Languages, automata and computation II Tutorial 2 – WQO

Winter semester 2024/2025

A binary relation \leq on a set X is **quasi-order** if it is reflexive and transitive. An antichain is a subset of X whose elements are pairwise incomparable w.r.t. \leq .

Exercise 1. Let (X, \preceq) be a quasi-order. Prove that the following conditions are equivalent.

- (a) Any infinite sequence a_1, a_2, \ldots of elements of X contains a domination $a_i \leq a_j$ for some indexes i < j.
- (b) X does not contain any infinite antichain and is well-founded (i.e., there are no infinite strictly decreasing sequences).
- (c) Any infinite sequence a_1, a_2, \ldots of elements of X contains an infinite nondecreasing subsequence, i.e., there are indices $i_1 < i_2 < \ldots$ with $a_{i_1} \leq a_{i_2} \leq \ldots$
- (d) Every upward closed set is the upward closure of a finite set. (The *upward closure* of a set X is the set of elements that dominate some element of X; X is *upward closed* if it is equal to its upward closure.)
- (e) Every nondecreasing chain of upward closed sets $U_1 \subseteq U_2 \subseteq \cdots \subseteq X$ is finite.

Whenever any of the conditions above holds we call the relation \leq a **well quasi-order** or **WQO**.

Exercise 2. Decide which of the following quasi orders are WQO.

- $(a) (\mathbb{N}, \leq);$
- $(b) (\mathbb{Z}, \leq);$
- (c) $(\mathbb{N}, |)$, where | is the divisibility relation;
- (d) (X, \preceq) where X is finite and \preceq is any quasi-order on X;
- (e) $(\mathbb{N}^2, \leq_{\text{lex}})$, where \leq_{lex} is the lexicographic order;
- (f) $(\{a,b\}^*, \leq_{\text{lex}})$, where \leq_{lex} is the lexicographic order on words;
- $(g) \{a,b\}^*$ with the prefix order;
- (h) $\{a,b\}^*$ with the infix order;

(i) intervals of $\mathbb N$ with the following order:

$$[a, b] \leq [c, d]$$
 if $b < c \lor (a = c \land b \leq d)$;

(j) graphs with the subgraph order (remove some edges and some vertices).

Exercise 3 (Dickson's lemma). Let (X, \leq_X) , (Y, \leq_Y) be WQOs and consider the product order $(X \times Y, \leq_{X \times Y})$ where

$$(x,y) \leq_{X \times Y} (x',y')$$
 iff $(x \leq_X x' \land y \leq_Y y')$.

Show that the product order is a WQO. Conclude that (\mathbb{N}^d, \leq) is a WQO, where \leq is the pointwise partial order.

Exercise 4. Let $\mathcal{P}_{\text{fin}}(X)$ be the set of all finite subsets of X. For a quasi-order \preceq on X, define a quasi-order $\widetilde{\preceq}$ on $\mathcal{P}_{\text{fin}}(X)$ as follows: $A\widetilde{\preceq}B$ if and only if there is an injection $f:A\to B$ such that $a\preceq f(a)$ for all $a\in A$. Prove that if (X,\preceq) is a WQO, then also $\left(\mathcal{P}_{\text{fin}}(X),\widetilde{\preceq}\right)$ is a WQO.

Exercise 5 (Higman's lemma). Let (X, \preceq) be a WQO and consider the *subword* order (a.k.a. scattered subsequence order) on words $\preceq^* \subseteq X^* \times X^*$. Prove that (X^*, \preceq^*) is a WQO.