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Problem Set 4

## Problem 1

a) We have (1): $\ell = \sum_{w \in V} \sum_{c \in V} \#(w,c)\left[\log(\sigma(v_w^T v_c)) + k \cdot \mathbb{E}_{c_n}\left[\log(\sigma(-v_w^T v_{c_n}))\right]\right]$

So with $P(c) = \frac{\# c}{|D|}$:

$\ell = \sum_{w \in V} \sum_{c \in V} \#(w,c)\log(\sigma(v_w^T v_c)) + \sum_{w \in V} \sum_{c \in V} \#(w,c) \cdot k \cdot \mathbb{E}_{c_n}\left[\log(\sigma(-v_w^T v_c))\right]$

$\Rightarrow \sum_{w \in V} \sum_{c \in V} \#(w,c)\log(\sigma(v_w^T v_c)) + \sum_{w \in V} \sum_{c \in V} \#(w,c) \cdot k \cdot \left(\sum_{c_n \in V} \log(\sigma(-v_w^T v_c)) \cdot \frac{\# c}{|D|}\right)$

By $\sum_{c \in V} \#(w,c) = \#(w)$:

$\ell = \sum_{w \in V} \sum_{c \in V} \#(w,c)\log(\sigma(v_w^T v_c)) + \sum_{w \in V} k \cdot \#(w) \cdot \left[\frac{\#(\hat{c})}{|D|}\log(\sigma(-v_w^T v_{\hat{c}})) + \sum_{c_n \in V - \{\hat{c}\}} \frac{\# c_n}{|D|} \log(\sigma(-v_w^T v_{c_n}))\right]$

So, if we limit the set V to the pair $(w,c)$:

$\ell(w,c) = \#(w,c)\log(\sigma(v_w^T v_c)) + k\,\#(w)\frac{\#(c)}{|D|}\log(\sigma(-v_w^T v_c))$

b) Using $x = v_w^T v_c$ we have:

$\ell(w,c) = \#(w,c)\log(\sigma(x)) + k\,\#(w)\frac{\#(c)}{|D|}\log(\sigma(-x))$

with $\sigma(x) \; \frac{1}{1+e^{-x}}$

So, $\frac{\partial \ell(w,c)}{\partial x} = \#(w,c)\frac{1}{\sigma(x)}\frac{\partial \sigma(x)}{\partial x} + k\,\#(w)\frac{\#(c)}{|D|}\frac{1}{\sigma(-x)}\frac{\partial \sigma(-x)}{\partial x} \cdot \frac{\partial(-x)}{\partial x} = 0$

$\Rightarrow \#(w,c)\cancel{(1+e^{-x})}\left[\frac{e^{-x}}{(1+e^{-x})^2}\right] - k\,\#(w)\frac{\#(c)}{|D|}\cancel{(1+e^{x})}\left[\frac{e^{x}}{(1+e^{x})^2}\right] = 0$

$\Rightarrow \#(w,c)\frac{e^{-x}}{1+e^{-x}} = k\,\#(w)\frac{\#(c)}{|D|}\frac{e^{x}}{1+e^{x}}$

$\Rightarrow \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}}(1+e^{x}) = e^{2x}(1+\frac{1}{e^{x}})$

$\Rightarrow \frac{-\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}}(1+e^{x}) = e^{2x} + e^{x}$

$\Rightarrow e^{2x} - \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}}e^{x} + e^{x} - \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} = 0$

$\Rightarrow e^{2x} - \left(\frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} - 1\right)e^{x} - \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}}$

c) We let $y = e^x$

So $e^{2x} - \left( \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} - 1 \right) e^x - \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} = 0$

$\Rightarrow y^2 - \left( \frac{\#(w,c)}{k\#(w)\frac{\#c}{|D|}} - 1 \right) y - \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} = 0$

Let $w = \frac{\#(w,c)}{k\#(w)\frac{\#c}{|D|}}$

So, $y = \left( (w-1) \pm \sqrt{(-w+1)^2 - 4(-w)} \right) / 2$

$= \frac{w-1 \pm \sqrt{w^2 - 2w + 4w + 1}}{2} \Rightarrow \frac{(w-1) \pm \sqrt{w^2 + 2w + 1}}{2}$

$= \frac{w-1 \pm w+1}{2}$

$y = w$ or $y = -1$

So, $e^x = \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}}$ or $e^x = -1$

For the former:

$e^{v_w^T v_c} = \frac{\#(w,c)}{k\#(w)\frac{\#(c)}{|D|}} \Rightarrow v_w^T v_c = \log\left[ \frac{\#(w,c)|D|}{k\#(w)\#(c)} \right]$ by log tranformation

So:

$v_w^T v_c = \log\left( \frac{\#(w,c)|D|}{\#(w)\#(c)} \right) - \log k$