1 Mittag-Leffler Theorem

Theorem: Suppose that $Q_n(z)$ are given polynomials for $n = 1, 2, ...$

Suppose also that we are given a sequence of complex numbers a_n without limit points. Prove that there exists a meromorphic function f(z) whose only poles are at a_n , and so that for each n, the difference $f(z) - Q_n(z)$ is holomorphic near a_n. In other words, prove that f has prescribed poles and principal parts at each of these poles.

Proof: We first rearrange the a_n in the sequence in ascending order such that $|a_1| \leq |a_2| \leq |a_3| \leq .....$ Now recall that we know from the fact that a_n has no limit points, that the only limit of a_n is infinity. So, for each n, we can choose an R_n greater than 0 such that $|a_n| \leq R_n$ such that $|a_n| > R_n$ and $|a_{n_k}| < R_n$, where $|a_{n-k+1}| = |a_{n-k+2}| = |a_{n-k+3}| = |a_n|$. (Obviously, if a_n had any other limit points, this would not hold true.)

Then, we have that $Q_n(\frac{1}{z-a_n})$ is clearly holomorphic inside D_R, where D_{Rn} is the open disc of radius R_n.

Now according, to Runge's Approximation Theorem, we can find a polynomial $P_n(z)$ such that $|Q_n(\frac{1}{z-a_n}) - P_n(z)| \leq \frac{1}{2^n}, \forall z \in D_{Rn}$.

Next, we define the following function f(z):
$f(z) = \sum_{n=1}^{\infty}(Q_n(\frac{1}{z-a_n}) - P_n(z))$

Now, we note that $\forall N \in \mathbb{N}, \sum_{i=1}^{N}(Q_n(\frac{1}{z-a_n}) - P_n(z))$ is the definition of a meromorphic function in R_n , whose poles are $a_1, a_2, ...a_n$.

Also, we note that the f(z) function we defined also satisfies the following: $f_z - Q_n(\frac{1}{z-a_n})$ is a holomorphic function near every pole.

Next, we have that $|\sum_{n=N+1}^{\infty}(Q_n(\frac{1}{z-a_n}) - P_n(z))| \leq \sum_{n=N+1}^{\infty}(Q_n(\frac{1}{z-a_n}) - P_n(z))$
$\leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^{N+1}}$.

Thus, this part of the sum (from N+1 to infinity) is convergent by the **Weierstrass M-Test**, and consequently is also holomorphic inside D_{Rn}.

Then, we have that f(z) as defined satisfies all the required properties in the theorem for D_{Rn}, and because R_n can be taken to be arbitrarily large, f(z) holds the required properties on $\mathbb{C}$, and is exactly the function we are looking for in the theorem statement. This proves the theorem.

$\square$