & -C_1 & a_3 C_1 C_3 \\ 0 & 0 & a_3 C_3 \\ 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} \dot{p}_3 \\ \omega_3 \end{bmatrix} = J_G \dot{q}$$

As expected, the rows of the geometric Jacobian related to linear velocity are the same ones found in the analytical Jacobian.

1.6 Relationship between JG and JA

The geometric and analytical Jacobians are related by the following relationship:

$$J_G = T_A(\Phi) J_A = \begin{bmatrix} I & 0 \\ 0 & T(\Phi) \end{bmatrix} J_A$$

Therefore:

$$\begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix} = T(\Phi) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies T(\Phi) = \begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & C_1 & 0 \\ 0 & S_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

2 Assignment 2 - Kinetic energy, potential energy

The object of the assignment is the computation of the kinetic energy and of the potential energy of the 3 DOF RPR manipulator.

To compute both the kinetic and potential energy, the positions of the centres of mass of each link are needed. With respect to each frame Σ_i attached to link i:

$$p^1_{l_1} = \begin{bmatrix} -\frac{h_1}{2} \\ 0 \\ 0 \end{bmatrix} \qquad p^2_{l_2} = \begin{bmatrix} 0 \\ \frac{a_2}{2} \\ 0 \end{bmatrix} \qquad p^3_{l_3} = \begin{bmatrix} -\frac{h_3}{2} \\ 0 \\ 0 \end{bmatrix}$$

Using the transformation matrices obtained with direct kinematics, the positions of the centres of mass with respect to the base frame Σ_b are:

$$p_{l_1} = R^1_b p^1_{l_1} + p_1 = \begin{bmatrix} \left(a_1 - \frac{h_1}{2}\right) C_1 \\ \left(a_1 - \frac{h_1}{2}\right) S_1 \\ d_0 \end{bmatrix} \qquad p_{l_2} = R^2_b p^2_{l_2} + p_2 = \begin{bmatrix} \left(q_2 + \overline{d}_2 - \frac{a_2}{2}\right) S_1 + a_1 C_1 \\ -\left(q_2 + \overline{d}_2 - \frac{a_2}{2}\right) C_1 + a_1 S_1 \\ d_0 \end{bmatrix}$$

$$p_{l_3} = R^3_b p^3_{l_3} + p_3 = \begin{bmatrix} (q_2 + \overline{d}_2) S_1 + a_1 C_1 + \left(a_3 - \frac{h_3}{2}\right) S_1 C_3 \\ -(q_2 + \overline{d}_2) C_1 + a_1 S_1 - \left(a_3 - \frac{h_3}{2}\right) C_1 C_3 \\ d_0 + \left(a_3 - \frac{h_3}{2}\right) S_3 \end{bmatrix}$$

2.1 Kinetic energy

The total kinetic energy for an open-chain manipulator is:

$$\mathcal{T}(q, \dot{q}) = \frac{1}{2} \dot{q}^T B(q) \dot{q} \qquad B(q) = \sum_{i=1}^{n} (m_{l_i} (J^{l_i}_P)^T J^{l_i}_P + (J^{l_i}_O)^T R^i_b I^i_{l_i} {R^i_b}^T J^{l_i}_O)$$

where $B(q)$ is the inertia matrix, $I^i_{l_i}$ are the inertia tensors with respect to Σ_i, $J^{l_i}_P$ and $J^{l_i}_O$ are the linear and angular partial Jacobian matrices and R^i_b are the rotation matrices that bring frame Σ_i to frame Σ_b.

The inertia tensors are obtained with Steiner's theorem:

$$\begin{aligned}
I^1_{l_1} &= I^{C_1}_{l_1} + m_{l_1} S^T(p^1_{l_1}) S(p^1_{l_1}) \\
&= m_{l_1} \begin{bmatrix} \frac{1}{2}(r_1^2) & 0 & 0 \\ 0 & \frac{1}{2}(3r_1^2 + h_1^2) & 0 \\ 0 & 0 & \frac{1}{2}(3r_1^2 + h_1^2) \end{bmatrix} + m_{l_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{h_1^2}{4} & 0 \\ 0 & 0 & \frac{h_1^2}{4} \end{bmatrix} \\
&= \frac{1}{2} m_{l_1} \begin{bmatrix} r_1^2 & 0 & 0 \\ 0 & 3r_1^2 + \frac{3h_1^2}{2} & 0 \\ 0 & 0 & 3r_1^2 + \frac{3h_1^2}{2} \end{bmatrix} \\
I^2_{l_2} &= I^{C_2}_{l_2} + m_{l_2} S^T(p^2_{l_2}) S(p^2_{l_2}) \\
&= m_{l_2} \begin{bmatrix} \frac{1}{12}(b_2^2 + c_2^2) & 0 & 0 \\ 0 & \frac{1}{12}(a_2^2 + c_2^2) & 0 \\ 0 & 0 & \frac{1}{12}(a_2^2 + b_2^2) \end{bmatrix} + m_{l_2} \begin{bmatrix} \frac{a_2^2}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{a_2^2}{4} \end{bmatrix} \\
&= \frac{1}{12} m_{l_2} \begin{bmatrix} 3a_2^2 + b_2^2 + c_2^2 & 0 & 0 \\ 0 & a_2^2 + c_2^2 & 0 \\ 0 & 0 & 4a_2^2 + b_2^2 \end{bmatrix} \\
I^3_{l_3} &= I^{C_3}_{l_3} + m_{l_3} S^T(p^3_{l_3}) S(p^3_{l_3}) \\
&= m_{l_3} \begin{bmatrix} \frac{1}{2}(r_3^2) & 0 & 0 \\ 0 & \frac{1}{2}(3r_3^2 + h_3^2) & 0 \\ 0 & 0 & \frac{1}{2}(3r_3^2 + h_3^2) \end{bmatrix} + m_{l_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{h_3^2}{4} & 0 \\ 0 & 0 & \frac{h_3^2}{4} \end{bmatrix} \\
&= \frac{1}{2} m_{l_3} \begin{bmatrix} r_3^2 & 0 & 0 \\ 0 & 3r_3^2 + \frac{3}{2}h_3^2 & 0 \\ 0 & 0 & 3r_3^2 + \frac{3}{2}h_3^2 \end{bmatrix}
\end{aligned}$$

The links are assumed to be made up of an homogeneous material, specifically aluminium. Therefore the mass m_{l_i} of link i is $m_{l_i} = \rho V_{l_i}$, where $\rho = 2710\ kg/m^3$ is the density of aluminium and V_{l_i} is the volume of link i.

The partial Jacobian matrices are constructed as follows:

$$J_P^{l_i} = \begin{bmatrix} j_{P1}^{l_i} & \dots & j_{Pj}^{l_i} & 0 & \dots & 0 \end{bmatrix} \text{ where } j_{Pj}^{l_i} = \begin{cases} z_{j-1} & \text{prismatic joint} \\ z_{j-1} \times (p_{l_i} - p_{j-1}) & \text{revolute joint} \end{cases}$$

$$J_O^{l_i} = \begin{bmatrix} j_{O1}^{l_i} & \dots & j_{Oj}^{l_i} & 0 & \dots & 0 \end{bmatrix} \text{ where } j_{Oj}^{l_i} = \begin{cases} 0 & \text{prismatic joint} \\ z_{j-1} & \text{revolute joint} \end{cases}$$

where p_{j-1} is the position vector of the origin of frame Σ_{j-1} and z_{j-1} is the unit vector of axis z of frame Σ_{j-1}, all with respect of Σ_b.

$$J_P^{l_1} = \begin{bmatrix} j_{P1}^{l_1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -S_1\left(a_1 - \frac{h_1}{2}\right) & 0 & 0 \\ C_1\left(a_1 - \frac{h_1}{2}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_1} = z_0 \times (p_{l_1} - p_0) = \begin{bmatrix} -S_1\left(a_1 - \frac{h_1}{2}\right) & C_1\left(a_1 - \frac{h_1}{2}\right) & 0 \end{bmatrix}^T$$

$$J_O^{l_1} = \begin{bmatrix} j_{O1}^{l_1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{O1}^{l_1} = z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$J_P^{l_2} = \begin{bmatrix} j_{P1}^{l_2} & j_{P2}^{l_2} & 0 \end{bmatrix} = \begin{bmatrix} \left(d_2 + q_2 - \frac{a_2}{2}\right) C_1 & S_1 & 0 \\ \left(d_2 + q_2 - \frac{a_2}{2}\right) S_1 & -C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$j_{P1}^{l_2} = z_0 \times (p_{l_2} - p_0) = \begin{bmatrix} \left(d_2 + q_2 - \frac{a_2}{2}\right) C_1 & \left(d_2 + q_2 - \frac{a_2}{2}\right) S_1 & 0 \end{bmatrix}^T$$

$$j_{P2}^{l_2} = z_1 = \begin{bmatrix} S_1 & -C_1 & 0 \end{bmatrix}^T$$

$$J_O^{l_2} = \begin{bmatrix} j_{O1}^{l_2} & j_{O2}^{l_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{O1}^{l_2} = z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$j_{O2}^{l_2} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

$$J_P^{l_3} = \begin{bmatrix} j_{P1}^{l_3} & j_{P2}^{l_3} & j_{P3}^{l_3} \end{bmatrix} = \begin{bmatrix} C_1C_3\left(a_3 - \frac{h_3}{2}\right) & S_1 & S_1S_3\left(a_3 - \frac{h_3}{2}\right) \\ S_1C_3\left(a_3 - \frac{h_3}{2}\right) & -C_1 & -C_1S_3\left(a_3 - \frac{h_3}{2}\right) \\ 0 & 0 & C_3\left(a_3 - \frac{h_3}{2}\right) \end{bmatrix}$$

$$j_{P1}^{l_3} = z_0 \times (p_{l_3} - p_0) = \begin{bmatrix} C_1C_3\left(a_3 - \frac{h_3}{2}\right) & S_1C_3\left(a_3 - \frac{h_3}{2}\right) & 0 \end{bmatrix}^T$$

$$j_{P2}^{l_3} = z_1 = \begin{bmatrix} S_1 & -C_1 & 0 \end{bmatrix}^T$$

$$j_{P3}^{l_3} = z_2 \times (p_{l_3} - p_2) = \begin{bmatrix} S_1S_3\left(a_3 - \frac{h_3}{2}\right) & -C_1S_3\left(a_3 - \frac{h_3}{2}\right) & C_3\left(a_3 - \frac{h_3}{2}\right) \end{bmatrix}^T$$

$$J_O^{l_3} = \begin{bmatrix} j_{O1}^{l_3} & j_{O2}^{l_3} & j_{O3}^{l_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$j_{O1}^{l_3} = z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$j_{O2}^{l_3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

$$j_{O3}^{l_3} = z_2 = \begin{bmatrix} -C_1 & -S_1 & 0 \end{bmatrix}^T$$

So the inertial matrices of each joint are:

$$\begin{aligned}
B_1(q) &= m_{l_1} (J_P^{l_1})^T J_P^{l_1} + (J_O^{l_1})^T R_b^1 I_{l_1}^1 {R_b^1}^T J_O^{l_1} \\
&= m_{l_1} \begin{bmatrix} a_1^2 - a_1h_1 + h_1^2 + \frac{3}{2}r_1^2 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \\
B_2(q) &= m_{l_2} (J_P^{l_2})^T J_P^{l_2} + (J_O^{l_2})^T R_b^2 I_{l_2}^2 {R_b^2}^T J_O^{l_2} \\
&= m_{l_2} \begin{bmatrix} \frac{1}{2}a_2^2 - a_2(d_2 + q_2) + \frac{1}{12}(b_2^2 + c_2^2) + d_2^2 + 2d_2q_2 + q_2^2 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{bmatrix} \\
B_3(q) &= m_{l_3} (J_P^{l_3})^T J_P^{l_3} + (J_O^{l_3})^T R_b^3 I_{l_3}^3 {R_b^3}^T J_O^{l_3} \\
&= m_{l_3} \begin{bmatrix} C_3^2\left(h_3^2 + a_3^2 - a_3h_3 + \frac{3}{2}r_3^2\right) + \frac{1}{2}r_3^2S_3^2 & 0 & 0 \\ * & 1 & -\frac{1}{2}S_3(2a_3 - h_3) \\ * & * & \frac{1}{2}(2a_3^2 - 2a_3h_3 + 2h_3^2 + 3r_3^2) \end{bmatrix}
\end{aligned}$$

So the overall inertia matrix is:

$$\begin{aligned}
B(q) &= B_1(q) + B_2(q) + B_3(q) \\
&= \begin{bmatrix} K & 0 & 0 \\ * & m_{l_2} + m_{l_3} & -\frac{1}{2}m_{l_3}S_3(2a_3 - h_3) \\ * & * & \frac{1}{2}(2a_3^2 - 2a_3h_3 + 2h_3^2 + 3r_3^2) \end{bmatrix}
\end{aligned}$$

with $K = m_{l_1}B_{1,11} + m_{l_2}B_{2,11} + m_{l_3}B_{3,11}$.

Finally, the kinetic energy is given by:

$$
\begin{aligned}
T &= \frac{1}{2}\dot{q}^T B(q)\dot{q} \\
&= \frac{1}{2}\left(m_{l_1}\left(a_1^2 + h_1^2 + \frac{3}{2}r_1^2\right) + m_{l_2}\left(\frac{1}{2}a_2^2 + \frac{1}{12}(b_2^2 + c_2^2) + (d_2 + q_2)^2\right) + m_{l_3}\left(a_3^2 + h_3^2 + \frac{3}{2}r_3^2 + a_3 h_3 S_3\right)\right)\dot{q}_1^2 \\
&+ \frac{1}{2}\left(m_{l_2} + m_{l_3}\right)\dot{q}_2^2 + \frac{1}{2}\left(\frac{1}{2}m_{l_3} h_3 S_3\right)\dot{q}_2\dot{q}_3 \\
&+ \frac{1}{2}\left(\frac{1}{2}m_{l_3}\left(2a_3^2 + 2h_3 + 3h_3^2 + 3r_3^2 + h_3 S_3\right)\right)\dot{q}_3^2
\end{aligned}
$$

2.2 Potential energy

The potential energy is given by:

$$
\mathcal{U}(q) = -\sum_{i=1}^{n} m_{l_i} g_0^T p_{l_i}
$$

where $g_0 = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$ is the gravity acceleration vector in the base frame Σ_b.
So:

$$
\begin{aligned}
\mathcal{U}_1 &= -m_{l_1} g d_0 \\
\mathcal{U}_2 &= -m_{l_2} g d_0 \\
\mathcal{U}_3(q) &= -m_{l_3} g \left(d_0 + \left(a_3 - \frac{h_3}{2}\right) S_3\right)
\end{aligned}
$$

Finally:

$$
\mathcal{U}(q) = -(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3(q)) = \left[(m_{l_1} + m_{l_2} + m_{l_3})d_0 + m_{l_3}\left(a_3 - \frac{h_3}{2}\right) S_3\right] g
$$

3 Assignment 3 - Equations of motion

The equations of motion for an open chain robotic manipulator are:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s sign(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

Ignoring the contributions related to friction (F_v, F_s) and the contribution related to an external wrench (h_e), the equations reduce to:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where τ is the command torque, $C(q,\dot{q})$ is the matrix of Christoffel symbols of the first type and $g(q)$ is the vector of gravity terms.

The gravity term is given by:

$$g_i(q) = -\sum_{j=1}^{n} m_{l_i} g_0^T j_{Pi}^{l_j}(q) \rightarrow g(q) = \begin{bmatrix} 0 \\ 0 \\ m_{l_3} g \left(a_3 - \frac{1}{2}h_3\right) C_3 \end{bmatrix}$$

The c_{ij} elements of $C(q,\dot{q})$ are:

$$c_{ij} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k$$

where b_{ij}, b_{ik} and b_{jk} are the elements of the inertial matrix $B(q)$. The derivatives of the $B(q)$ matrix are:

$$\frac{\partial B}{\partial q_1} = \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$
$$\frac{\partial B}{\partial q_2} = \begin{bmatrix} m_{l_2}(2d_2 - a_2 + 2q_2) & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$
$$\frac{\partial B}{\partial q_3} = \begin{bmatrix} 2m_{l_3} S_3 C_3 (a_3 h_3 - a_3^2 - h_3^2 - r_3^2) & 0 & 0 \\ * & 0 & -\frac{1}{2} m_{l_3} C_3 (2a_3 - h_3) \\ * & * & 0 \end{bmatrix}$$

So the c_{ij} components are:

$$\begin{aligned}
c_{11} &= \frac{1}{2}\left(\frac{\partial b_{11}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_1} - \frac{\partial b_{11}}{\partial q_1}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{11}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_1} - \frac{\partial b_{12}}{\partial q_1}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{11}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_1} - \frac{\partial b_{13}}{\partial q_1}\right)\dot{q}_3 \\
&= \frac{1}{2} m_{l_2}(2d_2 + 2q_2 - a_2)\dot{q}_2 + m_{l_3} S_3 C_3 (a_3 h_3 - a_3^2 - h_3^2 - r_3^2)\dot{q}_3 \\
c_{12} &= \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_1}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_1}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_2} - \frac{\partial b_{23}}{\partial q_1}\right)\dot{q}_3 \\
&= \frac{1}{2} m_{l_2}(2d_2 + 2q_2 - a_2)\dot{q}_1 \\
c_{13} &= \frac{1}{2}\left(\frac{\partial b_{13}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_1}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{13}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_1}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{13}}{\partial q_3} + \frac{\partial b_{13}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_1}\right)\dot{q}_3 \\
&= m_{l_3} S_3 C_3 (a_3 h_3 - a_3^2 - h_3^2 - r_3^2)\dot{q}_1 \\
c_{22} &= \frac{1}{2}\left(\frac{\partial b_{22}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_2} - \frac{\partial b_{21}}{\partial q_2}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{22}}{\partial q_2} + \frac{\partial b_{22}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_2}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{22}}{\partial q_3} + \frac{\partial b_{23}}{\partial q_2} - \frac{\partial b_{23}}{\partial q_2}\right)\dot{q}_3 \\
&= 0 \\
c_{23} &= \frac{1}{2}\left(\frac{\partial b_{23}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_2}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{23}}{\partial q_2} + \frac{\partial b_{22}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_2}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{23}}{\partial q_3} + \frac{\partial b_{23}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_2}\right)\dot{q}_3 \\
&= -\frac{1}{2} m_{l_3} C_3 (2a_3 - h_3)\dot{q}_3 \\
c_{33} &= \frac{1}{2}\left(\frac{\partial b_{33}}{\partial q_1} + \frac{\partial b_{31}}{\partial q_3} - \frac{\partial b_{31}}{\partial q_3}\right)\dot{q}_1 + \frac{1}{2}\left(\frac{\partial b_{33}}{\partial q_2} + \frac{\partial b_{32}}{\partial q_3} - \frac{\partial b_{32}}{\partial q_3}\right)\dot{q}_2 + \frac{1}{2}\left(\frac{\partial b_{33}}{\partial q_3} + \frac{\partial b_{33}}{\partial q_3} - \frac{\partial b_{33}}{\partial q_3}\right)\dot{q}_3 \\
&= 0
\end{aligned}$$

So the equations of motion are:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where $q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T, \dot{q} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{bmatrix}^T, \ddot{q} = \begin{bmatrix} \ddot{q}_1 & \ddot{q}_2 & \ddot{q}_3 \end{bmatrix}^T$ and:

$$B(q) = \begin{bmatrix} K & 0 & 0 \\ * & m_{l_2} + m_{l_3} & -\frac{1}{2}m_{l_3}S_3(2a_3 - h_3) \\ * & * & \frac{1}{2}(2a_3^2 - 2a_3h_3 + 2h_3^2 + 3r_3^2) \end{bmatrix}$$

$$K = m_{l_1}\left(a_1^2 - a_1h_1 + h_1^2 + \frac{3}{2}r_1^2\right) + m_{l_2}\left(\frac{1}{2}a_2^2 - a_2(d_2 + q_2) + \frac{1}{12}(b_2^2 + c_2^2) + d_2^2 + 2d_2q_2 + q_2^2\right)$$
$$+ m_{l_3}\left(C_3^2\left(h_3^2 + a_3^2 - a_3h_3 + \frac{3}{2}r_3^2\right) + \frac{1}{2}r_3^2S_3^2\right)$$

$$C(q,\dot{q}) = \begin{bmatrix} \frac{1}{2}m_{l_2}(2d_2 + 2q_2 - a_2)\dot{q}_2 + m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2)\dot{q}_3 & \frac{1}{2}m_{l_2}(2d_2 + 2q_2 - a_2)\dot{q}_1 & m_{l_3}S_3C_3(a_3h_3 - a_3^2 - h_3^2 - r_3^2)\dot{q}_1 \\ * & 0 & -\frac{1}{2}m_{l_3}C_3(2a_3 - h_3)\dot{q}_3 \\ * & * & 0 \end{bmatrix}$$

$$g(q) = \begin{bmatrix} 0 \\ 0 \\ m_{l_3}g\left(a_3 - \frac{1}{2}h_3\right)C_3 \end{bmatrix}$$

4 Assignment 4 - Recursive Newton-Euler formulation

5 Assignment 5 - Operational space dynamic model