

Languages, automata and computation II

Homework 1

Problems: deadline ~~22/11/2024~~ 24/11/2024

Problem 1. Consider a finite alphabet $\Sigma = \{a, b\}$. An ω -word is a sequence $a_0a_1 \dots$ of letters from Σ . Let Σ^ω be the set of all ω -words over Σ .

1. For two ω -words $u, v \in \Sigma^\omega$ we say that u *embeds into* v , written $u \sqsubseteq v$, if we can obtain u by deleting finitely or infinitely many letters from v . Show that embedding of ω -words is a well-quasi order.
2. Now consider the variant where only finitely many letters from v can be deleted. Is the resulting relation still a well-quasi order?

Problem 2. Consider the structure (\mathbb{N}^d, \leq) , where the order “ \leq ” is element-wise. Recall that a set $X \subseteq \mathbb{N}^d$ is *downward closed* if for every $x \in X$ and $y \leq x$ we have $y \in X$.

1. Show that any downward-closed set $X \subseteq \mathbb{N}^d$ is semilinear.
2. Provide an algorithm that solves the following problem: Given a VAS (without states) of dimension d and an initial configuration $x \in \mathbb{N}^d$, construct a semilinear representation for the downward closure of the set of configurations reachable from x .

Problem 3. Consider first-order logic over the structure (\mathbb{Q}, \leq) of the rational numbers together with their natural order. Let $\mathbb{Q}^{d\uparrow}$ be the set of *increasing d -tuples* of rational numbers (x_1, \dots, x_d) with $x_1 < \dots < x_d$. Find all first-order definable total orders on $\mathbb{Q}^{3\uparrow}$. (Such an order is a binary relation on triples, and therefore it is described by a first-order formula with six free variables.)

Problem 4. Consider the structures of *arithmetic* $(\mathbb{N}, +, \times)$ and of the *free monoid* $(\{0, 1\}^*, \cdot, 0, 1)$. Show that for every sentence of first-order logic φ of the free monoid one can compute a sentence of first-order logic ψ of arithmetic s.t. φ is true in the free monoid if and only if ψ is true in arithmetic.