Decidability of Timed Communicating Automata

Lorenzo Clemente[1]
University of Warsaw, Warsaw, Poland
clementelorenzo@gmail.com

Abstract

We study the reachability problem for networks of timed communicating processes. Each process is a timed automaton communicating with other processes by exchanging messages over unbounded FIFO channels. Messages carry clocks which are checked at the time of transmission and reception with suitable timing constraints. Each automaton can only access its set of local clocks and message clocks of sent/received messages. Time is dense and all clocks evolve at the same rate. We show a complete characterisation of decidable and undecidable communication topologies generalising and unifying previous work. From a technical point of view, we use quantifier elimination and a reduction to counter automata with registers.

2012 ACM Subject Classification Theory of computation → Distributed computing models. Theory of computation → Timed and hybrid models. Theory of computation → Logic

Keywords and phrases timed communicating automata, quantifier elimination

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

Timed automata (TA) were introduced almost thirty years ago by Alur and Dill [7, 8] as a decidable model of real-time systems elegantly combining finite automata with timing constraints over a densely timed domain. TA are still an extremely active research area, as testified by recent works on the reachability problem [28], a novel analysis technique based on tree automata [6], the binary reachability relation [40], and extensions with counters [12, 1], stacks [15, 25, 43, 10, 4, 39, 23, 24, 22], and lossy FIFO channels [3].

We study *systems of timed communicating automata* (TCA) [32], which are networks of TA exchanging messages over unbounded FIFO channels[2]. Messages are equipped with densely-valued clocks which elapse at the same rate as local TA clocks. Message transmissions/receptions are guarded by logical constraints between local and message clocks. We consider *classical*[3] $\mathbb{Q}$, *integral* $\mathbb{N}$, and *fractional clocks* $\mathbb{I} := \mathbb{Q} \cap [0, 1)$. All clocks evolve at the same rate. For classical and integral clocks $\mathtt{x}, \mathtt{y}$, we consider *inequality* $\mathtt{x} - \mathtt{y} \sim k$ and *modulo* $\mathtt{x} - \mathtt{y} \equiv_m k$ constraints; for fractional clocks $\mathtt{x}, \mathtt{y} : \mathbb{I}$ we consider *order* constraints $\mathtt{x} \sim \mathtt{y}$, where $\sim \in \{<, \leqslant, \geqslant, >\}$. The *non-emptiness problem* asks whether there exists a run where all processes start and end in predefined control locations and with empty channels. Already in the untimed setting of communicating automata (CA), non-emptiness is undecidable [16]. Decidability can be regained by restricting the *communication topology*, i.e., the graph where vertices are processes $\mathtt{p}$, and there is an edge $\mathtt{p} \to \mathtt{q}$ whenever there is a channel from process $\mathtt{p}$ to process $\mathtt{q}$ [38, 42]. A *polytree* is a topology whose underlying undirected graph is a tree;

[1] Partially supported by Polish NCN grant 2017/26/D/ST6/00201.

[2] The original name *communicating timed automata* [32] refers to a version of TCA with untimed channels. In order to stress that we consider timed channels, we speak about timed communicating automata.

[3] Considering the reals $\mathbb{R}$ would be equivalent.

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

a *polyforest* is a disjoint union of polytrees. Our main result is a complete characterisation of the decidable TCA topologies in dense time.

▸ **Theorem 1.** *Non-emptiness of* TCA *is decidable if, and only if, the communication topology is a polyforest s.t. in each polytree there is at most one channel with inequality tests.*

Note that neither fractional clocks nor modulo constraints affect decidability. This subsumes analogous characterisations for TCA with untimed channels in discrete [20, Theorem 3] and dense time [20, Theorem 5]. The only work considering timed channels that we are aware of is [9], which however considers only discrete time: with (integral) non-diagonal inequality tests of the form $\mathtt{x} \sim k$, the topology $\mathtt{p} \to \mathtt{q}$ is decidable [9, Theorem 4], while $\mathtt{p} \to \mathtt{q} \to \mathtt{r}$ is undecidable [9, Theorem 3]. Since our undecidability result holds already in discrete time, it follows from Theorem 1 that $\mathtt{p} \to \mathtt{q} \to \mathtt{r}$ is undecidable; additionally, new undecidable topologies can be deduced, such as $\mathtt{p} \to_1 \mathtt{q} \to_2 \mathtt{r} \to_3 \mathtt{s}$ with $\to_1, \to_3$ with integral inequality tests and $\to_2$ untimed. Regarding decidability, Theorem 1 vastly generalises previous results, since we consider: 1) timed channels, 2) arbitrary topologies, 3) a richer set of clocks comprising both classical, integral, and fractional clocks, which allows us to isolate the kind of clocks leading to undecidability, 4) diagonal constraints between channel and local clocks (not previously considered), 5) the more general setting of dense time.

Technical contribution. While our undecidability results are inherited from [20], new ideas are needed to show decidability with densely-timed channels. First, we show that diagonal constraints between channel and local clocks reduce to non-diagonal ones. Removal of diagonal constraints is well-known for timed automata [8]. In our context, this is a nontrivial result, which we prove with the method of *quantifier elimination* (cf. Lemma 2 in Sec. 4), recently applied to timed pushdown automata [22]. Quantifier elimination is not used as a black box, since we start from formulas which are not TCA constraints, and thus we need not obtain a constraint after the quantifiers are eliminated. Nonetheless, we show a hand-tailored procedure to carefully eliminate quantifiers yielding constraints in the syntactic form required by TCA. We believe that the use of quantifier elimination to the study of timed models has independent interest and its applicability should be further investigated.

Our second technical contribution is the encoding of fractional clocks into $\mathbb{I}$-valued registers over the *cyclic order* $K \subseteq \mathbb{I}^3$, i.e., the ternary relation $K(a, b, c)$ that holds whenever going clockwise on the unit circle starting at a, we first visit b, and then c. Since in our reduction we allow processes to elapse time independently, it would be insufficient to record the combined total order of all fractional clocks: we would need to additionally record also differences of fractional values. We avoid this issue by using registers recording the timestamps of the last clock resets (reminiscent of the *reset-point semantics* of [26, 11]). Since local time elapse is modelled by updating a single reference register (instead of elapsing time for a subset of the clocks), this greatly simplifies the analysis.

With the two technical tools above in hand, we reduce a TCA over a polyforest topology to a *register automaton with counters* (RAC), which is in turn reduced to a counter automaton. Since integral inequality tests inside the same polytree correspond to zero tests, we reduce to Petri nets if there is no inequality test (which are decidable by [36, 31, 33]; cf. also [34]), and to Petri nets with one zero test if there is at most one inequality test per polytree (which are decidable by [41, 14]). Converse reductions were provided in [20] already with untimed channels, showing that TCA reachability is Petri net-hard. Omitted proofs are in Sec. A.

Related work. *Communicating automata* (CA) are a fundamental model of concurrency [16, 38]. Methods to ensure decidability, other than restricting the communication topology, include: lossy messages [18, 5, 19] (cf. also [27]); half-duplex [17] and mutex communication

[30]); bounded context switching [42]; bag semantics [21]. The model of CA has been extended in diverse directions, such as CA with counters [29], with stacks [30]; and lossy CA with data [2] and time [3].

2 Preliminaries

Let $\mathbb{N}$ be the set of natural, $\mathbb{Z}$ the integer, $\mathbb{Q}$ the rational, and $\mathbb{Q}_{\geqslant 0}$ the nonnegative rational numbers. Let $\mathbb{I} := \mathbb{Q} \cap [0, 1)$ be the rational unit interval. For $a \in \mathbb{Q}$, let $\lfloor a \rfloor \in \mathbb{Z}$ and $\{a\} \in \mathbb{I}$ denote its integral and, resp., fractional part; for $b \in \mathbb{Q}$, let the *cyclic difference* be $a \ominus b = \{a - b\}$ and the *cyclic addition* be $a \oplus b = \{a + b\}$. For $a, k \in \mathbb{Z}$, let $a \equiv_m k$ denote the congruence modulo $m \in \mathbb{N} \backslash \{0\}$, which we extend to $a \in \mathbb{Q}$ by $a \equiv_m k$ iff $\lfloor a \rfloor \equiv_m k$. For a set of variables X and a domain A, let A^X be the set of valuations for variables in X taking values in A. For a valuation $\mu \in A^X$, a variable $x \in X$, and a new value $a \in A$, let $\mu[x \mapsto a]$ be the new valuation which assigns a to x, and agrees with μ on $X \backslash \{x\}$. For a subset of variables $Y \subseteq X$, let $\mu|_Y \in A^Y$ be the restricted valuation agreeing with μ on Y. For two disjoint domains X, Y and $\mu \in A^X, \nu \in A^Y$, let $(\mu \cup \nu) \in A^{X \cup Y}$ be the valuation which agrees with μ on X and with ν on Y.

Labelled transition systems. A *labelled transition system* (LTS) $\mathcal{A}$ is a tuple $\langle C, c_I, c_F, A, \rightarrow \rangle$ where C is a set of *configurations*, with $c_I, c_F \in C$ two distinguished *initial* and *final* configurations, resp., A a set of *actions*, and $\rightarrow \subseteq C \times A \times C$ a *labelled transition relation*. For simplicity, we write $c \xrightarrow{a} d$ instead of $(c, a, d) \in \rightarrow$, and for a sequence of actions $w = a_1 \cdots a_n \in A^*$ we overload this notation as $c \xrightarrow{w} d$ if there exist intermediate states $c = c_0, c_1, \ldots, c_n = d$ s.t., for every $1 \leqslant i \leqslant n$, $c_{i-1} \xrightarrow{a_i} c_i$. For a given LTS $\mathcal{A}$, the *non-emptiness problem* asks whether there is a sequence of actions $w \in A^*$ s.t. $c_I \xrightarrow{w} c_F$.

Clock constraints. Let $\mathtt{X}$ be a set of *classical* $\mathtt{x} : \mathbb{Q}$, *integral* $\mathtt{x} : \mathbb{N}$, or *fractional* $\mathtt{x} : \mathbb{I}$ clocks. A *clock constraint* over $\mathtt{X}$ is a boolean combination of the following *atomic constraints*

$$\begin{array}{lccc} & \text{(inequality)} & \text{(modular)} & \text{(order)} \\ \text{(non-diagonal)} & \mathtt{x}_0 \leqslant k & \mathtt{x}_0 \equiv_m k & \mathtt{y}_0 = 0 \\ \text{(diagonal)} & \mathtt{x}_0 - \mathtt{x}_1 \leqslant k & \mathtt{x}_0 - \mathtt{x}_1 \equiv_m k & \mathtt{y}_0 \leqslant \mathtt{y}_1, \end{array}$$

where $\mathtt{x}_0, \mathtt{x}_1$ are either both classical or integral clocks, $\mathtt{y}_0, \mathtt{y}_1$ fractional clocks, $m \in \mathbb{N}$, and $k \in \mathbb{Z}$. As syntactic sugar we also allow **true** and variants with any $\sim \in \{\leqslant, <, \geqslant, >\}$ in place of $\leqslant$. A *clock valuation* is a mapping $\mu \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}}$ assigning a non-negative rational number to every clock in $\mathtt{X}$. Let $\overline{0}$ be the clock valuation μ s.t. $\mu(\mathtt{x}) = 0$ for every clock $\mathtt{x} \in \mathtt{X}$. For a valuation μ and a clock constraint φ, μ *satisfies* φ, written $\mu \models \varphi$, if φ is satisfied when classical clocks $\mathtt{x} : \mathbb{Q}$ are evaluated as $\mu(\mathtt{x})$, integral clocks $\mathtt{x} : \mathbb{N}$ as $\lfloor \mu(\mathtt{x}) \rfloor$, and fractional clocks $\mathtt{y} : \mathbb{I}$ as $\{\mu(\mathtt{y})\}$. In particular, $\mu \models (\mathtt{x}_0 - \mathtt{x}_1 \equiv_m k)$ is equivalent to $\lfloor \mu(\mathtt{x}_0) - \mu(\mathtt{x}_1) \rfloor \equiv_m k$ if $\mathtt{x}_0, \mathtt{x}_1 : \mathbb{Q}$ are classical clocks, and to $\lfloor \mu(\mathtt{x}_0) \rfloor - \lfloor \mu(\mathtt{x}_1) \rfloor \equiv_m k$ if $\mathtt{x}_0, \mathtt{x}_1 : \mathbb{N}$ are integral clocks.

Timed communicating automata. A *communication topology* is a directed graph $\mathcal{T} = \langle \mathtt{P}, \mathtt{C} \rangle$ with nodes $\mathtt{P}$ representing *processes* and edges $\mathtt{C} \subseteq \mathtt{P} \times \mathtt{P}$ representing *channels* $\mathtt{pq} \in \mathtt{C}$ whenever $\mathtt{p}$ can send messages to $\mathtt{q}$. We do not allow multiple channels from $\mathtt{p}$ to $\mathtt{q}$ since such a topology would have an undecidable non-emptiness problem (stated below). A *system of timed communicating automata* (TCA) is a tuple $\mathcal{S} = \left\langle \mathcal{T}, \mathtt{M}, (\mathtt{X}^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}, (\mathcal{A}^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}} \right\rangle$ where $\mathcal{T} = \langle \mathtt{P}, \mathtt{C} \rangle$ is a communication topology, $\mathtt{M}$ a finite set of *messages*, $\mathtt{X}^{\mathtt{c}}$ a set of *channel clocks* for messages sent on channel $\mathtt{c} \in \mathtt{C}$, and, for every $\mathtt{p} \in \mathtt{P}$, $\mathcal{A}^{\mathtt{p}} = \langle \mathtt{L}^{\mathtt{p}}, \ell_I^{\mathtt{p}}, \ell_F^{\mathtt{p}}, \mathtt{X}^{\mathtt{p}}, \mathsf{Op}^{\mathtt{p}}, \Delta^{\mathtt{p}} \rangle$ is a *timed communicating automaton* with the following components: $\mathtt{L}^{\mathtt{p}}$ is a finite set of *control locations*, with $\ell_I^{\mathtt{p}}, \ell_F^{\mathtt{p}} \in \mathtt{L}^{\mathtt{p}}$ two distinguished *initial* and *final* locations therein, $\mathtt{X}^{\mathtt{p}}$ a set of *local clocks*, and $\rightarrow^{\mathtt{p}} \subseteq \mathtt{L}^{\mathtt{p}} \times \mathsf{Op}^{\mathtt{p}} \times \mathtt{L}^{\mathtt{p}}$ a set of transitions of the form $\ell \xrightarrow{\mathsf{op}}{}^{\mathtt{p}} \not\ell$, where $\mathsf{op} \in \mathsf{Op}^{\mathtt{p}}$ determines the kind of transition: $\mathsf{op} = \mathsf{nop}$ is a local operation without side effects; $\mathsf{op} = \mathsf{elapse}$ is a global time elapse operation which is executed by all processes at the same time; all local and channel clocks evolve at the same rate; $\mathsf{op} = \mathsf{test}(\varphi)$ is an operation testing the values of clocks $\mathtt{X}^{\mathtt{p}}$ against the *test constraint* φ; $\mathsf{op} = \mathsf{reset}(\mathtt{x}^{\mathtt{p}})$ resets clock $\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}$ to zero; $\mathsf{op} = \mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi)$ sends message $\mathtt{m} \in \mathtt{M}$ to process $\mathtt{q}$ over channel $\mathtt{pq} \in \mathtt{C}$; the *send constraint* ψ over $\mathtt{X}^{\mathtt{p}} \cup \mathtt{X}^{\mathtt{pq}}$ specifies the initial values of channel clocks; $\mathsf{op} = \mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi)$ receives message $\mathtt{m} \in \mathtt{M}$ from process $\mathtt{q}$ via channel $\mathtt{qp} \in \mathtt{C}$; the *receive constraint* ψ over $\mathtt{X}^{\mathtt{p}} \cup \mathtt{X}^{\mathtt{qp}}$ specifies the final values of channel clocks. We write $\mathtt{p} \xrightarrow{\mathsf{op}_1; \ldots; \mathsf{op}_n} \mathtt{q}$ as syntactic sugar. We assume w.l.o.g. that test constraints φ's are atomic, that M is the maximal constant used in any inequality or modulo constraint, that all modular constraints $\equiv_M$ are over the same modulus M, that all the sets of local $\mathtt{X}^{\mathtt{P}} := (\mathtt{X}^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}$ and channel clocks $\mathtt{X}^{\mathtt{C}} := (\mathtt{X}^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}$ are disjoint, and similarly for the sets of locations $\mathtt{L}^{\mathtt{p}}$ and thus operations $\mathsf{Op}^{\mathtt{p}}$; consequently, we can just write $\ell \xrightarrow{\mathsf{op}} \not\ell$ without risk of confusion. A TCA has *untimed channels* if $\mathtt{X}^{\mathtt{C}} = \varnothing$. A channel $\mathtt{c} \in \mathtt{C}$ has *inequality tests* if there exists at least one operation $\mathsf{send}(\mathtt{c}, \mathtt{m} : \psi)$ or $\mathsf{receive}(\mathtt{c}, \mathtt{m} : \psi)$ where ψ is an inequality constraint $\mathtt{x}_0 \sim k$ or $\mathtt{x}_0 - \mathtt{x}_1 \sim k$ over (classical or integral) channel clocks $\mathtt{x}_0, \mathtt{x}_1 \in \mathtt{X}^{\mathtt{C}}$.

Semantics. A *channel valuation* is a family $w = (w^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}$ of sequences $w^{\mathtt{c}} \in (\mathtt{M} \times \mathbb{Q}_{\geqslant 0}^{\mathtt{X}^{\mathtt{c}}})^*$ of pairs $(\mathtt{m}, \mu)$, where $\mathtt{m}$ is a message and μ is a valuation for channel clocks in $\mathtt{X}^{\mathtt{c}}$. For $\delta \in \mathbb{Q}_{\geqslant 0}$, let $\mu + \delta$ be the clock valuation μ' s.t. $\mu'(\mathtt{x}) := \mu(\mathtt{x}) + \delta$, and for a channel valuation $w = (w^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}$ with $w^{\mathtt{c}} = (\gamma_1^{\mathtt{c}}, \mu_1^{\mathtt{c}}) \cdots (\gamma_{k_{\mathtt{c}}}^{\mathtt{c}}, \mu_{k_{\mathtt{c}}}^{\mathtt{c}})$ let $w + \delta = (w'^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}$ where $w'^{\mathtt{c}} = (\gamma_1^{\mathtt{c}}, \mu_1^{\mathtt{c}} + \delta) \cdots (\gamma_{k_{\mathtt{c}}}^{\mathtt{c}}, \mu_{k_{\mathtt{c}}}^{\mathtt{c}} + \delta)$. The semantics of a TCA $\mathcal{S}$ is the infinite LTS $[\![\mathcal{S}]\!] = \langle C, c_I, c_F, A, \rightarrow \rangle$, where C is the set of triples $\left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu, (w^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}} \right\rangle$ of control locations $\ell^{\mathtt{p}}$ for every process $\mathtt{p} \in \mathtt{P}$, a local clock valuation $\mu \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}}$, and channel valuations $w^{\mathtt{c}}$'s for every channel $\mathtt{c}$; the initial configuration is $c_I = \left\langle (\ell_I^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \overline{0}, (\varepsilon)_{\mathtt{c} \in \mathtt{C}} \right\rangle$, where $\ell_I^{\mathtt{p}}$ is the initial location of $\mathtt{p}$, all local clocks are initially 0, and all channels are initially empty; similarly, the final configuration is $c_F = \left\langle (\ell_F^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \overline{0}, (\varepsilon)_{\mathtt{c} \in \mathtt{C}} \right\rangle$; the set of actions is $A = \bigcup_{\mathtt{p} \in \mathtt{P}} \mathsf{Op}^{\mathtt{p}} \cup \mathbb{Q}_{\geqslant 0}$; for a duration $\delta \in \mathbb{Q}_{\geqslant 0}$ we have a transition

$$\left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu, u \right\rangle \xrightarrow{\delta} \left\langle (\not\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \nu, v \right\rangle \tag{†}$$

if *for all* processes $\mathtt{p}$ there is a time elapse transition $\ell^{\mathtt{p}} \xrightarrow{\mathsf{elapse}} \not\ell^{\mathtt{p}}$, $\nu = \mu + \delta$, and $v = u + \delta$. For $\mathsf{op} \in \mathsf{Op}^{\mathtt{p}}$, we have a transition $\left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu, u = (u^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}} \right\rangle \xrightarrow{\mathsf{op}} \left\langle (\not\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \nu, v = (v^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}} \right\rangle$ whenever $\mathtt{p}$ has a transition $\ell^{\mathtt{p}} \xrightarrow{\mathsf{op}} \not\ell^{\mathtt{p}}$, for every other process $\mathtt{q} \neq \mathtt{p}$ the control location $\not\ell^{\mathtt{q}} = \ell^{\mathtt{q}}$ stays the same, and ν, v are determined by op: if $\mathsf{op} = \mathsf{nop}$, then $\nu = \mu$, and $v = u$; if $\mathsf{op} = \mathsf{test}(\varphi)$, then $\mu \models \varphi$, $\nu = \mu$, and $v = u$; if $\mathsf{op} = \mathsf{reset}(\mathtt{x}^{\mathtt{p}})$, then $\nu = \mu[\mathtt{x}^{\mathtt{p}} \mapsto 0]$, and $v = u$; if $\mathsf{op} = \mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi)$, then $\nu = \mu$, there exists a valuation for clock channels $\mu^{\mathtt{pq}} \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}^{\mathtt{pq}}}$ s.t. $\mu \cup \mu^{\mathtt{pq}} \models \psi$, message $\mathtt{m}$ is added to this channel $v^{\mathtt{pq}} = (\mathtt{m}, \mu^{\mathtt{pq}}) \cdot u^{\mathtt{pq}}$, and every other channel $\mathtt{c} \in \mathtt{C} \setminus \{\mathtt{pq}\}$ is unchanged $v^{\mathtt{c}} = u^{\mathtt{c}}$; if $\mathsf{op} = \mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi)$, then $\nu = \mu$, message $\mathtt{m}$ is removed from this channel $u^{\mathtt{qp}} = v^{\mathtt{qp}} \cdot (\mathtt{m}, \mu^{\mathtt{qp}})$ provided that clock channels satisfy $\mu \cup \mu^{\mathtt{qp}} \models \psi$, and every other channel $\mathtt{c} \in \mathtt{C} \setminus \{\mathtt{qp}\}$ is unchanged $v^{\mathtt{c}} = u^{\mathtt{c}}$. TCA $\mathcal{S}, \mathcal{S}'$ are *equivalent* if the non-emptiness problem has the same answer for $[\![\mathcal{S}]\!]$, $[\![\mathcal{S}']\!]$.

3 Main result

We characterise completely which TCA topologies have a decidable non-emptiness problem.

▸ **Theorem 1.** *Non-emptiness of* TCA *is decidable if, and only if, the communication topology is a polyforest s.t. in each polytree there is at most one channel with inequality tests.*

▸ **Remark (Inequality vs. emptiness tests).** A similar characterisation for *untimed* channels showed that non-emptiness of discrete-time TCA is decidable iff the topology is a polyforest where in each polytree there is at most one channel which can be tested for emptiness [20]. Since a timed channel with inequality tests can simulate an untimed channel with emptiness tests, our decidability result generalises [20] to the more general case of timed channels, and our undecidability result follows from their characterisation. The simulation is done as follows. Process $\mathtt{q}$ tests whether the channel $\mathtt{pq}$ is empty with the cooperation of $\mathtt{p}$. Time instants are split between even and odd instants. All standard operations of $\mathtt{p}, \mathtt{q}$ are performed at odd instants. At even instants, $\mathtt{p}$ sends to $\mathtt{q}$ a special message $\hat{\mathtt{m}}$ with initial age 0 by $\mathsf{send}(\mathtt{pq}, \hat{\mathtt{m}} : \mathtt{x}^{\mathtt{pq}} = 0)$. Process $\mathtt{q}$ simulates an emptiness test on $\mathtt{pq}$ by $\mathsf{receive}(\mathtt{pq}, \hat{\mathtt{m}} : \mathtt{x}^{\mathtt{pq}} = 0)$.

The rest of the paper is devoted to the decidability proof. In Sec. 4 we simplify the form of constraints. In Sec. 5 we define a more flexible *desynchronised* semantics [32] for the elapse of time and then a more restrictive *rendezvous* semantics [38] for the exchange of messages. Applying these two semantics allows us to remove channels at the cost of introducing counters (cf. [20]). Fractional constraints are so far kept unchanged. In Sec. 6 we introduce *register automata with counters* (RAC) where registers are used to handle fractional values, and counters for integer values; we show that reachability is decidable for RAC. Finally, in Sec. 7 we simulate the rendezvous semantics of TCA by RAC, thus showing decidability of TCA.

4 Simple TCA

A TCA is *simple* if: it contains only integral and fractional clocks; send constraints are of the form $\mathtt{x}^{\mathtt{c}} = 0$ (for $\mathtt{x}^{\mathtt{c}}$ a channel clock); receive constraints of the form $\mathtt{x}^{\mathtt{c}} \sim k$, $\mathtt{x}^{\mathtt{c}} \equiv_M k$ for an integral clock $\mathtt{x}^{\mathtt{c}} : \mathbb{N}$, and of the form $\mathtt{y}^{\mathtt{pq}} = \mathtt{y}^{\mathtt{q}}$ for fractional clocks $\mathtt{y}^{\mathtt{pq}}, \mathtt{y}^{\mathtt{q}} : \mathbb{I}$. We present a non-emptiness preserving transformation of a given TCA into a simple one.

Remove integral clocks. For every integral clock $\mathtt{x} : \mathbb{N}$, we introduce a classical $\mathtt{x}_{\mathbb{Q}} : \mathbb{Q}$ and a fractional clock $\mathtt{x}_{\mathbb{I}} : \mathbb{I}$ which are reset at the same moment as $\mathtt{x}$. A constraint $\mathtt{x} - \mathtt{y} \leqslant k$ on clocks $\mathtt{x}, \mathtt{y} : \mathbb{N}$ is replaced by the equivalent $(\mathtt{x}_{\mathbb{Q}} - \mathtt{y}_{\mathbb{Q}} \leqslant k \wedge \mathtt{x}_{\mathbb{I}} \leqslant \mathtt{y}_{\mathbb{I}}) \vee \mathtt{x}_{\mathbb{Q}} - \mathtt{y}_{\mathbb{Q}} \leqslant k + 1$. The same technique can handle modulo constraints and channel clocks.

Copy-send. A TCA is *copy-send* if channel clocks are copies of local clocks of the sender process, i.e., $\mathtt{X}^{\mathtt{pq}} = \{\hat{\mathtt{x}}_i^{\mathtt{pq}} \mid \mathtt{x}_i^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}\}$, and the only send constraint of $\mathtt{p}$ is $\psi^{\mathtt{p}}_{\mathsf{copy}} \equiv \bigwedge_{\mathtt{x}_i^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}} \hat{\mathtt{x}}_i^{\mathtt{pq}} = \mathtt{x}_i^{\mathtt{p}}$.

▸ **Lemma 2.** *Non-emptiness of* TCA*'s reduces to non-emptiness of copy-send* TCA*'s.*

Proof. Let $\mathcal{S}$ be a TCA. We construct an equivalent copy-send TCA $\mathcal{S}'$ by letting sender processes $\mathtt{p}$'s send copies of their local clocks to receiver processes $\mathtt{q}$'s; the latter verifies at the time of reception whether there existed suitable initial values for channel clocks of $\mathcal{S}$. This transformation relies on the method of *quantifier elimination* to show that the guessing of the receiver processes $\mathtt{q}$ can be implemented as constraints. We perform the following transformation for every channel $\mathtt{pq} \in \mathtt{C}$. Let classical local and channel clocks be of the form $\mathtt{x}_i^{\mathtt{p}}, \mathtt{x}_i^{\mathtt{q}}, \mathtt{x}_i^{\mathtt{pq}} : \mathbb{Q}$, and fractional clocks of the form $\mathtt{y}_i^{\mathtt{p}}, \mathtt{y}_i^{\mathtt{q}}, \mathtt{y}_i^{\mathtt{pq}} : \mathbb{I}$. Consider a pair of transmission (of $\mathtt{p}$) and reception (of $\mathtt{q}$) transitions $t^{\mathtt{p}} = (\ell^{\mathtt{p}} \xrightarrow{\mathsf{send}(\mathtt{pq},\mathtt{m}:\psi^{\mathtt{p}})} \ell'^{\mathtt{p}})$ and $t^{\mathtt{q}} = (\ell^{\mathtt{q}} \xrightarrow{\mathsf{receive}(\mathtt{qp},\mathtt{m}:\psi^{\mathtt{q}})} \ell'^{\mathtt{q}})$,

$$\begin{aligned}
\psi^{\mathtt{p}} \equiv & \bigwedge_{(i,j)\in I^{\mathtt{p}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{p}} \sim_{ij}^{\mathtt{p}} k_{ij}^{\mathtt{p}} \ \wedge \bigwedge_{(i,j)\in I^{\mathtt{pq}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{pq}} \sim_{ij}^{\mathtt{pq}} k_{ij}^{\mathtt{pq}} \ \wedge && \text{(inequality)}\\
& \bigwedge_{(i,j)\in J^{\mathtt{p}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{p}} \equiv_M h_{ij}^{\mathtt{p}} \ \wedge \bigwedge_{(i,j)\in J^{\mathtt{pq}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{pq}} \equiv_M h_{ij}^{\mathtt{pq}} \ \wedge && \text{(modular)}\\
& \bigwedge_{(i,j)\in K^{\mathtt{p}}} \mathtt{y}_i^{\mathtt{pq}} \approx_{ij}^{\mathtt{p}} \mathtt{y}_j^{\mathtt{p}} \ \wedge \bigwedge_{(i,j)\in K^{\mathtt{pq}}} \mathtt{y}_i^{\mathtt{pq}} \approx_{ij}^{\mathtt{pq}} \mathtt{y}_j^{\mathtt{pq}}, \ \text{and} && \text{(order)}\\
\psi^{\mathtt{q}} \equiv & \bigwedge_{(i,j)\in I^{\mathtt{q}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{q}} \sim_{ij}^{\mathtt{q}} k_{ij}^{\mathtt{q}} \ \wedge \bigwedge_{(i,j)\in J^{\mathtt{q}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{q}} \equiv_M h_{ij}^{\mathtt{q}} \ \wedge \bigwedge_{(i,j)\in K^{\mathtt{q}}} \mathtt{y}_i^{\mathtt{pq}} \approx_{ij}^{\mathtt{q}} \mathtt{y}_j^{\mathtt{q}},
\end{aligned}$$

with $\sim_{ij}^{\mathtt{p}}, \sim_{ij}^{\mathtt{pq}}, \sim_{ij}^{\mathtt{q}}, \approx_{ij}^{\mathtt{p}}, \approx_{ij}^{\mathtt{pq}}, \approx_{ij}^{\mathtt{q}} \in \{<, \leqslant, \geqslant, >\}$, $I^{\mathtt{p}}, I^{\mathtt{pq}}, J^{\mathtt{p}}, J^{\mathtt{pq}}, K^{\mathtt{p}}, K^{\mathtt{pq}}, I^{\mathtt{q}}, J^{\mathtt{q}}, K^{\mathtt{q}}$ sets of pairs of clock indices, and $k_{ij}^{\mathtt{p}}, k_{ij}^{\mathtt{pq}}, h_{ij}^{\mathtt{p}}, h_{ij}^{\mathtt{pq}}, k_{ij}^{\mathtt{q}}, h_{ij}^{\mathtt{q}} \in \mathbb{Z}$ integer constants. (It suffices to consider diagonal constraints since non-diagonal ones can be simulated. We don't consider reception constraints on $\mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{pq}}$ since they are invariant under time elapse and can be checked directly at the time of transmission; thence the asymmetry between $\psi^{\mathtt{p}}$ and $\psi^{\mathtt{q}}$.) In the new copy-send TCA $\mathcal{S}'$, we have a classical channel clock $\hat{\mathtt{x}}_i^{\mathtt{pq}} : \mathbb{Q}$ for every classical local clock $\mathtt{x}_i^{\mathtt{p}} : \mathbb{Q}$ of $\mathtt{p}$, and similarly a new fractional clock $\hat{\mathtt{y}}_i^{\mathtt{pq}} : \mathbb{I}$ for every $\mathtt{y}_i^{\mathtt{p}} : \mathbb{I}$. Let μ, ν be clock valuations at the time of transmission and reception, respectively. The initial value of $\hat{\mathtt{x}}_i^{\mathtt{pq}}$ is $\mu(\hat{\mathtt{x}}_i^{\mathtt{pq}}) = \mu(\mathtt{x}_i^{\mathtt{p}})$. We assume the existence of two special clocks $\mathtt{x}_0^{\mathtt{p}} : \mathbb{Q}, \mathtt{y}_0^{\mathtt{p}} : \mathbb{I}$ which are always zero upon send, i.e., $\mu(\mathtt{x}_0^{\mathtt{p}}) = \mu(\hat{\mathtt{x}}_0^{\mathtt{pq}}) = \mu(\mathtt{y}_0^{\mathtt{p}}) = \mu(\hat{\mathtt{y}}_0^{\mathtt{pq}}) = 0$, and thus when the message is received $\nu(\hat{\mathtt{x}}_0^{\mathtt{pq}}), \nu(\hat{\mathtt{y}}_0^{\mathtt{pq}})$ equal the total integer, resp., fractional time that elapsed between transmission and reception. This allows us to recover, at reception time, the initial value of local clocks $\mu(\mathtt{x}_i^{\mathtt{p}}), \mu(\mathtt{y}_i^{\mathtt{p}})$ and the final value of channel clocks $\nu(\mathtt{x}_i^{\mathtt{pq}}), \nu(\mathtt{y}_i^{\mathtt{pq}})$ as follows:

$$\mu(\mathtt{x}_i^{\mathtt{p}}) = \nu(\hat{\mathtt{x}}_i^{\mathtt{pq}}) - \nu(\hat{\mathtt{x}}_0^{\mathtt{pq}}), \qquad \nu(\mathtt{x}_i^{\mathtt{pq}}) = \mu(\mathtt{x}_i^{\mathtt{pq}}) + \nu(\hat{\mathtt{x}}_0^{\mathtt{pq}}), \tag{1}$$

$$\mu(\mathtt{y}_i^{\mathtt{p}}) = \nu(\hat{\mathtt{y}}_i^{\mathtt{pq}}) \ominus \nu(\hat{\mathtt{y}}_0^{\mathtt{pq}}), \qquad \nu(\mathtt{y}_i^{\mathtt{pq}}) = \mu(\mathtt{y}_i^{\mathtt{pq}}) \oplus \nu(\hat{\mathtt{y}}_0^{\mathtt{pq}}). \tag{2}$$

We replace transitions $t^{\mathtt{p}}, t^{\mathtt{q}}$ with $\ell^{\mathtt{p}} \xrightarrow{\mathsf{send}(\mathtt{pq}, \langle \mathtt{m}, \psi^{\mathtt{p}}, \psi^{\mathtt{q}} \rangle : \psi_{\mathsf{copy}}^{\mathtt{p}})} \ell'^{\mathtt{p}}$, resp., $\ell^{\mathtt{q}} \xrightarrow{\mathsf{receive}(\mathtt{qp}, \langle \mathtt{m}, \psi^{\mathtt{p}}, \psi^{\mathtt{q}} \rangle : \psi_0^{\mathtt{q}})} \ell'^{\mathtt{q}}$, where the original message $\mathtt{m}$ is replaced by $\langle \mathtt{m}, \psi^{\mathtt{p}}, \psi^{\mathtt{q}} \rangle$ (thus guessing and verifying the correct pair of send-receive constraints $\psi^{\mathtt{p}}, \psi^{\mathtt{q}}$), the send constraint is the copy constraint $\psi_{\mathsf{copy}}^{\mathtt{p}}$, and the new reception formula is $\psi_0^{\mathtt{q}} \equiv \exists \bar{\mathtt{x}}^{\mathtt{pq}}, \bar{\mathtt{y}}^{\mathtt{pq}} \cdot \psi'^{\mathtt{p}} \wedge \psi'^{\mathtt{q}}$, where, following (1), (2), $\psi'^{\mathtt{p}}$ is obtained from $\psi^{\mathtt{p}}$ by performing the substitution $\mathtt{x}_i^{\mathtt{p}} \mapsto \hat{\mathtt{x}}_i^{\mathtt{pq}} - \hat{\mathtt{x}}_0^{\mathtt{pq}}$, $\mathtt{y}_i^{\mathtt{p}} \mapsto \hat{\mathtt{y}}_i^{\mathtt{pq}} \ominus \hat{\mathtt{x}}_0^{\mathtt{pq}}$, and $\psi'^{\mathtt{q}}$ from $\psi^{\mathtt{q}}$ by $\mathtt{x}_i^{\mathtt{pq}} \mapsto \mathtt{x}_i^{\mathtt{pq}} + \hat{\mathtt{x}}_0^{\mathtt{pq}}$, $\mathtt{y}_i^{\mathtt{pq}} \mapsto \mathtt{y}_i^{\mathtt{pq}} \oplus \hat{\mathtt{x}}_0^{\mathtt{pq}}$. We can rearrange the conjuncts as $\psi_0^{\mathtt{q}} \equiv (\exists \bar{\mathtt{x}}^{\mathtt{pq}} \cdot \psi_{\bar{\mathtt{x}}^{\mathtt{pq}}}^{\mathtt{q}}) \wedge (\exists \bar{\mathtt{y}}^{\mathtt{pq}} \cdot \psi_{\bar{\mathtt{y}}^{\mathtt{pq}}}^{\mathtt{q}})$, where

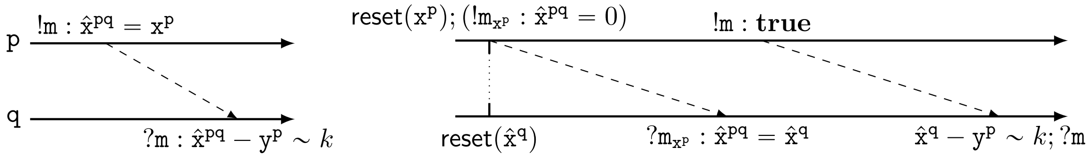
$$\begin{aligned}
\psi_{\bar{\mathtt{x}}^{\mathtt{pq}}}^{\mathtt{q}} \ \equiv & \bigwedge_{(i,j)\in I^{\mathtt{p}}} \mathtt{x}_i^{\mathtt{pq}} - (\hat{\mathtt{x}}_j^{\mathtt{pq}} - \hat{\mathtt{x}}_0^{\mathtt{pq}}) \sim_{ij}^{\mathtt{p}} k_{ij}^{\mathtt{p}} \wedge \bigwedge_{(i,j)\in I^{\mathtt{pq}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{pq}} \sim_{ij}^{\mathtt{pq}} k_{ij}^{\mathtt{pq}} \wedge \bigwedge_{(i,j)\in I^{\mathtt{q}}} (\mathtt{x}_i^{\mathtt{pq}} + \hat{\mathtt{x}}_0^{\mathtt{pq}}) - \mathtt{x}_j^{\mathtt{q}} \sim_{ij}^{\mathtt{q}} k_{ij}^{\mathtt{q}} \wedge \\
& \bigwedge_{(i,j)\in J^{\mathtt{p}}} \mathtt{x}_i^{\mathtt{pq}} - (\hat{\mathtt{x}}_j^{\mathtt{pq}} - \hat{\mathtt{x}}_0^{\mathtt{pq}}) \equiv_M h_{ij}^{\mathtt{p}} \wedge \bigwedge_{(i,j)\in J^{\mathtt{pq}}} \mathtt{x}_i^{\mathtt{pq}} - \mathtt{x}_j^{\mathtt{pq}} \equiv_M h_{ij}^{\mathtt{pq}} \wedge \bigwedge_{(i,j)\in J^{\mathtt{q}}} (\mathtt{x}_i^{\mathtt{pq}} + \hat{\mathtt{x}}_0^{\mathtt{pq}}) - \mathtt{x}_j^{\mathtt{q}} \equiv_M h_{ij}^{\mathtt{q}} \\
\psi_{\bar{\mathtt{y}}^{\mathtt{pq}}}^{\mathtt{q}} \ \equiv & \bigwedge_{(i,j)\in K^{\mathtt{p}}} \mathtt{y}_i^{\mathtt{pq}} \approx_{ij}^{\mathtt{p}} \hat{\mathtt{y}}_j^{\mathtt{pq}} \ominus \hat{\mathtt{y}}_0^{\mathtt{pq}} \wedge \bigwedge_{(i,j)\in K^{\mathtt{pq}}} \mathtt{y}_i^{\mathtt{pq}} \approx_{ij}^{\mathtt{pq}} \mathtt{y}_j^{\mathtt{pq}} \wedge \bigwedge_{(i,j)\in K^{\mathtt{q}}} \mathtt{y}_i^{\mathtt{pq}} \oplus \hat{\mathtt{y}}_0^{\mathtt{pq}} \approx_{ij}^{\mathtt{q}} \mathtt{y}_j^{\mathtt{q}}.
\end{aligned}$$

The formula $\psi_0^{\mathtt{q}}$ above is not a clock constraint due to the quantifiers. Thanks to quantifier elimination, we show that it is equivalent to a quantifier-free formula $\widetilde{\psi}^{\mathtt{q}}$, i.e., a constraint.

Classical clocks. We show that $\psi_1^{\mathtt{q}} \equiv \exists \bar{\mathtt{x}}^{\mathtt{pq}} \cdot \psi_{\bar{\mathtt{x}}^{\mathtt{pq}}}^{\mathtt{q}}$ is equivalent to a quantifier-free formula $\widetilde{\psi}_{\bar{\mathtt{x}}^{\mathtt{pq}}}^{\mathtt{q}}$. By highlighting $\mathtt{x}_1^{\mathtt{pq}}$, we can put $\psi_1^{\mathtt{q}}$ in the form (we avoid the indices for readability)

$$\psi_1^{\mathtt{q}} \ \equiv \ \exists \bar{\mathtt{x}}^{\mathtt{pq}} \cdot \psi' \wedge \bigwedge u \lesssim \mathtt{x}_1^{\mathtt{pq}} \wedge \bigwedge \mathtt{x}_1^{\mathtt{pq}} \lesssim v \wedge \bigwedge \mathtt{x}_1^{\mathtt{pq}} \equiv_M t,$$

where ψ' does not contain $\mathtt{x}_1^{\mathtt{pq}}$, the u, v's are of one of the three types: $(I^{\mathtt{p}})$ $k_{1j}^{\mathtt{p}} + \hat{\mathtt{x}}_j^{\mathtt{pq}} - \hat{\mathtt{x}}_0^{\mathtt{pq}}$, $(I^{\mathtt{pq}})$ $k_{1j}^{\mathtt{pq}} + \mathtt{x}_j^{\mathtt{pq}}$, or $(I^{\mathtt{q}})$ $k_{1j}^{\mathtt{q}} + \mathtt{x}_j^{\mathtt{q}} - \hat{\mathtt{x}}_0^{\mathtt{pq}}$, and similarly the t's are of one of the three types


p
q
$!\mathtt{m} : \hat{\mathtt{x}}^{\mathtt{pq}} = \mathtt{x}^{\mathtt{p}}$
$?\mathtt{m} : \hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{y}^{\mathtt{p}} \sim k$
$\mathsf{reset}(\mathtt{x}^{\mathtt{p}}); (!\mathtt{m}_{\mathtt{x}^{\mathtt{p}}} : \hat{\mathtt{x}}^{\mathtt{pq}} = 0)$
$!\mathtt{m} : \mathbf{true}$
$\mathsf{reset}(\hat{\mathtt{x}}^{\mathtt{q}})$
$?\mathtt{m}_{\mathtt{x}^{\mathtt{p}}} : \hat{\mathtt{x}}^{\mathtt{pq}} = \hat{\mathtt{x}}^{\mathtt{q}}$
$\hat{\mathtt{x}}^{\mathtt{q}} - \mathtt{y}^{\mathtt{p}} \sim k; ?\mathtt{m}$

(a) $\mathtt{p}$ sends $\mathtt{m}$ to $\mathtt{q}$.

(b) $\mathtt{q}$ guesses the last reset of $\mathtt{x}^{\mathtt{p}}$ before $!\mathtt{m}$.

Figure 1 Channel constraints of the form $\hat{\mathtt{x}}^{\mathtt{pq}} = 0$ (transmission) and $\hat{\mathtt{x}}^{\mathtt{pq}} = \hat{\mathtt{x}}^{\mathtt{q}}$ (reception) suffice.

$(J^{\mathtt{p}})$ $h^{\mathtt{p}}_{1j} + \hat{\mathtt{x}}^{\mathtt{pq}}_j - \hat{\mathtt{x}}^{\mathtt{pq}}_0$, $(J^{\mathtt{pq}})$ $h^{\mathtt{pq}}_{1j} + \mathtt{x}^{\mathtt{pq}}_j$, or $(J^{\mathtt{q}})$ $h^{\mathtt{q}}_{1j} + \mathtt{x}^{\mathtt{q}}_j - \hat{\mathtt{x}}^{\mathtt{pq}}_0$. We can now eliminate the existential quantifier on $\mathtt{x}^{\mathtt{pq}}_1$ and obtain the equivalent formula $\psi^{\mathtt{q}}_2 \equiv \exists \mathtt{x}^{\mathtt{pq}}_2 \cdots \mathtt{x}^{\mathtt{pq}}_n \cdot \psi' \wedge \bigwedge u \lesssim v \wedge \bigwedge t \equiv_M t'$. Atomic formulas $u \lesssim v$ in $\psi^{\mathtt{q}}_2$ are again of the same types as above: If $u : (I^{\mathtt{p}}), v : (I^{\mathtt{pq}})$, then $v - u : (I^{\mathtt{p}})$. If $u, v : (I^{\mathtt{pq}})$, then $v - u : I^{\mathtt{pq}}$. If $u : (I^{\mathtt{q}}), v : (I^{\mathtt{pq}})$, then $v - u : (I^{\mathtt{q}})$. In any other case, i.e., if $u : (I^{\mathtt{p}}), (I^{\mathtt{q}})$ and $v : (I^{\mathtt{p}}), (I^{\mathtt{q}})$, then $u \lesssim v$ is already a constraint not containing any $\mathtt{x}^{\mathtt{pq}}_i$'s ($\hat{x}^{\mathtt{pq}}_0$ appears on both side of each inequality and we can remove it) and thus does not participate anymore in the quantifier elimination process. The same reasoning applies to modulo constraints. We can thus repeat this process for the other variables $\mathtt{x}^{\mathtt{pq}}_2, \ldots, \mathtt{x}^{\mathtt{pq}}_n$, and we finally get a constraint equivalent to $\psi^{\mathtt{q}}_1$ of the form $\psi^{\mathtt{q}}_n \equiv \bigwedge u \lesssim v \wedge \bigwedge t \equiv_M t'$, where the u, v's are of the form $h^{\mathtt{p}}_{1j} + \hat{\mathtt{x}}^{\mathtt{pq}}_j$ or $k^{\mathtt{q}}_{1j} + \mathtt{x}^{\mathtt{q}}_j$, and similarly the t, t''s are of the form $h^{\mathtt{p}}_{1j} + \hat{\mathtt{x}}^{\mathtt{pq}}_j$ or $h^{\mathtt{q}}_{1j} + \mathtt{x}^{\mathtt{q}}_j$. Thus, $\psi^{\mathtt{q}}_n$ is the constraint $\widetilde{\psi}^{\mathtt{q}}_{\overline{\mathtt{x}}^{\mathtt{pq}}}$ we are after; it speaks only about new channel clocks $\hat{\mathtt{x}}^{\mathtt{pq}}_j$'s and local $\mathtt{q}$-clocks $\mathtt{x}^{\mathtt{q}}_j$'s.

Fractional clocks. With a similar argument we can show that $\exists \overline{\mathtt{y}}^{\mathtt{pq}} \cdot \psi^{\mathtt{q}}_{\overline{\mathtt{y}}^{\mathtt{pq}}}$ is equivalent to a quantifier-free formula $\widetilde{\psi}^{\mathtt{q}}_{\overline{\mathtt{y}}^{\mathtt{pq}}}$; the details are presented in App. A.1. To conclude, we have shown that the reception formula $\psi^{\mathtt{q}}_0$ is equivalent to the constraint $\widetilde{\psi}^{\mathtt{q}}_{\overline{\mathtt{x}}^{\mathtt{pq}}} \wedge \widetilde{\psi}^{\mathtt{q}}_{\overline{\mathtt{y}}^{\mathtt{pq}}}$, as required. ◂

Atomic channel constraints $\hat{\mathtt{x}}^{\mathtt{pq}} = \mathtt{x}^{\mathtt{p}}$, $\hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{x}^{\mathtt{q}} \sim k$, $\hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{x}^{\mathtt{q}} \equiv_M k$, $\hat{\mathtt{y}}^{\mathtt{pq}} \sim \mathtt{y}^{\mathtt{q}}$. Cf. App. A.1.

Atomic channel constraints $\hat{\mathtt{x}}^{\mathtt{pq}} = 0$, $\hat{\mathtt{x}}^{\mathtt{pq}} = \hat{\mathtt{x}}^{\mathtt{q}}$. We further simplify atomic channel constraints by only sending channel clocks $\hat{\mathtt{x}}^{\mathtt{pq}}$ initialised to 0, and having receive constraints of the form of equalities $\hat{\mathtt{x}}^{\mathtt{pq}} = \hat{\mathtt{x}}^{\mathtt{q}}$ between a channel and a local clock; this holds for both classical and fractional clocks. Consider a send/receive pair (S) $\ell^{\mathtt{p}} \xrightarrow{\mathsf{send}(\mathtt{pq}, \mathtt{m} : \hat{\mathtt{x}}^{\mathtt{pq}} = \mathtt{x}^{\mathtt{p}})} \ell'^{\mathtt{p}}$ and (R) $\ell^{\mathtt{q}} \xrightarrow{\mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi^{\mathtt{q}})} \ell'^{\mathtt{q}}$, where $\mathtt{x}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{pq}}$ are either classical or fractional clocks, and $\psi^{\mathtt{q}}$ is an atomic constraint of the form $\hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{y}^{\mathtt{q}} \sim k$ or $\hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{y}^{\mathtt{q}} \equiv_M k$ for classical clocks, or $\hat{\mathtt{x}}^{\mathtt{pq}} \sim \mathtt{y}^{\mathtt{q}}$ for fractional clocks; cf. Fig 1. Process $\mathtt{p}$ communicates to $\mathtt{q}$ every time clock $\mathtt{x}^{\mathtt{p}}$ is reset by replacing every reset $\ell^{\mathtt{p}}_0 \xrightarrow{\mathsf{reset}(\mathtt{x}_{\mathtt{p}})} \ell'^{\mathtt{p}}_0$ with $\ell^{\mathtt{p}}_0 \xrightarrow{\mathsf{reset}(\mathtt{x}^{\mathtt{p}}); \mathsf{send}(\mathtt{pq}, \mathtt{m}_{\mathtt{x}^{\mathtt{p}}} : \hat{\mathtt{x}}^{\mathtt{pq}} = 0)} \ell'^{p}_0$ where after the reset $\mathtt{p}$ sends a special message $\mathtt{m}_{\mathtt{x}^{\mathtt{p}}}$ to $\mathtt{q}$ with initial age 0. To simplify the rest of the construction, we assume that $\mathtt{p}$ has a preliminary initialisation phase whereby all of its local clocks are reset (this guarantees that at least one reset message above is sent for every clock). We add to process $\mathtt{q}$ a copy $\hat{\mathtt{x}}^{\mathtt{q}}$ of every clock $\mathtt{x}^{\mathtt{p}}$ of $\mathtt{p}$; let $\hat{\mathtt{X}}^{\mathtt{q}}$ be the set of these new clocks $\hat{\mathtt{x}}^{\mathtt{q}}$'s. Process $\mathtt{q}$ guesses the last reset of $\mathtt{x}^{\mathtt{p}}$ before the transmission (S) by resetting its corresponding local clock $\hat{\mathtt{x}}^{\mathtt{q}}$ and later verifying the guess by receiving message $\mathtt{m}_{\mathtt{x}^{\mathtt{p}}}$ with age equal to $\hat{\mathtt{x}}^{\mathtt{q}}$. Control locations of $\mathtt{q}$ are now of the form $(\ell^{\mathtt{q}}, \mathtt{Y})$, where $\mathtt{Y} \subseteq \hat{\mathtt{X}}^{\mathtt{q}}$ is the set of new clocks $\hat{\mathtt{x}}^{\mathtt{q}}$'s for which the reset has correctly been verified. Initially, $\mathtt{Y} = \varnothing$, i.e., no guess has been verified. For every location $(\ell^{\mathtt{q}}, \mathtt{Y})$ of $\mathtt{q}$ we have a transition $(\ell^{\mathtt{q}}, \mathtt{Y}) \xrightarrow{\mathsf{reset}(\hat{\mathtt{x}}^{\mathtt{q}})} (\ell^{\mathtt{q}}, \mathtt{Y} \setminus \{\hat{\mathtt{x}}^{\mathtt{q}}\})$ that allows $\mathtt{q}$ to reset $\hat{\mathtt{x}}^{\mathtt{q}}$; removing $\hat{\mathtt{x}}^{\mathtt{q}}$ from $\mathtt{Y}$ enforces that this reset must later be verified. For every control location $(\ell^{\mathtt{q}}, \mathtt{Y})$ of $\mathtt{q}$ s.t. $\hat{\mathtt{x}}^{\mathtt{q}} \notin \mathtt{Y}$ needs to be verified, we have a transition $(\ell^{\mathtt{q}}, \mathtt{Y}) \xrightarrow{\mathsf{receive}(\mathtt{qp}, \mathtt{m}_{\mathtt{x}^{\mathtt{p}}} : \hat{\mathtt{x}}^{\mathtt{q}} = \hat{\mathtt{x}}^{\mathtt{pq}})} (\ell^{\mathtt{q}}, \mathtt{Y} \cup \{\hat{\mathtt{x}}^{\mathtt{q}}\})$ which checks that $\hat{\mathtt{x}}^{\mathtt{q}}$ correctly guessed the last reset of

$\mathtt{x}^{\mathtt{p}}$, and a transition $(\ell^{\mathtt{q}}, \mathtt{Y}) \xrightarrow{\mathsf{receive}(\mathtt{qp},\mathtt{m}_{\mathtt{x}^{\mathtt{p}}}:\mathbf{true})} (\ell^{\mathtt{q}}, \mathtt{Y})$ which allows to drop the control messages $\mathtt{m}_{\mathtt{x}^{\mathtt{p}}}$ corresponding to previous resets of $\mathtt{x}^{\mathtt{p}}$ which the automaton guesses to be overridden by a more recent reset, and thus are no longer relevant. Once $\hat{\mathtt{x}}^{\mathtt{q}} \in \mathtt{Y}$, no receptions of $\mathtt{m}_{\mathtt{x}^{\mathtt{p}}}$ are allowed: This ensures that $\mathtt{q}$ indeed correctly guessed the *last* reset of $\mathtt{x}^{\mathtt{p}}$ before the original message $\mathtt{m}$ is sent. The original send transition (S) becomes $\ell^{\mathtt{p}} \xrightarrow{\mathsf{send}(\mathtt{pq},\mathtt{m}:\mathbf{true})} \not\ell^{\mathtt{p}}$ with the trivial timing constraint **true**, and the original receive transition (R) becomes an untimed reception $(\ell^{\mathtt{q}}, \mathtt{Y}) \xrightarrow{\mathsf{test}(\widetilde{\psi}^{\mathtt{q}});\mathsf{receive}(\mathtt{qp},\mathtt{m}:\mathbf{true})} (\not\ell^{\mathtt{q}}, \mathtt{Y})$ with $\hat{\mathtt{x}}^{\mathtt{q}} \in \mathtt{Y}$, together with a test on local clocks $\widetilde{\psi}^{\mathtt{q}} \equiv \hat{\mathtt{x}}^{\mathtt{q}} - \mathtt{y}^{\mathtt{q}} \sim k$ or, resp., $\widetilde{\psi}^{\mathtt{q}} \equiv \hat{\mathtt{x}}^{\mathtt{q}} - \mathtt{y}^{\mathtt{q}} \equiv_M k$ for classical clocks, or $\widetilde{\psi}^{\mathtt{q}} \equiv \hat{\mathtt{x}}^{\mathtt{q}} \sim \mathtt{y}^{\mathtt{q}}$ for fractional clocks. Constraint $\widetilde{\psi}^{\mathtt{q}}$ is now a test on local $\mathtt{q}$-clocks.

Atomic channel constraints $\hat{\mathtt{x}}^{\mathtt{pq}} = 0$, $\hat{\mathtt{x}}^{\mathtt{pq}} \sim k$, $\hat{\mathtt{x}}^{\mathtt{pq}} \equiv_M k$, $\hat{\mathtt{y}}^{\mathtt{pq}} = \hat{\mathtt{y}}^{\mathtt{q}}$. First, we eliminate local diagonal constraints $\mathtt{x}^{\mathtt{q}} - \mathtt{y}^{\mathtt{q}} \sim k, \mathtt{x}^{\mathtt{q}} - \mathtt{y}^{\mathtt{q}} \equiv_M k$ for classical clocks $\mathtt{x}^{\mathtt{q}}, \mathtt{y}^{\mathtt{q}} : \mathbb{Q}$ by their non-diagonal counterparts $\mathtt{x}^{\mathtt{q}} \sim k, \mathtt{x}^{\mathtt{q}} \equiv_M k$ [8]. By the previous part, receive channel classical constraints are of the form $\hat{\mathtt{x}}^{\mathtt{pq}} = \hat{\mathtt{x}}^{\mathtt{q}}$, and since now the local clock $\hat{\mathtt{x}}^{\mathtt{q}}$ participates only in non-diagonal constraints, what only matters is that $\hat{\mathtt{x}}^{\mathtt{pq}}$ and $\hat{\mathtt{x}}^{\mathtt{q}}$ are threshold equivalent for inequality constraints, and modulo equivalent for modular constraints. Two clock valuations μ, ν are *M-threshold equivalent*, written $\mu \approx_{\mathtt{M}} \nu$ if, for every $\mathtt{x} \in \mathtt{X}^{\mathtt{P}}$, $\mu(\mathtt{x}) = \nu(\mathtt{x})$ if $\mu(\mathtt{x}), \nu(\mathtt{x}) \leqslant M$, and $\mu(\mathtt{x}) \geqslant M$ iff $\nu(\mathtt{x}) \geqslant M$. Clearly, if $\mu \approx_{\mathtt{M}} \nu$, then $\mu \models \varphi$ iff $\nu \models \varphi$ for every constraint $\varphi \equiv \mathtt{x} \sim k$ using constants $k \leqslant M$. We can check that $\mathtt{x}, \mathtt{y}$ belong to the same M-threshold equivalence class with the non-diagonal inequality constraint $\varphi_{\approx_{\mathtt{M}}}(\mathtt{x}, \mathtt{y}) \equiv \bigvee_{k \in \{0,\ldots,M\}} (\mathtt{x} = k \wedge \mathtt{y} = k \;\vee\; \mathtt{x} \geqslant M \wedge \mathtt{y} \geqslant M)$. We handle modulo constraints in the same spirit. Two clock valuations μ, ν are *M-modulo equivalent*, written $\mu \equiv_M \nu$ if, for every $\mathtt{x} \in \mathtt{X}^{\mathtt{P}}$, $\mu(\mathtt{x}) \equiv_M \nu(\mathtt{x})$. Clearly, if $\mu \equiv_M \nu$, then $\mu \models \varphi$ iff $\nu \models \varphi$ for every constraint $\varphi \equiv (\mathtt{x} \equiv_M k)$. Moreover, we can check that $\mathtt{x}, \mathtt{y}$ belong to the same M-modulo equivalence class with the non-diagonal modular constraint $\varphi_{\equiv_M}(\mathtt{x}, \mathtt{y}) \equiv \bigvee_{k \in \{0,\ldots,M-1\}} (\mathtt{x} \equiv_M k \wedge \mathtt{y} \equiv_M k)$. Our objective is achieved by replacing classical diagonal reception constraints $\hat{\mathtt{x}}^{\mathtt{pq}} = \hat{\mathtt{x}}^{\mathtt{q}}$ with the non-diagonal $\varphi_{\approx_{\mathtt{M}}}(\hat{\mathtt{x}}^{\mathtt{pq}}, \hat{\mathtt{x}}^{\mathtt{q}}) \wedge \varphi_{\equiv_{\mathtt{M}}}(\hat{\mathtt{x}}^{\mathtt{pq}}, \hat{\mathtt{x}}^{\mathtt{q}})$. Fractional constraints are untouched in this step.

Remove classical clocks. We convert all constraints on classical clocks into equivalent constraints on integral and fractional clocks, thus undoing the first step of this section. For every classical clock $\mathtt{x} : \mathbb{Q}$, we introduce an integral $\mathtt{x}_{\mathbb{N}} : \mathbb{N}$ and a fractional clock $\mathtt{x}_{\mathbb{I}} : \mathbb{I}$ which are reset at the same moment as $\mathtt{x}$. Constraints of the form $\mathtt{x} < k$ are replaced with $\mathtt{x}_{\mathbb{N}} < k$, of the form $\mathtt{x} = k$ by $\mathtt{x}_{\mathbb{N}} = k \wedge \mathtt{x}_{\mathbb{I}} = 0$, and of the form $\mathtt{x} > k$ by $\mathtt{x}_{\mathbb{N}} \geqslant k+1 \vee (\mathtt{x}_{\mathbb{N}} \geqslant k \wedge \mathtt{x}_{\mathbb{I}} > 0)$. It is easy to see that we obtain simple constraints, as required.

5 Desynchronised and rendezvous semantics

Desynchronised semantics. We introduce an alternative run-preserving semantics for TCA, called *desynchronised semantics*, where time elapse transitions are *local* within processes; channels $\mathtt{pq}$'s elapse time together with receiving processes $\mathtt{q}$'s. In order to guarantee that messages are received only after they are sent, for every channel $\mathtt{pq}$ we allow $\mathtt{q}$ to be ahead of $\mathtt{p}$, but not the other way around. We make no assumptions on the underlying topology. Let $\mathcal{S} = \left\langle \mathcal{T} = \langle \mathtt{P}, \mathtt{C} \rangle, \mathtt{M}, (\mathtt{X}^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}, (\mathcal{A}^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}} \right\rangle$ be a TCA. For every process $\mathtt{p} \in \mathtt{P}$ there is a special clock $\mathtt{x}_0^{\mathtt{p}}$ which is never reset. The *desynchronised semantics* is the LTS $[\![\mathcal{S}]\!]^{\mathsf{de}} = \left\langle C^{\mathsf{de}}, c_I, c_F^{\mathsf{de}}, A, \rightarrow^{\mathsf{de}} \right\rangle$ where everything is defined as in the standard semantics $[\![S]\!] = \langle C, c_I, c_F, A, \rightarrow \rangle$, except C^{de}, which is defined as $C^{\mathsf{de}} = \left\{ \left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu, u \right\rangle \in C \;\middle|\; \forall \mathtt{pq} \in \mathtt{C} \cdot \mu(\mathtt{x}_0^{\mathtt{p}}) \leqslant \mu(\mathtt{x}_0^{\mathtt{q}}) \right\}$, the final configuration is $c_F^{\mathsf{de}} = \left\langle (\ell_F^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu_1, (\varepsilon)_{\mathtt{c} \in \mathtt{C}} \right\rangle$ where $\mu_1(\mathtt{x}^{\mathtt{p}}) = 0$ for every $\mathtt{x} \in \mathtt{X} \backslash \{\mathtt{x}_0^{\mathtt{p}} \mid \mathtt{p} \in \mathtt{P}\}$, and for the desynchronised transition relation $\rightarrow^{\mathsf{de}}$, which is defined as $\rightarrow$, except for the rules for time elapse and transmissions. For time elapse, (†) is replaced by $\left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu, (u^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}} \right\rangle \xrightarrow{\delta}$

$\left\langle(\ell'^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \nu, (v^{\mathtt{c}})_{\mathtt{c}\in\mathtt{C}}\right\rangle$ whenever *there exists* a process $\mathtt{q}\in\mathtt{P}$ s.t. there is a time elapse transition $\ell^{\mathtt{q}} \xrightarrow{\mathsf{elapse}} \ell'^{\mathtt{q}}$, $\nu|_{\mathtt{X}^{\mathtt{q}}} = \mu|_{\mathtt{X}^{\mathtt{q}}} + \delta$, $v^{\mathtt{pq}} = u^{\mathtt{pq}} + \delta$ for every channel $\mathtt{pq}\in C$, for every other process $\mathtt{p}\neq\mathtt{q}$, $\ell'^{\mathtt{p}} = \ell^{\mathtt{p}}$, $\nu|_{\mathtt{X}^{\mathtt{p}}} = \mu|_{\mathtt{X}^{\mathtt{p}}}$, and $v^{\mathtt{c}} = u^{\mathtt{c}}$ for every channel $\mathtt{c}$ not of the form $\mathtt{pq}$. For transmissions: if $\mathsf{op} = \mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi)$, $\nu = \mu$, then there exists a valuation for channel clocks $\mu^{\mathtt{pq}} \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}^{\mathtt{pq}}}$ s.t. $(\mu, \mu^{\mathtt{pq}}) \models \psi$, $v^{\mathtt{pq}} = (\mathtt{m}, \mu^{\mathtt{pq}} + \delta)\cdot u^{\mathtt{pq}}$ where we additionally increase the initial valuation $\mu^{\mathtt{pq}}$ by the desynchronisation $\delta := \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \geqslant 0$; every other channel $\mathtt{c}\in\mathtt{C}\setminus\{\mathtt{pq}\}$ is unchanged $v^{\mathtt{c}} = u^{\mathtt{c}}$. Correctness is proved as in [32, Lemma 1] and [20, Proposition 1].

▸ **Lemma 3.** *The standard semantics $[\![S]\!]$ is equivalent to the desynchronised semantics $[\![S]\!]^{\mathsf{de}}$.*

Rendezvous semantics. The main advantage of the desynchronised semantics introduced in the previous section is that, over polyforest topologies, channel operations can be scheduled as too keep the channels always empty. Moreover, doing this preserves the existence of runs. This is formalised via the following *rendezvous semantics*: For a TCA $\mathcal{S} = \left\langle\mathcal{T} = \langle\mathtt{P},\mathtt{C}\rangle, \mathtt{M}, (\mathtt{X}^{\mathtt{c}})_{\mathtt{c}\in\mathtt{C}}, (\mathcal{A}^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}\right\rangle$ define its rendezvous semantics $[\![\mathcal{S}]\!]^{\mathsf{rv}} = \langle C^{\mathsf{rv}}, c_I, c_F, A^{\mathsf{rv}}, \to^{\mathsf{rv}}\rangle$ to be the restriction of the desynchronised semantics $[\![\mathcal{S}]\!]^{\mathsf{de}} = \left\langle C, c_I, c_F, A, \to^{\mathsf{de}}\right\rangle$ where channels are always empty, $C^{\mathsf{rv}} = \left\{\left\langle(\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \mu, (u^{\mathtt{c}})_{\mathtt{c}\in\mathtt{C}}\right\rangle \in C \;\middle|\; \forall\mathtt{c}\in\mathtt{C}\cdot u^{\mathtt{c}} = \varepsilon\right\}$, and the transition relation $\to^{\mathsf{rv}}$ is obtained from $\to^{\mathsf{de}}$ by replacing the two rules for send and receive by the rendezvous transition $\left\langle(\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \mu, (\varepsilon)_{\mathtt{c}\in\mathtt{C}}\right\rangle \xrightarrow{(\mathsf{op}^{\mathtt{p}},\mathsf{op}^{\mathtt{q}})}{}^{\mathsf{rv}} \left\langle(\ell'^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \mu, (\varepsilon)_{\mathtt{c}\in\mathtt{C}}\right\rangle$ whenever there exists a channel $\mathtt{pq}\in\mathtt{C}$, a matching pair of send $\ell^{\mathtt{p}} \xrightarrow{\mathsf{op}^{\mathtt{p}}} \ell'^{\mathtt{p}}$ and receive transitions $\ell^{\mathtt{q}} \xrightarrow{\mathsf{op}^{\mathtt{q}}} \ell'^{\mathtt{q}}$ with $\mathsf{op}^{\mathtt{p}} = \mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{p}})$, $\mathsf{op}^{\mathtt{q}} = \mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi^{\mathtt{q}})$, and a valuation for clock channels $\mu^{\mathtt{pq}} \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}^{\mathtt{pq}}}$ s.t. $(\mu, \mu^{\mathtt{pq}}) \models \psi^{\mathtt{p}}$ and $(\mu, \mu^{\mathtt{pq}} + \delta) \models \psi^{\mathtt{q}}$, where, as in the desynchronised semantics, $\delta = \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \geqslant 0$ measures the amount of desynchronisation between $\mathtt{p}$ and $\mathtt{q}$; for every other $\mathtt{r}\in\mathtt{P}\setminus\{\mathtt{p},\mathtt{q}\}$, $\ell'^{\mathtt{r}} = \ell^{\mathtt{r}}$; the set of actions A^{rv} extends A accordingly.

▸ **Lemma 4** (cf. [30]). *Over polyforest topologies, $[\![S]\!]^{\mathsf{de}}$ is equivalent to $[\![S]\!]^{\mathsf{rv}}$.*

6 Register automata with counters

The integer (unbounded) part of the desynchronisation introduced by the rendezvous semantics is modelled by counters; the fractional part, by registers over $\mathbb{I} = \mathbb{Q}\cap[0,1)$.

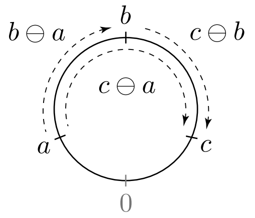
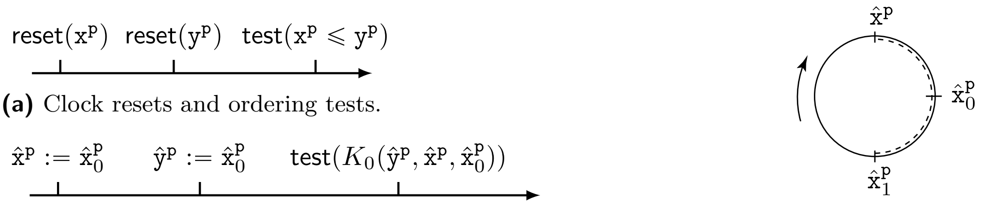


Figure 2 Cyclic order $K(a,b,c)$ vs. cyclic difference $\ominus$. The position of 0 is irrelevant.

Register constraints. Let $\mathtt{R}$ be a finite set of *registers*. We model fractional values by the *cyclic order* structure $\mathcal{K} = (\mathbb{I}, K)$, where $K \subseteq \mathbb{I}^3$ is the (strict) ternary cyclic order between rational points $a, b, c \in \mathbb{I}$ in the unit interval, defined as $K(a,b,c) \equiv a < b < c \vee b < c < a \vee c < a < b$. For $c\in\mathbb{Q}$, we have (cf. Fig. 2): $b\ominus a \leqslant c\ominus a$ iff $c\ominus b \leqslant c\ominus a$ iff $K_0(a,b,c)$, where $K_0(a,b,c) \equiv K(a,b,c) \vee a = b \vee b = c$. A *register constraint* is a quantifier-free formula φ with variables from $\mathtt{R}$ over the vocabulary of $\mathcal{K}$; since $\mathcal{K}$ admits elimination of quantifiers [35], we could allow arbitrary first-order formulas as register constraints without changing the expressiveness of the model. For a constraint φ and a register valuation $r\in\mathbb{I}^{\mathtt{R}}$, we write $r\models\varphi$ if the formula holds when variables are interpreted according to r.

Register automata with counters. A *register automaton with counters* (RAC) is a tuple $\mathcal{R} = \langle\mathtt{L}, \mathtt{l}_I, \mathtt{l}_F, \mathtt{R}, \mathtt{N}, \Delta\rangle$ where $\mathtt{L}$ is a finite set of locations, $\mathtt{l}_I, \mathtt{l}_F \in \mathtt{L}$ two distinguished initial and final locations therein, $\mathtt{R}$ a finite set of *registers*, $\mathtt{N}$ a finite set of non-negative integer *counters*, and Δ a finite set of rules of the form $\mathtt{l} \xrightarrow{\mathsf{op}} \mathtt{m}$ with $\mathtt{l}, \mathtt{m} \in \mathtt{L}$, where op is either $\mathtt{nop}$, an *increment* $\mathtt{n{+}{+}}$ of counter $\mathtt{n}$, a *decrement* $\mathtt{n{-}{-}}$ of counter $\mathtt{n}$, a *counter inequality*


reset($\mathtt{x}^{\mathtt{p}}$) reset($\mathtt{y}^{\mathtt{p}}$) test($\mathtt{x}^{\mathtt{p}} \leqslant \mathtt{y}^{\mathtt{p}}$)
$\hat{\mathtt{x}}^{\mathtt{p}} := \hat{\mathtt{x}}^{\mathtt{p}}_0$ $\hat{\mathtt{y}}^{\mathtt{p}} := \hat{\mathtt{x}}^{\mathtt{p}}_0$ test($K_0(\hat{\mathtt{y}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}_0)$)
$\hat{\mathtt{x}}^{\mathtt{p}}$
$\hat{\mathtt{x}}^{\mathtt{p}}_0$
$\hat{\mathtt{x}}^{\mathtt{p}}_1$

(a) Clock resets and ordering tests.

(b) Corresponding register assignments and tests.

(c) Fractional wrapping.

Figure 3 Fractional clocks $\mathtt{x}^{\mathtt{p}}, \mathtt{y}^{\mathtt{p}} : \mathbb{I}$ vs. cyclic registers $\hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{y}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}_0$.

$\mathsf{test}(\mathtt{n} \sim k)$ or *modular test* $\mathsf{test}(\mathtt{n} \equiv_m k)$, a *guess* $\mathsf{guess}(\mathtt{r})$ assigning a new non-deterministic value to register $\mathtt{r}$, or a *register test* $\mathsf{test}(\varphi)$ with φ a register constraint. We allow sequences of operations $\mathsf{op} = (\mathsf{op}_1; \cdots; \mathsf{op}_k)$ and group updates $\mathtt{N}'\mathtt{++}, \mathtt{N}'\mathtt{--}$ for $\mathtt{N}' \subseteq \mathtt{N}$ as syntactic sugar.

Semantics. The semantics of a RAC $\mathcal{R}$ as above is the infinite LTS $[\![\mathcal{R}]\!] = \langle C, c_I, c_F, A, \rightarrow \rangle$ where the set of configurations C consists of tuples $\langle \mathtt{l}, n, r \rangle$ with $\mathtt{l} \in \mathtt{L}$ a control location of $\mathcal{R}$, $n \in \mathbb{N}^{\mathtt{N}}$ a *counter valuation*, and $r \in \mathbb{I}^{\mathtt{R}}$ a *register valuation*, where the initial configuration is $c_I = \langle \mathtt{l}_I, \bar{0}, \bar{0} \rangle$ with $\bar{0}$ the initial counter and (overloaded) register valuation, and the final configuration is $c_F = \langle \mathtt{l}_F, \bar{0}, \bar{0} \rangle$. There is a transition $\langle \mathtt{l}, n, r \rangle \xrightarrow{\mathsf{op}} \langle \mathtt{m}, m, s \rangle$ just in case there is a rule $\mathtt{l} \xrightarrow{\mathsf{op}} \mathtt{m}$ s.t. (a) if $\mathsf{op} = \mathsf{nop}$, then $m = n$ and $s = r$; (b) if $\mathsf{op} = \mathtt{n++}$, then $m = n[\mathtt{n} \mapsto n(\mathtt{n}) + 1]$ and $s = r$; (c) if $\mathsf{op} = \mathtt{n--}$, then $n(\mathtt{n}) > 0$, $m = n[\mathtt{n} \mapsto n(\mathtt{n}) - 1]$, and $s = r$; (d) if $\mathsf{op} = \mathsf{test}(\mathtt{n} \sim k)$, then $n(\mathtt{n}) \sim k$, $m = n$, and $s = r$; (e) if $\mathsf{op} = \mathsf{test}(\mathtt{n} \equiv_m k)$, then $n(\mathtt{n}) \equiv_m k$, $m = n$, and $s = r$; (f) if $\mathsf{op} = \mathsf{guess}(\mathtt{r})$, then $m = n$ and there exists $x \in \mathbb{I}$ s.t. $s = r[\mathtt{r} \mapsto x]$; (g) if $\mathsf{op} = \mathsf{test}(\varphi)$ with φ a register constraint, then $r \models \varphi$, $m = n$, and $s = r$. A counter $\mathtt{n}$ appearing in some $\mathsf{test}(\mathtt{n} \sim k)$ is said to have *inequality tests*. These can be converted to the well-known zero-tests. Modular tests $\mathsf{test}(\mathtt{n} \equiv_m k)$ can be removed by storing in the control location the modulo class of $\mathtt{n}$. Register tests $\mathsf{test}(\varphi)$ can be removed by bookkeeping a symbolic description of the current register valuation called *orbit* (similarly as in the region construction for timed automata) [13].

▸ **Theorem 8.** *Non-emptiness is decidable for* RAC *with* $\leqslant 1$ *counter with inequality tests.*

7 Simulating the rendezvous semantics in RAC

Let $\mathcal{S} = \langle \mathcal{T} = \langle \mathtt{P}, \mathtt{C} \rangle, \mathtt{M}, (\mathtt{X}^{\mathtt{c}})_{\mathtt{c} \in \mathtt{C}}, (\mathcal{A}^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}} \rangle$ be a simple TCA with $\mathcal{A}^{\mathtt{p}} = \langle L^{\mathtt{p}}, \ell^{\mathtt{p}}_I, \ell^{\mathtt{p}}_F, \mathtt{X}^{\mathtt{p}}, A^{\mathtt{p}}, \Delta^{\mathtt{p}} \rangle$. We assume that there are neither local diagonal inequality nor modular constraints—they can be converted to their non-diagonal counterparts with a standard construction [8]. For every process $\mathtt{p}$, let $\mathtt{x}^{\mathtt{p}}_0$ be a *reference clock* which is never reset representing the “now” instant. We construct a RAC $\mathcal{R} = \langle \mathtt{L}, \mathtt{l}_I, \mathtt{l}_F, \mathtt{R}, \mathtt{N}, \Delta \rangle$ simulating the rendezvous semantics of $\mathcal{S}$.

From clocks to registers. Let $\hat{\mathtt{x}}^{\mathtt{p}}_0$ be a *reference register* representing the fractional part of the current absolute time of process $\mathtt{p}$; an auxiliary copy $\hat{\mathtt{x}}^{\mathtt{p}}_1$ of the reference register is additionally included to perform the simulation. For every fractional clock $(\mathtt{x}^{\mathtt{p}} : \mathbb{I}) \in \mathtt{X}^{\mathtt{p}}$ there is a corresponding register $\hat{\mathtt{x}}^{\mathtt{p}} \in \mathtt{R}$. While a clock $\mathtt{x}^{\mathtt{p}}$ stores the time elapsed since its last reset, the corresponding register $\hat{\mathtt{x}}^{\mathtt{p}}$ stores the value of $\hat{\mathtt{x}}^{\mathtt{p}}_0$ when $\mathtt{x}^{\mathtt{p}}$ was last reset. In this way, we can express a fractional clock $\mathtt{x}^{\mathtt{p}}$ as $\mathtt{x}^{\mathtt{p}} = \hat{\mathtt{x}}^{\mathtt{p}}_0 \ominus \hat{\mathtt{x}}^{\mathtt{p}}$. Local and channel fractional constraints are translated as the following constraints on registers, for $\mathtt{x}^{\mathtt{p}}, \mathtt{y}^{\mathtt{p}}, \mathtt{x}^{\mathtt{pq}}, \mathtt{x}^{\mathtt{q}} : \mathbb{I}$:

$$\text{[local]} \qquad \mathtt{x}^{\mathtt{p}} \leqslant \mathtt{y}^{\mathtt{p}} \quad \text{iff} \quad \hat{\mathtt{x}}^{\mathtt{p}}_0 \ominus \hat{\mathtt{x}}^{\mathtt{p}} \leqslant \hat{\mathtt{x}}^{\mathtt{p}}_0 \ominus \hat{\mathtt{y}}^{\mathtt{p}} \quad \text{iff} \quad K_0(\hat{\mathtt{y}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}_0), \tag{3}$$

$$\text{[send-receive]} \qquad \mathtt{x}^{\mathtt{pq}} \leqslant \mathtt{x}^{\mathtt{q}} \quad \text{iff} \quad \hat{\mathtt{x}}^{\mathtt{q}}_0 \ominus \hat{\mathtt{x}}^{\mathtt{p}}_0 \leqslant \hat{\mathtt{x}}^{\mathtt{q}}_0 \ominus \hat{\mathtt{x}}^{\mathtt{q}} \quad \text{iff} \quad K_0(\hat{\mathtt{x}}^{\mathtt{q}}, \hat{\mathtt{x}}^{\mathtt{p}}_0, \hat{\mathtt{x}}^{\mathtt{q}}_0). \tag{4}$$

Intuitively, $\mathtt{x}^{\mathtt{p}} \leqslant \mathtt{y}^{\mathtt{p}}$ holds iff the last reset of $\mathtt{y}^{\mathtt{p}}$ happened *before* that of $\mathtt{x}^{\mathtt{p}}$, i.e., $K_0(\hat{\mathtt{y}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{x}}_0^{\mathtt{p}})$; cf. Fig. 3a, 3b. For (4), when $\mathtt{p}$ sends a message with initial age 0, its age at the time of reception is $\mathtt{x}^{\mathtt{pq}} = \hat{\mathtt{x}}_0^{\mathtt{q}} \ominus \hat{\mathtt{x}}_0^{\mathtt{p}}$, i.e., the fractional desynchronisation between $\mathtt{p}$ and $\mathtt{q}$.

Unary equivalence. We abstract the integral value of clocks into a finite domain called *unary equivalence class* (akin to the well-known region construction for timed automata). Let $M \in \mathbb{N}$ be the maximal constant used in any clock constraint of $\mathcal{S}$. Two clock valuations $\mu, \nu \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}}$ are *M-unary equivalent*, written $\mu \approxeq_M \nu$, if their integral values are threshold $\lfloor \mu \rfloor \approx_M \lfloor \nu \rfloor$ and modular equivalent $\lfloor \mu \rfloor \equiv_M \lfloor \nu \rfloor$; cf. Sec. 4. Let Λ_M be the (finite) set of M-unary equivalence classes of clock valuations; for a clock valuation $\mu \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}}$, let $[\mu] \in \Lambda_M$ be its equivalence class. For a set of clocks $\mathtt{Y} \subseteq \mathtt{X}$, we write $\lambda[\mathtt{Y} \mapsto \mathtt{Y} + 1]$ for the unary class $[\mu']$ of valuations μ' obtained by taking some valuation $\mu \in \lambda$ and increasing it by 1 on $\mathtt{Y}$. If $\mu \approxeq_M \nu$ and φ contains only inequality and modular constraints on integral clocks with modulus M and maximal constant M, then $\mu \models \varphi$ iff $\nu \models \varphi$. We thus overload the notation and for $\lambda \in \Lambda_M$ we write $\lambda \models \varphi$ whenever there exists $\mu \in \lambda$ s.t. $\mu \models \varphi$.

The translation. Control locations in $\mathtt{L}$ are pairs $\mathtt{l} = \left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda \right\rangle$ of control locations $\ell^{\mathtt{p}}$ for every $\mathcal{A}^{\mathtt{p}}$ and a unary equivalence class $\lambda \in \Lambda_M$ abstracting away the values of local integral clocks, plus additional temporary locations. The initial location is $\mathtt{l}_I = \left\langle (\ell_I^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, [\bar{0}] \right\rangle$ and the final location is $\mathtt{l}_F = \left\langle (\ell_F^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, [\bar{0}] \right\rangle$. For each channel $\mathtt{pq} \in \mathtt{C}$ there is a corresponding counter $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}$ measuring the amount of integral desynchronisation $\mathtt{n}^{\mathtt{pq}} = \lfloor \mathtt{x}_0^{\mathtt{q}} - \mathtt{x}_0^{\mathtt{p}} \rfloor$ between the sender process $\mathtt{p}$ and the receiver process $\mathtt{q}$; the fractional desynchronisation is measured by $\hat{\mathtt{x}}_0^{\mathtt{q}} \ominus \hat{\mathtt{x}}_0^{\mathtt{p}} = \{\mathtt{x}_0^{\mathtt{q}} - \mathtt{x}_0^{\mathtt{p}}\}$. Transition rules in Δ are defined as follows. **(1)** A transition $\ell^{\mathtt{p}} \xrightarrow{\mathsf{nop}} \ell'^{\mathtt{p}}$ is simulated by $\left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda \right\rangle \xrightarrow{\mathsf{nop}} \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda \right\rangle$ with $\ell'^{\mathtt{q}} = \ell^{\mathtt{q}}\ \forall \mathtt{q} \neq \mathtt{p}$. **(2)** A local time elapse transition $t = \ell^{\mathtt{p}} \xrightarrow{\mathsf{elapse}} \ell'^{\mathtt{p}}$ in $\Delta^{\mathtt{p}}$ is simulated as follows. **(2a)** We go to a temporary location $\bullet_\lambda$ implicitly depending on t: $\left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda \right\rangle \xrightarrow{\mathsf{nop}} \bullet_\lambda$. **(2b)** We simulate an arbitrary integer time elapse for process $\mathtt{p}$. Let $\mathtt{N}^+ = \{\mathtt{n}^{\mathtt{qp}} \mid \mathtt{qp} \in \mathtt{C}\}$ be the set of counters corresponding to channels incoming to $\mathtt{p}$ and let $\mathtt{N}^- = \{\mathtt{n}^{\mathtt{pq}} \mid \mathtt{pq} \in \mathtt{C}\}$ for outgoing channels. We increase counters $\mathtt{n}^{\mathtt{qp}} \in \mathtt{N}^+$ by an arbitrary amount, and decrease counters $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^-$ by the same amount; the unary class λ of clocks of $\mathtt{p}$ is updated accordingly: For every λ, we have a transition $\bullet_\lambda \xrightarrow{\mathtt{N}^+\mathtt{++};\mathtt{N}^-\mathtt{--}} \bullet_{\lambda'}$, where $\lambda' = \lambda[\mathtt{X}^{\mathtt{p}} \mapsto \mathtt{X}^{\mathtt{p}} + 1]$. These transitions can be repeated an arbitrary number of times. **(2c)** We save the current local time of $\mathtt{p}$ in $\hat{\mathtt{x}}_1^{\mathtt{p}}$: $\bullet_\lambda \xrightarrow{\mathsf{guess}(\hat{x}_1^{\mathtt{p}});\mathsf{test}(\hat{\mathtt{x}}_1^{\mathtt{p}} = \hat{\mathtt{x}}_0^{\mathtt{p}})} \bullet_\lambda^1$. **(2d)** We simulate an arbitrary fractional time elapse for process $\mathtt{p}$ by guessing a new arbitrary value for the local reference register $\hat{\mathtt{x}}_0^{\mathtt{p}}$: $\bullet_\lambda^1 \xrightarrow{\mathsf{guess}(\hat{x}_0^{\mathtt{p}})} \bullet_\lambda^2$. **(2e)** We need to further increase by one the integral part of clocks $\mathtt{x}^{\mathtt{p}}$ whose fractional value was wrapped around 0 one time more than the fractional part of the reference clock $\mathtt{x}_0^{\mathtt{p}}$. Let

$$K_1(\hat{\mathtt{x}}, \hat{\mathtt{y}}, \hat{\mathtt{z}}) \equiv K(\hat{\mathtt{x}}, \hat{\mathtt{y}}, \hat{\mathtt{z}}) \vee \hat{\mathtt{y}} = \hat{\mathtt{z}} \quad \text{and} \quad K_2(\hat{\mathtt{x}}, \hat{\mathtt{y}}, \hat{\mathtt{z}}) \equiv K(\hat{\mathtt{x}}, \hat{\mathtt{y}}, \hat{\mathtt{z}}) \vee \hat{\mathtt{x}} = \hat{\mathtt{y}} \neq \mathtt{z}. \tag{5}$$

Cf. Fig. 3c, where register $\hat{\mathtt{x}}_1^{\mathtt{p}}$ stores the old fractional time: In the dashed arc ($\hat{\mathtt{x}}^{\mathtt{p}}$ included, $\hat{\mathtt{x}}_1^{\mathtt{p}}$ excluded) the fractional part of clock $\mathtt{x}^{\mathtt{p}}$ was wrapped around 0 one more time than $\mathtt{x}_0^{\mathtt{p}}$. This is the case precisely when $K_1(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{x}}_0^{\mathtt{p}})$ holds. The same adjustment is made for incoming channels $\mathtt{qp}$, where $\mathtt{n}^{\mathtt{qp}}$ must be increased by one whenever $K_1(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}_0^{\mathtt{q}}, \hat{\mathtt{x}}_0^{\mathtt{p}})$ holds. For outgoing channels $\mathtt{pq}$, counter $\mathtt{n}^{\mathtt{pq}}$ must be further decreased by one precisely when $K_2(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}_0^{\mathtt{q}}, \hat{\mathtt{x}}_0^{\mathtt{p}})$ holds. Let $\mathtt{S} = \mathtt{S}^+ \cup \mathtt{S}^-$, where $\mathtt{S}^+ = \{\hat{\mathtt{x}}^{\mathtt{p}} \mid \mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}\} \cup \{\hat{\mathtt{x}}_0^{\mathtt{q}} \mid \mathtt{qp} \in \mathtt{C}\}$ and $\mathtt{S}^- = \{\hat{\mathtt{x}}_0^{\mathtt{q}} \mid \mathtt{pq} \in \mathtt{C}\}$, be the set of registers that must be checked. The automaton guesses a partition $\mathtt{S} = \mathtt{S}_{\mathsf{yes}} \cup \mathtt{S}_{\mathsf{no}}$ of those registers corresponding to wrapped clocks. The guess is verified with the formula

$$\begin{aligned} \varphi \equiv\ & \forall \hat{\mathtt{x}} \in \mathtt{S}_{\mathsf{yes}} \cap \mathtt{S}^+ \cdot K_1(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}, \hat{\mathtt{x}}_0^{\mathtt{p}}) \wedge \forall \hat{\mathtt{x}} \in \mathtt{S}_{\mathsf{yes}} \cap \mathtt{S}^- \cdot K_2(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}, \hat{\mathtt{x}}_0^{\mathtt{p}}) \wedge \\ & \forall \hat{\mathtt{x}} \in \mathtt{S}_{\mathsf{no}} \cap \mathtt{S}^+ \cdot \neg K_1(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}, \hat{\mathtt{x}}_0^{\mathtt{p}}) \wedge \forall \hat{\mathtt{x}} \in \mathtt{S}_{\mathsf{no}} \cap \mathtt{S}^- \cdot \neg K_2(\hat{\mathtt{x}}_1^{\mathtt{p}}, \hat{\mathtt{x}}, \hat{\mathtt{x}}_0^{\mathtt{p}}). \end{aligned} \tag{6}$$

Let $\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} = \{\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}} \mid \hat{\mathtt{x}}^{\mathtt{p}} \in \mathtt{S}_{\mathrm{yes}}\}$ be the set of $\mathtt{p}$-clocks whose fractional values were wrapped around 0. The unary class for clocks in $\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}$ is updated by $\lambda' = \lambda[\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} \mapsto \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} + 1]$. Let $\mathtt{N}^{+}_{\mathrm{yes}} = \{\mathtt{n}^{\mathtt{qp}} \in \mathtt{N} \mid \hat{\mathtt{x}}^{\mathtt{q}}_0 \in \mathtt{S}_{\mathrm{yes}} \cap \mathtt{S}^{+}\}$ be the set of counters that need to be increased, and let $\mathtt{N}^{-}_{\mathrm{yes}} = \{\mathtt{n}^{\mathtt{pq}} \in \mathtt{N} \mid \hat{\mathtt{x}}^{\mathtt{q}}_0 \in \mathtt{S}_{\mathrm{yes}} \cap \mathtt{S}^{-}\}$ those that need to be decreased. For every λ and guessing as above, we have a transition $\bullet^2_\lambda \xrightarrow{\mathsf{test}(\varphi);(\mathtt{N}^{+}_{\mathrm{yes}})\text{++};(\mathtt{N}^{-}_{\mathrm{yes}})\text{--}} \bullet^3_{\lambda'}$. **(2f)** The simulation of time elapse terminates with a transition $\bullet^3_\lambda \xrightarrow{\mathsf{nop}} \langle(\not\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle$ for every λ, where $\not\ell^{\mathtt{q}} = \ell^{\mathtt{q}}$ for every other process $\mathtt{q} \neq \mathtt{p}$. **(3)** A test operation $\ell^{\mathtt{p}} \xrightarrow{\mathsf{test}(\varphi)} \not\ell^{\mathtt{p}}$ in $\Delta^{\mathtt{p}}$, is simulated by a corresponding transition in Δ $\langle(\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle \xrightarrow{\mathsf{op}} \langle(\not\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle$. An inequality $\varphi \equiv \mathtt{x}^{\mathtt{p}} \sim k$ or a modular $\varphi \equiv \mathtt{x}^{\mathtt{p}} \equiv_m k$ constraint is immediately checked by requiring $\lambda \models \varphi$ and $\mathsf{op} = \mathsf{nop}$. Here we use the fact that there are no diagonal inequality or modular constraints in $\mathcal{S}$. A fractional constraint $\varphi \equiv \mathtt{x}^{\mathtt{p}} \leqslant \mathtt{y}^{\mathtt{p}}$ on fractional clocks $\mathtt{x}^{\mathtt{p}}, \mathtt{y}^{\mathtt{p}} : \mathbb{I}$ is replaced by the corresponding constraint on fractional registers $\mathsf{op} = \mathsf{test}(K_0(\hat{\mathtt{y}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}_0))$; cf. (3). For every other $\mathtt{q} \neq \mathtt{p}$, $\not\ell^{\mathtt{q}} = \ell^{\mathtt{q}}$. **(4)** A reset operation $\ell^{\mathtt{p}} \xrightarrow{\mathsf{reset}(\mathtt{x}^{\mathtt{p}})} \not\ell^{\mathtt{p}}$ in $\Delta^{\mathtt{p}}$ with $\mathtt{x}^{\mathtt{p}} : \mathbb{N}$ an integral clock is simulated by updating the unary class with the transition $\langle(\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle \xrightarrow{\mathsf{nop}} \langle(\not\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda[\mathtt{x}^{\mathtt{p}} \mapsto [0]]\rangle$ in Δ. On the other hand, if $\mathtt{x}^{\mathtt{p}} : \mathbb{I}$ is a fractional clock, then the corresponding register $\hat{\mathtt{x}}^{\mathtt{p}}$ records the current timestamp $\hat{\mathtt{x}}^{\mathtt{p}}_0$ by executing $\langle(\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle \xrightarrow{\mathsf{guess}(\hat{\mathtt{x}}^{\mathtt{p}});\mathsf{test}(\hat{\mathtt{x}}^{\mathtt{p}}=\hat{\mathtt{x}}^{\mathtt{p}}_0)} \langle(\not\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda[\mathtt{x}^{\mathtt{p}} \mapsto 0]\rangle$ in Δ. For every other $\mathtt{q} \neq \mathtt{p}$, $\not\ell^{\mathtt{q}} = \ell^{\mathtt{q}}$. **(5)** A send-receive pair $\ell^{\mathtt{p}} \xrightarrow{\mathsf{send}(\mathtt{pq},\mathtt{m}:\psi^{\mathtt{p}})} \not\ell^{\mathtt{p}}$ in $\Delta^{\mathtt{p}}$ and $\ell^{\mathtt{q}} \xrightarrow{\mathsf{receive}(\mathtt{pq},\mathtt{m}:\psi^{\mathtt{q}})} \not\ell^{\mathtt{q}}$ in $\Delta^{\mathtt{q}}$ is simulated by a test transition $\langle(\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle \xrightarrow{\mathsf{test}(\varphi)} \langle(\not\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda\rangle$ in Δ, where $\not\ell^{\mathtt{r}} = \ell^{\mathtt{r}}$ for every other $\mathtt{r} \in \mathtt{P}\setminus\{\mathtt{p},\mathtt{q}\}$, provided that one of the following conditions holds: **(5a)** If it is an integral send-receive pair, since our TCA is simple, $\psi^{\mathtt{p}}, \psi^{\mathtt{q}}$ are (in)equality constraints of the form $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$ and $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \sim k$ with $\mathtt{x}^{\mathtt{pq}} : \mathbb{N}$ an integral clock. Since the counter $\mathtt{n}^{\mathtt{pq}}$ measures the integral desynchronisation between $\mathtt{p}$ and $\mathtt{q}$, it also measures final value of $\mathtt{x}^{\mathtt{pq}}$ at the time of reception. We take $\varphi \equiv \mathtt{n}^{\mathtt{pq}} \sim k$. **(5b)** If it is a modular send-receive pair, then, since our TCA is simple, $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$ and $\psi^{\mathtt{q}} \equiv (\mathtt{x}^{\mathtt{pq}} \equiv_M k)$ with $\mathtt{x}^{\mathtt{pq}} : \mathbb{N}$ an integral clock. Take $\varphi \equiv \mathtt{n}^{\mathtt{pq}} \equiv_M k$. **(5c)** The last case is a fractional send-receive pair. Since our TCA is simple, we can assume constraints are of the form $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$ and $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \leqslant \mathtt{x}^{\mathtt{q}}$ (the other inequality can be treated similarly) for fractional clocks $\mathtt{x}^{\mathtt{pq}}, \mathtt{x}^{\mathtt{q}} : \mathbb{I}$. By (4), take $\varphi \equiv K_0(\hat{\mathtt{x}}^{\mathtt{q}}, \hat{\mathtt{x}}^{\mathtt{p}}_0, \hat{\mathtt{x}}^{\mathtt{q}}_0)$. This concludes the description of RAC $\mathcal{R}$.

▸ **Lemma 5.** *The rendezvous semantics $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ and $[\![\mathcal{R}]\!]$ are equivalent.*

To sum up, we have so far reduced the non-emptiness problem of a TCA to that of a simple TCA (Sec. 4), then to its rendezvous semantics (Sec. 5 and 5), and in this section the latter is reduced to non-emptiness of RAC. In order to conclude by Theorem 8, we have to show that, if the communication topology has at most one channel with inequality tests per polytree, then a RAC with at most one inequality test suffices. We apply the translation of this section to each polytree (thus obtaining several RACs with at most one inequality test each), and then simulate the whole polyforest topology by sequentialising each polytree, which allows to reuse a single inequality test for the entire simulation; cf. Sec. A.6 for the details. To conclude, we are able to produce a single RAC with at most one inequality test equivalent to the original TCA. This finishes the proof of the "if" direction of Theorem 1.

8 Conclusions

We have presented a complete characterisation of decidable TCA topologies. For polyforest topologies, channel clock inequality constraints are the only source of undecidability; for future work, one could investigate decidable subclasses thereof, such as monotonic constraints of the form $\mathtt{x}^{\mathtt{c}} \geqslant k$. Moreover, it would be interesting to determine whether the set of reachable channel contents is a timed regular language, and effectively build a TA recognising it. Finally, the current notion of communication topology is overly pessimistic, since there might be potential cycles in the topology and still no actual execution of the system using them: This prompts a quest for finer notions of topology taking into account, at least to some extent, the local control structure of processes to rule out such cases.

References

1 P. Abdulla, P. Mahata, and R. Mayr. Dense-timed petri nets: Checking zenoness, token liveness and boundedness. *Logical Methods in Computer Science*, 3(1), 2007.

2 P. A. Abdulla, C. Aiswarya, and M. F. Atig. Data communicating processes with unreliable channels. In *Proc. of LICS'16*, pages 166–175, New York, NY, USA, 2016. ACM.

3 P. A. Abdulla, M. F. Atig, and J. Cederberg. Timed lossy channel systems. In D. D'Souza, T. Kavitha, and J. Radhakrishnan, editors, *Proc. of FSTTCS'12*, volume 18 of *LIPIcs*, pages 374–386, 2012.

4 P. A. Abdulla, M. F. Atig, and J. Stenman. Dense-timed pushdown automata. In *Proc. of LICS'12*, pages 35–44, June 2012.

5 P. A. Abdulla and B. Jonsson. Verifying programs with unreliable channels. *Information and Computation*, 127(2):91–101, 1996.

6 S. Akshay, P. Gastin, S. N. Krishna, and I. Sarkar. Towards an Efficient Tree Automata Based Technique for Timed Systems. In R. Meyer and U. Nestmann, editors, *In Proc. of CONCUR'17*, volume 85 of *LIPIcs*, pages 39:1–39:15, 2017.

7 R. Alur and D. Dill. Automata for modeling real-time systems. In *Proc. of ICALP'90*, volume 443 of *LNCS*, pages 322–335, 1990.

8 R. Alur and D. L. Dill. A theory of timed automata. *Theor. Comput. Sci.*, 126:183–235, April 1994.

9 P. Aziz Abdulla, M. Faouzi Atig, and S. Krishna. Communicating Timed Processes with Perfect Timed Channels. *ArXiv e-prints*, Aug. 2017.

10 M. Benerecetti, S. Minopoli, and A. Peron. Analysis of timed recursive state machines. In *Proc. TIME'10*, pages 61–68. IEEE, sept. 2010.

11 J. Bengtsson, B. Jonsson, J. Lilius, and W. Yi. Partial order reductions for timed systems. In *Proc. of CONUR'98*, CONCUR '98, pages 485–500, 1998.

12 B. Bérard, F. Cassez, S. Haddad, D. Lime, and O. Roux. Comparison of different semantics for time petri nets. In D. Peled and Y.-K. Tsay, editors, *In Proc. of ATVA'05*, volume 3707 of *LNCS*, pages 293–307. Springer, 2005.

13 M. Bojańczyk, B. Klin, and S. Lasota. Automata theory in nominal sets. *Log. Meth. Comput. Sci.*, 10(3:4), 2013.

14 R. Bonnet. The reachability problem for vector addition system with one zero-test. In *Proc. of MFCS'11*, pages 145–157. Springer, 2011.

15 A. Bouajjani, R. Echahed, and R. Robbana. On the automatic verification of systems with continuous variables and unbounded discrete data structures. In *Hybrid Systems'94*, pages 64–85, 1994.

16 D. Brand and P. Zafiropulo. On communicating finite-state machines. *J. ACM*, 30(2):323–342, Apr. 1983.

17 G. Cécé and A. Finkel. Verification of programs with half-duplex communication. *Information and Computation*, 202(2):166–190, 2005.

18 G. Cece, A. Finkel, and S. P. Iyer. Unreliable channels are easier to verify than perfect channels. *Information and Computation*, 124(1):20–31, 1996.

19 P. Chambart and P. Schnoebelen. Mixing lossy and perfect fifo channels. In F. van Breugel and M. Chechik, editors, *Proc. of CONCUR'08*, volume 5201 of *LNCS*, pages 340–355. Springer, 2008.

20 L. Clemente, F. Herbreteau, A. Stainer, and G. Sutre. Reachability of communicating timed processes. In F. Pfenning, editor, *Proc. of FOSSACS'13*, volume 7794 of *LNCS*, pages 810–96. Springer, 2013.

21 L. Clemente, F. Herbreteau, and G. Sutre. Decidable topologies for communicating automata with fifo and bag channels. In P. Baldan and D. Gorla, editors, *Proc. of CONCUR'14*, volume 8704 of *LNCS*, pages 281–296. Springer, 2014.

22 L. Clemente and S. Lasota. Binary reachability of timed pushdown automata via quantifier elimination. To appear in ICALP'18.

23 L. Clemente and S. Lasota. Timed pushdown automata revisited. In *Proc. LICS'15*, pages 738–749. IEEE, July 2015.

24 L. Clemente, S. Lasota, R. Lazić, and F. Mazowiecki. Timed pushdown automata and branching vector addition systems. In *Proc. of LICS'17*, 2017.

25 Z. Dang. Pushdown timed automata: a binary reachability characterization and safety verification. *Theor. Comput. Sci.*, 302(1-3):93–121, June 2003.

26 L. Fribourg. A closed-form evaluation for extended timed automata. Technical report, CNRS and Ecole Normale Superieure de Cachan, 1998.

27 C. Haase, S. Schmitz, and P. Schnoebelen. The power of priority channel systems. *Logical Methods in Computer Science*, 10(4), 2014.

28 F. Herbreteau, B. Srivathsan, and I. Walukiewicz. Better abstractions for timed automata. *Information and Computation*, 251:67–90, 2016.

29 A. Heußner, T. Le Gall, and G. Sutre. Safety verification of communicating one-counter machines. In D. D'Souza, T. Kavitha, and J. Radhakrishnan, editors, *Proc. of FSTTCS'12*, volume 18 of *LIPIcs*, pages 224–235, 2012.

30 A. Heußner, J. Leroux, A. Muscholl, and G. Sutre. Reachability analysis of communicating pushdown systems. *LMCS*, 8(3):1–20, September 2012.

31 S. R. Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *Proc of. STOC'82*, pages 267–281. ACM, 1982.

32 P. Krcal and W. Yi. Communicating timed automata: the more synchronous, the more difficult to verify. In *Proc. of CAV'06*, LNCS, pages 249–262. Springer, 2006.

33 J. L. Lambert. A structure to decide reachability in Petri nets. *Theor. Comput. Sci.*, 99(1):79–104, June 1992.

34 J. Leroux and S. Schmitz. Demystifying reachability in vector addition systems. In *In Proc. of LICS'15*, pages 56–67, July 2015.

35 D. Macpherson. A survey of homogeneous structures. *Discrete Mathematics*, 311(15):1599–1634, 2011.

36 E. W. Mayr. An algorithm for the general Petri net reachability problem. In *Proc. of STOC'81*, pages 238–246. ACM, 1981.

37 R. Milner. *Communication and Concurrency.* Prentice Hall, 1989.

38 J. Pachl. Reachability problems for communicating finite state machines. Technical Report CS-82-12, University of Waterloo, May 1982.

39 K. Quaas. Verification for Timed Automata extended with Unbounded Discrete Data Structures. *Logical Methods in Computer Science*, Volume 11, Issue 3, Sept. 2015.

40 K. Quaas, M. Shirmohammadi, and J. Worrell. Revisiting reachability in timed automata. In *Proc. of LICS'17*, pages 1–12, June 2017.

41 K. Reinhardt. Reachability in petri nets with inhibitor arcs. *Electron. Notes Theor. Comput. Sci.*, 223:239–264, Dec. 2008.

42 S. L. Torre, P. Madhusudan, and G. Parlato. Context-bounded analysis of concurrent queue systems. In *Proc. of TACAS'08*, volume 4963 of *LNCS*, pages 299–314, 2008.

43 A. Trivedi and D. Wojtczak. Recursive timed automata. In *Proc. ATVA'10*, volume 6252 of *LNCS*, pages 306–324. Springer, 2010.

A Appendix

Let $\mathbb{1}_{C?}$, for a condition C, be 1 if C holds, and 0 otherwise.

A.1 Missing proof for Sec. 4

We conclude the proof of Lemma 2 in the case of fractional clocks.

Second part of the proof of Lemma 2. *Fractional clocks.* Recall the definition of $\psi^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$:

$$\psi^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}} \equiv \bigwedge_{(i,j)\in K^{\mathsf{p}}} \mathrm{y}_i^{\mathsf{pq}} \approx_{ij}^{\mathsf{p}} \hat{\mathrm{y}}_j^{\mathsf{pq}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} \wedge \bigwedge_{(i,j)\in K^{\mathsf{pq}}} \mathrm{y}_i^{\mathsf{pq}} \approx_{ij}^{\mathsf{pq}} \mathrm{y}_j^{\mathsf{pq}} \wedge \bigwedge_{(i,j)\in K^{\mathsf{q}}} \mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}} \approx_{ij}^{\mathsf{q}} \mathrm{y}_j^{\mathsf{q}}.$$

We show that $\exists \bar{\mathrm{y}}^{\mathsf{pq}} \cdot \psi^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$ is equivalent to a quantifier-free formula $\widetilde{\psi}^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$. We replace the rightmost atomic formula $\mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}$ in $\psi^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$ by an equivalent formula using "$\ominus$" instead of "$\oplus$"; the other comparison operators can be dealt with in a similar manner. We would like to apply "$\ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$" to both sides of the inequality, using the obvious fact that $(\mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}}) \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} = \mathrm{y}_i^{\mathsf{pq}}$. This is safe to do if $\hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}}$ (and thus $\hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}$), which is equivalent to $\hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant 1 \ominus \mathrm{y}_i^{\mathsf{pq}}$, and we obtain $\mathrm{y}_i^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$ in this case. However, if $\mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}} < \hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}$, that is, $1 \ominus \mathrm{y}_i^{\mathsf{pq}} < \hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}$, then the inequality is inverted, and we obtain $\mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} < \mathrm{y}_i^{\mathsf{pq}}$ in this case. Finally, if $\mathrm{y}_j^{\mathsf{q}} < \hat{\mathrm{y}}_0^{\mathsf{pq}}$ (and thus $\mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}} < \hat{\mathrm{y}}_0^{\mathsf{pq}}$), then the inequality flips again, and we obtain again $\mathrm{y}_i^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$. Putting these three cases together, we have that $\mathrm{y}_i^{\mathsf{pq}} \oplus \hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}$ is equivalent to the formula

$$(\mathrm{y}_i^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}} \ominus \mathrm{y}_0^{\mathsf{pq}} \wedge (\hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant 1 \ominus \mathrm{y}_i^{\mathsf{pq}} \vee \mathrm{y}_j^{\mathsf{q}} < \hat{\mathrm{y}}_0^{\mathsf{pq}})) \vee (\mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} < \mathrm{y}_i^{\mathsf{pq}} \wedge 1 \ominus \mathrm{y}_i^{\mathsf{pq}} < \hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}).$$

We put the $\mathrm{y}_i^{\mathsf{pq}}$'s in positive positions obtaining the equivalent formula

$$(\mathrm{y}_i^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}} \ominus \mathrm{y}_0^{\mathsf{pq}} \wedge (\mathrm{y}_i^{\mathsf{pq}} \leqslant 1 \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} \vee \mathrm{y}_j^{\mathsf{q}} < \hat{\mathrm{y}}_0^{\mathsf{pq}})) \vee (\mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} < \mathrm{y}_i^{\mathsf{pq}} \wedge 1 \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} < \mathrm{y}_i^{\mathsf{pq}} \wedge \hat{\mathrm{y}}_0^{\mathsf{pq}} \leqslant \mathrm{y}_j^{\mathsf{q}}).$$

By distributing $\vee$ over $\wedge$, we can put $\psi^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$ in CNF. W.l.o.g. it suffices consider a single conjunct thereof, which has the general shape (we omit indices for readability)

$$\psi \wedge \exists \bar{\mathrm{y}}^{\mathsf{pq}} \cdot \bigwedge u \preceq v, \tag{7}$$

where ψ contains only constraints of the form $\mathrm{y}_j^{\mathsf{q}} \approx \hat{\mathrm{y}}_0^{\mathsf{pq}}$ with $\approx \in \{<, \leqslant, \geqslant, >\}$; $\preceq \in \{\leqslant, <\}$; and the lower u's and upper bound constraints v's are of one the forms $\mathrm{y}_i^{\mathsf{pq}}$, $\hat{\mathrm{y}}_j^{\mathsf{pq}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$, $\mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$, or $1 \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$. By solving (7) w.r.t. $\mathrm{y}_1^{\mathsf{pq}}$, we obtain a formula of the form

$$\psi \wedge \exists \mathrm{y}_2^{\mathsf{pq}} \cdots \mathrm{y}_n^{\mathsf{pq}} \cdot \varphi \wedge \exists \mathrm{y}_1^{\mathsf{pq}} \cdot \bigwedge u \preceq \mathrm{y}_1^{\mathsf{pq}} \wedge \bigwedge \mathrm{y}_1^{\mathsf{pq}} \preceq v,$$

where ψ is as in (7) and φ does not contain $\mathrm{y}_1^{\mathsf{pq}}$. By removing the existential quantifier on $\mathrm{y}_1^{\mathsf{pq}}$ we obtain

$$\psi \wedge \exists \mathrm{y}_2^{\mathsf{pq}} \cdots \mathrm{y}_n^{\mathsf{pq}} \cdot \varphi \wedge \bigwedge u \preceq v.$$

This formula is in the same form as (7), but with one quantifier less. We can repeat the process and remove all the quantifiers w.r.t. $\mathrm{y}_2^{\mathsf{pq}} \dots \mathrm{y}_n^{\mathsf{pq}}$, and obtain a quantifier-free formula of the form $\psi' \wedge \bigwedge u' \preceq v'$ where ψ' contains only constraints of the form $\mathrm{y}_j^{\mathsf{q}} \approx \hat{\mathrm{y}}_0^{\mathsf{pq}}$ with $\approx \in \{<, \leqslant, \geqslant, >\}$, and the u', v''s are of one of the forms $\hat{\mathrm{y}}_j^{\mathsf{pq}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$, $\mathrm{y}_j^{\mathsf{q}} \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$, or $1 \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$. Thus every $u' \preceq v'$ is of the form $a \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}} \preceq b \ominus \hat{\mathrm{y}}_0^{\mathsf{pq}}$, and by (8) it can be expressed purely in terms of order constraints on a, b. We have thus obtained the quantifier-free formula $\widetilde{\psi}^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$ we were after. Notice that $\widetilde{\psi}^{\mathsf{q}}_{\bar{\mathrm{y}}^{\mathsf{pq}}}$ speaks only about local q-clocks $\mathrm{y}_j^{\mathsf{q}}$'s and new channel clocks $\hat{\mathrm{y}}_j^{\mathsf{pq}}$'s (which hold copies of p-clocks $\mathrm{y}_j^{\mathsf{p}}$'s). ◂

Atomic channel constraints $\hat{\mathtt{x}}^{\mathtt{pq}} = \mathtt{x}^{\mathtt{p}}$, $\hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{x}^{\mathtt{q}} \sim k$, $\hat{\mathtt{x}}^{\mathtt{pq}} - \mathtt{x}^{\mathtt{q}} \equiv_M k$, $\hat{\mathtt{y}}^{\mathtt{pq}} \sim \mathtt{y}^{\mathtt{q}}$. Recall that channel clocks are copies of local clocks. As a consequence, we can assume w.l.o.g. that send and receive constraints are atomic. Let $\mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{p}}_{\mathsf{copy}})$, $\mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi^{\mathtt{q}}_1 \wedge \cdots \wedge \psi^{\mathtt{q}}_n)$ be a send-receive pair, where the $\psi^{\mathtt{q}}_i$'s are atomic. By sending n times in a row the same message $\mathtt{m}$ as $\mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{p}}_{\mathsf{copy}}); \ldots ; \mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{p}}_{\mathsf{copy}})$, we can split the receive operation into $\mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi^{\mathtt{q}}_1); \ldots ; \mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi^{\mathtt{q}}_n)$. Moreover, if a receive constraints uses only $\hat{\mathtt{x}}^{\mathtt{pq}}$, or $\hat{\mathtt{y}}^{\mathtt{pq}}$ resp., then we can assume that the corresponding send constraint is just $\hat{\mathtt{x}}^{\mathtt{pq}} = \mathtt{x}^{\mathtt{p}}$ or, resp., $\hat{\mathtt{y}}^{\mathtt{pq}} = \mathtt{y}^{\mathtt{p}}$—all other channel clocks are irrelevant. Consequently, all channel constraints can in fact be assumed to be atomic.

A.2 Missing proofs for Sec. 5

▶ **Lemma 6.** *The standard semantics $[\![S]\!]$ is equivalent to the desynchronised semantics $[\![S]\!]^{\mathsf{de}}$.*

Proof. Every run in $[\![S]\!]$ is also a run in $[\![S]\!]^{\mathsf{de}}$, since the latter semantics is a weakening of the former. For the other direction, a run in $[\![S]\!]^{\mathsf{de}}$ can be re-synchronised by rescheduling all processes $\mathtt{p}$'s to execute elapse transitions at the same time in order for the local now value $\mu_{\mathtt{p}}(\mathtt{x}^{\mathtt{p}}_0)$ to be the same for every process. Processes $\mathtt{p}, \mathtt{q}$ in the same polytree are in fact already synchronised $\mu_{\mathtt{p}}(\mathtt{x}^{\mathtt{p}}_0) = \mu_{\mathtt{q}}(\mathtt{x}^{\mathtt{q}}_0)$ by the definition of $[\![S]\!]^{\mathsf{de}}$. In order to re-synchronise a sender process $\mathtt{p}$ with a receiver process $\mathtt{q}$ from another polytree with $\mathtt{pq} \in \mathtt{C}$, since $\mathtt{q}$ is always ahead of $\mathtt{p}$ in $[\![S]\!]^{\mathsf{de}}$, in general we need to anticipate the actions of $\mathtt{p}$, and in particular transmissions actions. This comes at the cost of potentially increasing the length of the contents of channels outgoing from $\mathtt{p}$. Since in $[\![S]\!]^{\mathsf{de}}$ the initial value of channel clocks $\mu^{\mathtt{pq}}$ is automatically advanced by the amount of desynchronisation $\delta = \mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0) \geqslant 0$ between sender $\mathtt{p}$ and receiver $\mathtt{q}$, in the synchronised run we have $\delta = 0$ and the initial value of channel clocks sent is just $\mu^{\mathtt{pq}}$. ◀

A.3 Missing proofs for Sec. 5

▶ **Lemma 7** (cf. [30])**.** *Over polyforest topologies, $[\![S]\!]^{\mathsf{de}}$ is equivalent to $[\![S]\!]^{\mathsf{rv}}$.*

Proof. Every run in $[\![S]\!]^{\mathsf{rv}}$ is (essentially) a run in $[\![S]\!]^{\mathsf{de}}$ since the former semantics is a strengthening of the latter; "essentially" means that we need to split atomic send/receive operations into a send followed by a receive operation in order to properly get a run in $[\![S]\!]^{\mathsf{de}}$. For the other direction, it has been shown that on polyforest topologies a run in any system of communicating possibly infinite state automata (and in particular in $[\![S]\!]^{\mathsf{de}}$) can be rescheduled in order for transmissions to be immediately followed by matching receptions [30]. By executing these pairs of matching send/receive operations atomically we obtain rendezvous synchronisation. ◀

A.4 Missing proofs for Sec. 6

In this section we prove the following theorem.

▶ **Theorem 8.** *Non-emptiness is decidable for* RAC *with $\leqslant 1$ counter with inequality tests.*

First, we introduce some concepts used in the proof. An *automorphism* of the cyclic order structure $\mathcal{K} = (\mathbb{I}, K)$ is a bijection $\alpha : \mathbb{I} \to \mathbb{I}$ that preserves and reflects K, i.e., $K(a, b, c)$ iff $K(\alpha(a), \alpha(b), \alpha(c))$; automorphisms are extended point-wise to register valuations $\mathbb{I}^{\mathcal{R}}$. The *orbit* of a register valuation $r \in \mathbb{I}^{\mathcal{R}}$ is the set of valuations s s.t. there exists an automorphism α transforming r into $s = \alpha(r)$; the orbit of r is denoted $\mathrm{O}(r) \subseteq \mathbb{I}^{\mathrm{R}}$. The structure $\mathcal{K}$ is homogeneous [35], and thus the set of valuations $\mathbb{I}^{\mathtt{R}}$ is partitioned into exponentially many distinct orbits, denoted $\mathrm{O}(\mathbb{I}^{\mathtt{R}})$. We extend the satisfaction relation from valuations $r \models \varphi$ to orbits of valuations $o \in \mathrm{O}(\mathbb{I}^{\mathtt{R}})$, and write $o \models \varphi$ whenever there exists $r \in o$ s.t. $r \models \varphi$; by the definition of orbit, the choice of representative r does not matter. An orbit, like a region for clock valuations, is an equivalence class of valuations which are indistinguishable from the point of view of $\mathcal{K}$; for instance $(0.2, 0.3, 0.7)$, $(0.7, 0.2, 0.3)$, and $(0.8, 0.2, 0.3)$ belong to the same orbit, while $(0.2, 0.3, 0.3)$ belongs to a different orbit. We are now ready to prove the theorem above.

Proof. Let $\mathcal{R} = \langle \mathtt{L}, \mathtt{l}_I, \mathtt{l}_F, \mathtt{R}, \mathtt{N}, \Delta \rangle$ be a RAC with maximal constant M, where we assume w.l.o.g. that all modular tests are over the same modulus M. We construct a RAC without registers $\mathcal{R}'$ where counters can only be incremented, decremented, and tested for zero (i.e., an ordinary counter machine). Let $\mathcal{R}' = \langle \mathtt{L}', \mathtt{l}'_I, \mathtt{l}'_F, \mathtt{R}', \mathtt{N}', \Delta' \rangle$, where the set of locations is $\mathtt{L}' = \mathtt{L} \times \mathrm{O}(\mathbb{I}^{\mathtt{R}}) \times \{0, \dots, M-1\}^{\mathtt{N}}$, the initial location is $\mathtt{l}'_I = (\mathtt{l}_I, \mathrm{O}(\bar{0}), \bar{0})$, the final location is $\mathtt{l}'_F = (\mathtt{l}_F, \mathrm{O}(\bar{0}), \bar{0})$, the set of registers is empty $\mathtt{R}' = \varnothing$, the set of counters does not change $\mathtt{N}' = \mathtt{N}$, and the set of transition rules Δ' is defined as follows. Let $\mathtt{l} \xrightarrow{\mathsf{op}} \mathtt{l}'$ be a transition in Δ. Then we have one or more transitions in Δ' of the form $(\mathtt{l}, o, \lambda) \xrightarrow{\mathsf{op}'} (\mathtt{l}', o', \lambda')$ if any of the following conditions is satisfied. If $\mathsf{op} = \mathsf{nop}$, then $\mathsf{op}' = \mathsf{nop}, o' = o, \lambda' = \lambda$. If $\mathsf{op} = \mathtt{n}\texttt{++}$, then $\mathsf{op}' = \mathsf{op}, o' = o, \lambda' = \lambda[\mathtt{n} \mapsto (\lambda(\mathtt{n}) + 1) \bmod M]$, and if $\mathsf{op} = \mathtt{n}\texttt{--}$, then $\mathsf{op}' = \mathsf{op}, o' = o, \lambda' = \lambda[\mathtt{n} \mapsto (\lambda(\mathtt{n}) - 1) \bmod M]$. If $\mathsf{op} = \mathsf{test}(\mathtt{n} \leqslant k)$, then we have the following sequence of transitions for every $0 \leqslant h \leqslant k$: $\mathsf{op}' = \big((\mathtt{n}\texttt{--})^h; \mathsf{test}(\mathtt{n} = 0); (\mathtt{n}\texttt{++})^h\big)$, $o' = o, \lambda' = \lambda$. Upper bound constraints are thus reduced to ordinary zero tests. If $\mathsf{op} = \mathsf{test}(\mathtt{n} \geqslant k)$, then we have a sequence of transitions $\mathsf{op}' = \big((\mathtt{n}\texttt{--})^k; (\mathtt{n}\texttt{++})^k\big)$, $o' = o, \lambda' = \lambda$. If $\mathsf{op} = \mathsf{test}(\mathtt{n} \equiv_M k)$, then we have a transition $\mathsf{op}' = \mathsf{nop}$, $o' = o, \lambda' = \lambda$, provided that $\lambda \models \mathtt{n} \equiv_M k$. If $\mathsf{op} = \mathsf{guess}(\mathtt{r})$, then $\mathsf{op}' = \mathsf{nop}$, $\lambda' = \lambda$, and there is a transition for every orbit $o' \in \mathrm{O}(\mathbb{I}^{\mathtt{R}})$ which agrees with o on $\mathtt{R} \backslash \{\mathtt{r}\}$, and takes an arbitrary value on $\mathtt{r}$, i.e., for every $o' \in \mathrm{O}(\{r' \mid r \in o, r'[\mathtt{r} \mapsto r(\mathtt{r})] = r\})$. Finally, if $\mathsf{op} = \mathsf{test}(\varphi)$, then there is a transition $\mathsf{op}' = \mathsf{nop}$, $o' = o, \lambda' = \lambda$, provided that $o \models \varphi$. It is standard to show that $[\![\mathcal{R}]\!], [\![\mathcal{R}']\!]$ are equivalent [13]. Moreover, if $\mathcal{R}$ has at most one counter with inequality tests, then we obtain a counter machine $\mathcal{R}'$ where at most one counter can be tested for zero, and the latter model is decidable [41, 14]. ◂

A.5 Missing proofs for Sec. 7

▸ **Lemma 9.** *The rendezvous semantics $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ and $[\![\mathcal{R}]\!]$ are equivalent.*

Before delving into the proof, we recall the following basic relationship between $\ominus$ and K_0:

$$b \ominus a \leqslant c \ominus a \qquad \text{iff} \qquad c \ominus b \leqslant c \ominus a \qquad \text{iff} \qquad K_0(a, b, c). \tag{8}$$

Proof. We show that the rendezvous semantics of the TCA $\mathcal{S}$ and the semantics of the RAC $\mathcal{R}$ are related by a variant of weak bisimulation [37]. For a configuration $c \in [\![\mathcal{S}]\!]^{\mathsf{rv}}$ of the form $c = \big\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu \big\rangle$ (we ignore the contents of the channels because they are always empty by the definition of rendezvous semantics) and a configuration $d \in [\![\mathcal{R}]\!]$ of the form $d = \big\langle \big\langle (\ell'^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda \big\rangle, n, r \big\rangle$, we say that they are *equivalent*, written $c \approx d$, if

(1) Control locations are the same: $\ell^{\mathtt{p}} = \ell'^{\mathtt{p}}$ for every $\mathtt{p} \in \mathtt{P}$.

(2) The abstraction λ is the unary class of the local clock valuation:

$$\lambda(\mathtt{x}^{\mathtt{p}}) = [\mu(\mathtt{x}^{\mathtt{p}})], \qquad \text{for every clock } \mathtt{x}^{\mathtt{p}} \in \mathtt{X}. \tag{9}$$

(3) Register $\hat{\mathtt{x}}^{\mathtt{p}}$ keeps track of the fractional part of clock $\mathtt{x}^{\mathtt{p}}$:

$$r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}) = \{\mu(\mathtt{x}^{\mathtt{p}})\}\,, \qquad \text{for every clock } \mathtt{x}^{\mathtt{p}} \in \mathtt{X}. \tag{10}$$

(4) Counter $\mathtt{n}^{\mathtt{pq}}$ measures the integral desynchronisation between $\mathtt{p}$ and $\mathtt{q}$:

$$n(\mathtt{n}^{\mathtt{pq}}) = \lfloor \mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0)\rfloor, \qquad \text{for every channel } \mathtt{pq} \in \mathtt{C}. \tag{11}$$

(5) The fractional desynchronisation between $\mathtt{p}$ and $\mathtt{q}$ is expressed as:

$$r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0) = \{\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0)\}\,, \qquad \text{for every channel } \mathtt{pq} \in \mathtt{C}. \tag{12}$$

We show that $c \approx d$ implies that the two configurations c, d have the same set of runs starting therein. Since the two initial configurations are equivalent $c_I \approx d_I$, it follows that $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ is non-empty iff $[\![\mathcal{R}]\!]$ is non-empty, as required.

Assume $c \approx d$. We show two properties of $\approx$. Let successor configurations c', d' be of the form

$$\begin{aligned} c' &= \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \mu' \right\rangle, \text{ and} \\ d' &= \left\langle \mathtt{l}' = \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda' \right\rangle, n', r' \right\rangle. \end{aligned}$$

- [**Forth property**] For every transition $c \xrightarrow{\mathsf{op}}{}^{\mathsf{rv}} c'$, there is a sequence of transitions $d \rightarrow^* d'$ s.t. again $c' \approx d'$.
- [**Back property**] For every *minimal* sequence of transitions $d \rightarrow^* d'$ there is a transition $c \xrightarrow{\mathsf{op}}{}^{\mathsf{rv}} c'$ s.t. again $c' \approx d'$. Minimality is w.r.t. the length of any sequence of transitions from d to any configuration of the form d' above. (For instance, we do not allow/take in consideration $d \rightarrow^* \langle \bullet, n', r' \rangle$ where the latter is an internal state used during the simulation.)

It is clear that the initial and final configurations of the two systems are $\approx$-equivalent, and thus by the forth and back properties, $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ and $[\![\mathcal{R}]\!]$ are equivalent.

Proof of the forth property. Let $c \xrightarrow{\mathsf{op}}{}^{\mathsf{rv}} c'$. We proceed by case analysis on op.

(1) If $\mathsf{op} = \mathsf{nop}$, then $\mu' = \mu$. We take $\lambda' = \lambda$, $n' = n$, $r' = r$. Clearly $d \xrightarrow{\mathsf{nop}} d'$ with $c' \approx d'$.

(2) Let $\mathsf{op} = \delta \in \mathbb{Q}_{\geqslant 0}$ be a local time elapse operation for process $\mathtt{p}$. Let the amount of time elapsed by $\mathtt{p}$ be $\delta = \mu'(\mathtt{x}^{\mathtt{p}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0) \geqslant 0$. By the definition of desynchronised semantics, $\mu'(\mathtt{x}^{\mathtt{p}}) = \mu(\mathtt{x}^{\mathtt{p}}) + \delta$ for every clock $\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}$ of $\mathtt{p}$, and $\mu'(\mathtt{x}) = \mu(\mathtt{x})$ for every other clock $\mathtt{x} \in \mathtt{X}\backslash\mathtt{X}^{\mathtt{p}}$. We show how to update λ, n, r accordingly. According to the definition of $\mathcal{R}$, we start by taking transition

$$\left\langle \left\langle (\ell^{\mathtt{p}})_{\mathtt{p}\in\mathtt{P}}, \lambda \right\rangle, n, r \right\rangle \xrightarrow{\mathsf{nop}} \left\langle \bullet_\lambda, n, r \right\rangle.$$

We first simulate the integer time elapse $\lfloor \delta \rfloor$. Recall that $\mathtt{N}^+ = \{\mathtt{n}^{\mathtt{qp}} \mid \mathtt{qp} \in \mathtt{C}\}$ is the set of counters corresponding to channels incoming to $\mathtt{p}$ and $\mathtt{N}^- = \{\mathtt{n}^{\mathtt{pq}} \mid \mathtt{pq} \in \mathtt{C}\}$ for outgoing channels. We increase integer values $\lfloor \delta \rfloor$ times, obtaining

$$\left\langle \bullet_\lambda, n, r \right\rangle \xrightarrow{(\mathtt{N}^+\mathtt{++};\mathtt{N}^-\mathtt{--})^{\lfloor \delta \rfloor}} \left\langle \bullet_{\lambda''}, n'', r \right\rangle,$$

where $\lambda'' = \lambda[\mathtt{X}^{\mathtt{p}} \mapsto \mathtt{X}^{\mathtt{p}} + \lfloor \delta \rfloor]$ and $n'' = n[\mathtt{N}^+ \mapsto \mathtt{N}^+ + \lfloor \delta \rfloor, \mathtt{N}^- \mapsto \mathtt{N}^- - \lfloor \delta \rfloor]$. In order for this transition to be legal, it must be the case that for every counter $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^-$, $n(\mathtt{n}^{\mathtt{pq}}) \geqslant \lfloor \delta \rfloor$. By the definition of δ we have $\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0) = \mu'(\mathtt{x}^{\mathtt{q}}_0) - (\mu'(\mathtt{x}^{\mathtt{p}}_0) - \delta) = \mu'(\mathtt{x}^{\mathtt{q}}_0) - \mu'(\mathtt{x}^{\mathtt{p}}_0) + \delta$. By the definition of desynchronised semantics, $\mu'(\mathtt{x}^{\mathtt{q}}_0) \geqslant \mu'(\mathtt{x}^{\mathtt{p}}_0)$, and thus we conclude

$$\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0) \geqslant \delta \tag{13}$$

By (11), $n(\mathtt{n}^{\mathtt{pq}}) = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor$, and thus in particular $n(\mathtt{n}^{\mathtt{pq}}) \geqslant \lfloor \delta \rfloor$.

We now simulate the fractional time elapse $\{\delta\}$. We save the previous value of $\hat{\mathtt{x}}_0^{\mathtt{p}}$ in $\hat{\mathtt{x}}_1^{\mathtt{p}}$ and we guess a new fractional "now" for process $\mathtt{p}$:

$$\left\langle \bullet_{\lambda''}, n'', r \right\rangle \xrightarrow{\mathsf{guess}(\hat{x}_1^{\mathtt{p}});\mathsf{test}(\hat{\mathtt{x}}_1^{\mathtt{p}} = \hat{\mathtt{x}}_0^{\mathtt{p}});\mathsf{guess}(\hat{x}_0^{\mathtt{p}})} \left\langle \bullet^2_{\lambda''}, n'', r' \right\rangle,$$

where $r' = r[\hat{\mathtt{x}}_1^{\mathtt{p}} \mapsto r(\hat{\mathtt{x}}_0^{\mathtt{p}}), \hat{\mathtt{x}}_0^{\mathtt{p}} \mapsto r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \oplus \{\delta\}]$. Eq. (10) is satisfied for μ', r', since

$$\begin{aligned} \left\{\mu'(\mathtt{x}^{\mathtt{p}})\right\} &= \{\mu(\mathtt{x}^{\mathtt{p}}) + \delta\} = \{\mu(\mathtt{x}^{\mathtt{p}})\} \oplus \{\delta\} = && \text{by (10)} \\ &= r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}) \oplus \{\delta\} = r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}) = r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}). \end{aligned}$$

Also Eq. (12) is satisfied, since

$$\begin{aligned} r'(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) &= r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus (r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \oplus \delta) = (r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}})) \ominus \delta = && \text{by (12)} \\ &= \{\mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}})\} \ominus \delta = \left\{\mu'(\mathtt{x}_0^{\mathtt{q}}) - (\mu'(\mathtt{x}_0^{\mathtt{p}}) - \delta)\right\} \ominus \delta = \\ &= \mu'(\mathtt{x}_0^{\mathtt{q}}) \ominus \mu'(\mathtt{x}_0^{\mathtt{p}}) \oplus \delta \ominus \delta = \mu'(\mathtt{x}_0^{\mathtt{q}}) \ominus \mu'(\mathtt{x}_0^{\mathtt{p}}) = \\ &= \left\{\mu'(\mathtt{x}_0^{\mathtt{q}}) - \mu'(\mathtt{x}_0^{\mathtt{p}})\right\}. \end{aligned}$$

We now fix the integer value of those clocks whose fractional value was wrapped around zero one more time than the fractional value of $\mathtt{x}_0^{\mathtt{p}}$. Let $\mathtt{X} = \mathtt{X}^{\mathtt{p}} \cup \{\mathtt{x}^{\mathtt{pq}} \mid \mathtt{x}^{\mathtt{pq}} \in \mathtt{C}\} \cup \{\mathtt{x}^{\mathtt{qp}} \mid \mathtt{x}^{\mathtt{qp}} \in \mathtt{C}\}$ be the set of all possibly affected clocks. The set of local clocks of $\mathtt{p}$ to be further increased by one is $\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}$, the set of counters for incoming channels to be further increased by one is $\mathtt{N}^{+}_{\mathrm{yes}}$, and the set of counters for outgoing channels to be further decreased by one is $\mathtt{N}^{-}_{\mathrm{yes}}$, where:

$$\begin{aligned} \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} &= \left\{\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}} \;\middle|\; \lfloor \mu'(\mathtt{x}^{\mathtt{p}}) \rfloor = \lfloor \mu(\mathtt{x}^{\mathtt{p}}) \rfloor + \lfloor \delta \rfloor + 1\right\}, \\ \mathtt{N}^{+}_{\mathrm{yes}} &= \left\{\mathtt{n}^{\mathtt{qp}} \in \mathtt{N}^{+} \;\middle|\; \lfloor \mu'(\mathtt{x}_0^{\mathtt{p}}) - \mu'(\mathtt{x}_0^{\mathtt{q}}) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{p}}) - \mu(\mathtt{x}_0^{\mathtt{q}}) \rfloor + \lfloor \delta \rfloor + 1\right\}, \text{ and} \\ \mathtt{N}^{-}_{\mathrm{yes}} &= \left\{\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^{-} \;\middle|\; \lfloor \mu'(\mathtt{x}_0^{\mathtt{q}}) - \mu'(\mathtt{x}_0^{\mathtt{p}}) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor - \lfloor \delta \rfloor - 1\right\}. \end{aligned}$$

The set of registers to be checked $\mathtt{S} = \mathtt{S}^{+} \cup \mathtt{S}^{-}$ with $\mathtt{S}^{+} = \{\hat{\mathtt{x}}^{\mathtt{p}} \mid \mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}\} \cup \{\hat{\mathtt{x}}_0^{\mathtt{q}} \mid \mathtt{qp} \in \mathtt{C}\}$, $\mathtt{S}^{-} = \{\hat{\mathtt{x}}_0^{\mathtt{q}} \mid \mathtt{pq} \in \mathtt{C}\}$ is thus partitioned into $\mathtt{S} = \mathtt{S}_{\mathrm{yes}} \cup \mathtt{S}_{\mathrm{no}}$, where $\mathtt{S}_{\mathrm{yes}} = \{\hat{\mathtt{x}}^{\mathtt{p}} \mid \mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}\} \cup \left\{\hat{\mathtt{x}}_0^{\mathtt{q}} \;\middle|\; \mathtt{n}^{\mathtt{qp}} \in \mathtt{N}^{+}_{\mathrm{yes}} \text{ or } \mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^{-}_{\mathrm{yes}}\right\}$ and $\mathtt{S}_{\mathrm{no}} = \mathtt{S} \backslash \mathtt{S}_{\mathrm{yes}}$. We take transition

$$\left\langle \bullet^2_{\lambda''}, n'', r' \right\rangle \xrightarrow{\mathsf{test}(\varphi);(\mathtt{N}^{+}_{\mathrm{yes}})\mathtt{++};(\mathtt{N}^{-}_{\mathrm{yes}})\mathtt{--};\mathsf{nop}} \left\langle \left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda' \right\rangle, n', r' \right\rangle, \tag{14}$$

where φ was defined in (6), $n' = n''[\mathtt{N}^{+}_{\mathrm{yes}} \mapsto \mathtt{N}^{+}_{\mathrm{yes}} + 1, \mathtt{N}^{-}_{\mathrm{yes}} \mapsto \mathtt{N}^{-}_{\mathrm{yes}} - 1]$, and $\lambda' = \lambda''[\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} \mapsto \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} + 1]$. We need to argue that this transition can in fact be taken, and that equations (9) and (11) hold again for λ', n'.

First of all, we argue that $r' \models \varphi$ holds. There are three cases to consider.

1. If $\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} \subseteq \mathtt{S}_{\mathrm{yes}}$, then the integral value of $\mathtt{x}^{\mathtt{p}}$ after time elapse equals $\lfloor \mu'(\mathtt{x}^{\mathtt{p}}) \rfloor = \lfloor \mu(\mathtt{x}^{\mathtt{p}}) + \delta \rfloor = \lfloor \mu(\mathtt{x}^{\mathtt{p}}) \rfloor + \lfloor \delta \rfloor + 1$, which holds precisely when $\{\mu(\mathtt{x}^{\mathtt{p}})\} + \{\delta\} \geqslant 1$. By (10) and by the definition of δ, $(r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}})) + (r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}})) \geqslant 1$. By the definition of r', $(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}})) + (r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}_1^{\mathtt{p}})) \geqslant 1$. This is equivalent to say that the distance on the unit circle of going from $r'(\hat{\mathtt{x}}^{\mathtt{p}})$ to $r'(\hat{\mathtt{x}}_1^{\mathtt{p}})$ and then from the former to $r'(\hat{\mathtt{x}}_0^{\mathtt{p}})$, is at least one. This is the same as saying $K_1(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}), r'(\hat{\mathtt{x}}^{\mathtt{p}}), r'(\hat{\mathtt{x}}_0^{\mathtt{p}}))$ as defined in (5).
2. If $\hat{\mathtt{x}}_0^{\mathtt{q}} \in \mathtt{S}_{\mathrm{yes}} \cap \mathtt{S}^{+}$ (i.e. $\mathtt{n}^{\mathtt{qp}} \in \mathtt{N}^{+}_{\mathrm{yes}}$), then $\lfloor \mu'(\mathtt{x}_0^{\mathtt{p}}) - \mu'(\mathtt{x}_0^{\mathtt{q}}) \rfloor = \lfloor (\mu(\mathtt{x}_0^{\mathtt{p}}) + \delta) - \mu(\mathtt{x}_0^{\mathtt{q}}) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{p}}) - \mu(\mathtt{x}_0^{\mathtt{q}}) + \delta \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{p}}) - \mu(\mathtt{x}_0^{\mathtt{q}}) \rfloor + \lfloor \delta \rfloor + 1$, and the last equality holds precisely when $\{\mu(\mathtt{x}_0^{\mathtt{p}}) - \mu(\mathtt{x}_0^{\mathtt{q}})\} + \{\delta\} \geqslant 1$. By (12) and by the definition of δ, this is equivalent to $(r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{q}})) + (r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}})) \geqslant 1$. By the definition of r', $(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}_0^{\mathtt{q}})) + (r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}_1^{\mathtt{p}})) \geqslant 1$, which, similarly as before, is equivalent to $K_1(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}), r'(\hat{\mathtt{x}}_0^{\mathtt{q}}), r'(\hat{\mathtt{x}}_0^{\mathtt{p}}))$.

3. The argument for $\hat{\mathtt{x}}_0^{\mathtt{q}} \in \mathtt{S}_{\mathrm{yes}} \cap \mathtt{S}^-$ (i.e. $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^-_{\mathrm{yes}}$) is analogous: $\lfloor \mu'(\mathtt{x}_0^{\mathtt{q}}) - \mu'(\mathtt{x}_0^{\mathtt{p}}) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - (\mu(\mathtt{x}_0^{\mathtt{p}}) + \delta) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) - \delta \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor - \lfloor \delta \rfloor - 1$. Since by the desynchronised semantics $\mu'(\mathtt{x}_0^{\mathtt{q}}) - \mu'(\mathtt{x}_0^{\mathtt{p}}) \geqslant 0$ and thus $\mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \geqslant \delta$, the equality $\lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) - \delta \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor - \lfloor \delta \rfloor - 1$ holds precisely when $\{\mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}})\} < \{\delta\}$. By (12) and the definition of δ, this is equivalent to $r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}}) < r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}})$, which by the definition of r' is the same as $r'(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r'(\hat{\mathtt{x}}_1^{\mathtt{p}}) < r'(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}_1^{\mathtt{p}})$. This is the same as saying that, when going along the unit circle, the distance from $r'(\hat{\mathtt{x}}_1^{\mathtt{p}})$ to $r'(\hat{\mathtt{x}}_0^{\mathtt{q}})$ is strictly smaller than the distance from the same $r'(\hat{\mathtt{x}}_1^{\mathtt{p}})$ to $r'(\hat{\mathtt{x}}_0^{\mathtt{p}})$, i.e., $K_2(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}), r'(\hat{\mathtt{x}}_0^{\mathtt{q}}), r'(\hat{\mathtt{x}}_0^{\mathtt{p}}))$ as defined in (5).

Since the three arguments in the previous paragraph are equivalences, for $\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}} \backslash \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}$ $K_1(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}), r'(\hat{\mathtt{x}}^{\mathtt{p}}), r'(\hat{\mathtt{x}}_0^{\mathtt{p}}))$ does not hold. Similarly, for $\mathtt{x}_0^{\mathtt{q}} \in \mathtt{S}_{\mathrm{no}} \cap \mathtt{S}^+$, $K_1(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}), r'(\hat{\mathtt{x}}_0^{\mathtt{q}}), r'(\hat{\mathtt{x}}_0^{\mathtt{p}}))$ does not hold, and for $\hat{\mathtt{x}}_0^{\mathtt{q}} \in \mathtt{S}_{\mathrm{no}} \cap \mathtt{S}^-$, $K_2(r'(\hat{\mathtt{x}}_1^{\mathtt{p}}), r'(\hat{\mathtt{x}}_0^{\mathtt{q}}), r'(\hat{\mathtt{x}}_0^{\mathtt{p}}))$ does not hold. This concludes showing that $r' \models \varphi$ holds.

In order to conclude that the transition (14) can be taken, we need to show that counters in $\mathtt{N}^-_{\mathrm{yes}}$ can be decremented by one, i.e., that for every counter $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^-_{\mathrm{yes}}$, $n''(\mathtt{n}^{\mathtt{pq}}) > 0$. By the definition of n'', this is the same as $n(\mathtt{n}^{\mathtt{pq}}) > \lfloor \delta \rfloor$, and by (11) this is equivalent to $\lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor > \lfloor \delta \rfloor$. By the definition of $\mathtt{N}^-_{\mathrm{yes}}$ above, $\lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor = \lfloor \mu'(\mathtt{x}_0^{\mathtt{q}}) - \mu'(\mathtt{x}_0^{\mathtt{p}}) \rfloor + \lfloor \delta \rfloor + 1$, and by the definition of desynchronised semantics $\mu'(\mathtt{x}_0^{\mathtt{q}}) \geqslant \mu'(\mathtt{x}_0^{\mathtt{p}})$, and thus $\lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor \geqslant \lfloor \delta \rfloor + 1 > \lfloor \delta \rfloor$ as required.

We finally show that (9) and (11) hold again for λ', n'. Consider λ' and we need to show $\lambda'(\mathtt{x}) = [\lfloor \mu'(\mathtt{x}) \rfloor]$ for every clock $\mathtt{x} \in \mathtt{X}$. By the definition of λ', 1) if $\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}$, then $\lambda'(\mathtt{x}^{\mathtt{p}}) = \lambda(\mathtt{x}^{\mathtt{p}}) + \lfloor \delta \rfloor + 1$, 2) if $\mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}} \backslash \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}$, then $\lambda'(\mathtt{x}^{\mathtt{p}}) = \lambda(\mathtt{x}^{\mathtt{p}}) + \lfloor \delta \rfloor$, and 3) for every other $\mathtt{x}^{\mathtt{q}} \in \mathtt{X} \backslash \mathtt{X}^{\mathtt{p}}$, $\lambda'(\mathtt{x}^{\mathtt{q}}) = \lambda(\mathtt{x}^{\mathtt{q}})$. By (9) applied to λ, $\lambda(\mathtt{x}) = [\mu(\mathtt{x})]$. Case 3) is immediate since $\mu'(\mathtt{x}^{\mathtt{q}}) = \mu(\mathtt{x}^{\mathtt{q}})$. Regarding case 1), by definition of $\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}}$ we have $\lfloor \mu'(\mathtt{x}^{\mathtt{p}}) \rfloor = \lfloor \mu(\mathtt{x}^{\mathtt{p}}) \rfloor + \lfloor \delta \rfloor + 1$, and by taking the unary class we have $[\mu'(\mathtt{x}^{\mathtt{p}})] = [\mu(\mathtt{x}^{\mathtt{p}}) + \lfloor \delta \rfloor + 1] = [\mu(\mathtt{x}^{\mathtt{p}})] + \lfloor \delta \rfloor + 1 = \lambda(\mathtt{x}^{\mathtt{p}}) + \lfloor \delta \rfloor + 1 = \lambda'(\mathtt{x}^{\mathtt{p}})$. Case 2) is analogous.

Now consider n', and we need to show $n'(\mathtt{n}^{\mathtt{qr}}) = \lfloor \mu'(\mathtt{x}_0^{\mathtt{r}}) - \mu'(\mathtt{x}_0^{\mathtt{q}}) \rfloor$ for every channel $\mathtt{qr} \in \mathtt{C}$. For every channel $\mathtt{qr}$ not mentioning $\mathtt{p} \notin \{\mathtt{q}, \mathtt{r}\}$, the claim is immediate since $n'(\mathtt{n}^{\mathtt{qr}}) = n(\mathtt{n}^{\mathtt{qr}})$ by the definition of n', and μ', μ take the same value on clocks $\mathtt{x}_0^{\mathtt{r}}$ and, resp., $\mathtt{x}_0^{\mathtt{q}}$. For every counter $\mathtt{n}^{\mathtt{qp}} \in \mathtt{N}^+_{\mathrm{yes}}$ corresponding to an incoming channel $\mathtt{qp}$, by definition of $\mathtt{N}^+_{\mathrm{yes}}$, n', and n'' we have $\lfloor \mu'(\mathtt{x}_0^{\mathtt{p}}) - \mu'(\mathtt{x}_0^{\mathtt{q}}) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{p}}) - \mu(\mathtt{x}_0^{\mathtt{q}}) \rfloor + \lfloor \delta \rfloor + 1 = n(\mathtt{n}^{\mathtt{qp}}) + \lfloor \delta \rfloor + 1 = n'(\mathtt{n}^{\mathtt{qp}})$, as required. If $\mathtt{n}^{\mathtt{qp}} \in \mathtt{N}^+ \backslash \mathtt{N}^+_{\mathrm{yes}}$, then $\lfloor \mu'(\mathtt{x}_0^{\mathtt{p}}) - \mu'(\mathtt{x}_0^{\mathtt{q}}) \rfloor = \lfloor \mu(\mathtt{x}_0^{\mathtt{p}}) - \mu(\mathtt{x}_0^{\mathtt{q}}) \rfloor + \lfloor \delta \rfloor = n(\mathtt{n}^{\mathtt{qp}}) + \lfloor \delta \rfloor = n'(\mathtt{n}^{\mathtt{qp}})$. The two cases $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^-_{\mathrm{yes}}$ and $\mathtt{n}^{\mathtt{pq}} \in \mathtt{N}^- \backslash \mathtt{N}^-_{\mathrm{yes}}$ are similar.

(3) If $\mathsf{op} = \mathsf{test}(\varphi)$ is a test transition on local $\mathtt{p}$-clocks, then $\mu' = \mu$ and $\mu \models \varphi$. Take $\lambda' = \lambda$, $n' = n$, $r' = r$, and thus $c' \approx d'$. It remains to establish $d \xrightarrow{\mathsf{op}'} d'$. There are two cases to consider.

1. In the first case, φ is a non-diagonal inequality or modular constraint. By the definition of $\approx$, the unary class of μ is $[\mu] = \lambda$, and, by the definition of unary equivalence, $\lambda \models \varphi$, and thus the constraint can be checked by reading the local control state. By the definition of $\mathcal{R}$, $d \xrightarrow{\mathsf{op}'} d'$ with $\mathsf{op}' = \mathsf{nop}$.
2. In the second case, $\varphi = \mathtt{x}^{\mathtt{p}} \leqslant \mathtt{y}^{\mathtt{p}}$ is a fractional constraint with $\mathtt{x}^{\mathtt{p}}, \mathtt{y}^{\mathtt{p}} : \mathbb{I}$ fractional clocks. By assumption, $\{\mu(\mathtt{x}^{\mathtt{p}})\} \leqslant \{\mu(\mathtt{y}^{\mathtt{p}})\}$ holds. By (10), $r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}) \leqslant r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \ominus r(\hat{\mathtt{y}}^{\mathtt{p}})$. By the definition of K_0, it holds that $K_0(r(\hat{\mathtt{y}}^{\mathtt{p}}), r(\hat{\mathtt{x}}^{\mathtt{p}}), r(\hat{\mathtt{x}}_0^{\mathtt{p}}))$; cf. (3). Thus $d \xrightarrow{\mathsf{op}'} d'$ with $\mathsf{op}' = \mathsf{test}(K_0(\hat{\mathtt{y}}^{\mathtt{p}}, \hat{\mathtt{x}}^{\mathtt{p}}, \hat{\mathtt{x}}_0^{\mathtt{p}}))$.

(4) If $\mathsf{op} = \mathsf{reset}(\mathtt{x}^{\mathtt{p}})$ is a reset transition, then $\mu' = \mu[\mathtt{x}^{\mathtt{p}} \mapsto 0]$. We update the unary class as $\lambda' = \lambda[\mathtt{x}^{\mathtt{p}} \mapsto 0]$. Counters are unchanged $n' = n$. There are two cases to consider.

1. If $\mathtt{x}^{\mathtt{p}} : \mathbb{N}$ is an integral clock, then also registers are unchanged $r' = r$ and we directly have $d \xrightarrow{\mathsf{nop}} d'$ with $c' \approx d'$.
2. If $\mathtt{x}^{\mathtt{p}} : \mathbb{I}$ is a fractional clock, then we need to update its corresponding register $\hat{\mathtt{x}}^{\mathtt{p}}$ by taking $r' = r[\hat{\mathtt{x}}^{\mathtt{p}} \mapsto r(\hat{\mathtt{x}}^{\mathtt{p}}_0)]$. We execute $\mathsf{op}' = (\mathsf{guess}(\hat{\mathtt{x}}^{\mathtt{p}}); \mathsf{test}(\hat{\mathtt{x}}^{\mathtt{p}} = \hat{\mathtt{x}}^{\mathtt{p}}_0))$ as per the definition of $\mathcal{R}$, and we have $d \xrightarrow{\mathsf{op}'} d'$. After the transitions, (10) is satisfied since $0 = \{\mu'(\mathtt{x}^{\mathtt{p}})\} = r'(\hat{\mathtt{x}}^{\mathtt{p}}) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}_0) = r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0) = 0$.

(5) If $\mathsf{op} = (\mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{p}}); \mathsf{receive}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{q}}))$ is a send-receive pair, then $\mathsf{op}^{\mathtt{p}} = \mathsf{send}(\mathtt{pq}, \mathtt{m} : \psi^{\mathtt{p}})$, $\mathsf{op}^{\mathtt{q}} = \mathsf{receive}(\mathtt{qp}, \mathtt{m} : \psi^{\mathtt{q}})$, and clocks are unchanged $\mu' = \mu$. Thus, the unary abstraction $\lambda' = \lambda$, counters $n' = n$, and registers $r' = r$ are also unchanged. It is clear that $c' \approx d'$. It remains to establish $d \xrightarrow{\mathsf{op}'} d'$ for a suitable choice of op'.

By the definition of rendezvous semantics, there exists a valuation for clock channels $\mu^{\mathtt{pq}} \in \mathbb{Q}^{\mathtt{X}^{\mathtt{pq}}}_{\geqslant 0}$ s.t. $(\mu, \mu^{\mathtt{pq}}) \models \psi^{\mathtt{p}}$ and $(\mu, \mu^{\mathtt{pq}} + \delta) \models \psi^{\mathtt{q}}$ with $\delta = \mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0)$. By (11) and (12),

$$\lfloor \delta \rfloor = n(\mathtt{n}^{\mathtt{pq}}) \qquad \text{and} \qquad \{\delta\} = r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0). \tag{15}$$

There are now three cases to consider.

- **(5a)** In the first case, $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$, $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \sim k$, and thus $\mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) = 0$ and $\lfloor \delta \rfloor \sim k$. By (15), $n(\mathtt{n}^{\mathtt{pq}}) \sim k$. We take $\mathsf{op}' = \mathsf{test}(\mathtt{n}^{\mathtt{pq}} \sim k)$.
- **(5b)** In the second case, $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$, $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \equiv_M k$, and thus $\mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) = 0$ and $\lfloor \delta \rfloor \equiv_M k$. By (15), $n(\mathtt{n}^{\mathtt{pq}}) \equiv_M k$. we take $\mathsf{op}' = \mathsf{test}(\mathtt{n}^{\mathtt{pq}} \equiv_M k)$.
- **(5c)** In the third case, $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$, $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \leqslant \mathtt{x}^{\mathtt{q}}$ for fractional clocks $\mathtt{x}^{\mathtt{pq}}, \mathtt{x}^{\mathtt{q}} : \mathbb{I}$, and thus $\mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) = 0$ and $\{\delta\} \leqslant \{\mu(\mathtt{x}^{\mathtt{q}})\}$. By (10) and (15), this is the same as $r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \leqslant r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{q}})$ which by (8) is equivalent to $K_0(r(\hat{\mathtt{x}}^{\mathtt{q}}), r(\hat{\mathtt{x}}^{\mathtt{p}}_0), r(\hat{\mathtt{x}}^{\mathtt{q}}_0))$. We take $\mathsf{op}' = \mathsf{test}(K_0(\hat{\mathtt{x}}^{\mathtt{q}}, \hat{\mathtt{x}}^{\mathtt{p}}_0, \hat{\mathtt{x}}^{\mathtt{q}}_0))$.

Proof of the back property. Let $d \to^* d'$ be a minimal sequence of transitions. By minimality, no intermediate configuration when going from d to d' is of the form $d' = \left\langle \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda' \right\rangle, n', r' \right\rangle$. By inspection of the definition of $\mathcal{R}$, we need to consider five distinct cases.

(1) In the first case, $\mathcal{R}$ is simulating a nop transition $\ell^{\mathtt{p}} \xrightarrow{\mathsf{nop}}_{\mathtt{p}} \ell'^{\mathtt{p}}$ of $\mathcal{S}$, and thus by minimality $d \xrightarrow{\mathsf{nop}} d'$ in just one step, with $\lambda' = \lambda$, $n' = n$, and $r' = r$. Consequently, $c \xrightarrow{\mathsf{nop}} c'$ in $[\![\mathcal{S}]\!]$ with $c' = \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu \right\rangle$ where $\ell'^{\mathtt{q}} = \ell^{\mathtt{q}}$ for every $\mathtt{q} \in \mathtt{P} \setminus \{\mathtt{p}\}$, and thus $c' \approx d'$ as required.

(2) In the second case, $\mathcal{R}$ is simulating a local elapse transition $\ell^{\mathtt{p}} \xrightarrow{\mathsf{elapse}}_{\mathtt{p}} \ell'^{\mathtt{p}}$ of process $\mathtt{p}$. This is the most involved case. By the definition of $\mathcal{R}$ and by minimality, transitions in $d \to^* d'$ decompose as follows:

$$\begin{aligned} d = \left\langle \left\langle (\ell^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda \right\rangle, n, r \right\rangle \xrightarrow{\mathsf{nop}} \left\langle \bullet_{\lambda}, n, r \right\rangle \xrightarrow{(\mathtt{N}^{+}\texttt{++};\mathtt{N}^{-}\texttt{--})^{\lfloor \delta \rfloor}} \left\langle \bullet_{\lambda''}, n'', r \right\rangle \longrightarrow \\ \xrightarrow{\mathsf{guess}(\hat{x}^{\mathtt{p}}_1);\mathsf{test}(\hat{\mathtt{x}}^{\mathtt{p}}_1 = \hat{\mathtt{x}}^{\mathtt{p}}_0)} \left\langle \bullet^{1}_{\lambda''}, n'', r'' \right\rangle \xrightarrow{\mathsf{guess}(\hat{x}^{\mathtt{p}}_0)} \left\langle \bullet^{2}_{\lambda''}, n'', r' \right\rangle \longrightarrow \\ \xrightarrow{\mathsf{test}(\varphi);(\mathtt{N}^{+}_{\mathrm{yes}})\texttt{++};(\mathtt{N}^{-}_{\mathrm{yes}})\texttt{--}} \left\langle \bullet^{3}_{\lambda'}, n', r' \right\rangle \xrightarrow{\mathsf{nop}} \left\langle \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \lambda' \right\rangle, n', r' \right\rangle = d' \end{aligned}$$

where $\delta \in \mathbb{Q}_{\geqslant 0}$ it the total elapsed timed that is simulated, split into its discrete and fractional part $\delta = \lfloor \delta \rfloor + \{\delta\}$, $\lambda'' = \lambda[\mathtt{X}^{\mathtt{p}} \mapsto \mathtt{X}^{\mathtt{p}} + \lfloor \delta \rfloor]$, $n'' = n[\mathtt{N}^{+} \mapsto \mathtt{N}^{+} + \lfloor \delta \rfloor, \mathtt{N}^{-} \mapsto \mathtt{N}^{-} - \lfloor \delta \rfloor]$, $r'' = r[\hat{x}^{\mathtt{p}}_1 \mapsto r(\hat{x}^{\mathtt{p}}_0)]$, $r' = r''[\hat{x}^{\mathtt{p}}_0 \mapsto r(\hat{x}^{\mathtt{p}}_0) \oplus \{\delta\}]$, $r' \models \varphi$, $\lambda' = \lambda''[\mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} \mapsto \mathtt{X}^{\mathtt{p}}_{\mathrm{yes}} + 1]$, and $n' = n''[\mathtt{N}^{+}_{\mathrm{yes}} \mapsto \mathtt{N}^{+}_{\mathrm{yes}} + 1, \mathtt{N}^{-}_{\mathrm{yes}} \mapsto \mathtt{N}^{-}_{\mathrm{yes}} - 1]$. This is simulated in $\mathcal{S}$ by letting process $\mathtt{p}$ elapse δ time units and thus go to $c' = \left\langle (\ell'^{\mathtt{p}})_{\mathtt{p} \in \mathtt{P}}, \mu' \right\rangle$, with $\ell'^{\mathtt{q}} = \ell^{\mathtt{q}}$ for every $\mathtt{q} \in \mathtt{P} \setminus \{\mathtt{p}\}$, where $\mu' = \mu[\forall \mathtt{x}^{\mathtt{p}} \in \mathtt{X}^{\mathtt{p}} \cdot \mathtt{x}^{\mathtt{p}} \mapsto \mu(\mathtt{x}^{\mathtt{p}}) + \delta]$ (including the reference clock $\mathtt{x}^{\mathtt{p}}_0$). We need to show that the time elapse transition above is legal in $\mathcal{S}$, which by the desynchronised semantics amounts to establish that for every channel $\mathtt{qr} \in \mathtt{C}$, $\mu'(\mathtt{x}^{\mathtt{q}}_0) \leqslant \mu'(\mathtt{x}^{\mathtt{r}}_0)$. Since the value of $\mathtt{x}^{\mathtt{p}}_0$ increased during the time elapse transition, $\mu(\mathtt{x}^{\mathtt{q}}_0) = \mu'(\mathtt{x}^{\mathtt{q}}_0) \leqslant \mu'(\mathtt{x}^{\mathtt{p}}_0) = \mu(\mathtt{x}^{\mathtt{p}}_0) + \delta$ is immediately satisfied for incoming channels $\mathtt{qp} \in \mathtt{C}$ since $\mu(\mathtt{x}^{\mathtt{q}}_0) \leqslant \mu(\mathtt{x}^{\mathtt{p}}_0)$ follows from the fact that c is a legal configuration in $[\![\mathcal{S}]\!]$. Let $\mathtt{pq} \in \mathtt{C}$ be an outgoing channel and we need to establish $\mu'(\mathtt{x}^{\mathtt{q}}_0) - \mu'(\mathtt{x}^{\mathtt{p}}_0) \geqslant 0$. The latter inequality will follow immediately from establishing (11) and (12) for n', r', μ'. For fractional parts, we have

$$\begin{aligned}
\left\{\mu'(\mathtt{x}^{\mathtt{q}}_0) - \mu'(\mathtt{x}^{\mathtt{p}}_0)\right\} &= \{\mu(\mathtt{x}^{\mathtt{q}}_0) - (\mu(\mathtt{x}^{\mathtt{p}}_0) + \delta)\} = \{\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0)\} \ominus \delta = && \text{(by (12))} \\
&= (r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0)) \ominus \delta = (r'(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus (r'(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus \delta)) \ominus \delta = \\
&= r'(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}_0),
\end{aligned}$$

and thus (12) is again satisfied for r', μ'. For integral parts, we consider two cases, depending on whether the channel is incoming or outgoing.

1. For an outgoing channel $\mathtt{pq}$,

$$\begin{aligned}
&\lfloor\mu'(\mathtt{x}^{\mathtt{q}}_0) - \mu'(\mathtt{x}^{\mathtt{p}}_0)\rfloor = \lfloor\mu(\mathtt{x}^{\mathtt{q}}_0) - (\mu(\mathtt{x}^{\mathtt{p}}_0) + \delta)\rfloor = \\
&= \lfloor\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0) - \delta\rfloor = \\
&= \lfloor\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0)\rfloor - \lfloor\delta\rfloor - \mathbb{1}_{\left\{\mu(\mathtt{x}^{\mathtt{q}}_0) - \mu(\mathtt{x}^{\mathtt{p}}_0)\right\} < \{\delta\}?} = && \text{(by (11), (12))} \\
&= n(\mathtt{n}^{\mathtt{pq}}) - \lfloor\delta\rfloor - \mathbb{1}_{r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0) < \{\delta\}?} = && \text{(by the def. of } \delta) \\
&= n(\mathtt{n}^{\mathtt{pq}}) - \lfloor\delta\rfloor - \mathbb{1}_{r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0) < r'(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0)?} = && \text{(by the def. of } r') \\
&= n(\mathtt{n}^{\mathtt{pq}}) - \lfloor\delta\rfloor - \mathbb{1}_{r'(\hat{\mathtt{x}}^{\mathtt{q}}_0) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}_1) < r'(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}_1)?} = && \text{(by the def. of } K_2) \\
&= n(\mathtt{n}^{\mathtt{pq}}) - \lfloor\delta\rfloor - \mathbb{1}_{K_2(r'(\hat{\mathtt{x}}^{\mathtt{p}}_1), r'(\hat{\mathtt{x}}^{\mathtt{q}}_0), r'(\hat{\mathtt{x}}^{\mathtt{p}}_0))?} = && \text{(by the def. of } n') \\
&= n'(\mathtt{n}^{\mathtt{pq}}),
\end{aligned}$$

thus showing that (11) is again satisfied for n', μ' for outgoing channels $\mathtt{pq}$.

2. For an incoming channel $\mathtt{qp}$,

$$\begin{aligned}
&\lfloor\mu'(\mathtt{x}^{\mathtt{p}}_0) - \mu'(\mathtt{x}^{\mathtt{q}}_0)\rfloor = \lfloor\mu(\mathtt{x}^{\mathtt{p}}_0) + \delta - \mu(\mathtt{x}^{\mathtt{q}}_0)\rfloor = \\
&= \lfloor\mu(\mathtt{x}^{\mathtt{p}}_0) - \mu(\mathtt{x}^{\mathtt{q}}_0)\rfloor + \lfloor\delta\rfloor + \mathbb{1}_{\left\{\mu(\mathtt{x}^{\mathtt{p}}_0) - \mu(\mathtt{x}^{\mathtt{q}}_0)\right\} + \{\delta\} \geqslant 1?} = && \text{(by (11), (12))} \\
&= n(\mathtt{n}^{\mathtt{qp}}) + \lfloor\delta\rfloor + \mathbb{1}_{r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{q}}_0) + \{\delta\} \geqslant 1?} = && \text{(by the def. of } \delta) \\
&= n(\mathtt{n}^{\mathtt{qp}}) + \lfloor\delta\rfloor + \mathbb{1}_{(r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{q}}_0)) + (r'(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}_0)) \geqslant 1?} = \\
&= n(\mathtt{n}^{\mathtt{qp}}) + \lfloor\delta\rfloor + \mathbb{1}_{r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{q}}_0) \geqslant r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}_0)?} = && \text{(by the def. of } r') \\
&= n(\mathtt{n}^{\mathtt{qp}}) + \lfloor\delta\rfloor + \mathbb{1}_{r'(\hat{\mathtt{x}}^{\mathtt{p}}_1) \ominus r'(\hat{\mathtt{x}}^{\mathtt{q}}_0) \geqslant r'(\hat{\mathtt{x}}^{\mathtt{p}}_1) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}_0)?} = && \text{(by the def. of } K_1) \\
&= n(\mathtt{n}^{\mathtt{qp}}) + \lfloor\delta\rfloor + \mathbb{1}_{K_1(r'(\hat{\mathtt{x}}^{\mathtt{p}}_1), r'(\hat{\mathtt{x}}^{\mathtt{q}}_0), r'(\hat{\mathtt{x}}^{\mathtt{p}}_0))?} = && \text{(by def. of } n') \\
&= n'(\mathtt{n}^{\mathtt{pq}}).
\end{aligned}$$

thus showing that (11) is again satisfied for n', μ' for incoming channels $\mathtt{qp}$.

Also (10) holds:

$$\begin{aligned}
\left\{\mu'(\mathtt{x}^{\mathtt{p}})\right\} &= \{\mu(\mathtt{x}^{\mathtt{p}}) + \delta\} = \{\mu(\mathtt{x}^{\mathtt{p}})\} \oplus \delta = && \text{(by (10))} \\
&= (r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}})) \oplus \delta = (r(\hat{\mathtt{x}}^{\mathtt{p}}_0) \oplus \delta) \ominus r(\hat{\mathtt{x}}^{\mathtt{p}}) = \\
&= r'(\hat{\mathtt{x}}^{\mathtt{p}}_0) \ominus r'(\hat{\mathtt{x}}^{\mathtt{p}}).
\end{aligned}$$

Finally, also (9) holds:

$$\begin{aligned}
[\mu'(\mathtt{x}^{\mathsf{p}})] &= [\mu(\mathtt{x}^{\mathsf{p}}) + \delta] = \\
&= [\mu(\mathtt{x}^{\mathsf{p}})] + \lfloor\delta\rfloor + \mathbb{1}_{\{\mu(\mathtt{x}^{\mathsf{p}})\}+\{\delta\}\geqslant 1?} = && \text{(by (9), (10))} \\
&= \lambda(\mathtt{x}^{\mathsf{p}}) + \lfloor\delta\rfloor + \mathbb{1}_{r(\hat{\mathtt{x}}^{\mathsf{p}}_0)\ominus r(\hat{\mathtt{x}}^{\mathsf{p}})+\{\delta\}\geqslant 1?} = && \text{(by the def. of } \delta) \\
&= \lambda(\mathtt{x}^{\mathsf{p}}) + \lfloor\delta\rfloor + \mathbb{1}_{(r(\hat{\mathtt{x}}^{\mathsf{p}}_0)\ominus r(\hat{\mathtt{x}}^{\mathsf{p}}))+(r'(\hat{\mathtt{x}}^{\mathsf{p}}_0)\ominus r(\hat{\mathtt{x}}^{\mathsf{p}}_0))\geqslant 1?} = && \text{(by the def. of } r') \\
&= \lambda(\mathtt{x}^{\mathsf{p}}) + \lfloor\delta\rfloor + \mathbb{1}_{(r'(\hat{\mathtt{x}}^{\mathsf{p}}_1)\ominus r'(\hat{\mathtt{x}}^{\mathsf{p}}))+(r'(\hat{\mathtt{x}}^{\mathsf{p}}_0)\ominus r'(\hat{\mathtt{x}}^{\mathsf{p}}_1))\geqslant 1?} = \\
&= \lambda(\mathtt{x}^{\mathsf{p}}) + \lfloor\delta\rfloor + \mathbb{1}_{r'(\hat{\mathtt{x}}^{\mathsf{p}}_1)\ominus r'(\hat{\mathtt{x}}^{\mathsf{p}})\geqslant r'(\hat{\mathtt{x}}^{\mathsf{p}}_1)\ominus r'(\hat{\mathtt{x}}^{\mathsf{p}}_0)?} = && \text{(by the def. of } K_1) \\
&= \lambda(\mathtt{x}^{\mathsf{p}}) + \lfloor\delta\rfloor + \mathbb{1}_{K_1(r'(\hat{\mathtt{x}}^{\mathsf{p}}_1), r'(\hat{\mathtt{x}}^{\mathsf{p}}), r'(\hat{\mathtt{x}}^{\mathsf{p}}_0))?} = && \text{(by def. of } \lambda') \\
&= \lambda'(\mathtt{x}^{\mathsf{p}}).
\end{aligned}$$

Altogether, this establishes that the transition $c \xrightarrow{\delta} c'$ is legal and that $c' \approx d'$, as required.

(3) In the third case, $\mathcal{R}$ is simulating a test transition $\ell^{\mathsf{p}} \xrightarrow{\mathsf{test}(\varphi)}_{\mathsf{p}} \not\ell^{\mathsf{p}}$. By minimality, $d \xrightarrow{\mathsf{op}} d'$ with $d' = \left\langle\left\langle(\ell'^{\mathsf{p}})_{\mathsf{p}\in\mathsf{P}}, \lambda\right\rangle, n, r\right\rangle$ where $\ell'^{\mathsf{q}} = \ell^{\mathsf{q}}$ for every $\mathsf{q} \in \mathsf{P}\setminus\{\mathsf{p}\}$. Accordingly, we take $c' = \left\langle(\ell'^{\mathsf{p}})_{\mathsf{p}\in\mathsf{P}}, \mu\right\rangle$ and thus $c' \approx d'$ follows immediately from $c \approx d$. We need to show that $c \xrightarrow{\mathsf{test}(\varphi)} c'$. Following the definition of $\mathcal{R}$, we proceed by a case analysis on op.

1. If $\mathsf{op} = \mathsf{nop}$, then $\varphi \equiv \mathtt{x}^{\mathsf{p}} \sim k$ or $\varphi \equiv \mathtt{x}^{\mathsf{p}} \equiv_m k$ and it holds that $\lambda \models \varphi$. In the first case, this means that $\lambda(\mathtt{x}^{\mathsf{p}}) \sim k$ holds. By (9), $\lambda(\mathtt{x}^{\mathsf{p}}) = [\mu(\mathtt{x}^{\mathsf{p}})]$ and since the unary equivalence is sound when computed w.r.t. the maximal constant, $\lfloor\mu(\mathtt{x}^{\mathsf{p}})\rfloor \sim k$ holds. Since $\mathtt{x}^{\mathsf{p}} : \mathbb{N}$ is an integral clock, $\mu \models \varphi$ holds, as required. The reasoning for the second case is analogous.

2. If $\mathsf{op} = \mathsf{test}(K_0(\hat{\mathtt{y}}^{\mathsf{p}}, \hat{\mathtt{x}}^{\mathsf{p}}, \hat{\mathtt{x}}^{\mathsf{p}}_0))$, then $\varphi \equiv \mathtt{x}^{\mathsf{p}} \leqslant \mathtt{y}^{\mathsf{p}}$ for fractional clocks $\mathtt{x}^{\mathsf{p}}, \mathtt{y}^{\mathsf{p}} : \mathbb{I}$. Thus $K_0(r(\hat{\mathtt{y}}^{\mathsf{p}}), r(\hat{\mathtt{x}}^{\mathsf{p}}), r(\hat{\mathtt{x}}^{\mathsf{p}}_0))$ holds. By (3), $r(\hat{\mathtt{x}}^{\mathsf{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathsf{p}}) \leqslant r(\hat{\mathtt{x}}^{\mathsf{p}}_0) \ominus r(\hat{\mathtt{y}}^{\mathsf{p}})$. By (10), $\{\mu(\mathtt{x}^{\mathsf{p}})\} \leqslant \{\mu(\mathtt{y}^{\mathsf{p}})\}$, and thus $\mu \models \varphi$ holds, as required.

(4) In the fourth case, $\mathcal{R}$ is simulating a reset transition $\ell^{\mathsf{p}} \xrightarrow{\mathsf{reset}(\mathtt{x}^{\mathsf{p}})}_{\mathsf{p}} \not\ell^{\mathsf{p}}$ for a clock $\mathtt{x}^{\mathsf{p}}$ of process p. By minimality, $d \xrightarrow{\mathsf{op}} d'$ with $d' = \left\langle\left\langle(\ell'^{\mathsf{p}})_{\mathsf{p}\in\mathsf{P}}, \lambda'\right\rangle, n, r'\right\rangle$ where $\ell'^{\mathsf{q}} = \ell^{\mathsf{q}}$ for every $\mathsf{q} \in \mathsf{P}\setminus\{\mathsf{p}\}$. We take $c' = \left\langle(\ell'^{\mathsf{p}})_{\mathsf{p}\in\mathsf{P}}, \mu'\right\rangle$ with $\mu' = \mu[\mathtt{x}^{\mathsf{p}} \mapsto 0]$. Clearly, $c \xrightarrow{\mathsf{reset}(\mathtt{x}^{\mathsf{p}})} c'$ holds. In order to show that (9), (10) hold again for λ', r', μ', we do a case analysis on op.

1. In the first case, $\mathsf{op} = \mathsf{nop}$. By the definition of $\mathcal{R}$, $\mathtt{x}^{\mathsf{p}} : \mathbb{N}$ is an integral clock and $\lambda' = \lambda[\mathtt{x}^{\mathsf{p}} \mapsto 0]$ and $r' = r$. Obviously $\lambda'(\mathtt{x}^{\mathsf{p}}) = [0] = [\mu'(\mathtt{x}^{\mathsf{p}})]$, and for every other clock $\mathtt{x}^{\mathsf{q}} \neq \mathtt{x}^{\mathsf{p}}$, $\lambda'(\mathtt{x}^{\mathsf{p}}) = \lambda(\mathtt{x}^{\mathsf{p}}) = \text{(by (9))} = [\mu(\mathtt{x}^{\mathsf{q}})] = [\mu'(\mathtt{x}^{\mathsf{q}})]$. Thus, (9) holds again for λ', μ'. (That (10) holds is trivial since $r' = r$ and $\mu'(\mathtt{x}^{\mathsf{q}}) = \mu(\mathtt{x}^{\mathsf{q}})$ for fractional clocks $\mathtt{x}^{\mathsf{q}} : \mathbb{I}$.)

2. In the second case, $\mathsf{op} = (\mathsf{guess}(\hat{\mathtt{x}}^{\mathsf{p}}); \mathsf{test}(\hat{\mathtt{x}}^{\mathsf{p}} = \hat{\mathtt{x}}^{\mathsf{p}}_0))$. Consequently, $r' = r[\hat{\mathtt{x}}^{\mathsf{p}} \mapsto r(\hat{\mathtt{x}}^{\mathsf{p}}_0)]$. Therefore, $r'(\hat{\mathtt{x}}^{\mathsf{p}}_0) \ominus r'(\hat{\mathtt{x}}^{\mathsf{p}}) = r(\hat{\mathtt{x}}^{\mathsf{p}}_0) \ominus r(\hat{\mathtt{x}}^{\mathsf{p}}_0) = 0 = \{\mu'(\mathtt{x}^{\mathsf{p}})\}$. Thus, (10) holds again for r', μ'. (That (9) holds is trivial since $\lambda' = \lambda$ and $\mu'(\mathtt{x}^{\mathsf{q}}) = \mu(\mathtt{x}^{\mathsf{q}})$ for integral clocks $\mathtt{x}^{\mathsf{q}} : \mathbb{N}$.)

(5) In the fifth, and last case, $\mathcal{R}$ simulates a send-receive pair of transitions $\ell^{\mathsf{p}} \xrightarrow{\mathsf{op}^{\mathsf{p}}}_{\mathsf{p}} \not\ell^{\mathsf{p}}$ of p with $\mathsf{op}^{\mathsf{p}} = \mathsf{send}(\mathsf{pq}, m : \psi^{\mathsf{p}})$ and $\ell^{\mathsf{q}} \xrightarrow{\mathsf{op}^{\mathsf{q}}}_{\mathsf{q}} \not\ell^{\mathsf{q}}$ of q with $\mathsf{op}^{\mathsf{q}} = \mathsf{receive}(\mathsf{pq}, m : \psi^{\mathsf{q}})$. By the definition of $\mathcal{R}$ and by minimality, $d \xrightarrow{\mathsf{test}(()\varphi)} d'$ with $d' = \left\langle\left\langle(\ell'^{\mathsf{p}})_{\mathsf{p}\in\mathsf{P}}, \lambda\right\rangle, n, r\right\rangle$ where $\ell'^{\mathtt{r}} = \ell^{\mathtt{r}}$ for every $\mathtt{r} \in \mathsf{P}\setminus\{\mathsf{p}, \mathsf{q}\}$. We take $c' = \left\langle(\ell'^{\mathsf{p}})_{\mathsf{p}\in\mathsf{P}}, \mu\right\rangle$ and we need to argue that in $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ we can take the rendezvous transition $c \xrightarrow{(\mathsf{op}^{\mathsf{p}}, \mathsf{op}^{\mathsf{q}})} c'$. Let $\delta = \mu(\mathtt{x}^{\mathsf{q}}_0) - \mu^(\mathtt{x}^{\mathsf{p}}_0) \geqslant 0$ be the desynchronisation between sender and receiver. Following the definition of desynchronised semantics, we need to show that there exists a valuation for clock channels $\mu^{\mathsf{pq}} \in \mathbb{Q}_{\geqslant 0}^{\mathtt{X}^{\mathsf{pq}}}$ s.t. $(\mu, \mu^{\mathsf{pq}}) \models \psi^{\mathsf{p}}$ and $(\mu, \mu^{\mathsf{pq}} + \delta) \models \psi^{\mathsf{q}}$. We proceed by a case analysis on the condition φ.

(5a) In the first case, $\varphi \equiv \mathtt{n}^{\mathtt{pq}} \sim k$ is an inequality counter constraint, and thus $n(\mathtt{n}^{\mathtt{pq}}) \sim k$ holds. Then, $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$ and $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \sim k$ with $\mathtt{x}^{\mathtt{pq}} : \mathbb{N}$ an integral clock. Take $\mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) = 0$. Clearly $(\mu, \mu^{\mathtt{pq}}) \models \psi^{\mathtt{p}}$ is satisfied. By (11), $n(\mathtt{n}^{\mathtt{pq}}) = \lfloor \mu(\mathtt{x}_0^{\mathtt{q}}) - \mu(\mathtt{x}_0^{\mathtt{p}}) \rfloor = \lfloor \delta \rfloor$ and thus $\lfloor \delta \rfloor = \lfloor \mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) + \delta \rfloor \sim k$ holds. Since $\mathtt{x}^{\mathtt{pq}} : \mathbb{N}$ is an integral clock, the latter is equivalent to $\mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) + \delta \models \mathtt{x}^{\mathtt{pq}} \sim k$, thus showing $(\mu, \mu^{\mathtt{pq}} + \delta) \models \mathtt{x}^{\mathtt{pq}} \sim k$, as required.

(5b) In the second case, $\varphi \equiv \mathtt{n}^{\mathtt{pq}} \equiv_M k$ is a modular counter constraint, and we reason as above.

(5c) In the last case, $\varphi \equiv K_0(\hat{\mathtt{x}}^{\mathtt{q}}, \hat{\mathtt{x}}_0^{\mathtt{p}}, \hat{\mathtt{x}}_0^{\mathtt{q}})$ is a register constraint; thus $K_0(r(\hat{\mathtt{x}}^{\mathtt{q}}), r(\hat{\mathtt{x}}_0^{\mathtt{p}}), r(\hat{\mathtt{x}}_0^{\mathtt{q}}))$ holds. Then, $\psi^{\mathtt{p}} \equiv \mathtt{x}^{\mathtt{pq}} = 0$ and $\psi^{\mathtt{q}} \equiv \mathtt{x}^{\mathtt{pq}} \leqslant \mathtt{x}^{\mathtt{q}}$ with $\mathtt{x}^{\mathtt{pq}}, \mathtt{x}^{\mathtt{q}} : \mathbb{I}$ two fractional clocks. Take $\mu^{\mathtt{pq}}(\mathtt{x}^{\mathtt{pq}}) = 0$. Clearly $(\mu, \mu^{\mathtt{pq}}) \models \psi^{\mathtt{p}}$ is satisfied. By the definition of K_0, $r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}}) \leqslant r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r(\hat{\mathtt{x}}^{\mathtt{q}})$ (cf. (4)). By (12), $r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r(\hat{\mathtt{x}}_0^{\mathtt{p}}) = \{\delta\}$, and by (10), $r(\hat{\mathtt{x}}_0^{\mathtt{q}}) \ominus r(\hat{\mathtt{x}}^{\mathtt{q}}) = \{\mu(\mathtt{x}^{\mathtt{q}})\}$. Thus, $\{\delta\} \leqslant \{\mu(\mathtt{x}^{\mathtt{q}})\}$, that is $(\mu, \mu^{\mathtt{pq}} + \delta) \models \mathtt{x}^{\mathtt{pq}} \leqslant \mathtt{x}^{\mathtt{q}}$, as required. ◀

A.6 Missing proofs for Sec. 3

Proof of the "only if" direction. If the topology is not a polyforest, i.e., it contains an undirected cycle, then it is well-known that non-emptiness is undecidable already in the untimed setting [16, 38]. If the topology is a polyforest, but it contains a polytree with more than one timed channel with integral inequality tests, then undecidability follows from [20, Theorem 3] already in discrete time, since non-emptiness tests (on the side of the receiver) can be simulated by timed channels with inequality tests as remarked above. ◀

Proof of the "if" direction of Theorem 1. Let $\mathcal{S}$ be a TCA over a polyforest topology, where in each polytree there is at most one channel with integral inequality tests. By Lemma 6 the standard semantics $[\![\mathcal{S}]\!]$ is equivalent to the desynchronised one $[\![\mathcal{S}]\!]^{\mathsf{de}}$, which in turn is equivalent to the rendezvous one $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ by Lemma 7. By the transformations of Sec. 4 we an assume that the TCA is simple. This allows us to apply the construction of this section in order to build a RAC $[\![\mathcal{R}]\!]$ s.t. the rendezvous semantics $[\![\mathcal{S}]\!]^{\mathsf{rv}}$ is equivalent to $[\![\mathcal{R}]\!]$ by Lemma 9. Suppose the topology $\mathcal{T}$ decomposes into n disjoint polytrees $\mathcal{T}_1, \ldots, \mathcal{T}_n$, where by assumption in each of the $\mathcal{T}_i$'s there is at most one channel with integral inequality tests. We obtain a RAC $\mathcal{R}$ with counters of which n have threshold tests, and thus unless $n = 1$ we cannot apply immediately Theorem 8 to obtain decidability of the non-emptiness problem. With a small modification of the construction of $\mathcal{R}$ instead of simulating all polytrees $\mathcal{T}_1, \ldots, \mathcal{T}_n$ in parallel, we can simulate them sequentially by running $\mathcal{T}_1$ first, followed by $\mathcal{T}_2$, ..., till $\mathcal{T}_n$ (cf. [20, Theorem 3]). In order for the sequential simulation to be faithful, we need to ensure that the same total amount of time elapses when simulating any of the $\mathcal{T}_i$'s. For the integral part of the elapsed time, we can add an extra counter $\mathtt{n}_{\mathcal{T}_i}$ for each component which is increased by one every time some fixed process $\mathtt{p}$ therein elapses 1 time unit; at the end of all simulations, we additionally check that $\mathtt{n}_{\mathcal{T}_1} = \cdots = \mathtt{n}_{\mathcal{T}_n}$ by decreasing all such counters by 1 until they all hit 0. (Notice that at the end of the simulation of $\mathcal{T}_i$ all processes therein elapse the same amount of time since we require all counters $\mathtt{n}^{\mathtt{pq}}$ to be 0 at the end of the run.) For the fractional part of the elapsed time no additional check is needed, since reference registers $\hat{\mathtt{x}}_0^{\mathtt{p}} = 0$ at the end of the run by construction. In this way it suffices to have only one counter with threshold tests which is reused in the subsequent simulations, and we obtain decidability by Theorem 8. ◀