MTH 101: Calculus I

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Problem Set for Tutorial 2 on January 11, 2025 (combined)

Problems (Implications)

1. Write the converse of the following implication:
   "If it is a holiday, then I will not go to the class".
2. Write the following statement as a conjunction of two implications:
   "$|x| < a$ if and only if $x \in (-a, a)$".
3. Write the negation of the following implication:
   "For an integer n, if n^2 is divisible by 10, then n is divisible by 10".
4. Write the contrapositive of the following implication:
   "For an integer x, if $x^2 - 6x + 5$ is even, then x is odd".

Answers:

1. If I will not go to the class then it is a holiday.
2. If $|x| < a$ then $x \in (-a, a)$ <u>and</u> if $x \in (-a, a)$ then $|x| < a$.
3. There is an integer n for which n^2 is divisible by 10 and n is not divisible by 10.
4. For an integer x, if x is even, then $x^2 - 6x + 5$ is odd.

5. Write the contrapositive of the following statement:
S: "For an integer n, if $n^3 - 1$ is even, then n is odd".

Prove S using its contrapositive.

Proof:

- The contrapositive of S is: For an integer n, if n is even, then $n^3 - 1$ is odd.
- Let us prove the contrapositive.
- If n is even, then $n = 2k$, for some integer k.
- $n^3 - 1 = 8k^3 - 1$, which is odd.

$\square$

6. Write the negation of the following statement:
   S: "There is no rational number x such that $x^2 = 2$".

   Prove S by contradiction.

Proof:

- The negation of S is: There is a rational number x such that $x^2 = 2$.
- First, $x \neq 0$. Since x is a rational number, $x = p/q$, for two nonzero integers p and q with no common factors.
- Substitute it in $x^2 = 2$ to get $p^2 = 2q^2$.
- Arrive at a contradiction.
- Thus 2 divides p^2. Hence p^2 is even. Since square of an odd number is odd, p itself has to be an even number (otherwise its square p^2 is odd).
- We now have 2 divides p, thus $p = 2k$.
  Hence LHS $p^2 = 4k^2$.
  Thus $4k^2 = 2q^2$ which implies $2k^2 = q^2$.
  Therefore 2 divides q.
  Thus 2 is a common factor of both p and q, which contradicts our assumption about p and q.

$\square$

7. Give a direct proof that the following statements are equivalent for two given sets A and B.
(i) $A \subseteq B$.
(ii) $A \cap B = A$.
(iii) $A \cup B = B$.

Prove $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$.

Solution:
$(i) \Rightarrow (ii)$: Let $A \subseteq B$. To show: (ii), let $x \in A$, then $x \in B$. Hence $x \in A \cap B$. Thus $A \subseteq A \cap B \subseteq A$.

8. Give a counterexample to disprove the following statement:
"For real numbers x and y, $|x| > |y|$ if $x > y$".

Solution: Take $x = -1$, $y = -2$. Then $x > y$ but $|x| < |y|$. Thus the pair $x = -1$, $y = -2$ is a counterexample.

9. Prove that $(0,1)$ and $\mathbb{R}$ have the same cardinality.
Solution: Since $(0,1)$ and $(-1,1)$ have the same cardinality, and $(-1,1)$ and $\mathbb{R}$ have the same cardinality, $(0,1)$ and $\mathbb{R}$ have the same cardinality.

10. Prove that if X, Y are countably infinite sets, then the set $X \cup Y$ is also a countably infinite set.
Solution:
Case 1: $X \cap Y = \emptyset$.
There are bijections $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$.
Define $h : \mathbb{N} \to X \cup Y$ as

$$h(1) = f(1), h(3) = f(2), h(5) = f(3), \ldots, h(2n+1) = f(n), n \geq 0$$
$$h(2) = g(1), h(4) = g(2), h(6) = g(3), \ldots, h(2n) = g(n), n \geq 1.$$

Then h is a bijection.
Case 2: $X \cap Y \neq \emptyset$. (Not discussed in the tutorial)
Write $X \cup Y = (X \cap Y^c) \cup (X \cap Y) \cap (Y \cap X^c)$. Since $X \cap Y^c$ and $X \cap Y$ are subsets of X, they is countable (Why?).
Since $Y \cap X^c$ is a subset of Y, it is countable.
Thus $X \cup Y$ is a disjoint union of three sets. Thus it is countable (by Case 1). It is not finite since it contains a countably infinite set X. $\square$

11. Prove that the set of rational numbers, $\mathbb{Q}$, is countably infinite.

Proof:

$$\begin{array}{ccccccc}
\boxed{\frac{1}{1}} & \boxed{\frac{1}{2}} & \boxed{\frac{1}{3}} & \boxed{\frac{1}{4}} & \boxed{\frac{1}{5}} & \boxed{\frac{1}{6}} & \cdots \\
\boxed{\frac{2}{1}} & \frac{2}{2} & \boxed{\frac{2}{3}} & \frac{2}{4} & \boxed{\frac{2}{5}} & \frac{2}{6} & \cdots \\
\boxed{\frac{3}{1}} & \boxed{\frac{3}{2}} & \frac{3}{3} & \boxed{\frac{3}{4}} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\boxed{\frac{4}{1}} & \frac{4}{2} & \boxed{\frac{4}{3}} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\boxed{\frac{5}{1}} & \boxed{\frac{5}{2}} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Problems (Sets)

Use the Pigeonhole Principle to prove the following statements.

14. Among any 13 people, at least two share a birth month.

15. Suppose $a_1, \ldots, a_n$ are integers. Then some "consecutive sum" $a_k + a_{k+1} + a_{k+2} + \cdots + a_{k+m}$ is divisible by n.

    Proof: Consider

$$\begin{aligned} b_1 &= a_1 \\ b_2 &= a_1 + a_2 \\ \cdots \\ b_n &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

Divide each of $b_1, b_2, \ldots, b_n$ by n.
If one of the remainders $r_1, r_2, \ldots, r_n$ is 0, we are done.
Otherwise, if all the remainders are non-zero, the possible values are $1, 2, \ldots, n-1$.
Since there are n remainders and $n-1$ possible values, for some $i > j$, $r_i = r_j$.
Hence $b_i - b_j = a_i + a_{i-1} + \cdots + a_{j+1}$ is divisible by n. □

Problems (Sets)

This topic are yet to be covered in the class.

12. Prove that the power set of $\mathbb{N}$, denoted as $\mathcal{P}(\mathbb{N})$, is uncountable. Moreover, $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.

13. Prove that if X is an uncountable set, then $|\mathbb{N}| < |X|$.