Epidemic model and Covid-19

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Abstract. A few basic computations about SIR model, namely in the exponential regime

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1 The SIR model

The SIR model is a model of a population made up of Susceptibles (S), Infectious (I) and Recovered with Immunity (R). The total population N is constant.

The equations governing the population are the following:

$$N(t) = S(t) + I(t) + R(t) \tag{1}$$

$$S'(t) = dS(t)/dt = -\beta S(t)I(t)/N(t) \tag{2}$$

$$I'(t) = dI(t)/dt = \beta S(t)I(t)/N(t) - \gamma I(t) \tag{3}$$

$$R'(t) = dR(t)/dt = \gamma I(t) \tag{4}$$

The parameter β describes the rate of "a susceptible individual gets sick"; the parameter γ is the rate of "an infectious individual gets recovered and immune".

The average infectious period is $1/\gamma$. In this model there is no incubation period.

1.1 Exponential regime

At the onset of the epidemic, $S \approx N$, so equation 3 reads:

$$I'(t) = (\beta - \gamma)I(t) \tag{5}$$

whose solution is

$$I(t) = I_0 e^{t/\tau} \tag{6}$$

where the fundamental time constant is

$$\tau = 1/(\beta - \gamma) \tag{7}$$

Similarly to $I(t)$, also S and R follow an exponential trend during the on-set of the epidemic with the same time constant tau

$$R'(t) = \gamma I(t) = \gamma I_0 e^{t/\tau} \tag{8}$$

whose solution is

$$R(t) = \gamma\tau I_0 e^{t/\tau} = I_0 e^{(t+\tau_{RI})/\tau} = I(t + \tau_{RI}) \tag{9}$$

So $R(t)$ follows $I(t)$ with a delay τ_{RI} defined such that

$$\gamma\tau = e^{\tau_{RI}/\tau} \tag{10}$$

that means

$$\tau_{RI} = \tau \ln(\gamma\tau) \tag{11}$$

1.2 Basic Reproduction Number

The basic reproduction number R_0 is defined as the ratio β/γ and relates to τ as

$$\tau = \frac{1}{\gamma(R_0 - 1)} \tag{12}$$

$$R_0 = \frac{1}{\gamma\tau} + 1 \tag{13}$$

1.3 Social rarefaction and stopping of the epidemic

The social rarefaction affects the parameter β while is not affecting the parameter γ: in the case of complete segregation $\beta = 0$ and S' drops to zero.

Let's assume at $t = \hat{t}$ a complete segregation of the population is enforced. What would be the final number of recovered $R(t = \infty)$? Obviously, every infected $I(\hat{t})$ will eventually become recovered, so that (using 9)

$$R(\infty) = R(\hat{t}) + I(\hat{t}) = R(\hat{t}) \left(1 + \frac{1}{\gamma\tau}\right) = R_0 R(\hat{t}) \tag{14}$$

As a second example, the minimal rarefaction to stop the exponentially increasing epidemic is $R_0 = 1$ i.e. $\beta = \gamma$ that gives $I'(t > \hat{t}) = 0$ and

$$R'(t > \hat{t}) = \gamma I(\hat{t}) \tag{15}$$

giving a linear rise of the Recovered with time. In this condition, every Infectious is on-average infecting a Single susceptible.

2 The SIRCD model

During the epidemic event we face the fact that it is hard to have a proper estimate of infectious and recovered. Asymptomatic cases can let infectious events to be undetected and a proper testing for immunity can be not readily available. On the other side, two observables correlated to the epidemic are typically available: the number of confirmed cases (C) and the number of Deaths (D).

We modify the SIR model to relate these 2 observables to the SIR parameters.

The governing equations (Eq 1 to 4) are complemented with:

$$C'(t) = dC(t)/dt = -\epsilon S'(t) \tag{16}$$

$$D'(t) = dD(t)/dt = \delta R'(t) \tag{17}$$

We assume that only a fraction ϵ of those that become infectious is detected and tagged as confirmed (without any incubation period, consistently with the SIR model) and we assume that a fraction δ of the recovered are dead. With this approach, Deaths is indeed a subset of "Recovered with immunity": although it sounds weird, it's just a way of correlating an observable (the number of deaths) with the R of SIR through a known proportional factor delta. Note that D does not affect the total population N.

In the exponential regime the same approach of chapter 1.1 applies and we have

$$C(t) = \epsilon\beta\tau I(t) = I_0 e^{(t+\tau_{CI})/\tau} = I(t + \tau_{CI}) \tag{18}$$

So $C(t)$ follows $I(t)$ with a delay τ_{CI}:

$$\tau_{CI} = \tau \ln(\epsilon\beta\tau) \tag{19}$$

Similarly, for the Deaths population

$$D(t) = \delta\gamma\tau I(t) = I_0 e^{(t+\tau_{DI})/\tau} = I(t + \tau_{DI}) \tag{20}$$

Analogously to the other quantities, $D(t)$ follows $I(t)$ with a delay τ_{DI}:

$$\tau_{DI} = \tau \ln(\delta\gamma\tau) \tag{21}$$

2.1 Delay Confirmed to Death in SIRCD

At the onset of epidemic, the two observables C and D both follow an exponential grow with time constant τ with a delay τ_{CD} given by:

$$\tau_{CD} = \tau_{CI} - \tau_{DI} = \tau \ln\left(\frac{\epsilon\beta}{\delta\gamma}\right) = \tau \ln\left(\frac{\epsilon R_0}{\delta}\right) \tag{22}$$

Equation 22 shows that it is possible to estimate the ratio ϵ/δ from the delay between the curve $C(t)$ and $D(t)$, but not the values of the individual parameters. From the observed delay τ_{CD} we

can estimate:

$$\frac{\epsilon}{\delta} = \frac{1}{R_0} e^{\tau_{CD}/\tau} \tag{23}$$

2.2 Estimation of mortality rate

The mortality rate defined as $D(t=\infty)/N$ is δ in the approximation that $R(\infty) \approx N$. The Case Fatality Risk is $\frac{D(t)}{R(t)}\big|_{t=\infty} = \delta$

Note that the observable ratio

$$D(t)/C(t) = \frac{\delta}{\epsilon R_0} \tag{24}$$

provides a wrong estimate of the CFR. Obviously it must be corrected for R_0 (that can be estimated by exponential time constant and by the average infective period), but it is also affected by the unknown value of the detectability ϵ.

2.3 Numerical example

An example of evolution of a SIRCD model[1] for $R_0 = 3$, $\tau = 3.11$ ($\beta = 0.48$ and $\gamma = 0.16$) and $\epsilon/\delta = 10$ is given in Figure 1. As expected, during the exponential grow the delay between Confirmed and Deaths curves is 10.6 consistent with Eq. 22. Similarly the delay between Recovered and Infectious is -2.2 in agreement with Eq. 11 The actual CFR in this example is 0.0001, but a measurement of D/C during the exponential rise would give 0.033 in accordance with Eq. 24

3 SIRCD model on COVID-19 data

Figure 2 shows a rough fit of confirmed and deaths values for COVID-19 data in the province of Hubei, China that has an approximate population of 60M persons.

[1] A python package implementing the SIRCD model is available at https://github.com/lbusoni/epidemic_models. The package contains also the methods to retrieve COVID-19 data from repositories https://github.com/pcm-dpc/COVID-19 and https://github.com/CSSEGISandData/COVID-19

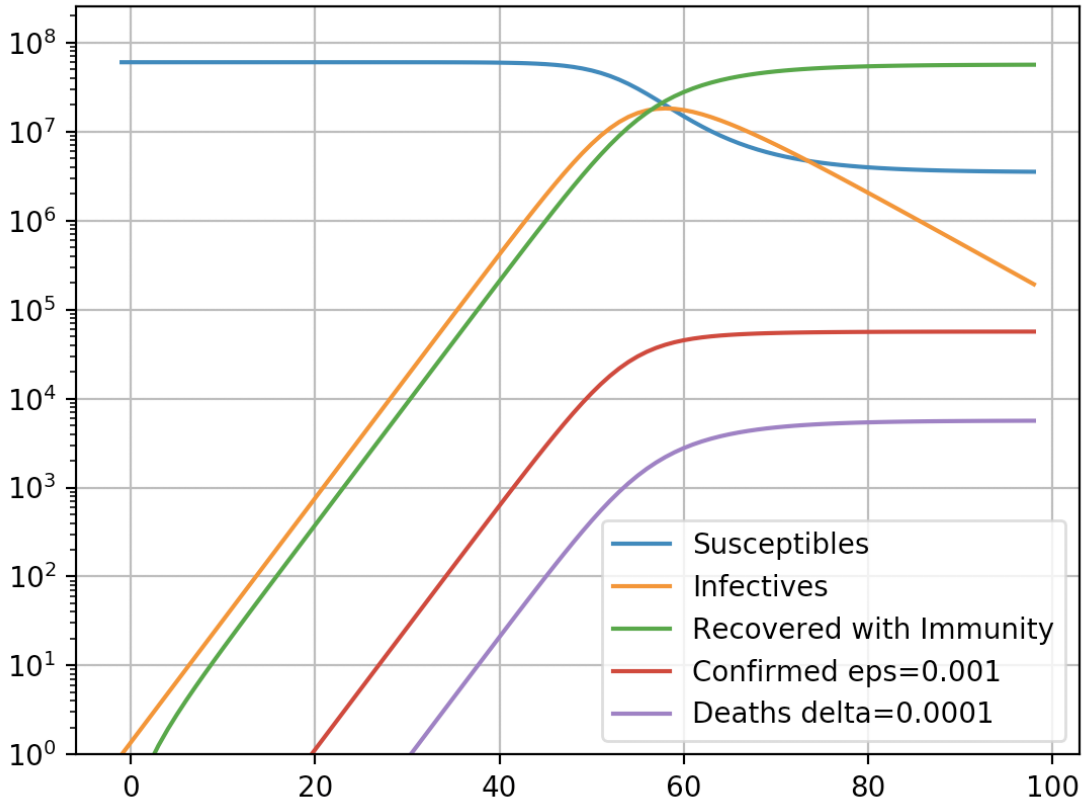

Fig 1 Example of a SIRCD epidemic. $S(0) = 60 \cdot 10^6$, $I(0) = 1$, $R(0) = C(0) = D(0) = 0$, $\beta = 0.48$, $\gamma = 0.16$, $\epsilon = 10^{-3}$, $\delta = 10^{-4}$

Both $C(t)$ and $D(t)$ trends can be roughly reproduced assuming $\epsilon = 0.001$ and $\delta = 5 \cdot 10^{-5}$. Note the very small value of ϵ: out of 1000 infected people only 1 develops symptoms such that he can be detected and confirmed.[2]

The SIRCD model is also capable of reproducing $C(t)$ and $D(t)$ trends introducing a "social rarefaction" effect. In Figure 3 the same Hubei data are fitted with a model that has a detectability $\epsilon = 0.75$ and a mortality $\delta = 0.03$ and with R_0 decreased from 3 to 0.8 at $t = 32$ days

Note that in both cases the apparent mortality $D(\infty)/C(\infty)$ is about 0.05. Instead, the fraction of Immune population (green curve) is very different, with most of the population not immune in

[2]This is incompatible with similar estimate from independent data: in Lodi's county (230k people) the number of Confirmed is 1320 on Mar, 15. This constrains the detectability parameter to $\epsilon > 0.0057$

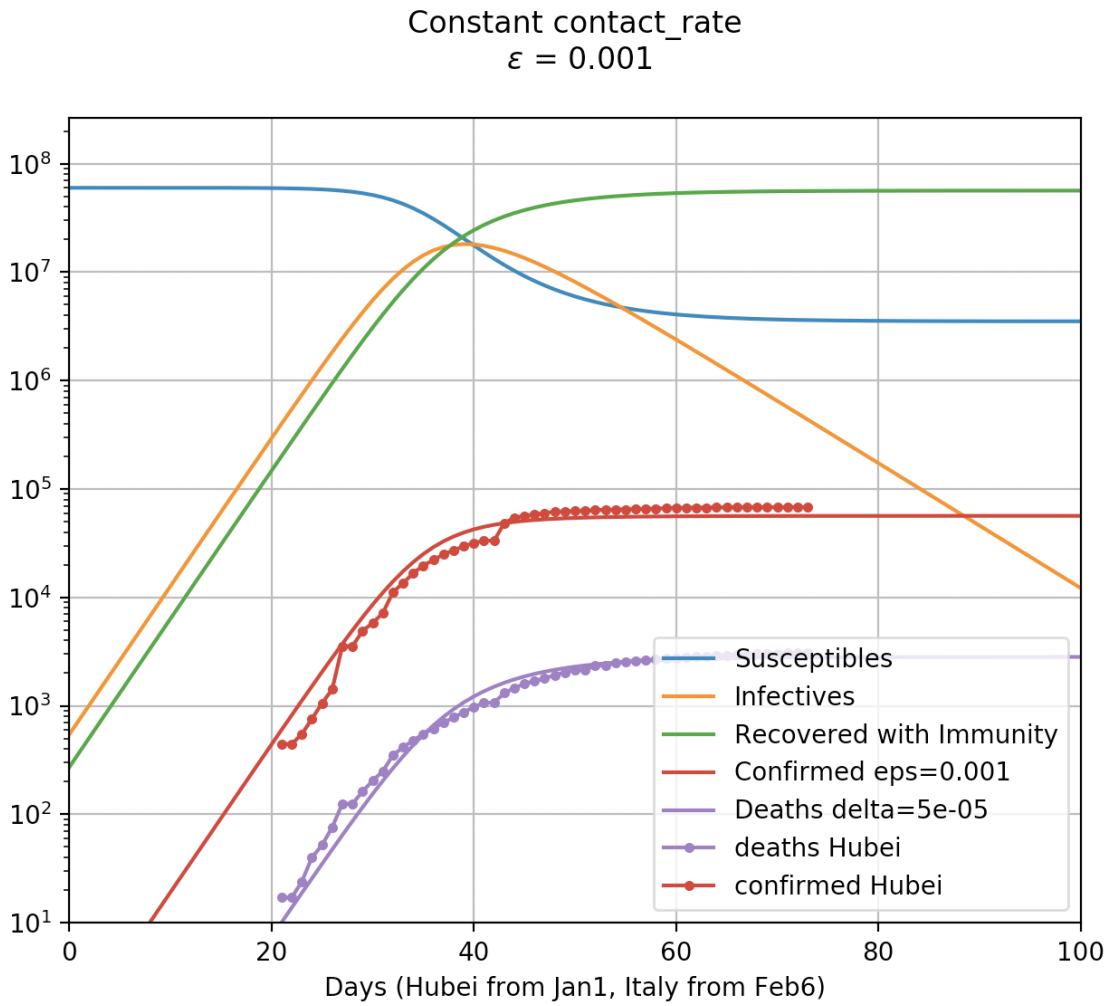

Fig 2 COVID-19 Hubei data fitted with a SIRCD model with very small detectability. $S(0) = 60 \cdot 10^6$, $\beta(t) = 0.48$, $\gamma(t) = 0.16$, $\epsilon = 10^{-3}$, $\delta = 5 \cdot 10^{-5}$

the case of Figure 3.

Obviously, several combination of R_0 and ϵ/δ parameters are possible in between the 2 considered example cases. So, we conclude that the number of "Recovered with Immunity" in Hubei at the end of the rarefaction period in Mar 2020 cannot be determined without an independent assessment of the detectability parameter ϵ.

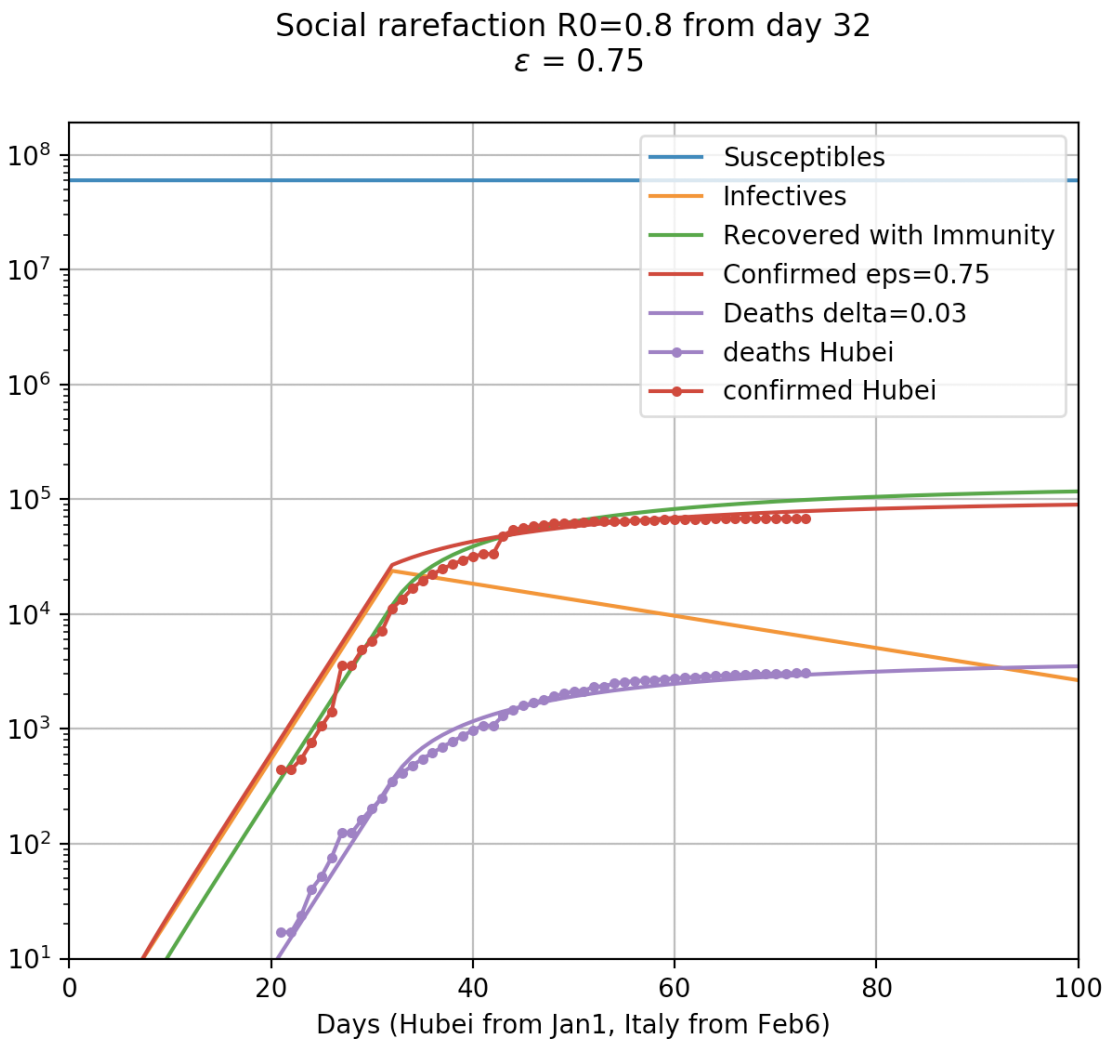

Fig 3 COVID-19 Hubei data fitted with a SIRCD model with social rarefaction. $S(0) = 60 \cdot 10^6$, $\gamma(t) = 0.16$, $\epsilon = 0.75$, $\delta = 0.03$. The contact rate is modified from $\beta(t < 32) = 0.48$ to $\beta(t > 32) = 0.128$