Languages, automata and computation II
Tutorial 7 - Games

Winter semester 2023/2024

Exercise 1. We say that a game is *finite* if every play eventually reaches a dead end (no player can move). Show that finite games are determined.

Solution: Consider a game $G = (V, V_0, V_1, E, W)$, where $V = V_0 \cup V_1$ is a partition of the game graph into vertices beloning to Player 0, resp., Player 1, $E \subseteq V \times V$ is the set of moves, and $W \subseteq V^\omega$ is the winning condition. The set of vertices V is in general infinite, even uncountable. Define the *predecessor operator* for Player $i \in \{0, 1\}$ as follows:

$$\mathsf{Pre}^i(_) : V \to V$$
$$\mathsf{Pre}^i(X) := \{u \in V_i \mid \exists (u \to v).v \in X\} \cup \{u \in V_{1-i} \mid \forall (u \to v).v \in X\}.$$

We now construct the winning region W^0 for Player 0. We build a sequence of sets W_α by induction on ordinals α:

$$\begin{aligned} W_0 &:= \left\{u \in V_1 \mid \neg\exists (u \xrightarrow{v})\right\}, \\ W_{\alpha+1} &:= W_\alpha \cup \mathsf{Pre}^0(W_\alpha), \\ W_\lambda &:= \bigcup_{\alpha<\lambda} W_\alpha. \end{aligned}$$

There exists an ordinal β s.t. $W_\beta = W_{\beta+1}$ and we have $W^0 := W_\beta$. Clearly, positions in W^0 are winning for Player 0 (by ordinal induction). On the other hand, positions in $W^1 := V \setminus W^0$ are winning for Player 1: By construction this is a trap for Player 0 (Player 1 can force the game to stay forever in W^1), and by the finiteness assumption the game eventually reaches a position where Player 0 cannot move. □

Exercise 2. Show that one player parity games on finite graphs can be solved in polynomial time. Conclude that two player parity games on finite graphs can be solved in $\mathsf{NP} \cap \mathsf{coNP}$.

Solution: First, one shows how to solve in PTIME Büchi games. Then, one-player parity games reduce in PTIME to one-player Büchi games, by guessing the maximal even priority. The player wins iff there is a reachable strongly connected component where the maximal priority is even.

If there are two players, then by memoryless determinacy one can guess a winning strategy for one player and check that the induced one-player game is losing for the opponent. This gives a NP bound. The coNP bound follows from the symmetry between the two players. □

Exercise 3. Are Muller games on finite graphs positionally determined? Finite-memory determined?

Solution: No, they are not positionally determined. Consider a one player game with three vertices v_0, v_1, v_2 and edges $v_0 \to v_1, v_0 \to v_2, v_1 \to v_0, v_2 \to v_0$. The winning condition is "v_1 and v_2 are visited infinitely often". Clearly the player wins, however no memoryless winning strategy suffices.

They are finite-memory determined, by a product construction of the game graph G and a deterministic parity automaton recognising the winning condition. $\square$

Exercise 4. Are all games on finite graphs finite-memory determined?

Solution: No, however by the Büchi-Landweber theorem a counter example needs to use a non-ω-regular winning condition. For instance consider a one-player two-vertices game $V = \{u, v\}$ with edges between every pair of nodes, and the non-ω-regular winning condition $W = \left\{uvu^2v^2u^3v^3 \cdots\right\}$. The player does not win with any finite amount of memory. $\square$