On Lazy Training in Differentiable Programming

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Abstract

Context. Recent theory shows that training **wide neural networks** amounts to doing regression with a positive-definite **kernel**.

Contributions. This *lazy training* phenomenon:

- is not intrinsically due to width but to a degenerate relative **scale**
  - $\to$ depends on **early stopping**, **initialization** and **normalization**
- removes some benefits of depth and may **hinder generalization**

Lazy Training

Setting. Adjust parameters of a differentiable model $h : \mathbb{R}^p \to \mathcal{F}$ by minimizing a loss $R : \mathcal{F} \to \mathbb{R}_+$ using gradient flow on the objective

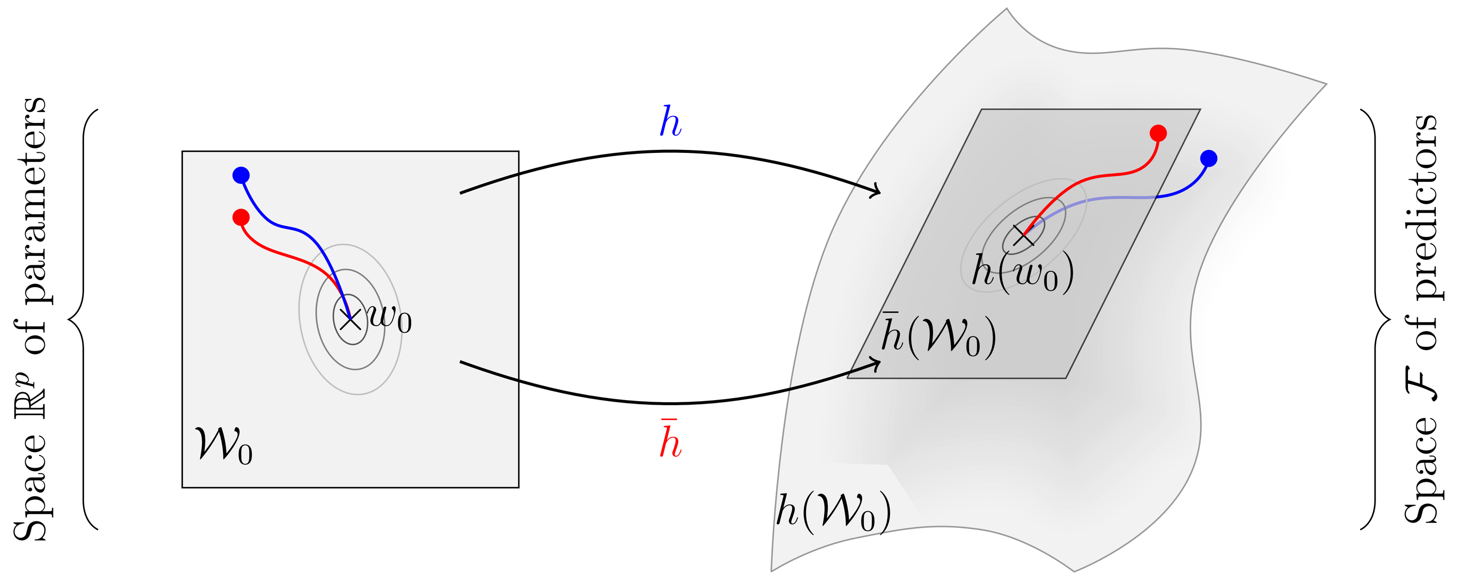
$$F(w) = R(\alpha h(w))/\alpha^2.$$

- $\mathcal{F}$ is a Hilbert space of predictors, R typically the empirical or population risk, h typically a neural network
- $\alpha > 0$ is a scale, often implicitly present
- gradient flows approximate (stochastic, accelerated) gradient descent

Training paths. For initialization w_0 and stopping time T, let

- $(w_\alpha(t))_{t\in[0,T]}$ be the *original* optimization path
- $(\bar{w}_\alpha(t))_{t\in[0,T]}$ be the *tangent* optimization path, for the tangent model

$$\bar{h}(w) = h(w_0) + Dh(w_0)(w - w_0)$$



Lazy Training (definition)

When the *original* and *tangent* optimization paths are close

Consequences. Lazy training is a type of implicit bias for gradient descent that leads to strong guarantees:

- on optimization speed (theory of convex optimization)
- on generalization (theory of kernel regression)

Lazy Training Theorems

Finite horizon

If $h(w_0) = 0$ and R potentially non-convex then for any $T > 0$,

$$\lim_{\alpha\to 0} \sup_{t\in[0,T]} \|\alpha h(w_\alpha(t)) - \alpha\bar{h}(\bar{w}_\alpha(t))\| = 0.$$

Infinite horizon

If $h(w_0) = 0$, and R is strongly convex, then

$$\lim_{\alpha\to 0} \sup_{t\geq 0} \|\alpha h(w_\alpha(t)) - \alpha\bar{h}(\bar{w}_\alpha(t))\| = 0.$$

- over-parameterization is not needed
- see paper for precise statements

When does it occur?

A sufficient criterion. For the square loss $R(y) = \frac{1}{2}\|y - y^\star\|^2$ and $\alpha = 1$, the relative difference $\Delta := \|h(w(t)) - \bar{h}(\bar{w}(t))\|/\|y^\star - h(w_0)\|$ is controlled by

$$\Delta \lesssim \tilde{t}^2 \cdot \kappa_h(w_0) \quad \text{where} \quad \kappa_h(w_0) := \frac{\|h(w_0) - y^\star\|\|D^2h(w_0)\|}{\|Dh(w_0)\|^2}$$

where $\tilde{t} = t\|Dh(w_0)\|^2$ is the normalized time ($\approx$ iteration number).

Case 1: Rescaled models

For $\alpha > 0$, one has $\kappa_{\alpha h}(w_0) \lesssim \|h(w_0) - y^\star/\alpha\|$

$\to$ lazy if $h(w_0)$ small and α large

Case 2: Homogeneous models

If $h(\lambda w) = \lambda^q h(w)$, one has $\kappa_h(\lambda w_0) \lesssim \|h(w_0) - y^\star/\lambda^q\|$

$\to$ lazy if $h(w_0)$ small and λ large

Case 3: Wide neural networks

If $h_m(w) = \alpha \sum_{i=1}^m \phi(\theta_i)$ where $w = (\theta_1, \dots, \theta_m)$ are i.i.d. and satisfy $\mathbb{E}\phi(\theta_i) = 0$ (two-layer neural network), then

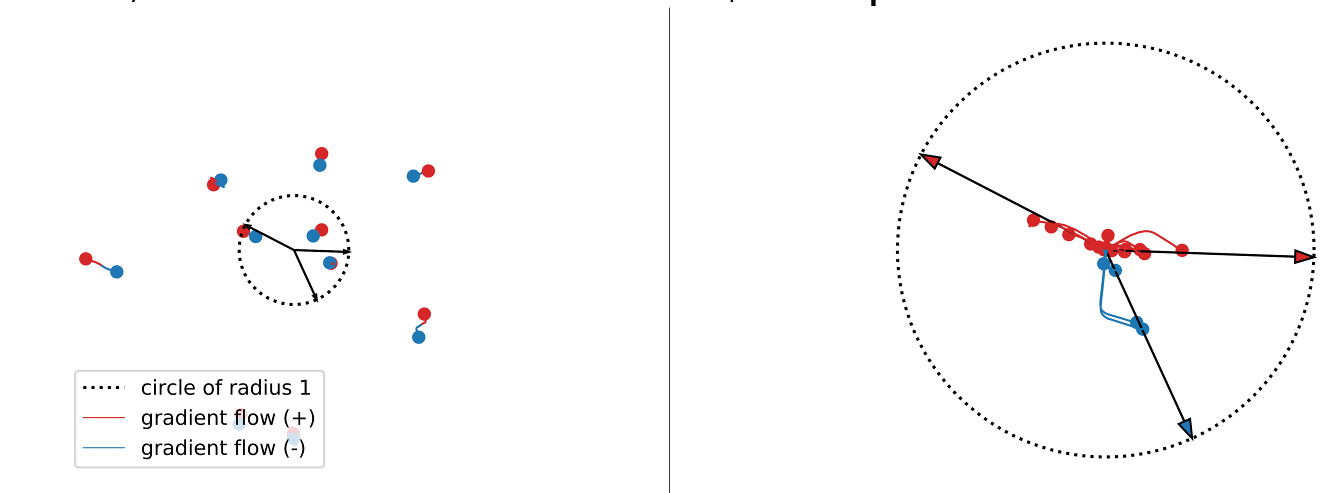
$$\kappa_{h_m}(w_0) \lesssim m^{-1/2} + (\alpha m)^{-1}$$

$\to$ lazy if $\lim_{m\to\infty} \alpha m = \infty$ (e.g. $\alpha = 1/\sqrt{m}$)

$\to$ can be extended to deep networks (Jacot et al.)

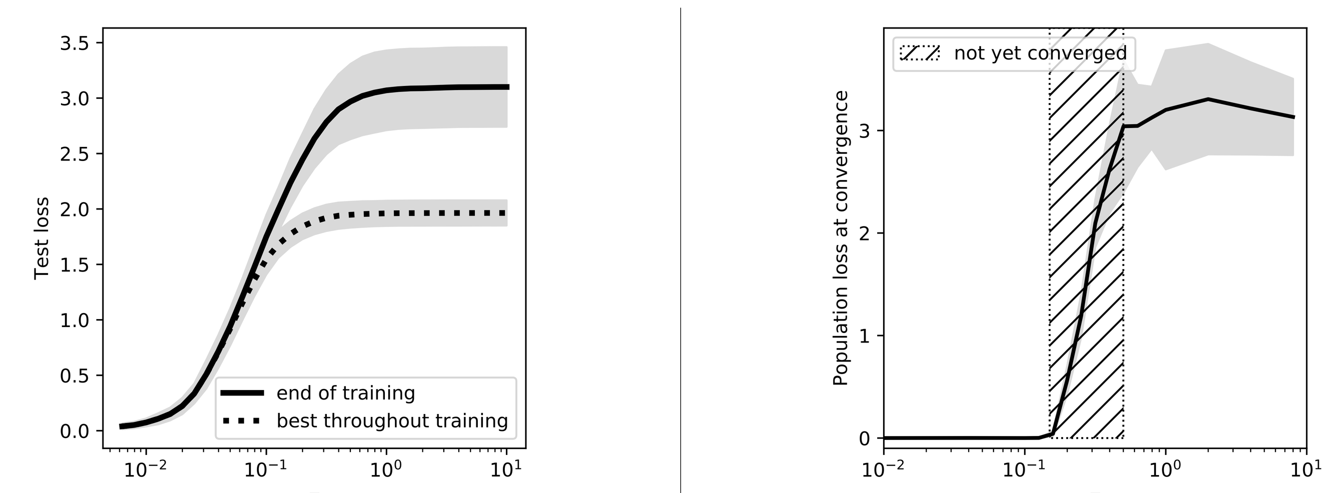
Is it desirable in practice?

Synthetic experiments. Two-layer ReLU neural network, square loss, initialized with variance τ, best predictor has 3 neurons.



Lazy Training ($\tau = 0.1$) Non-Lazy Training ($\tau = 2$)

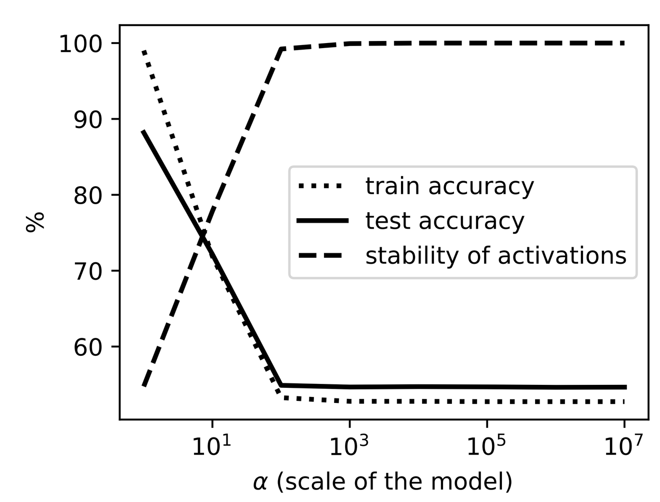
Trajectory of each "hidden" neuron during training (2-D input)



Over-parameterized (GD on train loss until 0 loss) Under-parameterized (SGD on population loss)

Impact of laziness on performance (100-D input)

Image recognition. Does lazy training explain deep learning?



Model	Train acc.	Test acc.
ResNet wide, linearized	55.0	56.7
VGG-11 wide, linearized	61.0	61.7
Prior features (Oyallon et al.)	-	82.3
Random features (Recht et al.)	-	84.2
VGG-11 wide, standard	99.9	89.7
ResNet wide, standard	99.4	91.0

Effect on laziness (VGG11 model) Linear vs. lazy vs. deep models

Theoretical arguments. Neural networks can be superior to kernel/fixed features methods, thanks to their adaptivity (Bach 2017).

Main references

- Jacot et al., *Neural Tangent Kernel: Convergence and Generalization in Neural Networks.* 2018.
- Du et al., *Gradient Descent Provably Optimizes Over-parameterized Neural Networks.* 2018.
- Bach. *Breaking the Curse of Dimensionality with Convex Neural Networks).* 2017.