# Languages, automata and computation II
# Tutorial 3 – Applications of well-quasi orders

Winter semester 2024/2025

## Regular languages

**Exercise 1.** Consider the set of finite words well-quasi ordered by the subword relation $(\Sigma^*, \sqsubseteq)$. Show that *every* downward closed language over $\Sigma$ is regular.

## Lossy rewrite systems

**Exercise 2.** A *rewrite system* over a finite alphabet $\Sigma$ is a finite set of pairs $u \to v$ with $u, v \in \Sigma^*$. Consider the least reflexive and transitive congruence $\to^*$ on $\Sigma^*$ containing $\to$. A rewrite system is *lossy* if it contains transitions $a \to \varepsilon$ for every $a \in \Sigma$. Show that the relation $\to^*$ is decidable when $\to$ is lossy.

## Vector addition systems

**Exercise 3.** Let $\mathcal{V}$ be a $d$-dimensional VASS and consider a target configuration $t \in P \times \mathbb{N}^d$, where $P$ is the set of states. Show that one can compute the set of *all configurations* $s$ which can cover $t$.

**Exercise 4.** Let $\mathcal{V}$ be a $d$-dimensional VAS and consider a source configuration $s \in \mathbb{N}^d$. Show that one decide whether there are only *finitely many* configurations reachable from $s$.

**Exercise 5.** Let $\mathcal{V}$ be a $d$-dimensional VAS and consider a source configuration $s \in \mathbb{N}^d$. Show that for any coordinate $k \in \{1, \ldots, d\}$ it is decidable whether there exists a number $n \in \mathbb{N}$ such that every configuration $t$ reachable from $s$ has the $k$th coordinate bounded by $n$.

**Exercise 6.** Show that a VASS of dimension $d$ can be simulated by a VAS (without states) of dimension $d + 3$.

## Vector addition systems over $\mathbb{Z}$

This part is about VASSes but unrelated with well quasi orders. We show that reachability in VASSes is considerably simpler if we relax the requirement that counters cannot become negative.

**Exercise 7.** Let a $\mathbb{Z}$-*VASS* of dimension $d \in \mathbb{N}$ be a pair $(Q, T)$ where $Q$ is a finite set of states and $T \subseteq Q \times \mathbb{Z}^d \times Q$ be a finite set of transitions. The semantics is as in VASS, except that now configurations are in $Q \times \mathbb{Z}^d$ (instead of the more restrictive $Q \times \mathbb{N}^d$). Show that reachability is decidable for $\mathbb{Z}$-VASSes.

# Strassen's matrix multiplication algorithm

This section is unrelated with well-quasi orders. We begin with a simpler problem.

**Problem 1.** Consider three complex numbers $a = a_1 + a_2 i, b = b_1 + b_2 i, c = c_1 + c_2 i \in \mathbb{C}$. The naive multiplication algorithm would compute the product $c = a \cdot b$ as

$$\begin{aligned} c_1 &= a_1 \cdot b_1 - a_2 \cdot b_2, \\ c_2 &= a_1 \cdot b_2 + a_2 \cdot b_1, \end{aligned}$$

which uses four multiplications in $\mathbb{R}$. Find a more efficient algorithm that uses only three multiplications and any number of additions.

Consider $2 \times 2$ matrices of rational numbers

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathbb{Q}^{2\times 2}.$$

The naive multiplication algorithm would compute the product $C = A \cdot B$ as

$$\begin{aligned} c_{11} &= a_{11} \cdot b_{11} + a_{12} \cdot b_{21}, \\ c_{12} &= a_{11} \cdot b_{12} + a_{12} \cdot b_{22}, \\ c_{21} &= a_{21} \cdot b_{11} + a_{22} \cdot b_{21}, \\ c_{22} &= a_{21} \cdot b_{12} + a_{22} \cdot b_{22}, \end{aligned}$$

which uses 8 multiplications (we do not care about additions). When applied recursively, this yields the following formula for the number of ring multiplications used in order to compute the product of two $n \times n$ matrices $A, B$

$$M(n) \le O(n^2) + 8 \cdot M(n/2).$$

From this we obtain a complexity upper bound $M(n) \le O(n^3)$ for naive matrix multiplication. Strassen's algorithm uses only 7 multiplications by computing the following products:

$$\begin{aligned} m_1 &= (a_{11} + a_{22}) \cdot (b_{11} + b_{22}), \\ m_2 &= (a_{21} + a_{22}) \cdot b_{11}, \\ m_3 &= a_{11} \cdot (b_{12} - b_{22}), \\ m_4 &= a_{22} \cdot (b_{21} - b_{11}), \\ m_5 &= (a_{11} + a_{12}) \cdot b_{22}, \\ m_6 &= (a_{21} - a_{11}) \cdot (b_{11} + b_{12}), \\ m_7 &= (a_{12} - a_{22}) \cdot (b_{21} + b_{22}). \end{aligned}$$

The number of ring multiplications for the improved algorithm satisfies

$$M(n) \le O(n^2) + 7 \cdot M(n/2).$$

and thus $M(n) \le O(n^{\log_2 7})$, where $\log_2 7 \approx 2.81$.