

# Languages, automata and computation II

## Homework 1

### Problems: deadline ~~17/11/2023~~ 24/11/2023

**Problem 1** (Learning regular separators). Consider the following problem. Teacher knows two disjoint regular languages  $L, M \subseteq \Sigma^*$  and Learner wants to find a regular separator, i.e., a language  $S \subseteq \Sigma^*$  including  $L$  and disjoint from  $M$ . There are two kind of queries. 1) Learner gives a word  $w \in \Sigma^*$  to Teacher, who answers “in  $L$ ”, “in  $M$ ” or “not in  $L \cup M$ ”. 2) Learner gives a (DFA recognising a) separator candidate  $S$  to Teacher, who answers either “yes” if it separates  $L, M$ , or in case it doesn’t, Teacher answers “no” and provides either a counter-example to  $L \subseteq S$  or to  $M \cap S = \emptyset$ . We assume that counter-examples returned by Teacher are of minimal size.

Give a learning protocol for this problem, with polynomially many queries in the sizes of minimal DFAs for  $L, M$ . Observe that in the special case when  $L$  is the complement of  $M$ , this problem corresponds to the original learning setup for the Angluin algorithm.

**Problem 2** (Finite-valued rational functions, part 1). Let  $L_1, \dots, L_k$  a partition of  $\Sigma^*$  into regular languages, and consider weights  $q_1, \dots, q_k \in \mathbb{Q}$ . Show that the following function  $f : \Sigma^* \rightarrow \mathbb{Q}$  is rational:

$$\text{for every } w \in \Sigma^*: \quad f(w) = \begin{cases} q_1 & \text{if } w \in L_1, \\ \vdots & \\ q_k & \text{if } w \in L_k. \end{cases}$$

**Problem 3.** Recall that a word  $w = a_0 \dots a_n \in \{0, 1\}^*$  encodes a number  $[w]_2$  under the least binary digit first encoding

$$[w]_2 = a_0 + a_1 \cdot 2 + \dots + a_n \cdot 2^n.$$

Let  $f : \mathbb{N} \rightarrow \mathbb{Q}$  be a rational function. Show that the following function  $g : \{0, 1\}^* \rightarrow \mathbb{Q}$  is recognisable by a polynomial automaton:

$$g(w) = f([w]_2), \quad \text{for every } w \in \{0, 1\}^*.$$

### Star problems: deadline 22/12/2023

(\*) **Problem 4** (Finite-valued rational functions, part 2). (This is the converse of Problem 2.) Let  $f$  be a rational function taking only finitely many values. Show that for each value  $q \in \mathbb{Q}$ , the inverse image  $f^{-1}(q)$  (the set of words which  $f$  maps to  $q$ ) is a regular language. *Hint: Consider the  $q$ -finite representation of  $f$ .*

## Open problems: end of the semester

**Open problem 1.** Prove or disprove: For every function  $f : \Sigma^* \rightarrow \mathbb{Q}$  computed by a polynomial automaton, there is a polynomial automaton that computes the reverse function defined by

$$a_1 \cdots a_n \mapsto f(a_n \cdots a_1).$$

Disclaimer: we do not know the answer, although we believe that the statement is most likely false, i.e. there is a counterexample.

**Open problem 2.** Is the following problem is decidable? We are given a polynomial automaton computing a function  $f : \Sigma^* \rightarrow \mathbb{Q}$ , and we want to know if the following function gives only zero outputs:

$$a_1 \cdots a_n \mapsto f(a_1 \cdots a_n a_n \cdots a_1).$$

Disclaimer: we do not know the answer, although we believe that the problem is decidable.