Languages, automata and computation II
Tutorial 2 – WQO

Winter semester 2024/2025

A binary relation $\preceq$ on a set X is **quasi-order** if it is reflexive and transitive. An antichain is a subset of X whose elements are pairwise incomparable w.r.t. $\preceq$.

Exercise 1. Let $(X, \preceq)$ be a quasi-order. Prove that the following conditions are equivalent.

(a) Any infinite sequence $a_1, a_2, \ldots$ of elements of X contains a *domination* $a_i \preceq a_j$ for some indexes $i < j$.

(b) X does not contain any infinite antichain and is *well-founded* (i.e., there are no infinite strictly decreasing sequences).

(c) Any infinite sequence $a_1, a_2, \ldots$ of elements of X contains an infinite nondecreasing subsequence, i.e., there are indices $i_1 < i_2 < \ldots$ with $a_{i_1} \preceq a_{i_2} \preceq \ldots$.

(d) Every upward closed set is the upward closure of a finite set. (The *upward closure* of a set X is the set of elements that dominate some element of X; X is *upward closed* if it is equal to its upward closure.)

(e) Every nondecreasing chain of upward closed sets $U_1 \subseteq U_2 \subseteq \cdots \subseteq X$ is finite.

Whenever any of the conditions above holds we call the relation $\preceq$ a **well quasi-order** or **WQO**.

Exercise 2. Decide which of the following quasi orders are WQO.

(a) $(\mathbb{N}, \leq)$;

(b) $(\mathbb{Z}, \leq)$;

(c) $(\mathbb{N}, |)$, where $|$ is the divisibility relation;

(d) $(X, \preceq)$ where X is finite and $\preceq$ is any quasi-order on X;

(e) $(\mathbb{N}^2, \leq_{\text{lex}})$, where $\leq_{\text{lex}}$ is the lexicographic order;

(f) $(\{a, b\}^*, \preceq_{\text{lex}})$, where $\preceq_{\text{lex}}$ is the lexicographic order on words;

(g) $\{a, b\}^*$ with the prefix order;

(h) $\{a, b\}^*$ with the infix order;

(i) intervals of $\mathbb{N}$ with the following order:

$$[a, b] \preceq [c, d] \quad \text{if } b < c \vee (a = c \wedge b \leq d);$$

(j) graphs with the subgraph order (remove some edges and some vertices).

Exercise 3 (Dickson's lemma)**.** Let $(X, \leq_X)$, $(Y, \leq_Y)$ be WQOs and consider the *product order* $(X \times Y, \leq_{X \times Y})$ where

$$(x, y) \leq_{X \times Y} (x', y') \quad \text{iff} \quad (x \leq_X x' \ \wedge \ y \leq_Y y').$$

Show that the product order is a WQO. Conclude that $(\mathbb{N}^d, \leq)$ is a WQO, where $\leq$ is the pointwise partial order.

Exercise 4. Let $\mathcal{P}_{\text{fin}}(X)$ be the set of all finite subsets of X. For a quasi-order $\preceq$ on X, define a quasi-order $\widetilde{\preceq}$ on $\mathcal{P}_{\text{fin}}(X)$ as follows: $A \widetilde{\preceq} B$ if and only if there is an injection $f : A \to B$ such that $a \preceq f(a)$ for all $a \in A$. Prove that if $(X, \preceq)$ is a WQO, then also $\left(\mathcal{P}_{\text{fin}}(X), \widetilde{\preceq}\right)$ is a WQO.

Exercise 5 (Higman's lemma)**.** Let $(X, \preceq)$ be a WQO and consider the *subword order* (a.k.a. *scattered subsequence order*) on words $\preceq^* \subseteq X^* \times X^*$. Prove that $(X^*, \preceq^*)$ is a WQO.