

Languages, automata and computation II

Tutorial 11 – Ordinal numbers, register automata, orbit-finite sets

Winter semester 2024/2025

Ordinal numbers

Ordinal numbers generalize natural numbers and can be seen as representations of well-founded total orders. For a set A ordered by \leq we denote by $\text{tp}(A, \leq)$ (or just $\text{tp}(A)$) the ordinal number representing this relation. Two well-founded total orders (A, \leq_A) , (B, \leq_B) are isomorphic if and only if $\text{tp}(A) = \text{tp}(B)$. A number $n \in \mathbb{N}$ represents a totally ordered finite set with n elements, i.e. $\text{tp}(A) = n$ if $|A| = n$. By ω we denote the ordinal type of the set of natural numbers with the usual order relation, i.e. $\text{tp}(\mathbb{N}) = \omega$. The class of ordinal numbers is denoted by **ON**. They have precise set theoretic definition which we will not mention it here.

Let (A, \leq) , (B, \leq) be well-founded total orders represented by respectively $\alpha, \beta \in \mathbf{ON}$. We write $\alpha \leq \beta$ if can (A, \leq) be embedded into (B, \leq) . We also define the following operations:

- $\alpha + \beta = \text{tp}(\{0\} \times A \cup \{1\} \times B, \preceq)$, where $(i, x) \preceq (j, y)$ if $i < j$ or $(i = j \text{ and } x \leq y)$
- $\alpha \cdot \beta = \text{tp}(B \times A, \preceq_{lex})$
- $\alpha^\beta = \text{tp}(F(A, B), \preceq)$, where $F(A, B)$ is the set of all functions $B \rightarrow A$ with finite support and $f \preceq g$ if $f = g$ or $f(\xi) < g(\xi)$, where $\xi = \max\{\eta \in B : f(\eta) \neq g(\eta)\}$

Consecutive ordinal numbers are: $0, 1, 2, \dots, \omega, \omega+1, \dots, \omega \cdot 2, \dots, \omega^2, \dots, \omega^\omega, \dots$

Exercise 1. Prove that for $\alpha, \beta, \gamma \in \mathbf{ON}$

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
3. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
4. $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
5. $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$

Exercise 2. Show the following equalities/inequalities:

1. $1 + \omega = \omega < \omega + 1$
2. $2 \cdot \omega = \omega < \omega \cdot 2$
3. $2^\omega = 3^\omega = \omega$

Register automata

Let $(\mathbb{A}, =)$ be the set of data values with equality as the only relation (one can consider other data structures). An *nondeterministic register automaton (NRA)* is a tuple $A = (\Sigma, R, L, I, F, \Delta)$ where Σ is an orbit-finite *input alphabet* (often $\Sigma = \mathbb{A}$), R is a finite set of registers, L is a finite set of *locations*, $I, F \subseteq L$ are *initial*, resp., *final* locations, and $\Delta \subseteq (L \times \mathbb{A}^R) \times \Sigma \times (L \times \mathbb{A}^R)$ is an equivariant set of *transitions* (thus orbit-finite). A *state* of a register automaton is a location together with register valuation, i.e. $Q = L \times \mathbb{A}^R$.

Exercise 3. Show that following languages are recognizable by register automata:

1. the first data value is the same as the last data value;
2. some data value appears twice;
3. the last data value appears again;
4. every three consecutive data values are pairwise distinct.

Exercise 4. Show that the class of languages recognizable by NRA is not closed under complement.

Exercise 5. Show that a nondeterministic register automaton can recognize the language "some data value appears twice", but a deterministic one cannot.

Exercise 6. Call a nondeterministic register automaton *guessing* if there exists a transition $t \in \Delta$ such that some data value in the target state appears neither in the source state nor in the input. Give an example of a language that needs guessing to be recognized.

A corollary of the above two exercises is that:

$$\text{deterministic} \subsetneq \text{nondeterministic without guessing} \subsetneq \text{nondeterministic}$$

Exercise 7. Prove the following (non)-closure properties of languages of finite data words recognised by NRA and its deterministic variant.

	\cup	\cap	--^R	$\Sigma^* \setminus \text{--}$
Deterministic NRA	✓	✓	×	✓
Nondeterministic NRA	✓	✓	✓	×

Exercise 8. The complexity of the emptiness problem for nondeterministic register automata depends on how the size $|A|$ of the input automaton is measured. Show that emptiness is:

- PSPACE-complete if $|A|$ is the number of locations and registers;

- NP-complete if $|A|$ is the number of reachable orbits of states;
- polynomial time if $|A|$ is the number of orbits of transitions.

Exercise 9. Consider the set of atoms (\mathbb{Q}, \leq) . Show that there exists a language closed under permutation of atoms (not only automorphism of \mathbb{Q}) that is recognizable by an NRA over (\mathbb{Q}, \leq) without guessing, but is not recognizable by an NRA over $(\mathbb{Q}, =)$ without guessing.

Sets with atoms

In this tutorial we explore sets with atoms. Fix a countable set A and consider the relational structure $\mathbb{A} = (A, =)$ over the signature consisting only of equality. Elements of \mathbb{A} are called *atoms* and automorphisms α of this structure are the bijections of A . A *set with atoms* is any set built from set constructors and atoms from \mathbb{A} . For an automorphism α of \mathbb{A} and a set with atoms x we extend α to x as $\alpha(x) = \{\alpha(y) \mid y \in x\}$. An automorphism α *fixes* a set with atoms x if $\alpha(x) = x$. Fix a tuple $\bar{a} \in A^n$. An \bar{a} -*automorphism* is an automorphism fixing \bar{a} . A set with atoms x is *supported by \bar{a}* if x is fixed by every \bar{a} -automorphism, and it is *equivariant* if it is supported by the empty set. A set with atoms is *finitely supported* if it is supported by some finite tuple of atoms, and it is *legal* if it is finitely supported and all its elements are legal (this is a recursive definition).

The *orbit* of element x is the set $\text{orbit}(x) = \{\alpha(x) \mid \text{automorphism } \alpha\}$ of elements which can be obtained by applying some automorphism to x . Thus an equivariant set x is a union of orbits (of its elements), and an equivariant set is *orbit finite* if this union is finite. The set of orbits of a set x is $\text{Orbits}(x) = \{\text{orbit}(y) \mid y \in x\}$ (this is a partition of x). For instance, for given $a, b \in \mathbb{A}$ with $a \neq b$, we have $\text{orbit}(a, a) = \{(c, d) \in \mathbb{A}^2 \mid c = d\}$ and $\text{orbit}(a, b) = \{(c, d) \in \mathbb{A}^2 \mid c \neq d\}$. Since there are no more orbits in \mathbb{A}^2 , we have $\text{Orbits}(\mathbb{A}^2) = \{\text{orbit}(a, a), \text{orbit}(a, b)\}$ and \mathbb{A}^2 is the union of two orbits.

Exercise 10. Fix the equality atoms $(\mathbb{A}, =)$. For each of the following sets with atoms decide whether they are 1) legal, 2) equivariant, 3) orbit finite.

1. \mathbb{A}^2 .
2. $\mathbb{A}^* := \mathbb{A}^0 \cup \mathbb{A}^1 \cup \mathbb{A}^2 \cup \dots$.
3. $\mathbb{A}^\omega := \{a_1 a_2 \dots \mid a_1, a_2, \dots \in \mathbb{A}\}$.
4. $2^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}\}$ (powerset).
5. $2_{\text{fin}}^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}, B \text{ finite}\}$ (finite powerset).
6. $2_{\text{fs}}^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}, B \text{ finitely supported}\}$ (finitely supported powerset).
7. $2_{\text{eq}}^{\mathbb{A}} := \{B \mid B \subseteq \mathbb{A}, B \text{ equivariant}\}$ (equivariant powerset).

Exercise 11. An atom structure \mathbb{A} is called *oligomorphic* if \mathbb{A}^n is orbit-finite for every $n \in \mathbb{N}$. Are the following atom structures oligomorphic?

1. $(\mathbb{N}, =)$.
2. (\mathbb{Z}, \leq) .

3. (\mathbb{Q}, \leq) .

4. $(\mathbb{Q}, +)$.