

# Languages, automata and computation II

## Tutorial 7 – $\text{Ra(n)do(m)}$ graph and zero-one laws

Winter semester 2024/2025

**Exercise 1.** Prove that the property: "the graph has an even number of edges" is not expressible in first order logic.

**Exercise 2.** Show that the following conditions are equivalent for every first-order formula:

1. its limiting probability is one for finite graphs;
2. it is true in the infinite random graph.

**Exercise 3.** Prove that the random graph contains any finite graph as an induced subgraph.

**Exercise 4.** Show that the quantifier elimination for the random graph is computable, i.e. the equivalent quantifier-free formula not only exists, but can be effectively computed.

**Exercise 5.** Prove that the infinite random graph is connected.

**Exercise 6.** Prove that the random graph is self-complementary, i.e. its complement is isomorphic to it.

**Exercise 7.** The goal of this exercise is to construct the random graph in a non-random way. Let  $P$  be the set of prime numbers that are congruent to 1 modulo 4. Let  $E \subseteq P \times P$  be the set of pairs  $(p, q)$  such that  $p$  is a quadratic residue modulo  $q$ , i.e. there exists some  $a \in \mathbb{N}$  such that  $p \equiv a^2 \pmod{q}$ .

1. Prove that the relation  $E$  is symmetric.
2. Prove that the graph  $(P, E)$  is isomorphic to the random graph.

*Remark: this exercise may require nontrivial tools from number theory.*