Languages, automata and computation II
Tutorial 1 – Semilinear sets

Winter semester 2024/2025

We consider $\mathbb{N}^d$ as a finitely generated free commutative additive monoid. We can definine several operations on subsets A, B of $\mathbb{N}^d$:

- Union $A \cup B$, intersection $A \cap B$, and complement $\mathbb{N}^d \setminus A$.
- Addition $A + B := \{a + b \mid a \in A, b \in B\}$.
- Iteration $\mathbb{N}A := A^* := \bigcup_{n \in \mathbb{N}} nA$, where $nA = A + A + \cdots + A$ (n times).
- Projection $\pi_i(A) := \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) \mid (a_1, \ldots, a_d) \in A\}$.

By convention, $0A = \{0\}$ if A is nonempty, otherwise $0\emptyset = \emptyset$.

Semilinear sets

A subset L of $\mathbb{N}^d$ is *linear* if it is of the form $b + P^*$ for some *basis vector* $b \in \mathbb{N}^d$ and a finite set of *periods* $P \subseteq \mathbb{N}^d$. A subset S of $\mathbb{N}^d$ is *semilinear* if it is a finite union of linear sets, $S = L_1 \cup \cdots \cup L_n$.

Exercise 1 (Closure properties)**.** Show that the semilinear sets are closed under union, addition, iteration, and projection.

Exercise 2. Find a subset of $\mathbb{N}$ which is not semilinear.

Exercise 3. Show that the set $\left\{(n, 2^n) \in \mathbb{N}^2 \mid n \in \mathbb{N}\right\}$ is not semilinear.

Exercise 4. Let $X \subseteq \mathbb{N}$ be an arbitrary (finite or infinite) set of natural numbers. Show that its iteration X^* is a semilinear set.

Exercise 5. Find a set $X \subseteq \mathbb{N}^2$ such that X^* is not semilinear.

Rational sets

In analogy with the rational subsets of the free monoid Σ^*, we say that a subset R of the free commutative monoid $\mathbb{N}^d$ is *rational* if it can be generated from finite sets using union "$\cup$", addition "$+$" and iteration "$_^*$".

Exercise 6 (Rational = semilinear)**.** Show that a subset of $\mathbb{N}^d$ is rational if, and only if, it is semilinear.

In view of the exercise, we can see the basis-periods representation of a semilinear set as a normal form for rational sets.

Büchi sets

We explore an automata-based approach to represent subsets of $\mathbb{N}^d$. For simplicity we deal with $d = 1$, however the whole approach generalizes to arbitrary d. Consider the alphabet $\Sigma = \{0, 1\}$. We see a word $w = a_0 \cdots a_n \in \Sigma^*$ as a binary encoding (least digit first) of the natural number

$$[w]_2 = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots + a_n \cdot 2^n \in \mathbb{N}. \tag{1}$$

Clearly every natural number as a (nonunique) binary encoding. Similarly, for every dimension $d \geq 1$ let $\Sigma_d := \Sigma^d$; then, a word $w \in \Sigma_d^*$ encodes a d-tuple of natural numbers $[w]_2 \in \mathbb{N}^d$. For instance, the word $w = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$ encodes the pair $(6, 5)$. In this way a language $L \subseteq \Sigma_d^*$ encodes a subset $[L]_2$ of $\mathbb{N}^d$, and reciprocally every subset X of $\mathbb{N}^d$ can be encoded as $X = [L]_2$ for some language $L \subseteq \Sigma_d^*$. A *Büchi subset of* $\mathbb{N}^d$ is one which is encoded by some regular language.

Exercise 7. Show that the set of numbers of the form 2^n with $n \in \mathbb{N}$ is a Büchi set.

Exercise 8. Show that the set of triples of natural numbers (a, b, c) s.t. $a+b = c$ is a Büchi subset of $\mathbb{N}^3$. Generalise the argument to triples of d-tuple of natural numbers.

Exercise 9. Show that Büchi sets are closed under union, addition, intersection, complement, and projection.

Exercise 10. Show that $P^* \subseteq \mathbb{N}^d$ is a Büchi set for any finite set of periods $P \subseteq \mathbb{N}^d$.

As a consequence, semilinear sets are a (strict) subset of Büchi sets.