Assessing and Mitigating Algorithmic Bias in Criminal Risk Scores

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Introduction

Across the nation, we see an increasing reliance on algorithms to assess a defendant's likelihood of becoming a recidivist (a term used to describe convicted criminals who reoffend). These risk assessment scores significantly impact the severity of the sentence for defendants. For example, Broward County in Florida uses the COMPAS scoring algorithm to help determine pretrial release decisions.

ProPublica is an independent, nonprofit newsroom whose mission is to "expose abuses of power and betrayals of the public trust by government, business, and other institutions, using the moral force of investigative journalism to spur reform through the sustained spotlighting of wrongdoing." Through their analysis of the COMPAS recidivism algorithm for defendants in Broward County, they discovered that the algorithm correctly predicted recidivism for black and white defendants at approximately the same rate (around 60 percent). However, the accuracy itself paints a very misleading picture. The algorithm over-predicted black defendants to be at a high risk of recidivating, while the algorithm predicted white defendants to be at a lower risk than they actually were.

ProPublica's analysis explores how scores are distributed across different demographics, creates a logistic regression to model the odds of getting a higher COMPAS score based on demographic features such as race, and uses a Cox proportional hazards model to measure how accurate the COMPAS predictions are (i.e. how often did people actually recidivate after receiving a score). In this project, we perform our own exploration and modeling of criminal risk scores and, using ProPublica's methodology for measuring bias, we compare our model's bias to the bias COMPAS scores exhibit.

Problem Statement

We want to build our own model that uses the existing features from an individual's profile to predict the likelihood of recidivating (a model that is not created in ProPublica's analysis). After developing our own logistic regression model, we want to (1) compare the predictive accuracy of our model to the COMPAS model and (2) explore what kinds of bias exist in our model using the approach outlined by ProPublica and see if there are techniques we can employ to minimize our own algorithmic bias.

Data Description

To construct their current dataset, ProPublica started with a database of COMPAS scores and built a profile of each individual's criminal history, both before and after they were scored. They then collected public criminal records from the Broward County Clerk's Office website through April 1, 2016, and merged these two datasets into one by matching an individual's first and last names as well as date of birth.

In the data set we look at, they considered only people who either "recidivated within two years of their crime or recidivated in two years, or had at least tow years outside of a correctional facility."

Data Dictionary

ProPublica did not provide a data dictionary explaining their variables. Through some manual exploration, we came up with the following descriptions for our best guess at what each variable measures.

Variable	Description
id	unque identifier for each individual
name	first and last name
first	first name
last	last name
compas_screening_da	atlate on which decile_score was given
sex	sex (male or female)
dob	date of birth
age	age in years
age_cat	age category (less than 25, 25-45, greater than 45)
race	race (African-American, Asian, Caucasian, Hispanic, Native American, Other)
juv_fel_count	juvenile felony count
decile_score	COMPAS Risk of Recidivism score from 1 to 10
juv_misd_count	juvenile misdemeanor count
juv_other_count	juvenile other offenses count
priors_count	prior offenses count
days_b_screening_ar	encentrober of days between COMPAS screening and arrest
c_jail_in	jail entry date for original crime
c_jail_out	jail exit date for original crime
c_case_number	case number for original crime
$c_offense_date$	offense date of original crime
c_arrest_date	arrest date for original crime
$c_days_from_compa$	sdays between COMPAS screening and original crime offense date
c_charge_degree	charge degree of original crime
c_charge_desc	description of charge for original crime
is_recid	binary indicator of recidivation (1=individual recidivated, 0=individual did not recidivate)
r_case_number	case number of follow-up crime
r_charge_degree	charge degree of follow-up crime
r_days_from_arrest	number of days between follow-up crime and arrest date
r_offense_date	date of follow-up crime
r_charge_desc	description of charge for follow-up crime
r_jail_in	jail entry date for follow-up crime
r_jail_out	jail exit date for follow-up crime
violent recid	values are all NA. This column is ignored.
is_voilent_recid	binary indicator of violent follow-up crime (1=follow-up crime was violent,
	0=follow-up crime was non-violent)
vr_case_number	case number for violent follow-up crime
vr charge degree	charge degree for violent follow-up crime
vr_offense_date	date of offense for violent follow-up crime
vr_charge_desc	description of charge for violent follow-up crime
type_of_assessment	the type of COMPAS score given for decile_score (here all values are Risk of Recidivism)
decile_score.1	repeat column of decile_score
score_text	ProPublica-defined category of decile_score (High=8-10, Medium=5-7, Low=1-4)
screening date	repeat column of compas_screening_date
<u> </u>	nthe type of COMPAS score given for v_decile_score (here all values are
v_decile_score	Risk_of_Violence) COMPAS Risk of Violence score from 1 to 10

Variable	Description
v_score_text	ProPublica-defined category of v_decile_score (High=8-10, Medium=5-7,
	Low=1-4)
$v_screening_date$	date on which v_decile_score was given
in_custody	date on which individual was brought into custody
out_custody	date on which individual was released from custody
priors_count.1	repeat column of priors_count
two_year_recid	binary indicator of recidivation within two years of scoring (1=individual
	recidivated, 0=individual did not recidivate)
start	unclear definition but not used in our analysis
end	unclear definition but not used in our analysis
event	unclear definition but not used in our analysis

ProPublica obtained this data with the goal of analyzing Northpointe Inc.'s commercial recidivism modeling tool – COMPAS. Aggregating data from public records, they collected data on 18,610 individuals who received COMPAS scores from 2013 to 2014, including demographic information, public criminal records, and incarceration records.

How are COMPAS scores used?

ProPublica describes that at least in Broward County, they "primarily [use] the score to determine whether to release or detain a defendant before his or her trial." 11,757 of the individuals in the database had their COMPAS scores used to assess whether or not they should be released before their trial.

What are COMPAS scores?

There are three types of COMPAS score. Each measures a type of 'risk' associated with a criminal re-offending in some way on a scale of 1 (low) to 10 (high). As ProPublica describes, these include

- Risk of Recidivism: ProPublica defines this as the person in question committing a "criminal offense that [results] in a jail booking and [takes] place after the crime for which the person was COMPAS scored." Northpointe hopes to use this score to predict "a new misdemeanor or felony offense within two years of the COMPAS administration date."
- Risk of Violence: They use the FBI definition of violent crime:
 - In the FBI's Uniform Crime Reporting (UCR) Program, violent crime is composed of four offenses: murder and nonnegligent manslaughter, forcible rape, robbery, and aggravated assault. Violent crimes are defined in the UCR Program as those offenses which involve force or threat of force. ucr.fbi.gov
- Risk of Failure to Appear: As evidenced by the name, this describes a failure to appear at the court hearing.

Exploratory Data Analysis

We first load in the data provided by ProPublica, and take a quick look at summaries of each variable to get a sense of distribution and potential outliers. We'll look at half of the columns at a time to keep things manageable.

```
compas_data <- read.csv('compas-scores-two-years.csv')
summary(compas_data[1:(ncol(compas_data)/2)])</pre>
```

```
##
                                    name
                                                       first
##
   Min.
                1
                     anthony smith
                                           3
                                               michael
                                                           : 149
                     angel santiago
                                           2
                                               christopher: 109
   1st Qu.: 2735
   Median : 5510
                     anthony gonzalez :
                                               james
```

```
Mean
           : 5501
                    anthony louis
                                          2
                                              anthony
                                                             83
                                      :
##
    3rd Qu.: 8246
                    brandon whitfield:
                                                             76
                                          2
                                              robert
                    carlos vasquez
##
          :11001
                                          2
                                              john
                                                             74
##
                                                          :6639
                     (Other)
                                      :7201
                                               (Other)
##
          last
                    compas_screening_date
                                                sex
                                                                  dob
##
                    2013-02-20: 32
                                           Female:1395
                                                          1987-02-04:
    williams: 83
                                                                        5
    iohnson:
               76
                    2013-03-20:
                                           Male :5819
                                                          1987-12-21:
                                  32
    brown
##
            :
               68
                    2013-02-07:
                                  31
                                                          1989-04-27:
##
    smith
            :
               65
                    2013-04-20:
                                  30
                                                          1989-08-31:
                                                                        5
##
                                  29
                                                                         5
    jones
               57
                    2013-01-03:
                                                          1990-02-22:
    davis
            : 46
                    2013-04-25:
                                  28
                                                          1990-05-02:
                                                                         5
##
    (Other):6819
                               :7032
                                                          (Other)
                     (Other)
                                                                    :7184
##
                                                          race
         age
                               age_cat
##
                                   :4109
    Min.
           :18.0
                   25 - 45
                                           African-American:3696
##
    1st Qu.:25.0
                   Greater than 45:1576
                                           Asian
                                                            : 32
##
    Median:31.0
                   Less than 25
                                   :1529
                                           Caucasian
                                                            :2454
##
    Mean
          :34.8
                                                            : 637
                                           Hispanic
##
    3rd Qu.:42.0
                                           Native American: 18
##
    Max. :96.0
                                           Other
                                                            : 377
##
##
    juv_fel_count
                      decile_score
                                      juv_misd_count
                                                        juv_other_count
    Min. : 0.000
                     Min. : 1.00
                                      Min.
                                            : 0.000
                                                        Min. : 0.000
    1st Qu.: 0.000
                     1st Qu.: 2.00
                                      1st Qu.: 0.000
                                                        1st Qu.: 0.000
##
##
    Median : 0.000
                     Median: 4.00
                                      Median : 0.000
                                                        Median : 0.000
   Mean : 0.067
##
                     Mean : 4.51
                                      Mean
                                                        Mean : 0.109
                                            : 0.091
    3rd Qu.: 0.000
                                                        3rd Qu.: 0.000
                     3rd Qu.: 7.00
                                      3rd Qu.: 0.000
##
    Max.
          :20.000
                     Max.
                            :10.00
                                      Max.
                                             :13.000
                                                        Max. :17.000
##
##
    priors_count
                    days_b_screening_arrest
                                                            c_jail_in
##
   Min.
          : 0.00
                    Min.
                           :-414.0
                                                                 : 307
##
    1st Qu.: 0.00
                    1st Qu.: -1.0
                                             2013-01-01 01:31:55:
##
    Median: 2.00
                    Median :
                              -1.0
                                             2013-01-01 03:16:15:
##
    Mean
          : 3.47
                    Mean
                                3.3
                                             2013-01-01 03:28:03:
##
    3rd Qu.: 5.00
                    3rd Qu.:
                                             2013-01-01 04:17:22:
                                0.0
##
    Max. :38.00
                    Max.
                            :1057.0
                                             2013-01-01 04:29:04:
##
                    NA's
                            :307
                                             (Other)
                                                                 :6902
##
                  c jail out
                                      c case number
                                                         c offense date
##
                        : 307
                                             : 22
                                                                :1159
##
    2013-09-12 10:31:00:
                           3
                                00004068CF10A:
                                                  1
                                                      2013-01-14:
                                                                   26
##
    2013-09-14 05:58:00:
                            3
                                00022077MM10A:
                                                      2013-02-22:
                                                  1
    2013-09-28 02:10:00:
                                01004839CF10A:
                                                      2013-03-01:
                                                  1
##
    2013-02-06 10:01:51:
                            2
                                01006487CF10D:
                                                      2013-01-11:
                                                  1
    2013-06-13 10:32:00:
                            2
                                01007205MM10A:
                                                      2013-02-16:
##
                                                  1
##
    (Other)
                        :6894
                                (Other)
                                             :7187
                                                      (Other)
                                                                :5933
##
       c_arrest_date    c_days_from_compas c_charge_degree
##
              :6077
                                  0
                      Min.
                                          F:4666
                            :
##
    2013-02-06:
                  9
                      1st Qu.:
                                  1
                                          M:2548
##
    2013-03-22:
                      Median:
                  8
                                  1
##
    2013-05-15:
                  8
                      Mean
                                 58
##
    2013-01-10:
                  7
                      3rd Qu.:
                              :9485
##
    2013-01-11:
                  7
                      Max.
##
                      NA's
    (Other)
              :1098
                              :22
##
                           c_charge_desc
                                                                 r_case_number
                                             is_recid
##
    Battery
                                  :1156
                                          Min.
                                                 :0.000
                                                                         :3743
```

```
arrest case no charge
                                   :1137
                                           1st Qu.:0.000
                                                             13000349MM10A:
##
    Possession of Cocaine
                                   : 474
                                           Median : 0.000
                                                             13000445MM20A:
                                                                               1
                                                   :0.481
                                                            13000677MM20A:
   Grand Theft in the 3rd Degree: 425
                                           Mean
                                                                               1
   Driving While License Revoked: 200
                                           3rd Qu.:1.000
                                                            13000758MM30A:
                                                                               1
    Driving Under The Influence
                                   : 135
                                           Max.
                                                   :1.000
                                                             13000785MM30A:
                                                                               1
##
    (Other)
                                   :3687
                                                             (Other)
                                                                           :3466
```

Here we notice there may be large outliers in many of the crime count variables, such as <code>juv_fel_count</code>, <code>juv_misd_count</code>, <code>juv_other_count</code>, and <code>priors_count</code>. We expect these are simply accurate observations corresponding to individuals with high numbers of prior offenses. Thus we will not remove these individuals from the data but will be aware of them as potential influential points when we later fit any models. We also note that the values for the 'days_from' variables are quite variable which may be relevant if we use those variables in later analysis. Looking at the second half of the columns, we have:

summary(compas_data[(ncol(compas_data)/2 + 1):ncol(compas_data)])

```
r_charge_degree r_days_from_arrest
##
                                              r_offense_date
##
            :3743
                     Min.
                             : -1
                                                     :3743
                      1st Qu.:
                                0
##
    (M1)
            :1201
                                          2014-12-08:
                                                        12
##
    (M2)
            :1107
                     Median :
                                0
                                          2015-01-28:
                                                        11
##
    (F3)
            : 892
                     Mean
                             : 20
                                          2014-09-15:
    (F2)
                                                        10
##
            : 168
                     3rd Qu.:
                                          2014-10-17:
                                1
##
    (F1)
               51
                     Max.
                             :993
                                          2015-02-10:
            :
                             :4898
##
    (Other):
               52
                     NA's
                                           (Other)
                                                     :3418
##
                                r charge desc
                                                      r jail in
                                        :3801
##
                                                            :4898
    Driving License Suspended
                                        : 258
##
                                                 2014-05-27:
##
    Possess Cannabis/20 Grams Or Less: 253
                                                 2013-11-22:
    Resist/Obstruct W/O Violence
                                        : 201
                                                 2014-06-05:
##
    Battery
                                        : 192
                                                 2014-07-10:
                                                                8
##
    Operating W/O Valid License
                                        : 172
                                                 2014-10-17:
                                                                8
##
    (Other)
                                        :2337
                                                 (Other)
                                                            :2275
##
         r_jail_out
                        violent_recid
                                        is_violent_recid
                                                                 vr_case_number
##
                        Mode:logical
                                        Min.
                                                :0.000
                                                                         :6395
               :4898
##
    2014-02-18:
                   9
                        NA's:7214
                                        1st Qu.:0.000
                                                           13001383CF10A:
                                                                             1
##
    2014-12-09:
                   9
                                        Median : 0.000
                                                           13001876CF10A:
                                                                             1
##
    2015-05-15:
                   9
                                        Mean
                                                :0.114
                                                           13002119CF10A:
                                                                             1
##
    2013-11-13:
                   8
                                        3rd Qu.:0.000
                                                           13002546CF10A:
                                                                             1
##
    2014-07-11:
                   8
                                        Max.
                                                :1.000
                                                           13003421CF10A:
                                                                             1
##
    (Other)
               :2273
                                                           (Other)
                                                                         : 814
##
    vr_charge_degree
                         vr_offense_date
                                                                    vr_charge_desc
##
            :6395
                                  :6395
                                                                           :6395
##
    (M1)
            : 344
                       2015-08-15:
                                      6
                                          Battery
                                                                            : 329
    (F3)
            : 228
                       2013-11-14:
                                          Battery on Law Enforc Officer:
##
    (F2)
                                          Felony Battery (Dom Strang)
##
            : 162
                       2014-02-18:
                                      4
    (F1)
                       2014-10-29:
                                          Aggravated Assault W/Dead Weap:
##
               38
##
    (M2)
               19
                       2014-12-26:
                                      4
                                          Aggrav Battery w/Deadly Weapon:
                                                                              34
##
    (Other):
                       (Other)
                                  : 797
                                           (Other)
                                                                           : 343
##
              type_of_assessment decile_score.1
                                                     score_text
##
    Risk of Recidivism:7214
                                   Min.
                                          : 1.00
                                                    High :1403
##
                                   1st Qu.: 2.00
                                                    Low
                                                           :3897
##
                                   Median: 4.00
                                                    Medium: 1914
##
                                   Mean
                                          : 4.51
##
                                   3rd Qu.: 7.00
##
                                   Max.
                                          :10.00
```

```
##
##
       screening_date
                              v_type_of_assessment v_decile_score
                       Risk of Violence:7214
##
    2013-02-20:
                  32
                                                    Min.
    2013-03-20:
                                                    1st Qu.: 1.00
##
                  32
##
    2013-02-07:
                                                    Median: 3.00
                                                            : 3.69
##
    2013-04-20:
                  30
                                                    Mean
                                                    3rd Qu.: 5.00
    2013-01-03:
##
    2013-04-25:
                  28
                                                    Max.
                                                            :10.00
##
    (Other)
               :7032
##
    v_score_text
                     v_screening_date
                                             in_custody
                                                               out_custody
    High : 714
                   2013-02-20:
                                 32
                                                  : 236
                                                                      : 236
          :4761
                                                     20
##
    Low
                   2013-03-20:
                                 32
                                       2013-02-22:
                                                           2020-01-01:
                                                                         61
##
    Medium: 1739
                   2013-02-07:
                                 31
                                       2013-12-12:
                                                     20
                                                           2013-05-14:
##
                   2013-04-20:
                                 30
                                       2014-01-04:
                                                     20
                                                           2014-02-04:
                                                                         24
##
                   2013-01-03:
                                 29
                                       2014-01-22:
                                                                         23
                                                     20
                                                           2013-11-26:
##
                   2013-04-25:
                                 28
                                       2013-01-27:
                                                     19
                                                           2013-02-15:
                                                                         21
                              :7032
                                                  :6879
##
                   (Other)
                                        (Other)
                                                           (Other)
                                                                      :6824
    priors_count.1
                                                           event
##
                         start
                                            end
##
   Min.
          : 0.00
                                0.0
                                              :
                                                  0
                                                              :0.000
                     \mathtt{Min}.
                                      Min.
                                                      Min.
    1st Qu.: 0.00
                     1st Qu.:
                                0.0
                                      1st Qu.: 148
                                                       1st Qu.:0.000
                     Median :
##
  Median: 2.00
                               0.0
                                      Median: 530
                                                      Median : 0.000
           : 3.47
                                              : 553
                                                              :0.383
   Mean
                     Mean
                             : 11.5
                                      Mean
                                                      Mean
    3rd Qu.: 5.00
                                      3rd Qu.: 914
##
                     3rd Qu.: 1.0
                                                       3rd Qu.:1.000
            :38.00
                             :937.0
##
    Max.
                     Max.
                                      Max.
                                              :1186
                                                       Max.
                                                              :1.000
##
```

With the second half we have similar characteristics as before. We remove the violent_recid column given that all values are NA (as mentioned in the data dictionary). Apart from that column, we make no other changes.

ProPublica's Bias-Assessment Model

This section is meant to recreate ProPublica's logistic regression model for assessing bias in COMPAS scores. The code and explanations in this section are adapted from their published methodlogy and code.

Using the same data set, they first filter rows based on the following criteria.

- 1. consider only individuals with a COMPAS score
- 2. assure the COMPAS score corresponds to the correct crime i.e. the score was given within 30 days of the arrest
- 3. do not include ordinary traffic offenses

Next we use their code to perform this filtering:

[1] 6172

In order to use this data to assess racial bias in scoring, the ProPublica analysts first create several factor variables from the existing columns.

For the age_factor they make "25 - 45" the reference level, for race_factor "Caucasian" is the reference level, for gender_factor "Male" is the reference level.

Next they fit a logistic regression model to predict score_factor from the other variables.

```
# code from https://github.com/propublica/compas-analysis
pp_model <- glm(</pre>
 score_factor ~ gender_factor + age_factor + race_factor + priors_count +
   crime_factor + two_year_recid,
 family="binomial",
 data=df
)
summary(pp_model)
##
## Call:
  glm(formula = score factor ~ gender factor + age factor + race factor +
      priors_count + crime_factor + two_year_recid, family = "binomial",
##
      data = df
##
## Deviance Residuals:
     Min
              1Q Median
                              3Q
                                     Max
## -2.997 -0.792 -0.330
                           0.812
                                   2.602
##
## Coefficients:
##
                              Estimate Std. Error z value Pr(>|z|)
## (Intercept)
                               -1.5255
                                           0.0785 -19.43 < 2e-16 ***
## gender_factorFemale
                                0.2213
                                           0.0795
                                                     2.78 0.00539 **
                                           0.0991 -13.68 < 2e-16 ***
## age_factorGreater than 45
                               -1.3556
## age factorLess than 25
                                1.3084
                                           0.0759
                                                   17.23 < 2e-16 ***
## race_factorAfrican-American
                                           0.0693
                                                   6.88 5.9e-12 ***
                               0.4772
## race_factorAsian
                               -0.2544
                                           0.4782
                                                   -0.53 0.59472
                                                   -3.34 0.00083 ***
## race_factorHispanic
                               -0.4284
                                           0.1281
## race factorNative American
                                                    1.82 0.06878 .
                                1.3942
                                           0.7661
## race factorOther
                               -0.8263
                                           0.1621
                                                  -5.10 3.4e-07 ***
## priors_count
                               0.2689
                                           0.0111 24.22 < 2e-16 ***
                                                   -4.68 2.9e-06 ***
## crime_factorM
                               -0.3112
                                           0.0665
## two_year_recid
                                0.6859
                                           0.0640
                                                   10.71 < 2e-16 ***
## ---
## Signif. codes: 0 '***' 0.001 '**' 0.05 '.' 0.1 ' ' 1
```

```
## (Dispersion parameter for binomial family taken to be 1)
##
## Null deviance: 8483.3 on 6171 degrees of freedom
## Residual deviance: 6168.4 on 6160 degrees of freedom
## AIC: 6192
##
## Number of Fisher Scoring iterations: 5
```

Calculating Relative Risk by Demographic

With a logistic regression model to measure the likelihood of getting a high COMPAS score based on demographic features, they compute how much more likely different demographic groups are to receive a higher score than others. The logistic regression model allows us to measure this difference after correcting for the other variables included in the model. The quantity ProPublica uses to compare black defendants to white defendants (or men to women, old defendants to young defendants, etc.) is called **relative risk**. ProPublica does not explain where this quantity comes from in their analysis, so we'll provide some quick background on logistic regression to justify the calculation. The following explanation is inspired by USC professor Sandy Eckel's slides here.

Logistic regression models a linear relationship between the log odds ratio for the probability of interest and the given predictor variables. An odds ratio measures the odds of success

odds ratio =
$$\frac{\text{probability of success}}{\text{probability of failure}} = \frac{P}{1 - P}$$

where P is the probability of success. The log odds ratio is simply the log of this quantity. Thus the logistic regression model for observation x_i is

$$\log(\frac{P_{x_i}}{1 - P_{x_i}}) = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}$$

where the probability of success for x_i is P_{x_i} , and we have p predictors with corresponding coeffecients β_j and observed values x_{ij} for $j=1,\ldots,p$. Given that our model here uses categorical predictors (factors), the coeffecients we estimate give us the **change in log odds** for the corresponding variable. Thus if we let P_{x_i} be the probability that individual x_i gets a high COMPAS score, then with coefficient β_1 for gender_factor, we would have

 β_0 : the log odds of getting a high COMPAS score for men

 β_1 : the difference in log odds of getting a high COMPAS score for women compared to men

The important observation here is that because men are the reference level for the **gender_factor** categorical variable, β_1 measures a **difference** relative to men. Thus if we want to answer the question, "How much more likely are women to get a high COMPAS score than men?" we'll want to use

 $\beta_0 + \beta_1$: the log odds of getting a high COMPAS score for women

to get the comparison. One other observation will also be helpful to calculate relative risk. We solve for P_{x_i} as follows.

$$\log(\frac{P}{1-P}) = x \to P = \frac{e^x}{1+e^x} = \mathtt{sigmoid}(x)$$

Thus we calculate relative risk for the categorical variable corresponding to β_1 as:

$$\mathtt{relative\ risk} = \frac{P_1}{P_2} = \frac{\mathtt{sigmoid}(\beta_0 + \beta_1)}{\mathtt{sigmoid}(\beta_0)}$$

ProPublica computes relative risk to compare blacks to whites, men to women, and people under 25 to middle-aged people in terms of COMPAS scores. They get the following results.

```
# code adapted from https://github.com/propublica/compas-analysis
model_intercept <- coef(pp_model)['(Intercept)']
black_coef <- coef(pp_model)['race_factorAfrican-American']
(relative_risk_black_v_white <- sigmoid(model_intercept + black_coef) / sigmoid(model_intercept))
## (Intercept)
## 1.453</pre>
```

As ProPublica states, this shows us that "Black defendants are 45% more likely than white defendants to receive a higher [COMPAS] score correcting for the seriousness of their crime, previous arrests, and future criminal behavior." Similarly, women are 19.4% more likely than men and people under 25 are 2.5 times as likely as middle aged people to get a higher score:

```
# code adapted from https://github.com/propublica/compas-analysis
woman_coef <- coef(pp_model)['gender_factorFemale']
(relative_risk_woman_v_man <- sigmoid(model_intercept + woman_coef) / sigmoid(model_intercept))
## (Intercept)
## 1.195
# code adapted from https://github.com/propublica/compas-analysis
age_coef <- coef(pp_model)['age_factorLess than 25']
(relative_risk_young_v_middleage <- sigmoid(model_intercept + age_coef) / sigmoid(model_intercept))
## (Intercept)
## (Intercept)</pre>
```

Our Logistic Regression Model

Now that we have gone through ProPublica's analysis, we will create our own logistic regression model to predict recidivism using ProPublica's dataset. To keep the scope of the project manageable, our focus here will be minimizing racial bias. For our model, we chose to include age, sex, juvenile infraction/misdemeanor/felony counts, prior offenses count, current charge degree (felony or misdemeanor), and time spent in jail for the current crime in our model. Although the dataset contains additional information about each individual, we did not include those in our model because it is unique to an individual (name, id, etc.) or related to a follow-up crime (and we cannot use follow-up crime information in predicting whether or not an individual will recidivate).

Additionally, we focused primarily on racial bias in our analysis, but we could imagine removing age and/or sex from our model if we wanted to investigate those biases as well.

Comparing Our Model's Predictive Performance to the COMPAS Model

Here we fit our model and compare its predictive accuracy to the COMPAS model. To get estimates of our predictive accuracy from the training data, we use 10-fold cross-validation.

```
set.seed(123)
all_predictions <- data.frame(id = compas_data$id, predicted_score = NA)</pre>
```

```
# split the data into 10 folds
all_indices <- 1:nrow(compas_data)</pre>
folds <- createFolds(all indices, k = 10)
for (i in 1:10) {
  # make train/test split
  test_indices <- folds[[i]]</pre>
  training indices <- all indices[-test indices]</pre>
  test_data <- compas_data[test_indices,]</pre>
  training_data <- compas_data[training_indices,]</pre>
  # build model on training data
  vars <- training_data %>%
    select(sex, age, ends_with('count'), c_charge_degree, is_recid)
  regmod <- glm(is_recid ~., family = binomial(link='logit'), data = vars)</pre>
  # store predictions on test data
  predicted_scores <- predict(regmod, newdata = test_data, type = 'response')</pre>
  all_predictions[names(predicted_scores),]$predicted_score <- predicted_scores</pre>
predicted_scores <- all_predictions$predicted_score</pre>
confusionMatrix(data = as.numeric(predicted_scores > 0.5), reference = compas_data$is_recid)
## Confusion Matrix and Statistics
##
##
             Reference
## Prediction
                 0
##
            0 2744 1322
            1 999 2149
##
##
##
                   Accuracy: 0.678
##
                     95% CI: (0.667, 0.689)
##
       No Information Rate: 0.519
       P-Value [Acc > NIR] : < 2e-16
##
##
##
                      Kappa: 0.353
##
    Mcnemar's Test P-Value : 2.33e-11
##
               Sensitivity: 0.733
##
               Specificity: 0.619
##
            Pos Pred Value: 0.675
##
##
            Neg Pred Value: 0.683
##
                 Prevalence: 0.519
            Detection Rate: 0.380
##
##
      Detection Prevalence: 0.564
##
         Balanced Accuracy: 0.676
##
##
          'Positive' Class: 0
##
```

The COMPAS Model has a predictive accuracy of roughly 62%. Using 10-fold cross validation, we were able to gather predictions for each individual in our dataset and achieve an accuracy of approximately 68%. Although the difference is not drastic, it is still a significant improvement over the COMPAS scoring model's performance. Additionally, our algorithm only uses a handful of features for each individual, while the COMPAS algorithm leverages data from a far more involved, detailed assessment.

Using ProPublica's Bias-Assessment Model on Our Model

We can see if our model suffers from the same bias using ProPublica's bias-assessment methodology. We'll set up the same model they did, but this time instead of predicting score_factor for the COMPAS score, we'll create a score_factor variable from our model's predictions. We create a new data frame with new_score as our risk of recidivism score. We also create a new_score_text variable that defines score categories in the same way, but this time based on a 0-1 scale instead of a 0-10 scale. Recall ProPublica defined this category as: High=8-10, Medium=5-7, Low=1-4. Thus we'll do: High=0.75-1.0, Medium=0.45-0.75, Low=0-0.45.

```
our_scores <- compas_data %>%
  mutate(new_score = predicted_scores) %>%
  mutate(
    new_score_text = as.factor(
    ifelse(new_score >= 0.75, 'High',
        ifelse(new_score >= 0.45, 'Medium', 'Low')))
)
```

Now we train the same bias-assessment model, this time using new_score instead of decile_score and new_score_text instead of score_text.

```
new df <- our scores %>%
  select(age, c_charge_degree, race, age_cat, new_score_text, sex,
         priors_count, days_b_screening_arrest, new_score, is_recid,
         two_year_recid, c_jail_in, c_jail_out) %>%
  filter(days b screening arrest <= 30) %>%
  filter(days_b_screening_arrest >= -30) %>%
  filter(is recid != -1) %>%
  filter(c_charge_degree != "0") %>%
  filter(new_score_text != 'N/A') %>%
  mutate(crime_factor = factor(c_charge_degree)) %>%
  mutate(age_factor = as.factor(age_cat)) %>%
  within(age_factor <- relevel(age_factor, ref = "25 - 45")) %>%
  mutate(race_factor = factor(race)) %>%
  within(race_factor <- relevel(race_factor, ref = "Caucasian")) %>%
  mutate(gender_factor = factor(sex, labels= c("Female", "Male"))) %>%
  within(gender_factor <- relevel(gender_factor, ref = "Male")) %>%
  mutate(score_factor = factor(new_score_text != "Low", labels = c("LowScore", "HighScore")))
new_pp_model <- glm(</pre>
  score_factor ~ gender_factor + age_factor + race_factor + priors_count +
    crime_factor + two_year_recid,
  family="binomial",
  data=new df
)
```

Our model's bias against black vs. white people

Now that we have re-fit ProPublica's model after using our own risk of recidivism scores, we can see what the relative risk for African-Americans relative to Caucasians is.

```
model_intercept <- coef(new_pp_model)['(Intercept)']
black_coef <- coef(new_pp_model)['race_factorAfrican-American']
(relative_risk_black_v_white <- sigmoid(model_intercept + black_coef) / sigmoid(model_intercept))
## (Intercept)</pre>
```

1.209

Here we see that with our model, there is still a bias against blacks. They are still 21% more likely than whites to get a high risk of recidivism score after controlling for other features. However, this is a significant improvement over the level of bias in COMPAS scores, and our estimate of predictive accuracy was still higher than that of the COMPAS scores!

Mitigating Bias Directly

Much work has been done exploring techniques that allow for fair modeling and prediction in the presence of biased data or algorithms, and this area of study is rich and actively researched. Researchers Faisal Kamiran and Toon Calders, in their 2011 paper "Data preprocessing techniques for classification without discrimination," describe several techniques that alter the given dataset to (ideally) eliminate the source of the bias. Several other techniques deal with modifying classification/regression algorithms themselves to make fairer predictions. Here, the general idea is to instead clean the bias out of the data, after which normal classification methods can be used. Before we discuss the techniques we employ for dealing with bias, we first introduce a few key concepts.

Algorithmic Bias

This Wikipedia article provides a nice overview of different types of Algorithmic bias and several examples. Bias can manifest in an algorithm in various ways and can have multiple causes. The Wikipedia article linked above uses to term "pre-existing bias" to loosely describe a type of bias resulting from baking social or institutional biases into algorithms. Here the source of bias is not necessarily the algorithm but instead the values encoded into it by its creators or by the data it sees. For our brief foray into algorithmic bias here, we focus on this type of bias under the assumption that some underlying bias does exist in the United States criminal justice system and thus in our data set.

Protected/Sensitive Attributes

The term *protected attribute* or *sensitive attribute* typically refers to a discriptor of an individual upon which it is illegal to discriminate under the Fair Work Act. These include characteristics such as sexual orientation, gender identity, and race.

Combatting Algorithmic Bias

Broadly speaking, research in algorithmic bias works to (1) identify where and in what way bias is present (ProPublica's analysis is one such example), (2) come up with ways to constrain models to enforce fair predictions, or (3) alter the underlying data to minimize bias in the data itself. For our recidivism predictions, we'll employ some techniques from category 3 – altering the underlying data – based on Kamiran and Calders' work.

Mitigating Bias Through Data Preprocessing

Kamiran and Calders outline four types of techniques for preprocessing to mitigate bias, described at length in their paper:

- 1. **Suppression:** Given a sensitive attribute S, find and remove S and the other features most correlated with S.
- 2. Massaging the dataset: Given a sensitive attribute S, swap the labels of some observations with differing values for S to decrease discrimination while maintaining the same overall class distribution.

- 3. **Reweighing:** Up- or down-weight observations in the training data based on whether they have an under- or over-represented combinations of sensitive attribute S and the response variable.
- 4. **Sampling:** Calculate sample sizes for combinations of sensitive attribute S and the response that would, as Kamiran and Calders put it, "make the dataset descrimination-free." Then under- or over-sample observations accordingly to create a data set with those proportions.

Here we focus on Sampling.

Applying Sampling to the COMPAS Data to Combat Racial Bias

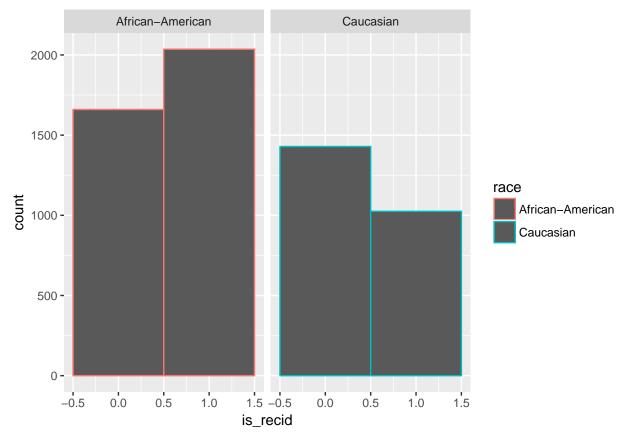
As mentioned before, to narrow the scope of this report, we focus on *racial bias* against black people as compared to white people, and we implement sampling to combat this type of bias only. If our example words for racial bias, we could theoretically extend it to each of the protected attributes in the data.

First we need to establish what the class imbalance is between blacks and whites in our data. We look for an imbalance in terms of recidivism rates, since recidivism is what our model predicts.

```
(recid rates <- compas data %>%
  select(race, is_recid) %>%
  group_by(race) %>%
  summarize(yes_recid = sum(is_recid == 1), no_recid = sum(is_recid == 0)))
## # A tibble: 6 x 3
##
                  race yes_recid no_recid
##
               <fctr>
                           <int>
                                     <int>
## 1 African-American
                            2036
                                      1660
## 2
                Asian
                              11
                                       21
## 3
            Caucasian
                            1025
                                      1429
## 4
                                       392
             Hispanic
                             245
## 5
      Native American
                              11
                                         7
                                       234
## 6
                Other
                             143
```

Focusing on African-Americans and Caucasians and plotting this table as a bar chart, we have:

```
compas_data %>%
  filter(race %in% c('African-American', 'Caucasian')) %>%
  ggplot(aes(x = is_recid, color = race)) +
   geom_histogram(bins = 2) +
   facet_wrap(~ race)
```



Kamiran and Calders define four types of observation with respect to a protected attribute and a class label. Their definitions are as follows:

- Deprived community with positive class labels (DP)
- Deprived community with negative class labels (DN)
- Favored community with positive class labels (FP)
- Favored community with negative class labels (FN)

In our case these correspond to:

- DP: African-Americans with is_recid = 1
- DN: African-Americans with is recid = 0
- FP: Caucasians with is_recid = 1
- FN: Caucasians with is_recid = 0

We create these partitions for our data.

```
DP_data <- compas_data %>%
  filter(race == 'African-American' & is_recid == 1)
DN_data <- compas_data %>%
  filter(race == 'African-American' & is_recid == 0)
FP_data <- compas_data %>%
  filter(race == 'Caucasian' & is_recid == 1)
FN_data <- compas_data %>%
  filter(race == 'Caucasian' & is_recid == 0)
```

The process for sampling to get a non-discrimatory distribution is as follows.

- 1. Compute the expected size for each group if the data were non-discriminatory
- 2. Under- and over-sample accordingly to eliminate descrimination

3. Train the model on the sampled (now bias-free) data

In order to get an estimation of how biased our model is after making this adjustment to the data, we nest this process in cross-validation. Thus our overall process will be the following (using k-fold cross-validation).

- 1. Split the data into k folds
- 2. Hold out one fold as test data, use the remaining k-1 folds as training data
- 3. Perform sampling on the training data
 - Compute the expected size for each group if the data were non-discriminatory
 - Under- and over-sample accordingly to eliminate descrimination
- 4. Train our model on the training data
- 5. Predict recidivism scores on the test data and store predictions
- 6. Repeat (2) (5) for each fold, getting one prediction for each data point
- 7. Compute relative risk with new predictions

First we walk through an example of the sampling process on the whole data set before embedding the enitre process in cross-validation.

Demonstration of the sampling process

1. Compute the expected size for each group if the data were non-discriminatory

Here we have the following counts for each group:

```
recid_rates %>% filter(race %in% c('African-American', 'Caucasian'))
## # A tibble: 2 x 3
##
                  race yes_recid no_recid
##
                <fctr>
                           <int>
                                     <int>
## 1 African-American
                             2036
                                      1660
            Caucasian
                            1025
                                      1429
Thus we'll say the expected number of observations for each group is the mean of the four counts:
(expected_num_observations <- mean(c(nrow(DP_data), nrow(DN_data), nrow(FP_data), nrow(FN_data))))
## [1] 1538
```

2. Under- and over-sample to eliminate descrimination

We want equal representation for each group (DP, DN, FP, and FN). This means getting 1538 observations of each. We under- or over-sample accordingly. Kamiran and Calders describe two methods for performing this sampling: *Preferential Sampling*, which is more likely to sample boundary observations (with a high probability of being in either class); and *Uniform Sampling*, which samples all points with equal probability. Here we perform Uniform Sampling with replacement.

```
set.seed(123)
# sample indices

DP_sampled_indices <- sample(1:nrow(DP_data), size = expected_num_observations, replace = TRUE)

DN_sampled_indices <- sample(1:nrow(DN_data), size = expected_num_observations, replace = TRUE)

FP_sampled_indices <- sample(1:nrow(FP_data), size = expected_num_observations, replace = TRUE)

FN_sampled_indices <- sample(1:nrow(FN_data), size = expected_num_observations, replace = TRUE)</pre>
```

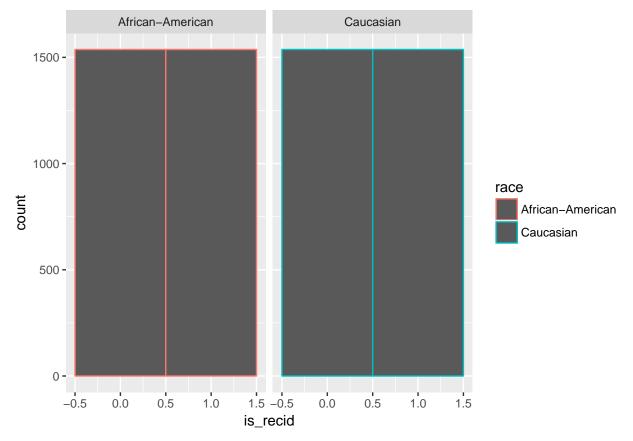
And we build a new data set from the sampled observations.

```
# build a new data set
sampled_compas_data <- rbind(
   DP_data[DP_sampled_indices,],</pre>
```

```
DN_data[DN_sampled_indices,],
FP_data[FP_sampled_indices,],
FN_data[FN_sampled_indices,])
```

We double check visually that our sampling successfully leveled the classes for each partition.

```
sampled_compas_data %>%
  filter(race %in% c('African-American', 'Caucasian')) %>%
  ggplot(aes(x = is_recid, color = race)) +
   geom_histogram(bins = 2) +
  facet_wrap(~ race)
```



With Uniform Sampling applied to our data for the black/white protected race attributes, we're ready to train our model again and see (1) if its level of bias has decreased and also see (2) how its predictive accuracy has changed.

3. Train the model on the sampled data

Here we'll train the model following the same process from above and see how our bias (measured by ProPublica's methodology) compares to the bias before sampling.

We train our logistic regression model.

```
vars <- sampled_compas_data %>%
   select(sex, age, ends_with('count'), c_charge_degree, is_recid)
post_sample_regmod <- glm(is_recid ~., family = binomial(link='logit'), data = vars)</pre>
```

Next we would predict scores for observations we haven't seen. Given that we didn't hold out a test set,

we don't have any such observations. However, we have now seen how sampling works and are thus ready to do so within cross-validation.

Sampling with Cross-validation

Given that we have seen most of the following process step by step in previous sections, we implement this section without explanation of those parts we have seen already.

```
set.seed(123)
all_predictions <- data.frame(id = compas_data$id, predicted_score = NA)
#### 1. Split the data into 10 folds
all_indices <- 1:nrow(compas_data)</pre>
folds <- createFolds(all_indices, k = 10)</pre>
#### 2. Hold out one fold as test data, use the remaining k-1 folds as training data
for (i in 1:10) {
 test indices <- folds[[i]]</pre>
 training_indices <- all_indices[-test_indices]</pre>
  test data <- compas data[test indices,]
  training_data <- compas_data[training_indices,]</pre>
  #### 3. Perform sampling on the training data
  # create partitions
  DP_data <- training_data %>%
   filter(race == 'African-American' & is_recid == 1)
  DN_data <- training_data %>%
    filter(race == 'African-American' & is_recid == 0)
  FP_data <- training_data %>%
    filter(race == 'Caucasian' & is recid == 1)
  FN_data <- training_data %>%
    filter(race == 'Caucasian' & is_recid == 0)
  # get expected number of observations
  expected_num_observations <- mean(c(nrow(DP_data), nrow(DN_data), nrow(FP_data), nrow(FN_data)))</pre>
  # under- and over-sample to eliminate descrimination
  DP_sampled_indices <- sample(1:nrow(DP_data), size = expected_num_observations, replace = TRUE)
  DN_sampled_indices <- sample(1:nrow(DN_data), size = expected_num_observations, replace = TRUE)
  FP_sampled_indices <- sample(1:nrow(FP_data), size = expected_num_observations, replace = TRUE)
  FN_sampled_indices <- sample(1:nrow(FN_data), size = expected_num_observations, replace = TRUE)
  # build new data set
  sampled_training_data <- rbind(</pre>
    DP_data[DP_sampled_indices,],
    DN_data[DN_sampled_indices,],
    FP_data[FP_sampled_indices,],
    FN_data[FN_sampled_indices,]
  )
  #### 4. Train our model on the training data
  vars <- sampled_training_data %>%
    select(sex, age, ends_with('count'), c_charge_degree, is_recid)
  post_sample_regmod <- glm(is_recid ~., family = binomial(link='logit'), data = vars)</pre>
  #### 5. Predict recidivism scores on the test data and store predictions
  predicted_scores <- predict(post_sample_regmod, newdata = test_data, type = 'response')</pre>
```

```
}
#### 7. Compute relative risk with new predictions
# turn predictions from log odds ratios into probabilities of recidivating
recidivism_scores <- all_predictions$predicted_score</pre>
# create new scores and score categories
post_sample_scores <- compas_data %>%
 mutate(new score = recidivism scores) %>%
 mutate(
   new score text = as.factor(
     ifelse(new_score >= 0.75, 'High',
            ifelse(new_score >= 0.45, 'Medium', 'Low'))))
# build ProPublica's data frame with our scores
post_sample_df <- post_sample_scores %>%
 select(age, c_charge_degree, race, age_cat, new_score_text, sex, priors_count,
        days_b_screening_arrest, new_score, is_recid, two_year_recid,
        c_jail_in, c_jail_out) %>%
 filter(days_b_screening_arrest <= 30) %>%
 filter(days_b_screening_arrest >= -30) %>%
 filter(is_recid != -1) %>%
 filter(c_charge_degree != "0") %>%
 filter(new_score_text != 'N/A') %>%
 mutate(crime_factor = factor(c_charge_degree)) %>%
 mutate(age_factor = as.factor(age_cat)) %>%
 within(age_factor <- relevel(age_factor, ref = "25 - 45")) %>%
 mutate(race_factor = factor(race)) %>%
 within(race factor <- relevel(race factor, ref = "Caucasian")) %>%
 mutate(gender_factor = factor(sex, labels= c("Female","Male"))) %>%
 within(gender_factor <- relevel(gender_factor, ref = "Male")) %>%
 mutate(score_factor = factor(new_score_text != "Low", labels = c("LowScore", "HighScore")))
# train ProPublica's model
post_sample_pp_model <- glm(</pre>
 score_factor ~ gender_factor + age_factor + race_factor + priors_count +
    crime_factor + two_year_recid,
 family="binomial",
 data=post_sample_df
)
```

Now we can measure the predictive accuracy and amount of bias in our model after applying sampling and compare to the bias before.

Predictive Accuracy Before and After Sampling

Looking at our confusion matrix after sampling, we get:

```
predicted_scores <- all_predictions$predicted_score
confusionMatrix(data = as.numeric(predicted_scores > 0.5), reference = compas_data$is_recid)

## Confusion Matrix and Statistics
##
## Reference
## Prediction 0 1
## 0 2533 1159
## 1 1210 2312
```

```
##
##
                  Accuracy: 0.672
                    95% CI: (0.661, 0.682)
##
       No Information Rate: 0.519
##
##
       P-Value [Acc > NIR] : <2e-16
##
##
                     Kappa: 0.343
##
   Mcnemar's Test P-Value: 0.304
##
##
               Sensitivity: 0.677
##
               Specificity: 0.666
            Pos Pred Value: 0.686
##
##
            Neg Pred Value: 0.656
                Prevalence: 0.519
##
##
            Detection Rate: 0.351
##
      Detection Prevalence: 0.512
##
         Balanced Accuracy: 0.671
##
##
          'Positive' Class: 0
##
```

Here we see our predictive accuracy is comparable to the accuracy before sampling.

Bias Before and After Sampling

We can re-compute the relative risk for blacks vs. whites and compare it to the relative risk for our model before sampling to see if sampling improved the level of bias in our mode.

```
model_intercept <- coef(post_sample_pp_model)['(Intercept)']
black_coef <- coef(post_sample_pp_model)['race_factorAfrican-American']
(post_sample_risk_black_v_white <- sigmoid(model_intercept + black_coef) / sigmoid(model_intercept))
## (Intercept)
## 1.26</pre>
```

Taking the ratio and subtracting from 1, we compute the change in relative risk before and after:

```
pre_sample_risk_black_v_white <- relative_risk_black_v_white
1 - post_sample_risk_black_v_white / pre_sample_risk_black_v_white
## (Intercept)
## -0.04212</pre>
```

Unfortunately, it looks like sampling did not help with our level of bias. The level of bias is about the same before and after – the slight difference we see here could simply be due to variation in the cross-validation process. Though our preliminary bias-mitigation technique was not successful in this instance, this fact still gives us insight into bias in the COMPAS data. There are still some important takeaways we can draw from our experimentation, discussed in the next section.

Takeaways

In this section we discuss what we think are the major takeaways from the above analysis. Some of these observations here are based on direct results from above while others are based on our best hypotheses about bias in this data given the results we saw.

One important takeaway from our use of sampling and the fact that the level of bias did not change before and after is that bias can permeate features other than protected attributes. We performed sampling for one variable – the race variable – but were unable to decrease racial bias. This could suggest that other features in the data served as proxies for race, so despite the fact that we created balance between recidivate and non-recidivate observations between blacks and whites, the presence racial bias across other features still drove the racial bias in our model. This occurrence is consistent with some techniques from the literature suggested for minimizing algorithmic bias, such as suppression, which we saw among Kamiran and Calders' four suggested techniques.

Another takeaway is that using only the most basic features about defendants (instead of a detailed survey of variables, as used by COMPAS), as compared to the COMPAS risk of recidivism scores, we were able to produce a model with higher accuracy and lower levels of bias (both estimated via cross-validation). Our analysis certainly did not provide a comprehensive comparison, and COMPAS scores may be superior in many ways to our risk scores; however, the difference in performance and bias that we saw, given how simple our model is, suggests that COMPAS scores could be significantly improved.

Future Work

To continue this project, there are several areas upon which we would expand. In addition to generating risk of recidivism scores, we would also explore risk of violent recidivism and perform a similar comparison to COMPAS scores as above. We would also work to improve our risk of recidivism models in terms of feature generation and accuracy and try other modeling approaches in addition to logistic regression. There is also much work to be done in eliminating bias from these models. While our attempt via uniform sampling did not work, there are several other techniques for data preprocessing (such as those mentioned in the report) to eliminate bias that could be more fruitful. Furthermore, we did not explore any methods that impose fairness constraints on the models themselves. A hybrid approach that does both may be necessary to successfully minimize bias. It would also be interesting to do a more in depth exploration of the trade-off between bias and performance and see how this trade-off manifests in predictions from the COMPAS data.

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