1 Proof of the Schwarz Lemma

First we prove the Schwarz Lemma, which is used several times in the proof of the Riemann Mapping Theorem.

Let $f : D \to D$ be holomorphic with f(0) = 0

Proof: First, we note that $f(z) = a_0 + a_1 + a_2z^2 + ...$ by the power series expansion, where $a_0 = 0$ since $f(0) = 0$.

Second, we define $g(z) = f(z)/z$. Then $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic function (since it only has one removable singularity at 0).

Now, for any $0 < \epsilon < 1$, we have by the maximum modulus principle that $|g(z)|$ must attain its maximum on the boundary of the disk $z : |z| = 1 - \epsilon$. We take z_ϵ to be this point. But then, we have $g(z) <= g(z_\epsilon)$, for any $|z| <= 1 - \epsilon$. Take the infimum of ϵ by taking the limit as $\epsilon \to 0$, and we have that the values of g are bounded, i.e.$|g(z)| <= 1$. Therefore we have that $|f(z)| <= |z|$.

Additionally, $f'(0) = g(0)$, and thus we have that $|f'(0)| = |g(0)| \leq 1$ which completes the proof of the first part of the lemma.

Now recall that a **rotation** is a map of the form $z \to az$, where $|a| = 1$, namely $a = e^{i\theta}$, and thus we just need to show that f(z) is a map of this form when $|f'(0)| = 1$.

We suppose that $|f'(0)| = 1$ for some f in $\mathbb{D}, \forall r > 0$. Then g(0) = 1. In this case, g clearly obtains a maximum at $0 \in \mathbb{D}$, implying that g is constant from the maximum modulus principle. That is to say , there exists some $a \in \mathbb{D}$, such that g(z) = a. Now note that we then have $|g(0)| = |a| = 1$.

So for $z \in D \setminus 0$, we therefore have $f(z)/z = a \implies f(z) = za$. As f(0) = 0, we then have $f(z) = az$, for all $z \in \mathbb{D}$ by definition, for some $a \in \mathbb{C}$ with $|a| = 1$, which concludes the proof.

□

2 Proof of the Riemann Mapping Theorem

Suppose Ω is proper (non-empty, and not the whole of $\mathbb{C}$, and simply connected. If $z_0 \in \Omega$, then $\exists$ a unique conformal map such that

$$F(z_0) = 0 \text{ and } F'(z_0) > 0$$

Proof of the Uniqueness of the Conformal Map $F : \Omega \to \mathbb{D}$:

Proof: This is straightforward. If F and G are conformal maps from Ω to $\mathbb{D}$ that satisfies the two conditions in the theorem, then clearly $H = F \circ G^{-1}$, is an automorphism of the disc that fixes the origin. Therefore, $H(z) = e^{i\theta}z$, and because $H'(0) > 0$, we must have $e^{i\theta} = 1$, so we can conclude H(z) = z which means F = G. □

Step 1: Prove that there exists a conformal map F_1 from Ω to $\tilde{\Omega}$.

Proof: Choose $\alpha \in \mathbb{C} \setminus \Omega$.

Then, $(z - \alpha)$ is nowhere vanishing on Ω □

Lemma:

Let Ω be a simply connected proper open subset of $\mathbb{C}$. Then Ω is conformally equivalent to an open subset of the unit disc $\mathbb{D}$ which contains the origin.

Step 2: Assume that Ω is an open subset of $\mathbb{D}$

Proof: Consider the family F of all injective holomorphic functions on Ω that map onto the unit disc, and fix the origin:

$\mathbb{F} = \{f : \Omega \to \mathbb{D} :$ f is holomorphic, injective, and f(0) = 0$\}$

First, note that $\mathbb{F}$ is nonempty, since it contains the identity map. Also, this family is uniformly bounded by construction, since all the functions are required to map onto the unit disc.

Now, we want to find a function f $\in$ which maximizes $|f'(0)|$. First, observe that —f'(0)— is uniformly bounded as f ranges in $\mathbb{F}$. This follows from the Cauchy inequalities for f', applied to a small disc centered at the origin. $\square$