Languages, automata and computation II
Tutorial 7 – Ra(n)do(m) graph and zero-one laws

Winter semester 2024/2025

Exercise 1. Prove that the property: "the graph has an even number of edges" is not expressible in first order logic.

Exercise 2. Show that the following conditions are equivalent for every first-order formula:

1. its limiting probability is one for finite graphs;
2. it is true in the infinite random graph.

Exercise 3. Prove that the random graph contains any finite graph as an induced subgraph.

Exercise 4. Show that the quantifier elimination for the random graph is computable, i.e. the equivalent quantifier-free formula not only exists, but can be effectively computed.

Exercise 5. Prove that the infinite random graph is connected.

Exercise 6. Prove that the random graph is self-complementary, i.e. its complement is isomorphic to it.

Exercise 7. The goal of this exercise is to construct the random graph in a non-random way. Let P be the set of prime numbers that are congruent to 1 modulo 4. Let $E \subseteq P \times P$ be the set of pairs (p, q) such that p is a quadratic residue modulo q, i.e. there exists some $a \in \mathbb{N}$ such that $p \equiv a^2 \mod q$.

1. Prove that the relation E is symmetric.
2. Prove that the graph (P, E) is isomorphic to the random graph.

Remark: this exercise may require nontrivial tools from number theory.