Languages, automata and computation II
Homework 1

Problems: deadline XX/11/2024

Problem 1. Consider a finite alphabet $\Sigma = \{a, b\}$. An *ω-word* is a sequence $a_0 a_1 \cdots$ of letters from Σ. Let Σ^ω be the set of all infinite words over Σ.

1. For two ω-words $u, v \in \Sigma^\omega$ we say that *u embeds into v*, written $u \sqsubseteq v$, if we can obtain u by deleting finitely or infinitely many letters from v. Show that embedding of ω-words is a well-quasi order.

2. Now consider the variant where only finitely many letters from v can be deleted. Is the resulting relation still a well-quasi order?

Problem 2. Consider the structure $(\mathbb{N}^d, \leq)$, where the order "$\leq$" is element-wise. Recall that a set $X \subseteq \mathbb{N}^d$ is *downward closed* if for every $x \in X$ and $y \leq x$ we have $y \in X$.

1. Show that any downward-closed set $X \subseteq \mathbb{N}^d$ is semilinear.

2. Provide an algorithm that solves the following problem: Given a VAS (without states) of dimension d and an initial configuration $x \in \mathbb{N}^d$, construct a semilinear representation for the downward closure of the set of configurations reachable from x.

Problem 3. Consider first-order logic over the structure $(\mathbb{Q}, \leq)$ of the rational numbers together with their natural order. Let $\mathbb{Q}^{d\uparrow}$ be the set of *increasing d-tuples* of rational numbers $(x_1, \ldots, x_d)$ with $x_1 < \cdots < x_d$. Find all first-order definable total orders on $\mathbb{Q}^{3\uparrow}$. (Such an order is a binary relation on triples, and therefore it is described by a first-order formula with six free variables.)

Problem 4. Consider the following two structures:

$$A = \left(\{0,1\}^*, \cdot, 0, 1\right) \quad \text{and} \quad B = (\mathbb{N}, +, \cdot)\,.$$

In structure A, the binary operation $\cdot$ is the concatenation of words.

1. Provide an algorithm s.t. given in input a first-order sentence φ over A it produces in output a first-order sentence ψ over B s.t. φ is true in A iff ψ is true in B.

2. Provide an algorithm as in the previous point with the roles of A and B swapped.

Problem 5. Consider the structures $(\mathbb{N}, +, \times)$ and $\left(\{0,1\}^*, \cdot, 0, 1\right)$. Show reductions in both directions, in the following sense: for every formula φ in one of the structures, one can compute a formula φ' in the other structure such that φ is true in the first structure if and only if φ' is true in the second structure. (Hint: show that Turing machines can be simulated by the free monoid.)