

# Relating complementation constructions for Büchi automata

(draft)

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**Abstract.**

## 1 Preliminaries

Fix a finite alphabet  $\Sigma$  and a finite set of states  $Q$ . A *transition profile* (over  $Q$ )  $t$  is a pair  $(\rightarrow_t, \twoheadrightarrow_t)$ , where  $\rightarrow_t \subseteq Q \times Q$  and  $\twoheadrightarrow_t \subseteq \rightarrow_t$ . Let  $\mathcal{P}$  be the set of transition profiles. Intuitively,  $p \rightarrow_t q$  if it is possible to go from  $p$  to  $q$  via a transition in  $t$ , and  $p \twoheadrightarrow_t q$  if, additionally, an accepting such transition is taken. A *Büchi Automaton* (BA)  $\mathcal{A}$  is a tuple  $(\Sigma, Q, I, (t_a)_{a \in \Sigma})$  where  $\Sigma$  is a finite alphabet,  $Q$  is a finite set of states,  $I \subseteq Q$  is a non-empty set of *initial* states, and, for each input symbol  $a \in \Sigma$ ,  $t_a$  is a transition profile over  $Q$ . For simplicity, instead of writing  $p \rightarrow_{t_a} q$  or  $p \twoheadrightarrow_{t_a} q$ , we just write  $p \xrightarrow{a} q$  and  $p \twoheadrightarrow^a q$ , respectively.

A *infinite trace* of  $\mathcal{A}$  on a word  $w = a_0 a_1 \dots \in \Sigma^\omega$  starting in a state  $q_0 \in Q$  is an infinite sequence of transitions  $\pi = q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} \dots$ . An infinite trace is *initial* if it starts in an initial state  $q_0 \in I$ , and it is *fair* iff  $q_i \twoheadrightarrow^{a_i} q_{i+1}$  for infinitely many  $i$ . The *language* of  $\mathcal{A}$  is  $\mathcal{L}(\mathcal{A}) = \{w \mid \mathcal{A} \text{ has an infinite, initial and fair trace on } w\}$ .

## 2 Ramsey

Fix an automaton  $\mathcal{A} = (\Sigma, Q, I, F, \delta)$ . For two profiles  $s$  and  $t$ , their product  $st$  is the pair  $(\rightarrow, \twoheadrightarrow)$  defined as follows:

- $p \rightarrow q$  iff there exists  $r \in Q$  s.t.  $p \rightarrow_s r \rightarrow_t q$ , and
- $p \twoheadrightarrow q$  iff there exists  $r \in Q$  s.t. either  $p \twoheadrightarrow_s r \rightarrow_t q$  or  $p \rightarrow_s r \twoheadrightarrow_t q$ .

Clearly, product of profiles is an associative operation.

For a finite word  $u = a_0 a_1 \dots a_m \in \Sigma^*$ , the induced profile is  $t_u = t_{a_0} t_{a_1} \dots t_{a_m}$ . The language of a profile  $t$  is the set of words  $\mathcal{L}(t)$  inducing that profile, i.e.,  $\mathcal{L}(t) = \{u \in \Sigma^* \mid t_u = t\}$ , and a profile is *valid* iff it has nonempty language (i.e., if it can be generated from profiles in  $(t_a)_{a \in \Sigma}$ ). Let  $\mathcal{P}^v$  be the set of valid profiles. The relationship between multiplication of profiles and their language is as follows.

**Lemma 1.** *For two profiles  $s$  and  $t$ ,  $\mathcal{L}(s)\mathcal{L}(t) \subseteq \mathcal{L}(st)$ , and the inclusion is strict in general.*

The fundamental property of profiles is that they only yield trivial intersections with  $A$ .

generalize the test  
to every pair?

**Lemma 2.** For two profiles  $s$  and  $t$ , if  $\mathcal{L}(s)(\mathcal{L}(t))^\omega \cap A \neq \emptyset$ , then  $\mathcal{L}(s)(\mathcal{L}(t))^\omega \subseteq A$ .

A pair of profiles  $(s, t)$  is *linked* iff  $st = s$  and  $tt = t$ . We define a test operation on linked profiles: For two linked profiles  $s$  and  $t$ , let  $?(s, t)$  iff  $p \rightarrow_s q \rightarrow_t q$  with  $p \in I$ .

**Lemma 3.** For two linked profiles  $s$  and  $t$ ,  $?(s, t)$  holds iff  $\mathcal{L}(s)(\mathcal{L}(t))^\omega \subseteq \mathcal{L}(\mathcal{A})$ .

Profiles define an action on sets of states: For a set of states  $P \subseteq Q$  and a profile  $t$ , let  $P \cdot t$  be those states which are reachable through transitions in  $t$  originating from states in  $P$ ; i.e.,  $P \cdot t = \{q \mid \exists(p \in P) \cdot p \rightarrow_t q\}$ . Clearly,  $P \cdot (st) = (P \cdot s) \cdot t$ . For simplicity, if  $w \in \mathcal{L}(t)$ , then we also just write  $I \cdot w$  instead of  $I \cdot t$ .

Describe Ramsey-based complementation.

### 3 Ranks

Let  $w = a_0 a_1 \dots \in \Sigma^\omega$  be an infinite word. The infinite traces of  $w$  on  $\mathcal{A}$  can be arranged by juxtaposition of transition profiles into an infinite transition DAG  $G = \langle V, T \rangle$ , where

- $V \subseteq Q \times \omega$  is the set of vertices s.t.  $(q, i) \in V$  iff  $q \in I \cdot (a_0 a_1 \dots a_{i-1})$ , and
- $T$  is a transition profile over  $V$  s.t., for every level  $i \geq 0$ ,  $\langle p, i \rangle \rightarrow_T \langle q, i+1 \rangle$  iff  $p \xrightarrow{a_i} q$  and  $\langle p, i \rangle \rightarrow_T \langle q, i+1 \rangle$  iff  $p \xrightarrow{a_i} q$ .

Then,  $w$  is not accepted by  $\mathcal{A}$  iff every infinite path in  $G$  eventually ceases taking accepting transitions. This is witnessed with the notion of ranking.

A ranking for a DAG  $G = \langle V, T \rangle$  is a mapping from  $V$  to  $\omega$  s.t. ranks along transitions do not increase, and odd ranks along accepting transitions are strictly decreasing. Clearly, every path in  $G$  gets eventually trapped in some rank, and, if this rank is odd for every path in  $G$ , then the ranking is called an *odd ranking*. Since odd ranks are strictly decreasing on accepting transitions, this implies that if  $G$  has an odd ranking, then every infinite path in  $G$  must eventually cease taking accepting transitions, and, therefore,  $G$  is a rejecting DAG.

Kupferman and Vardi have shown that bounded rankings in fact suffice. Let  $D^l = \{0, 1, \dots, l, \perp\}$  be the set of rank values bounded by  $l$  plus the additional undefined value  $\perp$ , where the order is extended as  $0 < 1 < \dots < \perp$ . We define a lifting function  $\lfloor \cdot \rfloor_{\text{even}}$  on rank values s.t.  $\lfloor n \rfloor_{\text{even}}$  is the largest even rank not larger than  $n$ ; i.e.,

$$\lfloor n \rfloor_{\text{even}} = \begin{cases} \perp & \text{if } n = \perp \\ n & \text{if } n \text{ is even} \\ n-1 & \text{otherwise} \end{cases}$$

The function  $\lfloor \cdot \rfloor_{\text{odd}}$  is defined analogously. A *l-level ranking* is a function  $f : Q \mapsto D^l$ . Let  $\mathcal{R}^l$  be the set of  $l$ -level rankings. For two level rankings  $f$  and  $g$ , let  $f \geq g$  iff, for every state  $p$ ,  $f(p) \geq g(p)$ . Transition profiles induce a successor relation on level rankings. For two rankings  $f$  and  $g$  and a profile  $t$ , we say that  $g$  is a *t-successor* of  $f$ , written  $f \xrightarrow{t} g$ , iff, for every transition  $p \rightarrow_t q$ ,  $f(p) \geq g(q)$ , and if  $p \rightarrow_t q$ , then  $\lfloor f(p) \rfloor_{\text{even}} \geq g(q)$ . If  $w \in \mathcal{L}(t)$ , we also just write  $f \xrightarrow{w} g$  instead of  $f \xrightarrow{t} g$ .

For a level ranking  $f$ , let  $\text{even}(f) = \{p \mid f(p) \text{ is even}\}$ , and similarly for  $\text{odd}(f)$ .

**Definition 1.** For an NBA  $\mathcal{A} = (\Sigma, Q, I, (t_a)_{a \in \Sigma})$  and a bound  $l \in \omega$ , define  $KV^l(\mathcal{A})$  to be the NBA  $(\Sigma, \mathcal{R}^l \times 2^Q, \{\langle f_0, \emptyset \rangle\}, (t'_a)_{a \in \Sigma})$ , where

- $f_0(p) = l$  if  $p \in I$ , and  $f_0(p) = \perp$  otherwise.
- $\langle f, O \rangle \xrightarrow{a} \langle f', (O \cdot a) \setminus \text{odd}(f') \rangle$  iff  $f \xrightarrow{a} f'$  and  $O \neq \emptyset$ , and
- $\langle f, \emptyset \rangle \xrightarrow{a} \langle f', \text{even}(f') \rangle$  iff  $f \xrightarrow{a} f'$ .

Profiles induce an action on level rankings. Intuitively, for a level ranking  $f$  and a profile  $t$ ,  $f \cdot t$  is the largest level ranking that can be obtained from  $f$  by following transitions in  $t$ . Formally,

$$f \cdot t = \max_{f \xrightarrow{t} g} g$$

It can be computed from  $f$  and  $t$  in the following way: for every  $q$ ,

$$(f \cdot t)(q) = \min_{p \rightarrow_t q} \begin{cases} \lfloor f(p) \rfloor_{\text{even}} & \text{if } p \rightarrow_t q \\ f(p) & \text{otherwise} \end{cases}$$

Note that, for two profiles  $s$  and  $t$ ,  $f \cdot (st) = (f \cdot s) \cdot t$ .

### 3.1 Periodic rankings

Level rankings and profiles are strongly related. We regard a profile as a system of (strict) inequalities between values assigned to states by level rankings. If a ranking complies with all these constraints, we say that it satisfies the profile. Formally, for a level ranking  $f$  and a profile  $t$ ,  $f$  *satisfies*  $t$ , written  $f \models t$ , iff the following two conditions hold:

$$f \models t \quad \text{iff} \quad \begin{array}{ll} f \cdot t \geq f, \text{ and} & \text{[Safety]} \\ \text{for all } p \rightarrow_t q, \lfloor f(p) \rfloor_{\text{odd}} \geq f(q) & \text{[Liveness]} \end{array}$$

The safety condition ensures that the level ranking  $f$  complies with the inequalities defined by  $t$ , while the liveness condition ensures even ranks in  $f$  are strictly decreasing along transitions in  $t$ . Intuitively, ranks get sufficiently small by liveness, and not too small by safety.

Given an idempotent profile  $t$ , we can associate to it a canonical level ranking  $f_t$ . First, we perform a SCC decomposition of  $t$ . For a state  $p$ , let  $[p]_t = \{q \mid p \rightarrow_t q \rightarrow_t p\}$  be the set of states which are inter-reachable from  $p$ ; notice that  $[p]_t$  is either empty, or it contains  $p$  (and there is a self-loop  $p \rightarrow_t p$ ). Then, we assign an index to each state  $p$  counting how many different non-empty classes  $[q]_t$  are reachable from  $p$  (via  $t$ ): We define a function  $\alpha_t : Q \mapsto \{0, 1, \dots, n\}$  ( $n$  is the number of states of the automaton) s.t., for every state  $p$ ,  $\alpha_t(p) = |\{[q] \neq \emptyset \mid p \rightarrow_t q\}|$ .

**Lemma 4.** Let  $t$  be an idempotent profile. Then, for any states  $p$  and  $q$ ,

1. If  $p \rightarrow_t q$ , then  $\alpha_t(p) \geq \alpha_t(q)$ .
2. If  $p \rightarrow_t q$  and  $[q]$  is empty,  $\alpha_t(p) \geq \alpha_t(q) + 1$ .
3. If  $t$  is rejecting,  $p \rightarrow_t q$  and  $[p]$  is not empty, then  $\alpha_t(p) \geq \alpha_t(q) + 1$ .

*Proof.* The first point follows immediately from the definition of  $\alpha_t$ . For the second point, notice that, by idempotence, a transition  $p \rightarrow_t q$  can always be split into two transitions  $p \rightarrow_t r \rightarrow_t q$ , for some state  $r$ . By repeating the process on the first transition until a duplicate state appears, there exists a state  $r'$  s.t.  $p \rightarrow_t r' \rightarrow_t q$  and  $r' \rightarrow_t r'$ . Therefore  $[r']$  is non-empty and  $q \notin [r']$  (otherwise one would have  $[q] = [r']$ ), thus  $\alpha_t(p) \geq \alpha_t(r') \geq \alpha_t(q) + 1$ . For the third point, notice that, since  $t$  is rejecting,  $[p]$  is different from  $[q]$  (one cannot have  $q \rightarrow_t p$ ). Therefore,  $\alpha_t(p) \geq \alpha_t(q) + 1$ .

Finally, given an idempotent profile  $t$ , we define the induced level ranking  $f_t$  as follows: For every state  $p$ ,

$$f_t(p) = \begin{cases} 2\alpha_t(p) & \text{if } [p] = \emptyset \\ 2\alpha_t(p) - 1 & \text{otherwise} \end{cases} \quad (1)$$

Notice that, if  $t$  is rejecting, then  $f_t$  is a valid level ranking: Indeed, if  $p$  is accepting, then  $[p]$  is necessarily empty (otherwise  $t$  would contain an accepting loop, which is impossible since it is rejecting), therefore  $f_t(p)$  is even. We have the following crucial property of  $f_t$ .

**Lemma 5.** *For an idempotent profile  $t$ , let  $f_t$  be the canonical level ranking  $t$  constructed as above. If  $t$  is rejecting, then  $f_t$  satisfies  $t$ .*

*Proof.* It suffices to prove that

- for every  $p \rightarrow_t q$ ,  $\lfloor f_t(p) \rfloor_{\text{odd}} \geq f_t(q)$ , and
- for every  $p \rightarrow_t q$ ,  $\lfloor f_t(p) \rfloor_{\text{even}} \geq f_t(q)$ .

For the first point, assume that  $p \rightarrow_t q$ . By Lemma 4,  $\alpha_t(p) \geq \alpha_t(q)$ . If  $[p]$  and  $[q]$  are both non-empty, clearly  $f_t(p) \geq f_t(q)$ . If  $[p]$  is non-empty and  $[q]$  is empty, by Lemma 4,  $\alpha_t(p) \geq \alpha_t(q) + 1$ , therefore  $f_t(p) = 2\alpha_t(p) - 1 \geq 2(\alpha_t(q) + 1) - 1 = 2\alpha_t(q) + 1 = f_t(q) + 1 \geq f_t(q)$ . If  $[p]$  and  $[q]$  are both empty, by Lemma 4,  $\alpha_t(p) \geq \alpha_t(q) + 1$ , therefore  $f_t(p) = 2\alpha_t(p) \geq 2(\alpha_t(q) + 1) = 2\alpha_t(q) + 2 = f_t(q) + 2$ . Finally, if  $[p]$  is empty and  $[q]$  is non-empty,  $f_t(p) = 2\alpha_t(p) \geq 2\alpha_t(q) = f_t(q) + 1$ .

For the second point, additionally assume that  $p \rightarrow_t q$  and that  $f_t(p)$  is odd, i.e.,  $[p]$  is non-empty. Then, by Lemma 4,  $\alpha_t(p) \geq \alpha_t(q) + 1$ , and one can proceed as above.

For a profile  $t$ , we say that  $t$  is *consistent* iff there exists a ranking  $f$  satisfying  $t$ . The following is the basic relation between rankings and profiles.

**Lemma 6.** *Let  $t$  be an idempotent profile. Then,  $t$  is consistent iff  $t$  is rejecting.*

*Proof.* Let  $t$  be a profile. For the “only if” direction, assume that  $t$  is consistent, and, by way of contradiction, that  $t$  is not rejecting. Thus,  $t$  has a loop  $p \rightarrow_t p$ . By consistency, there exists a ranking  $f$  satisfying  $t$ . By the liveness condition,  $f(p)$  cannot be even, otherwise  $f(p) > f(p)$ . So  $f(p)$  has to be odd. By the safety condition,  $(f \cdot t)(p) \geq f(p)$ . But  $p \rightarrow_t p$ , therefore, by the definition of  $f \cdot t$ ,  $(f \cdot t)(p) < \lfloor f(p) \rfloor_{\text{even}}$ , but  $f(p)$  is odd, thus  $(f \cdot t)(p) < f(p)$ , which is a contradiction. Therefore,  $t$  is rejecting. For the “if” direction, given a rejecting profile  $t$  one applies Lemma 5 for the canonical ranking  $f_t$  to show that  $f_t$  satisfies  $t$ .

Let  $w = a_0 a_1 \dots \in \Sigma^\omega$ . If  $w \notin \mathcal{L}(\mathcal{A})$ , then there exists two profiles  $s$  and  $t$  s.t.  $w$  factorizes as  $w = w_0 w_1 \dots$ , with  $w_0 \in \mathcal{L}(s)$  and  $w_i \in \mathcal{L}(t)$  for  $i \geq 1$ . Let  $i_j = |w_0 w_1 \dots w_j|$ . From this factorization, we extract the following canonical ranking for  $w$ , which we call *periodic ranking*:

$$r(q, i) = \begin{cases} m & \text{if } i < i_0 \\ f_t(q) & \text{if } i = i_j \text{ for some } j \geq 0 \\ (f_t \cdot w)(q) & \text{if } i_j < i < i_{j+1} \text{ for some } j \geq 0 \text{ and } w = w[i_j..i] \end{cases} \quad (2)$$

Notice that  $r$  is indeed a ranking function; in particular, it is non-increasing, since, at boundary indices  $i_j$ 's,  $f_t \cdot w[i_j..i_{j+1}] \geq f_t$  by  $w[i_j..i_{j+1}] \in \mathcal{L}(t)$  and since  $f_t$  satisfies  $t$  by definition. Furthermore, if  $t$  is a rejecting profile, then  $r$  is an odd ranking by Lemma 5.

## 4 Slices

We show that `slice` simulates `ramsey`.

We define an update operation of a slice preorder by a profile. Let  $\succeq$  be a total preorder on  $Q$ , and let  $t$  be a profile. For any state  $q$ ,  $\text{min-pre}^{\succeq, t}(q)$  is the set of  $\succeq$ -minimal  $t$ -predecessors of  $q$ ; i.e.,  $\text{min-pre}^{\succeq, t}(q) = \{p \mid p \rightarrow_t q \wedge \forall (r \rightarrow_t q) \cdot r \succeq p\}$ . Then,  $\succeq' = \succeq \cdot t$  is a new total preorder on  $Q$  defined as follows: For every states  $p, p', q, q'$  with  $p \in \text{min-pre}^{\succeq, t}(p')$  and  $q \in \text{min-pre}^{\succeq, t}(q')$ ,

- If  $p \succ q$ , then  $p' \succ' q'$ . Otherwise, if  $p \approx q$ :
  - If  $q \rightarrow_t q'$  but  $\neg(p \rightarrow_t p')$ , then  $p' \succ' q'$ .
  - Otherwise,  $p' \approx' q'$ .

Notice that the update operation above is not an action on preorders, since big-step updates can lose information. In fact, one can prove that it is a pre-action, in the following sense: For any preorder  $\succeq$  and profiles  $s$  and  $t$ ,  $\succeq \cdot (st) \supseteq (\succeq \cdot s) \cdot t$ . This means that the small-step update  $(\succeq \cdot s) \cdot t$  is *finer* than the corresponding big-step one  $\succeq \cdot (st)$ .

**Lemma 7.** *Let  $\succeq_0$  and  $\succeq_1$  be two total preorders, with  $\succeq_1$  finer than  $\succeq_0$ , and let  $t$  be a profile. Then,  $\succeq_1 \cdot t$  is finer than  $\succeq_0 \cdot t$ .*

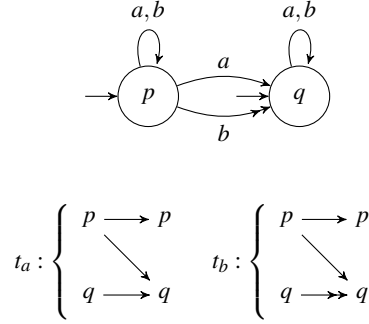
*Proof.*

**Lemma 8.** *Let  $t$  be an idempotent profile, let  $\succeq$  be a total preorder, and let  $\succeq' = \succeq \cdot t$  be its update. Then,*

- If  $p \rightarrow_t q$ , then  $p \succeq' q$ .
- If  $p \rightarrow_t q$  and  $t$  is additionally rejecting, then  $p \succ' q$ .

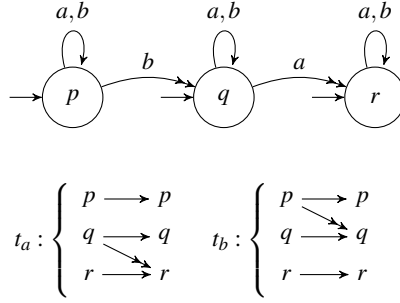
*Proof.*

With the two lemmata above, one can show a simulation from the `ramsey`-automaton  $\mathcal{A}$  to the `slice`-automaton  $\mathcal{B}$ . The simulation requires an intermediate modified `slice` construction (automaton  $\mathcal{C}$ ), where the preorder is introduced after a finite prefix has been read, and it is updated in big-steps. Clearly,  $\mathcal{B}$  simulates  $\mathcal{C}$  by Lemma 7.

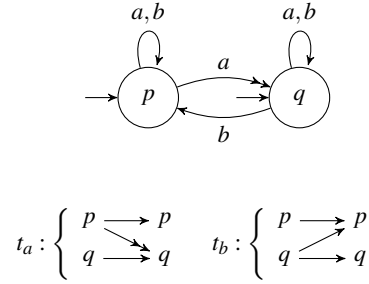


**Fig. 1.** `slice` does not simulate `ramsey-pre`. Let  $t_b$  and  $t_a$  be the two profiles above. Notice that  $t_b$  is (strictly) finer than  $t_a$ . The `ramsey-pre`-automaton commits to any total preorder realizing  $t_b$ , and then starts playing arbitrarily many  $a$ 's and resetting each time to  $t_\varepsilon$ . The `slice`-automaton is thus in a state  $\langle \{p, q\}, p \approx q \rangle$ , and to visit an accepting state it has to eventually commit to a level  $k$ . At this point, the `ramsey-pre`-automaton plays a single  $b$  (which can be extended to a rejected word), and the `slice`-automaton chokes since there is an accepting transition  $p \xrightarrow{b} q$  past the commit level  $k$  which is never subsumed by a better transition in the future.

We now argue that  $C$  simulates  $\mathcal{A}$ . Initially,  $C$  just tracks reachable states. When  $\mathcal{A}$  commits to profiles  $s, t$ ,  $C$  begins updating the preorder starting from the identity  $id$ . When  $\mathcal{A}$  loops for the first time after having read a word in  $t$ ,  $C$  updates its preorder to  $id \cdot t$ . At this point,  $C$  commits to the current level  $k$ . From now on, by Lemma 8, every time  $\mathcal{A}$  reads a word in  $t$  and resets (thus visiting an accepting state), ...



**Fig. 2.** `ramsey` does not simulate `slice`. After playing  $ab$ , the `slice`-automaton reaches the ordering  $p > q > r$ . From there, it keeps playing  $a$ 's. So the `ramsey`-automaton has to eventually commit to the profile  $t_a$ , at which point the `slice`-automaton switches to playing  $b$ 's, and the `ramsey`-automaton is no longer accepting.



**Fig. 3.** `ramsey-pre` does not simulate `slice`. After playing  $a$ , the `slice`-automaton reaches the ordering  $p > q$ , from which it keeps playing  $a$ 's. So the `ramsey`-automaton has to eventually commit to the unique preorder  $p > q$  compatible with profile  $t_a$ . At this point, the `slice`-automaton plays  $b$ 's (reaching the new ordering  $p \approx q$ ) and the `ramsey`-automaton chokes since  $t_b$  is not compatible with the preorder it previously committed to.