UNTITLED LEAN THESIS

A Thesis Submitted to the Faculty of

Georgetown College

In Partial Fulfillment of the Requirements for the

Honors Program

By

Logan Johnson

Georgetown, Kentucky

May 2024

Abstract

UNTITLED LEAN THESIS

Logan C. Johnson

idk dr burch?? maybe dr white???

Here is the text of your abstract. It goes on and on and on. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. The rest of this paragraph is a filler. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. It goes on like this for about 150 words, so it should all fit on this page. Note that the Abstract comes before the title page and has no page number. If your abstract is more than 250 words, consider shortening it.

APPROVED BY THE DIRECTOR OF HONORS THESES:

____

Dr. Homer White, Department of Mathematics

APPROVED BY THE HONORS PROGRAM:

____

Dr. Barbara Burch, Director

DATE:____________________

Table of contents

Preface

I will not do my homework today.

```
sum(4, 7, 3)
```

```
[1] 14
```

Hello World

Making Chapters

I am using this section to figure out how to incorporate a table of contents and different sections/chapters of the paper. This should prove useful in the final thesis and allow readers to quickly jump to important or interesting sections.

Incorporating Some Code

I will also be able to use some LaTeX equations within the document which could halp to make the paper look quite nice.

If $a < b$ and $c \leq d$, prove that $a + c \leq b + d$. It just so happens that I was able to prove this using lean!

```
example (a b c d : ℝ) (h1: a < b) (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by
    apply Ne.lt_of_le h3 h2
  apply add_lt_add h1 h4
  done
```

Now to display the benefits of the lean infoview!

Showing the Infoview with Picture Sequences

Not going to do this now as I think the columns are far superior.

Showing Infoview with Columns

As we can clearly see, this is the first step of the code and it can now be explained with great ease. Now onto the next step!

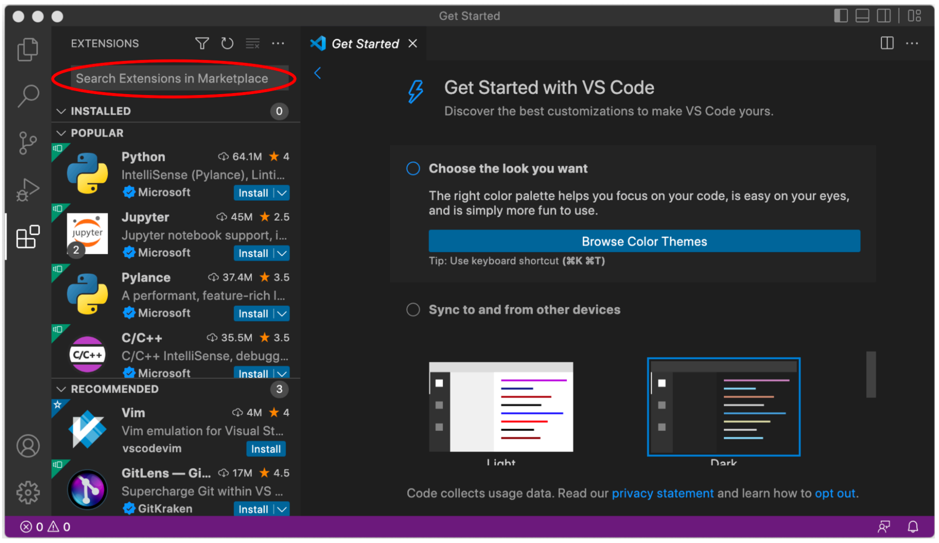
Lean File

```
theorem Example_3_2_4_v2 (P Q R : Prop)
    (h : P → (Q → R)) : ¬R → (P → ¬Q) := by
  assume h2 : ¬R
  assume h3 : P
  done
```

Tactic State in Infoview

```
P Q R : Prop
h : P → Q → R
h2 : ¬R
h3 : P
⊢ ¬Q
```

Click on the *Extensions* icon on the left side of the window, which is circled in red in the image above. That will bring up a list of available extensions:



Get Started with VS Code

Discover the best customizations to make VS Code yours.

Choose the look you want

The right color palette helps you focus on your code, is easy on your eyes, and is simply more fun to use.

Tip: Use keyboard shortcut (⌘K ⌘T)

Code collects usage data. Read our privacy statement and learn how to opt out.

Acknowledgments

1 Real Analysis

Symbol	Meaning
$\neg$	not
$\wedge$	and
$\vee$	or
$\rightarrow$	if ... then
$\leftrightarrow$	iff (that is, if and only if)

Name	Equivalence		
De Morgan's Laws	$\neg(P \wedge Q)$	is equivalent to	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	is equivalent to	$\neg P \wedge \neg Q$
Double Negation Law	$\neg\neg P$	is equivalent to	P
Conditional Laws	$P \rightarrow Q$	is equivalent to	$\neg P \vee Q$
	$P \rightarrow Q$	is equivalent to	$\neg(P \wedge \neg Q)$
Contrapositive Law	$P \rightarrow Q$	is equivalent to	$\neg Q \rightarrow \neg P$

$$A \cap B = \{x \mid x \in A \wedge x \in B\} = \text{ the } \textit{intersection} \text{ of } A \text{ and } B,$$
$$A \cup B = \{x \mid x \in A \vee x \in B\} = \text{ the } \textit{union} \text{ of } A \text{ and } B,$$

$A \setminus B = \{x \mid x \in A \wedge x \notin B\} =$ the *difference* of A and B,

$A \triangle B = (A \setminus B) \cup (B \setminus A) =$ the *symmetric difference* of A and B.

2 Functional Programming

$\forall x\, P(x)$ means “for all x, $P(x)$,”

Quantifier Negation Laws		
$\neg\exists x\, P(x)$	is equivalent to	$\forall x\, \neg P(x)$
$\neg\forall x\, P(x)$	is equivalent to	$\exists x\, \neg P(x)$

3 Lean as a Theorem Prover

Differences From Paragraph Style Proofs

Despite the incredible power that lean could provide in the verification of mathematical proofs, this does pose some difficulties, namely the ease with which the aforementioned proofs can be written up. Typically, proofs are simply written up in a paragraph style, where the steps being taken and the theorems being applied are laid out in plain terms so that it can be easily understood by fellow mathematicians. There are often times when mathematicians will take things for granted or skip over steps that they think the reader will either already know to be fact or can easily reason out for themselves when writing out typical proofs. This lax approach for conveying information simply does not work when trying to communicate with technology, and a much more specific and methodical approach must be adopted in order to take advantage of the logical verification benefits. Thankfully, lean has a community working to create libraries of previously proven theorems that can be applied to speed up the writing and verification of future proofs. This thankfully means that all proofs do not need to be taken all the way back to basic axioms: Users can save time by avoiding proving adjacent theorems and instead focus only on the immediately relevant steps of their proof.

For each of the following proofs, I will first provide a typical "paragraph style" version of the proof, so the differences between the two can easily be compared. ## Inequality Addition

Theorem. *Whatever the problem is*

Proof. whatever the solution to that problem is. □

Step 1 Putting the initial theorem we want to show in lean code and observing the final goal in the infoview.

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
⊢ a + c < b + d
```

Step 2

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c = d
⊢ a + c < b + d
```

Step 3

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c = d
⊢ a + d < b + d
```

Step 4

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: ¬c = d
⊢ a + c < b + d
```

Step 5

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c ≠ d
⊢ a + c < b + d
```

Step 6a We can see here that the lean infoview is now displaying my new hypothesis as the current goal.

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c ≠ d
⊢ c < d
```

Step 6b

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by
    apply Ne.lt_of_le h3 h2

  done
```

Tactic State in Infoview

```
No goals
```

Step 7

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by
    apply Ne.lt_of_le h3 h2
  apply add_lt_add h1 h4
  done
```

Tactic State in Infoview

```
No goals
```

Absolute Convergence

I will now display how I used to lean to demonstrate that a sequence converges if and only if the absolute value of that sequence converges.

Laying out the goal

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by

  done
```

Tactic State in Infoview

```
s1: ℕ → ℝ
⊢ ConvergesTo s1 0
  ↔ ConvergesTo |s1| 0
```

Step 1

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]

  done
```

Tactic State in Infoview

```
s1: ℕ → ℝ
⊢ (∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |s1 n - 0| < ε) ↔
    ∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |abs s1 n - 0| < ε
```

Step 2

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by

    done

  done
```

Tactic State in Infoview

```
s1: ℕ → ℝ
x: ℕ
⊢ |s1 x| = |abs s1 x|
```

Step 3

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done

  done
```

Step 4

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
⊢ (∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |s1 n - 0| < ε) ↔
    ∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |abs s1 n - 0| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards

  · --Reverse

  done
```

Step 5

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
⊢ (∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |s1 n - 0| < ε) →
    ∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |abs s1 n - 0| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1

  · --Reverse

  done
```

Step 6

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
h1: ∀ (ε : ℝ), ε > 0 →
  ∃ N, ∀ (n : ℕ),
  n ≥ N → |s1 n - 0| < ε
⊢ ∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |abs s1 n - 0| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1
    simp

  · --Reverse

  done
```

Step 7

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
h1: ∀ (ε : ℝ), ε > 0 →
  ∃ N, ∀ (n : ℕ),
  n ≥ N → |s1 n - 0| < ε
⊢ ∀ (ε : ℝ), 0 < ε →
    ∃ N, ∀ (n : ℕ),
    N ≤ n →
    |abs s1 n| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1
    simp
    simp at h1

  · --Reverse

  done
```

Step 8

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
h1: ∀ (ε : ℝ), 0 < ε →
  ∃ N, ∀ (n : ℕ),
  N ≤ n → |s1 n| < ε
⊢ ∀ (ε : ℝ), 0 < ε →
    ∃ N, ∀ (n : ℕ),
    N ≤ n →
    |abs s1 n| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1
    simp
    simp at h1
    simp [← h3]

  · --Reverse

  done
```

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
h1: ∀ (ε : ℝ), 0 < ε →
  ∃ N, ∀ (n : ℕ),
  N ≤ n → |s1 n| < ε
⊢ ∀ (ε : ℝ), 0 < ε →
    ∃ N, ∀ (n : ℕ),
    N ≤ n →
    |s1 n| < ε
```

Step 9

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1
    simp
    simp at h1
    simp [← h3]
    apply h1
  · --Reverse

  done
```

Step 10

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
⊢ (∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |abs s1 n - 0| < ε) →
    ∀ (ε : ℝ), ε > 0 →
    ∃ N, ∀ (n : ℕ),
    n ≥ N →
    |s1 n - 0| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1
    simp
    simp at h1
    simp [← h3]
    apply h1
  · --Reverse
    intro h1
    simp
    simp at h1
    simp_rw [h3]

  done
```

Step 11

Tactic State in Infoview

```
s1: ℕ → ℝ
h3: ∀ (x : ℕ), |s1 x| =
  |abs s1 x|
h1: ∀ (ε : ℝ), 0 < ε →
  ∃ N, ∀ (n : ℕ),
  N ≤ n → |abs s1 n| < ε
⊢ ∀ (ε : ℝ), 0 < ε →
    ∃ N, ∀ (n : ℕ),
    N ≤ n →
    |abs s1 n| < ε
```

Lean File

```
example (s1 : ℕ → ℝ) :
    ConvergesTo s1 (0 : ℝ) ↔
    ConvergesTo (abs s1) (0 : ℝ)
    := by
  rw [ConvergesTo]
  rw [ConvergesTo]
  have h3 (x : ℕ) : |s1 x| =
      |abs s1 x| := by
    simp [abs]
    done
  apply Iff.intro
  · --Forwards
    intro h1
    simp
    simp at h1
    simp [← h3]
    apply h1
  · --Reverse
    intro h1
    simp
    simp at h1
    simp_rw [h3]
    apply h1
  done
```

Tactic State in Infoview

```
No goals
```

a is Less Than or Equal to b

Setting up the problem

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
⊢ a ≤ b
```

Step 1

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: ¬a ≤ b
⊢ False
```

Step 2

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
⊢ False
```

Step 3

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by

    done

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
⊢ ε > 0
```

Step 4

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h

      done

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
⊢ 0 < a - b
```

Step 5

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h
    exact Iff.mpr sub_pos h2
    done

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
h3: ε > 0
⊢ False
```

Step 6

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h
    exact Iff.mpr sub_pos h2
    done
  have h4 : a ≤ b + ε := by

    done

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
h3: ε > 0
⊢ a ≤ b + ε
```

Step 7

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h
    exact Iff.mpr sub_pos h2
    done
  have h4 : a ≤ b + ε := by
    apply h1

    done

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
h3: ε > 0
⊢ ε > 0
```

Step 8

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h
    exact Iff.mpr sub_pos h2
    done
  have h4 : a ≤ b + ε := by
    apply h1
    apply h3
    done

  done
```

Step 9

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
h3: ε > 0
h4: a ≤ b + ε
⊢ False
```

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h
    exact Iff.mpr sub_pos h2
    done
  have h4 : a ≤ b + ε := by
    apply h1
    apply h3
    done
  dsimp at h4

  done
```

Step 10

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
ab: ℝ
h1: ∀ (ε : ℝ), ε > 0 →
  a ≤ b + ε
h2: b < a
ε: ℝ := (a - b) / 2
h3: ε > 0
h4: a ≤ b + (a - b) / 2
⊢ False
```

Lean File

```
example (a b : ℝ) (h1 : ∀ ε : ℝ,
    ε > 0 → a ≤ b + ε) :
    a ≤ b := by
  by_contra h2
  push_neg at h2
  let ε := (a - b) / 2
  have h3 : ε > 0 := by
    refine half_pos ?h
    exact Iff.mpr sub_pos h2
    done
  have h4 : a ≤ b + ε := by
    apply h1
    apply h3
    done
  dsimp at h4
  linarith
  done
```

Tactic State in Infoview

```
No goals
```

3.3. Proofs Involving Quantifiers

Theorem. *Suppose B is a set and $\mathscr{F}$ is a family of sets. If $\bigcup\mathscr{F} \subseteq B$ then $\mathscr{F} \subseteq \mathscr{P}(B)$.*

Proof. Suppose $\bigcup\mathscr{F} \subseteq B$. Let x be an arbitrary element of $\mathscr{F}$. Let y be an arbitrary element of x. Since $y \in x$ and $x \in \mathscr{F}$, by the definition of $\bigcup\mathscr{F}$, $y \in \bigcup\mathscr{F}$. But then since $\bigcup\mathscr{F} \subseteq B$, $y \in B$. Since y was an arbitrary element of x, we can conclude that $x \subseteq B$, so $x \in \mathscr{P}(B)$. But x was an arbitrary element of $\mathscr{F}$, so this shows that $\mathscr{F} \subseteq \mathscr{P}(B)$, as required. □

Theorem 3.4.7. *For every integer n, $6 \mid n$ iff $2 \mid n$ and $3 \mid n$.*

Proof. Let n be an arbitrary integer.

($\rightarrow$) Suppose $6 \mid n$. Then we can choose an integer k such that $6k = n$. Therefore $n = 6k = 2(3k)$, so $2 \mid n$, and similarly $n = 6k = 3(2k)$, so $3 \mid n$.

($\leftarrow$) Suppose $2 \mid n$ and $3 \mid n$. Then we can choose integers j and k such that $n = 2j$ and $n = 3k$. Therefore $6(j - k) = 6j - 6k = 3(2j) - 2(3k) = 3n - 2n = n$, so $6 \mid n$. $\square$

4 Conclusions

$$[x]_R = \{y \in A \mid yRx\}.$$

The set whose elements are all of these equivalence classes is called A *mod* R. It is written A/R, so

$$A/R = \{[x]_R \mid x \in A\}.$$

Note that A/R is a set whose elements are sets: for each $x \in A$, $[x]_R$ is a subset of A, and $[x]_R \in A/R$.

5 Works Cited

This work had been formatted and styled from the book *How To Prove It With Lean*, written by Daniel J. Velleman. *How To Prove It With Lean* contains short excerpts from *How To Prove It: A Structured Approach, 3rd Edition*, by Daniel J. Velleman and published by Cambridge University Press.

```
example : square1 = square2 := by rfl

#eval square1 7     --Answer: 49
```

6 Additional space

Extra chapter to write more things if needed!!