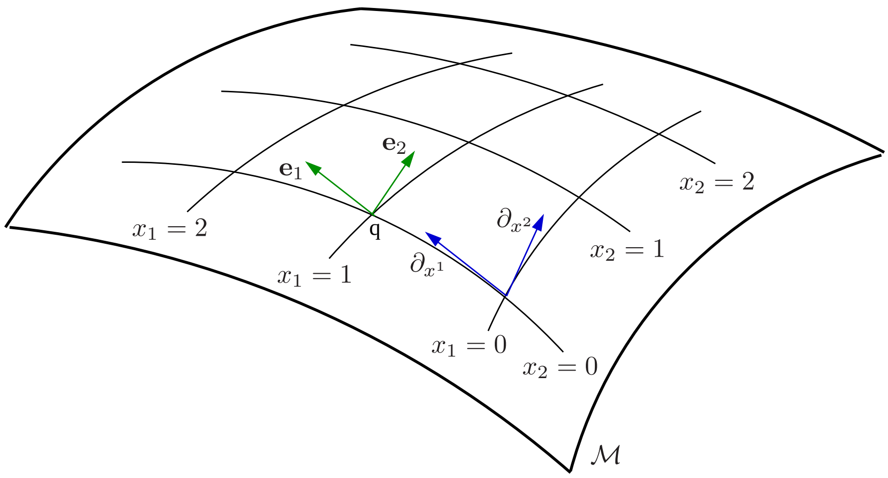
Catalogue of Spacetimes



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Chapter 1

Introduction and Notation

The *Catalogue of Spacetimes* is a collection of four-dimensional Lorentzian spacetimes in the context of the General Theory of Relativity (GR). The aim of the catalogue is to give a quick reference for students who need some basic facts of the most well-known spacetimes in GR. For a detailed discussion of a metric, the reader is referred to the standard literature or the original articles. Important resources for exact solutions are the book by Stephani et al[SKM+03] and the book by Griffiths and Podolský[GP09].

Most of the metrics in this catalogue are implemented in the Motion4D-library[MG09] and can be visualized using the GeodesicViewer[MG10]. Except for the Minkowski and Schwarzschild spacetimes, the metrics are sorted by their names.

1.1 Notation

The notation we use in this catalogue is as follows:

Indices: Coordinate indices are represented either by Greek letters or by coordinate names. Tetrad indices are indicated by Latin letters or coordinate names in brackets.

Einstein sum convention: When an index appears twice in a single term, once as lower index and once as upper index, we build the sum over all indices:

$$\zeta_\mu \zeta^\mu \equiv \sum_{\mu=0}^{3} \zeta_\mu \zeta^\mu. \tag{1.1.1}$$

Vectors: A coordinate vector in x^μ direction is represented as $\partial_{x^\mu} \equiv \partial_\mu$. For arbitrary vectors, we use boldface symbols. Hence, a vector $\mathbf{a}$ in coordinate representation reads $\mathbf{a} = a^\mu \partial_\mu$.

Derivatives: Partial derivatives are indicated by a comma, $\partial\psi/\partial x^\mu \equiv \partial_\mu \psi \equiv \psi_{,\mu}$, whereas covariant derivatives are indicated by a semicolon, $\nabla\psi = \psi_{;\mu}$.

Symmetrization and Antisymmetrization brackets:

$$a_{(\mu} b_{\nu)} = \frac{1}{2}\left(a_\mu b_\nu + a_\nu b_\mu\right), \qquad a_{[\mu} b_{\nu]} = \frac{1}{2}\left(a_\mu b_\nu - a_\nu b_\mu\right) \tag{1.1.2}$$

1.2 General remarks

The Einstein field equation in the most general form reads[MTW73]

$$G_{\mu\nu} = \varkappa T_{\mu\nu} - \Lambda g_{\mu\nu}, \qquad \varkappa = \frac{8\pi G}{c^4}, \tag{1.2.1}$$

R_{mu, nu}

with the symmetric and divergence-free Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R, the metric tensor $g_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$, the cosmological constant Λ, Newton's gravitational constant G, and the speed of light c. Because the Einstein tensor is divergence-free, the conservation equation ${T^{\mu\nu}}_{;\nu} = 0$ is automatically fulfilled.

the squared spacetime interval between two infinitesimally close events

A solution to the field equation is given by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.2.2}$$

with the symmetric, covariant metric tensor $g_{\mu\nu}$. The contravariant metric tensor $g^{\mu\nu}$ is related to the covariant tensor via $g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu$ with the Kronecker-δ. Even though $g_{\mu\nu}$ is only a component of the metric tensor $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$, we will also call $g_{\mu\nu}$ the metric tensor.

Note that, in this catalogue, we mostly use the convention that the signature of the metric is $+2$. In general, we will also keep the physical constants c and G within the metrics.

1.3 Basic objects of a metric

The basic objects of a metric are the Christoffel symbols, the Riemann and Ricci tensors as well as the Ricci and Kretschmann scalars which are defined as follows:

Christoffel symbols of the first kind:[1]

$$\Gamma_{\nu\lambda\mu} = \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right) \tag{1.3.1}$$

[g_{mu,lamda,nu]

with the relation

$$g_{\nu\lambda,\mu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} \tag{1.3.2}$$

Christoffel symbols of the second kind:

g^{mu, rho]

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \tag{1.3.3}$$

which are related to the Christoffel symbols of the first kind via

$$\Gamma^\mu_{\nu\lambda} = g^{\mu\rho}\Gamma_{\nu\lambda\rho} \tag{1.3.4}$$

Riemann tensor:

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho} \tag{1.3.5}$$

or

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma} = \Gamma_{\nu\sigma\mu,\rho} - \Gamma_{\nu\rho\mu,\sigma} + \Gamma^\lambda_{\nu\rho}\Gamma_{\mu\sigma\lambda} - \Gamma^\lambda_{\nu\sigma}\Gamma_{\mu\rho\lambda} \tag{1.3.6}$$

with symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \qquad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \qquad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \tag{1.3.7}$$

and

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \tag{1.3.8}$$

Ricci tensor:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R^\rho{}_{\mu\rho\nu} \tag{1.3.9}$$

Ricci and Kretschmann scalar:

$$\mathscr{R} = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu, \qquad \mathscr{K} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = R^{\gamma\delta}{}_{\alpha\beta} R^{\alpha\beta}{}_{\gamma\delta} \tag{1.3.10}$$

[1] The notation of the Christoffel symbols of the first kind differs from the one used by Rindler[Rin01], $\Gamma^{\text{Rindler}}_{\mu\nu\lambda} = \Gamma^{\text{CoS}}_{\nu\lambda\mu}$.

Weyl tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu} \tag{1.3.11}$$

If we change the signature of a metric, these basic objects transform as follows:

$$\Gamma^{\mu}_{\nu\lambda} \mapsto \Gamma^{\mu}_{\nu\lambda}, \qquad R_{\mu\nu\rho\sigma} \mapsto -R_{\mu\nu\rho\sigma}, \qquad C_{\mu\nu\rho\sigma} \mapsto -C_{\mu\nu\rho\sigma}, \tag{1.3.12a}$$

$$R_{\mu\nu} \mapsto R_{\mu\nu}, \qquad \mathscr{R} \mapsto -\mathscr{R}, \qquad \mathscr{K} \mapsto \mathscr{K}. \tag{1.3.12b}$$

Covariant derivative nabla/del - gradient, divergence

$$\nabla_{\lambda} g_{\mu\nu} = g_{\mu\nu;\lambda} = 0. \tag{1.3.13}$$

Covariant derivative of the vector field ψ^{μ}:

$$\nabla_{\nu}\psi^{\mu} = \psi^{\mu}_{;\nu} = \partial_{\nu}\psi^{\mu} + \Gamma^{\mu}_{\nu\lambda}\psi^{\lambda} \tag{1.3.14}$$

Covariant derivative of a r-s-tensor field:

$$\begin{aligned}\nabla_c T^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \partial_c T^{a_1\ldots a_r}{}_{b_1\ldots b_s} + \Gamma^{a_1}_{dc} T^{d\ldots a_r}{}_{b_1\ldots b_s} + \ldots + \Gamma^{a_r}_{dc} T^{a_1\ldots a_{r-1}d}{}_{b_1\ldots b_s} \\ - \Gamma^{d}_{b_1 c} T^{a_1\ldots a_r}{}_{d\ldots b_s} - \ldots - \Gamma^{d}_{b_s c} T^{a_1\ldots a_r}{}_{b_1\ldots b_{s-1}d}\end{aligned} \tag{1.3.15}$$

Killing equation:

tetrad - a set of four linearly independent vectors that form a local basis for the tangent space at each point in spacetime

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{1.3.16}$$

1.4 Natural local tetrad and initial conditions for geodesics

We will call a local tetrad natural if it is adapted to the symmetries or the coordinates of the spacetime. The four base vectors $\mathbf{e}_{(i)} = e^{\mu}_{(i)}\partial_{\mu}$ are given with respect to coordinate directions $\partial/\partial x^{\mu} = \partial_{\mu}$, compare Nakahara[Nak90] or Chandrasekhar[Cha06] for an introduction to the tetrad formalism. The inverse or dual tetrad is given by $\theta^{(i)} = \theta^{(i)}_{\mu} dx^{\mu}$ with

$$\theta^{(i)}_{\mu} e^{\mu}_{(j)} = \delta^{(i)}_{(j)} \qquad \text{and} \qquad \theta^{(i)}_{\mu} e^{\nu}_{(i)} = \delta^{\nu}_{\mu}. \tag{1.4.1}$$

Note that we us Latin indices in brackets for tetrads and Greek indices for coordinates.

1.4.1 Orthonormality condition

To be applicable as a local reference frame (Minkowski frame), a local tetrad $\mathbf{e}_{(i)}$ has to fulfill the orthonormality condition

$$\left\langle \mathbf{e}_{(i)}, \mathbf{e}_{(j)} \right\rangle_{\mathbf{g}} = \mathbf{g}\left(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right) = g_{\mu\nu} e^{\mu}_{(i)} e^{\nu}_{(j)} \stackrel{!}{=} \eta_{(i)(j)}, \tag{1.4.2}$$

where $\eta_{(i)(j)} = \mathrm{diag}(\mp 1, \pm 1, \pm 1, \pm 1)$ depending on the signature $\mathrm{sign}(\mathbf{g}) = \pm 2$ of the metric. Thus, the line element of a metric can be written as

$$ds^2 = \eta_{(i)(j)}\theta^{(i)}\theta^{(j)} = \eta_{(i)(j)}\theta^{(i)}_{\mu}\theta^{(j)}_{\nu} dx^{\mu} dx^{\nu}. \tag{1.4.3}$$

To obtain a local tetrad $\mathbf{e}_{(i)}$, we could first determine the dual tetrad $\theta^{(i)}$ via Eq. (1.4.3). If we combine all four dual tetrad vectors into one matrix Θ, we only have to determine its inverse Θ^{-1} to find the tetrad vectors,

$$\Theta = \begin{pmatrix} \theta^{(0)}_0 & \theta^{(0)}_1 & \theta^{(0)}_2 & \theta^{(0)}_3 \\ \theta^{(1)}_0 & \theta^{(1)}_1 & \theta^{(1)}_2 & \theta^{(1)}_3 \\ \theta^{(2)}_0 & \theta^{(2)}_1 & \theta^{(2)}_2 & \theta^{(2)}_3 \\ \theta^{(3)}_0 & \theta^{(3)}_1 & \theta^{(3)}_2 & \theta^{(3)}_3 \end{pmatrix} \quad \Rightarrow \quad \Theta^{-1} = \begin{pmatrix} e^0_{(0)} & e^0_{(1)} & e^0_{(2)} & e^0_{(3)} \\ e^1_{(0)} & e^1_{(1)} & e^1_{(2)} & e^1_{(3)} \\ e^2_{(0)} & e^2_{(1)} & e^2_{(2)} & e^2_{(3)} \\ e^3_{(0)} & e^3_{(1)} & e^3_{(2)} & e^3_{(3)} \end{pmatrix}. \tag{1.4.4}$$

There are also several useful relations:

$$e_{(a)\mu} = g_{\mu\nu} e^{\nu}_{(a)}, \qquad \eta_{(a)(b)} = e^{\mu}_{(a)} e_{(b)\mu}, \qquad e_{(b)\mu} = \eta_{(a)(b)} \theta^{(a)}_{\mu}, \tag{1.4.5a}$$

$$\theta^{(b)}_{\mu} = \eta^{(a)(b)} e_{(a)\mu}, \qquad g_{\mu\nu} = e_{(a)\mu} \theta^{(a)}_{\nu}, \qquad \eta^{(a)(b)} = \theta^{(a)}_{\mu} \theta^{(b)}_{\nu} g^{\mu\nu}. \tag{1.4.5b}$$

1.4.2 Tetrad transformations

Instead of the above found local tetrad that was directly constructed from the spacetime metric, we can also use any other local tetrad

$$\hat{\mathbf{e}}_{(i)} = A^{k}_{i} \mathbf{e}_{(k)}, \tag{1.4.6}$$

where $\mathbf{A}$ is an element of the Lorentz group $O(1,3)$. Hence $\mathbf{A}^T \eta \mathbf{A} = \eta$ and $(\det \mathbf{A})^2 = 1$.
Lorentz-transformation in the direction $n^a = (\sin\chi\cos\xi, \sin\chi\sin\xi, \cos\xi)^T = n_a$ with $\gamma = 1/\sqrt{1-\beta^2}$,

$$\Lambda^0_0 = \gamma, \qquad \Lambda^0_a = -\beta\gamma n_a, \qquad \Lambda^a_0 = -\beta\gamma n^a, \qquad \Lambda^a_b = (\gamma-1) n^a n_b + \delta^a_b. \tag{1.4.7}$$

1.4.3 Ricci rotation-, connection-, and structure coefficients

The Ricci rotation coefficients $\gamma_{(i)(j)(k)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\gamma_{(i)(j)(k)} := g_{\mu\lambda} e^{\mu}_{(i)} \nabla_{\mathbf{e}_{(k)}} e^{\lambda}_{(j)} = g_{\mu\lambda} e^{\mu}_{(i)} e^{\nu}_{(k)} \nabla_{\nu} e^{\lambda}_{(j)} = g_{\mu\lambda} e^{\mu}_{(i)} e^{\nu}_{(k)} \left(\partial_{\nu} e^{\lambda}_{(j)} + \Gamma^{\lambda}_{\nu\beta} e^{\beta}_{(j)} \right). \tag{1.4.8}$$

They are antisymmetric in the first two indices, $\gamma_{(i)(j)(k)} = -\gamma_{(j)(i)(k)}$, which follows from the definition, Eq. (1.4.8), and the relation

$$0 = \partial_{\mu} \eta_{(i)(j)} = \nabla_{\mu} \left(g_{\beta\nu} e^{\beta}_{(i)} e^{\nu}_{(j)} \right), \tag{1.4.9}$$

where $\nabla_{\mu} g_{\beta\nu} = 0$, compare [Cha06]. Otherwise, we have

$$\gamma^{(i)}{}_{(j)(k)} = \theta^{(i)}_{\lambda} e^{\nu}_{(k)} \nabla_{\nu} e^{\lambda}_{(j)} = -e^{\lambda}_{(j)} e^{\nu}_{(k)} \nabla_{\nu} \theta^{(i)}_{\lambda}. \tag{1.4.10}$$

The contraction of the first and the last index is given by

$$\gamma_{(j)} = \gamma^{(k)}{}_{(j)(k)} = \eta^{(k)(i)} \gamma_{(i)(j)(k)} = -\gamma_{(0)(j)(0)} + \gamma_{(1)(j)(1)} + \gamma_{(2)(j)(2)} + \gamma_{(3)(j)(3)} = \nabla_{\nu} e^{\nu}_{(j)}. \tag{1.4.11}$$

The connection coefficients $\omega^{(m)}_{(j)(n)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\omega^{(m)}_{(j)(n)} := \theta^{(m)}_{\mu} \nabla_{\mathbf{e}_{(j)}} e^{\mu}_{(n)} = \theta^{(m)}_{\mu} e^{\alpha}_{(j)} \nabla_{\alpha} e^{\mu}_{(n)} = \theta^{(m)}_{\mu} e^{\alpha}_{(j)} \left(\partial_{\alpha} e^{\mu}_{(n)} + \Gamma^{\mu}_{\alpha\beta} e^{\beta}_{(n)} \right), \tag{1.4.12}$$

compare Nakahara[Nak90]. They are related to the Ricci rotation coefficients via

$$\gamma_{(i)(j)(k)} = \eta_{(i)(m)} \omega^{(m)}_{(k)(j)}. \tag{1.4.13}$$

Furthermore, the local tetrad has a non-vanishing Lie-bracket $[X,Y]^{\nu} = X^{\mu} \partial_{\mu} Y^{\nu} - Y^{\mu} \partial_{\mu} X^{\nu}$. Thus,

$$\left[\mathbf{e}_{(i)}, \mathbf{e}_{(j)} \right] = c^{(k)}_{(i)(j)} \mathbf{e}_{(k)} \qquad \text{or} \qquad c^{(k)}_{(i)(j)} = \theta^{(k)} \left[\mathbf{e}_{(i)}, \mathbf{e}_{(j)} \right]. \tag{1.4.14}$$

The structure coefficients $c^{(k)}_{(i)(j)}$ are related to the connection coefficients or the Ricci rotation coefficients via

$$c^{(k)}_{(i)(j)} = \omega^{(k)}_{(i)(j)} - \omega^{(k)}_{(j)(i)} = \eta^{(k)(m)} \left(\gamma_{(m)(j)(i)} - \gamma_{(m)(i)(j)} \right) = \gamma^{(k)}{}_{(j)(i)} - \gamma^{(k)}{}_{(i)(j)}. \tag{1.4.15}$$

1.4.4 Riemann-, Ricci-, and Weyl-tensor with respect to a local tetrad

The transformations between the coordinate representations of the Riemann-, Ricci-, and Weyl-tensors and their representation with respect to a local tetrad $\mathbf{e}_{(i)}$ are given by

$$R_{(a)(b)(c)(d)} = R_{\mu\nu\rho\sigma} e^{\mu}_{(a)} e^{\nu}_{(b)} e^{\rho}_{(c)} e^{\sigma}_{(d)}, \tag{1.4.16a}$$

$$R_{(a)(b)} = R_{\mu\nu} e^{\mu}_{(a)} e^{\nu}_{(b)}, \tag{1.4.16b}$$

$$\begin{aligned} C_{(a)(b)(c)(d)} &= C_{\mu\nu\rho\sigma} e^{\mu}_{(a)} e^{\nu}_{(b)} e^{\rho}_{(c)} e^{\sigma}_{(d)} \\ &= R_{(a)(b)(c)(d)} - \frac{1}{2}\left(\eta_{(a)[(c)}R_{(d)](b)} - \eta_{(b)[(c)}R_{(d)](a)}\right) + \frac{R}{3}\eta_{(a)[(c)}\eta_{(d)](b)}. \end{aligned} \tag{1.4.16c}$$

1.4.5 Null or timelike directions

A null or timelike direction $\upsilon = \upsilon^{(i)}\mathbf{e}_{(i)}$ with respect to a local tetrad $\mathbf{e}_{(i)}$ can be written as

$$\upsilon = \upsilon^{(0)}\mathbf{e}_{(0)} + \psi\left(\sin\chi\cos\xi\,\mathbf{e}_{(1)} + \sin\chi\sin\xi\,\mathbf{e}_{(2)} + \cos\chi\,\mathbf{e}_{(3)}\right) = \upsilon^{(0)}\mathbf{e}_{(0)} + \psi\mathbf{n}. \tag{1.4.17}$$

In the case of a null direction we have $\psi = 1$ and $\upsilon^{(0)} = \pm 1$. A timelike direction can be identified with an initial four-velocity $\mathbf{u} = c\gamma(\mathbf{e}_0 + \beta\mathbf{n})$, where

$$\mathbf{u}^2 = \langle\mathbf{u},\mathbf{u}\rangle_{\mathbf{g}} = c^2\gamma^2\left\langle\mathbf{e}_{(0)} + \beta\mathbf{n}, \mathbf{e}_{(0)} + \beta\mathbf{n}\right\rangle = c^2\gamma^2\left(-1+\beta^2\right) = \mp c^2, \qquad \operatorname{sign}(\mathbf{g}) = \pm 2. \tag{1.4.18}$$

Thus, $\psi = c\beta\gamma$ and $\upsilon^0 = \pm c\gamma$. The sign of $\upsilon^{(0)}$ determines the time direction.

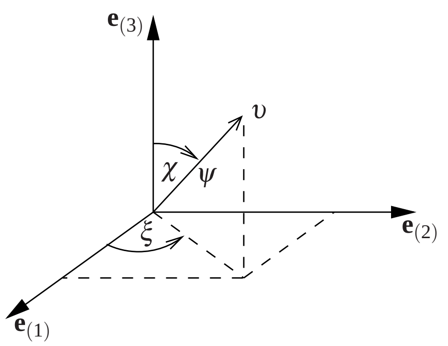

Figure 1.1: Null or timelike direction υ with respect to the local tetrad $\mathbf{e}_{(i)}$.

The transformations between a local direction $\upsilon^{(i)}$ and its coordinate representation υ^{μ} read

$$\upsilon^{\mu} = \upsilon^{(i)} e^{\mu}_{(i)} \qquad \text{and} \qquad \upsilon^{(i)} = \theta^{(i)}_{\mu}\upsilon^{\mu}. \tag{1.4.19}$$

1.4.6 Local tetrad for diagonal metrics

If a spacetime is represented by a diagonal metric

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \tag{1.4.20}$$

the natural local tetrad reads

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{g_{00}}}\partial_0, \quad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{11}}}\partial_1, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{22}}}\partial_2, \quad \mathbf{e}_{(3)} = \frac{1}{\sqrt{g_{33}}}\partial_3, \tag{1.4.21}$$

given that the metric coefficients are well behaved. Analogously, the dual tetrad reads

$$\theta^{(0)} = \sqrt{g_{00}}\,dx^0, \quad \theta^{(1)} = \sqrt{g_{11}}\,dx^1, \quad \theta^{(2)} = \sqrt{g_{22}}\,dx^2, \quad \theta^{(3)} = \sqrt{g_{33}}\,dx^3. \tag{1.4.22}$$

1.4.7 Local tetrad for stationary axisymmetric spacetimes

The line element of a stationary axisymmetric spacetime is given by

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}\,dt\,d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2, \tag{1.4.23}$$

where the metric components are functions of r and ϑ only.
The local tetrad for an observer on a stationary circular orbit, $(r = \text{const}, \vartheta = \text{const})$, with four velocity $\mathbf{u} = c\Gamma\left(\partial_t + \zeta\partial_\varphi\right)$ can be defined as, compare Bini[BJ00],

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{rr}}}\partial_r, \qquad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{\vartheta\vartheta}}}\partial_\vartheta, \tag{1.4.24a}$$

$$\mathbf{e}_{(3)} = \Delta\Gamma\left[\pm(g_{t\varphi} + \zeta g_{\varphi\varphi})\partial_t \mp (g_{tt} + \zeta g_{t\varphi})\partial_\varphi\right], \tag{1.4.24b}$$

where

$$\Gamma = \frac{1}{\sqrt{-\left(g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}\right)}} \qquad \text{and} \qquad \Delta = \frac{1}{\sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}. \tag{1.4.25}$$

The angular velocity ζ is limited due to $g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi} < 0$

$$\zeta_{\min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \qquad \text{and} \qquad \zeta_{\max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{1.4.26}$$

with $\omega = -g_{t\varphi}/g_{\varphi\varphi}$.
For $\zeta = 0$, the observer is static with respect to spatial infinity. The locally non-rotating frame (LNRF) has angular velocity $\zeta = \omega$, see also MTW[MTW73], exercise 33.3.
Static limit: $\zeta_{\min} = 0 \quad \Rightarrow \quad g_{tt} = 0$.
The transformation between the local direction $\upsilon^{(i)}$ and the coordinate direction υ^μ reads

$$\upsilon^0 = \Gamma\left(\upsilon^{(0)} \pm \upsilon^{(3)}\Delta w_1\right), \qquad \upsilon^1 = \frac{\upsilon^{(1)}}{\sqrt{g_{rr}}}, \qquad \upsilon^2 = \frac{\upsilon^{(2)}}{\sqrt{g_{\vartheta\vartheta}}}, \qquad \upsilon^3 = \Gamma\left(\upsilon^{(0)}\zeta \mp \upsilon^{(3)}\Delta w_2\right), \tag{1.4.27}$$

with

$$w_1 = g_{t\varphi} + \zeta g_{\varphi\varphi} \qquad \text{and} \qquad w_2 = g_{tt} + \zeta g_{t\varphi}. \tag{1.4.28}$$

The back transformation reads

$$\upsilon^{(0)} = \frac{1}{\Gamma}\frac{\upsilon^0 w_2 + \upsilon^3 w_1}{\zeta w_1 + w_2}, \qquad \upsilon^{(1)} = \sqrt{g_{rr}}\,\upsilon^1, \quad \upsilon^{(2)} = \sqrt{g_{\vartheta\vartheta}}\,\upsilon^2, \qquad \upsilon^{(3)} = \pm\frac{1}{\Delta\Gamma}\frac{\zeta\upsilon^0 - \upsilon^3}{\zeta w_1 + w_2}. \tag{1.4.29}$$

Note, to obtain a right-handed local tetrad, $\det\left(e^\mu_{(i)}\right) > 0$, the upper sign has to be used.

1.5 Newman-Penrose tetrad and spin-coefficients

The Newman-Penrose tetrad consists of four null vectors $\mathbf{e}^\star_{(i)} = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$, where $\mathbf{l}$ and $\mathbf{n}$ are real and $\mathbf{m}$ and $\overline{\mathbf{m}}$ are complex conjugates; see Penrose and Rindler[PR84] or Chandrasekhar[Cha06] for a thorough discussion. The Newman-Penrose (NP) tetrad has to fulfill the orthonormality relation

$$\left\langle \mathbf{e}^\star_{(i)}, \mathbf{e}^\star_{(j)} \right\rangle = \eta^\star_{(i)(j)} \qquad \text{with} \qquad \eta^\star_{(i)(j)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1.5.1}$$

A straightforward relation between the NP tetrad and the natural local tetrad, as discussed in Sec. 1.4, is given by

$$\mathbf{l} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_{(0)} + \mathbf{e}_{(1)}\right), \quad \mathbf{n} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_{(0)} - \mathbf{e}_{(1)}\right), \quad \mathbf{m} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_{(2)} + i\mathbf{e}_{(3)}\right), \tag{1.5.2}$$

where the upper/lower sign has to be used for metrics with positive/negative signature. The Ricci rotation-coefficients of a NP tetrad are now called *spin coefficients* and are designated by specific symbols:

$$\kappa = \gamma_{(2)(1)(1)}, \qquad \rho = \gamma_{(2)(0)(3)}, \qquad \varepsilon = \frac{1}{2}\left(\gamma_{(1)(0)(0)} + \gamma_{(2)(3)(0)}\right), \tag{1.5.3a}$$

$$\sigma = \gamma_{(2)(0)(2)}, \qquad \mu = \gamma_{(1)(3)(2)}, \qquad \gamma = \frac{1}{2}\left(\gamma_{(1)(0)(1)} + \gamma_{(2)(3)(1)}\right), \tag{1.5.3b}$$

$$\lambda = \gamma_{(1)(3)(3)}, \qquad \tau = \gamma_{(2)(0)(1)}, \qquad \alpha = \frac{1}{2}\left(\gamma_{(1)(0)(3)} + \gamma_{(2)(3)(3)}\right), \tag{1.5.3c}$$

$$\nu = \gamma_{(1)(3)(1)}, \qquad \pi = \gamma_{(1)(3)(0)}, \qquad \beta = \frac{1}{2}\left(\gamma_{(1)(0)(2)} + \gamma_{(2)(3)(2)}\right). \tag{1.5.3d}$$

1.6 Coordinate relations

1.6.1 Spherical and Cartesian coordinates

The well-known relation between the spherical coordinates (r, ϑ, φ) and the Cartesian coordinates (x, y, z), compare Fig. 1.2, are

$$x = r\sin\vartheta\cos\varphi, \qquad y = r\sin\vartheta\sin\varphi, \qquad z = r\cos\vartheta, \tag{1.6.1}$$

and

$$r = \sqrt{x^2+y^2+z^2}, \qquad \vartheta = \arctan 2(\sqrt{x^2+y^2}, z), \qquad \varphi = \arctan 2(y, x), \tag{1.6.2}$$

where $\arctan 2()$ ensures that $\varphi \in [0, 2\pi)$ and $\vartheta \in (0, \pi)$.

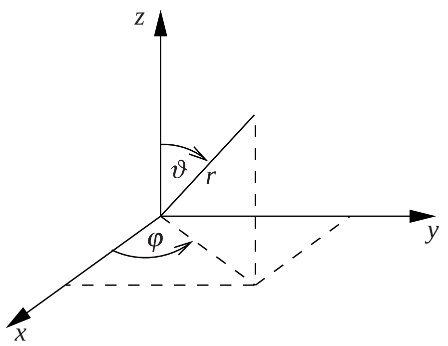

Figure 1.2: Relation between spherical and Cartesian coordinates.

The total differentials of the spherical coordinates read

$$dr = \frac{x\,dx + y\,dy + z\,dz}{r}, \qquad d\vartheta = \frac{xz\,dx + yz\,dy - (x^2+y^2)dz}{r^2\sqrt{x^2+y^2}}, \qquad d\varphi = \frac{-y\,dx + x\,dy}{x^2+y^2}, \tag{1.6.3}$$

whereas the coordinate derivatives read

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z \quad = \sin\vartheta\cos\varphi\,\partial_x + \sin\vartheta\sin\varphi\,\partial_y + \cos\vartheta\,\partial_z, \tag{1.6.4a}$$

$$\partial_\vartheta = \frac{\partial x}{\partial \vartheta}\partial_x + \frac{\partial y}{\partial \vartheta}\partial_y + \frac{\partial z}{\partial \vartheta}\partial_z = r\cos\vartheta\cos\varphi\,\partial_x + r\cos\vartheta\sin\varphi\,\partial_y - r\sin\vartheta\,\partial_z, \tag{1.6.4b}$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z \quad = -r\sin\vartheta\sin\varphi\,\partial_x + r\sin\vartheta\cos\varphi\,\partial_y, \tag{1.6.4c}$$

and

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \vartheta}{\partial x}\partial_\vartheta + \frac{\partial \varphi}{\partial x}\partial_\varphi = \sin\vartheta\cos\varphi\,\partial_r + \frac{\cos\vartheta\cos\varphi}{r}\,\partial_\vartheta - \frac{\sin\varphi}{r\sin\vartheta}\partial_\varphi, \tag{1.6.5a}$$

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial \vartheta}{\partial y}\partial_\vartheta + \frac{\partial \varphi}{\partial y}\partial_\varphi = \sin\vartheta\sin\varphi\,\partial_r + \frac{\cos\vartheta\sin\varphi}{r}\,\partial_\vartheta + \frac{\cos\varphi}{r\sin\vartheta}\partial_\varphi, \tag{1.6.5b}$$

$$\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial \vartheta}{\partial z}\partial_\vartheta + \frac{\partial \varphi}{\partial z}\partial_\varphi = \cos\vartheta\,\partial_r - \frac{\sin\vartheta}{r}\,\partial_\vartheta. \tag{1.6.5c}$$

1.6.2 Cylindrical and Cartesian coordinates

The relation between cylindrical coordinates (r,φ,z) and Cartesian coordinates (x,y,z) is given by

$$x = r\cos\varphi, \quad y = r\sin\varphi, \qquad \text{and} \qquad r = \sqrt{x^2+y^2}, \quad \varphi = \operatorname{arctan2}(y,x), \tag{1.6.6}$$

where $\operatorname{arctan2}()$ again ensures that the angle $\varphi \in [0,2\pi)$.

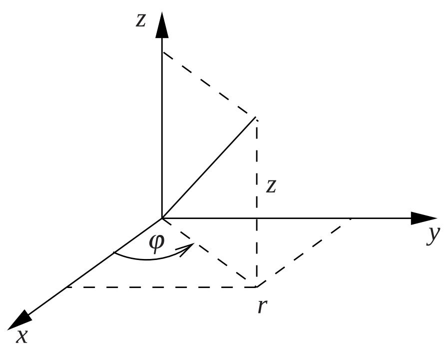

Figure 1.3: Relation between cylindrical and Cartesian coordinates.

The total differentials of the spherical coordinates are given by

$$dr = \frac{x\,dx + y\,dy}{r}, \qquad d\varphi = \frac{-y\,dx + x\,dy}{r^2}, \tag{1.6.7}$$

and

$$dx = \cos\varphi\,dr - r\sin\varphi\,d\varphi, \qquad dy = \sin\varphi\,dr + r\cos\varphi\,d\varphi. \tag{1.6.8}$$

The coordinate derivatives are

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y \quad = \cos\varphi\,\partial_x + \sin\varphi\,\partial_y, \tag{1.6.9a}$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y = -r\sin\varphi\,\partial_x + r\cos\varphi\,\partial_y m \tag{1.6.9b}$$

and

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \varphi}{\partial x}\partial_\varphi = \cos\varphi\,\partial_r - \frac{\sin\varphi}{r}\,\partial_y, \tag{1.6.10a}$$

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial \varphi}{\partial y}\partial_\varphi = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\,\partial_y. \tag{1.6.10b}$$

1.7 Embedding diagram

A two-dimensional hypersurface with line segment

$$d\sigma^2 = g_{rr}(r)dr^2 + g_{\varphi\varphi}(r)d\varphi^2 \tag{1.7.1}$$

can be embedded in a three-dimensional Euclidean space with cylindrical coordinates,

$$d\sigma^2 = \left[1 + \left(\frac{dz}{d\rho}\right)^2\right] d\rho^2 + \rho^2 d\varphi^2. \tag{1.7.2}$$

With $\rho(r)^2 = g_{\varphi\varphi}(r)$ and $dr = (dr/d\rho)d\rho$, we obtain for the embedding function $z = z(r)$,

$$\frac{dz}{dr} = \pm\sqrt{g_{rr} - \left(\frac{d\sqrt{g_{\varphi\varphi}}}{dr}\right)^2}. \tag{1.7.3}$$

If $g_{\varphi\varphi}(r) = r^2$, then $d\sqrt{g_{\varphi\varphi}}/dr = 1$.

1.8 Equations of motion and transport equations

1.8.1 Geodesic equation

The geodesic equation reads

$$\frac{D^2x^\mu}{d\lambda^2} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{1.8.1}$$

with the affine parameter λ. For timelike geodesics, however, we replace the affine parameter by the proper time τ.
The geodesic equation (1.8.1) is a system of ordinary differential equations of second order. Hence, to solve these differential equations, we need an initial position $x^\mu(\lambda = 0)$ as well as an initial direction $(dx^\mu/d\lambda)(\lambda = 0)$. This initial direction has to fulfill the constraint equation

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \kappa c^2, \tag{1.8.2}$$

where $\kappa = 0$ for lightlike and $\kappa = \mp 1$, $(\mathrm{sign}(\mathbf{g}) = \pm 2)$, for timelike geodesics.
The initial direction can also be determined by means of a local reference frame, compare sec. 1.4.5, that automatically fulfills the constraint equation (1.8.2). If we use the natural local tetrad as local reference frame, we have

$$\left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = \upsilon^\mu = \upsilon^{(i)}e^\mu_{(i)}. \tag{1.8.3}$$

1.8.2 Fermi-Walker transport

The Fermi-Walker transport, see e.g. Stephani[SS90], of a vector $\mathbf{X} = X^\mu\partial_\mu$ along the worldline $x^\mu(\tau)$ with four-velocity $\mathbf{u} = u^\mu(\tau)\partial_\mu$ is given by $\mathbb{F}_\mathbf{u}X^\mu = 0$ with

$$\mathbb{F}_\mathbf{u}X^\mu := \frac{dX^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}u^\rho X^\sigma + \frac{1}{c^2}\left(u^\sigma a^\mu - a^\sigma u^\mu\right)g_{\rho\sigma}X^\rho. \tag{1.8.4}$$

The four-acceleration follows from the four-velocity via

$$a^\mu = \frac{D^2x^\mu}{d\tau^2} = \frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}u^\rho u^\sigma. \tag{1.8.5}$$

1.8.3 Parallel transport

If the four-acceleration vanishes, the Fermi-Walker transport simplifies to the parallel transport $\mathbb{P}_\mathbf{u}X^\mu = 0$ with

$$\mathbb{P}_\mathbf{u}X^\mu := \frac{DX^\mu}{d\tau} = \frac{dX^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma}u^\rho X^\sigma. \tag{1.8.6}$$

1.8.4 Euler-Lagrange formalism

A detailed discussion of the Euler-Lagrange formalism can be found, e.g., in Rindler[Rin01]. The Lagrangian $\mathscr{L}$ is defined as

$$\mathscr{L} := g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \qquad \mathscr{L} \stackrel{!}{=} \kappa c^2, \tag{1.8.7}$$

where x^μ are the coordinates of the metric, and the dot means differentiation with respect to the affine parameter λ. For timelike geodesics, $\kappa = \mp 1$ depending on the signature of the metric, $\operatorname{sign}(\mathbf{g}) = \pm 2$. For lightlike geodesics, $\kappa = 0$.

The Euler-Lagrange equations read

$$\frac{d}{d\lambda}\frac{\partial\mathscr{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathscr{L}}{\partial x^\mu} = 0. \tag{1.8.8}$$

If $\mathscr{L}$ is independent of x^ρ, then x^ρ is a cyclic variable and

$$p_\rho = g_{\rho\nu}\dot{x}^\nu = \text{const}. \tag{1.8.9}$$

Note that $[\mathscr{L}]_{\text{U}} = \frac{\text{length}^2}{\text{time}^2}$ for timelike and $[\mathscr{L}]_{\text{U}} = 1$ for lightlike geodesics, see Sec. 1.9.

1.8.5 Hamilton formalism

The super-Hamiltonian $\mathscr{H}$ is defined as

$$\mathscr{H} := \frac{1}{2}g^{\mu\nu}p_\mu p_\nu, \qquad \mathscr{H} \stackrel{!}{=} \frac{1}{2}\kappa c^2, \tag{1.8.10}$$

where $p_\mu = g_{\mu\nu}\dot{x}^\nu$ are the canonical momenta, see e.g. MTW[MTW73], para. 21.1. As in classical mechanics, we have

$$\frac{dx^\mu}{d\lambda} = \frac{\partial\mathscr{H}}{\partial p_\mu} \qquad \text{and} \qquad \frac{dp_\mu}{d\lambda} = -\frac{\partial\mathscr{H}}{\partial x^\mu}. \tag{1.8.11}$$

1.9 Units

A first test in analyzing whether an equation is correct is to check the units. Newton's gravitational constant G, for example, has the following units

$$[G]_{\text{U}} = \frac{\text{length}^3}{\text{mass}\cdot\text{time}^2}, \tag{1.9.1}$$

where $[\cdot]_{\text{U}}$ indicates that we evaluate the units of the enclosed expression. Further examples are

$$[ds]_{\text{U}} = \text{length}, \qquad [\mathbf{u}]_{\text{U}} = \frac{\text{length}}{\text{time}}, \qquad [R^{\text{Schwarzschild}}_{trtr}]_{\text{U}} = \frac{1}{\text{time}^2}, \qquad \left[R^{\text{Schwarzschild}}_{\vartheta\varphi\vartheta\varphi}\right]_{\text{U}} = \text{length}^2. \tag{1.9.2}$$

1.10 Tools

1.10.1 Maple/GRTensorII

The Christoffel symbols, the Riemann- and Ricci-tensors as well as the Ricci and Kretschmann scalars in this catalogue were determined by means of the software Maple together with the GRTensorII package by Musgrave, Pollney, and Lake.[2]

A typical worksheet to enter a new metric may look like this:

[2]The commercial software Maple can be found here: http://www.maplesoft.com. The GRTensorII-package is free: http://grtensor.phy.queensu.ca.

```
> grtw();
> makeg(Schwarzschild);

Makeg 2.0: GRTensor metric/basis entry utility
To quit makeg, type 'exit' at any prompt.
Do you wish to enter a 1) metric [g(dn,dn)],
                       2) line element [ds],
                       3) non-holonomic basis [e(1)...e(n)], or
                       4) NP tetrad [l,n,m,mbar]?
> 2:

Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi]:

Enter the line element using d[coord] to indicate differentials.
(for example,  r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
ds^2 =

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
> {}:

You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Correct an element of the metric,
                  3) Re-enter the metric,
                  4) Add/change constraint equations,
                  5) Add a text description, or
                  6) Abandon this metric and return to Maple.
> 0:
```

The worksheets for some of the metrics in this catalogue can be found on the authors homepage. To determine the objects that are defined with respect to a local tetrad, the metric must be given as non-holonomic basis.
The various basic objects can be determined via

Christoffel symbols $\Gamma^{\mu}_{\nu\rho}$	`grcalc(Chr2);`	`grcalc(Chr(dn,dn,up));`
partial derivatives $\Gamma^{\mu}_{\nu\rho,\sigma}$		`grcalc(Chr(dn,dn,up,pdn));`
Riemann tensor $R_{\mu\nu\rho\sigma}$	`grcalc(Riemman);`	`grcalc(R(dn,dn,dn,dn));`
Ricci tensor $R_{\mu\nu}$	`grcalc(Ricci);`	`grcalc(R(dn,dn));`
Ricci scalar $\mathscr{R}$	`grcalc(Ricciscalar);`	
Kretschmann scalar $\mathscr{K}$	`grcalc(RiemSq);`	

1.10.2 Mathematica

The calculation of the Christoffel symbols, the Riemann- or Ricci-tensor within *Mathematica* could read like this:

```
Clearing the values of symbols:
    In[1]:=  Clear[coord, metric, inversemetric, affine,
                    t, r, Theta, Phi]

Setting the dimension:
    In[2]:=  n := 4

Defining a list of coordinates:
    In[3]:=  coord := {t, r, Theta, Phi}

Defining the metric:
    In[4]:=  metric := {{-(1 - rs/r) c^2, 0, 0, 0},
                        {0, 1/(1 - rs/r), 0, 0},
                        {0, 0, r^2, 0},
                        {0, 0, 0, r ^2 Sin[Theta]^2}}
    In[5]:=  metric // MatrixForm
```

```
Calculating the inverse metric:
    In[6]:=  inversemetric := Simplify[Inverse[metric]]

    In[7]:=  inversemetric // MatrixForm

Calculating the Christoffel symbols of the second kind:
    In[8]:=  affine :=  affine = Simplify[
      Table[(1/2) Sum[inversemetric[[Mu, Rho]] (
         D[metric[[Rho, Nu]], coord[[Lambda]]] +
         D[metric[[Rho, Lambda]], coord[[Nu]]] -
         D[metric[[Nu, Lambda]], coord[[Rho]]]),
      {Rho, 1, n}], {Nu, 1, n}, {Lambda, 1, n}, {Mu, 1, n}]]

Displaying the Christoffel symbols of the second kind:
    In[9]:=  listaffine :=
      Table[If[UnsameQ[affine[[Nu, Lambda, Mu]], 0],
        {Style[ Subsuperscript[\[CapitalGamma],
           Row[{coord[[Nu]], coord[[Lambda]]}], coord[[Mu]]], 18],
           "=",
           Style[affine[[Nu, Lambda, Mu]], 14]}],
        {Lambda, 1, n}, {Nu, 1, Lambda}, {Mu, 1, n}]

   In[10]:=  TableForm[Partition[DeleteCases[Flatten[listaffine],
                                                Null], 3],
                      TableSpacing -> {1, 2}]

Defining the Riemann tensor:
   In[11]:=  riemann :=  riemann =
    Table[D[affine[[Nu, Sigma, Mu]], coord[[Rho]]] -
          D[affine[[Nu, Rho, Mu]], coord[[Sigma]]] +
          Sum[affine[[Rho, Lambda, Mu]]
              affine[[Nu, Sigma, Lambda]] -
              affine[[Sigma, Lambda, Mu]]
              affine[[Nu, Rho, Lambda]],
            {Lambda, 1, n}],
    {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

Defining the Riemann tensor with lower indices:
   In[12]:= riemannDn :=  riemannDn =
     Table[Simplify[
        Sum[metric[[Mu, Kappa]] riemann[[Kappa, Nu, Rho, Sigma]],
        {Kappa, 1, n}]],
     {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

  In[13]:= listRiemann :=
     Table[If[UnsameQ[riemannDn[[Mu, Nu, Rho, Sigma]], 0],
    {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]], coord[[Rho]],
      coord[[Sigma]]}]], 16], "=",
      riemannDn[[Mu, Nu, Rho, Sigma]]}],
    {Nu, 1, n}, {Mu, 1, Nu}, {Sigma, 1, n}, {Rho, 1, Sigma}]

  In[14]:= TableForm[Partition[DeleteCases[Flatten[listRiemann],
                                              Null], 3],
                    TableSpacing -> {2, 2}]

Defining the Ricci tensor:
  In[15]:= ricci := ricci =
    Table[Simplify[
     Sum[riemann[[Rho, Mu, Rho, Nu]], {Rho, 1, n}]],
    {Mu, 1, n}, {Nu, 1, n}]

  In[16]:= listRicci :=
   Table[If[UnsameQ[ricci[[Mu, Nu]], 0],
      {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]]}]], 16],
      "=",
     Style[ricci[[Mu, Nu]], 16]}], {Nu, 1, 4}, {Mu, 1, Nu}]

  In[17]:= TableForm[Partition[DeleteCases[Flatten[listRicci],
                                              Null], 3],
                    TableSpacing -> {1, 2}]

Defining the Ricci scalar:
  In[18]:= ricciscalar :=  ricciscalar =
    Simplify[Sum[
```

```
      Sum[inversemetric[[Mu, Nu]] ricci[[Nu, Mu]],
      {Mu, 1, n}], {Nu, 1, n}]]

Defining the Kretschmann scalar:
  In[19]:= riemannUp :=  riemannUp =
    Table[Simplify[
     Sum[inversemetric[[Nu, Kappa]]
           riemann[[Mu, Kappa, Rho, Sigma]], {Kappa, 1, n}]],
    {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

  In[20]:= kretschmann := kretschmann =
     Simplify[Sum[ Sum[Sum[Sum[
        riemannUp[[Mu, Nu, Rho, Sigma]]
        riemannUp[[Rho, Sigma, Mu, Nu]],
       {Mu, 1, n}], {Nu, 1, n}], {Rho, 1, n}], {Sigma, 1, n}]]
```

Some example notebooks can be found on the authors homepage.

1.10.3 Maxima

Instead of using commercial software like *Maple* or *Mathematica,* Maxima also offers a tensor package that helps to calculate the Christoffel symbols etc. The above example for the Schwarzschild metric can be written as a maxima worksheet as follows:

```
/* load ctensor package */
load(ctensor);

/* define coordinates to use */
ct_coords:[t,r,theta,phi];

/* start with the identity metric */
lg:ident(4);
lg[1,1]:c^2*(1-rs/r);
lg[2,2]:-1/(1-rs/r);
lg[3,3]:-r^2;
lg[4,4]:-r^2*sin(theta)^2;
cmetric();

/* calculate the christoffel symbols of the second kind */
christof(mcs);

/* calculate the riemann tensor */
lriemann(mcs);

/* calculate the ricci tensor */
ricci(mcs);

/* calculate the ricci scalar */
scurvature();

/* calculate the Kretschmann scalar */
uriemann(mcs);
rinvariant();
ratsimp(%);
```

As you may have noticed, the Schwarzschild metric must be given with negative signature.

Chapter 2

Spacetimes

2.1 Minkowski

2.1.1 Cartesian coordinates

The Minkowski metric in Cartesian coordinates $\{t,x,y,z \in \mathbb{R}\}$ reads

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.} \tag{2.1.1}$$

All Christoffel symbols as well as the Riemann- and Ricci-tensor vanish identically. The natural local tetrad is trivial,

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(x)} = \partial_x, \qquad \mathbf{e}_{(y)} = \partial_y, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.2}$$

with dual

$$\theta^{(t)} = c\,dt, \qquad \theta^{(x)} = dx, \qquad \theta^{(y)} = dy, \qquad \theta^{(z)} = dz. \tag{2.1.3}$$

2.1.2 Cylindrical coordinates

The Minkowski metric in cylindrical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0,2\pi), z \in \mathbb{R}\}$,

$$\boxed{ds^2 = -c^2dt^2 + dr^2 + r^2d\varphi^2 + dz^2,} \tag{2.1.4}$$

has the natural local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.5}$$

Christoffel symbols:

$$\Gamma^r_{\varphi\varphi} = -r, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}. \tag{2.1.6}$$

Partial derivatives

$$\Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\varphi\varphi,r} = -1. \tag{2.1.7}$$

Ricci rotation coefficients:

$$\gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \qquad \text{and} \qquad \gamma_{(r)} = \frac{1}{r}. \tag{2.1.8}$$

2.1.3 Spherical coordinates

In spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$, the Minkowski metric reads

$$\boxed{ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).} \tag{2.1.9}$$

Christoffel symbols:

$$\Gamma^r_{\vartheta\vartheta} = -r, \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.1.10a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{2.1.10b}$$

Partial derivatives

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -1, \tag{2.1.11a}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.1.11b}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -\sin(2\vartheta). \tag{2.1.11c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.1.12}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.1.13}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.1.14}$$

2.1.4 Conform-compactified coordinates

The Minkowski metric in conform-compactified coordinates $\{\psi \in [-\pi,\pi], \xi \in (0,\pi), \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ reads[HE99]

$$\boxed{ds^2 = -d\psi^2 + d\xi^2 + \sin^2\xi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).} \tag{2.1.15}$$

This form follows from the spherical Minkowski metric (2.1.9) by means of the coordinate transformation

$$ct + r = \tan\frac{\psi+\xi}{2}, \qquad ct - r = \tan\frac{\psi-\xi}{2}, \tag{2.1.16}$$

resulting in the metric

$$d\tilde{s}^2 = \frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}} + \frac{\sin^2\xi}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{2.1.17}$$

and by the conformal transformation $ds^2 = \Omega^2 d\tilde{s}^2$ with $\Omega^2 = 4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}$.

Christoffel symbols:

$$\Gamma^\vartheta_{\xi\vartheta} = \cot\xi, \qquad \Gamma^\varphi_{\xi\varphi} = \cot\xi, \qquad \Gamma^\xi_{\vartheta\vartheta} = -\sin\xi\cos\xi, \tag{2.1.18a}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\xi_{\varphi\varphi} = -\sin\xi\cos\xi\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.1.18b}$$

Partial derivatives

$$\Gamma^{\vartheta}_{\xi\vartheta,\xi} = -\frac{1}{\sin^2\xi}, \qquad \Gamma^{\varphi}_{\xi\varphi,\xi} = -\frac{1}{\sin^2\xi}, \qquad \Gamma^{\xi}_{\vartheta\vartheta,\xi} = -\cos(2\xi), \tag{2.1.19a}$$

$$\Gamma^{\varphi}_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^{\xi}_{\varphi\varphi,\xi} = -\cos(2\xi)\sin^2\vartheta, \quad \Gamma^{\vartheta}_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.1.19b}$$

$$\Gamma^{\xi}_{\varphi\varphi,\vartheta} = -\frac{1}{2}\sin(2\xi)\sin(2\vartheta). \tag{2.1.19c}$$

Riemann-Tensor:

$$R_{\xi\vartheta\xi\vartheta} = \sin^2\xi, \qquad R_{\xi\varphi\xi\varphi} = \sin^2\xi\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = \sin^4\xi\sin^2\vartheta. \tag{2.1.20}$$

Ricci-Tensor:

$$R_{\xi\xi} = 2, \qquad R_{\vartheta\vartheta} = 2\sin^2\xi, \qquad R_{\varphi\varphi} = 2\sin^2\xi\sin^2\vartheta. \tag{2.1.21}$$

Ricci and Kretschmann scalars:

$$\mathscr{R} = 6, \qquad \mathscr{K} = 12. \tag{2.1.22}$$

The Weyl tensor vanishs identically.
Local tetrad:

$$\mathbf{e}_{(\psi)} = \partial_\psi, \qquad \mathbf{e}_{(\xi)} = \partial_\xi, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{\sin\xi}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\sin\xi\sin\vartheta}\partial_\varphi. \tag{2.1.23}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(\xi)(\vartheta)} = \gamma_{(\varphi)(\xi)(\varphi)} = \cot\xi, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sin\xi}. \tag{2.1.24}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\xi)} = 2\cot\xi, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sin\xi}. \tag{2.1.25}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\xi)(\vartheta)(\xi)(\vartheta)} = R_{(\xi)(\varphi)(\xi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = 1. \tag{2.1.26}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\xi)(\xi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 2. \tag{2.1.27}$$

2.1.5 Rotating coordinates

The transformation $d\varphi \mapsto d\varphi + \omega\, dt$ brings the Minkowski metric (2.1.4) into the rotating form[Rin01] with coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$\boxed{ds^2 = -\left(1 - \frac{\omega^2 r^2}{c^2}\right)[c\,dt - \Omega(r)d\varphi]^2 + dr^2 + \frac{r^2}{1-\omega^2 r^2/c^2}d\varphi^2 + dz^2} \tag{2.1.28}$$

with $\Omega(r) = (r^2\omega/c)/(1-\omega^2 r^2/c^2)$.
Metric-Tensor:

$$g_{tt} = -c^2 + \omega^2 r^2, \qquad g_{t\varphi} = \omega r^2, \qquad g_{rr} = g_{zz} = 1, \qquad g_{\varphi\varphi} = r^2. \tag{2.1.29}$$

Christoffel symbols:

$$\Gamma^r_{tt} = -\omega^2 r, \qquad \Gamma^\varphi_{tr} = \frac{\omega}{r}, \qquad \Gamma^r_{t\varphi} = -\omega r, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\varphi\varphi} = -r. \tag{2.1.30}$$

Partial derivatives

$$\Gamma^r_{tt,r} = -\omega^2, \quad \Gamma^\varphi_{tr,r} = -\frac{\omega}{r^2}, \quad \Gamma^r_{t\varphi,r} = -\omega, \quad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \quad \Gamma^r_{\varphi\varphi,r} = -1. \tag{2.1.31}$$

The local tetrad of the comoving observer is

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t - \frac{\omega}{c}\partial_\varphi, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.32}$$

whereas the static observer has the local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1-\omega^2 r^2/c^2}}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.33a}$$

$$\mathbf{e}_{(\varphi)} = \frac{\omega r}{c^2\sqrt{1-\omega^2 r^2/c^2}}\partial_t + \frac{\sqrt{1-\omega^2 r^2/c^2}}{r}\partial_\varphi. \tag{2.1.33b}$$

2.1.6 Rindler coordinates

The worldline of an observer in the Minkowski spacetime who moves with constant proper acceleration α along the x direction reads

$$x = \frac{c^2}{\alpha}\cosh\frac{\alpha t'}{c}, \qquad ct = \frac{c^2}{\alpha}\sinh\frac{\alpha t'}{c}, \tag{2.1.34}$$

where t' is the observer's proper time. The observer starts at $x = 1$ with zero velocity.
However, such an observer could also be described with Rindler coordinates. With the coordinate transformation

$$(ct, x) \mapsto (\tau, \rho): \qquad ct = \frac{1}{\rho}\sinh\tau, \qquad x = \frac{1}{\rho}\cosh\tau, \tag{2.1.35}$$

where $\rho = \alpha/c^2$, the Rindler metric reads

$$\boxed{ds^2 = -\frac{1}{\rho^2}d\tau^2 + \frac{1}{\rho^4}d\rho^2 + dy^2 + dz^2.} \tag{2.1.36}$$

Christoffel symbols:

$$\Gamma^\rho_{\tau\tau} = -\rho, \qquad \Gamma^\tau_{\tau\rho} = -\frac{1}{\rho}, \qquad \Gamma^\rho_{\rho\rho} = -\frac{2}{\rho}. \tag{2.1.37}$$

Partial derivatives

$$\Gamma^\rho_{\tau\tau,\rho} = -1, \qquad \Gamma^\tau_{\tau\rho,\rho} = \frac{1}{\rho^2}, \qquad \Gamma^\rho_{\rho\rho,\rho} = \frac{2}{\rho^2}. \tag{2.1.38}$$

The Riemann and Ricci tensors as well as the Ricci and Kretschmann scalar vanish identically.
Local tetrad:

$$\mathbf{e}_{(\tau)} = \rho\partial_\tau, \qquad \mathbf{e}_{(\rho)} = \rho^2\partial_\rho, \qquad \mathbf{e}_{(y)} = \partial_y, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.39}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(\rho)(\tau)} = \rho, \qquad \text{and} \qquad \gamma_{(\rho)} = -\rho. \tag{2.1.40}$$

2.2 Schwarzschild spacetime

2.2.1 Schwarzschild coordinates

In Schwarzschild coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$, the Schwarzschild metric reads

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{1}{1-r_s/r}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right),} \tag{2.2.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The critical point $r = 0$ is a real curvature singularity while the event horizon, $r = r_s$, is only a coordinate singularity, see e.g. the Kretschmann scalar.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s(r-r_s)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s}{2r(r-r_s)}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r(r-r_s)}, \tag{2.2.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r-r_s), \tag{2.2.2b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -(r-r_s)\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.2c}$$

Partial derivatives

$$\Gamma^r_{tt,r} = -\frac{(2r-3r_s)c^2 r_s}{2r^4}, \qquad \Gamma^t_{tr,r} = -\frac{(2r-r_s)r_s}{2r^2(r-r_s)^2}, \qquad \Gamma^r_{rr,r} = \frac{(2r-r_s)r_s}{2r^2(r-r_s)^2}, \tag{2.2.3a}$$

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -1, \tag{2.2.3b}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.2.3c}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -(r-r_s)\sin(2\vartheta). \tag{2.2.3d}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}\frac{c^2(r-r_s)r_s}{r^2}, \qquad R_{t\varphi t\varphi} = \frac{1}{2}\frac{c^2(r-r_s)r_s\sin^2\vartheta}{r^2}, \tag{2.2.4a}$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2}\frac{r_s}{r-r_s}, \qquad R_{r\varphi r\varphi} = -\frac{1}{2}\frac{r_s\sin^2\vartheta}{r-r_s}, \qquad R_{\vartheta\varphi\vartheta\varphi} = r r_s\sin^2\vartheta. \tag{2.2.4b}$$

As aspected, the Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathscr{K} = 12\frac{r_s^2}{r^6}. \tag{2.2.5}$$

Here, it becomes clear that at $r = r_s$ there is no real singularity.

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{1-\frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.6}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1-\frac{r_s}{r}}dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.7}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r-3r_s}{2r^2\sqrt{1-r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.2.9}$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \qquad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\varphi)}^{(\varphi)} = -\frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \qquad c_{(\vartheta)(\varphi)}^{(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.11a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.11b}$$

The covariant derivatives of the Riemann tensor read

$$R_{(t)(r)(t)(r);(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi);(r)} = \frac{3r_s}{r^5}\sqrt{r(r-r_s)}, \tag{2.2.12a}$$

$$\begin{aligned} R_{(t)(r)(r)(\vartheta);(\vartheta)} &= R_{(t)(r)(t)(\varphi);(\varphi)} = R_{(t)(\vartheta)(t)(\vartheta);(r)} = R_{(t)(\varphi)(t)(\varphi);(r)} = \\ &= R_{(r)(\varphi)(\vartheta)(\varphi);(\vartheta)} = -\frac{3r_s}{2r^5}\sqrt{r(r-r_s)}, \end{aligned} \tag{2.2.12b}$$

$$R_{(r)(\vartheta)(r)(\vartheta);(r)} = R_{(r)(\vartheta)(\vartheta)(\varphi);(\varphi)} = R_{(r)(\varphi)(r)(\varphi);(r)} = \frac{3r_s}{2r^5}\sqrt{r(r-r_s)}. \tag{2.2.12c}$$

Newman-Penrose tetrad:

$$\mathbf{l} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_{(t)} + \mathbf{e}_{(r)}\right), \qquad \mathbf{n} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_{(t)} - \mathbf{e}_{(r)}\right), \qquad \mathbf{m} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_{(\vartheta)} + i\mathbf{e}_{(\varphi)}\right). \tag{2.2.13}$$

Non-vanishing spin coefficients:

$$\rho = \mu = -\frac{1}{\sqrt{2}r}\sqrt{1-\frac{r_s}{r}}, \quad \gamma = \varepsilon = \frac{r_s}{4\sqrt{2}r^2\sqrt{1-r_s/r}}, \quad \alpha = -\beta = -\frac{\cot\vartheta}{2\sqrt{2}r}. \tag{2.2.14}$$

Embedding:
The embedding function reads

$$z = 2\sqrt{r_s}\sqrt{r-r_s}. \tag{2.2.15}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \tag{2.2.16}$$

with the constants of motion $k = (1-r_s/r)c^2\dot{t}$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.8.2). For timelike geodesics, the effective potential has the extremal points

$$r_\pm = \frac{h^2 \pm h\sqrt{h^2-3c^2r_s^2}}{c^2 r_s}, \tag{2.2.17}$$

where r_+ is a maximum and r_- is a minimum. The innermost timelike circular geodesic follows from $h^2 = 3c^2r_s^2$ and reads $r_{\text{itcg}} = 3r_s$. Null geodesics, however, have only a maximum at $r_{\text{po}} = \frac{3}{2}r_s$. The corresponding circular orbit is called photon orbit.

Further reading:
Schwarzschild[Sch16, Sch03], MTW[MTW73], Rindler[Rin01], Wald[Wal84], Chandrasekhar[Cha06], Müller[Mül08b, Mül09].

2.2.2 Schwarzschild in pseudo-Cartesian coordinates

The Schwarzschild spacetime in pseudo-Cartesian coordinates (t,x,y,z) reads

$$\boxed{\begin{aligned} ds^2 &= -\left(1-\frac{r_s}{r}\right)c^2dt^2+\left(\frac{x^2}{1-r_s/r}+y^2+z^2\right)\frac{dx^2}{r^2}+\left(x^2+\frac{y^2}{1-r_s/r}+z^2\right)\frac{dy^2}{r^2} \\ &\quad +\left(x^2+y^2+\frac{z^2}{1-r_s/r}\right)\frac{dz^2}{r^2}+\frac{2r_s}{r^2(r-r_s)}\left(xy\,dxdy+xz\,dxdz+yz\,dydz\right), \end{aligned}} \tag{2.2.18}$$

where $r^2 = x^2+y^2+z^2$. For a natural local tetrad that is adapted to the x-axis, we make the following ansatz:

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_t, \qquad \mathbf{e}_{(1)} = A\partial_x, \qquad \mathbf{e}_{(2)} = B\partial_x + C\partial_y, \qquad \mathbf{e}_{(3)} = D\partial_x + E\partial_y + F\partial_z. \tag{2.2.19}$$

$$A = \frac{1}{\sqrt{g_{xx}}}, \qquad B = \frac{-g_{xy}}{g_{xx}\sqrt{-g_{xy}^2/g_{xx}+g_{yy}}}, \qquad C = \frac{1}{\sqrt{-g_{xy}^2/g_{xx}+g_{yy}}}, \tag{2.2.20a}$$

$$D = \frac{g_{xy}g_{yz}-g_{xz}g_{yy}}{\sqrt{NW}}, \qquad E = \frac{g_{xz}g_{xy}-g_{xx}g_{yz}}{\sqrt{NW}}, \qquad F = \frac{\sqrt{N}}{\sqrt{W}}, \tag{2.2.20b}$$

with

$$N = g_{xx}g_{yy} - g_{xy}^2, \tag{2.2.21a}$$

$$W = g_{xx}g_{yy}g_{zz} - g_{xz}^2 g_{yy} + 2g_{xz}g_{xy}g_{yz} - g_{xy}^2 g_{zz} - g_{xx}g_{yz}^2. \tag{2.2.21b}$$

2.2.3 Isotropic coordinates

Spherical isotropic coordinates

The Schwarzschild metric (2.2.1) in spherical isotropic coordinates $(t,\rho,\vartheta,\varphi)$ reads

$$\boxed{ds^2 = -\left(\frac{1-\rho_s/\rho}{1+\rho_s/\rho}\right)^2 c^2dt^2 + \left(1+\frac{\rho_s}{\rho}\right)^4\left[d\rho^2+\rho^2\left(d\vartheta^2+\sin^2\vartheta\, d\varphi^2\right)\right],} \tag{2.2.22}$$

where

$$r = \rho\left(1+\frac{\rho_s}{\rho}\right)^2 \qquad \text{or} \qquad \rho = \frac{1}{4}\left(2r-r_s\pm 2\sqrt{r(r-r_s)}\right) \tag{2.2.23}$$

is the coordinate transformation between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ, see e.g. MTW[MTW73] page 840. The event horizon is given by $\rho_s = r_s/4$. The photon orbit and the innermost timelike circular geodesic read

$$\rho_{\text{po}} = \left(2+\sqrt{3}\right)\rho_s \qquad \text{and} \qquad \rho_{\text{itcg}} = \left(5+2\sqrt{6}\right)\rho_s. \tag{2.2.24}$$

Christoffel symbols:

$$\Gamma^{\rho}_{tt} = \frac{2(\rho-\rho_s)\rho^4\rho_s c^2}{(\rho+\rho_s)^7}, \qquad \Gamma^{t}_{t\rho} = \frac{2\rho_s}{\rho^2-\rho_s^2}, \qquad \Gamma^{\rho}_{\rho\rho} = -\frac{2\rho_s}{(\rho+\rho_s)\rho}, \tag{2.2.25a}$$

$$\Gamma^{\vartheta}_{\rho\vartheta} = \frac{\rho-\rho_s}{(\rho+\rho_s)\rho}, \qquad \Gamma^{\varphi}_{\rho\varphi} = \frac{\rho-\rho_s}{(\rho+\rho_s)\rho}, \qquad \Gamma^{\rho}_{\vartheta\vartheta} = -\rho\frac{\rho-\rho_s}{\rho+\rho_s}, \tag{2.2.25b}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\rho}_{\varphi\varphi} = -\frac{(\rho-\rho_s)\rho\sin^2\vartheta}{\rho+\rho_s}, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.25c}$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -4\frac{(\rho-\rho_s)^2\rho_s c^2}{(\rho+\rho_s)^4\rho}, \qquad R_{t\vartheta t\vartheta} = 2\frac{(\rho-\rho_s)^2\rho\rho_s c^2}{(\rho+\rho_s)^4}, \tag{2.2.26a}$$

$$R_{t\varphi t\varphi} = 2\frac{(\rho-\rho_s)^2\rho c^2\rho_s\sin^2\vartheta}{(\rho+\rho_s)^4}, \qquad R_{\rho\vartheta\rho\vartheta} = -2\frac{(\rho+\rho_s)^2\rho_s}{\rho^3}, \tag{2.2.26b}$$

$$R_{\rho\varphi\rho\varphi} = -2\frac{(\rho+\rho_s)^2\rho_s\sin^2\vartheta}{\rho^3}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{4(\rho+\rho_s)^2\rho_s\sin^2\vartheta}{\rho}. \tag{2.2.26c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathscr{K} = 192\frac{r_s^2}{\rho^6(1+\rho_s/\rho)^{12}} = 12\frac{r_s^2}{r(\rho)^6}. \tag{2.2.27}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1+\rho_s/\rho}{1-\rho_s/\rho}\frac{\partial_t}{c}, \qquad \mathbf{e}_{(r)} = \frac{1}{[1+\rho_s/\rho]^2}\partial_\rho, \tag{2.2.28a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\rho[1+\rho_s/\rho]^2}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\rho[1+\rho_s/\rho]^2\sin^2\vartheta}\partial_\varphi. \tag{2.2.28b}$$

Ricci rotation coefficients:

$$\gamma_{(\rho)(t)(t)} = \frac{2\rho_s\rho^2}{(\rho+\rho_s)^3(\rho-\rho_s)}, \qquad \gamma_{(\vartheta)(\rho)(\vartheta)} = \gamma_{(\varphi)(\rho)(\varphi)} = \frac{\rho(\rho-\rho_s)}{(\rho+\rho_s)^3}, \tag{2.2.29a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\rho\cot\vartheta}{(\rho+\rho_s)^2}. \tag{2.2.29b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\rho)} = \frac{2\rho(\rho^2-\rho\rho_s+\rho_s^2)}{(\rho+\rho_s)^3(\rho-\rho_s)}, \qquad \gamma_{(\vartheta)} = \frac{\rho\cot\vartheta}{(\rho+\rho_s)^2}. \tag{2.2.30}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \tag{2.2.31a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \tag{2.2.31b}$$

Further reading:
Buchdahl[Buc85].

Cartesian isotropic coordinates

The Schwarzschild metric (2.2.1) in Cartesian isotropic coordinates (t,x,y,z) reads,

$$\boxed{ds^2 = -\left(\frac{1-\rho_s/\rho}{1+\rho_s/\rho}\right)^2 c^2dt^2 + \left(1+\frac{\rho_s}{\rho}\right)^4\left[dx^2+dy^2+dz^2\right],} \tag{2.2.32}$$

where $\rho^2 = x^2+y^2+z^2$ and, as before,

$$r = \rho\left(1+\frac{\rho_s}{\rho}\right)^2. \tag{2.2.33}$$

Christoffel symbols:

$$\Gamma^x_{tt} = \frac{2c^2\rho^3\rho_s(\rho-\rho_s)x}{(\rho+\rho_s)^7}, \qquad \Gamma^y_{tt} = \frac{2c^2\rho^3\rho_s(\rho-\rho_s)y}{(\rho+\rho_s)^7}, \qquad \Gamma^z_{tt} = \frac{2c^2\rho^3\rho_s(\rho-\rho_s)z}{(\rho+\rho_s)^7}, \tag{2.2.34a}$$

$$\Gamma^t_{tx} = \frac{2\rho_s x}{\rho^3\left[1-\rho_s^2/\rho^2\right]}, \qquad \Gamma^t_{ty} = \frac{2\rho_s y}{\rho^3\left[1-\rho_s^2/\rho^2\right]}, \qquad \Gamma^t_{tz} = \frac{2\rho_s z}{\rho^3\left[1-\rho_s^2/\rho^2\right]}, \tag{2.2.34b}$$

$$\Gamma^x_{xx} = \Gamma^y_{xy} = \Gamma^z_{xz} = -\Gamma^x_{yy} = -\Gamma^x_{zz} = -\frac{2\rho_s}{\rho^3}\frac{x}{1+\rho_s/\rho}, \tag{2.2.34c}$$

$$\Gamma^y_{xx} = -\Gamma^x_{xy} = -\Gamma^y_{yy} = -\Gamma^z_{yz} = \Gamma^y_{zz} = \frac{2\rho_s}{\rho^3}\frac{y}{1+\rho_s/\rho}, \tag{2.2.34d}$$

$$\Gamma^z_{xx} = -\Gamma^x_{xz} = \Gamma^z_{yy} = -\Gamma^y_{yz} = -\Gamma^z_{zz} = \frac{2\rho_s}{\rho^3}\frac{z}{1+\rho_s/\rho}. \tag{2.2.34e}$$

2.2.4 Eddington-Finkelstein

The transformation of the Schwarzschild metric (2.2.1) from the usual Schwarzschild time coordinate t to the advanced null coordinate v with

$$cv = ct + r + r_s \ln(r - r_s) \tag{2.2.35}$$

leads to the ingoing Eddington-Finkelstein[Edd24, Fin58] metric with coordinates $(v, r, \vartheta, \varphi)$,

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dv^2 + 2c\,dv\,dr + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right).} \tag{2.2.36}$$

Metric-Tensor:

$$g_{vv} = -c^2\left(1-\frac{r_s}{r}\right), \qquad g_{vr} = c, \qquad g_{\vartheta\vartheta} = r^2, \qquad g_{\varphi\varphi} = r^2\sin^2\vartheta. \tag{2.2.37}$$

Christoffel symbols:

$$\Gamma^v_{vv} = \frac{cr_s}{2r^2}, \qquad \Gamma^r_{vv} = \frac{c^2r_s(r-r_s)}{2r^3}, \qquad \Gamma^r_{vr} = -\frac{cr_s}{2r^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.2.38a}$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^v_{\vartheta\vartheta} = -\frac{r}{c}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r-r_s), \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.2.38b}$$

$$\Gamma^v_{\varphi\varphi} = -\frac{r\sin^2\vartheta}{c}, \quad \Gamma^r_{\varphi\varphi} = -(r-r_s)\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.38c}$$

Partial derivatives

$$\Gamma^v_{vv,r} = -\frac{cr_s}{r^3}, \qquad \Gamma^r_{vv,r} = -\frac{(2r-3r_s)c^2r_s}{2r^4}, \qquad \Gamma^r_{vr,r} = \frac{cr_s}{r^3}, \tag{2.2.39a}$$

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^v_{\vartheta\vartheta,r} = -\frac{1}{c}, \tag{2.2.39b}$$

$$\Gamma^r_{\vartheta\vartheta,r} = -1, \qquad \Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^v_{\varphi\varphi,r} = -\frac{\sin^2\vartheta}{c}, \tag{2.2.39c}$$

$$\Gamma^v_{\varphi\varphi,\vartheta} = -\frac{r\sin(2\vartheta)}{c}, \qquad \Gamma^r_{\varphi\varphi,r} = -\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.2.39d}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -(r-r_s)\sin(2\vartheta). \tag{2.2.39e}$$

Riemann-Tensor:

$$R_{vrvr} = -\frac{c^2r_s}{r^3}, \qquad R_{v\vartheta v\vartheta} = \frac{c^2r_s(r-r_s)}{2r^2}, \qquad R_{v\vartheta r\vartheta} = -\frac{cr_s}{2r}, \tag{2.2.40a}$$

$$R_{v\varphi v\varphi} = \frac{c^2r_s(r-r_s)\sin^2\vartheta}{2r^2}, \qquad R_{v\varphi r\varphi} = -\frac{cr_s\sin^2\vartheta}{2r}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.40b}$$

While the Ricci tensor and the Ricci scalar vanish identically, the Kretschmann scalar is $\mathscr{K} = 12r_s^2/r^6$.

Static local tetrad:

$$\mathbf{e}_{(v)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v, \quad \mathbf{e}_{(r)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v + \sqrt{1-\frac{r_s}{r}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.41}$$

Dual tetrad:

$$\theta^{(v)} = c\sqrt{1-\frac{r_s}{r}}dv - \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(\vartheta)} = r\,d\vartheta, \quad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.42}$$

Ricci rotation coefficients:

$$\gamma_{(r)(v)(v)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.43}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r-3r_s}{2r^2\sqrt{1-r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.2.44}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(v)(r)(v)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.45a}$$

$$R_{(v)(\vartheta)(v)(\vartheta)} = R_{(v)(\varphi)(v)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.45b}$$

2.2.5 Kruskal-Szekeres

The Schwarzschild metric in Kruskal-Szekeres[Kru60, Wal84] coordinates $(T, X, \vartheta, \varphi)$ reads

$$\boxed{ds^2 = \frac{4r_s^3}{r}e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2,} \tag{2.2.46}$$

where $r \in \mathbb{R}_+ \setminus \{0\}$ is given by means of the LambertW-function $\mathscr{W}$,

$$\left(\frac{r}{r_s} - 1\right)e^{r/r_s} = X^2 - T^2 \qquad \text{or} \qquad r = r_s\left[\mathscr{W}\left(\frac{X^2-T^2}{e}\right) + 1\right]. \tag{2.2.47}$$

The Schwarzschild coordinate time t in terms of the Kruskal coordinates T and X reads

$$t = 2r_s \operatorname{arctanh}\frac{T}{X}, \qquad r > r_s, \tag{2.2.48a}$$

$$t = 2r_s \operatorname{arctanh}\frac{X}{T}, \qquad r < r_s, \tag{2.2.48b}$$

$$t = \infty, \qquad r = r_s. \tag{2.2.48c}$$

The transformations between Kruskal- and Schwarzschild coordinates read

$$X = \sqrt{1-\frac{r}{r_s}}e^{r/(2r_s)}\sinh\frac{ct}{2r_s}, \quad T = \sqrt{1-\frac{r}{r_s}}e^{r/(2r_s)}\cosh\frac{ct}{2r_s}, \qquad 0 < r < r_2, \tag{2.2.49a}$$

$$X = \sqrt{\frac{r}{r_s}-1}\,e^{r/(2r_s)}\cosh\frac{ct}{2r_s}, \quad T = \sqrt{\frac{r}{r_s}-1}\,e^{r/(2r_s)}\sinh\frac{ct}{2r_s}, \qquad r \geq r_s. \tag{2.2.49b}$$

Christoffel symbols:

$$\Gamma^{T}_{TT} = \Gamma^{X}_{TX} = \Gamma^{T}_{XX} = \frac{Tr_s(r+r_s)}{r^2}e^{-r/r_s}, \tag{2.2.50a}$$

$$\Gamma^{X}_{TT} = \Gamma^{T}_{TX} = \Gamma^{X}_{XX} = -\frac{Xr_s(r+r_s)}{r^2}e^{-r/r_s}, \tag{2.2.50b}$$

$$\Gamma^{\vartheta}_{T\vartheta} = -\frac{2r_s^2 T}{r^2}e^{-r/r_s}, \qquad \Gamma^{\vartheta}_{X\vartheta} = \frac{2r_s^2 X}{r^2}e^{-r/r_s}, \tag{2.2.50c}$$

$$\Gamma^{T}_{\vartheta\vartheta} = -\frac{r}{2r_s}T, \qquad \Gamma^{X}_{\vartheta\vartheta} = \frac{r}{2r_s}X, \tag{2.2.50d}$$

$$\Gamma^{T}_{\vartheta\vartheta} = -\frac{r}{2r_s}T\sin^2\vartheta, \qquad \Gamma^{X}_{\vartheta\vartheta} = \frac{r}{2r_s}X\sin^2\vartheta, \tag{2.2.50e}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.50f}$$

Riemann-Tensor:

$$R_{TXTX} = -16\frac{r_s^7}{r^5}e^{-2r/r_s}, \qquad R_{T\vartheta T\vartheta} = \frac{2r_s^4}{r^2}e^{-r/r_s}, \tag{2.2.51a}$$

$$R_{T\varphi T\varphi} = \frac{2r_s^4}{r^2}e^{-r/r_s}\sin^2\vartheta, \qquad R_{X\vartheta X\vartheta} = -\frac{2r_s^4}{r^2}e^{-r/r_s}, \tag{2.2.51b}$$

$$R_{X\varphi X\varphi} = -\frac{2r_s^4}{r^2}e^{-r/r_s}\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.51c}$$

The *Ricci-Tensor* as well as the *Ricci-scalar* vanish identically.
Kretschmann scalar:

$$\mathscr{K} = \frac{12r_s^2}{r^6}. \tag{2.2.52}$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}}e^{r/(2r_s)}\partial_T, \quad \mathbf{e}_{(X)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}}e^{r/(2r_s)}\partial_X, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi \tag{2.2.53}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(X)(T)(X)} = R_{(X)(\vartheta)(X)(\vartheta)} = R_{(X)(\varphi)(X)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.54a}$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.54b}$$

2.2.6 Tortoise coordinates

The Schwarzschild metric represented by tortoise coordinates $(t,\rho,\vartheta,\varphi)$ reads

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r(\rho)}\right)c^2dt^2 + \left(1-\frac{r_s}{r(\rho)}\right)d\rho^2 + r(\rho)^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right),} \tag{2.2.55}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The tortoise radial coordinate ρ and the Schwarzschild radial coordinate r are related by

$$\rho = r + r_s\ln\left(\frac{r}{r_s}-1\right) \qquad \text{or} \qquad r = r_s\left\{1+\mathscr{W}\left[\exp\left(\frac{\rho}{r_s}-1\right)\right]\right\}. \tag{2.2.56}$$

Christoffel symbols:

$$\Gamma^{\rho}_{tt} = \frac{c^2 r_s}{2r(\rho)^2}, \qquad \Gamma^{t}_{t\rho} = \frac{r_s}{2r(\rho)^2}, \qquad \Gamma^{\rho}_{\rho\rho} = \frac{r_s}{2r(\rho)^2}, \tag{2.2.57a}$$

$$\Gamma^{\vartheta}_{\rho\vartheta} = \frac{1}{r(\rho)} - \frac{1}{r_s}, \qquad \Gamma^{\varphi}_{\rho\varphi} = \frac{1}{r(\rho)} - \frac{1}{r_s}, \qquad \Gamma^{\rho}_{\vartheta\vartheta} = -r(\rho), \tag{2.2.57b}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\rho}_{\varphi\varphi} = -r(\rho)\sin^2\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.57c}$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -\frac{c^2 r_s}{r(\rho)^3}\left(1 - \frac{r_s}{r(\rho)}\right)^2, \qquad R_{t\vartheta t\vartheta} = \frac{c^2}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)}, \tag{2.2.58a}$$

$$R_{t\varphi t\varphi} = \frac{c^2\sin^2\vartheta}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)}, \qquad R_{\rho\vartheta\rho\vartheta} = -\frac{1}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)} \tag{2.2.58b}$$

$$R_{\rho\varphi\rho\varphi} = -\frac{\sin^2\vartheta}{2}\left(1 - \frac{r_s}{r(\rho)}\right)\frac{r_s}{r(\rho)}, \qquad R_{\vartheta\varphi\vartheta\varphi} = r(\rho) r_s \sin^2\vartheta. \tag{2.2.58c}$$

The Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathscr{K} = 12\frac{r_s^2}{r(\rho)^6}. \tag{2.2.59}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r(\rho)}}\partial_t, \quad \mathbf{e}_{(\rho)} = \frac{1}{\sqrt{1 - r_s/r(\rho)}}\partial_\rho, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r(\rho)}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r(\rho)\sin\vartheta}\partial_\varphi. \tag{2.2.60}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r(\rho)}}\,dt, \quad \theta^{(\rho)} = \sqrt{1 - \frac{r_s}{r(\rho)}}\,d\rho, \quad \theta^{(\vartheta)} = r(\rho)\,d\vartheta, \quad \theta^{(\varphi)} = r(\rho)\sin\vartheta\,d\varphi. \tag{2.2.61}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \tag{2.2.62a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \tag{2.2.62b}$$

Further reading:
MTW[MTW73]

2.2.7 Painlevé-Gullstrand

The Schwarzschild metric expressed in Painlevé-Gullstrand coordinates[MP01] reads

$$\boxed{ds^2 = -c^2 dT^2 + \left(dr + \sqrt{\frac{r_s}{r}}c\,dT\right)^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.2.63}$$

where the new time coordinate T follows from the Schwarzschild time t in the following way:

$$cT = ct + 2r_s\left(\sqrt{\frac{r}{r_s}} + \frac{1}{2}\ln\left|\frac{\sqrt{r/r_s} - 1}{\sqrt{r/r_s} + 1}\right|\right). \tag{2.2.64}$$

Metric-Tensor:

$$g_{TT} = -c^2\left(1-\frac{r_s}{r}\right), \qquad g_{Tr} = c\sqrt{\frac{r_s}{r}}, \qquad g_{rr} = 1, \qquad g_{\vartheta\vartheta} = r^2, \qquad g_{\varphi\varphi} = r^2\sin^2\vartheta. \tag{2.2.65}$$

Christoffel symbols:

$$\Gamma^T_{TT} = \frac{cr_s}{2r^2}\sqrt{\frac{r_s}{r}}, \qquad \Gamma^r_{TT} = \frac{c^2 r_s(r-r_s)}{2r^3}, \qquad \Gamma^T_{Tr} = \frac{r_s}{2r^2}, \tag{2.2.66a}$$

$$\Gamma^r_{Tr} = -\frac{cr_s}{2r^2}\sqrt{\frac{r_s}{r}}, \qquad \Gamma^T_{rr} = \frac{r_s}{2cr^2}\sqrt{\frac{r}{r_s}}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r^2}, \tag{2.2.66b}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^T_{\vartheta\vartheta} = -\frac{r}{c}\sqrt{\frac{r_s}{r}}, \tag{2.2.66c}$$

$$\Gamma^r_{\vartheta\vartheta} = -(r-r_s), \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^T_{\varphi\varphi} = -\frac{r}{c}\sqrt{\frac{r_s}{r}}\sin^2\vartheta, \tag{2.2.66d}$$

$$\Gamma^r_{\varphi\varphi} = -(r-r_s)\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.66e}$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{c^2 r_s}{r^3}, \qquad R_{T\vartheta T\vartheta} = \frac{c^2 r_s(r-r_s)}{2r^2}, \qquad R_{T\vartheta r\vartheta} = -\frac{cr_s}{2r}\sqrt{\frac{r_s}{r}}, \tag{2.2.67a}$$

$$R_{T\varphi T\varphi} = \frac{c^2 r_s(r-r_s)\sin^2\vartheta}{2r^2}, \qquad R_{T\varphi r\varphi} = -\frac{cr_s}{2r}\sqrt{\frac{r_s}{r}}\sin^2\vartheta, \qquad R_{r\vartheta r\vartheta} = -\frac{r_s}{2r}, \tag{2.2.67b}$$

$$R_{r\varphi r\varphi} = -\frac{r_s\sin^2\vartheta}{2r}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.67c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathscr{K} = 12r_s^2/r^6. \tag{2.2.68}$$

For the Painlevé-Gullstrand coordinates, we can define two natural local tetrads.

Static local tetrad:

$$\hat{\mathbf{e}}_{(T)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_T, \quad \hat{\mathbf{e}}_{(r)} = \frac{\sqrt{r_s}}{c\sqrt{r-r_s}}\partial_T + \sqrt{1-\frac{r_s}{r}}\partial_r, \quad \hat{\mathbf{e}}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \hat{\mathbf{e}}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi, \tag{2.2.69}$$

Dual tetrad:

$$\hat{\theta}^{(T)} = c\sqrt{1-\frac{r_s}{r}}dT - \frac{dr}{\sqrt{r/r_s-1}}, \quad \hat{\theta}^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \quad \hat{\theta}^{(\vartheta)} = r\,d\vartheta, \quad \hat{\theta}^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.70}$$

Freely falling local tetrad:

$$\mathbf{e}_{(T)} = \frac{1}{c}\partial_T - \sqrt{\frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.71}$$

Dual tetrad:

$$\theta^{(T)} = c\,dT, \qquad \theta^{(r)} = c\sqrt{\frac{r_s}{r}}dT + dr, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta d\varphi. \tag{2.2.72}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(r)(T)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.73a}$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.73b}$$

2.2.8 Israel coordinates

The Schwarzschild metric in Israel coordinates (x,y,ϑ,φ) reads[SKM+03]

$$\boxed{ds^2 = r_s^2\left[4dx\left(dy+\frac{y^2dx}{1+xy}\right)+(1+xy)^2\left(d\vartheta^2+\sin^2\vartheta\,d\varphi^2\right)\right],} \tag{2.2.74}$$

where the coordinates x and y follow from the Schwarzschild coordinates via

$$t = r_s\left(1+xy+\ln\frac{y}{x}\right) \qquad \text{and} \qquad r = r_s(1+xy). \tag{2.2.75}$$

Christoffel symbols:

$$\Gamma^x_{xx} = -\frac{y(2+xy)}{(1+xy)^2}, \quad \Gamma^y_{xx} = \frac{y^3(3+xy)}{(1+xy)^3}, \quad \Gamma^y_{xy} = \frac{y(2+xy)}{(1+xy)^2}, \tag{2.2.76a}$$

$$\Gamma^\vartheta_{x\vartheta} = \frac{y}{1+xy}, \quad \Gamma^\varphi_{x\varphi} = \frac{y}{1+xy}, \quad \Gamma^\vartheta_{y\vartheta} = \frac{x}{1+xy}, \tag{2.2.76b}$$

$$\Gamma^\varphi_{x\varphi} = \frac{x}{1+xy}, \quad \Gamma^x_{\vartheta\vartheta} = -\frac{x}{2}(1+xy), \quad \Gamma^y_{\vartheta\vartheta} = -\frac{y}{2}(1-xy), \tag{2.2.76c}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \quad \Gamma^x_{\varphi\varphi} = -\frac{x}{2}(1+xy)\sin^2\vartheta, \quad \Gamma^y_{\varphi\varphi} = -\frac{y}{2}(1-xy)\sin^2\vartheta, \tag{2.2.76d}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.76e}$$

Riemann-Tensor:

$$R_{xyxy} = -4\frac{r_s^2}{(1+xy)^3}, \quad R_{x\vartheta x\vartheta} = -2\frac{y^2r_s^2}{(1+xy)^2}, \quad R_{x\vartheta y\vartheta} = -\frac{r_s^2}{1+xy}, \tag{2.2.77a}$$

$$R_{x\varphi x\varphi} = -2\frac{r_s^2y^2\sin^2\vartheta}{(1+xy)^2}, \quad R_{x\varphi y\varphi} = -\frac{r_s^2\sin^2\vartheta}{1+xy}, \quad R_{\vartheta\varphi\vartheta\varphi} = (1+xy)r_s^2\sin^2\vartheta. \tag{2.2.77b}$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathscr{K} = \frac{12}{r_s^4(1+xy)^6}. \tag{2.2.78}$$

Local tetrad:

$$\mathbf{e}_{(0)} = -\frac{\sqrt{1+xy}}{2r_sy}\partial_x + \frac{y}{r_s\sqrt{1+xy}}\partial_y, \qquad \mathbf{e}_{(1)} = \frac{\sqrt{1+xy}}{2r_sy}\partial_x, \tag{2.2.79a}$$

$$\mathbf{e}_{(2)} = \frac{1}{r_s(1+xy)}\partial_\vartheta, \qquad \mathbf{e}_{(3)} = \frac{1}{r_s(1+xy)\sin\vartheta}\partial_\varphi. \tag{2.2.79b}$$

Dual tetrad:

$$\theta^{(0)} = \frac{r_s\sqrt{1+xy}}{y}dy, \qquad \theta^{(1)} = \frac{2r_sy}{\sqrt{1+xy}}dx + \frac{r_s\sqrt{1+xy}}{y}dy, \tag{2.2.80a}$$

$$\theta^{(2)} = r_s(1+xy)\,d\vartheta, \qquad \theta^{(3)} = r_s(1+xy)\sin\vartheta\,d\varphi. \tag{2.2.80b}$$

2.3 Alcubierre Warp

The Warp metric given by Miguel Alcubierre[Alc94] reads

$$\boxed{ds^2 = -c^2dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2} \tag{2.3.1}$$

where

$$v_s = \frac{dx_s(t)}{dt}, \tag{2.3.2a}$$

$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}, \tag{2.3.2b}$$

$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)}. \tag{2.3.2c}$$

The parameter $R > 0$ defines the radius of the warp bubble and the parameter $\sigma > 0$ its thickness.

Metric-Tensor:

$$g_{tt} = -c^2 + v_s^2 f(r_s)^2, \qquad g_{tx} = -v_s f(r_s), \qquad g_{xx} = g_{yy} = g_{zz} = 1. \tag{2.3.3}$$

Christoffel symbols:

$$\Gamma^t_{tt} = \frac{f^2 f_x v_s^3}{c^2}, \qquad \Gamma^z_{tt} = -f f_z v_s^2, \qquad \Gamma^y_{tt} = -f f_y v_s^2, \tag{2.3.4a}$$

$$\Gamma^x_{tt} = \frac{f^3 f_x v_s^4 - c^2 f f_x v_s^2 - c^2 f_t v_s}{c^2}, \qquad \Gamma^t_{tx} = -\frac{f f_x v_s^2}{c^2}, \qquad \Gamma^x_{tx} = -\frac{f^2 f_x v_s^3}{c^2}, \tag{2.3.4b}$$

$$\Gamma^y_{tx} = \frac{f_y v_s}{2}, \qquad \Gamma^z_{tx} = \frac{f_z v_s}{2}, \qquad \Gamma^t_{ty} = -\frac{f f_y v_s^2}{2c^2}, \tag{2.3.4c}$$

$$\Gamma^x_{ty} = -\frac{f^2 f_y v_s^3 + c^2 f_y v_s}{2c^2}, \qquad \Gamma^t_{tz} = -\frac{f f_z v_s^2}{2c^2}, \qquad \Gamma^x_{tz} = -\frac{f^2 f_z v_s^3 + c^2 f_z v_s}{2c^2}, \tag{2.3.4d}$$

$$\Gamma^t_{xx} = \frac{f_x v_s}{c^2}, \qquad \Gamma^x_{xx} = \frac{f f_x v_s^2}{c^2}, \qquad \Gamma^t_{xy} = \frac{f_y v_s}{2c^2}, \tag{2.3.4e}$$

$$\Gamma^x_{xy} = \frac{f f_y v_s^2}{2c^2}, \qquad \Gamma^t_{xz} = \frac{f_z v_s}{2c^2}, \qquad \Gamma^x_{xz} = \frac{f f_z v_s^2}{2c^2}, \tag{2.3.4f}$$

with derivatives

$$f_t = \frac{df(r_s)}{dt} = \frac{-v_s\sigma(x - x_s(t))}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.3.5a}$$

$$f_x = \frac{df(r_s)}{dx} = \frac{\sigma(x - x_s(t))}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.3.5b}$$

$$f_y = \frac{df(r_s)}{dy} = \frac{\sigma y}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.3.5c}$$

$$f_z = \frac{df(r_s)}{dz} = \frac{\sigma z}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.3.5d}$$

Riemann- and Ricci-tensor as well as Ricci- and Kretschman-scalar are shown only in the Maple worksheet.

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c}\left(\partial_t + v_s f \partial_x\right), \quad \mathbf{e}_{(1)} = \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \tag{2.3.6}$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{c^2 - v_s^2 f^2}}\partial_t, \quad \mathbf{e}_{(1)} = \frac{v_s f}{c\sqrt{c^2 - v_s^2 f^2}}\partial_t + \frac{\sqrt{c^2 - v_s^2 f^2}}{c}\partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \tag{2.3.7}$$

Further reading:

Pfenning[PF97], Clark[CHL99], Van Den Broeck[Bro99]

2.4 Barriola-Vilenkin monopol

The Barriola-Vilenkin metric describes the gravitational field of a global monopole[BV89]. In spherical coordinates (t,r,ϑ,φ), the metric reads

$$\boxed{ds^2 = -c^2dt^2 + dr^2 + k^2r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.4.1}$$

where k is the scaling factor responsible for the deficit/surplus angle.

Christoffel symbols:

$$\Gamma^r_{\vartheta\vartheta} = -k^2r, \qquad \Gamma^r_{\varphi\varphi} = -k^2r\sin^2\vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.4.2a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{2.4.2b}$$

Partial derivatives

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -k^2, \tag{2.4.3a}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -k^2\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta), \tag{2.4.3b}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -k^2r\sin(2\vartheta). \tag{2.4.3c}$$

Riemann-Tensor:

$$R_{\vartheta\varphi\vartheta\varphi} = (1-k^2)k^2r^2\sin^2\vartheta. \tag{2.4.4}$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{\vartheta\vartheta} = (1-k^2), \qquad R_{\varphi\varphi} = (1-k^2)\sin^2\vartheta, \qquad \mathscr{R} = 2\frac{1-k^2}{k^2r^2}, \qquad \mathscr{K} = 4\frac{(1-k^2)^2}{k^4r^4}. \tag{2.4.5}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(1-k^2)}{3k^2r^2}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{6}(1-k^2), \qquad C_{t\varphi t\varphi} = \frac{c^2}{6}(1-k^2)\sin^2\vartheta, \tag{2.4.6a}$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{6}(1-k^2), \quad C_{r\varphi r\varphi} = -\frac{1}{6}(1-k^2)\sin^2\vartheta, \; C_{\vartheta\varphi\vartheta\varphi} = \frac{k^2r^2}{3}(1-k^2)\sin^2\vartheta. \tag{2.4.6b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{kr}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{kr\sin\vartheta}\partial_\varphi. \tag{2.4.7}$$

Dual tetrad:

$$\theta^{(t)} = c\,dt, \qquad \theta^{(r)} = dr, \qquad \theta^{(\vartheta)} = kr\,d\vartheta, \qquad \theta^{(\varphi)} = kr\sin\vartheta\,d\varphi. \tag{2.4.8}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{kr}. \tag{2.4.9}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{kr}. \tag{2.4.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{1-k^2}{k^2r^2}. \tag{2.4.11}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{1-k^2}{k^2r^2}. \tag{2.4.12}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{1-k^2}{3k^2r^2}, \tag{2.4.13a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{1-k^2}{6k^2r^2}. \tag{2.4.13b}$$

Embedding:
The embedding function, see Sec. 1.7, for $k < 1$ reads

$$z = \sqrt{1-k^2}\,r. \tag{2.4.14}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{h_1^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(\frac{h_2^2}{k^2r^2} - \kappa c^2\right), \tag{2.4.15}$$

with the constants of motion $h_1 = c^2\dot{t}$ and $h_2 = k^2r^2\dot{\varphi}$.

The point of closest approach r_{pca} for a null geodesic that starts at $r = r_i$ with $\mathbf{y} = \pm\mathbf{e}_{(t)} + \cos\xi\,\mathbf{e}_{(r)} + \sin\xi\,\mathbf{e}_{(\varphi)}$ is given by $r = r_i \sin\xi$. Hence, the r_{pca} is independent of k. The same is also true for timelike geodesics.

Further reading:
Barriola and Vilenkin[BV89], Perlick[Per04].

2.5 Bertotti-Kasner

The Bertotti-Kasner spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ reads[Rin98]

$$\boxed{ds^2 = -c^2 dt^2 + e^{2\sqrt{\Lambda}ct} dr^2 + \frac{1}{\Lambda}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.5.1}$$

where the cosmological constant Λ must be positive.
Christoffel symbols:

$$\Gamma^r_{tr} = c\sqrt{\Lambda}, \qquad \Gamma^t_{rr} = \frac{\sqrt{\Lambda}}{c} e^{2\sqrt{\Lambda}ct}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.5.2}$$

Partial derivatives

$$\Gamma^t_{rr,t} = 2\Lambda e^{2\sqrt{\Lambda}ct}, \qquad \Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\cos(2\vartheta). \tag{2.5.3}$$

Riemann-Tensor:

$$R_{trtr} = -\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{\sin^2\vartheta}{\Lambda}. \tag{2.5.4}$$

Ricci-Tensor:

$$R_{tt} = -\Lambda c^2, \qquad R_{rr} = \Lambda e^{2\sqrt{\Lambda}ct}, \qquad R_{\vartheta\vartheta} = 1, \qquad R_{\varphi\varphi} = \sin^2\vartheta. \tag{2.5.5}$$

The Ricci and Kretschmann scalars read

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = 8\Lambda^2. \tag{2.5.6}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{2}{3}\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \qquad C_{t\vartheta t\vartheta} = \frac{c^2}{3}, \qquad C_{t\varphi t\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \tag{2.5.7a}$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \qquad C_{r\varphi r\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}\sin^2\vartheta, \qquad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3}\frac{\sin^2\vartheta}{\Lambda}. \tag{2.5.7b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = e^{-\sqrt{\Lambda}ct}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \sqrt{\Lambda}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{\sqrt{\Lambda}}{\sin\vartheta}\partial_\varphi. \tag{2.5.8}$$

Dual tetrad:

$$\theta^{(t)} = c\,dt, \qquad \theta^{(r)} = e^{\sqrt{\Lambda}ct}dr, \qquad \theta^{(\vartheta)} = \frac{1}{\sqrt{\Lambda}}d\vartheta, \qquad \theta^{(\varphi)} = \frac{\sin\vartheta}{\sqrt{\Lambda}}d\varphi. \tag{2.5.9}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \sqrt{\Lambda}, \qquad \gamma_{(\vartheta)(\varphi)(\varphi)} = -\sqrt{\Lambda}\cot\vartheta. \tag{2.5.10}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\sqrt{\Lambda}, \qquad \gamma_{(\vartheta)} = \sqrt{\Lambda}\cot\vartheta. \tag{2.5.11}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\Lambda. \tag{2.5.12}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = -R_{(\vartheta)(\vartheta)} = -R_{(\varphi)(\varphi)} = -\Lambda. \tag{2.5.13}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2\Lambda}{3}, \tag{2.5.14a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{\Lambda}{3}. \tag{2.5.14b}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$c^2\dot{t}^2 = h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa \tag{2.5.15}$$

with the constants of motion $h_1 = \dot{r}e^{2\sqrt{\Lambda}ct}$ and $h_2 = \dot{\varphi}/\Lambda$. Thus,

$$\lambda = \frac{1}{c\sqrt{\Lambda}\sqrt{\Lambda h_2^2 - \kappa}} \ln\left(\frac{1+q(t)}{1-q(t)}\frac{1-q(t_i)}{1+q(t_i)}\right), \qquad q(t) = \frac{h_1^2 e^{-2\sqrt{\Lambda}ct}}{\Lambda h_2^2 - \kappa} + 1, \tag{2.5.16}$$

where t_i is the initial time. We can also solve the orbital equation:

$$r(t) = w(t) - w(t_i) + r_i, \qquad w(t) = -\frac{\sqrt{h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa}}{h_1\sqrt{\Lambda}}, \tag{2.5.17}$$

where r_i is the initial radial position.

Further reading:
Rindler[Rin98]: *"Every spherically symmetric solution of the generalized vacuum field equations $R_{ij} = \Lambda g_{ij}$ is either equivalent to Kottler's generalization of Schwarzschild space or to the [...] Bertotti-Kasner space (for which Λ must be necessarily be positive)."*

2.6 Bessel gravitational wave

D. Kramer introduced in [Kra99] an exact gravitational wave solution of Einstein's vacuum field equations. According to [Ste03] we execute the substitution $x \to t$ and $y \to z$.

2.6.1 Cylindrical coordinates

The metric of the Bessel wave in cylindrical coordinates reads

$$\boxed{ds^2 = \mathrm{e}^{-2U}\left[\mathrm{e}^{2K}\left(d\rho^2 - dt^2\right) + \rho^2 d\varphi^2\right] + \mathrm{e}^{2U} dz^2.} \tag{2.6.1}$$

The functions U and K are given by

$$U := CJ_0(\rho)\cos(t), \tag{2.6.2}$$

$$K := \frac{1}{2}C^2\rho\left\{\rho\left[J_0(\rho)^2 + J_1(\rho)^2\right] - 2J_0(\rho)J_1(\rho)\cos^2(t)\right\}, \tag{2.6.3}$$

where $J_n(\rho)$ are the Bessel functions of the first kind.
Christoffel symbols:

$$\Gamma^t_{tt} = \Gamma^\rho_{t\rho} = \Gamma^t_{\rho\rho} = -\frac{\partial U}{\partial t} + \frac{\partial K}{\partial t}, \qquad \Gamma^\varphi_{t\varphi} = \Gamma^z_{tz} = -\frac{\partial U}{\partial t}, \qquad \Gamma^t_{\varphi\varphi} = -\mathrm{e}^{-2K}\rho^2\frac{\partial U}{\partial t}, \tag{2.6.4a}$$

$$\Gamma^\rho_{tt} = \Gamma^t_{t\rho} = \Gamma^\rho_{\rho\rho} = -\frac{\partial U}{\partial \rho} + \frac{\partial K}{\partial \rho}, \qquad \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho} - \frac{\partial U}{\partial \rho}, \qquad \Gamma^\rho_{zz} = -\mathrm{e}^{4U-2K}\frac{\partial U}{\partial \rho}, \tag{2.6.4b}$$

$$\Gamma^\rho_{\varphi\varphi} = \rho\mathrm{e}^{-2K}\left(\rho\frac{\partial U}{\partial \rho} - 1\right), \qquad \Gamma^z_{\rho z} = \frac{\partial U}{\partial \rho}, \qquad \Gamma^t_{zz} = \mathrm{e}^{4U-2K}\frac{\partial U}{\partial t}. \tag{2.6.4c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \mathrm{e}^{U-K}\partial_t, \quad \mathbf{e}_{(\rho)} = \mathrm{e}^{U-K}\partial_\rho, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho}\mathrm{e}^{U}\partial_\varphi, \quad \mathbf{e}_{(z)} = \mathrm{e}^{-U}\partial_z. \tag{2.6.5}$$

Dual tetrad:

$$\theta^{(t)} = \mathrm{e}^{K-U}dt, \quad \theta^{(\rho)} = \mathrm{e}^{K-U}d\rho, \quad \theta^{(\varphi)} = \rho\mathrm{e}^{-U}d\varphi, \quad \theta^{(z)} = \mathrm{e}^{U}dz. \tag{2.6.6}$$

2.6.2 Cartesian coordinates

In Cartesian coordinates with $\rho = \sqrt{x^2+y^2}$ the metric (2.6.1) reads

$$\boxed{\begin{aligned} ds^2 = -\mathrm{e}^{2(K-U)}dt^2 + \frac{\mathrm{e}^{-2U}}{x^2+y^2}\Big[&\left(\mathrm{e}^{2K}x^2+y^2\right)dx^2 + 2xy\left(\mathrm{e}^{2K}-1\right)dxdy \\ &+ \left(x^2+\mathrm{e}^{2K}y^2\right)dy^2\Big] + \mathrm{e}^{2U}dz^2. \end{aligned}} \tag{2.6.7}$$

Local tetrad:

$$\begin{aligned} &\mathbf{e}_{(t)} = \mathrm{e}^{U-K}\partial_t, \qquad \mathbf{e}_{(x)} = \mathrm{e}^{U}\sqrt{\frac{x^2+y^2}{\mathrm{e}^{2K}x^2+y^2}}\partial_x, \\ &\mathbf{e}_{(y)} = \mathrm{e}^{U-K}\sqrt{\frac{\mathrm{e}^{2K}x^2+y^2}{x^2+y^2}}\partial_y + xy\frac{\mathrm{e}^{U-K}\left(\mathrm{e}^{2K}-1\right)}{\sqrt{(x^2+y^2)\left(\mathrm{e}^{2K}x^2+y^2\right)}}\partial_x, \qquad \mathbf{e}_{(z)} = \mathrm{e}^{-U}\partial_z \end{aligned} \tag{2.6.8}$$

2.7 Cosmic string in Schwarzschild spacetime

A cosmic string in the Schwarzschild spacetime represented by Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ reads

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{1}{1-r_s/r}dr^2 + r^2\left(d\vartheta^2 + \beta^2\sin^2\vartheta d\varphi^2\right),} \tag{2.7.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and β is the string parameter, compare Aryal et al[AFV86].

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s(r-r_s)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s}{2r(r-r_s)}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r(r-r_s)}, \tag{2.7.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r-r_s), \tag{2.7.2b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -(r-r_s)\beta^2\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\beta^2\sin\vartheta\cos\vartheta. \tag{2.7.2c}$$

Partial derivatives

$$\Gamma^r_{tt,r} = -\frac{(2r-3r_s)c^2 r_s}{2r^4}, \qquad \Gamma^t_{tr,r} = -\frac{(2r-r_s)r_s}{2r^2(r-r_s)^2}, \qquad \Gamma^r_{rr,r} = \frac{(2r-r_s)r_s}{2r^2(r-r_s)^2}, \tag{2.7.3a}$$

$$\Gamma^\vartheta_{r\vartheta,r} = -\frac{1}{r^2}, \qquad \Gamma^\varphi_{r\varphi,r} = -\frac{1}{r^2}, \qquad \Gamma^r_{\vartheta\vartheta,r} = -1, \tag{2.7.3b}$$

$$\Gamma^\varphi_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^r_{\varphi\varphi,r} = -\beta^2\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi,\vartheta} = -\beta^2\cos(2\vartheta), \tag{2.7.3c}$$

$$\Gamma^r_{\varphi\varphi,\vartheta} = -(r-r_s)\beta^2\sin(2\vartheta). \tag{2.7.3d}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}\frac{c^2(r-r_s)r_s}{r^2}, \qquad R_{t\varphi t\varphi} = \frac{1}{2}\frac{c^2(r-r_s)r_s\beta^2\sin^2\vartheta}{r^2}, \tag{2.7.4a}$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2}\frac{r_s}{r-r_s}, \qquad R_{r\varphi r\varphi} = -\frac{1}{2}\frac{r_s\beta^2\sin^2\vartheta}{r-r_s}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\beta^2\sin^2\vartheta. \tag{2.7.4b}$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathscr{K} = 12\frac{r_s^2}{r^6}. \tag{2.7.5}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{1-\frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\beta\sin\vartheta}\partial_\varphi. \tag{2.7.6}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1-\frac{r_s}{r}}dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\beta\sin\vartheta\,d\varphi. \tag{2.7.7}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.7.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.7.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.7.10a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.7.10b}$$

Embedding:
The embedding function for $\beta^2 < 1$ reads

$$z = (r - r_s)\sqrt{\frac{r}{r - r_s} - \beta^2} - \frac{r_s}{2\sqrt{1-\beta^2}} \ln \frac{\sqrt{r/(r-r_s) - \beta^2} - \sqrt{1-\beta^2}}{\sqrt{r/(r-r_s) - \beta^2} + \sqrt{1-\beta^2}}. \tag{2.7.11}$$

If $\beta^2 = 1$, we have the embedding function of the standard Schwarzschild metric, compare Eq.(2.2.15).

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(\frac{h^2}{r^2\beta^2} - \kappa c^2\right) \tag{2.7.12}$$

with the constants of motion $k = (1 - r_s/r)c^2\dot{t}$ and $h = r^2\beta^2\dot{\varphi}$. The maxima of the effective potential V_{eff} lead to the same critical orbits $r_{\text{po}} = \frac{3}{2}r_s$ and $r_{\text{itcg}} = 3r_s$ as in the standard Schwarzschild metric.

2.8 Ernst spacetime

"The Ernst metric is a static, axially symmetric, electro-vacuum solution of the Einstein-Maxwell equations with a black hole immersed in a magnetic field."[KV92]

In spherical coordinates $(t, r, \vartheta, \varphi)$, the Ernst metric reads[Ern76] $(G = c = 1)$

$$\boxed{ds^2 = \Lambda^2 \left[-\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\vartheta^2 \right] + \frac{r^2 \sin^2\vartheta}{\Lambda^2} d\varphi^2,} \tag{2.8.1}$$

where $\Lambda = 1 + B^2 r^2 \sin^2\vartheta$. Here, M is the mass of the black hole and B the magnetic field strength.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{\left(2B^2r^3\sin^2\vartheta - 3MB^2r^2\sin^2\vartheta + M\right)(r-2M)}{r^3\Lambda}, \quad \Gamma^\vartheta_{tt} = \frac{2(r-2M)B^2\sin\vartheta\cos\vartheta}{r\Lambda}, \tag{2.8.2a}$$

$$\Gamma^t_{tr} = \frac{2B^2r^3\sin^2\vartheta - 3MB^2r^2\sin^2\vartheta + M}{r(r-2M)\Lambda}, \quad \Gamma^t_{t\vartheta} = \frac{2B^2r^2\sin\vartheta\cos\vartheta}{\Lambda}, \tag{2.8.2b}$$

$$\Gamma^r_{rr} = \frac{2B^2r^3\sin^2\vartheta - 5MB^2r^2\sin^2\vartheta - M}{r(r-2M)\Lambda}, \quad \Gamma^\vartheta_{rr} = -\frac{2B^2r\sin\vartheta\cos\vartheta}{(r-2M)\Lambda}, \tag{2.8.2c}$$

$$\Gamma^r_{r\vartheta} = \frac{2B^2r^2\sin\vartheta\cos\vartheta}{\Lambda}, \quad \Gamma^\vartheta_{r\vartheta} = \frac{3B^2r^2\sin^2\vartheta + 1}{r\Lambda}, \tag{2.8.2d}$$

$$\Gamma^\varphi_{r\varphi} = \frac{1 - B^2r^2\sin^2\vartheta}{r\Lambda}, \quad \Gamma^r_{\vartheta\vartheta} = \frac{\left(3B^2r^2\sin^2\vartheta + 1\right)(r-2M)}{\Lambda}, \tag{2.8.2e}$$

$$\Gamma^\vartheta_{\vartheta\vartheta} = \frac{2B^2r^2\sin\vartheta\cos\vartheta}{\Lambda}, \quad \Gamma^\varphi_{\vartheta\varphi} = \frac{\Xi\cos\vartheta}{\Lambda}, \tag{2.8.2f}$$

$$\Gamma^r_{\varphi\varphi} = \frac{(r-2M)\Xi\sin^2\vartheta}{\Lambda^5}, \tag{2.8.2g}$$

$$\Gamma^\vartheta_{\varphi\varphi} = \frac{\Xi\sin\vartheta\cos\vartheta}{\Lambda^5}. \tag{2.8.2h}$$

with $\Xi = 1 - B^2r^2\sin^2\vartheta$.

Riemann-Tensor:

$$R_{trtr} = \frac{2}{r^3}\left[B^4r^4\sin^4\vartheta(3M-r) - M + 2r^5B^4\sin^2\vartheta\cos^2\vartheta + B^2r^2\sin^2\vartheta(r-2M)\right], \tag{2.8.3a}$$

$$R_{trt\vartheta} = 2B^2\sin\vartheta\cos\vartheta\left[(3B^2r^2\sin^2\vartheta(2M-3r) + r - 2M\right], \tag{2.8.3b}$$

$$R_{t\vartheta t\vartheta} = \frac{1}{r^2}\left[B^4r^4(r-2M)(4r-9M)\sin^4\vartheta + 2\Xi B^2r^3(r-2M)\cos^2\vartheta + M(r-2M)\right], \tag{2.8.3c}$$

$$R_{t\varphi t\varphi} = \frac{1}{\Lambda^4r^2}\left[(2B^2r^3 - 3B^2Mr^2\sin^2\vartheta + M)\Xi(r-2M)\sin^2\vartheta\right], \tag{2.8.3d}$$

$$R_{r\vartheta r\vartheta} = -\frac{(2B^2r^3 - 3B^2Mr^2\sin^2\vartheta + M)\Xi}{r-2M}, \tag{2.8.3e}$$

$$R_{r\varphi r\varphi} = -\frac{\sin^2\vartheta}{\Lambda^4(r-2M)}\left[B^4r^4(4r-9M)\sin^4\vartheta + 2B^2r^2(8M-4r\vartheta)\sin^2\vartheta + 2\Xi B^2r^3\cos^2\vartheta + M\right], \tag{2.8.3f}$$

$$R_{r\varphi\vartheta\varphi} = -\frac{2B^2r^3\sin^3\vartheta\cos\vartheta\left(3B^2r^2\sin^2\vartheta - 5\right)}{\Lambda^4}, \tag{2.8.3g}$$

$$R_{\vartheta\varphi\vartheta\varphi} = \frac{r\sin^2\vartheta}{\Lambda^4}\left[2B^4r^4(r-3M)\sin^4\vartheta + 4B^2r^3\cos^2\vartheta(1+\Xi) + 2B^2r^2\sin^2\vartheta(2M-r) + 2M\right]. \tag{2.8.3h}$$

Ricci-Tensor:

$$R_{tt} = \frac{4B^2(r-2M)(r+2M\sin^2\vartheta)}{r^2\Lambda^2}, \quad R_{rr} = -\frac{4B^2[r\cos^2\vartheta-(r-2M)\sin^2\vartheta]}{(r-2M)\Lambda^2}, \tag{2.8.4a}$$

$$R_{r\vartheta} = \frac{8B^2 r\sin\vartheta\cos\vartheta}{\Lambda^2}, \quad R_{\vartheta\vartheta} = \frac{4B^2 r\left[r\cos^2\vartheta+(r-2M)\sin^2\vartheta\right]}{\Lambda^2}, \tag{2.8.4b}$$

$$R_{\varphi\varphi} = \frac{4B^2 r\sin^2\vartheta\left(r+2M\sin^2\vartheta\right)}{\Lambda^6}. \tag{2.8.4c}$$

Ricci and Kretschmann scalars:

$$R = 0, \tag{2.8.5a}$$

$$\begin{aligned}\mathscr{K} = \frac{16}{r^6\Lambda^8}\Big[&3B^8r^8\left(4r^2-18Mr+21M^2\right)\sin^8\vartheta \\ &+2B^4r^4\left(31M^2-37Mr-24B^2r^4\cos^2\vartheta+42B^2Mr^3\cos^2\vartheta+10r^2+6B^4r^6\cos^4\vartheta\right)\sin^6\vartheta \\ &+2B^2r^2\left(-3Mr+20B^2r^4\cos^2\vartheta+6M^2-46B^2Mr^3\cos^2\vartheta-12B^4r^6\cos^4\vartheta\right)\sin^4\vartheta \\ &-6B^6r^6\left(6B^2Mr^3\cos^2\vartheta+4r^2-4B^2r^4\cos^2\vartheta+18M^2-17Mr\right) \\ &+20B^4r^6\cos^4\vartheta+12B^2Mr^3\cos^2\vartheta+3M^2\Big].\end{aligned} \tag{2.8.5b}$$

Static local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{\Lambda\sqrt{1-2m/r}}\partial_t, \quad \mathbf{e}_{(r)} = \frac{\sqrt{1-2m/r}}{\Lambda}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\Lambda r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\Lambda}{r\sin\vartheta}\partial_\varphi. \tag{2.8.6}$$

Dual tetrad:

$$\theta^{(t)} = \Lambda\sqrt{1-\frac{2m}{r}}dt, \quad \theta^{(r)} = \frac{\Lambda}{\sqrt{1-2m/r}}dr, \quad \theta^{(\vartheta)} = \Lambda r\,d\vartheta, \quad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\Lambda}d\varphi. \tag{2.8.7}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{r}^2 + \frac{h^2(1-r_s/r)}{r^2} - \frac{k^2}{\Lambda^4} + \kappa\frac{1-r_s/r}{\Lambda^2} = 0 \tag{2.8.8}$$

with constants of motion $k = \Lambda^2(1-r_s/r)\dot{t}$ and $h = (r^2/\Lambda^2)\dot{\varphi}$.

Further reading:
Ernst[Ern76], Dhurandhar and Sharma[DS83], Karas and Vokrouhlicky[KV92], Stuchlík and Hledík[SH99].

2.9 Friedman-Robertson-Walker

The Friedman-Robertson-Walker metric describes a general homogeneous and isotropic universe. In a general form it reads:

$$ds^2 = -c^2dt^2 + R^2 d\sigma^2 \tag{2.9.1}$$

with $R = R(t)$ being an arbitrary function of time only and $d\sigma^2$ being a metric of a 3-space of constant curvature for which three explicit forms will be described here.
In all formulas in this section a dot denotes differentiation with respect to t, e.g. $\dot{R} = dR(t)/dt$.

2.9.1 Form 1

$$\boxed{ds^2 = -c^2dt^2 + R^2\left\{\frac{d\eta^2}{1-k\eta^2} + \eta^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right\}} \tag{2.9.2}$$

Christoffel symbols:

$$\Gamma^{\eta}_{t\eta} = \frac{\dot{R}}{R}, \qquad \Gamma^{\vartheta}_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^{\varphi}_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.3a}$$

$$\Gamma^{t}_{\eta\eta} = \frac{R\dot{R}}{c^2(1-k\eta^2)}, \quad \Gamma^{\eta}_{\eta\eta} = \frac{k\eta}{1-k\eta^2}, \qquad \Gamma^{\vartheta}_{\eta\vartheta} = \frac{1}{\eta}, \tag{2.9.3b}$$

$$\Gamma^{\varphi}_{\eta\varphi} = \frac{1}{\eta}, \qquad \Gamma^{t}_{\vartheta\vartheta} = \frac{R\eta^2\dot{R}}{c^2}, \qquad \Gamma^{\eta}_{\vartheta\vartheta} = (k\eta^2-1)\eta, \tag{2.9.3c}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{t}_{\varphi\varphi} = \frac{R\eta^2\sin^2\vartheta\dot{R}}{c^2}, \qquad \Gamma^{\eta}_{\varphi\varphi} = (k\eta^2-1)\eta\sin^2\vartheta, \tag{2.9.3d}$$

$$\Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.9.3e}$$

Riemann-Tensor:

$$R_{t\eta t\eta} = \frac{R\ddot{R}}{k\eta^2-1}, \qquad R_{t\vartheta t\vartheta} = -R\eta^2\ddot{R}, \tag{2.9.4a}$$

$$R_{t\varphi t\varphi} = -R\eta^2\sin^2\vartheta\ddot{R}, \qquad R_{\eta\vartheta\eta\vartheta} = -\frac{R^2\eta^2\left(\dot{R}^2+kc^2\right)}{c^2(k\eta^2-1)}, \tag{2.9.4b}$$

$$R_{\eta\varphi\eta\varphi} = -\frac{R^2\eta^2\sin^2\vartheta\left(\dot{R}^2+kc^2\right)}{c^2(k\eta^2-1)}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\eta^4\sin^2\vartheta\left(\dot{R}^2+kc^2\right)}{c^2}. \tag{2.9.4c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{\eta\eta} = \frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(1-k\eta^2)}, \tag{2.9.5a}$$

$$R_{\vartheta\vartheta} = \eta^2\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2}, \qquad R_{\varphi\varphi} = \eta^2\sin^2\vartheta\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2}. \tag{2.9.5b}$$

The *Ricci scalar and Kretschmann scalar read:*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+kc^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2kc^2+k^2c^4}{R^4c^4}. \tag{2.9.6}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\eta)} = \frac{\sqrt{1-k\eta^2}}{R}\partial_\eta, \qquad e_\vartheta = \frac{1}{R\eta}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\eta\sin\vartheta}\partial_\varphi. \tag{2.9.7}$$

Ricci rotation coefficients:

$$\gamma_{(\eta)(t)(\eta)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\eta)(\vartheta)} = \gamma_{(\varphi)(\eta)(\varphi)} = \frac{\sqrt{1-k\eta^2}}{R\eta}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{R\eta}. \tag{2.9.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = \frac{2\sqrt{1-k\eta^2}}{R\eta}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\eta}. \tag{2.9.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \tag{2.9.10a}$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+kc^2}{R^2c^2}. \tag{2.9.10b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2+2kc^2}{R^2c^2}. \tag{2.9.11}$$

2.9.2 Form 2

$$\boxed{ds^2 = -c^2dt^2 + \frac{R^2}{(1+\frac{k}{4}r^2)^2}\left\{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right\}} \tag{2.9.12}$$

Christoffel symbols:

$$\Gamma^r_{tr} = \frac{\dot{R}}{R}, \qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.13a}$$

$$\Gamma^t_{rr} = 16\frac{R\dot{R}}{c^2(4+kr^2)^2}, \qquad \Gamma^r_{rr} = -\frac{2kr}{4+kr^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{4-kr^2}{(4+kr^2)r}, \tag{2.9.13b}$$

$$\Gamma^\varphi_{r\varphi} = \frac{4-kr^2}{(4+kr^2)r}, \qquad \Gamma^t_{\vartheta\vartheta} = 16\frac{Rr^2\dot{R}}{c^2(4+kr^2)^2}, \qquad \Gamma^r_{\vartheta\vartheta} = \frac{r(kr^2-4)}{4+kr^2}, \tag{2.9.13c}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^t_{\varphi\varphi} = 16\frac{Rr^2\sin^2\vartheta\dot{R}}{c^2(4+kr^2)^2}, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{2.9.13d}$$

$$\Gamma^r_{\varphi\varphi} = \frac{r\sin^2\vartheta(kr^2-4)}{4+kr^2}. \tag{2.9.13e}$$

Riemann-Tensor:

$$R_{trtr} = -16\frac{R\ddot{R}}{(4+kr^2)^2}, \qquad R_{t\vartheta t\vartheta} = -16\frac{Rr^2\ddot{R}}{(4+kr^2)^2}, \tag{2.9.14a}$$

$$R_{t\varphi t\varphi} = -16\frac{Rr^2\sin^2\vartheta\ddot{R}}{(4+kr^2)^2}, \qquad R_{r\vartheta r\vartheta} = 256\frac{R^2r^2\left(\dot{R}^2+kc^2\right)}{c^2(4+kr^2)^4}, \tag{2.9.14b}$$

$$R_{r\varphi r\varphi} = 256\frac{R^2r^2\sin^2\vartheta\left(\dot{R}^2+kc^2\right)}{c^2(4+kr^2)^4}, \qquad R_{\vartheta\varphi\vartheta\varphi} = 256\frac{R^2r^4\sin^2\vartheta\left(\dot{R}^2+kc^2\right)}{c^2(4+kr^2)^4}. \tag{2.9.14c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{rr} = 16\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(4+kr^2)^2}, \tag{2.9.15a}$$

$$R_{\vartheta\vartheta} = 16r^2\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(4+kr^2)^2}, \qquad R_{\varphi\varphi} = 16r^2\sin^2\vartheta\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(4+kr^2)^2}. \tag{2.9.15b}$$

The *Ricci scalar and Kretschmann scalar read:*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+kc^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2kc^2+k^2c^4}{R^4c^4}. \tag{2.9.16}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(r)} = \frac{1+\frac{k}{4}r^2}{R}\partial_r, \qquad e_\vartheta = \frac{1+\frac{k}{4}r^2}{Rr}\partial_\vartheta, \qquad e_\varphi = \frac{1+{}^k\!/_4r^2}{Rr\sin\vartheta}\partial_\varphi. \tag{2.9.17}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = -\frac{\frac{k}{4}r^2-1}{Rr}, \tag{2.9.18a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{(\frac{k}{4}r^2+1)\cot\vartheta}{Rr}. \tag{2.9.18b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{1-\frac{k}{4}r^2}{Rr}, \qquad \gamma_{(\vartheta)} = \frac{(\frac{k}{4}r^2+1)\cot\vartheta}{Rr}. \tag{2.9.19}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \tag{2.9.20a}$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+kc^2}{R^2c^2}. \tag{2.9.20b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2+2kc^2}{R^2c^2}. \tag{2.9.21}$$

2.9.3 Form 3

The following forms of the metric are obtained from 2.9.2 by setting $\eta = \sin\psi, \psi, \sinh\psi$ for $k = 1, 0, -1$ respectively.

Positive Curvature

$$\boxed{ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \sin^2\psi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right\}} \tag{2.9.22}$$

Christoffel symbols:

$$\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, \qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.23a}$$

$$\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad \Gamma^\vartheta_{\psi\vartheta} = \cot\psi, \qquad \Gamma^\varphi_{\psi\varphi} = \cot\psi, \tag{2.9.23b}$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{R\sin^2\psi\dot{R}}{c^2}, \qquad \Gamma^\psi_{\vartheta\vartheta} = -\sin\psi\cos\psi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \tag{2.9.23c}$$

$$\Gamma^t_{\varphi\varphi} = \frac{R\sin^2\psi\sin^2\vartheta\dot{R}}{c^2}, \Gamma^\psi_{\varphi\varphi} = -\sin\psi\cos\psi\sin^2\vartheta, \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.9.23d}$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad R_{t\vartheta t\vartheta} = -R\sin^2\psi\ddot{R}, \tag{2.9.24a}$$

$$R_{t\varphi t\varphi} = -R\sin^2\psi\sin^2\vartheta\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sin^2\psi\left(\dot{R}^2+c^2\right)}{c^2}, \tag{2.9.24b}$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sin^2\psi\sin^2\vartheta\left(\dot{R}^2+c^2\right)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sin^4\psi\sin^2\vartheta\left(\dot{R}^2+c^2\right)}{c^2}. \tag{2.9.24c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{\psi\psi} = \frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}, \tag{2.9.25a}$$

$$R_{\vartheta\vartheta} = \sin^2\psi\frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}, \qquad R_{\varphi\varphi} = \sin^2\vartheta\sin^2\psi\frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}. \tag{2.9.25b}$$

The *Ricci scalar and Kretschmann read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+c^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2c^2+c^4}{R^4c^4}. \tag{2.9.26}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\sin\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\sin\psi\sin\vartheta}\partial_\varphi. \tag{2.9.27}$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\cot\psi}{R}, \tag{2.9.28a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sin\psi}. \tag{2.9.28b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{\cot\psi}{R}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sin\psi}. \tag{2.9.29}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \tag{2.9.30a}$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+c^2}{R^2c^2}. \tag{2.9.30b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2(\dot{R}^2+c^2)}{R^2c^2}. \tag{2.9.31}$$

Vanishing Curvature

$$\boxed{ds^2 = -c^2dt^2 + R^2\left\{d\psi^2+\psi^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\right\}} \tag{2.9.32}$$

Christoffel symbols:

$$\Gamma^{\psi}_{t\psi} = \frac{\dot{R}}{R}, \qquad \Gamma^{\vartheta}_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^{\varphi}_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.33a}$$

$$\Gamma^{t}_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad \Gamma^{\vartheta}_{\psi\vartheta} = \frac{1}{\psi}, \qquad \Gamma^{\varphi}_{\psi\varphi} = \frac{1}{\psi}, \tag{2.9.33b}$$

$$\Gamma^{t}_{\vartheta\vartheta} = \frac{R\psi^2\dot{R}}{c^2}, \qquad \Gamma^{\psi}_{\vartheta\vartheta} = -\psi, \qquad \Gamma^{\varphi}_{\vartheta\varphi} = \cot(\vartheta), \tag{2.9.33c}$$

$$\Gamma^{t}_{\varphi\varphi} = \frac{R\psi^2\sin^2\vartheta\dot{R}}{c^2}, \qquad \Gamma^{\psi}_{\varphi\varphi} = -\psi\sin^2\vartheta, \quad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.9.33d}$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad R_{t\vartheta t\vartheta} = -R\psi^2\ddot{R}, \tag{2.9.34a}$$

$$R_{t\varphi t\varphi} = -R\psi^2\sin^2\vartheta\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\psi^2\dot{R}^2}{c^2}, \tag{2.9.34b}$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\psi^2\sin^2\vartheta\dot{R}^2}{c^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\psi^4\sin^2\vartheta\dot{R}^2}{c^2}. \tag{2.9.34c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{\psi\psi} = \frac{R\ddot{R}+2\dot{R}^2}{c^2}, \tag{2.9.35a}$$

$$R_{\vartheta\vartheta} = \psi^2\frac{R\ddot{R}+2\dot{R}^2}{c^2}, \qquad R_{\varphi\varphi} = \sin^2\vartheta\,\psi^2\frac{R\ddot{R}+2\dot{R}^2}{c^2}. \tag{2.9.35b}$$

The *Ricci scalar and Kretschmann read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4}{R^4c^4}. \tag{2.9.36}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\psi\sin\vartheta}\partial_\varphi. \tag{2.9.37}$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{1}{R\psi}, \tag{2.9.38a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\vartheta)}{R\psi}. \tag{2.9.38b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = \frac{2}{R\psi}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\psi}. \tag{2.9.39}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \tag{2.9.40a}$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2}{R^2c^2}. \tag{2.9.40b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2}{R^2c^2}. \tag{2.9.41}$$

Negative Curvature

$$\boxed{ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \sinh^2\psi\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\right\}} \tag{2.9.42}$$

Christoffel symbols:

$$\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, \qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \tag{2.9.43a}$$

$$\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad \Gamma^\vartheta_{\psi\vartheta} = \coth\psi, \qquad \Gamma^\varphi_{\psi\varphi} = \coth\psi, \tag{2.9.43b}$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{R\sinh^2\psi\dot{R}}{c^2}, \qquad \Gamma^\psi_{\vartheta\vartheta} = -\sinh\psi\cosh\psi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.9.43c}$$

$$\Gamma^t_{\varphi\varphi} = \frac{R\sinh^2\psi\sin^2\vartheta\dot{R}}{c^2}, \qquad \Gamma^\psi_{\varphi\varphi} = -\sinh\psi\cosh\psi\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.9.43d}$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad R_{t\vartheta t\vartheta} = -R\sinh^2\psi\ddot{R}, \tag{2.9.44a}$$

$$R_{t\varphi t\varphi} = -R\sinh^2\psi\sin^2\vartheta\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sinh^2\psi\left(\dot{R}^2-c^2\right)}{c^2}, \tag{2.9.44b}$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sinh^2\psi\sin^2\vartheta\left(\dot{R}^2-c^2\right)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sinh\psi^4\sin^2\vartheta\left(\dot{R}^2-c^2\right)}{c^2}. \tag{2.9.44c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{\psi\psi} = \frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}, \tag{2.9.45a}$$

$$R_{\vartheta\vartheta} = \sinh^2\psi\frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}, \quad R_{\varphi\varphi} = \sin^2\vartheta\sin^2\psi\frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}. \tag{2.9.45b}$$

The *Ricci scalar and Kretschmann read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2-c^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4-2\dot{R}^2c^2+c^4}{R^4c^4}. \tag{2.9.46}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\sinh\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\sinh\psi\sin\vartheta}\partial_\varphi. \tag{2.9.47}$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\coth\psi}{R}, \tag{2.9.48a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sinh\psi}. \tag{2.9.48b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{\coth\psi}{R}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sinh\psi}. \tag{2.9.49}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \tag{2.9.50a}$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2-c^2}{R^2c^2}. \tag{2.9.50b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2(\dot{R}^2-c^2)}{R^2c^2}. \tag{2.9.51}$$

Further reading:
Rindler[Rin01]

2.10 Gödel Universe

Gödel introduced a homogeneous and rotating universe model in [Göd49]. We follow the notation of [KWSD04]

2.10.1 Cylindrical coordinates

The Gödel metric in cylindrical coordinates is

$$\boxed{ds^2 = -c^2dt^2 + \frac{dr^2}{1+[r/(2a)]^2} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]d\varphi^2 + dz^2 - 2r^2\frac{c}{\sqrt{2}a}dt\,d\varphi,} \tag{2.10.1}$$

where $2a$ is the Gödel radius.

Christoffel symbols:

$$\Gamma^t_{tr} = \frac{r}{2a^2}\frac{1}{1+[r/(2a)]^2}, \qquad \Gamma^\varphi_{tr} = -\frac{c}{\sqrt{2}ar}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.2a}$$

$$\Gamma^r_{t\varphi} = \frac{cr}{\sqrt{2}a}\left[1+\left(\frac{r}{2a}\right)\right]^2, \qquad \Gamma^r_{rr} = -\frac{r}{4a^2}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.2b}$$

$$\Gamma^t_{r\varphi} = \frac{r^3}{4\sqrt{2}ca^3}\frac{1}{1+[r/(2a)]^2}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.2c}$$

$$\Gamma^r_{\varphi\varphi} = r\left[1+\left(\frac{r}{2a}\right)^2\right]\left[1-\frac{1}{2}\left(\frac{r}{a}\right)^2\right]. \tag{2.10.2d}$$

Riemann-Tensor:

$$R_{trtr} = \frac{c^2}{2a^2}\frac{1}{1+[r/(2a)]^2}, \qquad R_{trr\varphi} = -\frac{cr^2}{2\sqrt{2}a^3}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.3a}$$

$$R_{t\varphi t\varphi} = \frac{c^2r^2}{2a^2}\frac{1}{1+[r/(2a)]^2}, \qquad R_{r\varphi r\varphi} = \frac{r^2}{2a^2}\frac{1+3[r/(2a)]^2}{1+[r/(2a)]^2}. \tag{2.10.3b}$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2}{a^2}, \qquad R_{t\varphi} = \frac{r^2c}{\sqrt{2}a^3}, \qquad R_{\varphi\varphi} = \frac{r^4}{2a^4}. \tag{2.10.4}$$

Ricci and Kretschmann scalar

$$\mathscr{R} = -\frac{1}{a^2}, \qquad \mathscr{K} = \frac{3}{a^4}. \tag{2.10.5}$$

cosmological constant:

$$\Lambda = \frac{R}{2} \tag{2.10.6}$$

Killing vectors:
An infinitesimal isometric transformation $x'^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu)$ leaves the metric unchanged, that is $g'_{\mu\nu}(x'^\sigma) = g_{\mu\nu}(x'^\sigma)$. A killing vector field ξ^μ is solution to the killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$. There exist five killing vector fields in Gödel's spacetime:

$$\underset{a}{\xi}^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \underset{b}{\xi}^\mu = \frac{1}{\sqrt{1+[r/(2a)]^2}}\begin{pmatrix}\frac{r}{\sqrt{2}c}\cos\varphi\\ a\left(1+[r/(2a)]^2\right)\sin\varphi\\ \frac{a}{r}\left(1+2[r/(2a)]^2\right)\cos\varphi\\ 0\end{pmatrix}, \quad \underset{c}{\xi}^\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \tag{2.10.7a}$$

$$\underset{d}{\xi}^\mu = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \underset{e}{\xi}^\mu = \frac{1}{\sqrt{1+[r/(2a)]^2}}\begin{pmatrix}\frac{r}{\sqrt{2}c}\sin\varphi\\ -a\left(1+[r/(2a)]^2\right)\cos\varphi\\ \frac{a}{r}\left(1+2[r/(2a)]^2\right)\sin\varphi\\ 0\end{pmatrix}. \tag{2.10.7b}$$

An arbitrary linear combination of killing vector fields is again a killing vector field.

Local tetrad:
For the local tetrad in Gödel's spacetime an ansatz similar to the local tetrad of a rotating spacetime in spherical coordinates (Sec. 1.4.7) can be used. After substituting $\vartheta \to z$ and swapping base vectors $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$ an orthonormalized and right-handed local tetrad is obtained.

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \quad \mathbf{e}_{(1)} = \sqrt{1+[r/(2a)]^2}\partial_r, \quad \mathbf{e}_{(2)} = \Delta\Gamma\left(A\partial_t + B\partial_\varphi\right), \quad \mathbf{e}_{(3)} = \partial_z, \tag{2.10.8a}$$

where

$$A = -\frac{r^2 c}{\sqrt{2}a} + \zeta r^2\left(1-[r/(2a)]^2\right), \qquad B = c^2 + \frac{\zeta r^2 c}{\sqrt{2}a}, \tag{2.10.9a}$$

$$\Gamma = \frac{1}{\sqrt{c^2+\zeta r^2 c\sqrt{2}/a - \zeta^2 r^2\left(1-[r/(2a)]^2\right)}}, \qquad \Delta = \frac{1}{rc\sqrt{1+[r/(2a)]^2}}. \tag{2.10.9b}$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ:

$$y^0 = y^{(0)}\Gamma + y^{(2)}\Delta\Gamma A, \quad y^1 = y^{(1)}\sqrt{1+[r/(2a)]^2}, \quad y^2 = y^{(0)}\Gamma\zeta + y^{(2)}\Delta\Gamma B, \quad y^3 = y^{(3)}. \tag{2.10.10}$$

with the above abbreviations.

2.10.2 Scaled cylindrical coordinates

If we apply the simple transformation

$$T = \frac{t}{r_G}, \qquad R = \frac{r}{r_G}, \qquad \phi = \varphi, \qquad Z = \frac{z}{r_G}, \tag{2.10.11}$$

with $r_G = 2a$, we find a formulation for the metric scaling with r_G, which is

$$\boxed{ds^2 = r_G^2\left(-c^2 dT^2 + \frac{dR^2}{1+R^2} + R^2(1-R^2)D\phi^2 + dZ^2 - 2\sqrt{2}cR^2\,dT\,d\phi\right).} \tag{2.10.12}$$

Christoffel symbols:

$$\Gamma^T_{TR} = \frac{2R}{1+R^2}, \qquad \Gamma^\phi_{TR} = -\frac{\sqrt{2}c}{R(1+R^2)}, \tag{2.10.13a}$$

$$\Gamma^R_{T\phi} = \sqrt{2}cR(1+R^2), \qquad \Gamma^R_{RR} = -\frac{R}{1+R^2}, \tag{2.10.13b}$$

$$\Gamma^T_{R\phi} = \frac{\sqrt{2}R^3}{c(1+R^2)}, \qquad \Gamma^\phi_{R\phi} = \frac{1}{R(1+R^2)}, \tag{2.10.13c}$$

$$\Gamma^R_{\phi\phi} = R(1+R^2)(2R^2-1). \tag{2.10.13d}$$

Riemann-Tensor:

$$R_{TRTR} = \frac{2r_G^2c^2}{1+R^2}, \qquad R_{TRR\phi} = -\frac{2\sqrt{2}r_G^2 cR^2}{1+R^2}, \tag{2.10.14a}$$

$$R_{T\phi T\phi} = 2c^2 r_G^2 R^2(1+R^2), \quad R_{R\phi R\phi} = \frac{2r_G^2R^2(1+3R^2)}{1+R^2}. \tag{2.10.14b}$$

Ricci-Tensor:

$$R_{TT} = 4c^2, \quad R_{T\phi} = 4\sqrt{2}cR^2, \quad R_{\phi\phi} = 8R^4. \tag{2.10.15}$$

Ricci and Kretschmann scalar

$$\mathscr{R} = -\frac{4}{r_G^2}, \qquad \mathscr{K} = \frac{48}{r_G^4}. \tag{2.10.16}$$

cosmological constant:

$$\Lambda = \frac{R}{2} \tag{2.10.17}$$

Killing vectors:
The Killing vectors read

$$\underset{a}{\xi}^{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{b}{\xi}^{\mu} = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2}c}\cos\varphi \\ \frac{1}{2}(1+R^2)\sin\varphi \\ \frac{1}{2R}(1+2R^2)\cos\varphi \\ 0 \end{pmatrix}, \quad \underset{c}{\xi}^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{2.10.18a}$$

$$\underset{d}{\xi}^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underset{e}{\xi}^{\mu} = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2}c}\sin\varphi \\ -\frac{1}{2}(1+R^2)\cos\varphi \\ \frac{1}{2R}(1+2R^2)\sin\varphi \\ 0 \end{pmatrix}. \tag{2.10.18b}$$

Local tetrad:
After the transformation to scaled cylindrical coordinates, the local tetrad reads

$$\mathbf{e}_{(0)} = \frac{\Gamma}{r_G}\left(\partial_T + \zeta\partial_\phi\right), \quad \mathbf{e}_{(1)} = \frac{1}{r_G}\sqrt{1+R^2}\,\partial_R, \qquad \mathbf{e}_{(2)} = \frac{\Delta\Gamma}{r_G}\left(A\partial_T + B\partial_\phi\right), \quad \mathbf{e}_{(3)} = \frac{1}{r_G}\partial_Z, \tag{2.10.19a}$$

where

$$A = R^2\left[-\sqrt{2}c + (1-R^2)\zeta\right], \qquad B = c^2 + \sqrt{2}R^2 c\zeta, \tag{2.10.20a}$$

$$\Gamma = \frac{1}{\sqrt{c^2 + 2\sqrt{2}R^2c\zeta - R^2(1-R^2)\zeta^2}}, \qquad \Delta = \frac{1}{Rc\sqrt{1+R^2}}. \tag{2.10.20b}$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^{μ}:

$$y^0 = \frac{\Gamma}{r_G}y^{(0)} + \frac{\Delta\Gamma A}{r_G}y^{(2)}, \qquad y^1 = \frac{1}{r_G}\sqrt{1+R^2}y^{(1)}, \qquad y^2 = \frac{\Gamma\zeta}{r_G}y^{(0)} + \frac{\Delta\Gamma B}{r_G}y^{(2)}, \qquad y^3 = \frac{1}{r_G}y^{(3)}, \tag{2.10.21}$$

and the back transformation is given by

$$y^{(0)} = \frac{r_G}{\Gamma}\frac{By^0 - Ay^2}{B-\zeta A}, \qquad y^{(1)} = \frac{r_G}{\sqrt{1+R^2}}y^1, \qquad y^{(2)} = \frac{r_G}{\Delta\Gamma}\frac{y^2 - \zeta y^0}{B-\zeta A}, \qquad y^{(3)} = r_G y^3. \tag{2.10.22a}$$

2.11 Halilsoy standing wave

The standing wave metric by Halilsoy[Hal88] reads

$$\boxed{ds^2 = V\left[e^{2K}\left(d\rho^2 - dt^2\right) + \rho^2 d\varphi^2\right] + \frac{1}{V}\left(dz + A\,d\varphi\right)^2,} \tag{2.11.1}$$

where

$$V = \cosh^2\alpha e^{-2CJ_0(\rho)\cos(t)} + \sinh^2\alpha e^{2CJ_0(\rho)\cos(t)}, \tag{2.11.2a}$$

$$K = \frac{C^2}{2}\left[\rho^2\left(J_0(\rho)^2 + J_1(\rho)^2\right) - 2\rho J_0(\rho)J_1(\rho)\cos^2 t\right], \tag{2.11.2b}$$

$$A = -2C\sinh(2\alpha)\rho J_1(\rho)\sin(t). \tag{2.11.2c}$$

with spherical Bessel functions $J_{1,2}$ and parameters α and C.
Local tetrad:

$$\mathbf{e}_{(0)} = \frac{e^{-K}}{\sqrt{V}}\partial_t, \qquad \mathbf{e}_{(1)} = \frac{e^{-K}}{\sqrt{V}}\partial_\rho, \qquad \mathbf{e}_{(2)} = \frac{1}{\rho\sqrt{V}}\partial_\varphi - \frac{A}{\rho\sqrt{V}}\partial_z, \qquad \mathbf{e}_{(3)} = \sqrt{V}\partial_z. \tag{2.11.3}$$

dual tetrad:

$$\theta^{(0)} = \sqrt{V}e^K\,dt, \qquad \theta^{(2)} = \sqrt{V}e^K\,d\rho, \qquad \theta^{(2)} = \sqrt{V}\rho\,d\varphi, \qquad \theta^{(3)} = \frac{1}{\sqrt{V}}\left(dz + A\,d\varphi\right). \tag{2.11.4}$$

2.12 Janis-Newman-Winicour

The Janis-Newman-Winicour[JNW68] spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ is represented by the line element

$$\boxed{ds^2 = -\alpha^\gamma c^2 dt^2 + \alpha^{-\gamma} dr^2 + r^2 \alpha^{-\gamma+1}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.12.1}$$

where $\alpha = 1 - r_s/(\gamma r)$. The Schwarzschild radius $r_s = 2GM/c^2$ is defined by Newton's constant G, the speed of light c, and the mass parameter M. For $\gamma = 1$, we obtain the Schwarzschild metric (2.2.1).

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{r_s c^2}{2r^2}\alpha^{2\gamma-1}, \qquad \Gamma^t_{tr} = \frac{r_s}{2\gamma r^2 \alpha}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2\gamma r^2 \alpha}, \tag{2.12.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \qquad \Gamma^\varphi_{r\varphi} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \qquad \Gamma^r_{\vartheta\vartheta} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma}, \tag{2.12.2b}$$

$$\Gamma^r_{\varphi\varphi} = \Gamma^r_{\vartheta\vartheta}\sin^2\vartheta, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.12.2c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{r_s c^2\left[2\gamma r - r_s(\gamma+1)\right]\alpha^{\gamma-2}}{2\gamma r^4}, \qquad R_{t\vartheta t\vartheta} = \frac{r_s c^2\left[2\gamma r - r_s(\gamma+1)\right]\alpha^{\gamma-1}}{4\gamma r^2}, \tag{2.12.3a}$$

$$R_{t\varphi t\varphi} = \frac{r_s c^2\left[2\gamma r - r_s(\gamma+1)\right]\alpha^{\gamma-1}\sin^2\vartheta}{4\gamma r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s\left[2\gamma^2 r - r_s(\gamma+1)\right]}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \tag{2.12.3b}$$

$$R_{r\varphi r\varphi} = -\frac{r_s\left[2\gamma^2 r - r_s(\gamma+1)\right]\sin^2\vartheta}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s\left[4\gamma^2 r - r_s(\gamma+1)^2\right]\sin^2\vartheta}{4\gamma^2\alpha^\gamma}. \tag{2.12.3c}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{r_s c^2 \alpha^{\gamma-2}\beta}{6\gamma^2 r^4}, \qquad C_{t\vartheta t\vartheta} = \frac{r_s c^2 \alpha^{\gamma-1}\beta}{12\gamma^2 r^2}, \tag{2.12.4a}$$

$$C_{t\varphi t\varphi} = \frac{r_s c^2 \alpha^{\gamma-1}\beta\sin^2\vartheta}{12\gamma^2 r^2}, \qquad C_{r\vartheta r\vartheta} = -\frac{r_s\beta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \tag{2.12.4b}$$

$$C_{r\varphi r\varphi} = -\frac{r_s\beta\sin^2\vartheta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \qquad C_{\vartheta\varphi\vartheta\varphi} = \frac{r_s\beta\sin^2\vartheta}{6\gamma^2\alpha^\gamma}, \tag{2.12.4c}$$

where $\beta = 6\gamma^2 r - r_s(\gamma+1)(2\gamma+1)$.

Ricci-Tensor:

$$R_{rr} = \frac{r_s^2(1-\gamma^2)}{2\gamma^2 r^4 \alpha^2}. \tag{2.12.5}$$

The *Ricci scalar* reads

$$\mathscr{R} = \frac{r_s^2(1-\gamma^2)\alpha^{\gamma-2}}{2\gamma^2 r^4}, \tag{2.12.6}$$

whereas the *Kretschmann scalar* is given by

$$\mathscr{K} = \frac{r_s^2\alpha^{2\gamma-4}}{4\gamma^4 r^8}\left[7\gamma^2 r_s^2(2+\gamma^2) + 48\gamma^4 r^2\alpha + 8\gamma r_s(2\gamma^2+1)(r_s - 2\gamma r) + 3r_s^2\right]. \tag{2.12.7}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\alpha^{\gamma/2}}\partial_t, \qquad \mathbf{e}_{(r)} = \alpha^{\gamma/2}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{\alpha^{(\gamma-1)/2}}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{\alpha^{(\gamma-1)/2}}{r\sin\vartheta}\partial_\varphi. \tag{2.12.8}$$

Dual tetrad:

$$\theta^{(t)} = c\alpha^{\gamma/2}dt, \qquad \theta^{(r)} = \frac{dr}{\alpha^{\gamma/2}}, \qquad \theta^{(\vartheta)} = \frac{r}{\alpha^{(\gamma-1)/2}}d\vartheta, \qquad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\alpha^{(\gamma-1)/2}}d\varphi. \tag{2.12.9}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \tag{2.12.10a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.12.10b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4\gamma r - r_s(2+\gamma)}{2\gamma r^2}\alpha^{(\gamma-1)/2}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.12.11}$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \qquad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\varphi)}^{(\varphi)} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \tag{2.12.12a}$$

$$c_{(\vartheta)(\varphi)}^{(\varphi)} = -\frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.12.12b}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\alpha^{\gamma}\left(\frac{h^2\alpha^{\gamma-1}}{r^2} - \kappa c^2\right) \tag{2.12.13}$$

with the constants of motion $h = r^2\alpha^{-\gamma+1}\dot{\varphi}$ and $k = \alpha^{\gamma}c^2\dot{t}$. For null geodesics ($\kappa = 0$) and $\gamma > \frac{1}{2}$, there is an extremum at

$$r = r_s\frac{1+2\gamma}{2\gamma}. \tag{2.12.14}$$

Embedding:
The embedding function $z = z(r)$ for $r \in [r_s(\gamma+1)^2/(4\gamma^2), \infty)$ follows from

$$\frac{dz}{dr} = \sqrt{\frac{r_s\left[4r\gamma^2 - r_s(1+\gamma)^2\right]}{4r^2\gamma^2\alpha^{\gamma+1}}}. \tag{2.12.15}$$

However, the analytic solution

$$z(r) = 2\sqrt{r_s r}F_1\left(-\frac{1}{2};\frac{\gamma+1}{2},-\frac{1}{2};\frac{1}{2},\frac{r_s}{r\gamma},\frac{r_s(1+\gamma)^2}{4r\gamma^2}\right) - \frac{2\pi\gamma}{\gamma+1}{}_2F_1\left(-\frac{1}{2},\frac{\gamma+1}{2};1;\frac{4\gamma}{(\gamma+1)^2}\right), \tag{2.12.16}$$

depends on the Appell-F_1- and the Hypergeometric-${}_2F_1$-function.

2.13 Kasner

The Kasner spacetime in Cartesian coordinates (t,x,y,z) is represented by the line element[MTW73, Kas21] $(c=1)$

$$\boxed{ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}tz^2,} \tag{2.13.1}$$

where p_1, p_2, p_3 have to fulfill the two conditions

$$p_1 + p_2 + p_3 = 1 \qquad \text{and} \qquad p_1^2 + p_2^2 + p_3^2 = 1. \tag{2.13.2}$$

These two conditions can also be represented by the Khalatnikov-Lifshitz parameter u with

$$p_1 = -\frac{u}{1+u+u^2}, \qquad p_2 = \frac{1+u}{1+u+u^2}, \qquad p_3 = \frac{u(1+u)}{1+u+u^2}. \tag{2.13.3}$$

Christoffel symbols:

$$\Gamma^x_{tx} = \frac{p_1}{t}, \qquad \Gamma^y_{ty} = \frac{p_2}{t}, \qquad \Gamma^z_{tz} = \frac{p_3}{t}, \tag{2.13.4a}$$

$$\Gamma^t_{xx} = \frac{p_1 t^{2p_1}}{t}, \qquad \Gamma^t_{yy} = \frac{p_2 t^{2p_2}}{t}, \qquad \Gamma^t_{zz} = \frac{p_3 t^{2p_3}}{t}. \tag{2.13.4b}$$

Partial derivatives

$$\Gamma^x_{tx,t} = -\frac{p_1}{t^2}, \qquad \Gamma^t_{ty,t} = -\frac{p_2}{t^2}, \qquad \Gamma^z_{tz,t} = -\frac{p_3}{t^2}, \tag{2.13.5a}$$

$$\Gamma^t_{xx,t} = p_1(2p_1-1)t^{2p_1-2}, \qquad \Gamma^t_{yy,t} = p_2(2p_2-1)t^{2p_2-2}, \qquad \Gamma^t_{zz,t} = p_3(2p_3-1)t^{2p_3-2}. \tag{2.13.5b}$$

Riemann-Tensor:

$$R_{txtx} = \frac{p_1(1-p_1)t^{2p_1}}{t^2}, \qquad R_{tyty} = \frac{p_2(1-p_2)t^{2p_2}}{t^2}, \qquad R_{tztz} = \frac{p_3(1-p_3)t^{2p_3}}{t^2}, \tag{2.13.6a}$$

$$R_{xyxy} = \frac{p_1p_2t^{2p_1}t^{2p_2}}{t^2}, \qquad R_{xzxz} = \frac{p_1p_3t^{2p_1}t^{2p_3}}{t^2}, \qquad .R_{yzyz} = \frac{p_2p_3t^{2p_2}t^{2p_3}}{t^2}. \tag{2.13.6b}$$

The Ricci tensor as well as the Ricci scalar vanish identically. The Kretschmann scalar reads

$$\mathscr{K} = \frac{4}{t^4}\left(p_1^2 - 2p_1^3 + p_1^4 + p_2^2 - 2p_2^3 + p_2^4 + p_1^2p_3^2 + p_3^2 - 2p_3^3 + p_3^4 + p_1^2p_2^2 + p_2^2p_3^2\right) \tag{2.13.7a}$$

$$= \frac{16u^2(1+u)^2}{t^4(1+u+u^2)^3}. \tag{2.13.7b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \qquad \mathbf{e}_{(x)} = t^{-p_1}\partial_x, \qquad \mathbf{e}_{(y)} = t^{-p_2}\partial_y, \qquad \mathbf{e}_{(z)} = t^{-p_3}\partial_z. \tag{2.13.8}$$

Dual tetrad:

$$\theta^{(t)} = dt, \qquad \theta^{(x)} = t^{p_1}dx, \qquad \theta^{(y)} = t^{p_2}dy, \qquad \theta^{(z)} = t^{p_3}dz. \tag{2.13.9}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \frac{p_1}{t}, \qquad \gamma_{(t)(\vartheta)(\vartheta)} = \frac{p_2}{t}, \qquad \gamma_{(t)(\varphi)(\varphi)} = \frac{p_3}{t}. \tag{2.13.10}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\frac{1}{t}. \tag{2.13.11}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(y)(x)} = \frac{p_1(1-p_1)}{t^2}, \qquad R_{(t)(y)(t)(y)} = \frac{p_2(1-p_2)}{t^2}, \qquad R_{(t)(z)(t)(z)} = \frac{p_3(1-p_3)}{t^2}, \tag{2.13.12a}$$

$$R_{(x)(y)(x)(y)} = \frac{p_1p_2}{t^2}, \qquad R_{(x)(z)(x)(z)} = \frac{p_1p_3}{t^2}, \qquad R_{(y)(z)(y)(z)} = \frac{p_2p_3}{t^2}. \tag{2.13.12b}$$

2.14 Kerr

The Kerr spacetime, found by Roy Kerr in 1963[Ker63], describes a rotating black hole.

2.14.1 Boyer-Lindquist coordinates

The Kerr metric in Boyer-Lindquist coordinates

$$\boxed{\begin{aligned} ds^2 &= -\left(1-\frac{r_s r}{\Sigma}\right)c^2dt^2 - \frac{2r_s ar\sin^2\vartheta}{\Sigma}c\,dt\,d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 \\ &\quad + \left(r^2+a^2+\frac{r_s a^2 r\sin^2\vartheta}{\Sigma}\right)\sin^2\vartheta\,d\varphi^2, \end{aligned}} \tag{2.14.1}$$

with $\Sigma = r^2 + a^2\cos^2\vartheta$, $\Delta = r^2 - r_s r + a^2$, and $r_s = 2GM/c^2$, is taken from Bardeen[BPT72]. M is the mass and a is the angular momentum per unit mass of the black hole. The contravariant form of the metric reads

$$\partial_s^2 = -\frac{A}{c^2\Sigma\Delta}\partial_t^2 - \frac{2r_s ar}{c\Sigma\Delta}\partial_t\partial_\varphi + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\vartheta^2 + \frac{\Delta - a^2\sin^2\vartheta}{\Sigma\Delta\sin^2\vartheta}\partial_\varphi^2, \tag{2.14.2}$$

where $A = \left(r^2+a^2\right)^2 - a^2\Delta\sin^2\vartheta = \left(r^2+a^2\right)\Sigma + r_s a^2 r\sin^2\vartheta$.
The event horizon r_+ is defined by the outer root of Δ,

$$r_+ = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2}, \tag{2.14.3}$$

whereas the outer boundary r_0 of the ergosphere follows from the outer root of $\Sigma - r_s r$,

$$r_0 = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2\cos^2\vartheta}, \tag{2.14.4}$$

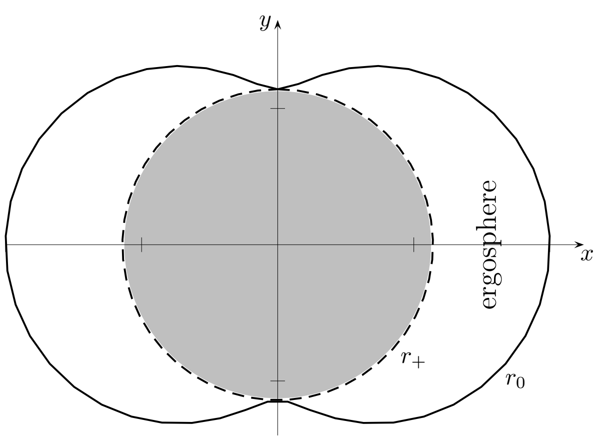

Figure 2.1: Ergosphere and horizon (dashed circle) for $a = 0.99\frac{r_s}{2}$.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s \Delta (r^2 - a^2\cos^2\vartheta)}{2\Sigma^3}, \qquad \Gamma^\vartheta_{tt} = -\frac{c^2 r_s a^2 r \sin\vartheta\cos\vartheta}{\Sigma^3}, \tag{2.14.5a}$$

$$\Gamma^t_{tr} = \frac{r_s(r^2+a^2)(r^2-a^2\cos^2\vartheta)}{2\Sigma^2\Delta}, \qquad \Gamma^\varphi_{tr} = \frac{c r_s a (r^2 - a^2\cos^2\vartheta)}{2\Sigma^2\Delta}, \tag{2.14.5b}$$

$$\Gamma^t_{t\vartheta} = -\frac{r_s a^2 r\sin\vartheta\cos\vartheta}{\Sigma^2}, \qquad \Gamma^\varphi_{t\vartheta} = -\frac{c r_s a r\cot\vartheta}{\Sigma^2}, \tag{2.14.5c}$$

$$\Gamma^r_{t\varphi} = -\frac{c\Delta r_s a \sin^2\vartheta (r^2 - a^2\cos^2\vartheta)}{2\Sigma^3}, \qquad \Gamma^\vartheta_{t\varphi} = \frac{c r_s a r (r^2+a^2)\sin\vartheta\cos\vartheta}{\Sigma^3}, \tag{2.14.5d}$$

$$\Gamma^r_{rr} = \frac{2ra^2\sin^2\vartheta - r_s(r^2-a^2\cos^2\vartheta)}{2\Sigma\Delta}, \qquad \Gamma^\vartheta_{rr} = \frac{a^2\sin\vartheta\cos\vartheta}{\Sigma\Delta}, \tag{2.14.5e}$$

$$\Gamma^r_{r\vartheta} = -\frac{a^2\sin\vartheta\cos\vartheta}{\Sigma}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{r}{\Sigma}, \tag{2.14.5f}$$

$$\Gamma^r_{\vartheta\vartheta} = -\frac{r\Delta}{\Sigma}, \qquad \Gamma^\vartheta_{\vartheta\vartheta} = -\frac{a^2\sin\vartheta\cos\vartheta}{\Sigma}, \tag{2.14.5g}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \frac{\cot\vartheta}{\Sigma^2}\left[\Sigma^2 + r_s a^2 r\sin^2\vartheta\right], \qquad \Gamma^t_{\vartheta\varphi} = \frac{r_s a^3 r\sin^3\vartheta\cos\vartheta}{c\Sigma^2}, \tag{2.14.5h}$$

$$\Gamma^t_{r\varphi} = \frac{r_s a\sin^2\vartheta\left[a^2\cos^2\vartheta(a^2-r^2) - r^2(a^2+3r^2)\right]}{2c\Sigma^2\Delta}, \tag{2.14.5i}$$

$$\Gamma^\varphi_{r\varphi} = \frac{2r\Sigma^2 + r_s\left[a^4\sin^2\vartheta\cos^2\vartheta - r^2(\Sigma + r^2 + a^2)\right]}{2\Sigma^2\Delta}, \tag{2.14.5j}$$

$$\Gamma^r_{\varphi\varphi} = \frac{\Delta\sin^2\vartheta}{2\Sigma^3}\left[-2r\Sigma^2 + r_s a^2\sin^2\vartheta(r^2 - a^2\cos^2\vartheta)\right], \tag{2.14.5k}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\frac{\sin\vartheta\cos\vartheta}{\Sigma^3}\left[A\Sigma + \left(r^2+a^2\right) r_s a^2 r\sin^2\vartheta\right], \tag{2.14.5l}$$

General local tetrad:

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \tag{2.14.6a}$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \qquad \mathbf{e}_{(3)} = \frac{\Gamma}{c}\left(\mp\frac{g_{t\varphi}+\zeta g_{\varphi\varphi}}{\sqrt{\Delta}\sin\vartheta}\partial_t \pm \frac{g_{tt}+\zeta g_{t\varphi}}{\sqrt{\Delta}\sin\vartheta}\partial_\varphi\right), \tag{2.14.6b}$$

where $-\Gamma^{-2} = g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}$,

$$\Gamma^{-2} = \left(1 - \frac{r_s r}{\Sigma}\right) + \frac{2r_s a r\sin^2\vartheta}{\Sigma}\frac{\zeta}{c} - \left(r^2 + a^2 + \frac{r_s a^2 r\sin^2\vartheta}{\Sigma}\right)\frac{\zeta^2}{c^2}\sin^2\vartheta \tag{2.14.7}$$

Non-rotating local tetrad ($\zeta = \omega$)**:**

$$\mathbf{e}_{(0)} = \sqrt{\frac{A}{\Sigma\Delta}}\left(\frac{1}{c}\partial_t + \omega\partial_\varphi\right), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \quad \mathbf{e}_{(3)} = \sqrt{\frac{\Sigma}{A}}\frac{1}{\sin\vartheta}\partial_\varphi, \tag{2.14.8}$$

where $\omega = -g_{t\varphi}/g_{\varphi\varphi} = r_s a r/A$.

Dual tetrad:

$$\theta^{(2)} = \sqrt{\frac{\Sigma\Delta}{A}}c\,dt, \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}}dr, \quad \theta^{(2)} = \sqrt{\Sigma}d\vartheta, \quad \theta^{(3)} = \sqrt{\frac{A}{\Sigma}}\sin\vartheta\left(d\varphi - \omega\, d\varphi\right). \tag{2.14.9}$$

The relation between the constants of motion E, L, Q, and μ (defined in Bardeen[BPT72]) and the initial direction υ, compare Sec. (1.4.5), with respect to the LNRF reads ($c = 1$)

$$\upsilon^{(0)} = \sqrt{\frac{A}{\Sigma\Delta}}E - \frac{r_s r a}{\sqrt{A\Sigma\Delta}}L, \qquad \upsilon^{(1)} = \sqrt{\frac{\Delta}{\Sigma}}p_r, \tag{2.14.10a}$$

$$\upsilon^{(2)} = \frac{1}{\sqrt{\Sigma}}\sqrt{Q - \cos^2\vartheta\left[a^2\left(\mu^2 - E^2\right) + \frac{L^2}{\sin^2\vartheta}\right]}, \qquad \upsilon^{(3)} = \sqrt{\frac{\Sigma}{A}}\frac{L}{\sin\vartheta}. \tag{2.14.10b}$$

Static local tetrad ($\zeta = 0$)**:**

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1 - r_s r/\Sigma}}\partial_t, \qquad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \qquad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \tag{2.14.11a}$$

$$\mathbf{e}_{(3)} = \pm\frac{r_s a r\sin\vartheta}{c\sqrt{1 - r_s r/\Sigma}\sqrt{\Delta\Sigma}}\partial_t \mp \frac{\sqrt{1 - r_s r/\Sigma}}{\sqrt{\Delta}\sin\vartheta}\partial_\varphi. \tag{2.14.11b}$$

Photon orbits:
The direct(-) and retrograd(+) photon orbits have radius

$$r_{\text{po}} = r_s\left[1 + \cos\left(\frac{2}{3}\arccos\frac{\mp 2a}{r_s}\right)\right]. \tag{2.14.12}$$

Marginally stable timelike circular orbits
are defined via

$$r_{\text{ms}} = \frac{r_s}{2}\left(3 + Z_2 \mp \sqrt{(3 - Z_1)(2 + Z_1 + 2Z_2)}\right), \tag{2.14.13}$$

where

$$Z_1 = 1 + \left(1 - \frac{4a^2}{r_s^2}\right)^{1/3}\left[\left(1 + \frac{2a}{r_s}\right)^{1/3} + \left(1 - \frac{2a}{r_s}\right)^{1/3}\right], \tag{2.14.14a}$$

$$Z_2 = \sqrt{\frac{12a^2}{r_s^2} + Z_1^2}. \tag{2.14.14b}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = 0 \tag{2.14.15}$$

with the effective potential

$$V_{\text{eff}} = \frac{1}{2r^3}\left\{h^2(r - r_s) + 2\frac{ahk}{c}r_s - \frac{k^2}{c^2}\left[r^3 + a^2(r + r_s)\right]\right\} - \frac{\kappa c^2\Delta}{r^2} \tag{2.14.16}$$

and the constants of motion

$$k = \left(1 - \frac{r_s}{r}\right)c^2\dot{t} + \frac{c r_s a}{r}\dot{\varphi}, \qquad h = \left(r^2 + a^2 + \frac{r_s a^2}{r}\right)\dot{\varphi} - \frac{c r_s a}{r}\dot{t}. \tag{2.14.17}$$

Further reading:
Boyer and Lindquist[BL67], Wilkins[Wil72], Brill[BC66].

2.15 Kottler spacetime

The Kottler spacetime is represented in spherical coordinates (t,r,ϑ,φ) by the line element[Per04]

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}-\frac{\Lambda r^2}{3}\right)c^2dt^2+\frac{1}{1-r_s/r-\Lambda r^2/3}dr^2+r^2d\Omega^2,} \tag{2.15.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and Λ is the cosmological constant. If $\Lambda > 0$ the metric is also known as Schwarzschild-deSitter metric, whereas if $\Lambda < 0$ it is called Schwarzschild-anti-deSitter.
For the following, we define the two abbreviations

$$\alpha = 1-\frac{r_s}{r}-\frac{\Lambda r^2}{3} \qquad \text{and} \qquad \beta = \frac{r_s}{r}-\frac{2\Lambda}{3}r^2. \tag{2.15.2}$$

The critical points of the Kottler metric follow from the roots of the cubic equation $\alpha = 0$. These can be found by means of the parameters $p = -1/\Lambda$ and $q = 3r_s/(2\Lambda)$. If $\Lambda < 0$, we have only one real root

$$r_1 = \frac{2}{\sqrt{-\Lambda}}\sinh\left[\frac{1}{3}\operatorname{arsinh}\left(\frac{3r_s}{2}\sqrt{-\Lambda}\right)\right]. \tag{2.15.3}$$

If $\Lambda > 0$, we have to distinguish whether $D \equiv q^2+p^3 = 9r_s^2/(4\Lambda^2)-\Lambda^{-3}$ is positive or negative. If $D > 0$, there is no real positive root. For $D < 0$, the two real positive roots read

$$r_\pm = \frac{2}{\sqrt{\Lambda}}\cos\left[\frac{\pi}{3}\pm\frac{1}{3}\arccos\left(\frac{3r_s}{2}\sqrt{\Lambda}\right)\right] \tag{2.15.4}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2\alpha\beta}{2r}, \qquad \Gamma^t_{tr} = \frac{\beta}{2r\alpha}, \qquad \Gamma^r_{rr} = -\frac{\beta}{2r\alpha}, \tag{2.15.5a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -\alpha r, \tag{2.15.5b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -\alpha r\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.15.5c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2\left(3r_s+\Lambda r^3\right)}{3r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}c^2\alpha\beta, \tag{2.15.6a}$$

$$R_{t\varphi t\varphi} = \frac{1}{2}c^2\alpha\beta\sin^2\vartheta, \qquad R_{r\vartheta r\vartheta} = -\frac{\beta}{2\alpha}, \tag{2.15.6b}$$

$$R_{r\varphi r\varphi} = -\frac{\beta}{2\alpha}\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = r\left(r_s+\frac{\Lambda r^3}{3}\right)\sin^2\vartheta. \tag{2.15.6c}$$

Ricci-Tensor:

$$R_{tt} = -c^2\alpha\Lambda, \qquad R_{rr} = \frac{\Lambda}{\alpha}, \qquad R_{\vartheta\vartheta} = \Lambda r^2, \qquad R_{\varphi\varphi} = \Lambda r^2\sin^2\vartheta. \tag{2.15.7}$$

The *Ricci scalar* and the *Kretschmann scalar* read

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = 12\frac{r_s^2}{r^6}+\frac{8\Lambda^2}{3}. \tag{2.15.8}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2r_s}{r^3}, \qquad C_{t\vartheta t\vartheta} = \frac{c^2\alpha r_s}{2r}, \qquad C_{t\varphi t\varphi} = \frac{c^2\alpha r_s\sin^2\vartheta}{2r}, \tag{2.15.9a}$$

$$C_{r\vartheta r\vartheta} = -\frac{r_s}{2r\alpha}, \qquad C_{r\varphi r\varphi} = -\frac{r_s\sin^2\vartheta}{2r\alpha}, \qquad C_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.15.9b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{\alpha}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\alpha}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.15.10}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{\alpha}\,dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{\alpha}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.15.11}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s - \frac{2}{3}\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{\alpha}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.15.12}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s - 2\Lambda r^3}{2r^2\sqrt{\alpha}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.15.13}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{\Lambda r^3 + 3r_s}{3r^3}, \tag{2.15.14a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{3r_s - 2\Lambda r^3}{6r^3}. \tag{2.15.14b}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.15.15a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.15.15b}$$

Embedding:
The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{r_s/r + \Lambda r^2/3}{1 - r_s/r - \Lambda r^2/3}}. \tag{2.15.16}$$

Euler-Lagrange:
The Euler-Lagrangian formalism[Rin01] yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \tag{2.15.17}$$

with the constants of motion $k = (1 - r_s/r - \Lambda r^2/3)c^2\dot{t}$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.8.2).
As in the Schwarzschild metric, the effective potential has only one extremum for null geodesics, the so called photon orbit at $r = \frac{3}{2}r_s$. For timelike geodesics, however, we have

$$\frac{dV_{\text{eff}}}{dr} = \frac{h^2(-6r + 9r_s) + c^2r^2(3r_s - 2r^3\Lambda)}{3r^4} \stackrel{!}{=} 0. \tag{2.15.18}$$

This polynomial of fifth order might have up to five extrema.

Further reading:
Kottler[Kot18], Weyl[Wey19], Hackmann[HL08], Cruz[COV05].

2.16 Morris-Thorne

The most simple wormhole geometry is represented by the metric of Morris and Thorne[MT88],

$$\boxed{ds^2 = -c^2dt^2 + dl^2 + (b_0^2 + l^2)\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.16.1}$$

where b_0 is the throat radius and l is the proper radial coordinate; and $\{t \in \mathbb{R}, l \in \mathbb{R}, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$.

Christoffel symbols:

$$\Gamma^{\vartheta}_{l\vartheta} = \frac{l}{b_0^2+l^2}, \qquad \Gamma^{\varphi}_{l\varphi} = \frac{l}{b_0^2+l^2}, \qquad \Gamma^{l}_{\vartheta\vartheta} = -l, \tag{2.16.2a}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{l}_{\varphi\varphi} = -l\sin^2\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.16.2b}$$

Partial derivatives

$$\Gamma^{\vartheta}_{l\vartheta,l} = -\frac{l^2-b_0^2}{(b_0^2+l^2)^2}, \qquad \Gamma^{\varphi}_{l\varphi,l} = -\frac{l^2-b_0^2}{(b_0^2+l^2)^2}, \qquad \Gamma^{l}_{\vartheta\vartheta,l} = -1, \tag{2.16.3a}$$

$$\Gamma^{\varphi}_{\vartheta\varphi,\vartheta} = -\frac{1}{\sin^2\vartheta}, \qquad \Gamma^{l}_{\varphi\varphi,l} = -\sin^2\vartheta, \qquad \Gamma^{l}_{\varphi\varphi,\vartheta} = -l\sin(2\vartheta), \tag{2.16.3b}$$

$$\Gamma^{\vartheta}_{\varphi\varphi,\vartheta} = -\cos(2\vartheta). \tag{2.16.3c}$$

Riemann-Tensor:

$$R_{l\vartheta l\vartheta} = -\frac{b_0^2}{b_0^2+l^2}, \qquad R_{l\varphi l\varphi} = -\frac{b_0^2\sin^2\vartheta}{b_0^2+l^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = b_0^2\sin^2\vartheta. \tag{2.16.4}$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{ll} = -2\frac{b_0^2}{\left(b_0^2+l^2\right)^2}, \qquad \mathscr{R} = -2\frac{b_0^2}{\left(b_0^2+l^2\right)^2}, \qquad \mathscr{K} = \frac{12b_0^4}{\left(b_0^2+l^2\right)^4}. \tag{2.16.5}$$

Weyl-Tensor:

$$C_{tltl} = -\frac{2}{3}\frac{c^2b_0^2}{\left(b_0^2+l^2\right)^2}, \qquad C_{t\vartheta t\vartheta} = \frac{1}{3}\frac{c^2b_0^2}{b_0^2+l^2}, \qquad C_{t\varphi t\varphi} = \frac{1}{3}\frac{c^2b_0^2\sin^2\vartheta}{b_0^2+l^2}, \tag{2.16.6a}$$

$$C_{l\vartheta l\vartheta} = -\frac{1}{3}\frac{b_0^2}{b_0^2+l^2}, \qquad C_{l\varphi l\varphi} = -\frac{1}{3}\frac{b_0^2\sin^2\vartheta}{b_0^2+l^2}, \qquad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3}b_0^2\sin^2\vartheta. \tag{2.16.6b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(l)} = \partial_l, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{\sqrt{b_0^2+l^2}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\sqrt{b_0^2+l^2}\,\sin\vartheta}\partial_\varphi. \tag{2.16.7}$$

Dual tetrad

$$\theta^{(t)} = c\,dt, \qquad \theta^{(l)} = dl, \qquad \theta^{(\vartheta)} = \sqrt{b_0^2+l^2}\,d\vartheta, \qquad \theta^{(\varphi)} = \sqrt{b_0^2+l^2}\sin\vartheta\,d\varphi. \tag{2.16.8}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{l}{b_0^2+l^2}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sqrt{b_0^2+l^2}}. \tag{2.16.9}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2l}{b_0^2+l^2}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sqrt{b_0^2+l^2}}. \tag{2.16.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(l)(\vartheta)(l)(\vartheta)} = R_{(l)(\varphi)(l)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{b_0^2}{\left(b_0^2+l^2\right)^2}. \tag{2.16.11}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(l)(l)} = -\frac{2b_0^2}{\left(b_0^2+l^2\right)^2}. \tag{2.16.12}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(l)(t)(l)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2b_0^2}{3\left(b_0^2+l^2\right)^2}, \tag{2.16.13a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(l)(\vartheta)(l)(\vartheta)} = -C_{(l)(\varphi)(l)(\varphi)} = \frac{b_0^2}{3\left(b_0^2+l^2\right)^2}. \tag{2.16.13b}$$

Embedding:
The embedding function reads

$$z(r) = \pm b_0 \ln\left[\frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0}\right)^2 - 1}\right] \tag{2.16.14}$$

with $r^2 = b_0^2 + l^2$.

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{l}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\left(\frac{h^2}{b_0^2+l^2} - \kappa c^2\right), \tag{2.16.15}$$

with the constants of motion $k = c^2\dot{t}$ and $h = (b_0^2+l^2)\dot{\varphi}$. The shape of the effective potential V_{eff} is independend of the geodesic type. The maximum of the effective potential is located at $l = 0$.

A geodesic that starts at $l = l_i$ with direction $\mathbf{y} = \pm\mathbf{e}_{(t)} + \cos\xi\,\mathbf{e}_{(l)} + \sin\xi\,\mathbf{e}_{(\varphi)}$ approaches the wormhole throat asymptotically for $\xi = \xi_{\text{crit}}$ with

$$\xi_{\text{crit}} = \arcsin\frac{b_0}{\sqrt{b_0^2 + l_i^2}}. \tag{2.16.16}$$

This critical angle is independent of the type of the geodesic.

Further reading:
Ellis[Ell73], Visser[Vis95], Müller[Mül04, Mül08a]

2.17 Oppenheimer-Snyder collapse

2.17.1 Outer metric

The metric of the outer spacetime, $R > R_b$, in comoving coordinates $(\tau, R, \vartheta, \varphi)$ with $(c = 1)$ is given by

$$\boxed{ds^2 = -d\tau^2 + \frac{R}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}} dR^2 + \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3} \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).} \tag{2.17.1}$$

Christoffel symbols:

$$\Gamma^R_{\tau R} = \frac{1}{2}\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\vartheta_{\tau\vartheta} = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \tag{2.17.2a}$$

$$\Gamma^\varphi_{\tau\varphi} = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\tau_{RR} = \frac{R\sqrt{r_s}}{2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{5/3}}, \tag{2.17.2b}$$

$$\Gamma^R_{RR} = -\frac{3\sqrt{r_s}\tau}{4\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)R}, \qquad \Gamma^\vartheta_{R\vartheta} = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \tag{2.17.2c}$$

$$\Gamma^\varphi_{R\varphi} = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\tau_{\vartheta\vartheta} = -\sqrt{r_s}\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}, \tag{2.17.2d}$$

$$\Gamma^R_{\vartheta\vartheta} = -\frac{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}{\sqrt{R}}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.17.2e}$$

$$\Gamma^\tau_{\varphi\varphi} = -\sqrt{r_s}\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{2.17.2f}$$

$$\Gamma^R_{\varphi\varphi} = -\frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)\sin^2\vartheta}{\sqrt{R}}. \tag{2.17.2g}$$

Riemann-Tensor:

$$R_{\tau R\tau R} = -\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{8/3}}, \qquad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{2/3}}, \tag{2.17.3a}$$

$$R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_s\sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{2/3}}, \qquad R_{R\vartheta R\vartheta} = -\frac{1}{2}\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{4/3}}, \tag{2.17.3b}$$

$$R_{R\varphi R\varphi} = -\frac{1}{2}\frac{Rr_s\sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{4/3}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{2/3} r_s\sin^2\vartheta. \tag{2.17.3c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathscr{K} = 12\frac{r_s^2}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^4}. \tag{2.17.4}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \qquad \mathbf{e}_{(R)} = \frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}}{\sqrt{R}}\partial_R, \tag{2.17.5a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}\sin\vartheta}\partial_\varphi. \tag{2.17.5b}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = -\frac{\sqrt{r_s}}{2R^{3/2}-3\sqrt{r_s}\tau}, \quad \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{2\sqrt{r_s}}{2R^{3/2}-3\sqrt{r_s}\tau}, \tag{2.17.6a}$$

$$\gamma_{(R)(\varphi)(\varphi)} = \gamma_{(R)(\vartheta)(\vartheta)} = -\left(R^{3/2}-\frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \tag{2.17.6b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}}{2R^{3/2}-3\sqrt{r_s}\tau}, \quad \gamma_{(R)} = 2\left(R^{3/2}-\frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}, \quad \gamma_{(\vartheta)} = \cot\vartheta\left(R^{3/2}-\frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \tag{2.17.7}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{4r_s}{\left(2R^{3/2}-3\sqrt{r_s}\tau\right)^2}, \tag{2.17.8a}$$

$$R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = -R_{(R)(\vartheta)(R)(\vartheta)} = -R_{(R)(\varphi)(R)(\varphi)} = \frac{2r_s}{\left(2R^{3/2}-3\sqrt{r_s}\tau\right)^2}. \tag{2.17.8b}$$

The Ricci tensor with respect to the local tetrad vanishes identically.

2.17.2 Inner metric

The metric of the inside, $R \le R_b$, reads

$$\boxed{ds^2 = -d\tau^2 + \left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{4/3}\left[dR^2 + R^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right)\right].} \tag{2.17.9}$$

For the following components, we define

$$A_{\text{Oin}} := 1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau. \tag{2.17.10}$$

Christoffel symbols:

$$\Gamma^R_{\tau R} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma^\vartheta_{\tau\vartheta} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma^\varphi_{\tau\varphi} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \tag{2.17.11a}$$

$$\Gamma^\tau_{RR} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}, \quad \Gamma^\vartheta_{R\vartheta} = \frac{1}{R}, \quad \Gamma^\varphi_{R\varphi} = \frac{1}{R}, \tag{2.17.11b}$$

$$\Gamma^R_{\vartheta\vartheta} = -R, \quad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \quad \Gamma^\tau_{\vartheta\vartheta} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2, \tag{2.17.11c}$$

$$\Gamma^R_{\varphi\varphi} = -R\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \quad \Gamma^\tau_{\varphi\varphi} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2\sin^2\vartheta. \tag{2.17.11d}$$

Riemann-Tensor:

$$R_{\tau R\tau R} = \frac{1}{2}\frac{r_s}{R_b^3A_{\text{Oin}}^{2/3}}, \quad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_sR^2}{R_b^3A_{\text{Oin}}^{2/3}}, \quad R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_sR^2\sin^2\vartheta}{R_b^3A_{\text{Oin}}^{2/3}}, \tag{2.17.12a}$$

$$R_{R\varphi R\varphi} = \frac{r_sR^2\sin^2\vartheta}{R_b^3}A_{\text{Oin}}^{2/3}, \quad R_{R\vartheta R\vartheta} = \frac{r_sR^2}{R_b^3}A_{\text{Oin}}^{2/3}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_sR^4\sin^2\vartheta}{R_b^3}A_{\text{Oin}}^{2/3}. \tag{2.17.12b}$$

Ricci-Tensor:

$$R_{\tau\tau} = \frac{3}{2}\frac{r_s}{R_b^3A_{\text{Oin}}^2}, \quad R_{RR} = \frac{3}{2}\frac{r_s}{R_b^3A_{\text{Oin}}^{2/3}}, \quad R_{\vartheta\vartheta} = \frac{3}{2}\frac{r_sR^2}{R_b^3A_{\text{Oin}}^{2/3}}, \quad R_{\varphi\varphi} = \frac{3}{2}\frac{r_sR^2\sin^2\vartheta}{R_b^3A_{\text{Oin}}^{2/3}}. \tag{2.17.13}$$

The *Ricci* and *Kretschmann* scalars read:

$$\mathscr{R} = \frac{3r_s}{R_b^3 A_{\text{Oin}}^2}, \qquad \mathscr{K} = 15\frac{r_s^2}{R_b^6 A_{\text{Oin}}^4}. \tag{2.17.14}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \qquad \mathbf{e}_{(R)} = \frac{1}{A_{\text{Oin}}^{2/3}}\partial_R, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{RA_{\text{Oin}}^{2/3}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{A_{\text{Oin}}^{2/3}R\sin\vartheta}\partial_\varphi. \tag{2.17.15}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \tag{2.17.16a}$$

$$\gamma_{(R)(\vartheta)(\vartheta)} = \gamma_{(R)(\varphi)(\varphi)} = -\frac{1}{RA_{\text{Oin}}^{2/3}}, \tag{2.17.16b}$$

$$\gamma_{(\vartheta)(\varphi)(\varphi)} = -\frac{\cot\vartheta}{RA_{\text{Oin}}^{2/3}}. \tag{2.17.16c}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \qquad \gamma_{(R)} = \frac{2}{RA_{\text{Oin}}^{2/3}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{RA_{\text{Oin}}^{2/3}}. \tag{2.17.17}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = \frac{r_s R_b^{-3}}{2A_{\text{Oin}}^2}, \tag{2.17.18a}$$

$$R_{(R)(\vartheta)(R)(\vartheta)} = R_{(R)(\varphi)(R)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s R_b^{-3}}{A_{\text{Oin}}^2}. \tag{2.17.18b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\tau)(\tau)} = R_{(R)(R)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{3r_s R_b^{-3}}{2A_{\text{Oin}}^2}. \tag{2.17.19}$$

Further reading:
Oppenheimer and Snyder[OS39].

2.18 Petrov-Type D – Levi-Civita spacetimes

The Petrov type D static vacuum spacetimes AI-C are taken from Stephani et al.[SKM+03], Sec. 18.6, with the coordinate and parameter ranges given in "Exact solutions of the gravitational field equations" by Ehlers and Kundt [EK62].

2.18.1 Case AI

In spherical coordinates, $(t, r, \vartheta, \varphi)$, the metric is given by the line element

$$\boxed{ds^2 = r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) + \frac{r}{r-b}dr^2 - \frac{r-b}{r}dt^2.} \tag{2.18.1}$$

This is the well known Schwarzschild solution if $b = r_s$, cf. Eq. (2.2.1). Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < \vartheta < \pi, \qquad \varphi \in [0, 2\pi), \qquad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{r}{r-b}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.18.2}$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{r-b}{r}}dt, \qquad \theta^{(r)} = \sqrt{\frac{r}{r-b}}dr, \qquad \theta^{(\vartheta)} = r\, d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\, d\varphi. \tag{2.18.3}$$

Effective potential:
With the Hamilton-Jacobi formalism it is possible to obtain an effective potential fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K\frac{r-b}{r^3} - \kappa\frac{r-b}{r} \tag{2.18.4}$$

and the constants of motion

$$C_0^2 = \dot{t}^2\left(\frac{r-b}{r}\right)^2, \tag{2.18.5a}$$

$$K = \dot{\vartheta}^2 r^4 + \dot{\varphi}^2 r^4 \sin^2\vartheta. \tag{2.18.5b}$$

2.18.2 Case AII

In cylindrical coordinates, the metric is given by the line element

$$\boxed{ds^2 = z^2\left(dr^2 + \sinh^2 r\, d\varphi^2\right) + \frac{z}{b-z}dz^2 - \frac{b-z}{z}dt^2.} \tag{2.18.6}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < r, \qquad \varphi \in [0, 2\pi), \qquad 0 < z < b.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{z}{b-z}}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{z\sinh r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}}\partial_z. \tag{2.18.7}$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{b-z}{z}}dt, \qquad \theta^{(r)} = z\, dr, \qquad \theta^{(\varphi)} = z\sinh r\, d\varphi, \qquad \theta^{(z)} = \sqrt{\frac{z}{b-z}}dz. \tag{2.18.8}$$

2.18.3 Case AIII

In cylindrical coordinates, the metric is given by the line element

$$\boxed{ds^2 = z^2\left(dr^2 + r^2 d\varphi^2\right) + z dz^2 - \frac{1}{z}dt^2.} \tag{2.18.9}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < r, \qquad \varphi \in [0, 2\pi), \qquad 0 < z.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{z}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{zr}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}}\partial_z. \tag{2.18.10}$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{\sqrt{z}}dt, \qquad \theta^{(r)} = z\,dr, \qquad \theta^{(\varphi)} = zr\,d\varphi, \qquad \theta^{(z)} = \sqrt{z}\,dz. \tag{2.18.11}$$

2.18.4 Case BI

In spherical coordinates, the metric is given by the line element

$$\boxed{ds^2 = r^2\left(d\vartheta^2 - \sin^2\vartheta\, dt^2\right) + \frac{r}{r-b}dr^2 + \frac{r-b}{r}d\varphi^2.} \tag{2.18.12}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad 0 < \vartheta < \pi, \qquad \varphi \in [0, 2\pi), \qquad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{r\sin\vartheta}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \sqrt{\frac{r}{r-b}}\partial_\varphi. \tag{2.18.13}$$

Dual tetrad:

$$\theta^{(t)} = r\sin\vartheta\, dt, \qquad \theta^{(r)} = \sqrt{\frac{r}{r-b}}dr, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = \sqrt{\frac{r-b}{r}}\,d\varphi. \tag{2.18.14}$$

Effective potential:
With the Hamilton-Jacobi formalism, an effective potential for the radial coordinate can be calculated fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K\frac{r-b}{r^3} - \kappa\frac{r-b}{r} \tag{2.18.15}$$

and the constants of motion

$$C_0^2 = \dot{\varphi}^2\left(\frac{r-b}{r}\right)^2, \tag{2.18.16a}$$
$$K = \dot{\vartheta}^2 r^4 - \dot{t}^2 r^4 \sin^2\vartheta. \tag{2.18.16b}$$

Note that the metric is not spherically symmetric. Particles or light rays fall into one of the poles if they are not moving in the $\vartheta = \frac{\pi}{2}$ plane.

2.18.5 Case BII

In cylindrical coordinates, the metric is given by the line element

$$\boxed{ds^2 = z^2\left(dr^2 - \sinh^2 r\,dt^2\right) + \frac{z}{b-z}dz^2 + \frac{b-z}{z}d\varphi^2.} \tag{2.18.17}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad \varphi \in [0, 2\pi), \qquad 0 < z < b, \qquad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{z\sinh r}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \sqrt{\frac{z}{b-z}}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}}\partial_z. \tag{2.18.18}$$

Dual tetrad:

$$\theta^{(t)} = z\sinh r\,dt, \qquad \theta^{(r)} = z\,dr, \qquad \theta^{(\varphi)} = \sqrt{\frac{b-z}{z}}d\varphi, \qquad \theta^{(z)} = \sqrt{\frac{z}{b-z}}dz. \tag{2.18.19}$$

2.18.6 Case BIII

In cylindrical coordinates, the metric is given by the line element

$$\boxed{ds^2 = z^2\left(dr^2 - r^2 dt^2\right) + z dz^2 + \frac{1}{z}d\varphi^2.} \tag{2.18.20}$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \qquad \varphi \in [0, 2\pi), \qquad 0 < z, \qquad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{zr}\partial_t, \qquad \mathbf{e}_{(r)} = \frac{1}{z}\partial_r, \qquad \mathbf{e}_{(\varphi)} = \sqrt{z}\,\partial_\varphi, \qquad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}}\partial_z. \tag{2.18.21}$$

Dual tetrad:

$$\theta^{(t)} = zr\,dt, \qquad \theta^{(r)} = z\,dr, \qquad \theta^{(\varphi)} = \frac{1}{\sqrt{z}}d\varphi, \qquad \theta^{(z)} = \sqrt{z}\,dz. \tag{2.18.22}$$

2.18.7 Case C

The metric is given by the line element

$$\boxed{ds^2 = \frac{1}{(x+y)^2}\left(\frac{1}{f(x)}dx^2 + f(x)d\varphi^2 - \frac{1}{f(-y)}dy^2 + f(-y)dt^2\right)} \tag{2.18.23}$$

with $f(u) := \pm(u^3 + au + b)$. Coordinates and parameters are restricted to

$$0 < x + y, \qquad f(-y) > 0, \qquad 0 > f(x).$$

Local tetrad:

$$\mathbf{e}_{(t)} = (x+y)\frac{1}{\sqrt{-y^3 - ay + b}}\partial_t, \qquad \mathbf{e}_{(x)} = (x+y)\sqrt{x^3 + ax + b}\,\partial_x, \tag{2.18.24a}$$

$$\mathbf{e}_{(y)} = (x+y)\sqrt{-y^3 - ay + b}\,\partial_y, \qquad \mathbf{e}_{(\varphi)} = (x+y)\frac{1}{\sqrt{x^3 + ax + b}}\partial_\varphi, \tag{2.18.24b}$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{x+y}\sqrt{-y^3-ay+b}\,dt, \qquad \theta^{(x)} = \frac{1}{x+y}\frac{1}{\sqrt{x^3+ax+b}}dx, \tag{2.18.25a}$$

$$\theta^{(y)} = \frac{1}{x+y}\frac{1}{\sqrt{-y^3-ay+b}}dy, \qquad \theta^{(\varphi)} = \frac{1}{x+y}\sqrt{x^3+ax+b}\,d\varphi, \tag{2.18.25b}$$

A coordinate change can eliminate the linear term in the polynom f generating a quadratic term instead. This brings the line element to the form

$$\boxed{ds^2 = \frac{1}{A(x+y)^2}\left[\frac{1}{f(x)}dx^2 + f(x)dp^2 - \frac{1}{f(-y)}dy^2 + f(-y)dq^2\right]} \tag{2.18.26}$$

with $f(u) := \pm(-2mAu^3 - u^2 + 1)$ given in [PP01].
Furthermore, coordinates can be adapted to the boost-rotation symmetry with the line element in [PP01] from in [Bon83]

$$\boxed{ds^2 = \frac{1}{z^2-t^2}\left[e^{\rho}r^2(z\,dt - t\,dz)^2 - e^{\lambda}(z\,dz - t\,dt)^2\right] - e^{\lambda}\,dr^2 - r^2e^{-\rho}\,d\varphi^2} \tag{2.18.27}$$

with

$$\begin{aligned}
e^{\rho} &= \frac{R_3+R+Z_3-r^2}{4\alpha^2\,(R_1+R+Z_1-r^2)},\\
e^{\lambda} &= \frac{2\alpha^2\left[R(R+R_1+Z_1)-Z_1r^2\right]\left[R_1R_3+(R+Z_1)(R+Z_3)-(Z_1+Z_3)r^2\right]}{R_iR_3\left[R(R+R_3+Z_3)-Z_3r^2\right]},\\
R &= \frac{1}{2}\left(z^2-t^2+r^2\right),\\
R_i &= \sqrt{(R+Z_i)^2-2Z_ir^2},\\
Z_i &= z_i - z_2,\\
\alpha^2 &= \frac{1}{4}\frac{m^2}{A^6(z_2-z_1)^2(z3-z1)^2},\\
q &= \frac{1}{4\alpha^2},
\end{aligned}$$

and $z_3 < z_1 < z_2$ the roots of $2A^4z^3 - A^2z^2 + m^2$.
Local tetrad:

Case $z^2-t^2>0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{z^2-t^2}}\left(qze^{-\rho/2}\,\partial_t + te^{-\lambda/2}\,\partial_z,\right), \qquad \mathbf{e}_{(r)} = e^{-\lambda/2}\,\partial_r, \tag{2.18.28a}$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{z^2-t^2}}\left(qte^{-\rho/2}\,\partial_t + ze^{-\lambda/2}\,\partial_z,\right), \qquad \mathbf{e}_{(\varphi)} = re^{\rho/2}\,\partial_\varphi. \tag{2.18.28b}$$

Case $z^2-t^2<0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{t^2-z^2}}\left(qte^{-\rho/2}\,\partial_t + ze^{-\lambda/2}\,\partial_z,\right), \qquad \mathbf{e}_{(r)} = e^{-\lambda/2}\,\partial_r, \tag{2.18.29a}$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{t^2-z^2}}\left(qze^{-\rho/2}\,\partial_t + te^{-\lambda/2}\,\partial_z,\right), \qquad \mathbf{e}_{(\varphi)} = re^{\rho/2}\,\partial_\varphi. \tag{2.18.29b}$$

Dual tetrad:

Case $z^2 - t^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^{\rho}}{z^2 - t^2}} \frac{1}{q} \left(z\, dt + t\, dz \right), \qquad \theta^{(r)} = e^{\lambda}\, dr, \tag{2.18.30a}$$

$$\theta^{(z)} = \sqrt{\frac{e^{\lambda}}{z^2 - t^2}} \left(t\, dt + z\, dz \right), \qquad \theta^{(\varphi)} = \frac{1}{re^{\rho}}\, d\varphi. \tag{2.18.30b}$$

Case $z^2 - t^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^{\lambda}}{t^2 - z^2}} \left(t\, dt + z\, dz \right), \qquad \theta^{(r)} = e^{\lambda}\, dr, \tag{2.18.31a}$$

$$\theta^{(z)} = \sqrt{\frac{e^{\rho}}{t^2 - z^2}} \frac{1}{q} \left(z\, dt + t\, dz \right), \qquad \theta^{(\varphi)} = \frac{1}{re^{\rho}}\, d\varphi. \tag{2.18.31b}$$

2.19 Plane gravitational wave

W. Rindler described in [Rin01] an exact plane gravitational wave which is bounded between two planes. The metric of the so called 'sandwich wave' with $u := t - x$ reads

$$\boxed{ds^2 = -dt^2 + dx^2 + p^2(u)\,dy^2 + q^2(u)\,dz^2.} \tag{2.19.1}$$

The functions $p(u)$ and $q(u)$ are given by

$$p(u) := \begin{cases} p_0 = \text{const.} & u < -a \\ 1-u & 0 < u \\ L(u)\,\mathrm{e}^{m(u)} & \text{else} \end{cases} \quad \text{and} \quad q(u) := \begin{cases} q_0 = \text{const.} & u < -a \\ 1-u & 0 < u \\ L(u)\,\mathrm{e}^{-m(u)} & \text{else} \end{cases}, \tag{2.19.2}$$

where a is the longitudinal extension of the wave. The functions $L(u)$ and $m(u)$ are

$$L(u) = 1 - u + \frac{u^3}{a^2} + \frac{u^4}{2a^3}, \qquad m(u) = \pm 2\sqrt{3}\int\sqrt{\frac{u^2+au}{2a^3u - 2au^3 - u^4 - 2a^3}}\,du. \tag{2.19.3}$$

Christoffel symbols:

$$\Gamma^y_{ty} = -\Gamma^y_{xy} = \frac{1}{p}\frac{\partial p}{\partial u}, \qquad \Gamma^t_{zz} = \Gamma^x_{zz} = q\frac{\partial q}{\partial u}, \qquad \Gamma^z_{tz} = -\Gamma^z_{xz} = \frac{1}{q}\frac{\partial q}{\partial u}, \qquad \Gamma^t_{yy} = \Gamma^x_{yy} = p\frac{\partial p}{\partial u}. \tag{2.19.4}$$

Riemann-Tensor:

$$R_{tyty} = R_{xyxy} = -R_{tyxy} = -p\frac{\partial^2 p}{\partial u^2}, \qquad R_{tztz} = R_{xzxz} = -R_{tzxz} = -q\frac{\partial^2 q}{\partial u^2}. \tag{2.19.5}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \quad \mathbf{e}_{(x)} = \partial_x, \quad \mathbf{e}_{(y)} = \frac{1}{p}\partial_y, \quad \mathbf{e}_{(z)} = \frac{1}{q}\partial_z. \tag{2.19.6}$$

Dual tetrad:

$$\theta^{(t)} = dt, \quad \theta^{(x)} = dx, \quad \theta^{(y)} = p\,dy, \quad \theta^{(z)} = q\,dz. \tag{2.19.7}$$

2.20 Reissner-Nordstrøm

The Reissner-Nordstrøm black hole in spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ is defined by the metric[MTW73]

$$\boxed{ds^2 = -A_{\mathrm{RN}} c^2 dt^2 + A_{\mathrm{RN}}^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.20.1}$$

where

$$A_{\mathrm{RN}} = 1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2} \tag{2.20.2}$$

with $r_s = 2GM/c^2$, the charge Q, and $\rho = G/(\varepsilon_0 c^4) \approx 9.33 \cdot 10^{-34}$. As in the Schwarzschild case, there is a true curvature singularity at $r = 0$. However, for $Q^2 < r_s^2/(4\rho)$ there are also two critical points at

$$r = \frac{r_s}{2} \pm \frac{r_s}{2}\sqrt{1 - \frac{4\rho Q^2}{r_s^2}}. \tag{2.20.3}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{A_{\mathrm{RN}} c^2 (r_s r - 2\rho Q^2)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s r - 2\rho Q^2}{2r^3 A_{\mathrm{RN}}}, \qquad \Gamma^r_{rr} = -\frac{r_s r - 2\rho Q^2}{2r^3 A_{\mathrm{RN}}}, \tag{2.20.4a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -rA_{\mathrm{RN}}, \tag{2.20.4b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -rA_{\mathrm{RN}}\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.20.4c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2(r_s r - 3\rho Q^2)}{r^4}, \qquad R_{t\vartheta t\vartheta} = \frac{A_{\mathrm{RN}} c^2 (r s_r - 2\rho Q^2)}{2r^2}, \tag{2.20.5a}$$

$$R_{t\varphi t\varphi} = \frac{A_{\mathrm{RN}} c^2 (r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2}, \qquad R_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\mathrm{RN}}}, \tag{2.20.5b}$$

$$R_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2 A_{\mathrm{RN}}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = (r_s r - \rho Q^2)\sin^2\vartheta. \tag{2.20.5c}$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2 \rho Q^2 A_{\mathrm{RN}}}{r^4}, \qquad R_{rr} = -\frac{\rho Q^2}{r^4 A_{\mathrm{RN}}}, \qquad R_{\vartheta\vartheta} = \frac{\rho Q^2}{r^2}, \qquad R_{\varphi\varphi} = \frac{\rho Q^2 \sin^2\vartheta}{r^2}. \tag{2.20.6}$$

While the Ricci scalar vanishes identically, the Kretschmann scalar reads

$$\mathscr{K} = 4\frac{3r_s^2 r^2 - 12 r_s r \rho Q^2 + 14\rho^2 Q^4}{r^8}. \tag{2.20.7}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(r_s r - 2\rho Q^2)}{r^4}, \qquad C_{t\vartheta t\vartheta} = -\frac{A_{\mathrm{RN}} c^2 (r s_r - 2\rho Q^2)}{2r^2}, \tag{2.20.8a}$$

$$C_{t\varphi t\varphi} = \frac{A_{\mathrm{RN}} c^2 (r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2}, \qquad C_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\mathrm{RN}}}, \tag{2.20.8b}$$

$$C_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2 A_{\mathrm{RN}}}, \qquad C_{\vartheta\varphi\vartheta\varphi} = (r_s r - 2\rho Q^2)\sin^2\vartheta. \tag{2.20.8c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{A_{\mathrm{RN}}}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{A_{\mathrm{RN}}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.20.9}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{A_{\mathrm{RN}}}\,dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{A_{\mathrm{RN}}}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.20.10}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{rr_s - 2\rho Q^2}{2r^3\sqrt{A_{\mathrm{RN}}}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{A_{\mathrm{RN}}}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.20.11}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r^2 - 3rr_s + 2\rho Q^2}{2r^3\sqrt{A_{\mathrm{RN}}}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.20.12}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -\frac{r_s r - 3\rho Q^2}{r^4}, \qquad R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s r - \rho Q^2}{r^4}, \tag{2.20.13a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \tag{2.20.13b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{\rho Q^2}{r^4}. \tag{2.20.14}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s r - 2\rho Q^2}{r^4}, \tag{2.20.15a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \tag{2.20.15b}$$

Embedding:
The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{1}{1 - r_s/r + \rho Q^2/r^2} - 1}. \tag{2.20.16}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\mathrm{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\mathrm{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \tag{2.20.17}$$

with constants of motion $k = A_{\mathrm{RN}} c^2 \dot{t}$ and $h = r^2\dot{\varphi}$. For null geodesics, $\kappa = 0$, there are two extremal points

$$r_\pm = \frac{3}{4}r_s\left(1 \pm \sqrt{1 - \frac{32\rho Q^2}{9r_s^2}}\right), \tag{2.20.18}$$

where r_+ is a maximum and r_- a minimum.

Further reading:
Eiroa[ERT02]

2.21 de Sitter spacetime

The de Sitter spacetime with $\Lambda > 0$ is a solution of the Einstein field equations with constant curvature. A detailed discussion can be found for example in Hawking and Ellis[HE99]. Here, we use the coordinate transformations given by Bičák[BK01].

2.21.1 Standard coordinates

The de Sitter metric in standard coordinates $\{\tau \in \mathbb{R}, \chi \in [-\pi,\pi], \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ reads

$$\boxed{ds^2 = -d\tau^2 + \alpha^2 \cosh^2\frac{\tau}{\alpha}\left[d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right],} \tag{2.21.1}$$

where $\alpha^2 = 3/\Lambda$.

Christoffel symbols:

$$\Gamma^{\chi}_{\tau\chi} = \frac{1}{\alpha}\tanh\frac{\tau}{\alpha}, \qquad \Gamma^{\vartheta}_{\tau\vartheta} = \frac{1}{\alpha}\tanh\frac{\tau}{\alpha}, \qquad \Gamma^{\varphi}_{\tau\varphi} = \frac{1}{\alpha}\tanh\frac{\tau}{\alpha}, \tag{2.21.2a}$$

$$\Gamma^{\tau}_{\chi\chi} = \alpha\sinh\frac{\tau}{\alpha}\cosh\frac{\tau}{\alpha}, \qquad \Gamma^{\vartheta}_{\chi\vartheta} = \cot\chi, \qquad \Gamma^{\varphi}_{\chi\varphi} = \cot\chi, \tag{2.21.2b}$$

$$\Gamma^{\tau}_{\vartheta\vartheta} = \alpha\sin^2\chi\sinh\frac{\tau}{\alpha}\cosh\frac{\tau}{\alpha}, \qquad \Gamma^{\chi}_{\vartheta\vartheta} = -\sin\chi\cos\chi, \qquad \Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \tag{2.21.2c}$$

$$\Gamma^{\tau}_{\varphi\varphi} = \alpha\sin^2\chi\sin^2\vartheta\sinh\frac{\tau}{\alpha}\cosh\frac{\tau}{\alpha}, \Gamma^{\chi}_{\varphi\varphi} = -\sin^2\vartheta\sin\chi\cos\chi, \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.21.2d}$$

Riemann-Tensor:

$$R_{\tau\chi\tau\chi} = -\cosh^2\frac{\tau}{\alpha}, \qquad R_{\tau\vartheta\tau\vartheta} = -\cosh^2\frac{\tau}{\alpha}\sin^2\chi, \tag{2.21.3a}$$

$$R_{\tau\varphi\tau\varphi} = -\cosh^2\frac{\tau}{\alpha}\sin^2\chi\sin^2\vartheta, \qquad R_{\chi\vartheta\chi\vartheta} = \alpha^2\left(1+\sinh^2\frac{\tau}{\alpha}\right)^2\sin^2\chi, \tag{2.21.3b}$$

$$R_{\chi\varphi\chi\varphi} = \alpha^2\left(1+\sinh^2\frac{\tau}{\alpha}\right)^2\sin^2\chi\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = \alpha^2\left(1+\sinh^2\frac{\tau}{\alpha}\right)^2\sin^4\chi\sin^2\vartheta. \tag{2.21.3c}$$

Ricci-Tensor:

$$R_{\tau\tau} = -\frac{3}{\alpha^2}, \quad R_{\chi\chi} = 3\cosh^2\frac{\tau}{\alpha}, \quad R_{\vartheta\vartheta} = 3\cosh^2\frac{\tau}{\alpha}\sin^2\chi, \quad R_{\varphi\varphi} = 3\cosh^2\frac{\tau}{\alpha}\sin^2\chi\sin^2\vartheta. \tag{2.21.4}$$

Ricci and Kretschmann scalars:

$$\mathscr{R} = \frac{12}{\alpha^2}, \qquad \mathscr{K} = \frac{24}{\alpha^4}. \tag{2.21.5}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(\chi)} = \frac{1}{\alpha\cosh\frac{\tau}{\alpha}}\partial_\chi, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\alpha\cosh\frac{\tau}{\alpha}\sin\chi}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\alpha\cosh\frac{\tau}{\alpha}\sin\chi\sin\vartheta}\partial_\varphi. \tag{2.21.6}$$

Dual tetrad:

$$\theta^{(\tau)} = d\tau, \quad \theta^{(\chi)} = \alpha\cosh\frac{\tau}{\alpha}d\chi, \quad \theta^{(\vartheta)} = \alpha\cosh\frac{\tau}{\alpha}\sin\chi\, d\vartheta, \quad \theta^{(\varphi)} = \alpha\cosh\frac{\tau}{\alpha}\sin\chi\sin\vartheta\, d\varphi. \tag{2.21.7}$$

2.21.2 Conformally Einstein coordinates

In conformally Einstein coordinates $\{\eta \in [0,\pi], \chi \in [-\pi,\pi], \vartheta \in [0,\pi], \varphi \in [0,2\pi)\}$, the de Sitter metric reads

$$\boxed{ds^2 = \frac{\alpha^2}{\sin^2\eta}\left[-d\eta^2 + d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right].} \tag{2.21.8}$$

It follows from the standard form (2.21.1) by the transformation

$$\eta = 2\arctan\left(e^{\tau/\alpha}\right). \tag{2.21.9}$$

2.21.3 Conformally flat coordinates

Conformally flat coordinates $\{T \in \mathbb{R}, r \in \mathbb{R}, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ follow from conformally Einstein coordinates by means of the transformations

$$T = \frac{\alpha \sin\eta}{\cos\chi + \cos\eta}, \quad r = \frac{\alpha \sin\chi}{\cos\chi + \cos\eta}, \quad \text{or} \quad \eta = \arctan\frac{2T\alpha}{\alpha^2 - T^2 + r^2}, \quad \chi = \arctan\frac{2r\alpha}{\alpha^2 + T^2 - r^2}. \tag{2.21.10}$$

For the transformation $(T,R) \to (\eta,\chi)$, we have to take care of the coordinate domains. In that case, if $\kappa^2 - T^2 + r^2 < 0$, we have to map $\eta \to \eta + \pi$. On the other hand, if $\kappa^2 + T^2 - r^2 < 0$, we have to consider the sign of r. If $r > 0$, then $\chi \to \chi + \pi$, otherwise $\chi \to \chi - \pi$.
The resulting metric reads

$$\boxed{ds^2 = \frac{\alpha^2}{T^2}\left[-dT^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right].} \tag{2.21.11}$$

Note that we identify points $(r < 0, \vartheta, \varphi)$ with $(r > 0, \pi - \vartheta, \varphi - \pi)$.
Christoffel symbols:

$$\Gamma^T_{TT} = \Gamma^r_{Tr} = \Gamma^\vartheta_{T\vartheta} = \Gamma^\varphi_{T\varphi} = \Gamma^T_{rr} = -\frac{1}{T}, \quad \Gamma^\vartheta_{r\vartheta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \quad \Gamma^T_{\vartheta\vartheta} = -\frac{r^2}{T}, \quad \Gamma^r_{\vartheta\vartheta} = -r, \tag{2.21.12a}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \quad \Gamma^T_{\varphi\varphi} = -\frac{r^2\sin^2\vartheta}{T}, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.21.12b}$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{\alpha^2}{T^4}, \qquad R_{T\vartheta T\vartheta} = -\frac{\alpha^2 r^2}{T^4}, \qquad R_{T\varphi T\varphi} = -\frac{\alpha^2 r^2 \sin^2\vartheta}{T^4}, \tag{2.21.13a}$$

$$R_{r\vartheta r\vartheta} = \frac{\alpha^2 r^2}{T^4}, \qquad R_{r\varphi r\varphi} = \frac{\alpha^2 r^2\sin^2\vartheta}{T^4}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{\alpha^2 r^4 \sin^2\vartheta}{T^4}. \tag{2.21.13b}$$

Ricci-Tensor:

$$R_{TT} = -\frac{3}{T^2}, \qquad R_{rr} = \frac{3}{T^2}, \qquad R_{\vartheta\vartheta} = \frac{3r^2}{T^2}, \qquad R_{\varphi\varphi} = \frac{3r^2\sin^2\vartheta}{T^2}. \tag{2.21.14}$$

The *Ricci and Kretschmann scalar read:*

$$\mathscr{R} = \frac{12}{\alpha^2}, \qquad \mathscr{K} = \frac{24}{\alpha^4}. \tag{2.21.15}$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{T}{\alpha}\partial_T, \qquad \mathbf{e}_{(r)} = \frac{T}{\alpha}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{T}{\alpha r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{T}{\alpha r\sin\vartheta}\partial_\varphi. \tag{2.21.16}$$

2.21.4 Static coordinates

The de Sitter metric in static spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0,\pi), \varphi \in [0,2\pi)\}$ reads

$$\boxed{ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)c^2dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right).} \tag{2.21.17}$$

It follows from the conformally Einstein form (2.21.8) by the transformations

$$t=\frac{\alpha}{2}\ln\frac{\cos\chi-\cos\eta}{\cos\chi+\cos\eta},\quad r=\alpha\frac{\sin\chi}{\sin\eta}. \tag{2.21.18}$$

Christoffel symbols:

$$\Gamma^r_{tt}=\frac{(\Lambda r^2-3)}{9}c^2\Lambda r,\qquad \Gamma^t_{tr}=\frac{\Lambda r}{\Lambda r^2-3},\qquad \Gamma^r_{rr}=\frac{\Lambda r}{3-\Lambda r^2}, \tag{2.21.19a}$$

$$\Gamma^\vartheta_{r\vartheta}=\frac{1}{r},\qquad \Gamma^\phi_{r\phi}=\frac{1}{r},\qquad \Gamma^r_{\vartheta\vartheta}=\frac{(\Lambda r^2-3)r}{3}, \tag{2.21.19b}$$

$$\Gamma^\phi_{\vartheta\phi}=\cot(\vartheta),\qquad \Gamma^r_{\phi\phi}=\frac{\Lambda r^2-3}{3}r\sin^2(\vartheta),\qquad \Gamma^\vartheta_{\phi\phi}=-\sin(\vartheta)\cos(\vartheta). \tag{2.21.19c}$$

Riemann-Tensor:

$$R_{trtr}=-\frac{\Lambda}{3}c^2,\qquad R_{t\vartheta t\vartheta}=-\frac{3-\Lambda r^2}{9}c^2\Lambda r^2,\qquad R_{t\varphi t\varphi}=-\frac{3-\Lambda r^2}{9}c^2\Lambda r^2\sin(\vartheta)^2, \tag{2.21.20a}$$

$$R_{r\vartheta r\vartheta}=\frac{\Lambda r^2}{-\Lambda r^2+3},\qquad R_{r\varphi r\varphi}=\frac{\Lambda r^2\sin(\theta)^2}{-\Lambda r^2+3},\qquad R_{\vartheta\varphi\vartheta\varphi}=\frac{r^4\sin^2(\theta)\Lambda}{3}. \tag{2.21.20b}$$

Ricci-Tensor:

$$R_{tt}=\frac{\Lambda r^2-3}{3}c^2\Lambda,\qquad R_{rr}=\frac{3\Lambda}{3-\Lambda r^2},\qquad R_{\vartheta\vartheta}=\Lambda r^2,\qquad R_{\varphi\varphi}=r^2\sin^2(\vartheta)\Lambda. \tag{2.21.21}$$

The *Ricci scalar and Kretschmann scalar read:*

$$\mathscr{R}=4\Lambda,\qquad \mathscr{K}=\frac{8}{3}\Lambda^2. \tag{2.21.22}$$

Local tetrad:

$$\mathbf{e}_{(t)}=\sqrt{\frac{3}{3-\Lambda r^2}}\frac{\partial_t}{c},\qquad \mathbf{e}_{(r)}=\sqrt{1-\frac{\Lambda r^2}{3}}\partial_r,\qquad \mathbf{e}_{(\vartheta)}=\frac{1}{r}\partial_\vartheta,\qquad \mathbf{e}_{(\varphi)}=\frac{1}{r\sin(\vartheta)}\partial_\varphi. \tag{2.21.23}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(t)}=-\frac{\Lambda r}{\sqrt{9-3\Lambda r^2}},\qquad \gamma_{(\vartheta)(r)(\vartheta)}=\gamma_{(\varphi)(r)(\varphi)}=\frac{\sqrt{9-3\Lambda r^2}}{3r},\qquad \gamma_{(\varphi)(\vartheta)(\varphi)}=\frac{\cot\vartheta}{r}. \tag{2.21.24}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)}=\frac{\sqrt{9-3\Lambda r^2}(\Lambda r^2-2)}{(\Lambda r^2-3)r},\qquad \gamma_{(\vartheta)}=\frac{\cot\vartheta}{r}. \tag{2.21.25}$$

Riemann-Tensor with respect to local tetrad:

$$-R_{(t)(r)(t)(r)}=-R_{(t)(\vartheta)(t)(\vartheta)}=-R_{(t)(\varphi)(t)(\varphi)}=R_{(r)(\vartheta)(r)(\vartheta)}=R_{(r)(\varphi)(r)(\varphi)}=R_{(\vartheta)(\varphi)(\vartheta)(\varphi)}=\frac{1}{3}\Lambda. \tag{2.21.26}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)}=R_{(r)(r)}=R_{(\vartheta)(\vartheta)}=R_{(\varphi)(\varphi)}=\Lambda. \tag{2.21.27}$$

2.21.5 Lemaître-Robertson form

The de Sitter universe in the Lemaître-Robertson form reads

$$\boxed{ds^2 = -c^2dt^2 + e^{2Ht}\left[dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right],} \tag{2.21.28}$$

with Hubble's Parameter $H = \sqrt{\frac{\Lambda c^2}{3}} = \frac{c}{\alpha}$, which is assumed here to be time-independent.
This a special case of the first and second form of the Friedman-Robertson-Walker metric defined in Eqs. (2.9.2) and (2.9.12) with $R(t) = e^{Ht}$ and $k = 0$.
Christoffel symbols:

$$\Gamma^r_{tr} = H, \qquad \Gamma^\vartheta_{t\vartheta} = H, \qquad \Gamma^\varphi_{t\varphi} = H, \tag{2.21.29a}$$

$$\Gamma^t_{rr} = \frac{e^{2Ht}H}{c^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \tag{2.21.29b}$$

$$\Gamma^t_{\vartheta\vartheta} = \frac{e^{2Ht}r^2H}{c^2}, \qquad \Gamma^r_{\vartheta\vartheta} = -r, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \tag{2.21.29c}$$

$$\Gamma^t_{\varphi\varphi} = \frac{e^{2Ht}r^2\sin^2(\theta)H}{c^2}, \qquad \Gamma^r_{\varphi\varphi} = -r\sin(\vartheta)^2, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta). \tag{2.21.29d}$$

Riemann-Tensor:

$$R_{trtr} = -e^{2Ht}H^2, \qquad R_{t\vartheta t\vartheta} = -e^{2Ht}r^2H^2, \tag{2.21.30a}$$

$$R_{t\varphi t\varphi} = -e^{2Ht}r^2\sin^2(\vartheta)H^2, \qquad R_{r\vartheta r\vartheta} = \frac{e^{4Ht}r^2H^2}{c^2}, \tag{2.21.30b}$$

$$R_{r\varphi r\varphi} = \frac{e^{4Ht}r^2\sin^2(\vartheta)H^2}{c^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{e^{4Ht}r^4\sin^2(\vartheta)H^2}{c^2}. \tag{2.21.30c}$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \qquad R_{rr} = 3\frac{e^{2Ht}H^2}{c^2}, \qquad R_{\vartheta\vartheta} = 3\frac{e^{2Ht}r^2H^2}{c^2}, \qquad R_{\varphi\varphi} = 3\frac{e^{2Ht}r^2\sin^2(\vartheta)H^2}{c^2}. \tag{2.21.31}$$

Ricci and Kretschmann scalars:

$$\mathscr{R} = \frac{12H^2}{c^2}, \qquad \mathscr{K} = \frac{24H^4}{c^4}. \tag{2.21.32}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = e^{-Ht}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{e^{-Ht}}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{e^{-Ht}}{r\sin\vartheta}\partial_\varphi. \tag{2.21.33}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{H}{c} \tag{2.21.34a}$$

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{e^{Ht}r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\theta)}{e^{Ht}r}. \tag{2.21.34b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}, \qquad \gamma_{(r)} = \frac{2}{e^{Ht}r}, \gamma_{(\vartheta)} = \frac{\cot(\theta)}{e^{Ht}r}. \tag{2.21.35}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{H^2}{c^2} \tag{2.21.36a}$$

$$R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{H^2}{c^2}. \tag{2.21.36b}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 3\frac{H^2}{c^2}. \tag{2.21.37}$$

2.21.6 Cartesian coordinates

The de Sitter universe in Lemaître-Robertson form can also be expressed in Cartesian coordinates:

$$\boxed{ds^2 = -c^2dt^2 + e^{2Ht}\left[dx^2 + dy^2 + dz^2\right].} \tag{2.21.38}$$

Christoffel symbols:

$$\Gamma^x_{tx} = H, \qquad \Gamma^y_{ty} = H, \qquad \Gamma^z_{tz} = H, \tag{2.21.39a}$$
$$\Gamma^t_{xx} = \frac{e^{2Ht}H}{c^2}, \qquad \Gamma^t_{yy} = \frac{e^{2Ht}H}{c^2}, \qquad \Gamma^t_{zz} = \frac{e^{2Ht}H}{c^2}. \tag{2.21.39b}$$
$$\tag{2.21.39c}$$

Partial derivatives

$$\Gamma^t_{xx,t} = \Gamma^t_{yy,t} = \Gamma^t_{zz,t} = \frac{2H^2e^{2Ht}}{c^2}. \tag{2.21.40}$$

Riemann-Tensor:

$$R_{txtx} = R_{txtx} = R_{tztz} = -e^{2Ht}H^2, \qquad R_{xyxy} = R_{xzxz} = R_{yzyz} = \frac{e^{4Ht}H^2}{c^2}. \tag{2.21.41}$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \qquad R_{xx} = R_{yy} = R_{zz} = 3\frac{e^{2Ht}H^2}{c^2}. \tag{2.21.42}$$

The Ricci and Kretschmann scalar read:

$$\mathscr{R} = 12\frac{H^2}{c^2}, \qquad \mathscr{K} = 24\frac{H^4}{c^4}. \tag{2.21.43}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(x)} = e^{-Ht}\partial_x, \qquad \mathbf{e}_{(y)} = e^{-Ht}\partial_y, \qquad \mathbf{e}_{(z)} = e^{-Ht}\partial_z. \tag{2.21.44}$$

Ricci rotation coefficients:

$$\gamma_{(x)(t)(x)} = \gamma_{(y)(t)(y)} = \gamma_{(z)(t)(z)} = \frac{H}{c}. \tag{2.21.45}$$

The only non-vanishing contraction of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}. \tag{2.21.46}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(t)(x)} = R_{(t)(y)(t)(y)} = R_{(t)(z)(t)(z)} = -\frac{H^2}{c^2}, \tag{2.21.47a}$$
$$R_{(x)(y)(x)(y)} = R_{(x)(z)(x)(z)} = R_{(y)(z)(y)(z)} = \frac{H^2}{c^2}. \tag{2.21.47b}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(x)(x)} = R_{(y)(y)} = R_{(z)(z)} = 3\frac{H^2}{c^2}. \tag{2.21.48}$$

Further reading:
Tolman[Tol34, sec. 142], Bičák[BK01]

2.22 Straight spinning string

The metric of a straight spinning string in cylindrical coordinates (t,ρ,φ,z) reads

$$\boxed{ds^2 = -(c\,dt - a\,d\varphi)^2 + d\rho^2 + k^2\rho^2 d\varphi^2 + dz^2,} \tag{2.22.1}$$

where $a \in \mathbb{R}$ and $k > 0$ are two parameters, see Perlick[Per04].

Metric-Tensor:

$$g_{tt} = -c^2, \qquad g_{t\varphi} = ac, \qquad g_{\rho\rho} = g_{zz} = 1, \qquad g_{\varphi\varphi} = k^2\rho^2 - a^2. \tag{2.22.2}$$

Christoffel symbols:

$$\Gamma^t_{\rho\varphi} = \frac{a}{c\rho}, \qquad \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho}, \qquad \Gamma^\rho_{\varphi\varphi} = -k^2\rho. \tag{2.22.3}$$

Partial derivatives

$$\Gamma^t_{\rho\varphi,\rho} = -\frac{\alpha}{c\rho^2}, \qquad \Gamma^\varphi_{\rho\varphi,\rho} = -\frac{1}{\rho^2}, \qquad \Gamma^\rho_{\varphi\varphi,\rho} = -k^2. \tag{2.22.4}$$

The Riemann-, Ricci-, and Weyl-tensors as well as the Ricci- and Kretschmann-scalar vanish identically.

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(1)} = \partial_\rho, \qquad \mathbf{e}_{(2)} = \frac{1}{k\rho}\left(\frac{a}{c}\partial_t + \partial_\varphi\right), \qquad \mathbf{e}_{(3)} = \partial_z. \tag{2.22.5}$$

Dual tetrad:

$$\theta^{(0)} = c\,dt - a\,d\varphi, \qquad \theta^{(1)} = d\rho, \qquad \theta^{(2)} = k\rho\,d\varphi, \qquad \theta^{(3)} = dz. \tag{2.22.6}$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(2)(1)(2)} = \frac{1}{\rho}, \qquad \gamma_{(0)} = \gamma_{(2)} = \gamma_{(3)} = 0, \qquad \gamma_{(1)} = \frac{1}{\rho}. \tag{2.22.7}$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{k^2\rho^2 - a^2}}{k\rho}\left(\frac{1}{c}\partial_t - \frac{a}{k^2\rho^2 - a^2}\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \partial_\rho, \tag{2.22.8a}$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{k^2\rho^2 - a^2}}\partial_\varphi, \qquad \mathbf{e}_{(3)} = \partial_z. \tag{2.22.8b}$$

Dual tetrad:

$$\theta^{(0)} = \frac{k\rho}{\sqrt{k^2\rho^2 - a^2}}c\,dt, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = \frac{ac\,dt}{\sqrt{k^2\rho^2 - a^2}} + \sqrt{k^2\rho^2 - a^2}\,d\varphi, \quad \theta^{(3)} = dz. \tag{2.22.9}$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(0)(1)(0)} = \frac{a^2}{\rho\,(k^2\rho^2 - a^2)}, \quad \gamma_{(2)(1)(0)} = \gamma_{(0)(2)(1)} = \gamma_{(0)(1)(2)} = \frac{ak}{k^2\rho^2 - a^2}, \tag{2.22.10a}$$

$$\gamma_{(2)(1)(2)} = \frac{k^2\rho}{k^2\rho^2 - a^2}, \tag{2.22.10b}$$

$$\gamma_{(1)} = \frac{1}{\rho}. \tag{2.22.10c}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{\rho}^2 + \frac{1}{k^2\rho^2}\left(h_2 - \frac{ah_1}{c}\right)^2 - \kappa c^2 = \frac{h_1^2}{c^2}, \tag{2.22.11}$$

with the constants of motion $h_1 = c(c\dot{t} - a\dot{\varphi})$ and $h_2 = a(c\dot{t} - a\dot{\varphi}) + k^2\rho^2\dot{\varphi}$.

The point of closest approach ρ_{pca} for a null geodesic that starts at $\rho = \rho_i$ with $\mathbf{y} = \pm\mathbf{e}_{(0)} + \cos\xi\,\mathbf{e}_{(1)} + \sin\xi\,\mathbf{e}_{(2)}$ with respect to the static tetrad is given by $\rho = \rho_i \sin\xi$. Hence, the ρ_{pca} is independent of a and k. The same is also true for timelike geodesics.

2.23 Sultana-Dyer spacetime

The Sultana-Dyer metric represents a black hole in the Einstein-de Sitter universe. In spherical coordinates $(t, r, \vartheta, \varphi)$, the metric reads[SD05] $(G = c = 1)$

$$\boxed{ds^2 = t^4\left[\left(1-\frac{2M}{r}\right)dt^2 - \frac{4M}{r}dt\,dr - \left(1+\frac{2M}{r}\right)dr^2 - r^2\,d\Omega^2\right],} \tag{2.23.1}$$

where M is the mass of the black hole and $\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the spherical surface element. Note that here, the signature of the metric is $\mathrm{sign}(\mathbf{g}) = -2$.

Christoffel symbols:

$$\Gamma^t_{tt} = \frac{2\left(r^3+4M^2r+M^2t\right)}{tr^3}, \quad \Gamma^r_{tt} = \frac{M(r-2M)(4r+t)}{tr^3}, \quad \Gamma^t_{tr} = \frac{M(r+2M)(4r+t)}{tr^3}, \tag{2.23.2a}$$

$$\Gamma^r_{tr} = \frac{2\left(r^3-4M^2r-M^2t\right)}{tr^3}, \quad \Gamma^\vartheta_{t\vartheta} = \frac{2}{t}, \quad \Gamma^\varphi_{t\varphi} = \frac{2}{t}, \tag{2.23.2b}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \quad \Gamma^t_{\vartheta\vartheta} = \frac{2\left(r^2+2Mr-Mt\right)}{t}, \tag{2.23.2c}$$

$$\Gamma^r_{\vartheta\vartheta} = -\frac{4Mr+tr-2Mt}{t}, \quad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{2.23.2d}$$

$$\Gamma^t_{rr} = \frac{2\left(r^3+4Mr^2+4M^2r+M^2t+Mtr\right)}{tr^3}, \quad \Gamma^r_{rr} = -\frac{M\left(4r^2+8Mr+2Mt+tr\right)}{tr^3}, \tag{2.23.2e}$$

$$\Gamma^t_{\varphi\varphi} = \frac{2\left(r^2+2Mr-Mt\right)\sin^2\vartheta}{t}, \quad \Gamma^r_{\varphi\varphi} = -\frac{(4Mr+tr-2Mt)\sin^2\vartheta}{t}. \tag{2.23.2f}$$

Riemann-Tensor:

$$R_{trtr} = \frac{2t^2\left(-2Mr^2-r^3+Mt^2+2Mtr\right)}{r^3}, \tag{2.23.3a}$$

$$R_{r\vartheta t\vartheta} = -\frac{t^2\left(2r^4+16M^2r^2+4Mtr^2-4M^2r^2t+Mt^2r-2M^2t^2\right)}{r^2}, \tag{2.23.3b}$$

$$R_{t\vartheta r\vartheta} = -\frac{2Mt^2(4r+t)(r^2+2Mr-Mt)}{r^2}, \tag{2.23.3c}$$

$$R_{r\varphi t\varphi} = -\frac{t^2\sin^2\vartheta\left(2r^4+16M^2r^2+4Mtr^2-4M^2r^2t+Mt^2r-2M^2t^2\right)}{r^2}, \tag{2.23.3d}$$

$$R_{t\varphi r\varphi} = -\frac{2Mt^2\sin^2\vartheta(4r+t)(r^2+2Mr-Mt)}{r^2}, \tag{2.23.3e}$$

$$R_{r\vartheta r\vartheta} = -\frac{t^2\left(4r^4+16Mr^4-4M^2tr+16M^2r^2-2M^2t^2-Mt^2r\right)}{r^2}, \tag{2.23.3f}$$

$$R_{r\varphi r\varphi} = -\frac{t^2\sin^2\vartheta\left(4r^4+16Mr^4-4M^2tr+16M^2r^2-2M^2t^2-Mt^2r\right)}{r^2}, \tag{2.23.3g}$$

$$R_{\vartheta\varphi\vartheta\varphi} = -2t^2r\sin^2\vartheta\left(2r^3+4Mr^2-4Mtr+mt^2\right). \tag{2.23.3h}$$

Ricci-Tensor:

$$R_{tt} = \frac{2\left(3r^2+12M^2+2Mt\right)}{t^2r^2}, \quad R_{tr} = \frac{4M(3r+t+6M)}{t^2r^2}, \tag{2.23.4a}$$

$$R_{rr} = \frac{2\left(3r^2+12Mr+2Mt+12M^2\right)}{t^2r^2}, \quad R_{\vartheta\vartheta} = \frac{6\left(r^2+2Mr-2Mt\right)}{t^2}, \tag{2.23.4b}$$

$$R_{\varphi\varphi} = \frac{6\left(r^2+2Mr-2Mt\right)\sin^2\vartheta}{t^2}. \tag{2.23.4c}$$

Ricci and Kretschmann scalars:

$$R = -\frac{12\left(r^2 + 2Mr - 2Mt\right)}{t^6 r^2}, \tag{2.23.5a}$$

$$\mathscr{K} = \frac{48\left(M^2t^4 + 20M^2r^4 + 20Mr^5 + 8M^2r^2t^2 - 4Mr^4t - 16M^2r^3t + 5r^6\right)}{t^12r^6}. \tag{2.23.5b}$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{1+2M/r}}{t^2}\partial_t - \frac{2M/r}{t^2\sqrt{1+2M/r}}\partial_r, \quad \mathbf{e}_{(1)} = \frac{1}{t^2\sqrt{1+2M/r}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r}\partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r\sin\vartheta}\partial_\varphi. \tag{2.23.6}$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{t^2\sqrt{1-2M/r}}\partial_t, \quad \mathbf{e}_{(1)} = \frac{2M/r}{t^2\sqrt{1-2M/r}}\partial_t + \frac{\sqrt{1-2M/r}}{t^2}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r}\partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r\sin\vartheta}\partial_\varphi. \tag{2.23.7}$$

Further reading:
Sultana and Dyer[SD05].

2.24 TaubNUT

The TaubNUT metric in Boyer-Lindquist like spherical coordinates $(t, r, \vartheta, \varphi)$ reads[BCJ02] $(G = c = 1)$

$$\boxed{ds^2 = -\frac{\Delta}{\Sigma}(dt + 2\ell\cos\vartheta\, d\varphi)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right),} \tag{2.24.1}$$

where $\Sigma = r^2 + \ell^2$ and $\Delta = r^2 - 2Mr - \ell^2$. Here, M is the mass of the black hole and ℓ the magnetic monopol strength.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{\Delta\rho}{\Sigma^3}, \qquad \Gamma^t_{tr} = \frac{\rho}{\Delta\Sigma}, \qquad \Gamma^t_{t\vartheta} = -2\ell^2\cos\vartheta\frac{\Delta}{\Sigma^2}, \tag{2.24.2a}$$

$$\Gamma^\varphi_{t\vartheta} = \frac{\ell\Delta}{\Sigma^2\sin\vartheta}, \quad \Gamma^r_{t\varphi} = \frac{2\ell\rho\Delta\cos\vartheta}{\Sigma^3}, \quad \Gamma^\vartheta_{t\varphi} = -\frac{\ell\Delta\sin\vartheta}{\Sigma^2}, \tag{2.24.2b}$$

$$\Gamma^r_{rr} = -\frac{\rho}{\Sigma\Delta}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{r}{\Sigma}, \qquad \Gamma^\varphi_{r\varphi} = \frac{r}{\Sigma}, \qquad \Gamma^r_{\vartheta\vartheta} = -\frac{r\Delta}{\Sigma}, \tag{2.24.2c}$$

$$\Gamma^t_{r\varphi} = \frac{-2\ell(r^3 - 3Mr^2 - 3r\ell^2 + M\ell^2)\cos\vartheta}{\Sigma\Delta}, \tag{2.24.2d}$$

$$\Gamma^t_{\vartheta\varphi} = -\frac{\ell\left[\cos^2\vartheta\left(6r^2\ell^2 - 8\ell^2Mr - 3\ell^4 + r^4\right) + \Sigma^2\right]}{\Sigma^2\sin\vartheta}, \tag{2.24.2e}$$

$$\Gamma^r_{\varphi\varphi} = \frac{\Delta}{\Sigma^3}\left[\cos^2\vartheta\left(9r\ell^4 + 4\ell^2Mr^2 - 4\ell^4M + r^5 + 2r^3\ell^2\right) - r\Sigma^2\right], \tag{2.24.2f}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \frac{\left(4r^2\ell^2 - 4Mr\ell^2 - \ell^4 + r^4\right)\cot\vartheta}{\Sigma^2}, \tag{2.24.2g}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\frac{\left(6r^2\ell^2 - 8Mr\ell^2 - 3\ell^4 + r^4\right)\sin\vartheta\cos\vartheta}{\Sigma^2}, \tag{2.24.2h}$$

where $\rho = 2r\ell^2 + Mr^2 - M\ell^2$.

Static local tetrad:

$$\mathbf{e}_{(0)} = \sqrt{\frac{\Sigma}{\Delta}}\partial_t, \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \quad \mathbf{e}_{(3)} = -\frac{2\ell\cot\vartheta}{\sqrt{\Sigma}}\partial_t + \frac{1}{\sqrt{\Sigma}\sin\vartheta}\partial_\varphi. \tag{2.24.3}$$

Dual tetrad:

$$\theta^{(0)} = \sqrt{\frac{\Delta}{\Sigma}}(dt + 2\ell\cos\vartheta\, d\varphi), \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}}dr, \quad \theta^{(2)} = \sqrt{\Sigma}d\vartheta, \quad \theta^{(3)} = \sqrt{\Sigma}\sin\vartheta\, d\varphi. \tag{2.24.4}$$

Euler-Lagrange:
The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \qquad V_{\text{eff}} = \frac{1}{2}\frac{\Delta}{\Sigma}\left(\frac{h^2}{\Sigma} - \kappa\right) \tag{2.24.5}$$

with the constants of motion $k = (\Delta/\Sigma)\dot{t}$ and $h = \Sigma\dot{\varphi}$. For null geodesics, we obtain a photon orbit at $r = r_{\text{po}}$ with

$$r_{\text{po}} = M + 2\sqrt{M^2 + \ell^2}\cos\left(\frac{1}{3}\arccos\frac{M}{\sqrt{M^2 + \ell^2}}\right) \tag{2.24.6}$$

Further reading:
Bini et al.[BCdMJ03].

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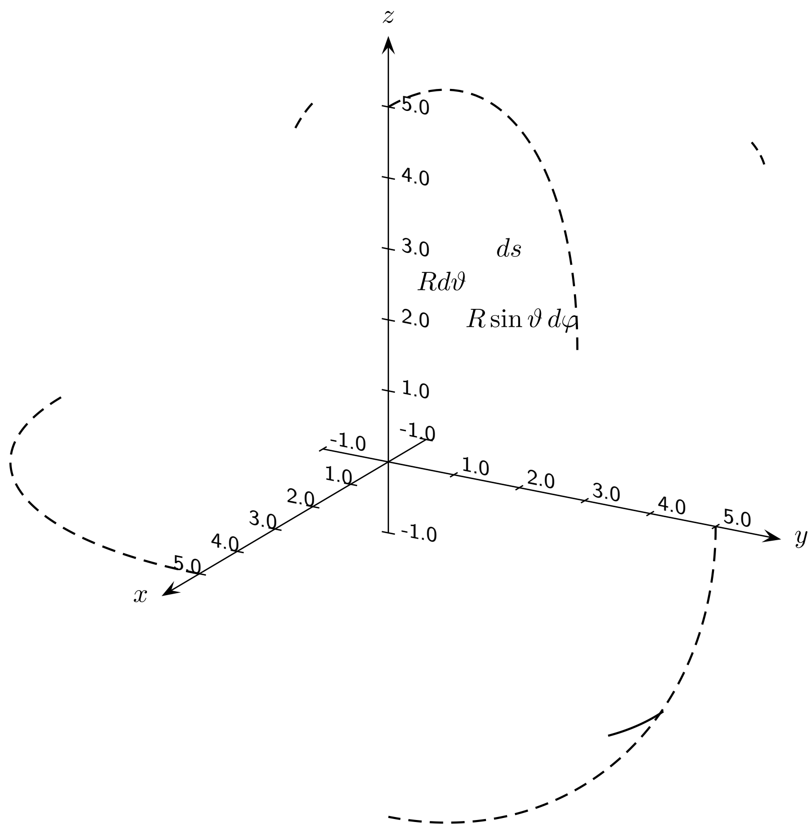
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