MTH 101: Calculus I

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Lecture 5
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Countability of Sets

Sets with same Cardinality

- Consider two sets X and Y.
- We say that X and Y have the same cardinality if
  a) either both are empty, or
  b) there is a bijection $f : X \to Y$.
- If the sets X and Y have the same cardinality, and the sets Y and Z have the same cardinality, then the sets X and Z have the same cardinality.

  Let $f : X \to Y$ and $g : Y \to Z$ be bijections, then $g \circ f : X \to Z$ is a bijection.

Examples

- The sets $X = \{n \ : \ 1 \leq n \leq 10\}$ and $Y = \{2n+1 \ : \ 1 \leq n \leq 10\}$ have the same cardinality.

  The function $f : X \to Y$ given by $f(n) = 2n+1$ is a bijection.

- Two sets X and Y both having n elements have the same cardinality.

  List the elements of X as $x_1, x_2, \ldots, x_n$, and list the elements of Y as $y_1, y_2, \ldots, y_n$.

  The function $f : X \to Y$ given by $f(x_i) = y_i$, for $1 \leq i \leq n$, is a bijection from X onto Y.

- The interval $(-1, 1)$ and $\mathbb{R}$ have the same cardinality.

  The function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \tan(\pi x/2)$, for all $x \in (-1, 1)$, is a bijection from $(-1, 1)$ onto $\mathbb{R}$.

More examples

- Intervals (a, b) and (c, d) have the same cardinality.

  The function $f : (a, b) \to (c, d)$ given by $f(x) = c + (d - c)\frac{x - a}{b - a}$, for all $x \in (a, b)$, is a bijection from (a, b) onto (c, d).

  Observe that even though $(0, 1)$ is a subset of $(-1, 1)$, the intervals $(0, 1)$ and $(-1, 1)$ have the same cardinality.

- Intervals $[a, b]$ and $[c, d]$ have the same cardinality.

  Same f as above works.

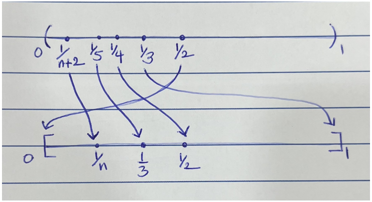
Another example

- Intervals $(0, 1)$ and $[0, 1]$ have the same cardinality.
  Hence intervals (a, b) and $[c, d]$ have the same cardinality.

  Define $f : (0, 1) \to [0, 1]$ as follows.

  Let $A = \{1/2, 1/3, 1/4, \ldots, 1/n, \ldots\}$.

  Define $f(1/2) = 0$, $f(1/3) = 1$, $f(1/(n+2)) = 1/n$, for all $n \geq 2$, and $f(x) = x$ for $x \in (0, 1) \setminus A$.



  Then f is a bijection.

  This technique is inspired from the Hilbert's hotel example.

Finite set

- A set X is said to be finite if
  a) either it is empty, or
  b) there is $n \in \mathbb{N}$ such that the sets X and $\{1, 2, \ldots, n\}$ have the same cardinality.

- We denote the number of elements of a finite set by $|X|$.
  Note that if X and $\{1, 2, \ldots, n\}$ have the same cardinality, then $|X| = n$.

- A set which is not finite is said to be an infinite set.

Two finite sets

Theorem

Let X be a set with m elements and Y be a set with n elements. Then X and Y have the same cardinality if and only if $m = n$.

Recall: Same cardinality means there is a bijection between X and Y.

Proof

- The statement has a bi-implication. We need to prove both implications.
- Step 1: Prove $(m = n) \Rightarrow X$ and Y have the same cardinality. Already discussed the proof.
- Step 2: Prove X and Y have the same cardinality $\Rightarrow (m = n)$.
  We prove this by showing that the contrapositive of this statement is true.
  The contrapositive is:
  $(m \neq n) \Rightarrow X$ and Y do not have the same cardinality.
  We prove the contrapositive in two parts:
  (a) Prove that if $m < n$, there is no onto function from X to Y.
  (b) Prove that if $m > n$, there is no one-one function from X to Y.

Proof continued

(a) We prove that if $m < n$, there is no onto function from X to Y.

- Let $f : X \to Y$ be a function.
  Let $X = \{x_1, x_2, \dots, x_m\}$.
  Then the set $\{f(x_1), f(x_2), \dots, f(x_m)\} \subseteq Y$ is the image set $f(X)$ having at most m elements.
  Since $m < n$, there is some element of Y which does not belong to $f(X)$.
  Hence, f is not onto.

(b) We prove that if $m > n$, there is no one-one function from X to Y.

- Let $f : X \to Y$ be a function.
  Let $X = \{x_1, x_2, \dots, x_m\}$.
  Then the set $\{f(x_1), f(x_2), \dots, f(x_m) \subseteq Y$ is the image set $f(X)$.
  Since Y has n elements and $n < m$, there is a pair $1 \leq i \neq j \leq m$ such that $f(x_i) = f(x_j)$.
  Hence, f is not one-one.

$\square$

Pigeonhole Principle

Let X be a set with m elements and Y be a set with n elements.If $m > n$ and $f : X \to Y$ is a function, then f is not one-one.

- f is not one-one means that there are two elements $x_1, x_2 \in X$ for which $f(x_1) = f(x_2)$.
- Why is it called Pigeonhole principle?
  Suppose there are m pigeons and n pigeonholes, and that each pigeon has to stay in a pigeonhole. Since there are more pigeons than pigeonholes, two pigeons will have to stay in the same pigeonhole.
- The Pigeonhole Principle is one of the most basic and powerful tool in Combinatorics.

An application of the Pigeonhole Principle

Let $k \geq 2$. Let $a_1, a_2, , \ldots, a_{k+1} \in \mathbb{Z}$. Prove that there exist $1 \leq i, j \leq k$, $i \neq j$ such that $a_i - a_j$ is divisible by k.

Proof:

- For each $1 \leq i \leq k+1$, let r_i be the remainder upon dividing a_i by k. That is, $a_i = kq_i + r_i$.
- The possible values of r_i are $0, 1, \ldots, k-1$ (k many possibilities).
- Since there are $k+1$ many elements $a_1, a_2, , \ldots, a_{k+1}$ and only k many possibilities for remainders $r_1, r_2, , \ldots, r_{k+1}$, there exist $1 \leq i, j \leq k$, $i \neq j$ such that $r_i = r_j$.
- Hence $a_i - a_j = kq_i + r_i - kq_j - r_j = k(q_i - q_j)$. Therefore $a_i - a_j$ is divisible by k.

$\square$

Countable set

- A set X is said to be countable if
  a) it is either finite, or
  b) it has the same cardinality as $\mathbb{N}$.

- If X has the same cardinality as $\mathbb{N}$, then X is said to be countably infinite.

- A set which is not countable is said to be uncountable.

Examples

- $\mathbb{Z}$ is countably infinite.
- If X, Y are countably infinite sets, then the set $X \cup Y$ is also a countably infinite set.
- If X, Y are countably infinite sets, then the set $X \times Y = \{(x, y) \ : \ x \in X, \ y \in Y\}$ is also a countably infinite set.
- The set of rational numbers, $\mathbb{Q}$, is countably infinite.
- The power set of $\mathbb{N}$, denoted as $\mathcal{P}(\mathbb{N})$, is uncountable.