# Relating complementation constructions for Büchi automata

(draft)

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Abstract.

### 1 Preliminaries

Fix a finite alphabet  $\Sigma$  and a finite set of states Q. A transition profile (over Q) t is a pair  $(\to_t, \to_t)$ , where  $\to_t \subseteq Q \times Q$  and  $\to_t \subseteq \to_t$ . Let  $\mathcal P$  be the set of transition profiles. Intuitively,  $p \to_t q$  if it is possible to go from p to q via a transition in t, and  $p \to_t q$  if, additionally, an accepting such transition is taken. A Büchi Automaton (BA)  $\mathcal A$  is a tuple  $(\Sigma, Q, I, (t_a)_{a \in \Sigma})$  where  $\Sigma$  is a finite alphabet, Q is a finite set of states,  $I \subseteq Q$  is a non-empty set of initial states, and, for each input symbol  $a \in \Sigma$ ,  $t_a$  is a transition profile over Q. For simplicity, instead of writing  $p \to_{t_a} q$  or  $p \to_{t_a} q$ , we just write  $p \to q$  and  $p \to q$ , respectively.

A infinite trace of  $\mathcal A$  on a word  $w=a_0a_1\cdots\in\Sigma^\omega$  starting in a state  $q_0\in Q$  is an infinite sequence of transitions  $\pi=q_0\stackrel{a_0}{\to}q_1\stackrel{a_1}{\to}\cdots$ . An infinite trace is initial if it starts in an initial state  $q_0\in I$ , and it is fair iff  $q_i\stackrel{a_i}{\to}q_{i+1}$  for infinitely many i. The language of  $\mathcal A$  is  $\mathcal L(\mathcal A)=\{w\mid \mathcal A \text{ has an infinite, initial and fair trace on }w\}$ .

## 2 Ramsey

Fix an automaton  $\mathcal{A} = (\Sigma, Q, I, F, \delta)$ . For two profiles s and t, their product st is the pair  $(\rightarrow, \twoheadrightarrow)$  defined as follows:

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- p \rightarrow q iff there exists r \in Q s.t. p \rightarrow_s r \rightarrow_t q, and

- p \rightarrow q iff there exists r \in Q s.t. either p \rightarrow_s r \rightarrow_t q or p \rightarrow_s r \rightarrow_t q.
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Clearly, product of profiles is an associative operation.

For a finite word  $u = a_0 a_1 \cdots a_m \in \Sigma^*$ , the induced profile is  $t_u = t_{a_0} t_{a_1} \cdots t_{a_m}$ . The language of a profile t is the set of words  $\mathcal{L}(t)$  inducing that profile, i.e.,  $\mathcal{L}(t) = \{u \in \Sigma^* \mid t_u = t\}$ , and a profile is valid iff it has nonempty language (i.e., if it can be generated from profiles in  $(t_a)_{a \in \Sigma}$ ). Let  $\mathcal{P}^v$  be the set of valid profiles. The relationship between multiplication of profiles and their language is as follows.

**Lemma 1.** For two profiles s and t,  $\mathcal{L}(s)\mathcal{L}(t) \subseteq \mathcal{L}(st)$ , and the inclusion is strict in general.

The fundamental property of profiles is that they only yield trivial intersections with *A*.

**Lemma 2.** For two profiles s and t, if  $\mathcal{L}(s)(\mathcal{L}(t))^{\omega} \cap A \neq \emptyset$ , then  $\mathcal{L}(s)(\mathcal{L}(t))^{\omega} \subseteq A$ .

generalize the test to every pair?

A pair of profiles (s,t) is *linked* iff st = s and tt = t. We define a test operation on linked profiles: For two linked profiles s and t, let ?(s,t) iff  $p \rightarrow_s q \rightarrow_t q$  with  $p \in I$ .

**Lemma 3.** For two linked profiles s and t, ?(s,t) holds iff  $\mathcal{L}(s)(\mathcal{L}(t))^{\omega} \subseteq \mathcal{L}(\mathcal{A})$ .

Profiles define an action on sets of states: For a set of states  $P \subseteq Q$  and a profile t, let  $P \cdot t$  be those states which are reachable through transitions in t originating from states in P; i.e.,  $P \cdot t = \{q \mid \exists (p \in P) \cdot p \rightarrow_t q\}$ . Clearly,  $P \cdot (st) = (P \cdot s) \cdot t$ . For simplicity, if  $w \in \mathcal{L}(t)$ , then we also just write  $I \cdot w$  instead of  $I \cdot t$ .

Describe Ramsey-based complementation.

### 3 Ranks

Let  $w = a_0 a_1 \cdots \in \Sigma^{\omega}$  be an infinite word. The infinite traces of w on  $\mathcal{A}$  can be arranged by juxtaposition of transition profiles into an infinite transition DAG  $G = \langle V, T \rangle$ , where

- $V \subseteq Q \times \omega$  is the set of vertices s.t.  $(q,i) \in V$  iff  $q \in I \cdot (a_0 a_1 \cdots a_{i-1})$ , and
- T is a transition profile over V s.t., for every level  $i \ge 0$ ,  $\langle p,i \rangle \to_T \langle q,i+1 \rangle$  iff  $p \xrightarrow{a_i} q$  and  $\langle p,i \rangle \to_T \langle q,i+1 \rangle$  iff  $p \xrightarrow{a_i} q$ .

Then, w is not accepted by  $\mathcal{A}$  iff every infinite path in G eventually ceases taking accepting transitions. This is witnessed with the notion of ranking.

A ranking for a DAG  $G = \langle V, T \rangle$  is a mapping from V to  $\omega$  s.t. ranks along transitions do not increase, and odd ranks along accepting transitions are strictly decreasing. Clearly, every path in G gets eventually trapped in some rank, and, if this rank is odd for every path in G, then the ranking is called an *odd ranking*. Since odd ranks are strictly decreasing on accepting transitions, this implies that if G has an odd ranking, then every infinite path in G must eventually cease taking accepting transitions, and, therefore, G is a rejecting DAG.

Kupferman and Vardi have shown that bounded rankings in fact suffice. Let  $D^l = \{0, 1, \dots, l, \bot\}$  be the set of rank values bounded by l plus the additional undefined value  $\bot$ , where the order is extended as  $0 < 1 < \dots < \bot$ . We define a lifting function  $|\cdot|_{\text{even}}$  on rank values s.t.  $|n|_{\text{even}}$  is the largest even rank not larger than n; i.e.,

$$\lfloor n \rfloor_{\text{even}} = \begin{cases} \bot & \text{if } n = \bot \\ n & \text{if } n \text{ is even} \\ n - 1 & \text{otherwise} \end{cases}$$

The function  $\lfloor \cdot \rfloor_{\text{odd}}$  is defined analogously. A *l-level ranking* is a function  $f: Q \mapsto D^l$ . Let  $\mathcal{R}^l$  be the set of *l*-level rankings. For two level rankings f and g, let  $f \geq g$  iff, for every state p,  $f(p) \geq g(p)$ . Transition profiles induce a successor relation on level rankings. For two rankings f and g and a profile t, we say that g is a *t-successor* of f, written  $f \stackrel{t}{\to} g$ , iff, for every transition  $p \to_t q$ ,  $f(p) \geq g(q)$ , and if  $p \to_t q$ , then  $|f(p)|_{\text{even}} \geq g(q)$ . If  $w \in \mathcal{L}(t)$ , we also just write  $f \stackrel{w}{\to} g$  instead of  $f \stackrel{t}{\to} g$ .

For a level ranking f, let  $even(f) = \{p \mid f(p) \text{ is even } \}$ , and similarly for odd(f).

**Definition 1.** For an NBA  $\mathcal{A} = (\Sigma, Q, I, (t_a)_{a \in \Sigma})$  and a bound  $l \in \omega$ , define  $KV^l(\mathcal{A})$  to be the NBA  $(\Sigma, \mathcal{R}^l \times 2^Q, \{\langle f_0, \emptyset \rangle\}, (t'_a)_{a \in \Sigma})$ , where

- $f_0(p) = l$  if  $p \in I$ , and  $f_0(p) = \bot$  otherwise.
- $-\langle f,O\rangle \xrightarrow{a}' \langle f',(O\cdot a) \setminus \operatorname{odd}(f')\rangle \text{ iff } f \xrightarrow{a}' f' \text{ and } O \neq \emptyset, \text{ and } -\langle f,\emptyset\rangle \xrightarrow{a}' \langle f',\operatorname{even}(f')\rangle \text{ iff } f \xrightarrow{a}' f'.$

Profiles induce an action on level rankings. Intuitively, for a level ranking f and a profile t,  $f \cdot t$  is the largest level ranking that can be obtained from f by following transitions in t. Formally,

$$f \cdot t = \max_{f \stackrel{t}{\to} g} g$$

It can be computed from f and t in the following way: for every q,

$$(f \cdot t)(q) = \min_{p \to tq} \begin{cases} \lfloor f(p) \rfloor_{\text{even}} & \text{if } p \to_t q \\ f(p) & \text{otherwise} \end{cases}$$

Note that, for two profiles s and t,  $f \cdot (st) = (f \cdot s) \cdot t$ .

#### 3.1 Periodic rankings

Level rankings and profiles are strongly related. We regard a profile as a system of (strict) inequalities between values assigned to states by level rankings. If a ranking complies with all these constraints, we say that it satisfies the profile. Formally, for a level ranking f and a profile t, f satisfies t, written  $f \models t$ , iff the following two conditions hold:

$$f \models t \qquad \text{iff} \qquad \begin{array}{ll} f \cdot t \geq f, \text{ and} & \text{[Safety]} \\ \text{for all } p \rightarrow_t q, \lfloor f(p) \rfloor_{\text{odd}} \geq f(q) & \text{[Liveness]} \end{array}$$

The safety condition ensures that the level ranking f complies with the inequalities defined by t, while the liveness condition ensures even ranks in f are strictly decreasing along transitions in t. Intuitively, ranks get sufficiently small by liveness, and not too small by safety.

Given an idempotent profile t, we can associate to it a canonical level ranking  $f_t$ . First, we perform a SCC decomposition of t. For a state p, let  $[p]_t = \{q \mid p \to_t q \to_t p\}$ be the set of states which are inter-reachable from p; notice that  $[p]_t$  is either empty, or it contains p (and there is a self-loop  $p \to_t p$ ). Then, we assign an index to each state p counting how many different non-empty classes  $[q]_t$  are reachable from p (via t): We define a function  $\alpha_t: Q \mapsto \{0, 1, \dots, n\}$  (*n* is the number of states of the automaton) s.t., for every state p,  $\alpha_t(p) = |\{[q] \neq \emptyset \mid p \rightarrow_t q\}|$ .

**Lemma 4.** Let t be an idempotent profile. Then, for any states p and q,

- 1. If  $p \to_t q$ , then  $\alpha_t(p) > \alpha_t(q)$ .
- 2. If  $p \rightarrow_t q$  and [q] is empty,  $\alpha_t(p) \ge \alpha_t(q) + 1$ .
- 3. If t is rejecting,  $p \rightarrow_t q$  and [p] is not empty, then  $\alpha_t(p) \ge \alpha_t(q) + 1$ .

*Proof.* The first point follows immediately from the definition of  $\alpha_t$ . For the second point, notice that, by idempotence, a transition  $p \to_t q$  can always be split into two transitions  $p \to_t r \to_t q$ , for some state r. By repeating the process on the first transition until a duplicate state appears, there exists a state r' s.t.  $p \to_t r' \to_t q$  and  $r' \to_t r'$ . Therefore [r'] is non-empty and  $q \notin [r']$  (otherwise one would have [q] = [r']), thus  $\alpha_t(p) \geq \alpha_t(r') \geq \alpha_t(q) + 1$ . For the third point, notice that, since t is rejecting, [p] is different from [q] (one cannot have  $q \to_t p$ ). Therefore,  $\alpha_t(p) \geq \alpha_t(q) + 1$ .

Finally, given an idempotent profile t, we define the induced level ranking  $f_t$  as follows: For every state p,

$$f_t(p) = \begin{cases} 2\alpha_t(p) & \text{if } [p] = \emptyset\\ 2\alpha_t(p) - 1 & \text{otherwise} \end{cases}$$
 (1)

Notice that, if t is rejecting, then  $f_t$  is a valid level ranking: Indeed, if p is accepting, then [p] is necessarily empty (otherwise t would contain an accepting loop, which is impossible since it is rejecting), therefore  $f_t(p)$  is even. We have the following crucial property of  $f_t$ .

**Lemma 5.** For an idempotent profile t, let  $f_t$  be the canonical level ranking t constructed as above. If t is rejecting, then  $f_t$  satisfies t.

Proof. It suffices to prove that

- for every  $p \to_t q$ ,  $\lfloor f_t(p) \rfloor_{\text{odd}} \geq f_t(q)$ , and
- for every  $p \rightarrow_t q$ ,  $[f_t(p)]_{\text{even}} \ge f_t(q)$ .

For the first point, assume that  $p \to_t q$ . By Lemma 4,  $\alpha_t(p) \ge \alpha_t(q)$ . If [p] and [q] are both non-empty, clearly  $f_t(p) \ge f_t(q)$ . If [p] is non-empty and [q] is empty, by Lemma 4,  $\alpha_t(p) \ge \alpha_t(q) + 1$ , therefore  $f_t(p) = 2\alpha_t(p) - 1 \ge 2(\alpha_t(q) + 1) - 1 = 2\alpha_t(q) + 1 = f_t(q) + 1 \ge f_t(q)$ . If [p] and [q] are both empty, by Lemma 4,  $\alpha_t(p) \ge \alpha_t(q) + 1$ , therefore  $f_t(p) = 2\alpha_t(p) \ge 2(\alpha_t(q) + 1) = 2\alpha_t(q) + 2 = f_t(q) + 2$ . Finally, if [p] is empty and [q] is non-empty,  $f_t(p) = 2\alpha_t(p) \ge 2\alpha_t(q) = f_t(q) + 1$ .

For the second point, additionally assume that  $p \twoheadrightarrow_t q$  and that  $f_t(p)$  is odd, i.e., [p] is non-empty. Then, by Lemma 4,  $\alpha_t(p) \ge \alpha_t(q) + 1$ , and one can proceed as above.

For a profile t, we say that t is consistent iff there exists a ranking f satisfying t. The following is the basic relation between rankings and profiles.

**Lemma 6.** Let t be an idempotent profile. Then, t is consistent iff t is rejecting.

*Proof.* Let t be a profile. For the "only if" direction, assume that t is consistent, and, by way of contradiction, that t is not rejecting. Thus, t has a loop  $p \to_t p$ . By consistency, there exists a ranking f satisfying t. By the liveness condition, f(p) cannot be even, otherwise f(p) > f(p). So f(p) has to be odd. By the safety condition,  $(f \cdot t)(p) \ge f(p)$ . But  $p \to_t p$ , therefore, by the definition of  $f \cdot t$ ,  $(f \cdot t)(p) < \lfloor f(p) \rfloor_{\text{even}}$ , but f(p) is odd, thus  $(f \cdot t)(p) < f(p)$ , which is a contradiction. Therefore, t is rejecting. For the "if" direction, given a rejecting profile t one applies Lemma 5 for the canonical ranking  $f_t$  to show that  $f_t$  satisfies t.

Let  $w = a_0 a_1 \cdots \in \Sigma^{\omega}$ . If  $w \notin \mathcal{L}(\mathcal{A})$ , then there exists two profiles s and t s.t. w factorizes as  $w = w_0 w_1 \cdots$ , with  $w_0 \in \mathcal{L}(s)$  and  $w_i \in \mathcal{L}(t)$  for  $i \ge 1$ . Let  $i_i = |w_0 w_1 \cdots w_i|$ . From this factorization, we extract the following canonical ranking for w, which we call periodic ranking:

$$r(q,i) = \begin{cases} m & \text{if } i < i_0 \\ f_t(q) & \text{if } i = i_j \text{ for some } j \ge 0 \\ (f_t \cdot w)(q) & \text{if } i_j < i < i_{j+1} \text{ for some } j \ge 0 \text{ and } w = w[i_j..i] \end{cases}$$
 (2)

Notice that r is indeed a ranking function; in particular, it is non-increasing, since, at boundary indices  $i_i$ 's,  $f_t \cdot w[i_i...i_{i+1}] \ge f_t$  by  $w[i_i...i_{i+1}] \in \mathcal{L}(t)$  and since  $f_t$  satisfies t by definition. Furthermore, if t is a rejecting profile, then r is an odd ranking by Lemma 5.

## **Slices**

We show that slice simulates ramsey.

We define an update operation of a slice preorder by a profile. Let  $\succeq$  be a total preorder on Q, and let t be a profile. For any state q, min-pre $^{\succeq,t}(q)$  is the set of  $\succeq$ minimal *t*-predecessors of *q*; i.e., min-pre $\succeq$ ,  $t(q) = \{p \mid p \to_t q \land \forall (r \to_t q) \cdot r \succeq p\}$ . Then,  $\succeq' = \succeq t$  is a new total preorder on Q defined as follows: For every states p, p', q, q' with  $p \in \text{min-pre}^{t,\succeq}(p') \text{ and } q \in \text{min-pre}^{\succeq,t}(q'),$ 

- If  $p \succ q$ , then  $p' \succ' q'$ . Otherwise, if  $p \approx q$ :
   If  $q \twoheadrightarrow_t q'$  but  $\neg (p \twoheadrightarrow_t p')$ , then  $p' \succ' q'$ .
   Otherwise,  $p' \approx' q'$ .

Notice that the update operation above is not an action on preorders, since bigstep updates can lose information. In fact, one can prove that it is a pre-action, in the following sense: For any preorder  $\succeq$  and profiles s and  $t, \succeq \cdot (st) \supseteq (\succeq \cdot s) \cdot t$ . This means that the small-step update  $(\succeq \cdot s) \cdot t$  is *finer* than the corresponding big-step one  $\succeq \cdot (st)$ .

**Lemma 7.** Let  $\succeq_0$  and  $\succeq_1$  be two total preorders, with  $\succeq_1$  finer than  $\succeq_0$ , and let t be a *profile. Then,*  $\succeq_1 \cdot t$  *is finer than*  $\succeq_0 \cdot t$ .

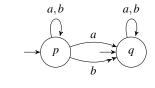
Proof.

**Lemma 8.** Let t be an idempotent profile, let  $\succeq$  be a total preorder, and let  $\succeq' = \succeq \cdot t$  be its update. Then,

- If p →<sub>t</sub> q, then p ≥' q.
  If p →<sub>t</sub> q and t is additionally rejecting, then p >' q.

Proof.

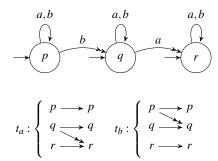
With the two lemmata above, one can show a simulation from the ramsey-automaton  $\mathcal{A}$  to the slice-automaton  $\mathcal{B}$ . The simulation requires an intermediate modified slice construction (automaton C), where the preorder is introduced after a finite prefix has been read, and it is updated in big-steps. Clearly,  $\mathcal{B}$  simulates  $\mathcal{C}$  by Lemma 7.



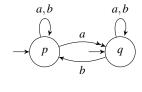
$$t_a: \left\{ \begin{array}{cc} p \longrightarrow p \\ \\ q \longrightarrow q \end{array} \right. \quad t_b: \left\{ \begin{array}{cc} p \longrightarrow p \\ \\ q \longrightarrow q \end{array} \right.$$

**Fig. 1.** slice does not simulate ramsey-pre. Let  $t_b$  and  $t_a$  be the two profiles above. Notice that  $t_b$  is (strictly) finer than  $t_a$ . The ramsey-pre-automaton commits to any total preorder realizing  $t_b$ , and then starts playing arbitrarily many a's and resetting each time to  $t_{\rm E}$ . The slice-automaton is thus in a state  $\langle \{p,q\},p\approx q\rangle$ , and to visit an accepting state it has to eventually commit to a level k. At this point, the ramsey-pre-automaton plays a single b (which can be extended to a rejected word), and the slice-automaton chokes since there is an accepting transition  $p\stackrel{b}{\twoheadrightarrow} q$  past the commit level k which is never subsumed by a better transition in the future.

We now argue that C simulates A. Initially, C just tracks reachable states. When A commits to profiles s,t, C begins updating the preorder starting from the identity id. When A loops for the first time after having read a word in t, C updates its preorder to  $id \cdot t$ . At this point, C commits to the current level k. From now on, by Lemma 8, every time A reads a word in t and resets (thus visiting an accepting state), ...



**Fig. 2.** ramsey does not simulate slice. After playing ab, the slice-automaton reaches the ordering p > q > r. From there, it keeps playing a's. So the ramsey-automaton has to eventually commit to the profile  $t_a$ , at which point the slice-automaton switches to playing b's, and the ramsey-automaton is no longer accepting.



$$t_a: \left\{ \begin{array}{c} p \longrightarrow p \\ q \longrightarrow q \end{array} \right. \quad t_b: \left\{ \begin{array}{c} p \longrightarrow p \\ q \longrightarrow q \end{array} \right.$$

**Fig. 3.** ramsey-pre does not simulate slice. After playing a, the slice-automaton reaches the ordering p>q, from which it keeps playing a's. So the ramsey-automaton has to eventually commit to the unique preorder p>q compatible with profile  $t_a$ . At this point, the slice-automaton plays b's (reaching the new ordering  $p\approx q$ ) and the ramsey-automaton chokes since  $t_b$  is not compatible with the preorder it previously committed to.