Languages, automata and computation II
Tutorial 1 – Semilinear sets

Winter semester 2024/2025

We consider $\mathbb{N}^d$ as a finitely generated free commutative additive monoid. We can definine several operations on subsets A, B of $\mathbb{N}^d$:

- Union $A \cup B$, intersection $A \cap B$, and complement $\mathbb{N}^d \setminus A$.
- Addition $A + B := \{a + b \mid a \in A, b \in B\}$.
- Iteration $\mathbb{N}A := A^* := \bigcup_{n \in \mathbb{N}} nA$, where $nA = A + A + \cdots + A$ (n times).
- Projection $\pi_i(A) := \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) \mid (a_1, \ldots, a_d) \in A\}$.

By convention, $0A = \{0\}$ if A is nonempty, otherwise $0\emptyset = \emptyset$.

Semilinear sets

A subset L of $\mathbb{N}^d$ is *linear* if it is of the form $b + P^*$ for some *basis vector* $b \in \mathbb{N}^d$ and a finite set of *periods* $P \subseteq \mathbb{N}^d$. A subset S of $\mathbb{N}^d$ is *semilinear* if it is a finite union of linear sets, $S = L_1 \cup \cdots \cup L_n$.

Exercise 1 (Closure properties)**.** Show that the semilinear sets are closed under union, addition, iteration, and projection.

Exercise 2. Find a subset of $\mathbb{N}$ which is not semilinear.

Exercise 3. Show that the set $\big\{(n, 2^n) \in \mathbb{N}^2 \mid n \in \mathbb{N}\big\}$ is not semilinear.

Exercise 4. Let $X \subseteq \mathbb{N}$ be an arbitrary (finite or infinite) set of natural numbers. Show that its iteration X^* is a semilinear set.

Exercise 5. Find a set $X \subseteq \mathbb{N}^2$ such that X^* is not semilinear.

Rational sets

In analogy with the rational subsets of the free monoid Σ^*, we say that a subset R of the free commutative monoid $\mathbb{N}^d$ is *rational* if it can be generated from finite sets using union "$\cup$", addition "$+$" and iteration "$\_^*$".

Exercise 6 (Rational = semilinear)**.** Show that a subset of $\mathbb{N}^d$ is rational if, and only if, it is semilinear.

In view of the exercise, we can see the basis-periods representation of a semilinear set as a normal form for rational sets.

Büchi sets

We explore an automata-based approach to represent subsets of $\mathbb{N}^d$. For simplicity we deal with $d = 1$, however the whole approach generalizes to arbitrary d.