VASS reachability algorithm

LC

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1 Introduction

Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ the set of natural numbers, and let $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$ be the extension thereof with a maximal element ω. A *vector addition system of dimension d* (VAS) is a finite set of actions $A \subseteq \mathbb{Z}^d$. An *ω-configuration* is a vector $c \in \mathbb{N}_\omega^d$. For two configurations c, d and an action $a \in \mathbb{Z}^d$, we have a *transition* $c \xrightarrow{a} d$ if $d = c + a$. Let $T \subseteq \mathbb{N}_\omega^d \times A \times \mathbb{N}_\omega^d$ be the set of such transitions. Moreover, we write $c \leqslant d$ for the component-wise ordering, and we write $c \sqsubseteq d$ if $c \leqslant d$ and, additionally, whenever $d(i) \neq \omega$, then $c(i) = d(i)$. In other words, $c \sqsubseteq d$ iff d can be obtained from c by making some components equal to ω.

A *witness graph* is a strongly connected graph $G = (S, E, s)$ where $S \subseteq \mathbb{N}_\omega^d$ is a non-empty finite set of configurations, $E \subseteq S \times A \times S$ is a (necessarily finite) set of transitions, and s is a distinguished state in S. Notice that, since G is strongly connected, all states in S have the same set of ω-components. Moreover, the total effect of cycles is unconstrained on ω-components, and necessarily 0 on finite components.

A *marked witness graph* is a triple $M = (s^{\mathsf{in}}, G, s^{\mathsf{out}})$ where $s^{\mathsf{in}}, s^{\mathsf{out}} \in \mathbb{N}_\omega^d$ are distinguished configurations s.t. $s^{\mathsf{in}}, s^{\mathsf{out}} \sqsubseteq s$. Therefore, if $s(i)$ is finite, then $s(i) = s^{\mathsf{in}}(i) = s^{\mathsf{out}}(i)$, while if $s(i)$ is infinite, then $s^{\mathsf{in}}(i), s^{\mathsf{out}}(i)$ are unconstrained—i.e., they can be of different value, and this value can be either finite or infinite. A marked witness graph is *forward pumpable* if there exists a cycle $s \xrightarrow{\sigma_+} s$ (thus $\sigma_+(i) = 0$ if $s(i)$ is finite) fireable from s^{in} s.t. $\sigma_+(i) > 0$ on those components i where $s^{\mathsf{in}}(i)$ is finite and $s(i)$ is infinite (thus for every n $s^{\mathsf{in}} \xrightarrow{\sigma_+^n} s_n$ for some s_n, and $s = \lim_n s_n$). Symmetrically, a marked witness graph is *backward pumpable* if there exists a cycle $s \xrightarrow{\sigma_-} s$ backward fireable from s^{out} s.t. $\sigma_-(i) < 0$ on those components i where $s^{\mathsf{out}}(i)$ is finite and $s(i)$ is infinite (thus for every n $s_n \xrightarrow{\sigma_-^n} s^{\mathsf{out}}$ for some s_n, and $s = \lim_n s_n$). Intuitively, if M is forward pumpable, then those components which are finite in s^{in} but infinite in s can be pumped to become arbitrarily large, and this can be done without modifying those components which are finite both in s^{in} and in s.

A *marked witness graph sequence* is a sequence

$$\xi = M_0, a_1, M_1, \ldots, a_k, M_k$$

s.t. $M_j = (s_j^{\mathsf{in}}, G_j, s_j^{\mathsf{out}})$ is a marked witness graph with $G_j = (S_j, E_j, s_j)$, and a_j is an action in A.

Let $\psi_j : E_j \to \mathbb{N}$ be a function counting how many times the edges in G_j are taken, and let its *effect* be

$$|\psi_j| := \sum_{e=(\_,a,\_)\in E_j} \psi_j(e) \cdot a.$$

We say that ψ_j is *total* if $\psi_j(e) \geqslant 1$ for every $e \in E_j$, and that it is *balanced* if it satisfies the following flow condition for every $s \in S_j$:

$$\sum_{e=(\_,\_,s)\in E_j} \psi_j(e) = \sum_{e=(s,\_,\_)\in E_j} \psi_j(e).$$

Operations on such functions ψ_j are defined component-wise. For two configurations $x_j, y_j \in \mathbb{N}^d$, let $x_j \xrightarrow{\psi_j} y_j$ if $y_j = x_j + |\psi_j|$. Let L_ξ be the set of those sequences

$$\pi := x_0 \xrightarrow{\psi_0} y_0 \xrightarrow{a_1} \cdots \xrightarrow{a_k} x_k \xrightarrow{\psi_k} y_k \tag{1}$$

s.t. $x_j \sqsubseteq s_j^{\mathsf{in}}$, $y_j \sqsubseteq s_j^{\mathsf{out}}$, and ψ_j is total and balanced. In particular, x_j agrees with s_j^{in} on the finite components thereof, and similarly for y_j and s_j^{out}.

A marked witness graph sequence ξ as above is *perfect* if, for every j:

- the witness graph M_j is both forward and backward pumpable,
- $s_j^{\mathsf{in}} = \sup X_j$ and $s_j^{\mathsf{out}} = \sup Y_j$, and
- for every $e \in E_j$, $\sup \Psi_j(e) = \omega$,

where X_j, Y_j, and Ψ_j are the sets of x_j, y_j, and ψ_j in the sequence L_ξ above.

Let M_ξ be the set of sequences

$$\rho := z_0 \xrightarrow{\varphi_0} z_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_k} z_{k+1} \tag{2}$$

with $z_j \in \mathbb{N}^d$ and $\varphi_j : E_j \to \mathbb{N}$, s.t. $z_j(i) = 0$ if $s_j^{\mathsf{in}}(i)$ is finite, $z_{j+1}(i) = 0$ if $s_j^{\mathsf{out}}(i)$ is finite, and φ_j is balanced. For two such sequences $\rho, \rho' \in M_\xi$, let $\rho + \rho'$ be the sequence $z_0 + z_0' \xrightarrow{\varphi_0+\varphi_0'} z_1 + z_1' \xrightarrow{\varphi_1+\varphi_1'} \cdots \xrightarrow{\varphi_k+\varphi_k'} z_{k+1} + z_{k+1}'$. Notice that the zero sequence $0 \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0$ is in M_ξ, and that M_ξ is *additive*, in the sense that $\rho, \rho' \in M_\xi$ implies $\rho + \rho' \in M_\xi$. Similarly, it is *subtractive* in the sense that $\rho \leqslant \rho'$ implies $\rho' - \rho \in M_\xi$. For $\pi \in L_\xi$ as in (1) and $\rho \in M_\xi$, let $\pi + \rho$ be the sequence

$$\pi + \rho := x_0 + z_0 \overset{\psi_0+\varphi_0}{\dashrightarrow} y_0 + z_1 \overset{a_1}{\dashrightarrow} x_1 + z_1 \overset{\psi_1+\varphi_1}{\dashrightarrow} \cdots \overset{\psi_k+\varphi_k}{\dashrightarrow} y_k + z_{k+1}.$$

We have that L_ξ is *additive relatively to* M_ξ in the sense that $\pi \in L_\xi$ and $\rho \in M_\xi$ implies $\pi + \rho \in L_\xi$.

Let $\pi_0 := x_0^0 \overset{\psi_0^0}{\dashrightarrow} y_0^0 \xrightarrow{a_1} \cdots \overset{\psi_k^0}{\dashrightarrow} y_k^0$ and $\pi_1 := x_0^1 \overset{\psi_0^1}{\dashrightarrow} y_0^1 \xrightarrow{a_1} \cdots \overset{\psi_k^1}{\dashrightarrow} y_k^1$ be two sequences in L_ξ. Notice that $y_j^1 - x_{j+1}^1 = a_{j+1} = y_j^0 - x_{j+1}^0$, and thus $y_j^1 - y_j^0 = x_{j+1}^1 - x_{j+1}^0$. For $\pi_0 \leqslant \pi_1$, let $\pi_1 - \pi_0$ be the sequence

$$\pi_1 - \pi_0 := z_0 \overset{\psi_0^1-\psi_0^0}{\dashrightarrow} z_1 \overset{\psi_1^1-\psi_1^0}{\dashrightarrow} z_2 \cdots \overset{\psi_k^1-\psi_k^0}{\dashrightarrow} z_{k+1},$$

where $z_j := x_j^1 - x_j^0$ for $j \leqslant k$ and $z_{k+1} := y_k^1 - y_k^0$ otherwise.

Lemma 1. *Let $\pi_0, \pi_1 \in L_\xi$ with $\pi_0 \leqslant \pi_1$. Then, $\pi_1 - \pi_0 \in M_\xi$.*

Proof. Since $y_j^1(i) = y_j^0(i) = s_j^{\mathsf{out}}(i)$ for those coordinates i's s.t. $s_j^{\mathsf{out}}(i)$ is finite, and $s_j^{\mathsf{out}}(i)$ is finite iff $s_{j+1}^{\mathsf{in}}(i)$ is finite, we have $z_j(i) = 0$ in this case, as required. Moreover, since ψ_j^0, ψ_j^1 are balanced and $\psi_j^0 \leqslant \psi_j^1$, then also $\psi_j^1 - \psi_j^0$ is balanced. ☐

Let $\hat{L}_\xi$ and $\hat{M}_\xi$ be the respective subsets of minimal elements of L_ξ and M_ξ. Those two sets are finite since $\leqslant$ is a wqo. We have the following decomposition result.

Lemma 2. $L_\xi = \hat{L}_\xi + \hat{M}_\xi^*$.

Proof. The right-to-left containment follows immediately by additivity $M_\xi + M_\xi \subseteq M_\xi$ and by relative additivity $L_\xi + M_\xi \subseteq L_\xi$. For the other direction, let $\pi \in L_\xi$. If π is minimal, we are done. Otherwise, there exists a minimal $\hat{\pi} \in \hat{L}_\xi$ s.t. $\hat{\pi} \leqslant \pi$. Let $\rho_0 := \pi - \hat{\pi}$, which is in M_ξ by Lemma 1. If ρ_0 is minimal, we are done since $\pi = \hat{\pi} + \rho_0$. Otherwise, there exists a minimal $\hat{\rho}_0 \in \hat{M}_\xi$ s.t. $\hat{\rho}_0 \leqslant \rho_0$ and $\rho_1 := \rho_0 - \hat{\rho}_0$ is in M_ξ by subtractivity. If ρ_1 is minimal, then we are done since $\rho_0 = \hat{\rho}_0 + \rho_1$. Otherwise, we can repeat this process and after a finite number of step we will get a finite sequence of minimal $\hat{\rho}_0, \ldots, \hat{\rho}_h \in \hat{M}_\xi$ s.t. $\rho_0 = \hat{\rho}_0 + \cdots + \hat{\rho}_h$, thus showing $\pi \in \hat{L}_\xi + \hat{M}_\xi^*$. ☐

We call a sequence in M_ξ as in (2) *diagonal* if

- $z_j(i) > 0$ if $s_j^{\mathsf{in}}(i) = \omega$,
- $z_{j+1}(i) > 0$ if $s_j^{\mathsf{out}}(i) = \omega$ (iff $s_{j+1}^{\mathsf{in}}(i) = \omega$), and
- φ_j is total.

Lemma 3. *If ξ is perfect, then there exists a diagonal solution in M_ξ.*

OLD INCOMPLETE PROOF. The following lemma allows us to extract *runs* from perfect marked witness graph sequences.

Lemma 4. *If ξ is a perfect marked witness graph sequence, then for every $n \in \mathbb{N}$ there are configurations $x'_{n,0}, y'_{n,0}, \ldots, x'_{n,k}, y'_{n,k} \in \mathbb{N}^d$ and sequences of actions $\delta_{n,0}, \delta_{n,1}, \ldots, \delta_{n,k} \in A^*$ admitting a run*

$$x_{n,0} \xrightarrow{\delta_{n,0}} y_{n,0} \xrightarrow{a_0} x_{n,1} \xrightarrow{\delta_{n,1}} \cdots \xrightarrow{a_k} x_{n,k} \xrightarrow{\delta_{n,k}} y_{n,k}, \tag{3}$$

s.t.

- $x_{n,j} \sqsubseteq s_j^{\mathsf{in}}$ *and* $y_{n,j} \sqsubseteq s_j^{\mathsf{out}}$, *and*
- $\lim_n x_{n,j} = s_j^{\mathsf{in}}$ *and* $\lim_n y_{n,j} = s_j^{\mathsf{out}}$.

Proof. By Lemma 3, let ρ be a diagonal sequence in M_ξ; cf. (2). Let $\sigma_{+,j}$ and $\sigma_{-,j}$ be two cycles on s_j witnessing that M_j is forward and backward pumpable, respectively. Thus, $\sigma_{+,j}$ pumps those finite components in s_j^{in} which are unbounded in s_j, and symmetrically $\sigma_{-,j}$ unpumps the finite components in s_j^{out} which are unbounded in s_j. However, those cycles can have a negative effect on the other infinite components of s_j (which are thus infinite also on s_j^{in}, or s_j^{out}, respectively), and we want to avoid it. Moreover, we would like to find a cycle on s_j which will "undo" the effect of $\sigma_{+,j}, \sigma_{-,j}$ except perhaps for some additional increase on the unbounded components in $s_j^{\mathsf{in}}, s_j^{\mathsf{out}}$. Since ρ is diagonal, φ_j is total. By summing up ρ sufficiently many times (since M_ξ is additive) we can assume w.l.o.g. that φ_j is total even after removing $\sigma_{+,j}, \sigma_{-,j}$, i.e.,

$$\varphi_j - |\sigma_{+,j}|_{E_j} - |\sigma_{-,j}|_{E_j} \geqslant 1. \tag{4}$$

Since $z_j(i) > 0$ on unbounded coordinates i of s_j^{in}, and similarly $z_{j+1}(i) > 0$ on unbounded coordinates i of s_j^{out} (equiv. of s_{j+1}^{in}), and since M_ξ is additive, by summing up ρ sufficiently many times we can further assume w.l.o.g. that for every prefix γ_+ of $\sigma_{+,j}$ and for every prefix γ_- of $\sigma_{-,j}$,

$$z_j(i) + |\gamma_+(i)| > 0 \qquad \text{if } s_j^{\mathsf{in}}(i) = \omega, \tag{5}$$

$$z_{j+1}(i) + |\gamma_-(i)| > 0 \qquad \text{if } s_j^{\mathsf{out}}(i) = \omega \text{ (iff } s_{j+1}^{\mathsf{in}}(i) = \omega). \tag{6}$$

By (4) and Lemma **??**, there exists a total cycle

$$s_j \xrightarrow{w_j} s_j$$

s.t. $|w_j|_{E_j}$ equals the quantity (4) above. By repeating $\sigma_{+,j}$ sufficiently many times we can assume w.l.o.g. that for every prefix γ of w_j we have

$$z_j(i) + |\sigma_{+,j}(i)| + |\gamma(i)| \geqslant 0 \qquad \text{if } s_j(i) = \omega. \tag{7}$$

Putting together (2), (4), and (7), we get... NO! you need to start from $x_j + z_j$!

$$z_j \xrightarrow{\sigma_{+,j}} z_j + |\sigma_{+,j}| \xrightarrow{w_j} z_j + |\sigma_{+,j}| + |w_j| = z_{j+1} - |\sigma_{-,j}|. \tag{8}$$

Let π be a solution in L_ξ; cf.(1). Then ψ_j is balanced and again by Lemma **??** there exists a cycle

$$s_j \xrightarrow{\alpha_j} s_j$$

s.t. $|\alpha_j|_{E_j} = \psi_j$. However, this does not necessarily imply $x_j \xrightarrow{\alpha_j}$. Since $x_j \sqsubseteq s_j^{\mathsf{in}} \sqsubseteq s_j$, this means that the executability of α_j depends only on whether unbounded components on s_j can be made arbitrarily large in x_j. There are two kinds of such unbounded components: The first kind are those components which are finite in s_j^{in} but unbounded in s_j (those can be pumped by $\sigma_{+,j}$), and the second kind are those components which unbounded in s_j^{in} (and thus also in s_j). We address here this second kind (the first kind will be addressed later), by choosing z_j to be large enough s.t., for every prefix γ of α_j,

$$z_j(i) + |\gamma(i)| \geqslant 0 \qquad\qquad \text{if } s_j^{\mathsf{in}}(i) = \omega \text{ (and thus } s_j(i) = \omega). \tag{9}$$

Since $x_j \sqsubseteq s_j^{\mathsf{in}}$ and $s_j^{\mathsf{in}} \xrightarrow{\sigma_{+,j}^n} \cdot$ by construction, the only obstacle to $x_j \xrightarrow{\sigma_{+,j}^n} \cdot$ is that unbounded components in s_j^{in} are "too small" in x_j. This is what z_j is for, since not only z_j is strictly positive on unbounded coordinates of s_j^{in}, but it remains so after executing every prefix of $\sigma_{+,j}$; cf. (5). Consequently,

$$x_j + z_j \xrightarrow{\sigma_{+,j}} x_j + z_j + |\sigma_{+,j}|. \tag{10}$$

In order to be able to execute α_j next, we need to run $\sigma_{+,j}$ sufficiently many times until finite components in s_j^{in} which are unbounded in s_j are large enough. Since $\sigma_{+,j}(i) > 0$ on those components, there exists n large enough s.t. for every prefix γ of α_j,

$$n \cdot |\sigma_{+,j}(i)| + |\gamma(i)| \geqslant 0 \qquad\qquad \text{if } s_j^{\mathsf{in}}(i) < \omega \text{ and } s_j(i) = \omega. \tag{11}$$

Combining (9) with (11), we get for every prefix γ of α_j

$$z_j(i) + n \cdot |\sigma_{+,j}(i)| + |\gamma(i)| \geqslant 0 \qquad\qquad \text{if } s_j(i) = \omega. \tag{12}$$

We are now ready to put all the pieces together. For every $n \in \mathbb{N}$, we define $x'_{n,j}$, $y'_{n,j}$, and $\delta_{n,j}$ as

$$x'_{n,j} := x_j + n \cdot z_j, \tag{13}$$

$$y'_{n,j} := y_j + n \cdot z_{j+1}, \text{ and} \tag{14}$$

$$\delta_{n,j} := \sigma_{+,j}^n \cdot \alpha_j \cdot w_j^n \cdot \sigma_{-,j}^n. \tag{15}$$

For sufficiently large n, there is a run as in (3) of the form:

$$\begin{aligned}
x'_{n,j} = x_j + n \cdot z_j && \xrightarrow{\sigma^n_{+,j}} && \text{(by (10))}\\
x_j + n \cdot z_j + n \cdot |\sigma_{+,j}| && \xrightarrow{\alpha_j} && \text{(by (1),(12))}\\
y_j + n \cdot z_j + n \cdot |\sigma_{+,j}| && \xrightarrow{w^n_j} && \text{(by (8))}\\
y_j + n \cdot z_{j+1} - n \cdot |\sigma_{-,j}| && \xrightarrow{\sigma^n_{-,j}} &&\\
y_j + n \cdot z_{j+1} = y'_{n,j}. &&&&
\end{aligned}$$

□

This implies the previous lemma and shows that preruns are adherent to runs.
Lemma 5. *Let ξ be a perfect marked witness graph sequence. For every $\pi \in L_\xi$ there exists a* run *$\pi' \in L_\xi$ s.t. $\pi \leqslant \pi'$.*

Proof. Let π be a solution in L_ξ as in (1), and let ρ be a diagonal sequence in M_ξ as in (2), which exists by Lemma 3. We partition components into the following types:

Type I Components which are finite (and equal) in $s^{\mathsf{in}}_j, s^{\mathsf{out}}_j, s_j$.

Type II$^{\text{in}}$ Components which are finite in s^{in}_j but infinite in s_j.

Type II$^{\text{out}}$ Components which are finite in s^{out}_j but infinite in s_j.

Type III$^{\text{in}}$ Components which are infinite in s^{in}_j (thus also in s_j).

Type III$^{\text{out}}$ Components which are infinite in s^{out}_j (thus also in s_j).

Notice that Type II$^{\text{in}}$ $\cup$ Type III$^{\text{in}}$ = Type II$^{\text{out}}$ $\cup$ Type III$^{\text{out}}$. Since ψ_j is balanced, by Lemma **??** there exists a cycle $s_j \xrightarrow{\tilde{\psi}_j} s_j$ s.t. $|\tilde{\psi}_j|_{E_j} = \psi_j$. Notice that $\tilde{\psi}_j$ is already executable on Type I components:

$$x_j(i) \xrightarrow{\tilde{\psi}_j(i)} x_j(i) = y_j(i) \quad \text{if } i \text{ is Type I.} \tag{16}$$

In order to execute $\tilde{\psi}_j$ we need to pump Type II$^{\text{in}}$ and Type III$^{\text{in}}$ components.

We first pump Type II$^{\text{in}}$ components. Let σ^+_j and σ^-_j be two cycles on s_j witnessing that M_j is forward and backward pumpable, respectively. By definition, σ^+_j can be executed from s^{in}_j, thus

$$x_j(i) \xrightarrow{\sigma^+_j(i)} x_j(i) + |\sigma^+_j(i)| \quad \text{if } i \text{ is Type I or Type II}^{\text{in}}. \tag{17}$$

Since σ^+_j is strictly positive on Type II$^{\text{in}}$ components, by pumping it sufficiently many times, we can assume w.l.o.g.

$$x_j(i) + |\sigma^+_j|(i) \xrightarrow{\tilde{\psi}_j(i)} y_j(i) + |\sigma^+_j|(i) \quad \text{if } i \text{ is Type II}^{\text{in}}. \tag{18}$$

We now pump Type III$^{\text{in}}$ components. On those components, z_j is strictly positive (and zero elsewhere). By pumping ρ (using additivity) we can assume w.l.o.g. that Type III$^{\text{in}}$ components in z_j are sufficiently large to enable both σ_j^+ and $\tilde{\psi}_j$:

$$z_j(i) \xrightarrow{\sigma_j^+(i)} z_j(i) + |\sigma_j^+|(i) \quad \text{if } i \text{ is Type III}^{\text{in}}, \tag{19}$$

$$z_j(i) \xrightarrow{\tilde{\psi}_j(i)} z_j(i) + |\psi_j|(i) \quad \text{if } i \text{ is Type III}^{\text{in}}. \tag{20}$$

We need one last assumption about ρ. We would like to find a cycle on s_j which will "undo" the effect of $\sigma_{+,j}, \sigma_{-,j}$ except perhaps for some additional increase on Type III$^{\text{in}}$ components. By pumping ρ we can assume w.l.o.g. that φ_j is total even after removing $\sigma_{+,j}$ and $\sigma_{-,j}$. Consequently, by Lemma **??**, there exists a total cycle $s_j \xrightarrow{w_j} s_j$ s.t. $|w_j|_{E_j} = \varphi_j - |\sigma_{+,j}|_{E_j} - |\sigma_{-,j}|_{E_j}$ and

$$x_j(i) \xrightarrow{w_j} y_j(i) \quad \text{if } i \text{ is Type I}, \tag{21}$$

$$z_j(i) + |\sigma_j^+|(i) \xrightarrow{w_j} z_{j+1}(i) - |\sigma_j^-|(i) \quad \text{otherwise.} \tag{22}$$

(Notice that $x_j(i) = y_j(i)$ and $z_j(i) = z_{j+1}(i) = |\sigma_j^+|(i) = |\sigma_j^-|(i) = |w_j| = 0$ if i is Type I.) We can now construct our run $\pi' = x'_0 \xrightarrow{\delta_0} y'_0 \xrightarrow{a_1} \cdots \xrightarrow{a_k} x_k \xrightarrow{\delta_k} y_k$ as follows:

$$\begin{aligned}
x'_j &:= x_j + z_j, \\
y'_j &:= y_j + z_{j+1}, \text{ and} \\
\delta_j &:= \sigma_j^+ \cdot \tilde{\psi}_j \cdot w_j \cdot \sigma_j^-.
\end{aligned}$$

Indeed, we have:

$$\begin{aligned}
x'_j = x_j + z_j & \xrightarrow{\sigma_j^+} & & \text{(by (17) and (19))} \\
x_j + z_j + |\sigma_j^+| & \xrightarrow{\tilde{\psi}_j} & & \text{(by (16), (18), and (20))} \\
y_j + z_j + |\sigma_j^+| & \xrightarrow{w_j} & & \text{(by (21) and (22))} \\
y_j + z_{j+1} - |\sigma_j^-| & \xrightarrow{\sigma_j^-} & & \\
y_j + z_{j+1} = y'_j. & & &
\end{aligned}$$

□