# Homework 2

M 383C - Method of Applied Mathematics

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**Problem 2.10.** Prove the following:

**Corollary.** *If $X$ and $Y$ are NLS's, $X$ finite dimensional, and $T : X \to Y$ linear, then $T$ is bounded. The dual space $X^* = B(X, \mathbb{F})$ is isomorphic and homeomorphic to $\mathbb{F}^d$.*

*Proof.* Let $d$ be the dimension of $X$ and $\{e_n\}_{n=1}^d$ be a basis. Then,

$$\|T(x)\|_Y = \left\| T(\sum_{n=1}^{d} x_n e_n) \right\|_Y \leq \sum_{n=1}^{d} |x_n| \, \|T(e_n)\|_Y \leq M \sum_{n=1}^{d} |x_n| = M \, \|x\|_{\ell^1}$$

where $M = \max_{n=1,2,\dots,d} \{\|T(e_n)\|_Y\} < \infty$. Since all norms on a finite dimensional vector space are equivalent, $\exists \, C > 0$ s.t. $\|T(x)\|_Y \leq M \|x\|_{\ell^1} \leq CM \|x\|_X$, where $\|\cdot\|_X$ is any norm on $X$. Therefore, T is bounded.

By Proposition 2.11, $X$ and $\mathbb{F}^d$ are isomorphic and homeomorphic. We need to show the same for $X$ and $X^*$ to complete the proof. For any basis $\{e_n\}_{n=1}^d$ in $X$, consider the set of linear functionals $\{e_n^*\}_{n=1}^d$ in $X^*$ s.t. $e_i^*(e_j) = \delta_{ij}$. We can show that it forms a basis in $X^*$:

For any $f \in X^*$, let $f_n := f(e_n)$, it can be represented by

$$f(x) = f(\sum_{n=1}^{d} x_n e_n) = \sum_{n=1}^{d} x_n f_n = \sum_{n=1}^{d} f_n e_n^*(x)$$

Therefore, $\{e_n^*\}_{n=1}^d$ spans $X^*$.

We also have

$$\sum_{n=1}^{d} \alpha_n e_n^* = \mathbf{0} \;\Rightarrow 0 = (\sum_{n=1}^{d} \alpha_n e_n^*) e_i = \sum_{n=1}^{d} \alpha_n \delta_{in} = \alpha_i \;\forall i = 1, 2, ..., n$$

Therefore, $\{e_n^*\}_{n=1}^d$ are linearly independent.

This concludes that the dimension of $X$ and $X^*$ are the same, thus they are isomorphic. Since the linear mappings between $X$ and $X^*$ in both way are bounded, as proved above, they must be continuous as well. Thus, $X$ and $X^*$ are also homeomorphic. $\square$

**Problem 2.12.** Consider the space $(\ell^p, \|\cdot\|_p)$.

(a) Prove that $\|\cdot\|_p$ is a norm for $1 \le p \le \infty$.

*Proof.* Let $x = \{x_n\}_{n=1}^\infty$ and $y = \{y_n\}_{n=1}^\infty$ be in $\ell^p$. Let $\lambda \in \mathbb{F}$.

(1) For $1 \le p < \infty$, $\|\lambda x\|_p = \big(\sum_{n=1}^\infty |\lambda x_n|^p\big)^{1/p} = |\lambda|\big(\sum_{n=1}^\infty |x_n|^p\big)^{1/p} = |\lambda|\,\|x\|_p$.
When $p = \infty$, $\|\lambda x\|_\infty = \sup_n |\lambda x_n| = |\lambda| \sup_n |x_n| = |\lambda|\,\|x\|_\infty$.

(2) $\|x\|_p = 0 \iff \big(\sum_{n=1}^\infty |x_n|^p\big)^{1/p}$ or $\sup_n |x_n| = 0 \iff x_n = 0\ \forall n \in \mathbb{N}$

(3) The triangle inequality for $p = 1$ and $p = \infty$ can be obtained easily by applying $|x_n + y_n| \le |x_n| + |y_n|$.
To prove the triangle inequality for $1 < p < \infty$, consider the following:

$$\|x+y\|_p^p = \sum_{n=1}^\infty |x_n + y_n|^p \le \sum_{n=1}^\infty |x_n + y_n|^{p-1}\big(|x_n| + |y_n|\big)$$
$$\le \Big(\sum_{n=1}^\infty |x_n + y_n|^{(p-1)q}\Big)^{1/q}\big(\,\|f\|_p + \|g\|_p\,\big)$$

by applying Hölder's Inequality twice. Since $(p-1)q = p$ and $1/q = (p-1)/p$, we have:

$$\|x+y\|_p^p \le \|x+y\|_p^{p-1}\left(\|x\|_p + \|y\|_p\right)$$

For $\|x+y\|_p = 0$, the triangle inequality is trivial. For $\|x+y\|_p > 0$, cancel out the power $p-1$ and the result follows.

□

(b) Prove that for $1 \le p \le \infty$, $\ell^p$ is a Banach space (using that $\mathbb{R}$ is complete).

*Proof.* Let $\{x^i\}_{i=1}^\infty$ be a Cauchy sequence in $\ell^p$ and it follows that

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ :\ \epsilon > \left\|x^i - x^k\right\|_p = \begin{cases} \big(\sum_{k=1}^\infty |x_n^i - x_n^k|^p\big)^{1/p} & 1 \le p < \infty \\ \\ \sup_n |x_n^i - x_n^k| & p = \infty \end{cases},\ \forall i, k > N$$

and this implies that $|x_n^i - x_n^k| < \epsilon$ for every $n \in N$ and $\{x_n^i\}_{i=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ . Since $\mathbb{R}$ is complete, $\lim_{i\to\infty} x_n^i = x_n \in \mathbb{R}$. Let $x = \{x_n\}_{n=1}^\infty$.

(1) For $1 \le p < \infty$,

$$\lim_{i\to\infty}\Big(\sum_{n=1}^\infty |x_n^i - x_n^k|^p\Big)^{1/p} = \Big(\sum_{n=1}^\infty |x_n - x_n^k|^p\Big)^{1/p} = \left\|x - x^k\right\|_p < \epsilon\ \ \forall k > N$$

For $p = \infty$, $\lim_{i\to\infty}\sup_n |x_n^i - x_n^k| = \sup_n |x_n - x_n^k| = \left\|x - x^k\right\|_\infty < \epsilon\ \ \forall k > N$
Therefore, $x = \lim_{i\to\infty} x^i$.

(2) $\|x\|_p \le \|x - x^k\|_p + \|x^k\|_p \le \epsilon + \|x^k\|_p < \infty$
Therefore, $x \in \ell^p$ and $\ell^p$ is a Banach space.

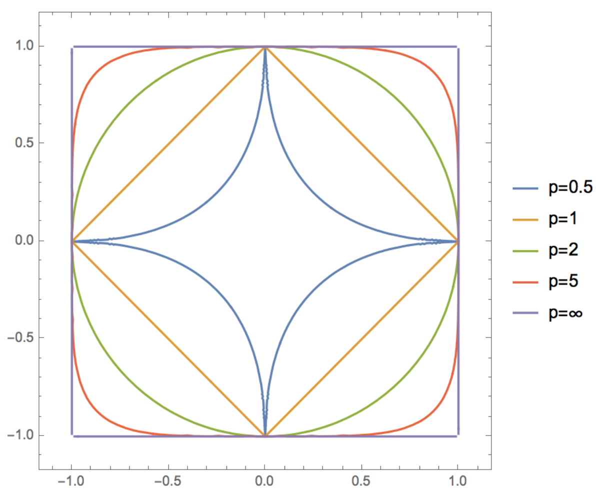
$\square$

(c) Show that $\|\cdot\|_p$ is not a norm for $0 < p < 1$ (by first showing the result on $\mathbb{R}^2$.)

*Proof.* Let $x = (1, 0), y = (0, 1)$ and $p = 1/2$, then $\|x + y\|_{1/2} = 4$ and $\|x\|_{1/2} = \|y\|_{1/2} = 1$. Therefore, $\|x + y\|_{1/2} > \|x\|_{1/2} + \|y\|_{1/2}$ and $\|\cdot\|_p$ is not a norm for $0 < p < 1$ on $\mathbb{R}^2$.
Similarly, for any $x = (1, 0, 0, 0, 0, ...) \in \ell^p$ and $y = (0, 1, 0, 0, ...) \in \ell^p$, we have $\|x + y\|_p = 2^{1/p} > \|x\|_p + \|y\|_p = 2 \ \forall p \in (0, 1)$ $\square$

(d) In Euclidean space $\mathbb{R}^2$, sketch the unit ball in the $\ell^p$-norm, for $1 \le p \le \infty$. What does the "unit ball" look like for $p < 1$?

The "unit ball" looks like a star (shown below).



**Problem 2.14.** If $f \in L^p(\Omega)$ show that

$$\|f\|_p = \sup\left|\int_\Omega fg\,dx\right| = \sup\int_\Omega |fg|\,dx$$

where the supremum is taken over all $g \in L^q(\Omega)$ such that $\|g\|_q \leq 1$, where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$.

*Proof.*

(1) $1 \leq p < \infty$
Let $h \in L^q(\Omega)$ and $\|h\|_q \neq 0$. Let $g = \epsilon h/\|h\|_q \in L^q$ where $\epsilon \in [0,1]$. Notice that $\|g\|_q = \epsilon \leq 1$. Then, we can replace the restriction on $g$ with a restriction on $\epsilon$:

$$\sup_{\substack{g\in L^q(\Omega)\\ 0<\|g\|_q\leq 1}} |\int_\Omega fg\,dx| = \sup_{\substack{\epsilon\in(0,1]\\ h\in L^q(\Omega)\\ \|h\|_q\neq 0}} \frac{\epsilon}{\|h\|_q}|\int_\Omega fh\,dx|$$

The supremum is reached when $\epsilon = 1$, which is when $\|g\|_q = 1$. In that case, we have

$$\sup_{\substack{g\in L^q(\Omega)\\ 0<\|g\|_q\leq 1}} |\int_\Omega fg\,dx| = \sup_{\substack{h\in L^q(\Omega)\\ \|h\|_q\neq 0}} \frac{|\int_\Omega fh\,dx|}{\|h\|_q} = \|f\|_p$$

Moreover, the Hölder Inequality gives:

$$\left|\int_\Omega fg\,dx\right| \leq \int_\Omega |fg|\,dx \leq \|f\|_p\,\|g\|_q\,.$$

which implies

$$\|f\|_p = \sup_{\substack{g\in L^q(\Omega)\\ 0<\|g\|_q\leq 1}} |\int_\Omega fg\,dx| \leq \sup_{\substack{g\in L^q(\Omega)\\ 0<\|g\|_q\leq 1}} \int_\Omega |fg|\,dx \leq \|f\|_p$$

Therefore,

$$\|f\|_p = \sup_{\substack{g\in L^q(\Omega)\\ 0<\|g\|_q\leq 1}} \left|\int_\Omega fg\,dx\right| = \sup_{\substack{g\in L^q(\Omega)\\ 0<\|g\|_q\leq 1}} \int_\Omega |fg|\,dx.$$

(2) $p = \infty$ and $q = 1$
Notice that if we restrict sign($g$)=sign($f$) at which the supremum is reached, we have

$$\sup_{\substack{g\in L^1(\Omega)\\ 0<\|g\|_1\leq 1}} \left|\int_\Omega fg\,dx\right| = \sup_{\substack{g\in L^1(\Omega)\\ 0<\|g\|_1\leq 1}} \int_\Omega |fg|\,dx$$

It's sufficient to prove that the latter is bounded by $\|f\|_\infty$ from above and below. Taking supremum over Hölder Inequality gives:

$$\sup_{\substack{g\in L^1(\Omega)\\ 0<\|g\|_1\leq 1}} \int_\Omega |fg|\,dx \leq \sup_{\substack{g\in L^1(\Omega)\\ 0<\|g\|_1\leq 1}} \|f\|_\infty \|g\|_1 = \|f\|_\infty .$$

Now for $\|f\|_\infty = 0$, i.e. $f = 0$ a.e. on $\Omega$, the result is obvious. When $\|f\|_\infty > 0$, let $\epsilon \in [0, \|f\|_\infty)$ and consider the set

$$\Omega_\epsilon = \{x \in \Omega : |f(x)| \geq \|f\|_\infty - \epsilon\}$$

According to the definition of infinity norm, $\mu(\Omega_\epsilon) > 0$. Thus, we have

$$\int_\Omega |fg|\,dx \geq (\|f\|_\infty - \epsilon)\Big(\int_\Omega |g|\,dx\Big) = (\|f\|_\infty - \epsilon)\,\|g\|_1$$

Take supremum on both sides:

$$\sup_{\substack{g\in L^1(\Omega)\\ 0<\|g\|_1\leq 1}} \int_\Omega |fg|\,dx \geq \|f\|_\infty - \epsilon$$

Since $\epsilon$ is arbiriarly small, the result follows.

$\square$

**Problem 2.15.** Suppose $\Omega \subset \mathbb{R}^d$ is measurable with finite measure and $1 \leq p \leq q \leq \infty$.

(a) Prove that if $f \in L^q(\Omega)$, then $f \in L^p(\Omega)$ and

$$\|f\|_p \leq (\mu(\Omega))^{1/p-1/q} \|f\|_q$$

*Proof.* The result is trivial for $p = q$. Consider the following cases:

(1) $1 \leq p < q < \infty$
Notice that $p/q < 1$. Apply Hölder Inequality to $|f|^p$ and $g = 1$ on $\Omega$:

$$\int_\Omega |f|^p \, dx \leq \Big(\int_\Omega 1 \, dx\Big)^{1-p/q} \Big(\int_\Omega |f|^{pq/p} \, dx\Big)^{p/q} = \big(\mu(\Omega)\big)^{1-p/q} \Big(\int_\Omega |f|^q \, dx\Big)^{p/q}$$

Now raising both side to the power $1/p$:

$$\|f\|_p = \Big(\int_\Omega |f|^p \, dx\Big)^{1/p} \leq \big(\mu(\Omega)\big)^{1/p-1/q} \Big(\int_\Omega |f|^q \, dx\Big)^{1/q} = \big(\mu(\Omega)\big)^{1/p-1/q} \|f\|_q$$

(2) $1 \leq p < q = \infty$
Let $1/q = 0$ and apply the same Hölder Inequality to $|f|^p$ and $g = 1$ on $\Omega$:

$$\int_\Omega |f|^p \, dx \leq \Big(\int_\Omega 1 \, dx\Big) \||f|^p\|_\infty = \mu(\Omega)\Big(\operatorname*{ess\,sup}_{x\in\Omega} |f(x)|^p\Big) = \mu(\Omega)\Big(\operatorname*{ess\,sup}_{x\in\Omega} |f(x)|\Big)^p$$

Raising both side to the power $1/p$:

$$\|f\|_p = \Big(\int_\Omega |f|^p \, dx\Big)^{1/p} \leq \mu(\Omega)^{1/p}\Big(\operatorname*{ess\,sup}_{x\in\Omega} |f(x)|\Big) = \big(\mu(\Omega)\big)^{1/p} \|f\|_\infty$$

□

(b) Prove that if $f \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

*Proof.*

(1) $\lim\limits_{p\to\infty} \|f\|_p \leq \|f\|_\infty$

From (a) above, $\|f\|_p \leq \big(\mu(\Omega)\big)^{1/p} \|f\|_\infty$. Pass both side to the limit as $p \to \infty$, the result follows.

(2) $\lim\limits_{p\to\infty} \|f\|_p \geq \|f\|_\infty$
If $\|f\|_\infty = 0$, then $f = 0$ a.e. in $\Omega$. Therefore, $\|f\|_p = 0$ and the equality holds. When $\|f\|_\infty > 0$, let $\epsilon \in [0, \|f\|_\infty)$ and consider the set

$$\Omega_\epsilon = \{x \in \Omega : |f(x)| \geq \|f\|_\infty - \epsilon\}$$

According to the definition of infinity norm, $\mu(\Omega_\epsilon) > 0$. Integrate on both side of the inequality:

$$\|f\|_p = \Big(\int_\Omega |f|^p \, dx\Big)^{1/p} \geq (\|f\|_\infty - \epsilon)\Big(\int_\Omega 1 \, dx\Big)^{1/p} = (\|f\|_\infty - \epsilon)\Big(\mu(\Omega)\Big)^{1/p}$$

Pass both side to the limit as $\to \infty$, then $\lim\limits_{p\to\infty} \|f\|_p \geq \|f\|_\infty - \epsilon$. Since $\epsilon$ can be arbitrarily small, the result follows.

$\square$

(c) Prove that if $f \in L^p(\Omega))$ for all $p$ with $1 \leq p < \infty$ and there is $K > 0$ such that $\|f\|_p \leq K$, then $f \in L^\infty(\Omega)$ and $\|f\|_\infty \leq K$.

*Proof.* Assume $f \notin L^\infty(\Omega)$, then $\operatorname*{ess\,sup}\limits_{x\in\Omega}|f| = \infty$. Consider the set that contain the essential supremum positions.

$$\Omega_\infty = \{x \in \Omega : |f(x)| = \inf_{\mu(A)=0} \sup_{x\in\Omega-A} |f(x)|\}$$

Then,

$$\int_\Omega |f|^p \, dx = \int_{\Omega\setminus\Omega_\infty} |f|^p \, dx \; + \int_{\Omega_\infty} |f|^p \, dx \; = \infty$$

This implies that $f \notin L^p(\Omega)$, a contradiction. Therefore, $f \in L^\infty(\Omega)$ and, as proved in (b), $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty \leq K$ $\square$

**Problem 2.16.** Let $1 \le p < \infty$ and define, for each $r \in \mathbb{R}^d$, the translation operator $\tau_r : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by

$$\tau_r(f)(x) = f(x + r).$$

(a) Verify that $\tau_r(f) \in L^p(\mathbb{R}^d)$ and that $\tau_r$ is bounded and linear. What is the norm of $\tau_r$?

*Proof.*

(1) Notice that

$$\int_{\mathbb{R}^d} \tau_r(f)(x)\,dx = \int_{\mathbb{R}^d} f(x+r)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$$

Therefore $f \in L^p(\mathbb{R}^d) \Rightarrow \tau_r(f) \in L^p(\mathbb{R}^d)$

(2) $\tau_r(\alpha f + \beta g)(x) = (\alpha f + \beta g)(x + r) = \alpha f(x+r) + \beta g(x+r) = \big(\alpha\tau_r(f) + \beta\tau_r(g)\big)(x)$. Therefore, $\tau_r$ is linear

(3) Apparently, $\tau_r$ is a continuous operator, as it preserves norms in $L^p(\mathbb{R}^d)$. Thus, it is bounded and the norm of $\tau_r$ is 1.

□

(b) Show that as $r \to s$, $\|\tau_r f - \tau_s f\|_{L^p} \to 0$. [Hint: use that the set of continuous functions with compact support are dense in $L^p(\mathbb{R}^d)$ for $p < \infty$.]

*Proof.* Let $g \in C_0(\Omega)$ s.t. $\|g - f\|_{L^p} < \frac{\epsilon}{3}$ for some $\epsilon > 0$. Since $\tau$ is continuous and translational, $\|\tau_r(g) - \tau_r(f)\|_{L^p} < \frac{\epsilon}{3}$ and $\|\tau_s(g) - \tau_s(f)\|_{L^p} < \frac{\epsilon}{3}$. Consequently:

$$\|\tau_r(f) - \tau_s(f)\|_{L^p} \le \|\tau_r(f) - \tau_r(g)\|_{L^p} + \|\tau_r(g) - \tau_s(g)\|_{L^p} + \|\tau_s(g) - \tau_s(f)\|_{L^p}$$

Due to continuity, $\exists\delta > 0$ s.t. $|r - s| < \delta \Rightarrow \|\tau_r(g) - \tau_s(g)\|_{L^p} \le \frac{\epsilon}{3}$. Therefore,

$$\|\tau_r(f) - \tau_s(f)\|_{L^p} \le \epsilon$$

Thus, $\|\tau_r f - \tau_s f\|_{L^p} \to 0$ as $r \to s$ □

**Problem 2.19.** If $X$ and $Y$ are NLS"s, then the product space $X \times Y$ is also an NLS with any of the norms

$$\|(x,y)\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\}$$

or, for any $1 \le p < \infty$,

$$\|(x,y)\|_{X\times Y} = (\|x\|_X^p + \|y\|_Y^p)^{1/p}$$

(a) Why are these norms equivalent?

$$\max\{\|x\|_X, \|y\|_Y\} \le (\|x\|_X^p + \|y\|_Y^p)^{1/p} \le 2^{1/p} \max\{\|x\|_X, \|y\|_Y\}$$

(b) if $X$ and $Y$ are Banach spaces, prove that $X \times Y$ is a Banach space.

*Proof.* Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a Cauchy sequence in $X \times Y$, then $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$\begin{aligned} \|(x_n, y_n) - (x_m, y_m)\|_{X\times Y} &= \|(x_n - x_m), (y_n - y_m)\|_{X\times Y} \\ &= \max\{\|x_n - x_m\|_X, \|y_n - y_m\|_Y\} \le \epsilon \ \ \forall n, m > N \end{aligned}$$

This implies that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $X$ and $Y$ respectively. Now let $x_n \to x \in X$ and $y_n \to y \in Y$, then $\forall N_\delta \in \mathbb{N} \ \exists \epsilon_1, \epsilon_2 > 0$ s.t. $\|x_n - x\|_X < \epsilon_1$ and $\|y_n - y\|_Y < \epsilon_2 \ \ \forall n \ge N$. Let $\delta = \max\{\epsilon_1, \epsilon_2\}$, then

$$\|(x_n, y_n) - (x, y)\|_{X\times Y} = \max\{\|x_n - x\|_X, \|y_n - y\|_Y\} \le \delta \ \ \forall n \ge N$$

which means that $(x_n, y_n) \to (x, y) \in X \times Y$ and $X \times Y$ is a Banach space.

□

**Problem 2.20.** Let $X$ be an NLS and $M$ a nonempty subset. The *annihilator* $M^a$ of $M$ is defined to be the set of all bounded linear functionals $f \in X^*$ such that $f$ restricted to $M$ is zero.

(a) Show that $M^a$ is a closed subspace of $X^*$.

*Proof.* Consider a sequence $\{f_n\}_{n=1}^{\infty} \in M^a$ s.t. $f_n \to f$ and $f \in \overline{M^a}$. This means that

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} : \|f - f_n\|_{X^*} < \epsilon \ \ \forall n \geq N.$$

Also notice that

$$|f(x) - f_n(x)| \leq \|f - f_n\|_{X^*} \|x\|_X \leq \epsilon \|x\|_X$$

Since $f_n(x) = 0 \ \ \forall x \in M$ and $\epsilon$ is arbitrarily small, we must have $f(x) = 0 \ \ \forall x \in M$. Therefore, $f \in M^a$ and $M^a$ is sequentially closed, which implies that $M^a$ is closed under $\|\cdot\|_{X^*}$.
Moreover, $\mathbf{0} \in X^*$ s.t. $\mathbf{0}(x) = 0 \ \ \forall x \in X$ is also in $M^a$. If $f_1, f_2 \in M^a$ and $\alpha, \beta \in \mathbb{F}$, then

$$(\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = 0 \ \ \forall x \in M.$$

Therefore, $M^a$ is a closed subspace of $X^*$ □

(b) What are $X^a$ and $0^a$?
$X^a = \mathbf{0}$ and $0^a = X^*$