Languages, automata and computation II
Tutorial 13 – ω-regular languages, distance automata

Winter semester 2024/2025

In this tutorial we explore ω-regular languages $L \subseteq \Sigma^*$. By definition, these are languages recognised by ω-regular expressions:

$$\begin{aligned} e, f &::= \varepsilon \mid a \mid e + f \mid e \cdot f \mid e^*, \\ g, h &::= g + h \mid e \cdot g \mid e^\omega. \end{aligned}$$

In the construction of e^ω we require that $\varepsilon \notin L(e)$.

Exercise 1. Show that a language $L \subseteq \Sigma^\omega$ is ω-regular iff it is recognised by a nondeterministic Büchi automaton (NBA).

Exercise 2. Are the following languages ω-regular?

1. ω-words with infinitely many prefixes in a fixed regular language of finite words $L \subseteq \Sigma^*$.
2. ω-words with infinitely many infixes of the form ab^pa with $p \in \mathbb{N}$ prime.
3. ω-words with infinitely many infixes of the form ab^na with $n \in \mathbb{N}_{>0}$ even.

Exercise 3. Show that a non-empty ω-regular language contains an ultimately periodic word.

Exercise 4 (Ultimately periodic words vs. runs)**.** An ultimately periodic run is labelled by an ultimately periodic word. Is it the case that if an NBA accepts an ultimately periodic word, then it has an accepting ultimately periodic *run* over this word?

Exercise 5. Are the following languages ω-regular?

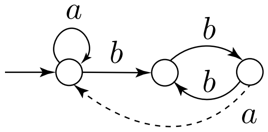

Figure 1: A deterministic Büchi automaton.

1. ω-words with arbitrarily long infixes from the regular language ab^*a.
2. ω-words with infinitely many prefixes from a fixed language $L \subseteq \Sigma^*$ (not necessarily regular).

Exercise 6. Show that "there exists a letter b" cannot be accepted by an NBA where all states are accepting (but some transitions may be missing).

Exercise 7. Show that the language "finitely many a's" cannot be accepted by a deterministic Büchi automaton.

Exercise 8. Are deterministic Büchi languages closed under union? intersection? complement?

Exercise 9. Show that nonemptiness is decidable for nondeterministic Büchi automata. Is the problem in PTIME? Can one do better in terms of computational complexity?

Exercise 10. Show that every language accepted by a nondeterministic Muller automaton is also accepted by some nondeterministic Büchi automaton.

Exercise 11. Show that if two ω-regular languages agree over the set of all ultimately periodic words, then they are equal.

Exercise 12. Is the set of all ultimately periodic words ω-regular?

Exercise 13. Show that nonemptiness is decidable for nondeterministic Muller automata.

Exercise 14. Consider the following binary operation on infinite words:

$$\begin{aligned} d &: \Sigma^\omega \times \Sigma^\omega \to \mathbb{Q} \\ d(u, v) &:= 2^{-n}, \end{aligned}$$

where n is the length of the longest common prefix of u, v.

- Show that d satisfies the axioms of a *metric*, and thus (Σ^ω, d) is a metric space.
- An *open set* in this space is a language $L \subseteq \Sigma^\omega$ s.t. for all $w \in L$ there is a radius $r > 0$ s.t. the open ball of radius r centered at w is also in L:

$$\{u \in \Sigma^\omega \mid d(w, u) < r\} \subseteq L.$$

  Find an open language which is not ω-regular and viceversa.
- Show that L is open iff $L = K \cdot \Sigma^\omega$ for some $K \subseteq \Sigma^*$.
- If additionally L is ω-regular, then show that K can be chosen to be regular. Conclude that an open ω-regular language is DBA recognisable.

Exercise 15. Reminiscent of the Myhill-Nerode characterisation of regular languages via congruences of finite index, we would like to characterise ω-regular languages L via equivalences $\sim_L$ s.t., for every language $L \subseteq \Sigma^\omega$,

$$L \text{ is } \omega\text{-regular iff } \sim_L \text{ has finite index.} \tag{1}$$

Which of the following candidates yields an equivalence satisfying (1)?

1. The equivalence $\sim_L$ on Σ^* defined as

$$u \sim_L v \quad \text{iff} \quad \forall w \in \Sigma^\omega : uw \in L \text{ iff } vw \in L.$$

2. The equivalence $\sim_L$ on Σ^ω defined as

$$u \sim_L v \quad \text{iff} \quad \forall w \in \Sigma^* : wu \in L \text{ iff } wv \in L.$$

3. The equivalence $\sim_L$ on Σ^+ defined as

   L: In the book it is on Σ^*, however in that case it is not clear what $s(ut)^\omega$ means when $ut = \varepsilon$.

$$u \sim_L v \quad \text{iff} \quad \begin{cases} \forall w \in \Sigma^\omega : uw \in L \text{ iff } vw \in L, \text{ and} \\ \forall s, t \in \Sigma^* : s(ut)^\omega \in L \text{ iff } s(vt)^\omega \in L. \end{cases}$$

Exercise 16. A language of ω-words is *prefix-independent* if

$$uv \in L \quad \text{iff} \quad v \in L, \qquad \text{for all } u \in \Sigma^*, v \in \Sigma^\omega.$$

1. Consider the definition of prefix-independence for finite words. Describe all the prefix-independent languages of finite words.

2. Describe all prefix-independent ω-regular languages.

Exercise 17 (Infinite Ramsey theorem)**.** Show that an infinite finitely coloured clique contains an infinite monochromatic clique. (The perhaps more familiar finite Ramsey theorem—any sufficiently large finitely colored clique contains a monochromatic clique—is implied by the infinite version by a compactness argument.)

Distance automata

Exercise 18. We say that a regular language L has the *finite power property* if there exists $n \in \mathbb{N}$ such that $L^* = L^0 \cup L^1 \cup \ldots \cup L^n$. Show that one can decide if a regular language has the finite power property.

Exercise 19. Show that it is decidable if a regular language is of star height one, i.e. it can be defined by a regular expression that uses Kleene star, maybe multiple times, but does not nest it.