Introduction

This worksheet walks through the Einstein field equations using the Schwarzschild metric as an example.

We derive the key tensors involved in general relativity using symbolic index notation and Python/SymPy code snippets.

Schwarzschild Metric

The Schwarzschild metric in natural units (G = c = 1) is given by:

ds^2 = -(1 - 2M/r) dt^2 + (1 - 2M/r)^(-1) dr^2 + r^2 dtheta^2 + r^2 sin^2(theta) dphi^2

In matrix form:

g_{mu nu} = diag(-(1 - 2M/r), (1 - 2M/r)^(-1), r^2, r^2 sin^2(theta))

Metric Tensor g_{mu nu}

Metric tensor components in Schwarzschild coordinates:

```
from sympy import symbols, diag, sin

M, r, theta = symbols('M r theta')

g = diag(-(1 - 2*M/r), 1/(1 - 2*M/r), r**2, r**2*sin(theta)**2)

g
```

Christoffel Symbols Gamma^lambda_{mu nu}

The Christoffel symbols are defined as:

Gamma^lambda_{mu nu} = 1/2 g^{lambda sigma} (partial_mu g_{nu sigma} + partial_nu g_{mu sigma} - partial_sigma g_{mu nu})

They represent the connection coefficients for parallel transport.

Riemann Tensor R^rho_{sigma mu nu}

The Riemann curvature tensor is defined as:

R^rho_{sigma mu nu} = partial_mu Gamma^rho_{nu sigma} - partial_nu Gamma^rho_{mu sigma} + Gamma^rho_{mu lambda} Gamma^lambda_{nu sigma} - Gamma^rho_{nu lambda} Gamma^lambda_{mu sigma}

It encodes the intrinsic curvature of spacetime.

Ricci Tensor R_{mu nu}

The Ricci tensor is obtained by contracting the Riemann tensor:

R_{mu nu} = R^lambda_{mu lambda nu}

Ricci Scalar R

The Ricci scalar is the trace of the Ricci tensor:

R = g^{mu nu} R_{mu nu}

Einstein Tensor G_{mu nu}

The Einstein tensor is defined as:

G_{mu nu} = R_{mu nu} - 1/2 g_{mu nu} R

It appears on the left-hand side of the Einstein field equations:

G_{mu nu} = 8pi T_{mu nu}