# Languages, automata and computation II
# Tutorial 8 – Weighted automata

Winter semester 2024/2025

## Weighted finite automata

A *weighted finite automaton* (WFA, or *linear representation*) over $\Sigma$ is a triple $A = (x, M, y)$ where the transition matrix $M : \Sigma \to \mathbb{Q}^{k \times k}$ maps each letter $a \in \Sigma$ to a $k \times k$ rational matrix $M_a$, $x : \mathbb{Q}^{1 \times k}$ is a row vector, and $y : \mathbb{Q}^{k \times 1}$ is a column vector. The transition matrix $M$ is extended homomorphically to a function $\Sigma^* \to \mathbb{Q}^{k \times k}$ (where matrices form a ring with the usual notions of matrix sum and product). The semantics of a WFA is the function $f : \Sigma^* \to \mathbb{Q}$ s.t.

$$f(w) = x \cdot M(w) \cdot y, \quad \text{for every } w \in \Sigma^*.$$

Call a function *rational* if it is of the form above.

**Exercise 1.** Construct weighted automata over a unary alphabet, which for a word of length $n$ output

1. $n$;

2. $n^2$;

3. $n^k$ for $k \in \mathbb{N}$;

4. $p(n)$ for any polynomial $\mathbb{Q}[x]$;

5. $2^n$;

6. the $n$-th Fibonacci number $F_n$.

**Exercise 2.** Construct a weighted automaton that computes an injective function from $\{a, b\}^*$ to the positive rational numbers.

**Exercise 3.** Call an NFA unambiguous if for every input there is at most one accepting run. Show that equivalence problem for unambiguous automata can be decided in polynomial time.

*Remark:* Contrast this with the fact that equivalence for NFA's is PSPACE-complete in general.

**Exercise 4.** Show that for weighted automata with 2 states over a unary alphabet, it is decidable whether the automaton assigns value 0 to some word.

*Remark:* for weighted automata over a unary alphabet with an arbitrary number of states, this is an important open problem, called the Skolem Problem.

**Exercise 5.** Show that for every weighted automaton there is an isomorphic (using the notion of isomorphism inherited from vector space automata) one which has one initial and one final state.

**Exercise 6** (name of the game)**.** Consider the special case of a unary alphabet $\Sigma = \{a\}$. Define the *generating series* of $f : \mathbb{N} \to \mathbb{Q}$ to be

$$f(x) = \sum_{n=0}^{\infty} f(n) \cdot x^n$$

Show that if $f$ is rational iff its generating series $f(x)$ is a rational power series. Recall that rational power series are those which can be written as $p(x)/q(x)$ for two polynomials $p, q \in \mathbb{Q}[x]$.

## Series

**Exercise 7.** Show that the set of functions $\Sigma^* \to \mathbb{Q}$ can be given the structure of a vector space over $\mathbb{Q}$. What is its dimension?

Given a function $f : \Sigma^* \to \mathbb{Q}$ and a word $u \in \Sigma^*$, let the *left quotient* $u^{-1}f : \Sigma^* \to \mathbb{Q}$ be the function defined as

$$(u^{-1}f)(w) = f(uw), \quad \text{for every } w \in \Sigma^*.$$

Call a function $f$ *q-finite* if the set of left quotients

$$\left\{ u^{-1}f \mid u \in \Sigma^* \right\}$$

span a finite-dimensional subspace of $\Sigma^* \to \mathbb{Q}$.

**Exercise 8.** Show that $f$ is q-finite iff it is rational.