Introduction to Scientific Computing I

Lecture 10

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Linear Algebra

Vector, Matrix

- Linear Algebra was invented to solve equations like this:

$$5x_1 + 12x_2 = -2$$
$$-2x_1 - 3x_2 = 12$$

- By representing them as matrices like this.

$$\begin{pmatrix} 5 & 12 & -2 \\ -2 & -3 & 12 \end{pmatrix}$$

- Or better yet:

$$A = \begin{pmatrix} 5 & 12 \\ -2 & -3 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} -2 \\ 12 \end{pmatrix}$$

- $$A\vec{x} = \vec{y} \longrightarrow \vec{x} = \underbrace{A^{-1}}_{\text{Inverse}}\vec{y}$$

Basics

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Columns

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

$$A_{:,j} = a_j$$

Rows

$$\begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix}$$

$$A_{i,:} = a_i^T$$

Transpose: $(A^T)_{ij} = A_{ji}$

Sum: Elementwise $C_{ij} = A_{ij} + B_{ij}$

Identity: $I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1}A = I = AA^{-1}$$

Products

Vector-Vector (inner)

aka: dot product

$$x \cdot y = $$

$$x^T y = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i.$$

Vector-Vector (outer)

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix}$$

Matrix-Vector

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \\ a_1 \\ \\ \end{bmatrix} x_1 + \begin{bmatrix} \\ a_2 \\ \\ \end{bmatrix} x_2 + \ldots + \begin{bmatrix} \\ a_n \\ \\ \end{bmatrix} x_n$$

$$y = Ax = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

Matrix-Matrix

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

$$(AB)C = A(BC)$$
$$A(B + C) = AB = AC$$
$$AB \neq BA$$

$$AI = A$$

Norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

l=1 Norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \qquad \|x\|_2^2 = x^T x$$

l=2 Norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

l=p Norm

$$\|x\|_\infty = \mathrm{max}_i \, |x_i|$$

l=∞ Norm

$$x \cdot y = ||x||_2 ||y||_2 \cos\theta$$

Law of cosines

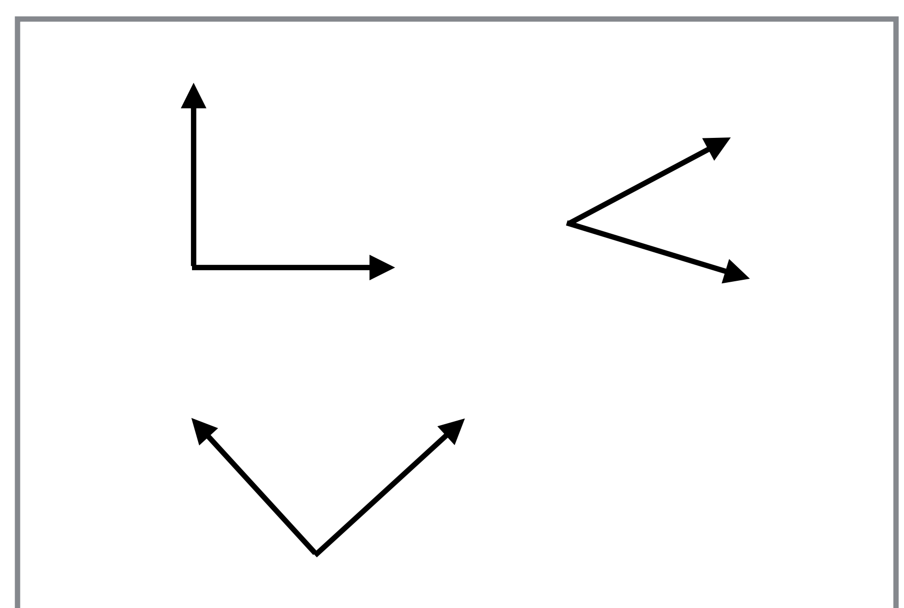
Linear (In)dependence

Consider a set of vectors: $\{x_1, x_2, \ldots x_n\}$

Can your write one as sum of others? $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$

Yes = Linear Dependent
No = Linear Independent

Orthogonal: $x^T y = 0$



Matrix Inverse

Inverse

$$A\vec{x} = \vec{y} \longrightarrow \vec{x} = A^{-1}\vec{y}$$

Step 1: Let $p = 0$, $d = 1$;
Step 2: $p \Leftarrow p + 1$
Step 3: If $a_{p,p} = 0$ then cannot calculate inverse, go to step 10.
Step 4: $d' \Leftarrow d \times a_{p,p}$

Step 5: Calculate the new elements of the pivot row by:

$$a'_{p,j} \Leftarrow \frac{a_{p,j}}{a_{p,p}}, \quad where \quad j = 1, \cdots, n, \quad j \neq p$$

Step 6: Calculate the new elements of the pivot column by:

$$a'_{i,p} \Leftarrow -\frac{a_{i,p}}{a_{p,p}}, \quad where \quad i = 1, \cdots, n, \quad i \neq p$$

Step 7: Calculate the rest of the new elements by:

$$a'_{i,j} \Leftarrow a_{i,j} + a_{p,j} \times a'_{i,p}, \quad where \quad i = 1, \cdots, n, j = 1, \cdots, n \;\&\; i, j \neq p$$

Step 8: Calculate the new value of the current pivot location:

$$a'_{p,p} \Leftarrow \frac{1}{a_{p,p}}$$

Step 9: If $p < n$ go to step 2 (n the dimension of the matrix A).
Step 10: Stop. If inverse exists- A contains the inverse and d is the determinant.

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