Languages, automata and computation II
Tutorial 8 - MSO and automata on infinite trees

Winter semester 2023/2024

Exercise 1. Determine whether the following languages of finite trees are regular:

1. Trees with an even number of nodes.
2. Trees with an even number of nodes labelled by a.
3. Trees where every subtree contains an a.
4. Trees encoding Boolean expressions evaluating to true. The alphabet contains two symbols $\top, \bot$ of arity 0, one symbol $\neg$ of arity 1, and two symbols $\wedge, \vee$ of arity 2.
5. Trees where every leaf is at the same depth (balanced trees).

Solution: □

Exercise 2.

1. Show that nondeterministic finite tree automata have the same expressive power as deterministic bottom-up finite tree automata.
2. Show that deterministic bottom-up finite tree automata are more expressive than deterministic top-down finite tree automata.
3. Show that deterministic bottom-up parity tree automata are less expressive than nondeterministic parity tree automata.

Solution:

1. The subset construction works straight away.
2. Consider the language $L = \{a(b, c), a(c, b)\}$. Any deterministic top-down finite tree automaton recognising L accepts also the tree $a(b, b)$, which is a contradiction.
3. The language of infinite trees containing finitely many occurrences of letter a is recognisable by a nondeterministic parity tree automaton, but not by a bottom-up deterministic one.

□

Exercise 3. Show that nonemptiness of nondeterministic parity tree automata is decidable. What is the computational complexity of this problem?

Solution: Given a NPTA A one can build in PTIME a finite game graph with parity winning condition. Viceversa, for every such game one can build in PTIME a NPTA A which has nonempty language iff Player 0 wins. Thus NPTA nonemptiness and parity games on finite graphs are PTIME-equivalent. $\square$

Exercise 4. Consider finite word structures $(\{1,\ldots,n\}, \leq, P_a, P_b)$ where P_a, P_b are two unary relation symbols. Show a class of finite word structures definable in existential second-order logic which is not MSO definable (thus not regular). Can this be done without unary relation symbols P_a, P_b?

Solution: The finite word structures corresponding to the nonregular language $a^n b^n$ are definable in existential second-order logic. It suffices to existentially quantify over a binary relation $R \subseteq \{1,\ldots,2n\}^2$ and axiomatise that it is a well-nested matching between the two half of the domain, where the first half has label a and the second has label b.

In fact we do not need P_a, P_b to define nonregular languages in existential second-order logic: The set of finite word structures of composite size (nonprime) is not regular, but definable in this logic. One guesses two subsets of the domain $X, Y \subseteq \{1,\ldots,n\}$, each of size at least 2, and a bijection from $\{1,\ldots,n\}$ to $X \times Y$. $\square$

Exercise 5. A tree is *regular* if it contains only finitely many distinct subtrees. Show that a nonempty regular language of infinite trees contains a regular tree.

Solution: Consider the nonemptiness game played on the infinite binary tree. Since Player 0 wins, she has a memoryless winning strategy. Applying this strategy constructs a regular tree accepted by the automaton. $\square$

Exercise 6. Fix a regular language of ω-words $L \subseteq \{a,b\}^\omega$. Show that the language of infinite trees where every branch belongs to L is regular.

Solution: From a DPA recognising L construct a top-down deterministic parity tree automaton that checks that every branch belongs to L. $\square$

Exercise 7. Construct a language of infinite trees recognised by a NPTA but not by a nondeterministic Büchi tree automaton, and prove that this is the case.

Solution: Consider the language of infinite trees s.t. every branch contains only finitely many a's. This language is even deterministic coBüchi recognisable. A suitable pumping argument shows that the Büchi acceptance condition cannot recognise this language. $\square$

Exercise 8.

1. Show that the structure $(\mathbb{N}^*, \preceq)$ has a decidable MSO theory, where $\preceq$ is the prefix order. *Hint: Reduce to the MSO theory of the infinite binary tree.*
2. Show that the structure $(\mathbb{N}^*, \preceq_{\text{lex}})$ has a decidable MSO theory, where $\preceq$ is the lexicographic order.
3. Conclude that $(\mathbb{Q}, \leq)$ has a decidable MSO theory.

Solution: We encode $w \in \mathbb{N}^*$ as a node of the binary tree $[w] \in \{L, R\}^*$. For instance $[\varepsilon] = \varepsilon$ and $[21] = LLRL$. In general, $[n_1 \cdots n_k] = L^{n_1} R L^{n_2} R \cdots L^{n_k}$. The image of the encoding is an MSO-definable subset of the binary tree. The prefix order $\preceq$ on $\mathbb{N}^*$ is the R-descendant relation on the binary tree, which is MSO-definable. Consequently, every MSO formula of the original structure can be encoded as an MSO formula of the binary tree.

The lexicographic order is encoded as the descendant relation.

The rationals with order are isomorphic $((0,1) \cap \mathbb{Q}, \leq)$, which can be MSO-encoded in $\left(\{0,1\}^*, \preceq_{\text{lex}}\right)$. □

Exercise 9. Show that Gale-Steward games with an ω-regular winning condition are decidable.

Solution: Reduce to MSO on infinite trees or nonemptiness of NPTA. □