

# Cost Automata, Safe Schemes, and Downward Closures

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## Abstract

Higher-order recursion schemes are an expressive formalism used to define languages of possibly infinite ranked trees. They extend regular and context-free grammars, and are equivalent to simply typed  $\lambda Y$ -calculus and collapsible pushdown automata. In this work we prove, under a syntactical constraint called safety, decidability of the model-checking problem for recursion schemes against properties defined by alternating B-automata, an extension of alternating parity automata for infinite trees with a boundedness acceptance condition. We then exploit this result to show how to compute downward closures of languages of finite trees recognized by safe recursion schemes.

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## 1 Introduction

Higher-order functions are nowadays widely used not only in functional programming languages such as Haskell and the OCAML family, but also in mainstream languages such as Java, JavaScript, Python, and C++. *Recursion schemes* are faithful and algorithmically manageable abstractions of the control flow of higher-order programs [35]. A deterministic recursion scheme normalises into a possibly infinite Böhm tree, and in this respect recursion schemes can equivalently be presented as simply-typed lambda-terms using a higher-order fixpoint combinator  $Y$  [49]. There are also nontrivial inter-reductions between recursion schemes and the equi-expressive collapsible higher-order pushdown automata [29] and ordered tree-pushdown automata [12]. In another semantics, also used in this paper, nondeterministic recursion schemes are recognisers of languages of finite trees, and in this view they are also known as higher-order OI grammars [22, 37], generalising indexed grammars [2] (which are recursion schemes of order two) and ordered multi-pushdown automata [7].

The most celebrated algorithmic result in the analysis of recursion schemes is the decidability of the model-checking problem against properties expressed in monadic second-order logic (MSO): given a recursion scheme  $\mathcal{G}$  and an MSO sentence  $\varphi$ , one can decide whether the Böhm tree generated by  $\mathcal{G}$  satisfies  $\varphi$  [42]. This fundamental result has been reproved several times, that is, using collapsible higher-order pushdown automata [29], intersection types [36], Krivine machines [47], and it has been extended in diverse directions such as global model checking [10], logical reflection [8], effective selection [11], and a transfer theorem via models of lambda-calculus [48]. When the input property is given as an MSO formula, the model-checking problem is non-elementary already for trees of order 0 (regular trees) [50]; when the input property is presented as a parity tree automaton (which is equi-expressive with MSO on trees, but less succinct), the MSO model-checking problem for recursion schemes of order  $n$  is complete for  $n$ -fold exponential time [42]. Despite these hardness results, the model-checking problem can be solved efficiently on multiple nontrivial examples, thanks to the development of several recursion-scheme model checkers [35, 34, 9, 46, 41].

**Unboundedness problems.** Recently, an increasing interest has arose for model checking quantitative properties going beyond the expressive power of MSO. The *diagonal problem* is an example of a quantitative property not expressible in MSO. Over words, the problem asks, for a given set of letters  $\Sigma$  and a language of finite words  $\mathcal{L}$ , whether for every  $n \in \mathbb{N}$  there is a word in  $\mathcal{L}$  where every letter from  $\Sigma$  occurs at least  $n$  times. Over full trios (classes of languages closed under regular transductions), decidability of the *diagonal problem* over finite words has interesting algorithmic consequences, such as computability of downward closures [53] and decidability of separability by piecewise testable languages [20]. The *diagonal problem* for languages of words recognised by recursion schemes is decidable [28, 13, 44].

Over full trios of finite words, the diagonal problem is equivalent to the computability of downward closures [21], which is an important problem in its own right. The *downward closure* of a language  $\mathcal{L}$  of finite trees is the set  $\mathcal{L}\downarrow$  of all trees that can be *homeomorphically embedded* into some tree in  $\mathcal{L}$ . By Higman’s lemma [30], the embedding relation on finite ranked trees is a well quasi-order. Consequently, the downward closure  $\mathcal{L}\downarrow$  of an arbitrary set of trees  $\mathcal{L}$  is always a regular language. The downward closure of a language offers a nontrivial regular abstraction thereof: even though the actual count of letters is lost, their limit properties are preserved, as well as their order of appearance.

We say that the *downward closure* is *computable* when a finite automaton for  $\mathcal{L}\downarrow$  can be effectively constructed (which is not true in general). Downward closures are computable for a

wide class of languages of finite words such as those recognised by context-free grammars [19, 40, 3], Petri nets [26], stacked counter automata [54], context-free FIFO rewriting systems and OL-systems [1], second-order pushdown automata [53], higher-order pushdown automata [28], and (possibly unsafe) recursion schemes over words [13]. Over finite trees, it is known that downward closures are computable for the class of regular tree languages [24]. We are not aware of other such computability results for other classes of languages of finite trees.

In another line of research, **B-automata**, and among them *alternating B-automata*, have been put forward as a quantitative extension to MSO [14, 17, 51, 38]. They extend alternating automata over infinite trees [25, Chapter 9] by nonnegative integer counters that can be incremented or reset to zero. The extra counters do not constrain the availability of transitions during a run (unlike in other superficially similar models, such as counter machines), but are used in order to define the acceptance condition: an infinite tree is *n-accepted* if  $n$  is a bound on the values taken by the counters during an accepting run of the automaton over it.

The *universality problem* consists in deciding whether for every tree there is a bound  $n$  for which it is *n-accepted*. The *boundedness problem* asks whether there exists a bound  $n$  for which all trees are *n-accepted*. These two problems are closely related. Their decidability is an important open problem in the field, and proving the decidability of the *boundedness problem* would solve the long standing nondeterministic Mostowski index problem [16]. However, though open in general, the *boundedness problem* is known to be decidable over finite words [14] and trees [17] and infinite words [38], as well as over infinite trees for its weak [52] and the more general quasi-weak [39] version.

Another expressive formalism expressing unboundedness properties beyond MSO is  $\text{MSO}+\text{U}$ , which extends MSO by a new quantifier “ $\text{UX}.\varphi$ ” [6] stating that there exist arbitrarily large finite sets  $X$  satisfying  $\varphi$ . This logic is incomparable with B-automata. The model-checking problem of recursion schemes against its weak fragment  $\text{WMSO}+\text{U}$ , where monadic second-order quantifiers are restricted to finite sets, is decidable [45].

**Contributions.** Our first contribution is the decidability of the *model-checking problem* of properties expressed by *alternating B-automata* for an expressive class of recursion schemes called *safe recursion schemes*. As generators of infinite trees, *safe recursion schemes* are equivalent to higher-order pushdown automata without the collapse operation [33] and are strictly less expressive than general (*unsafe*) recursion schemes [43, Corollary I.2]. Here, the *model-checking problem* asks whether a concrete infinite tree (the *Böhm tree generated* by the *safe recursion scheme*) is accepted by the *B-automaton* for some bound. This problem happens to be significantly simpler than the universality/boundedness problem above described. The proof goes by reducing the *order* of the *safe recursion scheme* similarly as done in Knapik, Niwiński, and Urzyczyn [33] to show decidability of the MSO model-checking problem, at the expense of making the property automaton two-way. We then rely on the fact that two-way alternating B-automata can be converted to equivalent one-way alternating B-automata [5]. Our result is incomparable with the result of Ong [42], since (1) alternating B-automata are strictly more expressive than MSO, however (2) we obtain it under the more restrictive safety assumption. Whether the safety assumption can be dropped while preserving decidability of the model-checking problem against *B-automata* properties remains open.

Our second contribution is to define the following generalization of the diagonal problem from words to trees: given a language of finite trees  $\mathcal{L}$  and a set of letters  $\Sigma$ , decide whether for every  $n \in \mathbb{N}$  there is a tree  $T \in \mathcal{L}$  such that every letter from  $\Sigma$  occurs at least  $n$  times on every branch of  $T$ . This generalization is designed in order to reduce the computation of *downward closures* to the *diagonal problem*, in the same fashion as for finite words. Our proof

strategy is to represent downward-closed sets of trees  $\mathcal{L}\downarrow$  by [simple tree regular expressions](#), which are a subclass of regular expressions for finite trees [23, 24]. By further analysing and simplifying the structure of these expressions, the computation of the downward closure can be reduced to finitely many instances of the diagonal problem. Unlike in the case of finite words, we do not know whether for full trios there exists a converse reduction from the diagonal problem to the problem of computing downward closures.

Our third contribution is decidability of the diagonal problem for languages of finite trees recognised by safe recursion schemes (and thus computability of downward closures of those languages). The diagonal problem can directly be expressed in a logic called [weak cost monadic second order logic](#) (WCMSO) [52], which extends weak MSO with atomic formulae of the form  $|X| \leq N$  stating that the cardinality of the [monadic variable](#)  $X$  is at most  $N$ . Since WCMSO can be translated to [alternating B-automata](#) [52], the diagonal problem reduces to the [model-checking problem](#) of [safe recursion schemes](#) against [alternating B-automata](#), which we have shown decidable in the first part. Note that it seems difficult to express the [diagonal problem](#) using [alternating B-automata](#) directly, and indeed the fact that alternating B-automata can express all WCMSO properties is nontrivial.

**Outline.** In Section 2, we define [recursion schemes](#) and [B-automata](#). In Section 3, we present our first result, namely decidability of model checking of safe recursion schemes against B-automata. In Section 4, we introduce the [diagonal problem](#), and we show how it can be used to compute [downward closures](#). In Section 5, we solve the diagonal problem for schemes. We conclude in Section 6 with some open problems.

## 2 Preliminaries

**Recursion schemes.** A [ranked alphabet](#) is a (usually finite) set  $\mathbb{A}$  of letters, together with a function  $\text{rank}: \mathbb{A} \rightarrow \mathbb{N}$ , assigning a [rank](#) to every letter. When we define trees below, we require that a node labeled by a letter  $a$  has exactly  $\text{rank}(a)$  children. In the sequel, we usually assume some fixed finite [ranked alphabet](#)  $\mathbb{A}$ . The set of [\(simple\) types](#) is constructed from a unique ground type  $\circ$  using a binary operation  $\rightarrow$ ; namely  $\circ$  is a type, and if  $\alpha$  and  $\beta$  are types, so is  $\alpha \rightarrow \beta$ . By convention,  $\rightarrow$  associates to the right, that is,  $\alpha \rightarrow \beta \rightarrow \gamma$  is understood as  $\alpha \rightarrow (\beta \rightarrow \gamma)$ . A type  $\circ \rightarrow \dots \rightarrow \circ$  with  $k$  occurrences of  $\rightarrow$  is also written as  $\circ^k \rightarrow \circ$ . The [order](#) of a type  $\alpha$ , denoted  $\text{ord}(\alpha)$  is defined by induction:  $\text{ord}(\circ) = 0$  and  $\text{ord}(\alpha_1 \rightarrow \dots \rightarrow \alpha_k \rightarrow \circ) = \max_i(\text{ord}(\alpha_i)) + 1$  for  $k \geq 1$ .

We coinductively define both [lambda-terms](#) and the two-argument relation “ $M$  is a [lambda-term of type](#)  $\alpha$ ” as follows (cf. [31, 4]):

- a letter  $a \in \mathbb{A}$  is a [lambda-term of type](#)  $\circ^{\text{rank}(a)} \rightarrow \circ$ ;
- for every [type](#)  $\alpha$  there is a countable set  $\{x, y, \dots\}$  of [variables of type](#)  $\alpha$  which can be used as [lambda-terms of type](#)  $\alpha$ ;
- if  $M$  is a [lambda-term of type](#)  $\beta$  and  $x$  a [variable of type](#)  $\alpha$ , then  $\lambda x.M$  is a [lambda-term of type](#)  $\alpha \rightarrow \beta$ ; this construction is called a [lambda-binder](#);
- if  $M$  is a [lambda-term of type](#)  $\alpha \rightarrow \beta$ , and  $N$  is a [lambda-term of type](#)  $\alpha$ , then  $M N$  is a [lambda-term of type](#)  $\beta$ , called an [application](#).

As usual, we identify [lambda-terms](#) up to [alpha-conversion](#) (i.e., renaming of bound variables). We use here the standard notions of [free variable](#), [subterm](#), (capture-avoiding) [substitution](#), and [beta-reduction](#) (see for instance [31, 4]). A [closed lambda-term](#) does not have free variables. For a [lambda-term](#)  $M$  of type  $\alpha$ , the [order](#) of  $M$ , denoted  $\text{ord}(M)$ , is defined as  $\text{ord}(\alpha)$ . It is [first-order](#) if its [order](#) is one. An [applicative term](#) is a [lambda-term](#) not

containing **lambda-binders** (it contains only letters, **applications**, and **variables**).

A **lambda-term**  $M$  is *superficially safe* if all its **free variables**  $x$  have **order**  $\text{ord}(x) \geq \text{ord}(M)$ . A **lambda-term**  $M$  is *safe* if it is *superficially safe*, and if for every **subterm** of the form  $K L_1 \dots L_k$ , where  $K$  is not an **application** and  $k \geq 1$ , all subterms  $K, L_1, \dots, L_k$  are *superficially safe*.

A *(higher-order, deterministic) recursion scheme* over the alphabet  $\mathbb{A}$  is a tuple  $\mathcal{G} = \langle \mathbb{A}, \mathcal{N}, X_0, \mathcal{R} \rangle$ , where  $\mathcal{N}$  is a finite set of typed **nonterminals**,  $X_0 \in \mathcal{N}$  is the *initial nonterminal*, and  $\mathcal{R}$  is a function assigning to every **nonterminal**  $X \in \mathcal{N}$  of type  $\alpha_1 \rightarrow \dots \rightarrow \alpha_k \rightarrow \circ$  a finite **lambda-term** of the form  $\lambda x_1. \dots \lambda x_k. K$ , of the same type  $\alpha_1 \rightarrow \dots \rightarrow \alpha_k \rightarrow \circ$ , in which  $K$  is an **applicative term** with **free variables** in  $\mathcal{N} \uplus \{x_1, \dots, x_k\}$ . We refer to  $\mathcal{R}(X)$  as the *rule* for  $X$ . The **order** of a recursion scheme  $\text{ord}(\mathcal{G})$  is the maximum **order** of its **nonterminals**.

The **lambda-term represented** by a recursion scheme  $\mathcal{G}$  as above, denoted  $\Lambda(\mathcal{G})$ , is the limit of applying recursively the following operation to  $X_0$ : take an occurrence of some **nonterminal**  $X$ , and replace it with  $\mathcal{R}(X)$  (the **nonterminals** should be chosen in a fair way, so that every **nonterminal** is eventually replaced). Thus,  $\Lambda(\mathcal{G})$  is a (usually infinite) regular **lambda-term** obtained by unfolding the **nonterminals** of  $\mathcal{G}$  according to their definition. We remark that when substituting  $\mathcal{R}(X)$  for a **nonterminal**  $X$  there is no need for any renaming of **variables** (capture-avoiding substitution), since  $\mathcal{R}(X)$  does not contain **free variables** other than **nonterminals**. We only consider recursion schemes for which  $\Lambda(\mathcal{G})$  is well-defined (e.g. by requiring that  $\mathcal{R}(X)$  is not a single **nonterminal**). A recursion scheme  $\mathcal{G}$  is *safe* if  $\Lambda(\mathcal{G})$  is *safe*.

A **tree** is a **closed applicative term** of type  $\circ$ . Such **lambda-terms** are coinductively of the form  $a M_1 \dots M_r$ , where  $a \in \mathbb{A}$  is of **rank**  $r$ , and where  $M_1, \dots, M_r$  are again trees. Thus, such a **lambda-term** can be identified with a tree understood in the traditional sense:  $a$  is the label of its root, and  $M_1, \dots, M_r$  describe subtrees attached in the  $r$  children of the root, from left to right. For **trees** we also use the notation  $a(M_1, \dots, M_r)$  instead of  $a M_1 \dots M_r$ . A **tree** is *regular* if it has finitely many subtrees (**subterms**) up to isomorphism.

The *Böhm tree* of a **lambda-term**  $M$  of type  $\circ$ , denoted  $BT(M)$ , is defined coinductively as follows: if there is a sequence of beta-reductions from  $M$  to a **lambda-term** of the form  $a M_1 \dots M_r$  (where  $a \in \mathbb{A}$  is a letter), then  $BT(M) = a(BT(M_1), \dots, BT(M_r))$ ; otherwise  $BT(M) = \perp()$ , where  $\perp \in \mathbb{A}$  is a distinguished letter of rank 0. It is a classical result that  $BT(M)$  exists, and is uniquely defined [31, 4]. Clearly,  $BT(M)$  is indeed a tree. The tree *generated* by a recursion scheme  $\mathcal{G}$ , denoted  $BT(\mathcal{G})$ , is  $BT(\Lambda(\mathcal{G}))$ .

A **lambda-term**  $N$  is *normalizing* if  $BT(N)$  does not contain the special letter  $\perp$ ; a recursion scheme  $\mathcal{G}$  is *normalizing* if  $\Lambda(\mathcal{G})$  is *normalizing*. In other words, in a normalizing recursion scheme/**lambda-term beta-reduction** always produces a letter. It is possible to transform every recursion scheme  $\mathcal{G}$  into a **normalizing recursion scheme**  $\mathcal{G}'$  *generating* the same tree as  $\mathcal{G}$ , up to renaming  $\perp$  into some non-special letter  $\perp'$  (cf. [27, Section 5]). Moreover, the construction preserves **safety** and the **order**.

**Recursion schemes as recognizers of languages of finite trees.** The standard semantics of a recursion scheme  $\mathcal{G} = \langle \mathbb{A}, \mathcal{N}, X_0, \mathcal{R} \rangle$  is the single infinite tree  $BT(\mathcal{G})$  *generated* by the scheme. An alternative view is to consider a recursion scheme as a **recognizer** of a language of finite trees  $\mathcal{L}(\mathcal{G})$ . This alternative view is relevant when discussing downward closures of languages of finite trees. We employ a special letter  $\text{nd} \in \mathbb{A}$  of rank 2 in order to represent  $\mathcal{L}(\mathcal{G})$  by resolving the nondeterministic choice of  $\text{nd}$  in the infinite tree  $BT(\mathcal{G})$  in all possible ways. Formally, for two trees  $T, U$ , we write  $T \rightarrow_{\text{nd}} U$  if  $U$  is obtained from  $T$  by choosing an  $\text{nd}$ -labeled node  $u$  of  $T$  and a child  $v$  thereof, and replacing the subtree rooted at  $u$  with

the subtree rooted at  $v$ . The relation  $\rightarrow_{\text{nd}}^*$  is the reflexive and transitive closure of  $\rightarrow_{\text{nd}}$ . We define the language of finite trees *recognized* by  $\mathcal{G}$  as  $\mathcal{L}(\mathcal{G}) = \mathcal{L}(BT(\mathcal{G}))$ , where

$$\mathcal{L}(T) = \{U \mid T \rightarrow_{\text{nd}}^* U, \text{ with } U \text{ finite and not containing “nd” or “}\perp\text{”}\}.$$

For an illustration of this encoding, and simultaneously for an example of a recursion scheme, consider the ranked alphabet  $\mathbb{A}$  containing a letter  $a$  of rank 2, two letters  $b_1, b_2$  of rank 1, and a letter  $c$  of rank 0. We use an initial nonterminal  $S$  of order-0 type  $\circ$ , and an additional nonterminal  $A$  of order-2 type  $(\circ \rightarrow \circ) \rightarrow (\circ \rightarrow \circ) \rightarrow \circ \rightarrow \circ \rightarrow \circ$ , together with the following two rules:

$$\begin{aligned} \mathcal{R}(S) &= A b_1 b_2 c c, \\ \mathcal{R}(A) &= \lambda f. \lambda g. \lambda x. \lambda y. \text{nd} (a x y) (A f g (f x) (g x)). \end{aligned}$$

Then,  $BT(\mathcal{G})$  is the infinite non-regular tree  $\text{nd}(a c c) (\text{nd}(a (b_1 c) (b_2 c)) (\cdots))$ , and  $\mathcal{L}(\mathcal{G})$  is the non-regular language of all finite trees of the form  $a (b_1^n c) (b_2^n c)$  with  $n \in \mathbb{N}$ .

**Alternating B-automata.** We introduce the model of automata used in this paper, namely *alternating one-way/two-way B-automata* over *trees* (over a *ranked alphabet*). We consider counters which can be *incremented*  $\mathbf{i}$ , *reset*  $\mathbf{r}$ , or left *unchanged*  $\varepsilon$ . Let  $\Gamma$  be a finite set of *counters* and let  $\mathbb{C} = \{\mathbf{i}, \mathbf{r}, \varepsilon\}$  be the *alphabet of counter actions*. Each *counter* starts with value zero, and the *value of a sequence of actions* is the supremum of the values achieved during this sequence. For instance  $\mathbf{iir}\varepsilon\mathbf{i}\varepsilon$  has value 2,  $(\mathbf{ir})^\omega$  has value 1, and  $\mathbf{iri}^2\mathbf{ri}^3\mathbf{r}\cdots$  has value  $\infty$ . For an infinite sequence of *counter actions*  $w \in \mathbb{C}^\omega$ , let  $\text{val}(w) \in \mathbb{N} \cup \{\infty\}$  be its *value*. In case of several *counters*,  $w = c_1 c_2 \cdots \in (\mathbb{C}^\Gamma)^\omega$ , we take the counter with the maximal value:  $\text{val}(w) = \sup_{c \in \Gamma} \text{val}(w(c))$ , where  $w(c) = c_1(c) c_2(c) \cdots$ .

An *(alternating, two-way) B-automaton* over a finite *ranked alphabet*  $\mathbb{A}$  is a tuple  $\langle \mathbb{A}, Q, q_0, \text{pr}, \Gamma, \delta \rangle$  consisting of a finite set of *states*  $Q$ , an *initial state*  $q_0 \in Q$ , a function  $\text{pr}: Q \rightarrow \mathbb{N}$  assigning *priorities* to states, a finite set  $\Gamma$  of *counters*, and a *transition function*

$$\delta: Q \times \mathbb{A} \rightarrow \mathcal{B}^+(\{\uparrow, \circlearrowleft, \downarrow_1, \downarrow_2, \dots\} \times \mathbb{C}^\Gamma \times Q)$$

mapping a *state* and a letter  $a$  to a (finite) positive Boolean combination of triples of the form  $(d, c, q)$ ; it is assumed that if  $d = \downarrow_i$  then  $i \leq \text{rank}(a)$ . Such a triple encodes the instruction to send the automaton to state  $q$  in direction  $d$  while performing action  $c$ . The direction  $\downarrow_i$  moves to the  $i$ -th child,  $\uparrow$  moves to the parent, and  $\circlearrowleft$  stays in place. We assume that  $\delta(q, a)$  is written in disjunctive normal form for all  $q$  and  $a$ .

The acceptance of an infinite input *tree*  $T$  by an *alternating B-automaton*  $\mathcal{A}$  is defined in terms of a game  $(\mathcal{A}, T)$  between two players, called Eve and Adam. Eve is in charge of disjunctive choices and tries to minimize the counter values while satisfying the parity condition. Adam, on the other hand, is in charge of conjunctive choices and tries to either maximize counter values, or to sabotage the parity condition. Since the *transition function* is given in disjunctive normal form, each turn of the game consists of Eve choosing a disjunct and Adam selecting a single tuple  $(d, c, q)$  thereof. In order to guarantee that from every position there is some move, we assume that each disjunction is nonempty and that each disjunct contains a tuple with some direction other than  $\uparrow$ . A *play* of  $\mathcal{A}$  on the tree  $T$  is a sequence  $q_0, (d_1, c_1, q_1), (d_2, c_2, q_2), \dots$  compatible with  $T$  and  $\delta$ :  $q_0$  is the *initial state*, and for all  $i \in \mathbb{N}$ ,  $(d_{i+1}, c_{i+1}, q_{i+1})$  appears in  $\delta(q_i, T(x_i))$  where  $x_i$  is the node of  $T$  after following the directions  $d_1 d_2 \dots d_i$  starting from the root. The value  $\text{val}(\pi)$  of a play  $\pi$  is the value  $\text{val}(c_1 c_2 \cdots)$  as defined above if the largest number appearing infinitely often among



the priorities  $pr(q_0), pr(q_1), \dots$  is even; otherwise,  $val(\pi) = \infty$ . We say that the play  $\pi$  is *n-winning* (for Eve) if  $val(\pi) \leq n$ .

A *strategy* for one of the players in the game  $(\mathcal{A}, T)$  is a function that returns the next choice given the history of the play. Note that choosing a strategy for Eve and a strategy for Adam fixes a play in  $(\mathcal{A}, T)$ . We say that a play  $\pi$  is *compatible* with a strategy  $\sigma$  if there is some strategy  $\sigma'$  for the other player such that  $\sigma$  and  $\sigma'$  together yield the play  $\pi$ . A *strategy* for Eve is *n-winning* if every play compatible with it is *n-winning*. We say that Eve *n-wins* the game if there is some *n-winning* strategy for Eve. A *B-automaton n-accepts* a tree  $T$  if Eve *n-wins* the game  $(\mathcal{A}, T)$ ; it *accepts*  $T$  if it *n-accepts*  $T$  for some  $n \in \mathbb{N}$ . The language *recognized* by  $\mathcal{A}$  is the set of all trees accepted by  $\mathcal{A}$ .

If no  $\delta(q, a)$  uses the direction  $\uparrow$ , then we call  $\mathcal{A}$  *one-way*. The following theorem essentially follows from a result of Blumensath, Colcombet, Kuperberg, Parys, and Vanden Boom [5, Theorem 6] modulo some cosmetic changes (c.f. Appendix A for more details).

► **Theorem 2.1** (c.f. [5, Theorem 6]). *Given an alternating two-way B-automaton, one can compute an alternating one-way B-automaton that recognizes the same language.* ◀

As a special case of a result by Colcombet and Göller [15] we obtain the following fact.

► **Fact 2.2.** *One can decide whether a given B-automaton accepts a given regular tree.* ◀

### 3 Model-checking safe recursion schemes against alternating B-automata

In this section we prove the first main theorem of our paper, the decidability of the *model-checking problem* of *safe recursion schemes* against properties described by *B-automata*:

► **Theorem 3.1.** *Given an alternating B-automaton  $\mathcal{A}$  and a safe recursion scheme  $\mathcal{G}$ , one can decide whether  $\mathcal{A}$  accepts the tree generated by  $\mathcal{G}$ .*

It is worth noticing that this theorem generalises the result of Knapik et al. [33] on safe recursion schemes from regular (MSO) properties to the more general quantitative realm of properties described by *B-automata*. On the other hand, our result is incomparable with the celebrated theorem of Ong [42] showing decidability of model checking regular properties of possibly unsafe recursion schemes. Whether model checking of possibly unsafe recursion schemes against properties described by *B-automata* is decidable remains an open problem.

By Theorem 2.1, every *B-automaton* can be effectively transformed into an equivalent *one-way B-automaton*, so it is enough to prove Theorem 3.1 for a *one-way B-automaton*  $\mathcal{A}$ . The proof of Theorem 3.1 is based on the following lemma, where we use in an essential way the assumption that the recursion scheme is safe.

► **Lemma 3.2.** *For every safe recursion scheme  $\mathcal{G}$  of order  $m$  and for every alternating one-way B-automaton  $\mathcal{A}$ , one can effectively construct a safe recursion scheme  $\mathcal{G}'$  of order  $m - 1$  and an alternating two-way B-automaton  $\mathcal{A}'$  such that*

$$\mathcal{A} \text{ accepts } BT(\mathcal{G}) \quad \text{if and only if} \quad \mathcal{A}' \text{ accepts } BT(\mathcal{G}').$$

Theorem 3.1 follows easily: Using Lemma 3.2 we can reduce the *order* of the considered *safe recursion scheme* by one. We obtain a *two-way B-automaton*, which we convert back to a *one-way B-automaton* using Theorem 2.1. It is then sufficient to repeat this process, until we end up with a recursion scheme of order 0. A *recursion scheme* of order 0 generates a regular tree and, by Fact 2.2, we can decide whether the resulting *B-automaton* accepts this tree, answering our original question.

**Lambda-trees.** We now come to the proof of Lemma 3.2. The construction of  $\mathcal{G}'$  from  $\mathcal{G}$  follows an analogous result for MSO [32, 33], which we generalise to **B-automata**. We represent some lambda-terms as trees. For a *finite* set  $\mathcal{X}$  of **variables of type o**, we define a new ranked alphabet  $\mathbb{A}_{\mathcal{X}}$  that contains 1) a letter  $\bar{a}$  of rank 0 for every letter  $a \in \mathbb{A}$ ; 2) a letter  $\bar{x}$  of rank 0 for every variable  $x \in \mathcal{X}$ ; 3) a letter  $\overline{\lambda x}$  of rank 1 for every variable  $x \in \mathcal{X}$ ; 4) a letter  $@$  of rank 2. We remark that  $\mathbb{A}_{\mathcal{X}}$  is a usual finite ranked alphabet. A *lambda-tree* is a tree over the alphabet  $\mathbb{A}_{\mathcal{X}}$ , where  $\mathcal{X}$  is clear from the context. Intuitively, a *lambda-tree* is a tree representation of a **first-order lambda-term**.

The semantics  $\llbracket T \rrbracket_{\mathcal{X},s}$  of a lambda-tree  $T$  is defined in such a way that if  $T$  “corresponds” to the **lambda-term**  $M$ , then  $\llbracket T \rrbracket_{\mathcal{X},s} = BT(M)$ . Since  $T$  uses only variables of type **o** we can read the resulting **Böhm tree** directly, without performing any reduction. Essentially, we walk down through  $T$ , skipping all **lambda-binders** and choosing the left branch in all **applications**. Whenever we reach some variable  $x$ , we go up to the corresponding **lambda-binder**, then up to the corresponding **application**, and then we again start going down in the argument of this application. Formally, let  $\mathcal{X}$  be a finite set of **variables** of type **o**, and let  $s \in \mathbb{N}$ . The intended meaning is that  $\mathcal{X}$  contains variables that may potentially appear in the considered *lambda-tree*  $T$ , and that  $s$  is a bound for the arity of types in the lambda-term represented by  $T$  (types of all its subterms should be of the form  $\mathbf{o}^k \rightarrow \mathbf{o}$  for  $k \leq s$ ). We take  $\text{Dirs}_{\mathcal{X},s} = \{\downarrow\} \cup \{\uparrow_x \mid x \in \mathcal{X}\} \cup \{\uparrow_i \mid 1 \leq i \leq s\}$ . Intuitively,  $\downarrow$  means to go down to the left child,  $\uparrow_x$  means that we are looking for the value of (lambda)variable  $x$ , and  $\uparrow_i$  means that we are looking for the  $i$ -th argument of an **application**. For a node  $v$  of  $T$  denote its parent by  $\text{par}(v)$ , and its  $i$ -th child by  $\text{ch}_i(v)$ . For  $d \in \text{Dirs}_{\mathcal{X},s}$ , and for a node  $v$  of  $T$  labeled by  $a \in \mathbb{A}$ , we define the  $(\mathcal{X}, s)$ -*successor* of  $(d, v)$  as

1.  $(\downarrow, \text{ch}_1(v))$  if  $d = \downarrow$  and  $a = \bar{\lambda x}$  (for some  $x$ ) or  $@$ ,
2.  $(\uparrow_x, v)$  if  $d = \downarrow$  and  $a = \bar{x}$  (for some  $x$ ),
3.  $(\uparrow_x, \text{par}(v))$  if  $d = \uparrow_x$  and  $a \neq \bar{\lambda x}$  (including the case when  $a = \bar{\lambda y}$  for  $y \neq x$ ),
4.  $(\uparrow_1, \text{par}(v))$  if  $d = \uparrow_x$  and  $a = \bar{\lambda x}$ ,
5.  $(\uparrow_{i+1}, \text{par}(v))$  if  $d = \uparrow_i$  for  $i < s$  and  $a = \bar{\lambda y}$  (for some  $y$ ),
6.  $(\uparrow_{i-1}, \text{par}(v))$  if  $d = \uparrow_i$  for  $i > 1$  and  $a = @$ ,
7.  $(\downarrow, \text{ch}_2(v))$  if  $d = \uparrow_1$  and  $a = @$ .

Rule 1 allows us go to down to the first child in the case of **lambda-binders** and **applications**. Rule 2 records that we have seen  $x$ , and thus we need to find its value by going up. Rule 3 climbs the tree upwards as long as we do not see the corresponding binder  $\bar{\lambda x}$ . Rule 4 records that we have seen  $\bar{\lambda x}$  and initialises its level to 1. We now need to find the corresponding application. Rule 5 increments the level and goes up when we encounter a binder  $\bar{\lambda y}$ , and Rule 6 decrements it for applications  $@$ . Finally, when we see an application at level 1 we apply Rule 7 which searches for the value of  $x$  in the right child. An  $(\mathcal{X}, s)$ -*maximal path* from  $(d_1, v_1)$  is a sequence of pairs  $(d_1, v_1), (d_2, v_2), \dots$  in which every  $(d_{i+1}, v_{i+1})$  is the  $(\mathcal{X}, s)$ -*successor* of  $(d_i, v_i)$ , and which is either infinite or ends in a pair that has no  $(\mathcal{X}, s)$ -*successor*. For  $d \in \text{Dirs}_{\mathcal{X},s}$ , and for a node  $v$  of  $T$ , we define the  $(\mathcal{X}, s)$ -*derived tree* from  $(T, d, v)$ , denoted by  $\llbracket T, d, v \rrbracket_{\mathcal{X},s}$ , by coinduction:

- if the  $(\mathcal{X}, s)$ -*maximal path* from  $(d, v)$  is finite and ends in  $(\downarrow, w)$  for a node  $w$  labeled by  $\bar{a}$ , then  $\llbracket T, d, v \rrbracket_{\mathcal{X},s} = a(\llbracket T, \uparrow_1, w \rrbracket_{\mathcal{X},s}, \dots, \llbracket T, \uparrow_{\text{rank}(a)}, w \rrbracket_{\mathcal{X},s})$ ;
- otherwise,  $\llbracket T, d, v \rrbracket_{\mathcal{X},s} = \perp$ .

The  $(\mathcal{X}, s)$ -*derived tree* from  $T$  is  $\llbracket T \rrbracket_{\mathcal{X},s} = \llbracket T, \downarrow, v_0 \rrbracket_{\mathcal{X},s}$ , where  $v_0$  is the root of  $T$ . We say that  $T$  is *normalizing* if  $\llbracket T \rrbracket_{\mathcal{X},s}$  does not contain  $\perp$ .

The following lemma performs the **order** reduction. It crucially relies on the safety assumption. It is a variant of results proved in Knapik et al. [32, 33] (c.f. Appendix C).



Intuitively, it says that a [lambda-tree](#) representation  $T$  of a [safe recursion scheme](#)  $\mathcal{G}$  of order  $m$  can be computed by a [safe recursion scheme](#) of order  $m - 1$  in a semantic-preserving way.

► **Lemma 3.3** ([32, 33]). *For every [safe recursion scheme](#)  $\mathcal{G}$  of order  $m \geq 1$  one can construct a [safe recursion scheme](#)  $\mathcal{G}'$  of order  $m - 1$ , a finite set of variables  $\mathcal{X}$ , and a number  $s \in \mathbb{N}$  such that*

$$\llbracket BT(\mathcal{G}') \rrbracket_{\mathcal{X},s} = BT(\mathcal{G}). \quad \blacktriangleleft$$

► **Remark 3.4.** In Knapik et al. [32, 33] the lambda-tree  $T$  is denoted  $\mathfrak{L}_{\mathcal{G}}(X_0)$  (where  $X_0$  is the starting nonterminal of  $\mathcal{G}$ ), and the recursion scheme  $\mathcal{G}'$  is denoted  $\mathcal{G}^\alpha$ . The set  $\mathcal{X}$  is just the set of variables appearing in the letters used in  $\mathcal{G}^\alpha$ ; the number  $s$  can be also read out of  $\mathcal{G}^\alpha$ . They prove that the  $(\mathcal{X}, s)$ -derived tree of  $\mathfrak{L}_{\mathcal{G}}(X_0)$  equals the tree generated by  $\mathcal{G}$  [32, Proposition 4], that  $\mathcal{G}^\alpha$  is safe [33, Lemma 3.5], and that  $\mathcal{G}^\alpha$  generates  $\mathfrak{L}_{\mathcal{G}}(X_0)$ .

In order to prove Lemma 3.3, one needs to replace in  $\mathcal{G}$  every variable  $x$  of type [o](#) by  $\bar{x}$ , every [lambda-binder](#) concerning such a variable by  $\bar{\lambda}x$ , and every application with an argument of type [o](#) by a construct creating a [@](#)-labeled node. Types of subterms change and the order of the recursion scheme decreases by one. Notice, however, that while computing  $BT(\mathcal{G})$  we may need to rename variables during capture-avoiding substitutions, while in the tree generated by the modified recursion scheme we leave original variable names. In general (i.e., when the transformation is applied to an arbitrary recursion scheme) this causes a problem of overlapping variable names. The assumption that  $\mathcal{G}$  is [safe](#) is crucial here and there is no need to rename variables when applying the transformation to a [safe recursion scheme](#).

Having Lemma 3.3, it remains to transform a [one-way B-automaton](#)  $\mathcal{A}$  operating on the tree generated by  $\mathcal{G}$  into a [two-way B-automaton](#)  $\mathcal{A}'$  operating on the lambda-tree generated by  $\mathcal{G}'$ , as described by the following lemma (as mentioned on page 5, we can assume that  $\mathcal{G}$  is normalizing, which implies that  $BT(\mathcal{G}')$  is normalizing: the tree  $\llbracket BT(\mathcal{G}') \rrbracket_{\mathcal{X},s} = BT(\mathcal{G})$  does not contain  $\perp$ ).

► **Lemma 3.5.** *Let  $\mathcal{A}$  be an [alternating one-way B-automaton](#) over a finite alphabet  $\mathbb{A}$ , let  $\mathcal{X}$  be a finite set of variables, and let  $s \in \mathbb{N}$ . One can construct an [alternating two-way B-automaton](#)  $\mathcal{A}'$  such that for every [normalizing lambda-tree](#)  $T$  over  $\mathbb{A}_{\mathcal{X}}$ ,*

$$\mathcal{A} \text{ accepts } \llbracket T \rrbracket_{\mathcal{X},s} \quad \text{if and only if} \quad \mathcal{A}' \text{ accepts } T.$$

**Proof.** The [B-automaton](#)  $\mathcal{A}'$  simulates  $\mathcal{A}$  on the [lambda-tree](#). Whenever  $\mathcal{A}$  wants to go down to the  $i$ -th child,  $\mathcal{A}'$  has to follow the  $(\mathcal{X}, s)$ -[maximal path](#) from  $(\uparrow_i, v)$  (where  $v$  is the current node). To this end, it has to remember the current pair  $(d, v)$ , and repeatedly find its  $(\mathcal{X}, s)$ -[successor](#). Here  $v$  is always just the current node visited by the [B-automaton](#); the  $d$  component comes from the (finite) set  $\text{Dirs}_{\mathcal{X},s}$ , and thus it can be remembered in the state. It is straightforward to encode the definition of an  $(\mathcal{X}, s)$ -[successor](#) in transitions of an automaton. We do not have to worry about infinite  $(\mathcal{X}, s)$ -[maximal paths](#), because by assumption the  $(\mathcal{X}, s)$ -derived tree does not contain  $\perp$ -labeled nodes.  $\blacktriangleleft$

## 4 Downward closures of tree languages

In this section we lay down a method for the computation of the [downward closure](#) for classes of languages of finite trees closed under [linear FTT transductions](#). This method is analogous to the one of Zetsche [53] for the case of finite words. In Section 4.1 we define the [downward](#)

closure of languages of finite ranked trees with respect to the embedding well-quasi order and in Section 4.2 we define the simultaneous unboundedness problem for trees and show how computing the downward closure reduces to it. In Section 4.3 we define the diagonal problem for finite trees and show how the previous problem reduces to it. We will then solve the diagonal problem for languages of finite trees recognized by safe recursion schemes in Section 5.

Let us emphasize that this section can be applied to any class of languages of finite trees closed under linear FTT transductions, not just those recognized by safe recursion schemes.

#### 4.1 Preliminaries

Given two finite trees  $S = a(S_1, \dots, S_k)$  and  $T = b(T_1, \dots, T_r)$ , we say that  $S$  *homeomorphically embeds into*  $T$ , written  $S \sqsubseteq T$ , if, either 1) there exists  $i \in \{1, \dots, r\}$  such that  $S \sqsubseteq T_i$ , or 2)  $a = b$ ,  $k = r$ , and  $S_i \sqsubseteq T_i$  for all  $i \in \{1, \dots, r\}$ . For a language of finite trees  $\mathcal{L}$ , its *downward closure*, denoted by  $\mathcal{L}\downarrow$ , is the set of trees  $S$  such that  $S \sqsubseteq T$  for some tree  $T \in \mathcal{L}$ .

**Pure products.** Goubault-Larrecq and Schmitz [24] describe downward-closed sets of trees using so-called *simple tree regular expressions*. Among those expressions they distinguish *products*, which describe *ideals* of trees. Because every downward-closed set of trees is a finite union of ideals, such a set can be described by a finite list of products. Since their definition of a product is rather indirect, we consider the stronger notion of *pure products*.

A *context* is a tree possibly containing one or more occurrences of a special leaf  $\square$ , called a *hole*. Given a context  $C$  and a set of trees  $\mathcal{L}$ , we write  $C[\mathcal{L}]$  for the set of trees obtained from  $C$  by replacing every occurrence of the hole  $\square$  by some tree from  $\mathcal{L}$ . Different occurrences of  $\square$  are replaced by possibly different trees from  $\mathcal{L}$ . The definition readily extends to a set of contexts  $\mathcal{C}$ , by writing  $\mathcal{C}[\mathcal{L}]$  for  $\bigcup_{C \in \mathcal{C}} C[\mathcal{L}]$ . If  $C$  does not have any  $\square$ , then  $C[\mathcal{L}]$  is just  $\{C\}$ . A *pure product* is defined according to the following abstract syntax:

$$\begin{aligned} P &::= a^?(P, \dots, P) \mid I^*.P, & C &::= a(P_{\square}, \dots, P_{\square}), \\ I &::= C + \dots + C, & P_{\square} &::= \square \mid P, \end{aligned}$$

where the sum of contexts is nonempty, and where in a context  $C = a(P_{\square,1}, \dots, P_{\square,r})$  it is required that at least one  $P_{\square,i}$  is a hole  $\square$ . A *pure product*  $P$  denotes a set of trees  $\llbracket P \rrbracket$  downward-closed for  $\sqsubseteq$ , which is defined recursively as follows:

$$\begin{aligned} \llbracket a^?(P_1, \dots, P_r) \rrbracket &= \{a(T_1, \dots, T_r) \mid \forall i. T_i \in \llbracket P_i \rrbracket\} \cup \llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_r \rrbracket, \\ \llbracket I^*.P \rrbracket &= \bigcup_{n \in \mathbb{N}} \underbrace{\llbracket I \rrbracket \dots \llbracket I \rrbracket}_{n} \llbracket P \rrbracket, \\ \llbracket C_1 + \dots + C_k \rrbracket &= \llbracket C_1 \rrbracket \cup \dots \cup \llbracket C_k \rrbracket, \\ \llbracket a(P_{\square,1}, \dots, P_{\square,r}) \rrbracket &= \{a(T_1, \dots, T_r) \mid \forall i. T_i \in \llbracket P_{\square,i} \rrbracket\} \cup \llbracket P_{\square,1} \rrbracket \cup \dots \cup \llbracket P_{\square,r} \rrbracket, \\ \llbracket \square \rrbracket &= \{\square\}. \end{aligned}$$

For example,  $\llbracket (a(b(), \square))^*.c^?() \rrbracket$  is the set of trees of the form either  $b()$ , or  $c()$ , or  $a(b(), a(b(), \dots a(b(), x) \dots))$  with  $x$  either  $b()$  or  $c()$ . Based on the results of Goubault-Larrecq and Schmitz [24] it is not difficult to deduce the following lemma (see Appendix D for a proof).

► **Lemma 4.1.** *Every set of trees  $\mathcal{L}$  downward-closed for  $\sqsubseteq$  can be represented as  $\mathcal{L} = \llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket$ , in which  $P_1, \dots, P_k$  are pure products.* ◀

This decomposition result strengthens the results of Goubault-Larrecq and Schmitz [24] by showing that **pure products** (instead of just products) suffice in order to represent downward-closed sets of trees.

**Transductions.** A (nondeterministic) *finite tree transducer* (FTT) is a tuple  $\mathcal{A} = (\mathbb{A}_{in}, \mathbb{A}_{out}, S, p^I, \delta)$ , where  $\mathbb{A}_{in}, \mathbb{A}_{out}$  are the input and output alphabets (finite, ranked),  $S$  is a finite set of *control states*,  $p^I \in S$  is an *initial state*, and  $\delta$  is a finite set of *transition rules* of the form either  $(p, a(x_1, \dots, x_r)) \rightarrow T$  or  $(p, x) \rightarrow T$ , where  $p \in S$  is a *control state*,  $a \in \mathbb{A}_{in}$  is a letter of rank  $r$ , and  $T$  is a finite tree over the alphabet  $\mathbb{A}_{out} \cup (S \times \{x_1, \dots, x_r\})$  or  $\mathbb{A}_{out} \cup (S \times \{x\})$ , respectively. The rank of all the pairs from  $S \times \{x_1, \dots, x_r\}$  or  $S \times \{x\}$  is 0. An FTT is *linear* if for each *rule* of the form  $(p, a(x_1, \dots, x_r)) \rightarrow T$  and for each  $i \in \{1, \dots, r\}$ , in  $T$  there is at most one letter from  $S \times \{x_i\}$ , and moreover for each *rule* of the form  $(p, x) \rightarrow T$ , in  $T$  there is at most one letter from  $S \times \{x\}$ . An FTT  $\mathcal{A}$  defines in a natural way a relation between finite trees, also denoted  $\mathcal{A}$  (c.f. Comon et al. [18]). For a language  $\mathcal{L}$  we write  $\mathcal{A}(\mathcal{L})$  for the set of trees  $U$  such that  $(T, U) \in \mathcal{A}$  for some  $T \in \mathcal{L}$ . A function that maps  $\mathcal{L}$  to  $\mathcal{A}(\mathcal{L})$  for some *linear FTT*  $\mathcal{A}$  is called a *linear FTT transduction*.

► **Fact 4.2.** The *downward closure* operation  $\mathcal{L} \mapsto \mathcal{L}\downarrow$  and the *regular restriction* operation  $\mathcal{L} \mapsto \mathcal{L} \cap \mathcal{R}$  (for every regular language  $\mathcal{R}$ ) are effectively *linear FTT transductions*. ◀

► **Lemma 4.3** (c.f. Appendix E). The class of languages of finite trees *recognized by safe recursion schemes* is effectively closed under *linear FTT transductions*. ◀

## 4.2 The simultaneous unboundedness problem for trees

We say that a *pure product*  $P$  is *diversified*, if no letter appears in  $P$  more than once. The *simultaneous unboundedness problem* (SUP) for a class  $\mathfrak{C}$  of finite trees asks, given a *diversified pure product*  $P$  and a language  $\mathcal{L} \in \mathfrak{C}$  such that  $\mathcal{L} \subseteq \llbracket P \rrbracket$ , whether  $\llbracket P \rrbracket \subseteq \mathcal{L}\downarrow$ .

► **Remark 4.4.** This is a generalization of SUP over finite words. In the latter problem, one is given a language of finite words  $\mathcal{L}$  such that  $\mathcal{L} \subseteq a_1^* \dots a_k^*$ , and must check whether  $a_1^* \dots a_k^* \subseteq \mathcal{L}\downarrow$ . A word in  $a_1^* \dots a_k^*$  can be represented as a linear tree by interpreting  $a_1, \dots, a_k$  as unary letters and by appending a new leaf  $e$  at the end. Thus  $a_1^* \dots a_k^*$  can be represented as the language of the *diversified pure product*  $(a_1(\square))^* \cdot (a_2(\square))^* \cdot \dots \cdot (a_k(\square))^* \cdot e(?)$ .

Following Zetsche [53], we can reduce the computation of the *downward closure* to SUP.

► **Theorem 4.5** (c.f. Appendix F). Let  $\mathfrak{C}$  be a class of languages of finite trees closed under *linear FTT transductions*. One can compute a finite tree automaton recognizing the downward closure of a given language from  $\mathfrak{C}$  if and only if SUP is decidable for  $\mathfrak{C}$ . ◀

► **Remark 4.6.** *Pure products* for trees correspond to expressions of the form  $a_0^? A_1^* a_1^? \dots A_k^* a_k^?$  for words (where  $A_i$  are sets of letters). In SUP for words simpler expressions of the form  $b_1^* \dots b_k^*$  suffice. This is not possible for trees: 1) expressions of the form  $a^?(\cdot, \cdot)$  cannot be removed since they are responsible for branching, and 2) reducing the two *contexts* in  $(a(P_1, \square) + b(P_2, \square))^* \cdot P_3$  to a single one would require changing trees of the form  $a(T_1, b(T_2, T_3))$  into trees of the form  $c(T_1, T_2, T_3)$ , which is not a *linear FTT transduction*.

## 4.3 The diagonal problem for trees

In SUP for words, instead of checking whether  $a_1^* \dots a_k^* \subseteq \mathcal{L}\downarrow$ , one can equivalently check whether, for each  $n \in \mathbb{N}$ , there is a word  $a_1^{x_1} \dots a_k^{x_k} \in \mathcal{L}$  such that  $x_1, \dots, x_k \geq n$ . The latter

problem is known as the *diagonal problem* for words. In this section, we define an analogous *diagonal problem for trees*, and we show how to reduce SUP to it.

Given a set of letters  $\Sigma$ , we say that a language of finite trees  $\mathcal{L}$  is  *$\Sigma$ -diagonal* if, for every  $n \in \mathbb{N}$ , there is a tree  $T \in \mathcal{L}$  such that for every letter  $a \in \Sigma$  and every branch  $B$  of  $T$  there are at least  $n$  occurrences  $a$  in  $B$ . The *diagonal problem* for a class  $\mathfrak{C}$  of finite trees asks, given a language  $\mathcal{L} \in \mathfrak{C}$  and a set of letters  $\Sigma$ , whether  $\mathcal{L}$  is  *$\Sigma$ -diagonal*.

**Versatile trees.** Contrary to the case of words, the presence of sums in our expressions creates some complications in reducing from SUP to the *diagonal problem*. We deal with these sums by introducing the notion of *versatile trees*. Intuitively, in order to obtain a *versatile tree* of a *pure product*  $P$ , for every sum  $I = C_1 + \dots + C_k$  in  $P$  we fix some order of the *contexts*  $C_1, \dots, C_k$ , and we allow the *contexts* to be appended in this order. Formally, the set  $\langle P \rangle$  of *versatile trees* of a *pure product*  $P$  is defined by structural induction on  $P$ :

$$\begin{aligned} \langle I^*.P \rangle &= \bigcup_{n \in \mathbb{N}} \langle I \rangle [\underbrace{(\langle I \rangle \cup \{\square\}) \dots (\langle I \rangle \cup \{\square\})}_{n}] [\langle P \rangle] \dots, \\ \langle a^?(P_1, \dots, P_r) \rangle &= \langle a(P_1, \dots, P_r) \rangle, \\ \langle C_1 + \dots + C_k \rangle &= \langle C_1 \rangle [\dots [\langle C_k \rangle] \dots], \\ \langle a(P_{\square,1}, \dots, P_{\square,r}) \rangle &= \{a(T_1, \dots, T_r) \mid \forall i. T_i \in \langle P_{\square,i} \rangle\}, \\ \langle \square \rangle &= \{\square\}. \end{aligned}$$

For example, if  $I = a(S_1, \square, \square) + b(\square, S_2)$ , then  $\langle I \rangle = \{a(S_1, b(\square, S_2), b(\square, S_2))\}$ . Notice that all trees in  $\langle P \rangle$  have the same root's label; denote this label by  $\text{root}(P)$ .

**From SUP to the diagonal problem.** Assuming that  $P$  is *diversified*, for a number  $n \in \mathbb{N}$  we say that a tree  $T$  is  *$n$ -large with respect to  $P$*  if, for every subexpression of  $P$  of the form  $I^*.P'$ , above every occurrence of  $\text{root}(P')$  in  $T$  there are at least  $n$  ancestors labeled by  $\text{root}(I^*.P')$ . In other words, for  $T \in \langle P \rangle$  this means that in  $T$  every *context* appearing in  $P$  was appended at least  $n$  times, on all branches where it was possible to append it. Clearly  $\langle P \rangle \subseteq \llbracket P \rrbracket$ . On the other hand, every tree from  $\llbracket P \rrbracket$  can be embedded into every large enough *versatile tree*. We thus obtain the following lemma.

► **Lemma 4.7.** *For every diversified pure product  $P$ , and for every sequence of trees  $T_1, T_2, \dots \in \langle P \rangle$  such that every  $T_n$  is  $n$ -large,  $\{T_n \mid n \in \mathbb{N}\} \downarrow = \llbracket P \rrbracket$ .* ◀

Using *versatile trees* we can reduce from SUP to the *diagonal problem*.

► **Lemma 4.8** (c.f. Appendix G). *Let  $\mathfrak{C}$  be a class of languages of finite trees closed under linear FTT transductions. SUP for  $\mathfrak{C}$  reduces to the diagonal problem for  $\mathfrak{C}$ .* ◀

► **Remark 4.9.** Another formulation of the diagonal problem for languages of finite trees [28, 13, 44] requires that, for every  $n \in \mathbb{N}$ , there is a tree  $T \in \mathcal{L}$  containing at least  $n$  occurrences of every letter  $a \in \Sigma$  (not necessarily on the same branch, unlike in our case). Such a formulation of the diagonal problem seems too weak to compute downward closures for languages of finite trees.

## 5 Languages of safe recursion schemes

In the previous section, we have developed a general machinery allowing one to compute downward closures for classes of languages of finite trees closed under linear FTT transductions.

In this section, we apply this machinery to the particular case of **languages recognized** by safe recursion schemes. The following is the main theorem of this section.

► **Theorem 5.1.** *Finite tree automata recognizing downward closures of languages of finite trees recognised by safe recursion schemes are computable.*

In order to prove the theorem we need to recall a formalism necessary to express the diagonal problem in logic.

**Cost logics.** *Cost monadic logic* (*CMSO*) was introduced in Colcombet [14] as a quantitative extension of monadic second-order logic. As usual, the logic can be defined over any relational structure, but we restrict our attention to *CMSO* over **trees**. In addition to *first-order variables* ranging over nodes of the tree and *monadic second-order variables* (also called *set variables*) ranging over sets of nodes, *CMSO* uses a single additional variable  $N$ , called the *numeric variable*, which ranges over  $\mathbb{N}$ . The atomic formulas in *CMSO* are those from *MSO* (the membership relation  $x \in X$  and relations  $a(x, x_1, \dots, x_r)$  asserting that  $a \in \mathbb{A}$  of rank  $r$  is the label at node  $x$  with children  $x_1, \dots, x_r$  from left to right), as well as a new predicate  $|X| \leq N$ , where  $X$  is any *set variable* and  $N$  is the numeric variable. Arbitrary *CMSO* formulas are built inductively by applying Boolean connectives and by quantifying (existentially or universally) over *first-order* or *set variables*. We require that any predicates of the form  $|X| \leq N$  appear positively in the formula (i.e., within the scope of an even number of negations). We regard  $N$  as a parameter. As usual, a *sentence* is a formula without *first-order* or *monadic free variables*; however, the parameter  $N$  is allowed to occur in a *sentence*. If we fix a value  $n \in \mathbb{N}$  for  $N$ , the semantics of  $|X| \leq N$  is what one would expect: the predicate holds when  $X$  has cardinality at most  $n$ . We say that a *sentence*  $\varphi$  *n-accepts* a tree  $T$  if it holds in  $T$  when  $n$  is used as value of  $N$ ; it *accepts*  $T$  if it *n-accepts*  $T$  for some  $n \in \mathbb{N}$ . The language *defined* by  $\varphi$  is the set of all trees (over a fixed alphabet  $\mathbb{A}$ ) *accepted* by  $\varphi$ .

*Weak cost monadic logic* (*WCMSO* for short) is the variant of *CMSO* where the second-order quantification is restricted to finite sets. Vanden Boom [52, Theorem 2] proves that *WCMSO* is effectively equivalent to a subclass of *alternating B-automata*, called *weak B-automata*. Thanks to Theorem 3.1, we obtain the following corollary.

► **Corollary 5.2.** *The model-checking problem of safe recursion schemes against WCMSO properties is decidable.* ◀

► **Remark 5.3.** The same holds for a more expressive logic called *quasi-weak cost monadic logic* (*QWCMSO*) [5], whose expressive power lies between *WCMSO* and the *CMSO*. Indeed, Blumensath et al. [5, Theorem 2] prove that *QWCMSO* is effectively equivalent to a subclass of *alternating B-automata* called *quasi-weak B-automata*, and thus by Theorem 3.1 even model checking of *safe recursion schemes* against *QWCMSO* properties is decidable.

**Solving the diagonal problem.** By Theorem 4.5 and Lemma 4.8, all we need to do is to show that the *diagonal problem* is decidable for **languages recognized** by *safe recursion schemes*, that is, that given a *safe recursion scheme*  $\mathcal{G}$  and a set of letters  $\Sigma$ , one can check whether for every  $n \in \mathbb{N}$  there is a tree  $T \in \mathcal{L}(\mathcal{G})$  such that there are at least  $n$  occurrences of every letter  $a \in \Sigma$  on every branch of  $T$  (we say that such a tree  $T$  is *n-large with respect to  $\Sigma$* ). In order to do this, given a set of letters  $\Sigma$ , we write a *WCMSO* sentence  $\varphi_\Sigma$  that  $n$ -accepts an (infinite) tree  $T$  if and only if no tree in  $\mathcal{L}(T)$  is *n-large with respect to  $\Sigma$* . Consequently,  $\varphi_\Sigma$  *accepts*  $T$  if for some  $n$  no tree in  $\mathcal{L}(T)$  is *n-large with respect to  $\Sigma$* , that is, if  $\mathcal{L}(T)$  is not  $\Sigma$ -diagonal. Thus, in order to solve the diagonal problem, it is enough to check

whether  $\varphi_\Sigma$  accepts  $BT(\mathcal{G})$  (recall that  $\mathcal{L}(\mathcal{G})$  is defined as  $\mathcal{L}(BT(\mathcal{G}))$ ), which is decidable by Corollary 5.2. It remains to construct the aforementioned sentence  $\varphi_\Sigma$ .

First, observe that the process of producing a finite tree recognized by  $\mathcal{G}$  from the infinite tree  $BT(\mathcal{G})$  generated by  $\mathcal{G}$  is expressible by a formula of **WCMSO** (actually, by a first-order formula). More precisely we can write a **WCMSO** formula  $\text{tree}(X)$  that holds in a tree  $T$  if and only if  $X$  is instantiated to a set of nodes of a tree  $T' \in \mathcal{L}(T)$ , together with their **nd**-labeled ancestors. See Appendix H for more details. Using  $\text{tree}(X)$  we now construct the desired formula  $\varphi_\Sigma$ , and thus we finish the proof of Theorem 5.1.

► **Lemma 5.4.** *Given a set of letters  $\Sigma$ , one can compute a **WCMSO** sentence  $\varphi_\Sigma$  that, for every  $n \in \mathbb{N}$ ,  $n$ -accepts a tree  $T$  if and only if no tree in  $\mathcal{L}(T)$  is  $n$ -large with respect to  $\Sigma$ .*

**Proof.** We can reformulate the property as follows: for every tree  $T' \in \mathcal{L}(T)$  there is a letter  $a \in \Sigma$ , and a leaf  $x$  that has less than  $n$   $a$ -labeled ancestors. This is expressed by the following formula of **WCMSO** (where  $\text{leaf}(x)$  states that the node  $x$  is a leaf,  $a(x)$  that  $x$  has label  $a$ , and  $z \leq x$  that  $z$  is an ancestor of  $x$ , all being easily expressible):

$$\forall X. \left( \text{tree}(X) \rightarrow \bigvee_{a \in \Sigma} \exists x \exists Z. (x \in X \wedge \text{leaf}(x) \wedge \forall z. (z \leq x \wedge a(z) \rightarrow z \in Z) \wedge |Z| < N) \right). \blacktriangleleft$$

## 6 Conclusions

A tantalising direction for further work is to drop the safety assumption from Theorem 3.1, that is, to establish whether the model-checking problem against B-automata is decidable for trees generated by (not necessarily safe) recursion schemes. We also leave open whether downward closures are computable for this more expressive class. Another direction for further work is to analyse the complexity of the considered model-checking problem. The related problem described in Remark 4.9 is  $k$ -EXP-complete for languages of finite trees recognised by recursion schemes of order  $k$  [44], and thus not harder than the nonemptiness problem [42]. Does the same upper bound hold for the more general diagonal problem that we consider in this paper? Zetzsche [55] has shown that the downward closure inclusion problem is **co- $k$ -NEXP**-hard for languages of finite trees recognised by safe recursion schemes of order  $k$ . Is it possible to obtain a matching upper bound?

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**A Remark on Theorem 2.1**

Due to some differences in definitions our Theorem 2.1 is weaker in two aspects and stronger in one aspect than the result of Blumensath et al. [5, Theorem 6]. We elaborate on these differences here.

First, Blumensath et al. [5] do not say that a one-way B-automaton  $\mathcal{A}$  and a two-way B-automaton  $\mathcal{B}$  recognize the same languages, but rather that the cost functions defined by these B-automata are equal (modulo domination equivalence). The latter means that there exists a non-decreasing function  $\alpha: \mathbb{N} \rightarrow \mathbb{N}$  such that if one of the B-automata ( $\mathcal{A}$  or  $\mathcal{B}$ )  $n$ -accepts some tree  $T$ , then the other B-automaton  $\alpha(n)$ -accepts this tree  $T$ . Clearly this is a stronger notion; it implies that the sets of accepted trees are equal.

Second, the definition of one-way B-automata in Blumensath et al. [5] forbids usage of the direction  $\circlearrowleft$  (along with  $\uparrow$ ), while we allow to use  $\circlearrowleft$  (only  $\uparrow$  is forbidden). A translation to one-way B-automata becomes only easier if their definition is less restrictive. We remark, however, that we actually need to allow usage of  $\circlearrowleft$  in order to correctly handle letters of rank 0—we do not want a one-way B-automaton to get stuck in a node without children.

Third, the B-automata of Blumensath et al. [5] work over binary trees, that is, all letters of the alphabet are assumed to be of rank 2, while we allow letters of arbitrary ranks. It is not difficult to believe that the assumption about a binary alphabet is just a technical simplification, and that all the proofs of Blumensath et al. [5] can be repeated for an arbitrary alphabet. Alternatively, it is possible to encode a tree over an arbitrary ranked alphabet in a binary tree, using the first-child next-sibling representation. Such an encoding can be easily incorporated into an B-automaton. Thus, in order to convert a two-way B-automaton  $\mathcal{A}$  into a one-way B-automaton  $\mathcal{B}$ , we can first convert it into a two-way B-automaton  $\mathcal{A}_2$  over a binary alphabet (reading the first-child next-sibling representation of a tree), then convert  $\mathcal{A}_2$  into a one-way B-automaton  $\mathcal{B}_2$  (using the results of Blumensath et al. [5, Theorem 6]), and then convert  $\mathcal{B}_2$  into  $\mathcal{B}$  reading the actual tree instead of its first-child next-sibling representation.

**B Remark on Fact 2.2**

In this short section we relate Fact 2.2 to results of Colcombet and Göller [15]. Recall that in Fact 2.2 we are given a B-automaton  $\mathcal{A}$  and a regular tree  $T$ , and the goal is to decide whether  $\mathcal{A}$  accepts  $T$ . Thanks to Theorem 2.1 we can assume that  $\mathcal{A}$  is one-way. Next, recall that acceptance of  $T$  is defined in terms of a game  $(\mathcal{A}, T)$ . When  $\mathcal{A}$  is one-way and  $T$  is regular, this game has actually a finite arena. Indeed, for the future of a play it does not matter what is the current node of  $T$ , it only matters which subtree starts in the current node—and in  $T$  we have finitely many different subtrees. As a consequence, games obtained this way are a special case of games considered by Colcombet and Göller [15], for which they prove decidability.

**C Remark on Lemma 3.3**

In this short section we relate Lemma 3.3 to results of Knapik et al. [32, 33]. Lemma 3.3 says that a lambda-tree representation  $T$  of a safe recursion scheme  $\mathcal{G}$  of order  $m$  can be computed by a safe recursion scheme  $\mathcal{G}'$  of order  $m - 1$ ; formally  $\llbracket T \rrbracket_{\mathcal{X}, s} = BT(\mathcal{G})$  where  $T = BT(\mathcal{G}')$ . In Knapik et al. [32, 33] the lambda-tree  $T$  is denoted  $\mathfrak{J}_{\mathcal{G}}(X_0)$  (where  $X_0$  is the starting nonterminal of  $\mathcal{G}$ ), and the recursion scheme  $\mathcal{G}'$  is denoted  $\mathcal{G}^\alpha$ . The set  $\mathcal{X}$  is just the set of variables appearing in the letters used in  $\mathcal{G}^\alpha$ ; the number  $s$  can be also read out of



$\mathcal{G}^\alpha$ . They prove that the  $(\mathcal{X}, s)$ -derived tree of  $\mathfrak{J}_{\mathcal{G}}(X_0)$  equals the tree generated by  $\mathcal{G}$  [32, Proposition 4], that  $\mathcal{G}^\alpha$  is safe [33, Lemma 3.5], and that  $\mathcal{G}^\alpha$  generates  $\mathfrak{J}_{\mathcal{G}}(X_0)$ .

Knapik et al. [32, 33] require that types of all nonterminals are **homogeneous** (a type  $\alpha_1 \rightarrow \dots \rightarrow \alpha_k \rightarrow \mathbf{o}$  is **homogeneous** if  $\text{ord}(\alpha_1) \geq \dots \geq \text{ord}(\alpha_k)$  and all  $\alpha_1, \dots, \alpha_k$  are **homogeneous**). However every **safe recursion scheme** can be converted into a **safe recursion scheme** of the same order, with all nonterminals having **homogeneous types** [58]. Additionally, they require that every nonterminal of positive order has at least one parameter of order 0. This can be easily ensured by adding to every nonterminal an additional parameter that is never used.

## D Proof of Lemma 4.1

**Simple tree regular expressions.** Let us recall the definition of *simple tree regular expressions* (*STREs*) from Goubault-Larrecq and Schmitz [24]. An *STRE* is defined according to the following abstract syntax:

$$\begin{aligned} S &::= P + \dots + P, & I &::= C + \dots + C, & S_\square &::= \square \mid S. \\ P &::= a^?(S, \dots, S) \mid I^*.S, & C &::= a(S_\square, \dots, S_\square), \end{aligned}$$

Unlike for our pure products, these expressions allow empty sums, which are denoted by 0, as well as **contexts**  $C$  with no **holes**. Subexpressions of the form  $P$  and  $I$  are called *pre-products* and *iterators*, respectively. The set of trees  $\llbracket S \rrbracket$  denoted by  $S$  is defined as for pure products, with the exception that

- $\llbracket P_1 + \dots + P_k \rrbracket = \llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket$ , and
- if  $\llbracket S_{\square, i} \rrbracket = \emptyset$  (or  $\llbracket S_i \rrbracket = \emptyset$ ) for some  $i \in \{1, \dots, r\}$ , then  $\llbracket a(S_{\square, 1}, \dots, S_{\square, r}) \rrbracket$  (and  $\llbracket a^?(S_1, \dots, S_r) \rrbracket$ , respectively) are defined to be  $\emptyset$  (notice that this does not change anything in the case of pure products described in Section 4, because  $\llbracket P \rrbracket \neq \emptyset$  for every pure product  $P$ ).

Two *STREs*  $S, T$  are *equivalent* if  $\llbracket S \rrbracket = \llbracket T \rrbracket$ . The following lemma is shown in Goubault-Larrecq and Schmitz [24, Proposition 18].

► **Lemma D.1.** *Every *STRE*  $S$  denotes a downward-closed set of trees  $\llbracket S \rrbracket$  and, vice versa, for every downward-closed set of trees  $\mathcal{L}$  there exists an *STRE*  $S$  such that  $\mathcal{L} = \llbracket S \rrbracket$ .* ◀

**Products.** Goubault-Larrecq and Schmitz [24] give a way of simplifying *STREs* by means of a rewrite relation  $\rightarrow_1$ . Intuitively, the idea is to move the operator “+” inside-out as much as possible. A **context**  $a(S_{\square, 1}, \dots, S_{\square, r})$  is **linear** if at most one  $S_{\square, i}$  is a hole  $\square$ , and it is **full** if  $r \geq 1$  and all the  $S_{\square, j}$ ’s are **holes**  $\square$ . An **iterator**  $C_1 + \dots + C_k$  is **linear** (full) if all the  $C_i$ ’s are **linear contexts** (full, respectively). Assuming that “+” is commutative and associative, we define the rewrite relation  $\rightarrow_1$  as follows:

$$P + P' \rightarrow_1 P' \quad \text{if } \llbracket P \rrbracket \subseteq \llbracket P' \rrbracket, \quad (1)$$

$$C + C' \rightarrow_1 C' \quad \text{if } \llbracket C \rrbracket \subseteq \llbracket C' \rrbracket, \quad (2)$$

$$0^*.S \rightarrow_1 S, \quad (3)$$

$$a^?(S_1, 0, S_2) \rightarrow_1 0, \quad (4)$$

$$a(S_{\square, 1}, 0, S_{\square, 2}) \rightarrow_1 0, \quad (5)$$

$$I^*.0 \rightarrow 0 \quad \text{if } I \text{ is full}, \quad (6)$$

$$(I + a(S_1, \dots, S_r))^*.S \rightarrow_1 I^*. (S + a^?(S_1, \dots, S_r)), \quad (7)$$

$$a^?(\vec{S}_1, S + S', \vec{S}_2) \rightarrow_1 a^?(\vec{S}_1, S, \vec{S}_2) + a^?(\vec{S}_1, S', \vec{S}_2), \quad (8)$$

$$a(\vec{S}_{\square,1}, S + S', \vec{S}_{\square,2}) \rightarrow_1 a(\vec{S}_{\square,1}, S, \vec{S}_{\square,2}) + a(\vec{S}_{\square,1}, S', \vec{S}_{\square,2}), \quad (9)$$

$$I^*.(S + S') \rightarrow_1 I^*.S + I^*.S' \quad \text{if } I \text{ is linear.} \quad (10)$$

A *product* is a pre-product that is in a normal form with respect to  $\rightarrow_1$ . We know that the rewrite relation  $\rightarrow_1$  preserves the denotation of **STRE** [24, Fact 19], and that every **STRE** has a normal form with respect to  $\rightarrow_1$  [24, Lemma 20]. The following corollary is immediate.

► **Corollary D.2.** *Every **STRE**  $S$  is equivalent to a sum of products  $P_1 + \dots + P_k$ .* ◀

**From products to pure products.** By definition, if every **STRE** subexpression of a product  $P$  is a single product (i.e., is neither 0 nor a sum of two or more products), then  $P$  is a pure product. In Lemma D.3 we show how to convert an arbitrary product into a pure product. Lemma 4.1 is then a direct consequence of Lemma D.1, Corollary D.2, and Lemma D.3.

► **Lemma D.3.** *For every product  $P$  one can create an equivalent pure product  $P'$ .*

**Proof.** The proof is by induction on the size of  $P$ . Before starting the actual proof, let us observe two facts, which we use implicitly below. First, every subterm of  $P$  that is a pre-product is actually a product (i.e., it cannot be rewritten by  $\rightarrow_1$ ). Second, if we replace a product subexpression of  $P$  by some equivalent product, then the resulting **STRE** is still a product (i.e., it cannot be rewritten by  $\rightarrow_1$ ).

Coming now to the proof, suppose that  $P$  is of the form  $a^?(S_1, \dots, S_r)$ . Then, because  $P$  is a product, that is, because it cannot be rewritten by  $\rightarrow_1$ , we can observe that all  $S_i$  are products. Indeed, if some  $S_i$  was a sum of two or more products (or 0), then  $P$  could be rewritten using Rule (8) (or Rule (4), respectively). By the induction assumption for every product  $S_i$  we can create an equivalent pure product  $S'_i$ ; as  $P'$  we take  $a^?(S'_1, \dots, S'_r)$ .

Next, suppose that  $P$  is of the form  $I^*.S$ . Consider a **context**  $C = a(S_{\square,1}, \dots, S_{\square,r})$ , being a component of  $I$ . We can observe that all  $S_{\square,i}$  are either  $\square$  or products. Indeed, if some  $S_{\square,i}$  was a sum of two or more products (or 0), then  $C$  could be rewritten using Rule (9) (or Rule (5), respectively). As previously, using the induction assumption we can replace every product  $S_{\square,i}$  that is not a **hole**  $\square$  by an equivalent pure product  $S'_{\square,i}$ . Applying this to every **context**  $C$  in  $I$ , we obtain a new **iterator**  $I_\circ$  in which every **STRE** subterm is a single product. Likewise, writing  $S = P_1 + \dots + P_k$ , we can replace every product  $P_i$  by an equivalent pure product  $P'_i$ . This way, we obtain a product  $P^\circ = I_\circ^*.(P'_1 + \dots + P'_k)$  equivalent to  $P$ . Observe also that there is at least one **context** in  $I_\circ$ , and that every **context** in  $I_\circ$  contains a **hole** (because Rules (3) and (7) cannot be applied to  $P^\circ$ ), as required in our definition of a pure product. Thus, when  $k = 1$ ,  $P^\circ$  is a pure product, hence it can be taken as  $P'$ . It remains to deal with the situation when  $k \neq 1$ .

One possibility is that  $k = 0$ . Then  $I_\circ$  is not full (otherwise Rule (6) could be applied to  $P^\circ$ ), which means that in  $I_\circ$  there is a **context**  $C' = a(S'_{\square,1}, \dots, S'_{\square,r})$  such that  $S'_{\square,j} \neq \square$  for some  $j \in \{1, \dots, r\}$ . Fix one such  $C'$  and  $j$ , and define  $P' := I_\circ^*.S'_{\square,j}$ . We easily see that no rule of  $\rightarrow_1$  can be applied to  $P'$  (recall that  $S'_{\square,j}$  is a single product), so  $P'$  is indeed a (pure) product. Clearly  $\llbracket P^\circ \rrbracket \subseteq \llbracket P' \rrbracket$ . On the other hand,  $\llbracket S'_{\square,i} \rrbracket \neq \emptyset$  for all  $i \in \{1, \dots, r\}$ , because  $S'_{\square,i}$  is either a hole or a pure product, and it can be easily seen (by induction on its structure) that a pure product always denotes a nonempty set; in consequence  $\llbracket S'_{\square,j} \rrbracket \subseteq \llbracket C' \rrbracket \subseteq \llbracket I_\circ \rrbracket$ , so also  $\llbracket P' \rrbracket \subseteq \llbracket P^\circ \rrbracket$ .

Another possibility is that  $k \geq 2$ . Then  $I_\circ$  is not linear (if  $I_\circ$  was linear, Rule (10) could be applied to  $P^\circ$ ), which means that in  $I_\circ$  there is a **context**  $C' = a(S'_{\square,1}, \dots, S'_{\square,r})$  with two or more **holes**. Fix one such  $C'$ . For simplicity, we show the proof assuming that the first  $k$

among  $S'_{\square,i}$  are *holes*, and the remaining  $r - k$  among  $S'_{\square,i}$  are products (i.e., are not *holes*); the general situation can be handled in the same way, but writing it down would require us to use intricate indices. We define

$$\begin{aligned} R_1 &= a^?(P'_1, P'_1, \dots, P'_1, S'_{\square,k+1}, \dots, S'_{\square,r}) & \text{and} \\ R_i &= a^?(R_{i-1}, P'_i, \dots, P'_i, S'_{\square,k+1}, \dots, S'_{\square,r}) & \text{for } i \in \{2, \dots, k\}, \end{aligned}$$

and we define  $P' := I_{\circ}^*.R_k$ . Notice that no rule of  $\rightarrow_1$  can be applied to  $P'$ ; it is a (pure) product. On the one hand,  $\llbracket P'_i \rrbracket \subseteq \llbracket R_k \rrbracket$  for every  $i \in \{1, \dots, k\}$  (it is important here that there are at least two *holes*, so  $P'_i$  actually appears in  $R_i$ ), so  $\llbracket P^\circ \rrbracket \subseteq \llbracket P' \rrbracket$ . On the other hand, we see that  $\llbracket R_k \rrbracket \subseteq \llbracket P^\circ \rrbracket$ , so also  $\llbracket P' \rrbracket \subseteq \llbracket P^\circ \rrbracket$ . ◀

### E Proof of Lemma 4.3

First, we restate Lemma 4.3 for convenience.

► **Lemma 4.3** (restated from page 11). *The class of languages of finite trees recognized by safe recursion schemes is effectively closed under linear FTT transductions.*

A very similar result has been proved in Clemente, Parys, Salvati, and Walukiewicz [13, Theorem 2.1].

► **Lemma E.1.** *The class of languages of finite trees recognized by recursion schemes is effectively closed under linear FTT transductions.* ◀

We need to strengthen the result and show that applying a linear FTT transduction to a language recognized by a safe recursion scheme preserves safety. Essentially the same construction as in the proof of Lemma E.1 [57, Appendix A] already achieves this, albeit some modifications are needed. We now argue how to modify the proof in three aspects:

1. The proof uses the fact that higher-order recursion schemes *with states* (as introduced in that proof) are convertible to equivalent higher-order recursion schemes by increasing the arity of nonterminals. It is a simple observation that such a translation preserves safety.
2. The proof of Clemente et al. [57, Appendix A] uses the notion of *normalised* recursion schemes, wherein every rule is assumed to be of the form

$$A x_1 \dots x_p \rightarrow h(B_1 x_1 \dots x_p) \dots (B_r x_1 \dots x_p),$$

where  $h$  is either a variable  $x_i$ , or a nonterminal, or a letter, and the  $B_j$ 's are nonterminals. This normal form is used only to simplify the presentation and is in no way essential. This is important since putting a recursion scheme in such a normal form does not preserve safety. Indeed, a subterm  $B_1 x_1 \dots x_p$  replaces some subterm  $M_i$  appearing originally on the right side of the rule for  $A$ ; if some variable  $x_j$  was not used in  $M_i$ , we could have  $\text{ord}(x_j) > \text{ord}(M_i)$  (the latter equals  $\text{ord}(B_1 x_1 \dots x_p)$ ), which violates safety of the normalized right side. Therefore, we modify the definition of the normal form to allow removal of selected variables on the right, that is, to allow rules of the form

$$A x_1 \dots x_p \rightarrow h(B_1 x_{i_1,1} \dots x_{i_1,k_1}) \dots (B_r x_{i_r,1} \dots x_{i_r,k_r}).$$

3. In order to handle non-complete transductions, it is important to make the recursion scheme *productive*, in the sense that the language of every subterm of the Böhm tree should be nonempty. One can effectively transform a given recursion scheme into a

productive one generating the same language. This is done in Clemente et al. [13, Lemma A.2] by using the *reflection property* of recursion schemes [8, Corollary 2]. Thus, in order to show that safety is preserved, we need to know that the reflection property holds also for *safe* recursion schemes. This is proved in Lemma E.2.

► **Lemma E.2.** *Safe recursion schemes enjoy the reflection property.*

**Proof.** Carayol and Wöhrle [56] prove that the class of trees generated by deterministic higher-order pushdown automata is closed under MSO-relabeling (i.e., reflection). This class coincides with the class of trees generated by safe recursion schemes [33, Theorems 5.1 and 5.3]. ◀

## F Proof of Theorem 4.5

► **Theorem 4.5** (restated from page 11). *Let  $\mathfrak{C}$  be a class of languages of finite trees closed under linear FTT transductions. One can compute a finite tree automaton recognizing the downward closure of a given language from  $\mathfrak{C}$  if and only if SUP is decidable for  $\mathfrak{C}$ .*

**Proof.** If downward closures are computable, then one can compute a finite tree automaton for  $\mathcal{L}\downarrow$  and check whether  $\llbracket P \rrbracket \subseteq \mathcal{L}\downarrow$  for a given (diversified) pure product  $P$ . The latter is possible since language inclusion for finite tree automata is decidable [18].

For the other direction, assume that SUP is decidable for  $\mathfrak{C}$  and let  $\mathcal{L} \in \mathfrak{C}$ . The downward closure  $\mathcal{L}^d := \mathcal{L}\downarrow$  is effectively in  $\mathfrak{C}$  since it can be obtained as a linear FTT transduction of  $\mathcal{L}$  (cf. Fact 4.2). Thus, it is enough to compute (a finite tree automaton recognizing) the downward closure of a downward-closed language  $\mathcal{L}^d$ . Furthermore, since by Lemma 4.1  $\mathcal{L}^d$  equals  $\llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket$  for some pure products  $P_1, \dots, P_k$ , it suffices to guess these pure products and check whether the equality  $\mathcal{L}^d = \llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket$  indeed holds.

We start by showing how to decide whether  $\mathcal{L}^d \subseteq \llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket$ . Firstly,  $\mathcal{R} := \llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket$  is (effectively) a regular language, and consequently its complement  $\mathcal{R}^c$  is also regular. In consequence,  $\mathcal{M} := \mathcal{L}^d \cap \mathcal{R}^c$  is effectively in  $\mathfrak{C}$ , because it can be obtained from  $\mathcal{L}^d$  by intersecting it with  $\mathcal{R}^c$ , which is a linear FTT transduction (cf. Fact 4.2). Secondly, emptiness of any language  $\mathcal{M} \in \mathfrak{C}$  is decidable by reducing to SUP, since it suffices to apply to it the linear FTT  $\mathcal{A}$  that ignores the input and outputs all trees of the form  $a(a(\dots(a(e()))\dots))$  (for some fixed letters  $a$  of rank 1 and  $e$  of rank 0), and to compare the result with the diversified pure product  $P := (a(\square))^*.e^?$ . Indeed,  $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathcal{M})\downarrow = \llbracket P \rrbracket$  if  $\mathcal{M}$  is nonempty, and  $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathcal{M})\downarrow = \emptyset$  if  $\mathcal{M}$  is empty; thus, on the one hand,  $\mathcal{A}(\mathcal{M}) \subseteq \llbracket P \rrbracket$  and, on the other hand,  $\mathcal{M}$  is nonempty if and only if  $\llbracket P \rrbracket \subseteq \mathcal{A}(\mathcal{M})\downarrow$ .

For the other inclusion  $\llbracket P_1 \rrbracket \cup \dots \cup \llbracket P_k \rrbracket \subseteq \mathcal{L}^d$  we can equivalently check whether  $\llbracket P_i \rrbracket \subseteq \mathcal{L}^d$  for all  $i \in \{1, \dots, k\}$ , which implies that it suffices to show decidability of checking the containment  $\llbracket P \rrbracket \subseteq \mathcal{L}^d$  for a single pure product  $P$ . We make  $P$  diversified by adding additional marks to letters appearing in  $P$ . Namely, for each letter  $a$  appearing  $k$  times in  $P$ , we consider “marked” letters  $a_1, \dots, a_k$ , and for each occurrence of  $a$  in  $P$  we substitute a different letter  $a_i$ . Call the resulting diversified pure product  $P'$ . We also create a corresponding linear FTT  $\mathcal{A}$ ; it replaces every label  $a$  in an input tree by an arbitrary among the letters  $a_i$  (for every occurrence of  $a$  we choose a mark  $i$  independently). We obtain the following equivalence.

▷ **Claim F.1.**  $\llbracket P \rrbracket \subseteq \mathcal{L}^d \iff \llbracket P' \rrbracket \subseteq \mathcal{A}(\mathcal{L}^d)$ .

**Proof.** For the left-to-right implication, assume that  $\llbracket P \rrbracket \subseteq \mathcal{L}^d$ , and take some  $T' \in \llbracket P' \rrbracket$ . Let  $T$  be the tree obtained from  $T'$  by removing marks from its labels; clearly  $T \in \llbracket P \rrbracket \subseteq \mathcal{L}^d$  and  $(T, T') \in \mathcal{A}$ , which implies that  $T' \in \mathcal{A}(\mathcal{L}^d)$ .

For the opposite implication, assume that  $\llbracket P' \rrbracket \subseteq \mathcal{A}(\mathcal{L}^d)$ , and take some  $T \in \llbracket P \rrbracket$ . By definition,  $T \in \llbracket P \rrbracket$  means that  $T$  can be “matched” into  $P$ , so that every node of  $T$  matches exactly one label in  $P$ . We fix one such matching, and we add marks to labels of  $T$  in the same way as it was done while creating  $P'$  from  $P$ ; call the resulting tree  $T'$ . We have that  $T' \in \llbracket P' \rrbracket \subseteq \mathcal{A}(\mathcal{L}^d)$  and  $(T, T') \in \mathcal{A}$ . Next, we observe that the only tree  $T_0$  such that  $(T_0, T') \in \mathcal{A}$  is  $T$  ( $T_0$  has to be obtainable from  $T'$  by removing marks). Thus,  $T' \in \mathcal{A}(\mathcal{L}^d)$  implies  $T \in \mathcal{L}^d$ .  $\triangleleft$

Thus, instead of checking whether  $\llbracket P \rrbracket \subseteq \mathcal{L}^d$ , we can check whether  $\llbracket P' \rrbracket \subseteq \mathcal{A}(\mathcal{L}^d)$ . Finally, we consider a language  $\mathcal{L}' := \mathcal{A}(\mathcal{L}^d) \cap \llbracket P' \rrbracket$ , which can be obtained from  $\mathcal{A}(\mathcal{L}^d)$  by a linear FTT transduction (cf. Fact 4.2), and thus which is effectively in  $\mathfrak{C}$ . Then, on the one hand,  $\mathcal{L}' \subseteq \llbracket P' \rrbracket$  and, on the other hand,  $\llbracket P' \rrbracket \subseteq \mathcal{A}(\mathcal{L}^d)$  if and only if  $\llbracket P' \rrbracket \subseteq \mathcal{L}'$ . Recall that  $\mathcal{L}^d$  and  $\llbracket P' \rrbracket$  are downward closed. It does not matter whether we first remove some parts of a tree and then we add marks to labels, or we first add marks to labels and then we remove some part of a tree, so  $\mathcal{A}(\mathcal{L}^d)$  and  $\mathcal{L}'$  are downward closed as well (and hence  $\mathcal{L}' \downarrow = \mathcal{L}'$ ). It follows that checking whether  $\llbracket P' \rrbracket \subseteq \mathcal{L}'$  is an instance of SUP.  $\blacktriangleleft$

## G Proof of Lemma 4.8

► **Lemma 4.8** (restated from page 12). *Let  $\mathfrak{C}$  be a class of languages of finite trees closed under linear FTT transductions. SUP for  $\mathfrak{C}$  reduces to the diagonal problem for  $\mathfrak{C}$ .*

**Proof.** In an instance of SUP we are given a diversified pure product  $P$  and a language  $\mathcal{L} \in \mathfrak{C}$ . Consider the language of trees  $\mathcal{L}' = \mathcal{L}_0 \downarrow \cap \llbracket P \rrbracket$ . Clearly  $\llbracket P \rrbracket$  is regular, so  $\mathcal{L}' \in \mathfrak{C}$  by Fact 4.2. The following claim is a direct consequence of Lemma 4.7.

▷ **Claim G.1.**  $\llbracket P \rrbracket \subseteq \mathcal{L} \downarrow$  if and only if for every  $n \in \mathbb{N}$  there is a tree in  $\mathcal{L}'$  that is  $n$ -large with respect to  $P$ .  $\triangleleft$

We have reduced to a problem which is very similar to the diagonal problem, except that we should put no requirement on the number of occurrences of  $\text{root}(I^*.P')$  for branches not containing an occurrence of  $\text{root}(P')$ . In order to fix this, let  $\mathcal{L}''$  be the set of trees  $T''$  obtained from some tree  $T'$  of  $\mathcal{L}'$  by the following procedure: whenever a branch of  $T'$  does not contain an occurrence of  $\text{root}(P')$ , then the leaf finishing this branch can be replaced by an arbitrarily large tree with internal nodes labeled by  $\text{root}(I^*.P')$ . Let  $\Sigma$  be the set of root labels of the form  $\text{root}(I^*.P')$  for every subexpression  $I^*.P'$  of  $P$ .

▷ **Claim G.2.**  $\mathcal{L}''$  is  $\Sigma$ -diagonal if and only if for every  $n \in \mathbb{N}$  there is a tree in  $\mathcal{L}'$  which is  $n$ -large with respect to  $P$ .  $\triangleleft$

The operation mapping  $\mathcal{L}'$  to  $\mathcal{L}''$  can be realized as a linear FTT transduction, and thus  $\mathcal{L}'' \in \mathfrak{C}$ . This completes the reduction from SUP to the diagonal problem.  $\blacktriangleleft$

## H Additional lemma for Section 5

► **Lemma H.1.** *There is a WCMSO formula  $\text{tree}(X)$  that holds in a tree  $T$  if and only if  $X$  is instantiated to a set of nodes of a tree  $T' \in \mathcal{L}(T)$ , together with their nd-labeled ancestors.*

**Proof.** The formula simply says that

- $X$  is finite,
- the root of the tree belongs to  $X$ ,

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- no node  $x \in X$  is  $\perp$ -labeled,
- for every **nd**-labeled node  $x \in X$ , exactly one among the children of  $x$  belongs to  $X$ ,
- for every node  $x \in X$  with label other than **nd**, all children of  $x$  belong to  $X$ , and
- if  $x \notin X$ , then no child of  $x$  belongs to  $X$ .

All the above statements can be easily expressed in WCMSO.

