SPHERE EVERSION

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Abstract. In this expository paper, we begin by using winding numbers and covering spaces to rigorously prove it impossible to evert a circle. Next, we provide a rough sketch of an indirect proof that sphere eversion is possible, followed by a modeling of an actual sphere eversion with the graphing calculator Desmos.

Introduction

In modern mathematics, mathematicians often find themselves asking increasingly complex questions that are nearly impossible to explain to others in simple terms. While ancient mathematicians like Eratosthenes and Pythagoras could simply say that they were calculating the circumference of the Earth, or had found a relationship between the side lengths of right triangles, modern mathematicians have to explain problems like the Riemann Hypothesis or the Poincaré Conjecture, for which even understanding the problems requires solid mathematical foundation in their respective areas. However, amidst all this complexity, mathematicians sometimes encounter beautiful problems that are not only incredibly challenging, but are also deceptively simple to explain.

One example of these problems is sphere eversion: Is it possible to turn a sphere inside-out under the following rules?

- The sphere can be bent, stretched, and shrunk in any way.
- The sphere can be pulled through itself.
- The sphere cannot be torn or glued together.
- The sphere cannot be "creased".

The first solution that may come to mind is to push two opposite ends of the sphere all the way through each other to form an everted sphere (see Figure 1). However, this strategy introduces a sharp crease around the middle of the sphere during the transformation, which

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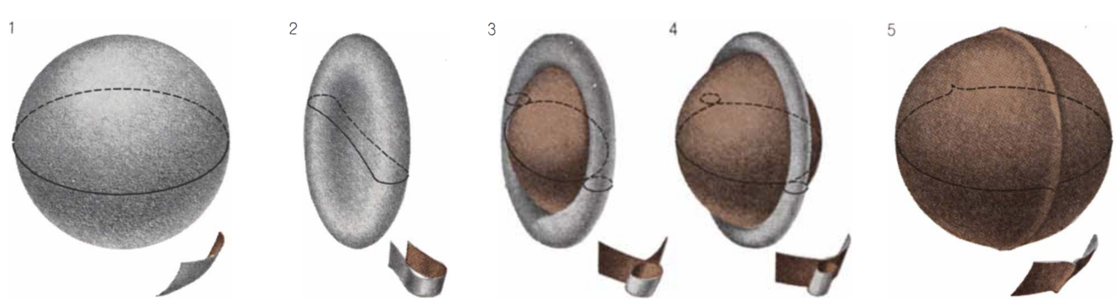

Figure 1. An example of an invalid sphere eversion.

is against the rules. As it turns out, the problem is incredibly challenging, and until the mid-20th century, most mathematicians believed it impossible to evert a sphere.

Nonetheless, in 1957, Stephen Smale finally indirectly proved that it was possible to evert a sphere by showing that all immersions of the sphere are reguarly homotopic [1]. However, his graduate advisor Raoul Bott read the paper and famously told him that it was fundamentally incorrect, believing it impossible to evert a sphere. Bott was later convinced the paper was valid and published it. After another 4 years, the first explicit example of an eversion was discovered by Arnold S. Shapiro in 1961. Shapiro did not publish this widely, and it was only after he described his eversion to mathematician Bernard Morin, who incidentally was blind at the time, that explicit sphere eversions became widely popularized. Eventually, a sphere eversion, along with a description of the problem, was published in *Scientific American* in 1966 [2], finally introducing the problem to a wide audience with one of the first published explicit examples of a sphere eversion.

1. Homotopies

The first step in proving eversions is rigorously defining what it means to "evert" something. We begin by defining smooth functions to and from Euclidean spaces.

Definition 1.1. Let X be some subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^m$ be a function. We say that f is *smooth* if it has continuous partial derivatives of all orders, or if there exists an extended function F of f with continuous partial derivatives of all orders.

In differential topology, surfaces are not thought of as collections of points, but are defined in terms of smooth parametrized functions across Euclidean spaces. The sphere and the everted sphere are examples of this. Usually, their parameterizations are functions $\psi, \psi' : [-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3$ where

$$\psi(\theta, \phi) = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi))$$
$$\psi'(\theta, \phi) = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi))$$

However, it is sometimes inconvenient or impossible to parametrize entire surfaces with a single mapping from Euclidean space. This leads us to the definition of manifolds, one of the most fundamental objects of topology.

Definition 1.2. A surface $X \in \mathbb{R}^N$ is an *n-dimensional manifold* (or simply an *n-manifold*) if it can be locally parameterized by smooth functions from $\mathbb{R}^n$. In other words, X is a manifold iff for all $x \in X$, there is a smooth parameterization $\phi : U \to V$ where $U \subseteq \mathbb{R}^n$, $x \in V$, and $V \subseteq X$.

Now, we define homotopies.

Definition 1.3. Let $f_0, f_1 : X \to \mathbb{R}^m$ be smooth maps. We say that f_0 and f_1 are *homotopic* if there exists a smooth map $H : X \times [0,1] \to \mathbb{R}^m$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in X$. We call H a *homotopy* between f_0 and f_1.

We can imagine homotopies as "movies", with their additional dimension $[0,1]$ thought of as time. In this analogy, a homotopy begins at time $t = 0$ as the initial map f_0, and then smoothly transitions into map f_1 by time $t = 1$.

Definition 1.4. An *immersion* is a differentiable function $f : X \to Y$ between manifolds X and Y, whose derivative is everywhere injective.

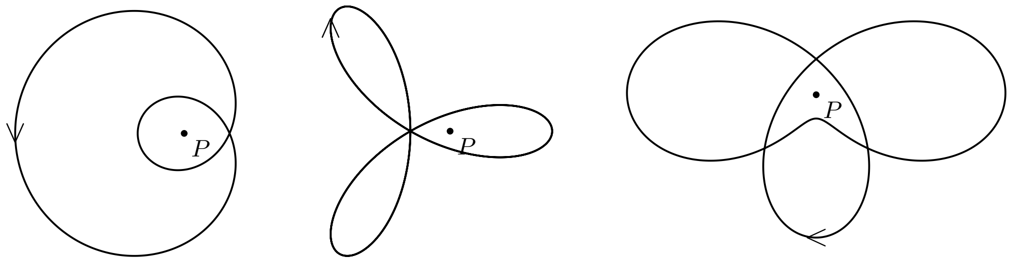

Figure 2. Curves with winding numbers 2, -2, and -1 around point P, respectively.

In other words, there should be no creasing, tearing, or gluing between manifolds X and Y in an immersion between X and Y.

Definition 1.5. A *regular homotopy* is a homotopy that remains an immersion of some manifold X throughout the homotopy. Two maps $f_0, f_1 : X \to \mathbb{R}^m$ are *regularly homotopic* if there exists a regular homotopy between them.

So, our goal is to show that the sphere and the everted sphere are regularly homotopic. We cannot crease, cut or glue the sphere anytime during the homotopy because that would prevent it from being smooth and regular. However, we are still allowed to pull the sphere through itself, bend it, stretch it, and shrink it in any way because the homotopy can still remain smooth and regular.

2. Winding Numbers

Before proving sphere eversion, it is helpful to first solve the problem of circle eversion. Note that, unlike the sphere, the circle cannot be everted. We begin by defining curves, which are a special kind of 1-manifold.

Definition 2.1. A *curve* is a smooth mapping $\gamma : [0, 1] \to \mathbb{C}$.

It is usually more convenient to work with complex numbers in the complex plane, but curves may also map to $\mathbb{R}^2$ instead of $\mathbb{C}$.

Definition 2.2. A *closed curve* is a curve that smoothly starts and ends at the same place. More formally, a curve γ is a closed curve iff $\gamma^{(k)}(0) = \gamma^{(k)}(1)$ for all $k \in \mathbb{Z}_{\geq 0}$, where $f^{(n)}$ denotes the nth derivative of f.

Because curves are defined as mappings from the real number line, all curves come with direction. To indicate this direction, curves are usually drawn with an arrow, as shown in Figure 2.

The circle is defined as the curve $\gamma(t) = e^{2\pi ti}$, and the everted circle as the curve $\gamma'(t) = e^{-2\pi ti}$. One strategy usually used to prove that the circle γ and the everted circle γ' are not homotopic is to find some characteristic of the two that does not change with homotopies, or in other words is "invariant" under homotopies, and show that this characteristic is different for the two curves. Such a characteristic is called a "homotopy invariant".

Remark 2.3. Note that the regular homotopy is just a stronger version of the homotopy. This means that by showing that the circle γ and the everted circle γ' are not homotopic, we are also showing that they are not regularly homotopic.

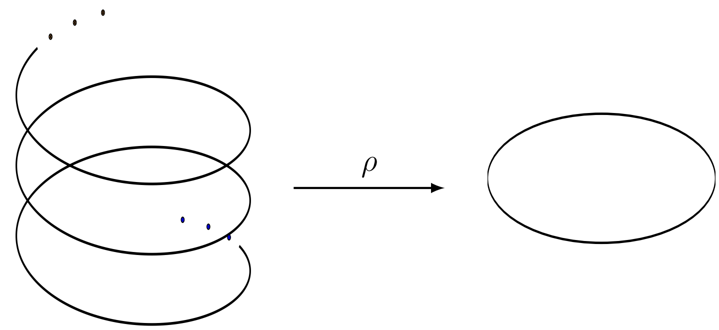

Figure 3. An example of a helix-shaped covering space of the unit circle $\mathbb{S}^1$, with the covering map $\rho : \mathbb{R} \to \mathbb{S}^1$, where $\rho(t) = e^{2\pi t}$.

In this paper, we will use a homotopy invariant called the winding number. Informally, the winding number is the number of times a curve "winds" counter-clockwise around some external point. Imagining a person fixed at a point, the winding number of a curve around that point would be the net number of times the person would have to fully turn around counter-clockwise when following the curve from start to end. Some examples are provided in Figure 2.

Remark 2.4. Another commonly used homotopy invariant for circle eversion is the *turning number*. The turning number is roughly defined as the number of times the normal vector (or the tangent vector) of a curve rotates counter-clockwise as it smoothly travels along the curve. This is equal to the total curvature of the curve divided by 2π.

It turns out that rigorously defining the winding number is far from trivial. Intuitively, the winding number is equal to the measure of the angle formed by $\gamma(0)$, $\gamma(1)$, and point P divided by 2π. However, it is difficult to differentiate between coterminal angles with algebraic functions, so this approach is far from simple. In this paper, we will do this using structures called "covering spaces". To define covering spaces, we must also define homeomorphisms.

Definition 2.5. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and let $f : X \to Y$ be a map between the two spaces. We say that f is a *homeomorphism* if it is bijective, continuous, and if its inverse function $f^{-1} : Y \to X$ is also continuous. If there exists a homeomorphism $f : X \to Y$, then we say that X and Y are *homeomorphic*.

Definition 2.6. For a space X, let $\rho : \tilde{X} \to X$ be a mapping such that for all points $x \in X$, x has an open neighborhood $U \subset X$ where $\rho^{-1}(U)$ is a union of disjointed sets that are all homeomorphic to U through ρ. ρ is called a *covering map* of X, and $\tilde{X}$ is called a *covering space* of X.

An example of a covering map for the unit circle $\mathbb{S}^1$ is provided in Figure 3. In order to apply covering maps in a useful way, we first need the following theorem.

Theorem 2.7. *Let $\rho : \tilde{X} \to X$ be a covering map. Given continuous maps $\gamma : Y \times [0,1] \to X$ and $\tilde{\gamma}_0 : Y \times \{0\} \to \tilde{X}$ such that $\rho \circ \tilde{\gamma}_0 = \gamma|_{Y \times \{0\}}$, there exists a unique and continuous map $\tilde{\gamma} : Y \times [0,1] \to \tilde{X}$ such that $\rho \circ \tilde{\gamma} = \gamma$. $\tilde{\gamma}$ is called the "lifted path of γ through ρ".*

Proof. While we will not prove this theorem, a relatively quick proof is provided in [3, pg.30–31]. Additionally, while it does appear complex at first, our use of this theorem should be fairly intuitive. ■

Theorem 2.8. *Any curve* $\gamma : [0,1] \to \mathbb{C}\backslash P$ *can be represented with unique and continuous polar functions* $r : [0,1] \to \mathbb{R}_{>0}$ *and* $\theta : [0,1] \to \mathbb{R}$ *such that* $\gamma(t) = P + r(t)e^{i\theta(t)}$ *for all* $t \in [0,1]$.

Proof. We first show that the map $\rho : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}\backslash P$ where $\rho(r,\theta) = P + re^{i\theta}$ is a covering map of $\mathbb{C}\backslash P$. Let the open neighborhoods U of $\mathbb{C}\backslash P$ be

$$\mathrm{Im}(z) > 0,\ \mathrm{Re}(z) < 0,\ \mathrm{Im}(z) < 0, \text{ and } \mathrm{Re}(z) > 0.$$

All points $z \in \mathbb{C}$ fall in at least one of these neighborhoods. The neighborhoods have the following inverses, respectively,

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n, n + \frac{\pi}{2}\right)$$

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n + \frac{\pi}{2}, n + \pi\right)$$

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n + \pi, n + \frac{3\pi}{2}\right)$$

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n + \frac{3\pi}{2}, n + 2\pi\right)$$

These inverses are all unions of disjoint open sets. Additionally, ρ is bijective when restricted to just these open sets, and the mappings ρ and ρ^{-1} are continuous, so the disjoint open sets of $\tilde{X}$ are all homeomorphic to their open neighborhoods in X. Thus, ρ is a covering map of $\mathbb{C}\backslash P$.

Let $\gamma : [0,1] \to \mathbb{C}\backslash P$ be a curve. Let Y be a single point $\{0\}$, and let $\tilde{\gamma}_0 : \{0\} \times \{0\} \to \mathbb{R}_{>0} \times \mathbb{R}$ be the continuous function $\tilde{\gamma}_0(y,t) = (|\gamma(t)|, \arg(\gamma(t)))$. If X is $\mathbb{C}\backslash P$ and $\tilde{X}$ is $\mathbb{R}_{>0} \times \mathbb{R}$, then by Theorem 2.7, there must exist a unique continuous $\tilde{\gamma}$ such that $\gamma = \rho \circ \tilde{\gamma}$. Let the component functions for $\tilde{\gamma}$ be continuous functions $r : [0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1] \to \mathbb{R}$. Because $\tilde{\gamma}$ is lifted through ρ, we know that $\gamma(t) = P + r(t)e^{i\theta(t)}$ for all $t \in [0,1]$. Therefore, we know that r and θ are valid polar functions, and that they exist, are unique, and are continuous for any curve γ. ■

Now, we can finally provide a rigorous definition of the winding number.

Definition 2.9. Let $\gamma : [0,1] \to \mathbb{C}\backslash P$ be a curve with external point P. From Theorem 2.8, let the corresponding polar function of γ be r and θ. The *winding number* $W(\gamma, P)$ of γ around point P is equal to

$$\frac{\theta(1) - \theta(0)}{2\pi}.$$

Remark 2.10. The winding number is also often defined in complex analysis as

$$W(\gamma, P) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t) - P} dt,$$

or in differential geometry as

$$W(\gamma, 0) = \frac{1}{2\pi} \int_\gamma \frac{x \cdot dy - y \cdot dx}{x^2 + y^2}.$$

In this paper, we use covering maps to define the winding number, since they are rooted in topology. However, the homotopy invariance of the winding number can also be proven using other definitions of the winding number.

3. Circle Eversion

We now prove it impossible to evert a circle by proving that the winding number is a homotopy invariant. First, we need to prove a couple lemmas.

Lemma 3.1. *Any homotopy between two curves $H : [0,1] \times [0,1] \to \mathbb{C} \backslash P$ can always be represented with unique and continuous polar functions $r : [0,1] \times [0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1] \times [0,1] \to \mathbb{R}$ such that $H(y,t) = P + r(t)e^{i\theta(t)}$ for all $t \in [0,1]$.*

Proof. This lemma and its proof are extensions of Theorem 2.8. We still use covering space ρ, but instead of letting Y be a single point $\{0\}$, we let it be the unit interval $[0,1]$. This makes the initial lifted path $\tilde{H}_0$ be $\tilde{H}_0 : [0,1] \times \{0\} \to \mathbb{R}_{>0} \times \mathbb{R}$, which itself is a lifted curve and can be continuous by Theorem 2.8. This makes the lifted path of $H : [0,1] \times [0,1] \to \mathbb{C} \backslash P$ through ρ be the function $\tilde{H} : [0,1] \times [0,1] \to \mathbb{R}_{>0} \times \mathbb{R}$, which is unique and continuous by Theorem 2.8. $\tilde{H}$ has continuous component functions $r : [0,1] \times [0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1] \times [0,1] \to \mathbb{R}$ such that $H(y,t) = P + r(y,t)e^{i\theta(y,t)}$ for all $(y,t) \in [0,1] \times [0,1]$. Therefore, we once again know that r and θ are valid polar functions, and that they exist, are unique, and are continuous for any homotopy H between two curves. ■

Lemma 3.2. *The winding number $W(\gamma, P)$ of a closed curve γ is always an integer.*

Proof. In a closed curve, $\gamma(0) = \gamma(1)$, so $P + r(0)e^{i\theta(0)} = P + r(1)e^{i\theta(1)}$. This implies that $\theta(0)$ and $\theta(1)$ must be coterminal angles, which means $\theta(1) = \theta(0) + 2\pi k$ for some $k \in \mathbb{Z}$. Now, calculating the winding number gets

$$W(\gamma, P) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{(\theta(0) + 2\pi k) - \theta(0)}{2\pi} = \frac{2\pi k}{2\pi} = k.$$

Therefore, $W(\gamma, P) = k \in \mathbb{Z}$ for any closed curve γ. ■

Now, we are finally ready to show that the winding number is homotopy invariant.

Theorem 3.3. *The winding number of a closed curve $\gamma : [0,1] \to \mathbb{C} \backslash P$ at point P is homotopy invariant.*

Proof. Let $H : [0,1] \times [0,1] \to \mathbb{C} \backslash P$ be a homotopy between two curves. By Lemma 3.1, the polar function $\theta : [0,1] \times [0,1] \to \mathbb{R}$ of H is continuous. This means that the winding number $W(H, P, t) = \frac{\theta(1,t) - \theta(0,t)}{2\pi}$ must be continuous throughout the homotopy.

Because the initial curve γ is closed, the curve must remain closed throughout H, since H would otherwise not be smooth (this would be like cutting the curve, which is against the rules of the homotopy). So, by Lemma 3.2, the winding number of the curve must remain an integer throughout H.

Thus, since the winding number is both continuous and discrete, it must be constant throughout H. This implies that the winding numbers of the initial curve and the final curve of any homotopy must be equal. Thus, the winding number is homotopy invariant. ■

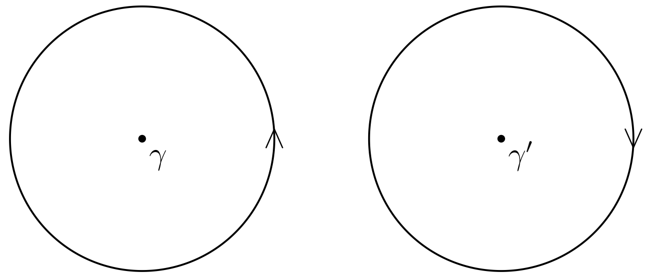

Figure 4. The circle $\gamma : [0, 1] \to \mathbb{C}$ and the everted circle $\gamma' : [0, 1] \to \mathbb{C}$, parameterized by maps $\gamma(t) = e^{2\pi t}$ and $\gamma'(t) = e^{-2\pi t}$.

Theorem 3.4. *The circle $\gamma(t) = e^{2\pi ti}$ and the everted circle $\gamma'(t) = e^{-2\pi ti}$ are not homotopic.*

Proof. By Theorem 2.8, the circle and the everted circle must be representable by unique polar functions r and θ. Let point P be the origin 0. For the circle, functions r and θ are $r(t) = 1$ and $\theta(t) = 2\pi t$ since $\gamma(t) = P + r(t)e^{i\theta(t)} = e^{2\pi ti}$. For the everted circle, functions r and θ are $r(t) = 1$ and $\theta(t) = -2\pi t$ since $\gamma'(t) = P + r(t)e^{i\theta(t)} = e^{-2\pi ti}$. So, calculating each of their winding numbers,

$$W(\gamma, 0) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{2\pi - 0}{2\pi} = 1$$

$$W(\gamma', 0) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{-2\pi - 0}{2\pi} = -1$$

Thus, because the winding numbers of the circle γ and the everted circle γ' are not equal, the two curves cannot be homotopic by Theorem 3.3. ■

This statement can be generalized to closed curves mapping to $\mathbb{C}$ rather than $\mathbb{C}\backslash P$ because any homotopy $H : [0, 1] \times [0, 1] \to \mathbb{C}$ can be transformed into some homotopy $H' : [0, 1] \times [0, 1] \to \mathbb{C}\backslash P$ by shifting the curve such that point P remains in its original region throughout the homotopy.

4. Turning Number

While sphere eversion is similar to circle eversion in many ways, it is much more difficult to prove. For this reason, the remaining sections of this paper will not contain any formal proofs. This section will instead provide a brief overview of the general method of indirectly proving sphere eversion using the turning number.

Intuitively, it is easier to prove that things are possible than it is to prove things are impossible, since we can just provide a proof of existence for things that are possible. However, the opposite is true in the case of circle and sphere eversion. It is much easier to prove that you *cannot* evert a circle than it is to prove that you *can* evert a sphere. This is partially because visualizing 3-dimensional surfaces, especially ones that pass through themselves, is much more challenging than visualizing 2-dimensional curves. The computer graphics to aid in this were available only after Smale's proof in 1957, and Shapiro's eversion in 1961. Additionally, we can use homotopy invariants to show that surfaces *are not* homotopic, but we cannot use them to show that surfaces *are* homotopic. In order to do this, we need a stronger version of the homotopy invariant.

A "complete homotopy invariant" is a homotopy invariant that, when constant between two surfaces, implies that they are homotopic. It turns out, the winding number is a complete

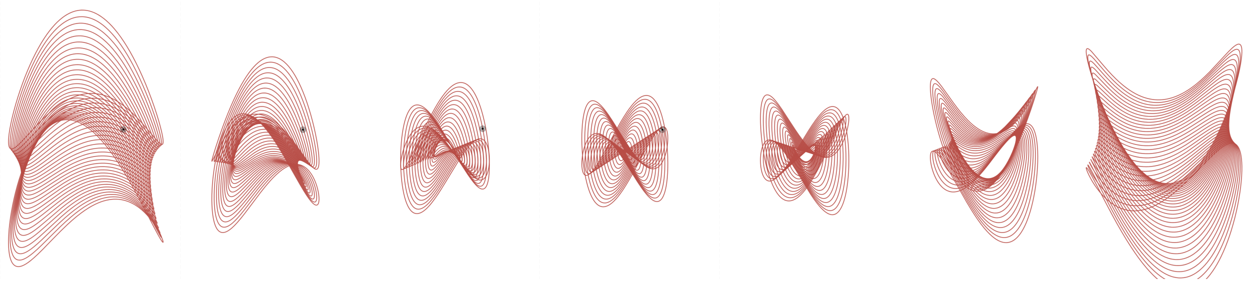

Figure 5. The restricted Bednorz-Bednorz cylindrical eversion (for $n = 2$) modeled in Desmos with 3-dimensional contour lines. The Desmos graph is available at `https://www.desmos.com/calculator/evacdlgwgo`.

homotopy invariant for curves. This follows closely from the Whitney-Graustein theorem, which is proven in [4]. However, the winding number is not applicable to the sphere.

Instead of the winding number, the turning number is used and is generalized to 3-dimensional surfaces. It turns out, the turning number is a complete homotopy invariant for spheres immersed in $\mathbb{R}^3$. A rough intuition for this can be found in [5]. The turning number of a closed surface is roughly defined as the number of "cups" and "bowls" in the surface minus the number of "saddles" in the surface. So, because both the sphere and the everted sphere have one cup, one bowl, and no saddles, both surfaces have the same turning number and are therefore regularly homotopic.

5. Bednorz-Bednorz Eversion

Another method in proving sphere eversion is providing an actual sphere eversion. This means providing the parameterizations for a regular homotopy between the sphere and the everted sphere and showing that they are smooth. This is possible with the recently discovered Bednorz-Bednorz sphere eversion [6], which, because of its simplicity, enables us to parametrize it with relatively simple equations. This not only makes the equations much easier to describe and to share, but also allows the eversion to be more easily modeled with common modeling software. One example of this is the website `https://rreusser.github.io/explorations/sphere-eversion/`, where the eversion is able to be modeled and described with both detail and speed.

The general idea for the Bednorz-Bednorz eversion is to first create a set of smooth parameterizations for a cylindrical eversion, and then extend this to a sphere. Note that the cylinder is infinitely tall and is parameterized by height and angle $(h, \phi) \in \mathbb{R} \times [0, 2\pi]$. We begin with the parameterizations for the cylindrical eversion:

$$x_1 = t\cos(\phi) + p\sin((n-1)\phi) - h\sin(\phi)$$

$$y_1 = t\sin(\phi) + p\cos((n-1)\phi) + h\cos(\phi)$$

$$z_1 = h\sin(n\phi) - \frac{t}{n}\cos(n\phi) - qth$$

Here, p and q are arbitrary values with $q \geq 0$, and t is the "timestamp" of the homotopy. We use $p = 0$ and $q = \frac{1}{2}$ in Figure 5.

Remark 5.1. We model this eversion in the 2-dimensional graphing calculator Desmos. This is both as a challenge and to demonstrate the simplicity of the equations presented by the

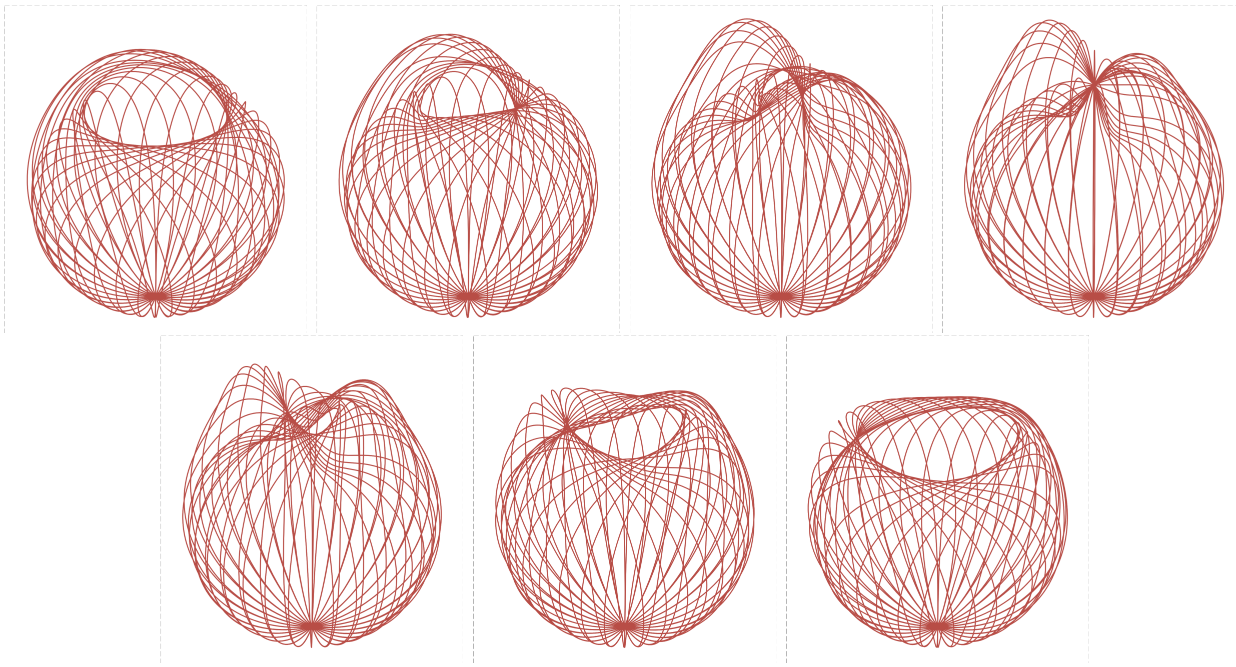

Figure 6. The Bednorz-Bednorz sphere eversion (for $n = 2$) modeled in Desmos with 3-dimensional contour lines. The Desmos graph is available at `https://www.desmos.com/calculator/2ru2ctxesf`.

Bednorz-Bednorz eversion. However, in general it is much more reasonable to use more sophisticated parametric modeling software to properly visualize the eversion. This is because Desmos is not built to render 3-dimensionally, and for this reason is not only unable to provide high quality visuals, but also cannot render complex curves quickly.

Now, similar to how a sphere can be thought of as a cylinder with the upper and lower bases smoothly "capped off", we can extend this eversion to the sphere. First, we define the following functions, which parametrize an intermediate "wormhole" that we need to define before we properly define the sphere:

$$x_2 = x_1(\xi + \eta(x^2 + y^2))^{-\kappa}$$
$$y_2 = y_1(\xi + \eta(x^2 + y^2))^{-\kappa}$$
$$z_2 = z_1(\xi + \eta(x^2 + y^2))^{-1}$$

Here, x_1, y_1, and z_1 are the equations from the cylinder eversion, ξ and η are arbitrary values with $\xi, \eta \geq 0$, and $\kappa = \frac{n-1}{2n}$. Now, we can finally parameterize the actual Bednorz-Bednorz sphere eversion

$$x_3 = \frac{x_2 e^{\gamma z_2}}{\alpha + \beta(x_2^2 + y + y_2^2)}$$
$$y_3 = \frac{y_2 e^{\gamma z_2}}{\alpha + \beta(x_2^2 + y_2^2)}$$
$$z_3 = \frac{\alpha - \beta(x_2^2 + y_2^2)}{\alpha + \beta(x_2^2 + y_2^2)} \cdot \frac{e^{\gamma z_2}}{\gamma} - \frac{\alpha - \beta}{\gamma(\alpha + \beta)}$$

where x_2, y_2, and z_2 are the previous equations, α and β are arbitrary values with $\alpha, \beta \geq 0$, and $\gamma = 2\sqrt{\alpha\beta}$. Most additional variables are used to adjust the eversion in ways that make it easier to see. Note that the sphere begins and ends with one of its poles twisted and pushed into the sphere towards its opposite pole. It is easy to see that this can be regularly homotopied back to the sphere and everted sphere. We model this in Figure 6.

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