

# Languages, automata and computation II

## Tutorial 11 – Ordinal numbers, register automata, orbit-finite sets

Winter semester 2024/2025

### Ordinal numbers

Ordinal numbers generalize natural numbers and can be seen as representations of well-founded total orders. For a set  $A$  ordered by  $\leq$  we denote by  $\text{tp}(A, \leq)$  (or just  $\text{tp}(A)$ ) the ordinal number representing this relation. Two well-founded total orders  $(A, \leq_A)$ ,  $(B, \leq_B)$  are isomorphic if and only if  $\text{tp}(A) = \text{tp}(B)$ . A number  $n \in \mathbb{N}$  represents a totally ordered finite set with  $n$  elements, i.e.  $\text{tp}(A) = n$  if  $|A| = n$ . By  $\omega$  we denote the ordinal type of the set of natural numbers with the usual order relation, i.e.  $\text{tp}(\mathbb{N}) = \omega$ . The class of ordinal numbers is denoted by **ON**. They have precise set theoretic definition which we will not mention it here.

Let  $(A, \leq)$ ,  $(B, \leq)$  be well-founded total orders represented by respectively  $\alpha, \beta \in \mathbf{ON}$ . We write  $\alpha \leq \beta$  if can  $(A, \leq)$  be embedded into  $(B, \leq)$ . We also define the following operations:

- $\alpha + \beta = \text{tp}(\{0\} \times A \cup \{1\} \times B, \preceq)$ , where  $(i, x) \preceq (j, y)$  if  $i < j$  or  $(i = j \text{ and } x \leq y)$
- $\alpha \cdot \beta = \text{tp}(B \times A, \preceq_{lex})$
- $\alpha^\beta = \text{tp}(F(A, B), \preceq)$ , where  $F(A, B)$  is the set of all functions  $B \rightarrow A$  with finite support and  $f \preceq g$  if  $f = g$  or  $f(\xi) < g(\xi)$ , where  $\xi = \max\{\eta \in B : f(\eta) \neq g(\eta)\}$

Consecutive ordinal numbers are:  $0, 1, 2, \dots, \omega, \omega+1, \dots, \omega \cdot 2, \dots, \omega^2, \dots, \omega^\omega, \dots$

**Exercise 1.** Prove that for  $\alpha, \beta, \gamma \in \mathbf{ON}$

1.  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2.  $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
3.  $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
4.  $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
5.  $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$

**Exercise 2.** Show the following equalities/inequalities:

1.  $1 + \omega = \omega < \omega + 1$
2.  $2 \cdot \omega = \omega < \omega \cdot 2$
3.  $2^\omega = 3^\omega = \omega$

## Register automata

Let  $(\mathbb{A}, =)$  be the set of data values with equality as the only relation (one can consider other data structures). An *nondeterministic register automaton (NRA)* is a tuple  $A = (\Sigma, R, L, I, F, \Delta)$  where  $\Sigma$  is an orbit-finite *input alphabet* (often  $\Sigma = \mathbb{A}$ ),  $R$  is a finite set of registers,  $L$  is a finite set of *locations*,  $I, F \subseteq L$  are *initial*, resp., *final* locations, and  $\Delta \subseteq (L \times \mathbb{A}^R) \times \Sigma \times (L \times \mathbb{A}^R)$  is an equivariant set of *transitions* (thus orbit-finite). A *state* of a register automaton is a location together with register valuation, i.e.  $Q = L \times \mathbb{A}^R$ .

**Exercise 3.** Show that following languages are recognizable by register automata:

1. the first data value is the same as the last data value;
2. some data value appears twice;
3. the last data value appears again;
4. every three consecutive data values are pairwise distinct.

**Exercise 4.** Show that the class of languages recognizable by NRA is not closed under complement.

**Exercise 5.** Show that a nondeterministic register automaton can recognize the language "some data value appears twice", but a deterministic one cannot.

**Exercise 6.** Call a nondeterministic register automaton *guessing* if there exists a transition  $t \in \Delta$  such that some data value in the target state appears neither in the source state nor in the input. Give an example of a language that needs guessing to be recognized.

A corollary of the above two exercises is that:

$$\text{deterministic} \subsetneq \text{nondeterministic without guessing} \subsetneq \text{nondeterministic}$$

**Exercise 7.** Prove the following (non)-closure properties of languages of finite data words recognised by NRA and its deterministic variant.

	$\cup$	$\cap$	$\text{--}^R$	$\Sigma^* \setminus \text{--}$
Deterministic NRA	✓	✓	×	✓
Nondeterministic NRA	✓	✓	✓	×

**Exercise 8.** The complexity of the emptiness problem for nondeterministic register automata depends on how the size  $|A|$  of the input automaton is measured. Show that emptiness is:

- PSPACE-complete if  $|A|$  is the number of locations and registers;

- NP-complete if  $|A|$  is the number of reachable orbits of states;
- polynomial time if  $|A|$  is the number of orbits of transitions.

**Exercise 9.** Consider the set of atoms  $(\mathbb{Q}, \leq)$ . Show that there exists a language closed under permutation of atoms (not only automorphism of  $\mathbb{Q}$ ) that is recognizable by an NRA over  $(\mathbb{Q}, \leq)$  without guessing, but is not recognizable by an NRA over  $(\mathbb{Q}, =)$  without guessing.

## Sets with atoms

In this tutorial we explore sets with atoms. Fix a countable set  $A$  and consider the relational structure  $\mathbb{A} = (A, =)$  over the signature consisting only of equality. Elements of  $\mathbb{A}$  are called *atoms* and automorphisms  $\alpha$  of this structure are the bijections of  $A$ . A *set with atoms* is any set built from set constructors and atoms from  $\mathbb{A}$ . For an automorphism  $\alpha$  of  $\mathbb{A}$  and a set with atoms  $x$  we extend  $\alpha$  to  $x$  as  $\alpha(x) = \{\alpha(y) \mid y \in x\}$ . An automorphism  $\alpha$  *fixes* a set with atoms  $x$  if  $\alpha(x) = x$ . Fix a tuple  $\bar{a} \in A^n$ . An  $\bar{a}$ -*automorphism* is an automorphism fixing  $\bar{a}$ . A set with atoms  $x$  is *supported by*  $\bar{a}$  if  $x$  is fixed by every  $\bar{a}$ -automorphism, and it is *equivariant* if it is supported by the empty set. A set with atoms is *finitely supported* if it is supported by some finite tuple of atoms, and it is *legal* if it is finitely supported and all its elements are legal (this is a recursive definition).

The *orbit* of element  $x$  is the set  $\text{orbit}(x) = \{\alpha(x) \mid \text{automorphism } \alpha\}$  of elements which can be obtained by applying some automorphism to  $x$ . Thus an equivariant set  $x$  is a union of orbits (of its elements), and an equivariant set is *orbit finite* if this union is finite. The set of orbits of a set  $x$  is  $\text{Orbits}(x) = \{\text{orbit}(y) \mid y \in x\}$  (this is a partition of  $x$ ). For instance, for given  $a, b \in \mathbb{A}$  with  $a \neq b$ , we have  $\text{orbit}(a, a) = \{(c, d) \in \mathbb{A}^2 \mid c = d\}$  and  $\text{orbit}(a, b) = \{(c, d) \in \mathbb{A}^2 \mid c \neq d\}$ . Since there are no more orbits in  $\mathbb{A}^2$ , we have  $\text{Orbits}(\mathbb{A}^2) = \{\text{orbit}(a, a), \text{orbit}(a, b)\}$  and  $\mathbb{A}^2$  is the union of two orbits.

**Exercise 10.** Fix the equality atoms  $(\mathbb{A}, =)$ . For each of the following sets with atoms decide whether they are 1) legal, 2) equivariant, 3) orbit finite.

1.  $\mathbb{A}^2$ .
2.  $\mathbb{A}^* := \mathbb{A}^0 \cup \mathbb{A}^1 \cup \mathbb{A}^2 \cup \dots$ .
3.  $\mathbb{A}^\omega := \{a_1 a_2 \dots \mid a_1, a_2, \dots \in \mathbb{A}\}$ .
4.  $2^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}\}$  (powerset).
5.  $2_{\text{fin}}^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}, B \text{ finite}\}$  (finite powerset).
6.  $2_{\text{fs}}^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}, B \text{ finitely supported}\}$  (finitely supported powerset).
7.  $2_{\text{eq}}^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}, B \text{ equivariant}\}$  (equivariant powerset).

**Exercise 11.** An atom structure  $\mathbb{A}$  is called *oligomorphic* if  $\mathbb{A}^n$  is orbit-finite for every  $n \in \mathbb{N}$ . Are the following atom structures oligomorphic?

1.  $(\mathbb{N}, =)$ .
2.  $(\mathbb{Z}, \leq)$ .

3.  $(\mathbb{Q}, \leq)$ .

4.  $(\mathbb{Q}, +)$ .