

Languages, automata and computation II

Tutorial 5

Winter semester 2023/2024

In this tutorial we study well-quasi orders with some applications in theoretical computer science.

Well-quasi orders

Exercise 1. Let (X, \preceq) be a preorder (a.k.a. quasi-order). Show that the following conditions are equivalent.

1. Every infinite sequence $x_1, x_2, \dots \in X$ contains a *domination*: $x_i \preceq x_j$ for some $i < j$.
2. Every infinite sequence $x_1, x_2, \dots \in X$ contains a *monotone infinite subsequence*: $x_{i_1} \preceq x_{i_2} \preceq \dots$ for some $i_1 < i_2 < \dots$.
3. (X, \preceq) is *well-founded* (no infinite strictly decreasing sequences $x_1 \succ x_2 \succ \dots$) and all *antichains are finite* (an antichain is a set of pairwise incomparable elements).
4. Every upward closed set is the upward closure of a finite set.
5. Every nondecreasing chain of upward closed sets $U_1 \subseteq U_2 \subseteq \dots \subseteq X$ is finite.

Solution: (1 \Rightarrow 2) Assume each x_i is dominated by finitely many elements to its right. We can then take the subsequence of elements which are not dominated by any element to their right. This sequence is infinite and does not have any domination by construction.

(2 \Rightarrow 1) This is trivial.

(1 \Rightarrow 3) Well-foundedness is obvious. Now consider an antichain $A \subseteq X$. If it is infinite, then we arrange its elements into an infinite sequence. By the assumption we can find (in particular) a domination, contradicting that A is an antichain.

(3 \Rightarrow 1) Consider the set of minimal elements of the infinite sequence. This is an antichain, so it must be finite. Take any element in the sequence not from this finite set, and it is preceded by a smaller element, yielding a domination.

(3 \Rightarrow 4) Let $U \subseteq X$ be upward closed and consider the subset of its elements M which are not dominated by any element in U (minimal elements). Clearly U is the upward closure of M and the latter set is an antichain, thus finite by assumption.

(4 \Rightarrow 3) Clearly antichains must be finite. Regarding well-foundedness, by way of contradiction assume we have an infinite strictly decreasing sequence $x_1 \succ x_2 \succ \dots$. Then the upward closure of its elements is not the upward closure of a finite set, which is a contradiction.

(4 \Rightarrow 5) The upward closed set $U_1 \cup U_2 \cup \dots$. By assumption it is the upward closure of finitely many elements $B = \{x_1, \dots, x_n\}$. Take the first U_n s.t. $B \subseteq U_n$ and we have $U_n = U_{n+1} = \dots$.

(5 \Rightarrow 4) Let $U \subseteq X$ be an upward closed set and consider the set of its minimal elements $M = \{x_1, x_2, \dots\}$ (if there are ties just select arbitrarily an element from each minimal equivalence class). Now consider the chain generated by the upward closure of $\{x_1\}$, $\{x_1, x_2\}$, etc. By assumption, this chain is finite, therefore there is n s.t. the upward closure of $\{x_1, \dots, x_n\}$ is the same as that of M itself, which is just X . \square

Exercise 2. Which of the following preorders are well-quasi orders?

1. \mathbb{N}^2 with the lexicographic order;
2. $\{a, b\}^*$ with the lexicographic order;
3. \mathbb{N} with the divisibility partial order;
4. $\{a, b\}^*$ with the prefix order;
5. $\{a, b\}^*$ with the infix order;
6. intervals of \mathbb{N} with the following order:

$$[a, b] \preceq [c, d] \quad \text{if } b < c \vee (a = c \wedge b \leq d);$$

7. graphs with the subgraph order (remove some edges and some vertices).

Solution: 1. Yes, by Dickson's lemma even the smaller pointwise order is a wqo.

2. No, we have the ill-founded sequence $b \succ ab \succ aab \succ \dots$.

3. No, the set of primes forms an infinite antichain.

4. No, we have the infinite antichain $\{a, ba, bba, \dots\}$.

5. No, we have the infinite antichain $\{aba, abba, abbbba, \dots\}$.

6. Yes. The order is well-founded since $[a, b] \prec [c, d]$ implies $a + b < c + d$ in \mathbb{N} . There are no infinite antichains: By way of contradiction, let A be an infinite antichain. Fix an element $[a, b] \in A$. Now look at some other $[c, d] \in A$. Since these two elements are incomparable, from the first condition we have $0 \leq c \leq b$. This yields only finitely many options for c , so there must be two elements $[c, d_1]$ and $[c, d_2]$ in A , which are comparable, yielding a contradiction.

7. No. The order is well-founded by it contains the infinite antichain consisting of cycles of length 2, of length 3, etc. \square

Exercise 3. Show that if (X, \preceq_X) and (Y, \preceq_Y) are well-quasi orders, then the following are also well-quasi orders:

1. The *product order* $(X \times Y, \preceq_{X \times Y})$, where

$$(x_0, y_0) \preceq_{X \times Y} (x_1, y_1) \quad \text{if } x_0 \preceq_X x_1 \text{ and } y_0 \preceq_Y y_1.$$

Deduce Dickson's lemma on \mathbb{N}^k .

2. The *subword order* (X^*, \preceq_{X^*}) , where

$$x_1 \cdots x_m \preceq_{X^*} y_1 \cdots y_n$$

if there exists a subsequence y_{i_1}, \dots, y_{i_m} s.t. $x_1 \preceq_X y_{i_1}, \dots, x_m \preceq_X y_{i_m}$.
Deduce Higman's lemma on Σ^* .

3. What about the suborder over infinite words $(X^\omega, \preceq_{X^\omega})$?

Solution: 1. Let $(x_0, y_0), (x_1, y_1), \dots$ be an infinite sequence in $X \times Y$. By looking at the first component we can find an infinite monotone subsequence $x_{i_0} \preceq_X x_{i_1} \preceq_X \dots$. Now we look at the second component $y_{i_0}, y_{i_1}, \dots \in Y$ and we can find a domination $y_{i_j} \preceq_Y y_{i_k}$. We thus get the required domination in the product order $(x_{i_j}, y_{i_j}) \preceq_{X \times Y} (x_{i_k}, y_{i_k})$.

2. TODO

3. The new quasi-order \preceq_{X^ω} needs not be a well-quasi order. TODO. □

Applications

Exercise 4. Let V be a d -dimensional VASS and consider a target configuration $t \in P \times \mathbb{N}^d$, where P is a finite set of control locations. Show that one can compute the set of *all configurations* s which can cover t .

Solution: Consider the following chain of sets:

$$U_0 \subseteq U_1 \subseteq \dots \subseteq \mathbb{N}^d,$$

where U_n is the set of all configurations that can cover s in $\leq n$ steps. Order configurations by $(p, \bar{x}) \leq (q, \bar{y})$ if $p = q$ and $\bar{x} \leq_{\mathbb{N}^d} \bar{y}$. This is a well quasi order. Clearly, U_n is upward closed w.r.t. " \leq ". It follows that the chain terminates, yielding finitely many minimally elements $B = \{s_1, \dots, s_m\}$ whose upward closure generates $\bigcup_n U_n$. One can thus guess B and for element $s_i \in B$ thereof check that the set of one-step predecessors of s_i (finitely many) is in the upward closure of B . □

Exercise 5. Let V be a d -dimensional VASS and consider a source configuration $s \in \mathbb{N}^d$. Show that one decide whether there are only *finitely many* configurations reachable from s .

Solution: Consider the following *reachability tree*. The root is labelled by the initial configuration $s \in \mathbb{N}^d$. Whenever a node is labelled by a configuration $x \in \mathbb{N}^d$ and there is a transition $x \rightarrow x + v$ with $v \in V$, then this node has a children node labelled with $x + v$. The reachability tree is infinite in general. We now prune the reachability tree as follows. On every infinite branch, stop as long as a configuration y has a smaller ancestor $x \geq y$. We call this a *domination*. In this way all branches become finite. Since the tree is finitely branching, overall we get a finite tree by König's lemma. There are infinitely many reachable configurations iff there is a strict domination $y \geq x$ with $y \neq x$. □

Exercise 6. A *rewrite system* over a finite alphabet Σ is a finite set of pairs $u \rightarrow v$ with $u, v \in \Sigma^*$. Consider the least reflexive and transitive congruence \rightarrow^* on Σ^* containing \rightarrow . A rewrite system is *lossy* if it contains transitions $a \rightarrow \varepsilon$ for every $a \in \Sigma$. Show that the relation \rightarrow^* is decidable when \rightarrow is lossy.

Solution: For a configuration $u \in \Sigma^*$, let $\text{pre}^*(u)$ be the set of all configurations v which can reach u , i.e., $v \rightarrow^* u$. We want to decide whether for given u, v we have $v \in \text{pre}^*(u)$. TODO \square

\mathbb{Z} -VASSes

Exercise 7. Let a \mathbb{Z} -VAS of dimension $d \in \mathbb{N}$ be a finite set of vectors $V \subseteq \mathbb{Z}^d$. The semantics is as in VAS, except that now configurations are in \mathbb{Z}^d (instead of the more restrictive \mathbb{N}^d). Show that reachability is decidable for \mathbb{Z} -VASSes.

Solution: \square