Solving the heat equation on a rectangle with Dirichlet conditions

Solve the heat equation $u_t = u_{xx} + u_{yy}$ in a rectangle

The temperature in a rectangular plate is described by a function $u(x, y, t)$ for $x \in [0, A]$, $y \in [0, B]$, $t \in [0, \infty)$.

We assume that the temperature is zero on all four sides of the rectangle (Dirichlet conditions). The initial temperature $u(x, y, 0) = g(x, y)$ is given.

The function $u(x, y, t)$ satisfies the **heat equation**:

$$u_t(x, y) = u_{xx}(x, y) + u_{yy}(x, y) \qquad x \in (0, A), \quad y \in (0, B) \quad \text{(PDE)}$$

We have **Dirichlet boundary conditions** on all four sides of the rectangle for all $t \in (0, \infty)$:

$$u(x, 0, t) = 0, \quad u(x, B, t) = 0 \qquad x \in (0, A)$$

$$u(0, y, t) = 0, \quad u(A, y, t) = 0 \qquad y \in (0, B)$$

We write the solution $u(x, y, t)$ as a sine series with respect to **both x and y**: With $\mu_k := k\pi/A$ and $\nu_\ell := \ell\pi/B$

$$u(x, y, t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} C_{k\ell}(t) \sin(\mu_k x) \sin(\nu_\ell y)$$

From this we get $u_t(x, y, t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} C'_{k\ell}(t) \sin(\mu_k x) \sin(\nu_\ell y)$ and

$$u_{xx}(x, y, t) + u_{yy}(x, y, t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} C_{k\ell}(t)(-\mu_k^2 - \nu_\ell^2) \sin(\mu_k x) \sin(\nu_\ell y)$$

This gives the following **Solution Formula**: First find the sine series of g with respect to **both x and y**:

$$g(x, y) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} b_{k\ell} \sin(\mu_k x) \sin(\nu_\ell y)$$

Then the solution is given by

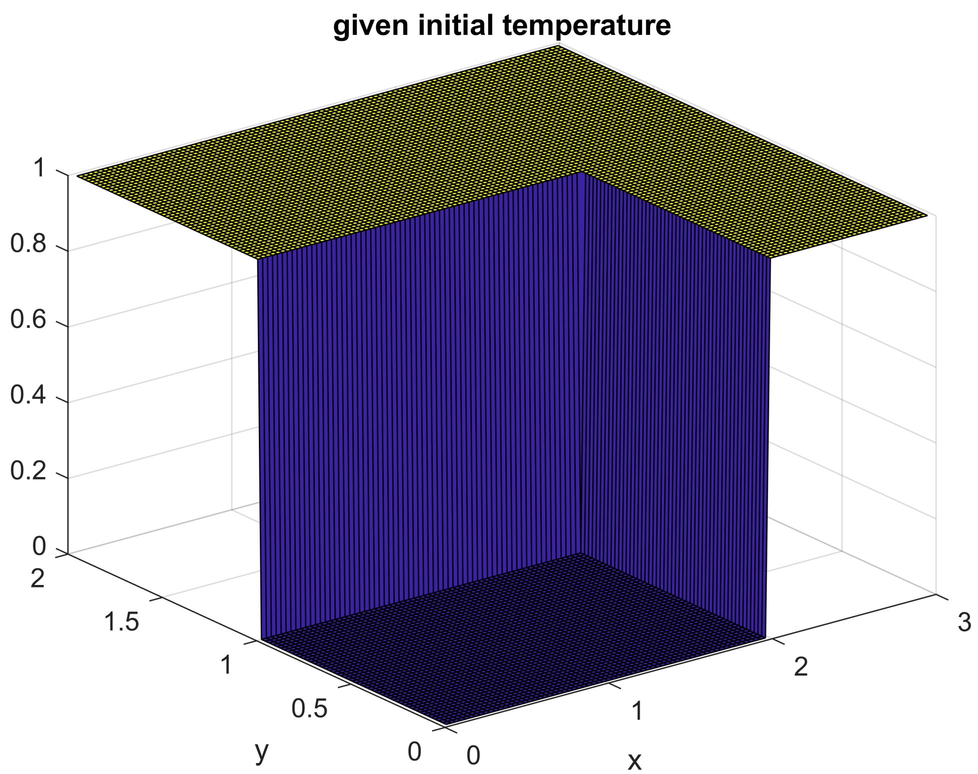
$$u(x, y, t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} b_{k\ell} e^{-(\mu_k^2 + \nu_\ell^2)t} \sin(\mu_k x) \sin(\nu_\ell y)$$

Example: For a rectangle with lengths $A = 3$ and $B = 2$ we are given the initial condition $u(x, y, 0) = g(x, y)$ with

$$g(x, y) = \begin{cases} 1 & \text{if } x > 2 \text{ or } y > 1 \\ 0 & \text{otherwise} \end{cases}$$

```
clearvars
A=3; B=2;                         % lengths of rectangle sides
g = @(x,y) double((x>2)|(y>1)); % given function g(x,y) for initial condition
```

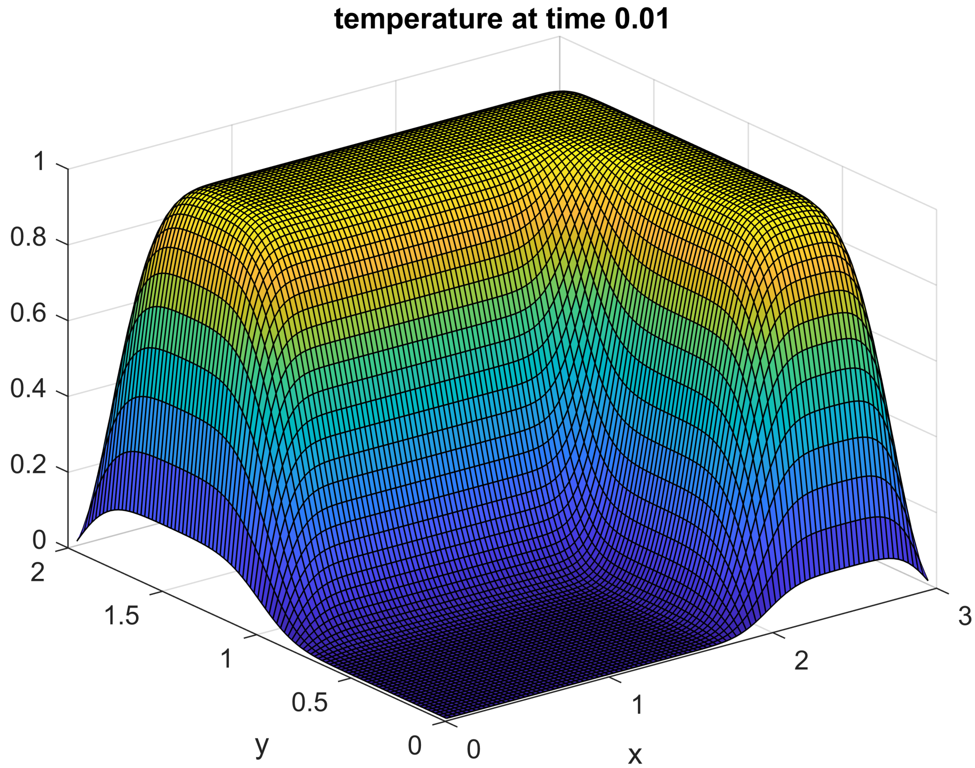
```
M = 100;                        % use spacing A/M for x
x = A*(1:M-1)/M;                % only interior nodes
N = 100;                        % use spacing B/N for y
y = B*(1:N-1)/N;                % only interior nodes

[X,Y] = ndgrid(x,y);            % X,Y are arrays of size M by N
G = g(X,Y);                     % G is array of values of g on grid:  G(j,k) = g(x_j,y_k)

surf(x,y,G');
xlabel('x'); ylabel('y'); title('given initial temperature')
```



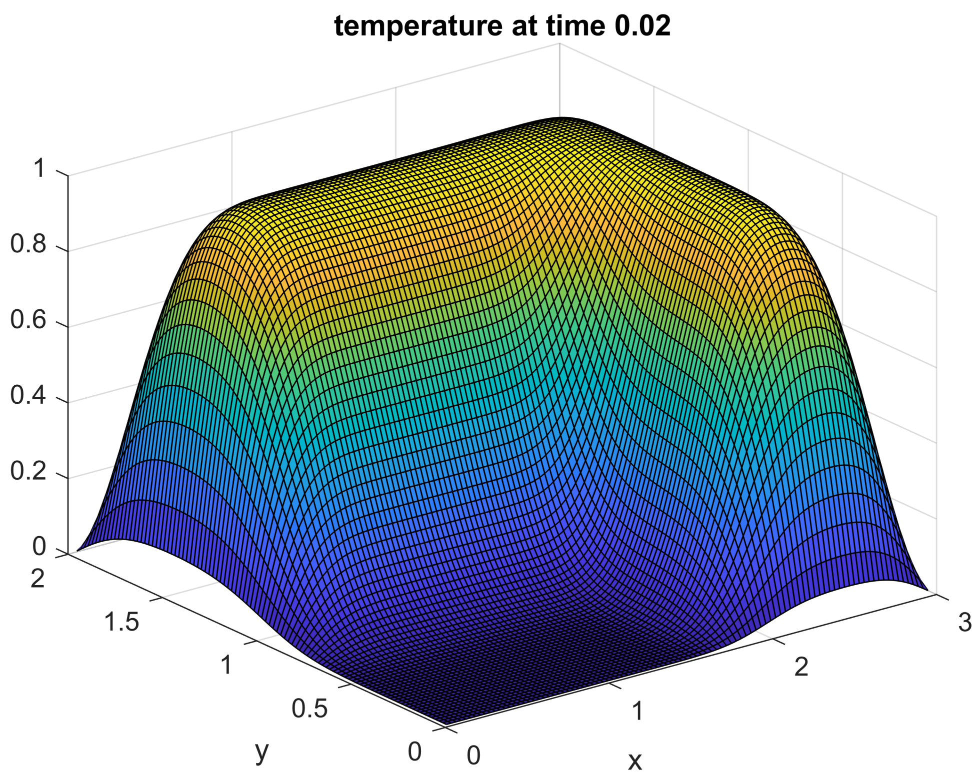
Use the discrete sine transform to compute the solution at several times

```
bx = sintr(G);                  % apply sine transform to columns
b = sintr(bx')';                % apply sine transform to rows

mu = (1:M-1)*pi/A;              % vector of mu_k
nu = (1:N-1)*pi/B;              % vector of nu_l
[Mu,Nu] = ndgrid(mu,nu);        % Mu,Nu are arrays of size M by N

for t=.01:.01:.04                 % for all times in vector tv
  C = b.*exp(-t*(Mu.^2+Nu.^2));   % solution formula in frequency domain
  Cx = isintr(C')';               % apply inverse sine transform to rows
  U = isintr(Cx);                 % apply inverse sine transform to columns
  figure
  surf(x,y,U');                   % make surface plot of solution
  xlabel('x'); ylabel('y'); title(sprintf('temperature at time %g',t))
```

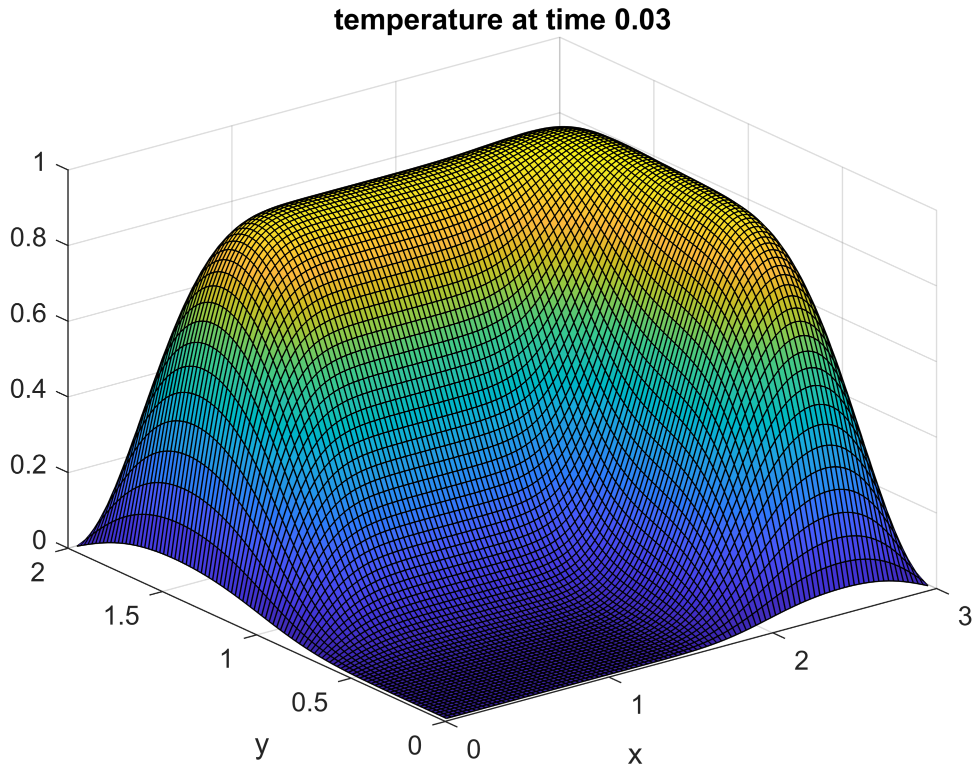
```
end
```

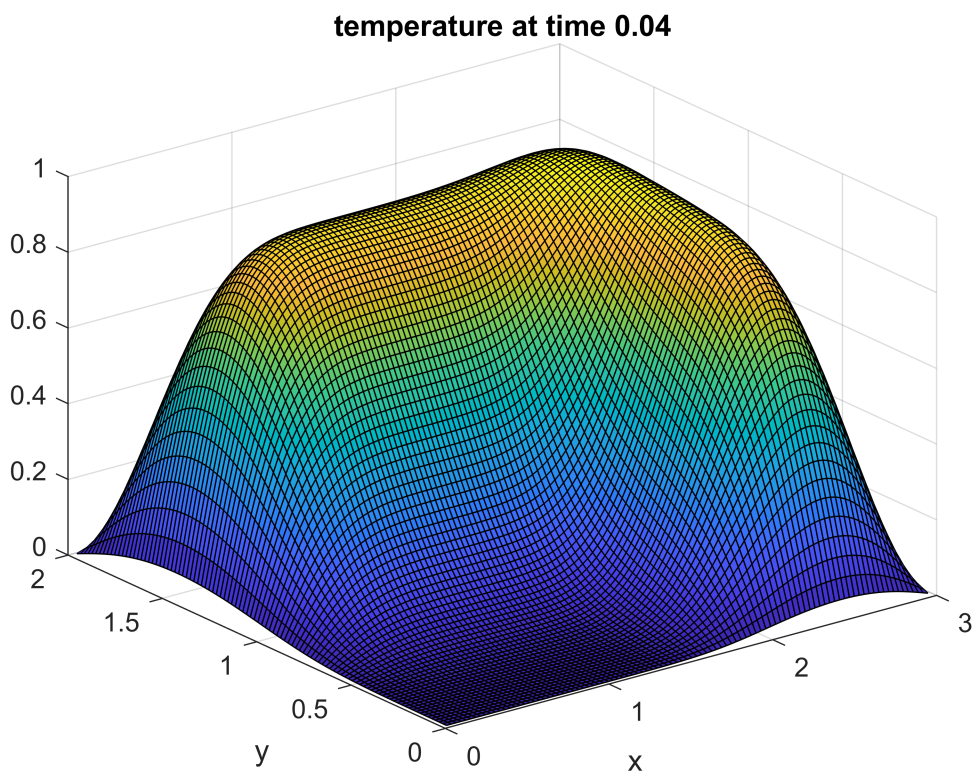


temperature at time 0.01
1
0.8
0.6
0.4
0.2
0
2
1.5
1
0.5
0
y
0
1
2
3
x



temperature at time 0.02
1
0.8
0.6
0.4
0.2
0
2
1.5
1
0.5
0
y
0
1
2
3
x


temperature at time 0.03
1
0.8
0.6
0.4
0.2
0
2
1.5
1
0.5
0
y
0
1
2
3
x


temperature at time 0.04
1
0.8
0.6
0.4
0.2
0
2
1.5
1
0.5
0
y
0
1
2
3
x