Energy in waves

Kinetic energy $= \frac{1}{2}(\rho_0 \delta x)\left(\frac{\partial \psi}{\partial t}\right)^2$

$$\mathcal{E}_k = \frac{1}{2}\rho_0 a^2 \omega^2 \sin^2(\omega t - kx) \quad \text{for a unit length}$$

$\frac{1}{2}$

$$\langle \mathcal{E}_k \rangle = \frac{1}{4}\rho_0 a^2 \omega^2$$

Potential energy

Longitudinal wave

$P + p$ $\qquad\qquad\qquad\qquad s = -\dfrac{\partial \psi}{\partial x}$

$1 - s$

Bulk modulus

$$\mathcal{E}_p = \int_{s=0}^{s} p\, ds' \qquad B = \frac{p}{s} \qquad p = Bs$$

$$= B\int_0^s s\, ds = \frac{1}{2}Bs^2 = \frac{1}{2}B\left(-\frac{\partial \psi}{\partial x}\right)^2$$

$$\mathcal{E}_p = \frac{1}{2} B a^2 k^2 \sin^2(\omega t - kx)$$

$$\langle \mathcal{E}_p \rangle = \frac{1}{4} B a^2 k^2$$

$$c = \sqrt{B/\rho_0} = \frac{\omega}{k}$$

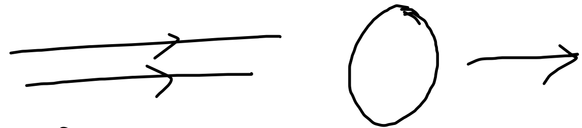
<u>Sound waves</u>

$$\langle \mathcal{E}_p \rangle = \frac{1}{4} \rho_0 a^2 \omega^2$$

$$\langle \mathcal{E} \rangle = \langle \mathcal{E}_k \rangle + \langle \mathcal{E}_p \rangle = \boxed{\frac{1}{2} \rho_0 a^2 \omega^2}$$

Energy density

Intensity



$$P = \frac{dE}{dt}$$

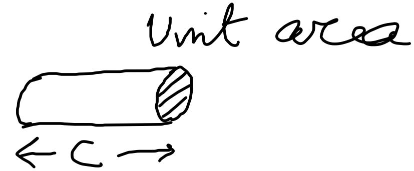
Average rate of energy flow through a unit area per unit time

Energy per unit time δE

$\delta S \rightarrow$ area element

$$I = \lim_{\delta S \to 0} \frac{\delta E}{\delta S}$$

Plane progressive wave



$$I = \frac{1}{2} \rho_0 c a^2 \omega^2$$

Practical aspects

Monochromatic wave $\quad \psi(x,t) = A e^{i(\omega t - kx)}$

Wave train $\qquad$ Frequency distribution

$$\psi(x,t) = \sum_{i=1}^{\infty} C_i \psi_i(x,t)$$

Group of waves

$\omega_0 \rightarrow$ Central frequency

$\underline{\omega_0 - \Delta\omega} \longrightarrow \underline{\omega_0 + \Delta\omega}$

Phase velocity

$$\psi(x,t) = A e^{i(\omega t - kx)}$$

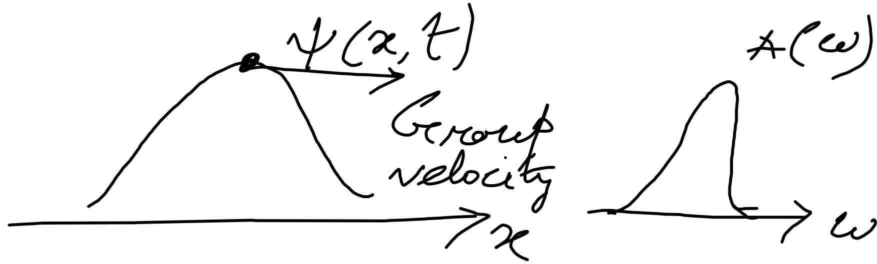
Constant phase

$$\omega t - kx = \text{constant}$$

$$\frac{d}{dt}(\omega t - kx) = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k} = c \quad \text{Phase velocity}$$

Group velocity



Dispersive medium

$$\omega_0 - \Delta\omega \longrightarrow \omega_0 + \Delta\omega$$

$\Delta\omega \ll \omega_0$ assumption

$$\psi(x,t) = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} A(\omega)\, e^{i(\omega t - kx)}\, d\omega$$

$\longrightarrow A_0$ assumption

$k(\omega) \rightarrow$ Dispersion relation

Taylor expansion

$$k = k(\omega_0 + \underbrace{(\omega - \omega_0)}_{\text{very small}})$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots$$

$$\Rightarrow k = k(\omega_0) + (\omega - \omega_0)\left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} + \cdots$$

$$\psi(x,t) = A_0 e^{i(\omega_0 t - k_0 x)} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} d\omega\, e^{i(\omega-\omega_0)\left[t - \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} x\right]}$$

$$\omega' = \omega - \omega_0$$

$$\boxed{t - \frac{dk}{d\omega}\Bigg|_{\omega=\omega_0} x = m}$$

$$\psi(x,t) = A_0 e^{i(\omega_0 t - k_0 x)} \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} d\omega' e^{im\omega'}$$

$$= A_0 e^{i(\omega_0 t - k_0 x)} \frac{e^{im\omega'}}{im}\Bigg|_{-\Delta\omega}^{+\Delta\omega}$$

$$= \frac{2\sin(m\Delta\omega)}{m} A_0 e^{i(\omega_0 t - k_0 x)}$$

$$\psi = \frac{2 \sin\left[\left\{ t - \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} x \right\} \Delta\omega\right]}{t - \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} x} A_0 e^{i(\omega_0 t - k_0 x)}$$