# Languages, automata and computation II
# Homework 1

## Problems: deadline ~~22/11/2024~~ 24/11/2024

**Problem 1.** Consider a finite alphabet $\Sigma = \{a, b\}$. An *$\omega$-word* is a sequence $a_0 a_1 \cdots$ of letters from $\Sigma$. Let $\Sigma^\omega$ be the set of all $\omega$-words over $\Sigma$.

1. For two $\omega$-words $u, v \in \Sigma^\omega$ we say that $u$ *embeds into* $v$, written $u \sqsubseteq v$, if we can obtain $u$ by deleting finitely or infinitely many letters from $v$. Show that embedding of $\omega$-words is a well-quasi order.

2. Now consider the variant where only finitely many letters from $v$ can be deleted. Is the resulting relation still a well-quasi order?

**Problem 2.** Consider the structure $\left(\mathbb{N}^d, \leq\right)$, where the order "$\leq$" is element-wise. Recall that a set $X \subseteq \mathbb{N}^d$ is *downward closed* if for every $x \in X$ and $y \leq x$ we have $y \in X$.

1. Show that any downward-closed set $X \subseteq \mathbb{N}^d$ is semilinear.

2. Provide an algorithm that solves the following problem: Given a VAS (without states) of dimension $d$ and an initial configuration $x \in \mathbb{N}^d$, construct a semilinear representation for the downward closure of the set of configurations reachable from $x$.

**Problem 3.** Consider first-order logic over the structure $(\mathbb{Q}, \leq)$ of the rational numbers together with their natural order. Let $\mathbb{Q}^{d\uparrow}$ be the set of *increasing $d$-tuples* of rational numbers $(x_1, \ldots, x_d)$ with $x_1 < \cdots < x_d$. Find all first-order definable total orders on $\mathbb{Q}^{3\uparrow}$. (Such an order is a binary relation on triples, and therefore it is described by a first-order formula with six free variables.)

**Problem 4.** Consider the structures of *arithmetic* $(\mathbb{N}, +, \times)$ and of the *free monoid* $\left(\{0,1\}^*, \cdot, 0, 1\right)$. Show that for every sentence of first-order logic $\varphi$ of the free monoid one can compute a sentence of first-order logic $\psi$ of arithmetic s.t. $\varphi$ is true in the free monoid if and only if $\psi$ is true in arithmetic.