MTH 101: Calculus I

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Proofs in Mathematics – Continued

Statements with existential quantifier

Consider the statement S: "$\exists x \in X(x$ has property $P)$".

- To prove S, we need to prove that there is some $x \in X$ which satisfies the property P.
- There are two standard ways to prove S.

  (a) Sometimes, it is possible to identify or construct an element $x \in X$, which satisfies the property P.

  (b) In many cases, you put a valid argument which guarantees existence of an element $x \in X$, which satisfies the property P.

Example 1

Let us prove the statement S: "There exists an integer m such that for every integer n, $m + n = n$".
Mathematically, "$\exists m \in \mathbb{Z}(\forall n \in \mathbb{Z}(m + n = n))$".

- This is a simple enough statement to prove.
- The only integer m which satisfied the equality $m + n = n$ for every integer n is $m = 0$.
- Thus we have been identified an element m of the set of integers which satisfies $m + n = n$ for every integer n. Hence S is true.

Example 2

Let us prove the statement S: "For every real number $x > 0$, there exists a real number y such that $0 < y < x$".
Mathematically, "$\forall x \in \mathbb{R}^+(\exists y \in \mathbb{R}^+(0 < y < x))$".

- Since $\mathbb{R}^+$ is an infinite set and we need to prove the statement for all $x \in \mathbb{R}^+$, it is not feasible to take elements of $\mathbb{R}^+$ one-by-one and prove S.
- Thus we take an arbitrary element $x \in \mathbb{R}^+$ and prove that there exists a real number y such that $0 < y < x$.
- Clearly $y = x/2$ is a real number satisfying $0 < y < x$. *Note that this y depends on x.*
- Hence we have provided an integer y for given x satisfying $0 < y < x$.
- Since x is an arbitrary positive real number, S is true.

Statements with implications

Consider the statement: "$S \Rightarrow T$".

- To prove the given statement "$S \Rightarrow T$", we need to prove that when S is true, then T is true.

- There are three standard ways to prove $S \Rightarrow T$.

  (a) Direct Proof: Give a sequence of statements $S_1, S_2, \ldots S_k$ such that $S \Rightarrow S_1$, $S_1 \Rightarrow S_2$, $\ldots$, $S_{k-1} \Rightarrow S_k$, and $S_k \Rightarrow T$ are known to be true. Hence $S \Rightarrow T$ is true.

  (b) Proof by Contradiction: We prove that the negation of the statement "$S \Rightarrow T$" is false. The negation of "$S \Rightarrow T$" is "S is true and T is false".
  Hence, assume that S is true but T is false, and arrive at a contradiction or a statement which you know for sure is false.

  (c) Prove the Contrapositive: The contrapositive of "$S \Rightarrow T$" is "$not - T \Rightarrow not - S$" which is same as the original statement "$S \Rightarrow T$".
  Prove that the contrapositive "$not - T \Rightarrow not - S$" is true, which implies that "$S \Rightarrow T$" is true.

Example 1 – Direct Proof

Prove that "For real numbers x, y, if $xy = 0$, then $x = 0$ or $y = 0$."

- Observe that the statement is symmetric in x and y. That is, if the roles of x and y are reversed, the statement remains the same.
- Suppose $xy = 0$ and $x \neq 0$. We need to prove that $y = 0$.
- Since $x \neq 0$, $1/x$ is a real number.
- Hence $y = \frac{1}{x}(xy) = \frac{1}{x}(0) = 0$.
- Thus the given statement is true.

Example 2 – Direct Proof

Prove that the following statements are equivalent for two given sets A and B.
(i) $A \subseteq B$.
(ii) $A \cap B = A$.
(iii) $A \cup B = B$.

- The standard technique to prove equivalence of multiple statements is to identify a suitable cyclic order of implications.
- For the given statements, one can prove $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$.

Example – Proof by Contradiction

Prove the statement "There is no greatest integer".

- Suppose the statement is false. That is, its negation "There is a greatest integer" is true.
- Let n be the greatest integer.
- Since n is an integer, $n + 1$ is also an integer with $n + 1 > n$.
- This contradicts our assumption that n is the greatest integer.
- Hence the given statement is true: "There is no greatest integer".

Example – Prove the Contrapositive

Prove the statement "For an integer n, if $n^2 < 20$, then $n < 5$".

- The contrapositive of the given statement is "For an integer n, if $n \geq 5$, then $n^2 \geq 20$".
- We prove that the contrapositive is true *(which implies that the original statement is true)*.
- If $n \geq 5$, then $n^2 \geq 5^2 = 25 > 20$. Hence $n^2 > 20$. This proves that the contrapositive is true.
- Thus the given statement "For an integer n, if $n^2 < 20$, then $n < 5$" is true.

Counterexamples

- Suppose we want to disprove the statement S: "$\forall x \in X(x$ has property $P)$". That is, to prove that S is false.
- The negation of S is not-S: "$\exists x \in X(x$ does not have property $P)$".
  To disprove S, we prove that not-S is true.
- Now to prove not-S, we need to show that there exists an element x in X which does not satisfy property P.
- Such an element x is known as a counterexample.

Example – Producing a Counterexample

Disprove S: "For $x, y \in \mathbb{R}$, $x^2 = y^2$ implies $x = y$".

- The negation of S is:

  not-S: "There exists $x, y \in \mathbb{R}$ such that $x^2 = y^2$ and $x \neq y$".

  We prove that not-S is true.

- Clearly for $x = 2$ and $y = -2$, $x^2 = y^2 = 4$.

- The pair $x = 2$ and $y = -2$ is a counterexample. (Note: There are many many counterexamples)

- Hence S is false.

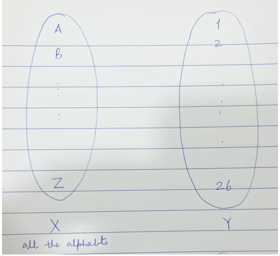
Countability of Sets

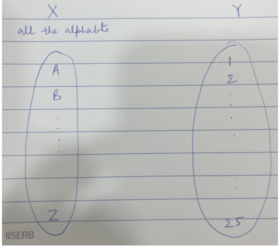
Sets with same Cardinality

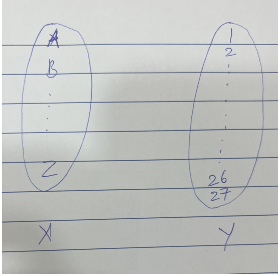
- Consider two sets X and Y.
- We say that X and Y have the same cardinality if
  a) either both are empty, or
  b) there is a bijection $f : X \to Y$.
- A bijection $f : X \to Y$ is a one-one (injective) and onto (surjective) function from X to Y.
  One-one/injective means $f(x_1) = f(x_2)$ implies $x_1 = x_2$, and
  Onto/surjective means for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
- Intuitively, the sets have "the same number of elements". This makes sense for finite sets but not for infinite sets.
- If the sets X and Y have the same cardinality, and the sets Y and Z have the same cardinality, then the sets X and Z have the same cardinality.

  Let $f : X \to Y$ and $g : Y \to Z$ be bijections, then $g \circ f : X \to Z$ is a bijection.

Identification of sets







In the first figure, the two sets have the same cardinality. Use the natural ordering to produce a bijection: $A \to 1, B \to 2, \ldots, Z \to 26$.

In the second figure, the two sets do not have the same cardinality. There is no injective (one-one) function from the set X to the set Y.

In the second figure, the two sets do not have the same cardinality. There is no surjection (onto) function from the set X to the set Y.