# Mobile Robotics, Localization: Motion Models

Material based on the book Probabilistic Robotics (Thrun, Burgard, Fox) [PR]; Chapter 5.3, 5.4

Part of the material is based on lectures from Cyrill Stachniss and Nived Chebrolu

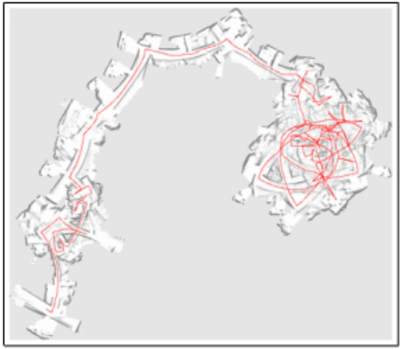
# Summary

- Introduction to probabilistic motion models
- Odometry Motion Models [Chapter 5.4]
- Velocity Motion Models [Chapter 5.3]

# Introduction to probabilistic motion models

# Uncertainty in Motion

- Motion is inherently uncertain
- Model this uncertainty using probability



Pure odometry, source [PR], courtesy Dirk Hähnel

Corrected trajectory, source [PR], courtesy Dirk Hähnel

# Probabilistic framework for state estimation



- Recursive Bayes Filter
  - $Bel(x_t) = P(x_t | u_{1:t}, z_{1:t})$
  - Bayes rule, Markov Assumption (and independence assumption)
  - $Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$

# Motion and Observation Model



- Prediction Step:

$$\overline{Bel(x_t)} = \int P(x_t|u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Correction Step:

$$Bel(x_t) = \eta P(z_t|x_t) \overline{Bel(x_t)}$$

- Probabilistic **Motion** model
  - Specifies a **posterior** probability that action $u$ carries the state from $x_{t-1}$ to $x_t$

$$P(x_t|u_t, x_{t-1})$$

# Typical Motion Models

- Odometry-based
  - require **encoders** that provide information on motion
  - calculate new pose based on encoder values
- Velocity-based (dead reckoning)
  - can be applied when no wheel encoders are available
  - calculate new pose based on velocity and time interval

# Typical Reasons for Motion uncertainty



◇ Violations of assumption we made to build the kinematic model

- Flat surface $\Rightarrow$ bumps
- Perfect geometry $\Rightarrow$ differences in wheel diameters
- Rolling constraint $\Rightarrow$ carpet or slippery floor
- No lateral movements for standard wheels $\Rightarrow$ drift due to high forces

# Odometry Motion Model

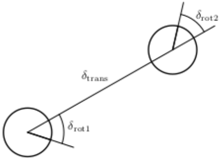


- Motion from $x_{t-1} = (\bar{x}, \bar{y}, \bar{\theta})$ to $x_t = (\bar{x}', \bar{y}', \bar{\theta}')$
- Odometry information: $u_t = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$

$$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$$

$$\delta_{rot1} = atan2(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$$

$$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$$




Odometry model, source [PR]

- Noise model: independent, zero means for **each component**

# Noise Model for Odometry



- Measured motion is given by the true motion corrupted with noise

$$\hat{\delta}_{rot1} = \delta_{rot1} + \epsilon_{\alpha_1|\delta_{rot1}|+\alpha_2|\delta_{trans}|}$$

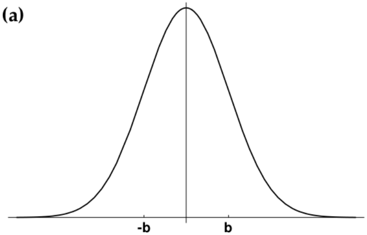
$$\hat{\delta}_{trans} = \delta_{trans} + \epsilon_{\alpha_3|\delta_{trans}|+\alpha_4(|\delta_{rot1}|+|\delta_{rot2}|)}$$

$$\hat{\delta}_{rot2} = \delta_{rot2} + \epsilon_{\alpha_1|\delta_{rot2}|+\alpha_2|\delta_{trans}|}$$

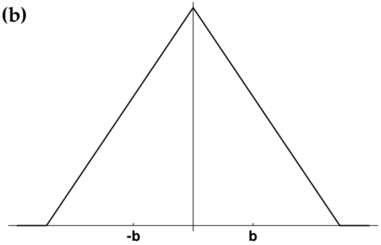
# Typical Distributions for Motion Models

$$\epsilon_{b^2}(x) = \frac{1}{\sqrt{2\pi\mathbf{b}^2}} e^{-\frac{1}{2}\frac{x^2}{\mathbf{b}^2}}$$

$$\epsilon_{b^2}(x) = \begin{cases} 0 & \text{if } |x| > \sqrt{6b^2} \\ \frac{\sqrt{6b^2} - |x|}{6b^2} & \text{otherwise} \end{cases}$$




Normal distribution, source [PR]




Triangular distribution, source [PR]

# Computing the Probability Density at a Query Point

- Query Point $a$ with given standard deviation $b$
- Normal distribution
  - prob_normal_distribution(a,b)
  - return: $\frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2}\frac{a^2}{b^2}}$
- Triangular distribution
  - prob_triangular_distribution(a,b)
  - return: $\max\left\{0, \frac{1}{\sqrt{6}b} - \frac{|a|}{6b^2}\right\}$

# Calculating the Posterior

- $x_{t-1} = (x, y, \theta)^T$ and $x_t = (x', y', \theta')^T$
- $u_t = (\bar{x}_{t-1}, \bar{x}_t)^T$ where $\bar{x}_{t-1} = (\bar{x}, \bar{y}, \bar{\theta})^T$ and $\bar{x}_t = (\bar{x}', \bar{y}', \bar{\theta}')^T$

**Data:** $x_{t-1}, x_t, u_t$

**Result:** $P(x_t | u_t, x_{t-1})$

$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$;

$\delta_{rot1} = atan2(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$;

$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$;

$\hat{\delta}_{trans} = \sqrt{(x' - x)^2 + (y' - y)^2}$;

$\hat{\delta}_{rot1} = atan2(y' - y, x' - x) - \theta$;

$\hat{\delta}_{rot2} = \theta' - \theta - \hat{\delta}_{rot1}$;

$P_1 = prob(\delta_{rot1} - \hat{\delta}_{rot1}, \alpha_1|\delta_{rot1}| + \alpha_2|\delta_{trans}|)$;

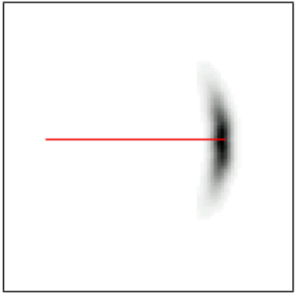
$P_2 = prob(\delta_{trans} - \hat{\delta}_{trans}, \alpha_3|\delta_{trans}| + \alpha_4(|\delta_{rot1}| + |\delta_{rot2}|))$;

$P_3 = prob(\delta_{rot2} - \hat{\delta}_{rot2}, \alpha_1|\delta_{rot2}| + \alpha_2|\delta_{trans}|)$;

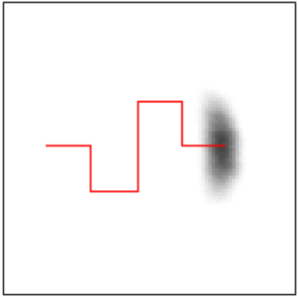
return $P_1 \cdot P_2 \cdot P_3$;

# Example: resulting distribution

- 2D projection of $P(x_t|u_t, x_{t-1})$ for different $u_t$
- Histogram representation



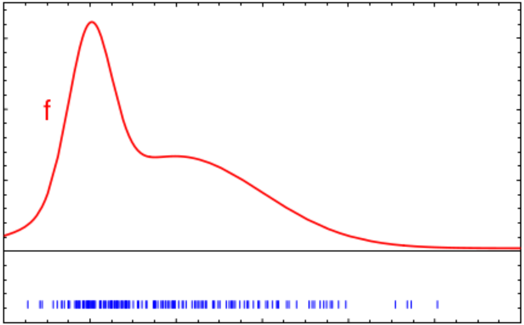
Posterior distribution after executing a move straight command, source [PR]



Posterior distribution after executing a series of turn and move straight commands, source [PR]

# Sample Based representation

- represent density function with a finite set of **samples**
- samples are **drawn** from the density function




Sample based representation of a density function $f$, source [PR]

# How to sample from a distribution



- Sampling from a Normal distribution (zero mean, variance $b^2$)
- Algorithm sample_normal_distribution($b^2$)
  - return $\frac{1}{2}\sum_{i=1}^{12} rand(-b, b)$

- Sampling from a Triangular distribution (zero mean, variance $b^2$)
- Algorithm sample_triangular_distribution($b^2$)
  - return $\frac{\sqrt{6}}{2}[rand(-b, b) + rand(-b, b)]$

- $rand(x, y)$ pseudo random number generator with uniform distribution in $[x, y]$

# Sample Odometry Motion Model

- $x_{t-1} = (x, y, \theta)^T$
- $u_t = (\bar{x}_{t-1}, \bar{x}_t)^T$ where $\bar{x}_{t-1} = (\bar{x}, \bar{y}, \bar{\theta})^T$ and $\bar{x}_t = (\bar{x}', \bar{y}', \bar{\theta}')^T$

**Data:** $x_{t-1}, u_t$

**Result:** $x_t$

$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$;

$\delta_{rot1} = atan2(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$;

$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$;

$\hat{\delta}_{trans} = \delta_{trans} + sample(\alpha_3|\delta_{trans}| + \alpha_4(|\delta_{rot1}| + |\delta_{rot2}|))$;

$\hat{\delta}_{rot1} = \delta_{rot1} + sample(\alpha_1|\delta_{rot1}| + \alpha_2|\delta_{trans}|)$;

$\hat{\delta}_{rot2} = \delta_{rot2} + sample(\alpha_1|\delta_{rot2}| + \alpha_2|\delta_{trans}|)$;

$x' = x + \hat{\delta}_{trans} \cos(\theta + \hat{\delta}_{rot1})$;

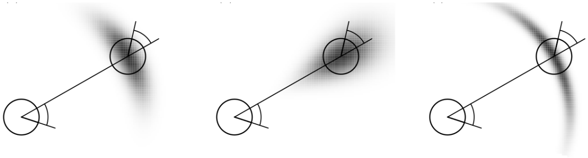
$y' = y + \hat{\delta}_{trans} \sin(\theta + \hat{\delta}_{rot1})$;

$\theta' = \theta + \hat{\delta}_{rot1} + \hat{\delta}_{rot2}$;

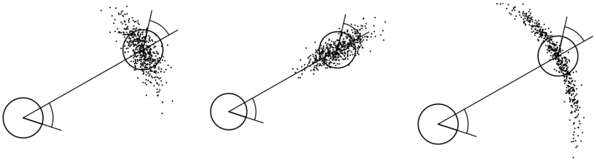
return $(x', y', \theta')^T$;

# Example: Odometry based resulting distributions





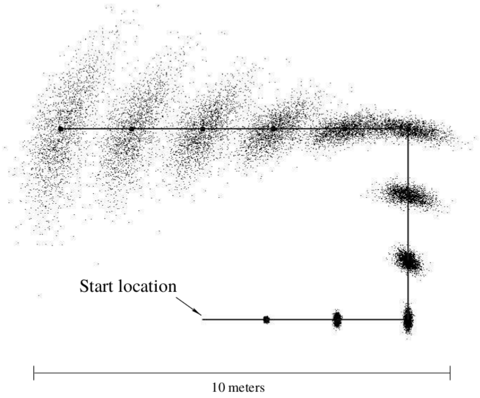
Histogram based representation of $P(x_t | u_t, x_{t-1})$, source [PR]



Sample based representation of $P(x_t | u_t, x_{t-1})$, source [PR]

# Sampling from Motion Model, No Sensing

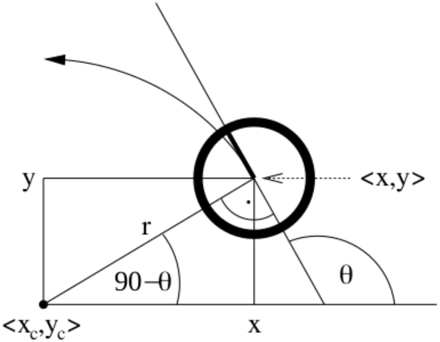





Sampling from $P(x_t | u_t, x_{t-1})$ without sensing, source [PR]

# Velocity-Based Model






Motion for a noise-free robot with $u = (v, \omega)^T$, starting from $(x, y, \theta)^T$, source [PR]

# Noise Model for Velocity-Based model



- Measured motion is given by true motion corrupted with noise

$$
\begin{aligned}
\hat{v} &= v + \epsilon_{\alpha_1 |v| + \alpha_2 |\omega|} \\
\hat{\omega} &= \omega + \epsilon_{\alpha_3 |v| + \alpha_4 |\omega|}
\end{aligned}
$$

- What is the drawback of this formulation ?

# Noise Model for Velocity-Based model



- $\hat{v}, \hat{\omega}$ constraint the final orientation

$$
\begin{aligned}
\hat{v} &= v + \epsilon_{\alpha_1|v|+\alpha_2|\omega|} \\
\hat{\omega} &= \omega + \epsilon_{\alpha_3|v|+\alpha_4|\omega|} \\
\hat{\gamma} &= \epsilon_{\alpha_5|v|+\alpha_6|\omega|}
\end{aligned}
$$

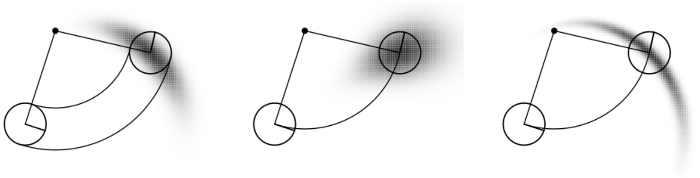
- $\hat{\gamma}$ account for final rotation

# Motion Model including $\hat{\gamma}$

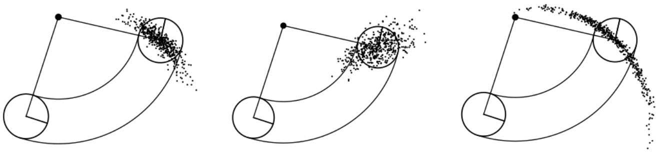
$$
\begin{aligned}
x' &= x - \tfrac{\hat{v}}{\hat{\omega}} \sin(\theta) + \tfrac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega}\Delta t) \\
y' &= y + \tfrac{\hat{v}}{\hat{\omega}} \cos(\theta) - \tfrac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega}\Delta t) \\
\theta' &= \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t
\end{aligned}
$$

# Example: Velocity based resulting distributions





Histogram based representation, source [PR]



Sample based representation, source [PR]

# Summary

- Motion models represent $P(x_t|u_t, x_{t-1})$
- Odometry-based and Velocity-Based motion models
- For both models there are ways to
  - Compute $P(x_t|u_t, x_{t-1})$
  - Sample from $P(x_t|u_t, x_{t-1})$
- Many other models exist that are customized for specific platforms/sensing systems