Languages, automata and computation II
Tutorial 5 – Quantifier elimination, Presburger arithmetic

Winter semester 2024/2025

Normal forms

Exercise 1. Prove that any first order formula is logically equivalent to some formula in prenex normal form, i.e. a formula with all quantifiers at the beginning.

Exercise 2. Prove that any quantifier-free first order formula is logically equivalent to some formula in conjunctive normal form (CNF), i.e. a conjunction of disjunctions of atomic formulas and negations of atomic formulas.

Quantifier elimination

Exercise 3. Prove that $(\mathbb{N}, =)$ admits quantifier elimination.

Exercise 4. Prove that $(\mathbb{N}, \leq)$ does not have quantifier elimination.

Presburger arithmetic

Exercise 5. Prove that multiplication cannot be defined in Presburger arithmetic, i.e. the set $\{(a, b, c) \in \mathbb{N}^3 : a \cdot b = c\}$ is not definable in $(\mathbb{N}, +)$.

Exercise 6. Recall[1] that subsets of $\mathbb{N}^d$ can be encoded in binary as languages over an alphabet $\{\begin{bmatrix} d_1 \\ \dots \\ d_n \end{bmatrix} : d_1, \dots, d_n \in \{0, 1\}\}$. For instance, the word $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ encodes the pair $(6, 5)$. Let $\phi(x_1, \dots, x_n)$ be a formula of Presburger arithmetic. Prove that the subset of $\mathbb{N}^d$ defined by ϕ can be encoded as a regular language. Conclude that the Presburger arithmetic is decidable.

[1] We mentioned this in the first tutorial