

# ROBOTICS, VISION AND CONTROL

## Trajectory Planning. Multipoint

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## Complements

# Complements

In several cases, only the interpolation points  $q_k$ ,  $k = 0, \dots, n$ , the initial time  $t_0$  and the final time  $t_n$  are known.

How to choose the *intermediate time instants*  $t_1, t_2, \dots, t_{n-1}$ ?

Let's focus on a normalized interval, i.e.  $t_0 = 0$  and  $t_n = 1$

The distribution of the intermediate time instants is

$$t_k = t_{k-1} + \frac{d_k}{d}, \quad d = \sum_{k=0}^{n-1} d_k$$

The problem is now: how to choose  $d_k$ ?

[Since  $t_n$  is fixed, different distributions in time of the interpolation points do not change the duration of the trajectory]

Three main options:

- ▶ equally spaced points

$$d_k = \frac{1}{n-1}$$

- ▶ cord length distribution

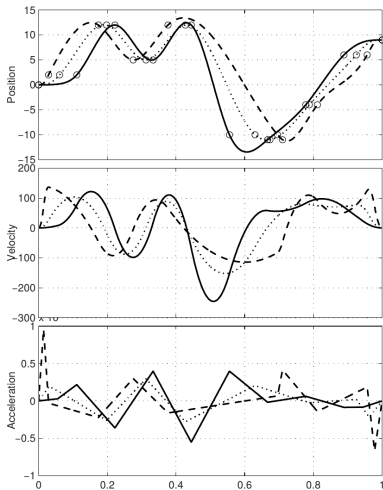
$$d_k = |q_{k+1} - q_k|$$

- ▶ centripetal distribution

$$d_k = \sqrt{|q_{k+1} - q_k|}$$

- ▶ or

$$d_k = |q_{k+1} - q_k|^\mu$$



$$\begin{aligned}q_0 &= 0, q_1 = 2, q_2 = 12, q_3 = 5, \\q_4 &= 12, q_5 = -10, q_6 = -11, q_7 = -4, \\q_8 &= 6, q_9 = 9\end{aligned}$$

uniformly spaced (solid),  
cord length (dashed),  
centripetal (dotted)

**Exercise.** Scale the trajectory duration to a generic  $t_f$

The total duration of a spline trajectory  $s(t)$  interpolating the points  $(t_k, q_k)$ ,  $k = 0, \dots, n$  is

$$T = \sum_{k=0}^{n-1} T_k, \quad T_k = t_{k+1} - t_k$$

and so

$$T = t_n - t_0$$

The goal is to minimize  $T$  such that the constraints on min/max velocity and min/max accelerations are satisfied

$$\{T_k^\circ\} = \arg \min_{T_k} T = \sum_{k=0}^{n-1} T_k$$

$$\begin{aligned} \text{s. to} \quad & |\dot{s}(t; T_0, T_1, \dots, T_{n-1})| < \dot{q}^{max} \\ & |\ddot{s}(t; T_0, T_1, \dots, T_{n-1})| < \ddot{q}^{max} \end{aligned}$$



It is a *nonlinear optimum problem* with a linear objective function, solvable with classical techniques of operational research.

## Suboptimal solution

Since the coefficients which determine the spline (and as a consequence the value of the velocity and of the acceleration along the trajectory) are computed as a function of the intervals  $T_k$ , the optimization problem can be solved in an iterative way, by *scaling in time the segments* which compose the spline

If the time interval  $T_k$  is replaced by

$$T_k \rightarrow \lambda T_k$$

then the velocity, acceleration and jerk are scaled by

$$\dot{\Pi}_k(t) \rightarrow \frac{1}{\lambda} \dot{\Pi}_k(t)$$

$$\ddot{\Pi}_k(t) \rightarrow \frac{1}{\lambda^2} \ddot{\Pi}_k(t)$$

$$\dddot{\Pi}_k(t) \rightarrow \frac{1}{\lambda^3} \dddot{\Pi}_k(t)$$

Then the optimal  $\lambda$  is

$$\lambda^\circ = \max\{\lambda_v, \lambda_a, \lambda_j\}$$

where

$$\begin{aligned}\lambda_v &= \max_k \lambda_{v,k}, & \lambda_{v,k} &= \max_{t \in [t_k, t_{k+1})} \frac{|\dot{\mathbf{p}}_k(t)|}{\dot{q}^{max}} \\ \lambda_a &= \max_k \lambda_{a,k}, & \lambda_{a,k} &= \max_{t \in [t_k, t_{k+1})} \sqrt{\frac{|\ddot{\mathbf{p}}_k(t)|}{\ddot{q}^{max}}} \\ \lambda_j &= \max_k \lambda_{j,k}, & \lambda_{j,k} &= \max_{t \in [t_k, t_{k+1})} \left( \frac{|\ddot{\mathbf{p}}_k(t)|}{\ddot{q}^{max}} \right)^{1/3}\end{aligned}$$

The spline  $s(t)$  will reach the maximum speed or the maximum acceleration or the maximum jerk, in at least a point of the interval  $[t_0, t_n]$

## Geometric Modification of a Trajectory

- ▶ *time-translation* of a trajectory  $q(t)$  from  $(0, 0)$  to  $(t_1, q_1)$

$$\bar{q}(t) = q(t) + q_0$$

$$\bar{q}(t) : (0, q_0) \mapsto (t_1, q_0 + q_1)$$

- ▶ *space-translation* of a trajectory  $q(t)$  from  $(0, 0)$  to  $(t_1, q_1)$

$$\bar{q}(t) = q(t - t_0)$$

$$\bar{q}(t) : (t_0, 0) \mapsto (t_0 + t_1, q_1)$$

- ▶ *space-reflection* of a trajectory  $q(t)$  from  $(0, 0)$  to  $(t_1, q_1)$

$$\bar{q}(t) = -q(t)$$

$$\bar{q}(t) : (0, 0) \mapsto (t_1, -q_1)$$

- *scaling in space* of a ‘unitary’ trajectory  $q(t)$  from  $(0, 0)$  to  $(1, 1)$

$$\bar{q}(t) = q_0 + h q(t), \quad h = q_1 - q_0$$

$$\bar{q}(t) : (0, q_0) \mapsto (1, q_1)$$

- *scaling in time* of a ‘unitary’ trajectory  $q(t)$  from  $(0, 0)$  to  $(1, 1)$

$$\bar{q}(t) = q(t/t_1)$$

$$\bar{q}(t) : (0, 0) \mapsto (t_1, 1)$$

**Exercise.** Starting from the ‘unitary’ cubic polynomial  $q(t)$  from  $(0, 0)$  to  $(1, 1)$  with zero initial and final velocities, design the multi-point trajectory passing through  $(0, 0)$ ,  $(2, 1)$ ,  $(3, -1)$ ,  $(5, )$  using the previous properties.

The scaling in time is useful when we have to satisfied the following constraints

- ▶ *Kinematic saturation*: limits on velocity and acceleration

$$|\dot{q}(t)| < \dot{q}^{max}, \quad |\ddot{q}(t)| < \ddot{q}^{max}$$

- ▶ *Dynamic saturation*: limits on the torques requested to the motors

$$|\tau(t)| < \tau^{max}$$

and the profile is already planned.

Given the original trajectory  $q(t)$ , define a strictly increasing function  $\sigma$

$$t = \sigma(\bar{t})$$

such that the scaled trajectory

$$\bar{q}(\bar{t}) = (q \circ \sigma)(\bar{t}) = q(\sigma(\bar{t}))$$

has its velocity and acceleration profiles

$$\begin{aligned}\dot{\bar{q}}(\bar{t}) &= \frac{dq(\sigma)}{d\sigma} \frac{d\sigma(\bar{t})}{d\bar{t}} \\ \ddot{\bar{q}}(\bar{t}) &= \frac{dq(\sigma)}{d\sigma} \frac{d^2\sigma(\bar{t})}{d\bar{t}^2} + \frac{d^2q(\sigma)}{d\sigma^2} \left( \frac{d\sigma(\bar{t})}{d\bar{t}} \right)^2\end{aligned}$$

that satisfied the kinematic constraints

A common  $\sigma$  function is the linear one

$$t = \lambda \bar{t}$$

with  $\lambda > 0$ . Then we have

$$\begin{aligned}\dot{\bar{q}}(\bar{t}) &= \frac{dq(\sigma)}{d\sigma} \lambda = \lambda \dot{q}(t) \\ \ddot{\bar{q}}(\bar{t}) &= \frac{d^2q(\sigma)}{d\sigma^2} \lambda^2 = \lambda^2 \ddot{q}(t) \\ \dddot{\bar{q}}(\bar{t}) &= \frac{d^3q(\sigma)}{d\sigma^3} \lambda^3 = \lambda^3 \dddot{q}(t) \\ &\vdots\end{aligned}$$

The choice

$$\lambda = \min \left\{ \frac{\dot{q}^{max}}{\max_t |\dot{q}(t)|}, \sqrt{\frac{\ddot{q}^{max}}{\max_t |\ddot{q}(t)|}}, \left( \frac{\dddot{q}^{max}}{\max_t |\dddot{q}(t)|} \right)^{1/3} \right\}$$

guarantees that the maximum values of speed, acceleration and jerk are never exceeded.



Let  $\tilde{q}(\tau)$  be the *normalized trajectory*

$$0 \leq \tilde{q}(\tau) \leq 1, \quad 0 \leq \tau \leq 1$$

A generic trajectory from  $(t_0, q_0)$  to  $(t_1, q_1)$  can be written as

$$\begin{aligned} q(t) &= q_0 + (q_1 - q_0) \tilde{q} \left( \frac{t - t_0}{t_1 - t_0} \right) \\ &= q_0 + \Delta q \tilde{q} \left( \frac{t - t_0}{\Delta T} \right) \end{aligned}$$

It is then possible to compute the time derivative of  $q$  as a function of the time derivative of  $\tilde{q}$

$$\dot{q}(t) = \frac{\Delta q}{\Delta T} \dot{\tilde{q}}(\tau), \quad \ddot{q}(t) = \frac{\Delta q}{(\Delta T)^2} \ddot{\tilde{q}}(\tau), \quad \dddot{q}(t) = \frac{\Delta q}{(\Delta T)^3} \dddot{\tilde{q}}(\tau), \quad \dots$$

Since the maximum values of velocity, acceleration, jerk of  $q(t)$  are obtained in correspondence of the maximum values of the velocity, acceleration, jerk of  $\tilde{q}(\tau)$ , it is easy to compute both these values and the corresponding time instants  $\tau$  from the a given parameterization  $\tilde{q}$

By scaling  $\Delta T$  we obtain motion profiles with maximum velocity/acceleration values equal to the saturation limits.

**Example.** Normalized trajectory = polynomial of order 3 with zero initial and final velocities

$$\begin{aligned}\tilde{q}(\tau) &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 \\ &= 3\tau^2 - 2\tau^3\end{aligned}$$

Then

$$\dot{\tilde{q}}(\tau) = 6\tau - 6\tau^2, \quad \ddot{\tilde{q}}(\tau) = 6 - 12\tau, \quad \dddot{\tilde{q}}(\tau) = -12$$

and the maximum velocity and acceleration are

$$\dot{\tilde{q}}^{max} = \max_{\tau} \dot{\tilde{q}}(\tau) = \dot{\tilde{q}}(0.5) = \frac{3}{2}, \quad \ddot{\tilde{q}}^{max} = \max_{\tau} \ddot{\tilde{q}}(\tau) = \ddot{\tilde{q}}(0) = 6,$$

According to the expression in the previous slide, we end up with

$$\dot{q}^{max} = \frac{3}{2} \frac{\Delta q}{\Delta T}, \quad \ddot{q}^{max} = 6 \frac{\Delta q}{(\Delta T)^2}$$

**Exercise.** Compute the expression for the maximum velocity, acceleration and jerk when the nominal trajectory is a polynomial of order 5 with zero initial and final velocities, and zero initial and final accelerations.

The equations of motion of a robotic manipulator with  $n$  DoF are

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} \quad (1)$$

The  $i$ -th row is

$$\mathbf{b}_i^T(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{L}_i(\mathbf{q})\dot{\mathbf{q}} + \mathbf{g}_i(\mathbf{q}) = \tau_i \quad (2)$$

where  $\frac{1}{2}\dot{\mathbf{q}}^T \mathbf{L}_i(\mathbf{q})\dot{\mathbf{q}}$  is an equivalent expression of the  $i$ -th component of  $\mathbf{C}(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}}$ .

Given the trajectory  $\mathbf{q}(t)$ ,  $t \in [0, T]$ , we have

$$\underbrace{\mathbf{b}_i^T(\mathbf{q}(t))\ddot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{L}_i(\mathbf{q}(t))\dot{\mathbf{q}}(t)}_{\tau_{s,i}(t)} + \underbrace{\mathbf{g}_i(\mathbf{q}(t))}_{\tau_{p,i}(t)} = \tau_i(t) \quad (3)$$

where  $\tau_{p,i}(t)$  depends only on the position.

We now consider a scaled version  $\bar{\mathbf{q}}(\bar{t})$ ,  $\bar{t} \in [0, \bar{T}]$  of  $\mathbf{q}(t)$ ,  $t \in [0, T]$  with

$$t = \sigma(\bar{t}) \quad (4)$$

The corresponding torques  $\bar{\tau}_i(t)$ ,  $i = 1, \dots, n$  are

$$\bar{\tau}_i(\bar{t}) = \mathbf{b}_i^T(\bar{\mathbf{q}}(\bar{t}))\ddot{\bar{\mathbf{q}}}(\bar{t}) + \frac{1}{2}\dot{\bar{\mathbf{q}}}^T(\bar{t})\mathbf{L}_i(\bar{\mathbf{q}}(\bar{t}))\dot{\bar{\mathbf{q}}}(\bar{t}) + \mathbf{g}_i(\bar{\mathbf{q}}(\bar{t})); \quad (5)$$

exploiting the relationships

$$\bar{\mathbf{q}}(\bar{t}) = (\bar{\mathbf{q}} \circ \sigma)(\bar{t}) \quad (6)$$

$$\dot{\bar{\mathbf{q}}}(\bar{t}) = \dot{\mathbf{q}}(t)\dot{\sigma}, \quad \dot{\sigma} = \frac{d\sigma}{d\bar{t}} \quad (7)$$

$$\ddot{\bar{\mathbf{q}}}(\bar{t}) = \ddot{\mathbf{q}}(t)\dot{\sigma}^2 + \dot{\mathbf{q}}(t)\ddot{\sigma}, \quad \ddot{\sigma} = \frac{d^2\sigma}{d\bar{t}^2} \quad (8)$$

we have

$$\bar{\tau}_i(\bar{t}) = \mathbf{b}_i^T(\mathbf{q}(t))\dot{\mathbf{q}}(t)\ddot{\sigma} + \left[ \mathbf{b}_i^T(\mathbf{q}(t))\ddot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{L}_i(\mathbf{q}(t))\dot{\mathbf{q}}(t) \right] \dot{\sigma}^2 + \mathbf{g}_i(\mathbf{q}(t)) \quad (9)$$

The contribution of the gravity term is independent of the time scaling.

Focusing on  $\bar{\tau}_{s,i}$

$$\bar{\tau}_{s,i}(\bar{t}) = \mathbf{b}_i^T(\mathbf{q}(t))\dot{\mathbf{q}}(t)\ddot{\sigma} + \left[ \mathbf{b}_i^T(\mathbf{q}(t))\ddot{\mathbf{q}}(t) + \frac{1}{2}\dot{\mathbf{q}}^T(t)\mathbf{L}_i(\mathbf{q}(t))\dot{\mathbf{q}}(t) \right] \dot{\sigma}^2 \quad (10)$$

$$= \mathbf{b}_i^T(\mathbf{q}(t))\dot{\mathbf{q}}(t)\ddot{\sigma} + \tau_{s,i}(t)\dot{\sigma}^2 \quad (11)$$

In the linear scaling case  $t = \sigma(\bar{t}) = \lambda \bar{t}$

$$\dot{\sigma}(\bar{t}) = \lambda, \quad \ddot{\sigma}(\bar{t}) = 0 \quad (12)$$

Then

$$\bar{\tau}_{s,i}(\bar{t}) = \tau_{s,i}(t)\lambda^2 \quad (13)$$

and finally

$$\bar{\tau}_i(\bar{t}) - \mathbf{g}_i(\bar{\mathbf{q}}(\bar{t})) = \lambda^2 [\tau_i(t) - \mathbf{g}_i(\mathbf{q}(t))] \quad (14)$$

The duration of the scaled trajectory is  $\bar{T} = T/\lambda$

Let  $\mathbf{q}(t)$  be the planned trajectory. The scaling factor can be used to increase ( $\lambda < 1$ ) or decrease ( $\lambda > 1$ ) the duration in order to allow for at least to one torque to be equal to the maximum value

$$\mathbf{q}(t) \rightarrow \boldsymbol{\tau}(t) \rightarrow \tau_i^{max}(t) = \max_t \tau_i \quad (15)$$

Then

$$\lambda^2 = \min \left\{ \frac{\tau_1^{max}}{|\max_t \tau_1(t)|}, \frac{\tau_2^{max}}{|\max_t \tau_2(t)|}, \dots, \frac{\tau_n^{max}}{|\max_t \tau_n(t)|} \right\} \quad (16)$$

and

$$\bar{t} = \frac{t}{\lambda} \quad (17)$$

$$\bar{T} = \frac{T}{\lambda} \quad (18)$$

**Remark 1.** The scaling factor  $\lambda$  is the same for the whole trajectory.

A more efficient approach could be to scale down the time only in the interval(s) where one torque is larger than the corresponding maximum.

→ *variable scaling*

**Remark 2.** the minimum time motion along a given path saturates the torque or the acceleration or the velocity of one of the actuators in at least a point of each segment.