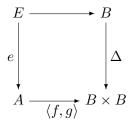
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Homework 6

1. (Equalizers by pullbacks and products) Show that a category with pullbacks and products has equalizers as follows: given arrows $f, g: A \to B$, take the pullback indicated below, where $\Delta = \langle 1_B, 1_B \rangle$:



Show that $e: E \to A$ is the equalizer of f and g.

2. Suppose the category \mathbf{C} has limits of type \mathbf{J} , for some index category \mathbf{J} . For diagrams F and G of type \mathbf{J} in \mathbf{C} , a morphism of diagrams $\theta: F \to G$ consists of arrows $\theta_i: Fi \to Gi$ for each $i \in \mathbf{J}$ such that for each $\alpha: i \to j$ in \mathbf{J} , one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square). This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).

Show that taking the vertex-objects of limiting cones determines a functor:

$$\varprojlim_{\mathbf{J}}:\mathbf{Diagrams}(\mathbf{J},\mathbf{C})\to\mathbf{C}$$

Conclude that for any set I, there is a product functor,

$$\prod_{i \in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

for *I*-indexed families of sets $(A_i)_{i \in I}$, generalizing the binary product functor $\times : \mathbf{Sets} \times \mathbf{Sets} \to \mathbf{Sets}$.

3. (a) Consider the sequence of posets $[0] \rightarrow [1] \rightarrow [2] \rightarrow \dots$, where

$$[n] = \{0 \le \dots \le n\},\$$

and the arrows $[n] \rightarrow [n+1]$ are the evident inclusions. Determine the limit and colimit posets of this sequence.

(b) Do the same for the sequence of powerset boolean algebras,

$$\mathcal{P}(0) \leftarrow \mathcal{P}(1) \leftarrow \mathcal{P}(2) \leftarrow \dots,$$

where the maps are determined by inverse image along the inclusions $0 \subseteq 1 \subseteq 2 \subseteq \dots$

4. Consider sequences of monoids,

$$M_0 \to M_1 \to M_2 \to \dots$$

 $N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \dots$

and the following limits and colimits, constructed in the category of monoids:

$$\underset{n}{\varinjlim} M_n, \quad \underset{n}{\varprojlim} M_n, \quad \underset{n}{\varinjlim} N_n, \quad \underset{n}{\varprojlim} N_n.$$

- (a) Suppose all M_n and N_n are abelian groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. is also an abelian group.
- (b) Suppose all M_n and N_n are finite groups. Determine whether each of the four (co)limits $\varinjlim_n M_n$ etc. has the following property: for every element x there is a number k such that $x^k = 1$ (the least such k is called the *order* of x).
- 5. * Show that the forgetful functor $U: \mathbf{Mon} \to \mathbf{Sets}$ creates all limits.