

ROBOTICS, VISION AND CONTROL

Trajectory Planning. A quick look at geometry

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2D space

2D space

Let $\mathbf{p}_0 = (x_0, y_0)$ be a point on the plane, and $\mathbf{u} = (l, m)$ a unit vector (i.e. a direction).

The unit vector (l, m) is called *direction vector*.

The parametric representation of the *straight line* on the plane passing by \mathbf{p}_0 along the direction \mathbf{u} is

$$\begin{cases} x(\sigma) &= x_0 + l\sigma \\ y(\sigma) &= y_0 + m\sigma \end{cases}$$

for $\sigma \in \mathbb{R}_{\geq 0}$.

According to our previous notation

$$\mathbf{p}(\sigma) = \mathbf{p}_0 + \mathbf{u}\sigma$$

Let $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$ be two distinct points on the plane.

The direction vector $\mathbf{u} = (l, m)$ is proportional to $\mathbf{p}_2 - \mathbf{p}_1$

$$\begin{aligned} l &= x_2 - x_1 \\ m &= y_2 - y_1 \end{aligned}$$

The parametric representation of the *straight line* passing by \mathbf{p}_1 and \mathbf{p}_2 is

$$\begin{cases} x(\sigma) &= x_0 + (x_2 - x_1) \sigma \\ y(\sigma) &= y_0 + (y_2 - y_1) \sigma \end{cases}$$

for $\sigma \in \mathbb{R}_{\geq 0}$.

The parametric equations are NOT unique.

Straight line – Implicit equation



The *implicit equation* or *Cartesian equation* of a straight line on a plane is

$$ax + by + c = 0$$

with a and b not both null.

Given two points $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$, the implicit equation is given by

$$\det \begin{bmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{bmatrix} = 0$$

or

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

The direction vector is $\mathbf{u} = (b, -a)$

Given a point $\bar{\mathbf{p}} = (\bar{x}, \bar{y})$ and the straight line

$$L : \begin{cases} x(t) &= x_0 + l\sigma \\ y(t) &= y_0 + m\sigma \end{cases}$$

The *distance* from a point $\bar{\mathbf{p}}$ to a line L is the shortest distance from $\bar{\mathbf{p}}$ to any point on an infinite straight line

It is the length of the segment connecting $\bar{\mathbf{p}}$ to its orthogonal projection \mathbf{h} on L

$$d(\bar{\mathbf{p}}, L) = d(\bar{\mathbf{p}}, \mathbf{h}) = \|\bar{\mathbf{p}} - \mathbf{h}\|$$

How can we compute \mathbf{h} ?

Given a point $\bar{\mathbf{p}} = (\bar{x}, \bar{y})$ and the straight line

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How can we compute \mathbf{h} ?

Proposition. Two lines

$$L: ax + by + c = 0$$

$$\bar{L}: \bar{a}x + \bar{b}y + \bar{c} = 0$$

are orthogonal if and only if

$$a\bar{a} + b\bar{b} = 0.$$



The point \mathbf{h} is the intersection of

$$L: \begin{cases} x(t) = x_0 + l\sigma \\ y(t) = y_0 + m\sigma \end{cases}, \bar{L}: \begin{cases} x(t) = \bar{x} - m\sigma \\ y(t) = \bar{y} + l\sigma \end{cases}$$

The distance is

$$d(\bar{\mathbf{p}}, L) = \frac{|a\bar{x} + b\bar{y} + c|}{\sqrt{a^2 + b^2}}$$

Exercise 1. Given a line $L : ax + by + c = 0$ and two points $\mathbf{p} \in L$ and $\mathbf{q} \notin L$. Find the point(s) \mathbf{h} on L such that the area of the triangle $\mathbf{p} - \mathbf{q} - \mathbf{h}$ is equal to 5.

Exercise 2. Given a line $L : ax + by + c = 0$ and two points $\mathbf{p} \in L$ and $\mathbf{q} \notin L$. Find the point \mathbf{h} on L such that the triangle $\mathbf{p} - \mathbf{q} - \mathbf{h}$ is equilateral.

The *circle* with centre coordinates (a, b) and radius r is the set of all points (x, y) such that

Cartesian coordinates (x, y)

$$C: (x - a)^2 + (y - b)^2 = r^2$$

Polar coordinates (ρ, θ)

$$C: \rho^2 - 2\rho\rho_0 \cos(\theta - \theta_0) + \rho_0^2 = r^2$$

where (ρ_0, θ_0) is the center of the circle

Parametric form

$$C: \begin{cases} x(\sigma) &= a + r \cos(\sigma) \\ y(\sigma) &= b + r \sin(\sigma) \end{cases}$$

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Parametric form

$$C: \begin{cases} x(\sigma) &= a + r \cos(\sigma) \\ y(\sigma) &= b + r \sin(\sigma) \end{cases}$$

Cartesian coordinates (3-point form)

Given three points not on a line (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , the equation of the circle is

$$\frac{(x - x_1)(x - x_2) + (y - y_1)(y - y_2)}{(y - y_1)(x - x_2) - (y - y_2)(x - x_1)} = \frac{(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2)}{(y_3 - y_1)(x_3 - x_2) - (y_3 - y_2)(x_3 - x_1)}$$

which is the solution of

$$\det \begin{bmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{bmatrix} = 0$$

Given the circle $C : (x - a)^2 + (y - b)^2 = r^2$ and a point $\mathbf{p} = (\bar{x}, \bar{y}) \in C$, the tangent line at \mathbf{p} is

$$(x - \bar{x})(\bar{x} - a) + (y - \bar{y})(\bar{y} - b) = 0$$

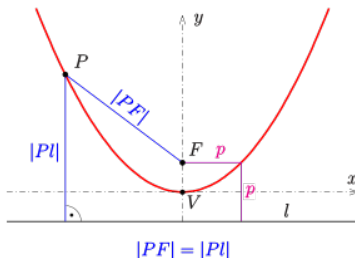
or

$$(\bar{x} - a)x + (\bar{y} - b)y = (\bar{x} - a)\bar{x} + (\bar{y} - b)\bar{y}$$

If $\bar{y} \neq 0$, the slope is

$$m = \frac{dy}{dx} = -\frac{\bar{x} - a}{\bar{y} - b}$$

Parabola: set of points \mathbf{p} such that the distance $\text{dist}(\mathbf{p}, \mathbf{f})$ to a fixed point \mathbf{f} , called the focus, is equal to the distance $\text{dist}(\mathbf{p}, L)$ to a fixed line L , called the directrix



The midpoint \mathbf{v} of the perpendicular from the focus \mathbf{f} onto the directrix L is the vertex.
The line along the vector $\mathbf{f} - \mathbf{v}$ is the axis of symmetry of the parabola.

If $\mathbf{f} = (0, f)$ the expression for the parabola in the picture is

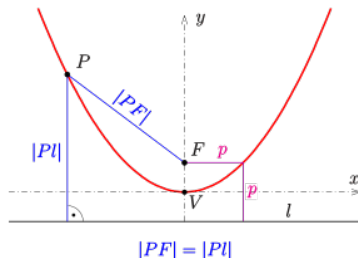
$$x^2 + (y - f)^2 = (y + f)^2$$

i.e.

$$y = \frac{1}{4f}x^2$$

If $f < 0$, the parabola has a downward opening.

If one exchanges x and y , the parabola is rotated by 90 deg.



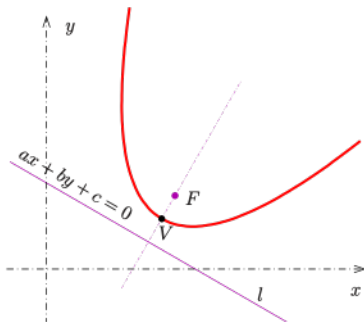
Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{f} = \mathbf{f} + (0, f) = (v_1, v_2 + f)$, the parabola has the equation

$$\begin{aligned} y &= \frac{1}{4f}(x - v_1)^2 + v_2 \\ &= \frac{1}{4f}x^2 - \frac{v_1}{2f}x + \frac{1}{4f}v_1^2 + v_2 \end{aligned}$$

The parametric representation of a normalized parabola $y = x^2$ is

$$P : \begin{cases} x(\sigma) &= \sigma \\ y(\sigma) &= \sigma^2 \end{cases}$$

What about the equation for a tilted parabola?



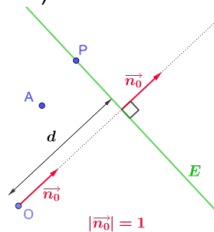
Let $\mathbf{f} = (f_1, f_2)$ and the directrix given by the line

$$ax + by + c = 0$$

The parabola equation is

$$\frac{(ax + by + c)^2}{a^2 + b^2} = (x - f_1)^2 + (y - f_2)^2$$

Hint. (Hesse normal form of a line or of a plane)



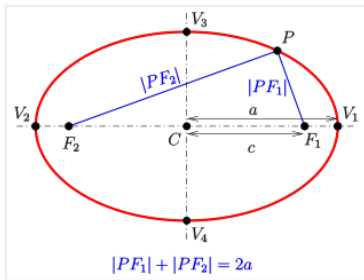
$\mathbf{p} \in E$ if

$$\langle \mathbf{p}, \mathbf{n}_0 \rangle - d = 0$$

where \mathbf{n}_0 is a normal unit vector to E and $d \geq 0$

Ellipse: Given two fixed points f_1, f_2 called the foci and a distance $2a$ larger than the distance between the foci, the ellipse E is the set of points p such that

$$\text{dist}(p, f_1) + \text{dist}(p, f_2) = 2a$$



The line through the foci is called the *major axis*

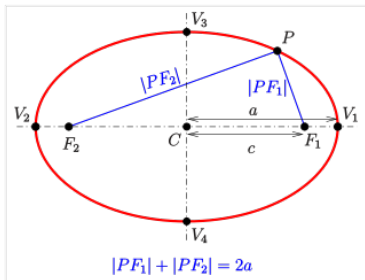
The line perpendicular to it through the center is the *minor axis*

The foci are $f_1 = (c, 0)$, $f_2 = (-c, 0)$ where

$$c = \sqrt{a^2 - b^2}$$

Eccentricity

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$



Standard ellipse centered at the origin with width $2a$ and height $2b$

$$\mathbf{f}_1 = (c, 0), \quad \mathbf{f}_2 = (-c, 0)$$

$$\mathbf{v}_1 = (a, 0), \quad \mathbf{v}_2 = (-a, 0)$$

$$\mathbf{v}_3 = (0, b), \quad \mathbf{v}_4 = (0, -b)$$

Cartesian coordinates (x, y)

$$E : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametric form

$$E : \begin{cases} x(\sigma) &= a \cos(\sigma) \\ y(\sigma) &= b \sin(\sigma) \end{cases}$$

The tangent at a point $\mathbf{p} = (x_1, y_1)$ of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has the coordinate equation:

$$\frac{x_1}{a^2}x + \frac{y_1}{b^2}y = 1$$

A parametric equation of the tangent is:

$$\begin{cases} x(\rho) &= x_1 - y_1 a^2 \rho \\ y(\rho) &= y_1 + x_1 b^2 \rho \end{cases}$$

with $\rho \in \mathbb{R}$.

Translated ellipse in (x_0, y_0)

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

Rotated ellipse by θ

$$\begin{cases} x(\sigma) &= a \cos \theta \cos \sigma - b \sin \theta \sin \sigma \\ y(\sigma) &= a \sin \theta \cos \sigma + b \cos \theta \sin \sigma \end{cases}$$

Translated ellipse in (x_0, y_0) and rotated by θ

$$\begin{aligned} X &= (x - x_0) \cos \theta + (y - y_0) \sin \theta \\ Y &= -(x - x_0) \sin \theta + (y - y_0) \cos \theta \end{aligned}$$

within

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

gives

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where

$$A = a^2 \sin^2 \theta + b^2 \cos^2 \theta$$

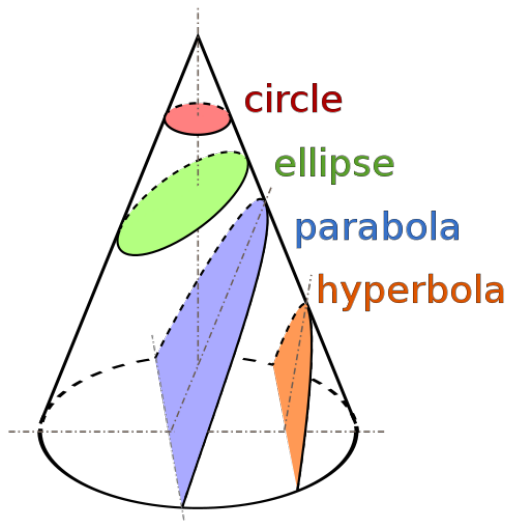
$$B = 2(b^2 - a^2) \sin \theta \cos \theta$$

$$C = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

$$D = -2Ax_0 - By_0$$

$$E = -Bx_0 - 2Cy_0$$

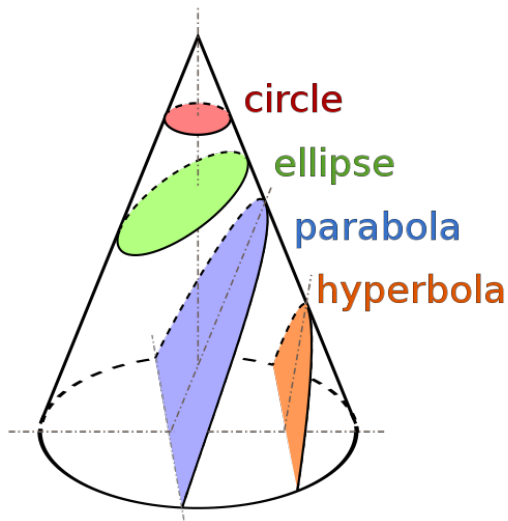
$$F = Ax_0^2 + Bx_0y_0 + Cy_0^2 - a^2b^2$$



The general equation of a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with all coefficients are real numbers and A , B , C not all zero.



The general equation of a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with all coefficients are real numbers and A , B , C not all zero.

If the conic is non-degenerate, then

- ▶ if $B^2 - 4AC < 0$, the equation represents an ellipse;
- ▶ if $B^2 - 4AC < 0$ and $A = C$ and $B = 0$, the equation represents a circle
- ▶ if $B^2 - 4AC = 0$, the equation represents a parabola;
- ▶ if $B^2 - 4AC > 0$, the equation represents a hyperbola;

Exercise. Evaluate the Frenet frames attached to a circle, a parabola and an ellipse