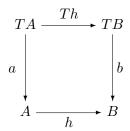
Homework 12

- 1. (a) A coHeyting algebra is a poset P such that P^{op} is a Heyting algebra. Determine the coHeyting implication operation a/b in a lattice L by adjointness (with respect to joins), and show that any Boolean algebra is a coHeyting algebra by explicitly defining this operation a/b in terms of the usual Boolean ones.
 - (b) In a coHeyting algebra, there are operations of coHeyting negation $\sim p = 1/p$ and coHeyting boundary $\partial p = p \wedge \sim p$. State logical rules of inference for these operations.
 - (c) A biHeyting algebra is a lattice that is both Heyting and coHeyting. Give an example of a biHeyting algebra that is not Boolean. (Hint: consider the lower sets in a poset.)
- 2. Let T be the equational theory with one constant symbol and one unary function symbol (no axioms). In any category with a terminal object, a natural numbers object (NNO) is just an initial T-model. Show that

$$(\mathbb{N}, 0 \in \mathbb{N}, n+1 : \mathbb{N} \to \mathbb{N})$$

is a NNO in \mathbf{Sets} , and that any NNO is uniquely isomorphic to it (as a \mathbb{T} -model).

3. (a) Let ${\bf C}$ be a category and $T:{\bf C}\to {\bf C}$ an endofunctor. A T-algebra consists of an object A and an arrow $a:TA\to A$ in ${\bf C}$. A morphism $h:(a,A)\to (b,B)$ of T-algebras is a morphism $h:A\to B$ in ${\bf C}$ such that $h\circ a=b\circ T(h)$.



Let \mathbf{C} be a category with a terminal object 1 and binary coproducts. Let $T: \mathbf{C} \to \mathbf{C}$ be the endofunctor

$$T(C) = 1 + C.$$

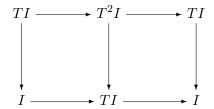
Show that the category of T-algebras in this sense is equivalent to the category of \mathbb{T} -algebras (i.e. models) for the algebraic theory \mathbb{T} with just one nullary and one unary operation,

$$T$$
-Alg $\simeq \mathbb{T}$ -Alg.

Conclude that free T-algebras exist in **Sets**, and that an initial T-algebra is the same thing as an NNO.

4. ("Lambek's Lemma") Show that for any endofunctor $T: \mathbf{C} \to \mathbf{C}$, if $i: TI \to I$ is an initial T-algebra, then i is an isomorphism.

Hint: Consider a diagram of the following form, with suitable arrows.



Conclude that for any NNO N in any category (with 1 and +), there is an isomorphism $N+1 \cong N$.

5. * (Lawvere's Hyperdoctrine Diagram)

Recall the inclusion functor $i: P(I) \to \mathbf{Sets}/I$ that takes a subset $U \subseteq I$ to its inclusion function $i(U): U \to I$. We know from last week that this functor has a left adjoint,

$$\sigma: \mathbf{Sets}/I \longrightarrow P(I),$$

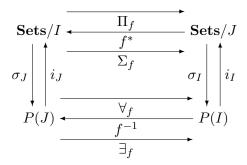
taking any $\alpha: A \to I$ to its image

$$\sigma(\alpha) = {\alpha(a) \in I \mid a \in A} \subseteq I.$$

Given any function

$$f: J \to I$$

consider the following diagram of functors.



There are adjunctions

$$\sigma\dashv i$$

(for both J and I), as well as

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

and

$$\exists_f\dashv f^{-1}\dashv \forall_f$$

where $f^*: \mathbf{Sets}/I \to \mathbf{Sets}/J$ is pullback, and $f^{-1}: P(I) \to P(J)$ is inverse image.

Consider, which of the 6 possible squares involving either i or σ and corresponding pairs of f^{-1} and f^* , etc., commute.