Languages, automata and computation II
Homework 2 (draft version)

Problems: deadline XX/12/2024

Problem 1. A *parametric finite weighted automaton* is a finite weighted automaton where the initial vector, the final vector, and the transition matrices instead of using values from the field $\mathbb{Q}$, use polynomials with one variable x. For every choice of $x \in \mathbb{Q}$, this gives us a finite weighted automaton in the usual sense, by evaluating the polynomials to get values in the field. Show that the set

$$\{x \in \mathbb{Q} \mid \text{the automaton with parameter } x \text{ has the zero semantics}\}$$

is either finite or equal to $\mathbb{Q}$. (Hint: consider short words)

Problem 2. Show that the following problem is decidable:

1. **Input.** A weighted automaton, which defines a function $f : \Sigma^* \to \mathbb{Q}$;
2. **Question.** Is the function f commutative, i.e. $f(w)$ does not depend on the order of the letters in the input word w.

Problem 3. Consider a finite field $\mathbb{F}$. Show that weighted automata and polynomial automata compute the same functions of type $\Sigma^* \to \mathbb{F}$.

Problem 4. Show that the following problem is decidable:

1. **Input.** A deterministic register automaton, defining language $L \subseteq \mathbb{A}^*$;
2. **Question.** Does the language satisfy

   $$w \in L \quad \Leftrightarrow \quad \sigma(w) \in L$$

   for every function $\sigma : \mathbb{A} \to \mathbb{A}$, not necessarily a permutation.

1 Star problems

The deadline for these problems is until the last week of the lectures.

Problem 5. Show that for every n there exists a finite undirected graph G with the following properties:

1. For every vertices v and w, there is an automorphism of G that maps v and w. Recall that an automorphism of a graph is a permutation of its vertices that preserves the edges.

2. Every graph with at most n vertices is an induced subgraph of G.

Observe that the infinite random graph has the two properties above; but we are looking for a finite one.

Problem 6. Consider a vector addition system with a distinguished initial and final vector. Define a *coverability run* to be a run of the vector addition system which begins in the initial vector, and ends in a vector that is coordinatewise greater or equal to the final vector. If we equip the system with a function f from transitions to some finite alphabet Σ, then we get a language

$$\{f(w) \in \Sigma^* \mid w \text{ is a coverability run}\}.$$

Such a language is called a *coverability language*. Show that

$$L \text{ is regular} \quad \Leftrightarrow \quad \text{both } L \text{ and its complement are coverability languages}$$

Problem 7. (We do not know the solution to this problem, could be easy, could be hard.) For a polynomial automaton, which computes a function $f : \Sigma^* \to \mathbb{Q}$, consider the function

$$n \in \{0, 1, \ldots\} \quad \mapsto \quad \sum_{w \in \Sigma^n} f(w).$$

Prove or disprove: the new function is also computed by a polynomial automaton, assuming that numbers are represented as word over a one-letter alphabet.