Languages, automata and computation II
Tutorial 2 – WQO

Winter semester 2024/2025

A binary relation $\preceq$ on a set X is **quasi-order** if it is reflexive and transitive. An antichain is a subset of X whose elements are pairwise incomparable by $\preceq$.

Exercise 1. Let $(X, \preceq)$ be a quasi-order. Prove that the following conditions are equivalent.

(a) For any infinite sequence $a_1, a_2, \dots$ of elements of X there are indexes $i < j$ such that $a_i \preceq a_j$.

(b) X does not contain any infinite antichain or any infinite strictly decreasing sequence.

(c) For any infinite sequence $a_1, a_2, \dots$ of elements of X there is an infinite increasing subsequence, i.e. there are indices $i_1 < i_2 < \dots$ with $a_{i_1} \preceq a_{i_2} \preceq \dots$.

Whenever any of the conditions (a)-(c) holds we call the relation $\preceq$ **well quasi-order** or **WQO**.

Exercise 2. Decide which of the following orders are WQO.

(a) $(\mathbb{N}, \leq)$

(b) $(\mathbb{Z}, \leq)$

(c) $(\mathbb{N}, |)$, where $|$ is the divisibility relation.

(d) $(X, \preceq)$ where X is finite and $\preceq$ is any quasi-order on X.

(e) $(\mathbb{N}^2, \leq_{lex})$, where $\leq_{lex}$ is the lexicographic order.

(f) $(\{a, b\}^*, \preceq_{lex})$, where $\preceq_{lex}$ is the lexicographic order on words.

Exercise 3 (Dickson's lemma)**.** Let $(X, \leq_X)$, $(Y, \leq_Y)$ be a WQO. Prove that $(X \times Y, \leq_{X \times Y})$ is a WQO, where $(x, y) \leq_{X \times Y} (x', y') \iff (x \leq_X x' \wedge y \leq_Y y')$. Conclude that $(\mathbb{N}^d, \leq)$ is a WQO, where $\leq$ is the pointwise partial order.

Exercise 4. Prove that any subset of $\mathbb{N}^d$ has finitely many minimal elements.

Exercise 5. Let $\mathcal{P}_{fin}(X)$ be the set of all finite subsets of X. For a quasi-order $\preceq$ on X we define a quasi-order $\preceq^*$ on $\mathcal{P}_{fin}(X)$ as follows: $A \preceq^* B$ if and only if there is an injection $f : A \to B$ such that $a \preceq f(a)$ for all $a \in A$. Prove that if $(X, \preceq)$ is a WQO, then also $(\mathcal{P}_{fin}(X), \preceq^*)$ is a WQO.

Exercise 6 (Higman's lemma)**.** Let Σ be a finite alphabet and $\preceq$ be the subsequence relation on words. Prove that $(\Sigma^*, \preceq)$ is s WQO.