Languages, automata and computation II
Tutorial 6 – Tarski's algebra

Winter semester 2024/2025

Exercise 1. 1. Show that the first-order theory of the complex numbers $(\mathbb{C}, +, \cdot, 0, 1)$ is decidable.

2. Does the first-order theory of the complex numbers admit elimination of quantifiers?

Exercise 2. Show that the first-order theory of the structure $(\mathbb{R}, +, \cdot, \sin)$ is undecidable.

Incidentally, decidability of the first-order theory of $(\mathbb{R}, +, \cdot, \exp)$ is an open problem. It has been shown to be decidabe by Maintyre and Wilkie subject to a number theoretic assumption known as *Schanuel's conjecture* [**?**].

Exercise 3. A structure $\mathbb{A} = (A, \leq, \dots)$ comprising a total order "$\leq$" on the domain A is called *o-minimal* if every first-order definable set $X \subseteq A$ is a finite union of single elements and intervals.

1. Show that the structure of the real numbers $(\mathbb{R}, +, \cdot)$ is o-minimal.
2. Is $(\mathbb{R}, +, \cdot, \sin)$ o-minimal?

Exercise 4. We say that a real number $a \in \mathbb{R}$ is *definable* if there exists a formula of the real numbers $\varphi(x)$ which is satisfied precisely when x is interpreted as a. Recall that a real number $a \in \mathbb{R}$ is *algebraic* if it satisfies a polynomial equation with rational number coefficients, i.e., $p(a) = 0$ for some univariate polynomial $\mathbb{Q}[x]$.

1. Show that algebraic real numbers are definable.
2. Show that definable real numbers are algebraic.
3. Show that the set of real algebraic numbers is a *subfield* of the set of real numbers. In other words: $0, 1$ are algebraic and algebraic numbers are closed under addition, multiplication, and multiplicative inverse (when defined).

Exercise 5. We show a natural example of a logic on $(\mathbb{N}, =)$ which has quantifier elimination however there is no algorithm that produces equivalent quantifier-free formulas. TODO.