Dynamic systems

Notes from the a.y. 2021/2022 course held by Prof. Paolo Fiorini

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1 Signals and systems

1.1 Signals

While studying dynamic systems we will only consider **deterministic** signals, which are functions in the continuous time case or sequences in the discrete time case.

Definition 1.1 (Continuous time signal)
A ***continuous time signal*** $s(t)$ *is a function:*

$$\begin{aligned} s : \mathbb{R} &\to \mathbb{C}^n \\ t &\mapsto s(t) \end{aligned}$$

Usually the variable t represents time. If the signal is valued in the real numbers, then $s(t) \in \mathbb{R}^n$.

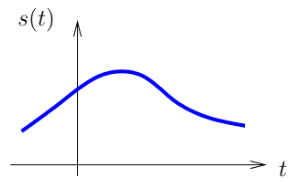

Figure 1: Continuous time signal

Definition 1.2 (Discrete time signal)
A ***discrete time signal*** $s(t)$ *is a sequence:*

$$\begin{aligned} s : \mathbb{Z} &\to \mathbb{C}^n \\ k &\mapsto s(k) \end{aligned}$$

The variable k usually represents a discrete time index. If the sequence is valued in the real numbers, then $s(k) \in \mathbb{R}^n$.

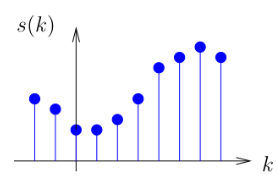

Figure 2: Discrete time signal

1.2 Systems

Definition 1.3 (System)
A system Σ *is a map from an input signal set* u *to an output signal set* y*.*

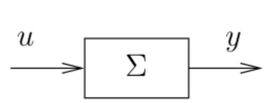

Figure 3: Representation of a system Σ

The system Σ is a mathematical representation of the behavior of a system or physical phenomenon. A **continuous time system** maps functions to functions, while a **discrete time system** maps sequences to sequences. In this course we will mainly deal with a specific class of systems that simplifies some theory: **linear time-invariant systems** (LTI systems).

Definition 1.4 (Linearity)
A dynamic system Σ *is* ***linear*** *if the superpositon principle holds true: for an system initially at rest, if the output values* y_1 *and* y_2 *correspond to the input values* u_1 *and* u_2*, then at the input* $a_1u_1 + a_2u_2$ *corresponds the output* $a_1y_1 + a_2y_2$*, with* $a_1, a_2 \in \mathbb{R}$*.*

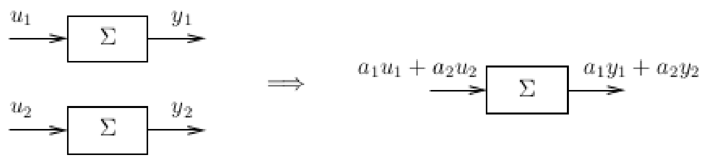

Figure 4: Linearity of a system Σ

Definition 1.5 (Time-invariance)

*A dynamic system Σ is **time-invariant** if a temporal translation of the input values $u(t)$ causes the same translation in the output values $y(t)$.*

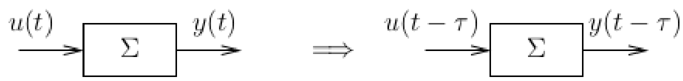

Figure 5: Time-invariance of a system Σ

Another important notion is causality, meaning that we do not deal with systems that require knowledge of future inputs to determine the current output.

Definition 1.6 (Causality)

*A dynamic system Σ is **causal** if the output at time t, $y(t)$, depends only on the inputs until time t, $u(t)$.*

We will now give some broad notions about the stability of systems without going too much into detail right away.

Definition 1.7 (Stability)

*A dynamic system Σ is **stable** if small variations in the input signal $u(t)$ produce small variations in the output signal $y(t)$.*

Definition 1.8 (Asymptotic stability)

*A dynamic system Σ is **asymptotically stable** if in the absence of an input signal, i.e. with $u(t) = 0\ \forall t \geq t_0$, the output signal converges asymptotically to zero.*

Definition 1.9 (BIBO stability)

*A continuous time system Σ is **BIBO stable** (Bounded Input Bounded Output) if for any positive constant M_u there exists another positive constant M_Y such that for any input signal $u(t)$ such that:*

$$|u(t)| \leq M_u \quad t \geq t_0$$

the output signal is such that:

$$|y(t)| \leq M_y \quad t \geq t_0$$

with $t \in \mathbb{R}$.

*A discrete time system Σ is **BIBO stable** (Bounded Input Bounded Output) if for any positive constant M_u there exists another positive constant M_Y such that for any input signal $u(k)$ such that:*

$$|u(k)| \leq M_u \quad k \geq k_0$$

the output signal is such that:

$$|y(k)| \leq M_y \quad k \geq k_0$$

with $k \in \mathbb{Z}$.

1.3 Input-Output representation of dynamic systems

The input-output (IO) behavior of a continuous time system Σ is described by a set of differential equations that define the relation between the output $y(t)$ and its n successive derivatives over time:

$$\frac{d^1y(t)}{dt^1} = y^{(1)}(t), \dots \frac{d^ky(t)}{dt^k} = y^{(k)}(t), \dots \frac{d^ny(t)}{dt^n} = y^{(n)}(t)$$

to the input $u(t)$ and its m successive derivatives:

$$\frac{d^1u(t)}{dt^1} = u^{(1)}(t), \dots \frac{d^ku(t)}{dt^k} = u^{(k)}(t), \dots \frac{d^nu(t)}{dt^n} = u^{(n)}(t)$$

For now we will only deal with single input single output (**SISO**) systems, in other words systems that map scalar inputs $u(t) \in \mathbb{R}$ to scalar outputs $y(t) \in \mathbb{R}$.

Definition 1.10 (IO representation)

The input-output relation for a continuous time LTI SISO system Σ is given by the following differential equation

$$\sum_{i=0}^{n} a_i \frac{d^iy(t)}{dt^i} = \sum_{j=0}^{m} b_i \frac{d^ju(t)}{dt^j} \tag{1.1}$$

with $\mathbb{R} \ni t \geq t_0$ and initial conditions (i.c.):

$$y(t_0^-), \left.\frac{dy(t)}{dt}\right|_{t=t_0^-}, \dots, \left.\frac{d^{n-1}y(t)}{dt^{n-1}}\right|_{t=t_0^-} \tag{1.2}$$

with $m \leq n, a_i, b_j \in \mathbb{R}$ such that $a_n \neq 0, b_m \neq 0, u(t) = 0\ \forall t < t_0$.

The input-output relation for a discrete time LTI SISO system Σ is given by the following difference equation

$$\sum_{i=0}^{n} a_i y(k-i) = \sum_{j=0}^{m} b_i u(k-j) \tag{1.3}$$

with $\mathbb{Z} \ni k \geq k_0$ and initial conditions (i.c.):

$$y(k_0-1), y(k_0-2), \dots, y(k_0-n) \tag{1.4}$$

with $m \leq n, a_i, b_j \in \mathbb{R}$ such that $a_n \neq 0, b_m \neq 0, u(k) = 0\ \forall k < k_0$.

The discrete time case has two interesting particular cases:

- $n = 0$: the system is described by the difference equation:

$$y(k) = \sum_{j=0}^{m} \frac{b_j}{a_0} u(k-j) \qquad k \in \mathbb{Z}$$

  whose output is the weighted average of the input on the mobile window $[k-m, k]$. Therefore this model is called **moving average model (MA)**.

- $m = 0$: the system is described by the difference equation:

$$\sum_{i=0}^{n} \frac{a_i}{b_0} y(k-i) = u(k) \qquad k \in \mathbb{Z}$$

  This model is called **autoregressive model (AR)**.

The general case given by definition 1.10 encapsulates both models, and is therefore referred as **autoregressive mobile average model (ARMA)**.

We can make a few observation on the input output representation of systems:

- For a continuous time system the initial conditions are the value of the output and the n derivatives evaluated in t_0, meanwhile for a discrete time system the initial conditions are the values of the input in the n time steps before k_0.
- The coefficients a_n and b_m must be non zero, because if they were null the values of n and m could be decreased. In the discrete time case a_0 mustn't be zero as well to maintain causality.
- In the continuous time case we evaluate the derivatives approaching t_0 from the left, so the model fits also systems with discontinuities and impulsive signals in t_0.

1.4 Free response

The solution to equations 1.1 and 1.3 can be expressed as a sum of two terms:

$$y(t) = y_l(t) + y_f(t)$$

where $y_l(t)$ is the **free response** of the system, that only depends on the system and its initial conditions, and $y_f(t)$ is the **forced response** of the system, which depends only on the system and the input $u(t)$.

Definition 1.11 (Free response)

Given the differential equation 1.1 and its initial conditions 1.2, the free response of a continuous time system is given by the solution of the associated homogeneous equation:

$$\sum_{i=0}^{n} a_i \frac{d^i y_l(t)}{dt^i} = 0 \tag{1.5}$$

with the same initial conditions as 1.2.

Given the difference equation 1.3 and its initial conditions 1.4, the free response of a discrete time system is given by the solution of the associated homogeneous equation:

$$\sum_{i=0}^{n} a_i y_l(k-i) = 0 \tag{1.6}$$

with the same initial conditions as 1.4.

Definition 1.12 (Forced response)

Given the differential equation 1.1, the forced response of a continuous time system is given by the solution of the original equation:

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{m} b_i \frac{d^j u(t)}{dt^j} \tag{1.7}$$

with initial conditions set to zero.

Given the difference equation 1.3, the forced response of a discrete time system is given by the solution to the original equation:

$$\sum_{i=0}^{n} a_i y(k-i) = \sum_{j=0}^{m} b_i u(k-j) \tag{1.8}$$

with initial conditions set to zero.

From mathematical analysis we know that the solution of a linear non homogeneous equation is given by the sum of a particular solution to the associated homogeneous equation and a solution of the complete equation. This is stated in the following theorem:

Theorem 1.1 (IO model solution)

The solution $y(t)$ of the differential equation 1.1, with initial conditions 1.2 and input $u(t)$ is given by the sum of its free response and its forced response:

$$y(t) = y_l(t) + y_f(t) \;\; \mathbb{R} \ni t \geq t_0$$

The solution $y(k)$ of the difference equation 1.3, with initial conditions 1.4 and input $u(k)$ is given by the sum of its free response and its forced response:

$$y(k) = y_l(k) + y_f(k) \;\; \mathbb{Z} \ni k \geq k_0$$

We will now focus on the free response of the system. To the associated homogeneous equation of the main differential (or difference) equation we can associate a polynomial called **characteristic polynomial**.

Definition 1.13 (Characteristic polynomial)

Given the homogeneous equation 1.5, its characteristic polynomial is given by the algebraic equation:

$$P(s) \coloneqq \sum_{i=0}^{n} a_i s^i = 0 \tag{1.9}$$

Given the homogeneous equation 1.6, its characteristic polynomial is given by the algebraic equation:

$$P(z) \coloneqq \sum_{i=0}^{n} a_{n-1} z^i = 0 \tag{1.10}$$

To obtain the characteristic polynomial for the continuous time case ($P(s) \in \mathbb{R}[s]$) we replace the derivatives of the homogeneous equation definition with the polynomial coefficients:

$$\begin{array}{ccccccccc}
a_n \frac{d^n y_l(t)}{dt^n} & + a_{n-1} & \frac{d^{n-1} y_l(t)}{dt^{n-1}} & + \ldots + a_1 & \frac{dy_l(t)}{dt} & + a_0 & y_l(t) & = 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
a_n \;\; s^n & + a_{n-1} & s^{n-1} & + \ldots + a_1 & s & + a_0 & 1 & = 0
\end{array}$$

To obtain the characteristic polynomial for the continuous time case ($P(s) \in \mathbb{R}[s]$) we first replace the difference terms of the homogeneous equation definition with the the monomial z^{-i}, and then we multiply by z^n to obtain a polynomial with positive or zero exponents:

$$\begin{array}{cccccccccccc}
a_n\, y_l(k-n) & + & a_{n-1}\, y_l(k-n+1)\ dots & + & a_1\, y_l(k-1) & + & a_0\, y_l(k) = 0 \\
a_n \quad \downarrow & & \downarrow & & \downarrow & & \downarrow \\
a_n \quad z^{-n} & + & a_{n-1} \quad z^{-(n-1)} & + \ldots + & a_1 \quad z^{-1} & + & a_0 \quad 1 & = 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
a_n \quad 1 & + & a_{n-1} \quad z & + \ldots + & a_1 \quad z^{n-1} & + & a_0 \quad z^n & = 0
\end{array}$$

Lastly, to obtain the solution to the associated homogeneous equation we need to define the form in which the solutions are expressed.

Definition 1.14 (System modes)

The elementary solutions to the associated homogeneous equation 1.5, expressed in the form:

$$m_{i,j}(t) = \frac{t^j}{j!} e^{p_i t} \quad t \in \mathbb{R} \tag{1.11}$$

for $i = 1, \ldots, h$ and $j = 0, \ldots, r_i$, where p_i is the i-th root of the characteristic polynomial 1.9, are called ***modes*** *of the system.*

The elementary solutions to the associated homogeneous equation 1.6, expressed in the form:

$$m_{i,j}(k) = \frac{k^j}{j!} p_i^k \quad k \in \mathbb{Z} \tag{1.12}$$

for $i = 1, \ldots, h$ and $j = 0, \ldots, r_i$, where p_i is the i-th root of the characteristic polynomial 1.10, are called ***modes*** *of the system.*

The free response is then given by the following theorem.

Theorem 1.2 (Free response)

The solution $y_l(t)$ of the homogeneous equation 1.5 is a linear combination of the system modes:

$$y_l(t) = \sum_{i=1}^{h} \sum_{j=0}^{\mu_i - 1} c_{ij} \frac{t^j}{j!} e^{p_i t}$$

where μ_i is the algebraic multiplicity of the p_i root of the characteristic polynomial 1.9 and the coefficients c_{ij} are determined via the initial conditions 1.2.

The solution $y_l(k)$ of the homogeneous equation 1.6 is a linear combination of the system modes:

$$y_l(k) = \sum_{i=1}^{h} \sum_{j=0}^{\mu_i - 1} c_{ij} \frac{k^j}{j!} p_i^k$$

where μ_i is the algebraic multiplicity of the p_i root of the characteristic polynomial 1.10 and the coefficients c_{ij} are determined via the initial conditions 1.4.

In both cases the determination of the coefficients c_{ij} requires the solution of a Cauchy problem.

Example 1.1

Consider the continuous time LTI system expressed by an IO model described by the differential equation:

$$\dot{y} + ay(t) = bu(t) \quad t \geq 0 \quad a, b \in \mathbb{R}$$

The associated homogeneous equation is obtained by setting $u(t) = 0\ \forall t \geq 0$:

$$\dot{y} + ay(t) = 0$$

The characteristic polynomial associated to the homogeneous equation:

$$P(s) = s + a$$

has only one solution $p = -a \in \mathbb{R}$ with algebraic multiplicity $\mu = 1$. Therefore its only mode is $m(t) = e^{-at}$, so the free response of the system is given by:

$$y_l(t) = ce^{-at} \quad t \geq 0$$

If the initial condition is $y_l(0) = y(0)$ then:

$$y_l(t) = y(0)e^{-at} \quad t \geq 0$$

Example 1.2

Consider the continuous time LTI system expressed by an IO model described by the differential equation:

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 0 \quad t \geq 0$$

The equation is already homogeneous, and its characteristic polynomial is:

$$P(s) = s^2 + 6s + 9$$

The polynomial has two real coincident roots $p_1 = p_2 = p = -3$, so we can consider a single real root with algebraic multiplicity $\mu = 2$. Therefore the system modes are:

$$m_{1,0}(t) = e^{-3t} \quad m_{1,1}(t) = te^{-3t} \quad t \geq 0$$

The free response is then given by:

$$y_l(t) = c_{1,0}e^{-3t} + c_{1,1}te^{-3t} \quad t \geq 0$$

where $c_{1,0}$ and $c_{1,1}$ are determined from the initial conditions, by solving the Cauchy problem:

$$\begin{cases} y_l(t_0) = c_1 \\ \dot{y}_l(t_0) = c_2 \end{cases} \quad \begin{cases} c_{1,0}e^{-3t_0} + c_{1,1}t_0e^{-3t_0} = c_1 \\ -3c_{1,0}e^{-3t_0} + c_{1,1}(e^{-3t_0} - 3t_0e^{-3t_0}) = c_2 \end{cases}$$

Assuming that the initial conditions are $c_1 = 1$ and $c_2 = 0$ the coefficients are $c_{1,0} = 1$ and $c_{1,1} = 3$, so the free response of the system considering the initial conditions is:

$$y_l(t) = e^{-3t} + 3te^{-3t} \quad t \geq 0$$

1.5 Stability and asymptotic stability

With the definition of system modes (1.14) we can give a more analytical definition of stability and asymptotic stability.

Proposition 1.1 (Mode characterization)

The elementary mode $m_{i,j}(t) = \frac{t^j}{j!}e^{p_i t} \quad t \in \mathbb{R}$, $j \in \mathbb{N}$, $p_i \in \mathbb{C}$:

1. *converges to zero if and only if $Re(p_i) < 0$,*
2. *is bounded to $[0, +\infty]$ if and only if $Re(p_i) \leq 0$ and the modes $p_i \in j\mathbb{R}$ are simple (i.e. $\mu_i = 1$),*
3. *diverges in any other case.*

The elementary mode $m_{i,j}(k) = \frac{k^j}{j!}p_i^k \quad k \in \mathbb{Z}$:

1. *converges to zero if and only if $|p_i| < 1$,*
2. *is bounded to $\mathbb{Z}_+$ if and only if $|p_i| \leq 0$ and the modes $p_i \in \partial\mathbb{D}$ are simple (i.e. $\mu_i = 1$),*
3. *diverges in any other case.*

With these definitions we can now characterize the stability of a system through the characterization of its modes.

Theorem 1.3 (Stability and asymptotic stability)

A causal LTI system Σ is:

1. *Stable, if and only if all its modes are bounded,*
2. *Asymptotically stable, if and only if all its modes converge to zero.*

Example 1.3

Consider the continuous time LTI system described by the IO model:

$$\ddot{y}(t) + 2\dot{y}(t) + (1 + \pi^2)y(t) = u(t) \quad t \geq 0$$

The characteristic polynomial of the associated homogeneous equation:

$$P(s) = s^2 + 2s + (1 + \pi^2)$$

has two complex conjugate roots $s_{1,2} = -1 \pm j\pi$ with a strictly negative real part, so the system is asymptotically stable.

Example 1.4

Consider the continuous time LTI system described by the IO model:

$$\ddot{y}(t) - 2\dot{y}(t) + (1+\omega^2)y(t) = u(t) \quad t \geq 0,\ \omega \in \mathbb{R}^+$$

The characteristic polynomial of the associated homogeneous equation:

$$P(s) = s^2 + 2s + (1+e^2)$$

has two complex conjugate roots $s_{1,2} = 1 \pm j\omega$ with a strictly positive real part, so the system is unstable.

Observation 1.1 (Stability vs asymptotic stability)

In general asymptotic stability implies stability, but stability does not imply asymptotic stability. Consider the homogeneous equation:

$$\ddot{y}(t) + 4\dot{y}(y) = 0 \quad t \geq 0$$

Its characteristic polynomial $P(s) = s^2 + 4s$ has two distinct real roots $p_1 = 0$ and $p_2 = -4$, both with algebraic multiplicity $\mu_1 = \mu_2 = \mu = 1$. The system modes therefore are:

$$m_1(t) = e^{0t} = 1 \quad m_2(t) = e = -4t \quad t \geq 0$$

Both modes are bounded, m_1 to the singular value 1 and m_2 to $(0, 1]$, so the system is stable. Since its value is constant, the mode $m_1(t)$ does not converge to zero for $t \to +\infty$, therefore the system is stable but not asymptotically so.

1.6 Forced response

To study the forced response of a system we study how it reacts to an impulsive signal supplied to it. First of all we define what an impulse signal is in both continuous and discrete time.

Definition 1.15 (Dirac's delta function)

Given the sequence of functions:

$$f_n(t) \coloneqq \begin{cases} \frac{n}{2} & -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 0 & otherwise \end{cases}$$

***Dirac's delta function** or **Dirac's impulse** is its limit for $n \to \infty$:*

$$\delta(t) \coloneqq \lim_{n\to\infty} f_n(t)$$

Definition 1.16 (Kronecker's delta function)

***Kronecker's delta function** is a sequence defined as:*

$$\delta(k) \coloneqq \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases} \qquad k \in \mathbb{Z}$$

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