ROBOTICS, VISION AND CONTROL

Trajectory Planning. A quick look at geometry

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Outline

3D space

PROJECT

PROJECT

3D space

Straight line – Point and direction

Let $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ be a point on the plane, and $\boldsymbol{u} = (l, m, n)$ a unit vector (i.e. a direction).

The unit vector $(l, m, n) \in \mathbb{R}^3$ is called *direction vector*.

The parametric representation of the *straight line* on the plane passing by $\boldsymbol{p}_0$ along the direction $\boldsymbol{u}$ is

$$\begin{cases} x(\sigma) & = & x_0 + l\,\sigma \\ y(\sigma) & = & y_0 + m\,\sigma \\ z(\sigma) & = & z_0 + n\,\sigma \end{cases}$$

for $\sigma \in \mathbb{R}$.

According to our previous notation

$$\boldsymbol{p}(\sigma) = \boldsymbol{p}_0 + \boldsymbol{u}\,\sigma$$

Straight line – Two points

Let $\boldsymbol{p}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{p}_2 = (x_2, y_2, z_2)$ be two distinct points on the plane.

The direction vector $\boldsymbol{u} = (l, m, n)$ is proportional to $\boldsymbol{p}_2 - \boldsymbol{p}_1$

$$
\begin{aligned}
l &= x_2 - x_1 \\
m &= y_2 - y_1 \\
n &= z_2 - z_1
\end{aligned}
$$

The parametric representation of the *straight line* passing by $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is

$$
\begin{cases}
x(\sigma) = x_0 + (x_2 - x_1)\,\sigma \\
y(\sigma) = y_0 + (y_2 - y_1)\,\sigma \\
z(\sigma) = z_0 + (z_2 - z_1)\,\sigma
\end{cases}
$$

for $\sigma \in \mathbb{R}$.

The parametric equations are NOT unique.

Two lines

$$L_1 : \begin{cases} x(\sigma) & = & x_1 + l_1\,\sigma \\ y(\sigma) & = & y_1 + m_1\,\sigma \\ z(\sigma) & = & z_1 + n_1\,\sigma \end{cases} , \qquad L_1 : \begin{cases} x(\sigma) & = & x_2 + l_2\,\sigma \\ y(\sigma) & = & y_2 + m_2\,\sigma \\ z(\sigma) & = & z_2 + n_2\,\sigma \end{cases} ,$$

are

- orthogonal if and only if

$$\langle \boldsymbol{u}_1 , \boldsymbol{u}_2 \rangle = l_1\,l_2 + m_1\,m_2 + n_1\,n_2 = 0$$

Two orthogonal lines can have or not a point in common.

Plane

The *Cartesian equation* (*implicit equation*) of a plane is

$$\pi : \qquad ax + by + cz + d = 0$$

where the parameters $a, b, c, d \in \mathbb{R}$ and a, b, c not all null.

If $d = 0$, then the plane passes by the origin.

A plane is uniquely determined by

- Three non-collinear points
- A line and a point not on that line
- Two distinct but intersecting lines
- Two distinct but parallel lines

Properties

- Two distinct planes are either parallel or they intersect in a line
- A line is either parallel to a plane, intersects it at a single point, or is contained in the plane
- Two distinct lines perpendicular to the same plane must be parallel to each other
- Two distinct planes perpendicular to the same line must be parallel to each other

Plane

Proposition. Let $\pi_1 : \quad a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : \quad a_2x + b_2y + c_2z + d_2 = 0$ be two planes, and consider the matrices

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix}$$

Then

- The planes π_1, π_2 are parallel and distinct if and only if $\text{rank}(A) = 1$ and $\text{rank}(\bar{A}) = 2$.
- The planes π_1, π_2 are coincident if and only if $\text{rank}(A) = 1$ and $\text{rank}(\bar{A}) = 1$.
- The planes π_1, π_2 have in common only a line if and only if $\text{rank}(A) = 2$.

Remark. The infinite set of all parallel planes to $ax + by + cz + d = 0$ is characterized by the equation

$$ax + by + cz + \bar{d} = 0$$

where $\bar{d} \in \mathbb{R}$

Proposition. A straight line can be represented as the intersection of two non parallel planes (*Cartesian representation*)

$$L : \begin{cases} ax + by + cz + d = 0 \\ \bar{a}x + \bar{b}y + \bar{c}z + \bar{d} = 0 \end{cases}$$

Proposition. The direction vector $\boldsymbol{u}$ of the line obtained as the intersection of two non parallel planes π and $\bar{\pi}$ has components (proportional) to (l, m, n) obtained as minors of order two of the matrix A with alternating sign

$$l = \det \begin{bmatrix} b & c \\ \bar{b} & \bar{c} \end{bmatrix}, \qquad m = -\det \begin{bmatrix} a & c \\ \bar{a} & \bar{c} \end{bmatrix}, \qquad n = \det \begin{bmatrix} a & b \\ \bar{a} & \bar{b} \end{bmatrix},$$

Planes and lines

Let $\pi : ax + by + cz + d = 0$ and L be a plane and a line. Then there are the following three situations

- π and L have only a point in common
- π and L have no intersection
- L belongs to π.

Proposition. The plane $\pi : ax + by + cz + d = 0$ and the line L with direction vector $\boldsymbol{u} = (l, m, n)$ are parallel if and only if

$$al + bm + cn = 0.$$

Plane – Three non-collinear points

Let $\boldsymbol{p}_1 = (x_1, y_1, z_1)$, $\boldsymbol{p}_2 = (x_2, y_2, z_2)$, $\boldsymbol{p}_3 = (x_3, y_3, z_3)$ be three non-collinear points, the equation of the plane is given by

$$\det \begin{bmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{bmatrix} = 0$$

or

$$\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

Plane – Point-normal form

Let $\boldsymbol{p}_0 \in \mathbb{R}^3$ be a point and $\boldsymbol{n} = (a, b, c)$ be a non-zero vector. The plane π determined by $(\boldsymbol{p}_0, \boldsymbol{n})$ is the set of points $\boldsymbol{p}$ that satisfy

$$\langle \boldsymbol{n}, \boldsymbol{p} - \boldsymbol{p}_0 \rangle = 0,$$

i.e. the points such that $\boldsymbol{p} - \boldsymbol{p}_0$ is perpendicular to $\boldsymbol{n}$

If $\boldsymbol{n} = (a, b, c)$, $\boldsymbol{p} = (x, y, z)$, $\boldsymbol{p}_0 = (x_0, y_0, z_0)$, then the equation of the plane is

$$ax + by + cz + d = 0$$

where $d = -ax_0 - by_0 - cz_0$

Plane – Point-normal form

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$$ax + by + cz + d = 0$$

where $d = -ax_0 - by_0 - cz_0$

The *Hessian normal form* is

$$\langle \hat{\boldsymbol{n}}, \boldsymbol{p} \rangle = -D,$$

where $\hat{\boldsymbol{n}}$ is unit vector and D is the distance of the plane to the origin.

Sheaf of Planes

Sheaf of Planes. The set of planes through a line.
The line is called the axis of the sheaf.

Let L be the line in Cartesian form

$$L : \begin{cases} ax + by + cz + d = 0 \\ \bar{a}x + \bar{b}y + \bar{c}z + \bar{d} = 0 \end{cases}$$

then the sheaf of planes is given by

$$\pi : h(ax + by + cz + d) + k(\bar{a}x + \bar{b}y + \bar{c}z + \bar{d}) = 0$$

where $h, k \in \mathbb{R}$ not both equal to zero.

Using the Hessian normal form, we get

$$\lambda(\langle \hat{\boldsymbol{n}}, \boldsymbol{p} \rangle + D) + \bar{\lambda}(\langle \bar{\hat{\boldsymbol{n}}}, \boldsymbol{p} \rangle + \bar{D}) = 0,$$

with $\hat{\boldsymbol{n}} \times \bar{\hat{\boldsymbol{n}}} \neq \boldsymbol{0}$

Plane – A line and a point not on that line

Let L be the line in Cartesian form

$$L : \begin{cases} ax + by + cz + d = 0 \\ \bar{a}x + \bar{b}y + \bar{c}z + \bar{d} = 0 \end{cases}$$

and $\boldsymbol{p} \notin L$. There exists only one plane through $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ containing L, i.e. the plane through $\boldsymbol{p}_0$ and parallel to (a, b, c) and $(\bar{a}, \bar{b}, \bar{c})$,

$$\pi : \qquad \det \begin{bmatrix} x - x_0 & y - y_0 & z - z_0 \\ a & b & c \\ \bar{a} & \bar{b} & \bar{c} \end{bmatrix} = 0$$

Bundle of Planes. Set of planes sharing a point in common.
Using the Hessian normal form, a bundle of planes can therefore be specified as

$$\lambda_1(\langle \hat{\boldsymbol{n}}_1, \boldsymbol{p} \rangle + D_1) + \lambda_2(\langle \hat{\boldsymbol{n}}_2, \boldsymbol{p} \rangle + D_2) + \lambda_3(\langle \hat{\boldsymbol{n}}_3, \boldsymbol{p} \rangle + D_3) = 0,$$

where $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, $\hat{\boldsymbol{n}}_3$ are three linearly independent unit vectors.

Plane – a point and two vectors lying on it

Let $\boldsymbol{p}_0 \in \mathbb{R}^3$ be a point and $\boldsymbol{v}$, $\boldsymbol{w}$ two linearly independent vectors defining the plane. Then the parametric equation of a plane is

$$\pi : \boldsymbol{p} = \boldsymbol{p}_0 + \sigma \boldsymbol{v} + \mu \boldsymbol{w}$$

where $\sigma, \mu \in \mathbb{R}$.

Distance from a point to a plane

Let $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ be a point not belonging to the plane π

$$\pi : \qquad ax + by + cz + d = 0$$

The distance of $\boldsymbol{p}_0$ to the plane π is

$$D = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$

Plane-plane intersection

Given the Hessian normal form of two planes

$$\begin{aligned} \pi_1 : \qquad \langle \hat{\boldsymbol{n}}_1, \boldsymbol{p} \rangle &= -D_1 \\ \pi_2 : \qquad \langle \hat{\boldsymbol{n}}_2, \boldsymbol{p} \rangle &= -D_2 \end{aligned}$$

The line of intersection between π_1 and π_2 is

$$\boldsymbol{p} = \boldsymbol{p}_0 + \sigma \hat{\boldsymbol{n}}_3$$

where $\boldsymbol{p}_0$ is a point on the line (i.e. a solution of the linear system $\begin{bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \end{bmatrix}^T \boldsymbol{p}_0 = \begin{bmatrix} -D_1 \\ -D_2 \end{bmatrix}$) and

$$\hat{\boldsymbol{n}}_3 = \hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2$$

Geometric interpretation of (a, b, c)

Let $\pi : \; ax + by + cz + d = 0$ be a plane. The vector

$$\boldsymbol{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is orthogonal to π.

The line

$$\begin{cases} x(\sigma) & = & x_0 + l\,\sigma \\ y(\sigma) & = & y_0 + m\,\sigma \\ z(\sigma) & = & z_0 + n\,\sigma \end{cases}$$

is perpendicular to π is and only if

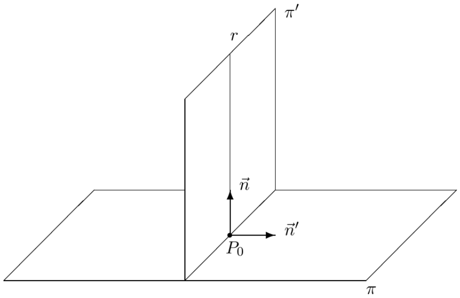
$$\text{rank} \begin{bmatrix} a & b & c \\ l & m & n \end{bmatrix} = 1$$

Orthogonal planes

Two non-parallel planes

$$\pi : \ ax + by + cz + d = 0$$
$$\bar{\pi} : \ \bar{a}x + \bar{b}y + \bar{c}z + \bar{d} = 0$$

are *orthogonal ($\pi \perp \bar{\pi}$)* if, given $\boldsymbol{p}_0 \in \pi \cap \bar{\pi}$, the line L through $\boldsymbol{p}_0$ and orthogonal to π belongs to $\bar{\pi}$



If $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$ are orthogonal vectors to π and $\bar{\pi}$, then

$$\begin{aligned} \pi \perp \bar{\pi} \quad &\Leftrightarrow \quad \boldsymbol{n} \perp \bar{\boldsymbol{n}} \\ &\Leftrightarrow \quad a\bar{a} + b\bar{b} + c\bar{c} = 0 \end{aligned}$$

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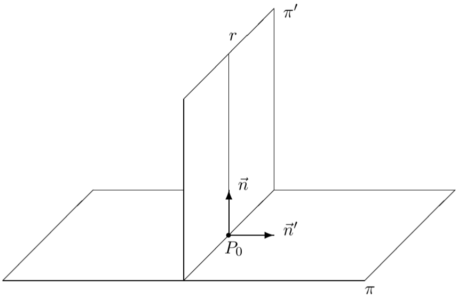

Remark 1. Given a plane π and a point $\boldsymbol{p}_0 \in \pi$, there exists an infinite number of planes $\bar{\pi}$ such that $\boldsymbol{p}_0 \in \bar{\pi}$ and $\pi \perp \bar{\pi}$.

Orthogonal planes

Two non-parallel planes

$$\pi : \; ax + by + cz + d = 0$$
$$\bar{\pi} : \; \bar{a}x + \bar{b}y + \bar{c}z + \bar{d} = 0$$

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If $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$ are orthogonal vectors to π and $\bar{\pi}$, then

$$\begin{aligned} \pi \perp \bar{\pi} \quad &\Leftrightarrow \quad \boldsymbol{n} \perp \bar{\boldsymbol{n}} \\ &\Leftrightarrow \quad a\bar{a} + b\bar{b} + c\bar{c} = 0 \end{aligned}$$

Remark 1. Given a plane π and a point $\boldsymbol{p}_0 \in \pi$, there exists an infinite number of planes $\bar{\pi}$ such that $\boldsymbol{p}_0 \in \bar{\pi}$ and $\pi \perp \bar{\pi}$.
Remark 2. Given a plane π and a line L, there always exists a plane $\bar{\pi}$ such that $L \in \bar{\pi}$ and $\pi \perp \bar{\pi}$. If L is not orthogonal to π, then $\bar{\pi}$ is unique.

Point-plane Distance

Given a point $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ and a plane

$$\pi : \; ax + by + cz + d = 0$$

The *distance* between the point and the plane, $d(\boldsymbol{p}_0, \pi)$, is the minimal distance between $\boldsymbol{p}_0$ and any point belonging to π

$$d(\boldsymbol{p}_0, \pi) = \min_{\boldsymbol{p} \in \pi} \|\boldsymbol{p} - \boldsymbol{p}_0\| = \|\boldsymbol{h} - \boldsymbol{p}_0\|$$

where $\boldsymbol{h} \in \pi$ is the *orthogonal projection* of $\boldsymbol{p}_0$ on π

Point-Plane Distance

To compute $d(\boldsymbol{p}_0, \pi)$ it is needed to

- find the line L passing through $\boldsymbol{p}_0$ and orthogonal to π
- determine the point of intersection $\boldsymbol{h}$ between the line L and the plane π
- compute the distance $d(\boldsymbol{h}, \boldsymbol{p}_0) = \|\boldsymbol{h} - \boldsymbol{p}_0\|$

The explicit expression is

$$d(\boldsymbol{p}_0, \pi) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Point-Lane Distance

Given a point $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ and a line L, distance $d(\boldsymbol{p}_0, L)$ is the minimal distance between $\boldsymbol{p}_0$ and any point $\boldsymbol{p} \in L$.

$$d(\boldsymbol{p}_0, L) = \min_{\boldsymbol{p} \in L} \|\boldsymbol{p} - \boldsymbol{p}_0\| = \|\boldsymbol{h} - \boldsymbol{p}_0\|$$

where $\boldsymbol{h} \in \pi$ is the *orthogonal projection* of $\boldsymbol{p}_0$ on L.

The point $\boldsymbol{h}$ is obtained as the intersection of the line L with the plane π passing through $\boldsymbol{p}_0$ and orthogonal to L.

Projection of a Line on a Plane

Let L and π be a line and a plane that are not orthogonal.

The projection of L on π is the line $\bar{L} \in \pi$ that contains the orthogonal projection $\bar{\boldsymbol{p}}$ of all the points $\boldsymbol{p} \in L$ on π.

The line $\pi \in L$ is called the *orthogonal projection* of L on π

$$\bar{L} = \bar{\pi} \cap \pi$$

where $\bar{\pi}$ is the plane perpendicular to π containing L.

Exercise. Compute the distance between are two lines that do not intersect and are not parallel (*skew lines*).

Sphere: the set of points that are all at the same distance R from a given point $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ in a three-dimensional space

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

or equivalently (*Cartesian representation*)

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0 \qquad (1)$$

Vice versa, (1) is the equation of a sphere if and only if

$$a^2 + b^2 + c^2 - 4d > 0;$$

then the center and the radius are

$$\boldsymbol{p}_0 = \left(-\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right), \qquad R = \frac{1}{2}\sqrt{a^2 + b^2 + c^2 - 4d}$$

Sphere

Given four not planar points, $\boldsymbol{p}_i = (x_i, y_i, z_i)$ there exists only one sphere passing through such points.

Spherical coordinate. The points $\boldsymbol{p} = (x, y, z)$ on the sphere with radius $r > 0$ and center $\boldsymbol{p}_0 = (x_0, y_0, z_0)$ can be parameterized via

$$\begin{aligned} x &= x_0 + R \sin\theta \; \cos\varphi \\ y &= y_0 + R \sin\theta \; \sin\varphi \qquad (0 \le \theta \le \pi, \; 0 \le \varphi < 2\pi) \\ z &= z_0 + R \cos\theta \end{aligned}$$

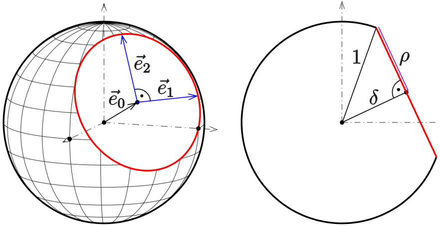
Spheres and Planes

The intersection of a sphere and a plane is a circle, or a point or empty.

A plane π is the *tangent plane* on the sphere at point $\boldsymbol{p}$ if the intersection between the plane and the sphere is just a point (i.e. $\boldsymbol{p}$).

The distance between the tangent plane and the origin of the sphere is its radius.

The normal vector $\boldsymbol{n}$ to the tangent plane π at $\boldsymbol{p}$ has direction along the line between the point $\boldsymbol{p}$ and the center of the sphere $\boldsymbol{p}_0$



[it is not the Death Star...]

Spheres and Planes

Given the sphere S

$$S: \quad x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$

to compute the tangent plane to the sphere at $\boldsymbol{p} \in S$, we have to

- determine the origin $\boldsymbol{p}_0$ of the sphere
- compute the vector $\boldsymbol{n}$ from $\boldsymbol{p}_0$ to $\boldsymbol{p}$
- determine the plane orthogonal to $\boldsymbol{n}$ and passing through $\boldsymbol{p}$


Project

PROJECT – Assignment # 9

To do

- Let $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ three points on a sphere of center $\boldsymbol{p}_0$ and radius R. Design the trajectory such that (1) the EE will pass through the three points along the shortest path, and (2) the z axis of the EE is always orthogonal to the sphere.


Project

To do

- Plan the pick-and-place task for the UR5 robot in ROS for three objects (cubes and parallelepipeds) using three different orientations for the end-effector.