MTH 101: Calculus I

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Implications and their converse

- For two statements S and T, a statement of the form

  "If S then T"

  is called an implication or a conditional statement.

- It says that *if S is true then T is true*.

- The implication obtained by interchanging S and T is:

  "If T then S"

  is called the converse of "If S then T".

Example

- The converse of the implication
  "If the hall is big, then I will use a microphone"
  is
  "If I will use a microphone, then the hall is big".

Bi-implications

- A conjunction of an implication and its converse is called a bi-implication.
- Consider the implication "If S then T" and its converse "If T then S".
- Their conjunction is

  "If S then T and If T then S"

  which can also be written as

  "T if S and T only if S"

  which is written as

  "T if and only if S"

  Mathematically written as, "$T \iff S$"
- A bi-implication is a statement of the last type: "T if and only if S".

Contrapositive of implications

- Consider the implication “If the apple is green, then it is sour”.
- Suppose it is a true statement. That is, a green apple is always sour.
- Now suppose there is an apple which is not sour. Can it be green? The answer is NO.
- We define a contrapositive of an implication to be another implication which conveys the same meaning as the given implication.
  The contrapositive of “If the apple is green, then it is sour” is
  “If the apple is not sour, then it is not green”
- In general, the contrapositive of “If S then T” is

  “If not-T, then not-S”.
- Since an implication and its contrapositive convey the same meaning, they are either both true or both false. *Sometimes, we use the contrapositive of an implication to prove the implication.*

Examples – Contrapositive of implications

The contrapositive of "If S then T" is "If not-T, then not-S".

- Consider the following statement:
  S1: For an integer n, if $\underline{n^2 < 20}$, then $\underline{n < 5}$.

  The contrapositive of S1 is:
  For an integer n, if $\underline{n \geq 5}$, then $\underline{n^2 \geq 20}$.

- Consider the following statement:
  S2: For two real numbers x and y, if <u>xy is an irrational number</u>, then <u>x is irrational or y is irrational</u>.

  S: xy is a irrational number, **not-S**: xy is a rational number.

  T: x is irrational or y is irrational, **not-T**: x is rational and y is rational.

  Thus the contrapositive of S2 is:
  For two real numbers x and y, if <u>x is rational and y is rational</u>, then <u>xy is a rational number</u>.

Clarifications based on the questions asked

Negation of an implication

- Consider the implication: S1: "If S then T".
- S1 is true means if S is true then T is true.
- Hence S1 is false if S is true but T is false.
- Thus not-S1: "S and not-T".
- Consider the statement: "For two subsets A and B of X, $A \subseteq B$".
  It is an implication: "For $x \in X$, if $x \in A$ then $x \in B$".
  This is false when there is an element $x \in A$ but $x \notin B$, in which case $A \subseteq B$ is false.

Negation of a bi-implication

- Consider the bi-implication: "S if and only if T".
- The bi-implication is a conjunction of two implications: "If S then T and If T then S".
- The negation of this conjunction is a disjunction:
  "S and not-T or T and not-S".

Proofs in Mathematics

Disclaimer: There can be multiple ways to prove a statement.

Some notations:

- $\mathbb{N}$: Set of natural numbers $1, 2, \ldots$
- $\mathbb{Z}$: Set of integers $0, \pm 1, \pm 2, \ldots$
- $\mathbb{R}$: Set of real numbers
- $\mathbb{R}^+$: Set of positive real numbers $x > 0$

Statements with universal quantifier

Consider the statement S: "$\forall x \in X$(x has property P)".

- To prove S, we need to prove that every element x of X satisfies the property P.
- If X is a finite set, we could check property P for each and every element of X, by hand.
- However, if X is an infinite set, it is not feasible to check property P for each and every element of X, by hand. What do we do then?
- There are some standard ways to prove S.

  (a) Take an arbitrary element x in X and prove the property P.

  (b) Principle of induction.

Example – Method (a) Prove for an arbitrary element x

Let us prove the statement S: "For every integer x, $x^2 \geq 2x - 1$".
Mathematically, "$\forall x \in \mathbb{Z}(x^2 \geq 2x - 1)$".

- Since there are infinitely many integers, we cannot check the inequality $x^2 \geq 2x - 1$ for each integer. What do we do then?
- We take an arbitrary integer x and consider $x^2 - 2x + 1$. We need to prove that it is ≥ 0.
- Note that $x^2 - 2x + 1 = (x - 1)^2$.
- Since $(x - 1)^2 \geq 0$, we get that $x^2 - 2x + 1 \geq 0$. Thus we have the required inequality for the x we started with.
- Since x is an arbitrary integer, the statement S is true.

Method (b) – Principle of Induction

- Suppose you are given a statement P(n) about the natural number n. Suppose we have to prove that P(n) is true for each natural number n.

- The Principle of Induction is the following:

  Suppose for each $n \in \mathbb{N}$, a statement $P(n)$ is given. Assume that:
  (i) Base case: $P(1)$ is true, and
  (ii) For $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.
  Then $P(n)$ is true for each $n \in \mathbb{N}$.

- This principle works because:

  We have assumed that $P(1)$ is true.
  Thus by (ii), $P(2)$ is true. Repeatedly applying (ii), we get $P(3)$, $P(4)$, $P(5)$, and so on, are all true.

- In (ii), the assumption that $P(k)$ is true is known as the induction hypothesis, and with this assumption, proving that $P(k+1)$ is true, is known as the inductive leap.

Example – Method (b) Principle of Induction

Prove that "$n^3 + 2n$ is divisible by 3", for each $n \in \mathbb{N}$.

- Here $P(n)$ is the statement "$n^3 + 2n$ is divisible by 3".
- Let us prove the above statement by the Principle of Induction.
- Base case: For $n = 1$, $n^3 + 2n = 3$, which is clearly divisible by 3. Hence $P(1)$ is true.
- Induction Hypothesis: Suppose for some $k \geq 1$, $P(k)$ is true. That is, suppose $k^3 + 2k$ is divisible by 3.
- Induction Leap: Consider

$$(k+1)^3 + 2(k+1) = (k^3 + 3k + 4) + (2k + 2) = (k^3 + 2k) + 3(k+2).$$

  Now $P(k)$ is true, $k^3 + 2k$ is divisible by 3, and also $3(k+2)$ is divisible by 3. Therefore, their sum is divisible by 3. Hence $P(k+1)$ is true.

- By the Principle of Induction, "$n^3 + 2n$ is divisible by 3", for each $n \in \mathbb{N}$.

Statements with existential quantifier

Consider the statement S: "$\exists x \in X(x \text{ has property } P)$".

- To prove S, we need to prove that there is some $x \in X$ which satisfies the property P.
- There are two standard ways to prove S.

  (a) Sometimes, it is possible to identify or construct an element $x \in X$, which satisfies the property P.

  (b) In many cases, you put a valid argument which guarantees existence of an element $x \in X$, which satisfies the property P.

The next two slides were not discussed in the lecture.
Go over them before you come to the next lecture.

Example 1

Let us prove the statement S: "There exists an integer m such that for every integer n, $m + n = n$".
Mathematically, "$\exists m \in \mathbb{Z}(\forall n \in \mathbb{Z}(m + n = n))$".

- This is a simple enough statement to prove.
- The only integer m which satisfied the equality $m + n = n$ for every integer n is $m = 0$.
- Thus we have been identified an element m of the set of integers which satisfies $m + n = n$ for every integer n. Hence S is true.

Example 2

Let us prove the statement S: "For every real number $x > 0$, there exists a real number y such that $0 < y < x$".
Mathematically, "$\forall x \in \mathbb{R}^+(\exists y \in \mathbb{R}^+(0 < y < x))$".

- Since $\mathbb{R}^+$ is an infinite set and we need to prove the statement for all $x \in \mathbb{R}^+$, it is not feasible to take elements of $\mathbb{R}^+$ one-by-one and prove S.
- Thus we take an arbitrary element $x \in \mathbb{R}^+$ and prove that there exists a real number y such that $0 < y < x$.
- Clearly $y = x/2$ is a real number satisfying $0 < y < x$. *Note that this y depends on x.*
- Hence we have provided an integer y for given x satisfying $0 < y < x$.
- Since x is an arbitrary positive real number, S is true.