

# Languages, automata and computation II

## Tutorial 5 – Quantifier elimination, Presburger arithmetic

Winter semester 2024/2025

### Normal forms

**Exercise 1.** Prove that any first order formula is logically equivalent to some formula in prenex normal form, i.e. a formula with all quantifiers at the beginning.

**Exercise 2.** Prove that any quantifier-free first order formula is logically equivalent to some formula in conjunctive normal form (CNF), i.e. a conjunction of disjunctions of atomic formulas and negations of atomic formulas.

### Quantifier elimination

**Exercise 3.** Prove that  $(\mathbb{N}, =)$  admits quantifier elimination.

**Exercise 4.** Prove that  $(\mathbb{N}, \leq)$  does not have quantifier elimination.

### Presburger arithmetic

**Exercise 5.** Prove that multiplication cannot be defined in Presburger arithmetic, i.e. the set  $\{(a, b, c) \in \mathbb{N}^3 : a \cdot b = c\}$  is not definable in  $(\mathbb{N}, +)$ .

**Exercise 6.** Recall<sup>1</sup> that subsets of  $\mathbb{N}^d$  can be encoded in binary as languages over an alphabet  $\left\{ \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} : d_1, \dots, d_n \in \{0, 1\} \right\}$ . For instance, the word  $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  encodes the pair  $(6, 5)$ . Let  $\phi(x_1, \dots, x_n)$  be a formula of Presburger arithmetic. Prove that the subset of  $\mathbb{N}^d$  defined by  $\phi$  can be encoded as a regular language. Conclude that the Presburger arithmetic is decidable.

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<sup>1</sup>We mentioned this in the first tutorial