Towards regular separability of Petri net languages

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At a glance

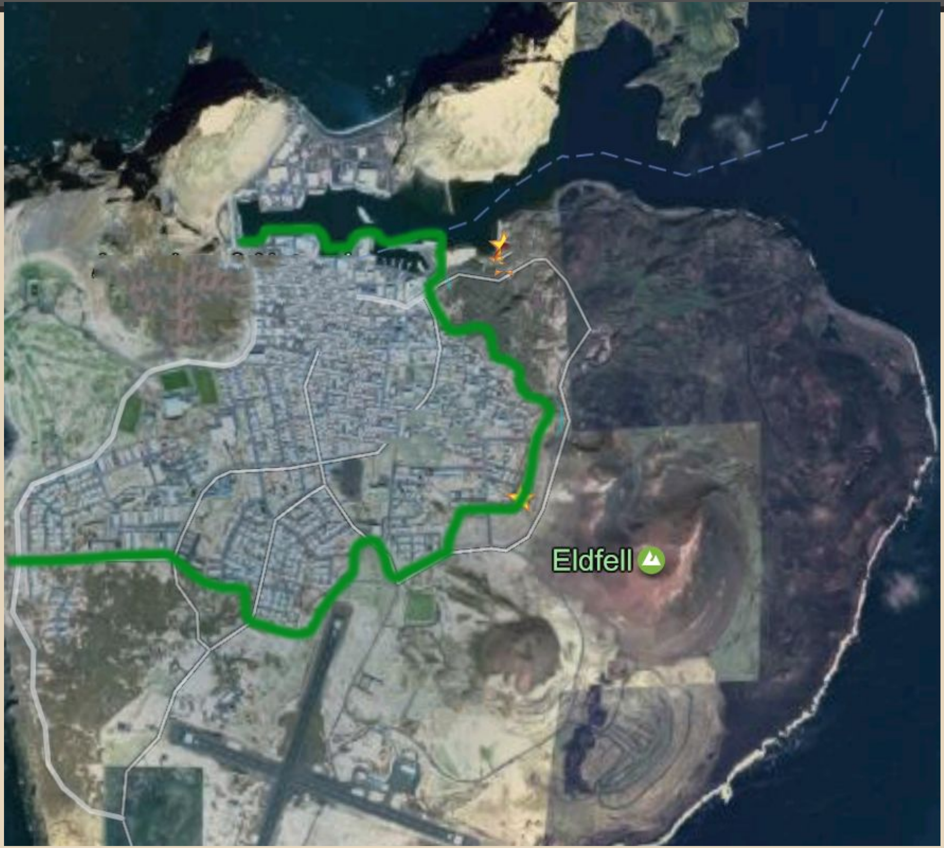
Problem:	Regular separability.
Model:	Languages recognised by Petri nets and related formalisms.
Results:	Decidability for restricted subclasses. Open for Petri nets.

Main technical ideas

Our results on regular separability are based on the following techniques:

1. Deterministic = nondeterministic.
2. Reduce to separability of bounded languages.
   The following problems are inter-reducible:
   - Separability of bounded languages.
   - Separability of commutative languages.
   - Separability of sets of vectors.
3. Regular partitioning.

Regular separability in Iceland



Regular separability in Iceland

A = Heimaey

Regular separability in Iceland

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B = lava from the volcano Eldfell winter 1973

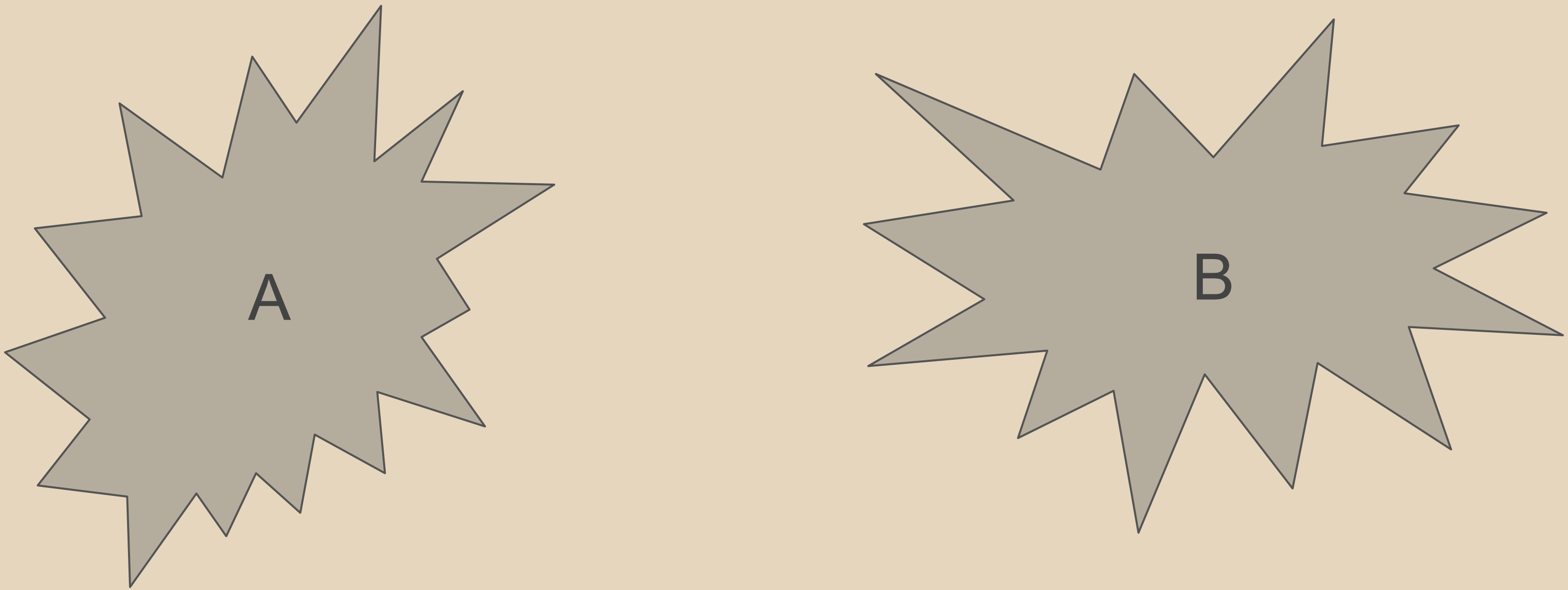
Regular separability in Iceland

A = Heimaey

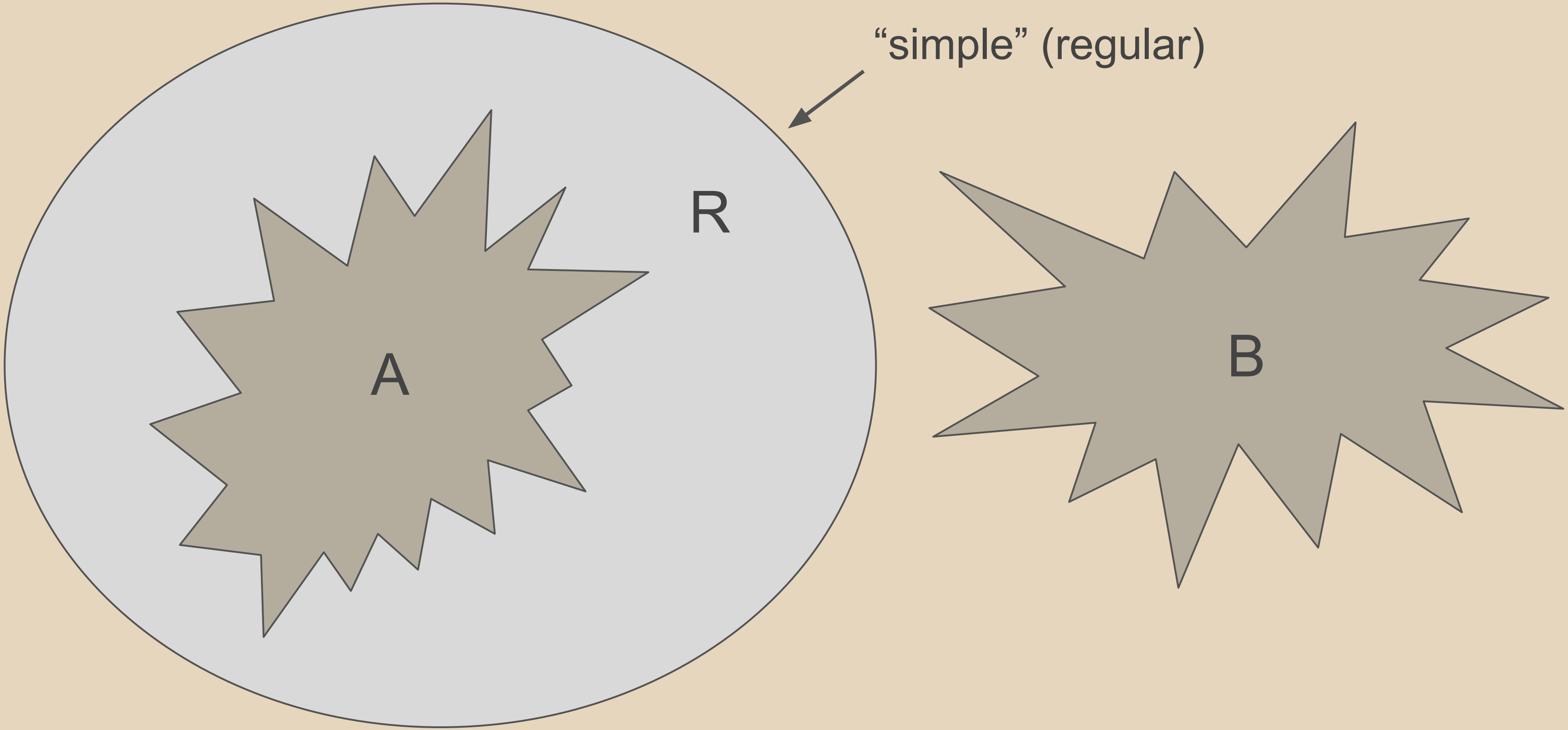
R = Sea water

B = lava from the volcano Eldfell winter 1973

Regular separability



Regular separability



Motivation

- Separability gives a certificate of disjointness.
  - Verifying that R is a separator is decidable*.
  - The separator yields a “simple reason” for disjointness.

*Under the assumption that the class of languages is closed under intersection with regular languages and has a decidable emptiness problem.

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- Separability can be used to circumvent undecidability of disjointness:
  - Disjointness of two CFLs is undecidable.
  - Separability by piecewise-testable languages of CFLs is decidable [Czerwiński, Martens, van Rooijen, Zeitoun '15].

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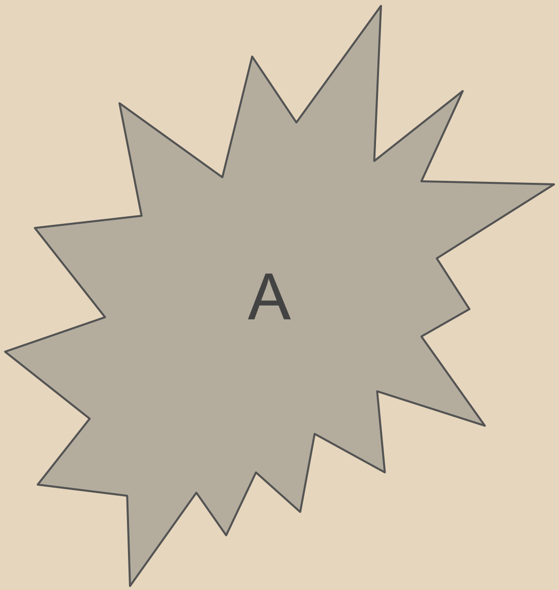
Regularity

is A regular?



Regularity

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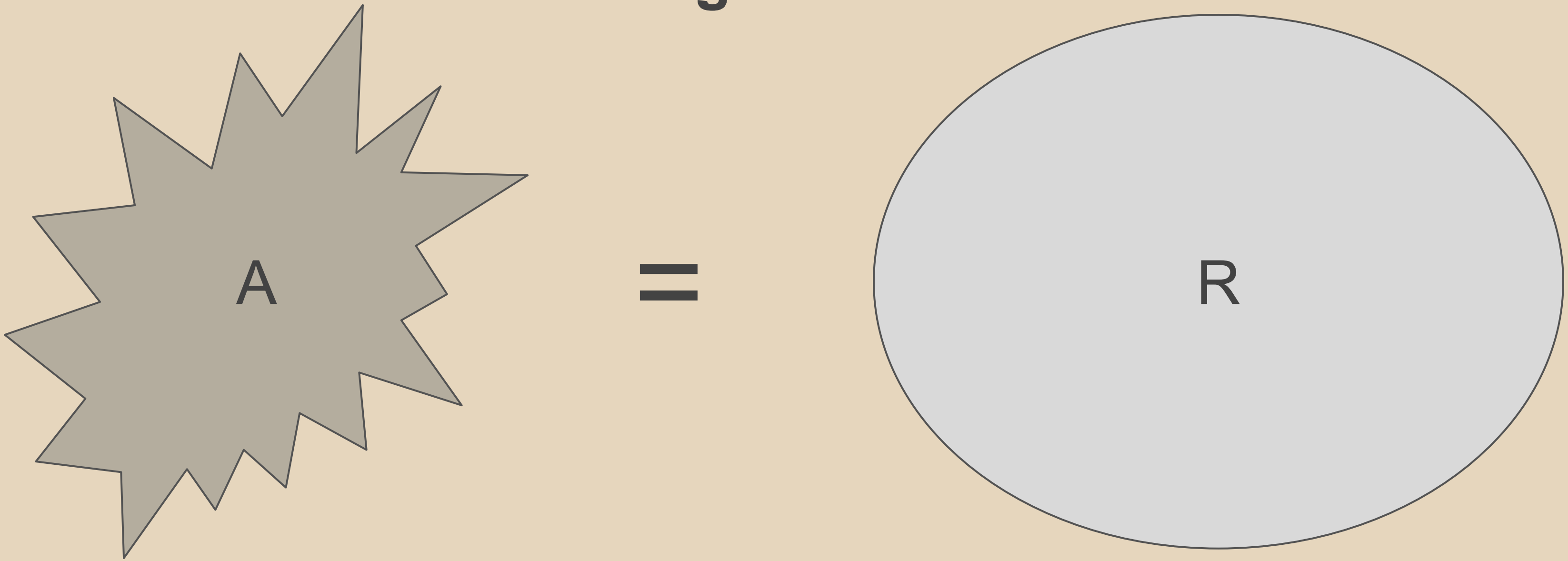
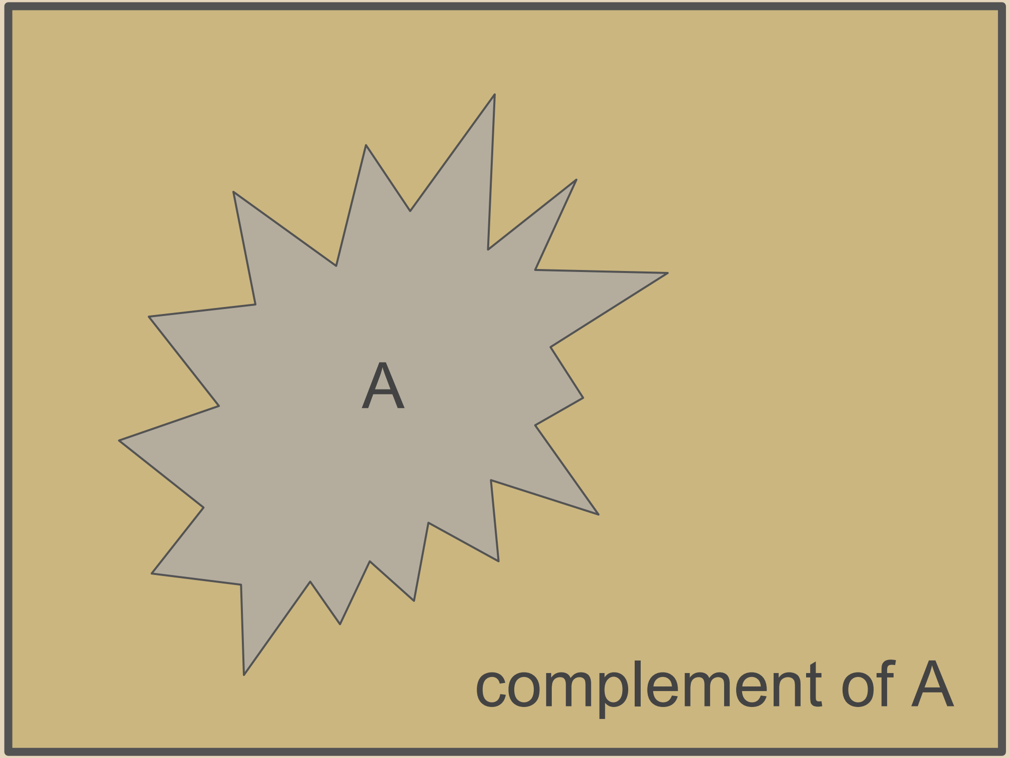


Regularity vs. Separability



A is regular

iff

A is regular separable from its complement

Regularity vs. Separability

- For classes closed under complement,
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Regularity vs. Separability

- For classes closed under complement,
  regular separability generalises the *regularity problem*.
- For classes *not* closed under complement,
  the two problems behave rather differently:
  - Regularity is very sensitive to
    determinism (decidable) vs. nondeterminism (undecidable).
  - Separability is insensitive in this respect:
    it always reduces to the deterministic case.

Regularity w.r.t. nondeterminism

Deterministic

DCFL	[Stearns ’67]
Petri nets	[Valk, Vidal-Niquet ’81]

decidable

Regularity w.r.t. nondeterminism

Deterministic			Nondeterministic
DCFL	[Stearns ’67]	$\supset$	1-counter nets (no zero test), even reversal bounded
Petri nets	[Valk, Vidal-Niquet ’81]		

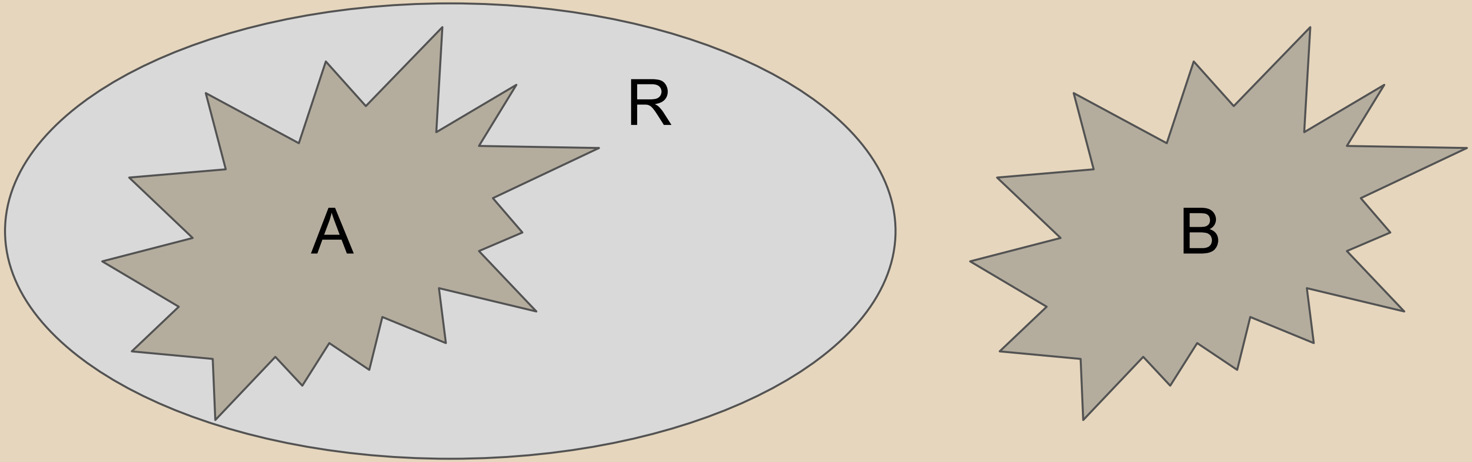
decidable

undecidable

(reduction from universality)

Separability w.r.t. nondeterminism

Separability is *insensitive* to nondeterminism:
It reduces to the deterministic case [C., Czerwiński, Lasota, Paperman ICALP'17].



$p \xrightarrow{a} q$

$r \xrightarrow{a} s$

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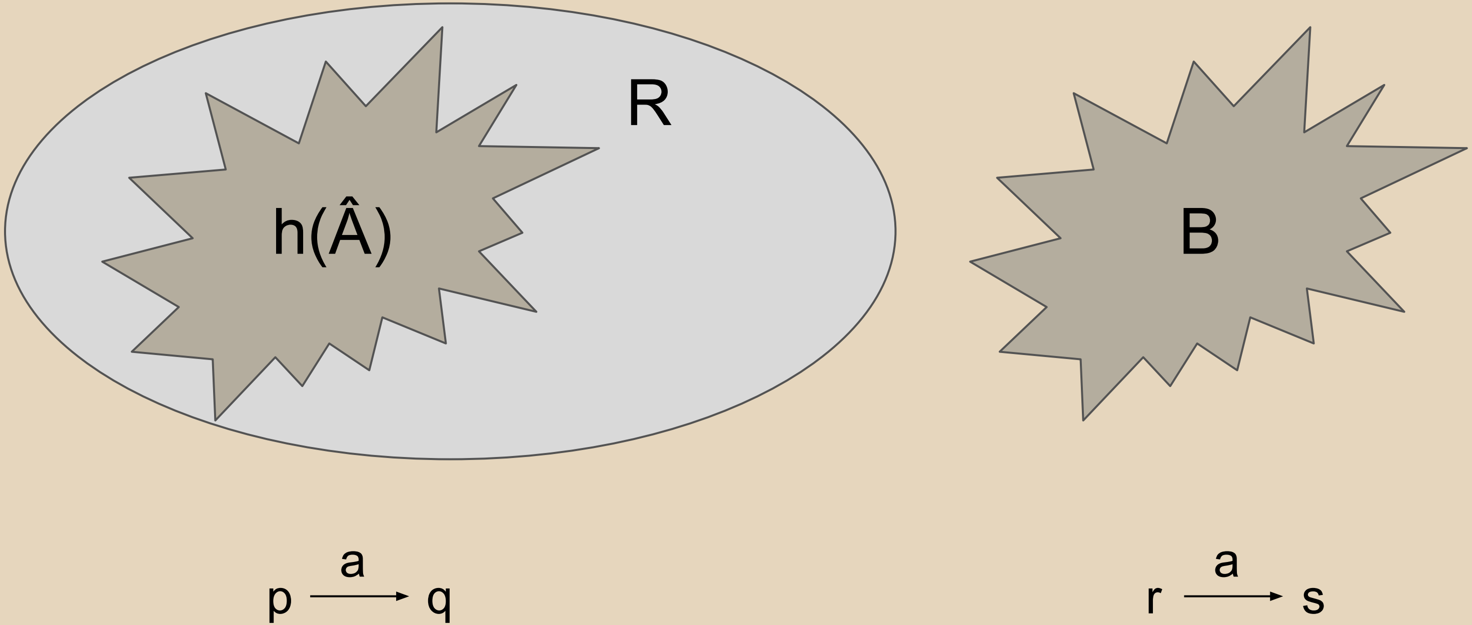

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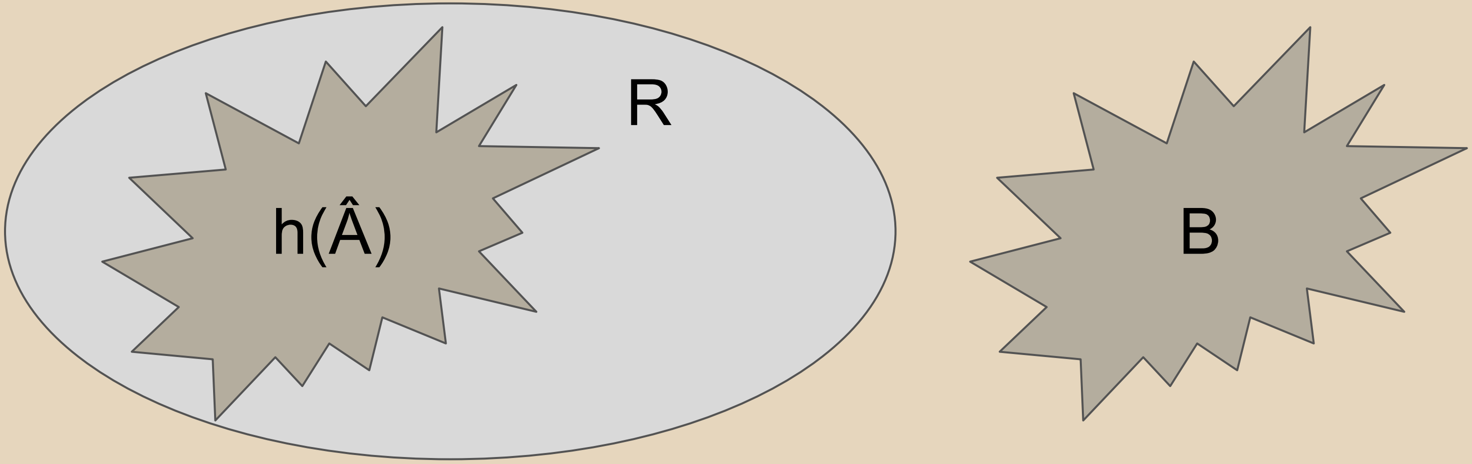

$p \xrightarrow{p \xrightarrow{a} q} q$ $\qquad$ $r \xrightarrow{a} s$

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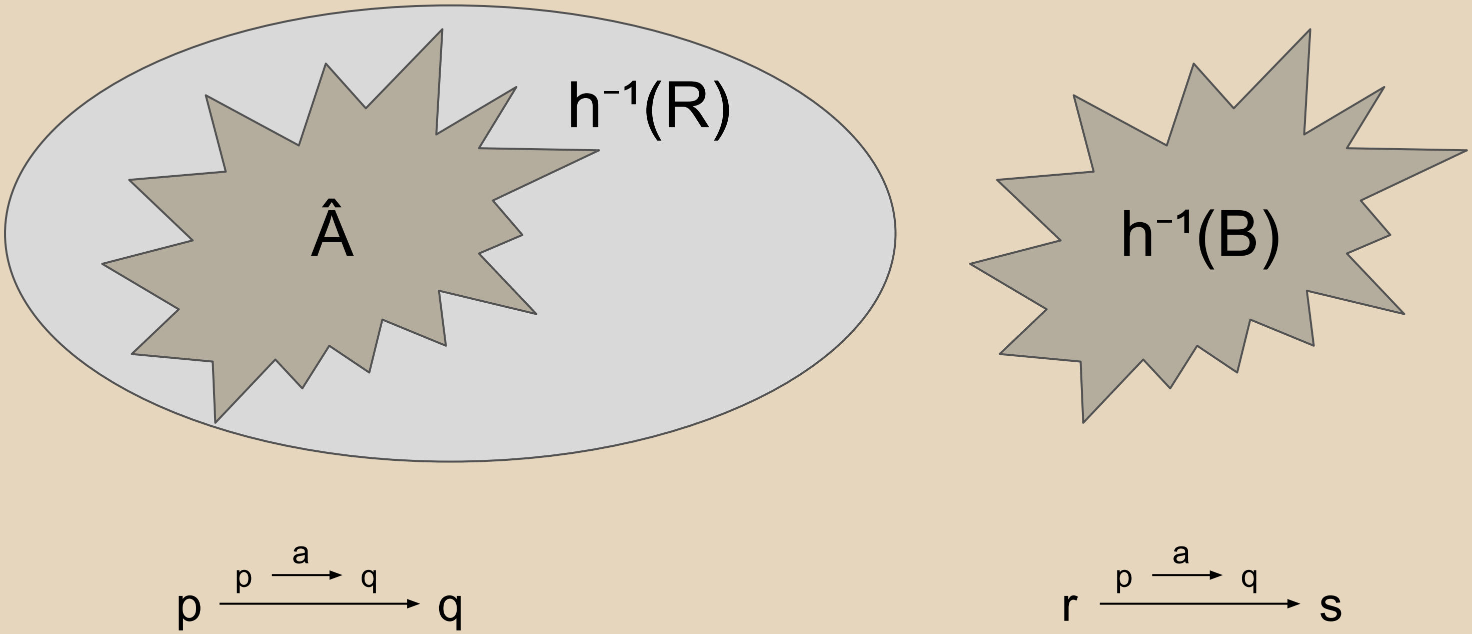

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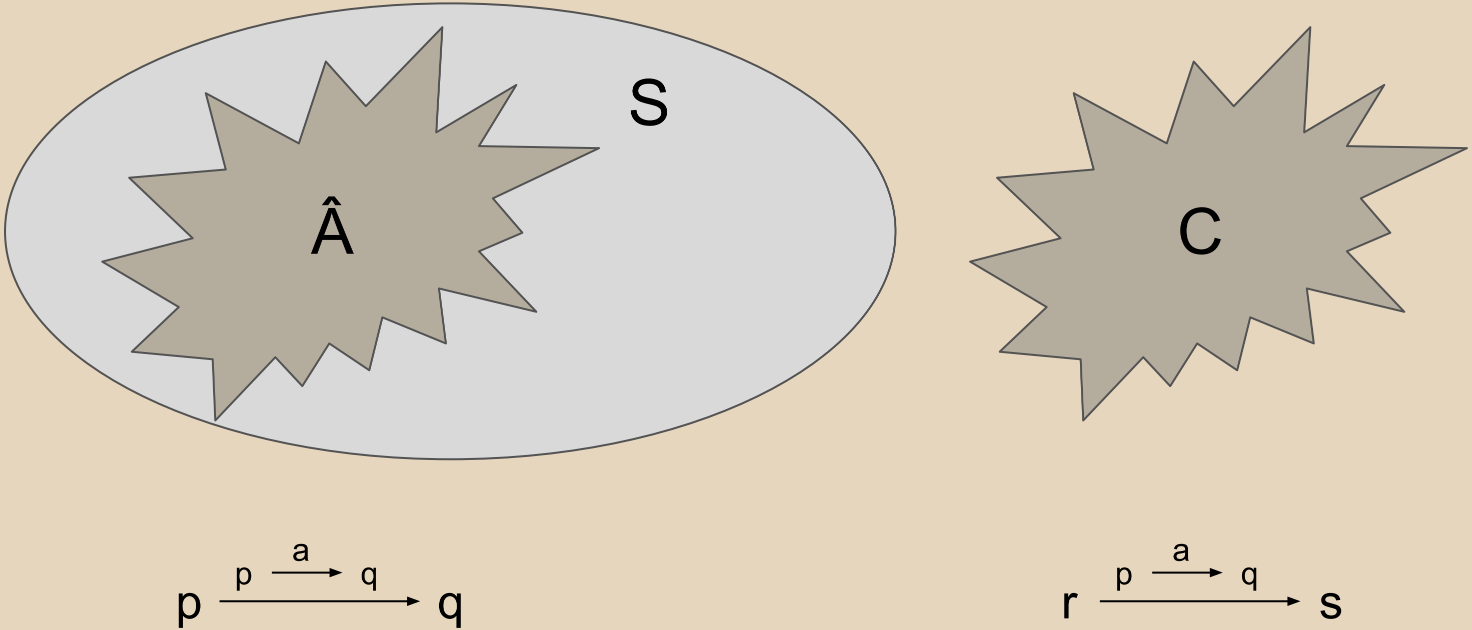

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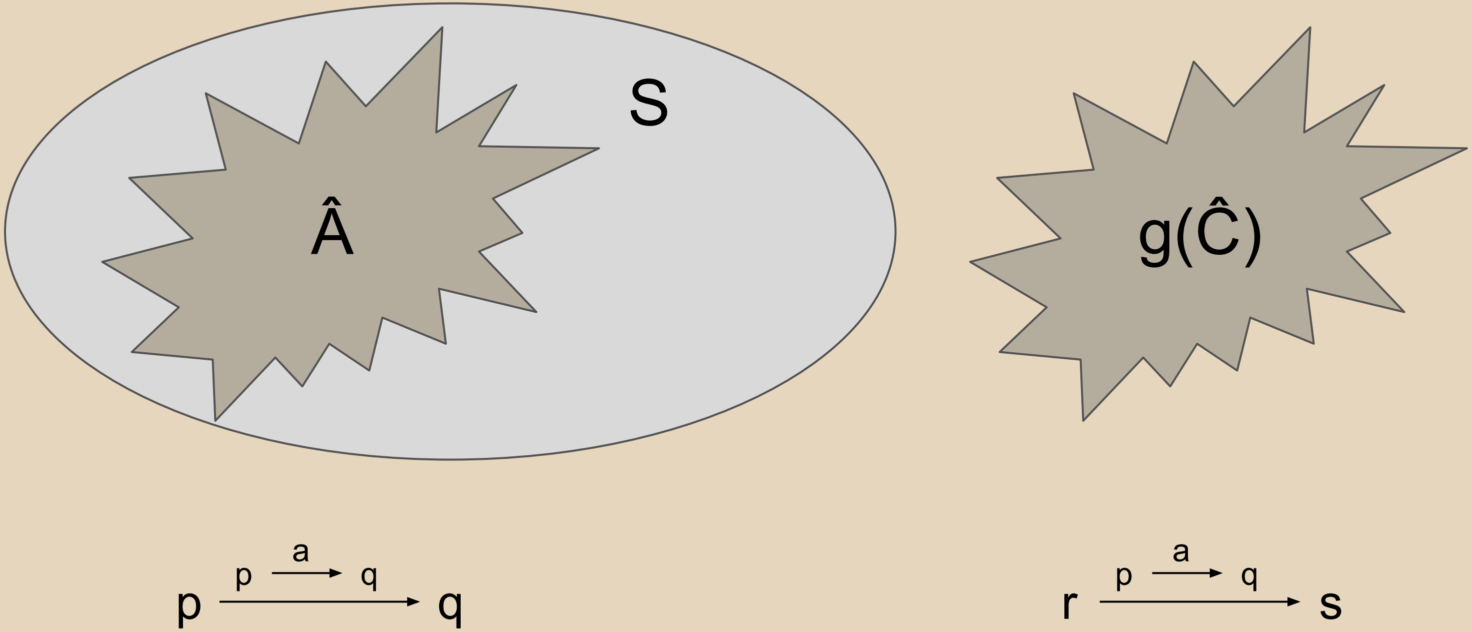

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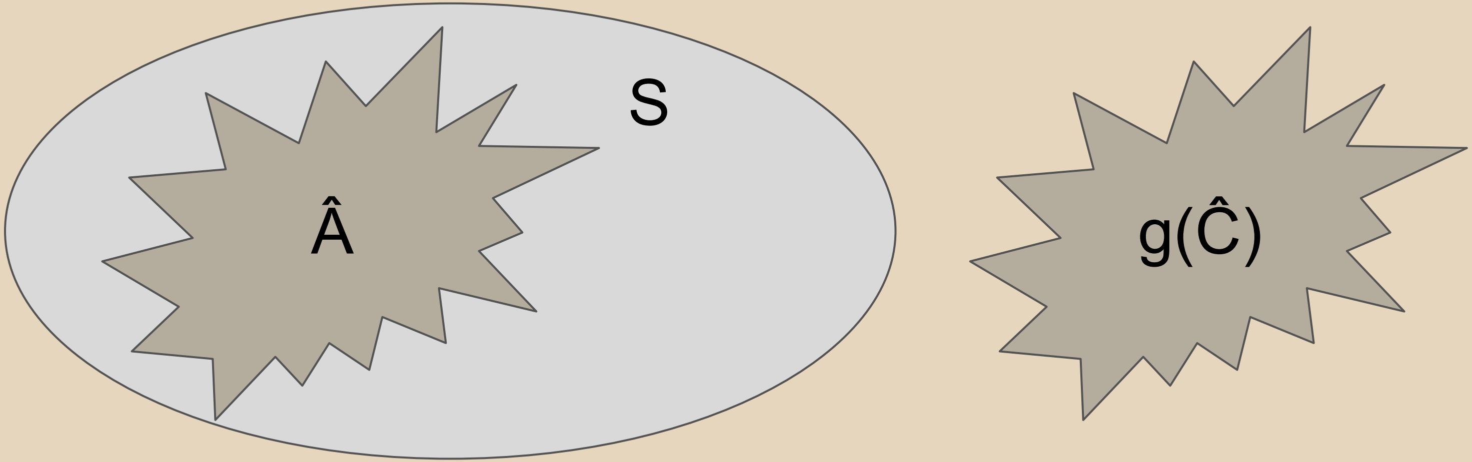

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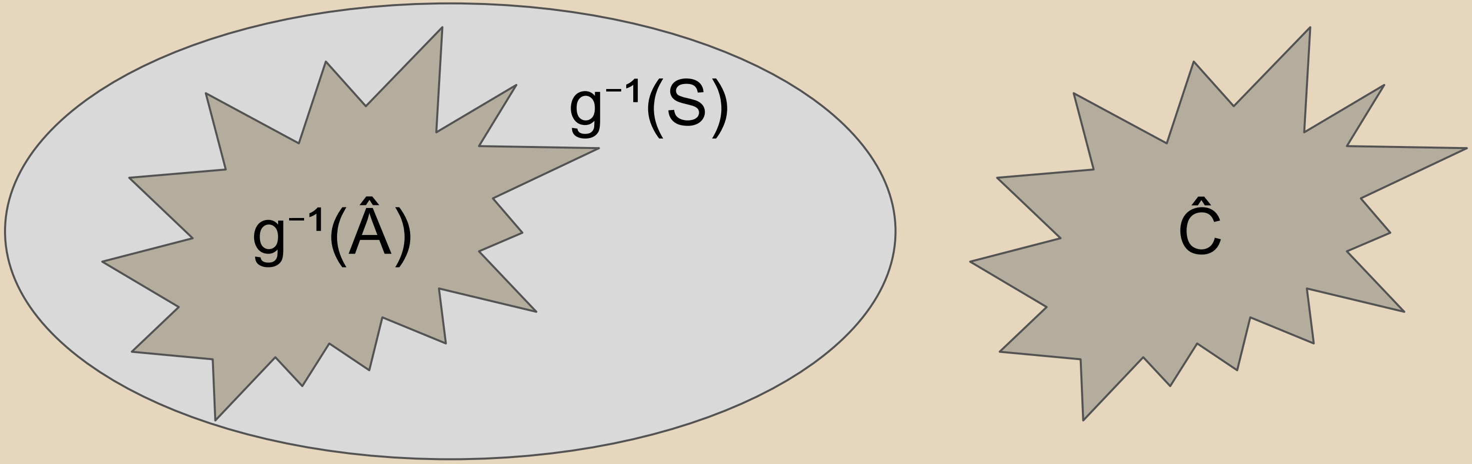

$$p \xrightarrow{r \xrightarrow{p \xrightarrow{a} q} s} q \qquad\qquad r \xrightarrow{r \xrightarrow{p \xrightarrow{a} q} s} s$$

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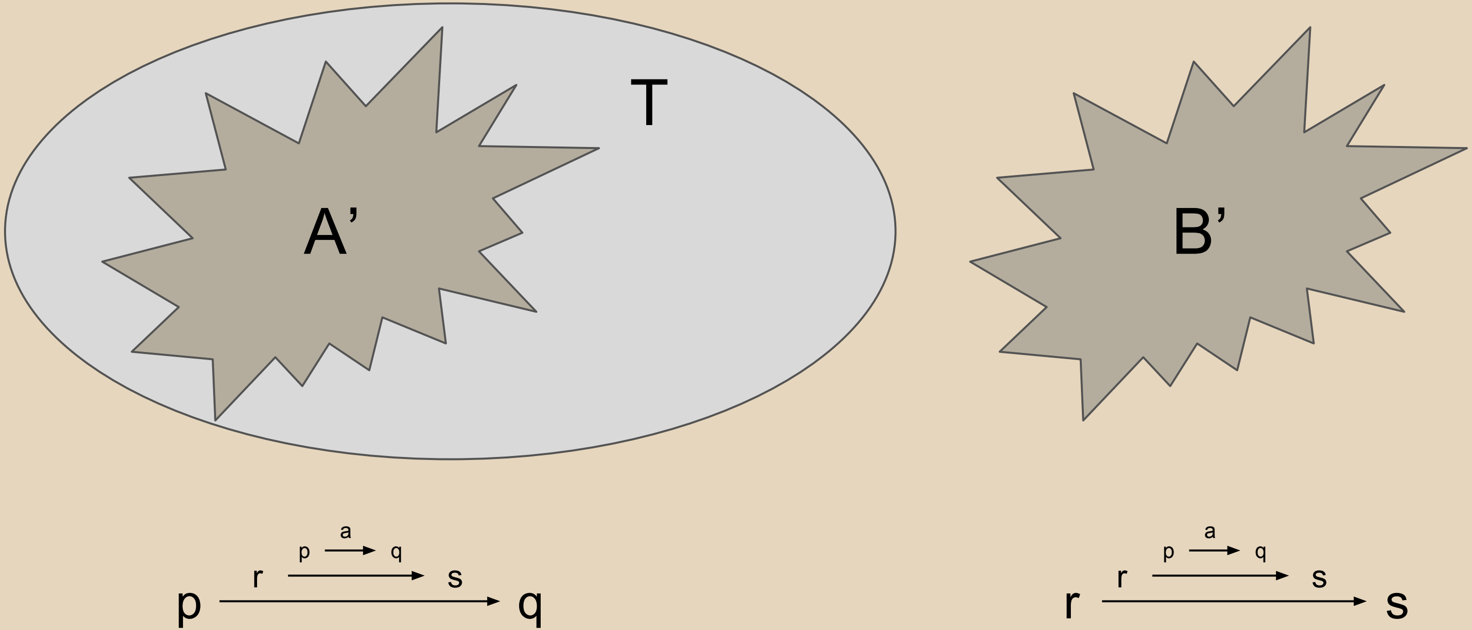

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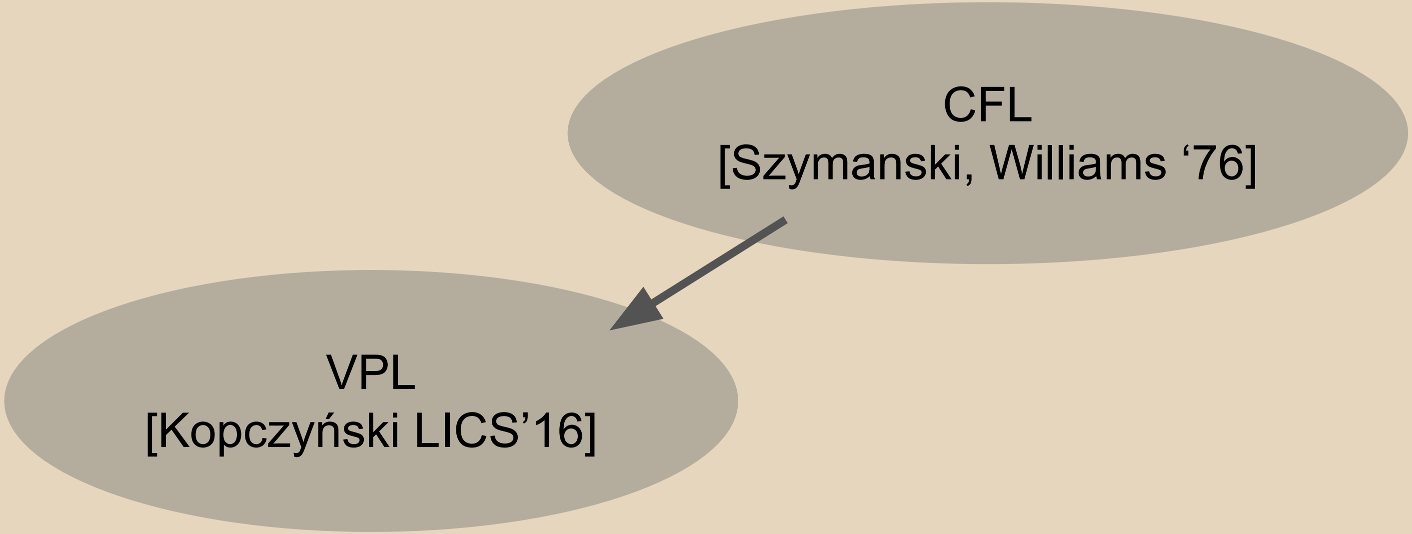
Ingredients:

- A is the homomorphic image of a *deterministic* Â. (The same for B.)
- Separability is invariant w.r.t. inverse homomorphic images.
- Regular languages are closed under inverse homomorphic images.
- A, B belong to a class closed under inverse homomorphic images.

Undecidable separability

CFL
[Szymanski, Williams ‘76]

Undecidable separability



Undecidable separability

- CFLs are the homomorphic images of deterministic CFLs.
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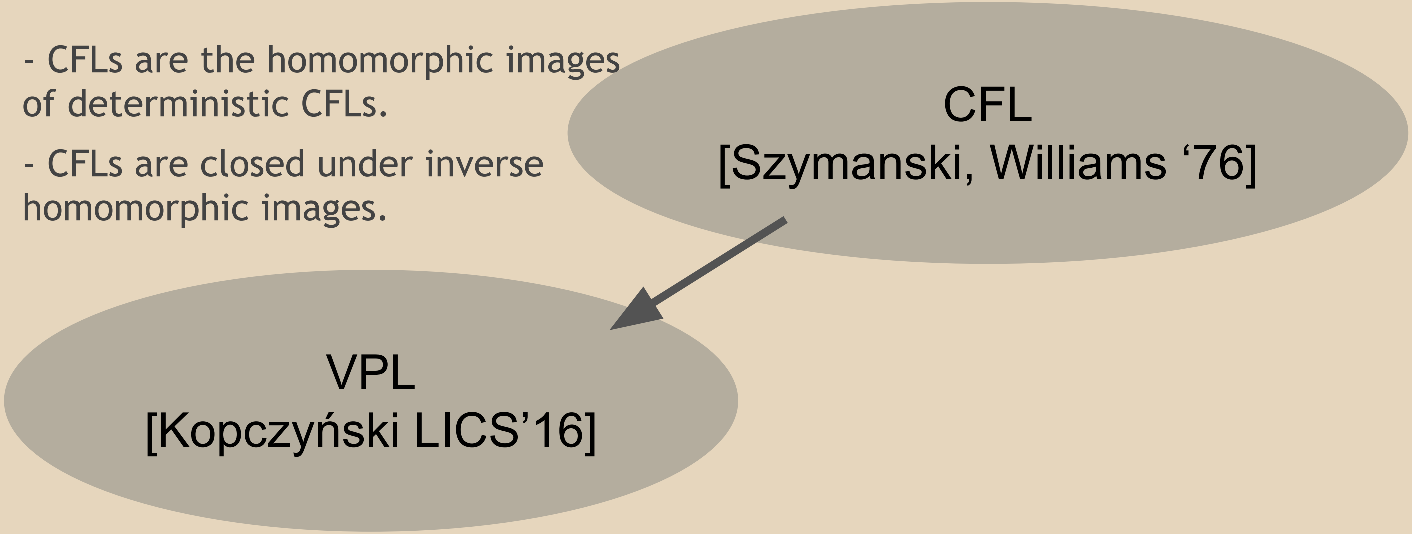

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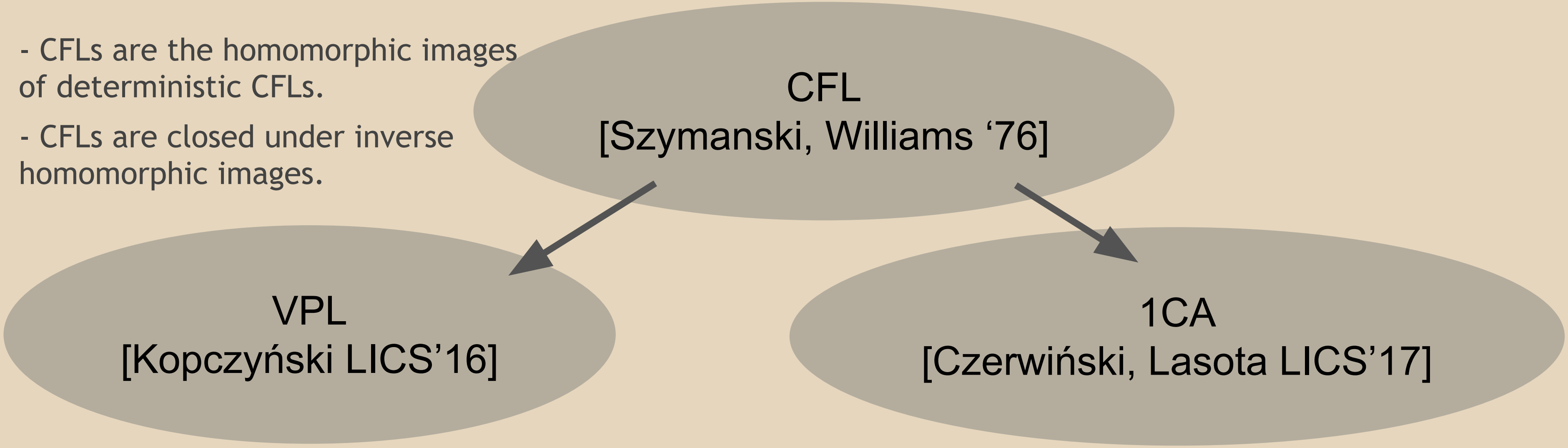

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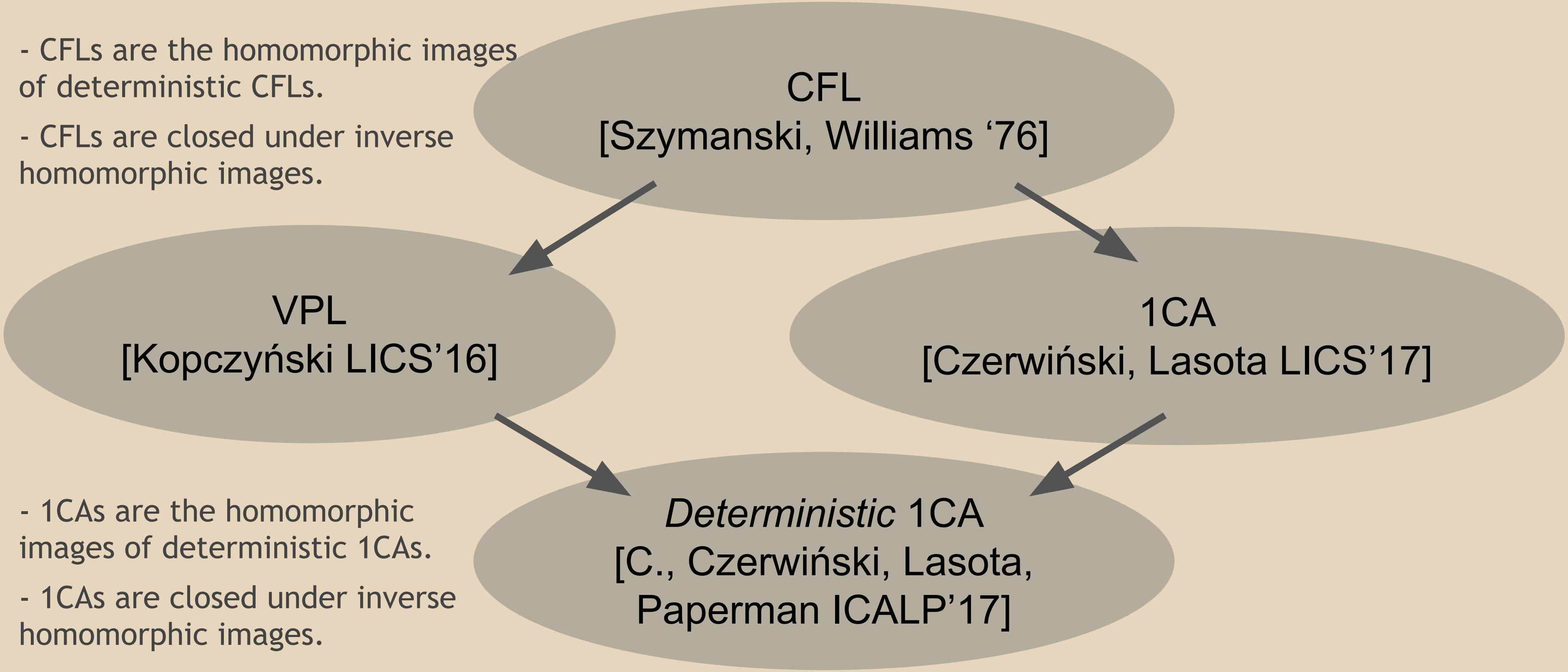

- 1CAs are the homomorphic images of deterministic 1CAs.
- 1CAs are closed under inverse homomorphic images.

Decidable separability

1. 1CN [Czerwiński, Lasota LICS'17].

   Via the *Regular Overapproximation* technique
   → **Wojtek Czerwiński's talk** on Tue 4A 2:05pm.

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   [C., Czerwiński, Lasota, Paperman STACS’17].

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   Reduction to the case of *bounded languages* $L \subseteq a^*b^*$.

Commutative ~ bounded ~ Parikh

The following problems are mutually inter-reducible:

1. Regular separability of *commutative* languages $\subseteq \{a, b\}^*$.
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What are unary sets?

Regular languages vs. Unary sets

Unary sets correspond to finite-memory counting separately in each coordinate.

- Let $i \in \mathbb{N}$. Vectors $u, v \in \mathbb{N}^d$ are *i-unary equivalent* $u \equiv_i v$ if $\forall\ 1 \leq k \leq d$,
    - Equivalent modulo i: $u[k] \equiv v[k] \bmod i$.
    - Either both big, or both small: $u[k] \geq i$ iff $v[k] \geq i$.
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Connections between unary sets and commutative/bounded regular languages:

- The Parikh image of a commutative regular language is unary.
- The Parikh image of a bounded regular language $\subseteq a^*b^*$ is unary.
- The inverse Parikh image of a unary set is a commutative regular language.

Unary sets correspond to commutative/bounded regular languages

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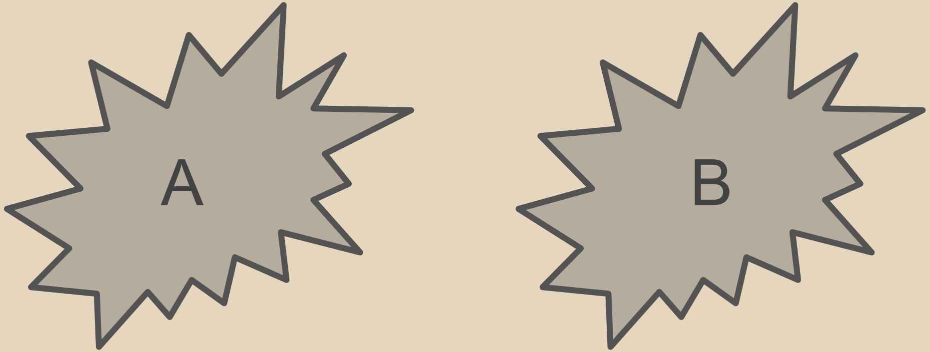

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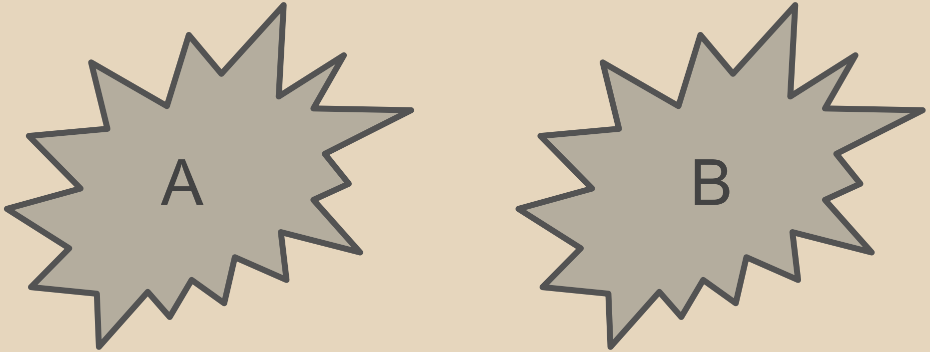
1→2:

Let A, B commutative.

Define:

$A' = A \cap a^*b^*$,

$B' = B \cap a^*b^*$.



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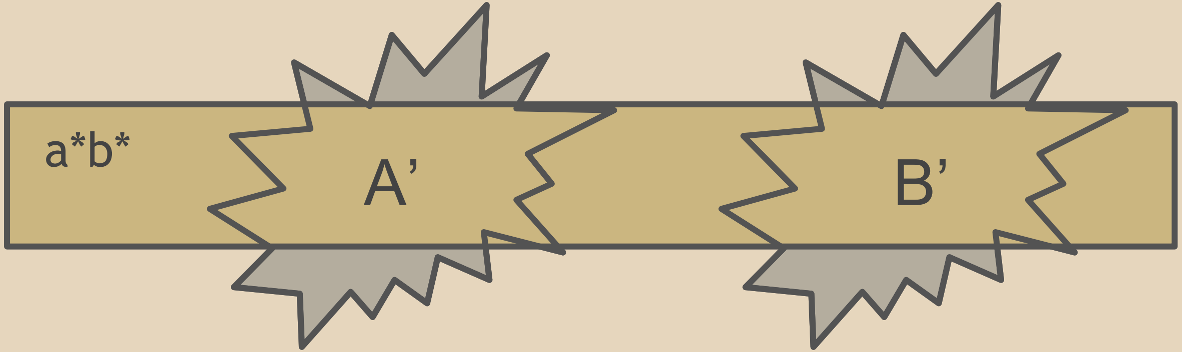

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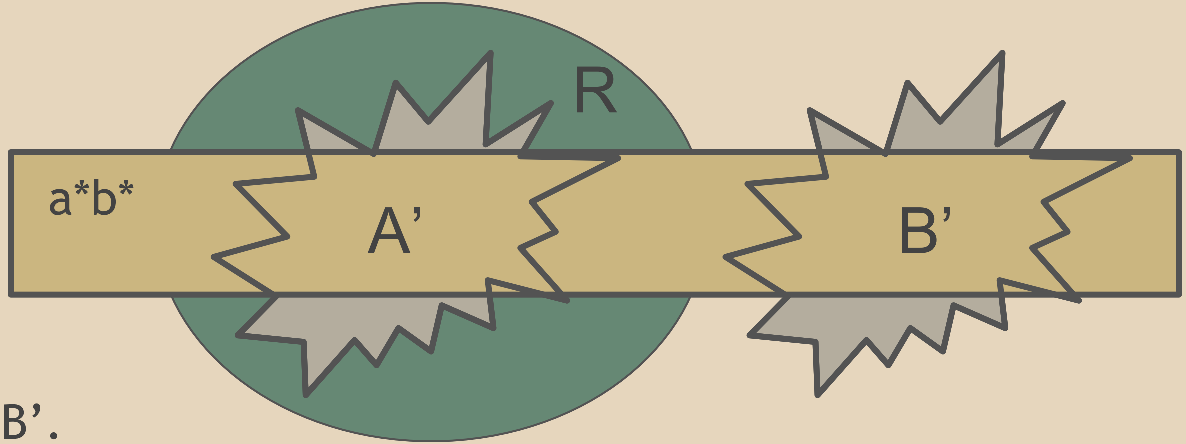

($\Rightarrow$) If R separates A, B,
then $R \cap a^*b^*$ separates A', B'.

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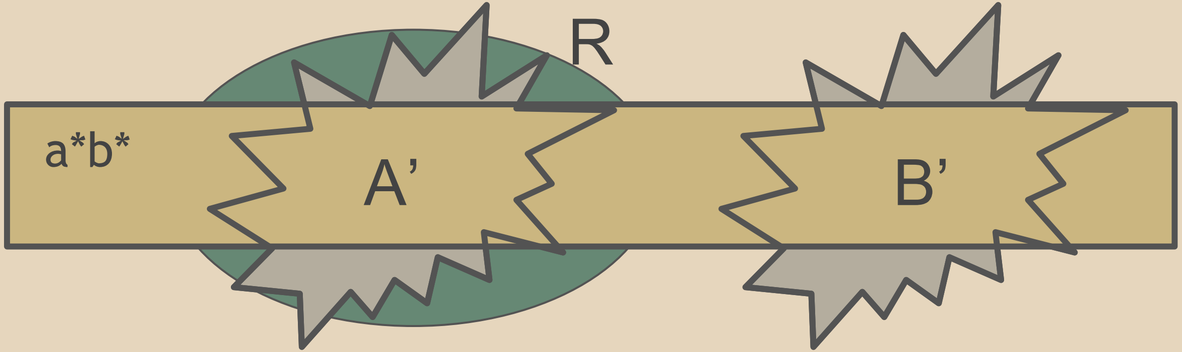


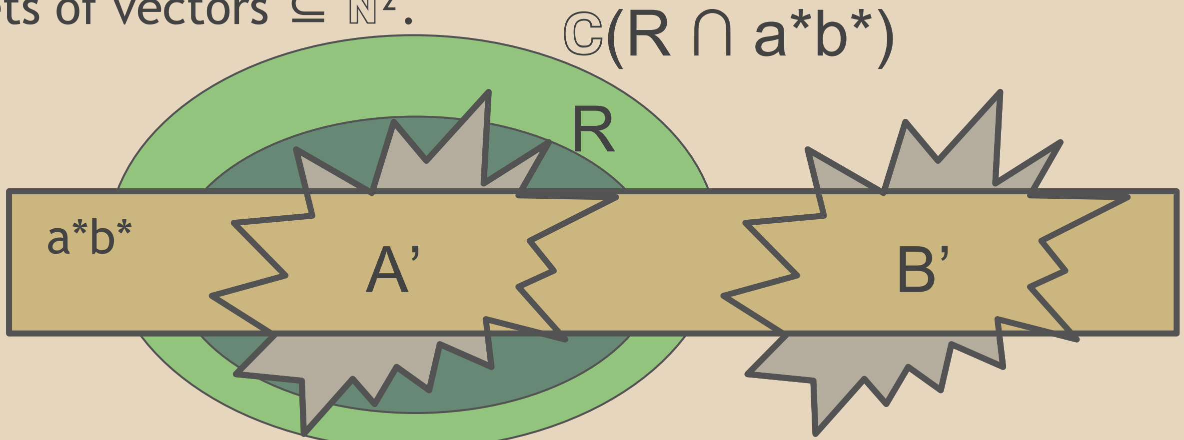
Commutative ~ bounded ~ Parikh

The following problems are mutually inter-reducible:

1. Regular separability of *commutative* languages ⊆ {a, b}*.
2. Regular separability of *bounded* languages ⊆ a*b*.
3. *Unary* separability of sets of vectors ⊆ $\mathbb{N}^2$.

1→2:
Let A, B commutative.

A' = A ∩ a*b*, B' = B ∩ a*b*.



(⟸) Let R separate A', B'.
Then the commutative closure of R ∩ a*b* separates A, B.
Hint: A equals the commutative closure of A'.

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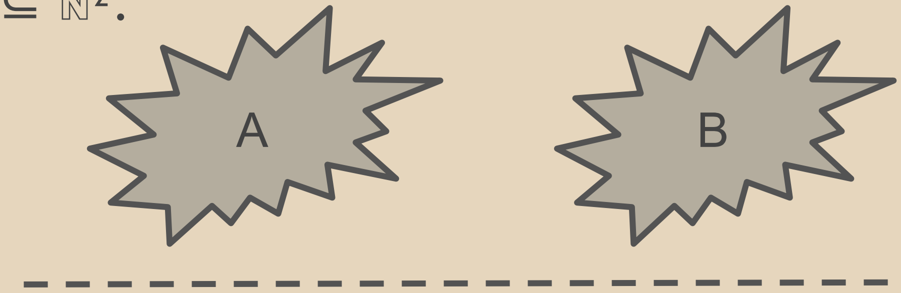

1→2 ✔◻. 2→3:
Let A, B ⊆ a*b* bounded.

Commutative ~ bounded ~ Parikh

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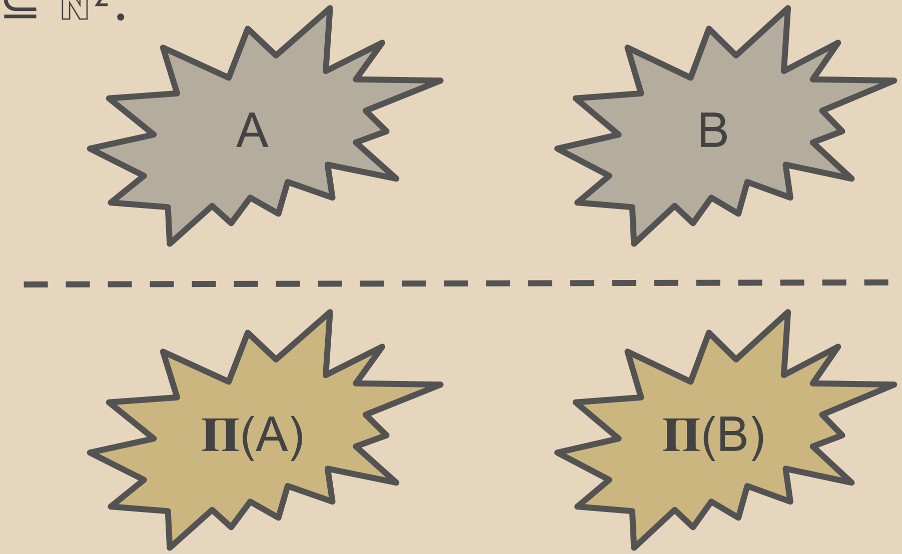
1→2 ✔□. 2→3:
Let A, B $\subseteq a^*b^*$ bounded.

Consider their Parikh images $\mathbf{\Pi}(A)$, $\mathbf{\Pi}(B)$.



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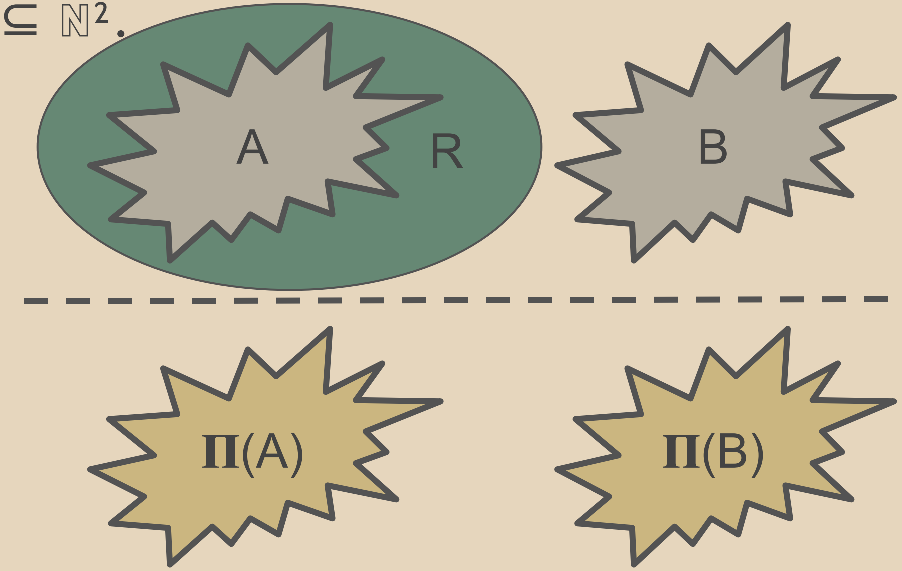

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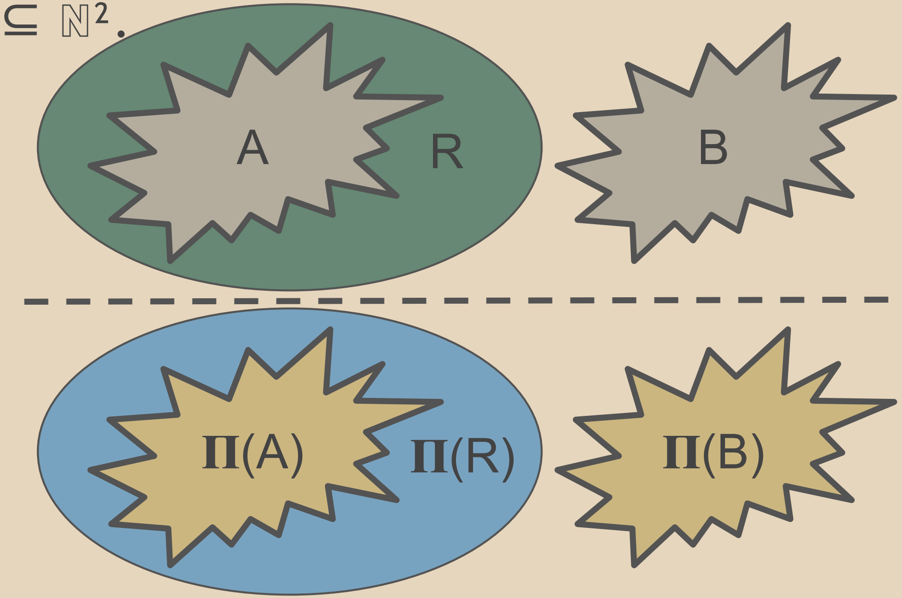

1→2 ✔□. 2→3:
Let A, B $\subseteq a^*b^*$ bounded.

Consider their Parikh images $\mathbf{\Pi}(A)$, $\mathbf{\Pi}(B)$.

($\Longrightarrow$) If R $\subseteq a^*b^*$ separates A, B, then $\mathbf{\Pi}(R)$ is unary and separates $\mathbf{\Pi}(A)$, $\mathbf{\Pi}(B)$.

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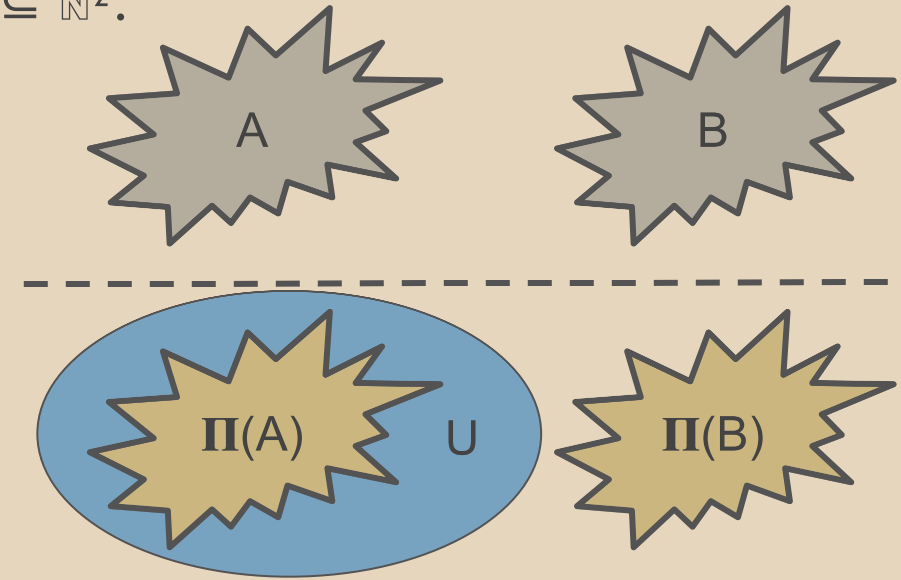
$1 \to 2$ ✔▫. $2 \to 3$:
Let $A, B \subseteq a^*b^*$ bounded.

Consider their Parikh images $\Pi(A)$, $\Pi(B)$.



($\Leftarrow$) If U unary separates $\Pi(A)$, $\Pi(B)$

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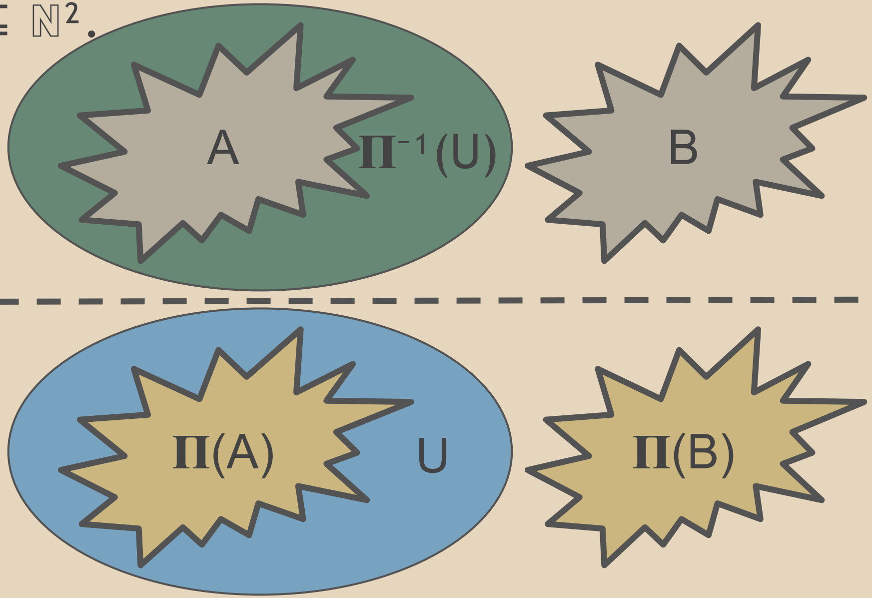

1→2 ✔□. 2→3:
Let A, B $\subseteq a^*b^*$ bounded.

Consider their Parikh images $\Pi(A)$, $\Pi(B)$.

(⟸) If U unary separates $\Pi(A)$, $\Pi(B)$,
then $\Pi^{-1}(U)$ is regular and separates A, B.

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Commutative ~ bounded ~ Parikh

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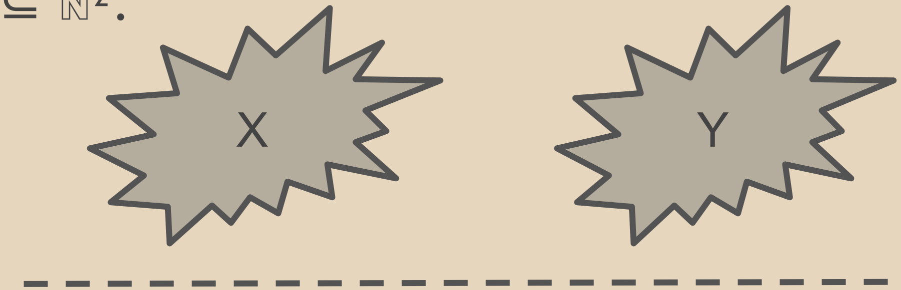

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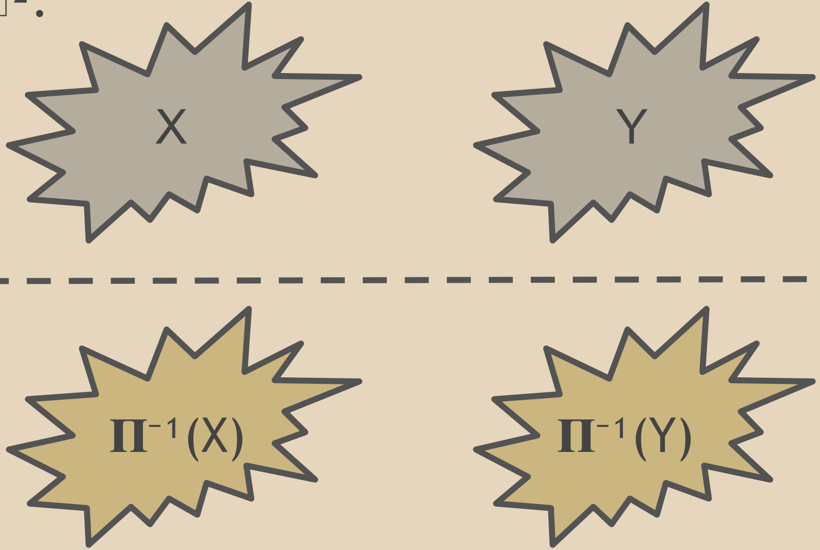

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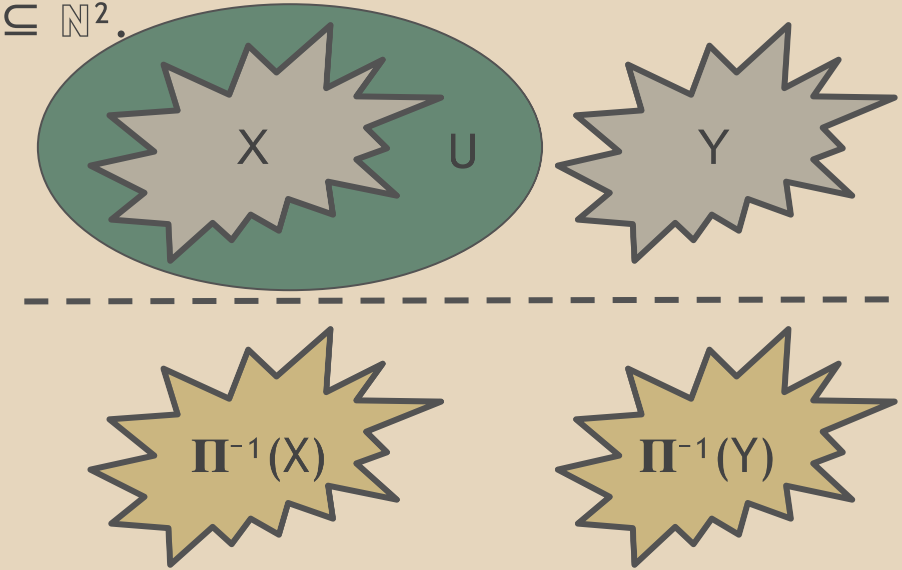

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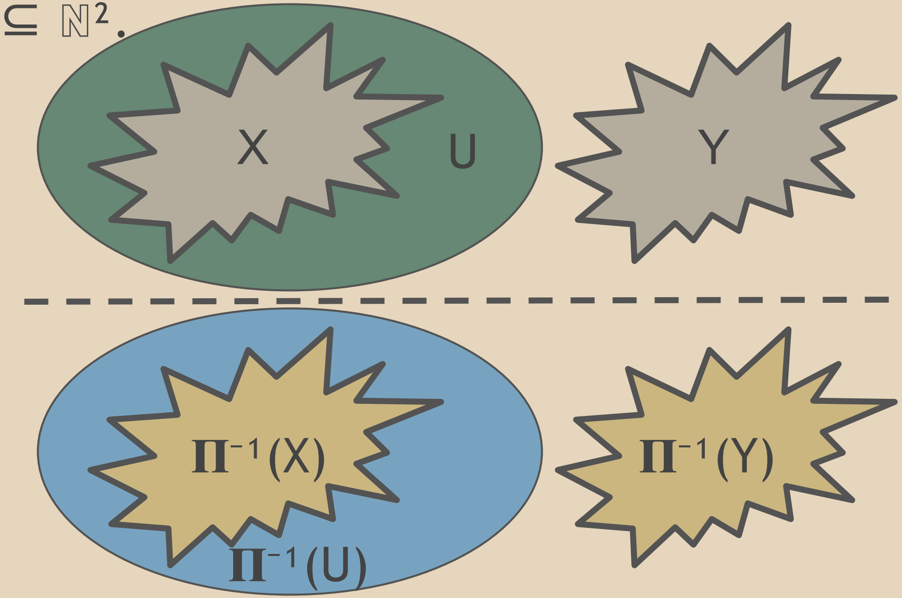

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($\Longrightarrow$) If U unary separates X, Y,
then $\Pi^{-1}(U)$ is regular
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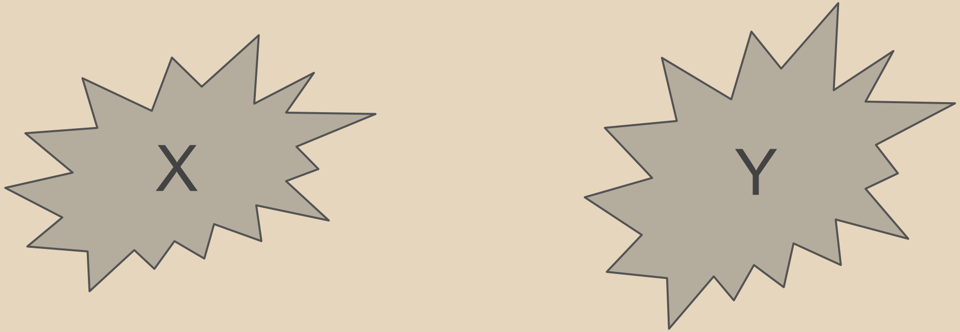

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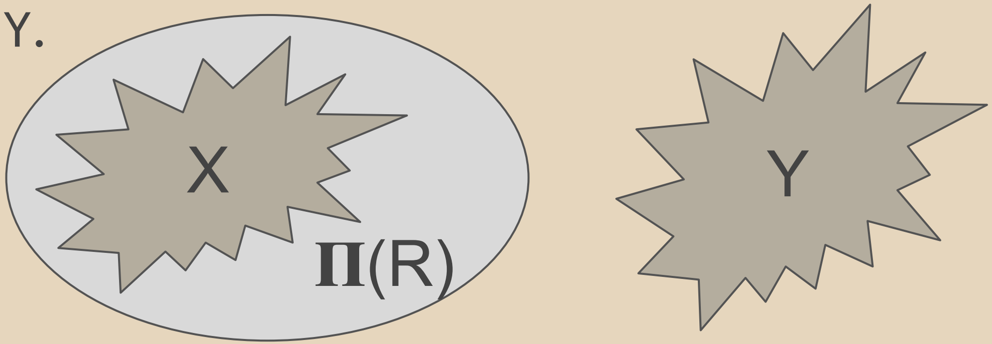

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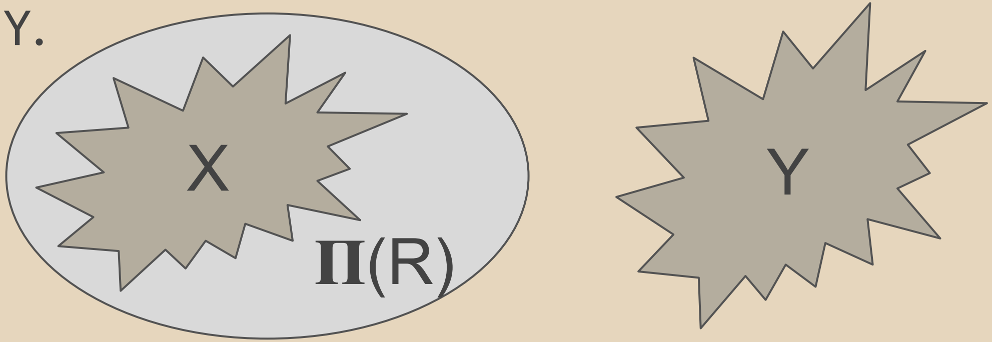

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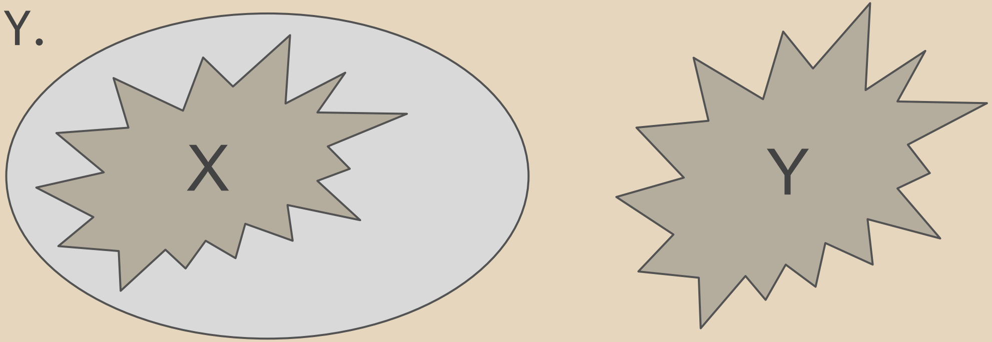

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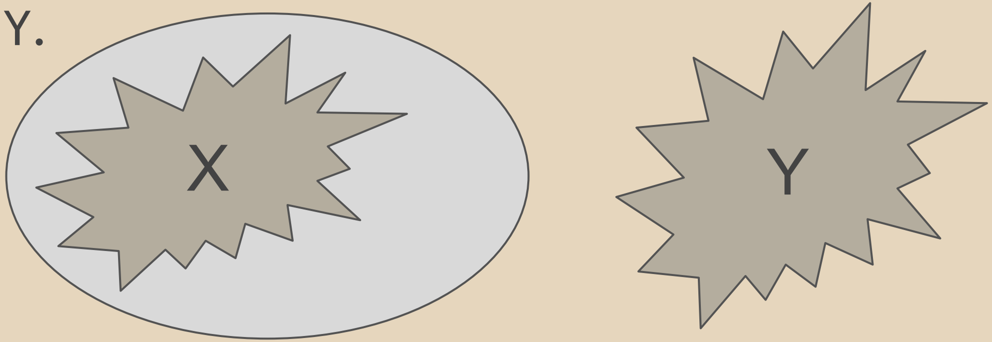

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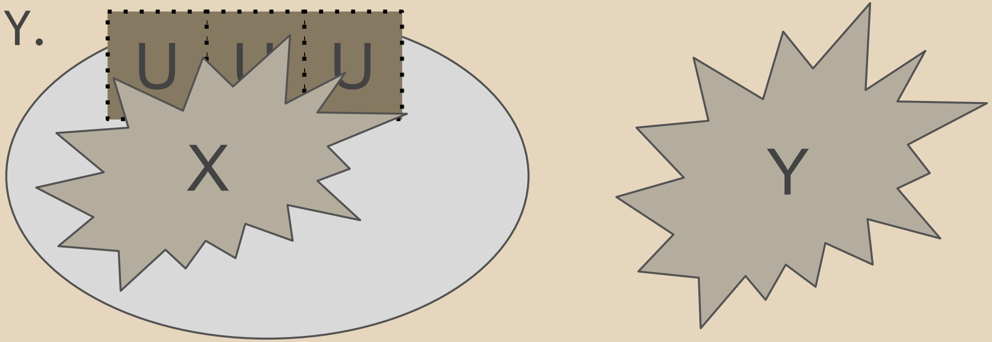

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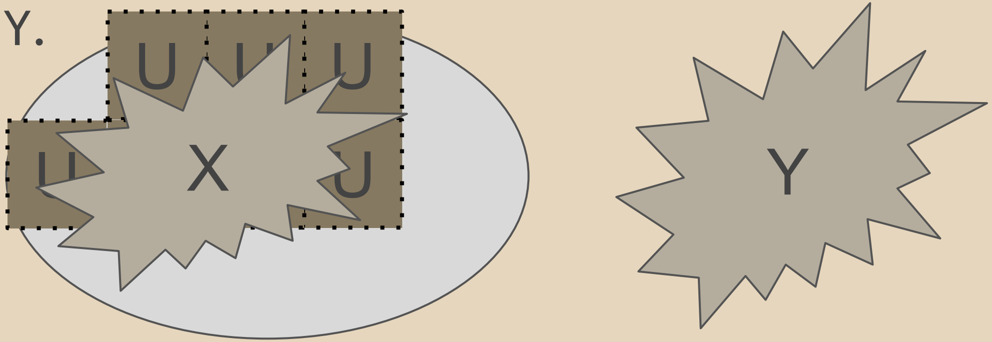

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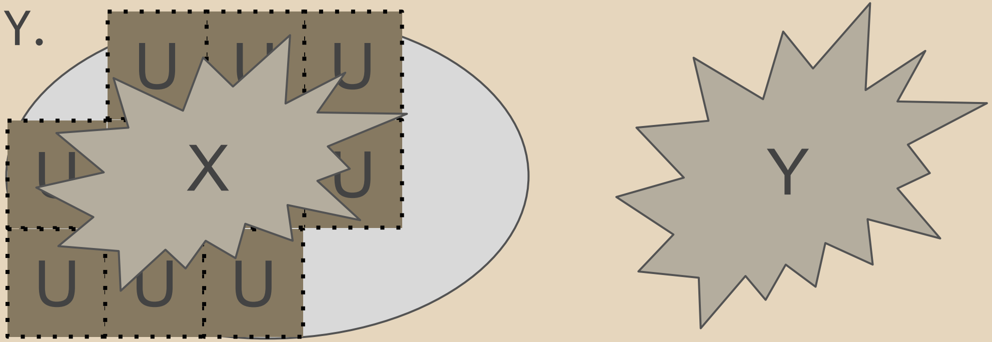

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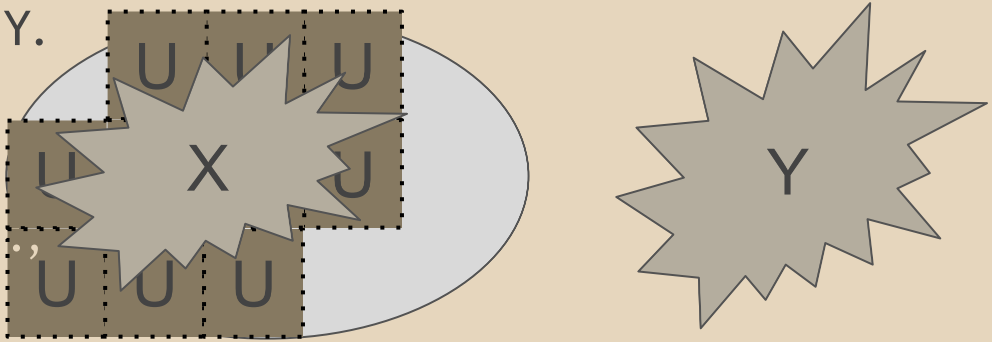

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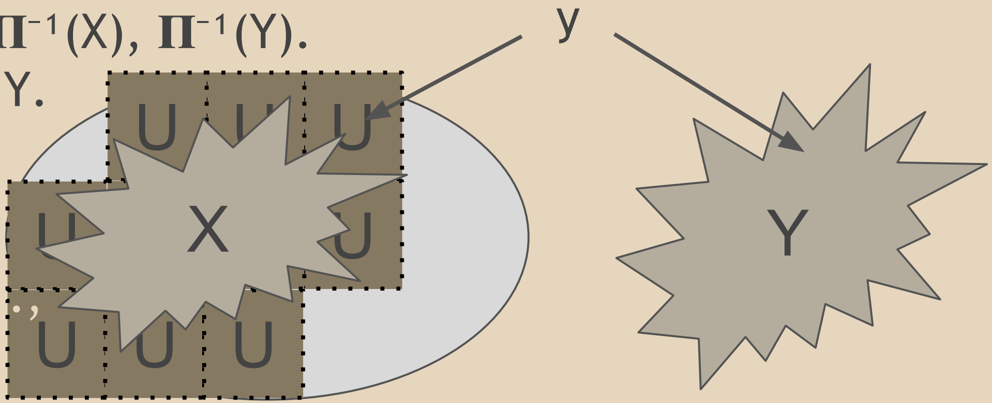

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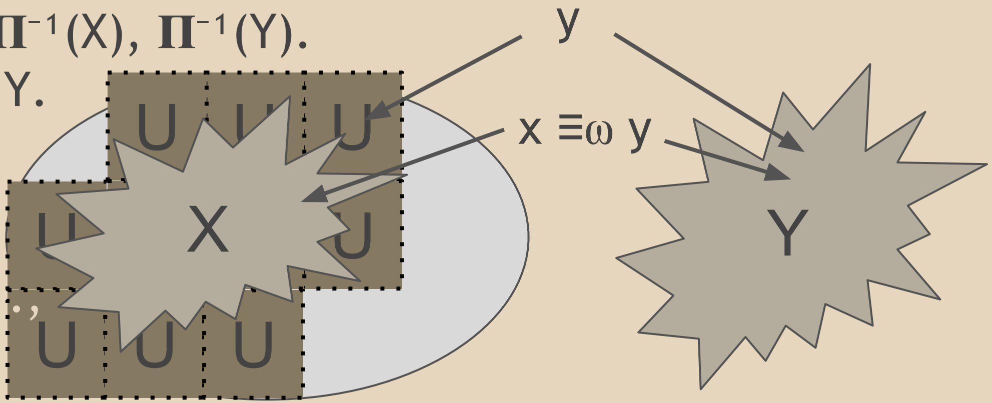

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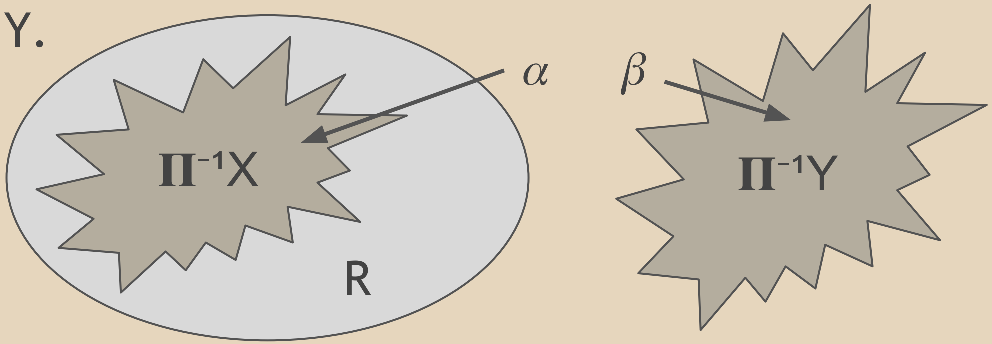

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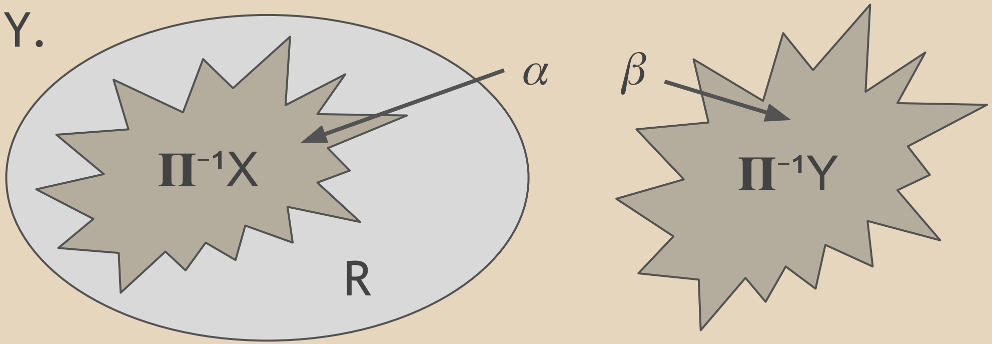

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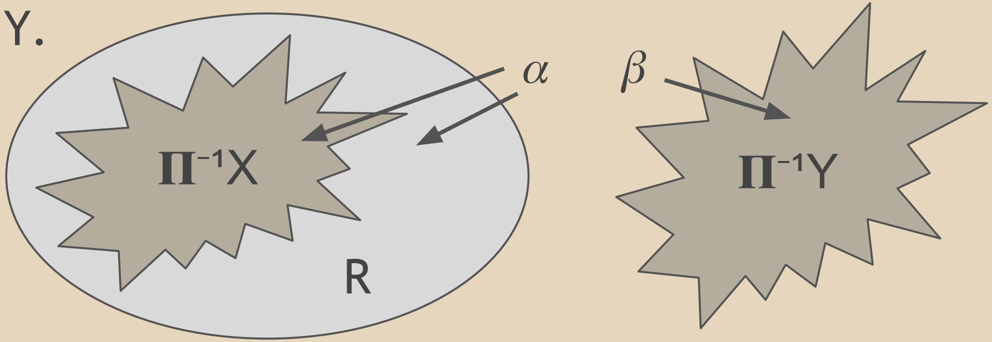

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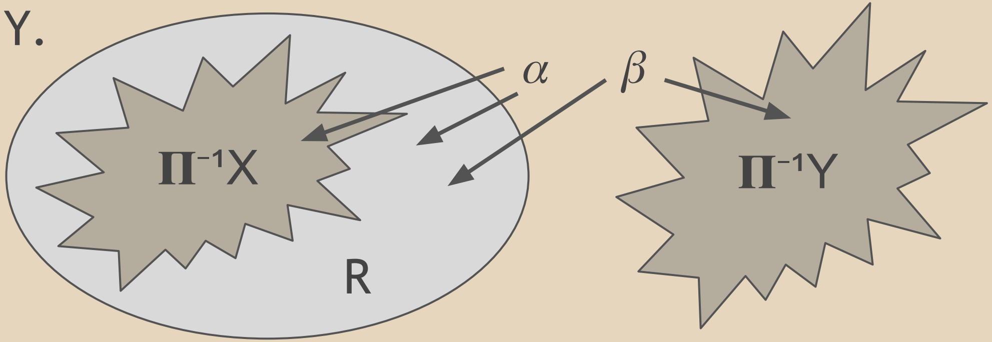

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Decidable separability

1. 1CN [Czerwiński, Lasota LICS’17].

   Via the *Regular Overapproximation* technique
   → **Wojtek Czerwiński’s talk** on Tue 4A 2:05pm.

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Regular separability of $\mathbb{C}$(PN)

Let A, B $\subseteq \Sigma^*$ be PN languages (acceptance by final configuration).

By the previous reduction:

regular separability of the commutative closures $\mathbb{C}(A), \mathbb{C}(B) \subseteq \Sigma^*$ $\xrightarrow{\text{reduces to}}$ unary separability of the Parikh images $\mathbf{\Pi}(A), \mathbf{\Pi}(B) \subseteq \mathbb{N}^d$

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- Add d extra components.
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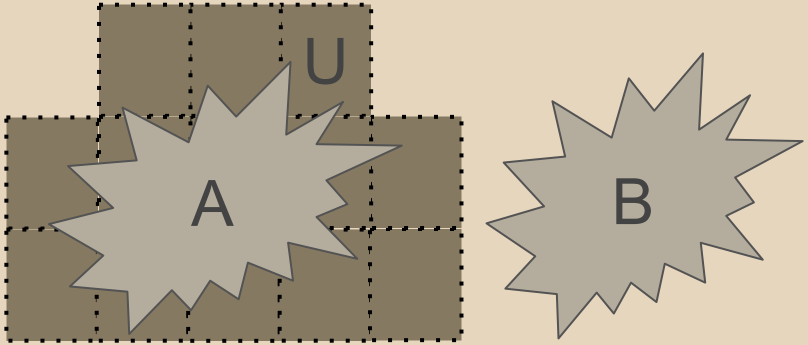
Unary separability of PN reach. sets

Positive separability witness: Unary separator.



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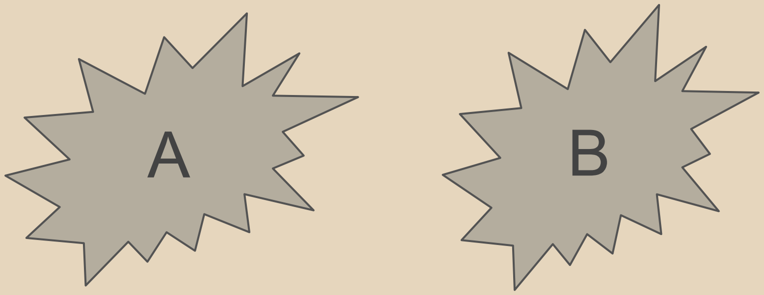

(Linear set: b + P*, for a base b and a finite set of periods P.)

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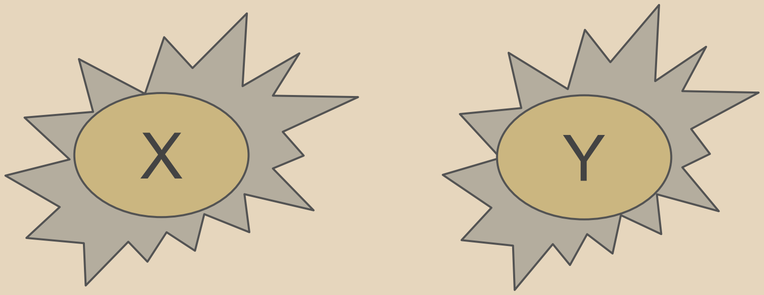

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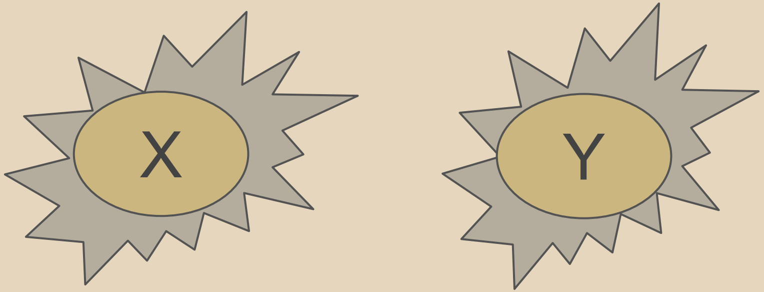

- Checking unary separability of linear sets is decidable [Choffrut, Grigorieff IPL’06].
- Checking inclusion of a linear set into a PN reachability set is decidable [Leroux LICS’13].

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Unary separability of PN reach. sets

Characterisation:

Two sets $A, B \subseteq \mathbb{N}^d$ are *not* unary separable iff
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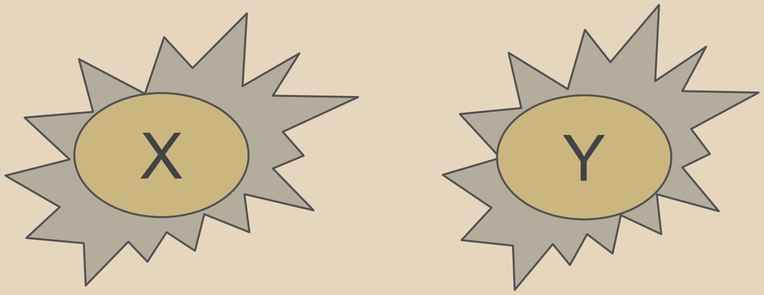

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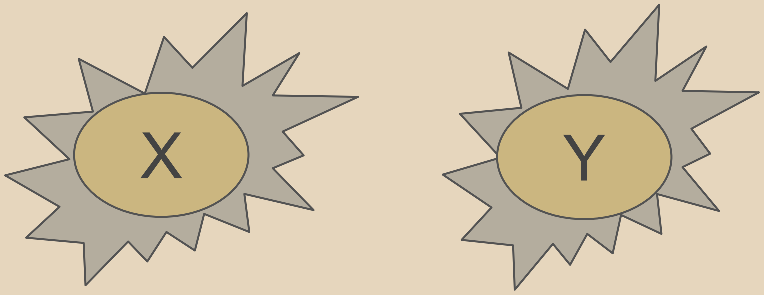
- X is obtained by "folding" the infinite sequence u_i into a linear set.

- Tool: wqo on PN runs
  to extract base + finitely many periods.



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Decidable separability

Theorem. Unary separability of PN reachability sets is decidable [C., Czerwiński, Lasota, Paperman STACS'17].

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Theorem. Regular separability of commutative closures of PN languages is decidable [C., Czerwiński, Lasota, Paperman STACS'17].

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Regular separability of Parikh autom.

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Parikh automaton (PA) [Klaedke, Rueß ICALP'03]:
NFA $A = (\Sigma, Q, I, F, \Delta)$ + semilinear acceptance condition $S \subseteq \mathbb{N}^{|\Delta|}$ on transitions.

- $L(A, S) = \pi(\mathrm{Runs}(A) \cap \mathbf{\Pi}^{-1}(S))$. ($\pi$ returns the word labelling a run.)

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Since *separability is insensitive to nondeterminism*, we can assume DFA.

- $L(A, S) = \mathrm{Runs}(A) \cap \boldsymbol{\Pi}^{-1}(S)$ = Regular ∩ Commutative.

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- $L(A, S) = \pi(\mathrm{Runs}(A) \cap \boldsymbol{\Pi}^{-1}(S))$. ($\pi$ returns the word labelling a run.)

Since *separability is insensitive to nondeterminism*, we can assume DFA.

- $L(A, S) = \mathrm{Runs}(A) \cap \boldsymbol{\Pi}^{-1}(S)$ = Regular $\cap$ Commutative.

Critical difference w.r.t. regularity:

- Regularity is undecidable for PA.
- Regularity is decidable for deterministic PA [Cadilhac, Finkel, McKenzie DLT'12].

Regular separability of Parikh autom.

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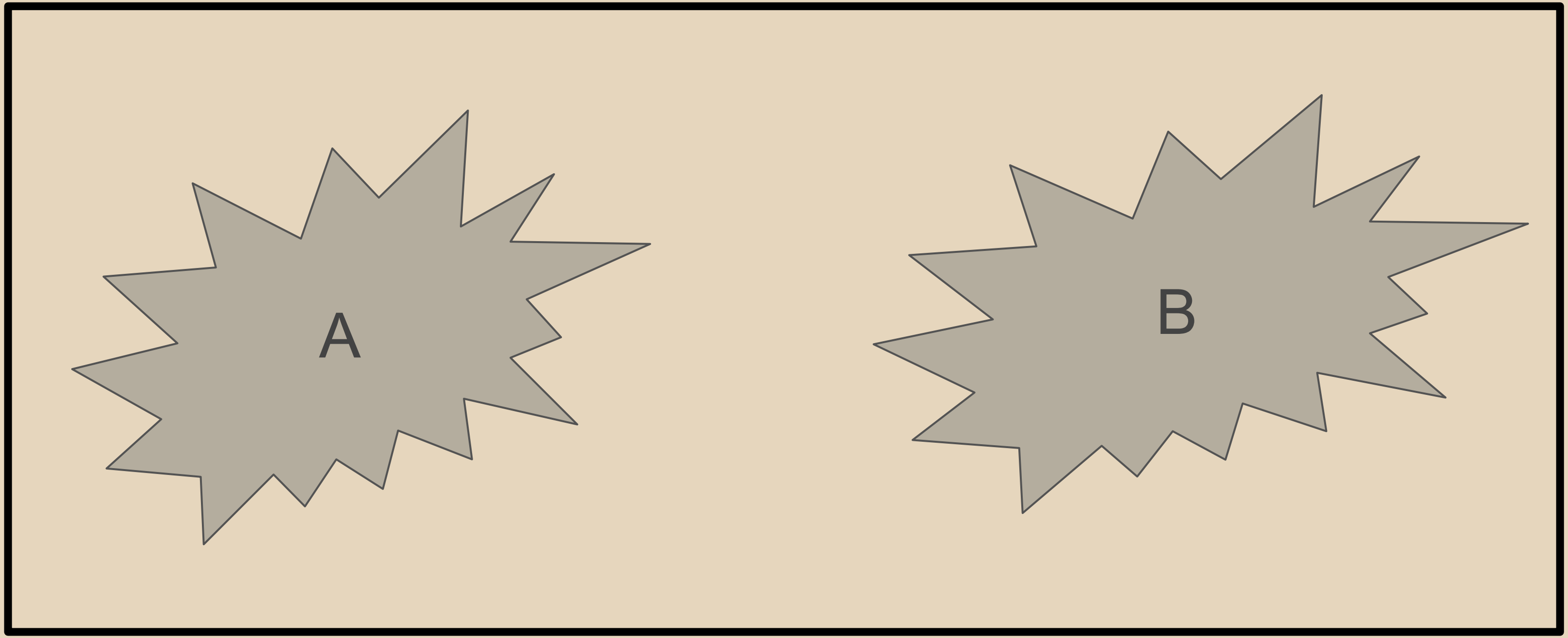
Theorem. Regular separability is decidable for PA/PN($\mathbb{Z}$).

Regular partitioning



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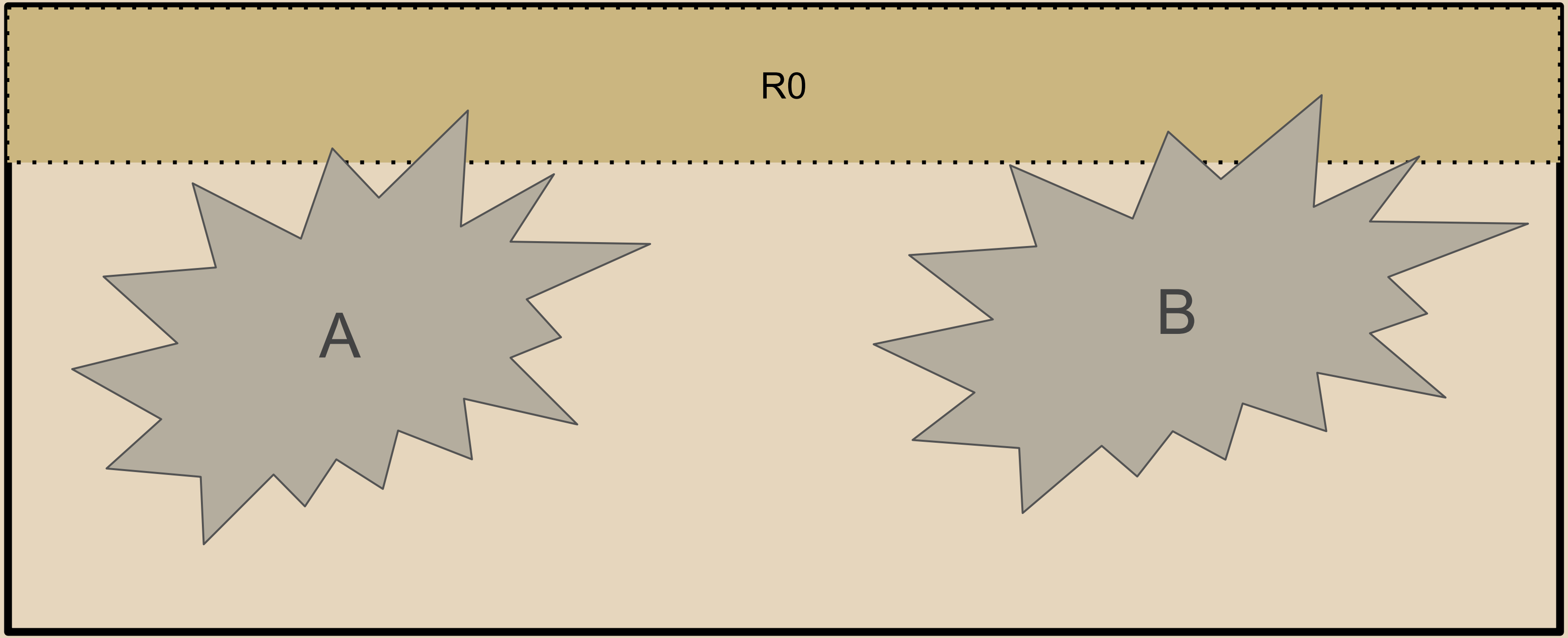

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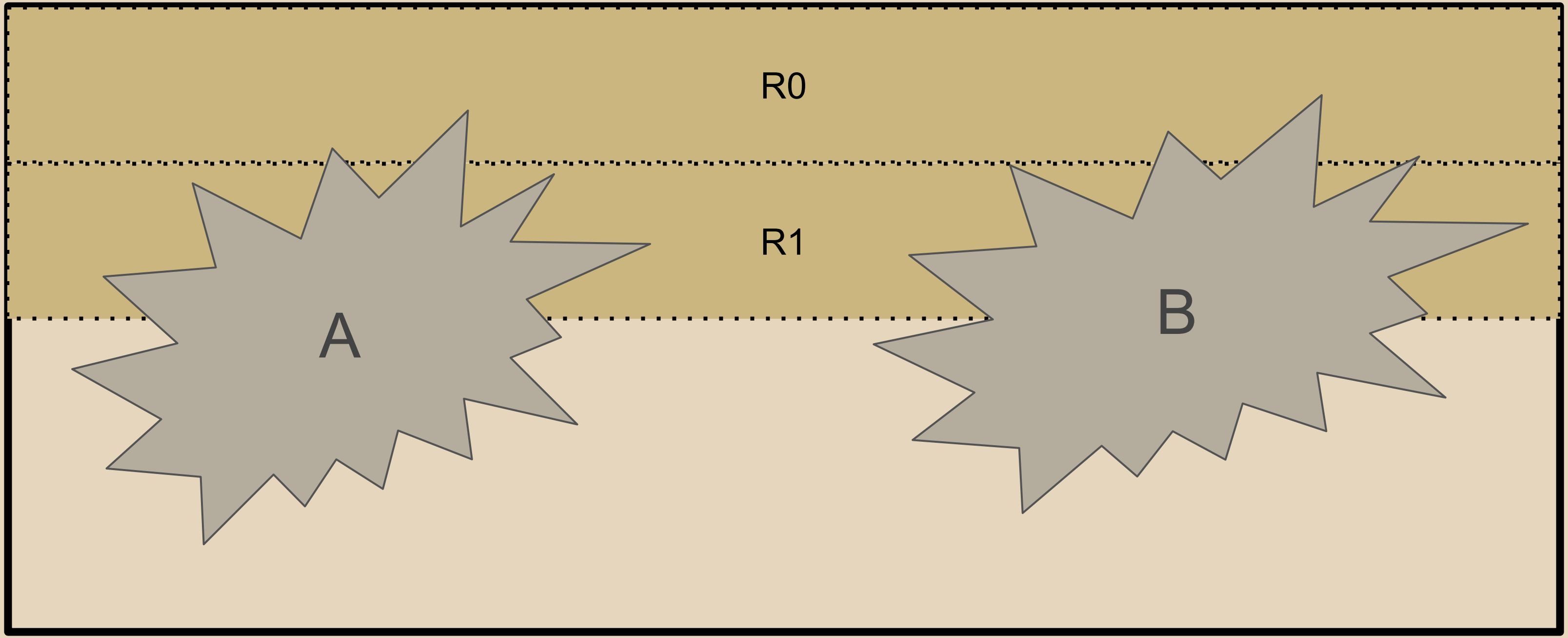

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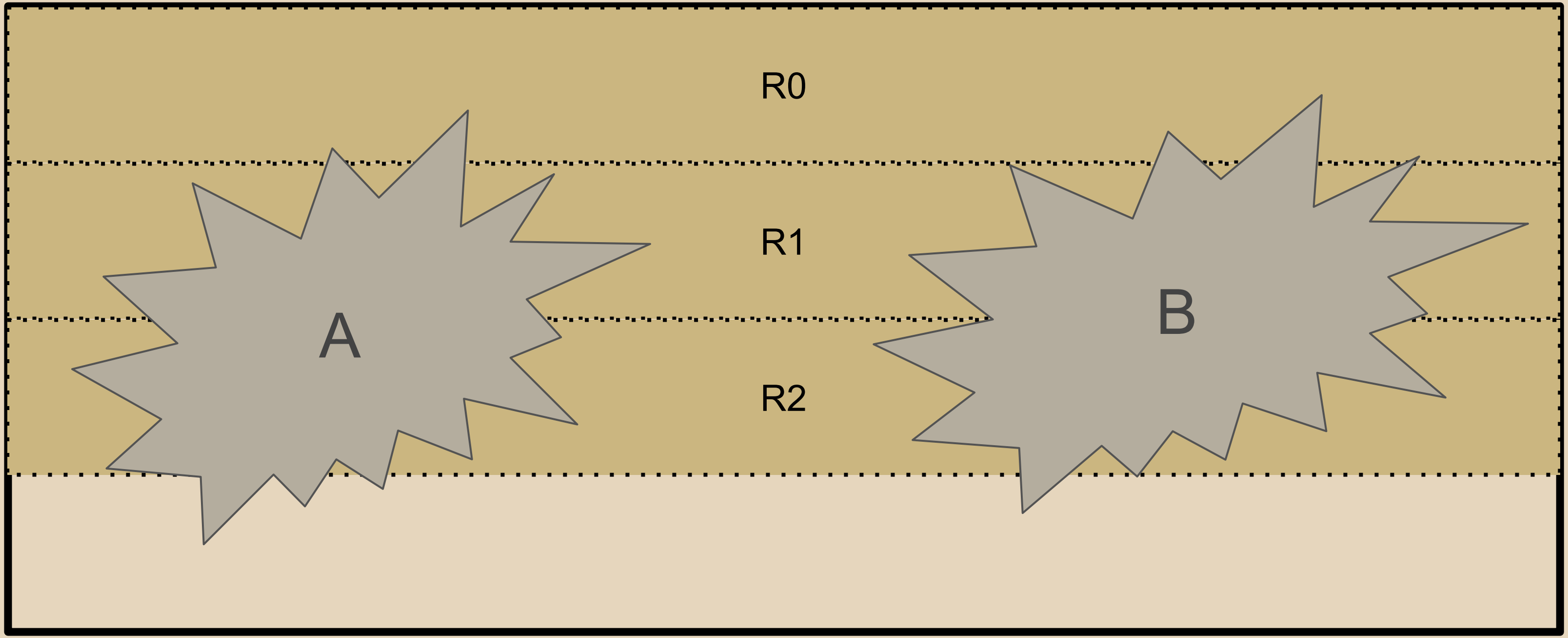

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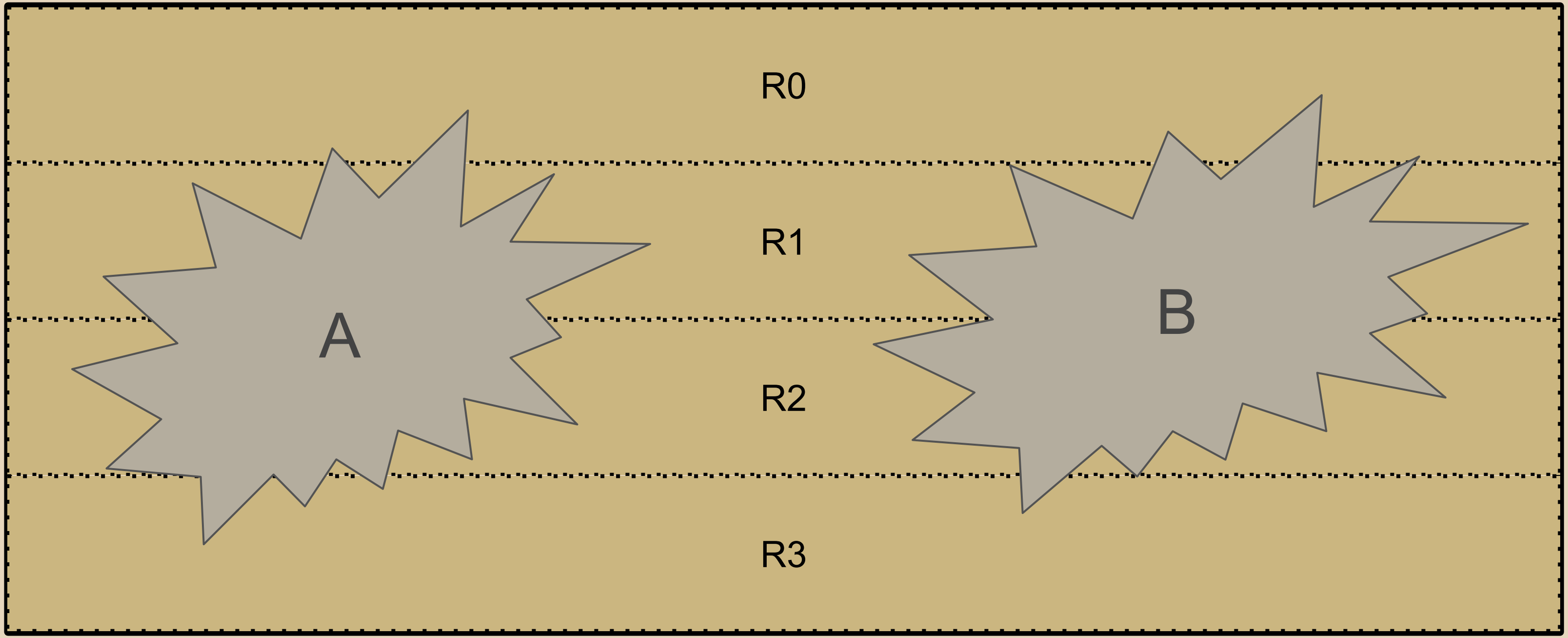

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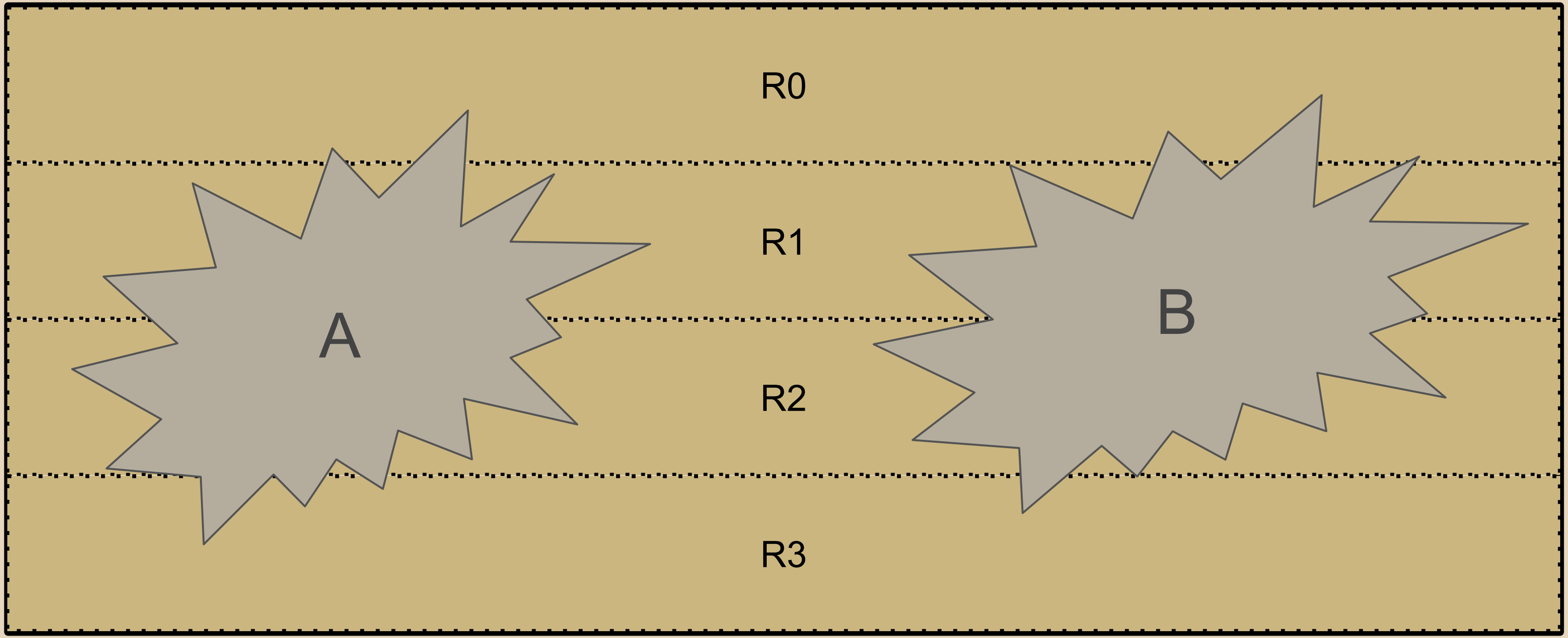
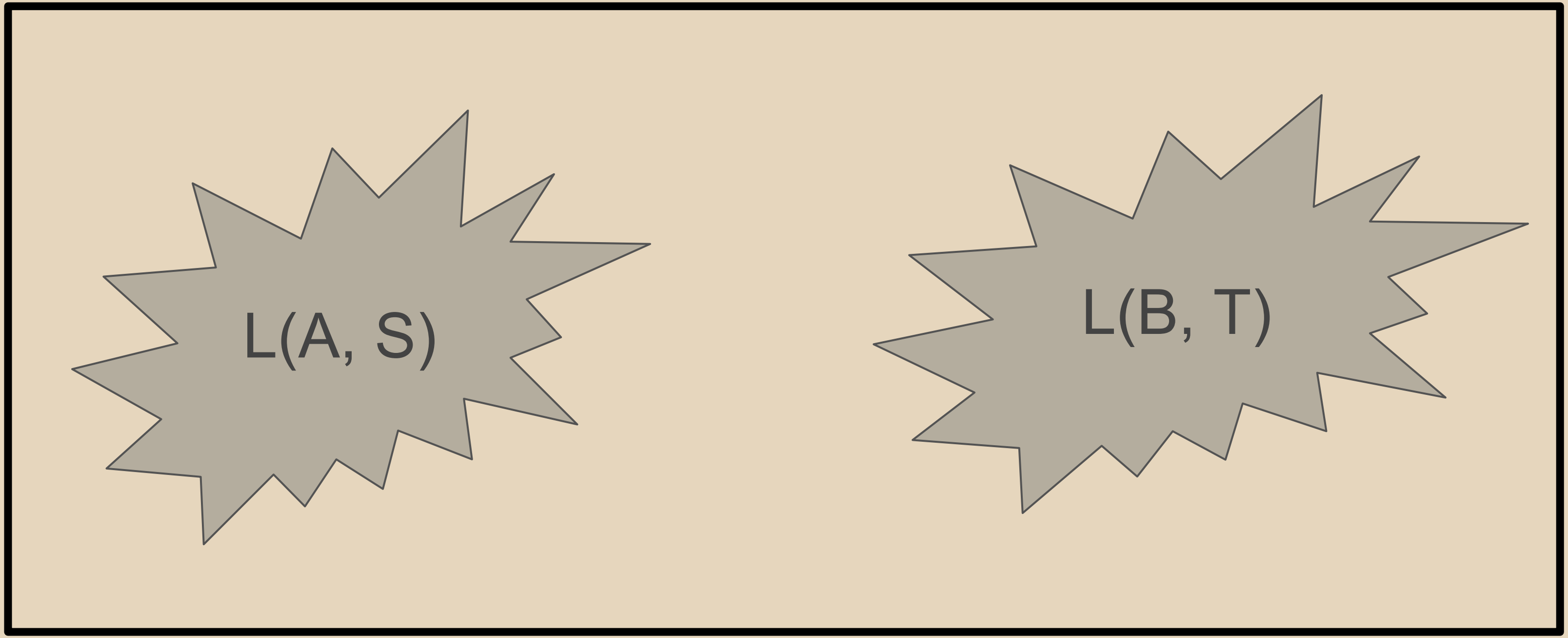


A, B separable iff, for every i, $A \cap R_i$, $B \cap R_i$ separable

Regular partitioning



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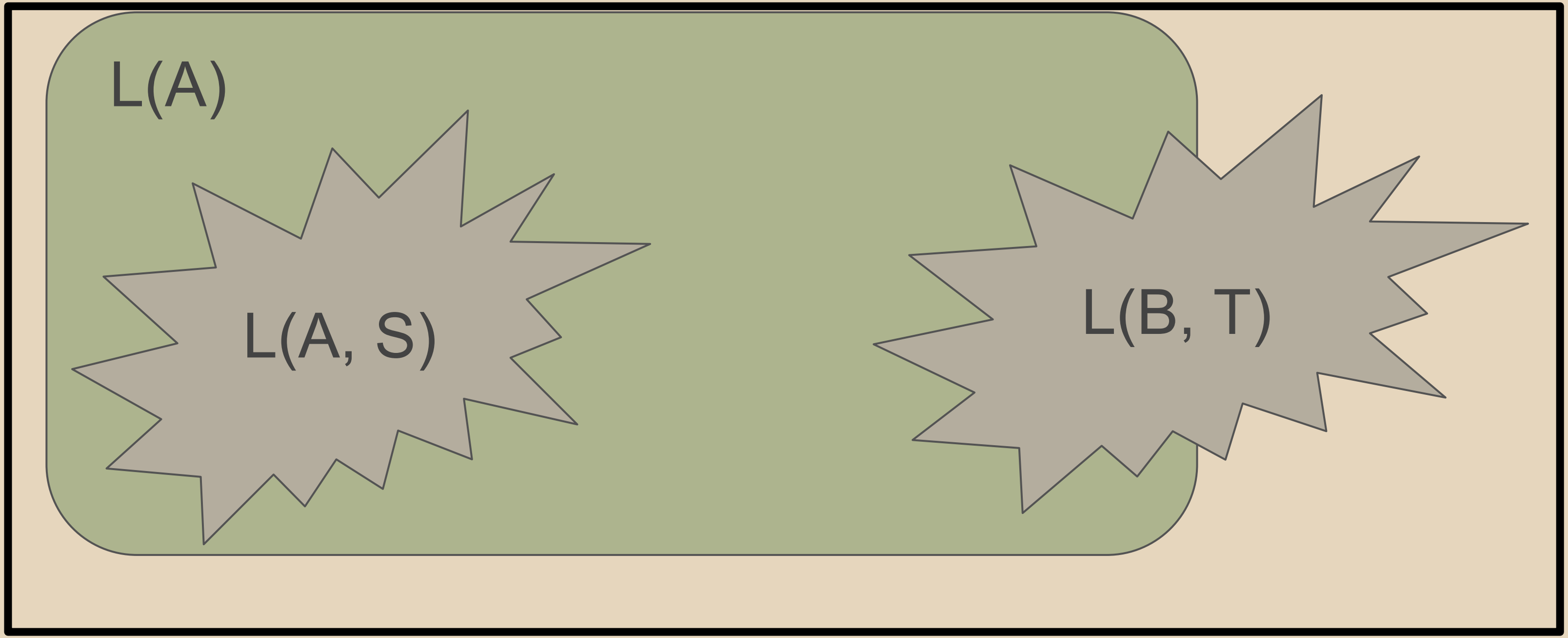

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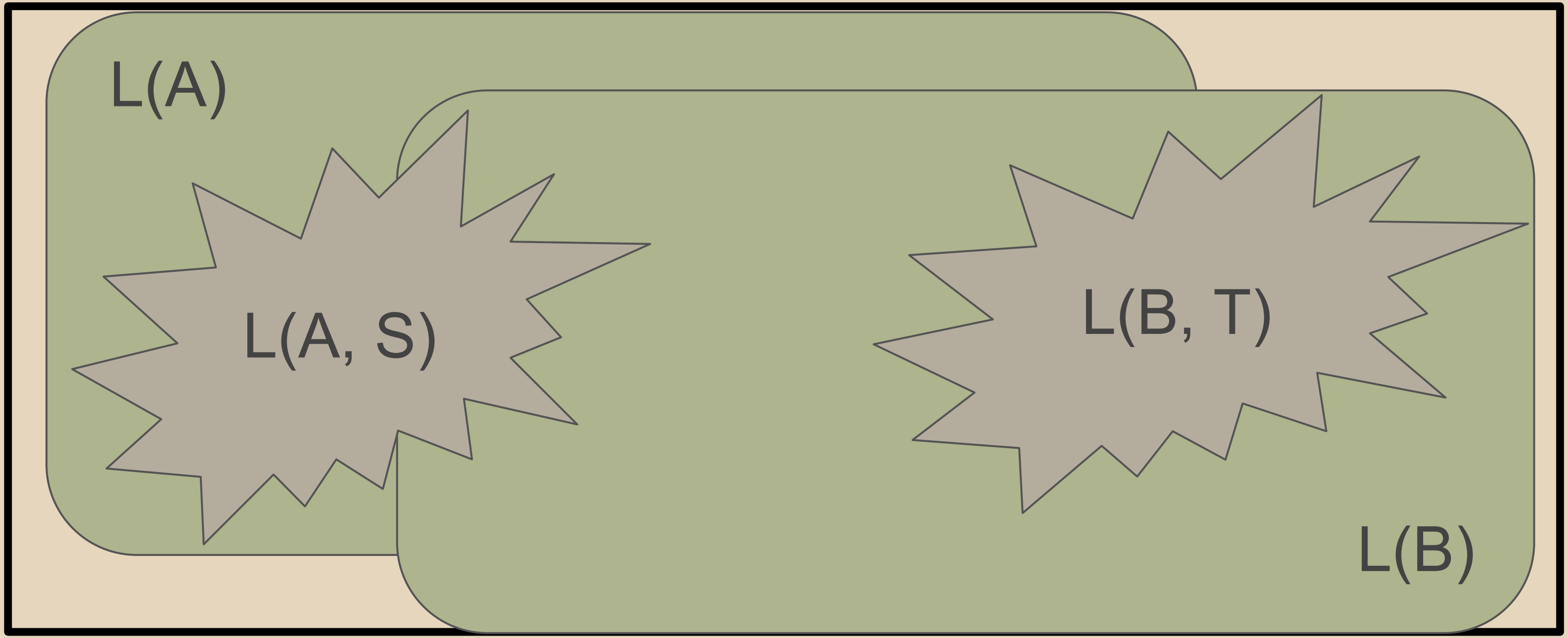

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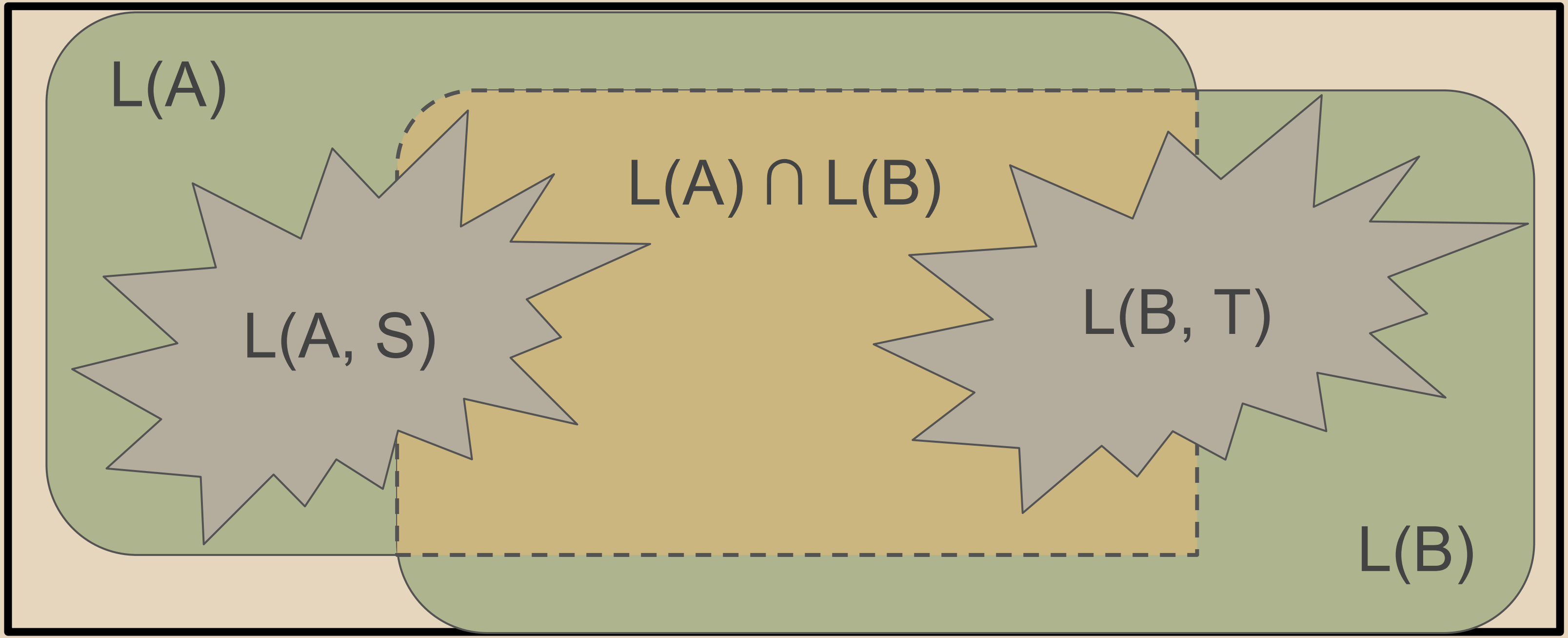

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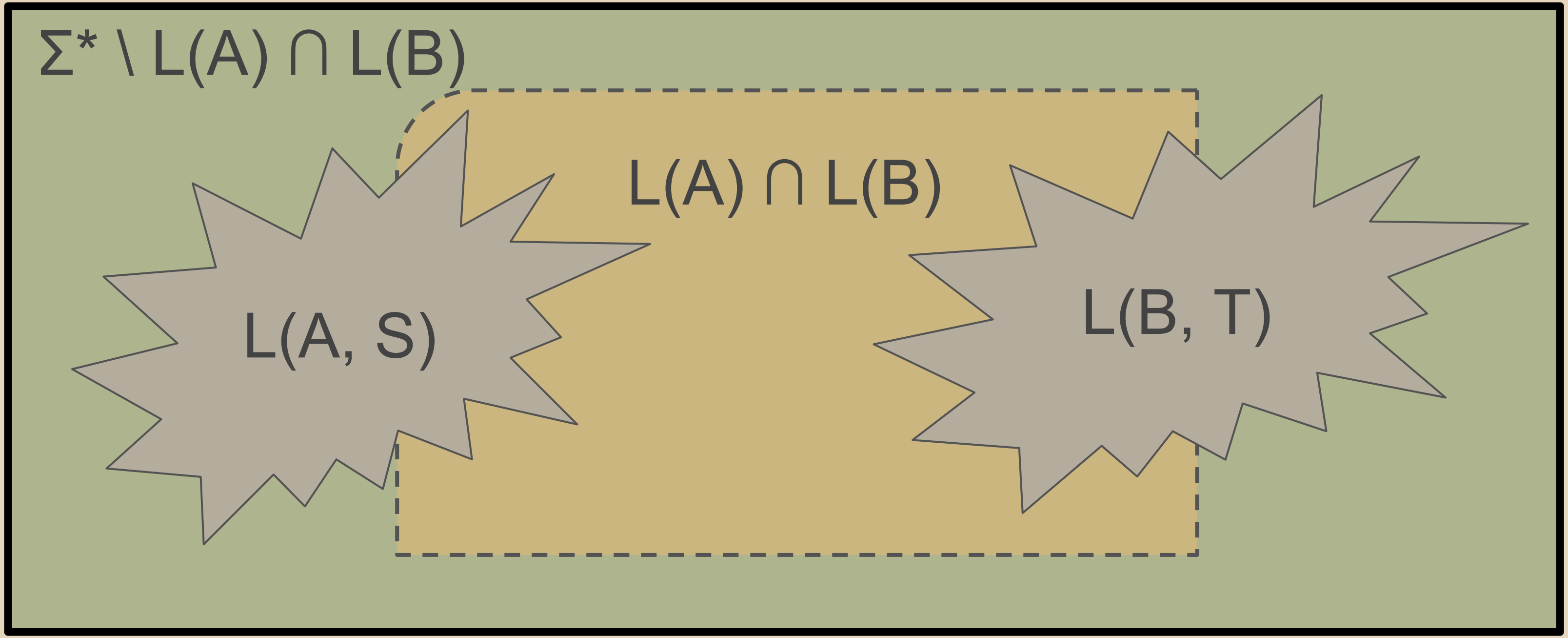

We can assume the *same underlying DFA* for the two PAs.

Regular separability of Parikh autom.

Basic idea: Count simple cycles instead of transitions.

- Once enough states have been visited,
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- This gives a bounded language of cycles.

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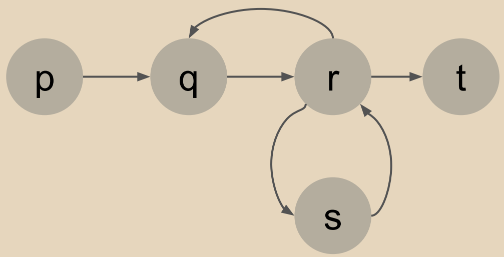
regular separability of PA languages $\xrightarrow{\text{reduces to}}$ regular separability of *bounded* PA languages $\xrightarrow{\text{reduces to}}$ unary separability of *semilinear sets* [Choffrut, Grigorieff ILP’06]

Counting cycles - Example



IDEA

Read a run from left to right
removing simple cycles
visiting only states which appeared so far.

Run: $p \to q \to r \to s \to r \to q \to r \to s \to r \to t$

Skeleton run: ε

Cycles count: $\#[q \to r \to q] = 0$

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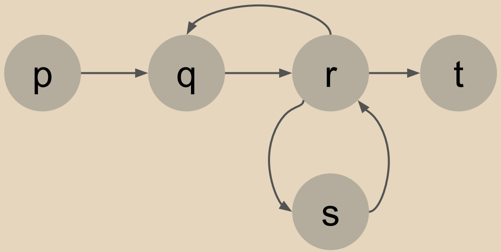

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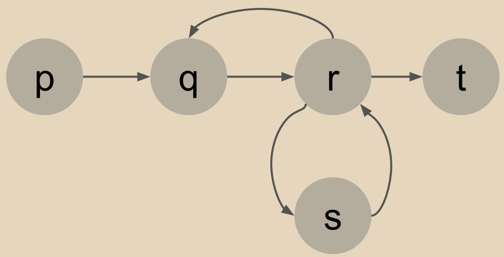

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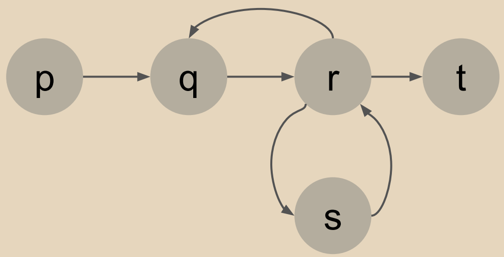

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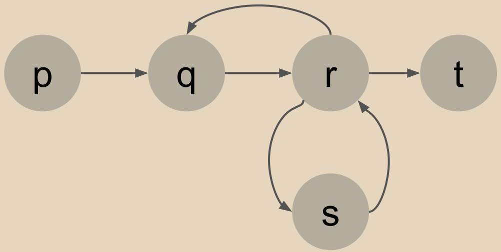

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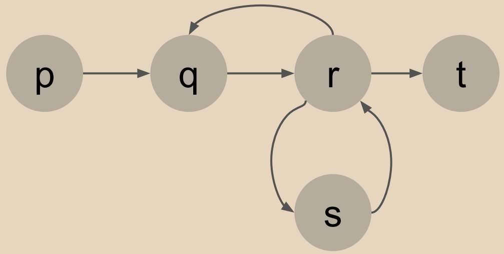

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Run: p→q→r→s→r→q→r→s→r→t

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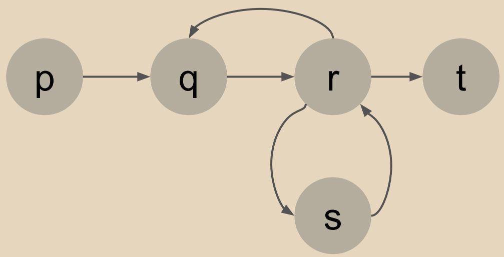

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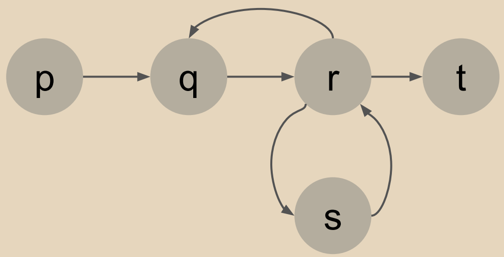

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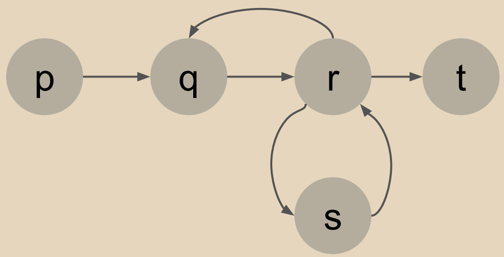

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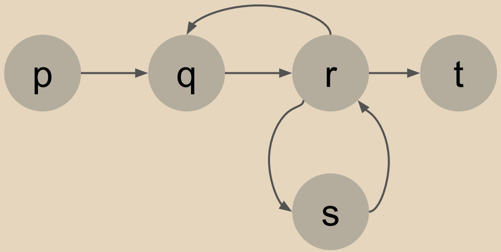

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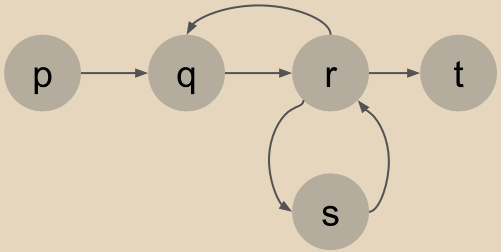

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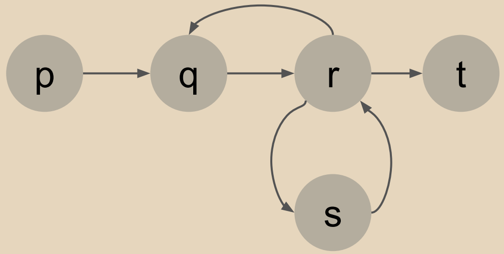

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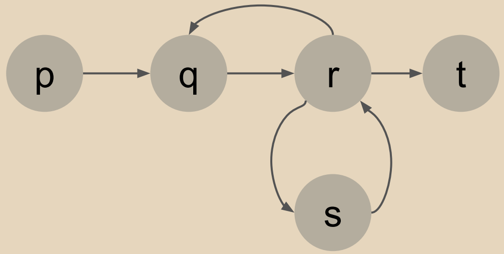

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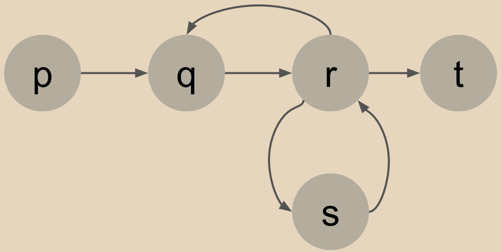

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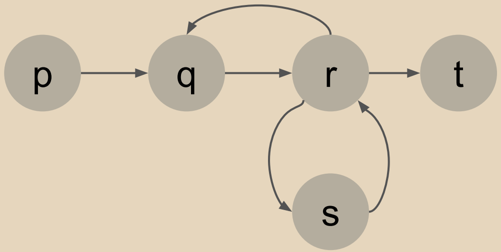

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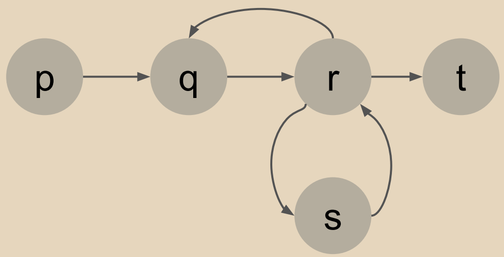

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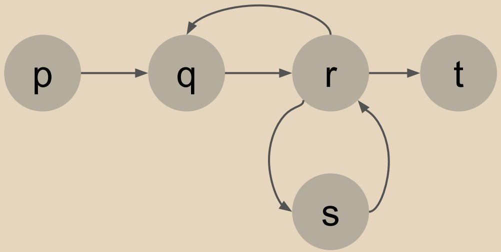

α(ρ) :=(ρ0, (i, j))

Canonical run (cycles appear in a fixed order):
p→q(→r→q)^i→r→s→r(→s→r)^j→t

Run: p→q→r→s→r→q→r→s→r→...→t =: ρ

Skeleton run: p→q→r→s→r→t =: ρ0

Cycles count: #[q→r→q] = i
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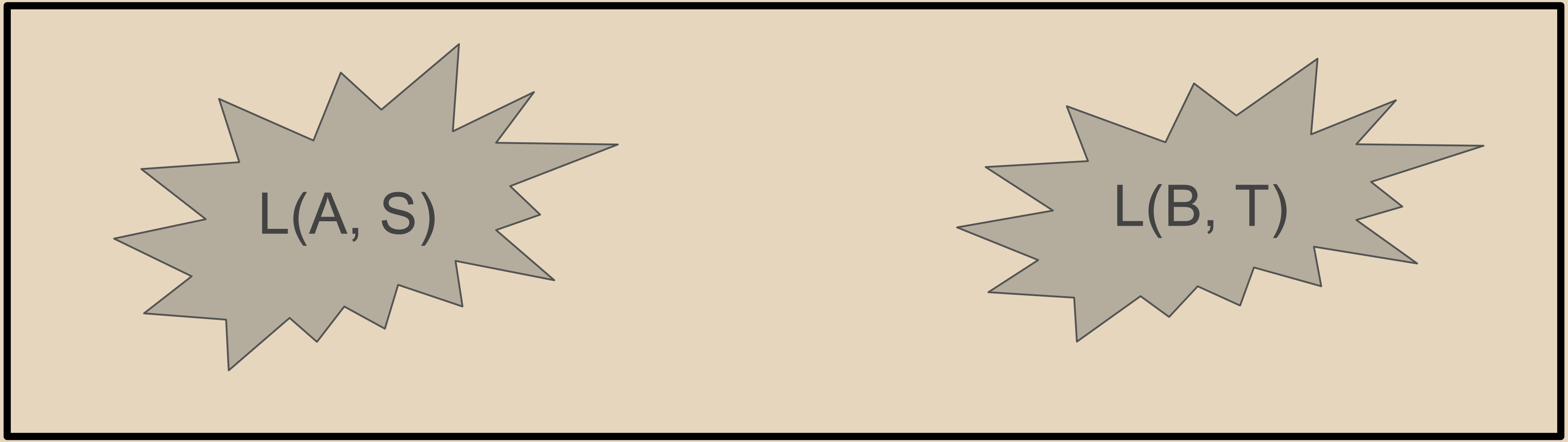
Regular partitioning



Three skeleton runs ρ0, ρ1, ρ2.

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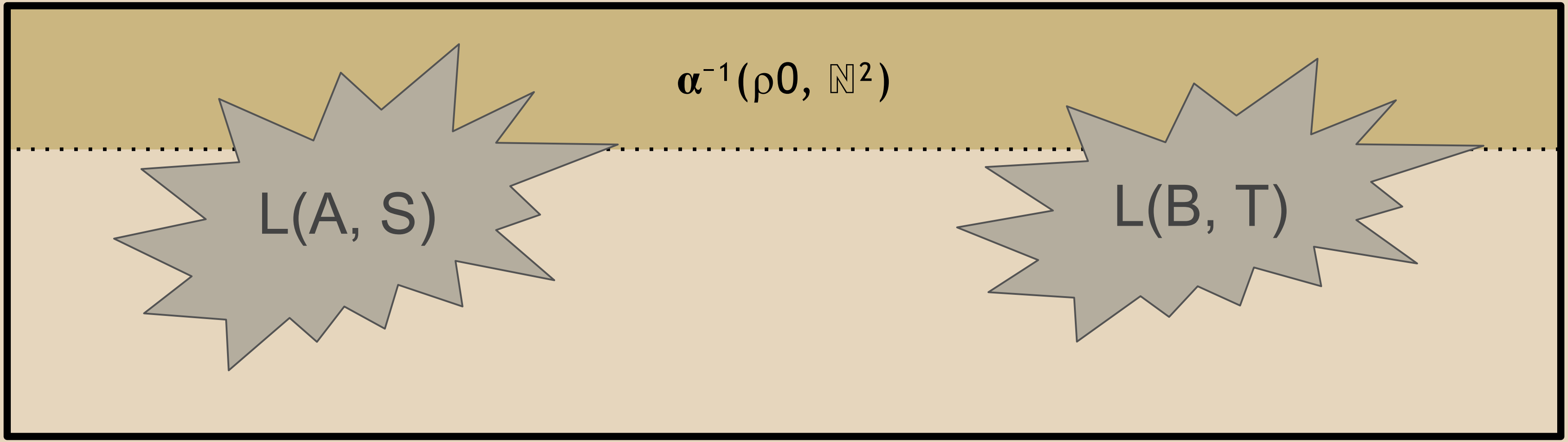

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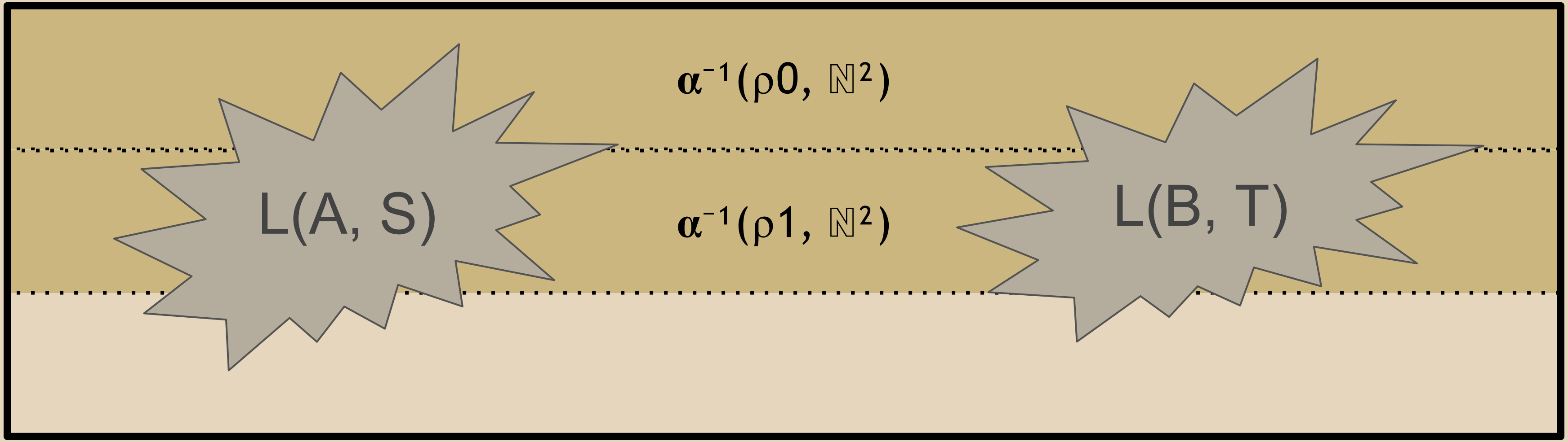

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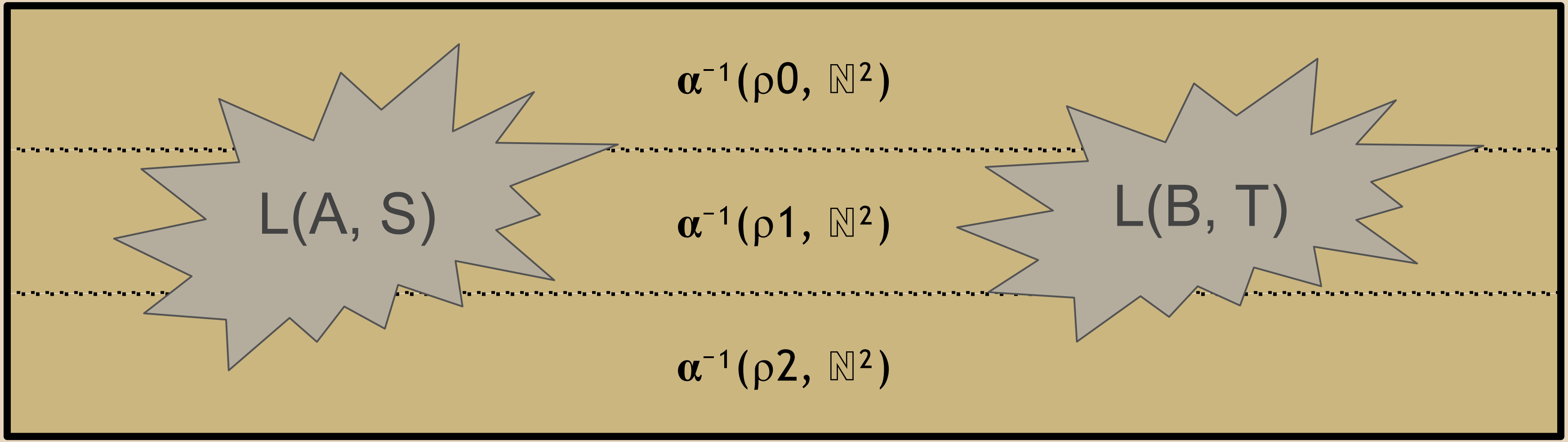

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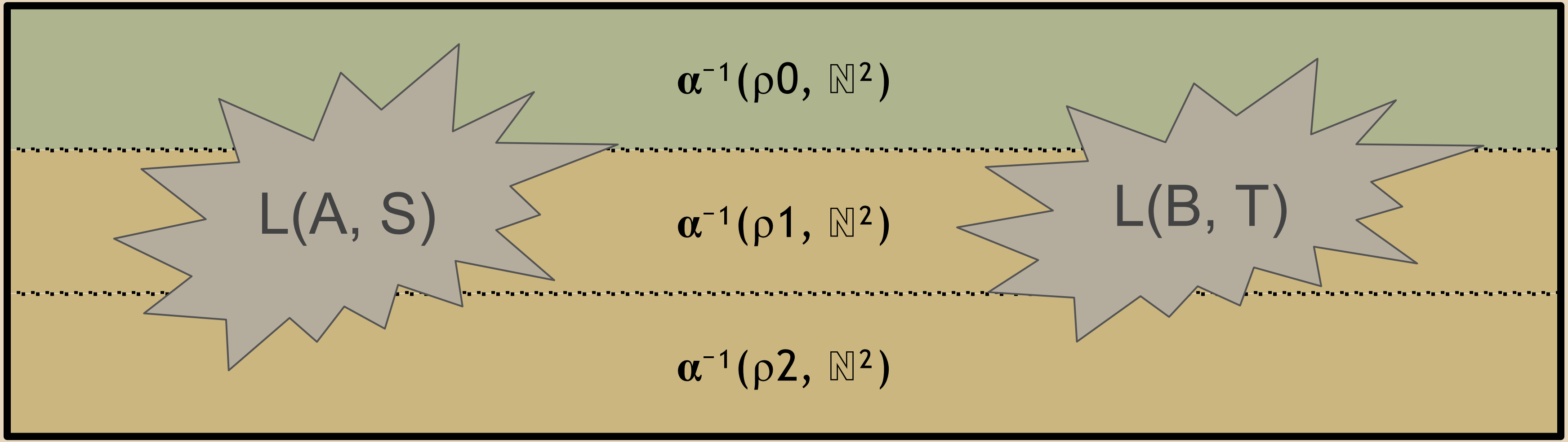
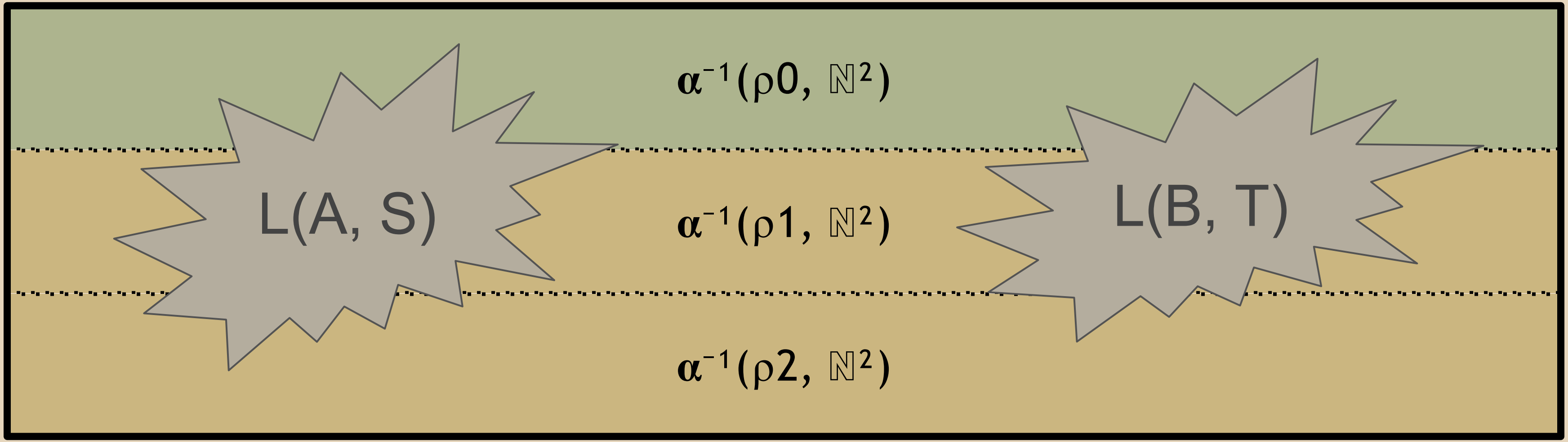


Fix a skeleton run $\rho 0$. Restrict to canonical runs $C := p \to c0^* \to r \to s \to c1^* \to t$.

$A, B \subseteq \Sigma^*$ are regular separable iff $A \cap C$, $B \cap C \subseteq \Sigma^*$ are regular separable

Regular partitioning



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→ Reduction to regular separability of *bounded* PA languages.

(→ Unary separability of semilinear sets.)

Decidable separability

1. 1CN [Czerwiński, Lasota LICS'17].

   Via the *Regular Overapproximation* technique
   → **Wojtek Czerwiński's talk** on Tue 4A 2:05pm.

2. $\mathbb{C}$(PN): commutative closure of PN languages
   [C., Czerwiński, Lasota, Paperman STACS'17]. ✔︎□

3. PA/PN($\mathbb{Z}$) [C., Czerwiński, Lasota, Paperman ICALP'17]. ✔︎□

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4. [Conjecture] Separability of PN languages is decidable.

Towards separability of Petri nets

Possible techniques:

- Regular over-approximations of Petri net languages?
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Other interesting directions

- Promise problems:
  - Given two CFGs for L and its complement, decide whether L is regular.
  - Cf. undecidability of regularity for CFLs.

DISCARDED SLIDES

Undecidable separability for 1CA

Technique: (polynomial) reduction from *every* decidable problem.

- Decidable problem (up to computable encoding):
  $L \subseteq \mathbb{N}$ recognised by deterministic, total 2CA (2 counters with zero test).
- If regular separability has time complexity $f(n)$,
  then every decidable problem has time complexity $f(p(n))$.
- This contradicts the time hierarchy theorem.

Reduction: From a 2CA and input k build two 1CA A, B s.t.

2CA accepts k iff A, B are regular separable.

- Each 1CA simulates 1 counter.
- If 2CA rejects, let n be the length of the rejecting computation.
- We can separate A, B by looking only at prefixes of length n:

$$L(A) \cap \Sigma^{(<n)} \cup \{ xy \mid x \in \mathrm{Prefix}(L(A), n), y \in \Sigma^* \}$$

Separability for $\mathbb{C}$(PN)

By reduction to (1) *unary separability* in $\mathbb{N}^d$ of *reachability sets* of PNs

$\mathbb{C}(A)$, $\mathbb{C}(B) \subseteq \Sigma^*$ are separable by a regular language

(2) iff

$\mathbb{C}(A)$, $\mathbb{C}(B) \subseteq \Sigma^*$ are separable by a *commutative* regular language

(3) iff

$\mathbf{\Pi}(A)$, $\mathbf{\Pi}(B) \subseteq \mathbb{N}^d$ are separable by a *unary set* → union of i-unary equivalence classes

Vectors u, v $\in \mathbb{N}^d$ are *i-unary equivalent* $u \equiv_i v$ if $\forall\ 1 \leq k \leq d$,

- Equivalent modulo n: $u[k] \equiv v[k] \bmod i$.
- Either both big, or both small: $u[k] \geq i$ iff $v[k] \geq i$.

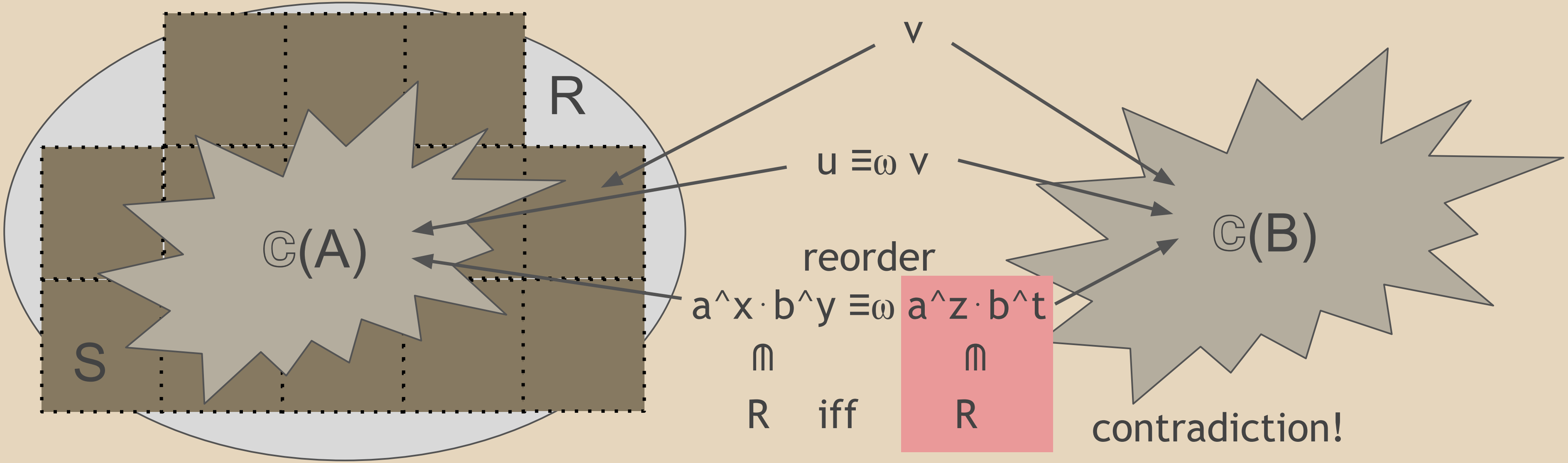
For two words u, v $\in \Sigma^*$, let $u \equiv_i v$ if $\mathbf{\Pi}(u) \equiv_i \mathbf{\Pi}(v)$.

Separability for $\mathbb{C}$(PN) - Step (2)

Let $\omega \in \mathbb{N}$ be the idempotent power of the syntactic monoid of R, i.e.,

$$(*)\ \forall x, y, z \in \Sigma^* .\ x \cdot y^\omega \cdot z \in R \text{ iff } x \cdot y^{2\omega} \cdot z \in R$$

- $S := \{ u \in \Sigma^* \mid \exists\, v \in \mathbb{C}(A) .\ u \equiv_\omega v \}$ separates $\mathbb{C}(A)$, $\mathbb{C}(B)$:



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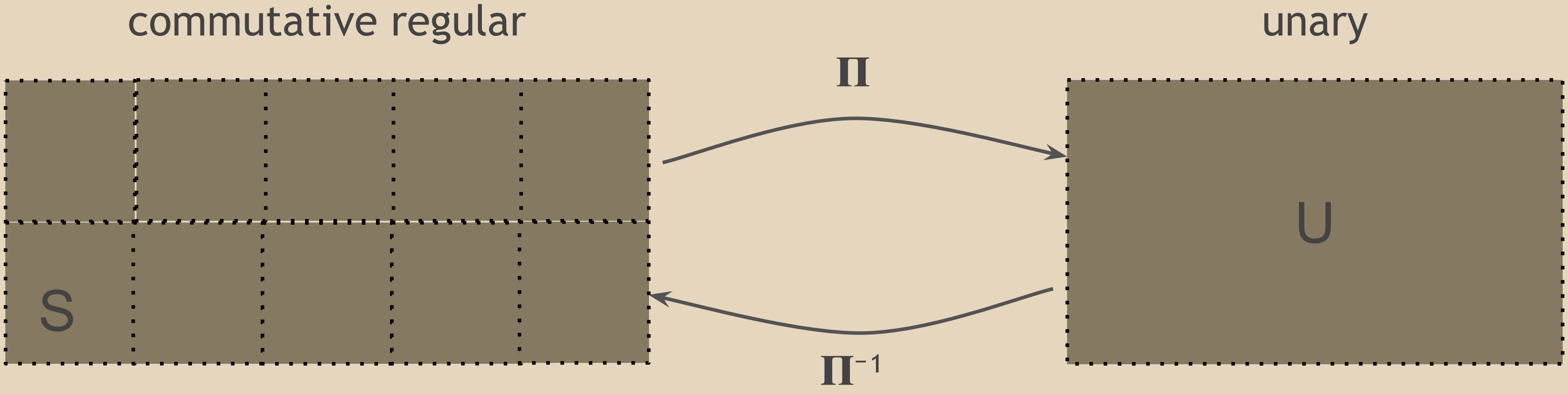
- S is commutative regular since $\mathbf{\Pi}(S)$ is ω-unary.

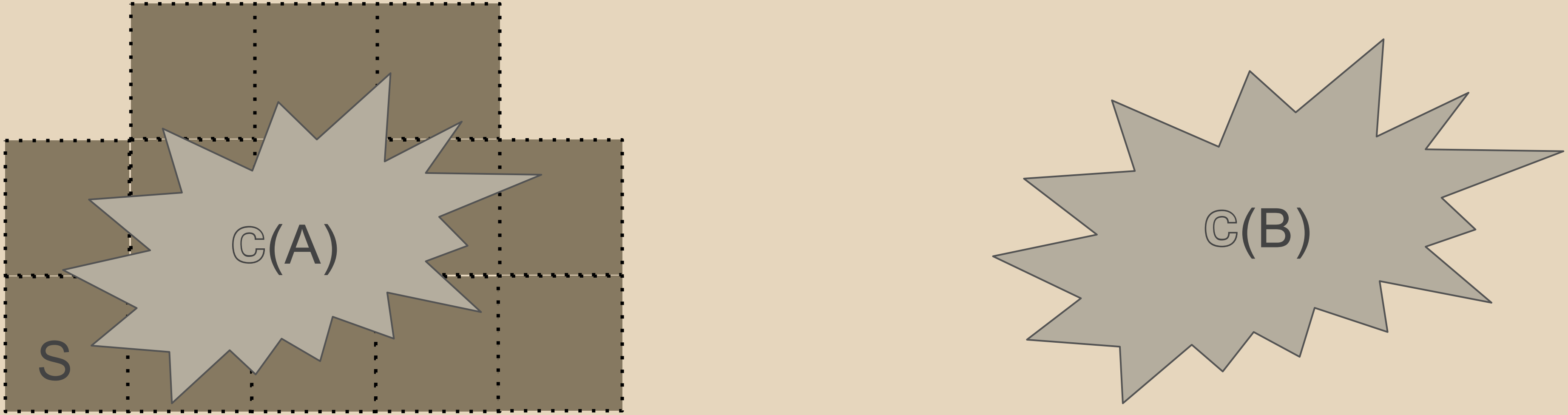


Separability for $\mathbb{C}$(PN) - Step (3)

$\mathbb{C}(A), \mathbb{C}(B) \subseteq \Sigma^*$ are separable by a *commutative* regular language S

(3) iff

$\mathbf{\Pi}(A), \mathbf{\Pi}(B) \subseteq \mathbb{N}^d$ are separable by a *unary set* $U = \mathbf{\Pi}(S)$

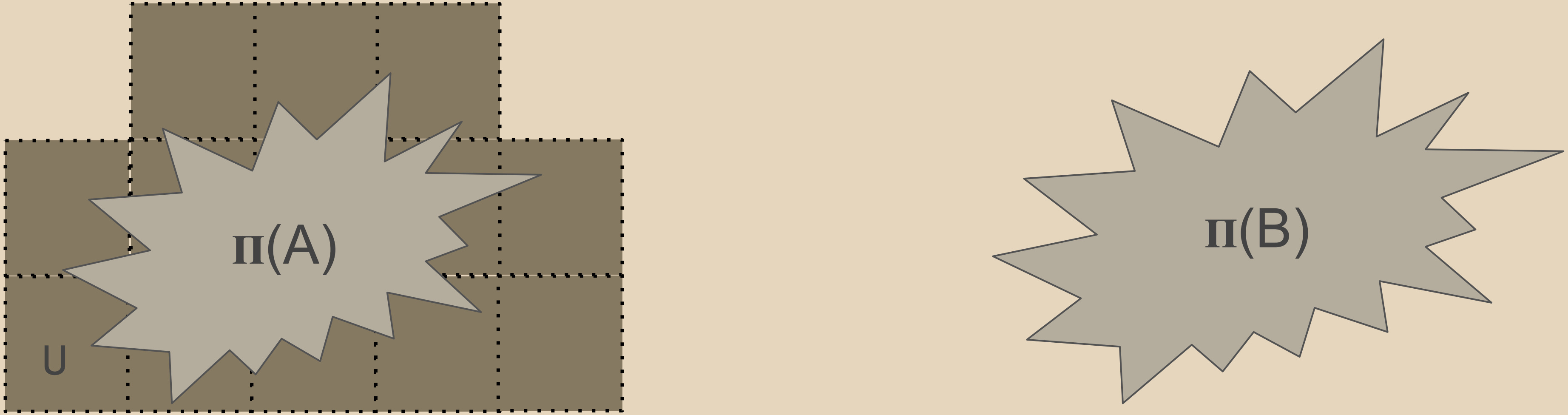


Separability for ℂ(PN) - Step (3)

ℂ(A), ℂ(B) $\subseteq \Sigma^*$ are separable by a *commutative* regular language S

(3) iff

$\mathbf{\Pi}$(A), $\mathbf{\Pi}$(B) $\subseteq \mathbb{N}^d$ are separable by a *unary set* U = $\mathbf{\Pi}$(S)



Separability for ©(PN) - Step (1)

$\mathbb{C}(A), \mathbb{C}(B) \subseteq \Sigma^*$ are separable by a *commutative* regular language S

(3) iff

$\boldsymbol{\Pi}(A), \boldsymbol{\Pi}(B) \subseteq \mathbb{N}^d$ are separable by a *unary set* $U = \boldsymbol{\Pi}(S)$

(projections of) PN reachability sets

- Add d extra components.
- Increment i-th extra component when reading a_i.
- Project away the other components.

(1) Reduction to *unary separability* in $\mathbb{N}^d$ of *projections of PN reachability sets*

Regular separability of Parikh autom.

NFA with transitions labelled by vectors in $\mathbb{Z}^d$.
Acceptance by reaching a final state with value $0 \in \mathbb{Z}^d$.

Since *separability is insensitive to nondeterminism*, we can assume DFA.
A deterministic PN($\mathbb{Z}$) language is the intersection of:

- A regular language (the one recognised by the underlying DFA),
- The inverse Parikh image of a semilinear set.

Critical difference w.r.t. regularity:

- Regularity is undecidable for PN($\mathbb{Z}$).
- Regularity is decidable for deterministic PN($\mathbb{Z}$) [Cadilhac, Finkel, McKenzie DLT'12].

Theorem. Regular separability is decidable for PN($\mathbb{Z}$).

Counting cycles

- Simple cycle: sequence transitions starting and ending in the same state, where no other state repeats.
- Two cycles are *equivalent* if one is a cyclic permutation of the other.
- Fix an enumeration of equivalence classes of simple cycles [c_1], …, [c_d].
- In order to freely reorder simple cycles, adding or removing a cycle must not change the set of visited states → need to visit enough states before cycles can be reordered independently.

IDEA

Read a run from left to right
removing simple cycles visiting only states which appeared so far.