Robotics Vision Control

venerdì 13 maggio 2022 11:45

EXPONENTIAL MATRIX

TAYLOR EXPANSION

THE TAYLOR EXPANSION OF A FUNCTION $f(x)$ THAT IS INFINITELY DIFFERENTIABLE AT A REAL NUMBER x_0 IS THE POWER SERIES:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots + \frac{f^{n}(x_0)(x-x_0)^{n}}{n!}$$

NATURAL EXPONENTIAL FUNCTION

$$f(x) = e^x \qquad x_0 = 0$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(y) = y - \frac{y^3}{3!} + y^5 = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}$$

$$\cos(y) = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} \dots = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!}$$

FOR SQUARE MATRICES M

$$e^M = I + M + \frac{M^2}{2!} + \dots + \frac{M^k}{k!} = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

GENERATING ROTATION MATRICES FROM SKEW-SYMMETRIC MATRICES

$$\begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

WE CAN SHOW THAT

$$\left(e^{A}\right)^{-1} = e^{-A}$$

MOREOVER

$$\left(e^{A}\right)^{T} = e^{(A)^{T}}$$

GIVEN A SKEW SYMMETRIC MATRIX S,

$\Rightarrow$ $S^{T} = -S$

WE CAN DEFINE $R = e^{S}$

$$R^{T} = \left(e^{S}\right)^{T} = e^{S^{T}} = e^{-S} = \left(e^{S}\right)^{-1} = R^{-1}$$

$\Rightarrow$ THIS MEANS THAT R MUST BE ORTHOGONAL

MOREOVER, THE CURVE OF ORTHOGONAL MATRIX e^{tS} IS A PATH CONNECTING I (WITH $t=0$) AND R (WITH $t=1$)

$\Rightarrow$ R AND I MUST HAVE THE SAME DETERMINANT $=1$

$\Rightarrow$ THIS MEANS THAT R MUST BE A ROTATION MATRIX

THE GENERIC SKEW-SYMMETRIC MATRIX IN 3D IS

$$S = \vartheta \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

WHERE ϑ, a, b, c are any real value numbers WITH $a^2 + b^2 + c^2 = 1$

CAYLEY-HAMILTON THEOREM

$$-S^3 - \vartheta^2 S = 0$$

$$\Rightarrow S^3 = -\vartheta^2 S$$

HIGHER POWER OF S

$$S^4 = S^3 S = -\vartheta^2 S^2$$

$$S^5 = S^4 S = -\vartheta^2 \underbrace{S^3}$$

$$= -\vartheta^2 (-\vartheta^2 S)$$

$$= +\vartheta^4 S$$

$$S^6 = \vartheta^4 S S$$

$$= \vartheta^4 S^2$$

NOW GO BACK TO

$$R = e^S$$

$$R = e$$

$$= I + S + \frac{S^2}{2} + \frac{S^3}{3!} + \frac{S^4}{4!} + \frac{S^5}{5!} + \frac{S^6}{6!} \dots$$

$$= I + S + \frac{S^2}{2} - \frac{\vartheta^2}{3!} S - \frac{\vartheta^2}{4!} S^2 + \frac{\vartheta^4}{5!} S + \frac{\vartheta^4 S^2}{6!} \dots$$

$$= I + \underbrace{\left(1 - \frac{\vartheta^2}{3!} + \frac{\vartheta^4}{5!} - \dots\right)}_{\frac{\sin\vartheta}{\vartheta}} S + \underbrace{\left(\frac{1}{2} - \frac{\vartheta^2}{4!} + \frac{\vartheta^4}{6!} + \dots\right)}_{\frac{1-\cos\vartheta}{\vartheta^2}} S^2$$

$$\boxed{R = I + \frac{\sin\vartheta}{\vartheta} S + \frac{1 - \cos\vartheta}{\vartheta^2} S^2}$$