

# Languages, automata and computation II

## Tutorial 7 - Games

Winter semester 2023/2024

**Exercise 1.** We say that a game is *finite* if every play eventually reaches a dead end (no player can move). Show that finite games are determined.

*Solution:* Consider a game  $G = (V, V_0, V_1, E, W)$ , where  $V = V_0 \cup V_1$  is a partition of the game graph into vertices belonging to Player 0, resp., Player 1,  $E \subseteq V \times V$  is the set of moves, and  $W \subseteq V^\omega$  is the winning condition. The set of vertices  $V$  is in general infinite, even uncountable. Define the *predecessor operator* for Player  $i \in \{0, 1\}$  as follows:

$$\text{Pre}^i(\_) : V \rightarrow V$$

$$\text{Pre}^i(X) := \{u \in V_i \mid \exists(u \rightarrow v).v \in X\} \cup \{u \in V_{1-i} \mid \forall(u \rightarrow v).v \in X\}.$$

We now construct the winning region  $W^0$  for Player 0. We build a sequence of sets  $W_\alpha$  by induction on ordinals  $\alpha$ :

$$\begin{aligned} W_0 &:= \left\{ u \in V_1 \mid \neg \exists(u \xrightarrow{v}) \right\}, \\ W_{\alpha+1} &:= W_\alpha \cup \text{Pre}^0(W_\alpha), \\ W_\lambda &:= \bigcup_{\alpha < \lambda} W_\alpha. \end{aligned}$$

There exists an ordinal  $\beta$  s.t.  $W_\beta = W_{\beta+1}$  and we have  $W^0 := W_\beta$ . Clearly, positions in  $W^0$  are winning for Player 0 (by ordinal induction). On the other hand, positions in  $W^1 := V \setminus W^0$  are winning for Player 1: By construction this is a trap for Player 0 (Player 1 can force the game to stay forever in  $W^1$ ), and by the finiteness assumption the game eventually reaches a position where Player 0 cannot move.  $\square$

**Exercise 2.** Show that one player parity games on finite graphs can be solved in polynomial time. Conclude that two player parity games on finite graphs can be solved in  $\text{NP} \cap \text{coNP}$ .

*Solution:* First, one shows how to solve in PTIME Büchi games. Then, one-player parity games reduce in PTIME to one-player Büchi games, by guessing the maximal even priority. The player wins iff there is a reachable strongly connected component where the maximal priority is even.

If there are two players, then by memoryless determinacy one can guess a winning strategy for one player and check that the induced one-player game is losing for the opponent. This gives a NP bound. The coNP bound follows from the symmetry between the two players.  $\square$

**Exercise 3.** Are Muller games on finite graphs positionally determined? Finite-memory determined?

*Solution:* No, they are not positionally determined. Consider a one player game with three vertices  $v_0, v_1, v_2$  and edges  $v_0 \rightarrow v_1, v_0 \rightarrow v_2, v_1 \rightarrow v_0, v_2 \rightarrow v_0$ . The winning condition is “ $v_1$  and  $v_2$  are visited infinitely often”. Clearly the player wins, however no memoryless winning strategy suffices.

They are finite-memory determined, by a product construction of the game graph  $G$  and a deterministic parity automaton recognising the winning condition.  $\square$

**Exercise 4.** Are all games on finite graphs finite-memory determined?

*Solution:* No, however by the Büchi-Landweber theorem a counter example needs to use a non- $\omega$ -regular winning condition. For instance consider a one-player two-vertices game  $V = \{u, v\}$  with edges between every pair of nodes, and the non- $\omega$ -regular winning condition  $W = \{uvu^2v^2u^3v^3\ldots\}$ . The player does not win with any finite amount of memory.  $\square$