

# Languages, automata and computation II

## Homework 2

### Problems: deadline 10/01/2025

**Problem 1.** A *parametric finite weighted automaton* is a finite weighted automaton where the initial vector, the final vector, and the transition matrices instead of using values from the field  $\mathbb{Q}$ , use polynomials with one variable  $x$ . For every choice of  $x \in \mathbb{Q}$ , this gives us a finite weighted automaton in the usual sense, by evaluating the polynomials to get values in the field. Show that the set

$$\{x \in \mathbb{Q} \mid \text{the automaton with parameter } x \text{ has the zero semantics}\}$$

is either finite or equal to  $\mathbb{Q}$ . (Hint: consider short words)

**Problem 2.** Show that the following problem is decidable:

1. **Input.** A weighted automaton, which defines a function  $f : \Sigma^* \rightarrow \mathbb{Q}$ ;
2. **Question.** Is the function  $f$  commutative, i.e.  $f(w)$  does not depend on the order of the letters in the input word  $w$ .

**Problem 3.** Consider a finite field  $\mathbb{F}$ . Show that weighted automata and polynomial automata compute the same functions of type  $\Sigma^* \rightarrow \mathbb{F}$ .

### Star problems

The deadline for these problems is until the last week of the lectures.

(\*) **Problem 4.** Show that for every  $n$  there exists a finite undirected graph  $G$  with the following properties:

1. For every vertices  $v$  and  $w$ , there is an automorphism of  $G$  that maps  $v$  and  $w$ . Recall that an automorphism of a graph is a permutation of its vertices that preserves the edges.
2. Every graph with at most  $n$  vertices is an induced subgraph of  $G$ .

Observe that the infinite random graph has the two properties above; but we are looking for a finite one.

(\*) **Problem 5.** Consider a vector addition system with states with a distinguished initial and final configuration. Define a *coverability run* to be a run

which begins in the initial state and ends in a configuration that is coordinate-wise greater or equal to the final configuration (and the state is the same). If we equip the system with a function  $f$  from transitions to some finite alphabet  $\Sigma$ , then we get a language

$$\{f(w) \in \Sigma^* \mid w \text{ is a coverability run}\}.$$

Such a language is called a *coverability language*. Show that

$$L \text{ is regular} \iff \text{both } L \text{ and its complement are coverability languages}$$

## Open problems

The deadline for these problems is until the last week of the lectures.

**Open problem 1.** (We do not know the solution to this problem, could be easy, could be hard.) For a polynomial automaton, which computes a function  $f : \Sigma^* \rightarrow \mathbb{Q}$ , consider the function

$$n \in \{0, 1, \dots\} \mapsto \sum_{w \in \Sigma^n} f(w).$$

Prove or disprove: The new function above is also computed by a polynomial automaton, assuming that numbers are represented as words over a one-letter alphabet.