

ROBOTICS, VISION AND CONTROL

Trajectory Planning. Operational Space

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Operational Space Trajectories

Motion primitives

PROJECT

Operational Space Trajectories

Operational Space Trajectories: trajectory planning in 3D, i.e. pose and orientation in the Cartesian space.

We have to take care of

- ▶ the *geometry* of the trajectory $\mathbf{x}_e(u)$ [Where]
 $\mathbf{p}(u) = [x(u) \ y(u) \ z(u)]^T \in \mathbb{R}^3, \phi(u) = [\varphi(u) \ \theta(u) \ \psi(u)]^T \in \mathbb{R}^3, u \in [u_i, u_f]$
→ *path*
- ▶ the *motion/timing law* $t = u(t), t \in [t_i, t_f]$ [How]
 $t \in \mathbb{R}, \mathbf{p}(t), \phi(t)$
→ path+motion law = *trajectory*

Given the trajectory, it is necessary to resort to the inverse kinematics to compute the corresponding joints trajectory $\mathbf{q}(t)$.

We will refer to *motion primitives* for the geometric features of the path and to *time primitives* for the timing law on the path itself.

n pairs $(\mathbf{x}_e(t_k), t_k)$: Interpolation component by component

The multi-dimensional problem can be decomposed in 6 scalar problems.

The *synchronization* among the different components is performed by imposing interpolation conditions at the *same time instants*.

Let $\mathbf{p} \in \mathbb{R}^3$ be a Cartesian point given by

$$\mathbf{p} = \mathbf{f}(u)$$

where $u \in [u_i, u_f]$ is the *parameter* of the function \mathbf{f} ; $\mathbf{p}(u)$ is the parametric representation on the *path* Γ with

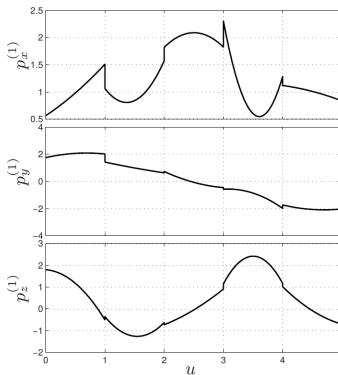
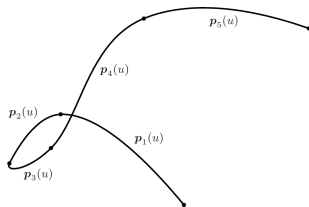
$$\mathbf{p}_i := \mathbf{p}(u_i) = \mathbf{f}(u_i) \quad \text{initial point}$$

$$\mathbf{p}_f := \mathbf{p}(u_f) = \mathbf{f}(u_f) \quad \text{final point}$$

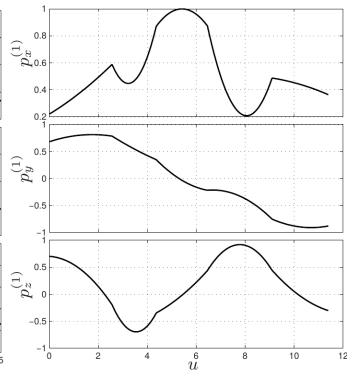
The path Γ has a direction.

$p(u)$

$\frac{dp(u)}{du}$



(a)



(b)

Geometric continuity (i.e. path) \neq *Parametric continuity* (vel. $\frac{dp(u)}{du}$ and accel. $\frac{d^2p(u)}{du^2}$)

Let $u \in [0, 1]$ and $\dot{\mathbf{p}}(u) = \frac{d\mathbf{p}(u)}{du}$, $\ddot{\mathbf{p}}(u) = \frac{d^2\mathbf{p}(u)}{du^2}$, \dots $\mathbf{p}^{(i)}(u) = \frac{d^i\mathbf{p}(u)}{du^i}$.

Two infinitely differentiable segments $\mathbf{p}_k(u)$, $u \in [0, 1]$, $\mathbf{p}_{k+1}(u)$, $u \in [0, 1]$ meeting at a common point

$$\mathbf{p}_k(1) = \mathbf{p}_{k+1}(0) \quad (1)$$

satisfy the *n -order parametric continuity, C^n* , if the first n parametric derivatives match at the common point

$$\begin{aligned} \dot{\mathbf{p}}_k(1) &= \dot{\mathbf{p}}_{k+1}(0) \\ \ddot{\mathbf{p}}_k(1) &= \ddot{\mathbf{p}}_{k+1}(0) \\ &\vdots \\ \mathbf{p}_k^{(n)}(1) &= \mathbf{p}_{k+1}^{(n)}(0) \end{aligned}$$

Remark. The derivative vectors are not intrinsic properties of a curve.

The *tangent unit vector*

$$\mathbf{t} = \frac{d\mathbf{p}}{du} / \left\| \frac{d\mathbf{p}}{du} \right\|$$

and the *curvature unit vector* (also known as *normal unit vector*)

$$\mathbf{n} = \frac{d^2\mathbf{p}}{d^2u} / \left\| \frac{d^2\mathbf{p}}{d^2u} \right\|$$

are intrinsic properties of the curve, and they lead to the notion of geometric continuity G .

Two parametric curves meet with a *first order geometric continuity G^1* if and only if they have a common tangent unit vector.

The tangent direction at the joint is preserved, but the continuity of the velocity vector is not guaranteed, since the tangent vectors may have different magnitude.

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Two parametric curves meet with *G^2 continuity* if and only if they have common unit tangent and curvature vectors.

Two parametric curves $\mathbf{p}_k(u)$, $\mathbf{p}_{k+1}(u)$ meet with *G^n continuity* if and only if there exists a parametrization \hat{u} equivalent to u such that $\hat{\mathbf{p}}_k(\hat{u})$, $\hat{\mathbf{p}}_{k+1}(\hat{u})$ meet with C^n continuity.

Two parameterizations u , \hat{u} are equivalent if there exists a regular C^n function $f : [\hat{u}_{min}, \hat{u}_{max}] \mapsto [u_{min}, u_{max}]$ such that:

1. $\hat{\mathbf{p}}(\hat{u}) = \mathbf{p}(f(\hat{u})) = \mathbf{p}(u)$
2. $f([\hat{u}_{min}, \hat{u}_{max}]) = [u_{min}, u_{max}]$
3. $\frac{df}{d\hat{u}} > 0$.

The *arc length* s of the generic point $\mathbf{p} \in \Gamma$ is the length of the arc of Γ with extremes \mathbf{p} and $\mathbf{p}_i \in \Gamma$.

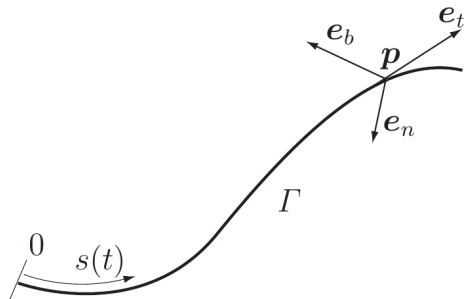
We can identify the point \mathbf{p} using the arc length

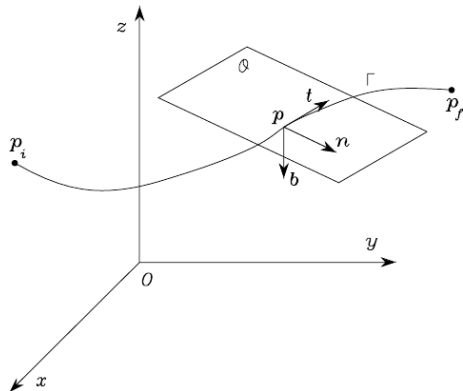
$$\Gamma : \mathbf{p} = \mathbf{p}(s), \quad s \in [0, L]$$

Goal: specify the orientation of the end effector on the basis of the orientation of the path at a given point

The *Frenet Frame* is a coordinate frame directly tied to the curve. It is represented by three unit vectors:

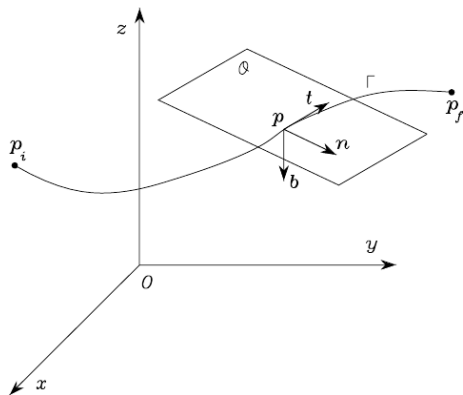
- ▶ the *tangent unit vector*
- ▶ the *normal unit vector*
- ▶ the *binormal unit vector*





tangent unit vector \mathbf{t} : vector oriented along the positive direction induced on the path Γ by s

$$\mathbf{t} = \frac{d\mathbf{p}}{ds}$$



normal unit vector \mathbf{n} : vector oriented along the line intersecting \mathbf{p} at a right angle with \mathbf{t} and lies in the so-called osculating plane \mathcal{O} ;

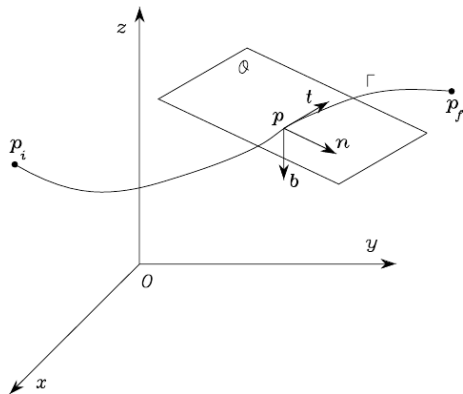
The plane \mathcal{O} is the limit plane containing \mathbf{t} and a point $\mathbf{p}' \in \Gamma$ when \mathbf{p}' tends to \mathbf{p} along the path;

The direction of \mathbf{n} is so that the path Γ , in the neighbourhood of \mathbf{p} with respect to the plane containing \mathbf{t} and normal to \mathbf{n} , lies on the same side of \mathbf{n}

Another definition

The normal unit vector \mathbf{n} , lying on the line passing through the point \mathbf{p} , and orthogonal to \mathbf{t} .

The orientation of \mathbf{n} is such that in a neighborhood of \mathbf{p} the curve is completely on the side of \mathbf{n} with respect to the plane passing through \mathbf{t} and normal to \mathbf{n} .



FRENET VECTORS

tangent unit vector t

$$t = \frac{dp}{ds}$$

normal unit vector n

$$n = \frac{d^2p}{ds^2} / \left\| \frac{d^2p}{ds^2} \right\|$$

binormal unit vector b : vector such that the frame (t, n, b) is right-handed

$$b = t \times n$$

Remarks.

- ▶ If the curve is characterized by the arc-length parameterization s and not by a generic parameter u , the tangent vector \mathbf{t} has unit length.
- ▶ In those applications in which the tool must have a fixed orientation with respect to the motion direction, the Frenet vectors implicitly define such an orientation
- ▶ Let $R_F(u) = [\mathbf{t}(u) \quad \mathbf{n}(u) \quad \mathbf{b}(u)]$ be the Frenet frame as a function of u (the same holds for s) and R_Δ be a constant matrix rotation for the tool with respect to $R_F(u)$; then the tool orientation is

$$R_T(u) = R_\Delta R_F(u)$$

Motion primitives

Rectilinear path: linear segment connecting initial point \mathbf{p}_i to the final point \mathbf{p}_f

- ▶ normalized parameterization

$$\mathbf{p}(u) = \mathbf{p}_i + (\mathbf{p}_f - \mathbf{p}_i)u, \quad u \in [0, 1]$$

- ▶ arc-length

$$\mathbf{p}(s) = \mathbf{p}_i + s \frac{\mathbf{p}_f - \mathbf{p}_i}{\|\mathbf{p}_f - \mathbf{p}_i\|}, \quad s \in [0, \|\mathbf{p}_f - \mathbf{p}_i\|]$$

The direction induced on Γ by the parametric representation s is from \mathbf{p}_i to \mathbf{p}_f .

Tangent unit vector

$$\mathbf{t} = \frac{d\mathbf{p}}{ds} = \frac{\mathbf{p}_f - \mathbf{p}_i}{\|\mathbf{p}_f - \mathbf{p}_i\|}$$

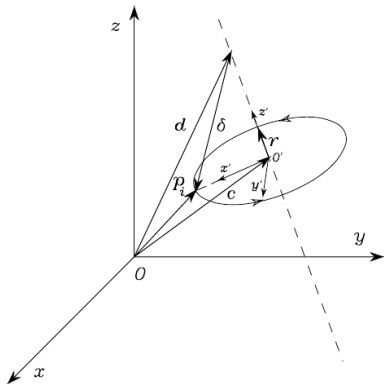
The normal unit vector \mathbf{n} and the binormal unit vector \mathbf{b} cannot be defined in a unique way since

$$\frac{d^2\mathbf{p}}{ds^2} = 0.$$

The trajectory composed by a set of linear segments is continuous but it is characterized by discontinuous derivatives at the intermediate points

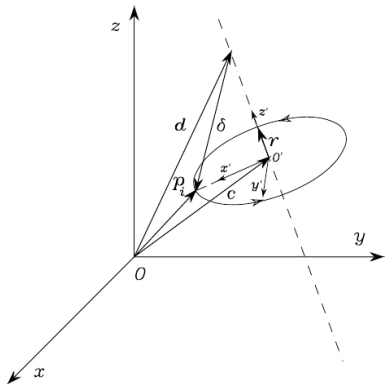
→ use *blending functions* to guarantee a smooth transition between consecutive segments (e.g. ...).

The circle Γ is specified by assigning:



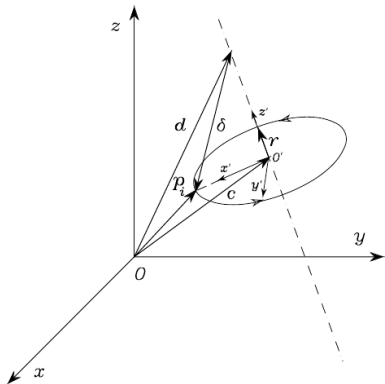
The circle Γ is specified by assigning:

- the unit vector of the circle axis \mathbf{r} ,



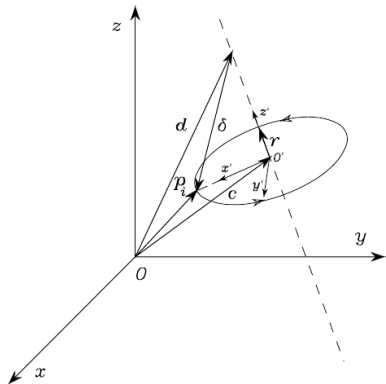
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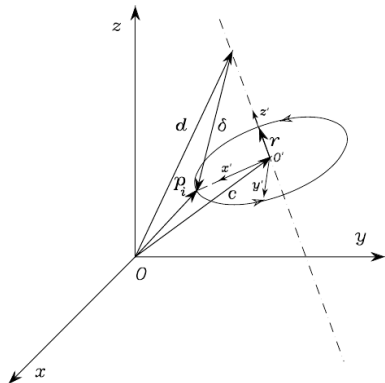
- ▶ the unit vector of the circle axis \mathbf{r} ,
- ▶ the position vector \mathbf{d} of a point along the circle axis,



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- ▶ the unit vector of the circle axis \mathbf{r} ,
- ▶ the position vector \mathbf{d} of a point along the circle axis,
- ▶ the position vector \mathbf{p}_i of a point on the circle.





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- ▶ the unit vector of the circle axis \mathbf{r} ,
- ▶ the position vector \mathbf{d} of a point along the circle axis,
- ▶ the position vector \mathbf{p}_i of a point on the circle.

The position vector \mathbf{c} of the centre of the circle is given by

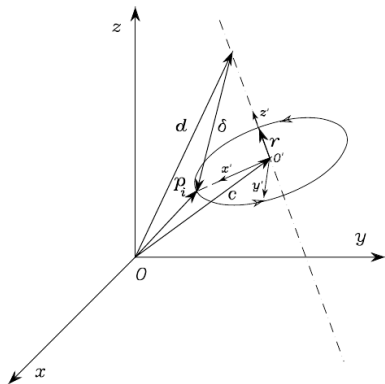
$$\mathbf{c} = \mathbf{d} + (\delta^T \mathbf{r}) \mathbf{r}$$

where

$$\delta = \mathbf{p}_i - \mathbf{d}$$

The radius is $\|\mathbf{p}_i - \mathbf{c}\|$

We need to find a parametric representation of the circle as a function of the arc length s .



$$\Sigma = \{O; x, y, z\}, \quad \Sigma' = \{O'; x', y', z'\}$$

x' oriented along the direction of the vector $\mathbf{p}_i - \mathbf{c}$
 z' along \mathbf{r}
 y' to have a right-handed frame

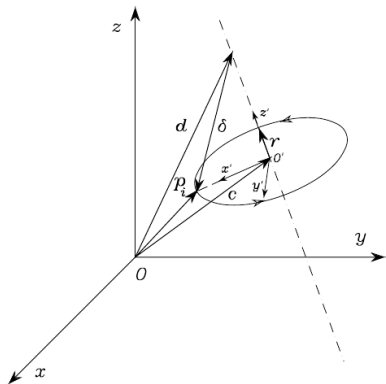
The parametric representation of the circle in Σ' is

$$\mathbf{p}'(s) = \begin{bmatrix} \rho \cos(s/\rho) \\ \rho \sin(s/\rho) \\ 0 \end{bmatrix}, \quad \rho := \|\mathbf{p}_i - \mathbf{c}\|$$

The parametric representation of the circle in Σ is

$$\mathbf{p}(s) = \mathbf{c} + R\mathbf{p}'(s)$$

where $R = \begin{bmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \end{bmatrix}$ is the rotation matrix of frame Σ' with respect to frame Σ



If we are interested in moving along a circular arc, then we can rely on the parametric representation

$$\mathbf{p}'(u) = \begin{bmatrix} \rho \cos(u + \varphi) \\ \rho \sin(u + \varphi) \\ 0 \end{bmatrix}, \quad \rho := \|\mathbf{p}_i - \mathbf{c}\|$$

$$\mathbf{p}(u) = \mathbf{c} + R\mathbf{p}'(u)$$

where $u \in [0, \theta]$ and

$$\mathbf{p}_0 = \mathbf{p}(0) = \mathbf{c} + R\mathbf{p}'(0)$$

$$\mathbf{p}_1 = \mathbf{p}(\theta) = \mathbf{c} + R\mathbf{p}'(\theta)$$

Tangent unit vector

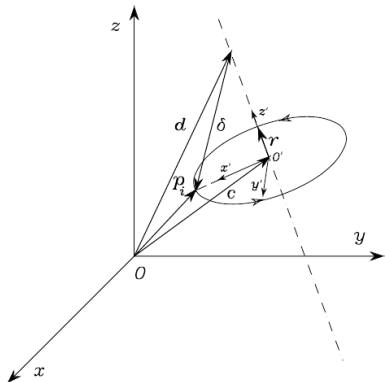
$$\mathbf{t} = \frac{d\mathbf{p}}{ds} = R \begin{bmatrix} -\sin(s/\rho) \\ \cos(s/\rho) \\ 0 \end{bmatrix}$$

The normal unit vector \mathbf{n} and the binormal unit vector \mathbf{b}

$$\mathbf{n} = \frac{d^2\mathbf{p}}{ds^2} / \left\| \frac{d^2\mathbf{p}}{ds^2} \right\|, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

where

$$\frac{d^2\mathbf{p}}{ds^2} = R \begin{bmatrix} -\frac{1}{\rho} \cos(s/\rho) \\ -\frac{1}{\rho} \sin(s/\rho) \\ 0 \end{bmatrix}$$





To do

- Compute the 3D trajectory (also also velocity, acceleration and jerk) in the picture as a combination of linear and circular motion primitives and compare it with the trajectory obtained using one of the multi-point methods.

[From $(0, 0, 0)$ to $(2, 0, 2)$ and back]

