Languages, automata and computation II
Tutorial 9 – Weighted automata (2)

Winter semester 2024/2025

Closure properties of rational series

Reminder on rational series. A *linear representation* over Σ is a triple $A = (x, M, y)$ where the transition matrix $M : \Sigma \to \mathbb{Q}^{k \times k}$ maps each letter $a \in \Sigma$ to a $k \times k$ rational matrix M_a, $x : \mathbb{Q}^{1 \times k}$ is a row vector, and $y : \mathbb{Q}^{k \times 1}$ is a column vector. The transition matrix M is extended homomorphically to a function $\Sigma^* \to \mathbb{Q}^{k \times k}$ (where matrices form a ring with the usual notions of matrix sum and product). The semantics of a WFA is the series $f : \Sigma^* \to \mathbb{Q}$ s.t.

$$f(w) = x \cdot M(w) \cdot y, \quad \text{for every } w \in \Sigma^*.$$

Call a function *rational* if it is of the form above.

Exercise 1. Show that the set of rational series is closed under the following operations.

1. Multiplication by a constant: $(c \cdot f)(w) := c \cdot f(w)$, $c \in \mathbb{Q}$.
2. Addition: $(f + g)(w) := f(w) + g(w)$.
3. Pointwise product (Hadamard product): $(f \cdot g)(w) := f(w) \cdot g(w)$.
4. Concatenation product (Cauchy product): $(f * g)(w) := \sum_{uv=w} f(u) \cdot g(v)$.
5. Iteration (w.r.t. concatenation), when $f(\varepsilon) = 0$: $f^* := f^0 + f^1 + f^2 + \cdots$, where $f^0(w)$ is 1 if $w = \varepsilon$ and 0 otherwise, and $f^{n+1} = f^n * f$ for every $n \geq 0$.

Exercise 2 (Concatenation inverses)**.** Under which condition does f have an inverse w.r.t. the concatenation product? In the positive case, find an expression for the inverse.

Regular expressions

Exercise 3 (Kleene-Schützenberger theorem)**.** Call a function *regular* if it can be generated by the following abstract grammar

$$f, g ::= p \mid \alpha \cdot f \mid f + g \mid f * g \mid f^*,$$

where p is a polynomial (function with finite support) and iteration f^* is only applied when defined. Show that a function is regular iff it is rational.

Supports

The *support* of a function $f : \Sigma^* \to \mathbb{Q}$, denoted $\mathsf{supp}\, f$, is the set of words where f is nonzero. A *rational support* is the support of a rational function. Since we do not consider any other kind of support, we just say "support" for "rational support" in the following.

Exercise 4.

1. Show that the class of supports includes all regular languages.
2. Are there nonregular supports?

Exercise 5. Are the following problems decidable for supports:

1. emptiness?
2. universality?
3. equivalence?
4. inclusion?

Exercise 6. Are supports closed under

1. intersection?
2. union?
3. concatenation?
4. Kleene star?
5. complement?