

# Languages, automata and computation II

## Homework 1

### Problems: deadline XX/11/2024

**Problem 1.** Consider a finite alphabet  $\Sigma = \{a, b\}$ . An  $\omega$ -word is a sequence  $a_0 a_1 \dots$  of letters from  $\Sigma$ . Let  $\Sigma^\omega$  be the set of all infinite words over  $\Sigma$ .

1. For two  $\omega$ -words  $u, v \in \Sigma^\omega$  we say that  $u$  *embeds into*  $v$ , written  $u \sqsubseteq v$ , if we can obtain  $u$  by deleting finitely or infinitely many letters from  $v$ . Show that embedding of  $\omega$ -words is a well-quasi order.
2. Now consider the variant where only finitely many letters from  $v$  can be deleted. Is the resulting relation still a well-quasi order?

**Problem 2.** Consider the structure  $(\mathbb{N}^d, \leq)$ , where the order “ $\leq$ ” is element-wise. Recall that a set  $X \subseteq \mathbb{N}^d$  is *downward closed* if for every  $x \in X$  and  $y \leq x$  we have  $y \in X$ .

1. Show that any downward-closed set  $X \subseteq \mathbb{N}^d$  is semilinear.
2. Provide an algorithm that solves the following problem: Given a VAS (without states) of dimension  $d$  and an initial configuration  $x \in \mathbb{N}^d$ , construct a semilinear representation for the downward closure of the set of configurations reachable from  $x$ .

**Problem 3.** Consider first-order logic over the structure  $(\mathbb{Q}, \leq)$  of the rational numbers together with their natural order. Let  $\mathbb{Q}^{d\uparrow}$  be the set of *increasing  $d$ -tuples* of rational numbers  $(x_1, \dots, x_d)$  with  $x_1 \leq \dots \leq x_d$ . Show that there are  $d! \cdot 2^d$  quantifier-free first-order definable total orders on  $\mathbb{Q}^{d\uparrow} \times \mathbb{Q}^{d\uparrow}$ .

**Problem 4.** Consider the following two structures:

$$A = (\{0, 1\}^*, \cdot, 0, 1) \quad \text{and} \quad B = (\mathbb{N}, +, \cdot).$$

In structure  $A$ , the binary operation  $\cdot$  is the concatenation of words.

1. Provide an algorithm s.t. given in input a first-order sentence  $\varphi$  over  $A$  it produces in output a first-order sentence  $\psi$  over  $B$  s.t.  $\varphi$  is true in  $A$  iff  $\psi$  is true in  $B$ .
2. Provide an algorithm as in the previous point with the roles of  $A$  and  $B$  swapped.