

# Languages, automata and computation II

## Tutorial 10 – Ideals, varieties, and polynomial automata

Winter semester 2024/2025

In this tutorial we explore ideals, varieties, and polynomial automata. Recall that an *ideal* of a ring  $R$  is a subset  $I \subseteq R$  which is 1) closed under sum  $I + I \subseteq I$ , and 2) closed under product with elements from the ring  $R \cdot I \subseteq I$ . For a set of vectors  $A \subseteq \overline{\mathbb{Q}}^k$ , let  $I(A) \subseteq \overline{\mathbb{Q}}[x_1, \dots, x_k]$  be the set of polynomials vanishing on  $A$ . (This is an ideal, justifying the notation). For a set of polynomials  $P \subseteq \overline{\mathbb{Q}}[x_1, \dots, x_k]$ , let  $V(P) \subseteq \overline{\mathbb{Q}}^k$  be the set of vectors where all polynomials in  $P$  vanish simultaneously. The *Zariski closure* of a set of vectors  $A$  is defined as

$$\overline{A} := V(I(A)).$$

**Exercise 1.** 1. Show that  $A \subseteq \overline{A}$ , for every  $A \subseteq \overline{\mathbb{Q}}^k$ .

2. Find a set of vectors  $A \subseteq \overline{\mathbb{Q}}^k$  where the inclusion in the previous point is strict. Can such a set  $A$  be finite?

**Exercise 2** (zero polynomial vs. zero polynomial function). Show that  $p : \overline{\mathbb{Q}}[x_1, \dots, x_k] \rightarrow \overline{\mathbb{Q}}$  is the zero polynomial iff as a function  $\overline{\mathbb{Q}}^k \rightarrow \overline{\mathbb{Q}}$  it is constantly zero. Is this true if we replace  $\overline{\mathbb{Q}}$  by  $\mathbb{F}_2$  (the field consisting just of the elements  $\{0, 1\}$ )?

**Exercise 3.** Show that  $I$  is an ideal of  $R[x]$  iff  $I$  is a vector subspace of  $R[x]$  over  $R$  (i.e.,  $I + I \subseteq I$  and  $aI \subseteq I$  for every  $a \in R$ ) s.t.  $xI \subseteq I$ .

**Exercise 4.** We have seen that  $I(A)$  is an ideal of  $\overline{\mathbb{Q}}[x_1, \dots, x_k]$  for every  $A \subseteq \overline{\mathbb{Q}}^d$ . Is every ideal of this ring of this form?

### Principal ideal rings

Recall that a ring  $R$  is a *principal ideal ring* if every ideal of  $R$  is generated by one element.

**Exercise 5.** Are the following principal ideal rings?

1. The field of rational numbers  $\mathbb{Q}$ .
2. The ring of integers  $\mathbb{Z}$ .

3. The ring of univariate polynomials over the rationals  $\mathbb{Q}[x]$ .
4. The ring of univariate polynomials over the integers  $\mathbb{Z}[x]$ .
5. The ring of bivariate polynomials over the rationals  $\mathbb{Q}[x, y]$ .
6. The quotient ring  $\mathbb{Q}[x, y]/\langle x - y \rangle$ .

**Exercise 6.** If  $R$  is a principal ideal ring, does the same hold for  $R[x]$ ?

### Noetherian rings

A ring  $R$  is *Noetherian* if every ideal  $I \subseteq R$  is finitely generated. In the following problem we explore ways to construct Noetherian rings.

**Exercise 7.** 1. Fields are Noetherian.

2. Finite rings are Noetherian.
3. Principal ideal rings are Noetherian.
4. If  $R$  is Noetherian and  $I \subseteq R$  is an ideal, then  $R/I$  is Noetherian.
5. If  $R$  is Noetherian, then  $R[x]$  is Noetherian. Does the converse hold?
6. If  $R$  is Noetherian, then  $R[[x]]$  is Noetherian.

**Exercise 8.** Are the following rings Noetherian?

1. Ring of polynomials with countably many variables:  $\mathbb{Q}[x_1, x_2, \dots]$ .
2. Ring of power series:  $\mathbb{Q}[[x]]$ .
3. Ring of rational power series:  $\mathbb{Q}[[x]] \cap \mathbb{Q}(x)$ .
4. Noncommutative ring of power series in noncommuting variables:  $R := \Sigma^* \rightarrow \mathbb{Q}$ ,  $|\Sigma| \geq 2$ , with sum and convolution (Cauchy) product.

**Exercise 9.** For every of the following sets  $A$  check that it is a ring. Is it Noetherian?

1.  $A = x \cdot \mathbb{Q}[x] \subseteq \mathbb{Q}[x]$ .
2.  $A = \mathbb{Z} + x \cdot \mathbb{Q}[x] \subseteq \mathbb{Q}[x]$ .

### Polynomial automata

Recall that a *polynomial automaton* is a tuple

$$A = (d, \Sigma, Q, q_I, p, F)$$

where  $d \in \mathbb{N}$  is the *dimension*,  $\Sigma$  is a finite alphabet,  $Q = \overline{\mathbb{Q}}^d$  is the set of *states*,  $q_I \in Q$  is the *initial state*,  $p : \Sigma \rightarrow \mathbb{Q}[d]^d$  is a collection of tuples of polynomials inducing a polynomial action on states

$$q \in Q \mapsto q \cdot a \in Q, \quad \text{for every } a \in \Sigma,$$

where  $q \cdot a := p^a(q) = (p_1^a(q), \dots, p_d^a(q))$ , and  $F : Q \rightarrow \overline{\mathbb{Q}}$  is the polynomial output function. The action of  $\Sigma$  is extended to words  $w \in \Sigma^*$  homomorphically:  $q \cdot \varepsilon := q$  and  $q \cdot (a \cdot w) := (q \cdot a) \cdot w$ . The semantics of state  $q \in Q$  is the mapping  $\llbracket q \rrbracket : \Sigma^* \rightarrow \overline{\mathbb{Q}}$  defined as

$$\llbracket q \rrbracket_w = F(q \cdot w), \quad \text{for every } w \in \Sigma^*.$$

The semantics of the automaton  $A$  is  $\llbracket A \rrbracket = \llbracket q_I \rrbracket$ . The automaton is *zero* if  $\llbracket A \rrbracket = 0$ .

The set  $\Sigma^* \rightarrow \overline{\mathbb{Q}}$  has the structure of a commutative ring w.r.t. addition and Hadamard product (pointwise product). This gives us an alternative presentation of the semantics of polynomial automata.

**Exercise 10.** A polynomial automaton  $A$  over a unary alphabet  $\{a\}$  computes a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  if  $\llbracket A \rrbracket_{a^n} = f(n)$  for  $n \in \mathbb{N}$ . Give examples of polynomial automata over a unary alphabet that compute the following functions:

- $f(n) = n!$
- $f(n) = 2^{2^n}$
- any rational function (computable by a unary weighted automaton).

**Exercise 11.** Show that a function  $f : \Sigma^* \rightarrow \overline{\mathbb{Q}}$  is recognisable by a polynomial automaton iff its reversal  $f^R$  belongs to a finitely generated subring of  $\Sigma^* \rightarrow \overline{\mathbb{Q}}$  closed under left quotients  $u^{-1}(\cdot)$ .

**Exercise 12.** Consider the following computational model  $B$ . States are tuples of polynomials  $S = \mathbb{Q}[d]^d$ , with  $p_I := (x_1, \dots, x_d)$  being the initial state and  $F : S \rightarrow \overline{\mathbb{Q}}$  a polynomial output function. The update function is described by a tuple of polynomials  $p^a = (p_1^a, \dots, p_d^a) \in \mathbb{Q}[d]^d$ , one for each  $a \in \Sigma$ , by polynomial substitution as follows:

$$p \mapsto p \cdot a := p(p^a) \in S, \quad \text{for every } a \in \Sigma.$$

In other words, in the current state  $p$  we replace  $x_1$  by  $p_1^a$ , ...,  $x_d$  by  $p_d^a$ . This is extended homomorphically to  $\Sigma^* \rightarrow S$ . For instance  $p_I \cdot ab = p^a(p^b)$  and  $p_I \cdot abc = p^a(p^b)(p^c) = p^a(p^b(p^c))$ . The output on reading  $w$  is  $\llbracket B \rrbracket_w = F(p_I \cdot w)$ . Decide zeroness for  $B$ .

A *variety* is a subset  $V \subseteq \overline{\mathbb{Q}}^d$  which is the set of common zeros of a set of polynomials:  $V = V(P)$  for some  $P \subseteq \mathbb{Q}[x_1, \dots, x_d]$ .

**Exercise 13.** For  $d = 2$  find a non-trivial infinite variety.

**Exercise 14.** Let  $V \subseteq \overline{\mathbb{Q}}^d$  be a variety.

1. Let  $g : \overline{\mathbb{Q}}^e \rightarrow \overline{\mathbb{Q}}^d$  be a polynomial map. Is  $g^{-1}(V) \subseteq \overline{\mathbb{Q}}^e$  a variety?
2. Let  $g : \overline{\mathbb{Q}}^d \rightarrow \overline{\mathbb{Q}}^e$  be a polynomial map. Is  $g(V) \subseteq \overline{\mathbb{Q}}^e$  a variety?

**Exercise 15.** Show that all real varieties  $V \subseteq \mathbb{R}^d$  are generated by a single polynomial. Is this true for varieties of  $\mathbb{C}^d$ ?

**Exercise 16.** Let  $U, V \subseteq \overline{\mathbb{Q}}^d$  be varieties. Are the following varieties?

1.  $U \cap V$ . What about possibly infinite intersections?
2.  $U \cup V$ . What about possibly infinite intersections?
3.  $U \setminus V$ .

**Exercise 17.** For each direction of the statements below, prove it if it holds, or find a counter-example otherwise.

1. For sets  $A, B \subseteq \overline{\mathbb{Q}}^d$ :  $A \subseteq B$  iff  $I(B) \subseteq I(A)$ .
2. For sets of polynomials  $P, Q \subseteq \overline{\mathbb{Q}}[x_1, \dots, x_d]$ :  $P \subseteq Q$  iff  $V(Q) \subseteq V(P)$ .
3. For varieties  $U, V \subseteq \overline{\mathbb{Q}}^d$ :  $U \subseteq V$  iff  $I(V) \subseteq I(U)$ .

In the next problem we explore an algorithm to decide zeroness of polynomial automata.

**Exercise 18.** Consider the set of states  $V_n \subseteq Q$  which give zero after reading words of length  $\leq n$ :

$$V_n = \{q \in Q \mid \forall w \in \Sigma^{\leq n} : q \cdot w = 0\}.$$

1. Show that the automaton is zero iff  $q_I \in \bigcap_n V_n$ .
2. Show that  $V_0 \supseteq V_1 \supseteq \dots \supseteq \bigcap_n V_n$  is a nonincreasing chain of varieties. Conclude that the chain stabilises at some finite level: There is  $N \in \mathbb{N}$  s.t.  $V_N = V_{N+1} = \dots = \bigcap_n V_n$ .
3. Show that for every  $n \in \mathbb{N}$ ,  $V_n = V_{n+1}$  implies  $n = N$ .
4. Let  $P_n$  be a finite set of polynomials s.t.  $V_n = V(P_n)$ . Show that we can compute a finite set of polynomials  $P_{n+1}$  s.t.  $V_{n+1} = V(P_{n+1})$ .
5. Show how to decide  $V(P) = V(Q)$  for two finite set of polynomials  $P, Q \subseteq \overline{\mathbb{Q}}[d]$ .
6. Conclude with an algorithm for zeroness.

**Exercise 19.** Provide a coRP algorithm (randomised polynomial time) for the following problem: Given a polynomial automaton  $A$  and an input word  $w \in \Sigma^*$ , decide whether  $\llbracket A \rrbracket_w = 0$ .

**Exercise 20.** Give an algorithm for the following problem: In input we are given a polynomial automaton  $A$  and a finite automaton  $B$  recognising a regular language  $L \subseteq \Sigma^*$ . In output we answer whether for every  $w \in L$  we have  $\llbracket A \rrbracket_w = 0$ .