HW Assignment 2

M 385C - Theory of Probability

Lianghao Cao

September 27, 2018

Problem 2.1. One can obtain the product σ-algebra $\mathcal{S}$ on $\{-1,1\}^{\mathbb{N}}$ as the Borel σ-algebra corresponding to a particular topology which makes $\{-1,1\}^{\mathbb{N}}$ compact. Here is how. Start by defining a mapping $d : \{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}} \to [0,\infty)$ by

$$d(\boldsymbol{s}^1, \boldsymbol{s}^2) = 2^{-i(\boldsymbol{s}^1,\boldsymbol{s}^2)}, \text{ where } i(\boldsymbol{s}^1, \boldsymbol{s}^2) = \inf\{i \in \mathbb{N} : s_i^1 \neq s_i^2\},$$

for $\boldsymbol{s}^j = (s_1^j, s_2^j, ...)$, $j = 1, 2$.

1. Show that d is a metric on $\{-1,1\}^{\mathbb{N}}$.

   Proof. Let $\boldsymbol{s}^1, \boldsymbol{s}^2 \in \{-1,1\}^{\mathbb{N}}$. By the definition of i and d, it is obvious that $d(\boldsymbol{s}^1, \boldsymbol{s}^2) \geq 0$ and $d(\boldsymbol{s}^1, \boldsymbol{s}^2) = d(\boldsymbol{s}^2, \boldsymbol{s}^1)$. If $\boldsymbol{s}^1 = \boldsymbol{s}^2$, their match begins at $i = 1$ and continues, thus $i(\boldsymbol{s}^1, \boldsymbol{s}^2) = \infty$ and $d(\boldsymbol{s}^1, \boldsymbol{s}^2) = 0$. Conversely, if $d(\boldsymbol{s}^1, \boldsymbol{s}^2) = 0$, we must have $\boldsymbol{s}^1 = \boldsymbol{s}^2$. Now let $\boldsymbol{s}^3 \in \{-1,1\}^{\mathbb{N}}$, and let $i(\boldsymbol{s}^1, \boldsymbol{s}^2) = n$, $i(\boldsymbol{s}^1, \boldsymbol{s}^3) = m_1$ and $i(\boldsymbol{s}^2, \boldsymbol{s}^3) = m_2$ If $n \leq m_1$, then $m_2 = n$. If $n > m_1$, then $m_1 = m_2$. The same the true if we replace m_1 with m_2. Therefore, $d(\boldsymbol{s}^1, \boldsymbol{s}^2) = 2^{-n} \leq d(\boldsymbol{s}^1, \boldsymbol{s}^3) + d(\boldsymbol{s}^2, \boldsymbol{s}^3) = 2^{-m_1} + 2^{-m_2}$, i.e., the triangle inequality holds. □

2. Show that $\{-1,1\}^{\mathbb{N}}$ is compact under d. (Hint: Use the diagonal argument.)

   Proof. Conisder the sequence $\{\boldsymbol{s}^n\}_{n=1}^{\infty}$ in $\{-1,1\}^{\mathbb{N}}$ and the sequence of all first elements: $\{s_1^n\}_{n=1}^{\infty}$. This sequence only takes value between $\{1,-1\}$, thus this sequence should at least either contain infinite 1 or infinite -1. Within this sequence, it's possible to find a constant subsequence $\{s_1^{n_k}\}_{k=1}^{\infty}$ that contains $b_1 \in \{-1,1\}$. Consequently, $\{\boldsymbol{s}^{n_k}\}_{k=1}^{n}$ all starts with b_1. Similarly, the sequence of all remaining second elements, $\{s_2^{n_k}\}_{k=1}^{\infty}$, should again have a subsequence eventually stabilized at $b_2 \in \{-1,1\}$, call it $\{s_2^{n_{k_l}}\}_{l=1}^{\infty}$, and $\{\boldsymbol{s}^{n_{k_l}}\}_{l=1}^{\infty}$ eventually starts with $\{b_1, b_2, ...\}$. Repeating this process for all $n \in \mathbb{N}$ and define $\boldsymbol{s} = \{b_n\}_{n=1}^{\infty} \in \{-1,1\}^{\mathbb{N}}$. Then, for all $\epsilon \in \mathbb{N}$, there exists infinite elements in the extracted subsequence that lies within $B(\boldsymbol{s}, 2^{-\epsilon})$. Therefore, $\{-1,1\}^{\mathbb{N}}$ is sequentially compact, which implies it is compact under d. □

3. Show that each cylinder of $\{-1,1\}^{\mathbb{N}}$ is both open and closed under d.

Proof. Notice a cylinder in $\{-1,1\}^{\mathbb{N}}$ defined by $n \in \mathbb{N}$ and a finite subset $B \in \{-1,1\}^n$:

$$C = \{\boldsymbol{s} \in \{-1,1\}^{\mathbb{N}} : \{s_1, s_2, ..., s_n\} \in B\}$$

can be represented by a finite union of product cylinders with the first n elements contained in B, i.e., $C = \bigcup\limits_{\{b_i\}_{i=1}^n \in B} C_{1,2,...,n,b_1,b_2,...b_n}$. The product cylinders are open, as proved in 4 below. Thus, C is open.
C is also closed, as the complement of C in B is

$$C' = \{\boldsymbol{s} \in \{-1,1\}^{\mathbb{N}} : \{s_1, s_2, ..., s_n\} \in \{-1,1\}^n \setminus B\} = \bigcup_{\{b_i\}_{i=1}^n \in \{-1,1\}^n \setminus B} C_{1,2,...,n,b_1,b_2,...b_n}$$

It is still a finite union of product cylinders and, therefore, is open.

□

4. Show that each open ball is a cylinder.

   Proof. Consider any $\boldsymbol{s}_0 = \{s_1, s_2, s_3, ...\} \in \{-1,1\}^{\mathbb{N}}$, the open ball centered at $\boldsymbol{s}$ is the set $B(\boldsymbol{s}, 2^{-n})$ for some $n \in \mathbb{N}$. This mean that any $\boldsymbol{s} \in \{-1,1\}^{\mathbb{N}}$ s.t. $d(\boldsymbol{s}, \boldsymbol{s}_0) < 2^{-n}$ must matches at least the first n elements of $\boldsymbol{s}$. This is equivalent to the definition of product cylinder, i.e., $B = C_{1,2...,n;s_1,s_2,...,s_n}$. □

5. Show that $\{-1,1\}^{\mathbb{N}}$ is separable, i.e., it admits a countable dense subset.

   Proof. Consider a dense subset $D \in \{-1,1\}^{\mathbb{N}}$ such that each element in D is arbitrarily close to a element in $\{-1,1\}^{\mathbb{N}} \setminus D$. In other words,

   $$\exists \boldsymbol{s}^D = \{s_1^D, s_2^D, ...\} \in D : d(\boldsymbol{s}^D, \boldsymbol{s}) < 2^{-k} \quad \forall k \in \mathbb{N}, \boldsymbol{s} = \{s_1, s_2, ...\} \in \{-1,1\}^N \setminus D$$

   A possible way to select the elements of D is to have $\boldsymbol{s}^D = \{s_1, s_2, ..., s_k, -1, -1, ...\}$ for every $k \in \mathbb{N}$. Clearly, the selected $\boldsymbol{s}^D$ must be arbitrarily close (as k gets larger) to some $\boldsymbol{s}$ in the complement of D that does not converge to -1. The dense set D is also countable, since we can find a bijection to a subset of $\mathbb{Q}$ by replacing -1 with 0 in $\boldsymbol{s}^D$ and realize $0.s_1^D s_2^D ....s_k^D \in \mathbb{Q}$. □

6. Conclude that $\mathcal{S}$ coincides with the Borel σ-algebra on $\{-1,1\}^{\mathbb{N}}$ under the metric d.

   Proof.

   (a) $\mathcal{S} \subseteq \mathcal{B}(S)$
   The σ-algebra, $\mathcal{S}$, is generated by all cylinders in $\{-1,1\}^{\mathbb{N}}$. Since all cylinders are open, thus set of all open sets contains all the cylinders. The Borel σ-algebra, $\mathcal{B}(S)$, is generated by all open sets. Therefore, $\mathcal{S} \subseteq \mathcal{B}(S)$

(b) $\mathcal{B}(S) \subseteq \mathcal{S}$

We already know that cylinders are open balls. However, the Borel σ-algebra is generated by all open sets. We know that all open sets can be represented by union of open balls. However, it remains to prove that all open sets can be represented by *countable* union of open balls, because the axiom of σ-algebra states that $\mathcal{B}(S)$ is only closed under countable union.

Indeed, there are countable open balls in $\{-1,1\}^{\mathbb{N}}$. Notice that every open ball $B(\boldsymbol{s}_0, 2^{-n})$, for some $\boldsymbol{s}_0 \in \{-1,1\}^{\mathbb{N}}$ and $n \in N$, must contain at least a $\boldsymbol{s}_0^D \in D$. In return, by the definition of d, every $\boldsymbol{s}_B \in B(\boldsymbol{s}_0, 2^{-n})$ must satisfy $\boldsymbol{s}_B \in B(\boldsymbol{s}_0^D, 2^{-n})$ (In 1, we proved that $n \leq m_1 \Rightarrow m_2 = n$.). This implies that all open balls are centered at countable points in D with countable possible radii, $\{2^{-n}\}_{n=1}^{\infty}$. Therefore, there are only countable open balls and thus $\mathcal{B}(S)$ can be generated by open balls, which is also cylinders. Therefore, $\mathcal{B}(S) \subseteq \mathcal{S}$.

$\square$

Problem 2.2. A measurable space $(T, \mathcal{T})$ is said to be countably separated if there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that for all $x, y \in T$ we have

$$x = y \iff \mathbf{1}_{A_n}(x) = \mathbf{1}_{A_n}(y) \;\; \forall n \in \mathbb{N}$$

such a family $\{A_n\}_{n\in\mathbb{N}}$ is called a separating family.

1. Let T be a separable metric space, and let $\mathcal{B}(T)$ be its Borel σ-algebra. Show that $(T, \mathcal{B}(T))$ is countably separated. In particular, $(R^d, \mathcal{B}(\mathbb{R}^d)$ is countably separated for each d.

   Proof. Since T is a separable metric space, then it admits a countable dense subset. Let $D = \{x_n\}_{n=1}^{\infty}$ denote the dense subset, then around every element $x_n \in D$ for some $n \in \mathbb{N}$, there one open ball A_n centered at x_n such that it has radius of the distance to the closest (measured by d) element in D for x_i. Therefore, there is only one element of D in one element of $\{A_n\}_{n=1}^{\infty}$.
   In the case of $\mathbb{R}$, for the dense set, $\mathbb{Q}$, the separating family is the all the open balls centered at each rational number with rational radius and end point at the closest rational number. $\square$

2. Show that $(T, \mathcal{T})$ is countably separable if and only if there exists a measurable injection $F : T \to ([0,1], \mathcal{B}([0,1]))$. (Hint: For the "if" direction, consider a function of the form $F(x) = \sum_{n=1}^{\infty} 3^{-n}\mathbf{1}_{A_n}(x)$, for some family $\{A_n\}_{n\in\mathbb{N}}$.)

   Proof.

   (a) "$\Rightarrow$"

   If $(T, \mathcal{T})$ is countably separable, there exists a separating family $\{A_n\}_{n=1}^{\infty} \in \mathcal{T}$. Consider the mapping:

$$F(x) = \sum_{n=1}^{\infty} 3^{-n}\mathbf{1}_{A_n}(x) \;\; \forall x \in T$$

Notice that this is a injective mapping from T to $[0,1]$, since each x only appears in one $A_n(x)$. If $x, y \in T$ appear in A_{n_x} and A_{n_y} respectively, the difference will be captured by 3^{-n} in $F(x)$. Thus,

$$F(x) = F(y) \Rightarrow \mathbf{1}_{A_n}(x) = \mathbf{1}_{A_n}(y) \Rightarrow x = y \ \ \forall x, y \in T$$

Also, F is measurable, since $\mathbf{1}_{A_n}$ is measurable.

(b) "$\Leftarrow$"
As proven above, $[0,1] \in R$ is countably separable and there exists a measurable separating family $\{B_n\}_{n=1}^{\infty} \in \mathcal{B}([0,1])$. Using a measurable injection, $F : T \to ([0,1], \mathcal{B}([0,1]))$, we can generate a family of sets in $\mathcal{T}$: $\{A_n = F^{-1}(B_n) \ \ \forall n \in \mathbb{N}\}$. If $x, y \in A_n$ for some $n \in \mathbb{N}$, we must have $F(x), F(y) \in B_n$, which implies that $F(x) = F(y)$. Since F is a injective map, we have $x = y$. Therefore, $\{A_n\}_{n=1}^{\infty}$ is a separating family in T and $(T, \mathcal{T})$ is countably separable.

□

3. Suppose that $(T, \mathcal{T})$ is countably separated. Show that the diagonal

$$D_T = \{(x,x) : x \in T\} \subseteq T \times T$$

is measurable in $\mathcal{T} \otimes \mathcal{T}$.

Proof. We know that the set $D_{[0,1]} = \{(x,x), x \in [0,1]\}$ is measurable in $([0,1], \mathcal{B}([0,1]))$, since it is a closed set that is contained in $\mathcal{B}([0,1])$. Therefore, we seek to pull D_T back to $D_{[0,1]} \in \mathcal{B}([0,1])$ by a measurable function. Consider the function

$$F^2 : T \times T \to [0,1]^2, \ \ F^2(x,y) = (F(x), F(y))$$

Then, F^2 is a measurable function, since it has measurable components. Notice that, by injectivity of F, i.e., $F(x) = F(y) \Rightarrow x = y$:

$$D_T = \{(x,y) \in T \times T : F(x) = F(y)\} = (F^2)^{-1}(D_{[0,1]})$$

Therefore, D_T is measurable. □

4. Find an example of (an obviously separated) measurable space $(T, \mathcal{T})$ such that D_T is not measurable in $\mathcal{T} \otimes \mathcal{T}$

Consider $T = \{0,1\}$ (not countably separated) and the trivial σ-algebra $\mathcal{T} = \{\emptyset, T\}$. Apparently, $D_T = \{(0,0), (1,1)\} \notin \mathcal{T} \otimes \mathcal{T} = \{\emptyset, T \times T\}$

5. Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces, and let $f : S \to T$ be a measurable function. Consider the following two statements:

(M) f is a measurable function.
(G) The graph $G_f = \{(x,y) \in S \times T : f(x) = y\}$ of f is a measurable set in $\mathcal{S} \otimes \mathcal{T}$.

Show that (M) implies (G) when T is countably separated, but not in general.

Proof. We have already proved that the diagonal set of a countable separated measure space is measurable. We can further show that G_f is measurable by pulling it back to $D_T \in \mathcal{T} \otimes \mathcal{T}$ with a measurable function.
Consider the map $G : S \times T \to T^2$ and $G(x, y) = (f(x), y)$. Since f is a measurable function, the components of G are all measurable. Therefore, G is a measurable function. Notice that

$$G^{-1}(D_T) = \{(x, f(x) \in S \times T\} = G_f$$

Therefore, G_f is measurable.
If T is not countably separated, consider again the example above. Let $T = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, T\}$ be the trivial σ-algebra. Let $S = T$ and $f(x) = x$ be the measurable map from S to T. $G_f = D_T$ is not measurable as stated in 4 above. □

Problem 2.3. Let $(S, \mathcal{S}, \mu)$ be a measure space and let $f, g \in \mathcal{L}^0(S, \mathcal{S}, \mu)$ satisfy $\mu\big(\{x \in S : f(x) < g(x)\}\big) > 0$. Prove or construct a counterexample for the following statement:

"There exist constants $a, b \in \mathbb{R}$ such that $\mu\big(\{x \in S : f(x) \leq a < b \leq g(x)\}\big) > 0$"

Proof. In contrary, assume the following:

$$\mu\big(\{x \in S : f(x) \leq a < b \leq g(x)\}\big) = 0 \ \ \forall a, b \in \mathbb{R}$$

It's possible to find two rationals, c and d, such that $c, d \in [a, b]$. If we replace a and b with c and d, the measure above will still be zero ($c, d \in \mathbb{R}$). Consequently, $A := \{x \in S : f(x) < g(x)\}$ can be represented by:

$$A = \bigcup_{c,d \in \mathbb{Q}} \{x \in S : f(x) \leq c < d \leq g(x)\}$$

Now we have a countable union and thus we can apply the property of measure:

$$\mu(A) \leq \sum_{c,d \in \mathbb{Q}} \mu\big(\{x \in S : f(x) \leq c < d \leq g(x)\}\big) = 0$$

Therefore, $\mu(A) = 0$, a contradiction

□