Relating complementation constructions for Büchi automata

(draft)

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Abstract.

1 Preliminaries

Fix a finite alphabet Σ and a finite set of states Q. A *transition profile (over Q) t* is a pair $(\rightarrow_t, \twoheadrightarrow_t)$, where $\rightarrow_t \subseteq Q \times Q$ and $\twoheadrightarrow_t \subseteq \rightarrow_t$. Let $\mathcal{P}$ be the set of transition profiles. Intuitively, $p \rightarrow_t q$ if it is possible to go from p to q via a transition in t, and $p \twoheadrightarrow_t q$ if, additionally, an accepting such transition is taken. A *Büchi Automaton (BA) $\mathcal{A}$* is a tuple $(\Sigma, Q, I, (t_a)_{a \in \Sigma})$ where Σ is a finite alphabet, Q is a finite set of states, $I \subseteq Q$ is a non-empty set of *initial* states, and, for each input symbol $a \in \Sigma$, t_a is a transition profile over Q. For simplicity, instead of writing $p \rightarrow_{t_a} q$ or $p \twoheadrightarrow_{t_a} q$, we just write $p \xrightarrow{a} q$ and $p \overset{a}{\twoheadrightarrow} q$, respectively.

A *infinite trace* of $\mathcal{A}$ on a word $w = a_0 a_1 \cdots \in \Sigma^\omega$ *starting* in a state $q_0 \in Q$ is an infinite sequence of transitions $\pi = q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} \cdots$. An infinite trace is *initial* if it starts in an initial state $q_0 \in I$, and it is *fair* iff $q_i \overset{a_i}{\twoheadrightarrow} q_{i+1}$ for infinitely many i. The *language of $\mathcal{A}$* is $\mathcal{L}(\mathcal{A}) = \{w \mid \mathcal{A}$ has an infinite, initial and fair trace on $w\}$.

2 Ramsey

Fix an automaton $\mathcal{A} = (\Sigma, Q, I, F, \delta)$. For two profiles s and t, their product st is the pair $(\rightarrow, \twoheadrightarrow)$ defined as follows:

- $p \rightarrow q$ iff there exists $r \in Q$ s.t. $p \rightarrow_s r \rightarrow_t q$, and
- $p \twoheadrightarrow q$ iff there exists $r \in Q$ s.t. either $p \twoheadrightarrow_s r \rightarrow_t q$ or $p \rightarrow_s r \twoheadrightarrow_t q$.

Clearly, product of profiles is an associative operation.

For a finite word $u = a_0 a_1 \cdots a_m \in \Sigma^*$, the induced profile is $t_u = t_{a_0} t_{a_1} \cdots t_{a_m}$. The language of a profile t is the set of words $\mathcal{L}(t)$ inducing that profile, i.e., $\mathcal{L}(t) = \{u \in \Sigma^* \mid t_u = t\}$, and a profile is *valid* iff it has nonempty language (i.e., if it can be generated from profiles in $(t_a)_{a \in \Sigma}$). Let $\mathcal{P}^{\mathrm{v}}$ be the set of valid profiles. The relationship between multiplication of profiles and their language is as follows.

Lemma 1. *For two profiles s and t, $\mathcal{L}(s)\mathcal{L}(t) \subseteq \mathcal{L}(st)$, and the inclusion is strict in general.*

The fundamental property of profiles is that they only yield trivial intersections with A.

Lemma 2. *For two profiles s and t, if $\mathcal{L}(s)(\mathcal{L}(t))^{\omega} \cap A \neq \emptyset$, then $\mathcal{L}(s)(\mathcal{L}(t))^{\omega} \subseteq A$.*

A pair of profiles (s,t) is *linked* iff $st = s$ and $tt = t$. We define a test operation on linked profiles: For two linked profiles s and t, let $?(s,t)$ iff $p \to_s q \twoheadrightarrow_t q$ with $p \in I$.

generalize the test to every pair?

Lemma 3. *For two linked profiles s and t, $?(s,t)$ holds iff $\mathcal{L}(s)(\mathcal{L}(t))^{\omega} \subseteq \mathcal{L}(\mathcal{A})$.*

Profiles define an action on sets of states: For a set of states $P \subseteq Q$ and a profile t, let $P \cdot t$ be those states which are reachable through transitions in t originating from states in P; i.e., $P \cdot t = \{q \mid \exists (p \in P) \cdot p \to_t q\}$. Clearly, $P \cdot (st) = (P \cdot s) \cdot t$. For simplicity, if $w \in \mathcal{L}(t)$, then we also just write $I \cdot w$ instead of $I \cdot t$.

Describe Ramsey-based complementation.

3 Ranks

Let $w = a_0 a_1 \cdots \in \Sigma^{\omega}$ be an infinite word. The infinite traces of w on $\mathcal{A}$ can be arranged by juxtaposition of transition profiles into an infinite transition DAG $G = \langle V, T \rangle$, where

- $V \subseteq Q \times \omega$ is the set of vertices s.t. $(q,i) \in V$ iff $q \in I \cdot (a_0 a_1 \cdots a_{i-1})$, and
- T is a transition profile over V s.t., for every level $i \geq 0$, $\langle p,i \rangle \to_T \langle q, i+1 \rangle$ iff $p \xrightarrow{a_i} q$ and $\langle p,i \rangle \twoheadrightarrow_T \langle q,i+1 \rangle$ iff $p \overset{a_i}{\twoheadrightarrow} q$.

Then, w is not accepted by $\mathcal{A}$ iff every infinite path in G eventually ceases taking accepting transitions. This is witnessed with the notion of ranking.

A ranking for a DAG $G = \langle V, T \rangle$ is a mapping from V to ω s.t. ranks along transitions do not increase, and odd ranks along accepting transitions are strictly decreasing. Clearly, every path in G gets eventually trapped in some rank, and, if this rank is odd for every path in G, then the ranking is called an *odd ranking*. Since odd ranks are strictly decreasing on accepting transitions, this implies that if G has an odd ranking, then every infinite path in G must eventually cease taking accepting transitions, and, therefore, G is a rejecting DAG.

Kupferman and Vardi have shown that bounded rankings in fact suffice. Let $D^l = \{0, 1, \ldots, l, \perp\}$ be the set of rank values bounded by l plus the additional undefined value $\perp$, where the order is extended as $0 < 1 < \cdots < \perp$. We define a lifting function $\lfloor \cdot \rfloor_{\text{even}}$ on rank values s.t. $\lfloor n \rfloor_{\text{even}}$ is the largest even rank not larger than n; i.e.,

$$\lfloor n \rfloor_{\text{even}} = \begin{cases} \perp & \text{if } n = \perp \\ n & \text{if } n \text{ is even} \\ n-1 & \text{otherwise} \end{cases}$$

The function $\lfloor \cdot \rfloor_{\text{odd}}$ is defined analogously. A *l-level ranking* is a function $f : Q \mapsto D^l$. Let $\mathcal{R}^l$ be the set of l-level rankings. For two level rankings f and g, let $f \geq g$ iff, for every state p, $f(p) \geq g(p)$. Transition profiles induce a successor relation on level rankings. For two rankings f and g and a profile t, we say that *g is a t-successor of f*, written $f \xrightarrow{t} g$, iff, for every transition $p \to_t q$, $f(p) \geq g(q)$, and if $p \twoheadrightarrow_t q$, then $\lfloor f(p) \rfloor_{\text{even}} \geq g(q)$. If $w \in \mathcal{L}(t)$, we also just write $f \xrightarrow{w} g$ instead of $f \xrightarrow{t} g$.

For a level ranking f, let $\text{even}(f) = \{p \mid f(p) \text{ is even }\}$, and similarly for $\text{odd}(f)$.

Definition 1. *For an NBA $\mathcal{A} = (\Sigma, Q, I, (t_a)_{a\in\Sigma})$ and a bound $l \in \omega$, define* $\mathrm{KV}^l(\mathcal{A})$ *to be the NBA* $(\Sigma, \mathcal{R}^l \times 2^Q, \{\langle f_0, \emptyset\rangle\}, (t'_a)_{a\in\Sigma})$*, where*

- $f_0(p) = l$ *if* $p \in I$*, and* $f_0(p) = \bot$ *otherwise.*
- $\langle f, O\rangle \overset{a}{\to}{}' \langle f', (O \cdot a) \setminus \mathrm{odd}(f')\rangle$ *iff* $f \xrightarrow{a} f'$ *and* $O \neq \emptyset$*, and*
- $\langle f, \emptyset\rangle \overset{a}{\twoheadrightarrow}{}' \langle f', \mathrm{even}(f')\rangle$ *iff* $f \xrightarrow{a} f'$.

Profiles induce an action on level rankings. Intuitively, for a level ranking f and a profile t, $f \cdot t$ is the largest level ranking that can be obtained from f by following transitions in t. Formally,

$$f \cdot t = \max_{f \xrightarrow{t} g} g$$

It can be computed from f and t in the following way: for every q,

$$(f \cdot t)(q) = \min_{p \to_t q} \begin{cases} \lfloor f(p) \rfloor_{\mathrm{even}} & \text{if } p \twoheadrightarrow_t q \\ f(p) & \text{otherwise} \end{cases}$$

Note that, for two profiles s and t, $f \cdot (st) = (f \cdot s) \cdot t$.

3.1 Periodic rankings

Level rankings and profiles are strongly related. We regard a profile as a system of (strict) inequalities between values assigned to states by level rankings. If a ranking complies with all these constraints, we say that it satisfies the profile. Formally, for a level ranking f and a profile t, f *satisfies* t, written $f \models t$, iff the following two conditions hold:

$$f \models t \qquad \text{iff} \qquad \begin{array}{ll} f \cdot t \geq f, \text{ and} & \text{[Safety]} \\ \text{for all } p \to_t q, \lfloor f(p) \rfloor_{\mathrm{odd}} \geq f(q) & \text{[Liveness]} \end{array}$$

The safety condition ensures that the level ranking f complies with the inequalities defined by t, while the liveness condition ensures even ranks in f are strictly decreasing along transitions in t. Intuitively, ranks get sufficiently small by liveness, and not too small by safety.

Given an idempotent profile t, we can associate to it a canonical level ranking f_t. First, we perform a SCC decomposition of t. For a state p, let $[p]_t = \{q \mid p \to_t q \to_t p\}$ be the set of states which are inter-reachable from p; notice that $[p]_t$ is either empty, or it contains p (and there is a self-loop $p \to_t p$). Then, we assign an index to each state p counting how many different non-empty classes $[q]_t$ are reachable from p (via t): We define a function $\alpha_t : Q \mapsto \{0, 1, \ldots, n\}$ (n is the number of states of the automaton) s.t., for every state p, $\alpha_t(p) = |\{[q] \neq \emptyset \mid p \to_t q\}|$.

Lemma 4. *Let t be an idempotent profile. Then, for any states p and q,*

1. *If $p \to_t q$, then $\alpha_t(p) \geq \alpha_t(q)$.*
2. *If $p \to_t q$ and $[q]$ is empty, $\alpha_t(p) \geq \alpha_t(q) + 1$.*
3. *If t is rejecting, $p \twoheadrightarrow_t q$ and $[p]$ is not empty, then $\alpha_t(p) \geq \alpha_t(q) + 1$.*

Proof. The first point follows immediately from the definition of α_t. For the second point, notice that, by idempotence, a transition $p \rightarrow_t q$ can always be split into two transitions $p \rightarrow_t r \rightarrow_t q$, for some state r. By repeating the process on the first transition until a duplicate state appears, there exists a state r' s.t. $p \rightarrow_t r' \rightarrow_t q$ and $r' \rightarrow_t r'$. Therefore $[r']$ is non-empty and $q \notin [r']$ (otherwise one would have $[q] = [r']$), thus $\alpha_t(p) \geq \alpha_t(r') \geq \alpha_t(q) + 1$. For the third point, notice that, since t is rejecting, $[p]$ is different from $[q]$ (one cannot have $q \rightarrow_t p$). Therefore, $\alpha_t(p) \geq \alpha_t(q) + 1$.

Finally, given an idempotent profile t, we define the induced level ranking f_t as follows: For every state p,

$$f_t(p) = \begin{cases} 2\alpha_t(p) & \text{if } [p] = \emptyset \\ 2\alpha_t(p) - 1 & \text{otherwise} \end{cases} \tag{1}$$

Notice that, if t is rejecting, then f_t is a valid level ranking: Indeed, if p is accepting, then $[p]$ is necessarily empty (otherwise t would contain an accepting loop, which is impossible since it is rejecting), therefore $f_t(p)$ is even. We have the following crucial property of f_t.

Lemma 5. *For an idempotent profile t, let f_t be the canonical level ranking t constructed as above. If t is rejecting, then f_t satisfies t.*

Proof. It suffices to prove that

- for every $p \rightarrow_t q$, $\lfloor f_t(p) \rfloor_{\mathrm{odd}} \geq f_t(q)$, and
- for every $p \twoheadrightarrow_t q$, $\lfloor f_t(p) \rfloor_{\mathrm{even}} \geq f_t(q)$.

For the first point, assume that $p \rightarrow_t q$. By Lemma 4, $\alpha_t(p) \geq \alpha_t(q)$. If $[p]$ and $[q]$ are both non-empty, clearly $f_t(p) \geq f_t(q)$. If $[p]$ is non-empty and $[q]$ is empty, by Lemma 4, $\alpha_t(p) \geq \alpha_t(q) + 1$, therefore $f_t(p) = 2\alpha_t(p) - 1 \geq 2(\alpha_t(q) + 1) - 1 = 2\alpha_t(q) + 1 = f_t(q) + 1 \geq f_t(q)$. If $[p]$ and $[q]$ are both empty, by Lemma 4, $\alpha_t(p) \geq \alpha_t(q) + 1$, therefore $f_t(p) = 2\alpha_t(p) \geq 2(\alpha_t(q) + 1) = 2\alpha_t(q) + 2 = f_t(q) + 2$. Finally, if $[p]$ is empty and $[q]$ is non-empty, $f_t(p) = 2\alpha_t(p) \geq 2\alpha_t(q) = f_t(q) + 1$.

For the second point, additionally assume that $p \twoheadrightarrow_t q$ and that $f_t(p)$ is odd, i.e., $[p]$ is non-empty. Then, by Lemma 4, $\alpha_t(p) \geq \alpha_t(q) + 1$, and one can proceed as above.

For a profile t, we say that t *is consistent* iff there exists a ranking f satisfying t. The following is the basic relation between rankings and profiles.

Lemma 6. *Let t be an idempotent profile. Then, t is consistent iff t is rejecting.*

Proof. Let t be a profile. For the "only if" direction, assume that t is consistent, and, by way of contradiction, that t is not rejecting. Thus, t has a loop $p \twoheadrightarrow_t p$. By consistency, there exists a ranking f satisfying t. By the liveness condition, $f(p)$ cannot be even, otherwise $f(p) > f(p)$. So $f(p)$ has to be odd. By the safety condition, $(f \cdot t)(p) \geq f(p)$. But $p \twoheadrightarrow_t p$, therefore, by the definition of $f \cdot t$, $(f \cdot t)(p) < \lfloor f(p) \rfloor_{\mathrm{even}}$, but $f(p)$ is odd, thus $(f \cdot t)(p) < f(p)$, which is a contradiction. Therefore, t is rejecting. For the "if" direction, given a rejecting profile t one applies Lemma 5 for the canonical ranking f_t to show that f_t satisfies t.

Let $w = a_0 a_1 \cdots \in \Sigma^\omega$. If $w \notin \mathcal{L}(\mathcal{A})$, then there exists two profiles s and t s.t. w factorizes as $w = w_0 w_1 \cdots$, with $w_0 \in \mathcal{L}(s)$ and $w_i \in \mathcal{L}(t)$ for $i \geq 1$. Let $i_j = |w_0 w_1 \cdots w_j|$. From this factorization, we extract the following canonical ranking for w, which we call *periodic ranking*:

$$r(q,i) = \begin{cases} m & \text{if } i < i_0 \\ f_t(q) & \text{if } i = i_j \text{ for some } j \geq 0 \\ (f_t \cdot w)(q) & \text{if } i_j < i < i_{j+1} \text{ for some } j \geq 0 \text{ and } w = w[i_j..i] \end{cases} \tag{2}$$

Notice that r is indeed a ranking function; in particular, it is non-increasing, since, at boundary indices i_j's, $f_t \cdot w[i_j..i_{j+1}] \geq f_t$ by $w[i_j..i_{j+1}] \in \mathcal{L}(t)$ and since f_t satisfies t by definition. Furthermore, if t is a rejecting profile, then r is an odd ranking by Lemma 5.

4 Slices

We show that `slice` simulates `ramsey`.

We define an update operation of a slice preorder by a profile. Let $\succeq$ be a total preorder on Q, and let t be a profile. For any state q, $\mathsf{min\text{-}pre}^{\succeq,t}(q)$ is the set of $\succeq$-minimal t-predecessors of q; i.e., $\mathsf{min\text{-}pre}^{\succeq,t}(q) = \{p \mid p \to_t q \wedge \forall (r \to_t q) \cdot r \succeq p\}$. Then, $\succeq' = \succeq \cdot t$ is a new total preorder on Q defined as follows: For every states p, p', q, q' with $p \in \mathsf{min\text{-}pre}^{t,\succeq}(p')$ and $q \in \mathsf{min\text{-}pre}^{\succeq,t}(q')$,

- If $p \succ q$, then $p' \succ' q'$. Otherwise, if $p \approx q$:
  - If $q \twoheadrightarrow_t q'$ but $\neg(p \twoheadrightarrow_t p')$, then $p' \succ' q'$.
  - Otherwise, $p' \approx' q'$.

Notice that the update operation above is not an action on preorders, since big-step updates can lose information. In fact, one can prove that it is a pre-action, in the following sense: For any preorder $\succeq$ and profiles s and t, $\succeq \cdot (st) \supseteq (\succeq \cdot s) \cdot t$. This means that the small-step update $(\succeq \cdot s) \cdot t$ is *finer* than the corresponding big-step one $\succeq \cdot (st)$.

Lemma 7. *Let $\succeq_0$ and $\succeq_1$ be two total preorders, with $\succeq_1$ finer than $\succeq_0$, and let t be a profile. Then, $\succeq_1 \cdot t$ is finer than $\succeq_0 \cdot t$.*

Proof.

Lemma 8. *Let t be an idempotent profile, let $\succeq$ be a total preorder, and let $\succeq' = \succeq \cdot t$ be its update. Then,*

- *If $p \to_t q$, then $p \succeq' q$.*
- *If $p \twoheadrightarrow_t q$ and t is additionally rejecting, then $p \succ' q$.*

Proof.

With the two lemmata above, one can show a simulation from the `ramsey`-automaton $\mathcal{A}$ to the `slice`-automaton $\mathcal{B}$. The simulation requires an intermediate modified `slice` construction (automaton $\mathcal{C}$), where the preorder is introduced after a finite prefix has been read, and it is updated in big-steps. Clearly, $\mathcal{B}$ simulates $\mathcal{C}$ by Lemma 7.

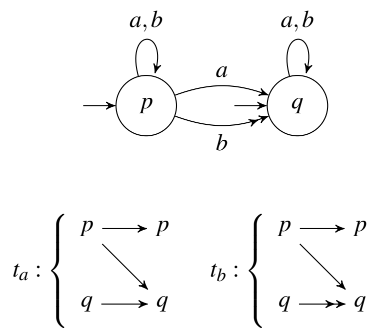

Fig. 1. `slice` does not simulate `ramsey-pre`. Let t_b and t_a be the two profiles above. Notice that t_b is (strictly) finer than t_a. The `ramsey-pre`-automaton commits to any total preorder realizing t_b, and then starts playing arbitrarily many a's and resetting each time to t_ε. The `slice`-automaton is thus in a state $\langle \{p,q\}, p \approx q \rangle$, and to visit an accepting state it has to eventually commit to a level k. At this point, the `ramsey-pre`-automaton plays a single b (which can be extended to a rejected word), and the `slice`-automaton chokes since there is an accepting transition $p \overset{b}{\twoheadrightarrow} q$ past the commit level k which is never subsumed by a better transition in the future.

We now argue that $\mathcal{C}$ simulates $\mathcal{A}$. Initially, $\mathcal{C}$ just tracks reachable states. When $\mathcal{A}$ commits to profiles s, t, $\mathcal{C}$ begins updating the preorder starting from the identity id. When $\mathcal{A}$ loops for the first time after having read a word in t, $\mathcal{C}$ updates its preorder to $id \cdot t$. At this point, $\mathcal{C}$ commits to the current level k. From now on, by Lemma 8, every time $\mathcal{A}$ reads a word in t and resets (thus visiting an accepting state), ...

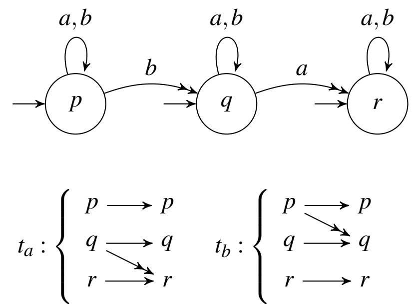

Fig. 2. `ramsey` does not simulate `slice`. After playing ab, the `slice`-automaton reaches the ordering $p > q > r$. From there, it keeps playing a's. So the `ramsey`-automaton has to eventually commit to the profile t_a, at which point the `slice`-automaton switches to playing b's, and the `ramsey`-automaton is no longer accepting.

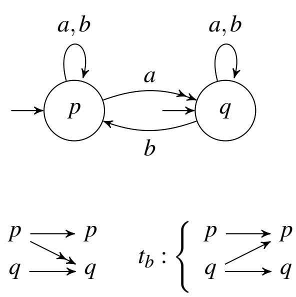

Fig. 3. `ramsey-pre` does not simulate `slice`. After playing a, the `slice`-automaton reaches the ordering $p > q$, from which it keeps playing a's. So the `ramsey`-automaton has to eventually commit to the unique preorder $p > q$ compatible with profile t_a. At this point, the `slice`-automaton plays b's (reaching the new ordering $p \approx q$) and the `ramsey`-automaton chokes since t_b is not compatible with the preorder it previously committed to.