

# VASS reachability algorithm

LC

May 1, 2019

## 1 Introduction

Let  $\mathbb{Z}$  be the set of integers,  $\mathbb{N}$  the set of natural numbers, and let  $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$  be the extension thereof with a maximal element  $\omega$ . A *vector addition system of dimension  $d$*  (VAS) is a finite set of actions  $A \subseteq \mathbb{Z}^d$ . An  $\omega$ -*configuration* is a vector  $c \in \mathbb{N}_\omega^d$ . For two configurations  $c, d$  and an action  $a \in \mathbb{Z}^d$ , we have a *transition*  $c \xrightarrow{a} d$  if  $d = c + a$ . Let  $T \subseteq \mathbb{N}_\omega^d \times A \times \mathbb{N}_\omega^d$  be the set of such transitions. Moreover, we write  $c \leq d$  for the component-wise ordering, and we write  $c \sqsubseteq d$  if  $c \leq d$  and, additionally, whenever  $d(i) \neq \omega$ , then  $c(i) = d(i)$ . In other words,  $c \sqsubseteq d$  iff  $d$  can be obtained from  $c$  by making some components equal to  $\omega$ .

A *witness graph* is a strongly connected graph  $G = (S, E, s)$  where  $S \subseteq \mathbb{N}_\omega^d$  is a non-empty finite set of configurations,  $E \subseteq S \times A \times S$  is a (necessarily finite) set of transitions, and  $s$  is a distinguished state in  $S$ . Notice that, since  $G$  is strongly connected, all states in  $S$  have the same set of  $\omega$ -components. Moreover, the total effect of cycles is unconstrained on  $\omega$ -components, and necessarily 0 on finite components.

A *marked witness graph* is a triple  $M = (s^{\text{in}}, G, s^{\text{out}})$  where  $s^{\text{in}}, s^{\text{out}} \in \mathbb{N}_\omega^d$  are distinguished configurations s.t.  $s^{\text{in}}, s^{\text{out}} \sqsubseteq s$ . Therefore, if  $s(i)$  is finite, then  $s(i) = s^{\text{in}}(i) = s^{\text{out}}(i)$ , while if  $s(i)$  is infinite, then  $s^{\text{in}}(i), s^{\text{out}}(i)$  are unconstrained—i.e., they can be of different value, and this value can be either finite or infinite. A marked witness graph is *forward pumpable* if there exists a cycle  $s \xrightarrow{\sigma_+} s$  (thus  $\sigma_+(i) = 0$  if  $s(i)$  is finite) fireable from  $s^{\text{in}}$  s.t.  $\sigma_+(i) > 0$  on those components  $i$  where  $s^{\text{in}}(i)$  is finite and  $s(i)$  is infinite (thus for every  $n$   $s^{\text{in}} \xrightarrow{\sigma_+^n} s_n$  for some  $s_n$ , and  $s = \lim_n s_n$ ). Symmetrically, a marked witness graph is *backward pumpable* if there exists a cycle  $s \xrightarrow{\sigma_-} s$  backward fireable from  $s^{\text{out}}$  s.t.  $\sigma_-(i) < 0$  on those components  $i$  where  $s^{\text{out}}(i)$  is finite and  $s(i)$  is infinite (thus for every  $n$   $s_n \xrightarrow{\sigma_-^n} s^{\text{out}}$  for some  $s_n$ , and  $s = \lim_n s_n$ ). Intuitively, if  $M$  is forward pumpable, then those components which are finite in  $s^{\text{in}}$  but infinite in  $s$  can be pumped to become arbitrarily large, and this can

be done without modifying those components which are finite both in  $s^{\text{in}}$  and in  $s$ .

A *marked witness graph sequence* is a sequence

$$\xi = M_0, a_1, M_1, \dots, a_k, M_k$$

s.t.  $M_j = (s_j^{\text{in}}, G_j, s_j^{\text{out}})$  is a marked witness graph with  $G_j = (S_j, E_j, s_j)$ , and  $a_j$  is an action in  $A$ .

Let  $\psi_j : E_j \rightarrow \mathbb{N}$  be a function counting how many times the edges in  $G_j$  are taken, and let its *effect* be

$$|\psi_j| := \sum_{e=(\cdot, a, \cdot) \in E_j} \psi_j(e) \cdot a.$$

We say that  $\psi_j$  is *total* if  $\psi_j(e) \geq 1$  for every  $e \in E_j$ , and that it is *balanced* if it satisfies the following flow condition for every  $s \in S_j$ :

$$\sum_{e=(\cdot, \cdot, s) \in E_j} \psi_j(e) = \sum_{e=(s, \cdot, \cdot) \in E_j} \psi_j(e).$$

Operations on such functions  $\psi_j$  are defined component-wise. For two configurations  $x_j, y_j \in \mathbb{N}^d$ , let  $x_j \xrightarrow{\psi_j} y_j$  if  $y_j = x_j + |\psi_j|$ . Let  $L_\xi$  be the set of those sequences

$$\pi := x_0 \xrightarrow{\psi_0} y_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} x_k \xrightarrow{\psi_k} y_k \quad (1)$$

s.t.  $x_j \sqsubseteq s_j^{\text{in}}$ ,  $y_j \sqsubseteq s_j^{\text{out}}$ , and  $\psi_j$  is total and balanced. In particular,  $x_j$  agrees with  $s_j^{\text{in}}$  on the finite components thereof, and similarly for  $y_j$  and  $s_j^{\text{out}}$ .

A marked witness graph sequence  $\xi$  as above is *perfect* if, for every  $j$ :

- the witness graph  $M_j$  is both forward and backward pumpable,
- $s_j^{\text{in}} = \sup X_j$  and  $s_j^{\text{out}} = \sup Y_j$ , and
- for every  $e \in E_j$ ,  $\sup \Psi_j(e) = \omega$ ,

where  $X_j, Y_j$ , and  $\Psi_j$  are the sets of  $x_j, y_j$ , and  $\psi_j$  in the sequence  $L_\xi$  above.

Let  $M_\xi$  be the set of sequences

$$\rho := z_0 \xrightarrow{\varphi_0} z_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_k} z_{k+1} \quad (2)$$

with  $z_j \in \mathbb{N}^d$  and  $\varphi_j : E_j \rightarrow \mathbb{N}$ , s.t.  $z_j(i) = 0$  if  $s_j^{\text{in}}(i)$  is finite,  $z_{j+1}(i) = 0$  if  $s_j^{\text{out}}(i)$  is finite, and  $\varphi_j$  is balanced. For two such sequences  $\rho, \rho' \in M_\xi$ , let  $\rho + \rho'$

be the sequence  $z_0 + z'_0 \xrightarrow{\varphi_0 + \varphi'_0} z_1 + z'_1 \xrightarrow{\varphi_1 + \varphi'_1} \dots \xrightarrow{\varphi_k + \varphi'_k} z_{k+1} + z'_{k+1}$ . Notice that the zero sequence  $0 \xrightarrow{0} 0 \xrightarrow{0} \dots \xrightarrow{0} 0$  is in  $M_\xi$ , and that  $M_\xi$  is *additive*, in the

sense that  $\rho, \rho' \in M_\xi$  implies  $\rho + \rho' \in M_\xi$ . Similarly, it is *subtractive* in the sense that  $\rho \leq \rho'$  implies  $\rho' - \rho \in M_\xi$ . For  $\pi \in L_\xi$  as in (1) and  $\rho \in M_\xi$ , let  $\pi + \rho$  be the sequence

$$\pi + \rho := x_0 + z_0 \xrightarrow{\psi_0 + \varphi_0} y_0 + z_1 \xrightarrow{a_1} x_1 + z_1 \xrightarrow{\psi_1 + \varphi_1} \dots \xrightarrow{\psi_k + \varphi_k} y_k + z_{k+1}.$$

We have that  $L_\xi$  is *additive relatively to*  $M_\xi$  in the sense that  $\pi \in L_\xi$  and  $\rho \in M_\xi$  implies  $\pi + \rho \in L_\xi$ .

Let  $\pi_0 := x_0^0 \xrightarrow{\psi_0^0} y_0^0 \xrightarrow{a_1} \dots \xrightarrow{\psi_k^0} y_k^0$  and  $\pi_1 := x_0^1 \xrightarrow{\psi_0^1} y_0^1 \xrightarrow{a_1} \dots \xrightarrow{\psi_k^1} y_k^1$  be two sequences in  $L_\xi$ . Notice that  $y_j^1 - x_{j+1}^1 = a_{j+1} = y_j^0 - x_{j+1}^0$ , and thus  $y_j^1 - y_j^0 = x_{j+1}^1 - x_{j+1}^0$ . For  $\pi_0 \leq \pi_1$ , let  $\pi_1 - \pi_0$  be the sequence

$$\pi_1 - \pi_0 := z_0 \xrightarrow{\psi_0^1 - \psi_0^0} z_1 \xrightarrow{\psi_1^1 - \psi_1^0} z_2 \dots \xrightarrow{\psi_k^1 - \psi_k^0} z_{k+1},$$

where  $z_j := x_j^1 - x_j^0$  for  $j \leq k$  and  $z_{k+1} := y_k^1 - y_k^0$  otherwise.

**Lemma 1.** *Let  $\pi_0, \pi_1 \in L_\xi$  with  $\pi_0 \leq \pi_1$ . Then,  $\pi_1 - \pi_0 \in M_\xi$ .*

*Proof.* Since  $y_j^1(i) = y_j^0(i) = s_j^{\text{out}}(i)$  for those coordinates  $i$ 's s.t.  $s_j^{\text{out}}(i)$  is finite, and  $s_j^{\text{out}}(i)$  is finite iff  $s_{j+1}^{\text{in}}(i)$  is finite, we have  $z_j(i) = 0$  in this case, as required. Moreover, since  $\psi_j^0, \psi_j^1$  are balanced and  $\psi_j^0 \leq \psi_j^1$ , then also  $\psi_j^1 - \psi_j^0$  is balanced.  $\square$

Let  $\hat{L}_\xi$  and  $\hat{M}_\xi$  be the respective subsets of minimal elements of  $L_\xi$  and  $M_\xi$ . Those two sets are finite since  $\leq$  is a wqo. We have the following decomposition result.

**Lemma 2.**  $L_\xi = \hat{L}_\xi + \hat{M}_\xi^*$ .

*Proof.* The right-to-left containment follows immediately by additivity  $M_\xi + M_\xi \subseteq M_\xi$  and by relative additivity  $L_\xi + M_\xi \subseteq L_\xi$ . For the other direction, let  $\pi \in L_\xi$ . If  $\pi$  is minimal, we are done. Otherwise, there exists a minimal  $\hat{\pi} \in \hat{L}_\xi$  s.t.  $\hat{\pi} \leq \pi$ . Let  $\rho_0 := \pi - \hat{\pi}$ , which is in  $M_\xi$  by Lemma 1. If  $\rho_0$  is minimal, we are done since  $\pi = \hat{\pi} + \rho_0$ . Otherwise, there exists a minimal  $\hat{\rho}_0 \in \hat{M}_\xi$  s.t.  $\hat{\rho}_0 \leq \rho_0$  and  $\rho_1 := \rho_0 - \hat{\rho}_0$  is in  $M_\xi$  by subtractivity. If  $\rho_1$  is minimal, then we are done since  $\rho_0 = \hat{\rho}_0 + \rho_1$ . Otherwise, we can repeat this process and after a finite number of step we will get a finite sequence of minimal  $\hat{\rho}_0, \dots, \hat{\rho}_h \in \hat{M}_\xi$  s.t.  $\rho_0 = \hat{\rho}_0 + \dots + \hat{\rho}_h$ , thus showing  $\pi \in \hat{L}_\xi + \hat{M}_\xi^*$ .  $\square$

We call a sequence in  $M_\xi$  as in (2) *diagonal* if

- $z_j(i) > 0$  if  $s_j^{\text{in}}(i) = \omega$ ,
- $z_{j+1}(i) > 0$  if  $s_j^{\text{out}}(i) = \omega$  (iff  $s_{j+1}^{\text{in}}(i) = \omega$ ), and
- $\varphi_j$  is total.

**Lemma 3.** *If  $\xi$  is perfect, then there exists a diagonal solution in  $M_\xi$ .*

OLD INCOMPLETE PROOF. The following lemma allows us to extract *runs* from perfect marked witness graph sequences.

**Lemma 4.** *If  $\xi$  is a perfect marked witness graph sequence, then for every  $n \in \mathbb{N}$  there are configurations  $x'_{n,0}, y'_{n,0}, \dots, x'_{n,k}, y'_{n,k} \in \mathbb{N}^d$  and sequences of actions  $\delta_{n,0}, \delta_{n,1}, \dots, \delta_{n,k} \in A^*$  admitting a run*

$$x_{n,0} \xrightarrow{\delta_{n,0}} y_{n,0} \xrightarrow{a_0} x_{n,1} \xrightarrow{\delta_{n,1}} \dots \xrightarrow{a_k} x_{n,k} \xrightarrow{\delta_{n,k}} y_{n,k}, \quad (3)$$

s.t.

- $x_{n,j} \subseteq s_j^{\text{in}}$  and  $y_{n,j} \subseteq s_j^{\text{out}}$ , and
- $\lim_n x_{n,j} = s_j^{\text{in}}$  and  $\lim_n y_{n,j} = s_j^{\text{out}}$ .

*Proof.* By Lemma 3, let  $\rho$  be a diagonal sequence in  $M_\xi$ ; cf. (2). Let  $\sigma_{+,j}$  and  $\sigma_{-,j}$  be two cycles on  $s_j$  witnessing that  $M_j$  is forward and backward pumpable, respectively. Thus,  $\sigma_{+,j}$  pumps those finite components in  $s_j^{\text{in}}$  which are unbounded in  $s_j$ , and symmetrically  $\sigma_{-,j}$  unpumps the finite components in  $s_j^{\text{out}}$  which are unbounded in  $s_j$ . However, those cycles can have a negative effect on the other infinite components of  $s_j$  (which are thus infinite also on  $s_j^{\text{in}}$ , or  $s_j^{\text{out}}$ , respectively), and we want to avoid it. Moreover, we would like to find a cycle on  $s_j$  which will “undo” the effect of  $\sigma_{+,j}, \sigma_{-,j}$  except perhaps for some additional increase on the unbounded components in  $s_j^{\text{in}}, s_j^{\text{out}}$ . Since  $\rho$  is diagonal,  $\varphi_j$  is total. By summing up  $\rho$  sufficiently many times (since  $M_\xi$  is additive) we can assume w.l.o.g. that  $\varphi_j$  is total even after removing  $\sigma_{+,j}, \sigma_{-,j}$ , i.e.,

$$\varphi_j - |\sigma_{+,j}|_{E_j} - |\sigma_{-,j}|_{E_j} \geq 1. \quad (4)$$

Since  $z_j(i) > 0$  on unbounded coordinates  $i$  of  $s_j^{\text{in}}$ , and similarly  $z_{j+1}(i) > 0$  on unbounded coordinates  $i$  of  $s_j^{\text{out}}$  (equiv. of  $s_{j+1}^{\text{in}}$ ), and since  $M_\xi$  is additive, by summing up  $\rho$  sufficiently many times we can further assume w.l.o.g. that for every prefix  $\gamma_+$  of  $\sigma_{+,j}$  and for every prefix  $\gamma_-$  of  $\sigma_{-,j}$ ,

$$z_j(i) + |\gamma_+(i)| > 0 \quad \text{if } s_j^{\text{in}}(i) = \omega, \quad (5)$$

$$z_{j+1}(i) + |\gamma_-(i)| > 0 \quad \text{if } s_j^{\text{out}}(i) = \omega \text{ (iff } s_{j+1}^{\text{in}}(i) = \omega). \quad (6)$$

By (4) and Lemma ??, there exists a total cycle

$$s_j \xrightarrow{w_j} s_j$$

s.t.  $|w_j|_{E_j}$  equals the quantity (4) above. By repeating  $\sigma_{+,j}$  sufficiently many times we can assume w.l.o.g. that for every prefix  $\gamma$  of  $w_j$  we have

$$z_j(i) + |\sigma_{+,j}(i)| + |\gamma(i)| \geq 0 \quad \text{if } s_j(i) = \omega. \quad (7)$$

Putting together (2), (4), and (7), we get... NO! you need to start from  $x_j + z_j$ !

$$z_j \xrightarrow{\sigma_{+,j}} z_j + |\sigma_{+,j}| \xrightarrow{w_j} z_j + |\sigma_{+,j}| + |w_j| = z_{j+1} - |\sigma_{-,j}|. \quad (8)$$

Let  $\pi$  be a solution in  $L_\xi$ ; cf.(1). Then  $\psi_j$  is balanced and again by Lemma ?? there exists a cycle

$$s_j \xrightarrow{\alpha_j} s_j$$

s.t.  $|\alpha_j|_{E_j} = \psi_j$ . However, this does not necessarily imply  $x_j \xrightarrow{\alpha_j}$ . Since  $x_j \subseteq s_j^{\text{in}} \subseteq s_j$ , this means that the executability of  $\alpha_j$  depends only on whether unbounded components on  $s_j$  can be made arbitrarily large in  $x_j$ . There are two kinds of such unbounded components: The first kind are those components which are finite in  $s_j^{\text{in}}$  but unbounded in  $s_j$  (those can be pumped by  $\sigma_{+,j}$ ), and the second kind are those components which unbounded in  $s_j^{\text{in}}$  (and thus also in  $s_j$ ). We address here this second kind (the first kind will be addressed later), by choosing  $z_j$  to be large enough s.t., for every prefix  $\gamma$  of  $\alpha_j$ ,

$$z_j(i) + |\gamma(i)| \geq 0 \quad \text{if } s_j^{\text{in}}(i) = \omega \text{ (and thus } s_j(i) = \omega). \quad (9)$$

Since  $x_j \subseteq s_j^{\text{in}}$  and  $s_j^{\text{in}} \xrightarrow{\sigma_{+,j}^n}$  by construction, the only obstacle to  $x_j \xrightarrow{\sigma_{+,j}^n}$  is that unbounded components in  $s_j^{\text{in}}$  are “too small” in  $x_j$ . This is what  $z_j$  is for, since not only  $z_j$  is strictly positive on unbounded coordinates of  $s_j^{\text{in}}$ , but it remains so after executing every prefix of  $\sigma_{+,j}$ ; cf. (5). Consequently,

$$x_j + z_j \xrightarrow{\sigma_{+,j}} x_j + z_j + |\sigma_{+,j}|. \quad (10)$$

In order to be able to execute  $\alpha_j$  next, we need to run  $\sigma_{+,j}$  sufficiently many times until finite components in  $s_j^{\text{in}}$  which are unbounded in  $s_j$  are large enough. Since  $\sigma_{+,j}(i) > 0$  on those components, there exists  $n$  large enough s.t. for every prefix  $\gamma$  of  $\alpha_j$ ,

$$n \cdot |\sigma_{+,j}(i)| + |\gamma(i)| \geq 0 \quad \text{if } s_j^{\text{in}}(i) < \omega \text{ and } s_j(i) = \omega. \quad (11)$$

Combining (9) with (11), we get for every prefix  $\gamma$  of  $\alpha_j$

$$z_j(i) + n \cdot |\sigma_{+,j}(i)| + |\gamma(i)| \geq 0 \quad \text{if } s_j(i) = \omega. \quad (12)$$

We are now ready to put all the pieces together. For every  $n \in \mathbb{N}$ , we define  $x'_{n,j}$ ,  $y'_{n,j}$ , and  $\delta_{n,j}$  as

$$x'_{n,j} := x_j + n \cdot z_j, \quad (13)$$

$$y'_{n,j} := y_j + n \cdot z_{j+1}, \text{ and} \quad (14)$$

$$\delta_{n,j} := \sigma_{+,j}^n \cdot \alpha_j \cdot w_j^n \cdot \sigma_{-,j}^n. \quad (15)$$

For sufficiently large  $n$ , there is a run as in (3) of the form:

$$\begin{array}{lll}
x'_{n,j} = x_j + n \cdot z_j & \xrightarrow{\sigma_{+,j}^n} & \text{(by (10))} \\
x_j + n \cdot z_j + n \cdot |\sigma_{+,j}| & \xrightarrow{\alpha_j} & \text{(by (1),(12))} \\
y_j + n \cdot z_j + n \cdot |\sigma_{+,j}| & \xrightarrow{w_j^n} & \text{(by (8))} \\
y_j + n \cdot z_{j+1} - n \cdot |\sigma_{-,j}| & \xrightarrow{\sigma_{-,j}^n} & \\
y_j + n \cdot z_{j+1} = y'_{n,j}. & & 
\end{array}$$

□

This implies the previous lemma and shows that preruns are adherent to runs.

**Lemma 5.** *Let  $\xi$  be a perfect marked witness graph sequence. For every  $\pi \in L_\xi$  there exists a run  $\pi' \in L_\xi$  s.t.  $\pi \leq \pi'$ .*

*Proof.* Let  $\pi$  be a solution in  $L_\xi$  as in (1), and let  $\rho$  be a diagonal sequence in  $M_\xi$  as in (2), which exists by Lemma 3. We partition components into the following types:

Type I Components which are finite (and equal) in  $s_j^{\text{in}}, s_j^{\text{out}}, s_j$ .

Type II<sup>in</sup> Components which are finite in  $s_j^{\text{in}}$  but infinite in  $s_j$ .

Type II<sup>out</sup> Components which are finite in  $s_j^{\text{out}}$  but infinite in  $s_j$ .

Type III<sup>in</sup> Components which are infinite in  $s_j^{\text{in}}$  (thus also in  $s_j$ ).

Type III<sup>out</sup> Components which are infinite in  $s_j^{\text{out}}$  (thus also in  $s_j$ ).

Notice that Type II<sup>in</sup>  $\cup$  Type III<sup>in</sup> = Type II<sup>out</sup>  $\cup$  Type III<sup>out</sup>. Since  $\psi_j$  is balanced, by Lemma ?? there exists a cycle  $s_j \xrightarrow{\tilde{\psi}_j} s_j$  s.t.  $|\tilde{\psi}_j|_{E_j} = \psi_j$ . Notice that  $\tilde{\psi}_j$  is already executable on Type I components:

$$x_j(i) \xrightarrow{\tilde{\psi}_j(i)} x_j(i) = y_j(i) \quad \text{if } i \text{ is Type I.} \quad (16)$$

In order to execute  $\tilde{\psi}_j$  we need to pump Type II<sup>in</sup> and Type III<sup>in</sup> components.

We first pump Type II<sup>in</sup> components. Let  $\sigma_j^+$  and  $\sigma_j^-$  be two cycles on  $s_j$  witnessing that  $M_j$  is forward and backward pumpable, respectively. By definition,  $\sigma_j^+$  can be executed from  $s_j^{\text{in}}$ , thus

$$x_j(i) \xrightarrow{\sigma_j^+(i)} x_j(i) + |\sigma_j^+(i)| \quad \text{if } i \text{ is Type I or Type II}^{\text{in}}. \quad (17)$$

Since  $\sigma_j^+$  is strictly positive on Type II<sup>in</sup> components, by pumping it sufficiently many times, we can assume w.l.o.g.

$$x_j(i) + |\sigma_j^+(i)| \xrightarrow{\tilde{\psi}_j(i)} y_j(i) + |\sigma_j^+(i)| \quad \text{if } i \text{ is Type II}^{\text{in}}. \quad (18)$$

We now pump Type III<sup>in</sup> components. On those components,  $z_j$  is strictly positive (and zero elsewhere). By pumping  $\rho$  (using additivity) we can assume w.l.o.g. that Type III<sup>in</sup> components in  $z_j$  are sufficiently large to enable both  $\sigma_j^+$  and  $\tilde{\psi}_j$ :

$$z_j(i) \xrightarrow{\sigma_j^+(i)} z_j(i) + |\sigma_j^+|(i) \quad \text{if } i \text{ is Type III}^{\text{in}}, \quad (19)$$

$$z_j(i) \xrightarrow{\tilde{\psi}_j(i)} z_j(i) + |\psi_j|(i) \quad \text{if } i \text{ is Type III}^{\text{in}}. \quad (20)$$

We need one last assumption about  $\rho$ . We would like to find a cycle on  $s_j$  which will “undo” the effect of  $\sigma_{+,j}, \sigma_{-,j}$  except perhaps for some additional increase on Type III<sup>in</sup> components. By pumping  $\rho$  we can assume w.l.o.g. that  $\varphi_j$  is total even after removing  $\sigma_{+,j}$  and  $\sigma_{-,j}$ . Consequently, by Lemma ??, there exists a total cycle  $s_j \xrightarrow{w_j} s_j$  s.t.  $|w_j|_{E_j} = \varphi_j - |\sigma_{+,j}|_{E_j} - |\sigma_{-,j}|_{E_j}$  and

$$x_j(i) \xrightarrow{w_j} y_j(i) \quad \text{if } i \text{ is Type I}, \quad (21)$$

$$z_j(i) + |\sigma_j^+|(i) \xrightarrow{w_j} z_{j+1}(i) - |\sigma_j^-|(i) \quad \text{otherwise.} \quad (22)$$

(Notice that  $x_j(i) = y_j(i)$  and  $z_j(i) = z_{j+1}(i) = |\sigma_j^+|(i) = |\sigma_j^-|(i) = |w_j| = 0$  if  $i$  is Type I.) We can now construct our run  $\pi' = x'_0 \xrightarrow{\delta_0} y'_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} x_k \xrightarrow{\delta_k} y_k$  as follows:

$$\begin{aligned} x'_j &:= x_j + z_j, \\ y'_j &:= y_j + z_{j+1}, \text{ and} \\ \delta_j &:= \sigma_j^+ \cdot \tilde{\psi}_j \cdot w_j \cdot \sigma_j^-. \end{aligned}$$

Indeed, we have:

$$\begin{array}{lll} x'_j = x_j + z_j & \xrightarrow{\sigma_j^+} & \text{(by (17) and (19))} \\ x_j + z_j + |\sigma_j^+| & \xrightarrow{\tilde{\psi}_j} & \text{(by (16), (18), and (20))} \\ y_j + z_j + |\sigma_j^+| & \xrightarrow{w_j} & \text{(by (21) and (22))} \\ y_j + z_{j+1} - |\sigma_j^-| & \xrightarrow{\sigma_j^-} & \\ y_j + z_{j+1} = y'_j. & & \end{array}$$

□