

Languages, automata and computation II

Tutorial 1 – Semilinear sets

Winter semester 2024/2025

We consider \mathbb{N}^d as a finitely generated free commutative additive monoid. We can define several operations on subsets A, B of \mathbb{N}^d :

- Union $A \cup B$, intersection $A \cap B$, and complement $\mathbb{N}^d \setminus A$.
- Addition $A + B := \{a + b \mid a \in A, b \in B\}$.
- Iteration $\mathbb{N}A := A^* := \bigcup_{n \in \mathbb{N}} nA$, where $nA = A + A + \dots + A$ (n times).
- Projection $\pi_i(A) := \{(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d) \mid (a_1, \dots, a_d) \in A\}$.

By convention, $0A = \{0\}$ if A is nonempty, otherwise $0\emptyset = \emptyset$.

Semilinear sets

A subset L of \mathbb{N}^d is *linear* if it is of the form $b + P^*$ for some *basis vector* $b \in \mathbb{N}^d$ and a finite set of *periods* $P \subseteq \mathbb{N}^d$. A subset S of \mathbb{N}^d is *semilinear* if it is a finite union of linear sets, $S = L_1 \cup \dots \cup L_n$.

Exercise 1 (Closure properties). Show that the semilinear sets are closed under union, addition, iteration, and projection.

Exercise 2. Show that the set $\{(n, 2^n) : n \in \mathbb{N}\}$ is not semilinear.

Exercise 3. Let $X \subseteq \mathbb{N}$ be an arbitrary (finite or infinite) set of natural numbers. Show that its iteration X^* is a semilinear set.

Exercise 4. Find a set $X \subseteq \mathbb{N}^2$ such that X^* is not semilinear.

Rational sets

In analogy with the rational subsets of the free monoid Σ^* , we say that a subset R of the free commutative monoid \mathbb{N}^d is *rational* if it can be generated from finite sets using union “ \cup ”, addition “ $+$ ” and iteration “ $_*$ ”.

Exercise 5 (Rational = semilinear). Show that a subset of \mathbb{N}^d is rational if, and only if, it is semilinear.

In view of the exercise, we can see the basis-periods representation of a semilinear set as a normal form for rational sets.

Complementation

Exercise 6. Let $p \in \mathbb{N}^d$ be a period and consider the linear set $L = \mathbb{N}p$. Show that the complement $M := \mathbb{N}^d \setminus L$ is semilinear.