

Languages, automata and computation II

Tutorial 7 - Games

Winter semester 2023/2024

Exercise 1. We say that a game is *finite* if every play eventually reaches a dead end (no player can move). Show that finite games are determined.

Solution: Consider a game $G = (V, V_0, V_1, E, W)$, where $V = V_0 \cup V_1$ is a partition of the game graph into vertices belonging to Player 0, resp., Player 1, $E \subseteq V \times V$ is the set of moves, and $W \subseteq V^\omega$ is the winning condition. The set of vertices V is in general infinite, even uncountable. Define the *predecessor operator* for Player $i \in \{0, 1\}$ as follows:

$$\text{Pre}^i(_) : V \rightarrow V$$

$$\text{Pre}^i(X) := \{u \in V_i \mid \exists(u \rightarrow v).v \in X\} \cup \{u \in V_{1-i} \mid \forall(u \rightarrow v).v \in X\}.$$

We now construct the winning region W^0 for Player 0. We build a sequence of sets W_α by induction on ordinals α :

$$\begin{aligned} W_0 &:= \left\{ u \in V_1 \mid \neg \exists(u \xrightarrow{v}) \right\}, \\ W_{\alpha+1} &:= W_\alpha \cup \text{Pre}^0(W_\alpha), \\ W_\lambda &:= \bigcup_{\alpha < \lambda} W_\alpha. \end{aligned}$$

There exists an ordinal β s.t. $W_\beta = W_{\beta+1}$ and we have $W^0 := W_\beta$. Clearly, positions in W^0 are winning for Player 0 (by ordinal induction). On the other hand, positions in $W^1 := V \setminus W^0$ are winning for Player 1: By construction this is a trap for Player 0 (Player 1 can force the game to stay forever in W^1), and by the finiteness assumption the game eventually reaches a position where Player 0 cannot move. \square

Exercise 2. Show that one player parity games on finite graphs can be solved in polynomial time. Conclude that two player parity games on finite graphs can be solved in $\text{NP} \cap \text{coNP}$.

Solution: First, one shows how to solve in PTIME Büchi games. Then, one-player parity games reduce in PTIME to one-player Büchi games, by guessing the maximal even priority. The player wins iff there is a reachable strongly connected component where the maximal priority is even.

If there are two players, then by memoryless determinacy one can guess a winning strategy for one player and check that the induced one-player game is losing for the opponent. This gives a NP bound. The coNP bound follows from the symmetry between the two players. \square

Exercise 3. Are Muller games on finite graphs positionally determined? Finite-memory determined?

Solution: No, they are not positionally determined. Consider a one player game with three vertices v_0, v_1, v_2 and edges $v_0 \rightarrow v_1, v_0 \rightarrow v_2, v_1 \rightarrow v_0, v_2 \rightarrow v_0$. The winning condition is “ v_1 and v_2 are visited infinitely often”. Clearly the player wins, however no memoryless winning strategy suffices.

They are finite-memory determined, by a product construction of the game graph G and a deterministic parity automaton recognising the winning condition. \square

Exercise 4. Are all games on finite graphs finite-memory determined?

Solution: No, however by the Büchi-Landweber theorem a counter example needs to use a non- ω -regular winning condition. For instance consider a one-player two-vertices game $V = \{u, v\}$ with edges between every pair of nodes, and the non- ω -regular winning condition $W = \{uvu^2v^2u^3v^3\cdots\}$. The player does not win with any finite amount of memory. \square