Topic 3: Fixed Income Valuation

(a) Bond definitions

Definition *A bond is a certificate that shows that a borrower owes a specified sum and the dates on which the interest and principal must be paid.*

Example

For instance, you might have a bond with:

- 5 years to maturity
- Face value $F = \$1000$
- Coupon rate 5%
- Interest(coupon) paid annually
  - Why called coupons? This comes from the days when you could actually tear them out and mail them in
- Bonds sometimes are called *Fixed Income* securities

(b) Valuation of Pure Discount Bonds

Definition *A pure discount bond is a bond that promises a single payment at maturity.*

These are also known as zero-coupon bonds or "zeros."

Example

1. Face Value=\$1000, maturing in 5 years and $r = 7\% \Rightarrow$ Price $P = \frac{\$1000}{(1+0.07)^5} = \$712.99$.
2. Face Value=\$1000, maturing in 3 years and $r = 7\% \Rightarrow$ Price $P = \frac{\$1000}{(1+0.07)^3} = \$816.30$.

Remarks:

1. We calculated the price of the bond using present value tools. However, bonds are not valued by us deciding the price; instead, buyers and sellers determine the price. Why is it that given the interest rate r, the 5-year bond price is equal to \$712.99?
   - $P >$ \$712.99 $\Rightarrow$ Nobody buys the bond. They instead deposit cash into a bank account. Bond dealers are left with an inventory of bonds, and so must lower the price.
   - $P <$ \$712.99 $\Rightarrow$ Everybody wants to buy the bond. Faced with high demand, bond dealers raise the price.
2. We can view this example as two different bonds at the same point of time, or as the same bond at different points of time:
   - Say we purchase a five-year bond with $P = 712.99$.
   - Two years later, assuming interest rates do not change, we will have a three-year bond, with $P = \$816.30$.
   - Three years later, the bond will mature, and $P = F = \$1000$.
   - Bond prices tend to rise toward their par (or face) value over time (at maturity, the price of the bond must equal the face value). Bonds are said to be "pulled to par."

We can ask: given a price (say, \$712.99), what r makes this price correct?

Notation:

- P = Price of the bond
- F = Face value of the bond
- t = # of yrs to maturity

Find r such that:

$$P = \frac{F}{(1+r)^t}$$

Multiplying both sides by $(1+r)^t$:

$$P(1+r)^t = F$$

Dividing and taking the root:

$$1 + r = (F/P)^{\frac{1}{t}}$$

Finally,

$$r = (F/P)^{\frac{1}{t}} - 1$$

Definition *The yield to maturity (YTM) is discount rate such that the present value of the bond's payments equals the price.*

When we apply this definition to zero-coupon bonds, we find:

$$\text{YTM} = (F/P)^{\frac{1}{t}} - 1$$

Example In the above example:

$$r = \left(\frac{\$1000}{\$712.99}\right)^{1/9} - 1 = 0.07$$

(c) Yield to Maturity vs. Holding Period Return

Bond investors frequently use the yield to maturity as a way of evaluating the investment.

We want to look closely at this practice. We ask: what is the rate of return you actually earn on a bond, and how does it relate to YTM? We will assume annual compounding.

First, what does return on investment mean? This discussion will start out kind of abstract and philosophical, but it will get concrete soon. Suppose you had an investment, perhaps in a stock, holdings in a mutual fund, or a piece of real estate. We want to derive a useful notion of the return on this investment between times 0 and t.

Say investment has value V_t at time t, V_0 at time 0. Here's one possibility:

$$R = \frac{V_t - V_0}{V_0} = \frac{V_t}{V_0} - 1 \qquad \text{(\% change in value)}$$

Is this a good measure of return?

5-year bond from Example 1:

$$R = \frac{\$1000}{\$712.99} - 1 = 0.4025$$

3-year bond:

$$R = \frac{\$1000}{\$816.30} - 1 = 0.2205$$

$\Rightarrow$ It looks like the 5-year bond is a better investment than the 3-year bond.

But is it? It took the 5-year bond 5 years to earn 40%, while it only took the 3-year bond 3 years to earn 22%. Thus, simple percent change, intuitive as it may be, is a WRONG notion of return. We need to fix it – we need to adjust for the length and time we have to hold the investment. We need to put returns over different periods on an equal footing, just like we needed to put different rates of compounding on an equal footing.

How do we do this? Find r such that:

$$\begin{aligned}(1+r)^t &= 1 + R \Rightarrow r = (1+R)^{1/t} - 1 \\ &= \left(\frac{V_t}{V_0}\right)^{1/t} - 1\end{aligned}$$

$\Rightarrow$The return calculated this way has two names. Depending on what one wants to emphasize, it is called *holding period return* or *annual return.*

Definition *The holding period return (HPR), or equivalently, the annual return on an investment with value V_0 at time 0 and value V_t at time t equals:*

$$\text{HPR} = \left(\frac{V_t}{V_0}\right)^{\frac{1}{t}} - 1$$

Returning to our bond examples, we find:

- For the 5-year bond:
$$r = (1 + .04025)^{1/5} - 1 = 0.07$$
- For the 3-year bond:
$$r = (1 + 0.225)^{1/3} - 1 = 0.07$$

How are holding period return and YTM related? It matters whether the bond is held to maturity. The HPR you earn on a zero-coupon bond when you hold it to maturity is the same as the YTM. *It turns out this is always true if the bond is held to maturity.* Compare the formulas:

$$\text{YTM} = \left(\frac{F}{P}\right)^{1/t} - 1$$

$\Rightarrow F$ is the face value, P is the price. For a bond you hold to maturity, $F = V_t$ and $P = V_0$, where t is the maturity of the bond. This is why above we simply got back the YTM.

Conclusion *If a zero-coupon bond is held to maturity:*

$$\text{YTM} = \text{ Holding Period Return}$$

Note that the notion of return here is annual return. However, we may want to sell a bond prior to maturity. Let's do a different example:

Example

Assume you have a 10-year bond: $F = \$1000$ and $P = \$450.11$. What is the YTM?

$$\begin{aligned}\text{YTM} &= \left(\frac{\$1000}{\$450.11}\right)^{\frac{1}{10}} - 1 \\ &= 0.0831\end{aligned}$$

Now, the question is: are you going to get this 8.31% return, as suggested by the YTM? *Suppose you decide to sell the bond after one year. However, the YTM may change by then* (investors may be more or less willing to buy bonds next year). Suppose first that the YTM falls to 8%. We want to calculate holding period return. That is what you actually earn:

$$\text{HPR} = \left(\frac{V_t}{V_0}\right)^{\frac{1}{t}} - 1$$

We know: $V_0 = \text{Price} = \$450.11$ We also know (because we only hold the bond for one year): $t = 1$. What's left? V_t, which is the price at which I sold the bond, if YTM = 8%.

What is that price? Note that after one year, the ten-year bond becomes a nine-year bond. *Price of a nine-year bond:*

$$V_t = \frac{\$1000}{(1+0.08)^9} = \$500.25$$

In this scenario, investors are more willing to hold bonds, so they have pushed up the price. Therefore, the HPR equals:

$$\text{HPR} = \left(\frac{\$500.25}{\$450.11}\right)^{\frac{1}{1}} - 1 = 0.1114$$

(I'm making a big deal about this one, because you might be faced with a problem where you hold the bond for more than one year). In this case:

$$\begin{aligned}\text{HPR} &> \text{YTM} \\ 11.14\% &> 8.31\%\end{aligned}$$

⇒ So the HPR (11.14%) we receive from holding the bond for one year is greater than the YTM (8.31%) we would have received if we held the bond to maturity. This is because yields were lower when you sold the bond than when you bought it.

Now, suppose you bought the *ten-year bond at a price of $450.11*, but when you sold it one year later, the *yield had risen to 8.6%*.

We know V_0 and t. Let's calculate V_t – *price of a nine-year bond:*

$$V_t = \frac{\$1000}{(1+0.086)^9} = \$475.92$$

Here, investors are less willing to buy bonds, so they have pushed down price. HPR equals:

$$\text{HPR} = \left(\frac{\$475.92}{\$450.11}\right)^{\frac{1}{1}} - 1 = 0.057$$

In this case:

$$\begin{aligned} \text{HPR} &< \text{YTM} \\ 5.7\% &< 8.31\% \end{aligned}$$

The HPR (5.7%) is now lower than the YTM (8.31%) that you would have received had you held the bond to maturity. You earned less than the YTM because yields were higher when you sold the bond than when you bought it – tough luck.

Of course, it is possible that *yields stay the same*. Consider the case of a bond that matures in 10 years (t=10), with $P = \$450.11$ per $1000 face value. Given the YTM calculation at the beginning of this section, YTM=8.31%. Suppose the bond is held til maturity.

We can see that HPR=YTM by directly applying the HPR formula. When it matures, the bond is worth F, its face value. Looking at the general HPR formula, and replacing F with V_t, and the YTM definition, we see this is exactly the YTM for a bond with a face value of F and price P, maturing in t years. So:

$$\begin{aligned} \text{HPR} &= \text{YTM} \\ 8.31\% &= 8.31\% \end{aligned}$$

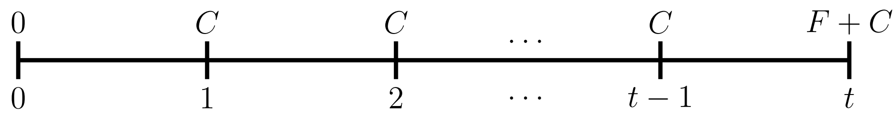
⇒ When yields stay the same, you actually earn the yield to maturity.

What can we conclude?

Unlike the YTM, HPR is *uncertain*. When we buy a bond, we do not know what price we will be able to sell it for next period, even if these bonds are a liability of the US government. Many think of government bonds as risk-less. However, it is these bond cash flows that are known with certainty, not the HPR.

(d) Prices and Returns on Coupon Bonds

It so happens that the government and corporations issue coupon bonds, so we need to know how to value them. The cash flows, generally, are:



Example

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^{t-1}} + \frac{F+C}{(1+r)^t} \tag{1}$$

There are two distinct interest rates associated with coupon bonds:

1. Coupon rate: C/F
2. YTM: the discount rate such that the present value of the bond's payments equals the price.

<u>Other Notation</u>:

- C: Coupon payment
- F: Face value of the bond
- P: Price of the bond

Something interesting happens when the two rates are equal:

Result. *Bond sells at par:* $P = F$ *if* $\text{YTM} = C/F$

Cute proof of result: Assume $C/F = \text{YTM} = r$, so $C = rF$. To show $P = F$:

- Assume for concreteness that $t = 4$ (can do general proof by induction):

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{F+C}{(1+r)^4}$$

- Multiply both sides of the formula by $(1+r)^4$. Then we have:

$$P(1+r)^4 = rF(1+r)^3 + rF(1+r)^2 + rF(1+r) + F + rF$$

- Now collect terms on the right hand side:

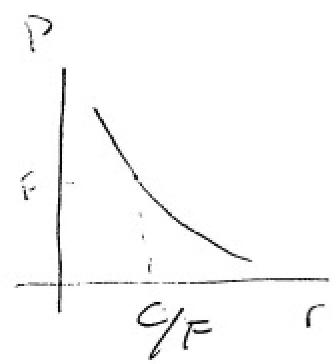
$$\begin{aligned}
P(1+r)^4 &= rF(1+r)^3 + rF(1+r)^2 + (rF+F)(1+r) \\
&= rF(1+r)^3 + rF(1+r)^2 + F(1+r)^2 \\
&= rF(1+r)^3 + (rF+F)(1+r)^2 \\
&= rF(1+r)^3 + F(1+r)^3 \\
&= (rF+F)(1+r)^3
\end{aligned}$$

$\Rightarrow P = F$!

What this says economically: you are indifferent between having \$1000 today, and an asset that pays interest at the market rate every year, plus \$1000 at the end. This intuition is in the proof: if you take the \$1000 today and invest it at the market rate, you have the same future value as if you take all the interest payments and invest them at the market rate.

Life would be simple if bonds sold at their face value ("at par") all the time. But all we need to do is to look at bond prices to see this is not the case. For coupon bonds, generally there is no analytical formula for YTM!

What can we say about YTM if the bond is not selling at par (or alternatively, about price if YTM does not equal the discount rate)? We can plot price against the discount rate r:



- Price P is decreasing in the YTM (represented on the graph by r). Note that this generally holds in present value calculations.
- When YTM is equal to C/F, $F = P$.

From this plot we can see:

- Bond sells at a discount: $P < F$ if YTM $> C/F$

- Bond sells at a premium: $P > F$ if YTM $< C/F$
- Bond sells at par: $P = F$ iff YTM $= C/F$.

Some examples (check at home):

Example Suppose a bond is selling at a discount, say $P = \$880.97$, $t = 4$ and $C/F = 8\%$.

Note: it is not possible to solve for YTM as in the case of a zero coupon bond. We need a calculator or spreadsheet and also will need to do trial and error. However, we can say something: for $P < \$1000$, we must have $r > C/F$.

Why? r has to work harder to get these cash flows back to this lower PV $\Rightarrow r = 11.24\%$

Example Suppose a bond is selling at premium, say $P = \$1122.41$.

Now r needs to work less hard. Discount back by less, so $r < 8\% \Rightarrow r = 5.16\%$ works here.

Another way to see r decreases: bonds payments are fixed. If the rate is higher, you are willing to pay less for these payments because you have better opportunities elsewhere.

BTW: Why are zero-coupon bonds called discount bonds? Because they *always* sell at a discount. Mathematically: $C/F = 0$, so YTM $> C/F \Rightarrow P < F$.

When do coupon bonds sell at par? When they are first issued.

Holding Period Returns on Coupon Bonds

How do we evaluate coupon bonds as an investment? Suppose we were comparing two different bonds – you may want to know which offers a higher return if held until maturity.

YTM is a *yardstick* for seeing how good return is. Is it flawless? If not, what are its flaws?

Previously, we showed:

Result. *For a zero-coupon bond that is held until maturity:*

$$\text{HPR} = \text{YTM}$$

$\Rightarrow$ Assuming we hold the bond to maturity, YTM is a flawless yardstick for zero-coupon bonds. It measures exactly the return we will receive.

Is the same result true for coupon bonds?

Example Assume: $t = 4$, YTM $= 8\%$, $F = \$1000$, and $C = \$80$. Therefore, the price of the bond $P = \$1000$. Because $C/F = YTM$, the bond sells at par. Therefore:

$$\text{HPR} = \left(\frac{V_t}{V_0}\right)^{1/t} - 1$$

$\Rightarrow V_0 = P$. The value in 4 years, V_4, depends on the reinvestment rate for coupons. Thus:

$$V_4 = \$1080 + ?????$$

We now need to know something about what happens to these cash flows after they are received. We could always put them under your mattress. Then:

$$V_4 = \$1080 + \$80 + \$80 + \$80$$

But this is too conservative! The point is that at time 0, the rate at which you can reinvest these cash flows is *uncertain*. Let's say we instead re-invest the coupons at the YTM:

1	2	3	4			
\$80	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\$80(1.08)^3$	=	\$100.78
	\$80	$\longrightarrow$	$\longrightarrow$	$\$80(1.08)^2$	=	\$93.31
		\$80	$\longrightarrow$	\$80(1.08)	=	\$86.40
			\$1080	\$1080	=	\$1080
				Total	=	\$1360.49

Therefore:

$$\begin{aligned} \text{HPR} &= \left(\frac{V_4}{V_0}\right)^{\frac{1}{4}} - 1 \\ &= \left(\frac{\$1360.49}{\$1000}\right)^{\frac{1}{4}} - 1 = 0.08 = \text{YTM} \end{aligned}$$

When the coupons are reinvested at the YTM, then HPR = YTM.

Note: This is not a coincidence:

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{F+C}{(1+r)^4}$$

Multiply by $(1+r)^4$:

$$P(1+r)^4 = C(1+r)^3 + C(1+r)^2 + C(1+r)^3 + F + C$$

Note that the right hand side is our V_4! Using this equation and substituting in to the HPR formula, we find:

$$\begin{aligned} \text{HPR} &= \left(\frac{P(1+r)^4}{P}\right)^{1/4} - 1 \\ &= r \end{aligned}$$

Result. *If the bond is held to maturity, and* if cash flows re-invested at the YTM*:*

$$\text{YTM} = \text{HPR}$$

What happens when the coupons are not reinvested at the YTM?

Suppose instead that the coupons are reinvested at 6%. Then:

1	2	3	4			
\$80	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\$80(1.06)^3$	=	\$95.28
	\$80	$\longrightarrow$	$\longrightarrow$	$\$80(1.06)^2$	=	\$89.89
		\$80	$\longrightarrow$	\$80(1.06)	=	\$84.80
			\$1080	\$1080	=	\$1080
				Total	=	\$1350

$$V_4 = 80(1.06)^3 + 80(1.06)^2 + 80(1.06) + 1080 < 1000(1.08)^4$$

So:

$$\text{HPR} < \left(\frac{1000(1.08)^4}{1000}\right)^{1/4} - 1 = \text{YTM}.$$

We can also show this using V_4=\$1350:

$$V_4 = \$1350$$

$$\text{HPR} = \left(\frac{\$1350}{\$1000}\right)^{1/4} - 1 = 7.8\% < \text{YTM}.$$

Because the coupons are reinvested at a rate less than YTM, we get a lower return. Even if we hold the bond to maturity, we may not get the YTM as our holding period return.

Suppose instead that we reinvest at a rate greater than the YTM. I leave it to you to show that HPR > YTM when the coupons are reinvested at a higher rate.

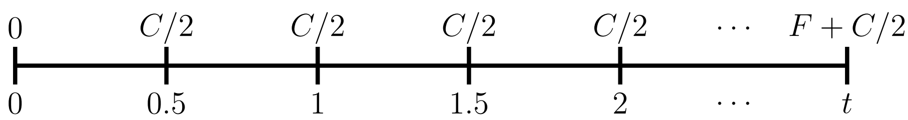
What can we conclude? YTM is a useful but flawed yardstick. We can only use YTM to compare holding period return on two coupon bonds if we assume that we re-invest the coupons at the relevant YTM. We can use the YTM as a yardstick, but we need to keep in mind that, even if we hold the bond to maturity, this is not the return we may actually get.

Question: Suppose we sold the coupon bond before maturity? How would we calculate HPR?

(e) Semi-Annual Bonds

We care about coupon bonds because these are what the U.S. Treasury and corporations actually issue. Unfortunately, there is one more complication when we deal with real-world bonds . The U.S. Treasury (and often corporations) issue bonds that pay semiannually. Instead of a coupon payment once a year, you receive one twice a year.

Semi-annual bond: Assume t yrs to maturity, coupon payment C, face value F, and YTM $= r_a$. The cash flows:



Price:

$$P = \frac{C/2}{1+\frac{r_a}{2}} + \frac{C/2}{(1+\frac{r_a}{2})^2} + \frac{C/2}{(1+\frac{r_a}{2})^3} + \cdots + \frac{F + C/2}{(1+\frac{r_a}{2})^{2t}}$$

Note: YTM is not an EAR, it is a SAIR. YTM for semi-annual bonds is quoted assuming semi-annual compounding.

Example Assume a semi-annual bond with $C/F = 8\%$; $YTM = 8\%$; $t = 10$, $F = \$1000$.

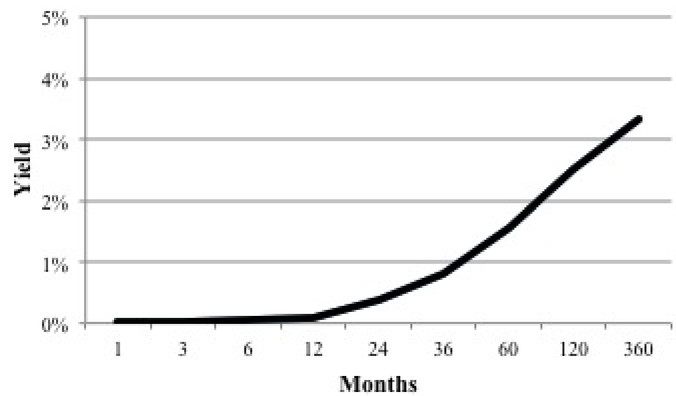
- Price:
$$P = \frac{\$40}{1.04} + \frac{\$40}{(1.04)^2} + \frac{\$40}{(1.04)^3} + \cdots + \frac{\$1040}{(1.04)^{20}}$$
In this case, we do not need to calculate the price of the bond. We can see that $P = \$1000$. Bond sells at par when coupon rate equals YTM

- EAR for this bond:
$$r = (1 + r_a/2)^2 - 1 = 8.16\%$$
Note: We could also have used the EAR in the price calculation:
$$P = \frac{\$40}{1.0816^{1/2}} + \frac{\$40}{(1.0816)} + \frac{\$40}{(1.0816)^{3/2}} + \cdots + \frac{\$1040}{(1.0816)^{10}}$$

(f) The Yield Curve

In our present value discussion, we assumed there was a single interest rate that we could use to discount cash flows across different periods. Now that we are talking about bonds, it is a good time to show how we can relax this assumption.

The yield curve is a *graphical depiction of the relation between maturity and yield to maturity prevailing in the market*



Note that this picture uses the 1-month, 3-month and 6-month Treasury Bills, and the 2, 3, 5, 10, and 30-year yields to maturity (as of May 18, 2014).

Aside: A Treasury Bill is a short-term zero-coupon bond.

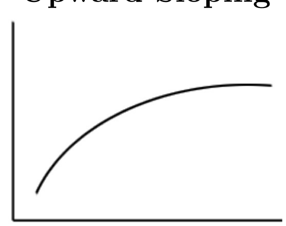
Recall that these are not the only bonds – there's also the 29-year bond (30-year bond from last year) and a 4-year bond (5-year bond from last year). But the yield curve typically uses nice round numbers. Because these bonds are the ones most recently issued, they are the most liquid, so their yields are the most reliable borrowing cost indicator.

The bonds that are most recently issued are called *on-the-run*; other bonds are *off-the-run*.

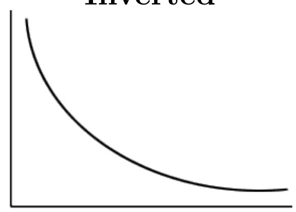
Note: the yield is higher, the longer the maturity $\Rightarrow$ *this is an upward sloping yield curve*

The yield curve is not always upward sloping. It can take on other shapes: downward sloping (inverted), hump-shaped, or even inverted hump-shape.

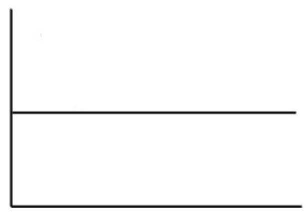
Upward Sloping



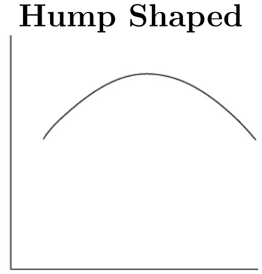
Inverted



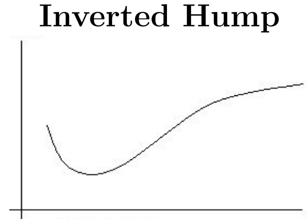
Flat



Hump Shaped



Inverted Hump



If we use the same r to discount cash flows at all maturities, we are implicitly assuming a flat yield curve.

The yield curve tells us the time value of money at various horizons. That is, they tell us the opportunity cost of receiving money at various times in the future versus today. We can use this information to help us value investments.

For simplicity, we will assume we have a *zero-couon yield curve*.

Notation:

- r_1 = YTM for a 1-year zero coupon bond. Then

$$P_1 = \frac{100}{1 + r_1}$$

is the price per \$100 of face value on this bond.

- r_2 = YTM for a 2-year zero coupon bond. Then

$$P_2 = \frac{100}{(1 + r_2)^2}$$

$\Rightarrow P_2 = F/(1 + r_2)^2$.

- In general, if r_t is the YTM for a $t-$year zero-coupon bond. Then

$$P_t = \frac{100}{(1 + r_t)^t}.$$

To understand how to use the zero-coupon yield curve, consider the following examples:

Example An investment costs 1M. It will pay 0.1M 1 year from now, 0.35M 2 years from now, and 0.6M 3 years from now. The zero-coupon yield curve is $r_1 = 1\%$, $r_2 = 1.5\%$, $r_3 = 4\%$. Should you undertake the investment?

We want to discount each cash flow using the appropriate rate, because this most accurately represents our opportunity cost. Thus

$$\text{NPV} = -1 + \frac{0.1}{1.01} + \frac{0.35}{(1.015)^2} + \frac{0.6}{(1.04)^3} = -0.028.$$

The firm should not undertake the investment.

Now consider a second example:

Example Consider the following zero-coupon bond prices below (per \$100 of face value):

- 1-yr: $P_1 = \$93.46$.
- 2-yr: $P_2 = \$89.00$
- 3-yr: $P_3 = \$83.96$

What is the price of a 3-year coupon bond with $F = \$100$ and $C = \$5$?

- Find the zero-coupon yield curve using the bond prices:

$$r_1 = \frac{100}{93.46} - 1 = 7\%$$
$$r_2 = \left(\frac{100}{89.00}\right)^{\frac{1}{2}} - 1 = 6\%$$
$$r_3 = \left(\frac{100}{83.96}\right)^{\frac{1}{3}} - 1 = 6\%$$

- Discount the bond's payments using the appropriate discount rate:

$$P = \frac{5}{1.07} + \frac{5}{(1.06)^2} + \frac{105}{(1.06)^3} = \$97.28$$

- We can still calculate the yield to maturity on this bond. Namely, we can solve for r such that

$$\$97.28 = \frac{5}{1+r} + \frac{5}{(1+r)^2} + \frac{105}{(1+r)^3}$$

  We know that this $r = \text{YTM}$ satisfies:

$$r > \frac{C}{F} = 5\%.$$

  Trial and error, or a calculator, will tell us that $r = 6.02\%$.

- We could have answered the question in the example above in an easier way. Note that \$93.46 is the price per \$100 of a 1-year zero-coupon bond. If we wanted to buy a 1-year zero with a face value of \$1, then we would pay \$0.9346.

  Applying this logic to all the coupon payments, we find:

$$P = 5(0.9346) + 5(0.8900) + 105(0.8396).$$

- According to this way of thinking, the 3-year coupon bonds is a basket of zero-coupon bonds, each with face value of \$1. These zero-coupon bonds are the basic building blocks for any security.

- Recall the definition of the discount factor from the lecture on present value. If we have a flat yield curve, the definition is

$$\text{DF}_t = \frac{1}{(1+r)^t}$$

  More generally,

$$\text{DF}_t = \frac{1}{(1+r_t)^t},$$

where r_t is the yield to maturity on the t-year zero-coupon bond. Note that the t-year discount factor is the same as a price on a t-year bond with face value of 1.

Written in terms of our discount factor notation, the formula for the price of the bond is given by:

$$P = 5\,(\mathrm{DF}_1) + 5\,(\mathrm{DF}_2) + 105\,(\mathrm{DF}_3) = \$97.28$$

One more question to take us into forward rates:

Consider two time points, s and t, with $s < t$:

Recall that for a flat term structure: $\mathrm{DF}_s \geq \mathrm{DF}_t$. Is this true more generally? Yes!

Why is $\mathrm{DF}_s \geq \mathrm{DF}_t$? This is the same as saying \$1 at time s is worth more than \$1 at t.

- Suppose rates were such that $\mathrm{DF}_t > \mathrm{DF}_s$:

$$\frac{1}{(1+r_t)^t} > \frac{1}{(1+r_s)^s}.$$

- This implies:

$$(1+r_s)^s > (1+r_t)^t \qquad t > s.$$

- Nobody would want to lend money all the way to time t. They would instead lend to s, then plan to put the money in the mattress between s and t.
- Another way to say this: you always prefer \$1 at time s than time t, because between s and t, you can put the \$1 under your mattress, and you will still have \$1 at time t

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(g) Forward Rates (OPTIONAL)

Previously, we implicitly made use of the concept of the investment rate between times s and t. Now let's make this explicit. Let r_t be the YTM on a t-year zero coupon bond. To understand forward rates, consider the following example:

Example Suppose $r_1 = 2\%$ and $r_2 = 3\%$. You will inherit \$1M next year from your distant cousin's estate. What rate can you lock in now?

By locking a future interest rate today, you do not need to bear the risk of future rate changes. This is an important way for corporations, who may anticipate future cash flows, to hedge their risks.

The rate that you can lock in today is called a forward rate. It satisfies:

$$\frac{(1+r_2)^2}{(1+r_1)} - 1 = \frac{1.03^2}{1.02} - 1 = 0.04$$

or 4%.

How will we do this? We want the following cash flows:

- Pay nothing now
- Pay $1M in 1 year
- Receive $1.04M in 2 years

Background: to short-sell a security means to borrow and then sell it. A short-sale accomplishes the same thing as selling a security: you get $. The difference is it leaves something left to do – you have to return the security by buying it back from the market.

Consider the following portfolio:

1. Buy a 2-year zero with $F = \$1.04M$. The cost of this bond today is:

$$P = \frac{1.04}{(1.03)^2} = 0.98M.$$

2. Short-sell a 1-year zero with $F = \$1M$. The proceeds of this sale today are:

$$P = \frac{1.0}{1.02} = 0.98M.$$

Have you achieved the desired cash flows? Yes, because your cash flows today are:

$$C_0 = -\frac{1.04}{(1.03)^2} + \frac{1}{(1.02)^2} = 0$$

(check the calculation!).

At end of year 1:

- Return borrowed 1-year zero by buying it back for face value, so you pay $1M$. Proceeds: $-1M$.

At end of year 2:

- Liquidate 2-year zero at face value. Proceeds $= 1.04M$.

⇒ We have satisfied our aim: we have locked in a rate of 4% between years 1 and 2.

Is this magic? How did I figure out these numbers? Let's work backwards:

- You know you will have \$1M to invest one year from now, so you can set up an investment where you will pay \$ 1 year from now and obtain money in two years
- So, you want to short-sell a one-year bond and use the proceeds to buy a two-year
- If you short-sell a 1-year bond w/\$1M face value, how much money will you get today? $\frac{1}{1+r_1} = \frac{1}{1.02}$ in our example, which means proceeds of $0.98M$
- What is the face value of the 2-year bond that you can then afford to buy? It solves the equation, where $r_1 = 2\%$ and $r_2 = 3\%$ in our example:

$$\frac{F}{(1+r_2)^2} = \frac{1}{1+r_1}$$

$$F = \frac{(1+r_2)^2}{1+r_1} = 1.04$$

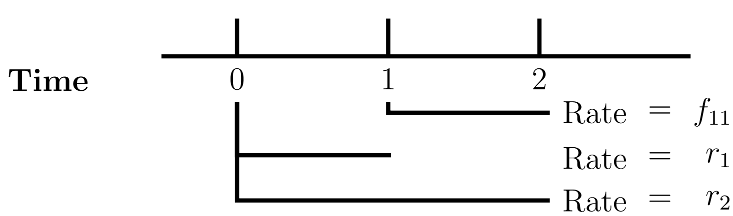
This leads us to our definition of a forward rate:

Definition *The forward rate f_{it} is the rate you can lock in now for lending between times t and $t+i$.*

$$f_{it} = \left[\frac{(1+r_{t+i})^{t+i}}{(1+r_t)^t}\right]^{1/i} - 1$$

In our example, we were interested in: $f_{11} = \frac{(1+r_2)^2}{1+r_1} - 1$.

Notice that the forward rate is the missing piece in the diagram.

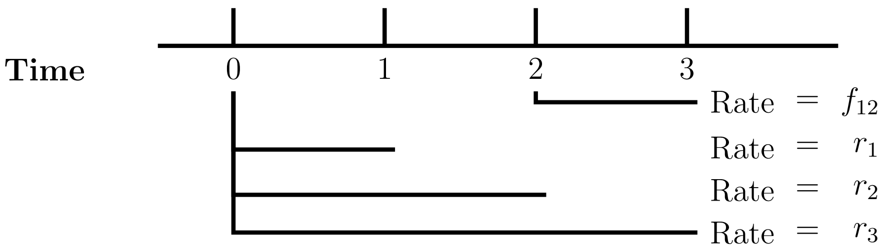


The diagram gives us an easy way to remember the formula:

$$(1+r_1)(1+f_{11}) = (1+r_2)^2$$

Behind this proof is the idea that the forward rate cannot give us a better way of lending money for two-years than the two-year rate. Now that we understand this formula for $t = 1$ and $i = 1$, let's try to understand it for a more general i and t.

Suppose the three-year rate was $r_3 = 3.5\%$. Could we use this same reasoning to lock in a rate between years 2 and 3? Or, in the diagram:



Yes. Similar reasoning shows that:

$$(1 + r_2)^2(1 + f_{12}) = (1 + r_3)^3$$

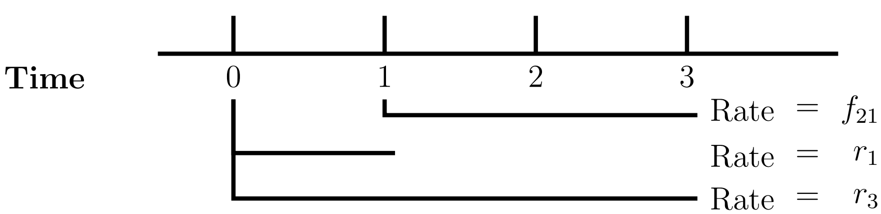
$$f_{12} = \frac{(1.035)^3}{1.03^2} - 1 = 4.5\%$$

How about a rate between years t and $t + 1$?

$$f_{1t} = \frac{(1 + r_{t+1})^{t+1}}{(1 + r_t)^t} - 1$$

$\Rightarrow$ f_{1t} is called a *forward rate*. It is to be distinguished from r_t, which is called the *spot rate*.

Suppose you wanted to lock in a rate between years 1 and 3? Diagram:



$$(1 + r_1)(1 + f_{21})^2 = (1 + r_3)^3$$

Lending at the 1-year rate then the forward rate has to equal lending at the 3-year rate.

$$f_{21} = \left(\frac{(1 + r_3)^3}{(1 + r_1)}\right)^{1/2} - 1$$

Note: we need to be careful to adjust this to make it an annual return!

In general: the formula tells that we can lock in an i period rate, t years in the future.

There's a neat formulation in terms of discount rates:

$$\begin{aligned} 1 + f_{it} &= \left[\frac{\frac{1}{(1+r_t)^t}}{\frac{1}{(1+r_{t+i})^{t+i}}} \right]^{1/i} \\ &= \left(\frac{\mathrm{DF}_t}{\mathrm{DF}_{t+i}} \right)^{1/i} \end{aligned}$$

Bottom line: forward rates allow you (or a corporation) to lock in a rate for an arbitrary time into the future, meaning that you do not need to bear the risk of future rate changes.