Social Choice Theory

- Intro: Voting and Social Choice

  Imagine our agents were going to vote. We have an odd number of them

  $$Ag = \{1, ..., n\}; |Ag| Mod 2 = 1$$

  And some outcomes

  $$\Omega = \{\omega_1, \omega_2, ...\}$$

  It could be a pairwise election

  $$|\Omega| = 2$$

  Or a general voting (with multiple candidates)

  $$|\Omega| > 2$$

  Agents have preferences about the outcome of elections. Pi over Omega is a set of all preference orderings for the agents

  $$\Pi(\Omega)$$

  In order to get all these preferences to form a group decision, we have multiple options

  - A Social Welfare function: we take the preference orderings, time them (x) and return a social outcome ranking

    $$f : \Pi(\Omega) \times \Pi(\Omega) \times ... \to \Pi(\Omega)$$

  - A Social Choice function: we take all the preferences and return a single outcome

    $$f : \Pi(\Omega) x \Pi(\Omega) x ... \to \Omega$$

- Voting
  - Plurality voting

All voters have a preference and submit what their preference over candidates is. Each time a candidate achieves position number 1 in a preference, counts as a vote for them. Candidate with more votes wins the election.

This is easy to implement and simple to understand.

Some interesting properties emerge when we have more than 2 candidates:

- ▼ Tactical Voting

  A group of voters who don't want a candidate to win can vote for their highest-voted preferred rival instead of their actually preferred candidate (since it wouldn't be elected anyway) in order to defeat the candidate they hate. These voters misrepresent their preferences in order to get a more favourable outcome.

- ▼ Condorcet's Paradox

  A winner cannot be chosen since candidates with the most votes are also the most disliked (last preference) of the most people. No matter the outcome, the majority of voters will be unhappy with the result.

- ▼ Sequential Majority Elections

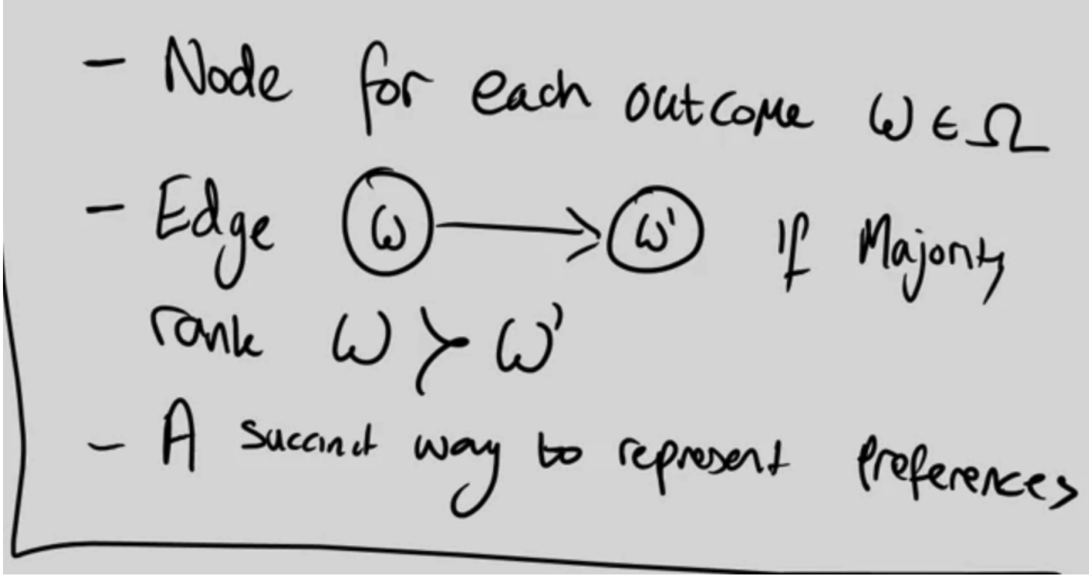
  To overcome the problems of plurality voting, we can consider holding a series of simple majority pairwise elections (winner stays on).

  One problem with this is that the outcome can not only depend on order of preference (taking away the impression of a wasted vote), but also on the order in which elections are held. We can order this as a Majority Graph

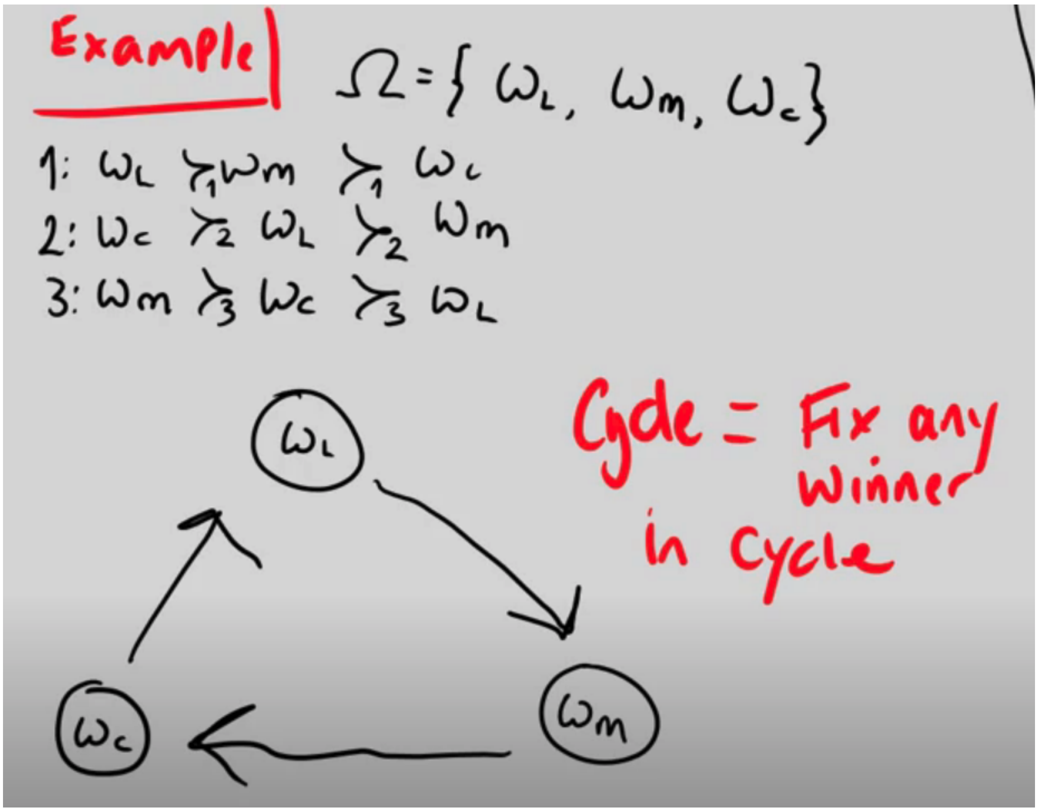
▼ Majority Graph

A Majority Graph tells us which outcomes have majorities in Sequential Majority Elections regarding the votes of preferences in regard to others

In the previous image, omega would win over omega'



▼ Example

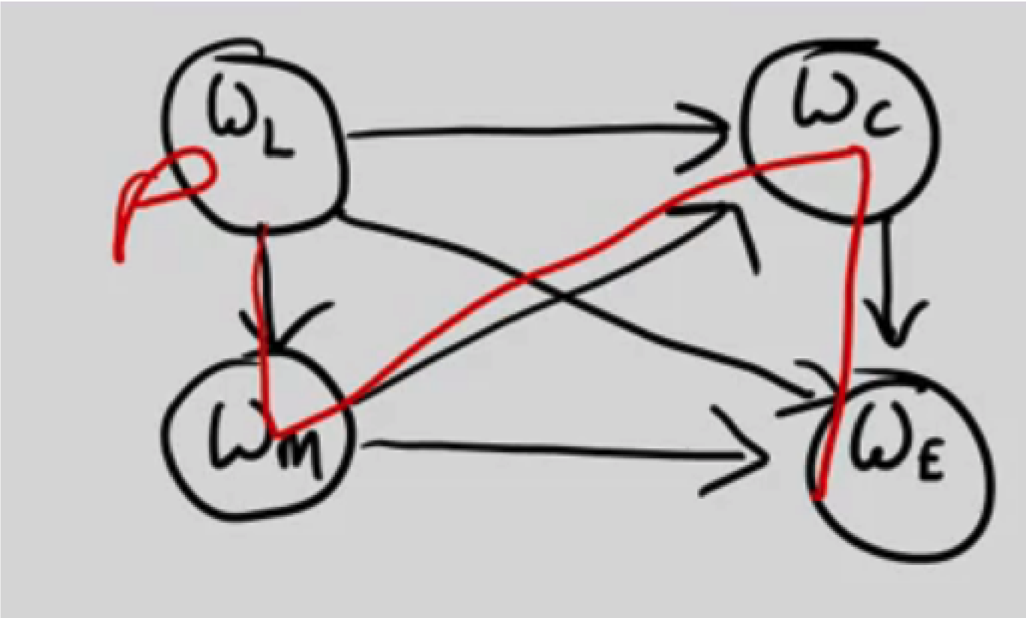


Here we could fix a specific agenda (order of elections) to make any omega in the cycle win.

▼ Possible winner and Condorcet Winner

A particular outcome omega is a possible winner if there exists an agenda that results in omega being the overall winner. In order to check for this we should pick a particular outcome and check if there is a path from it to all other particular outcomes.

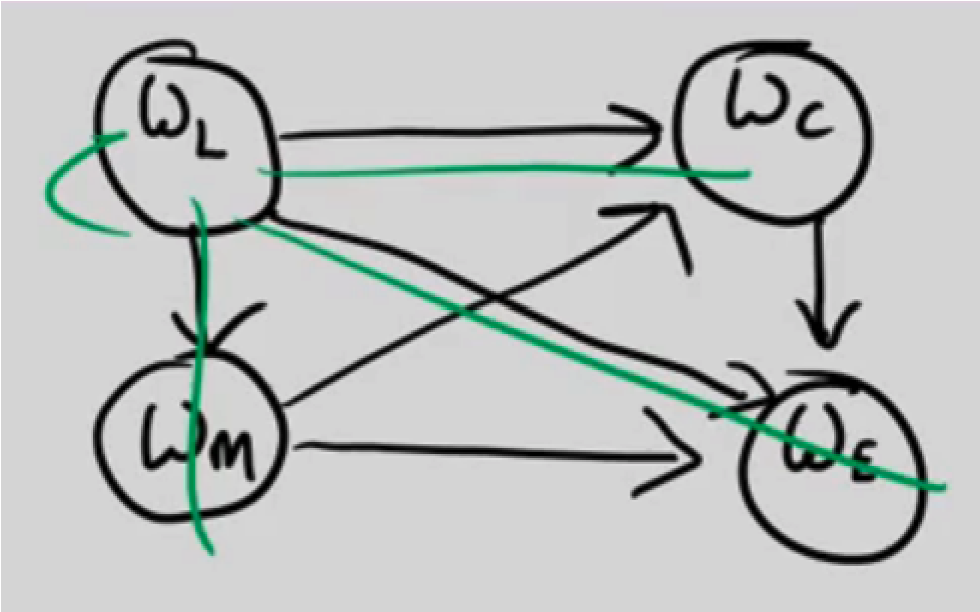
▼ Example



In this case, omega L is a possible winner since there is a path from it to all other outcomes, as we can see.

A particular outcome omega is a Condorcet Winner if it is the overall winner for every possible agenda. We can check this if there is an edge from it to every other node.

▼ Example



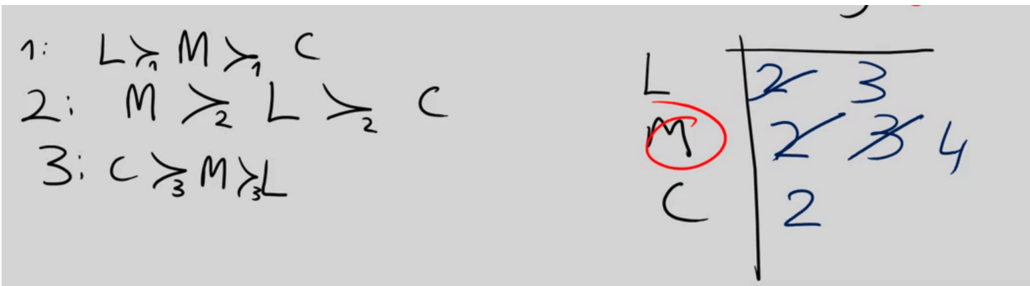
In this case, omega L is a Condorcet Winner because there is an edge from it to every other node.

- **Borda Count**

  A voting system that takes information from all the preference orderings (not just first rank).

  The elections count k candidates, with each candidate having a count starting at 0. If a candidate appears at position i, their count is increased by (k - i). This system is quite fair since it takes into account everybody's preference.

  - **Example**



- **Properties of Voting Systems**
  - **Pareto Condition**

    Pareto Optimality is where no other outcome will benefit an agent without making another agent suffer.

If we apply it to a voting system, such as a social preference order, and we rank omega over omega', and all voters indeed ranked them like so, that system has the Pareto Condition. If the social preference order ranked omega' over omega, then all the voters will be happy to change to the other outcome that applies to their preference.

This condition is satisfied by Borda Count and Plurality Voting, and not satisfied by Sequential Majority Elections.

- ▼ Condorcet Winner Condition

  Outcome omega is a Condorcet winner if it wins against all other outcomes (a very strong winner). This means omega is at the top of the preference order.

  Sequential Majority Elections are the only ones that will satisfy this condition, while Plurality Voting and Borda Count will not.

- ▼ Independence of Irrelevant Alternative

  Suppose omega is ranked over omega'. If you change your preference but still think omega should be ranked over omega' (like putting omega in position 2 and omega' in position 5 when before they were in positions 1 and 3), we expect that the social preference of omega > omega' also remains. This means that ranking in social preference should only depend on how outcomes are ranked within such preference relative to each other, not their actual values. (It only matters omega > omega').

  This condition is not satisfied by Borda Count, Plurality Voting or Sequential Majority Elections.

▼ Dictatorship

Let P_i be the preference order for voter i

A dictatorship is a social welfare function in the following form:

$$f : (P_1, ..., P_n) = P_i$$

This way, only voter i's preference matters.

This satisfies both the Pareto Condition and the Independence of Irrelevant Alternative conditions, although this doesn't mean it is a good system.

▼ Manipulation of Voting Systems

In previous voting systems we saw how people can misrepresent their preference in order to get an outcome that's more favourable to them.

We have a social choice function such that it takes everybody's preference over the outcomes and it returns an outcome.

$$f(\Pi(\omega) \times \dots \times \Pi(\omega)) \to \omega$$

We assume that the voters are expressing their true preference, but this is not always the case.

Social Choice can be thought of as a game

- Strategies: the preference I declare
- Outcomes may be more favourable (I might get more utility) if I misrepresent my preference

Can we engineer a voting system immune to manipulation?

We can say the Social Choice Function is manipulable if for the voter i, their utility is greater if they misrepresent their preference.

Manipulable Social Choice function:

$$f(P_1, P_2, \dots, P_i', \dots, P_n) \succ_i f(P_1, P_2, \dots, P_i, \dots, P_n)$$

▼ Complexity of Manipulation

There are two types of complexity here:

- Easy to compute: the Social Choice function can be run by an algorithm in polynomial time.
- Easy to manipulate: a voter can get a better outcome by misrepresenting their preferences, and this misrepresentation can be done in polynomial time

For example, there is a voting system called Second Copeland, which is easy to compute but not easy to manipulate. We can't compute, in reasonable time, how should we misrepresent our preferences.

The manipulation can be too difficult may be too computationally complex in practice, being NP-complete.

However, NP-completeness is a worst-case result, since "hard" instances of problems don't occur in practice and there are heuristics to reduce the difficulty.