Languages, automata and computation II
Tutorial 8 – Weighted automata

Winter semester 2024/2025

Weighted finite automata

A *weighted finite automaton* (WFA, or *linear representation*) over Σ is a triple $A = (x, M, y)$ where the transition matrix $M : \Sigma \to \mathbb{Q}^{k\times k}$ maps each letter $a \in \Sigma$ to a $k \times k$ rational matrix M_a, $x : \mathbb{Q}^{1\times k}$ is a row vector, and $y : \mathbb{Q}^{k\times 1}$ is a column vector. The transition matrix M is extended homomorphically to a function $\Sigma^* \to \mathbb{Q}^{k\times k}$ (where matrices form a ring with the usual notions of matrix sum and product). The semantics of a WFA is the function $f : \Sigma^* \to \mathbb{Q}$ s.t.

$$f(w) = x \cdot M(w) \cdot y, \quad \text{for every } w \in \Sigma^*.$$

Call a function *rational* if it is of the form above.

Exercise 1. Construct weighted automata over a unary alphabet, which for a word of length n output

1. n;
2. n^2;
3. n^k for $k \in \mathbb{N}$;
4. $p(n)$ for any polynomial $\mathbb{Q}[x]$;
5. 2^n;
6. the n-th Fibonacci number F_n.

Exercise 2. Construct a weighted automaton that computes an injective function from $\{a, b\}^*$ to the positive rational numbers.

Exercise 3. Call an NFA unambiguous if for every input there is at most one accepting run. Show that equivalence problem for unambiguous automata can be decided in polynomial time.

Remark: Contrast this with the fact that equivalence for NFA's is PSPACE-complete in general.

Exercise 4. Show that for weighted automata with 2 states over a unary alphabet, it is decidable whether the automaton assigns value 0 to some word.

Remark: for weighted automata over a unary alphabet with an arbitrary number of states, this is an important open problem, called the Skolem Problem.

Exercise 5. Show that for every weighted automaton there is an isomorphic (using the notion of isomorphism inherited from vector space automata) one which has one initial and one final state.

Exercise 6 (name of the game)**.** Consider the special case of a unary alphabet $\Sigma = \{a\}$. Define the *generating series* of $f : \mathbb{N} \to \mathbb{Q}$ to be

$$f(x) = \sum_{n=0}^{\infty} f(n) \cdot x^n$$

Show that if f is rational iff its generating series $f(x)$ is a rational power series. Recall that rational power series are those which can be written as $p(x)/q(x)$ for two polynomials $p, q \in \mathbb{Q}[x]$.