Cooperative Game Theory

▼ Intro: Cooperative Games

Game theory has been used to model multiagent interactions, but we came to some troubling conclusions:

- Cooperation can't occur in Prisoners' Dilemma (as it's irrational)
- Binding Agreements ("I'll cooperate if you'll cooperate") aren't possible since agents can't trust each other
- Utility is based in individual action rather than their actions as a group, so agents shouldn't care about social welfare

But in real life, this is not the case (humans are not all selfish and pessimistic):

- Legal contracts can bind parties to a behaviour
- Profits go to companies, not to individuals

For this to work we need to introduce the concept of Cooperative Games.

▼ Cooperative Games

A cooperative game is a tuple of a set of agents

$$G = \langle Ag, v \rangle$$

- A Coalition is a subset of agents

$$C \subset Ag$$

- A Grand Coalition is a group of agents that contains all agents

$$C = Ag$$

- A Singleton Coalition is a coalition of 1 agent

$$|C| = 1$$

- A Characteristic Function maps a coalition to a particular payoff

$$\nu : 2^{Ag} \rightarrow \mathbb{R}$$

- Example

  The coalition c can get utility k **in some way**, and distribute it to it's members

  $$v(c) = k$$

- Stages of Cooperative Action

  There are three stages we want to consider in cooperative action/games:

  1. Coalitional Structure Generation

     Agents must decide which other agents to work with (which coalitions will form). Characteristic functions can help, but all agents will want to maximise utility, so the problem lies in forming coalitions that no agent wants to leave (stable coalitions)

  2. Solving the Optimisation of Each Coalition

     It consists in solving the "joint problem" of maximising social welfare.

  3. Dividing Value of Solution for Each Coalition

     Deciding "who gets what" and how fair is that allocation.

- The Core

  The core is a set of feasible, efficient and acceptable utility distributions among the coalition's members. Agents that join a coalition will want to know what part of the benefits will correspond to them.

  No sub-coalition will reasonably object to any utility distribution within the core.

  It holds a coalition together, in the same way it holds a building together in construction.

  If no agent objects to the core of the Grand Coalition, a stable coalition is possible.

  - Example

    Are there any distributions of utility that mean no agent wants to defect from the grand coalition?

$$
\begin{aligned}
&Ag = \{1, 2\} c = Ag \\
&\nu(\{c\}) = 5 : \langle 5, 0\rangle, \langle 4, 1\rangle, \langle 3, 2\rangle, \langle 2, 3\rangle, \langle 1, 4\rangle, \langle 0, 5\rangle \\
&\nu(\{1\}) = 3 : \langle 3, 0\rangle \\
&\nu(\{2\}) = 2 : \langle 0, 2\rangle
\end{aligned}
$$

Everything in between angled brackets is a possible distribution. For example, in the first one, agent 1 could get 5 utility while agent 2 gets 0 utility and so on.

In this case, it would be the third distribution <3, 2> which no agent will want to defect, since both will get at least as much utility as they'd get by working individually.

In this case, that would be the core. A set of feasible, efficient and acceptable utility distributions.

An outcome X = <x1, x2...> for a coalition c = <1, 2, ..., k> in the game G = <Ag, nu> is a distribution of utility earned by the coalition to it's members. This way, x1 will be the payoff for agent 1, x2 for agent 2 and so on.

The outcome is feasible if it's possible for the coalition to earn nu(c) (whatever the characteristic function says).

The outcome is efficient if all the utility is being allocated, expressed as

$$\sum_{i \in x} x_i = \nu(c)$$

A coalition (any coalition) c = <1, 2, ..., k> will object to an outcome X = <x1, x2, ..., xk> if there exists another outcome X' = <x'1, x'2, ..., x'k> that it's doing strictly better, expressed as

$$\forall i \in c : x'_i > x_i$$

We want the core to be non-empty, since if it is, agents will have an incentive to join another group or work by themselves.

▼ Shapley Value

The final problem to solve in MAS revolves around stability. This stability does not guarantee fairness. For example consider the following distribution, which is stable since it's in the core of the grand coalition

$$\nu(\{1,2\}) = \langle 15, 5 \rangle$$
$$\nu(\{1\}) = 6$$
$$\nu(\{2\}) = 4$$

We can see no agent is incentivised to leave since both are getting more than if they worked on their own. However, this can be viewed as unfair by agent 2. Therefore, we need to create a fair distribution.

The Shapley Value formulates a way to divide utility according to an agent's contribution to the coalition's utility.

- ▼ Calculating Shapley Value
  - ▼ Marginal Contribution

    In order to calculate a SV we need a Marginal Contribution

    $$\mu_i(c)$$

    This tells us how much value an agent adds to the grand coalition.

    The Marginal Contribution of an agent i to the coalition c expressed as:

    $$\mu_i(c) = \nu(c \cup \{i\}) - \nu(c)$$

    This means "how much does the coalition earn when i joins it" minus "how much does the coalition earn without i". With this we know how much i brings to the coalition's utility.

    We expect agents to create synergy when joining coalitions (create added value when joining). If the agent does not create any value, the marginal contribution will be exactly the same as when he worked by himself.
  - ▼ Axioms

In order for distributions to be fair, there are some axioms that need to be known:

- Symmetry: agents with the same contribution get the same payoff. This is expressed as

$$\text{If } \mu_i(c) = \mu_j(c) \forall c \subset Ag - \{i, j\}, \text{ then } sh_i = sh_j$$

Being sh_j and sh_i the values j and i get respectively.

- Dummy Player: if a player never has synergy, it gets only what it earns when alone. This is expressed as

$$\text{If } \mu_i(c) = \nu(\{i\}) \forall c \subset Ag - \{i\} \text{ then } sh_i = \nu(\{i\})$$

- Additive Axiom: if you combine 2 games, the player should get the same utility for the combined game as if they played them individually. The player should not be penalised if they decide to play 2 games separately or together. Their characteristic functions should be added up, as seen in the third line here

$$\begin{aligned}&\text{2 games: } \mathcal{G}^1 = \langle Ag, \nu^1 \rangle, \mathcal{G}^2 = \langle Ag, \nu^2 \rangle \ \ldots \\ &\text{Now combined as: } \mathcal{G}^{1+2} = \langle Ag, \nu^{1+2} \rangle \\ &\text{The following should hold: } \nu^{1+2}(c) = \nu^1(c) + \nu^2(c) \\ &\text{and } sh_i^{1+2} = sh_i^1 + sh_i^2\end{aligned}$$

▼ Shapley Value main idea

An agent will be awarded the average marginal contribution that it makes to the coalition. However, this MC will differ, because if an agent joins early on in the coalition, they can make a bigger MC than if they join later on. So the order in which agents join has to be taken into account.

$$\begin{aligned}&Ag = \{1, 2, 3\} \\ &\text{All possible orderings: } \Pi(Ag) = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\} \\ &\text{For an ordering } o \in \Pi(Ag), C_i(o) \text{ is the set of agents that joined } \textbf{before } i.\end{aligned}$$

$$C_1((1,2,3)) = \emptyset$$
$$C_1((2,1,3)) = 2$$

- Formula

$$sh_i = \frac{\sum_{o \in \Pi(Ag)} \mu_i(C_i(o))}{|Ag|!}$$

We start by taking the Marginal Contribution of i to the coalition given a particular order of joining, summed over all possible orderings for this particular coalition (in our case this is the grand coalition). Then we divide that by the number of possible orderings. This gives us an average contribution to the coalition.

Summed over all possible join orderings

Marginal Contribution of i to Coalition, given order of Joining

$$sh_i = \frac{\sum_{o \in \Pi(Ag)} \overbrace{\mu_i(C_i(o))}}{|Ag|!}$$

Average Contribution to the Coalitions

Divided by # possible orderings

- Example

  We have the following 2 agents

$$Ag = \{1, 2\}; \nu(\{1, 2\}) = 20; \nu(\{1\}) = 5; \nu(\{2\}) = 5;$$
$$\Pi(Ag) = \{(1, 2), (2, 1)\}$$

We see the characteristic function for the coalition (in this case grand coalition) is 20. When both agents work by themselves, each of them gets a utility of 5.

First we need to calculate the marginal contribution of agent 1 with the empty set. At this point the coalition is just {1} and their characteristic function is 5, so

$$\mu_1(\emptyset) = 5$$

We minus from that the characteristic function of the set of agents that joined before agent 1, which are none (empty set). 5 - 0 = 5.

Now we look at the ordering {2, 1}, so the marginal contribution is

$$\mu_1(\{2\}) = 20 - 5 = 15$$

Next we calculate the value that is given to agent 1. We plug in our values and results as

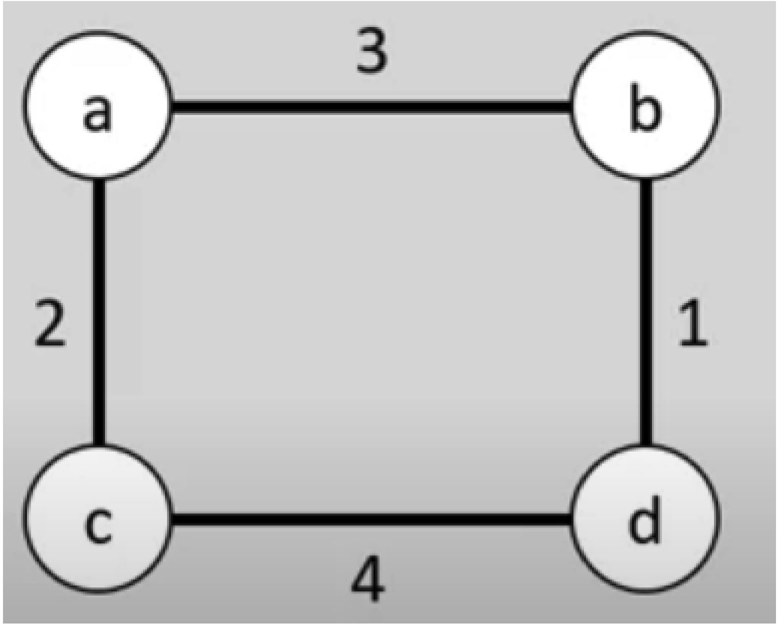
$$sh_1 = \frac{5 + 15}{2 * 1} = \frac{20}{2} = 10$$

- ▼ Representations

  There are several ways of representing Coalitional Games.

  - ▼ Induced Subgraph

    The graph can be used to represent value for a coalition and then calculate the Shapley Value in polynomial time. Nodes represent agents and edges represent a component game. The weight on the edges represents the value to the coalition of the connected agents. For example, here



we have agents a, b, c and d, and we can see when a and b form a coalition they get a value of 3, and when a and c form a coalition, they get a value of 2. To find the value for the grand coalition, we sum up the weight of the edges that create that coalition.

$$
\begin{aligned}
&\nu(\{a,b\}) = 3\\
&\nu(\{a,c\}) = 2\\
&\nu(\{c,d\}) = 4\\
&\nu(\{b,d\}) = 1\\
&\nu(\{a,b,c,d\}) = 3+2+4+1 = 10\\
&\nu(\{a,b,c\}) = 2+3 = 5
\end{aligned}
$$

However, this method, even though concise, it's not complete, since it cannot represent all coalitional games.

▼ Marginal Contribution Net

We represent the valuation function as a set of rules

$$\phi \to x$$

with phi being a conjunction of agents representing the members of a coalition and x being a real number (the value they get).

▼ Example

Let's say agents a and b will get a value of 5

$$a \land b \to 5$$

That can apply to coalitions

$$\{a, b\}, \{a, b, c\}$$

But it will not apply to

$$\{b\}$$

since that is not a subset of {a,b}.

The value of the coalition is calculated by summing up the value of all the rules that apply to the coalition.

▼ Example

We have the following rules

$$\begin{aligned} &a \land b \to 5 \\ &b \to 2 \\ &c \to 4 \\ &b \land \neg c \to -2 \end{aligned}$$

This is how values for coalitions would be calculated

$$\nu(\{a,b\})\ 5+2-2=5$$
$$\nu(\{a,b,c\})=5+2+4=11$$
$$\nu(\{d\})=0$$

▼ Calculating Shapley Value

We can use these representations to calculate the Shapley Value in polynomial time, not exponential.

▼ In Induced Subgraph

The Additive Axiom says the SV can be calculated by adding SV for a player from each component game. Each player gets half of ea \eqch edge it is connected to.

$$sh_i = \frac{\sum_{i!=j} w_{i,j}}{2}$$

▼ In Marginal Contribution Net

We apply a very similar reasoning as in the Induced Subgraph.

For every rule of the form

$$1 \wedge 2 \wedge ... \wedge n$$

that has L components, we assign a value of x/L

$$sh_i^{[... \wedge i \wedge ... \wedge l \rightarrow x]} = \frac{x}{l}$$

(we divide the value of the rule by the number of agents in that rule)

Then we sum up all the rules that apply to a particular agent

$$sh_i = \sum_{r \in \mathcal{R}} sh_i^r$$