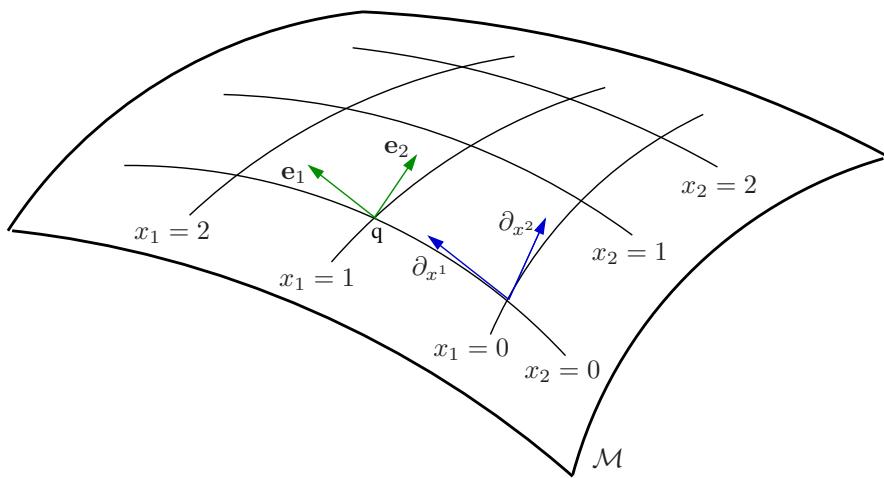


Catalogue of Spacetimes



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Contents

1	Introduction and Notation	1
1.1	Notation	1
1.2	General remarks	1
1.3	Basic objects of a metric	2
1.4	Natural local tetrad and initial conditions for geodesics	3
1.4.1	Orthonormality condition	3
1.4.2	Tetrad transformations	4
1.4.3	Ricci rotation-, connection-, and structure coefficients	4
1.4.4	Riemann-, Ricci-, and Weyl-tensor with respect to a local tetrad	5
1.4.5	Null or timelike directions	5
1.4.6	Local tetrad for diagonal metrics	5
1.4.7	Local tetrad for stationary axisymmetric spacetimes	6
1.5	Newman-Penrose tetrad and spin-coefficients	6
1.6	Coordinate relations	7
1.6.1	Spherical and Cartesian coordinates	7
1.6.2	Cylindrical and Cartesian coordinates	8
1.7	Embedding diagram	8
1.8	Equations of motion and transport equations	9
1.8.1	Geodesic equation	9
1.8.2	Fermi-Walker transport	9
1.8.3	Parallel transport	9
1.8.4	Euler-Lagrange formalism	10
1.8.5	Hamilton formalism	10
1.9	Units	10
1.10	Tools	10
1.10.1	Maple/GRTensorII	10
1.10.2	Mathematica	11
1.10.3	Maxima	13
2	Spacetimes	14
2.1	Minkowski	14
2.1.1	Cartesian coordinates	14
2.1.2	Cylindrical coordinates	14
2.1.3	Spherical coordinates	15
2.1.4	Conform-compactified coordinates	15
2.1.5	Rotating coordinates	16
2.1.6	Rindler coordinates	17
2.2	Schwarzschild spacetime	18
2.2.1	Schwarzschild coordinates	18
2.2.2	Schwarzschild in pseudo-Cartesian coordinates	20
2.2.3	Isotropic coordinates	20
2.2.4	Eddington-Finkelstein	22
2.2.5	Kruskal-Szekeres	23
2.2.6	Tortoise coordinates	24

2.2.7	Painlevé-Gullstrand	25
2.2.8	Israel coordinates	27
2.3	Alcubierre Warp	28
2.4	Barriola-Vilenkin monopol	29
2.5	Bertotti-Kasner	31
2.6	Bessel gravitational wave	33
2.6.1	Cylindrical coordinates	33
2.6.2	Cartesian coordinates	33
2.7	Cosmic string in Schwarzschild spacetime	34
2.8	Ernst spacetime	36
2.9	Friedman-Robertson-Walker	38
2.9.1	Form 1	38
2.9.2	Form 2	39
2.9.3	Form 3	40
2.10	Gödel Universe	44
2.10.1	Cylindrical coordinates	44
2.10.2	Scaled cylindrical coordinates	45
2.11	Halilsoy standing wave	47
2.12	Janis-Newman-Winicour	48
2.13	Kasner	50
2.14	Kerr	51
2.14.1	Boyer-Lindquist coordinates	51
2.15	Kottler spacetime	54
2.16	Morris-Thorne	56
2.17	Oppenheimer-Snyder collapse	58
2.17.1	Outer metric	58
2.17.2	Inner metric	59
2.18	Petrov-Type D – Levi-Civita spacetimes	61
2.18.1	Case AI	61
2.18.2	Case AII	61
2.18.3	Case AIII	62
2.18.4	Case BI	62
2.18.5	Case BII	63
2.18.6	Case BIII	63
2.18.7	Case C	63
2.19	Plane gravitational wave	66
2.20	Reissner-Nordstrøm	67
2.21	de Sitter spacetime	69
2.21.1	Standard coordinates	69
2.21.2	Conformally Einstein coordinates	69
2.21.3	Conformally flat coordinates	70
2.21.4	Static coordinates	70
2.21.5	Lemaître-Robertson form	72
2.21.6	Cartesian coordinates	73
2.22	Straight spinning string	74
2.23	Sultana-Dyer spacetime	76
2.24	TaubNUT	78
	Bibliography	79

Chapter 1

Introduction and Notation

The *Catalogue of Spacetimes* is a collection of four-dimensional Lorentzian spacetimes in the context of the General Theory of Relativity (GR). The aim of the catalogue is to give a quick reference for students who need some basic facts of the most well-known spacetimes in GR. For a detailed discussion of a metric, the reader is referred to the standard literature or the original articles. Important resources for exact solutions are the book by Stephani et al[SKM⁺03] and the book by Griffiths and Podolsky[GP09].

Most of the metrics in this catalogue are implemented in the Motion4D-library[MG09] and can be visualized using the GeodesicViewer[MG10]. Except for the Minkowski and Schwarzschild spacetimes, the metrics are sorted by their names.

1.1 Notation

The notation we use in this catalogue is as follows:

Indices: Coordinate indices are represented either by Greek letters or by coordinate names. Tetrad indices are indicated by Latin letters or coordinate names in brackets.

Einstein sum convention: When an index appears twice in a single term, once as lower index and once as upper index, we build the sum over all indices:

$$\zeta_\mu \zeta^\mu \equiv \sum_{\mu=0}^3 \zeta_\mu \zeta^\mu. \quad (1.1.1)$$

Vectors: A coordinate vector in x^μ direction is represented as $\partial_{x^\mu} \equiv \partial_\mu$. For arbitrary vectors, we use boldface symbols. Hence, a vector \mathbf{a} in coordinate representation reads $\mathbf{a} = a^\mu \partial_\mu$.

Derivatives: Partial derivatives are indicated by a comma, $\partial\psi/\partial x^\mu \equiv \partial_\mu \psi \equiv \psi_{,\mu}$, whereas covariant derivatives are indicated by a semicolon, $\nabla\psi = \psi_{;\mu}$.

Symmetrization and Antisymmetrization brackets:

$$a_{(\mu} b_{\nu)} = \frac{1}{2} (a_\mu b_\nu + a_\nu b_\mu), \quad a_{[\mu} b_{\nu]} = \frac{1}{2} (a_\mu b_\nu - a_\nu b_\mu) \quad (1.1.2)$$

1.2 General remarks

The Einstein field equation in the most general form reads[MTW73]

$$G_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4}, \quad R_{\{\mu, \nu\}} \quad (1.2.1)$$

with the symmetric and divergence-free Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R , the metric tensor $g_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$, the cosmological constant Λ , Newton's gravitational constant G , and the speed of light c . Because the Einstein tensor is divergence-free, the conservation equation $T^{\mu\nu}_{;\nu} = 0$ is automatically fulfilled.

the squared spacetime interval between two infinitesimally close events

A solution to the field equation is given by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.2.2)$$

with the symmetric, covariant metric tensor $g_{\mu\nu}$. The contravariant metric tensor $g^{\mu\nu}$ is related to the covariant tensor via $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$ with the Kronecker- δ . Even though $g_{\mu\nu}$ is only a component of the metric tensor $\mathbf{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu$, we will also call $g_{\mu\nu}$ the metric tensor.

Note that, in this catalogue, we mostly use the convention that the signature of the metric is +2. In general, we will also keep the physical constants c and G within the metrics.

1.3 Basic objects of a metric

The basic objects of a metric are the Christoffel symbols, the Riemann and Ricci tensors as well as the Ricci and Kretschmann scalars which are defined as follows:

Christoffel symbols of the first kind:¹

$$\Gamma_{\nu\lambda\mu} = \frac{1}{2} (g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \quad (1.3.1)$$

with the relation

$$g_{\nu\lambda,\mu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} \quad (1.3.2)$$

Christoffel symbols of the second kind:

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\rho} (g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}) \quad (1.3.3)$$

which are related to the Christoffel symbols of the first kind via

$$\Gamma_{\nu\lambda}^\mu = g^{\mu\rho} \Gamma_{\nu\lambda\rho} \quad (1.3.4)$$

Riemann tensor:

$$R^\mu_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\nu\rho} \quad (1.3.5)$$

or

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda_{\nu\rho\sigma} = \Gamma_{\nu\sigma\mu,\rho} - \Gamma_{\nu\rho\mu,\sigma} + \Gamma^\lambda_{\nu\rho} \Gamma_{\mu\sigma\lambda} - \Gamma^\lambda_{\nu\sigma} \Gamma_{\mu\sigma\lambda} \quad (1.3.6)$$

with symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad (1.3.7)$$

and

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \quad (1.3.8)$$

Ricci tensor:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R^\rho_{\mu\rho\nu} \quad (1.3.9)$$

Ricci and Kretschmann scalar:

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = R^\mu_{\mu}, \quad \mathcal{K} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = R^{\gamma\delta}_{\alpha\beta} R^{\alpha\beta}_{\gamma\delta} \quad (1.3.10)$$

¹The notation of the Christoffel symbols of the first kind differs from the one used by Rindler[Rin01], $\Gamma_{\mu\nu\lambda}^{\text{Rindler}} = \Gamma_{\nu\lambda\mu}^{\text{CoS}}$.

Weyl tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2} (g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}) + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu} \quad (1.3.11)$$

If we change the signature of a metric, these basic objects transform as follows:

$$\Gamma_{v\lambda}^\mu \mapsto \Gamma_{v\lambda}^\mu, \quad R_{\mu\nu\rho\sigma} \mapsto -R_{\mu\nu\rho\sigma}, \quad C_{\mu\nu\rho\sigma} \mapsto -C_{\mu\nu\rho\sigma}, \quad (1.3.12a)$$

$$R_{\mu\nu} \mapsto R_{\mu\nu}, \quad \mathcal{R} \mapsto -\mathcal{R}, \quad \mathcal{K} \mapsto \mathcal{K}. \quad (1.3.12b)$$

Covariant derivative

nabla/del - gradient, divergence

$$\nabla_\lambda g_{\mu\nu} = g_{\mu\nu;\lambda} = 0. \quad (1.3.13)$$

Covariant derivative of the vector field ψ^μ :

$$\nabla_v \psi^\mu = \psi_{;v}^\mu = \partial_v \psi^\mu + \Gamma_{v\lambda}^\mu \psi^\lambda \quad (1.3.14)$$

Covariant derivative of a r-s-tensor field:

$$\begin{aligned} \nabla_c T^{a_1 \dots a_r}_{\quad b_1 \dots b_s} &= \partial_c T^{a_1 \dots a_r}_{\quad b_1 \dots b_s} + \Gamma_{dc}^{a_1} T^{d \dots a_r}_{\quad b_1 \dots b_s} + \dots + \Gamma_{dc}^{a_r} T^{a_1 \dots a_{r-1} d}_{\quad b_1 \dots b_s} \\ &\quad - \Gamma_{b_1 c}^d T^{a_1 \dots a_r}_{\quad d \dots b_s} - \dots - \Gamma_{b_s c}^d T^{a_1 \dots a_r}_{\quad b_1 \dots b_{s-1} d} \end{aligned} \quad (1.3.15)$$

Killing equation:

$$\xi_{\mu;v} + \xi_{v;\mu} = 0. \quad (1.3.16)$$

tetrad - a set of four linearly independent vectors
that form a local basis for the tangent space at
each point in spacetime

1.4 Natural local tetrad and initial conditions for geodesics

We will call a local tetrad natural if it is adapted to the symmetries or the coordinates of the spacetime. The four base vectors $\mathbf{e}_{(i)} = e_{(i)}^\mu \partial_\mu$ are given with respect to coordinate directions $\partial/\partial x^\mu = \partial_\mu$, compare Nakahara[Nak90] or Chandrasekhar[Cha06] for an introduction to the tetrad formalism. The inverse or dual tetrad is given by $\theta^{(i)} = \theta_\mu^{(i)} dx^\mu$ with

$$\theta_\mu^{(i)} e_{(j)}^\mu = \delta_{(j)}^{(i)} \quad \text{and} \quad \theta_\mu^{(i)} e_{(i)}^\nu = \delta_\mu^\nu. \quad (1.4.1)$$

Note that we use Latin indices in brackets for tetrads and Greek indices for coordinates.

1.4.1 Orthonormality condition

To be applicable as a local reference frame (Minkowski frame), a local tetrad $\mathbf{e}_{(i)}$ has to fulfill the orthonormality condition

$$\langle \mathbf{e}_{(i)}, \mathbf{e}_{(j)} \rangle_g = g(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = g_{\mu\nu} e_{(i)}^\mu e_{(j)}^\nu \stackrel{!}{=} \eta_{(i)(j)}, \quad (1.4.2)$$

where $\eta_{(i)(j)} = \text{diag}(\mp 1, \pm 1, \pm 1, \pm 1)$ depending on the signature $\text{sign}(g) = \pm 2$ of the metric. Thus, the line element of a metric can be written as

$$ds^2 = \eta_{(i)(j)} \theta^{(i)} \theta^{(j)} = \eta_{(i)(j)} \theta_\mu^{(i)} \theta_\nu^{(j)} dx^\mu dx^\nu. \quad (1.4.3)$$

To obtain a local tetrad $\mathbf{e}_{(i)}$, we could first determine the dual tetrad $\theta^{(i)}$ via Eq. (1.4.3). If we combine all four dual tetrad vectors into one matrix Θ , we only have to determine its inverse Θ^{-1} to find the tetrad vectors,

$$\Theta = \begin{pmatrix} \theta_0^{(0)} & \theta_1^{(0)} & \theta_2^{(0)} & \theta_3^{(0)} \\ \theta_0^{(1)} & \theta_1^{(1)} & \theta_2^{(1)} & \theta_3^{(1)} \\ \theta_0^{(2)} & \theta_1^{(2)} & \theta_2^{(2)} & \theta_3^{(2)} \\ \theta_0^{(3)} & \theta_1^{(3)} & \theta_2^{(3)} & \theta_3^{(3)} \end{pmatrix} \Rightarrow \Theta^{-1} = \begin{pmatrix} e_{(0)}^0 & e_{(1)}^0 & e_{(2)}^0 & e_{(3)}^0 \\ e_{(0)}^1 & e_{(1)}^1 & e_{(2)}^1 & e_{(3)}^1 \\ e_{(0)}^2 & e_{(1)}^2 & e_{(2)}^2 & e_{(3)}^2 \\ e_{(0)}^3 & e_{(1)}^3 & e_{(2)}^3 & e_{(3)}^3 \end{pmatrix}. \quad (1.4.4)$$

There are also several useful relations:

$$e_{(a)\mu} = g_{\mu\nu} e_{(a)}^\nu, \quad \eta_{(a)(b)} = e_{(a)}^\mu e_{(b)\mu}, \quad e_{(b)\mu} = \eta_{(a)(b)} \theta_\mu^{(a)}, \quad (1.4.5a)$$

$$\theta_\mu^{(b)} = \eta^{(a)(b)} e_{(a)\mu}, \quad g_{\mu\nu} = e_{(a)\mu} \theta_v^{(a)}, \quad \eta^{(a)(b)} = \theta_\mu^{(a)} \theta_v^{(b)} g^{\mu\nu}. \quad (1.4.5b)$$

1.4.2 Tetrad transformations

Instead of the above found local tetrad that was directly constructed from the spacetime metric, we can also use any other local tetrad

$$\hat{\mathbf{e}}_{(i)} = A_i^k \mathbf{e}_{(k)}, \quad (1.4.6)$$

where \mathbf{A} is an element of the Lorentz group $O(1,3)$. Hence $\mathbf{A}^T \eta \mathbf{A} = \eta$ and $(\det \mathbf{A})^2 = 1$.

Lorentz-transformation in the direction $n^a = (\sin \chi \cos \xi, \sin \chi \sin \xi, \cos \xi)^T = n_a$ with $\gamma = 1/\sqrt{1-\beta^2}$,

$$\Lambda_0^0 = \gamma, \quad \Lambda_a^0 = -\beta \gamma n_a, \quad \Lambda_0^a = -\beta \gamma n^a, \quad \Lambda_b^a = (\gamma - 1)n^a n_b + \delta_b^a. \quad (1.4.7)$$

1.4.3 Ricci rotation-, connection-, and structure coefficients

The Ricci rotation coefficients $\gamma_{(i)(j)(k)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\gamma_{(i)(j)(k)} := g_{\mu\lambda} e_{(i)}^\mu \nabla_{\mathbf{e}_{(k)}} e_{(j)}^\lambda = g_{\mu\lambda} e_{(i)}^\mu e_{(k)}^\nu \nabla_\nu e_{(j)}^\lambda = g_{\mu\lambda} e_{(i)}^\mu e_{(k)}^\nu \left(\partial_\nu e_{(j)}^\lambda + \Gamma_{\nu\beta}^\lambda e_{(j)}^\beta \right). \quad (1.4.8)$$

They are antisymmetric in the first two indices, $\gamma_{(i)(j)(k)} = -\gamma_{(j)(i)(k)}$, which follows from the definition, Eq. (1.4.8), and the relation

$$0 = \partial_\mu \eta_{(i)(j)} = \nabla_\mu \left(g_{\beta\nu} e_{(i)}^\beta e_{(j)}^\nu \right), \quad (1.4.9)$$

where $\nabla_\mu g_{\beta\nu} = 0$, compare [Cha06]. Otherwise, we have

$$\gamma_{(i)(j)(k)}^{(i)} = \theta_{\lambda}^{(i)} e_{(k)}^\nu \nabla_\nu e_{(j)}^\lambda = -e_{(j)}^\lambda e_{(k)}^\nu \nabla_\nu \theta_{\lambda}^{(i)}. \quad (1.4.10)$$

The contraction of the first and the last index is given by

$$\gamma_{(j)} = \gamma_{(j)(k)}^{(k)} = \eta^{(k)(i)} \gamma_{(i)(j)(k)} = -\gamma_{(0)(j)(0)} + \gamma_{(1)(j)(1)} + \gamma_{(2)(j)(2)} + \gamma_{(3)(j)(3)} = \nabla_\nu e_{(j)}^\nu. \quad (1.4.11)$$

The connection coefficients $\omega_{(j)(n)}^{(m)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\omega_{(j)(n)}^{(m)} := \theta_\mu^{(m)} \nabla_{\mathbf{e}_{(j)}} e_{(n)}^\mu = \theta_\mu^{(m)} e_{(j)}^\alpha \nabla_\alpha e_{(n)}^\mu = \theta_\mu^{(m)} e_{(j)}^\alpha \left(\partial_\alpha e_{(n)}^\mu + \Gamma_{\alpha\beta}^\mu e_{(n)}^\beta \right), \quad (1.4.12)$$

compare Nakahara[Nak90]. They are related to the Ricci rotation coefficients via

$$\gamma_{(i)(j)(k)} = \eta_{(i)(m)} \omega_{(k)(j)}^{(m)}. \quad (1.4.13)$$

Furthermore, the local tetrad has a non-vanishing Lie-bracket $[X, Y]^\nu = X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu$. Thus,

$$[\mathbf{e}_{(i)}, \mathbf{e}_{(j)}] = c_{(i)(j)}^{(k)} \mathbf{e}_{(k)} \quad \text{or} \quad c_{(i)(j)}^{(k)} = \theta^{(k)} [\mathbf{e}_{(i)}, \mathbf{e}_{(j)}]. \quad (1.4.14)$$

The structure coefficients $c_{(i)(j)}^{(k)}$ are related to the connection coefficients or the Ricci rotation coefficients via

$$c_{(i)(j)}^{(k)} = \omega_{(i)(j)}^{(k)} - \omega_{(j)(i)}^{(k)} = \eta^{(k)(m)} (\gamma_{(m)(j)(i)} - \gamma_{(m)(i)(j)}) = \gamma_{(j)(i)}^{(k)} - \gamma_{(i)(j)}^{(k)}. \quad (1.4.15)$$

1.4.4 Riemann-, Ricci-, and Weyl-tensor with respect to a local tetrad

The transformations between the coordinate representations of the Riemann-, Ricci-, and Weyl-tensors and their representation with respect to a local tetrad $\mathbf{e}_{(i)}$ are given by

$$R_{(a)(b)(c)(d)} = R_{\mu\nu\rho\sigma} e_{(a)}^\mu e_{(b)}^\nu e_{(c)}^\rho e_{(d)}^\sigma, \quad (1.4.16a)$$

$$R_{(a)(b)} = R_{\mu\nu} e_{(a)}^\mu e_{(b)}^\nu, \quad (1.4.16b)$$

$$\begin{aligned} C_{(a)(b)(c)(d)} &= C_{\mu\nu\rho\sigma} e_{(a)}^\mu e_{(b)}^\nu e_{(c)}^\rho e_{(d)}^\sigma \\ &= R_{(a)(b)(c)(d)} - \frac{1}{2} (\eta_{(a)[(c]} R_{(d)](b)} - \eta_{(b)[(c]} R_{(d)](a)}) + \frac{R}{3} \eta_{(a)[(c)} \eta_{(d)](b)}. \end{aligned} \quad (1.4.16c)$$

1.4.5 Null or timelike directions

A null or timelike direction $v = v^{(i)} \mathbf{e}_{(i)}$ with respect to a local tetrad $\mathbf{e}_{(i)}$ can be written as

$$v = v^{(0)} \mathbf{e}_{(0)} + \psi (\sin \chi \cos \xi \mathbf{e}_{(1)} + \sin \chi \sin \xi \mathbf{e}_{(2)} + \cos \chi \mathbf{e}_{(3)}) = v^{(0)} \mathbf{e}_{(0)} + \psi \mathbf{n}. \quad (1.4.17)$$

In the case of a null direction we have $\psi = 1$ and $v^{(0)} = \pm 1$. A timelike direction can be identified with an initial four-velocity $\mathbf{u} = c\gamma(\mathbf{e}_0 + \beta \mathbf{n})$, where

$$\mathbf{u}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}} = c^2 \gamma^2 \langle \mathbf{e}_{(0)} + \beta \mathbf{n}, \mathbf{e}_{(0)} + \beta \mathbf{n} \rangle = c^2 \gamma^2 (-1 + \beta^2) = \mp c^2, \quad \text{sign}(\mathbf{g}) = \pm 2. \quad (1.4.18)$$

Thus, $\psi = c\beta\gamma$ and $v^0 = \pm c\gamma$. The sign of $v^{(0)}$ determines the time direction.

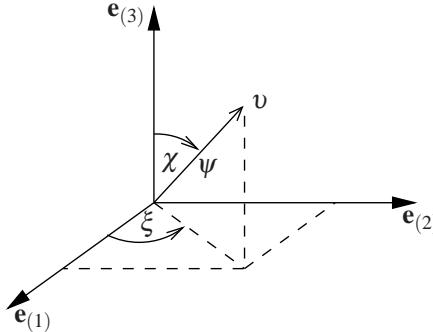


Figure 1.1: Null or timelike direction v with respect to the local tetrad $\mathbf{e}_{(i)}$.

The transformations between a local direction $v^{(i)}$ and its coordinate representation v^μ read

$$v^\mu = v^{(i)} e_{(i)}^\mu \quad \text{and} \quad v^{(i)} = \theta_\mu^{(i)} v^\mu. \quad (1.4.19)$$

1.4.6 Local tetrad for diagonal metrics

If a spacetime is represented by a diagonal metric

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \quad (1.4.20)$$

the natural local tetrad reads

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{g_{00}}} \partial_0, \quad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{11}}} \partial_1, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{22}}} \partial_2, \quad \mathbf{e}_{(3)} = \frac{1}{\sqrt{g_{33}}} \partial_3, \quad (1.4.21)$$

given that the metric coefficients are well behaved. Analogously, the dual tetrad reads

$$\theta^{(0)} = \sqrt{g_{00}} dx^0, \quad \theta^{(1)} = \sqrt{g_{11}} dx^1, \quad \theta^{(2)} = \sqrt{g_{22}} dx^2, \quad \theta^{(3)} = \sqrt{g_{33}} dx^3. \quad (1.4.22)$$

1.4.7 Local tetrad for stationary axisymmetric spacetimes

The line element of a stationary axisymmetric spacetime is given by

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dt d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2, \quad (1.4.23)$$

where the metric components are functions of r and ϑ only.

The local tetrad for an observer on a stationary circular orbit, ($r = \text{const}$, $\vartheta = \text{const}$), with four velocity $\mathbf{u} = c\Gamma(\partial_t + \zeta\partial_\varphi)$ can be defined as, compare Bini[BJ00],

$$\mathbf{e}_{(0)} = \Gamma(\partial_t + \zeta\partial_\varphi), \quad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{rr}}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{\vartheta\vartheta}}}\partial_\vartheta, \quad (1.4.24a)$$

$$\mathbf{e}_{(3)} = \Delta\Gamma[\pm(g_{t\varphi} + \zeta g_{\varphi\varphi})\partial_t \mp (g_{tt} + \zeta g_{t\varphi})\partial_\varphi], \quad (1.4.24b)$$

where

$$\Gamma = \frac{1}{\sqrt{-(g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi})}} \quad \text{and} \quad \Delta = \frac{1}{\sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}. \quad (1.4.25)$$

The angular velocity ζ is limited due to $g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi} < 0$

$$\zeta_{\min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \quad \text{and} \quad \zeta_{\max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \quad (1.4.26)$$

with $\omega = -g_{t\varphi}/g_{\varphi\varphi}$.

For $\zeta = 0$, the observer is static with respect to spatial infinity. The locally non-rotating frame (LNRF) has angular velocity $\zeta = \omega$, see also MTW[MTW73], exercise 33.3.

Static limit: $\zeta_{\min} = 0 \Rightarrow g_{tt} = 0$.

The transformation between the local direction $v^{(i)}$ and the coordinate direction v^μ reads

$$v^0 = \Gamma(v^{(0)} \pm v^{(3)}\Delta w_1), \quad v^1 = \frac{v^{(1)}}{\sqrt{g_{rr}}}, \quad v^2 = \frac{v^{(2)}}{\sqrt{g_{\vartheta\vartheta}}}, \quad v^3 = \Gamma(v^{(0)}\zeta \mp v^{(3)}\Delta w_2), \quad (1.4.27)$$

with

$$w_1 = g_{t\varphi} + \zeta g_{\varphi\varphi} \quad \text{and} \quad w_2 = g_{tt} + \zeta g_{t\varphi}. \quad (1.4.28)$$

The back transformation reads

$$v^{(0)} = \frac{1}{\Gamma} \frac{v^0 w_2 + v^3 w_1}{\zeta w_1 + w_2}, \quad v^{(1)} = \sqrt{g_{rr}} v^1, \quad v^{(2)} = \sqrt{g_{\vartheta\vartheta}} v^2, \quad v^{(3)} = \pm \frac{1}{\Delta\Gamma} \frac{\zeta v^0 - v^3}{\zeta w_1 + w_2}. \quad (1.4.29)$$

Note, to obtain a right-handed local tetrad, $\det(e_{(i)}^\mu) > 0$, the upper sign has to be used.

1.5 Newman-Penrose tetrad and spin-coefficients

The Newman-Penrose tetrad consists of four null vectors $\mathbf{e}_{(i)}^* = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, where \mathbf{l} and \mathbf{n} are real and \mathbf{m} and $\bar{\mathbf{m}}$ are complex conjugates; see Penrose and Rindler[PR84] or Chandrasekhar[Cha06] for a thorough discussion. The Newman-Penrose (NP) tetrad has to fulfill the orthonormality relation

$$\langle \mathbf{e}_{(i)}^*, \mathbf{e}_{(j)}^* \rangle = \eta_{(i)(j)}^* \quad \text{with} \quad \eta_{(i)(j)}^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (1.5.1)$$

A straightforward relation between the NP tetrad and the natural local tetrad, as discussed in Sec. 1.4, is given by

$$\mathbf{l} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_{(0)} + \mathbf{e}_{(1)}), \quad \mathbf{n} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_{(0)} - \mathbf{e}_{(1)}), \quad \mathbf{m} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_{(2)} + i\mathbf{e}_{(3)}), \quad (1.5.2)$$

where the upper/lower sign has to be used for metrics with positive/negative signature. The Ricci rotation-coefficients of a NP tetrad are now called *spin coefficients* and are designated by specific symbols:

$$\kappa = \gamma_{(2)(1)(1)}, \quad \rho = \gamma_{(2)(0)(3)}, \quad \varepsilon = \frac{1}{2} (\gamma_{(1)(0)(0)} + \gamma_{(2)(3)(0)}), \quad (1.5.3a)$$

$$\sigma = \gamma_{(2)(0)(2)}, \quad \mu = \gamma_{(1)(3)(2)}, \quad \gamma = \frac{1}{2} (\gamma_{(1)(0)(1)} + \gamma_{(2)(3)(1)}), \quad (1.5.3b)$$

$$\lambda = \gamma_{(1)(3)(3)}, \quad \tau = \gamma_{(2)(0)(1)}, \quad \alpha = \frac{1}{2} (\gamma_{(1)(0)(3)} + \gamma_{(2)(3)(3)}), \quad (1.5.3c)$$

$$\nu = \gamma_{(1)(3)(1)}, \quad \pi = \gamma_{(1)(3)(0)}, \quad \beta = \frac{1}{2} (\gamma_{(1)(0)(2)} + \gamma_{(2)(3)(2)}). \quad (1.5.3d)$$

1.6 Coordinate relations

1.6.1 Spherical and Cartesian coordinates

The well-known relation between the spherical coordinates (r, ϑ, φ) and the Cartesian coordinates (x, y, z) , compare Fig. 1.2, are

$$x = r \sin \vartheta \cos \varphi, \quad y = r \sin \vartheta \sin \varphi, \quad z = r \cos \vartheta, \quad (1.6.1)$$

and

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \vartheta = \arctan 2(\sqrt{x^2 + y^2}, z), \quad \varphi = \arctan 2(y, x), \quad (1.6.2)$$

where $\arctan 2()$ ensures that $\varphi \in [0, 2\pi]$ and $\vartheta \in (0, \pi)$.

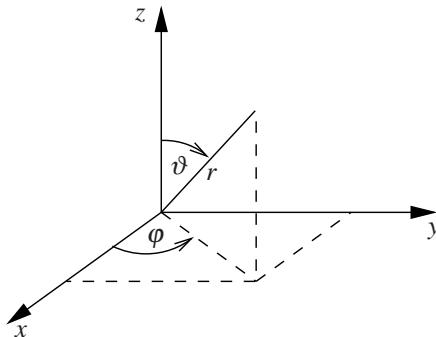


Figure 1.2: Relation between spherical and Cartesian coordinates.

The total differentials of the spherical coordinates read

$$dr = \frac{x dx + y dy + z dz}{r}, \quad d\vartheta = \frac{xz dx + yz dy - (x^2 + y^2) dz}{r^2 \sqrt{x^2 + y^2}}, \quad d\varphi = \frac{-y dx + x dy}{x^2 + y^2}, \quad (1.6.3)$$

whereas the coordinate derivatives read

$$\partial_r = \frac{\partial x}{\partial r} \partial_x + \frac{\partial y}{\partial r} \partial_y + \frac{\partial z}{\partial r} \partial_z = \sin \vartheta \cos \varphi \partial_x + \sin \vartheta \sin \varphi \partial_y + \cos \vartheta \partial_z, \quad (1.6.4a)$$

$$\partial_\vartheta = \frac{\partial x}{\partial \vartheta} \partial_x + \frac{\partial y}{\partial \vartheta} \partial_y + \frac{\partial z}{\partial \vartheta} \partial_z = r \cos \vartheta \cos \varphi \partial_x + r \cos \vartheta \sin \varphi \partial_y - r \sin \vartheta \partial_z, \quad (1.6.4b)$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi} \partial_x + \frac{\partial y}{\partial \varphi} \partial_y + \frac{\partial z}{\partial \varphi} \partial_z = -r \sin \vartheta \sin \varphi \partial_x + r \sin \vartheta \cos \varphi \partial_y, \quad (1.6.4c)$$

and

$$\partial_x = \frac{\partial r}{\partial x} \partial_r + \frac{\partial \vartheta}{\partial x} \partial_\vartheta + \frac{\partial \varphi}{\partial x} \partial_\varphi = \sin \vartheta \cos \varphi \partial_r + \frac{\cos \vartheta \cos \varphi}{r} \partial_\vartheta - \frac{\sin \varphi}{r \sin \vartheta} \partial_\varphi, \quad (1.6.5a)$$

$$\partial_y = \frac{\partial r}{\partial y} \partial_r + \frac{\partial \vartheta}{\partial y} \partial_\vartheta + \frac{\partial \varphi}{\partial y} \partial_\varphi = \sin \vartheta \sin \varphi \partial_r + \frac{\cos \vartheta \sin \varphi}{r} \partial_\vartheta + \frac{\cos \varphi}{r \sin \vartheta} \partial_\varphi, \quad (1.6.5b)$$

$$\partial_z = \frac{\partial r}{\partial z} \partial_r + \frac{\partial \vartheta}{\partial z} \partial_\vartheta + \frac{\partial \varphi}{\partial z} \partial_\varphi = \cos \vartheta \partial_r - \frac{\sin \vartheta}{r} \partial_\vartheta. \quad (1.6.5c)$$

1.6.2 Cylindrical and Cartesian coordinates

The relation between cylindrical coordinates (r, φ, z) and Cartesian coordinates (x, y, z) is given by

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad \text{and} \quad r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan 2(y, x), \quad (1.6.6)$$

where $\arctan 2()$ again ensures that the angle $\varphi \in [0, 2\pi)$.

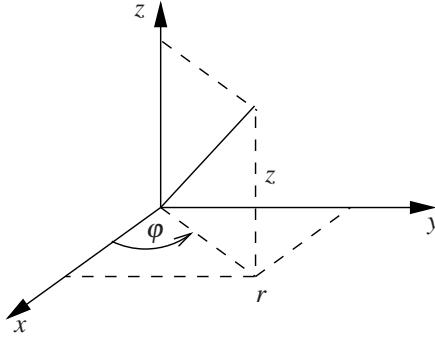


Figure 1.3: Relation between cylindrical and Cartesian coordinates.

The total differentials of the spherical coordinates are given by

$$dr = \frac{xdx + ydy}{r}, \quad d\varphi = \frac{-ydx + xdy}{r^2}, \quad (1.6.7)$$

and

$$dx = \cos \varphi dr - r \sin \varphi d\varphi, \quad dy = \sin \varphi dr + r \cos \varphi d\varphi. \quad (1.6.8)$$

The coordinate derivatives are

$$\partial_r = \frac{\partial x}{\partial r} \partial_x + \frac{\partial y}{\partial r} \partial_y = \cos \varphi \partial_x + \sin \varphi \partial_y, \quad (1.6.9a)$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi} \partial_x + \frac{\partial y}{\partial \varphi} \partial_y = -r \sin \varphi \partial_x + r \cos \varphi \partial_y, \quad (1.6.9b)$$

and

$$\partial_x = \frac{\partial r}{\partial x} \partial_r + \frac{\partial \varphi}{\partial x} \partial_\varphi = \cos \varphi \partial_r - \frac{\sin \varphi}{r} \partial_y, \quad (1.6.10a)$$

$$\partial_y = \frac{\partial r}{\partial y} \partial_r + \frac{\partial \varphi}{\partial y} \partial_\varphi = \sin \varphi \partial_r + \frac{\cos \varphi}{r} \partial_y. \quad (1.6.10b)$$

1.7 Embedding diagram

A two-dimensional hypersurface with line segment

$$d\sigma^2 = g_{rr}(r)dr^2 + g_{\varphi\varphi}(r)d\varphi^2 \quad (1.7.1)$$

can be embedded in a three-dimensional Euclidean space with cylindrical coordinates,

$$d\sigma^2 = \left[1 + \left(\frac{dz}{d\rho} \right)^2 \right] d\rho^2 + \rho^2 d\varphi^2. \quad (1.7.2)$$

With $\rho(r)^2 = g_{\varphi\varphi}(r)$ and $dr = (dr/d\rho)d\rho$, we obtain for the embedding function $z = z(r)$,

$$\frac{dz}{dr} = \pm \sqrt{g_{rr} - \left(\frac{d\sqrt{g_{\varphi\varphi}}}{dr} \right)^2}. \quad (1.7.3)$$

If $g_{\varphi\varphi}(r) = r^2$, then $d\sqrt{g_{\varphi\varphi}}/dr = 1$.

1.8 Equations of motion and transport equations

1.8.1 Geodesic equation

The geodesic equation reads

$$\frac{D^2x^\mu}{d\lambda^2} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \quad (1.8.1)$$

with the affine parameter λ . For timelike geodesics, however, we replace the affine parameter by the proper time τ .

The geodesic equation (1.8.1) is a system of ordinary differential equations of second order. Hence, to solve these differential equations, we need an initial position $x^\mu(\lambda = 0)$ as well as an initial direction $(dx^\mu/d\lambda)(\lambda = 0)$. This initial direction has to fulfill the constraint equation

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \kappa c^2, \quad (1.8.2)$$

where $\kappa = 0$ for lightlike and $\kappa = \mp 1$, $(\text{sign}(g) = \pm 2)$, for timelike geodesics.

The initial direction can also be determined by means of a local reference frame, compare sec. 1.4.5, that automatically fulfills the constraint equation (1.8.2). If we use the natural local tetrad as local reference frame, we have

$$\frac{dx^\mu}{d\lambda} \Big|_{\lambda=0} = v^\mu = v^{(i)} e_{(i)}^\mu. \quad (1.8.3)$$

1.8.2 Fermi-Walker transport

The Fermi-Walker transport, see e.g. Stephani[SS90], of a vector $\mathbf{X} = X^\mu \partial_\mu$ along the worldline $x^\mu(\tau)$ with four-velocity $\mathbf{u} = u^\mu(\tau) \partial_\mu$ is given by $\mathbb{F}_\mathbf{u} X^\mu = 0$ with

$$\mathbb{F}_\mathbf{u} X^\mu := \frac{dX^\mu}{d\tau} + \Gamma_{\rho\sigma}^\mu u^\rho X^\sigma + \frac{1}{c^2} (u^\sigma a^\mu - a^\sigma u^\mu) g_{\rho\sigma} X^\rho. \quad (1.8.4)$$

The four-acceleration follows from the four-velocity via

$$a^\mu = \frac{D^2x^\mu}{d\tau^2} = \frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma_{\rho\sigma}^\mu u^\rho u^\sigma. \quad (1.8.5)$$

1.8.3 Parallel transport

If the four-acceleration vanishes, the Fermi-Walker transport simplifies to the parallel transport $\mathbb{P}_\mathbf{u} X^\mu = 0$ with

$$\mathbb{P}_\mathbf{u} X^\mu := \frac{DX^\mu}{d\tau} = \frac{dX^\mu}{d\tau} + \Gamma_{\rho\sigma}^\mu u^\rho X^\sigma. \quad (1.8.6)$$

1.8.4 Euler-Lagrange formalism

A detailed discussion of the Euler-Lagrange formalism can be found, e.g., in Rindler[[Rin01](#)]. The Lagrangian \mathcal{L} is defined as

$$\mathcal{L} := g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad \mathcal{L} \stackrel{!}{=} \kappa c^2, \quad (1.8.7)$$

where x^μ are the coordinates of the metric, and the dot means differentiation with respect to the affine parameter λ . For timelike geodesics, $\kappa = \mp 1$ depending on the signature of the metric, $\text{sign}(\mathbf{g}) = \pm 2$. For lightlike geodesics, $\kappa = 0$.

The Euler-Lagrange equations read

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0. \quad (1.8.8)$$

If \mathcal{L} is independent of x^ρ , then x^ρ is a cyclic variable and

$$p_\rho = g_{\rho\nu}\dot{x}^\nu = \text{const.} \quad (1.8.9)$$

Note that $[\mathcal{L}]_u = \frac{\text{length}^2}{\text{time}^2}$ for timelike and $[\mathcal{L}]_u = 1$ for lightlike geodesics, see Sec. [1.9](#).

1.8.5 Hamilton formalism

The super-Hamiltonian \mathcal{H} is defined as

$$\mathcal{H} := \frac{1}{2}g^{\mu\nu}p_\mu p_\nu, \quad \mathcal{H} \stackrel{!}{=} \frac{1}{2}\kappa c^2, \quad (1.8.10)$$

where $p_\mu = g_{\mu\nu}\dot{x}^\nu$ are the canonical momenta, see e.g. MTW[[MTW73](#)], para. 21.1. As in classical mechanics, we have

$$\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\mu} \quad \text{and} \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu}. \quad (1.8.11)$$

1.9 Units

A first test in analyzing whether an equation is correct is to check the units. Newton's gravitational constant G , for example, has the following units

$$[G]_u = \frac{\text{length}^3}{\text{mass} \cdot \text{time}^2}, \quad (1.9.1)$$

where $[\cdot]_u$ indicates that we evaluate the units of the enclosed expression. Further examples are

$$[ds]_u = \text{length}, \quad [\mathbf{u}]_u = \frac{\text{length}}{\text{time}}, \quad [R_{trtr}^{\text{Schwarzschild}}]_u = \frac{1}{\text{time}^2}, \quad [R_{\vartheta\varphi\vartheta\varphi}^{\text{Schwarzschild}}]_u = \text{length}^2. \quad (1.9.2)$$

1.10 Tools

1.10.1 Maple/GRTensorII

The Christoffel symbols, the Riemann- and Ricci-tensors as well as the Ricci and Kretschmann scalars in this catalogue were determined by means of the software Maple together with the GRTensorII package by Musgrave, Pollney, and Lake.²

A typical worksheet to enter a new metric may look like this:

²The commercial software Maple can be found here: <http://www.maplesoft.com>. The GRTensorII-package is free: <http://grtensor.phy.queensu.ca>.

```

> grtw();
> makeg(Schwarzschild);

Makeg 2.0: GRTensor metric/basis entry utility
To quit makeg, type 'exit' at any prompt.
Do you wish to enter a 1) metric [g(dn,dn)],
                           2) line element [ds],
                           3) non-holonomic basis [e(1)...e(n)], or
                           4) NP tetrad [l,n,m,mbar]?
> 2:

Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi];

Enter the line element using d[coord] to indicate differentials.
(for example, r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
ds^2 =

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
> {}:

You may choose to 0) Use the metric WITHOUT saving it,
                           1) Save the metric as it is,
                           2) Correct an element of the metric,
                           3) Re-enter the metric,
                           4) Add/change constraint equations,
                           5) Add a text description, or
                           6) Abandon this metric and return to Maple.
> 0:

```

The worksheets for some of the metrics in this catalogue can be found on the authors homepage. To determine the objects that are defined with respect to a local tetrad, the metric must be given as non-holonomic basis.

The various basic objects can be determined via

Christoffel symbols $\Gamma_{\nu\rho}^\mu$	<code>grcalc(Chr2);</code>	<code>grcalc(Chr(dn,dn,up));</code>
partial derivatives $\Gamma_{\nu\rho,\sigma}^\mu$	<code>grcalc(Chr(dn,dn,up,pdn));</code>	
Riemann tensor $R_{\mu\nu\rho\sigma}$	<code>grcalc(Riemman);</code>	<code>grcalc(R(dn,dn,dn,dn));</code>
Ricci tensor $R_{\mu\nu}$	<code>grcalc(Ricci);</code>	<code>grcalc(R(dn,dn));</code>
Ricci scalar \mathcal{R}	<code>grcalc(Ricciscalar);</code>	
Kretschmann scalar \mathcal{K}	<code>grcalc(RiemSq);</code>	

1.10.2 Mathematica

The calculation of the Christoffel symbols, the Riemann- or Ricci-tensor within *Mathematica* could read like this:

```

Clearing the values of symbols:
In[1]:= Clear[coord, metric, inversemetric, affine,
           t, r, Theta, Phi]

Setting the dimension:
In[2]:= n := 4

Defining a list of coordinates:
In[3]:= coord := {t, r, Theta, Phi}

Defining the metric:
In[4]:= metric := {{-(1 - rs/r) c^2, 0, 0, 0},
                  {0, 1/(1 - rs/r), 0, 0},
                  {0, 0, r^2, 0},
                  {0, 0, 0, r^2 Sin[Theta]^2}}
In[5]:= metric // MatrixForm

```

```

Calculating the inverse metric:
In[6]:= inversemetric := Simplify[Inverse[metric]]

In[7]:= inversemetric // MatrixForm

Calculating the Christoffel symbols of the second kind:
In[8]:= affine := affine = Simplify[
  Table[(1/2) Sum[inversemetric[[Mu, Rho]] (
    D[metric[[Rho, Nu]], coord[[Lambda]]] +
    D[metric[[Rho, Lambda]], coord[[Nu]]] -
    D[metric[[Nu, Lambda]], coord[[Rho]]]),
  {Rho, 1, n}], {Nu, 1, n}, {Lambda, 1, n}, {Mu, 1, n}]]

Displaying the Christoffel symbols of the second kind:
In[9]:= listaffine :=
  Table[If[UnsameQ[affine[[Nu, Lambda, Mu]], 0],
  {Style[ Subsuperscript[\[CapitalGamma]],
  Row[{coord[[Nu]], coord[[Lambda]]}], coord[[Mu]]], 18},
  "=",
  Style[affine[[Nu, Lambda, Mu]], 14]}],
  {Lambda, 1, n}, {Nu, 1, Lambda}, {Mu, 1, n}]

In[10]:= TableForm[Partition[DeleteCases[Flatten[listaffine],
  Null], 3],
  TableSpacing -> {1, 2}]

Defining the Riemann tensor:
In[11]:= riemann := riemann =
  Table[D[affine[[Nu, Sigma, Mu]], coord[[Rho]]] -
  D[affine[[Nu, Rho, Mu]], coord[[Sigma]]] +
  Sum[affine[[Rho, Lambda, Mu]] *
  affine[[Nu, Sigma, Lambda]] -
  affine[[Sigma, Lambda, Mu]] *
  affine[[Nu, Rho, Lambda]],
  {Lambda, 1, n}],
  {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

Defining the Riemann tensor with lower indices:
In[12]:= riemannDn := riemannDn =
  Table[Simplify[
  Sum[metric[[Mu, Kappa]] riemann[[Kappa, Nu, Rho, Sigma]],
  {Kappa, 1, n}],
  {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]]

In[13]:= listRiemann :=
  Table[If[UnsameQ[riemannDn[[Mu, Nu, Rho, Sigma]], 0],
  {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]], coord[[Rho]],
  coord[[Sigma]]}]], 16], "=",
  riemannDn[[Mu, Nu, Rho, Sigma]]}],
  {Nu, 1, n}, {Mu, 1, Nu}, {Sigma, 1, n}, {Rho, 1, Sigma}]

In[14]:= TableForm[Partition[DeleteCases[Flatten[listRiemann],
  Null], 3],
  TableSpacing -> {2, 2}]

Defining the Ricci tensor:
In[15]:= ricci := ricci =
  Table[Simplify[
  Sum[riemann[[Rho, Mu, Rho, Nu]], {Rho, 1, n}],
  {Mu, 1, n}, {Nu, 1, n}]

In[16]:= listRicci :=
  Table[If[UnsameQ[ricci[[Mu, Nu]], 0],
  {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]]}]], 16],
  "=",
  Style[ricci[[Mu, Nu]], 16]}], {Nu, 1, 4}, {Mu, 1, Nu}]

In[17]:= TableForm[Partition[DeleteCases[Flatten[listRicci],
  Null], 3],
  TableSpacing -> {1, 2}]

Defining the Ricci scalar:
In[18]:= ricciscalar := ricciscalar =
  Simplify[Sum[

```

```

Sum[inversemetric[[Mu, Nu]] ricci[[Nu, Mu]],
{Mu, 1, n}], {Nu, 1, n}]

Defining the Kretschmann scalar:
In[19]:= riemannUp := riemannUp =
Table[Simplify[
Sum[inversemetric[[Nu, Kappa]],
riemannUp[[Mu, Kappa, Rho, Sigma]], {Kappa, 1, n}]],
{Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

In[20]:= kretschmann := kretschmann =
Simplify[Sum[ Sum[Sum[Sum[
riemannUp[[Mu, Nu, Rho, Sigma]],
riemannUp[[Rho, Sigma, Mu, Null]],
{Mu, 1, n}], {Nu, 1, n}], {Rho, 1, n}], {Sigma, 1, n}]]

```

Some example notebooks can be found on the authors homepage.

1.10.3 Maxima

Instead of using commercial software like *Maple* or *Mathematica*, Maxima also offers a tensor package that helps to calculate the Christoffel symbols etc. The above example for the Schwarzschild metric can be written as a maxima worksheet as follows:

```

/* load ctensor package */
load(ctensor);

/* define coordinates to use */
ct_coords:[t,r,theta,phi];

/* start with the identity metric */
lg:ident(4);
lg[1,1]:=c^2*(1-rs/r);
lg[2,2]:=-1/(1-rs/r);
lg[3,3]:=r^2;
lg[4,4]:=-r^2*sin(theta)^2;
cmetric();

/* calculate the christoffel symbols of the second kind */
christof(mcs);

/* calculate the riemann tensor */
lriemann(mcs);

/* calculate the ricci tensor */
ricci(mcs);

/* calculate the ricci scalar */
scurvature();

/* calculate the Kretschmann scalar */
uriemann(mcs);
rinvariant();
ratsimp(%);

```

As you may have noticed, the Schwarzschild metric must be given with negative signature.

Chapter 2

Spacetimes

2.1 Minkowski

2.1.1 Cartesian coordinates

The Minkowski metric in Cartesian coordinates $\{t, x, y, z \in \mathbb{R}\}$ reads

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \quad (2.1.1)$$

All Christoffel symbols as well as the Riemann- and Ricci-tensor vanish identically. The natural local tetrad is trivial,

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(x)} = \partial_x, \quad \mathbf{e}_{(y)} = \partial_y, \quad \mathbf{e}_{(z)} = \partial_z, \quad (2.1.2)$$

with dual

$$\theta^{(t)} = c dt, \quad \theta^{(x)} = dx, \quad \theta^{(y)} = dy, \quad \theta^{(z)} = dz. \quad (2.1.3)$$

2.1.2 Cylindrical coordinates

The Minkowski metric in cylindrical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\varphi^2 + dz^2, \quad (2.1.4)$$

has the natural local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r} \partial_\varphi, \quad \mathbf{e}_{(z)} = \partial_z. \quad (2.1.5)$$

Christoffel symbols:

$$\Gamma_{\varphi\varphi}^r = -r, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}. \quad (2.1.6)$$

Partial derivatives

$$\Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\varphi\varphi,r}^r = -1. \quad (2.1.7)$$

Ricci rotation coefficients:

$$\gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \quad \text{and} \quad \gamma_{(r)} = \frac{1}{r}. \quad (2.1.8)$$

2.1.3 Spherical coordinates

In spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$, the Minkowski metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.1.9)$$

Christoffel symbols:

$$\Gamma_{\vartheta\vartheta}^r = -r, \quad \Gamma_{\varphi\varphi}^r = -r \sin^2 \vartheta, \quad \Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad (2.1.10a)$$

$$\Gamma_{\varphi\vartheta}^\vartheta = -\sin \vartheta \cos \vartheta, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta. \quad (2.1.10b)$$

Partial derivatives

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -1, \quad (2.1.11a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta), \quad (2.1.11b)$$

$$\Gamma_{\varphi\vartheta,\vartheta}^r = -\sin(2\vartheta). \quad (2.1.11c)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \quad (2.1.12)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.1.13)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.1.14)$$

2.1.4 Conform-compactified coordinates

The Minkowski metric in conform-compactified coordinates $\{\psi \in [-\pi, \pi], \xi \in (0, \pi), \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads[HE99]

$$ds^2 = -d\psi^2 + d\xi^2 + \sin^2 \xi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.1.15)$$

This form follows from the spherical Minkowski metric (2.1.9) by means of the coordinate transformation

$$ct + r = \tan \frac{\psi + \xi}{2}, \quad ct - r = \tan \frac{\psi - \xi}{2}, \quad (2.1.16)$$

resulting in the metric

$$ds^2 = \frac{-d\psi^2 + d\xi^2}{4 \cos^2 \frac{\psi + \xi}{2} \cos^2 \frac{\psi - \xi}{2}} + \frac{\sin^2 \xi}{4 \cos^2 \frac{\psi + \xi}{2} \cos^2 \frac{\psi - \xi}{2}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.1.17)$$

and by the conformal transformation $ds^2 = \Omega^2 ds^2$ with $\Omega^2 = 4 \cos^2 \frac{\psi + \xi}{2} \cos^2 \frac{\psi - \xi}{2}$.

Christoffel symbols:

$$\Gamma_{\xi\vartheta}^\vartheta = \cot \xi, \quad \Gamma_{\xi\varphi}^\varphi = \cot \xi, \quad \Gamma_{\vartheta\vartheta}^\xi = -\sin \xi \cos \xi, \quad (2.1.18a)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\vartheta}^\xi = -\sin \xi \cos \xi \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.1.18b)$$

Partial derivatives

$$\Gamma_{\xi \vartheta, \xi}^{\vartheta} = -\frac{1}{\sin^2 \xi}, \quad \Gamma_{\xi \varphi, \xi}^{\varphi} = -\frac{1}{\sin^2 \xi}, \quad \Gamma_{\vartheta \vartheta, \xi}^{\xi} = -\cos(2\xi), \quad (2.1.19a)$$

$$\Gamma_{\vartheta \vartheta, \vartheta}^{\vartheta} = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi \varphi, \xi}^{\xi} = -\cos(2\xi) \sin^2 \vartheta, \quad \Gamma_{\varphi \varphi, \vartheta}^{\vartheta} = -\cos(2\vartheta), \quad (2.1.19b)$$

$$\Gamma_{\varphi \varphi, \vartheta}^{\xi} = -\frac{1}{2} \sin(2\xi) \sin(2\vartheta). \quad (2.1.19c)$$

Riemann-Tensor:

$$R_{\xi \vartheta \xi \vartheta} = \sin^2 \xi, \quad R_{\xi \varphi \xi \varphi} = \sin^2 \xi \sin^2 \vartheta, \quad R_{\vartheta \varphi \vartheta \varphi} = \sin^4 \xi \sin^2 \vartheta. \quad (2.1.20)$$

Ricci-Tensor:

$$R_{\xi \xi} = 2, \quad R_{\vartheta \vartheta} = 2 \sin^2 \xi, \quad R_{\varphi \varphi} = 2 \sin^2 \xi \sin^2 \vartheta. \quad (2.1.21)$$

Ricci and Kretschmann scalars:

$$\mathcal{R} = 6, \quad \mathcal{K} = 12. \quad (2.1.22)$$

The Weyl tensor vanishes identically.

Local tetrad:

$$\mathbf{e}_{(\psi)} = \partial_{\psi}, \quad \mathbf{e}_{(\xi)} = \partial_{\xi}, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\sin \xi} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\sin \xi \sin \vartheta} \partial_{\varphi}. \quad (2.1.23)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(\xi)(\vartheta)} = \gamma_{(\varphi)(\xi)(\varphi)} = \cot \xi, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{\sin \xi}. \quad (2.1.24)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\xi)} = 2 \cot \xi, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{\sin \xi}. \quad (2.1.25)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\xi)(\vartheta)(\xi)(\vartheta)} = R_{(\xi)(\varphi)(\xi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = 1. \quad (2.1.26)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\xi)(\xi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 2. \quad (2.1.27)$$

2.1.5 Rotating coordinates

The transformation $d\varphi \mapsto d\varphi + \omega dt$ brings the Minkowski metric (2.1.4) into the rotating form [Rin01] with coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$ds^2 = -\left(1 - \frac{\omega^2 r^2}{c^2}\right)[c dt - \Omega(r) d\varphi]^2 + dr^2 + \frac{r^2}{1 - \omega^2 r^2/c^2} d\varphi^2 + dz^2 \quad (2.1.28)$$

with $\Omega(r) = (r^2 \omega / c) / (1 - \omega^2 r^2 / c^2)$.

Metric-Tensor:

$$g_{tt} = -c^2 + \omega^2 r^2, \quad g_{t\varphi} = \omega r^2, \quad g_{rr} = g_{zz} = 1, \quad g_{\varphi\varphi} = r^2. \quad (2.1.29)$$

Christoffel symbols:

$$\Gamma_{tt}^r = -\omega^2 r, \quad \Gamma_{tr}^\varphi = \frac{\omega}{r}, \quad \Gamma_{t\varphi}^r = -\omega r, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\varphi\varphi}^r = -r. \quad (2.1.30)$$

Partial derivatives

$$\Gamma_{tt,r}^r = -\omega^2, \quad \Gamma_{tr,r}^\varphi = -\frac{\omega}{r^2}, \quad \Gamma_{t\varphi,r}^r = -\omega, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\varphi\varphi,r}^r = -1. \quad (2.1.31)$$

The local tetrad of the comoving observer is

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t - \frac{\omega}{c} \partial_\varphi, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r} \partial_\varphi, \quad \mathbf{e}_{(z)} = \partial_z, \quad (2.1.32)$$

whereas the static observer has the local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c \sqrt{1 - \omega^2 r^2 / c^2}} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(z)} = \partial_z, \quad (2.1.33a)$$

$$\mathbf{e}_{(\varphi)} = \frac{\omega r}{c^2 \sqrt{1 - \omega^2 r^2 / c^2}} \partial_t + \frac{\sqrt{1 - \omega^2 r^2 / c^2}}{r} \partial_\varphi. \quad (2.1.33b)$$

2.1.6 Rindler coordinates

The worldline of an observer in the Minkowski spacetime who moves with constant proper acceleration α along the x direction reads

$$x = \frac{c^2}{\alpha} \cosh \frac{\alpha t'}{c}, \quad ct = \frac{c^2}{\alpha} \sinh \frac{\alpha t'}{c}, \quad (2.1.34)$$

where t' is the observer's proper time. The observer starts at $x = 1$ with zero velocity.

However, such an observer could also be described with Rindler coordinates. With the coordinate transformation

$$(ct, x) \mapsto (\tau, \rho) : \quad ct = \frac{1}{\rho} \sinh \tau, \quad x = \frac{1}{\rho} \cosh \tau, \quad (2.1.35)$$

where $\rho = \alpha/c^2$, the Rindler metric reads

$$ds^2 = -\frac{1}{\rho^2} d\tau^2 + \frac{1}{\rho^4} d\rho^2 + dy^2 + dz^2. \quad (2.1.36)$$

Christoffel symbols:

$$\Gamma_{\tau\tau}^\rho = -\rho, \quad \Gamma_{\tau\rho}^\tau = -\frac{1}{\rho}, \quad \Gamma_{\rho\rho}^\rho = -\frac{2}{\rho}. \quad (2.1.37)$$

Partial derivatives

$$\Gamma_{\tau\tau,\rho}^\rho = -1, \quad \Gamma_{\tau\rho,\rho}^\tau = \frac{1}{\rho^2}, \quad \Gamma_{\rho\rho,\rho}^\rho = \frac{2}{\rho^2}. \quad (2.1.38)$$

The Riemann and Ricci tensors as well as the Ricci and Kretschmann scalar vanish identically.

Local tetrad:

$$\mathbf{e}_{(\tau)} = \rho \partial_\tau, \quad \mathbf{e}_{(\rho)} = \rho^2 \partial_\rho, \quad \mathbf{e}_{(y)} = \partial_y, \quad \mathbf{e}_{(z)} = \partial_z. \quad (2.1.39)$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(\rho)(\tau)} = \rho, \quad \text{and} \quad \gamma_{(\rho)} = -\rho. \quad (2.1.40)$$

2.2 Schwarzschild spacetime

2.2.1 Schwarzschild coordinates

In Schwarzschild coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi]\}$, the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.2.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The critical point $r = 0$ is a real curvature singularity while the event horizon, $r = r_s$, is only a coordinate singularity, see e.g. the Kretschmann scalar.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 r_s(r - r_s)}{2r^3}, \quad \Gamma_{tr}^r = \frac{r_s}{2r(r - r_s)}, \quad \Gamma_{rr}^r = -\frac{r_s}{2r(r - r_s)}, \quad (2.2.2a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad (2.2.2b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -(r - r_s) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.2c)$$

Partial derivatives

$$\Gamma_{tt,r}^r = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \quad \Gamma_{tr,r}^r = -\frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad \Gamma_{rr,r}^r = \frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad (2.2.3a)$$

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -1, \quad (2.2.3b)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta), \quad (2.2.3c)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -(r - r_s) \sin(2\vartheta). \quad (2.2.3d)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \quad R_{t\vartheta t\vartheta} = \frac{1}{2} \frac{c^2 (r - r_s) r_s}{r^2}, \quad R_{t\varphi t\varphi} = \frac{1}{2} \frac{c^2 (r - r_s) r_s \sin^2 \vartheta}{r^2}, \quad (2.2.4a)$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2} \frac{r_s}{r - r_s}, \quad R_{r\varphi r\varphi} = -\frac{1}{2} \frac{r_s \sin^2 \vartheta}{r - r_s}, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.4b)$$

As expected, the Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12 \frac{r_s^2}{r^6}. \quad (2.2.5)$$

Here, it becomes clear that at $r = r_s$ there is no real singularity.

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \quad (2.2.6)$$

Dual tetrad:

$$\theta^{(t)} = c \sqrt{1 - \frac{r_s}{r}} dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.2.7)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2 \sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.2.8)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.2.9)$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \quad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\varphi)}^{(\varphi)} = -\frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \quad c_{(\vartheta)(\varphi)}^{(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.2.10)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.11a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.11b)$$

The covariant derivatives of the Riemann tensor read

$$R_{(t)(r)(t)(r);(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi);(r)} = \frac{3r_s}{r^5}\sqrt{r(r - r_s)}, \quad (2.2.12a)$$

$$R_{(t)(r)(r)(\vartheta);(\vartheta)} = R_{(t)(r)(t)(\varphi);(\varphi)} = R_{(t)(\vartheta)(t)(\vartheta);(r)} = R_{(t)(\varphi)(t)(\varphi);(r)} = \\ = R_{(r)(\varphi)(\vartheta)(\varphi);(\vartheta)} = -\frac{3r_s}{2r^5}\sqrt{r(r - r_s)}, \quad (2.2.12b)$$

$$R_{(r)(\vartheta)(r)(\vartheta);(r)} = R_{(r)(\vartheta)(\vartheta)(\varphi);(\varphi)} = R_{(r)(\varphi)(r)(\varphi);(r)} = \frac{3r_s}{2r^5}\sqrt{r(r - r_s)}. \quad (2.2.12c)$$

Newman-Penrose tetrad:

$$\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{e}_{(t)} + \mathbf{e}_{(r)}), \quad \mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{e}_{(t)} - \mathbf{e}_{(r)}), \quad \mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_{(\vartheta)} + i\mathbf{e}_{(\varphi)}). \quad (2.2.13)$$

Non-vanishing spin coefficients:

$$\rho = \mu = -\frac{1}{\sqrt{2}r}\sqrt{1 - \frac{r_s}{r}}, \quad \gamma = \varepsilon = \frac{r_s}{4\sqrt{2}r^2\sqrt{1 - r_s/r}}, \quad \alpha = -\beta = -\frac{\cot \vartheta}{2\sqrt{2}r}. \quad (2.2.14)$$

Embedding:

The embedding function reads

$$z = 2\sqrt{r_s}\sqrt{r - r_s}. \quad (2.2.15)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}r^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \quad (2.2.16)$$

with the constants of motion $k = (1 - r_s/r)c^2t$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.8.2). For timelike geodesics, the effective potential has the extremal points

$$r_{\pm} = \frac{h^2 \pm h\sqrt{h^2 - 3c^2r_s^2}}{c^2r_s}, \quad (2.2.17)$$

where r_+ is a maximum and r_- is a minimum. The innermost timelike circular geodesic follows from $h^2 = 3c^2r_s^2$ and reads $r_{\text{itcg}} = 3r_s$. Null geodesics, however, have only a maximum at $r_{\text{po}} = \frac{3}{2}r_s$. The corresponding circular orbit is called photon orbit.

Further reading:

Schwarzschild [Sch16, Sch03], MTW [MTW73], Rindler [Rin01], Wald [Wal84], Chandrasekhar [Cha06], Müller [Mül08b, Mül09].

2.2.2 Schwarzschild in pseudo-Cartesian coordinates

The Schwarzschild spacetime in pseudo-Cartesian coordinates (t, x, y, z) reads

$$\boxed{ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(\frac{x^2}{1-r_s/r} + y^2 + z^2\right)\frac{dx^2}{r^2} + \left(x^2 + \frac{y^2}{1-r_s/r} + z^2\right)\frac{dy^2}{r^2} + \left(x^2 + y^2 + \frac{z^2}{1-r_s/r}\right)\frac{dz^2}{r^2} + \frac{2r_s}{r^2(r-r_s)}(xydxdy + xzdxdz + yzdydz),} \quad (2.2.18)$$

where $r^2 = x^2 + y^2 + z^2$. For a natural local tetrad that is adapted to the x-axis, we make the following ansatz:

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_t, \quad \mathbf{e}_{(1)} = A\partial_x, \quad \mathbf{e}_{(2)} = B\partial_x + C\partial_y, \quad \mathbf{e}_{(3)} = D\partial_x + E\partial_y + F\partial_z. \quad (2.2.19)$$

$$A = \frac{1}{\sqrt{g_{xx}}}, \quad B = \frac{-g_{xy}}{g_{xx}\sqrt{-g_{xy}^2/g_{xx} + g_{yy}}}, \quad C = \frac{1}{\sqrt{-g_{xy}^2/g_{xx} + g_{yy}}}, \quad (2.2.20a)$$

$$D = \frac{g_{xy}g_{yz} - g_{xz}g_{yy}}{\sqrt{NW}}, \quad E = \frac{g_{xz}g_{xy} - g_{xx}g_{yz}}{\sqrt{NW}}, \quad F = \frac{\sqrt{N}}{\sqrt{W}}, \quad (2.2.20b)$$

with

$$N = g_{xx}g_{yy} - g_{xy}^2, \quad (2.2.21a)$$

$$W = g_{xx}g_{yy}g_{zz} - g_{xz}^2g_{yy} + 2g_{xz}g_{xy}g_{yz} - g_{xy}^2g_{zz} - g_{xx}g_{yz}^2. \quad (2.2.21b)$$

2.2.3 Isotropic coordinates

Spherical isotropic coordinates

The Schwarzschild metric (2.2.1) in spherical isotropic coordinates $(t, \rho, \vartheta, \varphi)$ reads

$$\boxed{ds^2 = -\left(\frac{1-\rho_s/\rho}{1+\rho_s/\rho}\right)^2 c^2 dt^2 + \left(1 + \frac{\rho_s}{\rho}\right)^4 [d\rho^2 + \rho^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)]}, \quad (2.2.22)$$

where

$$r = \rho \left(1 + \frac{\rho_s}{\rho}\right)^2 \quad \text{or} \quad \rho = \frac{1}{4} \left(2r - r_s \pm 2\sqrt{r(r-r_s)}\right) \quad (2.2.23)$$

is the coordinate transformation between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ , see e.g. MTW[MTW73] page 840. The event horizon is given by $\rho_s = r_s/4$. The photon orbit and the innermost timelike circular geodesic read

$$\rho_{\text{po}} = (2 + \sqrt{3})\rho_s \quad \text{and} \quad \rho_{\text{itcg}} = (5 + 2\sqrt{6})\rho_s. \quad (2.2.24)$$

Christoffel symbols:

$$\Gamma_{tt}^\rho = \frac{2(\rho - \rho_s)\rho^4\rho_s c^2}{(\rho + \rho_s)^7}, \quad \Gamma_{t\rho}^t = \frac{2\rho_s}{\rho^2 - \rho_s^2}, \quad \Gamma_{\rho\rho}^\rho = -\frac{2\rho_s}{(\rho + \rho_s)\rho}, \quad (2.2.25a)$$

$$\Gamma_{\rho\vartheta}^\vartheta = \frac{\rho - \rho_s}{(\rho + \rho_s)\rho}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{\rho - \rho_s}{(\rho + \rho_s)\rho}, \quad \Gamma_{\vartheta\vartheta}^\rho = -\rho \frac{\rho - \rho_s}{\rho + \rho_s}, \quad (2.2.25b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\rho = -\frac{(\rho - \rho_s)\rho \sin^2 \vartheta}{\rho + \rho_s}, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.25c)$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -4 \frac{(\rho - \rho_s)^2 \rho_s c^2}{(\rho + \rho_s)^4 \rho}, \quad R_{t\vartheta t\vartheta} = 2 \frac{(\rho - \rho_s)^2 \rho \rho_s c^2}{(\rho + \rho_s)^4}, \quad (2.2.26a)$$

$$R_{t\varphi t\varphi} = 2 \frac{(\rho - \rho_s)^2 \rho c^2 \rho_s \sin^2 \vartheta}{(\rho + \rho_s)^4}, \quad R_{\rho \vartheta \rho \vartheta} = -2 \frac{(\rho + \rho_s)^2 \rho_s}{\rho^3}, \quad (2.2.26b)$$

$$R_{\rho \varphi \rho \varphi} = -2 \frac{(\rho + \rho_s)^2 \rho_s \sin^2 \vartheta}{\rho^3}, \quad R_{\vartheta \varphi \vartheta \varphi} = \frac{4(\rho + \rho_s)^2 \rho_s \sin^2 \vartheta}{\rho}. \quad (2.2.26c)$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = 192 \frac{r_s^2}{\rho^6 (1 + \rho_s/\rho)^{12}} = 12 \frac{r_s^2}{r(\rho)^6}. \quad (2.2.27)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1 + \rho_s/\rho}{1 - \rho_s/\rho} \frac{\partial_t}{c}, \quad \mathbf{e}_{(r)} = \frac{1}{[1 + \rho_s/\rho]^2} \partial_\rho, \quad (2.2.28a)$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\rho [1 + \rho_s/\rho]^2} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho [1 + \rho_s/\rho]^2 \sin^2 \vartheta} \partial_\varphi. \quad (2.2.28b)$$

Ricci rotation coefficients:

$$\gamma_{(\rho)(t)(t)} = \frac{2\rho_s \rho^2}{(\rho + \rho_s)^3 (\rho - \rho_s)}, \quad \gamma_{(\vartheta)(\rho)(\vartheta)} = \gamma_{(\varphi)(\rho)(\varphi)} = \frac{\rho(\rho - \rho_s)}{(\rho + \rho_s)^3}, \quad (2.2.29a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\rho \cot \vartheta}{(\rho + \rho_s)^2}. \quad (2.2.29b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\rho)} = \frac{2\rho(\rho^2 - \rho \rho_s + \rho_s^2)}{(\rho + \rho_s)^3 (\rho - \rho_s)}, \quad \gamma_{(\vartheta)} = \frac{\rho \cot \vartheta}{(\rho + \rho_s)^2}. \quad (2.2.30)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \quad (2.2.31a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \quad (2.2.31b)$$

Further reading:

Buchdahl [[Buc85](#)].

Cartesian isotropic coordinates

The Schwarzschild metric (2.2.1) in Cartesian isotropic coordinates (t, x, y, z) reads,

$$ds^2 = - \left(\frac{1 - \rho_s/\rho}{1 + \rho_s/\rho} \right)^2 c^2 dt^2 + \left(1 + \frac{\rho_s}{\rho} \right)^4 [dx^2 + dy^2 + dz^2], \quad (2.2.32)$$

where $\rho^2 = x^2 + y^2 + z^2$ and, as before,

$$r = \rho \left(1 + \frac{\rho_s}{\rho} \right)^2. \quad (2.2.33)$$

Christoffel symbols:

$$\Gamma_{tt}^x = \frac{2c^2\rho^3(\rho - \rho_s)x}{(\rho + \rho_s)^7}, \quad \Gamma_{tt}^y = \frac{2c^2\rho^3(\rho - \rho_s)y}{(\rho + \rho_s)^7}, \quad \Gamma_{tt}^z = \frac{2c^2\rho^3(\rho - \rho_s)z}{(\rho + \rho_s)^7}, \quad (2.2.34a)$$

$$\Gamma_{tx}^t = \frac{2\rho_s x}{\rho^3[1 - \rho_s^2/\rho^2]}, \quad \Gamma_{ty}^t = \frac{2\rho_s y}{\rho^3[1 - \rho_s^2/\rho^2]}, \quad \Gamma_{tz}^t = \frac{2\rho_s z}{\rho^3[1 - \rho_s^2/\rho^2]}, \quad (2.2.34b)$$

$$\Gamma_{xx}^x = \Gamma_{xy}^y = \Gamma_{xz}^z = -\Gamma_{yy}^x = -\Gamma_{zz}^x = -\frac{2\rho_s}{\rho^3} \frac{x}{1 + \rho_s/\rho},$$

$$\Gamma_{xx}^y = -\Gamma_{xy}^x = -\Gamma_{yy}^y = -\Gamma_{yz}^z = \Gamma_{zz}^y = \frac{2\rho_s}{\rho^3} \frac{y}{1 + \rho_s/\rho},$$

$$\Gamma_{xx}^z = -\Gamma_{xz}^x = \Gamma_{yz}^z = -\Gamma_{yz}^y = -\Gamma_{zz}^z = \frac{2\rho_s}{\rho^3} \frac{z}{1 + \rho_s/\rho}.$$

The Schwarzschild time coordinate (t) is a time coordinate in the Schwarzschild metric, but it is not the same as proper time (τ). It represents the time measured by a hypothetical observer infinitely far from the massive body, where the gravitational field is negligible. For an observer at rest, the interval of the Schwarzschild time coordinate (dt) is larger than their proper time interval ($d\tau$), which is always less than (dt) for any finite

2.2.4 Eddington-Finkelstein

The transformation of the Schwarzschild metric (2.2.1) from the usual Schwarzschild time coordinate t to the advanced null coordinate v with

$$cv = ct + r_s \ln(r - r_s) \quad (2.2.35)$$

leads to the ingoing Eddington-Finkelstein [Edd24, Fin58] metric with coordinates $(v, r, \vartheta, \varphi)$,

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dv^2 + 2cdvdr + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.2.36)$$

ds^2 represents an infinitesimal element of arc length along a curve.

Metric-Tensor:

$$g_{vv} = -c^2 \left(1 - \frac{r_s}{r}\right), \quad g_{vr} = c, \quad g_{\vartheta\vartheta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2 \vartheta. \quad (2.2.37)$$

Christoffel symbols:

$$\Gamma_{vv}^v = \frac{cr_s}{2r^2}, \quad \Gamma_{vv}^r = \frac{c^2 r_s(r - r_s)}{2r^3}, \quad \Gamma_{vr}^r = -\frac{cr_s}{2r^2}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad (2.2.38a)$$

$$\Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^v = -\frac{r}{c}, \quad \Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad (2.2.38b)$$

$$\Gamma_{\varphi\varphi}^v = -\frac{r \sin^2 \vartheta}{c}, \quad \Gamma_{\varphi\varphi}^r = -(r - r_s) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.38c)$$

Partial derivatives

$$\Gamma_{vv,r}^v = -\frac{cr_s}{r^3}, \quad \Gamma_{vv,r}^r = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \quad \Gamma_{vr,r}^r = \frac{cr_s}{r^3}, \quad (2.2.39a)$$

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^v = -\frac{1}{c}, \quad (2.2.39b)$$

$$\Gamma_{\vartheta\vartheta,r}^r = -1, \quad \Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^v = -\frac{\sin^2 \vartheta}{c}, \quad (2.2.39c)$$

$$\Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\frac{r \sin(2\vartheta)}{c}, \quad \Gamma_{\varphi\varphi,r}^r = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta), \quad (2.2.39d)$$

$$\Gamma_{\varphi\varphi,\vartheta}^\vartheta = -(r - r_s) \sin(2\vartheta). \quad (2.2.39e)$$

Riemann-Tensor:

$$R_{vrvr} = -\frac{c^2 r_s}{r^3}, \quad R_{v\vartheta v\vartheta} = \frac{c^2 r_s(r - r_s)}{2r^2}, \quad R_{v\vartheta r\vartheta} = -\frac{cr_s}{2r}, \quad (2.2.40a)$$

$$R_{v\varphi v\varphi} = \frac{c^2 r_s(r - r_s) \sin^2 \vartheta}{2r^2}, \quad R_{v\varphi r\varphi} = -\frac{cr_s \sin^2 \vartheta}{2r}, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.40b)$$

While the Ricci tensor and the Ricci scalar vanish identically, the Kretschmann scalar is $\mathcal{K} = 12r_s^2/r^6$.

Static local tetrad:

$$\mathbf{e}_{(v)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v, \quad \mathbf{e}_{(r)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v + \sqrt{1-\frac{r_s}{r}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \quad (2.2.41)$$

Dual tetrad:

$$\theta^{(v)} = c\sqrt{1-\frac{r_s}{r}}dv - \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\varphi)} = r\sin\vartheta d\varphi. \quad (2.2.42)$$

Ricci rotation coefficients:

$$\gamma_{(r)(v)(v)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \quad (2.2.43)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r-3r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \quad (2.2.44)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(v)(r)(v)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.45a)$$

$$R_{(v)(\vartheta)(v)(\vartheta)} = R_{(v)(\varphi)(v)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.45b)$$

2.2.5 Kruskal-Szekeres

The Schwarzschild metric in Kruskal-Szekeres [Kru60, Wal84] coordinates $(T, X, \vartheta, \varphi)$ reads

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} (-dT^2 + dX^2) + r^2 d\Omega^2, \quad (2.2.46)$$

where $r \in \mathbb{R}_+ \setminus \{0\}$ is given by means of the LambertW-function \mathcal{W} ,

$$\left(\frac{r}{r_s} - 1\right) e^{r/r_s} = X^2 - T^2 \quad \text{or} \quad r = r_s \left[\mathcal{W}\left(\frac{X^2 - T^2}{e}\right) + 1 \right]. \quad (2.2.47)$$

The Schwarzschild coordinate time t in terms of the Kruskal coordinates T and X reads

$$t = 2r_s \operatorname{arctanh} \frac{T}{X}, \quad r > r_s, \quad (2.2.48a)$$

$$t = 2r_s \operatorname{arctanh} \frac{X}{T}, \quad r < r_s, \quad (2.2.48b)$$

$$t = \infty, \quad r = r_s. \quad (2.2.48c)$$

The transformations between Kruskal- and Schwarzschild coordinates read

$$X = \sqrt{1 - \frac{r}{r_s}} e^{r/(2r_s)} \sinh \frac{ct}{2r_s}, \quad T = \sqrt{1 - \frac{r}{r_s}} e^{r/(2r_s)} \cosh \frac{ct}{2r_s}, \quad 0 < r < r_2, \quad (2.2.49a)$$

$$X = \sqrt{\frac{r}{r_s} - 1} e^{r/(2r_s)} \cosh \frac{ct}{2r_s}, \quad T = \sqrt{\frac{r}{r_s} - 1} e^{r/(2r_s)} \sinh \frac{ct}{2r_s}, \quad r \geq r_s. \quad (2.2.49b)$$

Christoffel symbols:

$$\Gamma_{TT}^T = \Gamma_{TX}^X = \Gamma_{XX}^T = \frac{Tr_s(r+r_s)}{r^2} e^{-r/r_s}, \quad (2.2.50a)$$

$$\Gamma_{TT}^X = \Gamma_{TX}^T = \Gamma_{XX}^X = -\frac{Xr_s(r+r_s)}{r^2} e^{-r/r_s}, \quad (2.2.50b)$$

$$\Gamma_{T\vartheta}^\vartheta = -\frac{2r_s^2 T}{r^2} e^{-r/r_s}, \quad \Gamma_{X\vartheta}^\vartheta = \frac{2r_s^2 X}{r^2} e^{-r/r_s}, \quad (2.2.50c)$$

$$\Gamma_{\vartheta\vartheta}^T = -\frac{r}{2r_s} T, \quad \Gamma_{\vartheta\vartheta}^X = \frac{r}{2r_s} X, \quad (2.2.50d)$$

$$\Gamma_{\vartheta\vartheta}^T = -\frac{r}{2r_s} T \sin^2 \vartheta, \quad \Gamma_{\vartheta\vartheta}^X = \frac{r}{2r_s} X \sin^2 \vartheta, \quad (2.2.50e)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\vartheta\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.50f)$$

Riemann-Tensor:

$$R_{TXTX} = -16 \frac{r_s^7}{r^5} e^{-2r/r_s}, \quad R_{T\vartheta T\vartheta} = \frac{2r_s^4}{r^2} e^{-r/r_s}, \quad (2.2.51a)$$

$$R_{T\varphi T\varphi} = \frac{2r_s^4}{r^2} e^{-r/r_s} \sin^2 \vartheta, \quad R_{X\vartheta X\vartheta} = -\frac{2r_s^4}{r^2} e^{-r/r_s}, \quad (2.2.51b)$$

$$R_{X\varphi X\varphi} = -\frac{2r_s^4}{r^2} e^{-r/r_s} \sin^2 \vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.51c)$$

The *Ricci*-Tensor as well as the *Ricci*-scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = \frac{12r_s^2}{r^6}. \quad (2.2.52)$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{\sqrt{r}}{2r_s \sqrt{r_s}} e^{r/(2r_s)} \partial_T, \quad \mathbf{e}_{(X)} = \frac{\sqrt{r}}{2r_s \sqrt{r_s}} e^{r/(2r_s)} \partial_X, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi \quad (2.2.53)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(X)(T)(X)} = R_{(X)(\vartheta)(X)(\vartheta)} = R_{(X)(\varphi)(X)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.54a)$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.54b)$$

2.2.6 Tortoise coordinates

The Schwarzschild metric represented by tortoise coordinates $(t, \rho, \vartheta, \varphi)$ reads

$$ds^2 = -\left(1 - \frac{r_s}{r(\rho)}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r(\rho)}\right) d\rho^2 + r(\rho)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.2.55)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The tortoise radial coordinate ρ and the Schwarzschild radial coordinate r are related by

$$\rho = r + r_s \ln \left(\frac{r}{r_s} - 1 \right) \quad \text{or} \quad r = r_s \left\{ 1 + \mathcal{W} \left[\exp \left(\frac{\rho}{r_s} - 1 \right) \right] \right\}. \quad (2.2.56)$$

Christoffel symbols:

$$\Gamma_{tt}^\rho = \frac{c^2 r_s}{2r(\rho)^2}, \quad \Gamma_{t\rho}^t = \frac{r_s}{2r(\rho)^2}, \quad \Gamma_{\rho\rho}^\rho = \frac{r_s}{2r(\rho)^2}, \quad (2.2.57a)$$

$$\Gamma_{\rho\vartheta}^\vartheta = \frac{1}{r(\rho)} - \frac{1}{r_s}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{1}{r(\rho)} - \frac{1}{r_s}, \quad \Gamma_{\vartheta\vartheta}^\rho = -r(\rho), \quad (2.2.57b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot\vartheta, \quad \Gamma_{\varphi\varphi}^\rho = -r(\rho) \sin^2\vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin\vartheta \cos\vartheta. \quad (2.2.57c)$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -\frac{c^2 r_s}{r(\rho)^3} \left(1 - \frac{r_s}{r(\rho)}\right)^2, \quad R_{t\vartheta t\vartheta} = \frac{c^2}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad (2.2.58a)$$

$$R_{t\varphi t\varphi} = \frac{c^2 \sin^2\vartheta}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad R_{\rho\vartheta\rho\vartheta} = -\frac{1}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad (2.2.58b)$$

$$R_{\rho\varphi\rho\varphi} = -\frac{\sin^2\vartheta}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad R_{\vartheta\varphi\vartheta\varphi} = r(\rho) r_s \sin^2\vartheta. \quad (2.2.58c)$$

The Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12 \frac{r_s^2}{r(\rho)^6}. \quad (2.2.59)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1-r_s/r(\rho)}} \partial_t, \quad \mathbf{e}_{(\rho)} = \frac{1}{\sqrt{1-r_s/r(\rho)}} \partial_\rho, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r(\rho)} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r(\rho) \sin\vartheta} \partial_\varphi. \quad (2.2.60)$$

Dual tetrad:

$$\theta^{(t)} = c \sqrt{1 - \frac{r_s}{r(\rho)}} dt, \quad \theta^{(\rho)} = \sqrt{1 - \frac{r_s}{r(\rho)}} d\rho, \quad \theta^{(\vartheta)} = r(\rho) d\vartheta, \quad \theta^{(\varphi)} = r(\rho) \sin\vartheta d\varphi. \quad (2.2.61)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \quad (2.2.62a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \quad (2.2.62b)$$

Further reading:

MTW[MTW73]

2.2.7 Painlevé-Gullstrand

The Schwarzschild metric expressed in Painlevé-Gullstrand coordinates[MP01] reads

$$ds^2 = -c^2 dT^2 + \left(dr + \sqrt{\frac{r_s}{r}} c dT\right)^2 + r^2 (d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (2.2.63)$$

where the new time coordinate T follows from the Schwarzschild time t in the following way:

$$cT = ct + 2r_s \left(\sqrt{\frac{r}{r_s}} + \frac{1}{2} \ln \left| \frac{\sqrt{r/r_s} - 1}{\sqrt{r/r_s} + 1} \right| \right). \quad (2.2.64)$$

Metric-Tensor:

$$g_{TT} = -c^2 \left(1 - \frac{r_s}{r}\right), \quad g_{Tr} = c \sqrt{\frac{r_s}{r}}, \quad g_{rr} = 1, \quad g_{\vartheta\vartheta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2 \vartheta. \quad (2.2.65)$$

Christoffel symbols:

$$\Gamma_{TT}^T = \frac{cr_s}{2r^2} \sqrt{\frac{r_s}{r}}, \quad \Gamma_{TT}^r = \frac{c^2 r_s (r - r_s)}{2r^3}, \quad \Gamma_{Tr}^T = \frac{r_s}{2r^2}, \quad (2.2.66a)$$

$$\Gamma_{Tr}^r = -\frac{cr_s}{2r^2} \sqrt{\frac{r_s}{r}}, \quad \Gamma_{rr}^T = \frac{r_s}{2cr^2} \sqrt{\frac{r}{r_s}}, \quad \Gamma_{rr}^r = -\frac{r_s}{2r^2}, \quad (2.2.66b)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^T = -\frac{r}{c} \sqrt{\frac{r_s}{r}}, \quad (2.2.66c)$$

$$\Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^T = -\frac{r}{c} \sqrt{\frac{r_s}{r}} \sin^2 \vartheta, \quad (2.2.66d)$$

$$\Gamma_{\varphi\varphi}^r = -(r - r_s) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.66e)$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{c^2 r_s}{r^3}, \quad R_{T\vartheta T\vartheta} = \frac{c^2 r_s (r - r_s)}{2r^2}, \quad R_{T\vartheta r\vartheta} = -\frac{cr_s}{2r} \sqrt{\frac{r_s}{r}}, \quad (2.2.67a)$$

$$R_{T\varphi T\varphi} = \frac{c^2 r_s (r - r_s) \sin^2 \vartheta}{2r^2}, \quad R_{T\varphi r\varphi} = -\frac{cr_s}{2r} \sqrt{\frac{r_s}{r}} \sin^2 \vartheta, \quad R_{r\vartheta r\vartheta} = -\frac{r_s}{2r}, \quad (2.2.67b)$$

$$R_{r\varphi r\varphi} = -\frac{r_s \sin^2 \vartheta}{2r}, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.67c)$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = 12r_s^2/r^6. \quad (2.2.68)$$

For the Painlevé-Gullstrand coordinates, we can define two natural local tetrads.

Static local tetrad:

$$\hat{\mathbf{e}}_{(T)} = \frac{1}{c\sqrt{1-r_s/r}} \partial_T, \quad \hat{\mathbf{e}}_{(r)} = \frac{\sqrt{r_s}}{c\sqrt{r-r_s}} \partial_T + \sqrt{1-\frac{r_s}{r}} \partial_r, \quad \hat{\mathbf{e}}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \hat{\mathbf{e}}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi, \quad (2.2.69)$$

Dual tetrad:

$$\hat{\theta}^{(T)} = c \sqrt{1 - \frac{r_s}{r}} dT - \frac{dr}{\sqrt{r/r_s - 1}}, \quad \hat{\theta}^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \quad \hat{\theta}^{(\vartheta)} = r d\vartheta, \quad \hat{\theta}^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.2.70)$$

Freely falling local tetrad:

$$\mathbf{e}_{(T)} = \frac{1}{c} \partial_T - \sqrt{\frac{r_s}{r}} \partial_r, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \quad (2.2.71)$$

Dual tetrad:

$$\theta^{(T)} = c dT, \quad \theta^{(r)} = c \sqrt{\frac{r_s}{r}} dT + dr, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.2.72)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(r)(T)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.73a)$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.73b)$$

2.2.8 Israel coordinates

The Schwarzschild metric in Israel coordinates $(x, y, \vartheta, \varphi)$ reads[SKM⁺03]

$$ds^2 = r_s^2 \left[4dx \left(dy + \frac{y^2 dx}{1+xy} \right) + (1+xy)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right], \quad (2.2.74)$$

where the coordinates x and y follow from the Schwarzschild coordinates via

$$t = r_s \left(1 + xy + \ln \frac{y}{x} \right) \quad \text{and} \quad r = r_s (1 + xy). \quad (2.2.75)$$

Christoffel symbols:

$$\Gamma_{xx}^x = -\frac{y(2+xy)}{(1+xy)^2}, \quad \Gamma_{xx}^y = \frac{y^3(3+xy)}{(1+xy)^3}, \quad \Gamma_{xy}^y = \frac{y(2+xy)}{(1+xy)^2}, \quad (2.2.76a)$$

$$\Gamma_{x\vartheta}^\vartheta = \frac{y}{1+xy}, \quad \Gamma_{x\varphi}^\varphi = \frac{y}{1+xy}, \quad \Gamma_{y\vartheta}^\vartheta = \frac{x}{1+xy}, \quad (2.2.76b)$$

$$\Gamma_{x\varphi}^\varphi = \frac{x}{1+xy}, \quad \Gamma_{\vartheta\vartheta}^x = -\frac{x}{2}(1+xy), \quad \Gamma_{\vartheta\vartheta}^y = -\frac{y}{2}(1-xy), \quad (2.2.76c)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^x = -\frac{x}{2}(1+xy) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^y = -\frac{y}{2}(1-xy) \sin^2 \vartheta, \quad (2.2.76d)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.76e)$$

Riemann-Tensor:

$$R_{xyxy} = -4 \frac{r_s^2}{(1+xy)^3}, \quad R_{x\vartheta x\vartheta} = -2 \frac{y^2 r_s^2}{(1+xy)^2}, \quad R_{x\vartheta y\vartheta} = -\frac{r_s^2}{1+xy}, \quad (2.2.77a)$$

$$R_{x\varphi x\varphi} = -2 \frac{r_s^2 y^2 \sin^2 \vartheta}{(1+xy)^2}, \quad R_{x\varphi y\varphi} = -\frac{r_s^2 \sin^2 \vartheta}{1+xy}, \quad R_{\vartheta\varphi\vartheta\varphi} = (1+xy)r_s^2 \sin^2 \vartheta. \quad (2.2.77b)$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = \frac{12}{r_s^4 (1+xy)^6}. \quad (2.2.78)$$

Local tetrad:

$$\mathbf{e}_{(0)} = -\frac{\sqrt{1+xy}}{2r_s y} \partial_x + \frac{y}{r_s \sqrt{1+xy}} \partial_y, \quad \mathbf{e}_{(1)} = \frac{\sqrt{1+xy}}{2r_s y} \partial_x, \quad (2.2.79a)$$

$$\mathbf{e}_{(2)} = \frac{1}{r_s (1+xy)} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{r_s (1+xy) \sin \vartheta} \partial_\varphi. \quad (2.2.79b)$$

Dual tetrad:

$$\theta^{(0)} = \frac{r_s \sqrt{1+xy}}{y} dy, \quad \theta^{(1)} = \frac{2r_s y}{\sqrt{1+xy}} dx + \frac{r_s \sqrt{1+xy}}{y} dy, \quad (2.2.80a)$$

$$\theta^{(2)} = r_s (1+xy) d\vartheta, \quad \theta^{(3)} = r_s (1+xy) \sin \vartheta d\varphi. \quad (2.2.80b)$$

2.3 Alcubierre Warp

The Warp metric given by Miguel Alcubierre[\[Alc94\]](#) reads

$$ds^2 = -c^2 dt^2 + (dx - v_s f(r_s) dt)^2 + dy^2 + dz^2 \quad (2.3.1)$$

where

$$v_s = \frac{dx_s(t)}{dt}, \quad (2.3.2a)$$

$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}, \quad (2.3.2b)$$

$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2 \tanh(\sigma R)}. \quad (2.3.2c)$$

The parameter $R > 0$ defines the radius of the warp bubble and the parameter $\sigma > 0$ its thickness.

Metric-Tensor:

$$g_{tt} = -c^2 + v_s^2 f(r_s)^2, \quad g_{tx} = -v_s f(r_s), \quad g_{xx} = g_{yy} = g_{zz} = 1. \quad (2.3.3)$$

Christoffel symbols:

$$\Gamma_{tt}^t = \frac{f^2 f_x v_s^3}{c^2}, \quad \Gamma_{tt}^z = -f f_z v_s^2, \quad \Gamma_{tt}^y = -f f_y v_s^2, \quad (2.3.4a)$$

$$\Gamma_{tt}^x = \frac{f^3 f_x v_s^4 - c^2 f f_x v_s^2 - c^2 f_t v_s}{c^2}, \quad \Gamma_{tx}^t = -\frac{f f_x v_s^2}{c^2}, \quad \Gamma_{tx}^x = -\frac{f^2 f_x v_s^3}{c^2}, \quad (2.3.4b)$$

$$\Gamma_{tx}^y = \frac{f_y v_s}{2}, \quad \Gamma_{tx}^z = \frac{f_z v_s}{2}, \quad \Gamma_{ty}^t = -\frac{f f_y v_s^2}{2c^2}, \quad (2.3.4c)$$

$$\Gamma_{ty}^x = -\frac{f^2 f_y v_s^3 + c^2 f_y v_s}{2c^2}, \quad \Gamma_{tz}^t = -\frac{f f_z v_s^2}{2c^2}, \quad \Gamma_{tz}^x = -\frac{f^2 f_z v_s^3 + c^2 f_z v_s}{2c^2}, \quad (2.3.4d)$$

$$\Gamma_{xx}^t = \frac{f_x v_s}{c^2}, \quad \Gamma_{xx}^x = \frac{f f_x v_s^2}{c^2}, \quad \Gamma_{xy}^t = \frac{f_y v_s}{2c^2}, \quad (2.3.4e)$$

$$\Gamma_{xy}^x = \frac{f f_y v_s^2}{2c^2}, \quad \Gamma_{xz}^t = \frac{f_z v_s}{2c^2}, \quad \Gamma_{xz}^x = \frac{f f_z v_s^2}{2c^2}, \quad (2.3.4f)$$

with derivatives

$$f_t = \frac{df(r_s)}{dt} = \frac{-v_s \sigma(x - x_s(t))}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5a)$$

$$f_x = \frac{df(r_s)}{dx} = \frac{\sigma(x - x_s(t))}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5b)$$

$$f_y = \frac{df(r_s)}{dy} = \frac{\sigma y}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5c)$$

$$f_z = \frac{df(r_s)}{dz} = \frac{\sigma z}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5d)$$

Riemann- and Ricci-tensor as well as Ricci- and Kretschman-scalar are shown only in the Maple worksheet.

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c} (\partial_t + v_s f \partial_x), \quad \mathbf{e}_{(1)} = \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.3.6)$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{c^2 - v_s^2 f^2}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{v_s f}{c \sqrt{c^2 - v_s^2 f^2}} \partial_t + \frac{\sqrt{c^2 - v_s^2 f^2}}{c} \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.3.7)$$

Further reading:

Pfenning[\[PF97\]](#), Clark[\[CHL99\]](#), Van Den Broeck[\[Bro99\]](#)

2.4 Barriola-Vilenkin monopol

The Barriola-Vilenkin metric describes the gravitational field of a global monopole[BV89]. In spherical coordinates $(t, r, \vartheta, \varphi)$, the metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + k^2 r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.4.1)$$

where k is the scaling factor responsible for the deficit/surplus angle.

Christoffel symbols:

$$\Gamma_{\vartheta\vartheta}^r = -k^2 r, \quad \Gamma_{\varphi\varphi}^r = -k^2 r \sin^2 \vartheta, \quad \Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad (2.4.2a)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta. \quad (2.4.2b)$$

Partial derivatives

$$\Gamma_{r\vartheta,\vartheta}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -k^2, \quad (2.4.3a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -k^2 \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta), \quad (2.4.3b)$$

$$\Gamma_{\varphi\varphi,\vartheta}^\vartheta = -k^2 r \sin(2\vartheta). \quad (2.4.3c)$$

Riemann-Tensor:

$$R_{\vartheta\varphi\vartheta\varphi} = (1 - k^2) k^2 r^2 \sin^2 \vartheta. \quad (2.4.4)$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{\vartheta\vartheta} = (1 - k^2), \quad R_{\varphi\varphi} = (1 - k^2) \sin^2 \vartheta, \quad \mathcal{R} = 2 \frac{1 - k^2}{k^2 r^2}, \quad \mathcal{K} = 4 \frac{(1 - k^2)^2}{k^4 r^4}. \quad (2.4.5)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(1 - k^2)}{3k^2 r^2}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{6}(1 - k^2), \quad C_{t\varphi t\varphi} = \frac{c^2}{6}(1 - k^2) \sin^2 \vartheta, \quad (2.4.6a)$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{6}(1 - k^2), \quad C_{r\varphi r\varphi} = -\frac{1}{6}(1 - k^2) \sin^2 \vartheta, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{k^2 r^2}{3}(1 - k^2) \sin^2 \vartheta. \quad (2.4.6b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{kr} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{kr \sin \vartheta} \partial_\varphi. \quad (2.4.7)$$

Dual tetrad:

$$\theta^{(t)} = c dt, \quad \theta^{(r)} = dr, \quad \theta^{(\vartheta)} = krd\vartheta, \quad \theta^{(\varphi)} = kr \sin \vartheta d\varphi. \quad (2.4.8)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{kr}. \quad (2.4.9)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{kr}. \quad (2.4.10)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\phi)(\vartheta)(\phi)} = \frac{1-k^2}{k^2 r^2}. \quad (2.4.11)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\vartheta)} = R_{(\phi)(\phi)} = \frac{1-k^2}{k^2 r^2}. \quad (2.4.12)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\phi)(\vartheta)(\phi)} = -\frac{1-k^2}{3k^2 r^2}, \quad (2.4.13a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\phi)(t)(\phi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\phi)(r)(\phi)} = \frac{1-k^2}{6k^2 r^2}. \quad (2.4.13b)$$

Embedding:

The embedding function, see Sec. 1.7, for $k < 1$ reads

$$z = \sqrt{1-k^2} r. \quad (2.4.14)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = \frac{1}{2} \frac{h_1^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(\frac{h_2^2}{k^2 r^2} - \kappa c^2 \right), \quad (2.4.15)$$

with the constants of motion $h_1 = c^2 i$ and $h_2 = k^2 r^2 \dot{\phi}$.

The point of closest approach r_{pca} for a null geodesic that starts at $r = r_i$ with $\mathbf{y} = \pm \mathbf{e}_{(t)} + \cos \xi \mathbf{e}_{(r)} + \sin \xi \mathbf{e}_{(\phi)}$ is given by $r = r_i \sin \xi$. Hence, the r_{pca} is independent of k . The same is also true for timelike geodesics.

Further reading:

Barriola and Vilenkin [BV89], Perlick [Per04].

2.5 Bertotti-Kasner

The Bertotti-Kasner spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ reads [Rin98]

$$ds^2 = -c^2 dt^2 + e^{2\sqrt{\Lambda}ct} dr^2 + \frac{1}{\Lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.5.1)$$

where the cosmological constant Λ must be positive.

Christoffel symbols:

$$\Gamma_{tr}^r = c\sqrt{\Lambda}, \quad \Gamma_{rr}^t = \frac{\sqrt{\Lambda}}{c} e^{2\sqrt{\Lambda}ct}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.5.2)$$

Partial derivatives

$$\Gamma_{rr,t}^t = 2\Lambda e^{2\sqrt{\Lambda}ct}, \quad \Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta). \quad (2.5.3)$$

Riemann-Tensor:

$$R_{trtr} = -\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{\sin^2 \vartheta}{\Lambda}. \quad (2.5.4)$$

Ricci-Tensor:

$$R_{tt} = -\Lambda c^2, \quad R_{rr} = \Lambda e^{2\sqrt{\Lambda}ct}, \quad R_{\vartheta\vartheta} = 1, \quad R_{\varphi\varphi} = \sin^2 \vartheta. \quad (2.5.5)$$

The Ricci and Kretschmann scalars read

$$\mathcal{R} = 4\Lambda, \quad \mathcal{K} = 8\Lambda^2. \quad (2.5.6)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{2}{3}\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{3}, \quad C_{t\varphi t\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \quad (2.5.7a)$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \quad C_{r\varphi r\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct} \sin^2 \vartheta, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3} \frac{\sin^2 \vartheta}{\Lambda}. \quad (2.5.7b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = e^{-\sqrt{\Lambda}ct} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \sqrt{\Lambda} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\sqrt{\Lambda}}{\sin \vartheta} \partial_\varphi. \quad (2.5.8)$$

Dual tetrad:

$$\theta^{(t)} = c dt, \quad \theta^{(r)} = e^{\sqrt{\Lambda}ct} dr, \quad \theta^{(\vartheta)} = \frac{1}{\sqrt{\Lambda}} d\vartheta, \quad \theta^{(\varphi)} = \frac{\sin \vartheta}{\sqrt{\Lambda}} d\varphi. \quad (2.5.9)$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \sqrt{\Lambda}, \quad \gamma_{(\vartheta)(\varphi)(\varphi)} = -\sqrt{\Lambda} \cot \vartheta. \quad (2.5.10)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\sqrt{\Lambda}, \quad \gamma_{(\vartheta)} = \sqrt{\Lambda} \cot \vartheta. \quad (2.5.11)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\Lambda. \quad (2.5.12)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = -R_{(\vartheta)(\vartheta)} = -R_{(\varphi)(\varphi)} = -\Lambda. \quad (2.5.13)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2\Lambda}{3}, \quad (2.5.14a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{\Lambda}{3}. \quad (2.5.14b)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$c^2 \dot{t}^2 = h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa \quad (2.5.15)$$

with the constants of motion $h_1 = \dot{r}e^{2\sqrt{\Lambda}ct}$ and $h_2 = \dot{\phi}/\Lambda$. Thus,

$$\lambda = \frac{1}{c\sqrt{\Lambda}\sqrt{\Lambda h_2^2 - \kappa}} \ln \left(\frac{1+q(t)}{1-q(t)} \frac{1-q(t_i)}{1+q(t_i)} \right), \quad q(t) = \frac{h_1^2 e^{-2\sqrt{\Lambda}ct}}{\Lambda h_2^2 - \kappa} + 1, \quad (2.5.16)$$

where t_i is the initial time. We can also solve the orbital equation:

$$r(t) = w(t) - w(t_i) + r_i, \quad w(t) = -\frac{\sqrt{h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa}}{h_1 \sqrt{\Lambda}}, \quad (2.5.17)$$

where r_i is the initial radial position.

Further reading:

Rindler[Rin98]: “Every spherically symmetric solution of the generalized vacuum field equations $R_{ij} = \Lambda g_{ij}$ is either equivalent to Kottler’s generalization of Schwarzschild space or to the [...] Bertotti-Kasner space (for which Λ must be necessarily be positive).”

2.6 Bessel gravitational wave

D. Kramer introduced in [Kra99] an exact gravitational wave solution of Einstein's vacuum field equations. According to [Ste03] we execute the substitution $x \rightarrow t$ and $y \rightarrow z$.

2.6.1 Cylindrical coordinates

The metric of the Bessel wave in cylindrical coordinates reads

$$\boxed{ds^2 = e^{-2U} [e^{2K} (d\rho^2 - dt^2) + \rho^2 d\varphi^2] + e^{2U} dz^2.} \quad (2.6.1)$$

The functions U and K are given by

$$U := CJ_0(\rho) \cos(t), \quad (2.6.2)$$

$$K := \frac{1}{2}C^2\rho \left\{ \rho \left[J_0(\rho)^2 + J_1(\rho)^2 \right] - 2J_0(\rho)J_1(\rho) \cos^2(t) \right\}, \quad (2.6.3)$$

where $J_n(\rho)$ are the Bessel functions of the first kind.

Christoffel symbols:

$$\Gamma_{tt}^t = \Gamma_{t\rho}^\rho = \Gamma_{\rho\rho}^t = -\frac{\partial U}{\partial t} + \frac{\partial K}{\partial t}, \quad \Gamma_{t\varphi}^\varphi = \Gamma_{tz}^z = -\frac{\partial U}{\partial t}, \quad \Gamma_{\varphi\varphi}^t = -e^{-2K}\rho^2 \frac{\partial U}{\partial t}, \quad (2.6.4a)$$

$$\Gamma_{tt}^\rho = \Gamma_{t\rho}^t = \Gamma_{\rho\rho}^\rho = -\frac{\partial U}{\partial \rho} + \frac{\partial K}{\partial \rho}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{1}{\rho} - \frac{\partial U}{\partial \rho}, \quad \Gamma_{zz}^\rho = -e^{4U-2K} \frac{\partial U}{\partial \rho}, \quad (2.6.4b)$$

$$\Gamma_{\varphi\varphi}^\rho = \rho e^{-2K} \left(\rho \frac{\partial U}{\partial \rho} - 1 \right), \quad \Gamma_{\rho z}^z = \frac{\partial U}{\partial \rho}, \quad \Gamma_{zz}^t = e^{4U-2K} \frac{\partial U}{\partial t}. \quad (2.6.4c)$$

Local tetrad:

$$\mathbf{e}_{(t)} = e^{U-K} \partial_t, \quad \mathbf{e}_{(\rho)} = e^{U-K} \partial_\rho, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho} e^U \partial_\varphi, \quad \mathbf{e}_{(z)} = e^{-U} \partial_z. \quad (2.6.5)$$

Dual tetrad:

$$\theta^{(t)} = e^{K-U} dt, \quad \theta^{(\rho)} = e^{K-U} d\rho, \quad \theta^{(\varphi)} = \rho e^{-U} d\varphi, \quad \theta^{(z)} = e^U dz. \quad (2.6.6)$$

2.6.2 Cartesian coordinates

In Cartesian coordinates with $\rho = \sqrt{x^2 + y^2}$ the metric (2.6.1) reads

$$\boxed{ds^2 = -e^{2(K-U)} dt^2 + \frac{e^{-2U}}{x^2 + y^2} \left[(e^{2K} x^2 + y^2) dx^2 + 2xy(e^{2K} - 1) dxdy + (x^2 + e^{2K} y^2) dy^2 \right] + e^{2U} dz^2.} \quad (2.6.7)$$

Local tetrad:

$$\begin{aligned} \mathbf{e}_{(t)} &= e^{U-K} \partial_t, & \mathbf{e}_{(x)} &= e^U \sqrt{\frac{x^2 + y^2}{e^{2K} x^2 + y^2}} \partial_x, \\ \mathbf{e}_{(y)} &= e^{U-K} \sqrt{\frac{e^{2K} x^2 + y^2}{x^2 + y^2}} \partial_y + xy \frac{e^{U-K} (e^{2K} - 1)}{\sqrt{(x^2 + y^2)(e^{2K} x^2 + y^2)}} \partial_x, & \mathbf{e}_{(z)} &= e^{-U} \partial_z \end{aligned} \quad (2.6.8)$$

2.7 Cosmic string in Schwarzschild spacetime

A cosmic string in the Schwarzschild spacetime represented by Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2(d\vartheta^2 + \beta^2 \sin^2 \vartheta d\varphi^2), \quad (2.7.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and β is the string parameter, compare Aryal et al [AFV86].

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 r_s(r - r_s)}{2r^3}, \quad \Gamma_{tr}^t = \frac{r_s}{2r(r - r_s)}, \quad \Gamma_{rr}^r = -\frac{r_s}{2r(r - r_s)}, \quad (2.7.2a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad (2.7.2b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -(r - r_s)\beta^2 \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\beta^2 \sin \vartheta \cos \vartheta. \quad (2.7.2c)$$

Partial derivatives

$$\Gamma_{tt,r}^r = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \quad \Gamma_{tr,r}^t = -\frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad \Gamma_{rr,r}^r = \frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad (2.7.3a)$$

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -1, \quad (2.7.3b)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -\beta^2 \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\beta^2 \cos(2\vartheta), \quad (2.7.3c)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -(r - r_s)\beta^2 \sin(2\vartheta). \quad (2.7.3d)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \quad R_{t\vartheta t\vartheta} = \frac{1}{2} \frac{c^2 (r - r_s) r_s}{r^2}, \quad R_{t\varphi t\varphi} = \frac{1}{2} \frac{c^2 (r - r_s) r_s \beta^2 \sin^2 \vartheta}{r^2}, \quad (2.7.4a)$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2} \frac{r_s}{r - r_s}, \quad R_{r\varphi r\varphi} = -\frac{1}{2} \frac{r_s \beta^2 \sin^2 \vartheta}{r - r_s}, \quad R_{\vartheta\varphi\vartheta\varphi} = r r_s \beta^2 \sin^2 \vartheta. \quad (2.7.4b)$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12 \frac{r_s^2}{r^6}. \quad (2.7.5)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\beta \sin \vartheta} \partial_\varphi. \quad (2.7.6)$$

Dual tetrad:

$$\theta^{(t)} = c \sqrt{1 - \frac{r_s}{r}} dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r \beta \sin \vartheta d\varphi. \quad (2.7.7)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2 \sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.7.8)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.7.9)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.7.10a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.7.10b)$$

Embedding:

The embedding function for $\beta^2 < 1$ reads

$$z = (r - r_s) \sqrt{\frac{r}{r - r_s} - \beta^2} - \frac{r_s}{2\sqrt{1 - \beta^2}} \ln \frac{\sqrt{r/(r - r_s) - \beta^2} - \sqrt{1 - \beta^2}}{\sqrt{r/(r - r_s) - \beta^2} + \sqrt{1 - \beta^2}}. \quad (2.7.11)$$

If $\beta^2 = 1$, we have the embedding function of the standard Schwarzschild metric, compare Eq.(2.2.15).

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(1 - \frac{r_s}{r}\right) \left(\frac{h^2}{r^2\beta^2} - \kappa c^2\right) \quad (2.7.12)$$

with the constants of motion $k = (1 - r_s/r)c^2t$ and $h = r^2\beta^2\dot{\varphi}$. The maxima of the effective potential V_{eff} lead to the same critical orbits $r_{\text{po}} = \frac{3}{2}r_s$ and $r_{\text{itcg}} = 3r_s$ as in the standard Schwarzschild metric.

2.8 Ernst spacetime

"The Ernst metric is a static, axially symmetric, electro-vacuum solution of the Einstein-Maxwell equations with a black hole immersed in a magnetic field." [KV92]

In spherical coordinates $(t, r, \vartheta, \varphi)$, the Ernst metric reads [Ern76] ($G = c = 1$)

$$ds^2 = \Lambda^2 \left[-\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\vartheta^2 \right] + \frac{r^2 \sin^2 \vartheta}{\Lambda^2} d\varphi^2, \quad (2.8.1)$$

where $\Lambda = 1 + B^2 r^2 \sin^2 \vartheta$. Here, M is the mass of the black hole and B the magnetic field strength.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{(2B^2 r^3 \sin^2 \vartheta - 3MB^2 r^2 \sin^2 \vartheta + M)(r - 2M)}{r^3 \Lambda}, \quad \Gamma_{tt}^\vartheta = \frac{2(r - 2M)B^2 \sin \vartheta \cos \vartheta}{r \Lambda}, \quad (2.8.2a)$$

$$\Gamma_{tr}^t = \frac{2B^2 r^3 \sin^2 \vartheta - 3MB^2 r^2 \sin^2 \vartheta + M}{r(r - 2M)\Lambda}, \quad \Gamma_{t\vartheta}^t = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \quad (2.8.2b)$$

$$\Gamma_{rr}^r = \frac{2B^2 r^3 \sin^2 \vartheta - 5MB^2 r^2 \sin^2 \vartheta - M}{r(r - 2M)\Lambda}, \quad \Gamma_{rr}^\vartheta = -\frac{2B^2 r \sin \vartheta \cos \vartheta}{(r - 2M)\Lambda}, \quad (2.8.2c)$$

$$\Gamma_{r\vartheta}^r = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{3B^2 r^2 \sin^2 \vartheta + 1}{r \Lambda}, \quad (2.8.2d)$$

$$\Gamma_{\vartheta\vartheta}^r = \frac{(3B^2 r^2 \sin^2 \vartheta + 1)(r - 2M)}{\Lambda}, \quad (2.8.2e)$$

$$\Gamma_{\vartheta\vartheta}^\vartheta = \frac{\Xi \cos \vartheta}{\Lambda}, \quad (2.8.2f)$$

$$\Gamma_{\varphi\varphi}^r = \frac{(r - 2M)\Xi \sin^2 \vartheta}{\Lambda^5}, \quad (2.8.2g)$$

$$\Gamma_{\varphi\varphi}^\vartheta = \frac{\Xi \sin \vartheta \cos \vartheta}{\Lambda^5}. \quad (2.8.2h)$$

with $\Xi = 1 - B^2 r^2 \sin^2 \vartheta$.

Riemann-Tensor:

$$R_{trtr} = \frac{2}{r^3} \left[B^4 r^4 \sin^4 \vartheta (3M - r) - M + 2r^5 B^4 \sin^2 \vartheta \cos^2 \vartheta + B^2 r^2 \sin^2 \vartheta (r - 2M) \right], \quad (2.8.3a)$$

$$R_{trt\vartheta} = 2B^2 \sin \vartheta \cos \vartheta [(3B^2 r^2 \sin^2 \vartheta (2M - 3r) + r - 2M)], \quad (2.8.3b)$$

$$R_{t\vartheta t\vartheta} = \frac{1}{r^2} [B^4 r^4 (r - 2M)(4r - 9M) \sin^4 \vartheta + 2\Xi B^2 r^3 (r - 2M) \cos^2 \vartheta + M(r - 2M)], \quad (2.8.3c)$$

$$R_{t\varphi t\varphi} = \frac{1}{\Lambda^4 r^2} [(2B^2 r^3 - 3B^2 M r^2 \sin^2 \vartheta + M)\Xi (r - 2M) \sin^2 \vartheta], \quad (2.8.3d)$$

$$R_{r\vartheta r\vartheta} = -\frac{(2B^2 r^3 - 3B^2 M r^2 \sin^2 \vartheta + M)\Xi}{r - 2M}, \quad (2.8.3e)$$

$$R_{r\varphi r\varphi} = -\frac{\sin^2 \vartheta}{\Lambda^4 (r - 2M)} [B^4 r^4 (4r - 9M) \sin^4 \vartheta + 2B^2 r^2 (8M - 4r\vartheta) \sin^2 \vartheta + 2\Xi B^2 r^3 \cos^2 \vartheta + M], \quad (2.8.3f)$$

$$R_{r\varphi\vartheta\varphi} = -\frac{2B^2 r^3 \sin^3 \vartheta \cos \vartheta (3B^2 r^2 \sin^2 \vartheta - 5)}{\Lambda^4}, \quad (2.8.3g)$$

$$R_{\vartheta\varphi\vartheta\varphi} = \frac{r \sin^2 \vartheta}{\Lambda^4} [2B^4 r^4 (r - 3M) \sin^4 \vartheta + 4B^2 r^3 \cos^2 \vartheta (1 + \Xi) + 2B^2 r^2 \sin^2 \vartheta (2M - r) + 2M]. \quad (2.8.3h)$$

Ricci-Tensor:

$$R_{tt} = \frac{4B^2(r-2M)(r+2M\sin^2\vartheta)}{r^2\Lambda^2}, \quad R_{rr} = -\frac{4B^2[r\cos^2\vartheta-(r-2M)\sin^2\vartheta]}{(r-2M)\Lambda^2}, \quad (2.8.4a)$$

$$R_{r\vartheta} = \frac{8B^2r\sin\vartheta\cos\vartheta}{\Lambda^2}, \quad R_{\vartheta r} = \frac{4B^2r[r\cos^2\vartheta+(r-2M)\sin^2\vartheta]}{\Lambda^2}, \quad (2.8.4b)$$

$$R_{\varphi\varphi} = \frac{4B^2r\sin^2\vartheta(r+2M\sin^2\vartheta)}{\Lambda^6}. \quad (2.8.4c)$$

Ricci and Kretschmann scalars:

$$R = 0, \quad (2.8.5a)$$

$$\begin{aligned} \mathcal{K} = & \frac{16}{r^6\Lambda^8} \left[3B^8r^8(4r^2-18Mr+21M^2)\sin^8\vartheta \right. \\ & + 2B^4r^4(31M^2-37Mr-24B^2r^4\cos^2\vartheta+42B^2Mr^3\cos^2\vartheta+10r^2+6B^4r^6\cos^4\vartheta)\sin^6\vartheta \\ & + 2B^2r^2(-3Mr+20B^2r^4\cos^2\vartheta+6M^2-46B^2Mr^3\cos^2\vartheta-12B^4r^6\cos^4\vartheta)\sin^4\vartheta \\ & - 6B^6r^6(6B^2Mr^3\cos^2\vartheta+4r^2-4B^2r^4\cos^2\vartheta+18M^2-17Mr) \\ & \left. + 20B^4r^6\cos^4\vartheta+12B^2Mr^3\cos^2\vartheta+3M^2 \right]. \end{aligned} \quad (2.8.5b)$$

Static local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{\Lambda\sqrt{1-2m/r}}\partial_t, \quad \mathbf{e}_{(r)} = \frac{\sqrt{1-2m/r}}{\Lambda}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\Lambda r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\Lambda}{r\sin\vartheta}\partial_\varphi. \quad (2.8.6)$$

Dual tetrad:

$$\theta^{(t)} = \Lambda\sqrt{1-\frac{2m}{r}}dt, \quad \theta^{(r)} = \frac{\Lambda}{\sqrt{1-2m/r}}dr, \quad \theta^{(\vartheta)} = \Lambda rd\vartheta, \quad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\Lambda}d\varphi. \quad (2.8.7)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{r}^2 + \frac{h^2(1-r_s/r)}{r^2} - \frac{k^2}{\Lambda^4} + \kappa\frac{1-r_s/r}{\Lambda^2} = 0 \quad (2.8.8)$$

with constants of motion $k = \Lambda^2(1-r_s/r)\dot{t}$ and $h = (r^2/\Lambda^2)\dot{\varphi}$.

Further reading:

Ernst [Ern76], Dhurandhar and Sharma [DS83], Karas and Vokrouhlický [KV92], Stuchlík and Hledík [SH99].

2.9 Friedman-Robertson-Walker

The Friedman-Robertson-Walker metric describes a general homogeneous and isotropic universe. In a general form it reads:

$$ds^2 = -c^2 dt^2 + R^2 d\sigma^2 \quad (2.9.1)$$

with $R = R(t)$ being an arbitrary function of time only and $d\sigma^2$ being a metric of a 3-space of constant curvature for which three explicit forms will be described here.

In all formulas in this section a dot denotes differentiation with respect to t , e.g. $\dot{R} = dR(t)/dt$.

2.9.1 Form 1

$$ds^2 = -c^2 dt^2 + R^2 \left\{ \frac{d\eta^2}{1-k\eta^2} + \eta^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right\} \quad (2.9.2)$$

Christoffel symbols:

$$\Gamma_{t\eta}^\eta = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.3a)$$

$$\Gamma_{\eta\eta}^t = \frac{R\ddot{R}}{c^2(1-k\eta^2)}, \quad \Gamma_{\eta\eta}^\eta = \frac{k\eta}{1-k\eta^2}, \quad \Gamma_{\eta\vartheta}^\vartheta = \frac{1}{\eta}, \quad (2.9.3b)$$

$$\Gamma_{\eta\varphi}^\varphi = \frac{1}{\eta}, \quad \Gamma_{\vartheta\vartheta}^t = \frac{R\eta^2\dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\eta = (k\eta^2 - 1)\eta, \quad (2.9.3c)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^t = \frac{R\eta^2 \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\eta = (k\eta^2 - 1)\eta \sin^2 \vartheta, \quad (2.9.3d)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.9.3e)$$

Riemann-Tensor:

$$R_{t\eta t\eta} = \frac{R\ddot{R}}{k\eta^2 - 1}, \quad R_{t\vartheta t\vartheta} = -R\eta^2 \ddot{R}, \quad (2.9.4a)$$

$$R_{t\varphi t\varphi} = -R\eta^2 \sin^2 \vartheta \ddot{R}, \quad R_{\eta\vartheta\eta\vartheta} = -\frac{R^2 \eta^2 (\dot{R}^2 + kc^2)}{c^2(k\eta^2 - 1)}, \quad (2.9.4b)$$

$$R_{\eta\varphi\eta\varphi} = -\frac{R^2 \eta^2 \sin^2 \vartheta (\dot{R}^2 + kc^2)}{c^2(k\eta^2 - 1)}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2 \eta^4 \sin^2 \vartheta (\dot{R}^2 + kc^2)}{c^2}. \quad (2.9.4c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\eta\eta} = \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(1-k\eta^2)}, \quad (2.9.5a)$$

$$R_{\vartheta\vartheta} = \eta^2 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2}, \quad R_{\varphi\varphi} = \eta^2 \sin^2 \vartheta \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2}. \quad (2.9.5b)$$

The Ricci scalar and Kretschmann scalar read:

$$\mathcal{R} = 6 \frac{R\ddot{R} + \dot{R}^2 + kc^2}{R^2 c^2}, \quad \mathcal{K} = 12 \frac{\dot{R}^2 R^2 + \dot{R}^4 + 2\dot{R}^2 kc^2 + k^2 c^4}{R^4 c^4}. \quad (2.9.6)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c} \partial_t, \quad e_{(\eta)} = \frac{\sqrt{1-k\eta^2}}{R} \partial_\eta, \quad e_\vartheta = \frac{1}{R\eta} \partial_\vartheta, \quad e_\varphi = \frac{1}{R\eta \sin \vartheta} \partial_\varphi. \quad (2.9.7)$$

Ricci rotation coefficients:

$$\begin{aligned}\gamma_{(\eta)(t)(\eta)} &= \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\phi)(t)(\phi)} = \frac{\dot{R}}{Rc} & \gamma_{(\vartheta)(\eta)(\vartheta)} &= \gamma_{(\phi)(\eta)(\phi)} = \frac{\sqrt{1-k\eta^2}}{R\eta}, \\ \gamma_{(\phi)(\vartheta)(\phi)} &= \frac{\cot\vartheta}{R\eta}.\end{aligned}\quad (2.9.8)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = \frac{2\sqrt{1-k\eta^2}}{R\eta}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\eta}. \quad (2.9.9)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\phi)(t)(\phi)} = -\frac{\ddot{R}}{Rc^2} \quad (2.9.10a)$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\phi)(\eta)(\phi)} = R_{(\vartheta)(\phi)(\vartheta)(\phi)} = \frac{\dot{R}^2 + kc^2}{R^2 c^2}. \quad (2.9.10b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\phi)(\phi)} = \frac{R\ddot{R} + 2\dot{R}^2 + 2kc^2}{R^2 c^2}. \quad (2.9.11)$$

2.9.2 Form 2

$$ds^2 = -c^2 dt^2 + \frac{R^2}{(1 + \frac{k}{4}r^2)^2} \{ dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2) \}$$

(2.9.12)

Christoffel symbols:

$$\Gamma_{tr}^r = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\phi}^\phi = \frac{\dot{R}}{R}, \quad (2.9.13a)$$

$$\Gamma_{rr}^t = 16 \frac{R\dot{R}}{c^2(4+kr^2)^2}, \quad \Gamma_{rr}^r = -\frac{2kr}{4+kr^2}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{4-kr^2}{(4+kr^2)r}, \quad (2.9.13b)$$

$$\Gamma_{r\phi}^\phi = \frac{4-kr^2}{(4+kr^2)r}, \quad \Gamma_{\vartheta\vartheta}^\vartheta = 16 \frac{Rr^2\dot{R}}{c^2(4+kr^2)^2}, \quad \Gamma_{\vartheta\vartheta}^r = \frac{r(kr^2-4)}{4+kr^2}, \quad (2.9.13c)$$

$$\Gamma_{\vartheta\phi}^\phi = \cot\vartheta, \quad \Gamma_{\phi\vartheta}^\vartheta = 16 \frac{Rr^2 \sin^2\vartheta \dot{R}}{c^2(4+kr^2)^2}, \quad \Gamma_{\phi\phi}^\vartheta = -\sin\vartheta \cos\vartheta, \quad (2.9.13d)$$

$$\Gamma_{\phi\phi}^\phi = \frac{r \sin^2\vartheta (kr^2-4)}{4+kr^2}. \quad (2.9.13e)$$

Riemann-Tensor:

$$R_{tttr} = -16 \frac{R\ddot{R}}{(4+kr^2)^2}, \quad R_{t\vartheta t\vartheta} = -16 \frac{Rr^2\ddot{R}}{(4+kr^2)^2}, \quad (2.9.14a)$$

$$R_{t\varphi t\varphi} = -16 \frac{Rr^2 \sin^2\vartheta \dot{R}}{(4+kr^2)^2}, \quad R_{r\vartheta r\vartheta} = 256 \frac{R^2 r^2 (\dot{R}^2 + kc^2)}{c^2 (4+kr^2)^4}, \quad (2.9.14b)$$

$$R_{r\varphi r\varphi} = 256 \frac{R^2 r^2 \sin^2\vartheta (\dot{R}^2 + kc^2)}{c^2 (4+kr^2)^4}, \quad R_{\vartheta\varphi\vartheta\varphi} = 256 \frac{R^2 r^4 \sin^2\vartheta (\dot{R}^2 + kc^2)}{c^2 (4+kr^2)^4}. \quad (2.9.14c)$$

Ricci-Tensor:

$$R_{tt} = -3 \frac{\ddot{R}}{R}, \quad R_{rr} = 16 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2 (4+kr^2)^2}, \quad (2.9.15a)$$

$$R_{\vartheta\vartheta} = 16r^2 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2 (4+kr^2)^2}, \quad R_{\varphi\varphi} = 16r^2 \sin^2\vartheta \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2 (4+kr^2)^2}. \quad (2.9.15b)$$

The Ricci scalar and Kretschmann scalar read:

$$\mathcal{R} = 6 \frac{R\ddot{R} + \dot{R}^2 + kc^2}{R^2 c^2}, \quad \mathcal{K} = 12 \frac{\dot{R}^2 R^2 + \dot{R}^4 + 2\dot{R}^2 kc^2 + k^2 c^4}{R^4 c^4}. \quad (2.9.16)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c} \partial_t, \quad e_{(r)} = \frac{1 + \frac{k}{4} r^2}{R} \partial_r, \quad e_\vartheta = \frac{1 + \frac{k}{4} r^2}{Rr} \partial_\vartheta, \quad e_\varphi = \frac{1 + \frac{k}{4} r^2}{Rr \sin \vartheta} \partial_\varphi. \quad (2.9.17)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = -\frac{\frac{k}{4} r^2 - 1}{Rr}, \quad (2.9.18a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{(\frac{k}{4} r^2 + 1) \cot \vartheta}{Rr}. \quad (2.9.18b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = 2 \frac{1 - \frac{k}{4} r^2}{Rr}, \quad \gamma_{(\vartheta)} = \frac{(\frac{k}{4} r^2 + 1) \cot \vartheta}{Rr}. \quad (2.9.19)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \quad (2.9.20a)$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + kc^2}{R^2 c^2}. \quad (2.9.20b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2 + 2kc^2}{R^2 c^2}. \quad (2.9.21)$$

2.9.3 Form 3

The following forms of the metric are obtained from 2.9.2 by setting $\eta = \sin \psi, \psi, \sinh \psi$ for $k = 1, 0, -1$ respectively.

Positive Curvature

$$ds^2 = -c^2 dt^2 + R^2 \{ d\psi^2 + \sin^2 \psi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \}$$

(2.9.22)

Christoffel symbols:

$$\Gamma_{t\psi}^\psi = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.23a)$$

$$\Gamma_{\psi\psi}^t = \frac{R\dot{R}}{c^2}, \quad \Gamma_{\psi\vartheta}^\vartheta = \cot \psi, \quad \Gamma_{\psi\varphi}^\varphi = \cot \psi, \quad (2.9.23b)$$

$$\Gamma_{\vartheta\vartheta}^t = \frac{R \sin^2 \psi \dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\psi = -\sin \psi \cos \psi, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot(\vartheta), \quad (2.9.23c)$$

$$\Gamma_{\varphi\varphi}^t = \frac{R \sin^2 \psi \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\psi = -\sin \psi \cos \psi \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.9.23d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \quad R_{t\vartheta t\vartheta} = -R\sin^2\psi\ddot{R}, \quad (2.9.24a)$$

$$R_{t\varphi t\varphi} = -R\sin^2\psi\sin^2\vartheta\ddot{R}, \quad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sin^2\psi(\dot{R}^2 + c^2)}{c^2}, \quad (2.9.24b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sin^2\psi\sin^2\vartheta(\dot{R}^2 + c^2)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sin^4\psi\sin^2\vartheta(\dot{R}^2 + c^2)}{c^2}. \quad (2.9.24c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\psi\psi} = \frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{c^2}, \quad (2.9.25a)$$

$$R_{\vartheta\vartheta} = \sin^2\psi\frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{c^2}, \quad R_{\varphi\varphi} = \sin^2\vartheta\sin^2\psi\frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{c^2}. \quad (2.9.25b)$$

The Ricci scalar and Kretschmann read

$$\mathcal{R} = 6\frac{R\ddot{R} + \dot{R}^2 + c^2}{R^2c^2}, \quad \mathcal{K} = 12\frac{\ddot{R}^2R^2 + \dot{R}^4 + 2\dot{R}^2c^2 + c^4}{R^4c^4}. \quad (2.9.26)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \quad e_{(\psi)} = \frac{1}{R}\partial_\psi, \quad e_{(\vartheta)} = \frac{1}{R\sin\psi}\partial_\vartheta, \quad e_{(\varphi)} = \frac{1}{R\sin\psi\sin\vartheta}\partial_\varphi. \quad (2.9.27)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \quad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\cot\psi}{R}, \quad (2.9.28a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sin\psi}. \quad (2.9.28b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = 2\frac{\cot\psi}{R}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sin\psi}. \quad (2.9.29)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \quad (2.9.30a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + c^2}{R^2c^2}. \quad (2.9.30b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{R^2c^2}. \quad (2.9.31)$$

Vanishing Curvature

$$ds^2 = -c^2dt^2 + R^2\{d\psi^2 + \psi^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\}$$

(2.9.32)

Christoffel symbols:

$$\Gamma_{t\psi}^\psi = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.33a)$$

$$\Gamma_{\psi\psi}^t = \frac{R\dot{R}}{c^2}, \quad \Gamma_{\psi\vartheta}^\vartheta = \frac{1}{\psi}, \quad \Gamma_{\psi\varphi}^\varphi = \frac{1}{\psi}, \quad (2.9.33b)$$

$$\Gamma_{\vartheta\vartheta}^t = \frac{R\psi^2\dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\psi = -\psi, \quad \Gamma_{\vartheta\vartheta}^\varphi = \cot\vartheta, \quad (2.9.33c)$$

$$\Gamma_{\varphi\varphi}^t = \frac{R\psi^2\sin^2\vartheta\dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\psi = -\psi\sin^2\vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin\vartheta\cos\vartheta. \quad (2.9.33d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \quad R_{t\vartheta t\vartheta} = -R\psi^2\ddot{R}, \quad (2.9.34a)$$

$$R_{t\varphi t\varphi} = -R\psi^2 \sin^2 \vartheta \ddot{R}, \quad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\psi^2\dot{R}^2}{c^2}, \quad (2.9.34b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\psi^2 \sin^2 \vartheta \dot{R}^2}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\psi^4 \sin^2 \vartheta \dot{R}^2}{c^2}. \quad (2.9.34c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\psi\psi} = \frac{R\ddot{R} + 2\dot{R}^2}{c^2}, \quad (2.9.35a)$$

$$R_{\vartheta\vartheta} = \psi^2 \frac{R\ddot{R} + 2\dot{R}^2}{c^2}, \quad R_{\varphi\varphi} = \sin^2 \vartheta \psi^2 \frac{R\ddot{R} + 2\dot{R}^2}{c^2}. \quad (2.9.35b)$$

The Ricci scalar and Kretschmann read

$$\mathcal{R} = 6\frac{R\ddot{R} + \dot{R}^2}{R^2 c^2}, \quad \mathcal{K} = 12\frac{\dot{R}^2 R^2 + \dot{R}^4}{R^4 c^4}. \quad (2.9.36)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \quad e_{(\psi)} = \frac{1}{R}\partial_\psi, \quad e_{\vartheta} = \frac{1}{R\psi}\partial_\vartheta, \quad e_{\varphi} = \frac{1}{R\psi \sin \vartheta}\partial_\varphi. \quad (2.9.37)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc}, \quad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{1}{R\psi}, \quad (2.9.38a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{R\psi}. \quad (2.9.38b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = \frac{2}{R\psi}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{R\psi}. \quad (2.9.39)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \quad (2.9.40a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2}{R^2 c^2}. \quad (2.9.40b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2}{R^2 c^2}. \quad (2.9.41)$$

Negative Curvature

$$ds^2 = -c^2 dt^2 + R^2 \{ d\psi^2 + \sinh^2 \psi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \}$$

(2.9.42)

Christoffel symbols:

$$\Gamma_{t\psi}^\psi = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.43a)$$

$$\Gamma_{\psi\psi}^\vartheta = \frac{R\dot{R}}{c^2}, \quad \Gamma_{\psi\vartheta}^\vartheta = \coth \psi, \quad \Gamma_{\psi\varphi}^\varphi = \coth \psi, \quad (2.9.43b)$$

$$\Gamma_{\vartheta\vartheta}^\vartheta = \frac{R \sinh^2 \psi \dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\psi = -\sinh \psi \cosh \psi, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad (2.9.43c)$$

$$\Gamma_{\varphi\varphi}^\vartheta = \frac{R \sinh^2 \psi \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\psi = -\sinh \psi \cosh \psi \sin^2 \vartheta, \quad \Gamma_{\vartheta\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.9.43d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \quad R_{t\vartheta t\vartheta} = -R \sinh^2 \psi \ddot{R}, \quad (2.9.44a)$$

$$R_{t\varphi t\varphi} = -R \sinh^2 \psi \sin^2 \vartheta \ddot{R}, \quad R_{\psi\vartheta\psi\vartheta} = \frac{R^2 \sinh^2 \psi (\dot{R}^2 - c^2)}{c^2}, \quad (2.9.44b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2 \sinh^2 \psi \sin^2 \vartheta (\dot{R}^2 - c^2)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2 \sinh \psi^4 \sin^2 \vartheta (\dot{R}^2 - c^2)}{c^2}. \quad (2.9.44c)$$

Ricci-Tensor:

$$R_{tt} = -3 \frac{\ddot{R}}{R}, \quad R_{\psi\psi} = \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{c^2}, \quad (2.9.45a)$$

$$R_{\vartheta\vartheta} = \sinh^2 \psi \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{c^2}, \quad R_{\varphi\varphi} = \sin^2 \vartheta \sin^2 \psi \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{c^2}. \quad (2.9.45b)$$

The Ricci scalar and Kretschmann read

$$\mathcal{R} = 6 \frac{R\ddot{R} + \dot{R}^2 - c^2}{R^2 c^2}, \quad \mathcal{K} = 12 \frac{\ddot{R}^2 R^2 + \dot{R}^4 - 2\dot{R}^2 c^2 + c^4}{R^4 c^4}. \quad (2.9.46)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c} \partial_t, \quad e_{(\psi)} = \frac{1}{R} \partial_\psi, \quad e_{(\vartheta)} = \frac{1}{R \sinh \psi} \partial_\vartheta, \quad e_{(\varphi)} = \frac{1}{R \sinh \psi \sin \vartheta} \partial_\varphi. \quad (2.9.47)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc}, \quad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\coth \psi}{R}, \quad (2.9.48a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \theta}{R \sinh \psi}. \quad (2.9.48b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = 2 \frac{\coth \psi}{R}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{R \sinh \psi}. \quad (2.9.49)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \quad (2.9.50a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 - c^2}{R^2 c^2}. \quad (2.9.50b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{R^2 c^2}. \quad (2.9.51)$$

Further reading:

Rindler [Rin01]

2.10 Gödel Universe

Gödel introduced a homogeneous and rotating universe model in [Göd49]. We follow the notation of [KWS04]

2.10.1 Cylindrical coordinates

The Gödel metric in cylindrical coordinates is

$$ds^2 = -c^2 dt^2 + \frac{dr^2}{1+[r/(2a)]^2} + r^2 \left[1 - \left(\frac{r}{2a} \right)^2 \right] d\varphi^2 + dz^2 - 2r^2 \frac{c}{\sqrt{2}a} dt d\varphi, \quad (2.10.1)$$

where $2a$ is the Gödel radius.

Christoffel symbols:

$$\Gamma_{tr}^t = \frac{r}{2a^2} \frac{1}{1+[r/(2a)]^2}, \quad \Gamma_{tr}^\varphi = -\frac{c}{\sqrt{2}ar} \frac{1}{1+[r/(2a)]^2}, \quad (2.10.2a)$$

$$\Gamma_{t\varphi}^r = \frac{cr}{\sqrt{2}a} \left[1 + \left(\frac{r}{2a} \right)^2 \right]^2, \quad \Gamma_{rr}^r = -\frac{r}{4a^2} \frac{1}{1+[r/(2a)]^2}, \quad (2.10.2b)$$

$$\Gamma_{r\varphi}^t = \frac{r^3}{4\sqrt{2}ca^3} \frac{1}{1+[r/(2a)]^2}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r} \frac{1}{1+[r/(2a)]^2}, \quad (2.10.2c)$$

$$\Gamma_{\varphi\varphi}^r = r \left[1 + \left(\frac{r}{2a} \right)^2 \right] \left[1 - \frac{1}{2} \left(\frac{r}{a} \right)^2 \right]. \quad (2.10.2d)$$

Riemann-Tensor:

$$R_{trtr} = \frac{c^2}{2a^2} \frac{1}{1+[r/(2a)]^2}, \quad R_{trr\varphi} = -\frac{cr^2}{2\sqrt{2}a^3} \frac{1}{1+[r/(2a)]^2}, \quad (2.10.3a)$$

$$R_{t\varphi t\varphi} = \frac{c^2 r^2}{2a^2} \frac{1}{1+[r/(2a)]^2}, \quad R_{r\varphi r\varphi} = \frac{r^2}{2a^2} \frac{1+3[r/(2a)]^2}{1+[r/(2a)]^2}. \quad (2.10.3b)$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2}{a^2}, \quad R_{t\varphi} = \frac{r^2 c}{\sqrt{2}a^3}, \quad R_{\varphi\varphi} = \frac{r^4}{2a^4}. \quad (2.10.4)$$

Ricci and Kretschmann scalar

$$\mathcal{R} = -\frac{1}{a^2}, \quad \mathcal{K} = \frac{3}{a^4}. \quad (2.10.5)$$

cosmological constant:

$$\Lambda = \frac{R}{2} \quad (2.10.6)$$

Killing vectors:

An infinitesimal isometric transformation $x'^\mu = x^\mu + \varepsilon \xi^\mu(x^\nu)$ leaves the metric unchanged, that is $g'_{\mu\nu}(x'^\sigma) = g_{\mu\nu}(x^\sigma)$. A killing vector field ξ^μ is solution to the killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$. There exist five killing vector fields in Gödel's spacetime:

$$\xi_a^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_b^\mu = \frac{1}{\sqrt{1+[r/(2a)]^2}} \begin{pmatrix} \frac{r}{\sqrt{2}c} \cos \varphi \\ a(1+[r/(2a)]^2) \sin \varphi \\ \frac{a}{r}(1+2[r/(2a)]^2) \cos \varphi \\ 0 \end{pmatrix}, \quad \xi_c^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (2.10.7a)$$

$$\xi_d^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi_e^\mu = \frac{1}{\sqrt{1+[r/(2a)]^2}} \begin{pmatrix} \frac{r}{\sqrt{2}c} \sin \varphi \\ -a(1+[r/(2a)]^2) \cos \varphi \\ \frac{a}{r}(1+2[r/(2a)]^2) \sin \varphi \\ 0 \end{pmatrix}. \quad (2.10.7b)$$

An arbitrary linear combination of killing vector fields is again a killing vector field.

Local tetrad:

For the local tetrad in Gödel's spacetime an ansatz similar to the local tetrad of a rotating spacetime in spherical coordinates (Sec. 1.4.7) can be used. After substituting $\vartheta \rightarrow z$ and swapping base vectors $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$ an orthonormalized and right-handed local tetrad is obtained.

$$\mathbf{e}_{(0)} = \Gamma (\partial_t + \zeta \partial_\phi), \quad \mathbf{e}_{(1)} = \sqrt{1 + [r/(2a)]^2} \partial_r, \quad \mathbf{e}_{(2)} = \Delta \Gamma (A \partial_t + B \partial_\phi), \quad \mathbf{e}_{(3)} = \partial_z, \quad (2.10.8a)$$

where

$$A = -\frac{r^2 c}{\sqrt{2a}} + \zeta r^2 (1 - [r/(2a)]^2), \quad B = c^2 + \frac{\zeta r^2 c}{\sqrt{2a}}, \quad (2.10.9a)$$

$$\Gamma = \frac{1}{\sqrt{c^2 + \zeta r^2 c \sqrt{2/a} - \zeta^2 r^2 (1 - [r/(2a)]^2)}}, \quad \Delta = \frac{1}{rc \sqrt{1 + [r/(2a)]^2}}. \quad (2.10.9b)$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ :

$$y^0 = y^{(0)}\Gamma + y^{(2)}\Delta\Gamma A, \quad y^1 = y^{(1)}\sqrt{1 + [r/(2a)]^2}, \quad y^2 = y^{(0)}\Gamma\zeta + y^{(2)}\Delta\Gamma B, \quad y^3 = y^{(3)}. \quad (2.10.10)$$

with the above abbreviations.

2.10.2 Scaled cylindrical coordinates

If we apply the simple transformation

$$T = \frac{t}{r_G}, \quad R = \frac{r}{r_G}, \quad \phi = \varphi, \quad Z = \frac{z}{r_G}, \quad (2.10.11)$$

with $r_G = 2a$, we find a formulation for the metric scaling with r_G , which is

$$ds^2 = r_G^2 \left(-c^2 dT^2 + \frac{dR^2}{1+R^2} + R^2(1-R^2) d\phi^2 + dZ^2 - 2\sqrt{2}cR^2 dTd\phi \right). \quad (2.10.12)$$

Christoffel symbols:

$$\Gamma_{TR}^T = \frac{2R}{1+R^2}, \quad \Gamma_{TR}^\phi = -\frac{\sqrt{2}c}{R(1+R^2)}, \quad (2.10.13a)$$

$$\Gamma_{T\phi}^R = \sqrt{2}cR(1+R^2), \quad \Gamma_{RR}^R = -\frac{R}{1+R^2}, \quad (2.10.13b)$$

$$\Gamma_{R\phi}^T = \frac{\sqrt{2}R^3}{c(1+R^2)}, \quad \Gamma_{R\phi}^\phi = \frac{1}{R(1+R^2)}, \quad (2.10.13c)$$

$$\Gamma_{\phi\phi}^R = R(1+R^2)(2R^2-1). \quad (2.10.13d)$$

Riemann-Tensor:

$$R_{TTRR} = \frac{2r_G^2 c^2}{1+R^2}, \quad R_{TRR\phi} = -\frac{2\sqrt{2}r_G^2 c R^2}{1+R^2}, \quad (2.10.14a)$$

$$R_{T\phi T\phi} = 2c^2 r_G^2 R^2 (1+R^2), \quad R_{R\phi R\phi} = \frac{2r_G^2 R^2 (1+3R^2)}{1+R^2}. \quad (2.10.14b)$$

Ricci-Tensor:

$$R_{TT} = 4c^2, \quad R_{T\phi} = 4\sqrt{2}cR^2, \quad R_{\phi\phi} = 8R^4. \quad (2.10.15)$$

Ricci and Kretschmann scalar

$$\mathcal{R} = -\frac{4}{r_G^2}, \quad \mathcal{K} = \frac{48}{r_G^4}. \quad (2.10.16)$$

cosmological constant:

$$\Lambda = \frac{R}{2} \quad (2.10.17)$$

Killing vectors:

The Killing vectors read

$$\xi_a^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_b^\mu = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2c}} \cos \varphi \\ \frac{1}{2}(1+R^2) \sin \varphi \\ \frac{1}{2R}(1+2R^2) \cos \varphi \\ 0 \end{pmatrix}, \quad \xi_c^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (2.10.18a)$$

$$\xi_d^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi_e^\mu = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2c}} \sin \varphi \\ -\frac{1}{2}(1+R^2) \cos \varphi \\ \frac{1}{2R}(1+2R^2) \sin \varphi \\ 0 \end{pmatrix}. \quad (2.10.18b)$$

Local tetrad:

After the transformation to scaled cylindrical coordinates, the local tetrad reads

$$\mathbf{e}_{(0)} = \frac{\Gamma}{r_G} (\partial_T + \zeta \partial_\phi), \quad \mathbf{e}_{(1)} = \frac{1}{r_G} \sqrt{1+R^2} \partial_R, \quad \mathbf{e}_{(2)} = \frac{\Delta \Gamma}{r_G} (A \partial_T + B \partial_\phi), \quad \mathbf{e}_{(3)} = \frac{1}{r_G} \partial_Z, \quad (2.10.19a)$$

where

$$A = R^2 \left[-\sqrt{2}c + (1-R^2)\zeta \right], \quad B = c^2 + \sqrt{2}R^2c\zeta, \quad (2.10.20a)$$

$$\Gamma = \frac{1}{\sqrt{c^2 + 2\sqrt{2}R^2c\zeta - R^2(1-R^2)\zeta^2}}, \quad \Delta = \frac{1}{Rc\sqrt{1+R^2}}. \quad (2.10.20b)$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ :

$$y^0 = \frac{\Gamma}{r_G} y^{(0)} + \frac{\Delta \Gamma A}{r_G} y^{(2)}, \quad y^1 = \frac{1}{r_G} \sqrt{1+R^2} y^{(1)}, \quad y^2 = \frac{\Gamma \zeta}{r_G} y^{(0)} + \frac{\Delta \Gamma B}{r_G} y^{(2)}, \quad y^3 = \frac{1}{r_G} y^{(3)}, \quad (2.10.21)$$

and the back transformation is given by

$$y^{(0)} = \frac{r_G}{\Gamma} \frac{By^0 - Ay^2}{B - \zeta A}, \quad y^{(1)} = \frac{r_G}{\sqrt{1+R^2}} y^1, \quad y^{(2)} = \frac{r_G}{\Delta \Gamma} \frac{y^2 - \zeta y^0}{B - \zeta A}, \quad y^{(3)} = r_G y^3. \quad (2.10.22a)$$

2.11 Halilsoy standing wave

The standing wave metric by Halilsoy[[Hal88](#)] reads

$$ds^2 = V [e^{2K} (d\rho^2 - dt^2) + \rho^2 d\varphi^2] + \frac{1}{V} (dz + Ad\varphi)^2, \quad (2.11.1)$$

where

$$V = \cosh^2 \alpha e^{-2CJ_0(\rho)\cos(t)} + \sinh^2 \alpha e^{2CJ_0(\rho)\cos(t)}, \quad (2.11.2a)$$

$$K = \frac{C^2}{2} [\rho^2 (J_0(\rho)^2 + J_1(\rho)^2) - 2\rho J_0(\rho) J_1(\rho) \cos^2 t], \quad (2.11.2b)$$

$$A = -2C \sinh(2\alpha) \rho J_1(\rho) \sin(t). \quad (2.11.2c)$$

with spherical Bessel functions $J_{1,2}$ and parameters α and C .

Local tetrad:

$$\mathbf{e}_{(0)} = \frac{e^{-K}}{\sqrt{V}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{e^{-K}}{\sqrt{V}} \partial_\rho, \quad \mathbf{e}_{(2)} = \frac{1}{\rho \sqrt{V}} \partial_\varphi - \frac{A}{\rho \sqrt{V}} \partial_z, \quad \mathbf{e}_{(3)} = \sqrt{V} \partial_z. \quad (2.11.3)$$

dual tetrad:

$$\theta^{(0)} = \sqrt{V} e^K dt, \quad \theta^{(1)} = \sqrt{V} e^K d\rho, \quad \theta^{(2)} = \sqrt{V} \rho d\varphi, \quad \theta^{(3)} = \frac{1}{\sqrt{V}} (dz + Ad\varphi). \quad (2.11.4)$$

2.12 Janis-Newman-Winicour

The Janis-Newman-Winicour[JNW68] spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ is represented by the line element

$$ds^2 = -\alpha^\gamma c^2 dt^2 + \alpha^{-\gamma} dr^2 + r^2 \alpha^{-\gamma+1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.12.1)$$

where $\alpha = 1 - r_s/(\gamma r)$. The Schwarzschild radius $r_s = 2GM/c^2$ is defined by Newton's constant G , the speed of light c , and the mass parameter M . For $\gamma = 1$, we obtain the Schwarzschild metric (2.2.1).

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{r_s c^2}{2r^2} \alpha^{2\gamma-1}, \quad \Gamma_{tr}^t = \frac{r_s}{2\gamma r^2 \alpha}, \quad \Gamma_{rr}^r = -\frac{r_s}{2\gamma r^2 \alpha}, \quad (2.12.2a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \quad \Gamma_{r\varphi}^\varphi = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \quad \Gamma_{\vartheta\vartheta}^\vartheta = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma}, \quad (2.12.2b)$$

$$\Gamma_{\varphi\varphi}^\varphi = \Gamma_{\vartheta\vartheta}^\vartheta \sin^2 \vartheta, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\vartheta}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.12.2c)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{r_s c^2 [2\gamma r - r_s(\gamma+1)] \alpha^{\gamma-2}}{2\gamma r^4}, \quad R_{t\vartheta t\vartheta} = \frac{r_s c^2 [2\gamma r - r_s(\gamma+1)] \alpha^{\gamma-1}}{4\gamma r^2}, \quad (2.12.3a)$$

$$R_{t\varphi t\varphi} = \frac{r_s c^2 [2\gamma r - r_s(\gamma+1)] \alpha^{\gamma-1} \sin^2 \vartheta}{4\gamma r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s [2\gamma^2 r - r_s(\gamma+1)]}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \quad (2.12.3b)$$

$$R_{r\varphi r\varphi} = -\frac{r_s [2\gamma^2 r - r_s(\gamma+1)] \sin^2 \vartheta}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s [4\gamma^2 r - r_s(\gamma+1)^2] \sin^2 \vartheta}{4\gamma^2 \alpha^\gamma}. \quad (2.12.3c)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{r_s c^2 \alpha^{\gamma-2} \beta}{6\gamma^2 r^4}, \quad C_{t\vartheta t\vartheta} = \frac{r_s c^2 \alpha^{\gamma-1} \beta}{12\gamma^2 r^2}, \quad (2.12.4a)$$

$$C_{t\varphi t\varphi} = \frac{r_s c^2 \alpha^{\gamma-1} \beta \sin^2 \vartheta}{12\gamma^2 r^2}, \quad C_{r\vartheta r\vartheta} = -\frac{r_s \beta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \quad (2.12.4b)$$

$$C_{r\varphi r\varphi} = -\frac{r_s \beta \sin^2 \vartheta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{r_s \beta \sin^2 \vartheta}{6\gamma^2 \alpha^\gamma}, \quad (2.12.4c)$$

where $\beta = 6\gamma^2 r - r_s(\gamma+1)(2\gamma+1)$.

Ricci-Tensor:

$$R_{rr} = \frac{r_s^2 (1 - \gamma^2)}{2\gamma^2 r^4 \alpha^2}. \quad (2.12.5)$$

The Ricci scalar reads

$$\mathcal{R} = \frac{r_s^2 (1 - \gamma^2) \alpha^{\gamma-2}}{2\gamma^2 r^4}, \quad (2.12.6)$$

whereas the Kretschmann scalar is given by

$$\mathcal{K} = \frac{r_s^2 \alpha^{2\gamma-4}}{4\gamma^4 r^8} [7\gamma^2 r_s^2 (2 + \gamma^2) + 48\gamma^4 r^2 \alpha + 8\gamma r_s (2\gamma^2 + 1) (r_s - 2\gamma r) + 3r_s^2]. \quad (2.12.7)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\alpha^{\gamma/2}} \partial_t, \quad \mathbf{e}_{(r)} = \alpha^{\gamma/2} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{\alpha^{(\gamma-1)/2}}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\alpha^{(\gamma-1)/2}}{r \sin \vartheta} \partial_\varphi. \quad (2.12.8)$$

Dual tetrad:

$$\theta^{(t)} = c\alpha^{\gamma/2}dt, \quad \theta^{(r)} = \frac{dr}{\alpha^{\gamma/2}}, \quad \theta^{(\vartheta)} = \frac{r}{\alpha^{(\gamma-1)/2}}d\vartheta, \quad \theta^{(\phi)} = \frac{r\sin\vartheta}{\alpha^{(\gamma-1)/2}}d\phi. \quad (2.12.9)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\phi)(r)(\phi)} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \quad (2.12.10a)$$

$$\gamma_{(\phi)(\vartheta)(\phi)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \quad (2.12.10b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4\gamma r - r_s(2+\gamma)}{2\gamma r^2}\alpha^{(\gamma-1)/2}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \quad (2.12.11)$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \quad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\phi)}^{(\phi)} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \quad (2.12.12a)$$

$$c_{(\vartheta)(\phi)}^{(\phi)} = -\frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \quad (2.12.12b)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\alpha^\gamma \left(\frac{h^2\alpha^{\gamma-1}}{r^2} - \kappa c^2 \right) \quad (2.12.13)$$

with the constants of motion $h = r^2\alpha^{-\gamma+1}\dot{\phi}$ and $k = \alpha^\gamma c^2 i$. For null geodesics ($\kappa = 0$) and $\gamma > \frac{1}{2}$, there is an extremum at

$$r = r_s \frac{1+2\gamma}{2\gamma}. \quad (2.12.14)$$

Embedding:

The embedding function $z = z(r)$ for $r \in [r_s(\gamma+1)^2/(4\gamma^2), \infty)$ follows from

$$\frac{dz}{dr} = \sqrt{\frac{r_s[4r\gamma^2 - r_s(1+\gamma)^2]}{4r^2\gamma^2\alpha^{\gamma+1}}}. \quad (2.12.15)$$

However, the analytic solution

$$z(r) = 2\sqrt{r_s r} F_1 \left(-\frac{1}{2}; \frac{\gamma+1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{r_s}{r\gamma}, \frac{r_s(1+\gamma)^2}{4r\gamma^2} \right) - \frac{2\pi\gamma}{\gamma+1} {}_2F_1 \left(-\frac{1}{2}, \frac{\gamma+1}{2}; 1; \frac{4\gamma}{(\gamma+1)^2} \right), \quad (2.12.16)$$

depends on the Appell- F_1 - and the Hypergeometric- ${}_2F_1$ -function.

2.13 Kasner

The Kasner spacetime in Cartesian coordinates (t, x, y, z) is represented by the line element [MTW73, Kas21] ($c = 1$)

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \quad (2.13.1)$$

where p_1, p_2, p_3 have to fulfill the two conditions

$$p_1 + p_2 + p_3 = 1 \quad \text{and} \quad p_1^2 + p_2^2 + p_3^2 = 1. \quad (2.13.2)$$

These two conditions can also be represented by the Khalatnikov-Lifshitz parameter u with

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}. \quad (2.13.3)$$

Christoffel symbols:

$$\Gamma_{tx}^x = \frac{p_1}{t}, \quad \Gamma_{ty}^y = \frac{p_2}{t}, \quad \Gamma_{tz}^z = \frac{p_3}{t}, \quad (2.13.4a)$$

$$\Gamma_{xx}^t = \frac{p_1 t^{2p_1}}{t}, \quad \Gamma_{yy}^t = \frac{p_2 t^{2p_2}}{t}, \quad \Gamma_{zz}^t = \frac{p_3 t^{2p_3}}{t}. \quad (2.13.4b)$$

Partial derivatives

$$\Gamma_{tx,t}^x = -\frac{p_1}{t^2}, \quad \Gamma_{ty,t}^t = -\frac{p_2}{t^2}, \quad \Gamma_{tz,t}^z = -\frac{p_3}{t^2}, \quad (2.13.5a)$$

$$\Gamma_{xx,t}^t = p_1(2p_1-1)t^{2p_1-2}, \quad \Gamma_{yy,t}^t = p_2(2p_2-1)t^{2p_2-2}, \quad \Gamma_{zz,t}^t = p_3(2p_3-1)t^{2p_3-2}. \quad (2.13.5b)$$

Riemann-Tensor:

$$R_{txtx} = \frac{p_1(1-p_1)t^{2p_1}}{t^2}, \quad R_{tyty} = \frac{p_2(1-p_2)t^{2p_2}}{t^2}, \quad R_{tztz} = \frac{p_3(1-p_3)t^{2p_3}}{t^2}, \quad (2.13.6a)$$

$$R_{xyxy} = \frac{p_1 p_2 t^{2p_1} t^{2p_2}}{t^2}, \quad R_{xzxz} = \frac{p_1 p_3 t^{2p_1} t^{2p_3}}{t^2}, \quad R_{yzyz} = \frac{p_2 p_3 t^{2p_2} t^{2p_3}}{t^2}. \quad (2.13.6b)$$

The Ricci tensor as well as the Ricci scalar vanish identically. The Kretschmann scalar reads

$$\mathcal{K} = \frac{4}{t^4} (p_1^2 - 2p_1^3 + p_1^4 + p_2^2 - 2p_2^3 + p_2^4 + p_3^2 - 2p_3^3 + p_3^4 + p_1^2 p_2^2 + p_2^2 p_3^2) \quad (2.13.7a)$$

$$= \frac{16u^2(1+u)^2}{t^4(1+u+u^2)^3}. \quad (2.13.7b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \quad \mathbf{e}_{(x)} = t^{-p_1} \partial_x, \quad \mathbf{e}_{(y)} = t^{-p_2} \partial_y, \quad \mathbf{e}_{(z)} = t^{-p_3} \partial_z. \quad (2.13.8)$$

Dual tetrad:

$$\theta^{(t)} = dt, \quad \theta^{(x)} = t^{p_1} dx, \quad \theta^{(y)} = t^{p_2} dy, \quad \theta^{(z)} = t^{p_3} dz. \quad (2.13.9)$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \frac{p_1}{t}, \quad \gamma_{(t)(\vartheta)(\vartheta)} = \frac{p_2}{t}, \quad \gamma_{(t)(\phi)(\phi)} = \frac{p_3}{t}. \quad (2.13.10)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\frac{1}{t}. \quad (2.13.11)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(y)(x)} = \frac{p_1(1-p_1)}{t^2}, \quad R_{(t)(y)(t)(y)} = \frac{p_2(1-p_2)}{t^2}, \quad R_{(t)(z)(t)(z)} = \frac{p_3(1-p_3)}{t^2}, \quad (2.13.12a)$$

$$R_{(x)(y)(x)(y)} = \frac{p_1 p_2}{t^2}, \quad R_{(x)(z)(x)(z)} = \frac{p_1 p_3}{t^2}, \quad R_{(y)(z)(y)(z)} = \frac{p_2 p_3}{t^2}. \quad (2.13.12b)$$

2.14 Kerr

The Kerr spacetime, found by Roy Kerr in 1963[Ker63], describes a rotating black hole.

2.14.1 Boyer-Lindquist coordinates

The Kerr metric in Boyer-Lindquist coordinates

$$\boxed{ds^2 = -\left(1 - \frac{r_s r}{\Sigma}\right)c^2 dt^2 - \frac{2r_s a r \sin^2 \vartheta}{\Sigma} c dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\vartheta^2 + \left(r^2 + a^2 + \frac{r_s a^2 r \sin^2 \vartheta}{\Sigma}\right) \sin^2 \vartheta d\varphi^2,} \quad (2.14.1)$$

with $\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta = r^2 - r_s r + a^2$, and $r_s = 2GM/c^2$, is taken from Bardeen[BPT72]. M is the mass and a is the angular momentum per unit mass of the black hole. The contravariant form of the metric reads

$$\partial_s^2 = -\frac{A}{c^2 \Sigma \Delta} \partial_t^2 - \frac{2r_s a r}{c \Sigma \Delta} \partial_t \partial_\varphi + \frac{\Delta}{\Sigma} \partial_r^2 + \frac{1}{\Sigma} \partial_\vartheta^2 + \frac{\Delta - a^2 \sin^2 \vartheta}{\Sigma \Delta \sin^2 \vartheta} \partial_\varphi^2, \quad (2.14.2)$$

where $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \vartheta = (r^2 + a^2) \Sigma + r_s a^2 r \sin^2 \vartheta$.

The event horizon r_+ is defined by the outer root of Δ ,

$$r_+ = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2}, \quad (2.14.3)$$

whereas the outer boundary r_0 of the ergosphere follows from the outer root of $\Sigma - r_s r$,

$$r_0 = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2 \cos^2 \vartheta}, \quad (2.14.4)$$

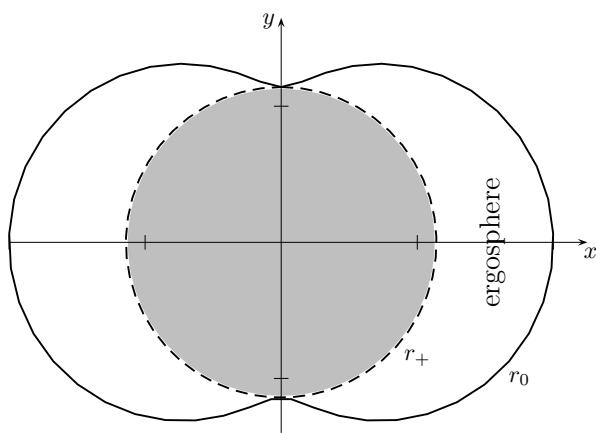


Figure 2.1: Ergosphere and horizon (dashed circle) for $a = 0.99\frac{r_s}{2}$.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 r_s \Delta (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^3}, \quad \Gamma_{tt}^\vartheta = -\frac{c^2 r_s a^2 r \sin \vartheta \cos \vartheta}{\Sigma^3}, \quad (2.14.5a)$$

$$\Gamma_{tr}^t = \frac{r_s (r^2 + a^2) (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^2 \Delta}, \quad \Gamma_{tr}^\varphi = \frac{c r_s a (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^2 \Delta}, \quad (2.14.5b)$$

$$\Gamma_{t\vartheta}^t = -\frac{r_s a^2 r \sin \vartheta \cos \vartheta}{\Sigma^2}, \quad \Gamma_{t\vartheta}^\varphi = -\frac{c r_s a r \cot \vartheta}{\Sigma^2}, \quad (2.14.5c)$$

$$\Gamma_{t\varphi}^r = -\frac{c \Delta r_s a \sin^2 \vartheta (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^3}, \quad \Gamma_{t\varphi}^\vartheta = \frac{c r_s a r (r^2 + a^2) \sin \vartheta \cos \vartheta}{\Sigma^3}, \quad (2.14.5d)$$

$$\Gamma_{rr}^r = \frac{2 r a^2 \sin^2 \vartheta - r_s (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma \Delta}, \quad \Gamma_{rr}^\vartheta = \frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma \Delta}, \quad (2.14.5e)$$

$$\Gamma_{r\vartheta}^r = -\frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{r}{\Sigma}, \quad (2.14.5f)$$

$$\Gamma_{\vartheta\vartheta}^r = -\frac{r \Delta}{\Sigma}, \quad \Gamma_{\vartheta\vartheta}^\vartheta = -\frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma}, \quad (2.14.5g)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \frac{\cot \vartheta}{\Sigma^2} [\Sigma^2 + r_s a^2 r \sin^2 \vartheta], \quad \Gamma_{\vartheta\varphi}^t = \frac{r_s a^3 r \sin^3 \vartheta \cos \vartheta}{c \Sigma^2}, \quad (2.14.5h)$$

$$\Gamma_{r\varphi}^t = \frac{r_s a \sin^2 \vartheta [a^2 \cos^2 \vartheta (a^2 - r^2) - r^2 (a^2 + 3r^2)]}{2c \Sigma^2 \Delta}, \quad (2.14.5i)$$

$$\Gamma_{r\varphi}^\varphi = \frac{2r \Sigma^2 + r_s [a^4 \sin^2 \vartheta \cos^2 \vartheta - r^2 (\Sigma + r^2 + a^2)]}{2\Sigma^2 \Delta}, \quad (2.14.5j)$$

$$\Gamma_{\varphi\varphi}^r = \frac{\Delta \sin^2 \vartheta}{2\Sigma^3} [-2r \Sigma^2 + r_s a^2 \sin^2 \vartheta (r^2 - a^2 \cos^2 \vartheta)], \quad (2.14.5k)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\frac{\sin \vartheta \cos \vartheta}{\Sigma^3} [A \Sigma + (r^2 + a^2) r_s a^2 r \sin^2 \vartheta], \quad (2.14.5l)$$

General local tetrad:

$$\mathbf{e}_{(0)} = \Gamma (\partial_t + \zeta \partial_\varphi), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad (2.14.6a)$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{\Gamma}{c} \left(\mp \frac{g_{t\varphi} + \zeta g_{\varphi\varphi}}{\sqrt{\Delta} \sin \vartheta} \partial_t \pm \frac{g_{tt} + \zeta g_{t\varphi}}{\sqrt{\Delta} \sin \vartheta} \partial_\varphi \right), \quad (2.14.6b)$$

where $-\Gamma^{-2} = g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}$,

$$\Gamma^{-2} = \left(1 - \frac{r_s r}{\Sigma}\right) + \frac{2r_s a r \sin^2 \vartheta}{\Sigma} \frac{\zeta}{c} - \left(r^2 + a^2 + \frac{r_s a^2 r \sin^2 \vartheta}{\Sigma}\right) \frac{\zeta^2}{c^2} \sin^2 \vartheta \quad (2.14.7)$$

Non-rotating local tetrad ($\zeta = \omega$):

$$\mathbf{e}_{(0)} = \sqrt{\frac{A}{\Sigma \Delta}} \left(\frac{1}{c} \partial_t + \omega \partial_\varphi \right), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \sqrt{\frac{\Sigma}{A}} \frac{1}{\sin \vartheta} \partial_\varphi, \quad (2.14.8)$$

where $\omega = -g_{t\varphi}/g_{\varphi\varphi} = r_s a r / A$.

Dual tetrad:

$$\theta^{(2)} = \sqrt{\frac{\Sigma \Delta}{A}} c dt, \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}} dr, \quad \theta^{(2)} = \sqrt{\Sigma} d\vartheta, \quad \theta^{(3)} = \sqrt{\frac{A}{\Sigma}} \sin \vartheta (d\varphi - \omega d\varphi). \quad (2.14.9)$$

The relation between the constants of motion E , L , Q , and μ (defined in Bardeen[[BPT72](#)]) and the initial direction v , compare Sec. ([1.4.5](#)), with respect to the LNRF reads ($c = 1$)

$$v^{(0)} = \sqrt{\frac{A}{\Sigma\Delta}} E - \frac{r_s r a}{\sqrt{A\Sigma\Delta}} L, \quad v^{(1)} = \sqrt{\frac{\Delta}{\Sigma}} p_r, \quad (2.14.10a)$$

$$v^{(2)} = \frac{1}{\sqrt{\Sigma}} \sqrt{Q - \cos^2 \vartheta \left[a^2 (\mu^2 - E^2) + \frac{L^2}{\sin^2 \vartheta} \right]}, \quad v^{(3)} = \sqrt{\frac{\Sigma}{A}} \frac{L}{\sin \vartheta}. \quad (2.14.10b)$$

Static local tetrad ($\zeta = 0$):

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1-r_s r/\Sigma}} \partial_t, \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad (2.14.11a)$$

$$\mathbf{e}_{(3)} = \pm \frac{r_s a \sin \vartheta}{c\sqrt{1-r_s r/\Sigma}\sqrt{\Delta\Sigma}} \partial_t \mp \frac{\sqrt{1-r_s r/\Sigma}}{\sqrt{\Delta} \sin \vartheta} \partial_\varphi. \quad (2.14.11b)$$

Photon orbits:

The direct(-) and retrograd(+) photon orbits have radius

$$r_{\text{po}} = r_s \left[1 + \cos \left(\frac{2}{3} \arccos \frac{\mp 2a}{r_s} \right) \right]. \quad (2.14.12)$$

Marginally stable timelike circular orbits

are defined via

$$r_{\text{ms}} = \frac{r_s}{2} \left(3 + Z_2 \mp \sqrt{(3 - Z_1)(2 + Z_1 + 2Z_2)} \right), \quad (2.14.13)$$

where

$$Z_1 = 1 + \left(1 - \frac{4a^2}{r_s^2} \right)^{1/3} \left[\left(1 + \frac{2a}{r_s} \right)^{1/3} + \left(1 - \frac{2a}{r_s} \right)^{1/3} \right], \quad (2.14.14a)$$

$$Z_2 = \sqrt{\frac{12a^2}{r_s^2} + Z_1^2}. \quad (2.14.14b)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. [1.8.4](#), for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = 0 \quad (2.14.15)$$

with the effective potential

$$V_{\text{eff}} = \frac{1}{2r^3} \left\{ h^2(r - r_s) + 2 \frac{ahk}{c} r_s - \frac{k^2}{c^2} [r^3 + a^2(r + r_s)] \right\} - \frac{\kappa c^2 \Delta}{r^2} \quad (2.14.16)$$

and the constants of motion

$$k = \left(1 - \frac{r_s}{r} \right) c^2 \dot{t} + \frac{cr_s a}{r} \dot{\varphi}, \quad h = \left(r^2 + a^2 + \frac{r_s a^2}{r} \right) \dot{\varphi} - \frac{cr_s a}{r} \dot{t}. \quad (2.14.17)$$

Further reading:

Boyer and Lindquist[[BL67](#)], Wilkins[[Wil72](#)], Brill[[BC66](#)].

2.15 Kottler spacetime

The Kottler spacetime is represented in spherical coordinates $(t, r, \vartheta, \varphi)$ by the line element[Per04]

$$ds^2 = -\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right)c^2 dt^2 + \frac{1}{1 - r_s/r - \Lambda r^2/3} dr^2 + r^2 d\Omega^2, \quad (2.15.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and Λ is the cosmological constant. If $\Lambda > 0$ the metric is also known as Schwarzschild-deSitter metric, whereas if $\Lambda < 0$ it is called Schwarzschild-anti-deSitter.

For the following, we define the two abbreviations

$$\alpha = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3} \quad \text{and} \quad \beta = \frac{r_s}{r} - \frac{2\Lambda}{3}r^2. \quad (2.15.2)$$

The critical points of the Kottler metric follow from the roots of the cubic equation $\alpha = 0$. These can be found by means of the parameters $p = -1/\Lambda$ and $q = 3r_s/(2\Lambda)$. If $\Lambda < 0$, we have only one real root

$$r_1 = \frac{2}{\sqrt{-\Lambda}} \sinh \left[\frac{1}{3} \operatorname{arsinh} \left(\frac{3r_s}{2} \sqrt{-\Lambda} \right) \right]. \quad (2.15.3)$$

If $\Lambda > 0$, we have to distinguish whether $D \equiv q^2 + p^3 = 9r_s^2/(4\Lambda^2) - \Lambda^{-3}$ is positive or negative. If $D > 0$, there is no real positive root. For $D < 0$, the two real positive roots read

$$r_{\pm} = \frac{2}{\sqrt{\Lambda}} \cos \left[\frac{\pi}{3} \pm \frac{1}{3} \arccos \left(\frac{3r_s}{2} \sqrt{\Lambda} \right) \right] \quad (2.15.4)$$

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 \alpha \beta}{2r}, \quad \Gamma_{tr}^t = \frac{\beta}{2r\alpha}, \quad \Gamma_{rr}^r = -\frac{\beta}{2r\alpha}, \quad (2.15.5a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -\alpha r, \quad (2.15.5b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -\alpha r \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.15.5c)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 (3r_s + \Lambda r^3)}{3r^3}, \quad R_{t\vartheta t\vartheta} = \frac{1}{2} c^2 \alpha \beta, \quad (2.15.6a)$$

$$R_{t\varphi t\varphi} = \frac{1}{2} c^2 \alpha \beta \sin^2 \vartheta, \quad R_{r\vartheta r\vartheta} = -\frac{\beta}{2\alpha}, \quad (2.15.6b)$$

$$R_{r\varphi r\varphi} = -\frac{\beta}{2\alpha} \sin^2 \vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = r \left(r_s + \frac{\Lambda r^3}{3} \right) \sin^2 \vartheta. \quad (2.15.6c)$$

Ricci-Tensor:

$$R_{tt} = -c^2 \alpha \Lambda, \quad R_{rr} = \frac{\Lambda}{\alpha}, \quad R_{\vartheta\vartheta} = \Lambda r^2, \quad R_{\varphi\varphi} = \Lambda r^2 \sin^2 \vartheta. \quad (2.15.7)$$

The Ricci scalar and the Kretschmann scalar read

$$\mathcal{R} = 4\Lambda, \quad \mathcal{K} = 12 \frac{r_s^2}{r^6} + \frac{8\Lambda^2}{3}. \quad (2.15.8)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2 r_s}{r^3}, \quad C_{t\vartheta t\vartheta} = \frac{c^2 \alpha r_s}{2r}, \quad C_{t\varphi t\varphi} = \frac{c^2 \alpha r_s \sin^2 \vartheta}{2r}, \quad (2.15.9a)$$

$$C_{r\vartheta r\vartheta} = -\frac{r_s}{2r\alpha}, \quad C_{r\varphi r\varphi} = -\frac{r_s \sin^2 \vartheta}{2r\alpha}, \quad C_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.15.9b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{\alpha}}\partial_t, \quad \mathbf{e}_{(r)} = \sqrt{\alpha}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\phi)} = \frac{1}{r\sin\vartheta}\partial_\phi. \quad (2.15.10)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{\alpha}dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{\alpha}}, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\phi)} = r\sin\vartheta d\phi. \quad (2.15.11)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s - \frac{2}{3}\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\phi)(r)(\phi)} = \frac{\sqrt{\alpha}}{r}, \quad \gamma_{(\phi)(\vartheta)(\phi)} = \frac{\cot\vartheta}{r}. \quad (2.15.12)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s - 2\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \quad (2.15.13)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\phi)(\vartheta)(\phi)} = -\frac{\Lambda r^3 + 3r_s}{3r^3}, \quad (2.15.14a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\phi)(t)(\phi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\phi)(r)(\phi)} = \frac{3r_s - 2\Lambda r^3}{6r^3}. \quad (2.15.14b)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\phi)(\vartheta)(\phi)} = -\frac{r_s}{r^3}, \quad (2.15.15a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\phi)(t)(\phi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\phi)(r)(\phi)} = \frac{r_s}{2r^3}. \quad (2.15.15b)$$

Embedding:

The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{r_s/r + \Lambda r^2/3}{1 - r_s/r - \Lambda r^2/3}}. \quad (2.15.16)$$

Euler-Lagrange:

The Euler-Lagrangian formalism[[Rin01](#)] yields the effective potential

$$V_{\text{eff}} = \frac{1}{2} \left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3} \right) \left(\frac{h^2}{r^2} - \kappa c^2 \right) \quad (2.15.17)$$

with the constants of motion $k = (1 - r_s/r - \Lambda r^2/3)c^2t$, $h = r^2\dot{\phi}$, and κ as in Eq. (1.8.2).

As in the Schwarzschild metric, the effective potential has only one extremum for null geodesics, the so called photon orbit at $r = \frac{3}{2}r_s$. For timelike geodesics, however, we have

$$\frac{dV_{\text{eff}}}{dr} = \frac{h^2(-6r + 9r_s) + c^2r^2(3r_s - 2r^3\Lambda)}{3r^4} \stackrel{!}{=} 0. \quad (2.15.18)$$

This polynomial of fifth order might have up to five extrema.

Further reading:

Kottler[[Kot18](#)], Weyl[[Wey19](#)], Hackmann[[HL08](#)], Cruz[[COV05](#)].

2.16 Morris-Thorne

The most simple wormhole geometry is represented by the metric of Morris and Thorne[MT88],

$$ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.16.1)$$

where b_0 is the throat radius and l is the proper radial coordinate; and $\{t \in \mathbb{R}, l \in \mathbb{R}, \vartheta \in (0, \pi), \varphi \in [0, 2\pi]\}$.

Christoffel symbols:

$$\Gamma_{l\vartheta}^\vartheta = \frac{l}{b_0^2 + l^2}, \quad \Gamma_{l\varphi}^\varphi = \frac{l}{b_0^2 + l^2}, \quad \Gamma_{\vartheta\vartheta}^l = -l, \quad (2.16.2a)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^l = -l \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.16.2b)$$

Partial derivatives

$$\Gamma_{l\vartheta,l}^\vartheta = -\frac{l^2 - b_0^2}{(b_0^2 + l^2)^2}, \quad \Gamma_{l\varphi,l}^\varphi = -\frac{l^2 - b_0^2}{(b_0^2 + l^2)^2}, \quad \Gamma_{\vartheta\vartheta,l}^l = -1, \quad (2.16.3a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,l}^l = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^l = -l \sin(2\vartheta), \quad (2.16.3b)$$

$$\Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta). \quad (2.16.3c)$$

Riemann-Tensor:

$$R_{l\vartheta l\vartheta} = -\frac{b_0^2}{b_0^2 + l^2}, \quad R_{l\varphi l\varphi} = -\frac{b_0^2 \sin^2 \vartheta}{b_0^2 + l^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = b_0^2 \sin^2 \vartheta. \quad (2.16.4)$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{ll} = -2 \frac{b_0^2}{(b_0^2 + l^2)^2}, \quad \mathcal{R} = -2 \frac{b_0^2}{(b_0^2 + l^2)^2}, \quad \mathcal{K} = \frac{12b_0^4}{(b_0^2 + l^2)^4}. \quad (2.16.5)$$

Weyl-Tensor:

$$C_{ttll} = -\frac{2}{3} \frac{c^2 b_0^2}{(b_0^2 + l^2)^2}, \quad C_{t\vartheta t\vartheta} = \frac{1}{3} \frac{c^2 b_0^2}{b_0^2 + l^2}, \quad C_{t\varphi t\varphi} = \frac{1}{3} \frac{c^2 b_0^2 \sin^2 \vartheta}{b_0^2 + l^2}, \quad (2.16.6a)$$

$$C_{l\vartheta l\vartheta} = -\frac{1}{3} \frac{b_0^2}{b_0^2 + l^2}, \quad C_{l\varphi l\varphi} = -\frac{1}{3} \frac{b_0^2 \sin^2 \vartheta}{b_0^2 + l^2}, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3} b_0^2 \sin^2 \vartheta. \quad (2.16.6b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(l)} = \partial_l, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\sqrt{b_0^2 + l^2}} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\sqrt{b_0^2 + l^2} \sin \vartheta} \partial_\varphi. \quad (2.16.7)$$

Dual tetrad

$$\theta^{(t)} = c dt, \quad \theta^{(l)} = dl, \quad \theta^{(\vartheta)} = \sqrt{b_0^2 + l^2} d\vartheta, \quad \theta^{(\varphi)} = \sqrt{b_0^2 + l^2} \sin \vartheta d\varphi. \quad (2.16.8)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{l}{b_0^2 + l^2}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{\sqrt{b_0^2 + l^2}}. \quad (2.16.9)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2l}{b_0^2 + l^2}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{\sqrt{b_0^2 + l^2}}. \quad (2.16.10)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(l)(\vartheta)(l)(\vartheta)} = R_{(l)(\phi)(l)(\phi)} = -R_{(\vartheta)(\phi)(\vartheta)(\phi)} = -\frac{b_0^2}{(b_0^2 + l^2)^2}. \quad (2.16.11)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(l)(l)} = -\frac{2b_0^2}{(b_0^2 + l^2)^2}. \quad (2.16.12)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(l)(t)(l)} = -C_{(\vartheta)(\phi)(\vartheta)(\phi)} = -\frac{2b_0^2}{3(b_0^2 + l^2)^2}, \quad (2.16.13a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\phi)(t)(\phi)} = -C_{(l)(\vartheta)(l)(\vartheta)} = -C_{(l)(\phi)(l)(\phi)} = \frac{b_0^2}{3(b_0^2 + l^2)^2}. \quad (2.16.13b)$$

Embedding:

The embedding function reads

$$z(r) = \pm b_0 \ln \left[\frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0} \right)^2 - 1} \right] \quad (2.16.14)$$

with $r^2 = b_0^2 + l^2$.

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{t}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(\frac{h^2}{b_0^2 + l^2} - \kappa c^2 \right), \quad (2.16.15)$$

with the constants of motion $k = c^2\dot{t}$ and $h = (b_0^2 + l^2)\dot{\phi}$. The shape of the effective potential V_{eff} is independent of the geodesic type. The maximum of the effective potential is located at $l = 0$.

A geodesic that starts at $l = l_i$ with direction $\mathbf{y} = \pm \mathbf{e}_{(t)} + \cos \xi \mathbf{e}_{(l)} + \sin \xi \mathbf{e}_{(\phi)}$ approaches the wormhole throat asymptotically for $\xi = \xi_{\text{crit}}$ with

$$\xi_{\text{crit}} = \arcsin \frac{b_0}{\sqrt{b_0^2 + l_i^2}}. \quad (2.16.16)$$

This critical angle is independent of the type of the geodesic.

Further reading:

Ellis[Ell73], Visser[Vis95], Müller[Mül04, Mül08a]

2.17 Oppenheimer-Snyder collapse

2.17.1 Outer metric

The metric of the outer spacetime, $R > R_b$, in comoving coordinates $(\tau, R, \vartheta, \phi)$ with $(c = 1)$ is given by

$$ds^2 = -d\tau^2 + \frac{R}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{2/3}} dR^2 + \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3} (d\vartheta^2 + \sin^2 \vartheta d\phi^2). \quad (2.17.1)$$

Christoffel symbols:

$$\Gamma_{\tau R}^R = \frac{1}{2} \frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad \Gamma_{\tau \vartheta}^\vartheta = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad (2.17.2a)$$

$$\Gamma_{\tau \phi}^\phi = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad \Gamma_{RR}^\tau = \frac{R\sqrt{r_s}}{2(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{5/3}}, \quad (2.17.2b)$$

$$\Gamma_{RR}^R = -\frac{3\sqrt{r_s}\tau}{4(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)R}, \quad \Gamma_{R\vartheta}^\vartheta = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad (2.17.2c)$$

$$\Gamma_{R\phi}^\phi = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad \Gamma_{\vartheta \vartheta}^\tau = -\sqrt{r_s} \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}, \quad (2.17.2d)$$

$$\Gamma_{\vartheta \vartheta}^R = -\frac{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}{\sqrt{R}}, \quad \Gamma_{\vartheta \phi}^\phi = \cot \vartheta, \quad (2.17.2e)$$

$$\Gamma_{\varphi \varphi}^\tau = -\sqrt{r_s} \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3} \sin^2 \vartheta, \quad \Gamma_{\vartheta \vartheta}^\vartheta = -\sin \vartheta \cos \vartheta, \quad (2.17.2f)$$

$$\Gamma_{\varphi \varphi}^R = -\frac{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau) \sin^2 \vartheta}{\sqrt{R}}. \quad (2.17.2g)$$

Riemann-Tensor:

$$R_{\tau R \tau R} = -\frac{R r_s}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{8/3}}, \quad R_{\tau \vartheta \tau \vartheta} = \frac{1}{2} \frac{r_s}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{2/3}}, \quad (2.17.3a)$$

$$R_{\tau \varphi \tau \varphi} = \frac{1}{2} \frac{r_s \sin^2 \vartheta}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{2/3}}, \quad R_{R \vartheta R \vartheta} = -\frac{1}{2} \frac{R r_s}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{4/3}}, \quad (2.17.3b)$$

$$R_{R \varphi R \varphi} = -\frac{1}{2} \frac{R r_s \sin^2 \vartheta}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{4/3}}, \quad R_{\vartheta \varphi \vartheta \varphi} = \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3} r_s \sin^2 \vartheta. \quad (2.17.3c)$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = 12 \frac{r_s^2}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^4}. \quad (2.17.4)$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(R)} = \frac{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{1/3}}{\sqrt{R}} \partial_R, \quad (2.17.5a)$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{2/3}} \partial_\vartheta, \quad \mathbf{e}_{(\phi)} = \frac{1}{(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau)^{2/3} \sin \vartheta} \partial_\phi. \quad (2.17.5b)$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = -\frac{\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{2\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad (2.17.6a)$$

$$\gamma_{(R)(\varphi)(\varphi)} = \gamma_{(R)(\vartheta)(\vartheta)} = -\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \quad (2.17.6b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad \gamma_{(R)} = 2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}, \quad \gamma_{(\vartheta)} = \cot\vartheta\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \quad (2.17.7)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{4r_s}{(2R^{3/2} - 3\sqrt{r_s}\tau)^2}, \quad (2.17.8a)$$

$$R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = -R_{(R)(\vartheta)(R)(\vartheta)} = -R_{(R)(\varphi)(R)(\varphi)} = \frac{2r_s}{(2R^{3/2} - 3\sqrt{r_s}\tau)^2}. \quad (2.17.8b)$$

The Ricci tensor with respect to the local tetrad vanishes identically.

2.17.2 Inner metric

The metric of the inside, $R \leq R_b$, reads

$$ds^2 = -d\tau^2 + \left(1 - \frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{4/3} [dR^2 + R^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)]. \quad (2.17.9)$$

For the following components, we define

$$A_{\text{Oin}} := 1 - \frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau. \quad (2.17.10)$$

Christoffel symbols:

$$\Gamma_{\tau R}^R = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma_{\tau\vartheta}^\vartheta = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma_{\tau\varphi}^\varphi = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad (2.17.11a)$$

$$\Gamma_{RR}^\tau = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}, \quad \Gamma_{R\vartheta}^\vartheta = \frac{1}{R}, \quad \Gamma_{R\varphi}^\varphi = \frac{1}{R}, \quad (2.17.11b)$$

$$\Gamma_{\vartheta\vartheta}^R = -R, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot\vartheta, \quad \Gamma_{\vartheta\vartheta}^\tau = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2, \quad (2.17.11c)$$

$$\Gamma_{\varphi\varphi}^R = -R\sin^2\vartheta, \quad \Gamma_{\varphi\vartheta}^\vartheta = -\sin\vartheta\cos\vartheta, \quad \Gamma_{\varphi\vartheta}^\tau = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2\sin^2\vartheta. \quad (2.17.11d)$$

Riemann-Tensor:

$$R_{\tau R \tau R} = \frac{1}{2} \frac{r_s}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\tau \vartheta \tau \vartheta} = \frac{1}{2} \frac{r_s R^2}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\tau \varphi \tau \varphi} = \frac{1}{2} \frac{r_s R^2 \sin^2\vartheta}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad (2.17.12a)$$

$$R_{R\varphi R\varphi} = \frac{r_s R^2 \sin^2\vartheta}{R_b^3} A_{\text{Oin}}^{2/3}, \quad R_{R\vartheta R\vartheta} = \frac{r_s R^2}{R_b^3} A_{\text{Oin}}^{2/3}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s R^4 \sin^2\vartheta}{R_b^3} A_{\text{Oin}}^{2/3}. \quad (2.17.12b)$$

Ricci-Tensor:

$$R_{\tau\tau} = \frac{3}{2} \frac{r_s}{R_b^3 A_{\text{Oin}}^2}, \quad R_{RR} = \frac{3}{2} \frac{r_s}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\vartheta\vartheta} = \frac{3}{2} \frac{r_s R^2}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\varphi\varphi} = \frac{3}{2} \frac{r_s R^2 \sin^2\vartheta}{R_b^3 A_{\text{Oin}}^{2/3}}. \quad (2.17.13)$$

The *Ricci* and *Kretschmann* scalars read:

$$\mathcal{R} = \frac{3r_s}{R_b^3 A_{\text{Oin}}^2}, \quad \mathcal{K} = 15 \frac{r_s^2}{R_b^6 A_{\text{Oin}}^4}. \quad (2.17.14)$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(R)} = \frac{1}{A_{\text{Oin}}^{2/3}} \partial_R, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{RA_{\text{Oin}}^{2/3}} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{A_{\text{Oin}}^{2/3} R \sin \vartheta} \partial_\varphi. \quad (2.17.15)$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{\sqrt{r_s} R_b^{-3/2}}{A_{\text{Oin}}}, \quad (2.17.16a)$$

$$\gamma_{(R)(\vartheta)(\vartheta)} = \gamma_{(R)(\varphi)(\varphi)} = -\frac{1}{RA_{\text{Oin}}^{2/3}}, \quad (2.17.16b)$$

$$\gamma_{(\vartheta)(\varphi)(\varphi)} = -\frac{\cot \vartheta}{RA_{\text{Oin}}^{2/3}}. \quad (2.17.16c)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s} R_b^{-3/2}}{A_{\text{Oin}}}, \quad \gamma_{(R)} = \frac{2}{RA_{\text{Oin}}^{2/3}}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{RA_{\text{Oin}}^{2/3}}. \quad (2.17.17)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = \frac{r_s R_b^{-3}}{2A_{\text{Oin}}^2}, \quad (2.17.18a)$$

$$R_{(R)(\vartheta)(R)(\vartheta)} = R_{(R)(\varphi)(R)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s R_b^{-3}}{A_{\text{Oin}}^2}. \quad (2.17.18b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\tau)(\tau)} = R_{(R)(R)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{3r_s R_b^{-3}}{2A_{\text{Oin}}^2}. \quad (2.17.19)$$

Further reading:

Oppenheimer and Snyder[OS39].

2.18 Petrov-Type D – Levi-Civita spacetimes

The Petrov type D static vacuum spacetimes AI-C are taken from Stephani et al.[SKM⁺03], Sec. 18.6, with the coordinate and parameter ranges given in "Exact solutions of the gravitational field equations" by Ehlers and Kundt [EK62].

2.18.1 Case AI

In spherical coordinates, $(t, r, \vartheta, \varphi)$, the metric is given by the line element

$$ds^2 = r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + \frac{r}{r-b} dr^2 - \frac{r-b}{r} dt^2. \quad (2.18.1)$$

This is the well known Schwarzschild solution if $b = r_s$, cf. Eq. (2.2.1). Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < \vartheta < \pi, \quad \varphi \in [0, 2\pi), \quad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{r}{r-b}} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \quad (2.18.2)$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{r-b}{r}} dt, \quad \theta^{(r)} = \sqrt{\frac{r}{r-b}} dr, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.18.3)$$

Effective potential:

With the Hamilton-Jacobi formalism it is possible to obtain an effective potential fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K \frac{r-b}{r^3} - \kappa \frac{r-b}{r} \quad (2.18.4)$$

and the constants of motion

$$C_0^2 = t^2 \left(\frac{r-b}{r} \right)^2, \quad (2.18.5a)$$

$$K = \dot{\vartheta}^2 r^4 + \dot{\varphi}^2 r^4 \sin^2 \vartheta. \quad (2.18.5b)$$

2.18.2 Case AII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 + \sinh^2 r d\varphi^2) + \frac{z}{b-z} dz^2 - \frac{b-z}{z} dt^2. \quad (2.18.6)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < r, \quad \varphi \in [0, 2\pi), \quad 0 < z < b.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{z}{b-z}} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{z \sinh r} \partial_\varphi, \quad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}} \partial_z. \quad (2.18.7)$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{b-z}{z}} dt, \quad \theta^{(r)} = z dr, \quad \theta^{(\varphi)} = z \sinh r d\varphi, \quad \theta^{(z)} = \sqrt{\frac{z}{b-z}} dz. \quad (2.18.8)$$

2.18.3 Case AIII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 + r^2 d\varphi^2) + zdz^2 - \frac{1}{z}dt^2. \quad (2.18.9)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < r, \quad \varphi \in [0, 2\pi), \quad 0 < z.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{z}\partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{zr}\partial_\varphi, \quad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}}\partial_z. \quad (2.18.10)$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{\sqrt{z}}dt, \quad \theta^{(r)} = zd\varphi, \quad \theta^{(\varphi)} = zrd\varphi, \quad \theta^{(z)} = \sqrt{z}dz. \quad (2.18.11)$$

2.18.4 Case BI

In spherical coordinates, the metric is given by the line element

$$ds^2 = r^2 (d\vartheta^2 - \sin^2 \vartheta dt^2) + \frac{r}{r-b}dr^2 + \frac{r-b}{r}d\varphi^2. \quad (2.18.12)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < \vartheta < \pi, \quad \varphi \in [0, 2\pi), \quad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{r \sin \vartheta}\partial_t, \quad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \sqrt{\frac{r}{r-b}}\partial_\varphi. \quad (2.18.13)$$

Dual tetrad:

$$\theta^{(t)} = r \sin \vartheta dt, \quad \theta^{(r)} = \sqrt{\frac{r}{r-b}}dr, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\varphi)} = \sqrt{\frac{r-b}{r}}d\varphi. \quad (2.18.14)$$

Effective potential:

With the Hamilton-Jacobi formalism, an effective potential for the radial coordinate can be calculated fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K \frac{r-b}{r^3} - \kappa \frac{r-b}{r} \quad (2.18.15)$$

and the constants of motion

$$C_0^2 = \dot{\varphi}^2 \left(\frac{r-b}{r} \right)^2, \quad (2.18.16a)$$

$$K = \dot{\vartheta}^2 r^4 - \dot{t}^2 r^4 \sin^2 \vartheta. \quad (2.18.16b)$$

Note that the metric is not spherically symmetric. Particles or light rays fall into one of the poles if they are not moving in the $\vartheta = \frac{\pi}{2}$ plane.

2.18.5 Case BII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 - \sinh^2 r dt^2) + \frac{z}{b-z} dz^2 + \frac{b-z}{z} d\varphi^2. \quad (2.18.17)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad \varphi \in [0, 2\pi), \quad 0 < z < b, \quad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{z \sinh r} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \sqrt{\frac{z}{b-z}} \partial_\varphi, \quad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}} \partial_z. \quad (2.18.18)$$

Dual tetrad:

$$\theta^{(t)} = z \sinh r dt, \quad \theta^{(r)} = z dr, \quad \theta^{(\varphi)} = \sqrt{\frac{b-z}{z}} d\varphi, \quad \theta^{(z)} = \sqrt{\frac{z}{b-z}} dz. \quad (2.18.19)$$

2.18.6 Case BIII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 - r^2 dt^2) + zdz^2 + \frac{1}{z} d\varphi^2. \quad (2.18.20)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad \varphi \in [0, 2\pi), \quad 0 < z, \quad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{zr} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \sqrt{z} \partial_\varphi, \quad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}} \partial_z. \quad (2.18.21)$$

Dual tetrad:

$$\theta^{(t)} = zr dt, \quad \theta^{(r)} = zdz, \quad \theta^{(\varphi)} = \frac{1}{\sqrt{z}} d\varphi, \quad \theta^{(z)} = \sqrt{z} dz. \quad (2.18.22)$$

2.18.7 Case C

The metric is given by the line element

$$ds^2 = \frac{1}{(x+y)^2} \left(\frac{1}{f(x)} dx^2 + f(x) d\varphi^2 - \frac{1}{f(-y)} dy^2 + f(-y) dt^2 \right) \quad (2.18.23)$$

with $f(u) := \pm(u^3 + au + b)$. Coordinates and parameters are restricted to

$$0 < x+y, \quad f(-y) > 0, \quad 0 > f(x).$$

Local tetrad:

$$\mathbf{e}_{(t)} = (x+y) \frac{1}{\sqrt{-y^3 - ay + b}} \partial_t, \quad \mathbf{e}_{(x)} = (x+y) \sqrt{x^3 + ax + b} \partial_x, \quad (2.18.24a)$$

$$\mathbf{e}_{(y)} = (x+y) \sqrt{-y^3 - ay + b} \partial_y, \quad \mathbf{e}_{(\varphi)} = (x+y) \frac{1}{\sqrt{x^3 + ax + b}} \partial_\varphi, \quad (2.18.24b)$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{x+y} \sqrt{-y^3 - ay + b} dt, \quad \theta^{(x)} = \frac{1}{x+y} \frac{1}{\sqrt{x^3 + ax + b}} dx, \quad (2.18.25a)$$

$$\theta^{(y)} = \frac{1}{x+y} \frac{1}{\sqrt{-y^3 - ay + b}} dy, \quad \theta^{(\varphi)} = \frac{1}{x+y} \sqrt{x^3 + ax + b} d\varphi, \quad (2.18.25b)$$

A coordinate change can eliminate the linear term in the polynom f generating a quadratic term instead. This brings the line element to the form

$$ds^2 = \frac{1}{A(x+y)^2} \left[\frac{1}{f(x)} dx^2 + f(x) dp^2 - \frac{1}{f(-y)} dy^2 + f(-y) dq^2 \right] \quad (2.18.26)$$

with $f(u) := \pm(-2mAu^3 - u^2 + 1)$ given in [PP01].

Furthermore, coordinates can be adapted to the boost-rotation symmetry with the line element in [PP01] from in [Bon83]

$$ds^2 = \frac{1}{z^2 - t^2} \left[e^\rho r^2 (z dt - t dz)^2 - e^\lambda (z dz - t dt)^2 \right] - e^\lambda dr^2 - r^2 e^{-\rho} d\varphi^2 \quad (2.18.27)$$

with

$$\begin{aligned} e^\rho &= \frac{R_3 + R + Z_3 - r^2}{4\alpha^2 (R_1 + R + Z_1 - r^2)}, \\ e^\lambda &= \frac{2\alpha^2 [R(R + R_1 + Z_1) - Z_1 r^2] [R_1 R_3 + (R + Z_1)(R + Z_3) - (Z_1 + Z_3)r^2]}{R_i R_3 [R(R + R_3 + Z_3) - Z_3 r^2]}, \\ R &= \frac{1}{2} (z^2 - t^2 + r^2), \\ R_i &= \sqrt{(R + Z_i)^2 - 2Z_i r^2}, \\ Z_i &= z_i - z_2, \\ \alpha^2 &= \frac{1}{4} \frac{m^2}{A^6 (z_2 - z_1)^2 (z_3 - z_1)^2}, \\ q &= \frac{1}{4\alpha^2}, \end{aligned}$$

and $z_3 < z_1 < z_2$ the roots of $2A^4 z^3 - A^2 z^2 + m^2$.

Local tetrad:

Case $z^2 - t^2 > 0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{z^2 - t^2}} (qze^{-\rho/2} \partial_t + te^{-\lambda/2} \partial_z), \quad \mathbf{e}_{(r)} = e^{-\lambda/2} \partial_r, \quad (2.18.28a)$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{z^2 - t^2}} (qte^{-\rho/2} \partial_t + ze^{-\lambda/2} \partial_z), \quad \mathbf{e}_{(\varphi)} = re^{\rho/2} \partial_\varphi. \quad (2.18.28b)$$

Case $z^2 - t^2 < 0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{t^2 - z^2}} (qte^{-\rho/2} \partial_t + ze^{-\lambda/2} \partial_z), \quad \mathbf{e}_{(r)} = e^{-\lambda/2} \partial_r, \quad (2.18.29a)$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{t^2 - z^2}} (qze^{-\rho/2} \partial_t + te^{-\lambda/2} \partial_z), \quad \mathbf{e}_{(\varphi)} = re^{\rho/2} \partial_\varphi. \quad (2.18.29b)$$

Dual tetrad:

Case $z^2 - t^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^\rho}{z^2 - t^2}} \frac{1}{q} (z dt + t dz), \quad \theta^{(r)} = e^\lambda dr, \quad (2.18.30a)$$

$$\theta^{(z)} = \sqrt{\frac{e^\lambda}{z^2 - t^2}} (t dt + z dz), \quad \theta^{(\varphi)} = \frac{1}{r e^\rho} d\varphi. \quad (2.18.30b)$$

Case $z^2 - t^2 < 0$:

$$\theta^{(t)} = \sqrt{\frac{e^\lambda}{t^2 - z^2}} (t dt + z dz), \quad \theta^{(r)} = e^\lambda dr, \quad (2.18.31a)$$

$$\theta^{(z)} = \sqrt{\frac{e^\rho}{t^2 - z^2}} \frac{1}{q} (z dt + t dz), \quad \theta^{(\varphi)} = \frac{1}{r e^\rho} d\varphi. \quad (2.18.31b)$$

2.19 Plane gravitational wave

W. Rindler described in [Rin01] an exact plane gravitational wave which is bounded between two planes. The metric of the so called 'sandwich wave' with $u := t - x$ reads

$$ds^2 = -dt^2 + dx^2 + p^2(u)dy^2 + q^2(u)dz^2. \quad (2.19.1)$$

The functions $p(u)$ and $q(u)$ are given by

$$p(u) := \begin{cases} p_0 = \text{const.} & u < -a \\ 1-u & 0 < u \\ L(u)e^{m(u)} & \text{else} \end{cases} \quad \text{and} \quad q(u) := \begin{cases} q_0 = \text{const.} & u < -a \\ 1-u & 0 < u \\ L(u)e^{-m(u)} & \text{else} \end{cases}, \quad (2.19.2)$$

where a is the longitudinal extension of the wave. The functions $L(u)$ and $m(u)$ are

$$L(u) = 1 - u + \frac{u^3}{a^2} + \frac{u^4}{2a^3}, \quad m(u) = \pm 2\sqrt{3} \int \sqrt{\frac{u^2 + au}{2a^3u - 2au^3 - u^4 - 2a^3}} du. \quad (2.19.3)$$

Christoffel symbols:

$$\Gamma_{ty}^y = -\Gamma_{xy}^y = \frac{1}{p} \frac{\partial p}{\partial u}, \quad \Gamma_{zz}^t = \Gamma_{zz}^x = q \frac{\partial q}{\partial u}, \quad \Gamma_{tz}^z = -\Gamma_{xz}^z = \frac{1}{q} \frac{\partial q}{\partial u}, \quad \Gamma_{yy}^t = \Gamma_{yy}^x = p \frac{\partial p}{\partial u}. \quad (2.19.4)$$

Riemann-Tensor:

$$R_{tyty} = R_{xyxy} = -R_{tyxy} = -p \frac{\partial^2 p}{\partial u^2}, \quad R_{tztz} = R_{xzxz} = -R_{tzxz} = -q \frac{\partial^2 q}{\partial u^2}. \quad (2.19.5)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \quad \mathbf{e}_{(x)} = \partial_x, \quad \mathbf{e}_{(y)} = \frac{1}{p} \partial_y, \quad \mathbf{e}_{(z)} = \frac{1}{q} \partial_z. \quad (2.19.6)$$

Dual tetrad:

$$\theta^{(t)} = dt, \quad \theta^{(x)} = dx, \quad \theta^{(y)} = pdy, \quad \theta^{(z)} = qdz. \quad (2.19.7)$$

2.20 Reissner-Nordstrøm

The Reissner-Nordstrøm black hole in spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ is defined by the metric[MTW73]

$$ds^2 = -A_{RN}c^2 dt^2 + A_{RN}^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.20.1)$$

where

$$A_{RN} = 1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2} \quad (2.20.2)$$

with $r_s = 2GM/c^2$, the charge Q , and $\rho = G/(\epsilon_0 c^4) \approx 9.33 \cdot 10^{-34}$. As in the Schwarzschild case, there is a true curvature singularity at $r = 0$. However, for $Q^2 < r_s^2/(4\rho)$ there are also two critical points at

$$r = \frac{r_s}{2} \pm \frac{r_s}{2} \sqrt{1 - \frac{4\rho Q^2}{r_s^2}}. \quad (2.20.3)$$

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{A_{RN}c^2(r_s r - 2\rho Q^2)}{2r^3}, \quad \Gamma_{tr}^r = \frac{r_s r - 2\rho Q^2}{2r^3 A_{RN}}, \quad \Gamma_{rr}^r = -\frac{r_s r - 2\rho Q^2}{2r^3 A_{RN}}, \quad (2.20.4a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -r A_{RN}, \quad (2.20.4b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -r A_{RN} \sin^2 \vartheta, \quad \Gamma_{\vartheta\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.20.4c)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2(r_s r - 3\rho Q^2)}{r^4}, \quad R_{t\vartheta t\vartheta} = \frac{A_{RN}c^2(r_s r - 2\rho Q^2)}{2r^2}, \quad (2.20.5a)$$

$$R_{t\varphi t\varphi} = \frac{A_{RN}c^2(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{RN}}, \quad (2.20.5b)$$

$$R_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2 A_{RN}}, \quad R_{\vartheta\varphi\vartheta\varphi} = (r_s r - \rho Q^2) \sin^2 \vartheta. \quad (2.20.5c)$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2 \rho Q^2 A_{RN}}{r^4}, \quad R_{rr} = -\frac{\rho Q^2}{r^4 A_{RN}}, \quad R_{\vartheta\vartheta} = \frac{\rho Q^2}{r^2}, \quad R_{\varphi\varphi} = \frac{\rho Q^2 \sin^2 \vartheta}{r^2}. \quad (2.20.6)$$

While the Ricci scalar vanishes identically, the Kretschmann scalar reads

$$\mathcal{K} = 4 \frac{3r_s^2 r^2 - 12r_s r \rho Q^2 + 14\rho^2 Q^4}{r^8}. \quad (2.20.7)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(r_s r - 2\rho Q^2)}{r^4}, \quad C_{t\vartheta t\vartheta} = -\frac{A_{RN}c^2(r_s r - 2\rho Q^2)}{2r^2}, \quad (2.20.8a)$$

$$C_{t\varphi t\varphi} = \frac{A_{RN}c^2(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2}, \quad C_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{RN}}, \quad (2.20.8b)$$

$$C_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2 A_{RN}}, \quad C_{\vartheta\varphi\vartheta\varphi} = (r_s r - 2\rho Q^2) \sin^2 \vartheta. \quad (2.20.8c)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{A_{RN}}}\partial_t, \quad \mathbf{e}_{(r)} = \sqrt{A_{RN}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\phi)} = \frac{1}{r\sin\vartheta}\partial_\phi. \quad (2.20.9)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{A_{RN}}dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{A_{RN}}}, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\phi)} = r \sin\vartheta d\phi. \quad (2.20.10)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{rr_s - 2\rho Q^2}{2r^3\sqrt{A_{RN}}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\phi)(r)(\phi)} = \frac{\sqrt{A_{RN}}}{r}, \quad \gamma_{(\phi)(\vartheta)(\phi)} = \frac{\cot\vartheta}{r}. \quad (2.20.11)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r^2 - 3rr_s + 2\rho Q^2}{2r^3\sqrt{A_{RN}}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \quad (2.20.12)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -\frac{r_s r - 3\rho Q^2}{r^4}, \quad R_{(\vartheta)(\phi)(\vartheta)(\phi)} = \frac{r_s r - \rho Q^2}{r^4}, \quad (2.20.13a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\phi)(t)(\phi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\phi)(r)(\phi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \quad (2.20.13b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\phi)(\phi)} = \frac{\rho Q^2}{r^4}. \quad (2.20.14)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\phi)(\vartheta)(\phi)} = -\frac{r_s r - 2\rho Q^2}{r^4}, \quad (2.20.15a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\phi)(t)(\phi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\phi)(r)(\phi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \quad (2.20.15b)$$

Embedding:

The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{1}{1 - r_s/r + \rho Q^2/r^2} - 1}. \quad (2.20.16)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \quad (2.20.17)$$

with constants of motion $k = A_{RN}c^2\dot{t}$ and $h = r^2\dot{\phi}$. For null geodesics, $\kappa = 0$, there are two extremal points

$$r_{\pm} = \frac{3}{4}r_s \left(1 \pm \sqrt{1 - \frac{32\rho Q^2}{9r_s^2}}\right), \quad (2.20.18)$$

where r_+ is a maximum and r_- a minimum.

Further reading:

Eiroa[ERT02]

2.21 de Sitter spacetime

The de Sitter spacetime with $\Lambda > 0$ is a solution of the Einstein field equations with constant curvature. A detailed discussion can be found for example in Hawking and Ellis[HE99]. Here, we use the coordinate transformations given by Bičák[BK01].

2.21.1 Standard coordinates

The de Sitter metric in standard coordinates $\{\tau \in \mathbb{R}, \chi \in [-\pi, \pi], \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads

$$ds^2 = -d\tau^2 + \alpha^2 \cosh^2 \frac{\tau}{\alpha} [d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \quad (2.21.1)$$

where $\alpha^2 = 3/\Lambda$.

Christoffel symbols:

$$\Gamma_{\tau\chi}^\chi = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \quad \Gamma_{\tau\vartheta}^\vartheta = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \quad \Gamma_{\tau\varphi}^\varphi = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \quad (2.21.2a)$$

$$\Gamma_{\chi\chi}^\tau = \alpha \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \quad \Gamma_{\chi\vartheta}^\vartheta = \cot \chi, \quad \Gamma_{\chi\varphi}^\varphi = \cot \chi, \quad (2.21.2b)$$

$$\Gamma_{\vartheta\vartheta}^\tau = \alpha \sin^2 \chi \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \quad \Gamma_{\vartheta\vartheta}^\chi = -\sin \chi \cos \chi, \quad \Gamma_{\vartheta\vartheta}^\varphi = \cot \vartheta, \quad (2.21.2c)$$

$$\Gamma_{\varphi\varphi}^\tau = \alpha \sin^2 \chi \sin^2 \vartheta \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \quad \Gamma_{\varphi\varphi}^\chi = -\sin^2 \vartheta \sin \chi \cos \chi, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.21.2d)$$

Riemann-Tensor:

$$R_{\tau\chi\tau\chi} = -\cosh^2 \frac{\tau}{\alpha}, \quad R_{\tau\vartheta\tau\vartheta} = -\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi, \quad (2.21.3a)$$

$$R_{\tau\varphi\tau\varphi} = -\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi \sin^2 \vartheta, \quad R_{\chi\vartheta\chi\vartheta} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha}\right)^2 \sin^2 \chi, \quad (2.21.3b)$$

$$R_{\chi\varphi\chi\varphi} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha}\right)^2 \sin^2 \chi \sin^2 \vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha}\right)^2 \sin^4 \chi \sin^2 \vartheta. \quad (2.21.3c)$$

Ricci-Tensor:

$$R_{\tau\tau} = -\frac{3}{\alpha^2}, \quad R_{\chi\chi} = 3 \cosh^2 \frac{\tau}{\alpha}, \quad R_{\vartheta\vartheta} = 3 \cosh^2 \frac{\tau}{\alpha} \sin^2 \chi, \quad R_{\varphi\varphi} = 3 \cosh^2 \frac{\tau}{\alpha} \sin^2 \chi \sin^2 \vartheta. \quad (2.21.4)$$

Ricci and Kretschmann scalars:

$$\mathcal{R} = \frac{12}{\alpha^2}, \quad \mathcal{K} = \frac{24}{\alpha^4}. \quad (2.21.5)$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(\chi)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha}} \partial_\chi, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha} \sin \chi} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha} \sin \chi \sin \vartheta} \partial_\varphi. \quad (2.21.6)$$

Dual tetrad:

$$\theta^{(\tau)} = d\tau, \quad \theta^{(\chi)} = \alpha \cosh \frac{\tau}{\alpha} d\chi, \quad \theta^{(\vartheta)} = \alpha \cosh \frac{\tau}{\alpha} \sin \chi d\vartheta, \quad \theta^{(\varphi)} = \alpha \cosh \frac{\tau}{\alpha} \sin \chi \sin \vartheta d\varphi. \quad (2.21.7)$$

2.21.2 Conformally Einstein coordinates

In conformally Einstein coordinates $\{\eta \in [0, \pi], \chi \in [-\pi, \pi], \vartheta \in [0, \pi], \varphi \in [0, 2\pi)\}$, the de Sitter metric reads

$$ds^2 = \frac{\alpha^2}{\sin^2 \eta} [-d\eta^2 + d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)]. \quad (2.21.8)$$

It follows from the standard form (2.21.1) by the transformation

$$\eta = 2 \arctan \left(e^{\tau/\alpha} \right). \quad (2.21.9)$$

2.21.3 Conformally flat coordinates

Conformally flat coordinates $\{T \in \mathbb{R}, r \in \mathbb{R}, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ follow from conformally Einstein coordinates by means of the transformations

$$T = \frac{\alpha \sin \eta}{\cos \chi + \cos \eta}, \quad r = \frac{\alpha \sin \chi}{\cos \chi + \cos \eta}, \quad \text{or} \quad \eta = \arctan \frac{2T\alpha}{\alpha^2 - T^2 + r^2}, \quad \chi = \arctan \frac{2r\alpha}{\alpha^2 + T^2 - r^2}. \quad (2.21.10)$$

For the transformation $(T, R) \rightarrow (\eta, \chi)$, we have to take care of the coordinate domains. In that case, if $\kappa^2 - T^2 + r^2 < 0$, we have to map $\eta \rightarrow \eta + \pi$. On the other hand, if $\kappa^2 + T^2 - r^2 < 0$, we have to consider the sign of r . If $r > 0$, then $\chi \rightarrow \chi + \pi$, otherwise $\chi \rightarrow \chi - \pi$.

The resulting metric reads

$$ds^2 = \frac{\alpha^2}{T^2} [-dT^2 + dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)]. \quad (2.21.11)$$

Note that we identify points $(r < 0, \vartheta, \varphi)$ with $(r > 0, \pi - \vartheta, \varphi - \pi)$.

Christoffel symbols:

$$\Gamma_{TT}^T = \Gamma_{Tr}^r = \Gamma_{T\vartheta}^\vartheta = \Gamma_{T\varphi}^\varphi = \Gamma_{rr}^T = -\frac{1}{T}, \quad \Gamma_{r\vartheta}^\vartheta = \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^T = -\frac{r^2}{T}, \quad \Gamma_{\vartheta\vartheta}^r = -r, \quad (2.21.12a)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^T = -\frac{r^2 \sin^2 \vartheta}{T}, \quad \Gamma_{\varphi\varphi}^r = -r \sin^2 \vartheta, \quad \Gamma_{\vartheta\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.21.12b)$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{\alpha^2}{T^4}, \quad R_{T\vartheta T\vartheta} = -\frac{\alpha^2 r^2}{T^4}, \quad R_{T\varphi T\varphi} = -\frac{\alpha^2 r^2 \sin^2 \vartheta}{T^4}, \quad (2.21.13a)$$

$$R_{r\vartheta r\vartheta} = \frac{\alpha^2 r^2}{T^4}, \quad R_{r\varphi r\varphi} = \frac{\alpha^2 r^2 \sin^2 \vartheta}{T^4}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{\alpha^2 r^4 \sin^2 \vartheta}{T^4}. \quad (2.21.13b)$$

Ricci-Tensor:

$$R_{TT} = -\frac{3}{T^2}, \quad R_{rr} = \frac{3}{T^2}, \quad R_{\vartheta\vartheta} = \frac{3r^2}{T^2}, \quad R_{\varphi\varphi} = \frac{3r^2 \sin^2 \vartheta}{T^2}. \quad (2.21.14)$$

The Ricci and Kretschmann scalar read:

$$\mathcal{R} = \frac{12}{\alpha^2}, \quad \mathcal{K} = \frac{24}{\alpha^4}. \quad (2.21.15)$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{T}{\alpha} \partial_T, \quad \mathbf{e}_{(r)} = \frac{T}{\alpha} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{T}{\alpha r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{T}{\alpha r \sin \vartheta} \partial_\varphi. \quad (2.21.16)$$

2.21.4 Static coordinates

The de Sitter metric in static spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads

$$ds^2 = - \left(1 - \frac{\Lambda}{3} r^2 \right) c^2 dt^2 + \left(1 - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.21.17)$$

It follows from the conformally Einstein form (2.21.8) by the transformations

$$t = \frac{\alpha}{2} \ln \frac{\cos \chi - \cos \eta}{\cos \chi + \cos \eta}, \quad r = \alpha \frac{\sin \chi}{\sin \eta}. \quad (2.21.18)$$

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{(\Lambda r^2 - 3)}{9} c^2 \Lambda r, \quad \Gamma_{tr}^t = \frac{\Lambda r}{\Lambda r^2 - 3}, \quad \Gamma_{rr}^r = \frac{\Lambda r}{3 - \Lambda r^2}, \quad (2.21.19a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\phi}^\phi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = \frac{(\Lambda r^2 - 3)r}{3}, \quad (2.21.19b)$$

$$\Gamma_{\vartheta\phi}^\phi = \cot(\vartheta), \quad \Gamma_{\phi\phi}^r = \frac{\Lambda r^2 - 3}{3} r \sin^2(\vartheta), \quad \Gamma_{\phi\phi}^\vartheta = -\sin(\vartheta) \cos(\vartheta). \quad (2.21.19c)$$

Riemann-Tensor:

$$R_{t\vartheta t\vartheta} = -\frac{\Lambda}{3} c^2, \quad R_{t\vartheta t\vartheta} = -\frac{3 - \Lambda r^2}{9} c^2 \Lambda r^2, \quad R_{t\vartheta t\vartheta} = -\frac{3 - \Lambda r^2}{9} c^2 \Lambda r^2 \sin(\vartheta)^2, \quad (2.21.20a)$$

$$R_{r\vartheta r\vartheta} = \frac{\Lambda r^2}{-\Lambda r^2 + 3}, \quad R_{r\vartheta r\vartheta} = \frac{\Lambda r^2 \sin(\theta)^2}{-\Lambda r^2 + 3}, \quad R_{r\vartheta r\vartheta} = \frac{r^4 \sin^2(\theta) \Lambda}{3}. \quad (2.21.20b)$$

Ricci-Tensor:

$$R_{tt} = \frac{\Lambda r^2 - 3}{3} c^2 \Lambda, \quad R_{rr} = \frac{3\Lambda}{3 - \Lambda r^2}, \quad R_{\vartheta\vartheta} = \Lambda r^2, \quad R_{\phi\phi} = r^2 \sin^2(\vartheta) \Lambda. \quad (2.21.21)$$

The Ricci scalar and Kretschmann scalar read:

$$\mathcal{R} = 4\Lambda, \quad \mathcal{K} = \frac{8}{3}\Lambda^2. \quad (2.21.22)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{3}{3 - \Lambda r^2}} \frac{\partial_t}{c}, \quad \mathbf{e}_{(r)} = \sqrt{1 - \frac{\Lambda r^2}{3}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\phi)} = \frac{1}{r \sin(\vartheta)} \partial_\phi. \quad (2.21.23)$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(t)} = -\frac{\Lambda r}{\sqrt{9 - 3\Lambda r^2}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\phi)(r)(\phi)} = \frac{\sqrt{9 - 3\Lambda r^2}}{3r}, \quad \gamma_{(\phi)(\vartheta)(\phi)} = \frac{\cot \vartheta}{r}. \quad (2.21.24)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{\sqrt{9 - 3\Lambda r^2} (\Lambda r^2 - 2)}{(\Lambda r^2 - 3)r}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.21.25)$$

Riemann-Tensor with respect to local tetrad:

$$-R_{(t)(r)(t)(r)} = -R_{(t)(\vartheta)(t)(\vartheta)} = -R_{(t)(\phi)(t)(\phi)} = R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\phi)(r)(\phi)} = R_{(\vartheta)(\phi)(\vartheta)(\phi)} = \frac{1}{3}\Lambda. \quad (2.21.26)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\phi)(\phi)} = \Lambda. \quad (2.21.27)$$

2.21.5 Lemaître-Robertson form

The de Sitter universe in the Lemaître-Robertson form reads

$$ds^2 = -c^2 dt^2 + e^{2Ht} [dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \quad (2.21.28)$$

with Hubble's Parameter $H = \sqrt{\frac{\Lambda c^2}{3}} = \frac{c}{\alpha}$, which is assumed here to be time-independent.

This is a special case of the first and second form of the Friedman-Robertson-Walker metric defined in Eqs. (2.9.2) and (2.9.12) with $R(t) = e^{Ht}$ and $k = 0$.

Christoffel symbols:

$$\Gamma_{tr}^r = H, \quad \Gamma_{t\vartheta}^\vartheta = H, \quad \Gamma_{t\varphi}^\varphi = H, \quad (2.21.29a)$$

$$\Gamma_{rr}^r = \frac{e^{2Ht} H}{c^2}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad (2.21.29b)$$

$$\Gamma_{\vartheta\vartheta}^r = \frac{e^{2Ht} r^2 H}{c^2}, \quad \Gamma_{\vartheta\vartheta}^r = -r, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot(\vartheta), \quad (2.21.29c)$$

$$\Gamma_{\varphi\varphi}^r = \frac{e^{2Ht} r^2 \sin^2(\theta) H}{c^2}, \quad \Gamma_{\varphi\varphi}^r = -r \sin(\vartheta)^2, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin(\vartheta) \cos(\vartheta). \quad (2.21.29d)$$

Riemann-Tensor:

$$R_{trtr} = -e^{2Ht} H^2, \quad R_{t\vartheta t\vartheta} = -e^{2Ht} r^2 H^2, \quad (2.21.30a)$$

$$R_{t\varphi t\varphi} = -e^{2Ht} r^2 \sin^2(\vartheta) H^2, \quad R_{r\vartheta r\vartheta} = \frac{e^{4Ht} r^2 H^2}{c^2}, \quad (2.21.30b)$$

$$R_{r\varphi r\varphi} = \frac{e^{4Ht} r^2 \sin^2(\vartheta) H^2}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{e^{4Ht} r^4 \sin^2(\vartheta) H^2}{c^2}. \quad (2.21.30c)$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \quad R_{rr} = 3\frac{e^{2Ht} H^2}{c^2}, \quad R_{\vartheta\vartheta} = 3\frac{e^{2Ht} r^2 H^2}{c^2}, \quad R_{\varphi\varphi} = 3\frac{e^{2Ht} r^2 \sin^2(\vartheta) H^2}{c^2}. \quad (2.21.31)$$

Ricci and Kretschmann scalars:

$$\mathcal{R} = \frac{12H^2}{c^2}, \quad \mathcal{K} = \frac{24H^4}{c^4}. \quad (2.21.32)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = e^{-Ht} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{e^{-Ht}}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{e^{-Ht}}{r \sin \vartheta} \partial_\varphi. \quad (2.21.33)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{H}{c} \quad (2.21.34a)$$

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{e^{Ht} r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\theta)}{e^{Ht} r}. \quad (2.21.34b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}, \quad \gamma_{(r)} = \frac{2}{e^{Ht} r}, \quad \gamma_{(\vartheta)} = \frac{\cot(\theta)}{e^{Ht} r}. \quad (2.21.35)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{H^2}{c^2} \quad (2.21.36a)$$

$$R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{H^2}{c^2}. \quad (2.21.36b)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\phi)(\phi)} = 3 \frac{H^2}{c^2}. \quad (2.21.37)$$

2.21.6 Cartesian coordinates

The de Sitter universe in Lemaître-Robertson form can also be expressed in Cartesian coordinates:

$$ds^2 = -c^2 dt^2 + e^{2Ht} [dx^2 + dy^2 + dz^2].$$

(2.21.38)

Christoffel symbols:

$$\Gamma_{tx}^x = H, \quad \Gamma_{ty}^y = H, \quad \Gamma_{tz}^z = H, \quad (2.21.39a)$$

$$\Gamma_{xx}^t = \frac{e^{2Ht} H}{c^2}, \quad \Gamma_{yy}^t = \frac{e^{2Ht} H}{c^2}, \quad \Gamma_{zz}^t = \frac{e^{2Ht} H}{c^2}. \quad (2.21.39b)$$

$$(2.21.39c)$$

Partial derivatives

$$\Gamma_{xx,t}^t = \Gamma_{yy,t}^t = \Gamma_{zz,t}^t = \frac{2H^2 e^{2Ht}}{c^2}. \quad (2.21.40)$$

Riemann-Tensor:

$$R_{txtx} = R_{txtx} = R_{tztz} = -e^{2Ht} H^2, \quad R_{xyxy} = R_{xzxz} = R_{yzyz} = \frac{e^{4Ht} H^2}{c^2}. \quad (2.21.41)$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \quad R_{xx} = R_{yy} = R_{zz} = 3 \frac{e^{2Ht} H^2}{c^2}. \quad (2.21.42)$$

The Ricci and Kretschmann scalar read:

$$\mathcal{R} = 12 \frac{H^2}{c^2}, \quad \mathcal{K} = 24 \frac{H^4}{c^4}. \quad (2.21.43)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(x)} = e^{-Ht} \partial_x, \quad \mathbf{e}_{(y)} = e^{-Ht} \partial_y, \quad \mathbf{e}_{(z)} = e^{-Ht} \partial_z. \quad (2.21.44)$$

Ricci rotation coefficients:

$$\gamma_{(x)(t)(x)} = \gamma_{(y)(t)(y)} = \gamma_{(z)(t)(z)} = \frac{H}{c}. \quad (2.21.45)$$

The only non-vanishing contraction of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3 \frac{H}{c}. \quad (2.21.46)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(t)(x)} = R_{(t)(y)(t)(y)} = R_{(t)(z)(t)(z)} = -\frac{H^2}{c^2}, \quad (2.21.47a)$$

$$R_{(x)(y)(x)(y)} = R_{(x)(z)(x)(z)} = R_{(y)(z)(y)(z)} = \frac{H^2}{c^2}. \quad (2.21.47b)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(x)(x)} = R_{(y)(y)} = R_{(z)(z)} = 3 \frac{H^2}{c^2}. \quad (2.21.48)$$

Further reading:

Tolman [[Tol34](#), sec. 142], Bičák [[BK01](#)]

2.22 Straight spinning string

The metric of a straight spinning string in cylindrical coordinates (t, ρ, φ, z) reads

$$ds^2 = -(c dt - ad\varphi)^2 + d\rho^2 + k^2 \rho^2 d\varphi^2 + dz^2, \quad (2.22.1)$$

where $a \in \mathbb{R}$ and $k > 0$ are two parameters, see Perlick[Per04].

Metric-Tensor:

$$g_{tt} = -c^2, \quad g_{t\varphi} = ac, \quad g_{\rho\rho} = g_{zz} = 1, \quad g_{\varphi\varphi} = k^2 \rho^2 - a^2. \quad (2.22.2)$$

Christoffel symbols:

$$\Gamma'_{\rho\varphi} = \frac{a}{c\rho}, \quad \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho}, \quad \Gamma^\rho_{\varphi\varphi} = -k^2 \rho. \quad (2.22.3)$$

Partial derivatives

$$\Gamma'_{\rho\varphi,\rho} = -\frac{\alpha}{c\rho^2}, \quad \Gamma^\varphi_{\rho\varphi,\rho} = -\frac{1}{\rho^2}, \quad \Gamma^\rho_{\varphi\varphi,\rho} = -k^2. \quad (2.22.4)$$

The Riemann-, Ricci-, and Weyl-tensors as well as the Ricci- and Kretschmann-scalar vanish identically.

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(1)} = \partial_\rho, \quad \mathbf{e}_{(2)} = \frac{1}{k\rho} \left(\frac{a}{c} \partial_t + \partial_\varphi \right), \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.22.5)$$

Dual tetrad:

$$\theta^{(0)} = c dt - ad\varphi, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = k\rho d\varphi, \quad \theta^{(3)} = dz. \quad (2.22.6)$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(2)(1)(2)} = \frac{1}{\rho}, \quad \gamma_{(0)} = \gamma_{(2)} = \gamma_{(3)} = 0, \quad \gamma_{(1)} = \frac{1}{\rho}. \quad (2.22.7)$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{k^2 \rho^2 - a^2}}{k\rho} \left(\frac{1}{c} \partial_t - \frac{a}{k^2 \rho^2 - a^2} \partial_\varphi \right), \quad \mathbf{e}_{(1)} = \partial_\rho, \quad (2.22.8a)$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{k^2 \rho^2 - a^2}} \partial_\varphi, \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.22.8b)$$

Dual tetrad:

$$\theta^{(0)} = \frac{k\rho}{\sqrt{k^2 \rho^2 - a^2}} c dt, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = \frac{ac dt}{\sqrt{k^2 \rho^2 - a^2}} + \sqrt{k^2 \rho^2 - a^2} d\varphi, \quad \theta^{(3)} = dz. \quad (2.22.9)$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(0)(1)(0)} = \frac{a^2}{\rho(k^2 \rho^2 - a^2)}, \quad \gamma_{(2)(1)(0)} = \gamma_{(0)(2)(1)} = \gamma_{(0)(1)(2)} = \frac{ak}{k^2 \rho^2 - a^2}, \quad (2.22.10a)$$

$$\gamma_{(2)(1)(2)} = \frac{k^2 \rho}{k^2 \rho^2 - a^2}, \quad (2.22.10b)$$

$$\gamma_{(1)} = \frac{1}{\rho}. \quad (2.22.10c)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{\rho}^2 + \frac{1}{k^2\rho^2} \left(h_2 - \frac{ah_1}{c} \right)^2 - \kappa c^2 = \frac{h_1^2}{c^2}, \quad (2.22.11)$$

with the constants of motion $h_1 = c(ct - a\phi)$ and $h_2 = a(ct - a\phi) + k^2\rho^2\dot{\phi}$.

The point of closest approach ρ_{pca} for a null geodesic that starts at $\rho = \rho_i$ with $\mathbf{y} = \pm\mathbf{e}_{(0)} + \cos\xi\mathbf{e}_{(1)} + \sin\xi\mathbf{e}_{(2)}$ with respect to the static tetrad is given by $\rho = \rho_i \sin\xi$. Hence, the ρ_{pca} is independent of a and k . The same is also true for timelike geodesics.

2.23 Sultana-Dyer spacetime

The Sultana-Dyer metric represents a black hole in the Einstein-de Sitter universe. In spherical coordinates (t, r, ϑ, ϕ) , the metric reads [SD05] ($G = c = 1$)

$$ds^2 = t^4 \left[\left(1 - \frac{2M}{r} \right) dt^2 - \frac{4M}{r} dt dr - \left(1 + \frac{2M}{r} \right) dr^2 - r^2 d\Omega^2 \right], \quad (2.23.1)$$

where M is the mass of the black hole and $\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\phi^2$ is the spherical surface element. Note that here, the signature of the metric is $\text{sign}(\mathbf{g}) = -2$.

Christoffel symbols:

$$\Gamma_{tt}^t = \frac{2(r^3 + 4M^2r + M^2t)}{tr^3}, \quad \Gamma_{tt}^r = \frac{M(r - 2M)(4r + t)}{tr^3}, \quad \Gamma_{tr}^t = \frac{M(r + 2M)(4r + t)}{tr^3}, \quad (2.23.2a)$$

$$\Gamma_{tr}^r = \frac{2(r^3 - 4M^2r - M^2t)}{tr^3}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{2}{t}, \quad \Gamma_{t\varphi}^\varphi = \frac{2}{t}, \quad (2.23.2b)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^t = \frac{2(r^2 + 2Mr - Mt)}{t}, \quad (2.23.2c)$$

$$\Gamma_{\vartheta\vartheta}^r = -\frac{4Mr + tr - 2Mt}{t}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta, \quad (2.23.2d)$$

$$\Gamma_{rr}^t = \frac{2(r^3 + 4Mr^2 + 4M^2r + M^2t + Mtr)}{tr^3}, \quad \Gamma_{rr}^r = -\frac{M(4r^2 + 8Mr + 2Mt + tr)}{tr^3}, \quad (2.23.2e)$$

$$\Gamma_{\varphi\varphi}^t = \frac{2(r^2 + 2Mr - Mt) \sin^2 \vartheta}{t}, \quad \Gamma_{\varphi\varphi}^r = -\frac{(4Mr + tr - 2Mt) \sin^2 \vartheta}{t}. \quad (2.23.2f)$$

Riemann-Tensor:

$$R_{trtr} = \frac{2t^2(-2Mr^2 - r^3 + Mt^2 + 2Mtr)}{r^3}, \quad (2.23.3a)$$

$$R_{r\vartheta t\vartheta} = -\frac{t^2(2r^4 + 16M^2r^2 + 4Mtr^2 - 4M^2r^2t + Mt^2r - 2M^2t^2)}{r^2}, \quad (2.23.3b)$$

$$R_{t\vartheta r\vartheta} = -\frac{2Mt^2(4r + t)(r^2 + 2Mr - Mt)}{r^2}, \quad (2.23.3c)$$

$$R_{r\varphi t\varphi} = -\frac{t^2 \sin^2 \vartheta (2r^4 + 16M^2r^2 + 4Mtr^2 - 4M^2r^2t + Mt^2r - 2M^2t^2)}{r^2}, \quad (2.23.3d)$$

$$R_{t\varphi r\varphi} = -\frac{2Mt^2 \sin^2 \vartheta (4r + t)(r^2 + 2Mr - Mt)}{r^2}, \quad (2.23.3e)$$

$$R_{r\vartheta r\vartheta} = -\frac{t^2(4r^4 + 16Mr^4 - 4M^2tr + 16M^2r^2 - 2M^2t^2 - Mt^2r)}{r^2}, \quad (2.23.3f)$$

$$R_{r\varphi r\varphi} = -\frac{t^2 \sin^2 \vartheta (4r^4 + 16Mr^4 - 4M^2tr + 16M^2r^2 - 2M^2t^2 - Mt^2r)}{r^2}, \quad (2.23.3g)$$

$$R_{\vartheta\varphi\vartheta\varphi} = -2t^2 r \sin^2 \vartheta (2r^3 + 4Mr^2 - 4Mtr + mt^2). \quad (2.23.3h)$$

Ricci-Tensor:

$$R_{tt} = \frac{2(3r^2 + 12M^2 + 2Mt)}{t^2 r^2}, \quad R_{tr} = \frac{4M(3r + t + 6M)}{t^2 r^2}, \quad (2.23.4a)$$

$$R_{rr} = \frac{2(3r^2 + 12Mr + 2Mt + 12M^2)}{t^2 r^2}, \quad R_{\vartheta\vartheta} = \frac{6(r^2 + 2Mr - 2Mt)}{t^2}, \quad (2.23.4b)$$

$$R_{\varphi\varphi} = \frac{6(r^2 + 2Mr - 2Mt) \sin^2 \vartheta}{t^2}. \quad (2.23.4c)$$

Ricci and Kretschmann scalars:

$$R = -\frac{12(r^2 + 2Mr - 2Mt)}{t^6 r^2}, \quad (2.23.5a)$$

$$\mathcal{K} = \frac{48(M^2 t^4 + 20M^2 r^4 + 20Mr^5 + 8M^2 r^2 t^2 - 4Mr^4 t - 16M^2 r^3 t + 5r^6)}{t^1 2 r^6}. \quad (2.23.5b)$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{1+2M/r}}{t^2} \partial_t - \frac{2M/r}{t^2 \sqrt{1+2M/r}} \partial_r, \quad \mathbf{e}_{(1)} = \frac{1}{t^2 \sqrt{1+2M/r}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r \sin \vartheta} \partial_\phi. \quad (2.23.6)$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{t^2 \sqrt{1-2M/r}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{2M/r}{t^2 \sqrt{1-2M/r}} \partial_t + \frac{\sqrt{1-2M/r}}{t^2} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r \sin \vartheta} \partial_\phi. \quad (2.23.7)$$

Further reading:

Sultana and Dyer [[SD05](#)].

2.24 TaubNUT

The TaubNUT metric in Boyer-Lindquist like spherical coordinates $(t, r, \vartheta, \varphi)$ reads[BCJ02] ($G = c = 1$)

$$ds^2 = -\frac{\Delta}{\Sigma} (dt + 2\ell \cos \vartheta d\varphi)^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right), \quad (2.24.1)$$

where $\Sigma = r^2 + \ell^2$ and $\Delta = r^2 - 2Mr - \ell^2$. Here, M is the mass of the black hole and ℓ the magnetic monopole strength.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{\Delta\rho}{\Sigma^3}, \quad \Gamma_{tr}^t = \frac{\rho}{\Delta\Sigma}, \quad \Gamma_{t\vartheta}^t = -2\ell^2 \cos \vartheta \frac{\Delta}{\Sigma^2}, \quad (2.24.2a)$$

$$\Gamma_{t\vartheta}^\varphi = \frac{\ell\Delta}{\Sigma^2 \sin \vartheta}, \quad \Gamma_{t\varphi}^r = \frac{2\ell\rho\Delta \cos \vartheta}{\Sigma^3}, \quad \Gamma_{t\varphi}^\vartheta = -\frac{\ell\Delta \sin \vartheta}{\Sigma^2}, \quad (2.24.2b)$$

$$\Gamma_{rr}^r = -\frac{\rho}{\Sigma\Delta}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{r}{\Sigma}, \quad \Gamma_{r\varphi}^\varphi = \frac{r}{\Sigma}, \quad \Gamma_{\vartheta\vartheta}^r = -\frac{r\Delta}{\Sigma}, \quad (2.24.2c)$$

$$\Gamma_{r\varphi}^t = \frac{-2\ell(r^3 - 3Mr^2 - 3r\ell^2 + M\ell^2) \cos \vartheta}{\Sigma\Delta}, \quad (2.24.2d)$$

$$\Gamma_{\vartheta\varphi}^r = -\frac{\ell [\cos^2 \vartheta (6r^2\ell^2 - 8\ell^2Mr - 3\ell^4 + r^4) + \Sigma^2]}{\Sigma^2 \sin \vartheta}, \quad (2.24.2e)$$

$$\Gamma_{\varphi\varphi}^r = \frac{\Delta}{\Sigma^3} \left[\cos^2 \vartheta \left(9r\ell^4 + 4\ell^2Mr^2 - 4\ell^4M + r^5 + 2r^3\ell^2 \right) - r\Sigma^2 \right], \quad (2.24.2f)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \frac{(4r^2\ell^2 - 4Mr\ell^2 - \ell^4 + r^4) \cot \vartheta}{\Sigma^2}, \quad (2.24.2g)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\frac{(6r^2\ell^2 - 8Mr\ell^2 - 3\ell^4 + r^4) \sin \vartheta \cos \vartheta}{\Sigma^2}, \quad (2.24.2h)$$

where $\rho = 2r\ell^2 + Mr^2 - M\ell^2$.

Static local tetrad:

$$\mathbf{e}_{(0)} = \sqrt{\frac{\Sigma}{\Delta}} \partial_t, \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = -\frac{2\ell \cot \vartheta}{\sqrt{\Sigma}} \partial_t + \frac{1}{\sqrt{\Sigma} \sin \vartheta} \partial_\varphi. \quad (2.24.3)$$

Dual tetrad:

$$\theta^{(0)} = \sqrt{\frac{\Delta}{\Sigma}} (dt + 2\ell \cos \vartheta d\varphi), \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}} dr, \quad \theta^{(2)} = \sqrt{\Sigma} d\vartheta, \quad \theta^{(3)} = \sqrt{\Sigma} \sin \vartheta d\varphi. \quad (2.24.4)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = \frac{1}{2} \frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \frac{\Delta}{\Sigma} \left(\frac{h^2}{\Sigma} - \kappa \right) \quad (2.24.5)$$

with the constants of motion $k = (\Delta/\Sigma)\dot{t}$ and $h = \Sigma\dot{\varphi}$. For null geodesics, we obtain a photon orbit at $r = r_{\text{po}}$ with

$$r_{\text{po}} = M + 2\sqrt{M^2 + \ell^2} \cos \left(\frac{1}{3} \arccos \frac{M}{\sqrt{M^2 + \ell^2}} \right) \quad (2.24.6)$$

Further reading:

Bini et al.[BCdMJ03].

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