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CMSC25025 ProblemSet 1

1. a) Density of Y: $Y = F(X) = F(F^{-1}(u)) = u$ where u is uniform.

So, density of Y is 1.

$F^{-1}(u) = X \Rightarrow \mathbb{P}(F^{-1}(u) \leq x) = \mathbb{P}(u \leq F(x)) = F(x) \Rightarrow F^{-1}(u) \sim F$

So, $X \sim F$.

b) $X, Y \sim \text{Uniform}(0,1)$ independent.

i) $Z = X - Y$, $-1 \leq z \leq 1$

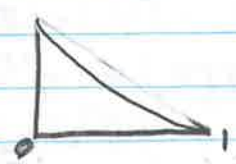
$$\mathbb{P}(Z \leq z) = \mathbb{P}(X - Y \leq z)$$

$$= \begin{cases} \int_0^{1+z} \int_{x-z}^{1} 1 \, dy \, dx & -1 \leq z < 0 \\ 1 - \int_z^1 \int_0^{x-z} 1 \, dy \, dx & 0 \leq z \leq 1 \end{cases}$$

$$\begin{cases} = \int_0^{1+z} 1 - x + z \, dx = x - \frac{x^2}{2} + zx \Big|_0^{1+z} = 1 + z - \frac{(1+z)^2}{2} + z + z^2 = \frac{z^2}{2} + z + \frac{1}{2} \\ = 1 - \int_z^1 \int_0^{x-z} 1 \, dy \, dx = 1 - \int_z^1 x - z \, dx = 1 - \left(\frac{x^2}{2} - zx\right)\Big|_z^1 = 1 - \left(\frac{1}{2} - z\right) + \left(\frac{z^2}{2} - z^2\right) \\ \quad = \frac{1}{2} + z - \frac{z^2}{2} \end{cases}$$

PDF:

$$\mathbb{P}(z) = \begin{cases} z + 1 & -1 \leq z < 0 \\ 1 - z & 0 \leq z \leq 1 \\ 0 & \text{ow} \end{cases}$$



ii) $Z = \min(X, Y)$, $0 \leq z \leq 1$.

$$\mathbb{P}(Z \leq z) = \mathbb{P}(\min(X,Y) \leq z) = 1 - \mathbb{P}(X > z, Y > z)$$
$$= 1 - \mathbb{P}(X > z)\,\mathbb{P}(Y > z) \text{ by iid}$$
$$= 1 - \left(\int_z^1 1\,dx\right)\left(\int_z^1 1\,dy\right)$$
$$= 1 - (1 - z)^2$$
$$= 1 - 1 + 2z - z^2$$
$$= 2z - z^2$$

PDF is: $\mathbb{P}(z) = \begin{cases} 2 - 2z & \text{if } 0 \leq z \leq 1 \\ 0 & \text{ow.} \end{cases}$