# Languages, automata and computation II
# Tutorial 1 – Semilinear sets

Winter semester 2024/2025

We consider $\mathbb{N}^d$ as a finitely generated free commutative additive monoid. We can definine several operations on subsets $A, B$ of $\mathbb{N}^d$:

- Union $A \cup B$, intersection $A \cap B$, and complement $\mathbb{N}^d \setminus A$.
- Addition $A + B = \{a + b \mid a \in A, b \in B\}$.
- Iteration $A^* = \bigcup_{n \in \mathbb{N}} nA$, where $nA = A + A + \cdots + A$ ($n$ times).
- Projection $\pi_i(A) = \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) \mid (a_1, \ldots, a_d) \in A\}$.

By convention, $0A = \{0\}$ if $A$ is nonempty, otherwise $0\emptyset = \emptyset$.

## Semilinear sets

A subset $L$ of $\mathbb{N}^d$ is *linear* if it is of the form $b + P^*$ for some *basis vector* $b \in \mathbb{N}^d$ and a finite set of *periods* $P \subseteq \mathbb{N}^d$. A subset $S$ of $\mathbb{N}^d$ is *semilinear* if it is a finite union of linear sets, $S = L_1 \cup \cdots \cup L_n$.

**Exercise 1** (Closure properties – Part 1)**.** Show that the semilinear sets are closed under union, addition, iteration, and projection.

We delay showing closure of intersection and complement, which are more difficult.

*Solution:* Closure under union is trivial. For addition, it suffices to show that the addition of two linear sets is linear. This follows at once from the following identity, which is valid for every set $A, B \subseteq \mathbb{N}^d$:

$$(A \cup B)^* = A^* + B^*. \tag{1}$$

For iteration, it suffices to show that the iteration $L^*$ of a linear set $L$ is semilinear. This follows at once from the following identity, which is valid for every set $A, B \subseteq \mathbb{N}^d$:

$$(A + B^*)^* = \{0\} \cup (A + (A \cup B)^*). \tag{2}$$

Indeed, $(A + B^*)^{\geq 1} = (A + B^*) \cup (2A + B^*) \cup \cdots = A + A^* + B^*$, to which we apply (1).

Closure under projection is easy: Just project the basis and periods of every linear component. $\square$

**Exercise 2.** Show that the set $\{(n, 2^n) \;:\; n \in \mathbb{N}\}$ is not semilinear.

*Solution:* For semilinear sets $\sup\{\frac{b}{a} : (a,b) \in S\}$ is finite. □

**Exercise 3.** Let $X \subseteq \mathbb{N}$ be an arbitrary (finite or infinite) set of natural numbers. Show that its iteration $X^*$ is a semilinear set.

*Solution:* TODO □

**Exercise 4.** Find a set $X \subseteq \mathbb{N}^2$ such that $X^*$ is not semilinear.

*Solution:* $X = \{(1, 2^n) \;:\; n \in \mathbb{N}\}$ □

# Rational sets

In analogy with the rational subsets of the free monoid $\Sigma^*$, we say that a subset $R$ of the free commutative monoid $\mathbb{N}^d$ is *rational* if it can be generated from finite sets using union "$\cup$", addition "$+$" and iteration "$\_^*$".

**Exercise 5** (Rational = semilinear)**.** Show that a subset of $\mathbb{N}^d$ is rational if, and only if, it is semilinear.

In view of the exercise, we can see the basis-periods representation of a semilinear set as a normal form for rational sets.

*Solution:* The "if" direction is trivial. For the "only if" direction, we need to show that semilinear sets are closed under union, addition, and iteration. We did this already in Exercise 1. □

# Systems of linear inequalities

**Exercise 6.** Prove that the set of nonnegative integer solutions of a system of linear inequalities with integer coefficients is semilinear.

*Solution:* We proceed in two steps.

- Start with proof for systems of equalities. Consider a system of equalities $Ax = b$, where $A \in \mathbb{N}^{d \times d}$ and $b \in \mathbb{N}^d$. Let $H$ be the set of minimal solutions of $Ax = 0$ and let $P$ be the set of minimal solutions of $Ax = b$. We argue that those sets are finite (Dickson lemma as blackbox?). Then the set of all solutions is $H + P^*$.
- Now we want to prove that set of nonnegative integer solutions of $Ax \geq b$ is semilinear. Observe that this set is the same as projection to first $d$ coordinates of the set of solutions of $\left[\begin{array}{c|c} A & -Id(d) \end{array}\right] \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = b$. From previous point we know that this is semilinear.

□

## Complementation

**Exercise 7.** Let $p \in \mathbb{N}^d$ be a period and consider the linear set $L = \mathbb{N}p$. Show that the complement $M := \mathbb{N}^d \setminus L$ is semilinear.

*Solution:* Consider the set $X$ of all nonzero vectors $u \in \mathbb{N}^d$ s.t. there is a component $1 \leq i \leq d$ with $u_i < p_i$. Clearly $X \subseteq M$ and $X$ is semilinear.

Consider the set $Y$ of all vectors $u \in \mathbb{N}^d$ s.t. $u \geq p$ and $u - p <_i p$ for some component $1 \leq i \leq d$. Clearly $Y \subseteq M$ and $Y$ is semilinear. In fact even $Z := Y + \mathbb{N}p \subseteq M$, which is also semilinear.

We claim that $M = X \cup Z$. We have already argued regarding the "$\supseteq$" inclusion. For the other inclusion, assume by way of contradiction that there is $u \in M$ but $u \notin X$ and $u \notin Z$. Take a $u$ minimal with this property w.r.t. the component-wise order. From $u \notin X$ we get $u \geq p$. From $u \notin Z$ we get in particular $u \notin Y$, and thus $u - p \geq p$. Clearly $u - p$ is also in $M$. Moreover, $u - p$ is not in $X$. Finally, $u - p$ is not in $Z$: If it were, then there would be $v \in Y$ s.t. $u - p \in v + \mathbb{N} \cdot p$, and thus $u \in v + \mathbb{N} \cdot p$, showing $u \in Z$, contrary to our assumption. Since $u - p$ is strictly smaller than $u$, we have contradicted the minimality of $u$. □

**Exercise 8** (Closure properties – Part 2)**.** Show that the semilinear sets are closed under intersection and complement.

*Solution:* Closure w.r.t. intersection follows from closure under union and complement. Complement: TODO. □