Holonomic generating functions and unambiguous register automata

LC

1 Questions

1. Is it the case that a generating function is holonomic iff the corresponding exponential generating function is holonomic? YES: Since holonomic is the same as P-recursiveness of the associated number sequence, we can go back an forth $a_n \leftrightarrow \frac{a_n}{n!}$ with polynomial manipulations.

   This fails for differentially algebraic functions. The EGF of the Bell numbers $f(x) = e^{e^x - 1}$ satisfies the following quadratic differential equation

   $$f'' \cdot f - (f')^2 - f' \cdot f = 0$$

   and thus is differentially algebraic (D-algebraic), however the OGF of Bell numbers is not D-algebraic

2. Connection with polynomial grammars? Are their generating functions holonomic? NO.

3. General model of Parikh grammar: The acceptance condition is given by a multidimensional P-recursive sequence, which counts the number of derivations yielding a given tuple $(n_1, ..., n_k)$ of numbers. Then, one can decide multiplicity inclusion of such a grammar into a Parikh automaton, or conversely the multiplicity inclusion of an automaton into a grammar.

2 P-recursive sequences

Definition: $w = a_0 a_1 \cdots \in \mathbb{N}^\omega$ is *P-recursive* if there is $d \in \mathbb{N}$ and $d + 1$ polynomials $p_0(x), \ldots, p_d(x)$ s.t., for every n, the following linear equation with polynomial coefficients is satisfied:

$$p_d(n) \cdot a_n \cdots p_0(n) \cdot a_{n-d} = 0.$$

3 Ryll-Nardzewski functions

Let $L \subseteq \mathbb{A}^*$ be a language of finite words over the infinite alphabet $\mathbb{A}$. The *exponential generating function* of L is

$$e_L(x) = \sum_{n=0}^{\infty} L_n \cdot \frac{x^n}{n!},$$

where $L_n = |\mathsf{orbits}(L \cap \mathbb{A}^n)|$ is the *Ryll-Nardzewski sequence of* L, i.e., the number of orbits of words in L of length n. Let $\phi_{\mathbb{A}}(n) = |\mathsf{orbits}(\mathbb{A}^n)|$ be the *Ryll-Nardzewski function* for atoms $\mathbb{A}$.

Equality atoms. For example, over equality atoms $(\mathbb{A}, =)$, $\phi_{\mathbb{A}}(n) = B_n$ is the n-th *Bell number*.

Let n be the length of the word and let k be the number of distinct data values therein. Over equality atoms, we obtain the following recurrence for the function $S(n,k)$ counting the number of orbits of words of length n with k distinct data values: $S(0,0) = 1$, $S(0,1) = S(1,0) = 0$, $S(n,k) = 0$ if $k > n$, and otherwise $(1 \leq k \leq n)$:

$$S(n,k) = \underbrace{S(n-1,k-1)}_{\text{fresh}} + \underbrace{k \cdot S(n-1,k)}_{\text{not fresh}}.$$

Those are the well-known *Stirling numbers of the second kind*. Thus, $\phi_{\mathbb{A}}(n) = B_n = \sum_{k=0}^{n} S(n,k)$. The ordinary generating function of $S(n,k)$ satisfies

$$\begin{aligned}
g_S(x,y) = & \sum_{n,k\geq 0} S(n,k) \cdot x^n y^k = \\
&= 1 + \sum_{n,k\geq 1} (S(n-1,k-1) + k \cdot S(n-1,k)) \cdot x^n y^k = \\
&= 1 + \sum_{n,k\geq 1} S(n-1,k-1) \cdot x^n y^k + \sum_{n,k\geq 1} k \cdot S(n-1,k) \cdot x^n y^k = \\
&= 1 + xy \cdot \sum_{n,k\geq 0} S(n,k) \cdot x^n y^k + y \cdot \sum_{n,k\geq 1} S(n-1,k) \cdot x^n \frac{\partial}{\partial y} y^k = \\
&= 1 + xy \cdot g_S(x,y) + xy \cdot \frac{\partial}{\partial y} \sum_{n,k\geq 0} S(n,k) \cdot x^n y^k = \\
&= 1 + xy \cdot g_S(x,y) + xy \cdot \frac{\partial}{\partial y} g_S(x,y).
\end{aligned}$$

We also need an equation about $\frac{\partial}{\partial x} g_S(x,y)$!

For the exponential generating function, we have

$$
\begin{aligned}
e_S(x,y) = \sum_{n,k\geq 0} S(n,k)\cdot\frac{x^n}{n!}\frac{y^k}{k!} =& \\
&= 1 + \sum_{n,k\geq 1}\left(S(n-1,k-1)+k\cdot S(n-1,k)\right)\cdot\frac{x^n}{n!}\frac{y^k}{k!} = \\
&= 1 + \sum_{n,k\geq 1} S(n-1,k-1)\cdot\frac{x^n}{n!}\frac{y^k}{k!} + \sum_{n,k\geq 1} k\cdot S(n-1,k)\cdot\frac{x^n}{n!}\frac{y^k}{k!} = \\
&= 1 + \sum_{n,k\geq 1} S(n-1,k-1)\cdot\int\frac{x^{n-1}}{(n-1)!}dx\int\frac{y^{k-1}}{(k-1)!}dy + \\
&\quad + \sum_{n,k\geq 1} S(n-1,k)\cdot\frac{x^n}{n!}\frac{y^k}{(k-1)!} = \\
&= 1 + \int\int e_S(x,y)dxdy + y\cdot\sum_{n,k\geq 1} S(n-1,k)\cdot\int\frac{x^{n-1}}{(n-1)!}dx\frac{y^{k-1}}{(k-1)!} = \\
&= 1 + \int\int e_S(x,y)dxdy + y\cdot\int\frac{\partial}{\partial y}\sum_{n,k\geq 0} S(n,k)\cdot\frac{x^n}{n!}\frac{y^k}{k!}dx = \\
&= 1 + \int\int e_S(x,y)dxdy + y\cdot\int\frac{\partial}{\partial y}e_S(x,y)dx.
\end{aligned}
$$

Not *so* pretty.

For the mixed bivariate generating function we have

$$
\begin{aligned}
f_S(x,y) = \sum_{n,k\geq 0} S(n,k)\cdot\frac{x^n}{n!}y^k =& \\
&= 1 + \sum_{n,k\geq 1} S(n-1,k-1)\cdot\frac{x^n}{n!}y^k + \sum_{n,k\geq 1} k\cdot S(n-1,k)\cdot\frac{x^n}{n!}y^k = \\
&= 1 + y\cdot\int\sum_{n,k\geq 1} S(n-1,k-1)\cdot\frac{x^{n-1}}{(n-1)!}y^{k-1}dx + \\
&\quad + y\cdot\int\sum_{n,k\geq 1} S(n-1,k)\cdot\frac{x^{n-1}}{(n-1)!}\frac{\partial}{\partial y}y^k dx = \\
&= 1 + y\cdot\int f_S(x,y)dx + y\cdot\int\frac{\partial}{\partial y}f_S(x,y)dx.
\end{aligned}
$$

If we take $\frac{\partial}{\partial x}$ on both sides, we obtain $\frac{\partial}{\partial x}f_S(x,y) = y\cdot f_S(x,y) + y\cdot\frac{\partial}{\partial y}f_S(x,y)$. A function satisfying the linear differential equation above is $f_S(x,y) = e^{y(e^x-1)}$. Notice that in this case the function is not holonomic since in the same equation we have both $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. However, it is *differentially algebraic* (D-algebraic).

Total order atoms. We now consider the case of homogeneous total order atoms $(\mathbb{Q},\leq)$. Let $P(n,k)$ count the number of orbits of words of length n with k distinct equivalence classes. Then, $P(0,0)=1$, $P(0,m)=P(m,0)=0$ for every $m \geq 1$, and for $n, k \geq 1$,

$$P(n,k) = \underbrace{k \cdot P(n-1,k-1)}_{\text{fresh}} + \underbrace{k \cdot P(n-1,k)}_{\text{not fresh}}.$$

Then, $\phi_{\mathbb{Q}}(n) = \sum_{k=0}^{n} P(n,k)$ is also known as the *ordered Bell numbers* or *Fubini numbers*. We now compute their generating function

$$\begin{aligned}
g_P(x,y) &= \sum_{n,k\geq 0} P(n,k) \cdot x^n y^k = \\
&= 1 + \sum_{n,k\geq 1} (k \cdot P(n-1,k-1) + k \cdot P(n-1,k)) \cdot x^n y^k = \\
&= 1 + x \cdot \sum_{n,k\geq 0} (k+1) \cdot P(n,k) \cdot x^n y^{k+1} + xy \cdot \sum_{n,k\geq 0} k \cdot P(n,k) \cdot x^n y^{k-1} = \\
&= 1 + xy \cdot \sum_{n,k\geq 0} P(n,k) \cdot x^n \frac{\partial}{\partial y} y^{k+1} + xy \cdot \sum_{n,k\geq 0} P(n,k) \cdot x^n \frac{\partial}{\partial y} y^k = \\
&= 1 + xy \cdot \frac{\partial}{\partial y}(y \cdot g_P(x,y)) + xy \cdot \frac{\partial}{\partial y} g_P(x,y) = \\
&= 1 + xy \cdot g_P(x,y) + xy^2 \cdot \frac{\partial}{\partial y} g_P(x,y) + xy \cdot \frac{\partial}{\partial y} g_P(x,y) = \\
&= 1 + xy \cdot g_P(x,y) + xy(1+y) \cdot \frac{\partial}{\partial y} g_P(x,y).
\end{aligned}$$

Also in this case we obtain a linear partial differential equation with polynomial coefficients, hence $g_P(x,y)$ is holonomic.

Equivalence relation atoms. We consider as the atom structure the set $(\mathbb{E}, \approx)$ where $\mathbb{E}$ is countable and $\approx$ is an equivalence relation on $\mathbb{E}$ of infinite (thus countable) index s.t. each equivalence class is also infinite. The corresponding Ryll-Nardzewski function $E(n,m,k)$ has three parameters: n is the total number of elements, m is the number of $\approx$-equivalence classes, and k is the number of distinct elements. We have $E(0,0,0) = 1$, $E(0,m,k) = E(n,0,k) = E(n,m,0) = 0$ whenever one of n, m, m is > 0, $E(n,m,k) = 0$ whenever it is not the case that $m \leq k \leq n$ holds. Finally, for $n, m, k \geq 1$,

$$E(n,m,k) = \underbrace{E(n-1,m-1,k-1)}_{\text{new class}} + \underbrace{m \cdot E(n-1,m,k-1) + k \cdot E(n-1,m,k)}_{\text{existing class}}.$$

The first case arises when the new element is fresh and not equivalent to any previous element, the second case when the new element is fresh and equivalent to some previous element, and finally the last case arises when the new element is not fresh. We obtain the following equation for the generating function:

$$
\begin{aligned}
g_E(x,y,z) = & \sum_{n,m,k\geq 0} E(n,m,k)\cdot x^n y^m z^k = \\
& = 1 + \sum_{n,m,k\geq 1} \left(E(n-1,m-1,k-1) + m\cdot E(n-1,m,k-1) + k\cdot E(n-1,m,k)\right)\cdot x^n y^m z^k = \\
& = 1 + xyz\cdot g_E(x,y,z) + \sum_{n,m,k\geq 1} m\cdot E(n-1,m,k-1)\cdot x^n y^m z^k + \\
& \quad + \sum_{n,m,k\geq 1} k\cdot E(n-1,m,k)\cdot x^n y^m z^k = \\
& = 1 + xyz\cdot g_E(x,y,z) + y\cdot \sum_{n,m,k\geq 1} E(n-1,m,k-1)\cdot x^n \frac{\partial}{\partial y} y^m z^k + \\
& \quad + z\cdot \sum_{n,m,k\geq 1} E(n-1,m,k)\cdot x^n y^m \frac{\partial}{\partial z} z^k = \\
& = 1 + xyz\cdot g_E(x,y,z) + xyz\cdot \frac{\partial}{\partial y} g_E(x,y,z) + x\cdot \frac{\partial}{\partial z} \sum_{n,k\geq 0, m\geq 1} E(n,m,k)\cdot x^n y^m z^k = \\
& = 1 + xyz\cdot g_E(x,y,z) + xyz\cdot \frac{\partial}{\partial y} g_E(x,y,z) + x\cdot \frac{\partial}{\partial z} \sum_{n,m,k\geq 0} E(n,m,k)\cdot x^n y^m z^k = \\
& = 1 + xyz\cdot g_E(x,y,z) + xyz\cdot \frac{\partial}{\partial y} g_E(x,y,z) + x\cdot \frac{\partial}{\partial z} g_E(x,y,z).
\end{aligned}
$$

Two-nested equivalence relations atoms. We consider the countable atom structure $(\mathbb{E}, \approx_1, \approx_2)$ generated by the Fraissee limit of all finite sets with two equivalence relations $\approx_1, \approx_2$ s.t. $\approx_2 \subseteq \approx_1$.

To simplify the presentation, we consider here the unique Ryll-Nardzewski function $F(n, m_1, m_2)$ counting the number of orbits of n *distinct atoms* s.t. m_1 is the number of $\approx_1$-equivalence classes and m_2 is the number of $\approx_2$-equivalence classes. We have $F(0,0,0) = 1$, $F(0, m_1, m_2) = F(n, 0, 0) = 0$ whenever $n > 0$ and one of m_1, m_2 is > 0, $F(n, m_1, m_2) = 0$ whenever it is not the case that $m_1 \leq m_2 \leq n$ holds. Finally, for $n, m_1, m_2 \geq 1$,

$$
F(n,m_1,m_2) = \underbrace{F(n-1,m_1-1,m_2-1)}_{\text{new 1, new 2}} + m_1\cdot \underbrace{F(n-1,m_1,m_2-1)}_{\text{old 1, new 2}} + m_2\cdot \underbrace{F(n-1,m_1,m_2)}_{\text{old 1, old 2}}.
$$

We obtain the following equation for the generating function:

$$
\begin{aligned}
g_F(x, y_1, y_2) = & \sum_{n,m_1,m_2} F(n, m_1, m_2) \cdot x^n y_1^{m_1} y_2^{m_2} = \\
= & 1 + \sum_{n,m_1,m_2 \geq 1} (F(n-1, m_1-1, m_2-1) + \\
& + m_1 \cdot F(n-1, m_1, m_2-1) + m_2 \cdot F(n-1, m_1, m_2)) \cdot x^n y_1^{m_1} y_2^{m_2} = \\
= & 1 + xy_1y_2 \cdot g_F(x, y_1, y_2) + \\
& + y_1 \cdot \sum_{n,m_1,m_2 \geq 1} m_1 \cdot F(n-1, m_1, m_2-1) \cdot x^n y_1^{m_1-1} y_2^{m_2} + \\
& + y_2 \cdot \sum_{n,m_1,m_2 \geq 1} m_2 \cdot F(n-1, m_1, m_2) \cdot x^n y_1^{m_1} y_2^{m_2-1} = \\
= & 1 + xy_1y_2 \cdot g_F(x, y_1, y_2) + \\
& + y_1 \cdot \frac{\partial}{\partial y_1} \sum_{n,m_1,m_2 \geq 1} F(n-1, m_1, m_2-1) \cdot x^n y_1^{m_1} y_2^{m_2} + \\
& + y_2 \cdot \frac{\partial}{\partial y_2} \sum_{n,m_1,m_2 \geq 1} F(n-1, m_1, m_2) \cdot x^n y_1^{m_1} y_2^{m_2} = \\
= & 1 + xy_1y_2 \cdot g_F(x, y_1, y_2) + \\
& + xy_1y_2 \cdot \frac{\partial}{\partial y_1} g_F(x, y_1, y_2) + \\
& + xy_2 \cdot \frac{\partial}{\partial y_2} (g_F(x, y_1, y_2) - g_F(x, 0, y_2)).
\end{aligned}
$$

In the last equation we have used

$$\sum_{n,m_2} F(n, 0, m_2) \cdot x^n y_2^{m_2} = g_F(x, 0, y_2).$$

Cyclic order atoms. Consider the structure $(\mathbb{Q}, K)$, where K is a ternary relation defined as $K(x, y, z)$ if either $x < y < z$, or $z < x < y$, or $y < z < x$. Let $C(n, k)$ be the number of orbits of words of length n with k distinct data values.

$$C(n, k) = \underbrace{(k-1) \cdot C(n-1, k-1)}_{\text{fresh}} + \underbrace{k \cdot C(n-1, k)}_{\text{not fresh}}.$$

This recurrence is very similar to total order atoms, with the difference that when adding a fresh data, there is one less option since adding a maximal element is the same as adding a minimal one. This yields the following equation for the ordinary generating function:

$$
\begin{aligned}
g_C(x,y) &= \sum_{n,k\geq 0} C(n,k)\cdot x^n y^k = \\
&= 1 + \sum_{n,k\geq 1} (k-1)\cdot C(n-1,k-1)\cdot x^n y^k + \sum_{n,k\geq 1} k\cdot C(n-1,k)\cdot x^n y^k = \\
&= 1 + xy^2 \cdot \sum_{n,k\geq 0} k\cdot C(n,k)\cdot x^n y^{k-1} + xy\cdot \sum_{n,k\geq 0} k\cdot C(n,k)\cdot x^n y^{k-1} = \\
&= 1 + xy^2 \cdot \sum_{n,k\geq 0} C(n,k)\cdot x^n \frac{\partial}{\partial y} y^k + xy\cdot \sum_{n,k\geq 0} C(n,k)\cdot x^n \frac{\partial}{\partial y} y^k = \\
&= 1 + xy(1+y)\cdot \frac{\partial}{\partial y} g_C(x,y).
\end{aligned}
$$

Betweenness order atoms. Consider the structure $(\mathbb{Q}, B)$, where B is a ternary relation defined as $B(x,y,z)$ if either $y < x < z$, or $z < x < y$. Let $B(n,k)$ be the number of orbits of words of length n with k distinct data values.

Total preorder atoms.

Partial order atoms.

Tree order atoms. Consider the structure $(\mathbb{T}, \leq, R)$ obtained as the Fraissee limit of finite trees where $\leq$ is a *tree order*[1] and $R(a,b,c)$ holds if the least upper bound of a, b is incomparable with c. Let $T(n,k)$ be the Ryll-Nardzewski function of tree order atoms. When adding a new vertex to a tree of size k, either we are adding a new root (1 case), or we are adding a new leaf (k cases), or we are subdividing a previous edge ($k-1$ cases). In total, we have $2k$ cases, explaining the following recurrence:

$$
T(n,k) = \underbrace{2(k-1)\cdot T(n-1,k-1)}_{\text{fresh}} + \underbrace{k\cdot T(n-1,k)}_{\text{not fresh}}.
$$

We proceed similarly as in the case of cyclic order atoms above, and we obtain

$$
g_T(x,y) = 1 + xy(2+y)\cdot \frac{\partial}{\partial y} g_T(x,y).
$$

Random graph atoms. Let the signature consist of a single binary relation R.

4 Closure properties

In this section, let $\mathbb{A} = \left(A, R_1^{\mathbb{A}}, \ldots, R_m^{\mathbb{A}}\right)$ and $\mathbb{B} = \left(B, S_1^{\mathbb{B}}, \ldots, S_n^{\mathbb{B}}\right)$ be two relational structures.

[1] A tree order is a partial order where every two elements have a common upper bound and the upward closure of every element is totally ordered.

Direct product. The *direct product* of $\mathbb{A}, \mathbb{B}$ is the structure

$$\mathbb{A} \times \mathbb{B} = \left(A \times B, R_1^{\mathbb{A}\times\mathbb{B}}, \dots, R_m^{\mathbb{A}\times\mathbb{B}}, S_1^{\mathbb{A}\times\mathbb{B}}, \dots, S_n^{\mathbb{A}\times\mathbb{B}}\right),$$

where

$$\begin{aligned} ((a_1, \_), \dots, (a_k, \_)) \in R_i^{\mathbb{A}\times\mathbb{B}} \quad &\text{iff} \quad (a_1, \dots, a_k) \in R_i^{\mathbb{A}}, \\ ((\_, b_1), \dots, (\_, b_k)) \in S_j^{\mathbb{A}\times\mathbb{B}} \quad &\text{iff} \quad (b_1, \dots, b_k) \in S_j^{\mathbb{B}}. \end{aligned}$$

Orbits of $\mathbb{A} \times \mathbb{B}$ are in bijective correspondence with pairs of orbits, one of $\mathbb{A}$ and one of $\mathbb{B}$. In other words, $\mathsf{orbits}(\mathbb{A} \times \mathbb{B}) \simeq \mathsf{orbits}(\mathbb{A}) \times \mathsf{orbits}(\mathbb{B})$. Consequently, $\phi_{\mathbb{A}\times\mathbb{B}}(n) = \phi_{\mathbb{A}}(n) \cdot \phi_{\mathbb{B}}(n)$.

If $g_{\mathbb{A}}(x)$ is the orbit generating function of $\mathbb{A}$ and $g_{\mathbb{B}}(x)$ that of $\mathbb{B}$, we have

$$g_{\mathbb{A}\times\mathbb{B}}(x) = \sum_n \phi_{\mathbb{A}\times\mathbb{B}}(n) \cdot x^n = \sum_n \phi_{\mathbb{A}}(n) \cdot \phi_{\mathbb{B}}(n) \cdot x^n =: (g_{\mathbb{A}} \odot g_{\mathbb{B}})(x).$$

The generating function $g_{\mathbb{A}} \odot g_{\mathbb{B}}$ is called the *Hadamard* product of $g_{\mathbb{A}}, g_{\mathbb{B}}$. In other words, it is the pointwise product of the two sequences. Moreover, if $g_{\mathbb{A}}, g_{\mathbb{B}}$ are holonomic, then so it is $g_{\mathbb{A}} \odot g_{\mathbb{B}}$ [2] (even in the multivariate case).

Disjoint union. The *disjoint union* of $\mathbb{A}, \mathbb{B}$ is the structure

$$\mathbb{A} \uplus \mathbb{B} = \left(\{1\} \times A \cup \{2\} \times B, A^{\mathbb{A}\uplus\mathbb{B}}, B^{\mathbb{A}\uplus\mathbb{B}}, R_1^{\mathbb{A}\uplus\mathbb{B}}, \dots, R_m^{\mathbb{A}\uplus\mathbb{B}}, S_1^{\mathbb{A}\uplus\mathbb{B}}, \dots, S_n^{\mathbb{A}\uplus\mathbb{B}}\right),$$

where

$$\begin{aligned} (p, \_) \in A^{\mathbb{A}\uplus\mathbb{B}} \quad &\text{iff} \quad p = 1, \\ (p, \_) \in B^{\mathbb{A}\uplus\mathbb{B}} \quad &\text{iff} \quad p = 2, \\ ((p_1, a_1), \dots, (p_k, a_k)) \in R_i^{\mathbb{A}\uplus\mathbb{B}} \quad &\text{iff} \quad p_1 = \cdots = p_k = 1 \text{ and } (a_1, \dots, a_k) \in R_i^{\mathbb{A}}, \\ ((p_1, b_1), \dots, (p_k, b_k)) \in S_j^{\mathbb{A}\uplus\mathbb{B}} \quad &\text{iff} \quad p_1 = \cdots = p_k = 2 \text{ and } (b_1, \dots, b_k) \in S_j^{\mathbb{B}}. \end{aligned}$$

Orbits of $\mathbb{A} \uplus \mathbb{B}$ are in bijective correspondence with the disjoint union of orbits of $\mathbb{A}$ and $\mathbb{B}$ (this is one reason to add the predicates $A^{\mathbb{A}\uplus\mathbb{B}}, B^{\mathbb{A}\uplus\mathbb{B}}$), $\mathsf{orbits}(\mathbb{A} \uplus \mathbb{B}) \simeq \mathsf{orbits}(\mathbb{A}) \uplus \mathsf{orbits}(\mathbb{B})$, and thus $\phi_{\mathbb{A}\uplus\mathbb{B}}(n) = \phi_{\mathbb{A}}(n) + \phi_{\mathbb{B}}(n)$. It follow that

$$g_{\mathbb{A}\uplus\mathbb{B}}(x) = \sum_n \phi_{\mathbb{A}\uplus\mathbb{B}}(n) \cdot x^n = \sum_n (\phi_{\mathbb{A}}(n) + \phi_{\mathbb{B}}(n)) \cdot x^n =: (g_{\mathbb{A}} + g_{\mathbb{B}})(x).$$

If $g_{\mathbb{A}}, g_{\mathbb{B}}$ are holonomic, then so it is $g_{\mathbb{A}} + g_{\mathbb{B}}$ (even in the multivariate case).

Wreath product. The *wreath product* (lexicographic product) of $\mathbb{A}, \mathbb{B}$ is the structure

$$\mathbb{A} \otimes \mathbb{B} = \left(A \times B, R_1^{\mathbb{A}\otimes\mathbb{B}}, \dots, R_m^{\mathbb{A}\otimes\mathbb{B}}, S_1^{\mathbb{A}\otimes\mathbb{B}}, \dots, S_n^{\mathbb{A}\otimes\mathbb{B}}\right)$$

where

$$\begin{aligned} ((a_1, \_), \dots, (a_k, \_)) \in R_i^{\mathbb{A}\otimes\mathbb{B}} \quad &\text{iff} \quad (a_1, \dots, a_k) \in R_i^{\mathbb{A}}, \\ ((a_1, b_1), \dots, (a_k, b_k)) \in S_j^{\mathbb{A}\otimes\mathbb{B}} \quad &\text{iff} \quad a_1 = \cdots = a_k \text{ and } (b_1, \dots, b_k) \in S_j^{\mathbb{B}}. \end{aligned}$$

$$\mathsf{orbits}((\mathbb{A} \otimes \mathbb{B})^n) \simeq \bigcup_{m_1+\cdots+m_k=n} \mathsf{orbits}(\mathbb{A}^{m_1,\dots,m_k}) \times \mathsf{orbits}(\mathbb{B}^{m_1}) \times \cdots \times \mathsf{orbits}(\mathbb{B}^{m_k}).$$

The set $\mathbb{A}^{m_1,\dots,m_k}$ contains all those $(m_1 + \cdots + m_k)$-tuples of elements of $\mathbb{A}$ with k distinct elements s.t. m_i is the multiplicity of the i-th uniquely occurring element. Let $\psi_{\mathbb{A}}(m_1,\dots,m_k)$ be the generalised Ryll-Nardzewski function of $\mathbb{A}$ which counts $|\mathsf{orbits}(\mathbb{A}^{m_1,\dots,m_k})|$. In one direction we have $\phi_{\mathbb{A}}(n) = \sum_{m_1+\cdots+m_k=n} \psi_{\mathbb{A}}(m_1,\dots,m_k)$. We have

$$\phi_{\mathbb{A}\otimes\mathbb{B}}(n) = \sum_{m_1+\cdots+m_k=n} \psi_{\mathbb{A}}(m_1,\dots,m_k) \cdot \phi_{\mathbb{B}}(m_1) \cdots \phi_{\mathbb{B}}(m_k).$$

From this observation one can derive an expression for the exponential generating funcion of $\mathbb{A} \otimes \mathbb{B}$ (easier) and for the ordinary generating function $g_{\mathbb{A}\otimes\mathbb{B}}$ (more difficult; c.f. *the cycle index* from Cameron's book [1, Sec. 3.7]).

5 Unambiguous register automata

Fix an unambiguous register automaton without guessing A with d registers over atoms $\mathbb{A}$. For a control location p and a d-tuple of atoms $\bar{a} \in \mathbb{A}^d$, let

$$\phi_{p,\bar{a}}(n) = |\mathsf{orbits}(L(p, \bar{a}) \cap \mathbb{A}^n)|$$

be the Ryll-Nardzewski function of configuration $(p, \bar{a})$. The value $\phi_{p,\bar{a}}(n)$ does not depend on the actual choice of $\bar{a}$, but just on its orbit: $\phi_{p,\bar{a}}(n) = \phi_{p,\bar{b}}(n)$ whenever $\mathsf{orbit}(\bar{a}) = \mathsf{orbit}(\bar{b})$. We can thus write $\phi_{p,o}(n)$ where $o \in \mathsf{orbits}(\mathbb{A}^d)$ is a d-orbit of atoms.

Equality atoms. Since the automaton does not have guessing, every register stores a data value only if it occurred in input. Let an (n, k)-*orbit* be an orbit of words of length n with k different equivalence classes (the *width* of the orbit), and let $\phi_c(n, k)$ be the number of (n, k)-orbits of words accepted from configuration $c = (p, o)$. Let $w(c)$, for a configuration $c = (p, o)$, denote the width of o. We have $\phi_c(0, 0) = 1$ if p is accepting, and $= 0$ otherwise, $\phi_c(n, 0) = \phi_c(0, n)$ for $n > 0$, $\phi_c(n, k) = 0$ for $k > n$, and, for $1 \leq k \leq n$, we recursively have

$$\phi_c(n,k) = \sum_{d \xrightarrow{=_i} c} \phi_d(n-1,k) + \sum_{d \xrightarrow{\neq, s_i} c} (\underbrace{\phi_d(n-1,k-1)}_{\text{fresh}} + \underbrace{(k-w(d)) \cdot \phi_d(n-1,k)}_{\text{not fresh}}).$$

The first sum follows from the fact that if the n-th data value a_n is in register i, then every $(n-1, k)$-orbit gives rise to exactly one (n, k)-orbit. The second sum consists of two parts. In the "fresh" case, the n-th data value a_n is globally fresh, and thus every $(n-1, k-1)$-orbit gives rise to a single (n, k)-orbit. In the "not fresh" case, a_n is not fresh, however it is different than the $w(o')$ distinct data values previously stored in the registers (holding values already occurring before), and thus every $(n-1, k)$-orbit gives rise to $k - w(d)$ many (n, k)-orbits. The last condition holds because there are $k - w(d)$ ways to select a_n amongst the k different previously occurring equivalence classes and avoiding $w(d)$ of them. The corresponding generating function $g_c(x, y)$ satisfies the recurrence:

$$
\begin{aligned}
g_c(x,y) = &\sum_{n,k} \phi_c(n,k) \cdot x^n y^k = \\
&= \phi_c(0,0) + \sum_{d \xrightarrow{=_i} c} \sum_{n,k\geq 1} \phi_d(n-1,k) \cdot x^n y^k + \sum_{d \xrightarrow{\neq} c} \Big(\sum_{n,k\geq 1} \phi_d(n-1,k-1) \cdot x^n y^k + \\
&\quad + \sum_{n,k\geq 1} (k - w(d)) \cdot \phi_d(n-1,k) \cdot x^n y^k \Big) = \\
&= \phi_c(0,0) + \sum_{d \xrightarrow{=_i} c} x \cdot (g_d(x,y) - \phi_d(0,0)) + \sum_{d \xrightarrow{\neq} c} \Big(xy \cdot g_d(x,y) + \\
&\quad + xy \cdot \frac{\partial}{\partial y} g_d(x,y) - w(d)x \cdot (g_d(x,y) - \phi_d(0,0)) \Big).
\end{aligned}
$$

This shows that the generating function of unambiguous register automata without guessing over equality atoms is holonomic.

Guessing. If the register values can be locally guessed, then this creates a problem because we cannot know whether the guessed value is globally fresh (appeared before) or not.

Let's now consider a globally fresh guessing action instead, guaranteeing that the new value is globally fresh (w.r.t. the input read so far). We can keep track, for each register r_i, whether its data value appeared already or not in the input word read so far. For a configuration c, let $w(c)$ be the number of distinct data values in the registers that already appeared so far. This quantity can be updated by the automaton. Then the same equations as in the previous paragraph hold with this new interpretation.

Total order atoms.

$$
\phi_c(n,k) = \sum_{d \xrightarrow{=_i} c} \phi_d(n-1,k) + \sum_{d \xrightarrow{\neq} c} \big(\underbrace{k \cdot \phi_d(n-1,k-1)}_{\text{fresh}} + \underbrace{(k - w(d)) \cdot \phi_d(n-1,k)}_{\text{not fresh}} \big).
$$

This is very similar to equality atoms, with the difference that in the "fresh" case, there are now k possible ways to add a fresh data into a total preorder with $k-1$ equivalence classes.

$$
\begin{aligned}
g_c(x,y) = &\sum_{n,k} \phi_c(n,k) \cdot x^n y^k = \\
&= \phi_c(0,0) + \sum_{d \xrightarrow{=_i} c} \sum_{n,k\geq 1} \phi_d(n-1,k) \cdot x^n y^k + \sum_{d \xrightarrow{\neq, s_i} c} \Big(\sum_{n,k\geq 1} k \cdot \phi_d(n-1,k-1) \cdot x^n y^k + \\
&\quad + \sum_{n,k\geq 1} (k - w(d)) \cdot \phi_d(n-1,k) \cdot x^n y^k \Big) = \\
&= \phi_c(0,0) + \sum_{d \xrightarrow{=_i} c} x \cdot (g_d(x,y) - \phi_d(0,0)) + \sum_{d \xrightarrow{\neq, s_i} c} \Big(xy \cdot \frac{\partial}{\partial y} g_d(x,y) + \\
&\quad + xy \cdot \frac{\partial}{\partial y} g_d(x,y) - w(d)x \cdot (g_d(x,y) - \phi_d(0,0)) \Big).
\end{aligned}
$$

References

[1] P. J. Cameron. Oligomorphic permutation groups. *London Math. Soc.*, 152, 1990.

[2] L. Lipshitz. The diagonal of a D-finite power series is D-finite. *Journal of Algebra*, 113(2):373–378, 1988.