

# Languages, automata and computation II

## Tutorial 2 – WQO

Winter semester 2024/2025

A binary relation  $\preceq$  on a set  $X$  is **quasi-order** if it is reflexive and transitive. An antichain is a subset of  $X$  whose elements are pairwise incomparable w.r.t.  $\preceq$ .

**Exercise 1.** Let  $(X, \preceq)$  be a quasi-order. Prove that the following conditions are equivalent.

- (a) Any infinite sequence  $a_1, a_2, \dots$  of elements of  $X$  contains a *domination*  $a_i \preceq a_j$  for some indexes  $i < j$ .
- (b)  $X$  does not contain any infinite antichain and is *well-founded* (i.e., there are no infinite strictly decreasing sequences).
- (c) Any infinite sequence  $a_1, a_2, \dots$  of elements of  $X$  contains an infinite nondecreasing subsequence, i.e., there are indices  $i_1 < i_2 < \dots$  with  $a_{i_1} \preceq a_{i_2} \preceq \dots$ .
- (d) Every upward closed set is the upward closure of a finite set. (The *upward closure* of a set  $X$  is the set of elements that dominate some element of  $X$ ;  $X$  is *upward closed* if it is equal to its upward closure.)
- (e) Every nondecreasing chain of upward closed sets  $U_1 \subseteq U_2 \subseteq \dots \subseteq X$  is finite.

Whenever any of the conditions above holds we call the relation  $\preceq$  a **well quasi-order** or **WQO**.

**Exercise 2.** Decide which of the following quasi orders are WQO.

- (a)  $(\mathbb{N}, \leq)$ ;
- (b)  $(\mathbb{Z}, \leq)$ ;
- (c)  $(\mathbb{N}, |)$ , where  $|$  is the divisibility relation;
- (d)  $(X, \preceq)$  where  $X$  is finite and  $\preceq$  is any quasi-order on  $X$ ;
- (e)  $(\mathbb{N}^2, \leq_{\text{lex}})$ , where  $\leq_{\text{lex}}$  is the lexicographic order;
- (f)  $(\{a, b\}^*, \preceq_{\text{lex}})$ , where  $\preceq_{\text{lex}}$  is the lexicographic order on words;
- (g)  $\{a, b\}^*$  with the prefix order;
- (h)  $\{a, b\}^*$  with the infix order;

(i) intervals of  $\mathbb{N}$  with the following order:

$$[a, b] \preceq [c, d] \quad \text{if } b < c \vee (a = c \wedge b \leq d);$$

(j) graphs with the subgraph order (remove some edges and some vertices).

**Exercise 3** (Dickson's lemma). Let  $(X, \leq_X)$ ,  $(Y, \leq_Y)$  be WQOs and consider the *product order*  $(X \times Y, \leq_{X \times Y})$  where

$$(x, y) \leq_{X \times Y} (x', y') \quad \text{iff} \quad (x \leq_X x' \wedge y \leq_Y y').$$

Show that the product order is a WQO. Conclude that  $(\mathbb{N}^d, \leq)$  is a WQO, where  $\leq$  is the pointwise partial order.

**Exercise 4.** Let  $\mathcal{P}_{\text{fin}}(X)$  be the set of all finite subsets of  $X$ . For a quasi-order  $\preceq$  on  $X$ , define a quasi-order  $\widetilde{\preceq}$  on  $\mathcal{P}_{\text{fin}}(X)$  as follows:  $A \widetilde{\preceq} B$  if and only if there is an injection  $f : A \rightarrow B$  such that  $a \preceq f(a)$  for all  $a \in A$ . Prove that if  $(X, \preceq)$  is a WQO, then also  $(\mathcal{P}_{\text{fin}}(X), \widetilde{\preceq})$  is a WQO.

**Exercise 5** (Higman's lemma). Let  $(X, \preceq)$  be a WQO and consider the *subword order* (a.k.a. *scattered subsequence order*) on words  $\preceq^* \subseteq X^* \times X^*$ . Prove that  $(X^*, \preceq^*)$  is a WQO.