Homework 3

M 383C - Methods for Applied Mathematics

Lianghao Cao

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Problem 2.26. Define the operator T by the formula

$$T(f)(x) = \int_a^b K(x,y)f(y)\,dy\ .$$

Suppose that $K \in L^q([a,b]\times[a,b])$, where q lies in the range $1 \le q \le \infty$. Determine the values of p for which T is necessarily a bounded linear operator from $L^p(a,b)$ to $L^q(a,b)$. In particular if a and b are both finite, show that $K \in L^\infty([a,b]\times[a,b])$ implies T to be bounded on all the L^p-spaces.

Proof. First, we can show that T is linear. Let $\alpha,\beta \in \mathbb{F}$ $f,g \in L^p$, then

$$\begin{aligned} T(\alpha f + \beta g)(x) &= \int_a^b K(x,y)\big(\alpha f(y) + \beta g(y)\big)\,dy \\ &= \alpha \int_a^b K(x,y)f(y)\,dy + \beta \int_a^b K(x,y)g(y)\,dy \\ &= \alpha T(f)(x) + \beta T(g)(x) \end{aligned}$$

Therefore T is linear.

T is bounded when $\|T(f)\|_q \le C\|f\|_p$ for some $C > 0$. To show the range of p for which T is bounded, consider the following cases:

1. $1 < p \le \infty$
   Apply Hölder's inequality:

$$\begin{aligned} \|T(f)\|_q^q &= \int_a^b \left| \int_a^b K(x,y)f(y)\,dy \right|^q dx \\ &\le \int_a^b \|f\|_p^q \Big(\int_a^b |K(x,y)|^q\,dy \Big)\,dx \\ &= \Big(\int_a^b \int_a^b |K(x,y)|^q dy\,dx \Big) \|f\|_p^q \\ &= \|K\|_{L^q([a,b]^2)}^q \|f\|_p^q \end{aligned}$$

   Since $K \in L^q([a,b]^2)$, $\exists C \in [0,\infty)$ s.t. $C^{1/q} = \|K\|_{L^q([a,b]^2)}$. Thus, $\|T(f)\|_q \le C\|f\|_p$ and T is bounded for $1 < p \le \infty$.

2. $p = 1$ and $q = \infty$
   Again, apply Hölder's inequality and let $C = \|K\|_{L^\infty([a,b]^2)} \in [0, \infty)$:

$$\begin{aligned}
\|T(f)\|_\infty &= \sup_{x\in[a,b]} \Big| \int_a^b K(x,y) f(y)\, dy \Big| \\
&\le \|f\|_1 \sup_{x\in[a,b]} \Big(\sup_{y\in[a,b]} \big| K(x,y) \big| \Big) \\
&= \|K\|_{L^\infty([a,b]^2)} \|f\|_1 \\
&= C \|f\|_1
\end{aligned}$$

   Therefore, T is bounded for $p = 1$.

Now let $K \in L^\infty([a,b]^2)$, and let a and b to be finite. We have already shown that T is bounded for $p = 1$ without the restriction on a and b. Consider $1 < p \le \infty$. Take the absolute value into the integral on x and apply Hölder's inequality to f and $g = 1$ a.e. on $[a, b]$:

$$\begin{aligned}
\|T(f)\|_q &= \Big(\int_a^b \Big| \int_a^b K(x,y) f(y)\, dy \Big|^q \, dx \Big)^{1/q} \\
&\le \Big(\int_a^b \big(\int_a^b \Big| K(x,y) f(y) \Big| \, dy \big)^q \, dx \Big)^{1/q} \\
&\le \|K\|_{L^\infty([a,b]^2)} \Big(\int_a^b \big(\int_a^b \Big| f(y) \Big| \, dy \big)^q \, dx \Big)^{1/q} \\
&\le \|K\|_{L^\infty([a,b]^2)} \Big(\int_a^b (b-a) \|f\|_p^q \, dx \Big)^{1/q} \\
&= (b-a)^{2/q} \|K\|_{L^\infty([a,b]^2)} \|f\|_p \\
&= C \|f\|_p
\end{aligned}$$

where $C = (b-a)^{2/q} \|K\|_{L^\infty([a,b]^2)} \in [0, \infty)$. Therefore C is bounded for $1 < p \le \infty$ as well. $\square$

Problem 2.27. Suppose that X, Y, and Z are Banach spaces and that $T : X \times Y \to Z$ is bilinear. By T being *bilinear*, we mean that it is linear in each of its arguments separately; that is, $T(x, y)$ is linear in $x \in X$ for each fixed $y \in Y$, and also linear in $y \in T$ for each fixed $x \in X$.

(a) If T is continuous, prove that there is a constant $M < \infty$ s.t.

$$\|T(x,y)\| \le M\|x\|\,\|y\| \quad \forall x \in X, y \in Y \ .$$

In this case, we say that T is bounded. Is completeness needed here?

Proof. T is continuous at $(0, 0)$ implies that $\forall \epsilon > 0 \ \ \exists \delta > 0$ s.t.

$$\|(x,y)\|_{X\times Y} \le \delta \Rightarrow \|T(x,y)\|_Z \le \epsilon \quad \forall (x,y) \in X \times Y \ .$$

Take the norm $\|(x,y)\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\}$ and it follows that

$$\begin{aligned}\|T(x,y)\|_Z &= \left\| \frac{\|x\|_X\|y\|_Y}{\delta^2} T\big(\frac{\delta}{\|x\|_X}x, \frac{\delta}{\|y\|_Y}y\big)\right\|_Z \\ &= \frac{1}{\delta^2}\left\|T\big(\frac{\delta}{\|x\|_X}x, \frac{\delta}{\|y\|_Y}y\big)\right\|_Z \|x\|_X\|y\|_Y \\ &\le \frac{\epsilon}{\delta^2}\|x\|_X\|y\|_Y \quad \forall (x,y) \in \{X\setminus 0\}\times\{Y\setminus 0\}\end{aligned}$$

The inequality is concluded from

$$\left\|\big(\frac{\delta}{\|x\|_X}x, \frac{\delta}{\|y\|_Y}y\big)\right\|_{X\times Y} = \max\{\left\|\frac{\delta}{\|x\|_X}x\right\|_X, \left\|\frac{\delta}{\|y\|_Y}y\right\|_Y\} = \delta$$

Let $M = \epsilon/\delta^2$ and the result follows.
When one of x and y is zero, instead of division by its norm, we set the component to zero directly and the same derivation will lead us to the desired result. When both x and y are zero, the equality is taken with 0 on both sides.
Completeness is not needed for the proof. □

(b) Prove that T is continuous iff it is continuous at the origin $(0, 0)$.

Proof. "$\Rightarrow$" is trivial. For "$\Leftarrow$", let T be continuous at $(0, 0)$, then, as proved above, $\exists M < \infty$ s.t. $\|T(x,y)\|_Z \le M\|x\|_X\|y\|_Y$. Now assume we have

$$\|(x_1,y_1) - (x_2 - y_2)\|_{X\times Y} = \|(x_1 - x_2, y_1 - y_2)\|_{X\times Y} \le \delta$$

for some $\delta > 0$ and $(x_1, y_1), (x_2, y_2) \in X \times Y$. Apply the triangle inequalities and boundedness/bilinearity of T:

$$\begin{aligned}\|T(x_1,y_1) - T(x_2,y_2)\|_Z &\le \|T(x_1,y_1) - T(x_1,y_2)\|_Z + \|T(x_1,y_2) - T(x_2,y_2)\|_Z \\ &= \|T(x_1, y_1 - y_2)\|_Z + \|T(x_1 - x_2, y_2)\|_Z \\ &\le M_1\|x_1\|_X\|y_1 - y_2\|_Y + M_2\|x_1 - x_2\|_X\|y_2\|_Y \\ &\le C\big(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y\big) \\ &\le 2C\|(x_1 - x_2, y_1 - y_2)\|_{X\times Y} \\ &\le 2C\delta \ .\end{aligned}$$

where $C = \max\{M_1 \|x_1\|_X, M_2 \|y_2\|_Y\}$. Therefore, T is continuous everywhere.

$\square$

(c) Prove that T is continuous iff it is bounded.

Proof. "$\Rightarrow$" is proved in (a). For "$\Leftarrow$" Assume that T is bounded, i.e., $\exists M < \infty$ s.t. $\|T(x,y)\|_Z \leq M \|x\|_X \|y\|_Y$. Assume that $\|(x,y)\|_{X\times Y} \leq \delta$, i.e., $\|x\|_X, \|y\|_Y \leq \delta$ for some $(x,y) \in X \times Y$ and $\delta > 0$. Then

$$\|T(x,y)\| \leq M\delta^2 .$$

Therefore, T is continuous at $(0,0)$. As proved in (b), T is continuous everywhere. $\square$

Problem 2.28. Suppose that X is a Banach space, M and N are linear subspaces, and that $X = M \oplus N$, which means that

$$X = M + N = \{m + n : m \in M, n \in N\}$$

and $M \cap N = \{0\}$ is the trivial linear subspace consisting only of the zero element. In this case, we say that X is the direct sum of M and N and denoted that fact by writing $X = M \oplus N$. Let P denote the projection of X onto M. That is, if $x = m + n$, then

$$P(x) = m \ .$$

Show that P is well defined and linear. Prove that P is bounded iff both M and N are closed.

Proof.

1. Assume x can be decomposed into two different representations:

   $$x = m_1 + n_1 = m_2 + n_2$$

   where $m_1, m_2 \in M$ and $n_1, n_2 \in N$. Then $M \ni m_1 - m_2 = n_2 - n_1 \in N$ and this implies that $M \cap N \neq \emptyset$, a contradiction. Therefore, the decomposition of x in M and N is unique and P is well define.

2. For any $\alpha, \beta \in \mathbb{F}$ as well as any $x_1 = m_1 + n_1$ and $x_2 = m_2 + n_2$ in X, we have

   $$P(\alpha x_1 + \beta x_2) = \alpha m_1 + \beta m_2 = \alpha P(x_1) + \beta P(x_2)$$

   Thus, P is linear.

3. "$\Rightarrow$"
   P is bounded implies P is continuous. Notice that $N = P^{-1}(\{0\})$. Since $\{0\}$ is closed, N is closed as well. Consider a sequence $\{m_k\}_{k=1}^{\infty}$ in M that converges to $m \in X$. Now apply the projection to the sequence and obtain another sequence $\{P(m_k)\}_{k=1}^{\infty}$ in M that converges to $P(m) \in M$, due to continuity of P. Notice that $P(m_k) = m_k \ \forall k \in \mathbb{N}$. Therefore, $P(m) = m$ and $m \in M$. Therefore, M is closed.

4. "$\Leftarrow$"
   Assume M and N are closed. To show that P is bounded, it is sufficient to show that P is a closed operator, i.e., graph(P) is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X that converges to $x \in X$. Let the corresponding sequence $\{P(x_n)\}_{n=1}^{\infty}$, in M converges to y. Since M is closed, $y \in M$. Consider another sequence $\{x_n - P(x_n)\}_{n=1}^{\infty} \in N$. Since N is closed, the sequence converges to $x - y \in N$. This implies that $P(x - y) = 0$, which further implies that $P(x) = P(y) = y$. Therefore, graph(P) is closed and P is bounded.

$\square$

Problem 2.29. Let X be a Banach space with closed linear subspaces Y and Z such that $Y \cap Z = \{0\}$ and $X = Y \oplus Z$. For $f \in Z^*$, show that $F : X \to \mathbb{F}$ is a well defined linear extension of f, where

$$F(y+z) = f(z) \ \ \forall y \in Y,\ z \in Z \ ,$$

and that F is bounded on X.

Proof. Consider the projection of X onto Z. If $X \ni x = y + z$, where $y \in Y$ and $z \in Z$, then $P(x) = z$. As we proved in Problem 2.28, P is well-defined, linear and continuous. Notice that

$$F(x) = F(y+z) = f(z) = f\big(P(x)\big)$$

for all $x \in X$. Since f is also well-defined, linear and continuous, F is well-defined, linear and continuous. This implies that F is bounded on X as well. □

Problem 2.32. Prove that $L^2([0,1])$ is of the first category in $L^1([0,1])$. [Hint: show that $A_k = \{ f \in L^1([0,1]) : \|f\|_{L^2} \leq k\}$ is closed in L^1 but has empty interior.]

Proof. Since, in $L^1([0,1])$, $L^2([0,1]) = \bigcup\limits_{k=1}^{\infty} A_k$, we only need to show that A_k $\forall k \in \mathbb{N}$ are nowhere dense, i.e., their closure has empty interior.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A_k for some $k \in \mathbb{N}$ and converges to $f \in L^1([0,1])$. Notice that $f = \lim\limits_{n\to\infty} f_n = \liminf\limits_{n\to\infty} f_n$. Apply Fatou's Lemma:

$$\int_0^1 |f|^2 \, dx \leq \liminf_{n\to\infty} \int_0^1 |f|^2 \, dx \leq k^2 \, .$$

Therefore, $f \in L^2([0,1])$ and A_k is closed.

A_k also has no interior point because all of its elements are limit points of some sequences in the complement of A_k. Consider $g \in L^1([0,1]) \setminus L^2([0,1])$ and any $f \in A_k$. The sequence $\{f + g/n\}_{n=1}^{\infty}$ is a sequence in $L^1([0,1]) \setminus L^2([0,1])$ that converges to f. Thus, A_k has empty interior.

$\square$

Problem 2.37. If a Banach space X is reflexive, show that X^* is also reflexive. Is the converse true?

Proof.

1. "$\Rightarrow$"
   Let $E_x \in X^{**}$ s.t. $E_x(f) = f(x)$ for any $x \in X$ and $f \in X^*$. Since X is reflexive, $\forall g \in X^{**}\ \exists x \in X$ s.t. $g = E_x$ and the function $F : X \to X^{**}$ defined by $F(x) = E_x$ is a bijection and an isometry.
   Now consider $G_f \in X^{***}$ s.t. $G_f(g) = G_f(E_x) = E_x(f) = f(x)$ for some $f \in X^*$ and $g \in X^{**}$ with its corresponding $x \in X$. Define $H : X^* \to X^{***}$ s.t. $H(f) = G_f$. We can show that

   (a) H is surjective, i.e., $\forall h \in X^{***}\ \exists f \in X^*$ s.t. $H(f) = h$.
   For any $h \in X^{***}$, define $f_h : X \to \mathbb{F}$ s.t. $f_h(x) := h(E_x) = h\big(F^{-1}(x)\big)\ \forall x \in X$. We can show that F is linear, since

   $$\begin{aligned} F(\alpha x + \beta y)(f) &= E_{\alpha x + \beta y}(f) \\ &= f(\alpha x + \beta y) \\ &= \alpha f(x) + \beta f(y) \\ &= (\alpha E_x + \beta E_y)(f) \\ &= \big(\alpha F(x) + \beta F(y)\big)(f) \end{aligned}$$

   where $x, y \in X$, $\alpha, \beta \in \mathbb{F}$, $f \in X^*$. Consequently, the inverse of the bijective map, F^{-1}, is also linear. As the composition of two linear functions, f_h is linear. f_h is also bounded as h is bounded. Therefore, $f_h \in X^*$. Notice that, $\forall g \in X^{**}\ \exists x \in X$ s.t.

   $$\big(H(f_h)\big)(g) = G_{f_h}(E_x) = E_x(f_h) = f_h(x) = h(E_x) = h(g)$$

   Therefore, $H(f_h) = h$ and H surjective.

   (b) H is an isometry.
   Define the norm in X^{***} as in Corollary 2.32., we have

   $$\|G_f\|_{X^{***}} = \sup_{\substack{g \in X^{**} \\ g \neq \mathbf{0}}} \frac{|G_f(g)|}{\|g\|_{X^{**}}} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|_X} = \|f\|_{X^*}$$

   Therefore, H is an isometry.

   (c) H is a bounded linear bijection, i.e., isomorphic.
   H is bounded, given the fact that

   $$\|H(f)\|_{X^{***}} = \|G_f\|_{X^{***}} = \|f\|_{X^*}$$

   for all $f \in X^*$. H is also injective. Let $f_1, f_2 \in X^*$. Since H is an isometry, we have

   $$\|H(f_1) - H(f_2)\|_{X^{***}} = \|f_1 - f_2\|_{X^*}$$

Therefore, $H(f_1) = H(f_2) \Rightarrow f_1 = f_2$ and H is injective. We can also show that H is linear:

$$\begin{aligned}\big(H(\alpha f_1 + \beta f_2)\big)(g) &= G_{(\alpha f_1+\beta f_2)}(E_x) \\ &= \alpha f_1(x) + \beta f_2(x) \\ &= \alpha G_{f_1}(E_x) + \beta G_{f_2}(E_x) \\ &= \big(\alpha H(f_1) + \beta H(f_2)\big)(g)\end{aligned}$$

where $\alpha, \beta \in \mathbb{F}$, $f_1, f_2 \in X^*$ and $g \in X^{**}$ with its corresponding $x \in X$.

(a) to (c) combined gives us that X^* is reflexive.

2. "$\Leftarrow$"
If the Banach space X^* is reflexive, then X^{**} is also reflexive, as proven above. Notice that X is embedded into X^{**} with the reasons that $\tilde{X} = \{E_x \in X^{**} : x \in X\} \subseteq X^{**}$ and X is isometrically isomorphic to $\tilde{X}$. We only need to show that **the closed subset (Banach) of a reflexive Banach space is reflexive** to conclude that X is also reflexive.

Let Y be a closed subspace of X, a reflexive Banach space. Consider $j_y \in Y^{**}$ s.t. $j_y(f) = f(y)\ \forall y \in Y, f \in Y^*$ and the corresponding map:

$$J : Y \to Y^{**},\ J(y) = j_y$$

We need to show that

$$\forall g \in Y^{**}\ \exists y \in Y \text{ s.t. } \big(J(y)\big)(f) = j_y(f) = f(y) = g(f)$$

for any $f \in Y^*$, i.e., J is a surjection. All other properties (isometrically isomorphic) can be derived following the similar proof for (b) and (c) in part 1.
For any $g \in Y^{**}$ and $F \in X^*$, define $G(F) := g(F|_Y)$. g is bounded and linear. The restriction to Y preserves linearity. Therefore, $G \in X^{**}$. Since X is reflexive, $\exists x_G \in X$ s.t. $G = E_{x_G}$ and $g(F|_Y) = E_{x_G}(F) = F(x_G)$. If $x_G \in Y$, then it gives us the desired conclusion.
To show that $x_G \in Y$, assume, on the contrary, that $x_G \notin Y$. By Mazure Separation Lemma I (Lemma 2.35), $\exists F_0 \in X^*$ s.t. $\|F_0\|_{X^*} \leq 1$, $F_0(y) = 0$ and $F_0(x) > 0\ \forall y \in Y, x \in X \setminus Y$. Using the properties of F_0, we have

$$g(F_0|_Y) = E_{x_G}(F_0) = F_0(x_G) > 0$$

However, since $F_0|_Y = \mathbf{0}$ and $g(F_0|_Y) = 0$, a contradiction. Therefore, $x_G \in Y$ and Y is reflexive.

$\square$