Languages, automata and computation II
Homework 1

Problems: deadline XX/11/2024

Problem 1. Consider a finite alphabet $\Sigma = \{a, b\}$.

1. For two infinite words $u, v \in \Sigma^\omega$ we say that u *embeds into* v, written $u \sqsubseteq v$, if we can obtain u by deleting finitely or infinitely many letters from v. Show that embedding of infinite words is a well-quasi order.

2. Now consider the variant where only finitely many letters from v can be deleted. Is the resulting relation still a well-quasi order?

Problem 2. Consider first order logic over the structure $(\mathbb{Q}, \leq)$ of the rational numbers together with their natural order. Let $\mathbb{Q}^{d\uparrow}$ be the set of *increasing d-tuples* of rational numbers $(x_1, \ldots, x_d)$ with $x_1 \leq \cdots \leq x_d$. Show that there are $d! \cdot 2^d$ quantifier-free definable total orders on $\mathbb{Q}^{d\uparrow} \times \mathbb{Q}^{d\uparrow}$.

Problem 3. Consider the following two structures:

$$A = \left(\{0, 1\}^*, \cdot, 0, 1\right) \quad \text{and} \quad B = (\mathbb{N}, +, \cdot)\,.$$

In structure A, the binary operation $\cdot$ is the concatenation of words.

1. Provide an algorithm s.t. given in input a first-order sentence φ over A it produces in output a first-order sentence ψ over B s.t. φ is true in A iff ψ is true in B.

2. Provide an algorithm as in the previous point with the roles of A and B swapped.

Problem 4. Consider the structure $(\mathbb{N}, +, |)$ where "$|$" is the binary divisibility relation on the natural numbers: For all $a, b \in \mathbb{N}$ we have $a \,|\, b$ iff there exists $c \in \mathbb{N}$ s.t. $a \cdot c = b$. Show that multiplication of natural numbers is definable in first-order logic in this structure.