HW Assignment 1

M 385C - Theory of Probability

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Problem 1.1. Show that:

1. Every σ-algebra is an algebra.

   Proof. Pick arbitrary $A, B \in \mathcal{S}$. Assume $\mathcal{S}$ is an σ-algebra, then $A_n \in \mathcal{S}\ \forall n \in \mathbb{N} \Rightarrow \bigcup_n A_n \in \mathcal{S}$ (A7). Let $A_1 = A$, $A_2 = B$ and $A_i = B$ for $i \in \{3, 4, 5, ...\}$. Thus, we have $\bigcup_{i=1}^{\infty} A_i = A \cup B \in \mathcal{S}$ (A4). $\square$

2. Each algebra is a π-system and each σ-algebra is an algebra and a λ-system.

   Proof.

   (a) Pick arbitrary $A, B \in \mathcal{S}$. Assume $\mathcal{S}$ is an algebra, then $A^c, B^c \in \mathcal{S}$ (A3) and $A^c \cup B^c \in \mathcal{S}$ (A4). Apply De Morgan's law and A3, we have $(A^c \cup B^c)^c = A \cap B \in \mathcal{S}$.

   (b) σ-algebra is an algebra is already proven above.
   To prove a σ-algebra is a λ-system, consider the following:

      i. If $\mathcal{S}$ is a σ-algebra and $A \subseteq B \in \mathcal{S}$ (A3), we have $A^c \in \mathcal{S}$. Notice that $B \setminus A = B \cap A^c \in \mathcal{S}$, as proven in (a).

      ii. For a increasing sequence $\mathcal{S} \ni A_n \nearrow A$, $A = \bigcup_n A_n \in \mathcal{S}$ (A7).

   $\square$

3. A family $\mathcal{S}$ is a σ-algebra if and only if it contains the empty set, is closed under finite intersections, complements and countable unions of pairwise disjoint sets.

   Proof.

   (a) " $\Rightarrow$ "
   It is proven in 2(a) that $\mathcal{S}$ is closed under finite intersections. Everything else are given by definition.

(b) $" \Leftarrow "$
We only need to prove that $\mathcal{S}$ is closed under countable union of any sets. Consider a countable collection $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{S}$. We can construct a collection of pairwise disjoint sets $\{B_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$ with $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus (A_1 \cup A_2)$, ..., $B_n = A_n \setminus (A_{n-1} \cup ... \cup A_1)$. Notice that $B_n \in \mathcal{S}\ \forall n \in \mathbb{N}$ by invoking De Morgan's law, A3 and A6, similar to the proof in 2(a). Also notice that, with this set up, $\bigcup_n B_n = \bigcup_n A_n$. Since $\mathcal{S}$ is closed under countable union of pairwise disjoint set, we have $\bigcup_n A_n \in \mathcal{S}$.

□

4. A λ-system which is a π-system is also a σ-algebra.

Proof. Assume $\mathcal{S}$ is a λ-system which is a π-system, i.e., $\mathcal{S}$ satisfies A2 ($S \in \mathcal{S}$), A5 ($A, B \in \mathcal{S}, A \subseteq B \Rightarrow B \setminus a \in \mathcal{S}$), A6 ($A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$) and A8 ($A_n \in \mathcal{S}\ \forall n \in \mathbb{N}$ and $A_n \nearrow A \Rightarrow A \in \mathcal{S}$). We seek to prove:

(a) A1: $\emptyset \in S$
For $A, B \in \mathcal{S}, A \subseteq B$, we have $B \setminus A \in \mathcal{S}$ (A5). Apply A6, we have $(B \setminus A) \cap A = \emptyset \in \mathcal{S}$.

(b) A3: $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$
Consider A2 and A5: for any $A \subset S \in \mathcal{S}$, $S \setminus A = A^c \in \mathcal{S}$. Also, notice that $\emptyset = S^c \in \mathcal{S}$.

(c) A7: $A_n \in \mathcal{S}\ \forall n \in \mathbb{N} \Rightarrow \bigcup_n A_n \in \mathcal{S}$
Similar to the proof in 2(a), we can conclude that $A, B \in \mathcal{S} \Rightarrow A \cup B$ (A4) from A3 and A6 by applying De Morgan's law: $A^c, B^c \in \mathcal{S}$ and $(A^c \cap B^c)^c = A \cup B \in \mathcal{S}$. Now, let $\{A_n\}_{n\in\mathbb{N}}$ be any countable collection in $\mathcal{S}$. We can create an increasing sequence B_n in S with $B_1 = A_1$, $B_2 = A_1 \cup A_2$ ..., $B_n = A_1 \cup A_2 \cup ... \cup A_n$, ... and $B_n \nearrow B \in \mathcal{S}$ (A8). Notice that, with this set up, $\bigcup_n B_n = \bigcup_n A_n \in \mathcal{S}$.

□

5. There are π systems which are not algebras.

Proof. Consider a collection of nested open balls in $\mathbb{R}^n$ with radius larger then r: $\mathcal{B} = \{B_R(0) | \mathbb{N} \ni R \geq r\}$. $\mathcal{B}$ is closed with respect to intersection, thus it is a π-algebra. However, $\emptyset \notin \mathcal{B}$ and it is not an algebra. □

6. There are algebras which are not σ-algebras.

Proof. Consider $X = \{1, 2, 3, 4, ...\}$, and let $\mathcal{A} = \{A \subseteq X : A \text{ is finite or } A^c \text{ is finite}\}$. $\mathcal{A}$ is a algebra on X. We know that $\emptyset$ is finite, thus $\emptyset \in \mathcal{A}$ (A1). By definition, $\mathcal{A}$ is closed under complements (A3). Let $A, B \in \mathcal{A}$, we have either $A \cup B \in \mathcal{A}$ is finite for finite A and B, or $A \cup B$ is infinite for infinite A or B. Notice that, if A is infinite, then A^c is finite and in $\mathcal{A}$. Therefore, $(A \cup B)^c = (A^c \cap B^c) \subseteq A^c$ is finite and $(A \cup B) \in \mathcal{A}$.

The same goes for infinite B ,or infinite A and B. (A4). However, $\mathcal{A}$ not a σ-algebra. Consider sets that each contains a single even number: $\{2n\}, n = 1, 2, ...$ All of these sets are in $\mathcal{A}$. However, $\bigcup_n \{2n\}$ is not in $\mathcal{A}$, since it is infinite and its complement, the set of all odd numbers, is infinite. $\square$

7. There are λ-systems which are not π-systems.

   Proof. Consider $X = \{1, 2, 3, 4\}$, a λ-system on this set is

   $$\{\emptyset, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 3\}, X\}$$

   This is not a π-system, because it is not closed under finite intersection, i.e., $\{1\}$, $\{2\}$,... are not in the collection. $\square$

Problem 1.2. A partition of a set S is a family $\mathcal{P}$ of non-empty subset of S with the property that each $\omega \in S$ belongs to exactly one $A \in \mathcal{P}$.

1. How many algebras are there on the set $S = \{1, 2, 3\}$?

   There are 4: $\big\{\emptyset, S\big\}$, $\big\{\emptyset, \{1\}, \{2, 3\}, S\big\}$, $\big\{\emptyset, \{2\}, \{1, 3\}, S\big\}$,$\big\{\emptyset, \{3\}, \{1, 2\}, S\big\}$.

2. By constructing a bijection between the two families, show that the number of different algebras in a finite set S is equal to the number of different partitions S.

   Proof. Let $\mathcal{P} = \{A_1, A_2, ..., A_k\}$ be a partition of S. Corresponding to this partition, we have an unique σ-algebra generated by $\mathcal{P}$: $\sigma(\mathcal{P})$, which consists of the empty set, $A_1, A_2, ..., A_k$ and all their countable unions.
   Conversely, if we have a σ-algebra $\mathcal{S} = \{B_1, B_2, ..., B_n\}$ on S, then for every x in S, we can find a set $A_x := \bigcap_{x \in B_i \in \mathcal{S}} B_i$. In this way, we have an unique collection of sets that generates $\mathcal{S}$ in which $\bigcup_{x \in S} A_x = S$ and each element in S only appear in one set. Therefore, the collection of A_x $\forall x \in S$ is an unique partition of S. Because of this bijection, we can conclude that the number of different algebras and the number of different partitions, both defined on a finite set S, is the same. $\square$

3. Does there exist an algebra with 754 elements?

   No, there is no algebra with 754 elements. The number of elements must be $2^k, k = 1, 2, ..., n$, where n is the number of elements on the set.

Problem 1.3. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\bar{\mathbb{R}}$. We define

$$\liminf_n x_n = \sup_n \inf_{k \geq n} x_k \text{ and } \limsup_n x_n = \inf_n \sup_{k \geq n} x_k$$

where inf and sup are extended to subset of $\bar{\mathbb{R}}$. Prove the following:

1. $a \in \bar{\mathbb{R}}$ satisfies $a \geq \limsup_n x_n$ if and only if for any $\epsilon \in (0, \infty)$ there exits $n_\epsilon \in \mathbb{N}$ such that $x_n \leq a + \epsilon$ for $n \geq n_\epsilon$. For $a = -\infty$, $a + \epsilon$ should be interpreted as $-1/\epsilon$.

   Proof.

   (a) "$\Rightarrow$"
   Let $y_n = \sup_{k \geq n} x_k \; \forall n \in \mathbb{N}$. Notice that it is a decreasing sequence, so $y = \lim_{n\to\infty} y_n = \limsup_n x_n$ and $a \geq y$.

   i. $a = -\infty$
   If $a = -\infty$, then $y = a = -\infty$. By definition of limit to $-\infty$, we have:

   $$\lim_{n\to\infty} y_n = -\infty \iff \forall c \in \mathbb{R} \; \exists n_c \in \mathbb{N} : \; n \geq n_c \Rightarrow y_n \leq c$$

   Since $y_n \geq x_n$, we have $x_n \leq c$. Let $c = -1/\epsilon \in (-\infty, 0)$ and $n_c = n_\epsilon$ to complete the proof.

   ii. $a > -\infty$
   Using definition of limit and $y_n \geq y \; \forall n \in \mathbb{N}$, we have:

   $$\lim_{n\to\infty} y_n = y \iff \forall \epsilon > 0 \; \exists n_\epsilon \in \mathbb{N} : \; n \geq n_\epsilon \Rightarrow y_n - y < \epsilon$$

   Therefore, $x_n \leq y_n < y + \epsilon \leq a + \epsilon$.

   (b) "$\Leftarrow$"

   i. $a = -\infty$
   Let $y_{n=} \sup_{k \geq n} x_k$. Given any $\epsilon > 0$, $y_n \leq -1/\epsilon \; \forall n \geq n_\epsilon$. Let $c = -1/\epsilon \in (-\infty, 0)$ and $n_c = n_\epsilon$, by definition of limit to $-\infty$, $\lim_{n\to\infty} y_n = -\infty$. Therefore, $a = y = \limsup_n x_n$.

   ii. $a > -\infty$
   Define y_n the same as in i. We have $y_n \leq a + \epsilon \; \forall \epsilon \in (0, \infty)$, since y_n is a decreasing sequence, it approaches its limit from above. Therefore, $\forall \epsilon > 0 \; \exists n_\epsilon \in \mathbb{N} : \; n \geq n_\epsilon \Rightarrow |y_n - a| = y_n - a < \epsilon$. By definition of limit, $\lim_{n\to\infty} y_n = \limsup_n x_n = a$.

   $\square$

2. $\liminf_n x_n \leq \limsup_n x_n$.

   Proof. By the definition of inf and sup, we have: $a_n := \inf_{k \geq n} x_k \leq \sup_{k \geq n} x_k =: b_n$. Notice that a_n is an increasing sequence and b_n is a decreasing sequence, since the inf and sup are taken on a shrinking set. Therefore, $\sup_n a_n = \lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n = \inf_n b_n$. Since $a_n \leq b_n \; \forall n \in \mathbb{N}$, we have $\sup_n a_n \leq \inf_n b_n$. $\square$

3. Let A be the set of accumulation (cluster) points of $\{x_n\}_{n\in\mathbb{N}}$ in $\bar{\mathbb{R}}$., i.e.,

$$A = \{\lim_k x_{n_k} : x_{n_k} \text{is a convergent (in } \bar{\mathbb{R}}\text{) subsequence of } \{x_n\}_{n\in\mathbb{N}}\}$$

Show that

$$\{\liminf_n x_n, \limsup_n x_n\} \subseteq A \subseteq [\,\liminf_n x_n, \limsup_n x_n\,]$$

Give an example in which both inclusions above are strict.

Proof.

(a) $\{\liminf_n x_n, \limsup_n x_n\} \subseteq A$

i. $\liminf_n x_n = \infty$ or $\limsup_n x_n = -\infty$:
Let $y_n = \sup_{k\geq n} x_n$ and $\lim_{n\to\infty} y_n = \limsup_n x_n = -\infty$. For $\epsilon_1 = -1$, there exists n_1 s.t. $n \geq n_1 \Rightarrow y_n < \epsilon$. By definition of sup, we can find $x_{n_1} < \epsilon_1$. Repeat this process to find x_{n_i} with $\epsilon_i = -i \forall i \in \mathbb{N}$, then $\lim_{k\to\infty} x_{n_k} = -\infty$. Therefore, $\limsup_n x_n = -\infty \in A$.
The same construction can be made for $\liminf_n x_n = \infty$. Let $\epsilon_i = i$, there exists n_i s.t. $n \geq n_i \Rightarrow y > \epsilon_i \Rightarrow \exists x_{n_i} : x_{n_i} > \epsilon_i$. Therefore, we have subsequence converges to infinity and $\liminf_n x_n = \infty \in A$.

ii. $\liminf_n x_n \neq \infty$ or $\limsup_n x_n \neq -\infty$:
Let $y_n = \sup_{k\geq n} x_n$ and $\lim_{n\to\infty} y_n = \limsup_n x_n = y$. For $\epsilon_1 = 1/2$, there exists n_1 s.t. $n \geq n_1 \Rightarrow y_{n_1} - y < \epsilon_1$. By definition of sup, we can find $x_{n_1} < y + \epsilon_1$. Repeat this process to find x_{n_i} with $\epsilon_i = (\frac{1}{2})^i$, then $\lim_{k\to\infty} x_{n_k} = y$. Therefore, $\limsup_n x_n = y \in A$.
Similar construction can be made for $\liminf_n x_n$. For $\epsilon_i = (\frac{1}{2})^i$, there exists n_i s.t. $n \geq n_i \Rightarrow y - \inf_{k\geq n_i} x_k < \epsilon_i \Rightarrow \exists x_{n_i} :\ x_{n_i} > y - \epsilon_i$. Again, $\lim_{k\to\infty} x_{n_k} = y$. Therefore, $\liminf_n x_n \in A$.

(b) $A \subseteq [\,\liminf_n x_n, \limsup_n x_n\,]$
If $\liminf_n x_n = \infty$ or $\liminf_n x_n = -\infty$ then they must be the upper/lower bound. Consider when they are finite:
We can first show that $\sup A = \limsup_n x_n$. Again, we define y_n and y as before. Assume $\exists a \in A : a > y = \limsup_n x_n$, then $\forall \epsilon > 0\ \exists n_\epsilon \in \mathbb{N} : n \geq n_\epsilon \Rightarrow x_n < y_n < y + \epsilon$. We can choose ϵ s.t. $y < y + \epsilon < a - \epsilon < a$. Notice that there are only finite subsequence (for $n < n_\epsilon$) of $\{x_n\}_{n\in\mathbb{N}}$ contained in ϵ neighbourhood of a, which implies that a is not a cluster point. Therefore, by contradiction, $a <= \limsup_n x_n\ \forall a \in A$ and $\sup A = \limsup_n x_n$.

Similar proof can be done to show that $\inf A = \liminf_n x_n$. Assume $\exists b \in A : b < y = \liminf_n x_n$ and choose ϵ s.t. $b > b + \epsilon > y - \epsilon > y$. By the same logic, we can conclude again that $b \notin A$. Therefore, $\inf A = \liminf_n x_n$.
Example:
Consider the sequence $\{x_n\}_{n\in\mathbb{N}} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, ...\}$ This sequence has $A = \mathbb{N}$, with $\liminf_n x_n = 1$ and $\limsup_n x_n = \infty$.

$\square$