Locality-Sensitive Hashing for f-Divergences and Kreĭn Kernels: Mutual Information Loss and Beyond

Lin Chen[1,2], Hossein Esfandiari[1], Thomas Fu[1], and Vahab Mirrokni[1]

[1]Google Research, [2]Yale University

1. Submodular Functions

- Submodularity is an intuitive diminishing returns property: a discrete set function $f : 2^V \to \mathbb{R}_+$ is **submodular** if for all sets $A \subseteq B \subseteq V$ and $x \notin B$, we have
$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$
Submodular function f is **monotone** if for all $A \subseteq B$: $f(A) \leq f(B)$
- Continuous analogue: a differentiable function $F : \mathcal{X} \to \mathbb{R}_{\geq 0}$ defined on a box $\mathcal{X} \triangleq \prod_{i=1}^{d} \mathcal{X}_i$, where each $\mathcal{X}_i$ is a closed interval of $\mathbb{R}_{\geq 0}$. We say that F is **Continuous DR-submodular** [?] if for every $x, y \in \mathcal{X}$ that satisfy $x \leq y$ and every $i \in [d] \triangleq \{1, \ldots, d\}$, we have
$$\frac{\partial F}{\partial x_i}(x) \geq \frac{\partial F}{\partial x_i}(y).$$

2. Bandit Optimization

Online optimization is a repeated two-player game. At each round t:

- The learner chooses an action x_t from a convex set $\mathcal{K} \subseteq \mathbb{R}^n$;
- The adversary chooses a reward function F_t from $\mathcal{F}$, a family of real-valued functions;
- The learner receives a reward $F_t(x_t)$, and observes feedback.

The aim is to maximize the **regret**: the gap between her accumulated reward and the reward of the best single choice in hindsight
$$\mathcal{R}_T \triangleq \max_{x \in \mathcal{K}} \left\{ \sum_{t=1}^{T} F_t(x) - \sum_{t=1}^{T} F_t(x_t) \right\}$$
In the **bandit** setting, the feedback is only a single real number $F_t(x_t)$. However, even in the offline scenario, continuous DR-submodular maximization problem cannot be approximated within a factor of $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$ in polynomial time, unless $RP = NP$ [?]. Therefore, we consider the **$(1 - 1/e)$-regret**
$$\mathcal{R}_{1-1/e,T} \triangleq (1 - 1/e) \max_{x \in \mathcal{K}} \left\{ \sum_{t=1}^{T} F_t(x) - \sum_{t=1}^{T} F_t(x_t) \right\}.$$

3. Our Contribution

We studied the following three problems:

- `OCSM` : the Online Continuous DR-Submodular Maximization problem,
- `BCSM` : the Bandit Continuous DR-Submodular Maximization problem, and
- `RBSM` : the Responsive Bandit Submodular Maximization problem.

Comparison of previous and our proposed algorithms:

Setting	Algorithm	Stochastic gradient	# of grad. evaluations	$(1 - 1/e)$-regret
OCSM	Meta-FW[?]	No	$T^{1/2}$	$O(\sqrt{T})$
	VR-FW[?]	Yes	$T^{3/2}$	$O(\sqrt{T})$
	Mono-FW (this work)	Yes	1	$O(T^{4/5})$
BCSM	**Bandit-FW (this work)**	-	-	$O(T^{8/9})$
RBSM	**Responsive-FW (this work)**	-	-	$O(T^{8/9})$

4. One-shot Online Continuous DR-Submodular Maximization

Offline **Frank-Wolfe (FW)** method for maximizing monotone continuous DR-submodular functions: at k-th iteration, solves a linear optimization problem
$$v^{(k)} \leftarrow \arg\max_{v \in \mathcal{K}} \langle v, \nabla F(x^{(k)}) \rangle$$
which is used to update $x^{(k+1)} \leftarrow x^{(k)} + \eta_k v^{(k)}$, where η_k is the step size. We extended it to online setting.

- **Obstacle 1:** To obtain $v_t^{(k)}$, we have to know the gradient in advance.
Solution: Use K no-regret online linear maximization oracles $\{\mathcal{E}^{(k)}\}, k \in [K]$, with $\langle \cdot, d_t^{(k)} \rangle$ being the objective function, and $v_t^{(k)}$ being the output, where $d_t^{(k)}$ is an estimation of $\nabla F_t(x_t(k))$.
- **Obstacle 2:** To obtain a **one-shot** algorithm, we need to reduce K from $T^{3/2}$ [? ?] to 1.
Solution: Proposed the **blocking procedure** and the **permutation methods**.

5. Bandit Continuous DR-Submodular Maximization

One-point Gradient Estimator: define the δ-smoothed version of a function F as $\hat{F}_\delta(x) \triangleq \mathbb{E}_{v \sim B^d}[F(x + \delta v)]$. Then we have
$$\nabla \hat{F}_\delta(x) = \mathbb{E}_{u \sim S^{d-1}} \left[\frac{d}{\delta} F(x + \delta u) u \right].$$

- **Obstacle:** The point $x + \delta u$ may fall outside of $\mathcal{K}$.
Solution: Introduce the notion of **δ-interior**. A set $\mathcal{K}'$ is a δ-interior of $\mathcal{K}$ if it is a *subset* of
$$\operatorname{int}_\delta(\mathcal{K}) = \{x \in \mathcal{K} | \inf_{s \in \partial \mathcal{K}} d(x, s) \geq \delta\}.$$
Also define the **discrepancy** between $\mathcal{K}$ and $\mathcal{K}'$ by
$$d(\mathcal{K}, \mathcal{K}') = \sup_{x \in \mathcal{K}} d(x, \mathcal{K}'),$$
When F_t is Lipschitz and $d(\mathcal{K}, \mathcal{K}')$ is small, we can **approximate** the optimal total reward on $\mathcal{K}$ by that on $\mathcal{K}'$.
Lemma 1: Under some regularization assumptions, $\mathcal{K}' = (1 - \alpha)\mathcal{K} + \delta 1$ is a δ-interior of $\mathcal{K}$ with $d(\mathcal{K}, \mathcal{K}') \leq [\sqrt{d}(\frac{R}{r} + 1) + \frac{R}{r}]\delta$.

6. Bandit Submodular Set Maximization

- **Obstacle:** If there is a rounding scheme $\mathrm{round}_\mathcal{I} : [0, 1]^d \to \mathcal{I}$ satisfying the following unbiasedness condition
$$\mathbb{E}[f(\mathrm{round}_\mathcal{I}(x))] = F(x), \quad \forall x \in [0, 1]^d$$
for any submodular set function f within the matroid constraint $\mathcal{I}$ and its multilinear extension F, we can solve the discrete bandit problem by applying **Bandit-FW** on F_t. However, this kind of rounding schemes does **NOT** exist (**Lemma 2**).
Solution: Study the **responsive** model. If $X_t \notin \mathcal{I}$, we can still observe the function value $f_t(X_t)$ as feedback, while the received reward at round t is 0.

7. References