Languages, automata and computation II
Tutorial 1 – Semilinear sets

Winter semester 2024/2025

We consider $\mathbb{N}^d$ as a finitely generated free commutative additive monoid. We can definine several operations on subsets A, B of $\mathbb{N}^d$:

- Union $A \cup B$, intersection $A \cap B$, and complement $\mathbb{N}^d \setminus A$.
- Addition $A + B = \{a + b \mid a \in A, b \in B\}$.
- Iteration $A^* = \bigcup_{n \in \mathbb{N}} nA$, where $nA = A + A + \cdots + A$ (n times).
- Projection $\pi_i(A) = \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) \mid (a_1, \ldots, a_d) \in A\}$.

By convention, $0A = \{0\}$ if A is nonempty, otherwise $0\emptyset = \emptyset$. In analogy with the rational subsets of the free monoid Σ^*, we say that a subset R of the free commutative monoid $\mathbb{N}^d$ is *rational* if it can be generated from finite sets using union "$\cup$", addition "$+$" and iteration "$\_^*$".

A subset L of $\mathbb{N}^d$ is *linear* if it is of the form $b + P^*$ for some *basis vector* $b \in \mathbb{N}^d$ and a finite set of *periods* $P \subseteq \mathbb{N}^d$. A subset S of $\mathbb{N}^d$ is *semilinear* if it is a finite union of linear sets, $S = L_1 \cup \cdots \cup L_n$.

Exercise 1 (TODO)**.** Show that a subset of $\mathbb{N}^d$ is rational if, and only if, it is semilinear.

Solution: The "if" direction is trivial. For the "only if" direction, we need to show that semilinear sets are closed under union, addition, and iteration. Union is trivial. For addition, it suffices to show that the sum of two linear sets is linear. This follows at once from the following identity, which is valid for every sets $A, B \subseteq \mathbb{N}^d$:

$$(A \cup B)^* = A^* + B^*. \tag{1}$$

For iteration, it suffices to show that the iteration L^* of a linear set L is semilinear. This follows at once from the following identity, which is valid for every sets $A, B \subseteq \mathbb{N}^d$:

$$(A + B^*)^* = \{0\} \cup (A + (A \cup B)^*). \tag{2}$$

Indeed, $(A + B^*)^{\geq 1} = (A + B^*) \cup (2A + B^*) \cup \cdots = A + A^* + B^*$, to which we apply (1) in reverse. $\square$

Exercise 2. Show that rational subsets of $\mathbb{N}^d$ are closed under complement.

Solution: TODO ☐

Exercise 3. Show that rational subsets of $\mathbb{N}^d$ are closed under intersection.

Solution: TODO ☐

Exercise 4. Show that rational subsets of $\mathbb{N}^d$ are closed under projection.

Solution: TODO ☐

References