Assignment
$Sol^{ns.}$

[PHY - 106]

Assignment - 1:

1. Given, $\psi(x,t) = f_1(ct-x) + f_2(ct+x)$

Let, $u = ct - x$, $v = ct + x$ } Light cone coordinates

1D wave eqn.: $\dfrac{\partial^2 \psi}{\partial t^2} = c^2 \dfrac{\partial^2 \psi}{\partial x^2}$

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\left(f_1(u) + f_2(v)\right) = \frac{df_1}{du}\cdot\underbrace{\frac{\partial u}{\partial t}}_{=c} + \frac{df_2}{dv}\cdot\underbrace{\frac{\partial v}{\partial t}}_{=c}$$

$$= c\left(\frac{df_1}{du} + \frac{df_2}{dv}\right)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}\right) = c\left[\frac{d^2 f_1}{du^2}\cdot\underbrace{\frac{\partial u}{\partial t}}_{=c} + \frac{d^2 f_2}{dv^2}\cdot\underbrace{\frac{\partial v}{\partial t}}_{=c}\right]$$

Chain Rule !!!

$$= c^2\left(\frac{d^2 f_1}{du^2} + \frac{d^2 f_2}{dv^2}\right). \quad \text{———} \quad (1)$$

Similarly,

$$\frac{\partial \psi}{\partial x} = \frac{df_1}{du}\underbrace{\frac{\partial u}{\partial x}}_{=+1} + \frac{df_2}{dv}\underbrace{\frac{\partial v}{\partial x}}_{=-1} = \frac{df_1}{du} - \frac{df_2}{dv}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{d^2 f_1}{du^2}\underbrace{\frac{\partial u}{\partial x}}_{=+1} - \frac{d^2 f_2}{dv^2}\underbrace{\frac{\partial v}{\partial x}}_{=-1} = \frac{d^2 f_1}{du^2} + \frac{d^2 f_2}{dv^2} \quad \text{———} \quad (2)$$

From (1) & (2) it is evident that the given funcn. $\psi(x,t)$ satisfies the 1D wave eqn.

2. $\dfrac{\partial^2 \psi}{\partial t^2} = c^2 \dfrac{\partial^2 \psi}{\partial x^2}$ (Given)

Let, $\psi(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)$ [Traveling wave Ansatz!]

Check: $\dfrac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$; $\dfrac{\partial^2 \psi}{\partial x^2} = -k^2 \psi$ where, $c = \dfrac{\omega}{k}$.

$$\Rightarrow \omega^2 = c^2 k^2 \Rightarrow c = \frac{\omega}{k}$$

2. Given, $\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$

Let, $\psi(x,t) = A \sin(kx - \omega t) + B \cos(kx - \omega t)$ [Traveling wave ansatz]

Check: $\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi$, $\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$

$\Rightarrow \underbrace{\omega^2 = c^2 k^2}_{\text{From wave eq}^n.} \Rightarrow c = \frac{\omega}{k}$ (Since, we assumed a traveling wave in +ve x-direction!)

Now, $\psi(x,0) = A \sin kx + B \cos kx = g(x)$. (given initial cond^n.)

and, $\dot{\psi}(x,t) = -\omega\left[A\cos(kx-\omega t) - B \sin(kx - \omega t)\right]$

$\dot{\psi}(x,0) = -\omega(A\cos kx - B \sin kx) = 0$ (given initial cond^n.)

$\Rightarrow -A\cos kx + B \sin kx = 0.$

Thus, we have, $\begin{pmatrix} \sin & \cos \\ -\cos & \sin \end{pmatrix}\begin{pmatrix} \sin & -\cos \\ \cos & \sin \end{pmatrix} = \begin{pmatrix} \sin^2 \end{pmatrix}$

(1) $A \sin kx + B \cos kx = g(x)$.

(2) $-A \cos kx + B \sin kx = 0$.

$\xRightarrow{\text{Matrix Form}} \underbrace{\begin{pmatrix} \sin kx & \cos kx \\ -\cos kx & \sin kx \end{pmatrix}}_{= M \text{ (say)}} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} g(x) \\ 0 \end{pmatrix}$

[$\det M = \sin^2 kx + \cos^2 kx = 1 \Rightarrow M^{-1}$ exists]

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sin kx & -\cos kx \\ \cos kx & \sin kx \end{pmatrix}\begin{pmatrix} g(x) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} g(x) \sin kx \\ g(x) \cos kx \end{pmatrix} \Rightarrow A = g(x) \sin kx \; ; \; B = g(x) \cos kx$$

$\therefore \psi(x,t) = g(x)\left[\sin kx \cdot \sin(kx - \omega t) + \cos kx \cdot \cos(kx - \omega t)\right]$

($\sim \sin\theta \sin\varphi + \cos\theta\cos\varphi = \cos(\varphi - \theta)$; $\theta = kx$, $\varphi = kx - \omega t$)

[But this turns out to be a standing wave!] $= g(x) \cos \omega t$.

(OR)

2. 1D Wave Eqn. : $\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$

Let, $\psi(x,t) = a \cdot f_1(x-ct) + b \cdot f_2(x+ct)$ Let, $u = x - ct$, $v = x + ct$

$\psi(x,0) = g(x) \Rightarrow a f_1(x) + b f_2(x) = g(x)$. $(a, b \rightarrow \text{consts.})$

——— (1)

$\dot{\psi}(x,t) = c\left[-a \frac{df_1}{dv} + b \frac{df_2}{du}\right]$

$\Rightarrow \dot{\psi}(x,0) = c\left[-a f_1'(x) + b f_2'(x)\right] = 0$. (at $t=0$, $u = x = v$)

$\Rightarrow a f_1'(x) = b f_2'(x)$. (where, $f'(x) = \frac{df(x)}{dx}$)

$\Rightarrow a f_1(x) = b f_2(x) + B$ ($B \rightarrow$ const. of integration!)

——— (2)

From (1) & (2) we have,

$2 b f_2(x) + B = g(x)$

$\Rightarrow b \cdot f_2(x) = \frac{1}{2}(g(x) - B)$ ——— (3)

Putting this back in (1) we have,

$a f_1(x) = \frac{1}{2}(g(x) + B)$ ——— (4)

$\therefore \psi(x,t) = a f_1(x-ct) + b f_2(x+ct)$

$= \frac{1}{2}\left[g(x-ct) + B + g(x+ct) - B\right]$ (From (3) & (4))

$= \frac{1}{2}\left[g(x-ct) + g(x+ct)\right]$. (~ Still this is a standing wave ?)

3. Given, $c_p = A\sqrt{\lambda}$; Now, $c_p = \frac{\omega}{k} \Rightarrow \frac{\omega}{k} = A\sqrt{\lambda} \Rightarrow \omega = A\sqrt{\lambda}\,.\,k$

$\therefore \quad c_g = \frac{d\omega}{dk} \qquad \Rightarrow \boxed{\omega(\lambda) = \frac{2\pi A}{\sqrt{\lambda}}} \qquad \left[\because k = \frac{2\pi}{\lambda}\right]$

$$= \frac{d\omega}{d\lambda}\cdot\frac{d\lambda}{dk}$$

$$\left[\because \frac{d\omega(\lambda)}{d\lambda} = -\frac{1}{2}\cdot\frac{2\pi A}{\lambda^{3/2}} \; ; \; \lambda = \frac{2\pi}{k}\right.$$

$$= -\frac{1}{2}\cdot\frac{\cancel{2\pi}A}{\lambda^{3/2}} \times \left(\frac{-\lambda^2}{\cancel{2\pi}}\right)$$

$$\Rightarrow \frac{d\lambda}{dk} = \frac{-2\pi}{k^2} = -\frac{\lambda^2}{2\pi}$$

$$= \frac{1}{2}\cdot A\sqrt{\lambda} = \frac{c_p}{2} \quad (\underline{\underline{Pd}}.) \qquad \left.\left(\text{as, } k = \frac{2\pi}{\lambda}\right)\right].$$

4. $c^2 = \frac{g\lambda}{2\pi} + \frac{2\pi S}{\rho\lambda} \Rightarrow \underset{\omega/k}{\underset{\parallel}{c}} = \left(\frac{g\lambda}{2\pi} + \frac{2\pi S}{\rho\lambda}\right)^{1/2} = \left(\frac{g}{k} + \frac{kS}{\rho}\right)^{1/2}$

$\boxed{\because k = \frac{2\pi}{\lambda}}$

$\therefore \quad \boxed{\omega(k) = k\left(\frac{g}{k} + \frac{kS}{\rho}\right)^{1/2}.}$

a) $$\frac{d\omega}{dk} = \left(\frac{g}{k} + \frac{kS}{\rho}\right)^{1/2} + k\cdot\frac{1}{2}\cdot\frac{\left[-g/k^2 + S/\rho\right]}{\left(\frac{g}{k} + \frac{kS}{\rho}\right)^{1/2}}$$

$$= \frac{\frac{g}{k} + \frac{kS}{\rho} + \frac{k}{2}\left(-\frac{g}{k^2} + \frac{S}{\rho}\right)}{\left(\frac{g}{k} + \frac{kS}{\rho}\right)^{1/2}} = \frac{\frac{g}{2k} + \frac{3}{2}\frac{kS}{\rho}}{\left(\frac{g}{k} + \frac{kS}{\rho}\right)^{1/2}}$$

Now,

$$\left.\frac{d\omega}{dk}\right|_{k=k_0} = \left.c\right|_{k=k_0} = \left(\frac{g}{k_0} + \frac{k_0 S}{\rho}\right)^{1/2} \Bigg\} \text{ For no dispersion !!!}$$

(where, $k_0 = \frac{2\pi}{\lambda_0}$)

$$\Rightarrow \cancel{\left(\frac{g}{k_0} + \frac{k_0 S}{\rho}\right)} + \frac{k_0}{2}\left(-\frac{g}{k_0^2} + \frac{S}{\rho}\right) = \cancel{\frac{g}{k_0} + \frac{k_0 S}{\rho}}$$

$$\Rightarrow \frac{S}{\rho} = \frac{g}{k_0^2} \Rightarrow \boxed{k_0 = \sqrt{\frac{g\rho}{S}}}. \quad \text{OR} \quad \boxed{\lambda_0 = \frac{2\pi}{k_0} = 2\pi\sqrt{\frac{S}{g\rho}}.}$$

$$\therefore \left.c_g\right|_{k=k_0} = \left(\frac{g}{k_0} + \frac{k_0 S}{\rho}\right)^{1/2}$$

4. b) if $\lambda \ll \lambda_0 \Rightarrow k \gg k_0$

$$\therefore \; c\Big|_{k \gg k_0} = \sqrt{\frac{kS}{\rho}} \quad \text{and,} \quad c_g\Big|_{k \gg k_0} = \frac{\frac{3}{2}\frac{kS}{\rho}}{\sqrt{\frac{kS}{\rho}}} = \frac{3}{2}\sqrt{\frac{kS}{\rho}}$$

$$\therefore \quad \frac{c_g}{c} = \frac{3}{2} \Rightarrow c_g = \frac{3c}{2}.$$

4. c) if $\lambda \gg \lambda_0 \Rightarrow k \ll k_0$ then,

$$c\Big|_{k \ll k_0} = \sqrt{\frac{g}{k}} \quad \text{and,} \quad c_g\Big|_{k \ll k_0} = \frac{\frac{1}{2}\frac{g}{k}}{\sqrt{\frac{g}{k}}} = \frac{1}{2}\sqrt{\frac{g}{k}}.$$

$$\therefore \quad \frac{c_g}{c} = \frac{1}{2} \Rightarrow c_g = \frac{c}{2}.$$

* Alternatively, [Part (a) Q.4]

$$\lambda = \frac{2\pi}{k} \Rightarrow \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

Now, $$c_g = \frac{d}{dk}(\omega = c_p k) = c_p + k\frac{dc_p}{dk} = c_p + k\frac{dc_p}{d\lambda}\cancel{\frac{d\lambda}{dk}} = -\frac{2\pi}{k^2}$$

$$= c_p - \frac{2\pi}{k}\frac{dc_p}{d\lambda} = c_p - \lambda\frac{dc_p}{d\lambda}$$

Non-dispersive $\Rightarrow c_g = c_p \Rightarrow \boxed{\frac{dc_p}{d\lambda} = 0.}$

$\Rightarrow$ We get g.s of this wave eqn by Saperation of Variable method

$$\psi(x,t) = A_1 e^{i(\omega t + kx)} + A_2 e^{-i(\omega t + kx)} + A_3 e^{-i(\omega t - kx)} + A_4 e^{i(\omega t - kx)}$$

[As per class of 8 jan]

$\Rightarrow \psi(x,0) = g(x)$

$$\psi(x,0) = A e^{ikx} + A_2 e^{-ikx} + A_3 e^{ikx} + A_4 e^{-ikx} \quad \text{——— (1)}$$

Now $\dot{\psi}(x,0) = 0$

$$\dot{\psi}(x,0) = A_1 e^{ikx} - A_2 e^{-ikx} - A_3 e^{ikx} + A_4 e^{-ikx} = 0 \quad \text{——— (2)}$$

$\Rightarrow$ (1) + (2)

$$2A_1 e^{ikx} + 2A_4 e^{-ikx} = g(x)$$

$$2A_1(\cos kx + i\sin kx) + 2A_4(\cos kx - i\sin kx) = g(x)$$

$\Rightarrow$ Equating real & imaginary part of LHS & RHS we get,

$$2A_1 \cos kx + 2A_4 \cos kx = g(x) \quad \text{——— (3)}$$

$$2A_1 \sin kx - 2A_4 \sin kx = 0 \quad \text{——— (4)}$$

$$2(A_1 - A_4)\sin kx = 0$$

So $\boxed{A_1 = A_4}$ ——— (5)

from eqn (3)

$$\boxed{4A_1 \cos kx = g(x)}$$

$e^{ikx} = \cos kx + i\sin kx$
$e^{-ikx} = \cos kx - i\sin kx$

from eqn (3)

$$(A_1 - A_3)\, e^{ikx} - (A_2 - A_4)\, e^{-ikx} = 0$$

$$(A_1 - A_3)(\cos kx + i\sin kx) - (A_2 - A_4)(\cos kx - i\sin kx) = 0$$

$$(A_1 - A_3)\cos kx - (A_2 - A_4)\cos kx = 0$$

$$(A_1 - A_3)\sin kx + (A_2 - A_4)\cos kx = 0$$

so $A_1 - A_3 - A_2 + A_4 = 0$ ——— (6)

$A_1 - A_3 + A_2 - A_4 = 0$ ——— (7)

$\Rightarrow$ so from eqn (5)

$$\cancel{A_1} - A_3 + A_2 - \cancel{A_1} = 0$$

$\boxed{A_3 = A_2}$ ——— (8)

$\Rightarrow$ put it in eqn (6)

$$A_1 - A_3 - A_3 + A_1 = 0$$

$\boxed{A_1 = A_3}$ $\longrightarrow$ (9)

$\Rightarrow$ So $A_1 = A_2 = A_3 = A_4$

$$\Rightarrow \Psi(x,t) = A_1 \left[e^{i(\omega t + kx)} + e^{-i(\omega t + kx)} + e^{-i(\omega t - kx)} + e^{i(\omega t - kx)} \right]$$

$$= 2A_1 \cos(\omega t + kx) + 2A_1 \cos(\omega t - kx)$$

$$\psi(x,t) = \cancel{4A_1 \cos}\ 4A_1 \left[\cos\omega t \ \cos kx \right]$$

$$= \underbrace{4A_1 \cos kx}_{g(x)} \cos\omega t$$

$$= g(x) \cos\omega t$$

$$\boxed{\psi(x,t) = \psi(x,0) \cos\omega t} \longrightarrow \text{solution}$$