When modelling variance-covariance matrices, we may be interested in making inference on the standard deviations (or marginal variances) and on correlations separately. To do so, consider the decomposition of the variance-covariance matrix Σ of order k as $\mathcal{S}R\mathcal{S}$, where $\mathcal{S} = \mathrm{diag}S$, $S = (\sigma_1, \dots \sigma_k)^\top$ is the vector of standard deviations and R is the matrix of correlations.

We can write a prior on the variance-covariance matrix Σ, then compute the prior on the standard deviations and correlations knowing that

$$\pi(S, R) = \pi(\Sigma) \mid J_{\Sigma(S,R)} \mid$$

Where we define $J_{Y(X)}$ as the Jacobian matrix of the transformation from X to Y, namely $\frac{\partial \mathrm{vec} Y}{\partial \mathrm{vec} X^\top}$, where the vec operator reduces to the half-vector vech operator for symmetric matrices.

Since Σ is symmetric, $\mathcal{S}$ is diagonal, and R has only $k(k-1)/2$ unique off-diagonal elements, $(S, \mathrm{vech}R) \to \mathrm{vech}\Sigma$ is an application from a set of size $k(k+1)/2$ to a set of size $k(k+1)/2$

Each column of the Jacobian matrix includes the derivatives of all the elements of the dependent variable with respect to one element of its arguments [1]. Let us consider a covariance matrix of order 4, for instance. We would have

$$\mathrm{vech}\Sigma = (\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \rho_{12}\sigma_1\sigma_2, \rho_{13}\sigma_1\sigma_3, \rho_{14}\sigma_1\sigma_4, \rho_{23}\sigma_2\sigma_3, \rho_{24}\sigma_2\sigma_4\rho_{34}\sigma_3\sigma_4)^\top$$

The derivative of this vector with respect to σ_1 would be

$$(2\sigma_1, 0, 0, 0, \rho_{12}\sigma_2, \rho_{13}\sigma_3, \rho_{14}\sigma_4, 0, 0, 0)^\top$$

Doing the same $\forall \sigma_i, \quad i = 1, \dots, k$, we obtain k vectors with 1 of the first k elements being $2\sigma_i$ and the other being zeroes; while $k-1$ of the remaining elements are of the form $\rho_{ij}\sigma_j \quad \forall j \neq i$. The derivative with respect to a correlation $\rho_{ij}, i \neq j$, has zeroes in the first k positions, and only one nonzero element in the remainder, equal to $\sigma_i\sigma_j$. This is the Jacobian matrix in the 4×4 case:

$$\begin{pmatrix}
2\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\rho_{12}\sigma_2 & \rho_{12}\sigma_1 & 0 & 0 & \sigma_1\sigma_2 & 0 & 0 & 0 & 0 & 0 \\
\rho_{13}\sigma_3 & 0 & \rho_{13}\sigma_1 & 0 & 0 & \sigma_1\sigma_3 & 0 & 0 & 0 & 0 \\
\rho_{14}\sigma_4 & 0 & 0 & \rho_{14}\sigma_1 & 0 & 0 & \sigma_1\sigma_4 & 0 & 0 & 0 \\
0 & \rho_{23}\sigma_3 & \rho_{23}\sigma_2 & 0 & 0 & 0 & 0 & \sigma_2\sigma_3 & 0 & 0 \\
0 & \rho_{24}\sigma_4 & 0 & \rho_{24}\sigma_2 & 0 & 0 & 0 & 0 & \sigma_2\sigma_4 & 0 \\
0 & 0 & \rho_{34}\sigma_4 & \rho_{34}\sigma_3 & 0 & 0 & 0 & 0 & 0 & \sigma_3\sigma_4
\end{pmatrix}$$

This is a block matrix of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are diagonals of order k and $k(k-1)/2$ respectively; B is a matrix of zeroes of dimension $k \times k(k-1)/2$ and C is the block including correlations, of dimension $k(k-1)/2 \times k$.

Since $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - C^{-1}AB| = |A| \cdot |D|$, we see the Jacobian determinant only depends on standard deviations. And specifically we have

$$| J_{\Sigma(S,R)} |= 2^k \prod_{i=1}^{k} \sigma_i^k \tag{1}$$

References

[1] Magnus, J. R., Neudecker, H. (2019). Matrix differential calculus with applications in statistics and econometrics. John Wiley and Sons.