

# Languages, automata and computation II

## Homework 1

### Problems: deadline ~~22/11/2024~~ 24/11/2024

**Problem 1.** Consider a finite alphabet  $\Sigma = \{a, b\}$ . An  $\omega$ -word is a sequence  $a_0a_1\cdots$  of letters from  $\Sigma$ . Let  $\Sigma^\omega$  be the set of all  $\omega$ -words over  $\Sigma$ .

1. For two  $\omega$ -words  $u, v \in \Sigma^\omega$  we say that  $u$  *embeds into*  $v$ , written  $u \sqsubseteq v$ , if we can obtain  $u$  by deleting finitely or infinitely many letters from  $v$ . Show that embedding of  $\omega$ -words is a well-quasi order.
2. Now consider the variant where only finitely many letters from  $v$  can be deleted. Is the resulting relation still a well-quasi order?

**Problem 2.** Consider the structure  $(\mathbb{N}^d, \leq)$ , where the order “ $\leq$ ” is element-wise. Recall that a set  $X \subseteq \mathbb{N}^d$  is *downward closed* if for every  $x \in X$  and  $y \leq x$  we have  $y \in X$ .

1. Show that any downward-closed set  $X \subseteq \mathbb{N}^d$  is semilinear.
2. Provide an algorithm that solves the following problem: Given a VAS (without states) of dimension  $d$  and an initial configuration  $x \in \mathbb{N}^d$ , construct a semilinear representation for the downward closure of the set of configurations reachable from  $x$ .

**Problem 3.** Consider first-order logic over the structure  $(\mathbb{Q}, \leq)$  of the rational numbers together with their natural order. Let  $\mathbb{Q}^{d\uparrow}$  be the set of *increasing  $d$ -tuples* of rational numbers  $(x_1, \dots, x_d)$  with  $x_1 < \dots < x_d$ . Find all first-order definable total orders on  $\mathbb{Q}^{3\uparrow}$ . (Such an order is a binary relation on triples, and therefore it is described by a first-order formula with six free variables.)

**Problem 4.** Consider the structures of *arithmetic*  $(\mathbb{N}, +, \times)$  and of the *free monoid*  $(\{0, 1\}^*, \cdot, 0, 1)$ . Show that for every sentence of first-order logic  $\varphi$  of the free monoid one can compute a sentence of first-order logic  $\psi$  of arithmetic s.t.  $\varphi$  is true in the free monoid if and only if  $\psi$  is true in arithmetic.