

Information content

The information content is related to the number of *binary decisions* required to find the information.

The number of binary decisions (= number of questions whose answer is yes/no) required to find the correct element in a set of N elements is:

$$n_q = \log_2 N$$

Example: consider the 4 nucleotides, {A,C,G,T}.

John: I have a nucleotide in mind. Guess which one!

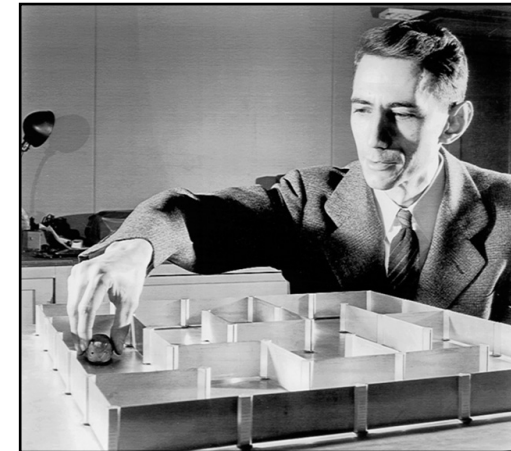
Mike: is it A or T?

John: no!

Mike: is it G?

John: yes!

Thus, $\log_2 4 = 2$ questions where necessary to find unambiguously the right nucleotide.



Claude Elwood Shannon
(1916-2001)

If we had N elements, we first ask if the hidden element is among the $N/2$ first elements, then in the $N/4$ left elements, then in $N/8$ elements, etc, till we reach the 1 correct element. There are thus

$$\begin{aligned} n_q &= 1(N/2?) + 1(N/4?) + 1(N/8?) + \dots 1(N/N) \\ &= 1(N/2?) + 1(N/2^2?) + 1(N/2^3?) + \dots 1(N/2^{\log_2 N}) \\ &= \log_2 N \end{aligned}$$

questions to ask.

Information content

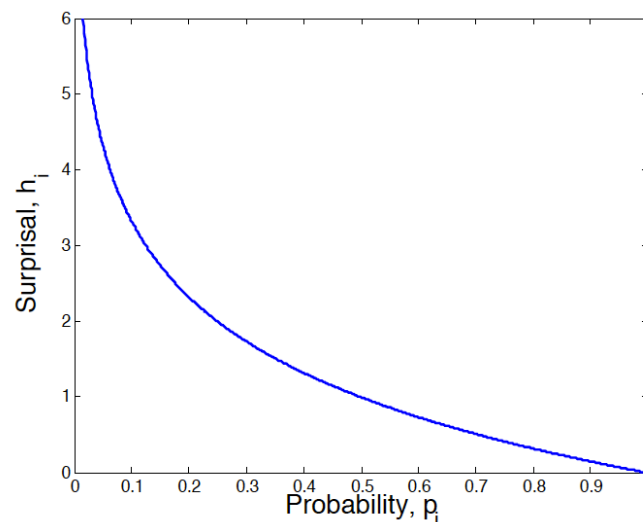
If each element has the same probability, then this probability is $p=1/N$, and

$$n_q = \log_2 N = -\log_2 p$$

In general the elements are not equally likely; they have different probabilities, p_i .

Tribus (1961) then generalizes the formula here above by introducing the concept of "*surprisal*" h_i :

$$h_i = -\log_2 p_i$$



Interpretation:

- if p_i approaches 0, then we will be very surprised to see the i th symbol (since it should almost never appear), and the formula says h_i approaches ∞ .
- if $p_i=1$, then we won't be surprised at all to see the i th symbol (because it should always appear) and $h_i = 0$.

Information content

If each element has the same probability, then this probability is $p=1/N$, and

$$n_q = \log_2 N = -\log_2 p$$

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Tribus (1961) then generalizes the formula here above by introducing the concept of "*surprisal*" h_i :

$$h_i = -\log_2 p_i$$

On this basis, Shannon introduced the "*uncertainty measure*" (also called "**entropy**"), which is the average of all surprisals h_i weighted by their occurrence p_i :

$$H = \sum_i p_i h_i = -\sum_i p_i \log_2 p_i$$

Special cases of uncertainty
(shown here for a 4 letter alphabet, {A,C,G,T})

$p=\{1,0,0,0\}$

$H=\min(H)=0$

No uncertainty at all: the nucleotide is completely specified.

$p=\{0.5,0,0,0.5\}$

$H=1$

Uncertainty between two letters

$p=\{0.25,0.25,0.25,0.25\}$

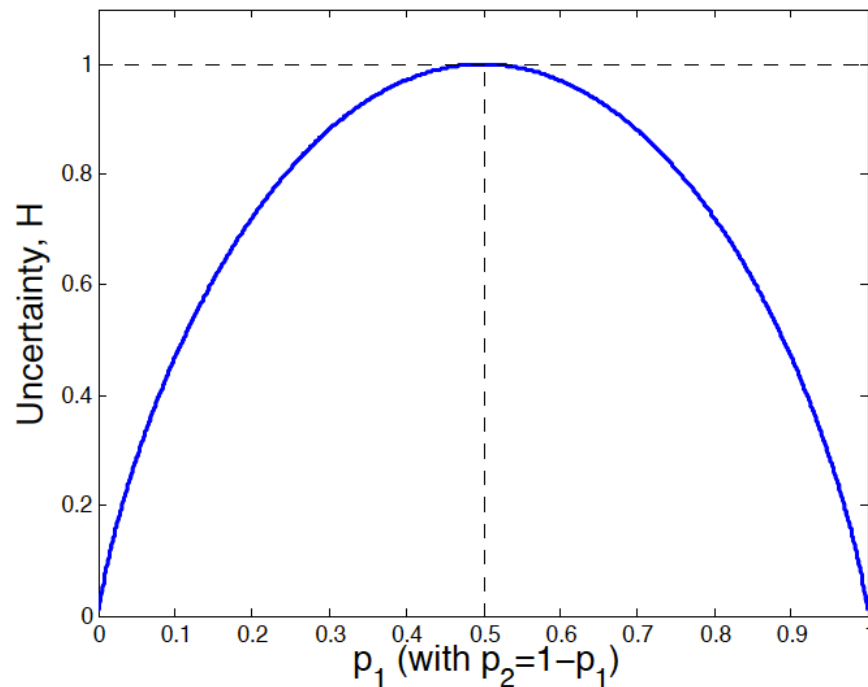
$H=\max(H)=2$

Complete uncertainty

Information content

Case of 2 symbols:

Uncertainty H for the case of 2 symbols, as a function of the probability p_1 of one symbol (with $p_2=1-p_1$).



Exercise

Calculate the uncertainty of the following probability sets:

$p=\{1/2, 1/4, 1/8, 1/8\}$

$p=\{1/2, 1/6, 1/6, 1/6\}$

NB: For the second case, you will "certainly" need a computer...

Check that the uncertainty is indeed comprised between 0 and 2 (= maximum uncertainty calculated above).

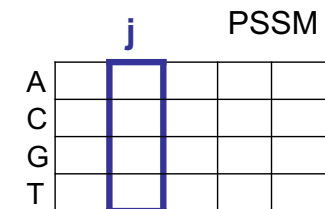
Information content

Shannon uncertainty applied to PSSM

Uncertainty of a column j of a PSSM:

$$H_s(j) = - \sum_{i=1}^A f_{i,j} \log_2(f_{i,j})$$

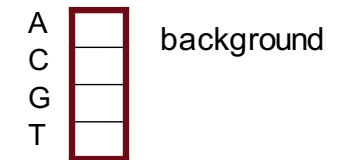
A is the number of possible elements (4 letter for DNA, 20 for aa) and f_{ij} is the frequency of letter i in column j .



Uncertainty of the background (e.g. the GC content of a genome):

$$H_g = - \sum_{i=1}^A p_i \log_2(p_i)$$

p_i is the background frequency of letter i .



Schneider (1986) defines an *information content* based on Shannon's uncertainty.

$$R_{seq}(j) = H_g - H_s(j) \quad \text{for column } j$$

$$R_{seq} = \sum_{j=1}^w R_{seq}(j) \quad \text{for the PSSM}$$

For skewed genomes (i.e. unequal residue probabilities), Schneider recommends an alternative formula for the information content.

$$R_{seq}^*(j) = \sum_{i=1}^A f_{ij} \log_2 \left(\frac{f_{ij}}{p_i} \right) \quad \text{for column } j$$

$$R_{seq}^* = \sum_{j=1}^w R_{seq}^*(j) \quad \text{for the PSSM}$$

This is the formula that is nowadays used.

Information content

Shannon uncertainty applied to PSSM: example

Count matrix (TRANSFAC matrix F\$PHO4_01)

Residue\position	1	2	3	4	5	6	7	8	9	10	11	12
A	1	3	2	0	8	0	0	0	0	0	1	2
C	2	2	3	8	0	8	0	0	0	2	0	2
G	1	2	3	0	0	0	8	0	5	4	5	2
T	4	1	0	0	0	0	0	8	3	2	2	2
Sum	8	8	8	8	8	8	8	8	8	8	8	8

Counts



$$f_{i,j} = \frac{n_{i,j}}{\sum_{i=1}^A n_{i,j}}$$

A alphabet size (=4)
 $n_{i,j}$ occurrences of residue i at position j
 p_i prior residue probability for residue i
 $f_{i,j}$ relative frequency of residue i at position j

Pos	1	2	3	4	5	6	7	8	9	10	11	12
A	0.13	0.38	0.25	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.13	0.25
C	0.25	0.25	0.38	1.00	0.00	1.00	0.00	0.00	0.00	0.25	0.00	0.25
G	0.13	0.25	0.38	0.00	0.00	0.00	1.00	0.00	0.63	0.50	0.63	0.25
T	0.50	0.13	0.00	0.00	0.00	0.00	0.00	1.00	0.38	0.25	0.25	0.25
Sum	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Frequency

Information content

Shannon uncertainty applied to PSSM: example

Pos	1	2	3	4	5	6	7	8	9	10	11	12
A	0.13	0.38	0.25	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.13	0.25
C	0.25	0.25	0.38	1.00	0.00	1.00	0.00	0.00	0.00	0.25	0.00	0.25
G	0.13	0.25	0.38	0.00	0.00	0.00	1.00	0.00	0.63	0.50	0.63	0.25
T	0.50	0.13	0.00	0.00	0.00	0.00	0.00	1.00	0.38	0.25	0.25	0.25
Sum	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Frequency

1st option: identically distributed pseudo-weight

$$f'_{i,j} = \frac{n_{i,j} + k/A}{\sum_{i=1}^A n_{i,j} + k}$$

2nd option: pseudo-weight distributed according to residue priors

$$f'_{i,j} = \frac{n_{i,j} + p_i k}{\sum_{i=1}^A n_{i,j} + k}$$

A
 $n_{i,j}$
 p_i
 $f_{i,j}$
 k
 $f'_{i,j}$

alphabet size (=4)
occurrences of residue i at position j
prior residue probability for residue i
relative frequency of residue i at position j
pseudo weight (arbitrary, 1 in this case)
corrected frequency of residue i at position j

Pos	1	2	3	4	5	6	7	8	9	10	11	12
A	0.15	0.37	0.26	0.04	0.93	0.04	0.04	0.04	0.04	0.04	0.15	0.26
C	0.24	0.24	0.35	0.91	0.02	0.91	0.02	0.02	0.02	0.24	0.02	0.24
G	0.13	0.24	0.35	0.02	0.02	0.02	0.91	0.02	0.58	0.46	0.58	0.24
T	0.48	0.15	0.04	0.04	0.04	0.04	0.04	0.93	0.37	0.26	0.26	0.26
Sum	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Frequency
(+pseudo-counts)

Information content

Shannon uncertainty applied to PSSM: example

Pos	1	2	3	4	5	6	7	8	9	10	11	12
A	0.15	0.37	0.26	0.04	0.93	0.04	0.04	0.04	0.04	0.04	0.15	0.26
C	0.24	0.24	0.35	0.91	0.02	0.91	0.02	0.02	0.02	0.24	0.02	0.24
G	0.13	0.24	0.35	0.02	0.02	0.02	0.91	0.02	0.58	0.46	0.58	0.24
T	0.48	0.15	0.04	0.04	0.04	0.04	0.04	0.93	0.37	0.26	0.26	0.26
Sum	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Frequency
(+pseudo-counts)



$$f'_{i,j} = \frac{n_{i,j} + p_i k}{\sum_{r=1}^A n_{r,j} + k}$$

$$W_{i,j} = \ln \left(\frac{f'_{i,j}}{p_i} \right)$$

A alphabet size (=4)

$n_{i,j}$ occurrences of residue i at position j

p_i prior residue probability for residue i

$f_{i,j}$ relative frequency of residue i at position j

k pseudo weight (arbitrary, 1 in this case)

$f'_{i,j}$ corrected frequency of residue i at position j

$W_{i,j}$ weight of residue i at position j

Prior	Pos	1	2	3	4	5	6	7	8	9	10	11	12
0.325	A	-0.79	0.13	-0.23	-2.20	1.05	-2.20	-2.20	-2.20	-2.20	-2.20	-0.79	-0.23
0.175	C	0.32	0.32	0.70	1.65	-2.20	1.65	-2.20	-2.20	-2.20	0.32	-2.20	0.32
0.175	G	-0.29	0.32	0.70	-2.20	-2.20	-2.20	1.65	-2.20	1.19	0.97	1.19	0.32
0.325	T	0.39	-0.79	-2.20	-2.20	-2.20	-2.20	-2.20	1.05	0.13	-0.23	-0.23	-0.23
1.000	Sum	-0.37	-0.02	-1.02	-4.94	-5.55	-4.94	-4.94	-5.55	-3.08	-1.13	-2.03	0.19

Position-weight
matrix

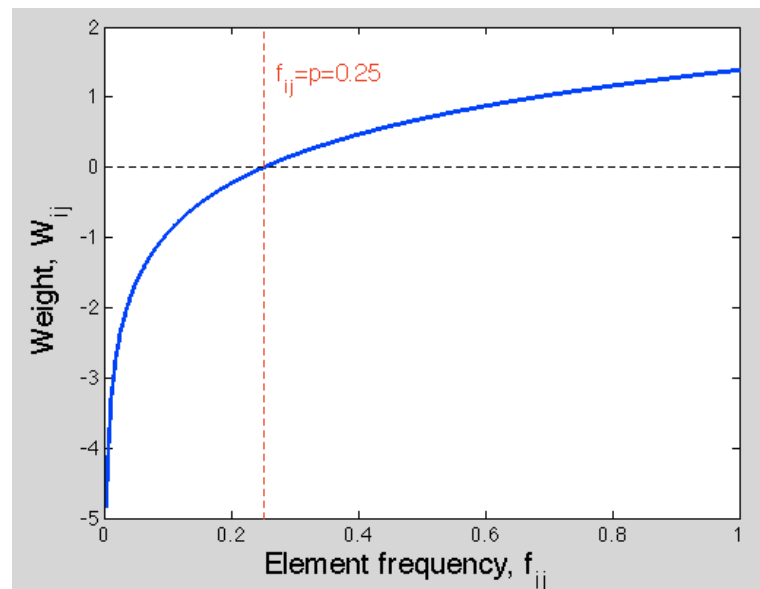
log (frequency /
background)

Information content

Shannon uncertainty applied to PSSM: example

Prior	Pos	1	2	3	4	5	6	7	8	9	10	11	12
0.325	A	-0.79	0.13	-0.23	-2.20	1.05	-2.20	-2.20	-2.20	-2.20	-2.20	-0.79	-0.23
0.175	C	0.32	0.32	0.70	1.65	-2.20	1.65	-2.20	-2.20	-2.20	0.32	-2.20	0.32
0.175	G	-0.29	0.32	0.70	-2.20	-2.20	-2.20	1.65	-2.20	1.19	0.97	1.19	0.32
0.325	T	0.39	-0.79	-2.20	-2.20	-2.20	-2.20	-2.20	1.05	0.13	-0.23	-0.23	-0.23
1.000	Sum	-0.37	-0.02	-1.02	-4.94	-5.55	-4.94	-4.94	-5.55	-3.08	-1.13	-2.03	0.19

Position-weight matrix



$$W_{i,j} = \ln \left(\frac{f'_{i,j}}{p_i} \right) \quad f'_{i,j} = \frac{n_{i,j} + p_i k}{\sum_{i=1}^A n_{i,j} + k} \quad \sum_{i=1}^A f'_{i,j} = 1$$

The weight W_{ij} is:

- positive when $f'_{i,j} > p_i$
(favourable positions for the binding of the transcription factor)
- negative when $f'_{i,j} < p_i$
(unfavourable positions)

Information content

Shannon uncertainty applied to PSSM: example

Prior	Pos	1	2	3	4	5	6	7	8	9	10	11	12
0.325	A	-0.79	0.13	-0.23	-2.20	1.05	-2.20	-2.20	-2.20	-2.20	-2.20	-0.79	-0.23
0.175	C	0.32	0.32	0.70	1.65	-2.20	1.65	-2.20	-2.20	-2.20	0.32	-2.20	0.32
0.175	G	-0.29	0.32	0.70	-2.20	-2.20	-2.20	1.65	-2.20	1.19	0.97	1.19	0.32
0.325	T	0.39	-0.79	-2.20	-2.20	-2.20	-2.20	-2.20	1.05	0.13	-0.23	-0.23	-0.23
1.000	Sum	-0.37	-0.02	-1.02	-4.94	-5.55	-4.94	-4.94	-5.55	-3.08	-1.13	-2.03	0.19

$$I_{i,j} = f'_{i,j} \ln \left(\frac{f'_{i,j}}{p_i} \right)$$

Element (i,j)
Information
Content

$$I_j = \sum_{i=1}^A I_{i,j}$$

Column (j)
Information
Content

$$I_{matrix} = \sum_{j=1}^w \sum_{i=1}^A I_{i,j}$$

PSSM
Information
Content

Prior	Pos	1	2	3	4	5	6	7	8	9	10	11	12
0.325	A	-0.12	0.05	-0.06	-0.08	0.97	-0.08	-0.08	-0.08	-0.08	-0.08	-0.12	-0.06
0.175	C	0.08	0.08	0.25	1.50	-0.04	1.50	-0.04	-0.04	-0.04	0.08	-0.04	0.08
0.175	G	-0.04	0.08	0.25	-0.04	-0.04	-0.04	1.50	-0.04	0.68	0.45	0.68	0.08
0.325	T	0.19	-0.12	-0.08	-0.08	-0.08	-0.08	-0.08	0.97	0.05	-0.06	-0.06	-0.06
1.000	Sum	0.11	0.09	0.36	1.29	0.80	1.29	1.29	0.80	0.61	0.39	0.47	0.04

Inform.
content

SUM = 7.54

Information content

Shannon uncertainty applied to PSSM: example

Prior	Pos	1	2	3	4	5	6	7	8	9	10	11	12
0.325	A	-0.12	0.05	-0.06	-0.08	0.97	-0.08	-0.08	-0.08	-0.08	-0.08	-0.12	-0.06
0.175	C	0.08	0.08	0.25	1.50	-0.04	1.50	-0.04	-0.04	-0.04	0.08	-0.04	0.08
0.175	G	-0.04	0.08	0.25	-0.04	-0.04	-0.04	1.50	-0.04	0.68	0.45	0.68	0.08
0.325	T	0.19	-0.12	-0.08	-0.08	-0.08	-0.08	-0.08	0.97	0.05	-0.06	-0.06	-0.06
1.000	Sum	0.11	0.09	0.36	1.29	0.80	1.29	1.29	0.80	0.61	0.39	0.47	0.04

$$f'_{i,j} = \frac{n_{i,j} + p_i k}{\sum_{i=1}^A n_{i,j} + k}$$

$$I_{i,j} = f'_{i,j} \ln \left(\frac{f'_{i,j}}{p_i} \right)$$

$$I_j = \sum_{i=1}^A I_{i,j}$$

$$I_{matrix} = \sum_{j=1}^w \sum_{i=1}^A I_{i,j}$$

- A alphabet size (=4)
- $n_{i,j}$ occurrences of residue i at position j
- w matrix width (=12)
- p_i prior residue probability for residue i
- $f_{i,j}$ relative frequency of residue i at position j
- k pseudo weight (arbitrary, 1 in this case)
- $f'_{i,j}$ corrected frequency of residue i at position j
- $W_{i,j}$ weight of residue i at position j
- $I_{i,j}$ information of residue i at position j

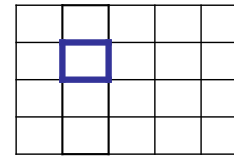
Reference: Hertz (1999).

Bioinformatics 15:563-577₁₃

Information content

Properties of the information content

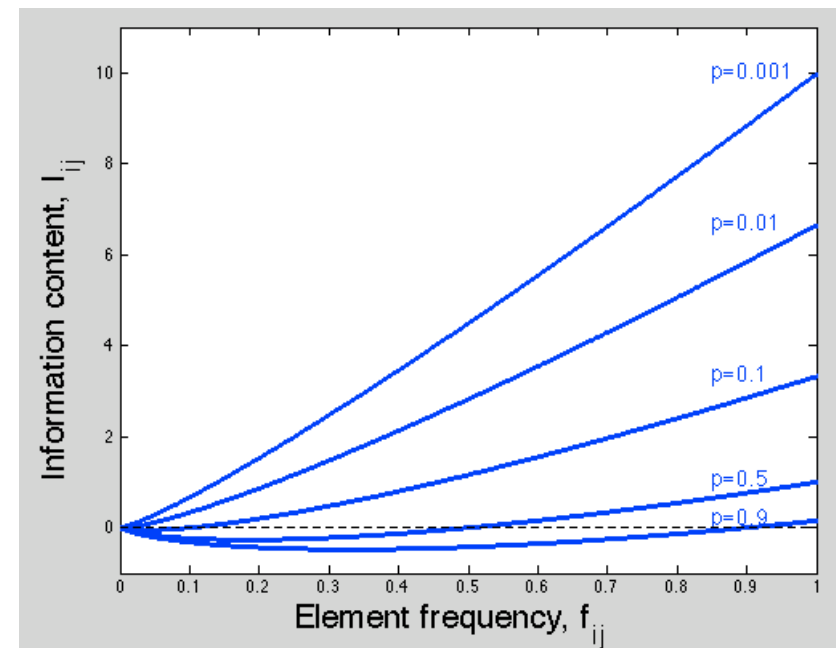
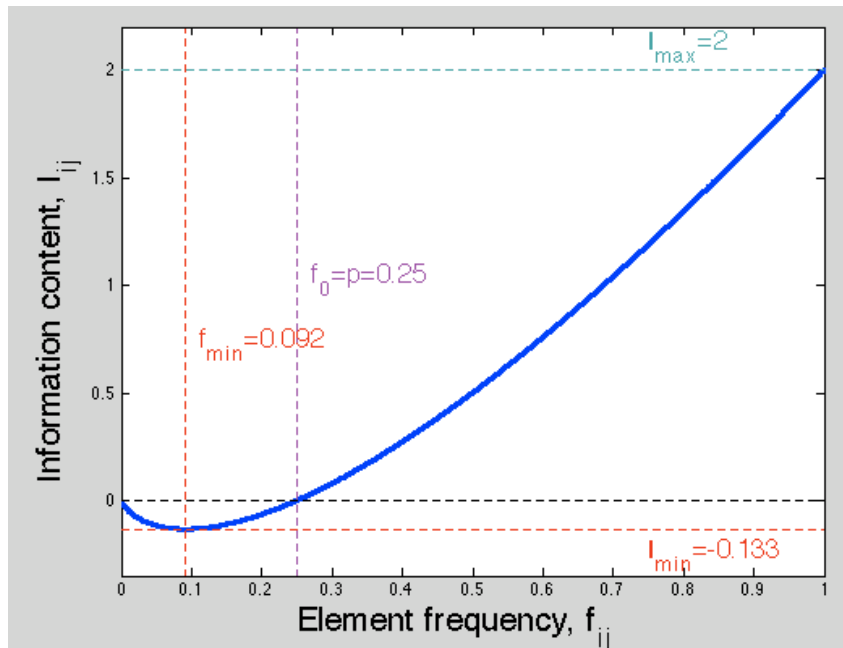
For a given cell (i,j) of the matrix:



$$I_{i,j} = f'_{i,j} \ln \left(\frac{f'_{i,j}}{p_i} \right)$$

Information content of a given cell (i,j)

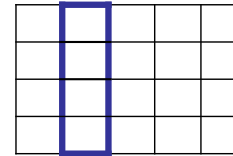
- $I_{ij} > 0$ when $f'_{ij} > p_i$ (i.e. when residue i is more frequent at position j than expected by chance)
- $I_{ij} < 0$ when $f'_{ij} < p_i$ (i.e. when residue i is less frequent at position j than expected by chance)
- I_{ij} tends towards 0 when $f'_{ij} \rightarrow 0$ (because $\lim_{x \rightarrow 0} x \ln(x) = 0$)



Remark: The upper bound of I_{ij} increases when p_i decreases ($I_{ij} \rightarrow \text{Inf}$ when $p_i \rightarrow 0$). The information content, as defined by Gerald Hertz, has thus no upper bound.

Information content

Properties of the information content



For a given column j of the matrix:

- The information content of the column (I_j) is the sum of information of its cells.
- I_j is always positive ($I_j > 0$)
- I_j is 0 when the frequency of all residues equal their prior probability ($I_j = 0$ if $f_{ij} = p_i$)
- I_j is maximal when the residue with the lowest prior probability has a frequency of 1 (all other residues have a frequency of 0); the pseudo-weight is 0

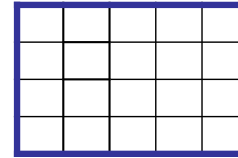
$$I_j = \sum_{i=1}^A I_{i,j} = \sum_{i=1}^A f'_{i,j} \ln \left(\frac{f'_{i,j}}{p_i} \right)$$

Information content of a given column (j)

$$\max(I_j) = 1 * \ln \left(\frac{1}{p_i} \right) = -\ln(p_i)$$

Information content

Properties of the information content



- The total information content represents the capability of the matrix to make the distinction between a binding site (represented by the matrix) and the background model.
- The information content also allows to estimate an upper limit for the expected frequency of the binding sites in random sequences.
- The pattern discovery program **consensus** (developed by Jerry Hertz) optimizes the information content in order to detect over-represented motifs.
- Note that this is not the case of all pattern discovery programs: the **gibbs sampler** algorithm optimizes a log-likelihood.

$$I_{matrix} = \sum_{j=1}^w \sum_{i=1}^A I_{i,j}$$

Information content of the PSSM

$$P(site) \leq e^{-I_{matrix}}$$

Information content

Sequences logo based on information content

Schneider (1990) proposes a graphical representation (logo) based on his previous entropy (H) for representing the importance of each residue at each position of an alignment.

He provides a new formula for R_{seq} :

$$H_s(j) = - \sum_{i=1}^A f_{ij} \log_2(f_{ij})$$
$$R_{seq}(j) = 2 - H_s(j) + e(n)$$
$$h_{ij} = f_{ij} R_{seq}(j)$$

$H_s(j)$ = uncertainty of column j

$R_{seq}(j)$ = “information content” of column j (NB: this definition differs from Hertz’ information content)

$e(n)$ = correction for small samples (pseudo-weight)



Remarks

This information content does not include any correction for the prior residue probabilities (p_i)

This information content is expressed in bits.

Boundaries

$\min(R_{seq})=0 \Leftrightarrow$ equiprobable residues

$\max(R_{seq})=2 \Leftrightarrow$ perfect conservation of 1 residue with a pseudo-weight of 0.

Sequence logos can be generated from aligned sequences on the Weblogo server

<http://weblogo.berkeley.edu/>

Information content

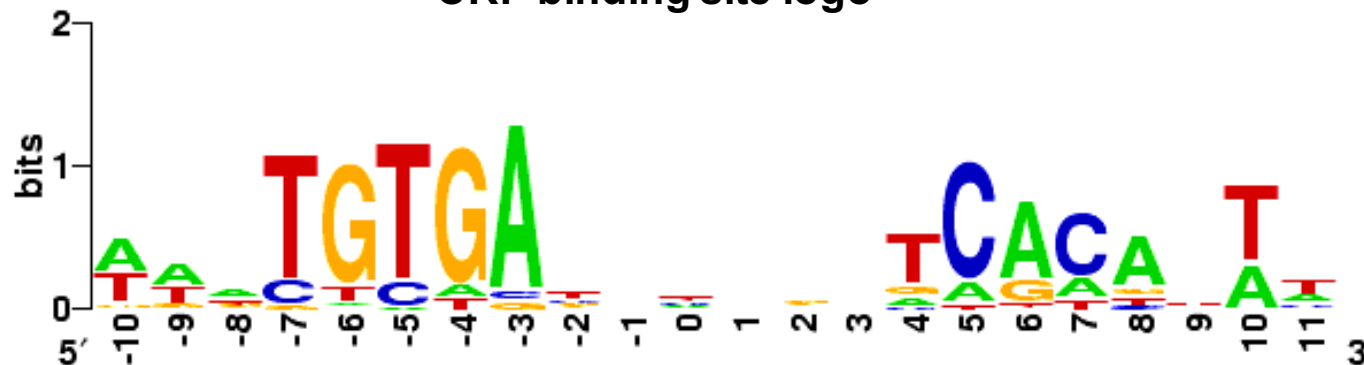
Example: CRP binding site.



- cAMP Receptor Protein (CRP) is a dimer of two identical subunits.
- cAMP-CRP activates expression of many genes in *E. coli*, by binding to specific sites on the DNA where it directly interacts with RNA Polymerase.
- Nucleotide sequencing and analysis of CRP binding sites established a consensus binding sequence consisting of an imperfect 5 bp palindrome:

TGTGA---N6---TCACA

CRP binding site logo



Information content

References

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- Schneider TD, Stormo GD, Gold L, Ehrenfeucht A (1986) Information content of binding sites on nucleotide sequences. *J Mol Biol* 188:415-431.
- Schneider TD, Stephens RM (1990) Sequence logos: a new way to display consensus sequences. *Nucleic Acids Res* 18: 6097-6100.
- Schneider TD (1997) Information content of individual genetic sequences. *J Theor Biol* 189:427-441.
- Hertz GZ & Stormo GD (1999) Identifying DNA and protein patterns with statistically significant alignments of multiple sequences. *Bioinformatics* 15:563-577.
- Stormo GD, Hartzell GW (1989) Identifying protein-binding sites from unaligned DNA fragments. *Proc Natl Acad Sci USA* 86:1183-7.

See also:

Schneider TD (2012) *Information Theory Primer*, Lecture notes
(available at <http://schneider.ncifcrf.gov/>)