Untitled Lean Thesis

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Invalid Date

Table of contents

Preface

I will not do my homework today.

```
sum(4, 7, 3)
```

```
[1] 14
```

Hello World

Making Chapters

I am using this section to figure out how to incorporate a table of contents and different sections/chapters of the paper. This should prove useful in the final thesis and allow readers to quickly jump to important or interesting sections.

Incorporating Some Code

I will also be able to use some LaTeX equations within the document which could halp to make the paper look quite nice.

If $a < b$ and $c \leq d$, prove that $a + c \leq b + d$. It just so happens that I was able to prove this using lean!

```
example (a b c d : ℝ) (h1: a < b) (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by
    apply Ne.lt_of_le h3 h2
  apply add_lt_add h1 h4
  done
```

Now to display the benefits of the lean infoview!

Showing the Infoview with Picture Sequences

Not going to do this now as I think the columns are far superior.

Showing Infoview with Columns

As we can clearly see, this is the first step of the code and it can now be explained with great ease. Now onto the next step!

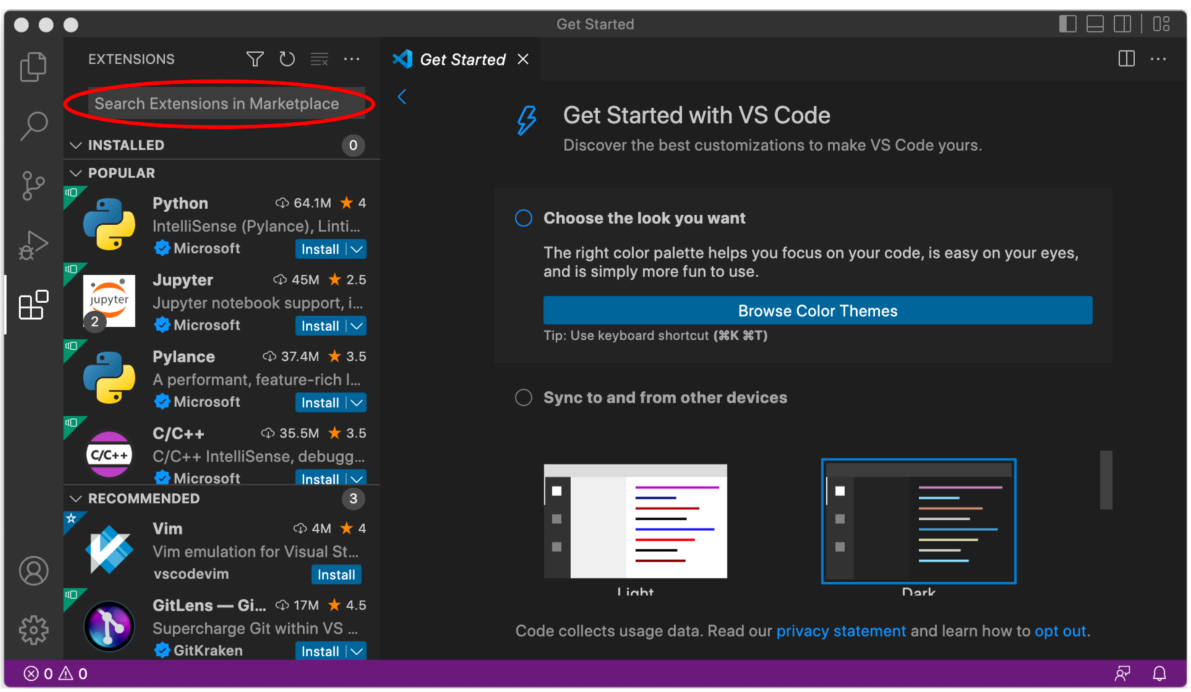
Lean File

```
theorem Example_3_2_4_v2 (P Q R : Prop)
    (h : P → (Q → R)) : ¬R → (P → ¬Q) := by
  assume h2 : ¬R
  assume h3 : P
  done
```

Tactic State in Infoview

```
P Q R : Prop
h : P → Q → R
h2 : ¬R
h3 : P
⊢ ¬Q
```

Click on the *Extensions* icon on the left side of the window, which is circled in red in the image above. That will bring up a list of available extensions:



Acknowledgments

1 Real Analysis

Symbol	Meaning
$\neg$	not
$\wedge$	and
$\vee$	or
$\rightarrow$	if ... then
$\leftrightarrow$	iff (that is, if and only if)

Name		Equivalence	
De Morgan's Laws	$\neg(P \wedge Q)$	is equivalent to	$\neg P \vee \neg Q$
	$\neg(P \vee Q)$	is equivalent to	$\neg P \wedge \neg Q$
Double Negation Law	$\neg\neg P$	is equivalent to	P
Conditional Laws	$P \rightarrow Q$	is equivalent to	$\neg P \vee Q$
	$P \rightarrow Q$	is equivalent to	$\neg(P \wedge \neg Q)$
Contrapositive Law	$P \rightarrow Q$	is equivalent to	$\neg Q \rightarrow \neg P$

$$
\begin{aligned}
A \cap B &= \{x \mid x \in A \wedge x \in B\} = \text{ the } \textit{intersection} \text{ of } A \text{ and } B, \\
A \cup B &= \{x \mid x \in A \vee x \in B\} = \text{ the } \textit{union} \text{ of } A \text{ and } B, \\
A \setminus B &= \{x \mid x \in A \wedge x \notin B\} = \text{ the } \textit{difference} \text{ of } A \text{ and } B, \\
A \bigtriangleup B &= (A \setminus B) \cup (B \setminus A) = \text{ the } \textit{symmetric difference} \text{ of } A \text{ and } B.
\end{aligned}
$$

2 Functional Programming

$\forall x\, P(x)$ means "for all x, $P(x)$,"

Quantifier Negation Laws		
$\neg\exists x\, P(x)$	is equivalent to	$\forall x\, \neg P(x)$
$\neg\forall x\, P(x)$	is equivalent to	$\exists x\, \neg P(x)$

3 Lean as a Theorem Prover

Inequality Addition

Step 1 Putting the initial theorem we want to show in lean code and observing the final goal in the infoview.

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
⊢ a + c < b + d
```

Step 2

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c = d
⊢ a + c < b + d
```

Step 3

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c = d
⊢ a + d < b + d
```

Step 4

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: ¬c = d
⊢ a + c < b + d
```

Step 5

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c ≠ d
⊢ a + c < b + d
```

Step 6a We can see here that the lean infoview is now displaying my new hypothesis as the current goal.

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by

  done
```

Tactic State in Infoview

```
R: Type u_1
inst†: Ring R
abcd: ℝ
h1: a < b
h2: c ≤ d
h3: c ≠ d
⊢ c < d
```

Step 6b

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by
    apply Ne.lt_of_le h3 h2

  done
```

Tactic State in Infoview

```
No goals
```

Step 7

Lean File

```
example (a b c d : ℝ) (h1: a < b)
    (h2 : c ≤ d) : a + c < b + d := by
  by_cases h3 : c = d
  rw [h3]
  apply add_lt_add_right h1
  push_neg at h3
  have h4 : c < d := by
    apply Ne.lt_of_le h3 h2
  apply add_lt_add h1 h4
  done
```

Tactic State in Infoview

```
No goals
```

To prove a goal of the form `P → Q`:

1. Assume `P` is true and prove `Q`.
2. Assume `Q` is false and prove that `P` is false.

Tactic State Before Using Strategy

```
 ⋮
⊢ P → Q
```

Tactic State After Using Strategy

```
 ⋮
h : P
⊢ Q
```

Q	¬Q	(Q	→	False)
F	T	F	T	F
T	F	T	F	F

Lean File

```
theorem Example_3_2_4_v2 (P Q R : Prop)
    (h : P → (Q → R)) : ¬R → (P → ¬Q) := by
  assume h2 : ¬R
  assume h3 : P
  done
```

Tactic State in Infoview

```
P Q R : Prop
h : P → Q → R
h2 : ¬R
h3 : P
⊢ ¬Q
```

Exercises

Fill in proofs of the following theorems. All of them are based on exercises in *HTPI*.

1.
```
theorem Exercise_3_2_1a (P Q R : Prop)
    (h1 : P → Q) (h2 : Q → R) : P → R := by

  done
```

2.
```
theorem Exercise_3_2_1b (P Q R : Prop)
    (h1 : ¬R → (P → ¬Q)) : P → (Q → R) := by

  done
```

3.
```
theorem Exercise_3_2_2a (P Q R : Prop)
    (h1 : P → Q) (h2 : R → ¬Q) : P → ¬R := by

  done
```

4.
```
theorem Exercise_3_2_2b (P Q : Prop)
    (h1 : P) : Q → ¬(Q → ¬P) := by

  done
```

3.3. Proofs Involving Quantifiers

Let `x` stand for an arbitrary object of type `U` and prove `P x`. If the letter `x` is already being used in the proof to stand for something, then you must choose an unused variable, say `y`, to stand for the arbitrary object, and prove `P y`.

`quant_neg` Tactic		
`¬∀ (x : U), P x`	is changed to	`∃ (x : U), ¬P x`
`¬∃ (x : U), P x`	is changed to	`∀ (x : U), ¬P x`
`∀ (x : U), P x`	is changed to	`¬∃ (x : U), ¬P x`
`∃ (x : U), P x`	is changed to	`¬∀ (x : U), ¬P x`

Theorem. *Suppose B is a set and $\mathcal{F}$ is a family of sets. If $\bigcup\mathcal{F} \subseteq B$ then $\mathcal{F} \subseteq \mathscr{P}(B)$.*

Proof. Suppose $\bigcup\mathcal{F} \subseteq B$. Let x be an arbitrary element of $\mathcal{F}$. Let y be an arbitrary element of x. Since $y \in x$ and $x \in \mathcal{F}$, by the definition of $\bigcup\mathcal{F}$, $y \in \bigcup\mathcal{F}$. But then since $\bigcup\mathcal{F} \subseteq B$, $y \in B$. Since y was an arbitrary element of x, we can conclude that $x \subseteq B$, so $x \in \mathscr{P}(B)$. But x was an arbitrary element of $\mathcal{F}$, so this shows that $\mathcal{F} \subseteq \mathscr{P}(B)$, as required. □

Theorem 3.4.7. *For every integer n, $6 \mid n$ iff $2 \mid n$ and $3 \mid n$.*

Proof. Let n be an arbitrary integer.

($\rightarrow$) Suppose $6 \mid n$. Then we can choose an integer k such that $6k = n$. Therefore $n = 6k = 2(3k)$, so $2 \mid n$, and similarly $n = 6k = 3(2k)$, so $3 \mid n$.

($\leftarrow$) Suppose $2 \mid n$ and $3 \mid n$. Then we can choose integers j and k such that $n = 2j$ and $n = 3k$. Therefore $6(j - k) = 6j - 6k = 3(2j) - 2(3k) = 3n - 2n = n$, so $6 \mid n$. □

For the next exercise you will need the following definitions:

4 Conclusions

$$[x]_R = \{y \in A \mid yRx\}.$$

The set whose elements are all of these equivalence classes is called A *mod* R. It is written A/R, so

$$A/R = \{[x]_R \mid x \in A\}.$$

Note that A/R is a set whose elements are sets: for each $x \in A$, $[x]_R$ is a subset of A, and $[x]_R \in A/R$.

5 Works Cited

This work had been formatted and styled from the book *How To Prove It With Lean*, written by Daniel J. Velleman. *How To Prove It With Lean* contains short excerpts from *How To Prove It: A Structured Approach, 3rd Edition*, by Daniel J. Velleman and published by Cambridge University Press.

```
example : square1 = square2 := by rfl

#eval square1 7     --Answer: 49
```

6 Additional space

Extra chapter to write more things if needed!!