HW Assignment 3

M 385C - Theory of Probability

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Problem 3.1. Let $(S, \mathcal{S}, \mu)$ be a measure space and suppose $f \in \mathcal{L}^1$. Show that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{S}$ and $\mu(A) < \delta$, then $|\int_A f \, d\mu| < \epsilon$.

Proof. Consider $g \in \mathcal{L}_+^{\text{Simp},0}$ s.t. $0 \le g \le |f|$ and

$$\int_A \big(|f| - g\big) \, d\mu < \frac{\epsilon}{2} \tag{1}$$

Since $g = \sum_{k=1}^{n} \beta_k \mathbf{1}_{B_k}$, where $\beta_1, \beta_2, ..., \beta_n \in \mathbb{R}$ and $B_1, B_2, ..., B_n \in \mathcal{S}$ are pairwise disjoint, we have

$$\int_A g \, d\mu = \sum_{k=1}^{n} \beta_k \mu(B_k \cap A) \le \mu(A) \sum_{k=1}^{n} \beta_k$$

If $\sum_{k=1}^{n} \beta_k = 0$, then the desired conclusion is reached with (1). Assume $\sum_{k=1}^{n} \beta_k \neq 0$, then let $\delta = \epsilon/(2 \sum_{k=1}^{n} \beta_k)$. Consequently,

$$\begin{aligned}
\mu(A) < \delta &\Rightarrow \mu(A) \sum_{k=1}^{n} \beta_k \le \frac{\epsilon}{2} \\
&\Rightarrow \mu(A) \sum_{k=1}^{n} \beta_k + \int_A \big(|f| - g\big) \, d\mu < \epsilon \\
&\Rightarrow \int_A g \, d\mu + \int_A \big(|f| - g\big) \, d\mu < \epsilon \\
&\Rightarrow \Big| \int_A f \, d\mu \Big| \le \int_A |f| \, d\mu < \epsilon
\end{aligned}$$

□

Problem 3.2. Let $(S, \mathcal{S}, \mu)$ be a finite measure space. For $f \in \mathcal{L}^0_+$, show that $f \in \mathcal{L}^1$ if and only if

$$\sum_{n\in\mathbb{N}} \mu(\{f \geq n\}) < \infty,$$

where, as usual, $\{f \geq n\} = \{x \in S : f(x) \geq n\}$.

Proof. Let $A_n = \{f \geq n\}$ and consider $B_k = \{x \in S : f(x) \in [k, k+1)\}, k \in \mathbb{Z}^+$. Notice that $B_n = A_{n+1} \setminus A_n \;\; \forall n \in \mathbb{N}$ and $B_k \in \mathcal{S} \;\; \forall k \in \mathbb{Z}^+$ are pairwise disjoint. Also, $A_n = \bigcup\limits_{k=n}^{\infty} B_k$ and $S = \bigcup\limits_{k=0}^{\infty} B_k$.

1. "$\Rightarrow$ "
   If $f \in \mathcal{L}^1$ and is non-negative, then $\int_S f \; d\mu < \infty$. Now consider the simple function $g = \sum\limits_{k=0}^{N} k\mathbf{1}_{B_k} = \sum\limits_{k=1}^{N} k\mathbf{1}_{B_k} \leq f \;\; \forall N \in \mathbb{N}$. By definition of Lebesgue integral and $\mathcal{L}^0_+$:

$$\begin{aligned}
\lim_{N\to\infty} \int_S g \; d\mu = \sum_{k=1}^{\infty} k\mu(B_k) &= \mu(A_1) + \sum_{k=2}^{\infty}(k-1)\mu(B_k) \\
&= \mu(A_1) + \mu(A_2) + \sum_{k=3}^{\infty}(k-2)\mu(B_k) \\
&= ... = \sum_{n\in\mathbb{N}} \mu(A_n) \\
&\leq \int_S f \; d\mu < \infty
\end{aligned}$$

2. "$\Leftarrow$"
   Assume $\sum\limits_{n\in\mathbb{N}} \mu(A_n) < \infty$. Since $f \in \mathcal{L}^0_+$ is bounded on $B_k \;\; \forall k \in \mathbb{Z}^+$, f is integratable on these sets. Thus, we can represent the Lebesgue integral of f by:

$$\int_S f \; d\mu = \sum_{k=0}^{\infty} \int_{B_k} f \; d\mu$$

   Notice that $(k+1)\mathbf{1}_{B_k} \geq f\mathbf{1}_{B_k}$. With a finite measure space, we have

$$\begin{aligned}
\int_S f \; d\mu \leq \sum_{k=0}^{\infty} \int_{B_k} (k+1)\mathbf{1}_{B_k} \; d\mu \leq \sum_{k=0}^{\infty}(k+1)\mu(B_k) &= \mu(S) + \sum_{k=1}^{\infty} k\mu(B_k) \\
&= \mu(S) + \sum_{n\in\mathbb{N}} \mu(A_k) < \infty
\end{aligned}$$

   Therefore, $f \in \mathcal{L}^1$.

$\square$

Problem 3.3. Let $(S, \mathcal{S}, \mu)$ and $(T, \mathcal{T}, \nu)$ be two measurable spaces, and let $F : S \to T$ be a measurable function with the property that $\nu = F_* \mu$ (i.e., ν is the push-forward of μ through F). Show that for every $f \in \mathcal{L}^0_+(T, \mathcal{T})$ or $f \in \mathcal{L}^1(T, \mathcal{T})$, we have

$$\int f \, d\nu = \int (f \circ F) \, d\mu$$

.

Proof. Apply the standard machine.

1. **Indicator functions**
   Let $B \in \mathcal{T}$ and $f = \mathbf{1}_B$, then

$$\int_T f \, d\nu = \nu(B) = \mu\big(F^{-1}(B)\big) = \int_S \mathbf{1}_{F^{-1}(B)} \, d\mu$$

   Notice that $\mathbf{1}_{F^{-1}(B)} = \mathbf{1}_B \circ F$. They only take $x \in S$ to 1 when $F(x) \in B$. Therefore,

$$\int_T f \, d\nu = \int_S (f \circ F) \, d\mu$$

2. **Simple functions**
   Let $f = \sum_{k=1}^{n} \beta_k \mathbf{1}_{B_k}$, where $\beta_1, \beta_2, ..., \beta_n \in \mathbb{R}$ and $B_1, B_2.., B_k \in \mathcal{T}$ are pairwise disjoint. Notice that $F^{-1}(B_1), F^{-1}(B_2), .., F^{-1}(B_n)$ are pairwise disjoint as well. By definition of Lebesgue integration for simple functions:

$$\begin{aligned}
\int_T f \, d\nu &= \int_T \Big(\sum_{k=1}^{n} \beta_k \mathbf{1}_{B_K}\Big) \, d\nu = \sum_{k=1}^{n} \beta_k \nu(B_k) = \sum_{k=1}^{n} \beta_k \mu\big(F^{-1}(B_k)\big) \\
&= \int_S \Big(\sum_{k=1}^{n} \beta_k \mathbf{1}_{F^{-1}(B_k)}\Big) \, d\mu = \int_S \Big(\sum_{k=1}^{n} \beta_k \big(\mathbf{1}_{B_k} \circ F\big)\Big) \, d\mu \\
&= \int_S \Big(\big(\sum_{k=1}^{n} \beta_k \mathbf{1}_{B_k}\big) \circ F\Big) \, d\mu = \int_S (f \circ F) \, d\mu.
\end{aligned}$$

3. **Non-negative measurable functions**
   Let $f \in \mathcal{L}^0_+(T, \mathcal{T})$, then $f \circ F \in \mathcal{L}^0_+(S, \mathcal{S})$. By approximation by simple functions (Prop. 3.10), $\exists \, \{g_n\}_{n \in \mathbb{N}} \in \mathcal{L}^{\text{Simp},0}_+$ s.t. $g_1 \leq g_2 \leq ... \leq g_n \leq ... \leq f$ and $f = \lim_{n\to\infty} g_n$. This also implies that $\{g_n \circ F\}_{n \in \mathbb{N}}$ is a non-decreasing sequence and $\lim_{n\to\infty} g_n \circ F = f \circ F$. Apply Monotone convergence theorem twice:

$$\begin{aligned}
\int_T f \, d\nu = \lim_{n\to\infty} \int_T g_n \, d\nu = \lim_{n\to\infty} \int_S (g_n \circ F) \, d\mu &= \int_S \big(\lim_{n\to\infty} g_n \circ F\big) \, d\mu \\
&= \int_S (f \circ F) \, d\mu
\end{aligned}$$

4. **All measurable functions**
Let $f \in \mathcal{L}^1(T, \mathcal{T})$, then $f = f^+ - f^-$, where $f^+, f^- \in \mathcal{L}^0_+(T, \mathcal{T})$, and $f \circ F \in \mathcal{L}^1(S, \mathcal{S})$. Notice that $(f \circ F)^+ = f^+ \circ F$ and $(f \circ F)^- = f^- \circ F$. Therefore, we have:

$$\begin{aligned} \int_T f \, d\nu = \int_T f^+ \, d\nu - \int_T f^- \, f\nu &= \int_S (f^+ \circ F) \, d\mu - \int_S (f^- \circ F) \, d\mu \\ &= \int_S (f \circ F)^+ \, d\mu - \int_S (f \circ F)^- \, d\mu \\ &= \int_S (f \circ F) \, d\mu \end{aligned}$$

$\square$

Problem 3.4. Let $S \neq \emptyset$ be a set and let $f : S \to \mathbb{R}$ be a function. Prove that a function $g : S \to \mathbb{R}$ is measurable with respect to the pair $(\sigma(f), \mathcal{B}(\mathbb{R}))$ if and only if there exists a Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $g = h \circ f$

Proof.

1. "$\Leftarrow$"
   $g = h \circ f$ is a composition of a $(\sigma(f), \mathcal{B}(\mathbb{R}))$-measurable function, f, and a Borel function, h. Thus g is $(\sigma(f), \mathcal{B}(\mathbb{R}))$-measurable.

2. "$\Rightarrow$"
   Assume $g : S \to \mathbb{R}$ is $(\sigma(f), \mathcal{B}(\mathbb{R}))$-measurable, i.e., $g^{-1}(B) \in \sigma(f)$ for each $B \in \mathcal{B}(\mathbb{R})$. Apply the standard machine.

   (a) **Indicator functions**
   Let $g = \mathbf{1}_A$ for some $A \in \sigma(f)$. This means that $\exists B \in \mathcal{B}(\mathbb{R}) : A = f^{-1}(B)$. Let $h = \mathbf{1}_B$ and h is a Borel function, since it is define with a set B in the Borel σ-algebra. Notice that

   $$g = \mathbf{1}_A = \mathbf{1}_{f^{-1}(B)} = \mathbf{1}_B \circ f = h \circ f.$$

   (b) **Simple functions**
   Let $g = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}$,where $\alpha_1, \alpha_2, .., \alpha_n \in \mathbb{R}$ and $A_1, A_2, ..., A_n \in \sigma(f)$ are pairwise disjoint. For each A_k, $k = 1, 2, ..., n$, there exists $B_k \in \mathcal{B}(\mathbb{R})$ s.t. $A_k = f^{-1}(B)$. Apparently, $\{B_k\}_{k=1}^n$ are pairwise disjoint. Consider $h = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{B_k}$. It is a simple function defined on sets in the Borel σ-algebra, therefore it is a Borel function. Notice that

   $$g = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k} = \sum_{k=1}^{n} \alpha_k (\mathbf{1}_{B_k} \circ f) = (\sum_{k=1}^{n} \alpha_k \mathbf{1}_{B_k}) \circ f = h \circ f.$$

   (c) **Non-negative measurable functions**
   For $g \in \mathcal{L}^0_+(S, \sigma(f))$, $\exists\ \{g_n\}_{n \in \mathbb{N}} \in \mathcal{L}_+^{\text{Simp},0}$ s.t. $g_1 \leq g_2 \leq ... \leq g_n \leq ... \leq g$ and $g = \lim_{n\to\infty} g_n$. This implies that we have sequence of simple functions $\{h_k\}_{k=1}^n$ s.t. $g_n = h_n \circ f$. Notice that

   $$g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} (h_n \circ f) = (\lim_{n\to\infty} h_n) \circ f.$$

   Let $h = \lim_{n\to\infty} h_n$. It is a Borel function, since limiting operation preserve measurability (Prop. 1.23). Therefore, $g = h \circ f$.

   (d) **All measurable functions**
   For $g \in \mathcal{L}^0$, consider having $g = g^+ - g^-$, where $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \min\{g(x), 0\}$ $\forall x \in S$ are all non-negative measurable functions. Thus, there exist Borel functions, h^+ and h^- s.t.

   $$g = g^+ - g^- = h^+ \circ g - h^- \circ g = (h^+ - h^-) \circ g$$

Let $h = h^+ - h^-$. Notice that h might be ill-defined if h^+ and h^- reach infinity simultaneously. Let $N = \{x \in \mathbb{R} : h^+(x) = \infty \wedge h^-(x) = \infty\}$. We have $N \cap f(S) = \emptyset$, since at least one of $h^+(f(x)) = g^+(x)$ and $h^-(f(x)) = g^-(x)$ is zero $\forall x \in S$. Consequently, we now define $h := \mathbf{1}_{\mathbb{R}\setminus N}(h^+ - h^-)$ and still have $g = h \circ f$. h is Borel because subtraction/composition of measurable functions is still measurable.

$\square$