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CMSC 25025
Assignment 6

Problem 1

Reference:

Smoothing Kernel

$$\hat{m}_h(x) = \frac{\sum_{i=1}^n Y_i K_h(X_i, x)}{\sum_{i=1}^n K_h(X_i, x)}$$

for some kernel $K_h(x, z)$

Mercer Kernel

$$\hat{m}(x) = \sum_{i=1}^n \hat{\alpha}_i K(X_i, x)$$

$$\hat{\alpha}_i = (\mathbb{K} + \lambda I)^{-1} Y_i$$

$$\mathbb{K}_{ij} = K(X_i, X_j)$$

a) We wish to show that $\hat{m}_n(X) = LY$ for $\hat{m}_n(X) = \begin{bmatrix} \hat{m}_n(X_1) \\ \vdots \\ \hat{m}_n(X_n) \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $L \in \mathbb{R}^{n \times n}$

Smoothing Kernel case

$$\hat{m}_n(X) = \begin{bmatrix} \hat{m}_h(X_1) \\ \vdots \\ m_h(X_n) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n \frac{Y_i K_h(X_i, X_1)}{\sum_{i=1}^n K_h(X_i, X_1)} \\ \vdots \\ \sum_{i=1}^n \frac{Y_i K_h(X_i, X_n)}{\sum_{i=1}^n K_h(X_i, X_n)} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{K_h(X_1, X_1)}{\sum_{i=1}^n K_h(X_i, X_1)} & \cdots & \frac{K_h(X_n, X_1)}{\sum_{i=1}^n K_h(X_i, X_1)} \\ \vdots & & \\ \frac{K_h(X_1, X_n)}{\sum_{i=1}^n K_h(X_i, X_n)} & \cdots & \frac{K_h(X_n, X_n)}{\sum_{i=1}^n K_h(X_i, X_n)} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}}_{Y}$$

Mercer Kernel Case

$$\hat{m}_n(x) = \begin{bmatrix} \hat{m}_n(x_1) \\ \vdots \\ \hat{m}_n(x_n) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n e_i^T[(\mathbb{K}+\lambda I)^{-1}Y]\, K(x_i, x_1) \\ \vdots \\ \sum_{i=1}^n e_i^T[(\mathbb{K}+\lambda I)^{-1}Y]\, K(x_i, x_n) \end{bmatrix}$$

where e_i is a standard basis vector

$$= \underbrace{\begin{bmatrix} e_1^T(\mathbb{K}+\lambda I)^{-1}K(x_1,x_1) & \cdots & e_n^T(\mathbb{K}+\lambda I)^{-1}K(x_n,x_1) \\ \vdots & \ddots & \vdots \\ e_1^T(\mathbb{K}+\lambda I)^{-1}K(x_1,x_n) & \cdots & e_n^T(\mathbb{K}+\lambda I)^{-1}K(x_n,x_n) \end{bmatrix}}_{= L} \underbrace{\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}}_{= Y}$$

b) $$\hat{R}(h) = \frac{1}{n}\sum_{i=1}^n \left(\frac{Y_i - \hat{m}_n(x_i)}{1 - L_{ii}} \right)^2 \qquad (1)$$

We wish to show that we can deduce (1) from $\hat{R}(h) = \frac{1}{n}\sum_{i=1}^n (Y_i - \hat{Y}_{(i)})^2$ for $\hat{Y}_{(i)} = \hat{m}_{n,(i)}(x_i)$. Let $1 < i < n$.

Then, $$\hat{m}_{n,(i)}(x) = \begin{bmatrix} \frac{K(x_1,x_1)}{\sum_{j\neq i} K(x_j,x_1)} & \cdots & \frac{K(x_n,x_1)}{\sum_{j\neq i} K(x_j,x_1)} \\ \vdots & \ddots & \vdots \\ \frac{K(x_1,x_n)}{\sum_{j\neq i} K(x_j,x_n)} & \cdots & \frac{K(x_n,x_n)}{\sum_{j\neq i} K(x_j,x_n)} \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

$$\hat{m}_{n,(i)}(x_i) = \frac{\sum_{j\neq i} Y_j K(x_j, x_i)}{\sum_{j\neq i} K(x_j, x_i)} \qquad \Bigg| \qquad \hat{m}_n(x_i) = \frac{\sum_j Y_j K(x_j, x_i)}{\sum_j K(x_j, x_i)}$$

$$m_{n,(i)}(x_i) - m_n(x_i) = \frac{\sum_{j\neq i} Y_j K(x_j, x_i)}{\sum_{j\neq i} K(x_j, x_i)} - \frac{\sum_j Y_j K(x_j, x_i)}{\sum_j K(x_j, x_i)}$$

$$= \frac{\sum_{j\neq i} Y_j K(x_j, x_i)}{\sum_{j\neq i} K(x_j, x_i)} - \frac{\sum_{j\neq i} K(x_j, x_i) \cancel{\sum K(x_j, x_i)}}{\sum_j K(x_j, x_i) \sum_{j\neq i} K(x_j, x_i)} \frac{\sum_j Y_j K(x_j, x_i)}{\cancel{\sum K(x_j, x_i)}}$$

$$= \left(\frac{\sum_{j\neq i} Y_j K(x_j, x_i)}{\sum_{j\neq i} K(x_j, x_i)} - \left(\frac{\sum_{j\neq i} K(x_j, x_i)}{\sum_j K(x_j, x_i)} \right) \frac{\sum_j Y_j K(x_j, x_i)}{\sum_{j\neq i} K(x_j, x_i)} \right)$$

$$= \frac{1}{\sum_{j\neq i} K(x_j, x_i)} \left[\sum_{j\neq i} Y_j K(x_j, x_i) - \left[1 - \frac{K(x_i, x_i)}{\sum_j K(x_j, x_i)} \right] \sum_j Y_j K(x_j, x_i) \right]$$

$$= \frac{1}{\sum_{j\neq i} K(x_j, x_i)} \left[\sum_{j\neq i} Y_j K(x_j, x_i) - \left[1 - L_{ii} \right] \sum_j Y_j K(x_j, x_i) \right]$$

So we can rewrite

$$\hat{Y}_i = \hat{m}_n(x_i) = [1 - L_{ii}] \frac{\sum_j Y_j K(x_j, x_i)}{\sum_{j\neq i} K(x_j, x_i)} = [1 - L_{ii}] \left(\hat{Y}_{(i)} + \frac{Y_i K(x_i, x_i)}{\sum_{j\neq i} K(x_j, x_i)} \right)$$

$$\Rightarrow \frac{\hat{Y}_i}{1 - L_{ii}} - \frac{Y_i L_{ii}}{1 - L_{ii}} = \hat{Y}_{(i)}$$

So, $\Rightarrow$

$$\hat{R}(h) = \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \hat{Y}_{(i)} \right)^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{Y_i (1 - L_{ii})}{1 - L_{ii}} - \frac{\hat{Y}_i}{1 - L_{ii}} + \frac{Y_i L_{ii}}{1 - L_{ii}} \right)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{Y_i - \hat{Y}_i}{1 - L_{ii}} \right) = \hat{R}(h)$$