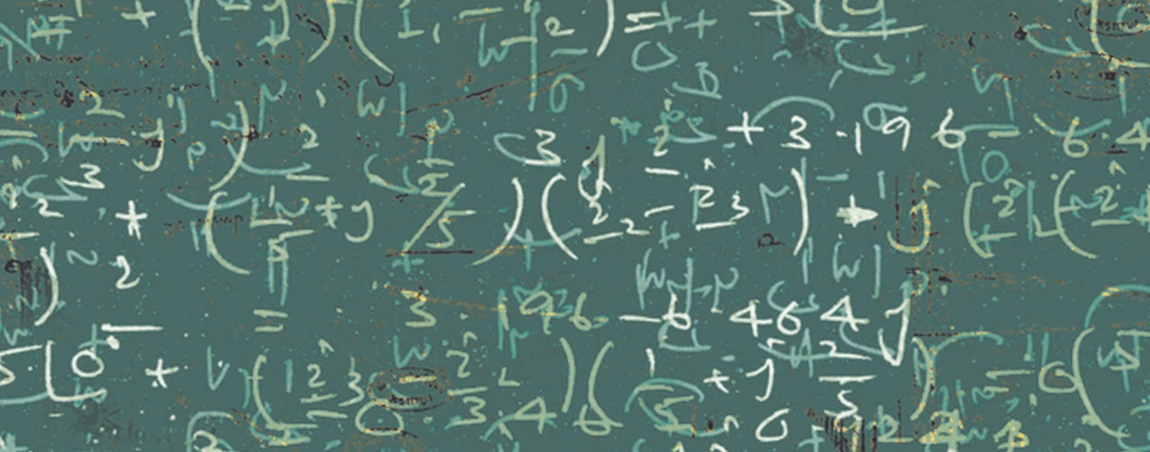

Fundamentals of Advanced Mathematics 2

Henri Bourlès

Field Extensions, Topology and Topological Vector Spaces, Functional Spaces, and Sheaves

ISTE PRESS

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Fundamentals of Advanced Mathematics 2

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Henri Bourlès

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Preface

The first volume [P1] of this *Précis* gave conditions for solving systems of equations; this included some polynomial equations and a few short detours into algebraic geometry, but we were first and foremost interested in linear differential equations. With the exception of the elementary case of constant coefficients, we arrived at conditions that were admittedly still formal, all too formal, just as Nietzsche once wrote "human, all too human". The goal of this second volume is to add flesh to the bones of this mathematical skeleton. One approach to finding solutions to equations with coefficients in the field $\mathbf{K}$ is to look in the field extension $\mathbf{L}/\mathbf{K}$. In the case of polynomial equations, this leads us to Galois theory, and, for differential equations, to differential Galois theory ("Picard-Vessiot" theory, whose most decisive results were established by E. Kolchin); there is an exact parallel between these two theories. Galois theory is presented in detail in Chapter 1, including a full proof of the Abel-Ruffini theorem, showing that the general quintic equation cannot be solved by radicals, a question that tortured mathematicians for three centuries. The main results of differential Galois theory demand long and difficult proofs, and so we shall typically omit them, as each result is somewhat justified by its counterpart in Galois theory.

Once we have vanquished this deeply algebraic first chapter, we can postpone the study of analysis no longer. More precisely, we shall focus on functional analysis, where the class of "functions" should be understood to include "generalized functions" in the sense of I. Gelfand [GEL 68] (essentially Schwartz distributions), or even some more exotic objects such as Sato's hyperfunctions. The latter will in fact provide us with the analytic tools needed to solve systems of differential equations with variable coefficients, as we shall see in section 5.4.6 at the end of this volume.

But before we arrive at this lofty goal, we must first lay the foundations of analysis, beginning with topology, which is presented in Chapter 2. We shall assume that elementary topological concepts in metric spaces are known. These concepts can be found in references such as the *Foundations of Modern Analysis* by J. Dieudonné, (1st volume of the *Elements of Analysis* [DIE 82]). Since most distribution spaces are not metrizable, we will need to study general topological spaces and their variants (uniform spaces, etc.). In metric spaces, sequences are a valuable instrument for proving all kinds of theorems and results; but in non-metrizable spaces we can no longer rely on them, and must replace them with "generalized sequences" such as the " nets" proposed by Moore-Smith, or the "filters" of H. Cartan. We will study these concepts at the start of Chapter 2, showing how nets and filters are equivalent in a strictly logical sense, but complementary on a psychological level. In section 2.5, it seemed useful to introduce the notion of bornology on a general Lipschitz space rather than just a locally convex topological vector space, in particular because the notion of bounded set in a metric space (without any vector-space structure) is familiar. It will be very convenient to have one single general framework. We used quantifiers, especially in topology, in an attempt to lighten the text (breaking with Bourbakian and other conventions), and we added more parenthesis and commas than the logicians would conventionally recommend, seeking clarity above all else.

"Generalized function spaces" (let us denote one such space by E' for now) may be constructed by taking the dual of a "test function space" E. Suppose that E is the L^2 space of square-integrable functions on the real line. Since E is a Hilbert space (Theorem 4.12), it is self-dual (Theorem 3.151). Now, if we choose a test function space E that is smaller than L^2, its dual E' will be larger, and the smaller we make E, the larger E' will become. This is what prompted L. Schwartz to choose $E = \mathcal{D}$, the space of infinitely differentiable functions with compact support. There exist non-zero functions which do in fact satisfy these criteria (Remark 4.70), a result that might seem surprising at first; in any case, $\mathcal{D}$ is much smaller than L^2, and so its dual, $\mathcal{D}'$, the space of distributions, is much larger.

Topological vector spaces, which include the theory of dual spaces, are studied in Chapter 3. The most important such spaces for applications are the locally convex spaces. The theory of dual spaces may be developed naturally on these spaces; we shall therefore focus on them for the majority of our presentation. The two most classic types of locally convex space are the Banach spaces (the complete normed vector spaces) and the Hilbert spaces, a special case of the former. We mentioned earlier that L^2 is a Hilbert space. We will see that the spaces L^p $(p \in [1, \infty])$ are in fact Banach spaces. But many of the spaces that we wish to study here are neither, and so we cannot restrict ourselves to Banach and Hilbert spaces alone. Fréchet spaces and their inductive limits, Montel spaces, and Schwartz spaces (section 3.4) will all have key roles to play; the latter two types of space are always non-metrizable except in finite dimensions. Furthermore, we will constantly require "weak topologies", for instance in distribution theory, and so we shall study them in detail in section 3.5. "Reflexivity" is also studied in section 3.7: given a space E, is E canonically isomorphic to its bidual (the dual of its dual when both are given the "strong topology")? If E is a finite-dimensional space, we already know that the answer is yes ([P1], section 3.1.3(**VI**)). We will see that L^p is reflexive for each $p \in]1, \infty[$, but is not reflexive in general for $p = 1$ or $p = \infty$ (Corollary 4.17 and Remark 4.18). The limited scope of this volume sadly did not allow the inclusion of the elegant theory of compact operators developed by F. Riesz, and related questions (Fredholm theory, Hilbert-Schmidt operators, Sturm-Liouville problem, etc.), but interested readers can find a comprehensive presentation in the *Foundations of Modern Analysis*. Nor was it possible to discuss the primary application of Grothendieck's theory of nuclear spaces (section 3.11), the Schwartz kernel theorem: see [TRE 67]. In short, the kernel theorem guarantees the existence of a canonical isomorphism $\mathcal{D}'(X \times Y) \cong \mathcal{L}(\mathcal{D}(Y); \mathcal{D}'(X))$ between the distribution space on $X \times Y$ (where X and Y are open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively) and the space of continuous linear mappings from $\mathcal{D}(Y)$ to $\mathcal{D}'(X)$ (called the space of $\mathcal{D}'(X)$-valued distributions on Y), a property which still holds true with $\mathcal{D}$ changed to $\mathcal{E}$ and $\mathcal{D}'$ changed to $\mathcal{E}'$ (see section 4.3.1). This result is due to the fact that $\mathcal{D}$ and $\mathcal{E}$ are both nuclear spaces (section 4.3.2(**III**), Remark 4.84), and has deep connections with Fredholm theory [GRO 56].

Armed with our discoveries from Chapters 2 and 3, we will finally be able to study "generalized" function spaces in Chapter 4. Our first rendez-vous is measure and integration theory, a truly fabulous tool crafted by E. Borel, H. Lebesgue, and their successors without which none of the rest would be possible. We will encounter our first type of "generalized function": Radon measures. There are two separate theories of integration: one founded on "abstract measures", the other founded on Radon measures. Bourbaki [BKI 69] adopted the second approach, for entirely defensible reasons, and was strongly criticized for doing so, for equally defensible reasons, most notably due to the significant difficulties involved in constructing any reasonable theory of probability using Radon measures ([BKI 69], Chapter IX), [SCW 73]. In section 4.1, rather than choosing sides and restricting ourselves to either one of two approaches, we shall instead examine the parallels between them and allow ourselves the luxury of selecting the most appropriate in any given case. Although the theory of functions in a single complex variable is covered by the *Foundations of Modern Analysis* (and is deeply embedded within engineering culture), we shall reiterate it here in section 4.2 using the concept of homology (following the approach initiated by Ahlfors [AHL 66]), rather than the concept of homotopy employed by the *Foundations*. As well as being more general ([P1], section 3.3.8(**II**)), the notion of homology is more geometric, and ultimately more convenient. The concept of a meromorphic function will be presented (section 4.2.6) using the Mittag-Leffler and Weierstrass theorems (whose proofs are omitted) to prepare for the subsequent chapter. We then move on to the "classic" function spaces (spaces of infinitely differentiable and analytic functions) and their duals (distributions and hyperfunctions). Distribution theory can be viewed as the culmination of the theory of topological vector spaces, whose developement spanned from Hilbert until the 1950s. M. Sato had the idea to use analytic functions as test functions in order to construct hyperfunctions; however, there are no non-zero analytic functions with compact support, which prevented the theory of hyperfunctions from being a more or less direct application of the "classic" theory of duality. Hyperfunctions are the "boundary values" of holomorphic functions, or alternatively cohomology spaces whose values belong to a sheaf of holomorphic functions. Hence, although it is relatively simple to define hyperfunctions in a single variable (section 4.4.2), we can only understand their deeper nature or attempt to generalize them to multiple variables once we have studied sheaf theory.

The latter, developed by J. Leray and later H. Cartan, is therefore the subject of our final chapter (Chapter 5); we will focus on sheaves of Modules on ringed spaces (section 5.3) and their cohomology (section 5.4). Coherent algebraic sheaves and coherent analytic sheaves, the vital ingredients of algebraic geometry and analytic geometry respectively (in the sense of J.P. Serre for the latter), will enjoy a privileged status in our considerations. We will consider two applications: meromorphic functions in several variables (section 5.4.4) and hyperfunctions (section 5.4.5). Finally, we will see how the latter may be applied to systems of differential equations with variable coefficients to obtain a result that is analogous (despite requiring computations that are effectively much more difficult) to the result obtained in ([P1], section 3.4.4) for systems of differential equations with constant coefficients; this final section will adopt a heuristic approach, aiming to gradually narrow down the possibilities.

Henri BOURLÈS
October 2017

Errata for Volume 1

p. IX, lines 10-11: omit "together with... [MAC 14]"

p. 24, lines 4 and 5: read *inductive* instead of *injective*

p. 54, line 9: read **(III),(IV)** instead of **(IV),(V)**

p. 63, line 22: read *of elements of* instead of $\in$

p. 68, line 11: after is, read *thus*. After *if*, read *and only if*

p. 71, line 1: delete *domain*

p. 148, line 12: read *induced by a function* instead of *one*

p. 165, lines 7 and 9: read $\mathrm{Hom}_{\mathbf{B}}$ instead of Hom_B

p. 180, line 14: read $\mathbf{A}$ instead of A

p. 216, line 4: read $\mathbf{K}$ instead of K

List of Notation

Chapter 1: Field Extensions and Differential Field extensions

$\mathcal{R}(f)$: set of roots of the polynomial f, p. 2

$\mathrm{Gal}(f)$: Galois group of the polynomial f, p. 2

$\mathfrak{S}_{\mathcal{R}}, \mathfrak{S}_n$: symmetric group of $\mathcal{R}$, of $\{1, ..., n\}$, p. 2

$[\mathbf{L} : \mathbf{K}] := \dim_{\mathbf{K}}(\mathbf{L})$, p. 3

$\mathbf{L}/\mathbf{K}$: field extension, p. 4

$\mathbf{K}(\alpha)$, $\mathbf{K}(E)$, p. 4

$\bar{\mathbf{K}}$: algebraic closure of $\mathbf{K}$, p. 4

$\bar{\mathbb{Q}}$: field of algebraic numbers, p. 4

$\mathrm{Gal}(\mathbf{L}/\mathbf{K})$: Galois group of the Galois extension $\mathbf{L}/\mathbf{K}$, p. 6

$\mathbf{M}^{\perp} = \mathrm{Gal}(\mathbf{N}/\mathbf{M})$, p. 6

$\Delta^{\perp}$: fixed field of the group Δ, p. 6

$(\mathfrak{G} : \mathfrak{H})$: index of the subgroup $\mathfrak{H}$ of $\mathfrak{G}$, p. 6

$\mathbf{K}(X_1, ..., X_n)$: field of rational fractions in $X_1, ..., X_n$, p. 8

$s_i\,(i = 1, ..., n)$: elementary symmetric polynomials, p. 8

$\sim\equiv$: not congruent to, p. 10

$\Sigma\left(f,g\right)$: Sylvester matrix of the polynomials f and g, p. 15

$\mathrm{Res}\left(f,g\right)$: resultant of the polynomials f and g, p. 15

$\Delta\left(f\right)$: discriminant of the polynomial f, p. 17

$\mathrm{GF}\left(p^{n}\right)$: Galois field, p. 18

$\boldsymbol{\mu}_{h}$: group of h-th roots of unity, p. 19

$\varphi\left(h\right)$: Euler's totient function evaluated at h, p. 19

$\left[\mathbf{L}:\mathbf{K}\right]_{s}$: separable degree of the algebraic extension $\mathbf{L}/\mathbf{K}$, p. 22

$\bar{\mathbf{K}}_{s}$: separable closure of $\mathbf{K}$, p. 24

$\mathbb{Q}\left(i\right)$: field of Gaussian rationals, p. 27

$\mathbb{Z}\left[i\right]$: ring of Gaussian integers, p. 27

$\deg\mathrm{tr}_{\mathbf{K}}\,\mathbf{E}$: transcendence degree of the extension $\mathbf{E}$ over $\mathbf{K}$, p. 39

$\mathbf{K}\left\langle v_{1},...,v_{m}\right\rangle$: differential field extension, p. 40

$W\left(u_{1},...,u_{n}\right)$: Wronksian, p. 42

$\mathrm{Gal}^{D}\left(\mathbf{M}/\mathbf{K}\right)$: differential Galois group, p. 44

$\boldsymbol{D}_{n}\left(\mathbf{C}\right)$: group of invertible diagonal matrices, p. 45

$\mathrm{G}_{m}\left(\mathbf{C}\right)$: multiplicative group of $\mathbf{C}^{\times}$, p. 45

$\mathrm{T}_{n}\left(\mathbf{C}\right)$: group of invertible upper triangular matrices, p. 45

$\mathrm{U}_{n}\left(\mathbf{C}\right)$: group of unipotent upper triangular matrices, p. 45

$\mathrm{G}_{a}\left(\mathbf{C}\right)$: additive group of $\mathbf{C}$, p. 45

$\mathrm{O}_{n}\left(S;\mathbf{C}\right),\mathrm{O}_{n}\left(S;\mathbf{C}\right)$: orthogonal group, p. 45

$\mathrm{SO}_{n}\left(\mathbf{C}\right)$: special orthogonal group, p. 45

Chapter 2: General Topology

X^{∞} : one-point compactification of the locally compact space X, p. 80

$A \Subset B$, p. 80

$\mathrm{supp}\,(f)$: support of the numerical function f, p. 83

$W \circ W'$: p. 84

$U_d\,(r)\,, U_d^c\,(r)$: p. 84, 88

$\mathbf{Topu}$: category of uniform spaces, p. 87

$B_d\,(x; r)\,, B_d^c\,(x; r)$: open semi-ball, closed semi-ball, p. 88

$[d]$: Lipschitz equivalence class of the pseudometric d, p. 90

$\mathbf{Topl}$: category of Lipschitz spaces, p. 90

$\hat{X}$: Hausdorff completion of the uniform space X, p. 93

$\mathbf{Topuc}$: category of uniform Hausdorff complete spaces, p. 93

$\mathbf{Bor}$: category of bornological sets, p. 98

$\mathfrak{S}$: bornology, p. 99

s : discrete bornology, p. 99

c : bornology of compact subsets, p. 99

pc : bornology of precompact subsets, p. 99

b : canonical bornology, p. 99

u : trivial bornology, p. 99

$\mathcal{F}\,(X; Y)$: set of mappings from X into Y, p. 103

$\mathcal{F}_{\mathfrak{S}}\,(X; Y)$: p. 103

$\mathcal{C}\,(X; Y)$: set of continuous mappings from X into Y.

$\mathcal{C}_{\mathfrak{S}}\,(X; Y)$: p. 105

$\mathcal{B}(E;F)$: set of bounded mappings from E into F, p. 109

$\mathfrak{B}(E;F)$: equibornology, p. 109

$\mathbf{Topgrp}$: category of topological groups, p. 109

$\mathfrak{U}_g, \mathfrak{U}_d$: left, right uniform structure of a topological group, p. 110

$\ker(u), \operatorname{im}(u), \operatorname{coim}(u)$: kernel, image, coimage, p. 113

$\mathbf{Topab}$: category of abelian topological groups, p. 115

$\operatorname{coker}(u)$: cokernel, p. 115

Chapter 3: Topological Vector Spaces

$\mathbb{K}$: field of real or complex numbers, p. 118

$\Re, \Im, |.|$: p. 118, 121

$\mathbf{Vec}$: category of $\mathbb{K}$-vector spaces, p. 118

$E[\mathfrak{T}]$: topological vector space (where $\mathfrak{T}$ denotes the topology on E), p. 118

$\sum_{i \in I} x_i$: (possibly infinite) sum of terms, p. 119

$\mathbf{Tvs}$: category of topological vector spaces, p. 120

$\mathcal{L}(E;F)$: space of continuous linear mappings from E into F, p. 120

$\operatorname{Hom}(E;F)$: space of linear mappings from E into F, p. 120

E^* : algebraic dual, p. 120

E' : dual, p. 120

$\mathcal{L}(E)$: ring of continuous endomorphisms of E, p. 120

$\mathbf{Tvsh}$: category of Hausdorff topological vector spaces, p. 120

$F_{(\mathbb{C})}, \left(F_{(\mathbb{C})}\right)_0$, p. 120

$[A]$: vector space generated by A, p. 120.

$\operatorname{codim}(M)$: codimension of M, p. 123

$[x, y]\,,]x, y[$: closed interval, open interval, p. 126

$\operatorname{epi}(f)$: epigraph of f, p. 126

$B_p(\alpha)\,, B_p^c(\alpha)\,$, p. 127

p_A : gauge of the absorbing disc A, p. 128

Lcs : category of locally convex spaces, p. 133

Lcsh : category of locally convex Hausdorff spaces, p. 133

$\bigoplus_{i \in I} E_i$: topological direct sum, p. 134

$\mathcal{L}_{\mathfrak{S}}(E; F)$: p. 136

$\mathcal{L}(E)\,$, p. 139

$\|u\|$: norm of the continuous linear mapping u, p. 139

$(\mathcal{F})\,, (\mathcal{LF})\,, (\mathcal{L}_s\mathcal{F})$: p. 141

E_V : normed vector space associated with V, p. 142

$\widehat{E_V}$: completion of E_V, p. 142

$B_p^c(\alpha; F)$: p. 144

$\mathcal{K}(E; F)$: space of compact operators from E into F, p. 148

$(\mathcal{S})\,, (\mathcal{FS})\,, (\mathcal{FM})\,, (\mathcal{MS})$: p. 150

$\sigma(E, F)$: weak topology of E, p. 151

${}^t u$: transpose of u, p. 153

E_s', E_s : weak* dual, space E equipped with the weak topology, p. 157

$\tau(E, F)$: Mackey topology, p. 158

E_τ, E'_τ : spaces E, E' equipped with the Mackey topology, p. 158

$\beta(E', E), E'_b$: strong topology, strong dual, p. 159

$E'_{\mathfrak{S}}$: dual equipped with the $\mathfrak{S}$-topology, p. 159

$E'', \mathbf{c}_E$: bidual, canonical mapping $E \to E''$, p. 163

$\gamma(E', E)$: topology of compact convergence of the dual, p. 168

$\|u\|$: norm of the multilinear mapping u, p. 177

$\mathcal{B}(E_1, ..., E_n; F), \mathcal{L}(E_1, ..., E_n; F)$, p. 177

$\langle x|y\rangle$: scalar product, p. 178

$A^\perp$: orthogonal of A, p. 179

$p_H(x)$: orthogonal projection of x onto H, p. 180

σ_E : anti-linear bijection $E' \to E$, p. 184

$\widetilde{\bigoplus}_{i \in I} E_i$: Hilbertian sum, p. 185

u^* : adjoint of the operator u, p. 188

$\mathcal{H}(E)$: space of Hermitian operators in E, p. 188

$\mathrm{U}(E), \mathrm{O}(E)$: group of unitary endomorphisms, of symmetric endomorphisms, p. 188

$\mathcal{H}_+(E)$: cone of positive operators in E, p. 188

Chapter 4: Measure, Integration, Function spaces

$\mathfrak{M}_+(X, \mathcal{T}), \mathfrak{M}_\mathbb{R}(X, \mathcal{T}), \mathfrak{M}(X, \mathcal{T}), \mathfrak{M}_\pm(X, \mathcal{T})$: set of positive, real, complex, signed measures, p. 199

λ : Lebesgue measure on $\mathbb{R}$, p. 200

δ_a : Dirac measure or distribution at the point a, p. 200, 213, 245

Υ : Heaviside function, p. 200

$\hat{\mu}$: completion of the measure μ, p. 200

$\sim_{\mu}$: equivalence $\mathrm{mod}\ .\mu$, p. 201

$\mathring{\mathbf{f}}$: equivalence class of $\mathbf{f}$ $(\mathrm{mod}\ .\mu)$, p. 201

$\int \mathbf{f}.d\mu$, $\int_X \mathbf{f}(x).d\mu(x)$: Bochner integral of $\mathbf{f}$, p. 202

ϖ : Dirac comb, p. 202, 213, 246

$\mathrm{ess\,sup}, \mathrm{ess\,inf}$: essential supremum, essential infimum, p. 204

$\mathcal{L}^p(X, \mu; F), L^p(X, \mu; F)$, p. 204

N_p : norm on $L^p(X, \mu; F)$, p. 204

$\mathcal{T}_1 \otimes \mathcal{T}_2, \mu_1 \otimes \mu_2$: tensor product of σ-algebras, of measures, p. 207

$\lambda^{n\otimes}$: Lebesgue measure on $\mathbb{R}^n$, p. 207

$\mathcal{K}_K(\Omega), \mathcal{C}(K), \mathcal{C}_0(\Omega), \mathcal{C}(\Omega), \mathcal{K}(\Omega)$, p. 209

$\mathcal{M}(\Omega) = \mathcal{K}'(\Omega), \mathcal{C}_0'(\Omega), \mathcal{C}'(\Omega)$: space of Radon measures, p. 210

$\mathrm{supp}(\mu), \mathrm{supp}(T)$: support of a measure, of a distribution, p. 211, 243

$\mathcal{L}^1_{loc}(\Omega, \mu), L^1_{loc}(\Omega, \mu), N_{1,K}(f), \mathcal{L}^1_{loc}(\Omega, dx), L^1_{loc}(\Omega, dx)$, p. 211

$\mu^+, \mu^-, |\mu|$: p. 213

$|\nu| \ll \mu$, p. 216

$\frac{d\nu}{d\mu}$: Radon-Nikodym derivative, p. 216

$h \downarrow 0$, p. 218

$NBV([a, b])$: space of functions of bounded variation, p. 219

$\mu \perp \nu$: disjoint measures, p. 221

$AC([a, b])$: space of absolutely continuous functions, p. 222

$\int_Z \mathbf{H}_z d\pi(z)$: continuous sum of Hilbert spaces, p. 223

$\mathcal{H}(\Omega), \mathcal{H}\left(\Omega; E_{(\mathbb{C})}\right)$: space of holomorphic functions, p. 225

$\int_\gamma \mathbf{f}(z).dz, \oint_\gamma \mathbf{f}(z).dz$: integral along a path, along a closed path, p. 225

$\mathcal{M}(\Omega)$: space of meromorphic functions, p. 233

$\mathcal{H}^*(U)$: multiplicative group of invertible elements of $\mathcal{H}(U)$, p. 232

$\mathcal{A}_E(\Omega)$: space of analytic functions with values in E, p. 239

$\alpha! := \alpha_1!...\alpha_n!, r^\alpha$, p. 240

$\mathcal{E}^{(m)}(\Omega), \mathcal{E}(\Omega)$, p. 235

$\mathcal{D}_K^{(m)}(\Omega), \mathcal{D}^{(m)}(\Omega), \mathcal{D}_K(\Omega), \mathcal{D}(\Omega)$, p. 236

$\mathcal{S}(\mathbb{R}^n)$, p. 237

C^ω : class of analytic functions, p. 239

$\mathcal{H}(S), \mathcal{A}(S)$: space of germs of holomorphic functions, of analytic functions, p. 242

$\mathcal{D}'(\Omega), \mathcal{E}'(\Omega), \mathcal{S}'(\mathbb{R}^n)$: distribution spaces, p. 243

$\mathcal{O}_M$: space of infinitely differentiable slowly growing functions, p. 247

$\operatorname{sing\,supp}$: singular support, p. 247

$\mathcal{B}[S]$: space of hyperfunctions on S, p. 250

$[f]_{z=x}, f(x \pm i0)$, p. 251

$\mathcal{B}_c[\mathfrak{u}]$: space of hyperfunctions on the open set $\mathfrak{u}$ with compact support, p. 253

Chapter 5: Sheaves

1

Field Extensions and Differential Field Extensions

1.1. Galois theory

1.1.1. *Introduction*

(I) The problem of "solving algebraic equations by radicals" (where by algebraic equations we mean those of the form $f(x) = 0$, for f a non-zero polynomial with rational coefficients) was one of the most important topics in mathematics from the earliest periods of antiquity until the 19th century. The Babylonians, and later the Greeks, already knew how to solve quadratic equations, although their formulas were more complex than those we use today, since their notation was inferior and they lacked the concepts of zero and negative numbers. Cubic equations were solved by S. del Ferro (around 1515) and N. Tartaglia (whose contributions were published in 1545 by G. Cardano and are often incorrectly attributed to the latter); quartic equations were solved by L. Ferrari using a method that was also published in Cardano's *Ars Magna* together with the method proposed by Tartaglia. In 1576, R. Bombelli compiled a summary of all of this work using simpler notation in *Algebra*[1]. However, all efforts to solve quintic equations were unsuccessful from the end of the 16th century until the early 19th century. J.-L. Lagrange was arguably the first to recognize the underlying reasons in his memoirs from 1770-1771. Finally, P. Ruffini attempted to show that the general quintic equation cannot be solved by radicals in a series of dense and

1 See the Wikipedia articles on Cardano*'s method* and *Quartic equations*.

controversial memoirs gradually published between 1799 and 1813. One proof that was entirely correct but weighed down by long calculations was provided by N. Abel in 1824; in 1826, he showed:

THEOREM 1.1.– *(Abel-Ruffini) The general algebraic equation of degree n with rational coefficients may be solved by radicals if and only if $n \leq 4$.*

(II) É. Galois discovered the deeper reasons underlying this theorem by deriving it from group theory after defining the notion of a *normal* subgroup ([P1], section 2.2.2**(I)**). Let $\mathcal{R} = \mathcal{R}(f)$ be the set of n roots of a non-zero polynomial f. Assume that there are no duplicate roots. We shall begin by giving a definition of the Galois group of f and stating Galois' theorem. Later, with the benefit of hindsight, this definition and this theorem will both reveal themselves to be provisional in nature (see below, Definition 1.10 and Theorem 1.15).

DEFINITION 1.2.– *The Galois group of f is the subgroup $\mathrm{Gal}(f)$ of the symmetric group $\mathfrak{S}_{\mathcal{R}}$ consisting of the permutations that fix every rational expression of the roots $r \in \mathcal{R}$ of f.*

THEOREM 1.3.– *(Galois) The algebraic equation $f(x) = 0$, where $f \in \mathbb{Q}[X]$ is an irreducible polynomial of degree $n > 0$, may be solved by radicals if and only if* $\mathrm{Gal}(f)$ *is* solvable *([P1], section 2.2.7**(I)**).*

These new ideas and Galois' theorem were presented in a memoir submitted to the *Académie des Sciences de Paris* in 1830, which was given to A.-L. Cauchy, who lost it. They were later retranscribed in the documents attached to the letter written by Galois the night before he died (May 31st, 1832). These documents were published in 1846 thanks to the efforts of J. Liouville [GAL 89]. The permutations of $\mathcal{R}$ had already been studied in detail by Lagrange (without any special focus on those which fix rational expressions), followed by Cauchy almost half a century later. The notion of solvable group was introduced by C. Jordan (1867) and its definitive form was established by O. Hölder (1889) in the theorem which bears both their names ([P1], section 2.2.5**(II)**, Theorem 2.15).

(III) The scope and ramifications of Galois' work extended much further than solving algebraic equations. His original manuscripts, so dense that they are difficult to decipher, were gradually pieced together throughout the second half of the 19th century by various authors, most notably E. Betti, J. Serret, and of course C. Jordan. Later, L. Kronecker, R. Dedekind, and H. Weber expanded upon Galois' ideas by placing them within a larger framework, namely that of (finite, normal, and separable) algebraic field extensions. This opened a path that led, by way of the work performed by Steinitz and other leading algebraists (notably including Hilbert and some of his colleagues: E. Noether, M. Deuring, etc.), to the fundamental theorem of finite Galois theory (Theorem 1.9 below). This theorem was established by E. Artin (in his lectures from 1926, published in 1931 by B.L. van der Waerden [WAE 91], and later by Artin himself between 1938 and 1942 [ART 42])[2]. The final hurdle was to extend this theory to infinite (normal and separable) algebraic extensions, which was accomplished by W. Krull using the "profinite topology" that he proposed in 1928 [KRU 28] (see section 1.1.12). Galois theory was generalized to division rings by H. Cartan in 1947, but this goes beyond the scope of this *Précis* (for a recent presentation of this topic, see [COH 95], section 3.3).

1.1.2. *Overview and terminology*

(I) This section aims to give a full proof of the Abel-Ruffini theorem using the tools and methods introduced by Galois and those who came after him. Given the complexity of Galois theory, which should come as no surprise in light of its extensive history, we shall begin with an overview.

(II) In the following, the term "extension" will always mean a "field extension" ([P1], section 2.3.5**(II)**). Every ring may be assumed to be commutative (and in particular every division ring is a field) unless otherwise stated. Recall the following: if $\mathbf{K}$ is a field and $\mathbf{L}$ is a $\mathbf{K}$-algebra, then the cardinal $\dim_{\mathbf{K}}(\mathbf{L})$ is denoted $[\mathbf{L} : \mathbf{K}]$ and is called the *degree* of $\mathbf{L}$ over $\mathbf{K}$ (*ibid.*). If the algebra $\mathbf{L}$ is entire and $[\mathbf{L} : \mathbf{K}] < +\infty$, then $\mathbf{L}$ is a field ([P1], section 3.2.5**(III)**, Lemma 3.90(2)). The expressions "finite extension" and "extension of finite degree" are synonymous. If $\mathbf{L}$ and $\mathbf{K}$ are fields, we can

2 Readers will find [EDW 84] to be an excellent commentary of Galois' work. For the complex history of the development of Galois theory, ranging over around a century from Galois to Artin (but which truly began with Lagrange, some 60 years before Galois), see [KIE 71] and [WAE 72] (the first reference for the period between Galois and Weber, and the second for the period thereafter). The history of solving algebraic equations from 9th century until Jordan is presented in the first part of [WAE 85]. The books by van der Waerden [WAE 91] and Artin [ART 42] differ in their presentations of Galois theory, but the criticism of [WAE 91] raised in [KIE 71] is invalid (see [WAE 72]).

either say that $\mathbf{L}/\mathbf{K}$ is an extension or that $\mathbf{L}$ is an extension of $\mathbf{K}$. A subextension $\mathbf{M}/\mathbf{K}$ of $\mathbf{L}/\mathbf{K}$ is a field $\mathbf{M}$ such that $\mathbf{K} \subset \mathbf{M} \subset \mathbf{L}$. An extension $\mathbf{L}/\mathbf{K}$ is said to be *algebraic* if every element of $\mathbf{L}$ is algebraic over $\mathbf{K}$ ([P1], section 2.3.5**(II)**). If $p = \mathrm{Char}\,(\mathbf{M})$ is the characteristic of $\mathbf{M}$, then $\mathrm{Char}\,(\mathbf{K}) = \mathrm{Char}\,(\mathbf{L}) = p$ ([P1], section 2.3.5**(I)**).

Let $(\mathbf{L}/\mathbf{K})$ and $\mathbf{M}/\mathbf{L}$ be two extensions, let $(l_i)_{i \in I}$ be a basis of the $\mathbf{K}$-vector space $\mathbf{L}$, and let $(m_j)_{j \in J}$ be a basis of the $\mathbf{L}$-vector space $\mathbf{M}$. Then $(l_i m_j)_{(i,j) \in I \times J}$ is a basis of the $\mathbf{K}$-vector space $\mathbf{K}$, and hence ([P1], section 1.1.2**(IV),** (1.3))

$$[\mathbf{M} : \mathbf{K}] = [\mathbf{M} : \mathbf{L}]\,[\mathbf{L} : \mathbf{K}]\,. \qquad [1.1]$$

(III) Let $\mathbf{L}/\mathbf{K}$ be a field extension. If $\alpha = (\alpha_j)_{j \in J}$ is a family of elements of $\mathbf{L}$, we define $\mathbf{K}\,(\alpha)$ to be the smallest field containing $\mathbf{K}$ and every α_j $(j \in J)$. Similarly, if $E \subset \mathbf{L}$, we define $\mathbf{K}\,(E)$ to be the smallest field containing $\mathbf{K}$ and every element of E. If E has only one element, α, then the extension $\mathbf{K}\,(\alpha)$ is said to be *monogenous* or *simple*.

Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathbf{K}$-algebras. A $\mathbf{K}$-*homomorphism* $u : \mathbf{A} \to \mathbf{B}$ is a morphism of $\mathbf{K}$-algebras ([P1], section 2.3.10**(I)**) that fixes every element of $\mathbf{K}$. We can define the concepts of $\mathbf{K}$-monomorphism, $\mathbf{K}$-epimorphism, $\mathbf{K}$-isomorphism, and $\mathbf{K}$-automorphism in a similar fashion.

DEFINITION 1.4.– *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be algebraic extensions of* $\mathbf{K}$ *contained in the same algebraic closure* $\bar{\mathbf{K}}$ *of* $\mathbf{K}$ *(see below, section 1.1.4). We say that* $\mathbf{E}$ *and* $\mathbf{F}$ *are* conjugate *(in* $\bar{\mathbf{K}}$*) if there exists a* $\mathbf{K}$*-automorphism* u *of* $\bar{\mathbf{K}}$ *such that* $u\,(\mathbf{E}) = \mathbf{F}$. *We say that two elements* x *and* y *of* $\bar{\mathbf{K}}$ *are* conjugate *over* $\mathbf{K}$ *if there exists a* $\mathbf{K}$*-automorphism* u *of* $\bar{\mathbf{K}}$ *such that* $u\,(x) = y$.

Clearly, *conjugation* of two elements is an equivalence relation. The corresponding equivalence classes are called the *conjugation classes* of $\mathbf{K}$. If x, x' are conjugate, then we have that $\mathbf{K}\,(x) \cong \mathbf{K}\,(x')$ (**exercise**).

(IV) Let $f \in \mathbf{K}\,[X]$ be a non-zero polynomial. In general, not every solution of the equation $f\,(x) = 0$ is necessarily contained in $\mathbf{K}$. For example, the solutions $\alpha := i\sqrt{2}, -\alpha$ of the algebraic equation $x^2 + 2 = 0$, which has coefficients in $\mathbb{Q}$, do not belong to $\mathbb{Q}$; however, they are in fact contained in "the" algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, namely the field of algebraic numbers ([P1], section 2.3.5(**II**)). These two solutions also belong to the field $\mathbb{Q}(\alpha) = \mathbb{Q}(-\alpha)$, which is strictly smaller than $\bar{\mathbb{Q}}$. A superfield of $\mathbb{Q}(\alpha)$ may be constructed by first adding $\sqrt{2}$ to $\mathbb{Q}$, which yields the field $\mathbb{Q}\left(\sqrt{2}\right)$, then adding i to $\mathbb{Q}\left(\sqrt{2}\right)$. Proceeding in the reverse order gives the same result, since $\mathbb{Q}\left(\sqrt{2}\right)(i) = \mathbb{Q}(i)\left(\sqrt{2}\right) = \mathbb{Q}\left(\sqrt{2}, i\right)$. Note that $\mathbb{Q}(\alpha) \subset \mathbb{Q}\left(\sqrt{2}, i\right)$. The polynomial $X^2 + 2$ is said to *split* in $\mathbb{Q}(\alpha)$, meaning that it may be written as the product of two linear polynomials: $X^2 + 2 = (X + \alpha)(X - \alpha)$; for the same reason, we say that $\mathbb{Q}(\alpha)$ is a *decomposition field* of $X^2 + 2$.

$\mathbb{Q}(\alpha)$ is also a decomposition field of $\left(X^2 + 2\right)^2$, but unlike the latter polynomial, $X^2 + 2$ only has simple roots in $\bar{\mathbb{Q}}$. We say that $X^2 + 2$ is *separable* (and that $\left(X^2 + 2\right)^2$ is inseparable). Since α is a simple root of its minimal polynomial $X^2 + 2$ ([P1], section 2.3.5(**II**)), it is said to be *separable* as an algebraic element. Finally, every element of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is separable (cf. below, Corollary 1.43). A finite extension such that every element is separable is said to be itself *separable*:

DEFINITION 1.5.–

i) A polynomial $f \in \mathbf{K}[X]^{\times}$ is said to be separable *if it only has simple roots in some algebraic closure $\bar{\mathbf{K}}$ of $\mathbf{K}$.*

ii) An element $x \in \bar{\mathbf{K}}$ is said to be separable *over $\mathbf{K}$ if it is the root of a separable polynomial $f \in \mathbf{K}[X]$.*

iii) An algebraic extension $\mathbf{L}/\mathbf{K}$ is said to be separable *if every element of $\mathbf{L}$ is separable.*

As we shall see (Theorem 1.39), if $\mathbf{K}$ is a field of characteristic 0, or more generally is a "perfect field" (Definition 1.36), then every algebraic extension $\mathbf{L}/\mathbf{K}$ is separable.

The conjugate roots (in the sense of the definition given above) $\alpha, -\alpha$ (under the $\mathbb{Q}$-isomorphism $x \mapsto -x$) generate the same extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Consider now the polynomial $X^3 - 2$. Its three roots $\beta = \sqrt[3]{2}, j\beta, j^2\beta$ $\left(j = e^{i2\pi/3}\right)$ in $\bar{\mathbb{Q}}$ are conjugate, since they have the same minimal polynomial $X^3 - 2$ (cf. below, Theorem 1.46); however, they generate different extensions over $\mathbb{Q}$, since $j\beta$ and $j^2\beta$ are complex and therefore do not belong to $\mathbb{Q}(\beta)$.

DEFINITION 1.6.– *An algebraic extension* $\mathbf{N}/\mathbf{K}$ *is said to be* normal *(or* quasi-*Galois) if it is algebraic and every irreducible polynomial in* $\mathbf{K}[X]$ *with at least one root in* $\mathbf{N}$ *splits into a product of (possibly non-distinct) linear polynomials in* $\mathbf{L}[X]$ *.*

DEFINITION 1.7.– *An algebraic extension* $\mathbf{N}/\mathbf{K}$ *is said to be Galois if it is separable and normal.*

(V) Suppose that $\mathbf{N}/\mathbf{K}$ is a *finite* Galois extension of degree n (i.e. $[\mathbf{N} : \mathbf{K}] = n$); then $\mathbf{N}$ has precisely n distinct $\mathbf{K}$-automorphisms (cf. below, Theorem 1.42).

DEFINITION 1.8.– *Let* $\mathbf{N}/\mathbf{K}$ *be a Galois extension. We define the Galois* group *of* $\mathbf{N}$ *over* $\mathbf{K}$*, written* $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ *, to be the group of* $\mathbf{K}$*-automorphisms of* $\mathbf{N}$*.*

If $[\mathbf{N} : \mathbf{M}] < \infty$, then the group $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ has order $(\mathrm{Gal}(\mathbf{N}/\mathbf{K}) : 1) = [\mathbf{N} : \mathbf{M}]$. Recall that, for a group $\mathfrak{G}$ and a subgroup $\mathfrak{H} \subseteq \mathfrak{G}$, the index of $\mathfrak{H}$ in $\mathfrak{G}$ is written $(\mathfrak{G} : \mathfrak{H})$ ([P1], section 2.2.1**(I)**).

Let $\mathbf{N}/\mathbf{K}$ be a Galois extension and let $\mathcal{K}$ be the set of fields $\mathbf{M}$ such that $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$ (namely the set of subextensions of $\mathbf{N}$); this is a lattice ordered by inclusion ([P1], section 2.1.3**(I)**), and, for any $\mathbf{M} \in \mathcal{K}$, the extension $\mathbf{N}/\mathbf{M}$ is Galois (cf. below, Lemma 1.54). Let $\mathcal{G}$ be the set of subgroups of $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$; this is once again an ordered lattice under inclusion, and, whenever $\mathbf{M} \in \mathcal{K}$, the group $\mathbf{M}^{\perp}$ of $\mathbf{M}$-automorphisms of $\mathbf{N}$ is a subgroup of $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$; therefore, $\mathbf{M}^{\perp} \in \mathcal{G}$. If $\Delta \in \mathcal{G}$, we write $\Delta^{\perp} \in \mathcal{K}$ for the *fixed field* of Δ, namely the largest field $\mathbf{M} \in \mathcal{K}$ such that $u(x) = x$, $\forall x \in \mathbf{M}, \forall u \in \Delta$. For each $\mathbf{M} \in \mathcal{K}$, we have that $x \in \mathbf{M} \Leftrightarrow \left[u(x) = x, \forall u \in \mathbf{M}^{\perp}\right]$, and hence $\mathbf{M} = \mathbf{M}^{\perp\perp}$. We will show the following result, originally established by Artin, in section 1.1.8:

THEOREM 1.9.– (*Artin*'s fundamental theorem of *Galois* theory)

1) The correspondence

$$\mathbf{M} \mapsto \mathbf{M}^{\perp}, \quad \Delta \mapsto \boldsymbol{\Delta}^{\perp} \qquad [1.2]$$

is a Galois connection *between* $\mathcal{K}$ *and* $\mathcal{G}$*, which therefore induces bijections from* $\Delta^{\perp}$ *onto* $\mathbf{M}$ *and from* $\mathbf{M}^{\perp}$ *onto the Galois closure* $\boldsymbol{\Delta}^{\perp\perp}$ *of* Δ *([P1], section 2.1.2**(II)**).*

2) $[\mathbf{M}:\mathbf{K}]<\infty$ *if and only if* $\left(\mathbf{M}^{\perp}:1\right)<\infty$; *whenever this condition is satisfied, the Galois connection (1.2) is a bijection, meaning that* Δ *is equal to its Galois closure* $\Delta^{\perp\perp}$, *in which case*

$$[\mathbf{N}:\mathbf{M}]=\left(\mathbf{M}^{\perp}:1\right), \quad (\mathrm{Gal}\,(\mathbf{N}/\mathbf{K}):\mathrm{Gal}\,(\mathbf{N}/\mathbf{M}))=[\mathbf{M}:\mathbf{K}]. \qquad [1.3]$$

3) The Galois connection (1.2) gives a correspondence between the normal subextensions $\mathbf{M}/\mathbf{K}$ *of* $\mathbf{N}/\mathbf{K}$ *and the* normal subgroups *([P1], section 2.2.2(**I**)) of* $\mathrm{Gal}\,(\mathbf{N}/\mathbf{K})$.

4) If $\mathbf{K}\subset\mathbf{M}\subset\mathbf{M}'\subset\mathbf{N}$, *then the extensions* $\mathbf{N}/\mathbf{M}$ *and* $\mathbf{N}/\mathbf{M}'$ *are Galois, and if* $[\mathbf{N}:\mathbf{K}]<\infty$, *then*

$$\left[\mathbf{M}':\mathbf{M}\right]=(\mathrm{Gal}\,(\mathbf{N}/\mathbf{M}):1)\,/\,\left(\mathrm{Gal}\,\left(\mathbf{N}/\mathbf{M}'\right):1\right).$$

5) Under the conditions stated in (4), $\mathbf{M}'/\mathbf{M}$ *is a* normal extension *if and only if* $\mathrm{Gal}\,(\mathbf{N}/\mathbf{M}')$ *is a* normal subgroup *of* $\mathrm{Gal}\,(\mathbf{N}/\mathbf{M})$, *in which case*

$$\boxed{\mathrm{Gal}\,(\mathbf{M}'/\mathbf{M})\cong\mathrm{Gal}\,(\mathbf{N}/\mathbf{M})\,/\,\mathrm{Gal}\,(\mathbf{N}/\mathbf{M}')}.$$

This theorem reduces the study of subextensions of $\mathbf{N}$ to that of the subgroups of $\mathrm{Gal}\,(\mathbf{N}/\mathbf{K})$.

(VI) In particular, let $f\in\mathbf{K}\,[X]^{\times}$ be a separable polynomial (which we may assume to be unitary without loss of generality) and let $\mathcal{R}=\{r_1,...,r_n\}$ be the set of its roots in some algebraic closure $\bar{\mathbf{K}}$ of $\mathbf{K}$. The field $\mathbf{N}=\mathbf{K}\,(\mathcal{R})$, called the *field of roots* of f, is a Galois extension of $\mathbf{K}$, and every $\mathbf{K}$-automorphism u of $\mathbf{N}$ satisfies $u\,(\mathcal{R})\subset\mathcal{R}$ (**exercise**); restriction $\sigma:=u\,|_{\mathcal{R}}$ is a bijection from $\mathcal{R}$ onto $\mathcal{R}$, and thus is an element of the symmetric group $\mathfrak{S}_{\mathcal{R}}$ of $\mathcal{R}$ ([P1], section 2.2.1(**I**)). The latter may be identified with $\mathfrak{S}_n$ via the bijection $r_i\mapsto i$ $(i=1,...,n)$.

DEFINITION 1.10.– $\mathrm{Gal}\,(\mathbf{N}/\mathbf{K})$ *is called the Galois group of* f, *denoted* $\mathrm{Gal}\,(f)$.

Let $\sigma\in\mathfrak{S}_n$ and let $g\in\mathbf{K}\,[X_1,...,X_n]$. Set $g_\sigma\,(X_1,...,X_n)=g\left(X_{\sigma(1)},...,X_{\sigma(n)}\right)$. The next result shows that Definitions 1.2 and 1.10 are consistent.

LEMMA 1.11.– *Let $\sigma \in \mathfrak{S}_n$. Then $\sigma \in \mathrm{Gal}(f)$ if and only if*

$$\forall g \in \mathbf{K}[X_1, ..., X_n], [g(r_1, ..., r_n) \in \mathbf{K} \Rightarrow g(r_1, ..., r_n) = g_\sigma(r_1, ..., r_n)].$$

PROOF.– The necessary condition is clear. For the converse, consider $\sigma \in \mathfrak{S}_n$. After identification, σ is a $\mathbf{K}$-endomorphism of $\mathbf{K}(\mathcal{R})$ (this can be seen by taking g to be an arbitrary constant polynomial), and so is a $\mathbf{K}$-monomorphism from $\mathbf{K}(\mathcal{R})$ onto $\mathbf{K}(\mathcal{R})$ ([P1], section 2.3.5**(I),** Lemma 2.33); $\mathcal{R}$ is finite, so σ is a bijection from $\mathcal{R}$ into $\mathcal{R}$, and $\sigma : \mathbf{K}(\mathcal{R}) \to \mathbf{K}(\mathcal{R})$ is surjective. ■

For example, if $f(X) = X^4 - 2 \in \mathbb{Q}[X]$, then, over $\bar{\mathbb{Q}}[X]$, we have that $f(X) = \prod_{i=1}^{4}(X - r_i)$ with $r_1 = \sqrt[4]{2}, r_2 = ir_1, r_3 = -r_1, r_4 = -ir_1$. The permutation $\sigma : r_1 \mapsto r_1, r_2 \mapsto r_3, r_3 \mapsto r_4, r_4 \mapsto r_2$, which, following in the footsteps of Cauchy, we shall write as

$$\sigma = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 \\ r_1 & r_3 & r_4 & r_2 \end{pmatrix},$$

is not a $\mathbb{Q}$-homomorphism, since $r_1 r_2 - r_3 r_4 = 0$ but $r_1 r_3 - r_4 r_2 \neq 0$. We have that $(\mathfrak{S}_n : 1) = n!$ and $(\mathrm{Gal}(\mathbf{L}/\mathbf{K}) : 1) = n$, so $(\mathfrak{S}_n : \mathrm{Gal}(\mathbf{L}/\mathbf{K})) = n!/n = (n-1)!$ by Lagrange's theorem ([P1], section 2.2.1**(I)**, (2.5)).

Let $X_1, ..., X_n$ and T be indeterminates (equivalently, we could assume that $X_1, ..., X_n$ are algebraically independent elements over $\mathbf{K}$ and that T is an indeterminate) and let $\mathbf{N} = \mathbf{K}(X_1, ..., X_n)$ be the field of rational fractions in $X_1, ..., X_n$, i.e. the field of fractions of $\mathbf{K}[X_1, ..., X_n]$ ([P1], section 3.1.10**(I)**). Let

$$\begin{aligned} s_i &= \sum_{1 \leq i_1 < ... < i_k = n} X_{i_1} ... X_{i_k} \in \mathbf{K}[X_1, ..., X_n], \quad \mathbf{E} := \mathbf{K}(s_1, ..., s_n), \\ f(T) &= \sum_{i=0}^{n} (-1)^i s_i T^{n-i} \in \mathbf{E}[T]. \end{aligned}$$

The s_i $(i = 1, ..., n)$ are called the *elementary symmetric polynomials* of degree n. A simple calculation suffices to show that $f(T) = \prod_{i=0}^{n}(T - X_i)$, and hence $\mathbf{N}$ is a decomposition field of $f \in \mathbf{E}[T]$.

DEFINITION 1.12.– *The equation* $f(T) = 0$ *is called the* general algebraic equation of degree n over $\mathbf{K}$.

The s_i belong to $\mathbf{N}$, so $\mathbf{E} \subset \mathbf{N}$. We will show the following result in section 1.1.11:

THEOREM 1.13.– *The Galois group* $\mathrm{Gal}(\mathbf{N}/\mathbf{E})$ *is isomorphic to* $\mathfrak{S}_n$.

(VII) SOLVING ALGEBRAIC EQUATIONS BY RADICALS Let $\mathbf{L}/\mathbf{K}$ be a field extension. This extension is said to be *radical*[3] if there exists an increasing sequence of fields $\mathbf{M}_0 \subset ... \subset \mathbf{M}_r = \mathbf{L}$ such that $\mathbf{M}_0 = \mathbf{K}$ and $\mathbf{M}_i = \mathbf{M}_{i-1}(\alpha_i)$, where α_i is a root of the equation $X^{n_i} - a_i = 0$, $a_i \in \mathbf{M}_{i-1}$ $(1 \leq i \leq n)$.

DEFINITION 1.14.– *An algebraic equation* $f(X) = 0$, $f \in \mathbf{K}[X]^{\times}$, *is said to be* solvable by radicals *if the decomposition field of* f *is contained in some radical extension of* $\mathbf{K}$.

It is sufficient to consider the case where f is irreducible in $\mathbf{K}[X]$ (because if $f = gh$, then $f(x) = 0$ if and only if $g(x) = 0$ or $h(x) = 0$). We will show the following generalization of Theorem 1.3 in section 1.1.10:

THEOREM 1.15.– *(Galois) Let* $\mathbf{K}$ *be a field and* $f \in \mathbf{K}[X]^{\times}$ *an irreducible polynomial of degree* $n > 0$ *such that* $\mathrm{Char}(\mathbf{K}) \nmid n!$.[4] *The algebraic equation* $f(X) = 0$ *is solvable by radicals if and only if the Galois group* $\mathrm{Gal}(f)$ *is solvable.*

The Abel-Ruffini theorem (Theorem 1.1) immediately follows from Theorems 1.13 and 1.15 by noting the fact, shown by Galois, that $\mathfrak{S}_n$ is solvable if and only if $n \leq 4$ ([P1], section 2.2.7**(I)**, Theorem 2.17).

1.1.3. *Decomposition fields*

Let $\mathbf{K}$ be a field. A polynomial in $\mathbf{K}[X]$ is said to be *irreducible* if it cannot be expressed as the product of two non-constant polynomials ([P1], section 2.1.1**(II)**).

3 This is not the same as a *p-radical* extension (see [BKI 12], Chapter V, section 5).

4 For finer conditions on $\mathrm{Char}(\mathbf{K})$, see Lemmas 1.63 and 1.65.

THEOREM 1.16.– (*Eisenstein*'s criterion)[5] *Let* $\mathbf{A}$ *be a commutative unique factorization domain ([P1], section 2.3.8(**II**)),* $\mathbf{K}$ *its field of fractions ([P1], section 3.1.10(**I**)), and* $f(X) = a_0 X^n + a_1 X^{n-1} + ... + a_n$ *a polynomial of degree* $n \geq 1$ *in* $\mathbf{A}[X]$ *([P1], section 2.3.9(**I**)). Suppose that* p *is a prime element in* $\mathbf{A}$ *([P1], sections 2.1.1(**II**) and 2.3.8(**II**)) and suppose further that*

$$a_0 \sim\equiv 0 \pmod{.p}, \quad a_i \equiv 0 \pmod{.p}, \forall i > 0, \quad a_n \sim\equiv 0 \pmod{.p^2}$$

(where $\sim\equiv$ *means "not congruent to"). Then* f *is irreducible in* $\mathbf{K}[X]$.

DEFINITION 1.17.–

1) Let f *be an irreducible polynomial in* $\mathbf{K}[X]$. *We say that the field* $\mathbf{L} \supset \mathbf{K}$ *is a* splitting field *of* f *if it is a simple extension* $\mathbf{L} = \mathbf{K}(x)$ *such that* x *is a root of* f.

2) Let $(f_i)_{i \in I}$ *be a family of non-constant polynomials in* $\mathbf{K}[X]$. *A field* $\mathbf{L} \supset \mathbf{K}$ *is called a* decomposition field *of* $(f_i)_{i \in I}$ *if each* f_i *decomposes into a product of linear factors in* $\mathbf{L}[X]$ *and if, writing* $\mathcal{R}_i$ *for the set of roots of* f_i, $\mathbf{L} = \mathbf{K}(\mathcal{R})$, *where* $\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i$.

LEMMA 1.18.– *Let* $f \in \mathbf{K}[X]$ *be a polynomial of degree* $n \geq 1$ *and suppose that* $\mathbf{L} \supset \mathbf{K}$ *is a field. Then* $a \in \mathbf{L}$ *is a root of* f *if and only if* $X - a$ *divides* $f(X)$ *in* $\mathbf{L}[X]$; f *has at most* n *distinct roots in* $\mathbf{L}$. *If* f *has precisely* n *distinct roots in* $\mathbf{L}[X]$, *then* f *is separable, and so its roots are also separable (Definition 1.5).*

PROOF.– Euclidean division of $f(X)$ by $X - a$ in $\mathbf{L}[X]$ yields $f(X) = (X - a) q(X) + r(X)$, where $d^\circ(r) < 1$, and so $r \in \mathbf{L}$. Hence, $f(a) = 0$ if and only if $r = 0$. Writing $a_1, ..., a_m$ for the distinct roots of f in $\mathbf{L}$, it follows by induction that the product $p(X) = \prod_{i=1}^{m} (X - a_i)$ divides $f(X)$. Since $d^\circ(p) = m$, we have that $n \geq m$, with equality if and only if the roots of f in $\mathbf{L}$ are all simple. ■

THEOREM 1.19.– *(Kronecker)*

1) Let $p \in \mathbf{K}[X]$ *be a unitary polynomial of degree* $n \geq 1$. *Then* $\mathbf{K}[X]/(p)$ *is a field if and only if* p *is irreducible. If so,* $\mathbf{K}[X]/(p)$ *is a*

5 See the Wikipedia article on Eisenstein*'s criterion* for the case where $\mathbf{A} = \mathbb{Z}$ and ([LAN 99], Theorem 3.1) for the general case.

splitting field of p that is also a monogenous extension $\mathbf{K}(x)$ of $\mathbf{K}$, and p is the minimal polynomial of x in $\mathbf{K}[X]$.

2) Let a be an algebraic element over $\mathbf{K}$. If the monogenous extension $\mathbf{K}(a)$ is a splitting field of p, then there exists a unique $\mathbf{K}$-isomorphism $\Phi : \mathbf{K}[X]/(p) \tilde{\rightarrow} \mathbf{K}(a)$, $a = \Phi(x)$, where x is the canonical image of X in $\mathbf{K}[X]/(p)$, and $\mathbf{K}(a)$ is the $\mathbf{K}$-vector space with basis $\left\{1, a, ..., a^{n-1}\right\}$. In particular, $[\mathbf{K}(a) : \mathbf{K}] = n$.

PROOF.–

1) The ideal $(p) \neq (0)$ in the principal ideal domain $\mathbf{K}[X]$ is prime if and only if p is irreducible. Thus, the ring $\mathbf{K}[X]/(p)$ is an integral domain if and only if p is irreducible ([P1], section 2.3.3(**II**)). If so, the ideal (p) is maximal and $\mathbf{K}[X]/(p)$ is a field ([P1], section 2.3.5(**I**)). Let $\varphi : \mathbf{K}[X] \twoheadrightarrow \mathbf{K}[X]/(p)$ be the canonical surjection. Then $\mathbf{K}[X]/(p)$ is the $\mathbf{K}$-vector space with basis $\left\{1, x, ..., x^{n-1}\right\}$, where $x = \varphi(X)$. Indeed, $\varphi(p) = 0$, which implies that

$$p := X^n - p_1 X^{n-1} ... - p_n \Rightarrow x^n = p_1 x^{n-1} + ... + p_n$$

and $1, x, ..., x^{n-1}$ are $\mathbf{K}$-linearly independent (after identifying $\mathbf{K}$ and $\varphi(\mathbf{K}) \subset \mathbf{L}$ via the monomorphism $\varphi|_{\mathbf{K}}$). Since p is irreducible, it is the minimal polynomial of x in $\mathbf{K}[X]$.

2) Let $\Psi : \mathbf{K}[X] \rightarrow \mathbf{K}(a) : f \mapsto f(a)$. This ring homomorphism is clearly surjective and $\ker(\Psi)$ is a principal ideal (q) in $\mathbf{K}[X]$ for some unitary q. Moreover, $\Psi(p(x)) = 0$, so $p \in (q)$, and since p is irreducible, $p = q$. Consequently, Ψ induces an isomorphism $\bar{\Psi} = \Phi : \mathbf{K}[X]/(p) \tilde{\rightarrow} \mathbf{K}(a)$. This isomorphism sends $X + (p)$ to a and $c + (p)$ to c $(c \in \mathbf{K})$, and is therefore a uniquely determined $\mathbf{K}$-isomorphism. Since $\mathbf{K}[X]/(p)$ has basis $\left\{1, x, ..., x^{n-1}\right\}$, $\mathbf{K}(a) = \Phi(\mathbf{K}[X]/(p))$ has basis $\left\{\Phi(1), \Phi(x), ..., \Phi\left(x^{n-1}\right)\right\} = \left\{1, a, ..., a^{n-1}\right\}$. ■

Each isomorphism of fields $\sigma : \mathbf{K} \tilde{\rightarrow} \mathbf{K}'$ canonically induces an isomorphism $\mathbf{K}[X] \tilde{\rightarrow} \mathbf{K}'[X] : f \mapsto f' = \sigma(f)$ as follows: if

$$f(X) = a_0 X^n + a_1 X^{n-1} + ... + a_n, \qquad [1.4]$$

then

$$\sigma f(X) = \sigma(a_0) X^n + \sigma(a_1) X^{n-1} + \ldots + \sigma(a_n). \qquad [1.5]$$

COROLLARY 1.20.– *Let $\sigma : \mathbf{K} \tilde{\rightarrow} \mathbf{K}'$ be an isomorphism of fields, let α be an algebraic element over $\mathbf{K}$, and suppose that $f \in \mathbf{K}[X]$ is the minimal polynomial of α. Let $f' = \sigma(f)$ and α' a root of f'. The isomorphism σ may be uniquely extended to an isomorphism $\mathbf{K}(\alpha) \tilde{\rightarrow} \mathbf{K}(\alpha')$ that sends α to α'.*

PROOF.– We have that $\mathbf{K}/(f) \cong \mathbf{K}'/(f')$, and, by Theorem 1.19, $\mathbf{K}/(f) \tilde{\rightarrow} \mathbf{K}(\alpha) : X + (f) \mapsto \alpha$, $\mathbf{K}'/(f') \tilde{\rightarrow} \mathbf{K}'(\alpha') : X + (f') \mapsto \alpha'$. Uniqueness is clear. ∎

THEOREM 1.21.– *Let $(f_i)_{i \in I}$ be a family of non-constant polynomials in $\mathbf{K}[X]$.*

1) There exists a decomposition field of this family.

2) Every isomorphism $\sigma : \mathbf{K} \tilde{\rightarrow} \mathbf{K}'$ canonically induces an isomorphism $\mathbf{K}[X] \tilde{\rightarrow} \mathbf{K}'[X] : f(X) \mapsto f'(X)$, as well as an isomorphism $\mathbf{K}(\mathcal{R}) \tilde{\rightarrow} \mathbf{K}'(\mathcal{R}')$, where $\mathbf{K}(\mathcal{R})$ and $\mathbf{K}'(\mathcal{R}')$ are respectively decomposition fields of $(f_i)_{i \in I}$ and of $(f_i')_{i \in I}$. Consequently, the decomposition field of $(f_i)_{i \in I}$ is unique up to $\mathbf{K}$-isomorphism.

PROOF.– [6]

1) Let $i \in I$. By Theorem 1.19(1), f_i has a splitting field $\mathbf{K}(x_i)$, hence $f_i(X) = (X - x_i) g_i(X)$, $g_i \in \mathbf{K}(x_i)[X]$. By induction, it follows that f_i has a decomposition field $\mathbf{K}(\mathcal{R}_i)$, and hence $(f_i)_{i \in I}$ has a decomposition field $\mathbf{K}\left(\bigcup_{i \in I} \mathcal{R}_i\right)$.

2) Let f be an element of the family $(f_i)_{i \in I}$ of the form (1.4) with $a_0 \neq 0$, and let $\mathbf{K}(\mathcal{R}) \subset \mathbf{K}\left(\bigcup_{i \in I} \mathcal{R}_i\right)$ be a decomposition field of f, $\mathcal{R} = \{\alpha_1, \ldots, \alpha_n\}$. The image of f under σ is of the form (1.5); set $a_i' = \sigma(a_i)$ $(i = 0, 1, \ldots, n)$. Let $\mathbf{K}'(\mathcal{R}')$ be a decomposition field of f', $\mathcal{R}' = \{\alpha_1', \ldots, \alpha_n'\}$. We will show by induction that there exists an isomorphism $\mathbf{K}(\mathcal{R}) \cong \mathbf{K}'(\mathcal{R}')$ that extends σ.

6 This proof, established by Artin, was published for the first time by van der Waerden [WAE 91] in 1930 (cf. section 6.5 in the 7th edition).

Let $\mathcal{R}_k = \{\alpha_1, ..., \alpha_k\}$, $\mathcal{R}'_k = \{\alpha'_1, ..., \alpha'_k\}$, $k \leq n$, $\mathcal{R}_0 = \mathcal{R}'_0 = \varnothing$. We have that $\mathbf{K}(\mathcal{R}_0) = \mathbf{K}$, $\mathbf{K}'(\mathcal{R}'_0) = \mathbf{K}'$, so $\sigma(\mathbf{K}(\mathcal{R}_0)) = \mathbf{K}'(\mathcal{R}'_0)$. Suppose that there exists an isomorphism $\mathbf{K}(\mathcal{R}_k) \cong \mathbf{K}'(\mathcal{R}'_k)$ that extends σ. Then

$$\begin{aligned} f(X) &= (X-\alpha_1)...(X-\alpha_k)\,\varphi_{k+1}(X)...\varphi_h(X), \\ f'(X) &= \left(X-\alpha'_1\right)...\left(X-\alpha'_k\right)\varphi'_{k+1}(X)...\varphi'_h(X), \end{aligned}$$

where the $\varphi_j(X)$ and the $\varphi'_j(X)$ $(j = k+1, ..., h)$ are irreducible in $\mathbf{K}(\mathcal{R}_k)$ and $\mathbf{K}'(\mathcal{R}'_k)$ respectively. By permuting the indices if necessary, we obtain

$$\begin{aligned} \varphi_{k+1}(X)...\varphi_h(X) &= (X-\alpha_{k+1})...(X-\alpha_n), \\ \varphi'_{k+1}(X)...\varphi'_h(X) &= \left(X-\alpha'_{k+1}\right)...\left(X-\alpha'_n\right), \end{aligned}$$

where α_{k+1} and α'_{k+1} are roots of $\varphi_{k+1}(X)$ and $\varphi'_{k+1}(X)$ respectively. By Corollary 1.20, there exists an isomorphism $\mathbf{K}(\mathcal{R}_k)(\alpha_{k+1}) \tilde{\rightarrow} \mathbf{K}'(\mathcal{R}'_k)\left(\alpha'_{k+1}\right)$ that extends σ and which sends α_{k+1} to α'_{k+1}. By induction, we therefore obtain the stated result. ■

EXAMPLE 1.22.–

1) The polynomial X^2+1 *is irreducible in* $\mathbb{R}[X]$ *. Let* i *be the canonical image of* X *in* $\mathbb{R}[X]/\left(X^2+1\right)$ *. Then* i *is a root of* X^2+1 *in the field of complex numbers* $\mathbb{C} := \mathbb{R}[X]/\left(X^2+1\right) \cong \mathbb{R}(i)$ *. We have that* $X^2+1 = (X+i)(X-i)$ *over* $\mathbb{C}[X]$*, so* $\mathbb{C}$ *is a decomposition field of* X^2+1 *over* $\mathbb{R}[X]$.

2) The polynomial X^3-2 *is irreducible over* $\mathbb{Q}[X]$*. Setting* $\beta = \sqrt[3]{2}$, $\mathbb{Q}(\beta)$ *is a splitting field but not a decomposition field of* X^3-2*. However,* $\mathbb{Q}\left(\beta, j\beta, j^2\beta\right)$ $\left(j = e^{i2\pi/3}\right)$ *is a decomposition field for this polynomial.*

1.1.4. *Algebraically closed fields*

(I) CONCEPT OF AN ALGEBRAICALLY CLOSED FIELD

THEOREM-DEFINITION 1.23.– *Let* $\mathbf{K}$ *be a field.*

1) The following conditions are equivalent:

i) Every non-constant polynomial in $\mathbf{K}[X]$ *is a product of linear polynomials.*

ii) Every irreducible polynomial in $\mathbf{K}[X]$ *is linear.*

iii) Every non-constant polynomial in $\mathbf{K}[X]$ *has a root in* $\mathbf{K}$.

2) The field $\mathbf{K}$ *is said to be* algebraically closed *if any one of these equivalent conditions is satisfied.*

PROOF.– (i)⇒(ii): This is clear. (ii)⇒(iii): Every non-constant polynomial f in $\mathbf{K}[X]$ is divisible by an irreducible polynomial, since the ring $\mathbf{K}[X]$ is a unique factorization domain by Gauss' lemma ([P1], section 2.3.9**(I)**, Lemma 2.60). If (ii) holds, then this polynomial is of the form $X - a$, and so a is a root of f. (iii)⇒(i) may be shown by induction. ■

The fundamental theorem of algebra is a result from the field of *analysis*:

THEOREM 1.24.– *(d'Alembert-Gauss'* fundamental theorem of algebra*) The field* $\mathbb{C}$ *of complex numbers is algebraically closed.*

PROOF.– Let $g \in \mathbb{C}[X]$ be a polynomial of degree ≥ 1, and suppose that this polynomial does not have a root in $\mathbb{C}$. Then the function $f = 1/g$ is entire (section 4.2.1); it is continuous, and so must be bounded on any compact set (Theorem 2.39); furthermore, $\lim_{|\zeta| \to +\infty} |f(\zeta)| = 0$, so f is bounded. It must therefore be constant by Liouville's theorem (Theorem-Definition 4.81), and the same is true for g : contradiction. This proves the property stated by Theorem-Definition 1.23. ■

DEFINITION 1.25.– *An* algebraic closure *of* $\mathbf{K}$ *is an algebraic extension of* $\mathbf{K}$ *that is algebraically closed.*

By Theorem-Definition 1.23, an algebraic closure Ω of $\mathbf{K}$ is an algebraic extension such that every non-constant polynomial in $\mathbf{K}[X]$ factors into a product of linear factors in $\Omega[X]$. By Theorem 1.21, we deduce:

THEOREM 1.26.– *(Steinitz) Every field* $\mathbf{K}$ *has an algebraic closure* $\bar{\mathbf{K}}$. *If* Ω *and* Ω' *are two algebraic closures of* $\mathbf{K}$, *then there exists a* $\mathbf{K}$*-isomorphism from* Ω *onto* Ω'.

DEFINITION 1.27.– *We say that a subfield* $\mathbf{K}$ *of a field* $\mathbf{E}$ *is* algebraically closed *in* $\mathbf{E}$ *if every element of* $\mathbf{E}$ *that is algebraic over* $\mathbf{K}$ *belongs to* $\mathbf{E}$. *The extension* $\mathbf{L}$ *of* $\mathbf{K}$ *given by the elements of* $\mathbf{E}$ *that are algebraic over* $\mathbf{K}$ *is called the* algebraic closure *of* $\mathbf{K}$ *in* $\mathbf{E}$.

Every field is algebraically closed in itself.

1.1.5. *Separability*

(I) Let $\mathbf{K}$ be a field, $\bar{\mathbf{K}}$ an algebraic closure of $\mathbf{K}$, and $f \in \mathbf{K}[X]$ a polynomial of degree ≥ 1.

LEMMA 1.28.– *The polynomial f is separable if and only if f and its derivative f' are coprime. In particular, an irreducible polynomial f is separable if and only if $f' \neq 0$.*

PROOF.– The polynomials f and f' are coprime if and only if they do not share a root in $\bar{\mathbf{K}}$. Moreover, f is separable if and only if (Definition 1.5(i)) f does not have a repeated root in $\bar{\mathbf{K}}$. Let $a \in \bar{\mathbf{K}}$ be a root of f of order n; we have that $f(X) = (X-a)^n g(X)$, where $g \in \bar{\mathbf{K}}[X]$ satisfies $g(a) \neq 0$, so by Leibniz' rule $f'(X) = n(X-a)^{n-1} g(X) + (X-a)^n g'(X)$. Hence, $f'(a) = n0^{n-1} g(a)$. If $n = 1$, then $f'(a) = g(a) \neq 0$; if $n > 1$, then $f'(a) = 0$. ■

(II) Let $f = a_0 X^n + a_1 X^{n-1} + \ldots + a_n$ and $g = b_0 X^m + b_1 X^{m-1} + \ldots + b_m$ be two polynomials in $\mathbf{K}[X]$, where $a_0, b_0 \neq 0$.

DEFINITION 1.29.– *If the polynomials f and g are both non-zero, their* resultant $\operatorname{Res}(f,g)$ *is defined as* $\det(\Sigma(f,g)) \in \mathbf{K}$, *where* $\Sigma(f,g) \in \mathfrak{M}_{n+m}(\mathbf{K})$ *is the Sylvester* matrix[7]

$$\Sigma(f,g) = \left[\begin{array}{ccccccc}
a_0 & 0 & & b_0 & 0 & & \vdots \\
a_1 & \ddots & & b_1 & b_0 & 0 & \vdots \\
a_2 & \ddots & a_0 & \vdots & b_1 & \ddots & 0 \\
\vdots & \ddots & a_1 & b_m & \vdots & \ddots & b_0 \\
a_n & \ddots & a_2 & 0 & b_m & \vdots & b_1 \\
0 & \ddots & \vdots & \vdots & 0 & \vdots & \vdots \\
\cdots & 0 & a_n & 0 & \cdots & 0 & b_m \\
\multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{m}} & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad\qquad}_{n}}
\end{array}\right].$$

7 Some authors define the Sylvester matrix as the transpose of the matrix given here.

If $f = 0$ *or* $g = 0$, *we set* $\operatorname{Res}(f, g) = 0$.

LEMMA 1.30.– *The polynomials* f *and* g *have a shared root in* $\bar{\mathbf{K}}[X]$ *if and only if there exist polynomials* $h, k \in \mathbf{K}[X]$ *such that* $d^\circ(h) \leq m-1, d^\circ(k) \leq n-1$ *and*

$$fh = gk. \tag{1.6}$$

PROOF.– If the equality (1.6) is satisfied, then the n linear divisors of f in $\bar{\mathbf{K}}[X]$ also divide $g(X)\,k(X)$, and, since $d^\circ(k) \leq n-1$, some of them must divide $g(X)$. Conversely, if f and g are coprime, k must necessarily be of degree at least n, which is a contradiction. ■

THEOREM 1.31.– *The polynomials* f *and* g *in* $\mathbf{K}[X]^\times$ *are coprime if and only if their resultant* $\operatorname{Res}(f, g)$ *is non-zero.*

PROOF.– Write $h(X) = \sum_{i=0}^{m-1} c_i X^{m-i}$, $k(X) = \sum_{i=0}^{n-1} d_i X^{n-i}$. By equating same-degree terms in (1.6), we obtain

$$\begin{cases} c_0 a_0 - d_0 b_0 = 0, \\ c_1 a_0 + c_0 a_1 - d_1 b_0 - d_0 b_1 = 0, \\ \ldots \\ c_{m-1} a_n - d_{n-1} b_m = 0. \end{cases}$$

This system of linear equations may be written in the form $\Sigma(f, g)\,E = 0$, where E is the matrix whose elements are the c_i $(i = 0, ..., m-1)$, the d_j $(j = 0, ..., n-1)$, and zeroes. This system has a non-zero solution E if and only if $\operatorname{Res}(f, g) = 0$. ■

By performing a few matrix calculations, the following result can be obtained ([ESC 07], Proposition 3.5.2 and Exercise 3.3)[8]:

LEMMA 1.32.– *If the polynomials* f *and* g *are non-zero, then*

$$\begin{aligned} \operatorname{Res}(f, g) &= a_0^m b_0^n \prod_{1 \leq i \leq n, 1 \leq j \leq m} (\alpha_i - \beta_j) = a_0^m \prod_{i=1}^{n} g(\alpha_i) \\ &= (-1)^{mn} \operatorname{Res}(g, f) \end{aligned} \tag{1.7}$$

8 See also the Wikipedia article on *Resultants*.

where the α_i $(1 \leq i \leq n)$ *and the* β_j $(1 \leq j \leq m)$ *respectively denote the roots of* f *and* g *in* $\bar{\mathbf{K}}$.

DEFINITION 1.33.– *Let* $f \in \mathbf{K}[X]$ *be a polynomial of degree* $n \geq 1$ *and let* a_0 *be the coefficient of its* n*-th degree term. The* discriminant *of this polynomial is*[9]

$$\boxed{\Delta(f) = \frac{(-1)^{n(n-1)/2}}{a_0} \operatorname{Res}(f, f')}. \qquad [1.8]$$

EXAMPLE 1.34.– *If* $f(X) = aX^2 + bX + c$, *then* $\Delta(f) = b^2 - 4ac$. *If* $f(X) = X^3 + px + q$, *then* $\Delta(f) = -\left(4p^3 + 27q^2\right)$ ***(exercise)****.*

Lemmas 1.28 and 1.32 together with Theorem 1.31 imply that:

THEOREM 1.35.–

1) The discriminant $\Delta(f) \in \mathbf{K}$ *of a polynomial* $f \in \mathbf{K}[X]$ *of degree* $n \geq 1$ *may be expressed in terms of the roots of* f *as follows in an algebraic closure* $\bar{\mathbf{K}}$ *of* $\mathbf{K}$*:*

$$\boxed{\Delta(f) = a_0^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2}. \qquad [1.9]$$

For all of the roots of this polynomial to be in $\mathbf{K}$*, it is necessary (and sufficient if* f *is quadratic) for* $\Delta(f)$ *to be a square in* $\mathbf{K}$*. For* f *to be separable, it is necessary and sufficient for* $\Delta(f)$ *to be non-zero.*

2) An element x *that is algebraic over* $\mathbf{K}$ *is separable (Definition 1.5) if and only if its minimal polynomial* φ_x *satisfies* $\varphi'_x = 0$. *If* $\mathbf{K}$ *has characteristic* 0, *this condition is always satisfied. If* $\mathbf{K}$ *has characteristic* $p > 0$ *([P1], section 2.3.5(**I**)), this condition is satisfied if and only if there exists a polynomial* $\psi_x \in \mathbf{K}[X]^\times$ *such that* $\varphi_x(X) = \psi_x(X^p)$.

9 Many sources incorrectly define the signs in (1.7) and (1.8).

PROOF.–

1) The expression (1.9) follows from Definition 1.33, (1.7), and the fact that

$$f(X) = a_0 \prod_{i=1}^{n} (X - \alpha_i) \Rightarrow f'(X) = a_0 \sum_{i=1}^{n} \prod_{i \neq j} (X - \alpha_j).$$

2) By setting $f(X) = \sum_{i=0}^{n} a_i' X^i$, we have that $f'(X) = \sum_{i=1}^{n} i a_i' X^{i-1}$. Therefore, $f'(X) = 0 \Leftrightarrow i a_i' = 0 \; (i = 1, ..., n)$. If $\mathbf{K}$ has characteristic 0, this condition cannot be satisfied, since $a_n' \neq 0$. If $\mathbf{K}$ has characteristic $p > 0$, this condition is equivalent to saying that $a_i' = 0$ for i coprime with p. ■

(III) Let $\mathbf{K}$ be a field of characteristic $p > 0$. The mapping $x \mapsto x^p$ is a field endomorphism, and therefore a monomorphism ([P1], section 2.3.5**(I)**, Lemma 2.33), since $(xy)^p = x^p y^p$ and $(x + y)^p = x^p + y^p$ by Newton's binomial formula. This mapping is called the Frobenius endomorphism.

DEFINITION 1.36.– *A field is said to be* perfect *if it has characteristic* 0, *or if it has characteristic* $p > 0$ *and the Frobenius endomorphism is surjective.*

LEMMA 1.37.– *Every finite field is perfect.*

PROOF.– Let $\mathbf{K} = \{x_1, ..., x_q\}$ be a finite field and let $p = \mathrm{Char}(\mathbf{K})$ be its characteristic. We necessarily have that $p > 0$; let $\mathbf{F} = \{x_1^p, ..., x_q^p\} \subseteq \mathbf{K}$. Since $x_i^p \neq x_j^p$ for $i \neq j$, we have that $\mathbf{F} = \mathbf{K}$. ■

THEOREM-DEFINITION 1.38.–

1) If $\mathrm{Card}(\mathbf{K}) = q$, *then* q *is of the form* p^n, *where* p *is a prime number.*

2) A field of p^n *elements is written* $\mathrm{GF}(p^n)$ *and is called the Galois* field *(with* p^n *elements).*

3) The Galois field $\mathrm{GF}(p^n)$ *is unique up to isomorphism and consists of the roots of the polynomial* $X^{p^n} - X$. *Every subfield of* $\mathrm{GF}(p^n)$ *is of the form* $\mathrm{GF}(p^m)$, *where* $m \mid n$ *and* $\mathrm{GF}(p) \cong \mathbb{Z}/p\mathbb{Z}$.

4) The (abelian) multiplicative group $\mathrm{GF}(p^n)^{\times}$ *consists of the* h-th roots of unity, *i.e. the roots of the polynomial* $X^h - 1$, *where, in this specific case,* $h = p^n - 1$.

5) Let $\mathbf{K}$ *be a field,* h *an arbitrary integer* > 0*, and* $\boldsymbol{\mu}_h$ *the group of* h*-th roots of unity in some algebraic closure* $\bar{\mathbf{K}}$ *of* $\mathbf{K}$*. This group is cyclic, and any generator* ζ *of this group is called a* primitive h-th root of unity. *The number of primitive* h*-th roots of unity is equal to the number* $\varphi(h)$ *of integers* k *such that* $0 \leq k < h$ *and* k *is coprime with* h*. The number* $\varphi(h)$ *is called Euler's totient function evaluated at* h*, satisfying* $\varphi(h) = h \prod_{p \in \mathfrak{D}(h)} (1 - 1/p)$*, where* $\mathfrak{D}(h)$ *is the set of prime divisors of* h*, and, in particular,* $\varphi(h) = h - 1$ *if* h *is a prime number.*

6) $\mathrm{GF}(p^n)$ *is a simple extension* $\mathrm{GF}(p^m)(\zeta)$ *of* $\mathrm{GF}(p^m)$ *whenever* $m \mid n$*.*

PROOF.–

1) The field $\mathbf{K}$ is a $\mathbb{Z}/p\mathbb{Z}$-algebra. Let $n = [\mathbf{K} : \mathbb{Z}/p\mathbb{Z}]$ and let $\{b_1, ..., b_n\}$ be a basis of the $\mathbb{Z}/p\mathbb{Z}$-vector space $\mathbf{K}$. Then $\mathbf{K}$ is the set of elements of the form $\sum_{1 \leq i \leq n} \alpha_i b_i$, $\alpha_i \in \mathbb{Z}/p\mathbb{Z}$, and there are p^n such elements.

3) Let $x \in \mathrm{GF}(p^n)$ and let $h = p^n - 1$ be the order of the multiplicative group $\mathrm{GF}(p^n)^\times$. Then $x^h = 1$ ([P1], section 2.2.4**(I)**), and hence $x^{p^n} = x$. This relation is also satisfied when $x = 0$. By Theorem 1.21, $\mathrm{GF}(p^n)$ is therefore unique up to isomorphism. If $\mathbf{F} \subseteq \mathrm{GF}(p^n)$, then $\mathrm{Char}(\mathbf{F}) = p$, so $\mathbf{F} = \mathrm{GF}(p^m)$, and by (1.1) it follows that $m \mid n$.

4) is obvious.

5) Let ε be the exponent of $\boldsymbol{\mu}_h$ ([P1], section 2.2.1**(I)**). Then $\varepsilon \mid h$, and the equation $X^\varepsilon - 1 = 0$ has at most ε solutions in $\mathbf{K}$, so $h \leq \varepsilon$, and thus $h = \varepsilon$. Since $\boldsymbol{\mu}_h$ is abelian, there exists an element ζ of order ε in $\boldsymbol{\mu}_h$ ([P1], section 2.2.4**(I)**). We have that $\left(\zeta^k\right)^l = \zeta^{kl}$, and so the following conditions are equivalent: (i) ζ^k generates $\boldsymbol{\mu}_h$; (ii) $k + h\mathbb{Z}$ is invertible in $\mathbb{Z}/h\mathbb{Z}$; (iii) there exists an integer x such that $kx \equiv 1 \pmod{.h}$; (iv) there exist integers x and y satisfying Bezout's equation $kx + hy = 1$; (v) k and h are coprime. Deriving the expression for $\varphi(h)$ is a classic **exercise*** ([BKI 12], Chapter V, section 11.3).

6) We have that $\mathbf{K} = \left\{0, \zeta, ..., \zeta^{h-1}, 1\right\} = \mathrm{GF}(p^m)(\zeta)$. ∎

Note that, in $\mathbb{C}$, the primitive h-th roots of unity are the $e^{i2\pi k/h}$ such that $0 \leq k < h$ and k is coprime with h.

THEOREM 1.39.– *Let* $\mathbf{K}$ *be a field. The following conditions are equivalent:*

i) The field $\mathbf{K}$ *is perfect.*

ii) Every algebraic extension of $\mathbf{K}$ *is separable.*

iii) Every irreducible polynomial in $\mathbf{K}[X]$ *is separable.*

PROOF.– The algebraic elements over $\mathbf{K}$ are the roots in $\bar{\mathbf{K}}$ of irreducible polynomials in $\mathbf{K}[X]$ ([P1, section 2.3.5**(II)**), so (ii)⇔(iii).

(i)⇒(ii): The result is clear when $\mathbf{K}$ has characteristic 0 by Theorem 1.35(2). Suppose that $\mathbf{K}$ has characteristic $p > 0$. Let x be an algebraic element over $\mathbf{K}$, and suppose that it is not separable, meaning that its minimal polynomial is of the form $\psi_x(X^p)$. Write $\psi_x(X) = c_0 + c_1 X + ... + c_r X^r$. There exist $d_i \in \mathbf{K}$ $(1 \leq i \leq r)$ such that $c_i = d_i^p$. Then $\psi_x(X^p) = \xi_x(X)^p$, where $\xi_x(X) = d_0 + d_1 X + ... + d_r X^r$, which is impossible, since the polynomial $\psi_x(X^p)$ is irreducible.

(iii)⇒(i): By Theorem 1.35(2), we may assume that $\mathbf{K}$ has characteristic $p > 0$. Supposing that (iii) holds, let $y \in \mathbf{K}$. We will show by contradiction that y has a p-th root in $\mathbf{K}$, which shows that the Frobenius endomorphism is surjective. Let $y^{1/p}$ be a p-th root of y in $\bar{\mathbf{K}}$ and let f be the minimal polynomial of $y^{1/p}$ in $\mathbf{K}[X]$. Since f is irreducible, it divides $X^p - y$ in $\mathbf{K}[X]$, and, since $y \notin \mathbf{K}$, we have that $d^\circ(f) > 1$. Given that $\bar{\mathbf{K}}$ has characteristic p, we have that $X^p - y = \left(X - y^{1/p}\right)^p$ in $\bar{\mathbf{K}}[X]$, so $f(X) = \left(X - y^{1/p}\right)^q$ with $2 \leq q \leq p$; this implies that f is not separable, which is a contradiction. ■

(IV) SEPARABLE DEGREE

THEOREM 1.40.–

1) Dedekind's lemma*: Let* $\mathbf{L}$ *and* $\mathbf{L}'$ *be two fields.*

i) If $\{u_i : i \in I\}$ *is a set of distinct homomorphisms from* $\mathbf{L}$ *into* $\mathbf{L}'$*, then these homomorphisms are linearly independent over* $\mathbf{L}'$*.*

ii) If $\mathbf{L}$ *and* $\mathbf{L}'$ *are two field extensions of* $\mathbf{K}$ *and* $[\mathbf{L} : \mathbf{K}] = n$*, then there are at most* n *distinct* $\mathbf{K}$*-homomorphisms from* $\mathbf{L}$ *into* $\mathbf{L}'$*.*

2) Artin's theorem[10]*: Let* $\mathbf{N}$ *be a field,* Δ *a group of automorphisms of* $\mathbf{N}$, *and* $\mathbf{M} = \Delta^{\perp}$ *the set of elements of* $\mathbf{N}$ *fixed by every automorphism in* Δ. *Then* $\mathbf{M}$ *is a field, and* $[\mathbf{N} : \mathbf{M}] < \infty$ *if and only if* $(\Delta : 1) < \infty$. *If so,* $[\mathbf{N} : \mathbf{M}] = (\Delta : 1)$.

PROOF.– i) Suppose that there exists a linearly dependent subset $\{u_i : j \in J\}$ over $\mathbf{L}'$. This set necessarily has a finite minimal subset $\{u_1, ..., u_r\}$. Therefore, there exist elements $\alpha_1, ..., \alpha_r$ of $\mathbf{L}'$ such that $u_1 = \sum_{i=2}^{r} \alpha_i u_i$ and $u_2, ..., u_r$ are linearly independent, so, if $x, y \in \mathbf{L}$, $x \neq 0$, then

$$\begin{aligned} u_1(x) &= \sum\nolimits_{i=2}^{r} \alpha_i u_i(x), \quad u_1(x)\, u_1(y) = \sum\nolimits_{i=2}^{r} \alpha_i u_i(x)\, u_i(y) \\ &\Rightarrow \sum\nolimits_{i=2}^{r} \alpha_i u_i(x)\,(u_i(y) - u_1(y)) = 0, \end{aligned}$$

where the second equality is obtained by replacing x by xy in the first; but $u_i(y) \neq u_1(y)$ for $i \neq 1$, so $\alpha_i = 0$, and $u_1(x) = 0$ by the first equality, which is a contradiction.

ii) Let $\{b_1, ..., b_n\}$ be a basis of the $\mathbf{K}$-vector space $\mathbf{L}$. Suppose that there exist $n+1$ distinct $\mathbf{K}$-homomorphisms $u_1, ..., u_{n+1}$ from $\mathbf{L}$ into $\mathbf{L}'$. The system of n equations in $n+1$ unknowns x_i

$$\sum\nolimits_{i=1}^{n+1} u_i(b_j)\, x_i = 0 \quad (j = 1, ..., n)$$

has a non-zero solution $x_i = c_i$ in $\mathbf{L}'$. Every element $a \in \mathbf{L}$ is of the form $a = \sum_{j=1}^{n} \alpha_j b_j$ $(\alpha_j \in \mathbf{K})$. Hence,

$$\sum_{i=1}^{n+1} u_i(a)\, c_i = \sum_{i=1}^{n+1} u_i \left(\sum_{j=1}^{n} \alpha_j b_j \right) c_i = \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n+1} u_i(b_j)\, c_i = 0,$$

and so $\{u_i : 1 \leq i \leq n+1\}$ is a linearly dependent set over $\mathbf{L}'$, which is impossible by (i).

10 This result is conventionally attributed to Artin, although it was already present in Dedekind's work.

2) $\mathbf{M}$ is a field (**exercise**). If $(\Delta : 1)$ is infinite, then $[\mathbf{N} : \mathbf{M}]$ is infinite by (i). Let $(\Delta : 1) = n$, $\Delta = \{\sigma_1, ..., \sigma_r\}$, and suppose that there exists a basis $\{b_1, ..., b_r\}$ of $\mathbf{N}$ over $\mathbf{M}$ with $r < n$. The system of r equations and $n > r$ unknowns $x_i \in \mathbf{N}$

$$\left\{ \begin{array}{c} \sigma_1 (b_1) x_1 + ... + \sigma_n (b_1) x_n = 0 \\ \cdots \\ \sigma_1 (b_r) x_1 + ... + \sigma_n (b_r) x_n = 0 \end{array} \right. \qquad [1.10]$$

has a non-trivial solution. Let $(x_1, ..., x_n)$ be one such solution. Let $a \in \mathbf{N}$. There exist $\alpha^j \in \mathbf{M}$ such that $a = \sum_{1 \leq i \leq r} \alpha^i b_j$. Multiply the i-th equation of (1.10) by $\sigma_1 \left(\alpha^i\right)$ for each $i \in \{1, ..., r\}$. Since $\alpha^i \in \mathbf{M}$, we have that $\sigma_1 \left(\alpha^i\right) = \sigma_j \left(\alpha^i\right) (1 \leq i \leq r)$, and so we obtain

$$\left\{ \begin{array}{c} \sigma_1 \left(\alpha^1 b_1\right) x_1 + ... + \sigma_n \left(\alpha^1 b_1\right) x_n = 0 \\ \cdots \\ \sigma_1 (\alpha^r b_r) x_1 + ... + \sigma_n (\alpha^r b_r) x_n = 0 \end{array} \right.$$

which, by adding together all terms on the left, implies that $\sum_{i=1}^{r} \sigma_i (a) x_i = 0$. Since a was an arbitrary element of $\mathbf{N}$, this implies that $\sum_{i=1}^{r} x_i \sigma_i = 0$, which is impossible by (i). Therefore, $r \geq n$, and, by (ii), $r = n$. ■

DEFINITION 1.41.– *Let* $\mathbf{L}/\mathbf{K}$ *be an algebraic extension and let* $\bar{\mathbf{K}} \supset \mathbf{L}$ *be an algebraic closure of* $\mathbf{K}$. *The order of the group of* $\mathbf{K}$*-homomorphisms from* $\mathbf{L}$ *into* $\bar{\mathbf{K}}$ *is called the* separable degree *of* $\mathbf{L}$, *and is written* $[\mathbf{L} : \mathbf{K}]_s$.

We have the relation

$$[\mathbf{M} : \mathbf{K}]_s = [\mathbf{M} : \mathbf{L}]_s [\mathbf{L} : \mathbf{K}]_s ,$$

which is analogous to (1.1) ([LAN 99], Theorem 4.1). For finite extensions, Dedekind's lemma (Theorem 1.40(1)) implies that $[\mathbf{L} : \mathbf{K}]_s \leq [\mathbf{L} : \mathbf{K}]$, hence $[\mathbf{M} : \mathbf{K}]_s = [\mathbf{M} : \mathbf{K}]$ if and only if $[\mathbf{M} : \mathbf{L}]_s = [\mathbf{M} : \mathbf{L}]$ and $[\mathbf{L} : \mathbf{K}]_s = [\mathbf{L} : \mathbf{K}]$.

THEOREM 1.42.– *A finite extension* $\mathbf{L}/\mathbf{K}$ *is separable if and only if* $[\mathbf{L} : \mathbf{K}]_s = [\mathbf{L} : \mathbf{K}]$.

PROOF.–

1) Suppose that $\mathbf{L}/\mathbf{K}$ is separable. We therefore have that $\mathbf{L} = \mathbf{K}(x_1, ..., x_n)$, where each x_i is separable over $\mathbf{K}$. Furthermore,

$$\mathbf{K} \subset \mathbf{K}(x_1) \subset \mathbf{K}(x_1, x_2) \subset ... \subset \mathbf{K}(x_1, ..., x_n) = \mathbf{L}.$$

Since each x_i is separable over $\mathbf{K}$, x_i is separable over $\mathbf{K}(x_1, ..., x_{i-1})$ for all $i \geq 2$. Let f_i be the minimal polynomial of x_i over $\mathbf{K}(x_1, ..., x_{i-1})$ and let $n_i = d^\circ(f_i)$. Since f_i is separable, it has n_i roots α_j $(j = 1, ..., n_i)$, and therefore there exist n_i $\mathbf{K}(x_1, ..., x_{i-1})$-homomorphisms $u_j : x_i \mapsto \alpha_j$; moreover, the α_j form a basis of $\mathbf{K}(x_1, ..., x_i)$ over $\mathbf{K}(x_1, ..., x_{i-1})$. Hence,

$$[\mathbf{K}(x_1, ..., x_i) : \mathbf{K}(x_1, ..., x_{i-1})]_s = [\mathbf{K}(x_1, ..., x_i) : \mathbf{K}(x_1, ..., x_{i-1})],$$

so $[\mathbf{L} : \mathbf{K}]_s = [\mathbf{L} : \mathbf{K}]$ by induction.

2) Conversely, suppose that $[\mathbf{L} : \mathbf{K}]_s = [\mathbf{L} : \mathbf{K}]$. Let $x \in \mathbf{L}$; then $\mathbf{K} \subset \mathbf{K}(x) \subset \mathbf{L}$. We necessarily have that $[\mathbf{K}(x) : \mathbf{K}]_s = [\mathbf{K}(x) : \mathbf{K}]$, so x is separable. ■

COROLLARY 1.43.– *Let* $\mathbf{L}$ *be an algebraic extension of* $\mathbf{K}$. *If* $\mathbf{L}$ *is generated by a family of elements* $(x_i)_{i \in I}$, *then this extension is separable if and only if each* x_i *is separable.*

PROOF.– The necessary condition is clear. To show the sufficient condition, observe that every element of $\mathbf{L}$ belongs to some finite extension $\mathbf{K}(x_{i_1}, ..., x_{i_n})$, which is separable by Theorem 1.42. ■

COROLLARY 1.44.– *Let* $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$ *be three fields such that* $\mathbf{N}/\mathbf{K}$ *is an algebraic extension. For* $\mathbf{N}$ *to be separable over* $\mathbf{K}$, *it is necessary and sufficient for* $\mathbf{N}$ *to be separable over* $\mathbf{M}$ *and for* $\mathbf{M}$ *to be separable over* $\mathbf{K}$.

PROOF.–

1) (a) If $\mathbf{N}$ is separable over $\mathbf{K}$, then every element x of $\mathbf{N}$ is separable over $\mathbf{K}$, and this is certainly true for every element of $\mathbf{M}$, so $\mathbf{M}$ is separable over $\mathbf{K}$. (b) Every $x \in \mathbf{N}$ is a simple root of its minimal polynomial f over $\mathbf{K}$ by Definition 1.5(i); moreover, $f \in \mathbf{M}[X]$, so x is separable over $\mathbf{M}$, and consequently $\mathbf{N}$ is separable over $\mathbf{M}$.

2) The converse may be shown by reasoning similarly to (a) above. ■

Let $\mathbf{K}$ be a field and $\mathbf{N}/\mathbf{K}$ an algebraic extension. The set of elements of $\mathbf{N}$ that are separable over $\mathbf{K}$ form a subfield (**exercise**) called the *separable closure* of $\mathbf{K}$ in $\mathbf{N}$.

DEFINITION 1.45.– *Let* $\mathbf{K}$ *be a field. The* separable closure $\bar{\mathbf{K}}_s$ *of* $\mathbf{K}$ *(unique up to* $\mathbf{K}$*-isomorphism) is the separable closure of* $\mathbf{K}$ *in some algebraic closure* $\bar{\mathbf{K}}$ *of* $\mathbf{K}$.

1.1.6. *Normality*

THEOREM 1.46.– *Let x and y be two elements of* $\bar{\mathbf{K}}$. *The following conditions are equivalent:*

i) x and y are conjugate over $\mathbf{K}$.

ii) x and y have the same minimal polynomial over $\mathbf{K}$.

PROOF.– (i)⇒(ii): Let x and y be two conjugate elements over $\mathbf{K}$, u a $\mathbf{K}$-automorphism of $\mathbf{K}$ such that $u(x) = y$, and f the minimal polynomial of x over $\mathbf{K}$. We have that $f(y) = f(u(x)) = u(f(x)) = 0$, and f is irreducible over $\mathbf{K}[X]$, so f is the minimal polynomial of y.

(ii)⇒(i): Suppose that x and y have the same minimal polynomial f. Kronecker's theorem (Theorem 1.19) implies the existence of a $\mathbf{K}$-isomorphism $u : \mathbf{K}[X]/(f) \tilde{\rightarrow} \mathbf{K}(x)$ and a $\mathbf{K}$-isomorphism $v : \mathbf{K}[X]/(f) \tilde{\rightarrow} \mathbf{K}(y)$, so $w := v \circ u^{-1}$ is a $\mathbf{K}$-isomorphism $\mathbf{K}(x) \tilde{\rightarrow} \mathbf{K}(y)$ such that $w(x) = y$. Therefore, the $\mathbf{K}$-isomorphism w may be extended to a $\mathbf{K}$-automorphism of $\bar{\mathbf{K}}$. Indeed, let $z \in \mathbf{K}(x)$, write g for its minimal polynomial over $\mathbf{K}$, and let Φ be the set of roots of g in $\mathbf{K}(x)$. Since Φ is finite, w induces a bijection from Φ onto $w(\Phi)$, so $z \in w(\Phi) \subset \mathbf{K}(y)$, and thus $\mathbf{K}(x) = \mathbf{K}(y)$. By Steinitz' theorem (Theorem 1.26), there exists a $\mathbf{K}(x)$-automorphism $\bar{w}$ of $\bar{\mathbf{K}}$, and, writing $\iota : \mathbf{K}(x) \hookrightarrow \bar{\mathbf{K}}$ for the canonical injection, we have that $\iota \circ w = \bar{w} \circ \iota$. Given that $\bar{w}(x) = w(x) = y$, we deduce that x and y are conjugate over $\mathbf{K}$ (Definition 1.4). ■

COROLLARY 1.47.– *Let* $x \in \mathbf{K}$ *have degree n over* $\mathbf{K}$.

i) The conjugates of x are the roots of the minimal polynomial f of x, of which there are at most n.

ii) The element x is separable over $\mathbf{K}$ if and only if it has precisely n conjugates, in which case all of its conjugates are also separable.

PROOF.– (i) is clear by Theorem 1.46. (ii): The roots of f are the conjugates of x over $\mathbf{K}$, and (ii) follows from Lemma 1.18. ■

THEOREM 1.48.– *Let $\mathbf{N} \subset \bar{\mathbf{K}}$ be an extension of $\mathbf{K}$. The following conditions are equivalent:*

i) $\mathbf{N}$ is a normal extension of $\mathbf{K}$.

ii) For all $x \in \mathbf{N}$, the conjugates of x over $\mathbf{K}$ in $\bar{\mathbf{K}}$ all belong to $\mathbf{N}$.

iii) Every $\mathbf{K}$-automorphism of $\bar{\mathbf{K}}$ induces an automorphism of $\mathbf{N}$.

iv) $\mathbf{N}$ is the decomposition field in $\bar{\mathbf{K}}$ of some family $(f_i)_{i\in I}$ of non-constant polynomials in $\mathbf{K}[X]$.

PROOF.– (i)⇒(iv) and (iii)⇒(ii): clear. (iv)⇒(iii): Assuming that (iv) holds, let u be a $\mathbf{K}$-automorphism of $\bar{\mathbf{K}}$. For any $i \in I$, we have that $u|_{\mathcal{R}_i} \in \mathfrak{S}_{\mathcal{R}_i}$ where $\mathcal{R}_i$ is the set of roots of f_i in $\mathbf{K}$. But $\mathbf{N} = \mathbf{K}\left(\bigcup_{i\in I} \mathcal{R}_i\right)$, so $u(\mathbf{N}) = \mathbf{N}$, which shows (iii).

(ii)⇒(i): Suppose that (ii) holds. Let f be an irreducible unitary polynomial in $\mathbf{K}[X]$ of degree $n \geq 1$ with at least one root a_1 in $\mathbf{N}$. Given that $\bar{\mathbf{K}}$ is algebraically closed, there exist elements $a_2, ..., a_n$ in $\bar{\mathbf{K}}$ such that $f(X) = \prod_{i=1}^{n}(X - a_i)$. But the a_i $(2 \leq i \leq n)$ are the conjugates of a_1 over $\mathbf{K}$ by Corollary 1.47, which are all in $\mathbf{N}$, and (i) follows. ■

REMARK 1.49.– *If $(\mathbf{N}_i)_{i\in I}$ is a family of normal extensions of $\mathbf{K}$, then $\bigcap_{i\in I} \mathbf{N}_i$ and $\mathbf{K}\left(\bigcup_{i\in I} \mathbf{N}_i\right)$ are normal extensions of $\mathbf{K}$ (**exercise***: cf. [BKI 12], Chapter V, section 9.4, Proposition 4). If $\mathbf{L}/\mathbf{K}$ is a finite extension, then the normal extension $\mathbf{N}$ generated by $\mathbf{L}$ is also finite, since $\mathbf{L} = \mathbf{K}(A)$, where A is finite, and $\mathbf{N} = \mathbf{K}(B)$, where B is the set of conjugates of the elements of A. In particular, every normal extension $\mathbf{N}$ of $\mathbf{K}$ is a union of finite normal subextensions of $\mathbf{N}$ over $\mathbf{K}$, namely $\mathbf{N} = \bigcup_{x\in\mathbf{L}} \mathbf{N}_x$, where $\mathbf{N}_x$ is the normal extension generated by $\mathbf{K}(x)$.*

COROLLARY 1.50.– *Let $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$ be three fields. If the extension $\mathbf{N}/\mathbf{K}$ is normal, then the extension $\mathbf{N}/\mathbf{M}$ is normal.*

PROOF.– Let $\Omega \supset \mathbf{N}$ be an algebraically closed field (Theorem 1.26). Let u be an $\mathbf{M}$-automorphism of Ω. Then u is a $\mathbf{K}$-automorphism of Ω. Since the extension $\mathbf{N}/\mathbf{K}$ is normal, we have that $u(\mathbf{N}) = \mathbf{N}$ by Theorem 1.48, so the extension $\mathbf{N}/\mathbf{M}$ is normal by the same theorem. ■

1.1.7. *Galois extensions and Galois groups*

THEOREM 1.51.– *Let* $\mathbf{N}$ *be an* algebraic *extension of* $\mathbf{K}$ *and let* $\mathfrak{G}$ *be the group of* $\mathbf{K}$*-automorphisms of* $\mathbf{N}$*. The following conditions are equivalent:*

i) $\mathbf{K}$ *is the fixed field of* $\mathfrak{G}$*.*

ii) $\mathbf{N}$ *is a Galois extension of* $\mathbf{K}$ *(Definition 1.7).*

iii) For every $x \in \mathbf{N}$*, the minimal polynomial of* x *over* $\mathbf{K}$ *factors into a product of distinct linear polynomials in* $\mathbf{N}[X]$*.*

PROOF.– (ii)⇔(iii) by Definitions 1.5 and 1.6.

(i)⇒(iii): Let $\mathbf{K}$ be the fixed field of $\mathfrak{G}$, $x \in \mathbf{N}$, f the minimal polynomial of x over $\mathbf{K}$, write $\mathcal{R}$ for the set of roots of f in $\mathbf{N}$, and set $g(X) = \prod_{a \in \mathcal{R}} (X - a) \in \mathbf{K}[X]$. Then $g \mid f$. Since f is irreducible, $g = f$.

(iii)⇒(i): Let $x \in \complement_{\mathbf{N}}\mathbf{K}$ and let Ω be an algebraic closure of $\mathbf{N}$ (Theorem 1.26). Let f be the minimal polynomial of x over $\mathbf{K}$. Since $x \notin \mathbf{K}$, we have that $d^\circ(f) \geq 2$. Let $\mathcal{R}$ be the set of roots of f in $\mathbf{N}$. Condition (iii) implies that $f(X) = \prod_{y \in \mathcal{R}} (X - y)$, and $\mathcal{R}$ is therefore the set of conjugates of x in Ω (Corollary 1.47). There exists an element $y \neq x$ in $\mathcal{R}$, and there also exists $u \in \mathfrak{G}$ such that $u(x) = y$ (Definition 1.4). Given that (ii)⇔(iii), $\mathbf{N}$ is a normal extension of $\mathbf{K}$, so condition (i) of Theorem 1.48 is satisfied, and consequently condition (iii) of this theorem is also satisfied, namely that $u(\mathbf{N}) = \mathbf{N}$. Let $\sigma = u|_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$. Thus, for every $x \notin \mathbf{K}$, there exists an automorphism σ of $\mathbf{N}$ such that $\sigma(x) \neq x$, hence $\mathbf{K}$ is the fixed field of $\mathfrak{G}$. ■

If one of the equivalent conditions in Theorem 1.51 is satisfied, then $\mathfrak{G}$ is the Galois group $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ (Definition 1.8).

REMARK 1.52.– *Remark 1.49 still holds if we replace "normal" by "Galois" (**exercise**).*

LEMMA 1.53.– *Let* $\mathbf{N}$ *be a Galois extension of* $\mathbf{K}$ *and let* $\mathbf{L}$ *be a subextension* $\mathbf{N}$ *that is finite over* $\mathbf{K}$. *Then there exists a Galois subextension* $\mathbf{M}$ *of* $\mathbf{N}$ *containing* $\mathbf{L}$ *that is finite over* $\mathbf{K}$.

PROOF.– The extension $\mathbf{N}/\mathbf{K}$ is normal and $[\mathbf{L}:\mathbf{K}] < \infty$, so there exists a finite normal subextension $\mathbf{M}/\mathbf{K}$ that contains $\mathbf{L}$ (Remark 1.49). As $\mathbf{N}$ is separable over $\mathbf{K}$, $\mathbf{M}$ is also separable over $\mathbf{K}$. ■

LEMMA 1.54.– *Consider three fields* $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$. *If the extension* $\mathbf{N}/\mathbf{K}$ *is Galois, then the extension* $\mathbf{N}/\mathbf{M}$ *is Galois.*

PROOF.– This follows from Corollaries 1.44 and 1.50. ■

Recall that $\mathcal{R}(f)$ denotes the set of roots in $\mathbf{K}$ of the polynomial $f \in \mathbf{K}[X]^{\times}$.

LEMMA 1.55.– *Let* x *and* y *be elements of* $\mathcal{R}(f)$.

1) The following conditions are equivalent:

i) x *and* y *are conjugate over* $\mathbf{K}$.

ii) x *and* y *belong to the same orbit of* $\mathrm{Gal}(f)$ *([P1], section 2.2.8(**I**)).*

iii) x *and* y *are roots of the same irreducible factor of* f.

2) In particular, f *is irreducible if and only if* $\mathrm{Gal}(f)$ *acts transitively on* $\mathcal{R}(f)$ *([P1], section 2.2.8(**II**)).*

PROOF.– The equivalence (i)⇔(ii) follows from the definitions, and (i)⇔(iii) follows from Theorem 1.46. ■

For example, if $f(X) = X^2 + 1 \in \mathbb{Q}[X]$, then $\mathcal{R}(f) = \{i, -i\}$, and $\mathrm{Gal}(f)$ is the group of permutations $\langle\sigma\rangle$, where $\sigma : i \leftrightarrow -i$, and thus acts transitively on $\mathcal{R}(f)$. The field $\mathbb{Q}(i)$ is called the field of *Gaussian rationals* (which is the field of fractions of the ring of *Gaussian integers* $\mathbb{Z}[i]$).

If the Galois group $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ is cyclic ([P1], section 2.2.4(**I**)), as is the case in the above example, we say that the extension $\mathbf{N}/\mathbf{K}$ is *cyclic*.

If $f(X) = (X^2+2)(X^2+1)$, then $\mathcal{R}(f) = \{\sqrt{2}, -\sqrt{2}, i, -i\}$ and $\mathrm{Gal}(f) = \langle\sigma, \mu\rangle$, where σ is the same permutation as above and $\mu : \sqrt{2} \leftrightarrow -\sqrt{2}$. In this case, $\sqrt{2}$ and i belong to separate orbits.

1.1.8. *Fundamental theorem of Galois theory*

In this section, we will give a proof of Theorem 1.9, then illustrate this theorem with an example.

(I) PROOF OF THEOREM 1.9 Let $\mathbf{N}/\mathbf{K}$ be a Galois extension. Every field $\mathbf{M}$ such that $\mathbf{K} \subseteq \mathbf{M} \subseteq \mathbf{N}$ uniquely determines the subset $\Delta = \mathbf{M}^{\perp}$ of $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ consisting of the $\mathbf{M}$-automorphisms of $\mathbf{N}$. Conversely, every subgroup Δ of $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ uniquely determines the subset $\mathbf{M} = \boldsymbol{\Delta}^{\perp}$ of $\mathbf{N}$ consisting of the elements fixed by every $\mathbf{K}$-automorphism in Δ. Furthermore, $\mathbf{M}$ is a field (Theorem 1.40(2)) and Δ is a group (**exercise**). The extension $\mathbf{N}/\mathbf{M}$ is Galois (Lemma 1.54). This proves claim (1) of the theorem.

For the rest of this section, every extension is finite. Therefore, $(\mathrm{Gal}(\mathbf{N}/\mathbf{M}) : 1) = [\mathbf{M} : \mathbf{N}]$ by Dedekind's lemma and $(\Delta : 1) = [\mathbf{M} : \mathbf{N}]$ by Artin's theorem (Theorem 1.40). Furthermore, $(\mathrm{Gal}(\mathbf{N}/\mathbf{K}) : \mathrm{Gal}(\mathbf{N}/\mathbf{M})) = [\mathbf{M} : \mathbf{K}]$ by Lagrange's theorem (cf. [P1], section 2.2.1**(I),** and above (1.1)), which proves claim (2) of Theorem 1.9.

LEMMA 1.56.– *Let* $\mathfrak{G} = \mathrm{Gal}(\mathbf{N}/\mathbf{K})$, $\beta \in \mathbf{N}$, *and* $\tau \in G$.

i) The set of $\mathbf{K}$*-automorphisms* $u \in \mathfrak{G}$ *such that* $u(\beta) = \beta$ *is the subgroup* $\mathfrak{H}_\beta = \mathrm{Gal}(\mathbf{K}(\beta)/\mathbf{K})$ *of* $\mathfrak{G}$.

ii) The set of $\mathbf{K}$*-automorphisms* $\rho \in G$ *such that* $\rho(\beta) = \tau(\beta)$ *is the left coset of* τ *mod.* $\mathfrak{H}_\beta$ *([P1], section 2.2.1***(I)***), namely* $\tau\mathfrak{H}_\beta \in \mathfrak{G}/\mathfrak{H}_\beta$.

PROOF.– (i) is clear, so we only need to prove (ii). Let $\rho : \beta \mapsto \tau(\beta)$. Then

$$\left(\tau^{-1} \circ \rho\right)(\beta) = \beta \Leftrightarrow \tau^{-1} \circ \rho \in \mathfrak{H}_\beta \Leftrightarrow \rho \in \tau\mathfrak{H}_\beta. \blacksquare$$

With the notation of Lemma 1.56, we have that $\rho \in \mathfrak{H}_{\tau(\beta)}$ if and only if

$$\rho(\tau(\beta)) = \tau(\beta) \Leftrightarrow \left(\tau^{-1} \circ \rho \circ \tau\right)(\beta) = \beta \Leftrightarrow \tau^{-1} \circ \rho \circ \tau \in \mathfrak{H}_\beta,$$

which implies that $\mathfrak{H}_{\tau(\beta)} = \tau \mathfrak{H}_\beta \tau^{-1}$. Let $\mathbf{M}/\mathbf{K}$ be a subextension of $\mathbf{N}/\mathbf{K}$. We have that $\mathbf{M} = \mathbf{K}\left(\bigcup_{\beta \in M} \beta\right)$ and $\mathrm{Gal}\left(\mathbf{M}/\mathbf{K}\right) = \bigcap_{\beta \in \mathbf{M}} \mathfrak{H}_\beta$. Let $\tau \in G$. Then $\tau\left(\mathbf{M}\right) = \mathbf{K}\left(\bigcup_{\beta \in \mathbf{M}} \tau\left(\beta\right)\right)$ and

$$\begin{aligned}\mathrm{Gal}\left(\tau\left(\mathbf{M}\right)/\mathbf{K}\right) &= \bigcap\nolimits_{\beta \in \mathbf{M}} \mathfrak{H}_{\tau(\beta)} = \bigcap\nolimits_{\beta \in \mathbf{M}} \tau \mathfrak{H}_\beta \tau^{-1} \\ &= \tau\left(\bigcap\nolimits_{\beta \in \mathbf{M}} \mathfrak{H}_\beta\right)\tau^{-1} = \tau\,\mathrm{Gal}\left(\mathbf{M}/\mathbf{K}\right)\tau^{-1}.\end{aligned}$$

From this lemma, we can deduce the following result, which shows claim (3) of Theorem 1.9:

LEMMA 1.57.– *The subextension* $\mathbf{M}/\mathbf{K}$ *of* $\mathbf{N}/\mathbf{K}$ *is normal if and only if* $\mathrm{Gal}\left(\mathbf{M}/\mathbf{K}\right)$ *is a normal subgroup of* $\mathrm{Gal}\left(\mathbf{N}/\mathbf{K}\right)$.

PROOF.– The extension $\mathbf{M}/\mathbf{K}$ is normal if and only if, for all $\tau \in G$, $\tau\left(\mathbf{M}\right) = \mathbf{M}$ (Theorem 1.48), which is equivalent to saying that $\tau\,\mathrm{Gal}\left(\mathbf{M}/\mathbf{K}\right)\tau^{-1} = \mathrm{Gal}\left(\mathbf{M}/\mathbf{K}\right)$ by the above. ■

Claim (5) of Theorem 1.9 directly follows from claims (3) and (4). Claim (4) may be shown as follows. We know that:

$$\begin{aligned}\left[\mathbf{M}' : \mathbf{M}\right] &= \left[\mathbf{M}' : \mathbf{K}\right]/\left[\mathbf{M} : \mathbf{K}\right] = \frac{\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{K}\right) : \mathrm{Gal}\left(\mathbf{N}/\mathbf{M}'\right)\right)}{\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{K}\right) : \mathrm{Gal}\left(\mathbf{N}/\mathbf{M}\right)\right)} \\ &= \frac{\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{K}\right) : 1\right)/\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{M}'\right) : 1\right)}{\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{K}\right) : 1\right)/\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{M}\right) : 1\right)} = \frac{\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{M}\right) : 1\right)}{\left(\mathrm{Gal}\left(\mathbf{N}/\mathbf{M}'\right) : 1\right)}.\end{aligned}$$

(II) EXAMPLE Let $a > 0$ be an integer that is not a square in $\mathbb{Z}$. The polynomial $f\left(X\right) = X^4 - a$ is irreducible over $\mathbf{K} = \mathbb{Q}$ by Eisenstein's criterion (Theorem 1.16). This means that it is separable, since $\mathbb{Q}$ has characteristic 0. The four roots of f in $\mathbb{C}$ are $r, ir, -r, -ir$, where $r = \sqrt[4]{a}$. The decomposition field of f is $\mathbf{N} = \mathbb{Q}\left(r, i\right)$, so the extension $\mathbf{N}/\mathbb{Q}$ is normal, and hence Galois. We have that $\left[\mathbb{Q}\left(r\right) : \mathbb{Q}\right] = 4$, because $\left\{1, r, r^2, r^3\right\}$ is a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}\left(r\right)$. Furthermore, $\left[\mathbb{Q}\left(r, i\right) : \mathbb{Q}\left(r\right)\right] = 2$, because $\left\{1, i\right\}$ is a basis of the $\mathbb{Q}\left(r\right)$-vector space

$\mathbb{Q}(r,i)$ (as well as of the $\mathbb{Q}$-vector space $\mathbb{Q}(i)$). Consequently, $[\mathbb{Q}(r,i):\mathbb{Q}] = 8$ by (1.1), which implies that $(\mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q}):1) = 8$ by the first equality in (1.3).

There exists a $\mathbb{Q}$-automorphism σ of $\mathbb{Q}(r,i)$ that sends r to ri and which induces the cycle $r \mapsto ir \mapsto -r \mapsto -ir \mapsto r$. This mapping fixes i, and so belongs to $\mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q}(i))$. But

$$\begin{aligned}(\mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q}(i)):1) &= (\mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q}):1)/\,\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \\ &= 8/2 = 4,\end{aligned}$$

so $\mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q}(i)) = \langle\sigma\rangle = \{1,\sigma,\sigma^3,\sigma^3\}$, and $(\langle\sigma\rangle:1) = 4$.

Similarly, the extension $\mathbb{Q}(r,i)/\mathbb{Q}(r)$ is cyclic and normal, since the two $\mathbb{Q}(r)$-automorphisms of $\mathbb{Q}(r,i)$ are $\tau^0 = 1$ and τ, where 1 is the neutral element of $\mathfrak{G} = \mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q})$ and $\tau : i \mapsto -i$; therefore $\mathrm{Gal}(\mathbb{Q}(r,i)/\mathbb{Q}(r)) = \langle\tau\rangle$, and $(\langle\tau\rangle:1) = 2$.

Finally, $\mathfrak{G} = \langle\tau,\sigma\rangle$. The 8 elements of this Galois group are listed in the table below (1 is the neutral element):

	1	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
image of r	r	ir	$-r$	$-ir$	r	ir	$-r$	$-ir$
image of i	i	i	i	i	$-i$	$-i$	$-i$	$-i$

The lattice of subgroups of $\mathfrak{G}$ is shown below (each arrow represents an inclusion):

$$\begin{array}{ccccccccc}
 & & \longleftarrow & & \langle 1\rangle & & \longrightarrow & & \\
 & \swarrow & & \swarrow & \downarrow & \searrow & & \searrow & \\
\langle\tau\rangle & & \langle\sigma^2\tau\rangle & & \langle\sigma^2\rangle & & \langle\sigma\tau\rangle & & \langle\sigma^3\tau\rangle \\
 & \searrow & \downarrow & & \downarrow & & \downarrow & \swarrow & \\
 & & \langle\tau,\sigma^2\tau\rangle & & \langle\sigma\rangle & & \langle\sigma\tau,\sigma^3\tau\rangle & & \\
 & & & \searrow & \downarrow & \swarrow & & & \\
 & & & & \mathfrak{G} & & & &
\end{array}$$

By Theorem 1.9, this lattice of subgroups corresponds to the lattice of subextensions of $\mathbf{N}/\mathbb{Q}$ (bearing in mind the equalities $(1+i)^2 = 2i = -(1-i)^2$):

$$\begin{tikzcd}[column sep=small]
 & & \mathbf{N} & & \\
\mathbb{Q}(r) \arrow[urr] & \mathbb{Q}(ir) \arrow[ur] & \mathbb{Q}\left(i, r^2\right) \arrow[u] & \mathbb{Q}(r(1+i)) \arrow[ul] & \mathbb{Q}(r(1-i)) \arrow[ull] \\
 & \mathbb{Q}\left(r^2\right) \arrow[ul] \arrow[u] & \mathbb{Q}(i) \arrow[u] & \mathbb{Q}\left(ir^2\right) \arrow[u] \arrow[ur] & \\
 & & \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

1.1.9. *Binomial equation*

Let $\mathbf{K}$ be a field of characteristic $p \geq 0$. The *binomial equation* in $\mathbf{K}$ is the equation

$$X^n - a = 0,$$

where $a \in \mathbf{K}^\times$ and n is an integer > 1. The solutions of the binomial equation are therefore the roots of the polynomial $f(X) = X^n - a \in \mathbf{K}[X]$.

LEMMA 1.58.– *The polynomial f is separable if and only if $p = 0$ or $p \nmid n$.*

PROOF.– We know that $f'(X) = nX^{n-1} \neq 0$. If $p = 0$ or $p \nmid n$, then f' is coprime with f, so f is separable (Lemma 1.28). If $p \neq 0$ and $p \mid n$, then $f' = 0$, so f and f' are not coprime and f is not separable. ■

If $p \neq 0$ and $p \mid n$, let $k = n/p$, and let r be a root of f in some algebraic closure $\bar{\mathbf{K}}$ of $\mathbf{K}$. Since $\bar{\mathbf{K}}$ has characteristic p (cf. proof of Theorem 1.39), we have that $f(X) = X^n - r^n = X^{kp} - r^{kp} = (X - r)^{kp}$, so r is the unique root (of order n) of f.

If $p = 0$ or $p \nmid n$, then f is separable, and the set $\mathcal{R}$ of roots of f in $\bar{\mathbf{K}}$ consists of the n distinct elements $r_1, r_2, ..., r_n$. The decomposition field of f is $\mathbf{K}(\mathcal{R})$. The quotients $1, r_2/r_1, ..., r_n/r_1$ are the solutions of $X^n - 1 = 0$; the set of these solutions is therefore the group $\boldsymbol{\mu}_n$ of n-th roots of unity, which is cyclic (Theorem-Definition 1.38(4)).

LEMMA 1.59.– *Consider the polynomial $f(X) = X^n - a \in \mathbf{K}[X]$, where n is a prime number. Either f has a root in $\mathbf{K}$, or f is irreducible over $\mathbf{K}$.*

PROOF.– (1) If $n = \mathrm{Char}(\mathbf{K})$ and $a^n \notin \mathbf{K}$, then f is irreducible. (2) Suppose now that $n \neq \mathrm{Char}(\mathbf{K})$. Then the roots of f in $\bar{\mathbf{K}}$ are $r, \zeta r, ..., \zeta^{n-1} r$, where r is a root of f in $\bar{\mathbf{K}}$ and ς is a primitive n-th root of unity. If f is reducible in $\mathbf{K}$, then there exists a non-constant polynomial $g \in \mathbf{K}[X]$ of degree $m < n$ that may be written as the product of factors $X - \zeta^i r$. Its constant term is a product of m factors $\zeta^i r$, so is of the form $b = \zeta^j r^m$. Therefore, $b^n = (\zeta^n)^j (r^n)^m = a^m$. Since $1 < m < n$ and given that n is prime, n and m are coprime, and so there exist integers u and v satisfying Bézout's equation $un + vm = 1$. Thus, $a = a^{un+vm} = a^{un} b^{vn} = (a^u b^v)^n = c^n$, where $c := a^u b^v \in \mathbf{K}$. Consequently, $f(X) = X^n - c^n$, so $f(c) = 0$, and c is a root of f that belongs to $\mathbf{K}$. ■

Let $\mathbf{N}$ be the decomposition field of $X^n - a$. Consider the diagram shown below, whose top row consists of the subextensions of $\mathbf{N}/\mathbf{K}$, and where each vertical arrow associates the field $\mathbf{L} \subset \mathbf{N}$ with the group $\mathrm{Gal}(\mathbf{N}/\mathbf{L})$:

$$\begin{array}{ccccccccc}
\mathbf{K}(\zeta, r) = \mathbf{N} & \supset & \mathbf{K}(\zeta) & \supset & \mathbf{K} & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\{1\} & \to & \Delta & \to & \mathfrak{G} & \to & \mathfrak{G}/\Delta & \to & \{1\}
\end{array}$$

LEMMA 1.60.– *The subgroup $\Delta = \mathrm{Gal}(\mathbf{N}/\mathbf{K}(\zeta))$ of $\mathfrak{G}$ is normal and $\Sigma = \mathrm{Gal}(\mathbf{K}(\zeta)/\mathbf{K}) \cong \mathfrak{G}/\Delta$ is a cyclic group whose order divides $n - 1$.*

PROOF.– The extension $\mathbf{K}(\zeta)/\mathbf{K}$ is a normal subextension of $\mathbf{N}/\mathbf{K}$, because it is the decomposition field of $X^n - 1$ (Theorem 1.48). Hence (by Theorem 1.9(4), setting $\mathbf{M}' = \mathbf{K}(\zeta)$ and $\mathbf{M} = \mathbf{K}$), $\Delta = \mathrm{Gal}(\mathbf{N}/\mathbf{K}(\zeta))$ is a normal subgroup of $\mathfrak{G}$. The bottom row of the diagram above is therefore a short exact sequence of groups ([P1], section 2.2.2(**II**)) and $\Sigma := \mathfrak{G}/\Delta \cong \mathrm{Gal}(\mathbf{K}(\zeta)/\mathbf{K})$. The group Σ is a subgroup of the group Θ of automorphisms of $\boldsymbol{\mu}_n = \left\{1, \zeta, ..., \zeta^{n-1}\right\}$. Every element of Θ may be written in the form $\zeta \mapsto \zeta^i$ $(i = 1, ..., n-1)$, so Θ is isomorphic to the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$, namely the group of units of $\mathbb{Z}/n\mathbb{Z}$, which is cyclic of order $\varphi(n) = n - 1$ (Theorem-Definition 1.38(5)). Consequently, $(\Sigma : 1)$ divides $n - 1$. ■

THEOREM 1.61.– *If n is a prime number that is not the characteristic p of $\mathbf{K}$, then the Galois group of $X^n - a$ is solvable.*

PROOF.– The group Δ is the Galois group of the extension $\mathbf{N}/\mathbf{K}(\zeta)$, and by Lemma 1.59 there are two possibilities: (1) $r \notin \mathbf{K}(\zeta)$; in this case, $X^n - a$ is irreducible over $\mathbf{K}(\zeta)$, so Δ is a group of order n by Artin's theorem (Theorem 1.40(2)), and is therefore isomorphic to the additive group $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, ..., n-1\} = \langle 1 \rangle$, which is cyclic of order n. (2) $r \in \mathbf{K}(\zeta)$, and thus $\Delta = \{1\}$. In both cases, $\mathfrak{G}$ is the extension of the cyclic group Σ by the normal cyclic subgroup Δ ([P1], section 2.2.2(**II**)). Since Σ and Δ are solvable, $\mathfrak{G}$ is also solvable ([P1], section 2.2.7(**I**)). ■

THEOREM 1.62.– *(Kummer) Let $\mathbf{N}/\mathbf{K}$ be a normal extension such that $[\mathbf{N} : \mathbf{K}]$ is a prime number n. If $\mathbf{K}$ contains a primitive n-th root of unity ζ, then $\mathbf{N}$ is the decomposition field of some irreducible polynomial $X^n - a \in \mathbf{K}[X]$.*

PROOF.– Let $\mathfrak{A} = \{x \in \mathbf{N}^\times : x^n \in \mathbf{K}\}$. The commutative group $\mathbf{K}^\times$ is a subgroup of $\mathfrak{A}$. Let $a \in \complement_{\mathfrak{A}} \mathbf{K}^\times$ and let $a\mathbf{K}^\times$ be its coset $(\mathrm{mod}\, .\mathbf{K}^\times)$. Since $a \in \mathfrak{A}$, we have that $a^n \in \mathbf{K}$, and, by the hypothesis, the $\zeta a, ..., \zeta^{n-1} a$ must all belong to $\mathbf{N}$. Therefore, $\mathbf{N}$ contains some decomposition field $\mathbf{M}$ of $X^n - a$. We know that $[\mathbf{N} : \mathbf{K}] = [\mathbf{N} : \mathbf{M}][\mathbf{M} : \mathbf{K}]$, $[\mathbf{M} : \mathbf{K}] \neq 1$ and $[\mathbf{N} : \mathbf{K}] = n$ is prime, so $[\mathbf{N} : \mathbf{M}] = 1$ and $\mathbf{M} = \mathbf{N}$. ■

1.1.10. *Solving algebraic equations by radicals*

The goal of this section is to prove Galois' theorem (Theorem 1.15) and give a few additional remarks. Let $\mathbf{K}$ be a field and suppose that $f \in \mathbf{K}[X]^\times$ is an irreducible polynomial in $\mathbf{K}[X]$.

LEMMA 1.63.– *If the equation $f(X) = 0$ may be solved by radicals (Definition 1.14) of the form $\sqrt[n_i]{.}$ $(1 \leq i \leq r)$ and if none of the exponents n_i are divisible by $p = \mathrm{Char}(\mathbf{K})$, then the group $\mathrm{Gal}(f)$ is solvable.*

PROOF.– Let $\mathbf{L}/\mathbf{K}$ be a Galois extension such that the decomposition field $\mathbf{N}$ of f is contained in $\mathbf{L}$ and $\mathbf{L} \supset \mathbf{M}_r$, with $\mathbf{M}_0 = \mathbf{K}$, $\mathbf{M}_i = \mathbf{M}_{i-1}(\alpha_i)$, $X^{n_i} - a_i = 0$, $a_i \in \mathbf{M}_{i-1}$ $(1 \leq i \leq r)$. We may assume without loss of generality that the n_i are prime numbers q_i (since if $n = q_1 q_2 ... q_k$ is a product of prime numbers, then $\sqrt[n]{.} = \sqrt[q_1]{\sqrt[q_2]{... \sqrt[q_k]{.}}}$); the q_i are not equal to p by the hypothesis. The chain of subfields of $\mathbf{L}$

$$\mathbf{K} = \mathbf{M}_0 \subset \mathbf{M}_1 \subset ... \subset \mathbf{M}_r = \mathbf{L}$$

corresponds to the chain of subgroups of $\mathfrak{G} = \mathrm{Gal}(\mathbf{L}/\mathbf{K})$

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_r = \{1\} \tag{1.11}$$

where $\mathfrak{G}_i \lhd \mathfrak{G}_{i-1}$ and $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ is cyclic of order q_i by the proof of Theorem 1.61. Hence, $\mathfrak{G}$ is solvable ([P1], section 2.2.7(**I**), Lemma 2.19) and (1.11) is a Jordan-Hölder series of $\mathfrak{G}$ ([P1], section 2.2.5(**II**)).

The extension $\mathbf{L}/\mathbf{N}$ is a normal subextension of $\mathbf{L}/\mathbf{K}$, and, by Theorem 1.9(3), the Galois group $\mathfrak{H} = \mathrm{Gal}(\mathbf{L}/\mathbf{N})$ is a normal subgroup of $\mathfrak{G}$. By Schreier's theorem ([P1], section 2.1.3(**II**), Theorem 2.9(1)), the normal series $\mathfrak{G} \supset \mathfrak{H} \supset \{1\}$ has a refinement

$$\mathfrak{G} = \mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \ldots \supset \mathfrak{H} \supset \ldots \supset \{1\}$$

that is isomorphic to (1.11), which implies that $\mathfrak{H}_{i-1}/\mathfrak{H}_i \cong \mathfrak{G}_{i-1}/\mathfrak{G}_i$. The Galois group of $\mathbf{N}/\mathbf{K}$, namely the Galois group of f (Definition 1.10), is isomorphic to $\mathfrak{G}/\mathfrak{H}$ (Theorem 1.9(5)), and by taking quotients we obtain the normal series

$$\mathfrak{G}/\mathfrak{H} \supset \mathfrak{H}_1/\mathfrak{H} \supset \mathfrak{H}_2/\mathfrak{H} \supset \ldots \supset \mathfrak{H}/\mathfrak{H} = \{1\}.$$

By Noether's third isomorphism theorem ([P1], section 2.2.3(**II**), Theorem 2.12(3)),

$$\frac{\mathfrak{H}_{i-1}/\mathfrak{H}}{\mathfrak{H}_i/\mathfrak{H}} \cong \frac{\mathfrak{H}_{i-1}}{\mathfrak{H}_i} \cong \frac{\mathfrak{G}_{i-1}}{\mathfrak{G}_i}.$$

Since these quotients are cyclic of prime order, $\mathfrak{G}/\mathfrak{H}$ is solvable. ■

Let $\mathbf{K}$ be a field, suppose that $f \in \mathbf{K}[X]^\times$ is a polynomial whose irreducible factors are separable, and let $\mathbf{N}$ be the decomposition field of f. Let $\mathbf{K}'/\mathbf{K}$ be an extension and let $\mathbf{N}'$ be the decomposition field of f viewed as an element of $\mathbf{K}'[X]$. Finally, let $\{\alpha_1, \ldots, \alpha_s\}$ be the roots of f in $\mathbf{N}'$; then $\mathbf{K}(\alpha_1, \ldots, \alpha_s)$ is a subfield of $\mathbf{N}'$, and $f(X) = \prod_{i=1}^{s}(X - \alpha_i) \in \mathbf{K}[X]$, so $\mathbf{K}(\alpha_1, \ldots, \alpha_s)$ is a decomposition field of f viewed as an element of $\mathbf{K}[X]$. Consequently, $\mathbf{N} \cong \mathbf{K}(\alpha_1, \ldots, \alpha_s)$ (Theorem 1.21(2)). These two fields may therefore be identified, and we can assume without loss of generality that $\mathbf{N} \subset \mathbf{N}'$. Hence, $\mathbf{N} \cap \mathbf{N}'$ is a field and $\mathbf{K} \subset \mathbf{N} \cap \mathbf{N}' \subset \mathbf{N}$.

THEOREM 1.64.– (of natural irrationalities) *Let* $\mathfrak{G} = \mathrm{Gal}\,(\mathbf{N}/\mathbf{K})$ *and suppose that* $\mathfrak{H} = \mathrm{Gal}\left(\mathbf{N}'/\mathbf{K}'\right)$. *Then* $\mathfrak{H} \cong (\mathbf{N} \cap \mathbf{K}')^{\perp} \subseteq \mathfrak{G}$.

PROOF.– Each element of $\mathrm{Gal}\,(\mathbf{N}'/\mathbf{K}')$ induces a permutation of $\alpha_1, ..., \alpha_s$ that fixes the elements of $\mathbf{K}'$ (in particular the elements of $\mathbf{K}$), and hence uniquely determines an element of $\mathrm{Gal}\,(\mathbf{N}/\mathbf{K})$. Hence, $\mathfrak{H} = \mathrm{Gal}\left(\mathbf{N}'/\mathbf{K}'\right)$ is a subgroup of $\mathfrak{G}$. Each element of $\mathfrak{H}$ fixes the elements of $\mathbf{N} \cap \mathbf{K}'$; since any element of $\mathbf{N}$ that is not in $\mathbf{N} \cap \mathbf{K}'$ is not in $\mathbf{K}'$, this element is not fixed by $\mathfrak{H}$, so $\mathfrak{H} = (\mathbf{N} \cap \mathbf{K}')^{\perp}$. ■

LEMMA 1.65.– *Let* $f \in \mathbf{K}[X]$ *be an irreducible polynomial of degree* $n > 0$. *If the Galois group* $\mathfrak{G} = \mathrm{Gal}\,(f)$ *is solvable, and if* $\mathrm{Char}\,(\mathbf{K}) = 0$ *or* $\mathrm{Char}\,(\mathbf{K})$ *is greater than the order of all quotients* $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ *in some Jordan-Hölder series of* $\mathfrak{G}$, *then the equation* $f(X) = 0$ *is solvable by radicals.*

PROOF.– Let $\mathbf{N}$ be the decomposition field of f and suppose that

$$\mathfrak{G} = \mathfrak{G}_0 \rhd \mathfrak{G}_1 \rhd ... \rhd \mathfrak{G}_r = \{1\}$$

is a Jordan-Hölder series of $\mathfrak{G} = \mathrm{Gal}\,(f) = \mathrm{Gal}\,(\mathbf{N}/\mathbf{K})$. Since $\mathfrak{G}$ is solvable, the quotients $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ $(i = 1, ..., r)$ are cyclic of prime order ([P1], section 2.2.7(**I**), Lemma 2.19). Let $\mathbf{N}_i = \mathfrak{G}_i^{\perp}$. The Jordan-Hölder series given above corresponds to a chain of field extensions

$$\mathbf{K} = \mathbf{K}_0 \subset \mathbf{K}_1 \subset ... \subset \mathbf{K}_r = \mathbf{N}$$

where $\mathfrak{G}_i = \mathbf{K}_i^{\perp} = \mathrm{Gal}\,(\mathbf{N}/\mathbf{K}_i)$. Therefore, $[\mathbf{K}_i : \mathbf{K}_{i-1}] = (\mathfrak{G}_{i-1} : 1) / (\mathfrak{G}_i : 1)$ (Theorem 1.9(3)) is a prime number q_i and $\mathbf{K}_i/\mathbf{K}_{i-1}$ is a normal extension. Thus, if $\mathbf{K}$ contains the q_i-th roots of unity, then there exists $a_i \in \mathbf{K}_i$ such that $\mathbf{K}_i = \mathbf{K}_{i-1}(a_i)$ and $a_i = a_{i-1}^{q_i}$, where $a_{i-1} \in \mathbf{K}_{i-1}$, by Kummer's theorem (Theorem 1.62). This proves the lemma on the condition that we may assume that $\mathbf{K}$ contains the q_i-th roots of unity $(1 \leq i \leq r)$.

If this is not the case, consider the field $\mathbf{K}' = \mathbf{K}(\zeta)$, where ζ is a primitive q-th root of unity with $q \geq \max\{q_i : 1 \leq i \leq r\}$. Let $\mathbf{N}'$ be a decomposition field of f viewed as an element of $\mathbf{K}'[X]$. Then $\mathfrak{H} = \mathrm{Gal}\left(\mathbf{N}'/\mathbf{K}'\right)$ is a subgroup of $\mathfrak{G}$ by the theorem of natural irrationalities (Theorem 1.64), so $\mathfrak{H}$ is solvable ([P1], section 2.2.7(**I**)); hence, the equation $f(X) = 0$ is solvable by radicals over $\mathbf{K}'$, and therefore also over $\mathbf{K}$. ■

This completes the full proof of Galois' theorem. The precise statement of this theorem consists of Lemmas 1.63 and 1.65 (see footnote 4 on page 9).

1.1.11. *General algebraic equations*

Let us now consider the general algebraic equation of degree n over $\mathbf{K}$ (Definition 1.12). We shall prove Theorem 1.13. Every permutation $\mathfrak{S}_n \ni \sigma : X_i \to X_{\sigma(i)}$ $(i = 1, ..., n)$ of the indeterminates X_i induces a $\mathbf{K}$-automorphism of $\mathbf{N}$ that fixes the symmetric polynomials $s_1, ..., s_n$ and therefore the field $\mathbf{E} = \mathbf{K}(s_1, ..., s_n)$. Conversely, if $u \in \mathrm{Gal}(\mathbf{N}/\mathbf{E})$, then its restriction $u\big|_{\{X_1,...,X_n\}}$ is a permutation of $\{X_1, ..., X_n\}$. Hence, there exists an isomorphism $\mathfrak{S}_n \cong \mathrm{Gal}(\mathbf{N}/\mathbf{E})$.

1.1.12. *Topology of the Galois group*

(I) Let $\mathbf{N}/\mathbf{K}$ be a Galois extension. The Galois connection gives a bijection between the subextensions $\mathbf{M}/\mathbf{K}$ and the subgroups Δ of $\mathfrak{G} = \mathrm{Gal}(\mathbf{N}/\mathbf{K})$ that are equal to their Galois closure $\Delta^{\perp\perp}$. For $\Delta = \Delta^{\perp\perp}$ to hold, it suffices that $[\mathbf{N} : \mathbf{M}] < \infty$ (Theorem 1.9(2)). This sufficient condition is not a necessary condition, an observation that we shall study in this section using concepts that will be introduced in Chapter 2.

(II) Let $(\mathbf{N}_i)_{i \in I}$ be the right directed family of finite Galois subextensions of $\mathbf{N}$ (where I is ordered by inclusion). For each $x \in \mathbf{N}$, there exists $i \in I$ such that $x \in \mathbf{N}_i$, so $\mathbf{N} = \bigcup_{i \in I} \mathbf{N}_i = \varinjlim_{i \in I} \mathbf{M}_i$. Each $\mathfrak{G}_i = \mathrm{Gal}(\mathbf{N}_i/\mathbf{K})$ is finite and $(\mathfrak{G}_i : 1) = [\mathbf{N}_i : \mathbf{K}]$. The Galois connection is antitone ([P1], section 2.1.2), so we must switch $\cap$ and $\cup$. Consequently ([P1], section 1.2.8**(II)**),

$$\mathfrak{G} = \bigcap\nolimits_{i \in I} \mathfrak{G}_i = \varprojlim_{i \in I} \mathfrak{G}_i. \qquad [1.12]$$

Given that each $\mathfrak{G}_i$ $(i \in I)$ is a finite set, the only Hausdorff topology that can exist on this set is the discrete topology (**exercise**), which is known as the *finite topology* in this particular context. By equipping the group $\mathfrak{G}$ with the projective limit topology of these finite topologies (called the *profinite topology* or Krull topology, and written $\mathfrak{T}_p$), the projective limit (1.12) is in the category $\mathbf{Topgrp}$ of topological groups (cf. below, section 2.8.1) and the resulting topological group $\mathfrak{G}$ is compact, since each $\mathfrak{G}_i$ is compact (cf. below, Corollary 2.44).

(III) Let $\mathcal{F} = \{\mathfrak{N} \lhd \mathfrak{G} : (\mathfrak{G} : \mathfrak{N}) < \infty\}$. If $\sigma \in \mathfrak{G}$, then $\{\sigma\mathfrak{N} : \mathfrak{N} \in \mathcal{F}\}$ is a filter base of the neighborhoods of σ in $\mathfrak{T}_p$ by the definition of this topology. We will show that the closure $\bar{\mathfrak{H}}$ of any subgroup $\mathfrak{H}$ of $\mathfrak{G}$ with respect to the topology $\mathfrak{T}_p$ is equal to its Galois closure.

If $\mathfrak{H}$ is an arbitrary subgroup of $\mathfrak{G}$ and $\mathfrak{N} \in \mathcal{F}$, then $\mathfrak{N}$ is open in $\mathfrak{T}_p$, so $\mathfrak{H}\mathfrak{N} = \bigcup_{h\in\mathfrak{H}} x\mathfrak{N}$ is open in $\mathfrak{T}_p$; moreover, $(\mathfrak{G} : \mathfrak{H}\mathfrak{N}) < \infty$, since $\mathfrak{N} \subseteq \mathfrak{H}\mathfrak{N} \subseteq \mathfrak{G}$. We have that $\complement_{\mathfrak{G}}\mathfrak{H}\mathfrak{N} = \bigcup_{g\neq 1} g\mathfrak{H}\mathfrak{N}$, which is open in $\mathfrak{T}_p$, so $\mathfrak{H}\mathfrak{N}$ is closed in $\mathfrak{T}_p$ (the subspace $\mathfrak{H}\mathfrak{N}$ is therefore both open and closed in $\mathfrak{T}_p$, which is not a contradiction because the topological space $(\mathfrak{G}, \mathfrak{T}_p)$ is not connected; it is even *totally disconnected*, i.e. the connected component of any element g is just $\{g\}$). Since $(\mathfrak{G} : \mathfrak{H}\mathfrak{N}) < \infty$, we have that $(\mathfrak{H}\mathfrak{N})^{\perp\perp} = \mathfrak{H}\mathfrak{N}$ by Theorem 1.9(2), so $\mathfrak{H}^{\perp\perp} \subseteq \mathfrak{H}\mathfrak{N}$, and hence $\mathfrak{H}^{\perp\perp} \subseteq \bigcap_{\mathfrak{N}\in\mathcal{F}} \mathfrak{H}\mathfrak{N}$. We will now show that this inclusion is an equality. If $\sigma \in \mathfrak{H}$ and $\sigma \notin \mathfrak{H}^{\perp\perp}$, then there exists $a \in \mathfrak{H}^{\perp}$ such that $\sigma(a) \neq a$. Let $\mathbf{M}/\mathbf{K}$ be a Galois extension such that $\mathbf{K}(a) \subset \mathbf{M} \subset \mathbf{N}$ and suppose that $[\mathbf{N} : \mathbf{M}] < \infty$. Then $\mathbf{M}^{\perp} \in \mathcal{F}$; but $\mathfrak{H}$ and $\mathbf{M}^{\perp}$ fix a, so $\sigma \notin \mathfrak{H}\mathbf{M}^{\perp} \supset \bigcap_{\mathfrak{N}\in\mathcal{F}} \mathfrak{H}\mathfrak{N}$. Therefore, $\sigma \notin \bigcap_{\mathfrak{N}\in\mathcal{F}} \mathfrak{H}\mathfrak{N}$, which implies that $\mathfrak{H}^{\perp\perp} = \bigcap_{\mathfrak{N}\in\mathcal{F}} \mathfrak{H}\mathfrak{N}$. Since $\bigcap_{\mathfrak{N}\in\mathcal{F}} \mathfrak{H}\mathfrak{N}$ is the intersection of all closed neighborhoods of $\mathfrak{H}$ in $\mathfrak{T}_p$, we have that $\bigcap_{\mathfrak{N}\in\mathcal{F}} \mathfrak{H}\mathfrak{N} = \bar{\mathfrak{H}}$.

Finally, we obtain the result stated below, which completes Theorem 1.9, using similar notation:

THEOREM 1.66.– *Let* $\mathbf{N}/\mathbf{K}$ *be a Galois extension and let* $\mathcal{K}$ *be the set of fields* $\mathbf{M}$ *such that* $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$. *Let* $\bar{\mathcal{G}}$ *be the set of* closed *subgroups* $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$ *with respect to the profinite topology. Then the correspondence*

$$\mathbf{M} \mapsto \mathbf{M}^{\perp}, \quad \Delta \mapsto \boldsymbol{\Delta}^{\perp}$$

is a bijective *Galois* connection *between* $\mathcal{K}$ *and* $\bar{\mathcal{G}}$.

1.2. Transcendental extensions

The theory of transcendental extensions, briefly presented below, was developed by E. Steinitz [STE 10].

1.2.1. *Pure extensions*

Let $\mathbf{K}$ be a field and let $\mathbf{E}$ be an extension of $\mathbf{K}$. The family $x = (x_i)_{i \in I}$ of elements of $\mathbf{E}$ is algebraically independent over $\mathbf{K}$ if and only if the relation $f((x_i)) = 0$, where $f \in \mathbf{K}\left[(X_i)_{i \in I}\right]$, implies that $f = 0$ ([P1], section 2.3.10(**IV**)).

DEFINITION 1.67.– *An algebraically independent family $(x_i)_{i \in I}$ of elements of $\mathbf{E}$ is said to be a* pure basis *of $\mathbf{E}$ if $\mathbf{E} = \mathbf{K}\left((x_i)_{i \in I}\right)$. We say that $\mathbf{E}$ is a* pure extension *of $\mathbf{K}$ if $\mathbf{E}$ has a pure basis.*

THEOREM 1.68.– *An extension $\mathbf{E}$ of $\mathbf{K}$ is pure if, and only if there exist a family $(x_i)_{i \in I}$ of elements of $\mathbf{E}$ such that $\mathbf{E} = \mathbf{K}\left((x_i)_{i \in I}\right)$ and a $\mathbf{K}$-isomorphism $\mathbf{K}\left((X_i)_{i \in I}\right) \to \mathbf{E} : X_i \mapsto x_i$.*

PROOF.– See ([P1], section 2.3.10(**IV**)). ■

For example, $\mathbb{Q}\left(\sqrt{2}, \pi\right)$ is a transcendental extension of $\mathbb{Q}$, because π is transcendental (von Lindemann's theorem). However, it is not pure, since $\sqrt{2} \notin \mathbb{Q}$, but $\sqrt{2}$ is algebraic.

1.2.2. *Transcendence bases*

DEFINITION 1.69.– *A subset B of an extension $\mathbf{E}$ over $\mathbf{K}$ is said to be a* transcendence basis *(over $\mathbf{K}$) if B is algebraically free over $\mathbf{K}$ and $\mathbf{E}$ is algebraic over $\mathbf{K}(B)$.*

For example, $\{\pi\}$ is a transcendence basis of $\mathbb{Q}\left(\sqrt{2}, \pi\right)$ over $\mathbb{Q}$, since $\mathbb{Q}\left(\sqrt{2}, \pi\right) = \mathbb{Q}(\pi)\left(\sqrt{2}\right)$ is algebraic over $\mathbb{Q}(\pi)$.

The next result is analogous to the basis extension theorem in a vector space ([P1], section 3.1.3(**IV**), Theorem 3.10):

THEOREM 1.70.– *(Steinitz) Let $\mathbf{E}$ be an extension of a field $\mathbf{K}$, S a subset of $\mathbf{E}$ such that $\mathbf{E}$ is algebraic over $\mathbf{K}(S)$, and T a subset of S that is algebraically independent over $\mathbf{K}$; then there exists a transcendence basis B of $\mathbf{E}$ such that $T \subset B \subset S$. In particular,* every extension $\mathbf{E}$ of a field $\mathbf{K}$ has a transcendence basis over $\mathbf{K}$.

PROOF.– From the definitions, it follows that a family $(x_i)_{i \in I}$ of $\mathbf{E}$ is algebraically independent over $\mathbf{K}$ if and only if every finite subfamily is algebraically independent over $\mathbf{K}$. Hence, the set of algebraically independent subsets of S containing T, ordered by inclusion, is of finite character ([P1], section 1.1.2(**III**)), and therefore has a maximal element B (*ibid*. Lemma 1.4). Given that $\mathbf{E}$ is algebraic over $\mathbf{K}(S)$, $\mathbf{E}$ is also algebraic over $\mathbf{K}(B)$ (cf. [BKI 12], Chapter V, section 14.2, Corollary of Proposition 4), and so B is a transcendence basis of $\mathbf{E}$ over $\mathbf{K}$. ■

COROLLARY 1.71.– *(*exchange theorem*) Let $\mathbf{E}$ be an extension of $\mathbf{K}$, S a subset of $\mathbf{E}$ such that $\mathbf{E}$ is algebraic over $\mathbf{K}(S)$, and T a subset of $\mathbf{E}$ that is algebraically independent over $\mathbf{K}$; then there exists a subset S' of S such that $T \cup S'$ is a transcendence basis of $\mathbf{E}$ over $\mathbf{K}$ and $T \cap S' = \varnothing$ (**exercise**).*

Just as any two bases of a $\mathbf{K}$-vector space are equipotent ([P1], section 3.1.7, Theorem 3.28(ii)), we can use a similar reasoning to show ([BKI 12], Chapter V, section 14.3, Theorem 3):

THEOREM 1.72.– *(Steinitz) All transcendence bases of $\mathbf{E}$ over $\mathbf{K}$ are equipotent.*

DEFINITION 1.73.– *Let $\mathbf{E}$ be an extension of a field $\mathbf{K}$. The cardinal of any transcendence basis of $\mathbf{E}$ over $\mathbf{K}$ is called the* transcendence degree *of $\mathbf{E}$ over $\mathbf{K}$, denoted $\deg \operatorname{tr}_{\mathbf{K}} \mathbf{E}$ or $\deg \operatorname{tr}(\mathbf{E}/\mathbf{K})$.*

For example, $\deg \operatorname{tr}_{\mathbb{Q}} \mathbb{Q}\left(\sqrt{2}, \pi\right) = 1$. It can be shown ([BKI 12], Chapter V, section 14.3, Corollary of Theorem 4) that, if $\mathbf{K}, \mathbf{E}$ and $\mathbf{F}$ are three fields satisfying $\mathbf{K} \subset \mathbf{E} \subset \mathbf{F}$, then

$$\deg \operatorname{tr}(\mathbf{F}/\mathbf{K}) = \deg \operatorname{tr}(\mathbf{E}/\mathbf{K}) + \deg \operatorname{tr}(\mathbf{F}/\mathbf{E}). \qquad [1.13]$$

According to a conjecture formulated by S. Schanuel in the 1960s, which has remained an open question ever since, $\pi = 3.141592...$ and the base of the natural logarithm $e = 2.71828...$ are algebraically independent. If this conjecture is true, then

$$\deg \operatorname{tr}_{\mathbb{Q}} \mathbb{Q}(\pi, e) = \underbrace{\deg \operatorname{tr}_{\mathbb{Q}} \mathbb{Q}(\pi)}_{1} + \underbrace{\deg \operatorname{tr}_{\mathbb{Q}(\pi)} \mathbb{Q}(\pi, e)}_{1} = 2.$$

1.3. Differential Galois theory

1.3.1. *Introduction*

(I) As we saw in section 1.1.1, Galois theory uses the Galois group and its subgroups to describe the nature of the field extensions generated by the solutions of the algebraic equation $f(x) = 0$, $f \in \mathbf{K}[X]^{\times}$, and in particular those that may be obtained using radicals. Now we shall study the "differential version" of this theory.

A differential field is a field $\mathbf{K}$ equipped with a derivation δ ([P1], section 2.3.12), that is, an additive mapping $\delta : \mathbf{K} \to \mathbf{K}$ that satisfies Leibniz' rule $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in \mathbf{K}$. We sometimes write $\delta(a) = \dot{a}$. One classic example is the field of rational fractions $\mathbb{C}(t)$, equipped with the usual derivation d/dt. An element a is said to be a constant if $\dot{a} = 0$. The set of constants of the field $\mathbf{K}$ is a subfield (**exercise**), called the subfield of constants of $\mathbf{K}$, denoted $\mathbf{C}$. Consider the ring of skew polynomials $\mathbf{A} = \mathbf{K}[D; \delta]$ ([P1], section 3.1.11**(I)**), where D is the intdeterminate. Leibniz' rule $Da - aD = \delta(a)$ holds for all $a \in \mathbf{K}$. Let $\mathbf{M}$ be an $\mathbf{A}$-module; then

$$\delta_{\mathbf{M}} : \mathbf{M} \to \mathbf{M} : m \mapsto Dm \qquad [1.14]$$

is a derivation of $\mathbf{M}$ that extends δ. If $\mathbf{M}$ is a field, it becomes a differential field equipped with the derivation $\delta_{\mathbf{M}}$, in which case $\mathbf{M}/\mathbf{K}$ is a *differential field extension*. More precisely, a morphism of differential fields is a morphism of fields $f : \mathbf{L} \to \mathbf{M}$ such that $\delta_{\mathbf{M}}(f(a)) = f(\delta_{\mathbf{L}}(a))$ for all $a \in \mathbf{L}$ (where $\delta_{\mathbf{L}}$ and $\delta_{\mathbf{M}}$ denote the derivations of $\mathbf{L}$ and $\mathbf{M}$ respectively). This defines the category of differential fields. A differential field extension $\mathbf{M}/\mathbf{K}$ is a field extension such that $\delta_{\mathbf{M}}|_{\mathbf{K}} = \delta_{\mathbf{K}}$. If $v_1, ..., v_m \in \mathbf{M}$, we write $\mathbf{K}\langle v_1, ..., v_m\rangle$ for the differential field generated over $\mathbf{K}$ by the elements $v_1, ..., v_m$ and their derivatives of all orders.

LEMMA 1.74.– *Let $(\mathbf{K}, \delta)$ be a differential field, $\mathbf{M}$ an extension of $\mathbf{K}$, $(x_i)_{i \in I}$ a transcendence basis of $\mathbf{M}$, and $(u_i)_{i \in I}$ a family of elements in $\mathbf{M}$. There exists precisely one derivation $\delta_{\mathbf{M}}$ of $\mathbf{M}$ extending δ that satisfies $\delta_{\mathbf{M}}(x_i) = u_i, \forall i \in I$. In particular, if the extension $\mathbf{M}/\mathbf{K}$ is algebraic, then there exists precisely one derivation of $\mathbf{M}$ extending δ.*

PROOF.– (1) Let $a \in \mathbf{M}$ be an algebraic element over $\mathbf{K}$, and let $f \in \mathbf{K}[X]$ be its minimal polynomial ([P1], section 2.3.5(**II**)), where $d^\circ(f) \geq 1$. Let $\delta_\mathbf{M}$ be a derivation of $\mathbf{M}$ that extends δ. We have that $f(a) = 0$, so $0 = \delta(f)(a) + \left(\frac{\partial f}{\partial X}\right)(a).\delta_\mathbf{M}(a)$ where $\delta(f) \in \mathbf{K}[X]$ is the polynomial with coefficients $\delta(f_j)$ (where the $f_j \in \mathbf{K}$ are the coefficients of f). But $0 \leq d^\circ\left(\frac{\partial f}{\partial X}\right) < d^\circ(f)$ since $\mathrm{Char}(\mathbf{K}) = 0$, and consequently $\left(\frac{\partial f}{\partial X}\right)(a) \neq 0$. Therefore, $\delta_\mathbf{M}(a) = -\delta(f)(a) / \left(\frac{\partial f}{\partial X}\right)(a)$ is fully determined.

(2) If B is a transcendence basis of $\mathbf{M}$, then $\mathbf{M}$ is algebraic over $\mathbf{K}(B)$ (Theorem 1.70). It is therefore sufficient to consider a pure transcendental extension $\mathbf{E} = \mathbf{K}(B)$, where $B = (x_i)_{i \in I}$ is a transcendence basis. Any such extension may be written as a union of finite pure extensions, and, setting $\mathbf{E}_i = \mathbf{K}(x_1, ..., x_i)$ $(1 \leq i \leq n)$, $\mathbf{E}_i/\mathbf{E}_{i-1}$ is a simple pure transcendental extension, since $\mathbf{E}_i = \mathbf{E}_{i-1}(x_i)$. This reduces the problem to the case of a simple pure transcendental extension $\mathbf{E} = \mathbf{K}(x)$, where $\delta_\mathbf{M}(x) = u$ $(u \in \mathbf{K})$. In any such case, $\delta_\mathbf{M}$ is fully determined; indeed, every $y \in \mathbf{E}$ is of the form $f(x)/g(x)$ such that $f, g \in \mathbf{K}[X]$ and $g(x) \neq 0$. We can deduce $\delta_\mathbf{M}(y)$ from the usual formula $\delta_\mathbf{M}(y) = (\delta(f)(x)g(x) - f(x)\delta(g)\ (x))u/g(x)^2$. ■

In light of (1.14), given a differential field extension $\mathbf{M}/\mathbf{K}$, we will henceforth write D for the derivation of $\mathbf{M}$, δ for the derivation of $\mathbf{K}$ and, for $u \in \mathbf{M}$, we define $\dot{u} = D(u), \ddot{u} = D^2(u), ..., u^{(n)} = D^n(u)$.

Any linear differential equation over $\mathbf{M}$ with coefficients in $\mathbf{K}$ may be written in the form

$$L(D)y = 0 \qquad [1.15]$$

where $L(D)$ is the linear differential operator

$$L(D) = a_0 D^n + a_1 D^{n-1} + ... + a_n, \quad a_i \in \mathbf{K},\ a_0 \neq 0. \qquad [1.16]$$

Consider the system of linear differential equations

$$R(D)x = 0, \qquad [1.17]$$

where $R \in \mathbf{A}^{q \times k}$ is a matrix of rank k. Then $T := \operatorname{coker}_{\mathbf{A}} (\bullet R)$ is a cyclic $\mathbf{A}$-torsion module ([P1], section 3.4.2**(III)**, Definition 3.206 and Corollary 3.207), since $\mathbf{A}$ is simple ([P1], section 3.1.11**(I)**, Theorem 3.51(1)), and so the study of the solutions of (1.17) in $\mathbf{M}^k$ may be reduced to the study of the solutions of (1.15) in $\mathbf{M}$.

(II) After noticing the analogy between (1.15) and an algebraic equation, S. Lie, and later, more decisively, C. Picard and E. Vessiot, attempted to construct a "differential Galois theory" to describe the nature of the differential field extensions generated by the solutions of (1.15) by finding an object analogous to the Galois group. Picard and Vessiot thought that this object might be a Lie group (cf. [P3]), but did not have time to fully explore their ideas before the foundations of differential algebra were laid by J.F. Ritt and E.R. Kolchin between 1930 and the late 1950s [RIT 48], [KOL 48]. Kolchin would go on to fully realize the program set out by Picard and Vessiot. His synthesis [KOL 73] on differential algebra and algebraic groups was published in 1973. The books [KAP 57] and [MAG 97] provide a good introduction, and [PUT 03] complements [KOL 73] in many ways, without entirely superseding it (see also [CRE 11], written in a similar spirit, but with fewer details). Adapting these ideas to difference algebras (and linear difference equations) currently presents a number of challenges [PUT 97].

1.3.2. *Picard-Vessiot extensions*

THEOREM-DEFINITION 1.75.– *Let $u_1, ..., u_n$ be solutions of* (1.15) .

1) These solutions are linearly independent over the field $\mathbf{C}$ of constants of $\mathbf{K}$ if and only if

$$W(u_1, ..., u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ \dot{u}_1 & \dot{u}_2 & \cdots & \dot{u}_n \\ \vdots & \vdots & & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \neq 0.$$

2) The determinant $W(u_1, ..., u_n)$ is called the Wronskian *of $u_1, ..., u_n$ (named after J. Wronski).*

PROOF.–

a) Suppose that $u_1, ..., u_n$ are related over $\mathbf{C}$. Then there exist constants $c^1, ..., c^n$, not all of which are zero, such that $\sum_{i=1}^{n} c^i u_i = 0$. By differentiating this relation $n-1$ times, we obtain n homogeneous linear equations in $c^1, ..., c^n$. The determinant of this linear system is $W(u_1, ..., u_n)$, which is therefore equal to zero.

b) Conversely, suppose that $W(u_1, ..., u_n) = 0$. Using the same reasoning as in (a), we can find elements $c^1, ..., c^n \in \mathbf{K}$, not all of which are zero, such that $\sum_{i=1}^{n} u_i^{(j)} c^i = 0$, $j = 0, ..., n-1$. Relabeling the c^i if necessary, we may assume that $c^1 \neq 0$, and similarly that $c^1 = 1$, by dividing by c^1. Differentiating yields

$$0 = D\left(\sum_{i=1}^{n} u_i^{(j)} c^i\right) = \underbrace{\sum_{i=1}^{n} u_i^{(j+1)} c^i}_{0} + \sum_{i=1}^{n} u_i^{(j+1)} \delta\left(c^i\right),$$

and, since $\delta\left(c^1\right) = 0$, we obtain $n-1$ homogeneous linear equations in the $\delta\left(c^i\right)$; the determinant of this system is $W(u_2, ..., u_n)$.

(α) If $W(u_2, ..., u_n) \neq 0$, then $\delta\left(c^i\right) = 0$ $(i = 2, ..., n)$ and the c^i all belong to $\mathbf{C}$, which shows that the solutions $u_1, ..., u_n$ are linearly dependent over $\mathbf{C}$.

(β) If $W(u_2, ..., u_n) = 0$, we may assume that $c^2 = 1$, and continue the same reasoning: by iterating, we will eventually arrive at a linear dependence relation for $u_1, ..., u_n$ over $\mathbf{C}$. ■

DEFINITION 1.76.– *Consider a homogeneous linear differential equation of type* (1.15), *where* $L(D)$ *is a linear differential operator of the form* (1.16) *of degree* n. *Let* $u_1, ..., u_n$ *be* $\mathbf{C}$*-linearly independent solutions, where* $\mathbf{C}$ *is the field of constants of* $\mathbf{K}$. *A differential field extension* $\mathbf{N}/\mathbf{K}$ *is called a Picard-Vessiot extension for the equation* (1.15) *if:*

i) $\mathbf{N} = \mathbf{K}\langle u_1, ..., u_n\rangle$, *where* $u_1, ..., u_n$ *are* n *solutions of* (1.15) *that are linearly independent over the field of constants.*

ii) $\mathbf{N}$ *has the same field of constants* $\mathbf{C}$ *as* $\mathbf{K}$.

If $\mathbf{K}/\mathbf{M}$, $\mathbf{K}/\mathbf{N}$ are differential field extensions, we say that $\tau : \mathbf{M} \to \mathbf{N}$ is a $\mathbf{K}^D$-morphism if τ is a field $\mathbf{K}$-homomorphism and a morphism of differential fields. We can define the concepts of $\mathbf{K}^D$-epimorphism, $\mathbf{K}^D$-monomorphism, $\mathbf{K}^D$-isomorphism, and $\mathbf{K}^D$-automorphism similarly. For a proof of the following result, see ([MAG 97], Theorems 3.4, 3.5) or ([CRE 11], Theorem 5.6.10).

THEOREM 1.77.– *If* $\mathbf{K}$ *has characteristic zero and* $\mathbf{C}$ *is algebraically closed, then there exists a Picard-Vessiot extension for* (1.15) *that is unique up to* $\mathbf{K}^D$-*isomorphism.*

DEFINITION 1.78.– *Let* $\mathbf{N}/\mathbf{K}$ *be a differential field extension. The differential Galois group* $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ *is the group of* $\mathbf{K}^D$*-automorphisms of* $\mathbf{N}$*.*

The reader may wish to check that $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ is indeed a group.

DEFINITION 1.79.– *Let* G *be a group and* $\mathbf{C}$ *a field. A* representation *of* G *in* $\mathbf{C}^n$ *is a group-homomorphism* $\rho : g \mapsto U(g)$ *from* G *into* $\mathrm{GL}_n(\mathbf{C})$. *We say that this representation is* faithful *if* ρ *is injective.*

With the notation of Definition 1.76, let $g \in \mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$. We can write that $g.u_i = \sum_{i=1}^n c_i^j(g).u_j$, where the $c_i^j(g)$ are elements of $\mathbf{C}$. The mapping $g \mapsto c(g) = \left(c_i^j(g)\right)$ is a faithful representation of $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ in $\mathbf{C}^n$. We will specify the nature of $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ more precisely in Theorem 1.86 below after giving a few remarks on the subject of algebraic groups.

1.3.3. *Algebraic groups*

DEFINITION 1.80.–

i) An algebraic group G *over a field* $\mathbf{C}$ *is an algebraic set over* $\mathbf{C}$ *([P1], section 3.2.7(**I**)) such that the mapping* $(x, y) \mapsto xy$ *from* $G \times G$ *into* G *and the mapping* $x \mapsto x^{-1}$ *from* G *into* G *are both morphisms of algebraic sets ([P1], section 3.2.7(**III**)).*

ii) A linear algebraic group *over* $\mathbf{C}$ *is a closed subgroup of the group* $\mathrm{GL}_n(\mathbf{C})$ *(with respect to the Zariski topology)*.

REMARK 1.81.– *Let* G *be an algebraic group over* $\mathbf{C}$ *(equipped with the Zariski topology). The mapping* $x \mapsto x^{-1}$ *is continuous with respect to this topology, but the Zariski topology on a product of sets is not the product of the Zariski topologies. Therefore, the operation* $(x, y) \mapsto xy$ *is not continuous,* *but only* separately *continuous (however, the mapping* $x \mapsto x^{-1}ax$ *is continuous for all* $a \in G$ *([KAP 57], Chapter IV)). An algebraic group is therefore not a topological group equipped with the Zariski topology (cf. below, section 2.8.1), and therefore is not a Lie group (cf. [P3]).*

It can be shown that any subgroup of a linear algebraic group that is closed with respect to the Zariski topology is itself a linear algebraic group ([HUM 75], section 8.6), which gives us the following examples:

EXAMPLE 1.82.– *The following subgroups of* $\mathrm{GL}_n(\mathbf{C})$ *are linear algebraic groups:*

i) The group $\mathrm{SL}_n(\mathbf{C}) = \{g \in \mathrm{GL}_n(\mathbf{C}) : \det(g) = 1\}$, *called the special linear group ([P1], section 2.3.11(**III**)). Indeed,* $\mathrm{SL}_n(\mathbf{C})$ *is a Zariski-closed subgroup of* $\mathrm{GL}_n(\mathbf{C})$*, since* $\det(g) = 1$ *is an algebraic equation involving elements of* g.

ii) The group $\boldsymbol{D}_n(\mathbf{C})$ *of* $n \times n$ *diagonal matrices of the form* $\mathrm{diag}(u_1, u_2, ..., u_n)$, $u_i \neq 0$. *We have that* $\boldsymbol{D}_n(\mathbf{C}) \cong \mathrm{GL}_1(\mathbf{C})^n \cong (\mathrm{G}_m(\mathbf{C}))^n$*, where* $\mathrm{G}_m(\mathbf{C}) := \mathbf{C}^\times$ *(multiplicative group of* $\mathbf{C}^\times$*).*

iv) The group $\mathrm{T}_n(\mathbf{C})$ *of invertible upper triangular matrices.*

v) The group $\mathrm{U}_n(\mathbf{C}) = (\mathrm{T}_n(\mathbf{C}))_u$ *of unipotent upper triangular matrices (the subscript* $(.)_u$ *denotes the property of unipotency), i.e. matrices with ones along the diagonal. For* $n = 2$, $\mathrm{U}_n(\mathbf{C})$ *is isomorphic to the additive group of* $\mathbf{C}$, $\mathrm{G}_a(\mathbf{C})$.

vi) The orthogonal group $\mathrm{O}_n(\mathbf{C}) = \left\{g \in \mathrm{GL}_n(\mathbf{C}) : g^T g = I_n\right\}$ *(or more generally* $\mathrm{O}_n(S; \mathbf{C}) = \left\{g \in \mathrm{GL}_n(\mathbf{C}) : g^T S g = S\right\}$*, where* $S \in \mathrm{GL}_n(\mathbf{C})$*).*

vii) The special orthogonal group $\mathrm{SO}_n(\mathbf{C}) := \mathrm{O}_n(\mathbf{C}) \cap \mathrm{SL}_n(\mathbf{C})$.

viii) The group $\boldsymbol{\mu}_h$ *of* h*-th roots of unity in* $\mathbf{C}$ *(Theorem-Definition 1.38).*

The reader may wish to check that $\mathrm{U}_n(\mathbf{C}) = (\mathrm{T}_n(\mathbf{C}), \mathrm{T}_n(\mathbf{C}))$ (derived subgroup of $\mathrm{T}_n(\mathbf{C})$: cf. [P1], section 2.2.6), which implies that $\mathrm{U}_n(\mathbf{C}) \lhd \mathrm{T}_n(\mathbf{C})$, and $\mathrm{T}_n(\mathbf{C}) = \mathrm{D}_n(\mathbf{C}) . \mathrm{U}_n(\mathbf{C}) := \{du : d \in \mathrm{D}_n(\mathbf{C}), u \in \mathrm{U}_n(\mathbf{C})\}$ ([P1], section 2.2.2(**I**)). It is possible to show the following lemma ([HUM 75], section 7.3):

LEMMA 1.83.– *Let* G *be a linear algebraic group with neutral element* e.

*i) The neutral element belongs to precisely one irreducible component ([P1], section 3.2.7(**IV**)), denoted G°.*

ii) G° is a normal subgroup of G of finite index, and the cosets gG° are simultaneously the irreducible components of G and the connected components of G (section 2.3.8).

iii) Every Zariski-closed subgroup of G of finite index contains G°.

The algebraic group $\mathrm{GL}_n(\mathbf{C})$ is connected; in the affine space $\mathbf{C}^{n\times n}$, it is in fact the principal open set Ω_f, where $f = \det$ ([P1], section 3.2.7(**II**)).

The following result can also be shown ([HUM 75], section 7.5, Corollary):

THEOREM 1.84.– *Let $G \subseteq \mathrm{GL}_n(\mathbf{C})$ be an algebraic group generated by a family H_i of closed connected subgroups. Then G is connected.*

COROLLARY 1.85.– *The algebraic groups $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{D}_n(\mathbf{C})$, $\mathrm{T}_n(\mathbf{C})$, $\mathrm{U}_n(\mathbf{C})$ (Example 1.82) are connected (**exercise***: cf. [CRE 11], Corollary 3.2.5).*

1.3.4. *Fundamental theorem of Picard-Vessiot theory*

(I) *Throughout the rest of this section*, $\mathbf{K}$ has characteristic zero and its field of constants $\mathbf{C}$ is algebraically closed. A Picard-Vessiot extension is therefore the differential algebraic analog of the concept of Galois extension. It is possible to prove the following result ([KAP 57], Theorem 5.5):

THEOREM 1.86.– *Let $\mathbf{N}/\mathbf{K}$ be a Picard-Vessiot extension for (1.15). Then $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ is a linear algebraic group over the field of constants $\mathbf{C}$ of $\mathbf{K}$; $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ is the group G of automorphisms $V \to V$, where V is the $\mathbf{C}$-vector space with basis given by the n $\mathbf{C}$-linearly independent solutions $u_1, ..., u_n$ (after identifying V with $\mathbf{C}^n$ in this basis, and identifying G with a subgroup of $\mathrm{GL}_n(\mathbf{C})$).*

We will reuse and adapt the notation of section 1.1.2(**V**). Let $\mathbf{N}/\mathbf{K}$ be a Picard-Vessiot extension and let $\mathcal{K}^D$ be the set of differential fields $\mathbf{M}$ such that $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$; this is a lattice ordered by inclusion ([P1], section 2.1.3(**I**)) and, for all $\mathbf{M} \in \mathcal{K}$, the extension $\mathbf{N}/\mathbf{M}$ is a Picard-Vessiot extension. Let $\mathcal{G}^D$ be the set of Zariski-closed subgroups of $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$;

this is also a lattice ordered by inclusion, and, if $\mathbf{M} \in \mathcal{K}^D$, then the group $\mathbf{M}^{\perp}$ of $\mathbf{M}^D$-automorphisms of $\mathbf{N}$ is a Zariski-closed subgroup of $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$; we therefore have that $\mathbf{M}^{\perp} \in \mathcal{G}^D$. If $\Delta \in \mathcal{G}^D$, we write $\Delta^{\perp} \in \mathcal{K}$ for the *differential fixed field* of Δ, namely the largest field $\mathbf{M} \in \mathcal{K}^D$ such that $u(x) = x, \forall x \in \mathbf{M}, \forall u \in \Delta$. For all $\mathbf{M} \in \mathcal{K}$, we have that $(\forall x), x \in \mathbf{M} \Leftrightarrow \left[u(x) = x, \forall u \in \mathbf{M}^{\perp}\right]$, so $\mathbf{M} = \mathbf{M}^{\perp\perp}$.

The following result, established by Kolchin, is analogous to the fundamental theorem of Galois theory (Theorem 1.9). A proof can for example be found in ([MAG 97], Theorem 6.5; [CRE 11], Corollary 6.2.8).

THEOREM 1.87.– *(Kolchin*'s fundamental theorem of *Picard-Vessiot* theory*)*

1) The correspondence

$$\mathbf{M} \mapsto \mathbf{M}^{\perp}, \quad \Delta \mapsto \mathbf{\Delta}^{\perp} \qquad [1.18]$$

is a bijective Galois connection *between $\mathcal{K}^D$ and $\mathcal{G}^D$, which therefore induces a bijection from $\Delta^{\perp}$ onto $\mathbf{M}$, and from $\mathbf{M}^{\perp}$ onto $\mathbf{\Delta}^{\perp\perp} = \Delta$ ([P1], section 2.1.2(**II**)).*

2) The Galois connection (1.18) gives a correspondence between the Picard-Vessiot subextensions $\mathbf{M}/\mathbf{K}$ *of* $\mathbf{N}/\mathbf{K}$ *and the Zariski-closed* normal *subgroups of* $\mathrm{Gal}(\mathbf{N}/\mathbf{K})$. *Furthermore (with $\mathbf{K} \subset \mathbf{M} \subset \mathbf{N}$),*

$$\boxed{\mathrm{Gal}^D(\mathbf{M}/\mathbf{K}) \cong \mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) / \mathrm{Gal}^D(\mathbf{N}/\mathbf{M})}.$$

3) The relation $\dim\left(\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})\right) = \deg\mathrm{tr}(\mathbf{N}/\mathbf{M})$ *holds (where* dim *denotes the Krull dimension ([P1], section 3.2.6(**I**))).*

4) Let $\mathbf{N}^{\circ} \subset \mathbf{N}$ be the differential field corresponding to $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})^{\circ}$ *(with the notation of Lemma 1.83) under* (1.18). *Then $\mathbf{N}^{\circ}$ is the algebraic closure of $\mathbf{K}$ in $\mathbf{N}$ (Definition 1.27), $\mathbf{N}^{\circ}$ is a finite Galois extension of $\mathbf{K}$ with Galois group* $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) / \mathrm{Gal}^D(\mathbf{N}/\mathbf{K})^{\circ}$, *and* $\deg\mathrm{tr}(\mathbf{N}/\mathbf{N}^{\circ}) = \dim\left(\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})^{\circ}\right)$; $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$ *is connected if and only if $\mathbf{K}$ is algebraically closed in* $\mathbf{N}$.

(II) GALOIS EXTENSIONS VS. PICARD-VESSIOT EXTENSIONS A proof of the following result is given in ([MAG 97], Proposition 3.20):

THEOREM 1.88.– *Let* $\mathbf{K}$ *be a differential field whose field of constants* $\mathbf{C}$ *is algebraically closed. A finite Galois extension* $\mathbf{N}/\mathbf{K}$ *is a Picard-Vessiot extension (after equipping* $\mathbf{N}$ *with the unique derivation that extends the derivation of* $\mathbf{K}$, *cf. Lemma 1.74) and* $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) = \mathrm{Gal}(\mathbf{N}/\mathbf{K})$.

1.3.5. *Simple adjunctions*

(I) ADJUNCTION OF THE ROOT OF A POLYNOMIAL This type of adjunction was already presented in section 1.1 and falls within the scope of Picard-Vessiot theory by Theorem 1.88.

(II) ADJUNCTION OF A PRIMITIVE Let $\mathbf{N} = \mathbf{K}\langle u\rangle$, where $\dot{u} = v \in \mathbf{K}$, and v is not the derivative of an element of $\mathbf{K}$. We say that $\mathbf{N}$ may be constructed by *adjunction of the primitive* (or, by abuse of language, of the integral) $u = \int v$. We will show the following properties: (i) u is transcendental over $\mathbf{K}$, (ii) $\mathbf{N}/\mathbf{K}$ is a Picard-Vessiot extension, (iii) $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) \cong \mathrm{G}_a(\mathbf{C}) \cong \mathrm{U}_2(\mathbf{C})$.

i) Suppose that u is algebraic over $\mathbf{K}$. Let $f(X) = X^n + \sum_{i=1}^{n} a_i X^{n-i}$ be its minimal polynomial. By differentiating the expression $f(u) = 0$, we obtain $nu^{n-1}v + \dot{a}_1 u^{n-1} + \varphi(u) = 0$, where $\varphi(u)$ is a polynomial in u of degree $< n-1$. Hence,

$$D(f(u)) = 0 \Rightarrow nu^{n-1}v + \dot{a}_1 u^{n-1} = 0 \Rightarrow v = -\delta(a_1/n),$$

which is a contradiction.

ii) We need to show that $\mathbf{K}\langle u\rangle$ does not contain any constants other than those in $\mathbf{K}$. But, since $\dot{u} \in \mathbf{K}$, we know that $\mathbf{K}\langle u\rangle = \mathbf{K}(u)$.

a) Suppose first that there exists a constant c which belongs to $\mathbf{K}[u]$ but not to $\mathbf{K}$. Then there must exist a polynomial $g(X) = \sum_{i=0}^{m} b_i X^{m-i} \in \mathbf{K}[X]$ of degree $m > 0$ such that $g(u)$ is constant, implying that

$$0 = \dot{b}_0 u^m + mb_0 u^{m-1} v + \dot{b}_1 u^{m-1} + \psi(u),$$

where $\psi(u)$ is a polynomial in u of degree $< n-2$. Since u is not algebraic over $\mathbf{K}$ by (i), we have that $\dot{b}_0 = 0$ and $mb_0 v + \dot{b}_1 = 0$, and hence $v = -\dot{b}_1/(mb_0) = \delta(-b_1/(mb_0))$, contradiction.

b) Suppose now that there exists a constant c which belongs to $\mathbf{K}(u)$ but not to $\mathbf{K}$. Then there must exist a rational fraction $f/g \in \mathbf{K}(X)$ such that $(f/g)(u) = c$. We may assume that g is unitary and of minimal degree ≥ 1. Therefore, reusing the notation from the proof of Lemma 1.74,

$$0 = \frac{\delta(f)(u)\, g(u)\, v - f(u)\, \delta(g)\, v}{g(u)^2} \Rightarrow c = \frac{f(u)}{g(u)} = \frac{\delta(f)(u)}{\delta(g)(u)},$$

where $d^\circ(\delta(g)) < d^\circ(g)$: contradiction. Since 1 and u are linearly independent over $\mathbf{K}$, they are also linearly independent over $\mathbf{C}$, which proves (ii).

iii) The $\mathbf{K}^D$-automorphisms of $\mathbf{K}\langle u\rangle$ are the mappings $u \mapsto u + c$, $c \in \mathbf{C}$, and are therefore the elements $\mathrm{G}_a(\mathbf{C}) \cong \mathrm{U}_2(\mathbf{C})$ (Example 1.82(v)).

(III) Adjunction of the exponential of a primitive Let $\mathbf{N} = \mathbf{K}\langle u\rangle$, where $\dot{u}/u = v \in \mathbf{K}^\times$. We write that $u = e^{\int v}$ (we do not define an exponentiation operation, but given $\mathbf{K} = \mathbb{C}(t)$, the relation $e^t = e^{\int 1.dt}$ holds). We say that $\mathbf{N}$ may be constructed from $\mathbf{K}$ by *adjunction of the exponential of a primitive*. Suppose that $\mathbf{N}$ does not contain any constants other than those in $\mathbf{K}$ and write $\mathbf{C}$ for the field of constants. We will show the following properties: (i) $\mathbf{N} = \mathbf{K}(u)$ and $\mathbf{N}/\mathbf{K}$ is a Picard-Vessiot extension. (ii) If u is algebraic over $\mathbf{K}$, then there exists an integer $h > 1$ such that $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) \cong \boldsymbol{\mu}_h$ (Example 1.82(viii)). (iii) If u is transcendental over $\mathbf{K}$, then $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) \cong \mathrm{G}_m(\mathbf{C})$ (Example 1.82(ii)).

i) $\dot{u} = uv \in \mathbf{K}(u)$, and, by induction, $u^{(n)} \in \mathbf{K}(u), \forall n \in \mathbb{N}$, so $\mathbf{N} = \mathbf{K}(u)$. Moreover, u is the only non-zero solution of the equation $\dot{y} - vy = 0$, so $\mathbf{N}/\mathbf{K}$ is a Picard-Vessiot extension.

ii) Suppose that u is algebraic over $\mathbf{K}$ and let $f(X) = X^h + \sum_{i=1}^h a_i X^{h-i}$ be its minimal polynomial. By differentiating the expression $f(u) = 0$, setting $a_0 = 1$, we obtain:

$$\begin{aligned} 0 &= (\delta f)(u) + \left(\frac{\partial f}{\partial X}\right)(u) .\dot{u} = (\delta f)(u) + \left(\frac{\partial f}{\partial X}\right)(u)\, vu \\ &= hvu^h + \sum\nolimits_{i=0}^{h-1} (\dot{a}_i + iva_i)\, u^i = hv\left(u^h + \sum\nolimits_{i=0}^{h-1} \frac{\dot{a}_i + iva_i}{hv} u^i\right). \end{aligned}$$

Therefore, f divides this polynomial, so $(\dot{a}_i + iva_i)/hv = a_i$, and finally $\dot{a}_i = v(h-i)a_i$ $(i = 0, ..., h-1)$. Hence, $\delta\left(u^{h-i}/a_i\right) = 0$. In particular, $u^h = c$ for $c \in \mathbf{C}$. We conclude that f divides $X^h - c$, and consequently $f(X) = X^h - c$.

Let $\sigma \in \mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$. We have that $\delta(\sigma(u)) = \sigma(\delta(u)) = \sigma(vu) = v\sigma(u)$, which implies that $\delta(\sigma(u)/u) = 0$, in turn implying that $\sigma(u) = \lambda u$, where $\lambda \in \mathbf{C}$. Therefore, σ is the $\mathbf{K}^D$-automorphism $u \mapsto \lambda u$. If $u^h = c$, then $\sigma(u)^h = \sigma\left(u^h\right) = \sigma(c) = c$; furthermore, $\sigma(u)^h = \lambda^h u^h = \lambda^h c$. Hence, $\lambda^h = 1$ and $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K}) \cong \boldsymbol{\mu}_h$.

iii) is clear from the above.

1.3.6. *Liouville extensions and elementary functions*

(I) LIOUVILLE EXTENSIONS Named after J. Liouville, Liouville extensions are analogous to radical extensions in Galois theory (section 1.1.2**(VII)**).

DEFINITION 1.89.– *A Picard-Vessiot extension* $\mathbf{N} = \mathbf{K}\langle\alpha_1, ..., \alpha_q\rangle$ *is said to be a Liouville extension if any one of the following conditions is satisfied for all* $i \in \{0, ..., q-1\}$:

i) α_{i+1} *is algebraic over* $\mathbf{K}\langle\alpha_1, ..., \alpha_i\rangle$;

ii) α_{i+1} *is a primitive of an element of* $\mathbf{K}\langle\alpha_1, ..., \alpha_i\rangle$;

iii) α_{i+1} *is an exponential of a primitive of an element of* $\mathbf{K}\langle\alpha_1, ..., \alpha_i\rangle$.

EXAMPLE 1.90.– *Let* $\mathbf{K} = \mathbb{C}(t)$.

1) If $\alpha_1(t) = \sqrt{1-t^2}$, *then* $\mathbf{K}\langle\alpha_1\rangle$ *is a Liouville extension of* $\mathbf{K}$, *since* α_1 *is a root of the polynomial* $X^2 + t^2 - 1 \in \mathbf{K}[X]$.

2) If $\alpha_2(t) = \arcsin(t)$, *then* $\mathbf{K}\langle\alpha_1, \alpha_2\rangle$ *is a Liouville extension of* $\mathbf{K}$, *as* $\alpha_2(t) = \int \frac{dt}{\alpha_1(t)}$.

3) If $\alpha_3(t) = e^{\arcsin(t)}$, *then* $\mathbf{K}\langle\alpha_1, \alpha_2, \alpha_3\rangle$ *is a Liouville extension of* $\mathbf{K}$.

4) $\mathbf{K}\langle\beta_1, \beta_2\rangle$ *with* $\beta_1(t) = e^{-t^2/2}$ *and* $\beta_2(t) = \int e^{-t^2/2}dt$ *is a Liouville extension of* $\mathbf{K}$.

The next theorem, which is proven in ([MAG 97], Proposition 6.7), is analogous to Theorem 1.15:

THEOREM 1.91.– *(Kolchin) A Picard-Vessiot extension* $\mathbf{N}/\mathbf{K}$ *is Liouville if and only if* $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})^\circ$ *is solvable.*

(II) ELEMENTARY FUNCTIONS

DEFINITION 1.92.– *Let* $\mathbf{C}$ *be an algebraically closed field and let* $\mathbf{K} = \mathbf{C}(t)$ *be equipped with its usual derivation (*$\delta(c) = 0$ *if* $c \in \mathbf{C}$, $\delta(t) = 1$*). A Picard-Vessiot extension* $\mathbf{E} = \mathbf{K}\langle \alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m \rangle$ *is called an* elementary function field *if, setting* $\mathbf{M}_j = \mathbf{K}\langle \beta_1, ..., \beta_{j-1} \rangle$,

i) $D(\alpha_i) \in \mathbf{K}$ $(i = 1, ..., n)$;

ii) $D(\beta_j)/\beta_j \in \mathbf{M}_j$ *or* β_j *is algebraic over* $\mathbf{M}_j$ $(j = 1, ..., m)$.

An elementary function field is therefore a special case of a Liouville extension of $\mathbf{C}(t)$. It can be shown that ([MAG 97], Proposition 6.12):

THEOREM 1.93.– *If* $\mathbf{E}$ *is an elementary function field over* $\mathbf{K} = \mathbf{C}(t)$, *then, for any Picard-Vessiot subextension* $\mathbf{N}/\mathbf{K}$ *of* $\mathbf{E}/\mathbf{K}$, *the group* $\mathrm{Gal}^D(\mathbf{N}/\mathbf{K})^\circ$ *is abelian.*

(III) GAUSSIAN FUNCTION We will show that the Gaussian function $u(t) = \int e^{-t^2/2} dt$ is not an elementary function over $\mathbf{K} = \mathbb{C}(t)$. Let $\mathbf{N} = \mathbf{K}\langle u \rangle$. Then $\mathbf{N}/\mathbf{K}$ is a Picard-Vessiot extension for the differential equation $\ddot{y} + t\dot{y} = 0$. Since 1 is a solution of this equation, $\mathcal{B} = \{1, u\}$ is a basis of the $\mathbb{C}$-vector space V generated by its solutions (Theorem 1.86). Note that the Wronskian is indeed $w(1, u) = \dot{u} \neq 0$. Let $\sigma \in \mathrm{Gal}^D(\mathbf{N}/\mathbf{K})$. We have that $\sigma : 1 \mapsto 1$, $u \mapsto c1 + du$, $c, d \in \mathbf{C}$. The coefficients c and d are determined as follows: since $\delta(\sigma(u)) = \sigma(\dot{u}) = \sigma(w)$ and $\delta(\sigma(u)) = \delta(c + du) = d\dot{u}$, it follows that $d = \sigma(w)/w$; moreover, $c = \sigma(u) - du = u\left(\frac{\sigma(u)}{u} - \frac{\sigma(w)}{w}\right) = -\frac{u^2}{\dot{u}}\delta\left(\frac{\sigma(u)}{u}\right)$. With respect to the basis $\mathcal{B}$, σ is represented by the matrix

$$\begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

The group G generated by the matrices on the left (called the Borel subgroup of $\mathrm{GL}_2(\mathbf{C})$) is not abelian, and the factorization on the right shows (by

Theorem 1.84) that G is connected as an algebraic group, since it is generated by the connected groups $\mathrm{G}_m(\mathbf{C})$ and $\mathrm{G}_a(\mathbf{C})$, which implies that $G = G^\circ$. Therefore, by Theorem 1.93, u is not an elementary function.

1.4. Differentially transcendental extensions

We can now adapt the definitions and results from section 1.2 to the context of differential algebra. The following discussion retraces the presentation given by Kolchin ([KOL 73], Chapters I and II), omitting proofs and simplifying by only considering one single derivation and assuming that the differential fields are perfect, which avoids the need to distinguish between algebraic extensions and separable algebraic extensions (Theorem 1.39). The ideas presented below were used by M. Fliess to build a framework for a non-linear systems theory (in the sense that it is used in the field of automation) [FLI 89].

1.4.1. *Differentially algebraic dependence*

Let $\mathbf{K}$ be a differential field, suppose that $\mathbf{M}/\mathbf{K}$ is a differential field extension (section 1.3.2), and write D for derivation in $\mathbf{M}$ (section 1.3.1(**I**)). When $D = 0$ (trivial derivation), the following concepts and results are identical to those studied in section 1.2 (in which case the adverb *differentially* may be omitted).

DEFINITION 1.94.–

i) We say that a family $(u_i)_{i \in I}$ of elements of $\mathbf{M}$ is differentially algebraically dependent *over $\mathbf{K}$ if the family $\left(D^j u_i\right)_{(i,j) \in I \times \mathbb{N}}$ is algebraically dependent over $\mathbf{K}$, i.e. is not algebraically free over $\mathbf{K}$ (section 1.2.1);*

ii) An element $u \in \mathbf{M}$ is differentially algebraic *over $\mathbf{K}$ if there exists a non-zero polynomial $f \in \mathbf{K}\left[(X_i)_{i \in I}\right]$ such that $f\left(\left(D^i(u)\right)_{i \in I}\right) = 0$;*

iii) The extension $\mathbf{M}/\mathbf{K}$ is differentially algebraic *if every element of M is differentially algebraic over $\mathbf{K}$.*

It is possible to show the following result ([KOL 73], Chapter II, section 9, Proposition 10):

LEMMA 1.95.– *Let $\mathbf{M}/\mathbf{K}$ be a differential field extension and let $B \subset \mathbf{M}$. The following conditions on B are equivalent:*

i) B is differentially algebraically independent over $\mathbf{K}$ and $\mathbf{M}$ is differentially algebraic over $\mathbf{K}\langle B\rangle$.

ii) B is a minimal subset of $\mathbf{M}$ such that $\mathbf{M}$ is differentially algebraic over $\mathbf{K}\langle B\rangle$.

iii) B is a maximal subset of $\mathbf{M}$ that is differentially algebraically free over $\mathbf{K}$.

DEFINITION 1.96.– *A set $B \subset M$ satisfying any one of the equivalent conditions in Lemma 1.95 is called a* differential transcendental basis *of* $\mathbf{M}$ *over* $\mathbf{K}$.

1.4.2. *Differential transcendence degree*

The result stated below ([KOL 73], Chapter II, section 9, Theorem 4) is analogous to the theorems established by Steinitz (Theorems 1.70 and 1.72).

THEOREM 1.97.– *(Kolchin)*

i) Let $\mathbf{M}/\mathbf{K}$ be a differential field extension. Let S be a subset of $\mathbf{M}$ such that $\mathbf{M}$ is differentially algebraic over $\mathbf{K}\langle S\rangle$, and suppose that T is a subset of S that is differentially algebraically free over $\mathbf{K}$; then there exists a differential transcendence basis B of $\mathbf{E}$ such that $T \subset B \subset S$.

ii) Every differential field extension $\mathbf{M}/\mathbf{K}$ has a differential transcendence basis over $\mathbf{K}$.

iii) All differential transcendence bases of $\mathbf{M}$ over $\mathbf{K}$ are equipotent.

DEFINITION 1.98.– *Let $\mathbf{M}/\mathbf{K}$ be a differential field extension. The cardinal of a differential transcendence basis of $\mathbf{M}$ over $\mathbf{K}$ is called the* differential transcendence degree *of* $\mathbf{M}$ *over* $\mathbf{K}$, *and is written* $\deg \operatorname{tr}^D(\mathbf{M}/\mathbf{K})$.

If $\mathbf{K}, \mathbf{E}$, and $\mathbf{F}$ are three differential fields such that $\mathbf{K} \subset \mathbf{E} \subset \mathbf{F}$, then the following relation holds (cf. [KOL 73], Chapter II, section 9, Corollary 2), analogously to (1.13) ,

$$\boxed{\deg \operatorname{tr}^D(\mathbf{F}/\mathbf{K}) = \deg \operatorname{tr}^D(\mathbf{E}/\mathbf{K}) + \deg \operatorname{tr}^D(\mathbf{F}/\mathbf{E})}.$$

2

General Topology

2.1. Introduction to general topology

2.1.1. *Metric spaces*

(I) We shall assume that the elementary concepts of topology in metric spaces are known (open sets, neighborhoods, distance, open and closed balls, and so on): cf. ([DIE 82], Volume 1). A metric on a set $X \neq \varnothing$ is a mapping $d : X \times X \to \mathbb{R}_+$ that satisfies the following conditions for all $x, y, z \in X$:

(**D**$_1$) $d(x, x) = 0$.

(**D**$_2$) $d(x, y) = d(y, x)$.

(**D**$_3$) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

(**D**$_4$) $d(x, y) = 0 \Rightarrow x = y$ (*separation condition*).

(**D**$_3$) implies the relation (**exercise**)

$$\left|d(x, y) - d\left(x', y'\right)\right| \leq d\left(x, x'\right) + d\left(y, y'\right). \qquad [2.1]$$

(II) In a metric space X equipped with a metric d, we write $B(a; r)$ (resp. $B^c(a; r)$) for the open ball (resp. closed ball) with center a and radius $r > 0$; in other words, $B(a; r) = \{x \in X : d(a, x) < r\}$ and $B^c(a; r) = \{x \in X : d(a, x) \leq r\}$. An *open set* is a union of open balls ($\varnothing$ is therefore an open set as it is the empty union). If $x \in X$ and $A \subset X$, we define $d(x, A) = \inf\{d(x, y) : y \in A\}$. The function $x \mapsto d(x, A)$ is continuous (**exercise**). If $A, B \subset X$, we define $d(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\} = \inf\{d(x, B) : x \in A\}$. The *diameter* of a set $A \subset X$ is $\delta(A) = \sup\{d(x, y) : (x, y) \in A \times A\}$.

The usual metrics on $\mathbb{K}^n$, where $\mathbb{K}$ is the field of real numbers (resp. complex numbers), may be defined as follows:

$$d_p(x, y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}\ (p \in [1, \infty[)\,;\ d_\infty(x, y) = \sup_{1 \le i \le n} |x_i - y_i|\,;$$

d_2 is the Euclidean (resp. Hermitian) distance function and, for all $(x, y) \in \mathbb{K}^n \times \mathbb{K}^n$, $d_\infty(x, y) = \lim_{p \to +\infty} d_p(x, y)$. The proof of $(\mathbf{D}_1), (\mathbf{D}_2), (\mathbf{D}_4)$ is immediate. The triangle inequality $(\mathbf{D}_3)$ is obvious for $p = 1$ and $p = \infty$. For $p \in]1, \infty[$, the triangle inequality follows from Minkowski's inequality, which in turn follows from Hölder's inequality[1]:

DEFINITION 2.1.– *We say that the numbers* $p, q \in [1, \infty]$ *are* conjugate exponents *if* $\frac{1}{p} + \frac{1}{q} = 1$.[2]

LEMMA 2.2.– *Let* $(a_i)_{1 \le i \le n}$ *and* $(b_i)_{1 \le i \le n}$ *be sequences of* n *elements in* $\mathbb{K}$. *For every pair of conjugate exponents* $p, q \in]1, \infty[$*, the following inequalities hold (Hölder*'s inequality *and Minkowski*'s inequality*):*

$$\sum_{i=1}^{n} |a_i b_i| \le \sqrt[p]{\sum_{i=1}^{n} |a_i|^p} . \sqrt[q]{\sum_{i=1}^{n} |b_i|^q}, \qquad [2.2]$$

$$\sqrt[p]{\sum_{i=1}^{n} |a_i + b_i|^p} \le \sqrt[p]{\sum_{i=1}^{n} |a_i|^p} + \sqrt[p]{\sum_{i=1}^{n} |b_i|^p}. \qquad [2.3]$$

All of these metrics d_p $(p \in [1, \infty])$ define the same topology on $\mathbb{K}^n$, known as the *canonical topology* of $\mathbb{K}^n$ (cf. below, Theorem 3.17(2)). Hölder's inequality also holds for the conjugate exponents $p = 1$ and $q = \infty$, since

$$\sum_{i=1}^{n} |a_i b_i| \le \sum_{i=1}^{n} |a_i| . \sup_{1 \le i \le n} |b_i|. \qquad [2.4]$$

1 See the Wikipedia articles on Hölder*'s inequality* and Minkowski*'s inequality*.

2 In the following, we will write ∞ for $+\infty$ whenever it is not ambiguous to do so.

2.1.2. *Concept of topology*

In general, a *topology* $\mathfrak{T}$ on a set X is a subset of $\mathfrak{P}(X)$ such that

($\mathbf{O}_{\mathrm{I}}$) If $(O_{i \in I})$ is a family of elements of $\mathfrak{T}$, then $\cup_{i \in I} O_i \in \mathfrak{T}$.

($\mathbf{O}_{\mathrm{II}}$) If $O_1, ..., O_n$ are (finitely many) elements of $\mathfrak{T}$, then their intersection belongs to $\mathfrak{T}$.

The axiom $(\mathbf{O}_{\mathrm{I}})$ implies that $\varnothing \in \mathfrak{T}$, and $(\mathbf{O}_{\mathrm{II}})$ implies that $X \in \mathfrak{T}$, since the empty union in $\mathfrak{P}(X)$ is the empty set and the empty intersection is X **(exercise)**. The elements of $\mathfrak{T}$ are called the *open sets* of the *topological space* $(X, \mathfrak{T})$. A set U is called a *neighborhood* of the point x if there exists an open subset of U that contains x. A set $\Omega \subset X$ is open if and only if it is a neighborhood for each of its points (**exercise**). If $\mathfrak{B} \subset \mathfrak{T}$, we say that $\mathfrak{B}$ is a *base* of $\mathfrak{T}$ if every open set in $(X, \mathfrak{T})$ may be written as a union of elements of $\mathfrak{B}$. In particular, in a metric space, the set of open balls is one possible base for the underlying topology.

2.1.3. *General topology vs. topology in metric spaces*

The theory of metric spaces, summarized by F. Hausdorff between 1914 and 1927 [HAU 27], ultimately revealed itself to be an overly narrow framework in the late 1920s, especially once the notion of weak topology began to emerge in the work performed by S. Banach and his colleagues [BAN 32]. There are significant differences between topology in general ([BKI 74], Chapters I and II) and the special case of topology in metric spaces:

(I) General topology has various separation axioms, traditionally denoted T_0, T_1, T_2, etc.[3], or $\mathrm{O_{III}}$,$\mathrm{O_{IV}}$,$\mathrm{O_V}$, etc. by Bourbaki. We will use both notations to avoid needing to write T_i in cases where i is a fraction.

– T_0 is Kolmogoroff's axiom. A topological space is T_0 if, given two distinct points $x_1, x_2 \in X$, *at least one of them* has a neighborhood that does not contain the other.

3 The letter T comes from the German term *Trennungsaxiom* (separation axiom). The classification given here was reproduced from [ALE 35]. Although these conventions are the most traditional, some authors choose other conventions and notation. For more information, see the Wikipedia article on *Separation axioms (topology)*.

– T_1 is Fréchet's axiom. A topological space is T_1 if, given two distinct points $x_1, x_2 \in X$, *both points* have a neighborhood that does not contain the other.

– T_2 is Hausdorff's axiom. A topological space is T_2 if, given two distinct points $x_1, x_2 \in X$, these points have *disjoints* neighborhoods U_1, U_2. Any such space is said to be a Hausdorff space.

– T_3 (or $\mathrm{O_{III}}$) is the axiom of regularity (cf. section 2.3.9**(I)**). A topological space is said to be *regular* if it is T_1 and T_3.

– $\mathrm{O_{IV}}$ (or $T_{3\frac{1}{2}}$) is the axiom of complete regularity (cf. section 2.4.1**(II)**). A space satisfies the axiom $\mathrm{O_{IV}}$ if and only if it is uniformizable. A topological space satisfying $\mathrm{O_{IV}}$ and T_0 is said to be *completely regular* (or a Tychonov $\mathbf{T}$-space).

– T_4 (or $\mathrm{O_V}$) is the axiom of normality (section 2.3.11). A topological space is *normal* if it is T_4 and T_1. The T_4 property is not hereditary, i.e. the subspaces of a T_4 space are not necessarily T_4. This motivates the following axiom.

– T_5 is the axiom of complete normality. A topological space is T_5 if all of its subspaces are T_4. A space that is both T_1 and T_5 is said to be *completely normal*.

It is clear that $T_2 \Rightarrow T_1 \Longrightarrow T_0$. The singleton $\{x\}$ is closed in the topological space X if and only if $\complement_X \{x\}$ is a neighborhood of each of its points, and this condition is satisfied for every point x if and only if X is T_1 (**exercise**). The Hausdorff separation axiom is the most important in most practical contexts, with the exception of algebraic geometry and commutative algebra:

– Local non-Noetherian rings equipped with the $\mathfrak{m}$-adic topology are not necessarily T_0 ([P1], section 3.1.8**(I)**).

– The Zariski topology on the prime spectrum $\mathrm{Spec}\,(\mathbf{R})$ of a commutative ring $\mathbf{R}$ ([P1], section 2.3.3**(III)**) is T_0 but not T_1, as the singleton $\{\mathfrak{a}\}$ in $\mathrm{Spec}\,(\mathbf{R})$ is not closed whenever the ideal $\mathfrak{a}$ is not maximal.

– The Zariski topology on an algebraic variety ([P1], section 3.2.7**(II)**) is T_1 (since the points of this variety are closed sets) but not necessarily T_2.

– If (X, d) is a metric space, consider two distinct points x_1, x_2 in X and let $\delta = d(x_1, x_2) > 0$; then the balls $B(x_1; \delta/2)$ and $B(x_2; \delta/2)$ are disjoint neighborhoods of x_1 and x_2 respectively, so (X, d) is T_2. In section 2.3.11, we will show that every metrizable space is completely normal. Furthermore, we will see later that

$$\begin{aligned}(T_1 \ \& \ T_3) \quad &\Leftrightarrow \quad (T_2 \ \& \ T_3), \quad (T_0 \ \& \ \mathrm{O_{IV}}) \Leftrightarrow (T_2 \ \& \ \mathrm{O_{IV}}), \\ (T_1 \ \& \ T_4) \quad &\Leftrightarrow \quad (T_2 \ \& \ T_4) \Rightarrow \mathrm{O_{IV}} \Rightarrow T_3, \quad T_5 \Rightarrow T_4.\end{aligned}$$

In particular, in the class of Hausdorff T_2 spaces, $T_5 \Rightarrow T_4 \Rightarrow \mathrm{O_{IV}} \Rightarrow T_3$. In the class of uniformizable $\mathrm{O_{IV}}$ spaces, $T_0 \Leftrightarrow T_1 \Leftrightarrow T_2 \Leftrightarrow T_3$ (Lemma 2.66), so a space is completely regular if and only if it is a uniformizable Hausdorff space (this type of space is the most important in practice).

(II) In a metric space X, every point x has a *countable* fundamental system $(V_n)_{n \in \mathbb{N}}$ of neighborhoods. Hence (see below, section 2.3.1**(I)**), for any neighborhood U of x, there exists V_n such that $V_n \subset U$: we can simply pick $V_n = B\left(x; \frac{1}{n}\right)$ for n sufficiently large. This is precisely why studying convergence and sequences is so useful in the context of metric spaces. For example, to show that a function is continuous at the point a, we simply need to show that, for any sequence (x_n) converging to a, the sequence $(f(x_n))$ converges to $f(a)$. This is no longer the case in general topological spaces. One attempt to overcome this issue is to replace sequences with the notion of *generalized sequences* or *nets* (analogous to sequences but indexed by uncountable sets). This idea was proposed by E.H. Moore and H.L. Smith as early as 1922 [MOO 22], but was not sufficient to fully resolve all of their problems. In particular, the question of how to define a *subnet* to replace the notion of an extracted sequence proved difficult. Another approach involving the notion of *filter* was introduced by H. Cartan in 1937 [CAR 37], and immediately adopted by N. Bourbaki in the first edition of *Topologie générale* in 1940. This new concept allowed the most relevant definition for a subnet to be identified in 1972 [AAR 72]. As we will see in section 2.2, nets and filters are equivalent approaches to topology, and so we may freely choose the simplest in any given situation. The advantage of nets is that they can be used like sequences in metric spaces (with a few precautions), which in the author's opinion makes them extremely useful.

(III) Metric spaces allow us not only to say whether the point x' is near to the point x, but also whether the two points x and x' are close to each other: specifically, whenever the distance $d(x, x')$ is sufficiently small. In a topological space, this symmetry between x and x' (which in particular is required to study completeness) does not exist. To replace it, we must introduce some other structure; the notion of uniform space was proposed accordingly by A. Weil [WEI 37]. This alternative structure is built upon the notion of *entourage*: two points x and x' are considered to be "close" whenever $(x, x') \in W$, where W is a "sufficiently small" entourage. Much like Cartan's proposal that very same year, Weil's invention (or discovery?) was immediately adopted by Bourbaki and integrated into the first edition of *Topologie générale*.

2.2. Filters and nets

2.2.1. *Filters and ultrafilters*

(I) FILTERS A *filter* $\mathfrak{F}$ on a set X is a subset of $\mathfrak{P}(X)$ such that: (a) $\mathfrak{F} \neq \varnothing$; (b) $\varnothing \notin \mathfrak{F}$; (c) if $A \in \mathfrak{F}$ and $B \supset A$, then $B \in \mathfrak{F}$; if $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$. A filter is therefore a directed set for the order relation $\supset$ ([P1], section 1.2.8(**I**)). A subset $\mathfrak{B}$ of $\mathfrak{P}(X)$ is a *filter base* if the set $\mathfrak{F} = \{A \in \mathfrak{P}(X) : \exists B \in \mathfrak{B}, A \supset B\}$ is a filter. If so, we say that $\mathfrak{B}$ is a base of the filter $\mathfrak{F}$, or that $\mathfrak{F}$ is the filter generated by $\mathfrak{B}$. For $\mathfrak{B}$ to be a filter base, it is necessary and sufficient for the following three conditions to be satisfied (**exercise**): (1) $\mathfrak{B} \neq \varnothing$; (2) $\varnothing \notin \mathfrak{B}$; (3) if $A, B \in \mathfrak{B}$, then $\exists C \in \mathfrak{B} : A \cap B \supset C$.

The filter of complements of finite subsets of $\mathbb{N}$ is called the Fréchet filter.

(II) ULTRAFILTERS A filter $\mathfrak{F}_1$ is said to be *finer* than $\mathfrak{F}_2$ if $\mathfrak{F}_1 \supset \mathfrak{F}_2$. With this order relation, the set of filters that are finer than a given filter $\mathfrak{F}$ is inductive, and so has a maximal element by Zorn's lemma ([P1], section 1.1.2(**III**), Lemma 1.3). This maximal element is called an *ultrafilter*. This implies:

THEOREM 2.3.– *Given any filter $\mathfrak{F}$, there exists an ultrafilter that is finer than $\mathfrak{F}$.*

Let X be a set, $\mathfrak{G} \subset \mathfrak{P}(X)$, and $A \subset X$. The *trace* of $\mathfrak{G}$ on A is defined as $\{A \cap B : B \in \mathfrak{G}\}$.

(III) Filter eventually or frequently in a set

DEFINITION 2.4.– *We say that the filter $\mathfrak{F}$ on X is* eventually *(resp.* frequently*) in the set $A \subset X$ if A contains some set $B \in \mathfrak{F}$ (resp. if A intersects non-trivially with every set in $\mathfrak{F}$).*

Note that for $\mathfrak{F}$ to be eventually (resp. frequently) in A, it is necessary and sufficient for there to exist a base $\mathfrak{B}$ of $\mathfrak{F}$ such that A contains some $B \in \mathfrak{B}$ (resp. intersects non-trivially with every element of $\mathfrak{B}$); whenever this condition is satisfied, we say that $\mathfrak{B}$ is eventually (resp. frequently) in A. For $\mathfrak{F}$ to be eventually in A, it is necessary and sufficient for it to not be frequently in $\complement_X A$; conversely, for $\mathfrak{F}$ to be frequently in A, it is necessary and sufficient for it to not be eventually in $\complement_X A$ (**exercise**). Moreover:

LEMMA 2.5.– *Let $A \subset X$. For the filter $\mathfrak{F}$ on X to be frequently in A, it is necessary and sufficient for there to exist a filter $\mathfrak{F}'$ on X that is finer than $\mathfrak{F}$ and that is eventually in A.*

PROOF.– (1) Suppose that $\mathfrak{F}$ is frequently in A, and let $\mathfrak{F}'$ be the trace on A of the filter $\mathfrak{F}$. Then $\mathfrak{F}'$ is a filter that is finer than $\mathfrak{F}$ and which is eventually in A. (2) Conversely, if there exists a filter $\mathfrak{F}'$ that is finer than $\mathfrak{F}$ and which is eventually in A, then there exists $B' \in \mathfrak{F}'$ such that $B' \supset A$. For every $B \in \mathfrak{F}$, we have that $B \cap B' \neq \varnothing$, since $B' \in \mathfrak{F}$, so $B \cap A \neq \varnothing$. ■

LEMMA 2.6.– *Let $\mathfrak{U}$ be a filter on X. The following conditions are equivalent:*

i) $\mathfrak{U}$ is an ultrafilter.

ii) For every set $A \subset X$, $\mathfrak{U}$ is either eventually in A or eventually in $\complement_X A$.

PROOF.– (i)⇒(ii): We shall argue by contradiction; suppose that $A \notin \mathfrak{U}$ and $\complement_X A \notin \mathfrak{U}$. Let $\mathfrak{G}$ be the set of subsets M of X such that $A \cup M \in \mathfrak{U}$. Then $\mathfrak{G}$ is a filter (**exercise**) that is strictly finer than $\mathfrak{U}$. Indeed, if $B \in \mathfrak{U}$, then $A \cup B \in \mathfrak{U}$, so $B \in \mathfrak{G}$, and hence $\mathfrak{F} \subset \mathfrak{G}$. Moreover, $\complement_X A \notin \mathfrak{U}$ and $\complement_X A \in \mathfrak{G}$, since $A \cup \left(\complement_X A\right) = X \in \mathfrak{U}$. (ii)⇒(i): Let $\mathfrak{F}$ be a filter that is finer than $\mathfrak{U}$. If $B \in \mathfrak{F}$, then $\complement_X B \notin \mathfrak{F}$, since otherwise we would have $\varnothing = B \cap \complement_X B \in \mathfrak{F}$, which contradicts the definition of a filter. Therefore, $\complement_X B \notin \mathfrak{U}$, and so $\complement_X B \in \mathfrak{U}$. Hence, $\mathfrak{F} = \mathfrak{U}$. ■

(IV) For a filter $\mathfrak{F}'$ with base $\mathfrak{B}'$ to be finer than the filter $\mathfrak{F}$ with base $\mathfrak{B}$ on the set X, it is necessary and sufficient for one of the following equivalent conditions to be satisfied: (a) every element of $\mathfrak{B}$ contains some element of $\mathfrak{B}'$; (b) for every $A \subset X$, if $\mathfrak{F}$ is eventually in A, then $\mathfrak{F}'$ is eventually in A. If $\mathfrak{U}$ is an ultrafilter on X and $A \subset X$, then $\mathfrak{U}$ is frequently in X if and only if it is eventually in X (**exercise**).

Let X, Y be two sets and let $f : X \to Y$ be a mapping. Let $\mathfrak{B}$ be a base of the filter $\mathfrak{F}$ on X. The reader may wish to check that its image under f, namely $f(\mathfrak{B}) := \{f(A) : A \in \mathfrak{B}\}$, is a filter base on Y (**exercise**). The filter that it generates is sometimes called, by abuse of language, the *image* of $\mathfrak{F}$ under f ([WIL 70], Definition 12.7). If $\mathfrak{B}$ is an ultrafilter base, then $f(\mathfrak{B})$ is an ultrafilter base (**exercise**).

2.2.2. *Nets and ultranets*

(I) NETS We define a *net* in X to be a family $\mathfrak{x} = (x_i)_{i \in I}$, where I is a directed set of indices ([P1], section 1.2.8(**I**)). Let $(x_i)_{i \in I}$ be a net in X, and, for all $i \in I$, let $\mathfrak{B}_i = \{x_k : k \succeq i\}$. The set $\mathfrak{B} = \{\mathfrak{B}_i : i \in I\}$ is a filter base on X (**exercise**), called the *elementary filter base* associated with the net $\mathfrak{x}$. The filter that it generates is called the *elementary filter* associated with $\mathfrak{x}$. Conversely, let $\mathfrak{F}$ be a filter on X and let $\mathfrak{B}$ be a base of $\mathfrak{F}$, ordered by inclusion. For all $A \in \mathfrak{B}$, A is non-empty; therefore, by the axiom of choice ([P1], section 1.1.2(**III**)), there exists $x_A \in A$ for all $A \in \mathfrak{B}$. The net $(x_A)_{A \in \mathfrak{B}}$ is said to be associated with $\mathfrak{F}$ (but is not necessarily unique). In particular, the elementary filter associated with the sequence (x_n) in X is the image of the Fréchet filter (section 2.2.1(**I**)) under the mapping $n \mapsto x_n$ from $\mathbb{N}$ into X.

LEMMA 2.7.– *Let $\mathfrak{F}$ be a filter on X and let $(x_A)_{A \in \mathfrak{B}}$ be a net associated with $\mathfrak{F}$. Then the elementary filter $\mathfrak{G}$ associated with $(x_A)_{A \in \mathfrak{B}}$ is finer than $\mathfrak{F}$.*

PROOF.– $E \in \mathfrak{G} \Leftrightarrow [\exists A \in \mathfrak{B} : E \supset \{x_{A'} : A' \subset A\}]$, $E \in \mathfrak{F} \Leftrightarrow [\exists A \in \mathfrak{B} : E \supset A]$, and $E \supset A \Rightarrow E \supset \{x_{A'} : A' \subset A\}$, so $\mathfrak{F} \subset \mathfrak{G}$. ■

LEMMA-DEFINITION 2.8.– *Consider two nets $(x_i)_{i \in I}$ and $\left(x'_j\right)_{i \in J}$ in X.*

1) The following conditions are equivalent:

i) The elementary filter associated with $\left(x'_j\right)_{i \in J}$ is finer than the elementary filter associated with $(x_i)_{i \in I}$.

ii) For all $i_0 \in I$, there exists $j_0 \in J$ such that $\left\{x'_j : j \succeq j_0\right\} \subset \{x_i : i \succeq i_0\}$.

2) The net $\left(x'_j\right)_{i\in J}$ *is called a* subnet *of* $(x_i)_{i\in I}$ *if the equivalent conditions (i) and (ii) above are satisfied.*

3) Two nets are said to be equivalent[4] *if each is a subnet of the other, or in other words if the elementary filters associated with them are identical.*

PROOF.– We need to show (1). Let $\mathfrak{F}$ and $\mathfrak{F}'$ be the elementary filters associated with the nets $(x_i)_{i\in I}$ and $\left(x'_j\right)_{i\in J}$ respectively. Then $\mathfrak{F}'$ is finer than $\mathfrak{F}$ if and only if (section 2.2.1(**IV**)) every element $\{x_i : i \succeq i_0\}$ of the base $\mathfrak{B}$ of $\mathfrak{F}$ contains some element $\left\{x'_j : j \succeq j_0\right\}$ of the base $\mathfrak{B}'$ of $\mathfrak{F}'$. ■

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a set is a net in that set, and any extracted sequence (x_{n_k}) is a subnet. However, the converse does not necessarily hold: subnets of $(x_n)_{n\in\mathbb{N}}$ are not extracted sequences in general. For example, consider the two sequences of real numbers $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ defined by $x_n = y_n = 0$ if $n > 0$, $x_0 = 1 = -y_0$. These yield two equivalent nets, but neither is an extracted sequence of the other. It is nonetheless possible to prove the following result ([AAR 72], Proposition 2.6):

LEMMA 2.9.– *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a set X and let $(y_k)_{k\in\mathbb{N}}$ be a subnet of $(x_n)_{n\in\mathbb{N}}$. Then there exists an extracted sequence $(x_{n_k})_{k\in\mathbb{N}}$ that is equivalent to $(y_k)_{k\in\mathbb{N}}$ (as a net).*

Henceforth, given any sequence $(x_n)_{n\in\mathbb{N}}$ in a set X, we will use the term *extracted sequence* to refer to any sequence $(x_{n_k})_{k\in\mathbb{N}}$, as usual, and the term *subsequence* of $(x_n)_{n\in\mathbb{N}}$ to refer to a subnet of the net $(x_n)_{n\in\mathbb{N}}$. Thus, every extracted sequence is a subsequence, but the converse may fail. In practice, the distinction is of little importance, thanks to Lemma 2.9.

4 This notion of equivalence is the one used in [AAR 72]. There are other types of equivalence of nets that do not translate so well to filters: see the Wikipedia article on *Generalized sequences*.

(II) NET EVENTUALLY OR FREQUENTLY IN A SET

DEFINITION 2.10.– *Let* $(x_i)_{i\in I}$ *be a net in* X. *We say that* $(x_i)_{i\in I}$ *is* eventually *(resp.* frequently*) in the set* $A \subset X$ *if there exists* $i_0 \in I$ *such that* $\{x_i : i \succeq i_0\} \subset A$ *(resp. for all* $i_0 \in I$, *there exists* $i \succeq i_0$ *such that* $x_i \in A$*).*

For the net $\mathfrak{x}$ to eventually be in A, it is necessary and sufficient for it to not be frequently in $\complement_X A$; conversely, for it to be frequently in A, it is necessary sufficient for it to not be eventually in $\complement_X A$ (**exercise**). A net $\left(x'_j\right)_{i\in J}$ is a subnet of $(x_i)_{i\in I}$ in X if and only if, for all $A \subset X$ such that $(x_i)_{i\in I}$ is eventually in A, $\left(x'_j\right)_{i\in J}$ is eventually in A; a net $(x_i)_{i\in I}$ is eventually in a set $A \subset X$ if and only if the elementary filter associated with this net is eventually in A (**exercise**).

THEOREM 2.11.– *Let* $\mathfrak{F}$ *be a filter on* X *and let* $A \subset X$. *The following conditions are equivalent:*

i) $\mathfrak{F}$ *is eventually (resp. frequently) in* A.

ii) Every elementary filter $\mathfrak{G}$ *that is finer than* $\mathfrak{F}$ *is eventually (resp. frequently) in* A.

iii) Every net $\mathfrak{x}$ *associated with* $\mathfrak{F}$ *is eventually (resp. frequently) in* A.

PROOF.– We know that $\mathfrak{F}$ (resp. $\mathfrak{G}$, resp. $\mathfrak{x}$) is frequently in A if and only if it is not eventually in $\complement_X A$. (i)⇒(ii): This is clear from section 2.2.1**(III)**. (ii)⇒(iii): Any net $\mathfrak{x}$ associated with $\mathfrak{F}$ is of the form $(x_A)_{A\in\mathfrak{B}}$, where $\mathfrak{B}$ is a base of $\mathfrak{F}$, and the elementary filter $\mathfrak{G}$ associated with $\mathfrak{x}$ is finer than $\mathfrak{F}$ (Lemma 2.7). Moreover, if $\mathfrak{G}$ is eventually in A, then so is $(x_A)_{A\in\mathfrak{B}}$. (iii)⇒(i): Suppose that $\mathfrak{F}$ is not eventually in A. Let $\mathfrak{B}$ be a base of the filter $\mathfrak{F}$. For every $B \in \mathfrak{B}$, we have that $A \not\supset B$, so (by the axiom of choice) there exists $x_B \in B$ such that $x_B \notin A$. Hence, $(x_B)_{B\in\mathfrak{B}}$ is a net associated with $\mathfrak{F}$ that is not eventually in A. ■

Given the correspondence between filters and nets, Lemma 2.5 implies:

COROLLARY 2.12.– *Let* $A \subset X$ *and let* $\mathfrak{x}$ *be a net in* X. *For* $\mathfrak{x}$ *to be frequently in* X, *it is necessary and sufficient for there to exist a subnet of* $\mathfrak{x}$ *that is eventually in* A.

(III) ULTRANETS A net $\mathfrak{u}$ is called an *ultranet* (or *universal net*) if every subnet of this net is in fact equal to it. From Lemma-Definition 2.8, Lemma 2.6, and Theorem 2.3, we deduce:

COROLLARY 2.13.–

1) Let $\mathfrak{u}$ be a net in X. The following conditions are equivalent:

i) The elementary filter associated with $\mathfrak{u}$ is an ultrafilter.

ii) For all $A \subset X$, $\mathfrak{u}$ is either eventually in A or eventually in $\complement_X A$.

2) Every net has a subnet that is an ultranet.

Let X, Y be two sets, $f : X \to Y$ a mapping, and $\mathfrak{x} = (x_i)_{i \in I}$ a net in X. The image of $\mathfrak{x}$ under f is $f(\mathfrak{x}) = (f(x_i))_{i \in I}$. If $\mathfrak{B}$ is the elementary filter base associated with $\mathfrak{x}$, then the elementary filter base associated with $f(\mathfrak{x})$ is $f(\mathfrak{B})$ (**exercise**).

2.3. Topological structures

2.3.1. *Elementary concepts*

(I) Let $(X, \mathfrak{T})$ be a topological space (which we shall simply write as X whenever doing so is not ambiguous). The complement of an open set (i.e. of an element of $\mathfrak{T}$) is said to be a *closed set*. Let $A \subset X$. The *closure* of A in X, denoted $\overline{A}$, is the smallest closed set containing A, which is also the intersection of all closed sets containing A. If (X, d) is a metric space, then $a \in \overline{A}$ if and only if $d(a, A) = 0$. The set A is said to be *dense* (or sometimes *everywhere dense*) in X if $\overline{A} = X$. A topological space X is said to be *separable* if it has a countable dense subset. The *interior* of A is the largest open subset $\overset{\circ}{A}$ of A. Any two subsets A and B of X satisfy $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, $\complement_X \overline{A} = \overset{\circ}{\overline{\complement_X A}}$, $\complement_X \overset{\circ}{A} = \overline{\complement_X A}$, and $\overset{\circ}{A} \cap \overset{\circ}{B} = \overset{\circ}{\overline{A \cap B}}$ (**exercise**). The *frontier* of A is $\operatorname{Fr}(A) = \complement_{\overline{A}} \overset{\circ}{A}$. In a general metric space, it is not always true that $B^c(a; r) = \overline{B(a; r)}$ (but this statement holds in normed vector spaces).

(II) LOCALLY CLOSED SETS A subset A of a topological space X is said to be *locally closed* in X if, for all $x \in A$, there exists a neighborhood U_x of x

in X such that $U_x \cap A$ is closed in U_x; this condition is satisfied if and only if A is the intersection of an open subset and a closed subset of X (**exercise**). We say that an open set U of x is an *open-neighborhood* of the locally closed set A if $A \subset U$ and A is closed in U. The set of open-neighborhoods of A is denoted $\mathfrak{N}_X(A)$; this set is ordered by $\supset$ and is directed for this order relation ([P1], section 1.2.8(**I**)), since, if $U, V \in \mathfrak{N}_X(A)$, then $W := U \cap V \in \mathfrak{N}_X(A)$ and $U, V \supset W$.

The hyphen in "open-neighborhood" (which is not standard notation) is essential to avoid ambiguity: the open disc with center 0 and radius ρ in the complex plane is an open neighborhood of the interval $]-1, 1[$ of the real axis for all $\rho \geq 1$, but is only an open-neighborhood of this segment when $\rho = 1$.

(III) METRIZABILITY A topology with a base (section 2.1.2) that can be expressed as the open balls of a metric space is said to be *metrizable*. If $\mathfrak{T}_1$, $\mathfrak{T}_2$ are topologies on X, then $\mathfrak{T}_2$ is said to be *finer* than $\mathfrak{T}_1$ if $\mathfrak{T}_1 \subset \mathfrak{T}_2$. The *discrete topology* on a set X is $\mathfrak{P}(X)$. This topology is Hausdorff, and is the finest topology on X (**exercise**). The set of closed sets with respect to this topology, which may be constructed by taking the complements of the open sets, is again $\mathfrak{P}(X)$. The discrete topology is metrizable, since it may always be defined by the metric $d : X \times X \to \mathbb{R}_+$ such that $d(x, y) = 1$ if $x \neq y$, $d(x, y) = 0$ if $x = y$. The coarsest (least fine) topology on X is $\mathfrak{T} = \{\varnothing, X\}$. Let X be a set and let d_1, d_2 be two metrics on X. These metrics are said to be *topologically equivalent* if they define the same topology.

(IV) COUNTABILITY AXIOMS In a metrizable topological space, every point a has a *countable* fundamental system of neighborhoods $(V_n)_{n \geq 1}$; any space with this property is said to satisfy the *first axiom of countability*. In any metrizable topological space, we may choose $V_n = B^c\left(a, \frac{1}{n}\right)$, which are closed sets. Every topological space with a countable base (we sometimes say that such spaces satisfy the *second axiom of countability*) is separable and necessarily satisfies the first axiom of countability (**exercise**); the converse does not hold in general ([ABR 83], Exercise I.11), but is true for topological groups (section 2.8.1). Moreover:

LEMMA 2.14.– *Let X be a metrizable space. Then X is separable if and only if its topology has a countable base.*

PROOF.– The necessary condition follows from the above. Let us prove the sufficient condition: consider a sequence $(a_n)_{n \geq 1}$ in X such that

$A = \{a_n : n \geq 1\}$ is dense in X. Then the family $(U_{n,m})_{n,m \geq 1}$, where $U_{n,m} = B\left(a_n; \frac{1}{m}\right)$, is countable ([P1], section 1.1.2(**IV**)); furthermore, for any open set O in X, there exists a pair $(n, m) \in \mathbb{N}^\times \times \mathbb{N}^\times$ such that $U_{n,m} \subset O$. ■

2.3.2. *Convergence*

(I) Limits of filters The set of neighborhoods of any point a in a topological space X is a filter $\mathfrak{V}(a)$.

DEFINITION 2.15.– *We say that the filter $\mathfrak{F}$ (resp. filter base $\mathfrak{B}$) converges to a (written $\mathfrak{F} \to a$, resp. $\mathfrak{B} \to a$) and that a is a* limit value *of $\mathfrak{F}$ (resp. $\mathfrak{B}$) whenever $\mathfrak{F}$ (resp. the filter generated by $\mathfrak{B}$) is finer than $\mathfrak{V}(a)$.*

Observe that $\mathfrak{F} \to a$ (resp. $\mathfrak{B} \to a$) if and only if $\mathfrak{F}$ (resp. $\mathfrak{B}$) is eventually in every neighborhood of a. The filter base $\mathfrak{V}(a)$ is called a *fundamental system of neighborhoods* of a.

(II) Cluster points of filters Let X be a topological space, $a \in X$, and let $\mathfrak{F}$ be a filter or filter base on X.

DEFINITION 2.16.– *The point a is said to be a* cluster point *of $\mathfrak{F}$ if every neighborhood of a intersects non-trivially with some element of $\mathfrak{F}$.*

The point a is a cluster point of the filter $\mathfrak{F}$ if and only if every neighborhood of a intersects non-trivially with some element of some filter base of $\mathfrak{F}$; this is equivalent to saying (**exercise**) that $\mathfrak{F}$ is frequently in every neighborhood of a (Definition 2.4). Hence, by Lemma 2.5:

COROLLARY 2.17.– *The point $a \in X$ is a cluster point of $\mathfrak{F}$ if and only if there exists a filter finer than $\mathfrak{F}$ that converges to a. In particular, the point a is a cluster point of the ultrafilter $\mathfrak{U}$ if and only if $\mathfrak{U}$ converges to a.*

(III) Limits and cluster points of nets Let X be a topological space, $a \in X$, and let $\mathfrak{x} = (x_i)_{i \in I}$ be a net in X.

DEFINITION 2.18.–

1) We say that $\mathfrak{x}$ converges to a (abbreviated to $\mathfrak{x} \to a$) and that a is a limit value *of $\mathfrak{x}$ if, for any neighborhood U of a, there exists an index i_0 such that $x_i \in U$ for all $i \succeq i_0$.*

2) We say that $a \in X$ is a cluster point *of $\mathfrak{x}$ if, for any neighborhood U of a and every $i_0 \in I$, there exists $i \succeq i_0$ such that $x_i \in U$.*

From Theorem 2.11 and Definitions 2.15, 2.16, we can deduce the next two corollaries:

COROLLARY 2.19.– *Let $\mathfrak{F}$ be a filter on a topological space X and let $a \in X$. The following conditions are equivalent:*

i) $\mathfrak{F}$ converges to a (resp. a is a cluster point of $\mathfrak{F}$).

ii) Every elementary filter finer than $\mathfrak{F}$ converges to a (resp. admits the cluster point a).

iii) Every net associated with $\mathfrak{F}$ converges to a (resp. admits the cluster point a).

COROLLARY 2.20.– *Let X be a topological space, $A \subset X$, and $a \in X$. The following conditions are equivalent:*

i) $a \in \overline{A}$.

ii) There exists a filter base $\mathfrak{B}$ on A that converges to a in X.

iii) There exists a net $\mathfrak{x}$ in A that converges to a in X.

Therefore, a subset A of a topological space is closed if and only if, for every net $(x_i)_{i \in I}$ such that $x_i \in A$ for all $i \in I$ that converges to some limit l in X, we also have that $l \in A$; this leads to the following definition:

DEFINITION 2.21.– *A subset A of a topological space is said to be* sequentially closed *if, given any sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in A$ for all $n \in \mathbb{N}$ which converges to some limit l in X, we also have that $l \in A$.*

(IV) UNIQUENESS OF LIMITS IN HAUSDORFF SPACES

LEMMA 2.22.– *Let X be a topological space. The following conditions are equivalent:*

i) Every filter $\mathfrak{F}$ (resp. filter base $\mathfrak{B}$, resp. net $\mathfrak{x}$) on X has at most one limit point.

ii) X is Hausdorff, i.e. satisfies the Hausdorff T_2 separation axiom.

If these conditions are satisfied, the unique limit is written $\lim \mathfrak{F}$ *(resp.* $\lim \mathfrak{B}$, *resp.* $\lim \mathfrak{x}$*).*

PROOF.– Sufficient condition: **exercise**; necessary condition: cf. ([BKI 74], Chapter I, section 8.1, Proposition 1). ■

(V) LIMITS AND CLUSTER POINTS (OF FUNCTIONS) WITH RESPECT TO A FILTER Let X and Y be two topological spaces, $\mathfrak{F}$ a filter on X, $f : X \to Y$ a mapping, and pick $y \in Y$.

DEFINITION 2.23.– *We say that* y *is a limit point of* f *with respect to the filter* $\mathfrak{F}$ *if the filter base* $f(\mathfrak{F})$ *converges to* y*. If so, we write* $y = \lim_{\mathfrak{F}} f$*.*

If $A \subset X$, $a \in \overline{A}$, and $\mathfrak{F}$ is the filter of traces on A of the filter $\mathfrak{V}(a)$, then $\lim_{\mathfrak{F}} f$ is written $\lim\limits_{x \to a, x \in A} f(x)$; if $X = \bar{\mathbb{R}}$, $a < +\infty$, and $A =]a, +\infty]$ (resp. $b > -\infty$ and $B = [-\infty, b[$), then we write $f(a+0)$ for $\lim\limits_{x \to a, x \in A} f(x)$ (resp. $f(b-0)$ for $\lim\limits_{x \to b, x \in B} f(x)$). If $\mathfrak{F}$ is the filter of traces on $\complement_X \{a\}$ of the filter $\mathfrak{V}(a)$, then $\lim_{\mathfrak{F}} f$ is written $\lim\limits_{x \to a, x \neq a} f(x)$, and so on.

Similarly, we say that $y \in Y$ is a *cluster point* of f with respect to the filter $\mathfrak{F}$ if y is a cluster point of the filter base $f(\mathfrak{F})$ (Definition 2.16).

(VI) UPPER AND LOWER LIMITS Let E be a set and $\mathfrak{F}$ a filter on E. Let $f : E \to \bar{\mathbb{R}}$ be a mapping and, for all $X \in E$, define $F(X) := \sup_{x \in X} f(x)$. The mapping $F : \mathfrak{F} \to \bar{\mathbb{R}}$ is decreasing when $\mathfrak{F}$ is equipped with the relation $\supset$, and therefore has a limit in $\bar{\mathbb{R}}$.

DEFINITION 2.24.– *The value* $\lim_{X \in \mathfrak{F}} (\sup_{x \in X} f(x))$ *is called the* upper limit *of* f *with respect to* $\mathfrak{F}$*, and is written* $\limsup_{\mathfrak{F}} f$*. Similarly, we define the* lower limit *of* f *with respect to* $\mathfrak{F}$*, which is written* $\liminf_{\mathfrak{F}} f$*. If* (x_n) *is a sequence in* E *and* $\mathfrak{F}$ *is the Fréchet filter (section 2.2.1*(**I**)*), then the upper and lower limits are written* $\limsup\limits_{n \to +\infty} x_n$ *and* $\liminf\limits_{n \to +\infty} x_n$ *respectively.*

The following result holds (**exercise***: cf. [BKI 74], Chapter IV, section 5.6, Theorem 3):

THEOREM 2.25.– *The upper (resp. lower) limit* $\limsup_{\mathfrak{F}} f$ *(resp.* $\liminf_{\mathfrak{F}} f$*) is equal to the greatest (resp. smallest) cluster point of* f *with respect to* $\mathfrak{F}$ *(cf. (*$\mathbf{V}$*)).*

2.3.3. *Continuity*

(I) Let X and Y be two topological spaces, suppose that $f : X \to Y$ is a mapping, and pick $a \in X$.

LEMMA 2.26.– *The following conditions are equivalent:*

i) For any neighborhood V of $f(a)$ in Y, there exists a neighborhood U of a in X such that $f(U) \subset V$.

ii) For every filter $\mathfrak{F}$ on X that converges to a, $f(\mathfrak{F})$ converges to $f(a)$.

iii) For every net $\mathfrak{x}$ in X that converges to a, $f(\mathfrak{x})$ converges to $f(a)$.

PROOF.– (i)⇒(ii): Let V be a neighborhood of $f(a)$ in Y and let U be a neighborhood of a such that $f(U) \subset V$. Suppose that $\mathfrak{F}$ is a filter that converges to a. Then $U \in \mathfrak{F}$, so $f(U) \in f(\mathfrak{F})$. Since $V \supset f(U)$, this implies that $V \in f(\mathfrak{F})$, and so $f(\mathfrak{F})$ converges to $f(a)$. The equivalence (ii)⇔(iii) holds by Theorem 2.11. (ii)⇒(i): If (ii) holds, let $\mathfrak{V}(a)$ be the filter of neighborhoods of a in X. Since $f(\mathfrak{V}(a))$ converges to $f(a)$, every neighborhood V of $f(a)$ in Y belongs to $f(\mathfrak{V}(a))$, so there exists $U \in \mathfrak{V}(a)$ such that $f(U) \subset V$. ■

DEFINITION 2.27.– *The mapping f is said to be* continuous *at the point a if one of the equivalent conditions in Lemma 2.26 is satisfied for $y = f(a)$. The mapping f is said to be* continuous *if it is continuous at every point $a \in X$.*

Condition (i) of Lemma 2.26 is the most common definition of continuity for a function at a point. Conditions (ii) and (iii) illustrate that continuity is the property of *preservation of convergence*. The reader may wish to show the next lemma as an **exercise*** (cf. [BKI 74], Chapter I, section 2.1, Theorem 1):

LEMMA 2.28.– *Let X, Y be topological spaces and let $f : X \to Y$ be a mapping. The following conditions are equivalent:*

i) f is continuous.

ii) For every open subset O_Y of Y, $f^{-1}(O_Y)$ is an open subset of X.

iii) For every closed subset F_Y of Y, $f^{-1}(F_Y)$ is a closed subset of X.

iv) For every subset A of X, $f\left(\overline{A}\right) \subset \overline{f(A)}$.

(II) CONSEQUENCES OF CONTINUITY

LEMMA 2.29.– *Let X, Y be two topological spaces. If $f : X \to Y$ is continuous and Y is Hausdorff, then the graph of f, which is defined as* $\mathrm{Gr}(f) = \{(x, f(x)) : x \in X\}$ *is closed.*

PROOF.– Suppose that the point (x, y) is in the closure of $\mathrm{Gr}(f)$ in $X \times Y$. By Corollary 2.20, there exists a net $(x_i, (f(x_i)))_{i \in I}$ in $\mathrm{Gr}(f)$ that converges to (x, y) in $X \times Y$. Therefore, $(x_i)_{i \in I} \to x$, and, since f is continuous, $(f(x_i))_{i \in I} \to f(x)$ (Lemma 2.26). Since Y is Hausdorff, $f(x) = y$, so $(x, y) \in \mathrm{Gr}(f)$, and thus this set is closed in $X \times Y$. ■

LEMMA 2.30.– *Let X, Y be two topological spaces and $f, g : X \to Y$ two continuous mappings. If Y is Hausdorff, then $A = \{x \in X : f(x) = g(x)\}$ is closed in X. Consequently, if A is dense in X, then $f = g$* (principle of extension of identities).

PROOF.– If $a \in \overline{A}$, then there exists a net $\mathfrak{x} = (x_i)_{i \in I}$ in A that converges to a in X (Corollary 2.20); therefore (Lemma 2.26), $f(\mathfrak{x}) \to f(a)$ and $g(\mathfrak{x}) \to g(a)$; but these limits are unique (Lemma 2.22), so $f(a) = g(a)$, which implies that $a \in A$. ■

(III) SEMI-CONTINUITY

DEFINITION 2.31.– *Let X be a topological space and f a mapping from X into $\bar{\mathbb{R}}$. We say that f is* lower *(resp.* upper*)* semi-continuous *if, $\forall a \in X$, $f^{-1}(]a, +\infty])$ (resp. $f^{-1}([-\infty, a[)$) is open in X.*

The following results are easy to check (**exercise**): if f, g are lower semi-continuous, then so are $f + g$ and fg (provided that these functions are everywhere defined). If $(f_i)_{i \in I}$ is a family of lower semi-continuous mappings from X into $\bar{\mathbb{R}}$, then $\sup_{i \in I} f_i$ is also lower semi-continuous. A subset A of X is open in X if and only if its characteristic function $\chi_A : X \to \{0, 1\}$ (defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$) is lower semi-continuous.

2.3.4. *Category of topological spaces*

(I) As in metric spaces, the composition of two continuous mappings is continuous (**exercise**). This property allows us to define the category **Top** of topological spaces, whose morphisms are the continuous mappings. The category $\mathbf{Top}$ is concrete ([P1], section 1.3.1), so each topology determines a (topological) structure; the topology $\mathfrak{T}_2$ on a set X is finer than the topology $\mathfrak{T}_1$ if and only if the (topological) structure determined by $\mathfrak{T}_2$ is finer than the structure determined by $\mathfrak{T}_1$ (cf. ([P1], section 1.3.1) and section 2.3.1**(III)**, above).

The general notions of initial structure and final structure ([P1], section 1.3.2) are realized in $\mathbf{Top}$ by the notions of initial topology and final topology. If $(X, \mathfrak{T})$ is a topological space and $\varphi : X \to X'$ is a mapping, the final topology X' for φ is the finest topology for which φ is continuous. If f is a mapping from a set X into a topological space $(X', \mathfrak{T}')$, the initial topology $\mathfrak{T}$ for $f : X \to X'$ is the coarsest topology X for which f is continuous, and $\mathfrak{T} = \left\{ f^{-1}(O') : O' \in \mathfrak{T}' \right\}$; the topology $\mathfrak{T}$ is also called the *inverse image* of the topology $\mathfrak{T}'$ under f.

(II) Induced topology In the case where X is a topological space and $A \subset X$, the *induced topology* on A is the set of traces on A of the open sets of X (the set of $A \cap O$ such that O is open in X); this is the initial topology for the inclusion mapping $\iota : A \hookrightarrow X$, and A is called a *subspace* of X when equipped with this topology.

(III) Open mappings, closed mappings, homeomorphisms Let X, Y be two topological spaces. A mapping $f : X \to Y$ is said to be *open* (resp. *closed*) if the image $f(A)$ of any open (resp. closed) set A in X is open (resp. closed) in Y. An isomorphism of topological spaces ([P1], section 1.1.1**(III)**), known as a *homeomorphism*, is a bijective and continuous mapping f such that f^{-1} is continuous, i.e. such that f is an open mapping (or, equivalently in this context, a closed mapping). Let $a \in X$; the mapping f is a *local homeomorphism* in a neighborhood of a if there exists a neighborhood U of a in X and a neighborhood V of $b = f(a)$ in Y such that f induces a homeomorphism from U onto V. The mapping $f : X \to Y$ is continuous if and only if $p : \mathrm{Gr}(f) \to X : (x, f(x)) \mapsto x$ is an open mapping (**exercise**).

(IV) Quotient topology Let X be a topological space and suppose that $\sim$ is an equivalence relation on X. The *quotient topology* on $X/\sim$ is the final topology for the surjection mapping $\pi : X \twoheadrightarrow X/\sim$ ([P1], section 1.1.2**(VI)**). When equipped with this topology, $X/\sim$ is called a *quotient* *space* of X. We say that the equivalence relation $\sim$ is Hausdorff if the quotient space $X/\sim$ is Hausdorff. The equivalence relation $\sim$ is said to be *open* (resp. closed) if π is open (resp. closed). For an open equivalence relation $\sim$ to be Hausdorff, it is necessary and sufficient for the graph $\mathrm{Gr}(\sim)$ of $\sim$ (defined by $\mathrm{Gr}(\sim) = \{(x,y) \in X \times X : x \sim y\}$) to be closed in $X \times X$ ([BKI 74], Chapter I, section 8.3, Proposition 8).

2.3.5. *Products and projective limits of topological spaces*

(I) PRODUCTS Let $(X_i)_{i \in I}$ be a family of non-empty topological spaces, and write $(|X|_i)_{i \in I}$ for the family of underlying sets, as well as $|X| = \prod_{i \in I} |X_i|$ for the product set of the family $(|X|_i)_{i \in I}$ ([P1], section 1.2.6(**I**)). The initial topology of $(\mathrm{pr}_i : |X| \twoheadrightarrow |X_i|)_{i \in I}$, which is the *coarsest* topology on the set $|X|$ for which the canonical projection mappings pr_i are continuous ([P1], section 1.3.2(**I**)), is called the *product topology*[5]. One base of this topology is given by the finite intersections of sets $\mathrm{pr}_i^{-1}(O_i)$ such that each O_i is open in X_i. With this topology, we write $|X|$ as $X = \prod_{i \in I} X_i$; products in the concrete category $(\mathbf{Top}, |.|, \mathbf{Set})$ are concrete ([P1], section 1.3.3(**I**)). The topological space $X = \prod_{i \in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff (the proof of the sufficient condition is an **exercise**; for the necessary condition, cf. [SCW 93], Volume I, Theorem 2.2.39). A *countable* product of separable topological spaces is separable (**exercise**). Given a family of non-empty sets $(X_i)_{i \in I}$ and a filter $\mathfrak{F}_i$ on X_i for each $i \in I$, we write $\prod_{i \in I} \mathfrak{F}_i$ for the filter on $\prod_{i \in I} X_i$ with base given by the set of subsets of the form $\prod_{i \in I} A_i$ such that $A_i \in \mathfrak{F}_i$ and $A_i = X_i$ except for finitely many indices. For $\prod_{i \in I} \mathfrak{F}_i$ to converge to $a = (a_i)_{i \in I}$, it is necessary and sufficient for $\mathfrak{F}_i$ to converge to a_i for all $i \in I$ (**exercise**).

Given two metric spaces X_1, X_2 with metrics d_1, d_2 respectively, the topological product space $X_1 \times X_2$ is also a metric space for either of the metrics d, d' defined as follows: for any pair of points $x = (x_1, x_2)$, $y = (y_1, y_2)$,

$$\begin{aligned} d(x,y) &= \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}, \\ d'(x,y) &= d_1(x_1,y_1) + d_2(x_2,y_2). \end{aligned} \qquad [2.5]$$

5 Readers who find this notation unhelpful (either here or in section 2.3.6) can safely ignore the $|.|$ (see footnote 5 on page 132).

(II) PROJECTIVE LIMITS Every double arrow $A \underset{g}{\overset{f}{\rightrightarrows}} B$ has a concrete equalizer; as was the case in $\mathbf{Set}$, this is the canonical injection $\kappa : K \hookrightarrow A$, where $K = \{x \in A : f(x) = g(x)\}$ is equipped with the topology induced by the topology of A. Hence, by ([P1], sections 1.2.8**(II)** and 1.3.3**(II)**):

PROPOSITION 2.32.– *The category* $\mathbf{Top}$ *admits arbitrary concrete* projective limits.

More precisely, the proof of ([P1], section 1.2.8**(II)**, Proposition 1.22) shows that

$$\boxed{\varprojlim_{i \in I} X_i = \left\{ x \in \prod_{i \in I} X_i : \psi_i^j \left(\mathrm{pr}_j(x)\right) = \mathrm{pr}_i(x), \forall i \preceq j \right\}, \ \psi_i : \varprojlim_{j \in I} X_j \to X_i,} \qquad [2.6]$$

where $\left\{X_i, \psi_i^j; I\right\}$ is the relevant inverse system of these topological spaces and $\psi_i = \psi_i^j \circ \psi_j$ $(j \succeq i)$; by Lemma 2.30, this implies:

COROLLARY 2.33.– *If each of the* X_i *is Hausdorff, then* $\varprojlim_{i \in I} X_i$ *is a closed subset of* $\prod_{i \in I} X_i$.

REMARK 2.34.–

1) The expression in (2.6) *also holds for projective limits in the category* $\mathbf{Vec}$ *of vector spaces; this expression will once again hold for projective limits in concrete categories on* $\mathbf{Vec}$ *that we shall consider later (*$\mathbf{Tvs}$ *and* $\mathbf{Lcs}$ *: cf. sections 3.2.2 and 3.3.5), equipping* $\prod_{i \in I} X_i$ *with the initial structure for the projections* pr_i *and* $\varprojlim_{i \in I} X_i$ *with the initial structure for the canonical injection* $\varprojlim_{i \in I} X_i \hookrightarrow \prod_{i \in I} X_i$.

2) In each of these categories (including $\mathbf{Top}$*), subspaces and products are in fact special cases of projective limits.*

2.3.6. *Coproducts and inductive limits of topological spaces*

We saw before that "dualizing" the notion of product in a category yields the notion of coproduct ([P1], section 1.2.6**(II)**). This construction also works in $\mathbf{Top}$, this time yielding the notion of coproduct (or sum) $X = \coprod_{i \in I} X_i$ of a family of topological spaces $(X_i)_{i \in I}$. The coproduct topology is the final topology for $(\mathrm{inj}_i : |X_i| \hookrightarrow |X|)_{i \in I}$, which is the finest topology on the set $|X|$ for which the canonical injections are continuous. The coproducts thus obtained are concrete. The space $\coprod_{i \in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff (**exercise**).

The coequalizer of a double arrow $A \overset{f}{\underset{g}{\rightrightarrows}} B$ may be constructed in the same way in $\mathbf{Top}$ as in $\mathbf{Set}$ ([P1], section 1.1.1**(III)**) by equipping the coequalizer $|\mathrm{coeq}\,(f, g)|$ in $\mathbf{Set}$ with the final topology for the canonical surjection $\gamma : |B| \twoheadrightarrow |\mathrm{coeq}\,(f, g)|$. By ([P1], section 1.2.8**(II)**, Proposition 1.22, and section 1.3.3**(II)**):

PROPOSITION 2.35.– *The category* $\mathbf{Top}$ *admits arbitrary concrete inductive limits.*

2.3.7. *Compactness*

Let X be a set and $\mathfrak{A} = (A_i)_{i \in I}$ a family of subsets of X. This family is said to be a *covering* of X if its union is X. If X is a topological space and the A_i are open (resp. closed), $\mathfrak{A}$ is said to be an *open covering* (resp. *closed covering*).

LEMMA 2.36.– *Let X be a topological space. The following conditions are equivalent:*

i) Every covering of X consisting of open sets has a finite subcovering (this is known as the Borel-Lebesgue property).

i') Every family of closed subsets of X with empty intersection contains a finite subset with empty intersection.

ii) Every filter on X has a cluster point.

iii) Every ultrafilter on X is convergent.

iv) Every net in X has a cluster point[6]*.*

v) Every ultranet on X is convergent.

6 The equivalence (i)⇔(iv) is known as the Bolzano-Weierstrass *theorem* in the case where X is a metric space, in which case this net is a sequence.

PROOF.– (i)⇔(i') by taking complements, and conditions (ii), (iii), (iv), (v) are all equivalent by Theorem 2.11, Corollary 2.17, and the definitions. For (i')⇒(iii) and (iv)⇒(i'), we argue by contradiction:

(i')⇒(iii): Let $\mathfrak{U}$ be a non-convergent ultrafilter and suppose that $x \in \bigcap_{A \in \mathfrak{U}} \overline{A}$. Then every neighborhood of x intersects non-trivially with the $A \in \mathfrak{U}$ and hence $\mathfrak{U} \to x$, which is a contradiction. Hence, $\bigcap_{A \in \mathfrak{U}} \overline{A} = \varnothing$. If there exist $A_1, ..., A_n$ such that $\bigcap_{i=1}^{n} \overline{A}_i = \varnothing$, then it is certainly true that $\bigcap_{i=1}^{n} A_i = \varnothing$, which is impossible, since $\mathfrak{U}$ is a filter. Therefore, condition (i') is not satisfied.

(iv)⇒(i'): Let $\mathfrak{G}$ be a family of closed sets with empty intersection. If every finite subfamily $\mathfrak{G}_\alpha$ $(\alpha \in A)$ has empty intersection, then, for each $\alpha \in A$, there exists an element x_α in the intersection of the elements of $\mathfrak{G}_\alpha$ (by the axiom of choice). By Zermelo's theorem ([P1], section 1.1.2**(III)**, Theorem 1.5), there exists a well-ordering relation on A. Therefore, $(x_\alpha)_{\alpha \in A}$is a net, which by (iv) has a cluster point that belongs to every set $A \in \mathfrak{G}$, contradiction.

■

DEFINITION 2.37.– *A topological space X is said to be* quasi-compact *if one of the equivalent conditions in Lemma 2.36 is satisfied. A topological space is said to be* compact *if it is both quasi-compact and Hausdorff.*

If X is a compact space and $A \subset X$ is closed in X, then the subspace A is compact, since every ultranet in A converges in A. If X is a Hausdorff topological space and the subspace $A \subset X$ is compact, then A is closed in X; indeed, if $a \in \overline{A}$, then there exists an ultranet $\mathfrak{x}$ in A that converges to a in X (Corollary 2.20), and $\mathfrak{x}$ converges in A (Lemma 2.36, Property (v)), so $a \in A$. A subset A of a Hausdorff topological space X is said to be *relatively compact* if its closure $\overline{A}$ in X is compact. We know that any closed and bounded interval in $\mathbb{R}$ (with the usual topology) is compact (Borel-Lebesgue theorem).

LEMMA 2.38.– *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in a topological space X that converges to some point a in X. Then $\{x_n : n \in \mathbb{N}\} \cup \{a\}$ is a quasi-compact subspace of X, which furthermore is compact if X is Hausdorff (**exercise**).*

THEOREM 2.39.– *Let X, Y be topological spaces, let A be a quasi-compact subset of X, and suppose that $f : X \to Y$ is a continuous mapping. Then $f(A)$ is quasi-compact in Y. If X and Y are also Hausdorff, and A is a* *compact (resp. relatively compact) subset of X, then $f(A)$ is a compact (resp. relatively compact) subset of Y. In particular, if X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then it has maximum and minimum values.*

PROOF.– Thanks to Lemma 2.28, we simply need to prove (without requiring any separation hypotheses) that $f(A)$ is quasi-compact whenever A is quasi-compact. If $(O_i)_{i \in I}$ is an open covering $f(A)$, then $\left(f^{-1}(O_i)\right)_{i \in I}$ is an open covering of A; thus, there exists a finite subcovering $\left(f^{-1}(O_{i_k})\right)_{1 \le k \le n}$ of A, and $(O_{i_k})_{1 \le k \le n}$ is a finite open covering of $f(A)$. ■

COROLLARY 2.40.– *Let X be a set, and let $\mathfrak{T}_1, \mathfrak{T}_2$ be two topologies on X such that $\mathfrak{T}_1$ is finer than $\mathfrak{T}_2$. Then any quasi-compact subset of $(X, \mathfrak{T}_1)$ is a quasi-compact subset of $(X, \mathfrak{T}_2)$. If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are also Hausdorff, then every compact (resp. relatively compact) subset of $(X, \mathfrak{T}_1)$ is a compact (resp. relatively compact) subset of $(X, \mathfrak{T}_2)$.*

COROLLARY 2.41.– *Let X, Y be two Hausdorff topological spaces and $f : X \to Y$ a continuous bijection. If X is compact, then f is a homeomorphism.*

PROOF.– If $A \subset X$ is closed in X, then it is compact, so $f(A)$ is compact, and hence closed in Y. Therefore, f^{-1} is continuous (Lemma 2.28). ■

THEOREM 2.42.– *Let X be a non-empty compact space and $f : X \to \bar{\mathbb{R}}$ a lower (resp. upper) semi-continuous function. Then f attains its lower (resp. upper) bound.*

PROOF.– For all $y \in f(X)$, the $A_y = f^{-1}([-\infty, y])$ are closed, and any intersection of finitely many of them is non-empty. Therefore, the intersection of all of them is non-empty (Lemma 2.36, Property (i')). Let a be an element in this intersection. Then, for all $x \in X$, $f(a) \le f(x)$. ■

THEOREM 2.43.– *(Tychonov) If $(X_i)_{i \in I}$ is a family of non-empty topological spaces, then $\prod_{i \in I} X_i$ is quasi-compact (resp. compact) if and only if X_i is quasi-compact (resp. compact).*

PROOF.– (H. Cartan [CAR 37]) We know that $\prod_{i \in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff. If $\prod_{i \in I} X_i$ is quasi-compact, then, for all $i \in I$, $X_i = \mathrm{pr}_i(X)$ is compact, since pr_i is continuous (Theorem 2.39). Conversely, suppose that the X_i are quasi-compact. Let $\mathfrak{U}$ be an ultrafilter on X. For every $i \in I$, $\mathrm{pr}_i(\mathfrak{U})$ is an ultrafilter base on X_i (section 2.3.5(**I**)), and so converges (Lemma 2.36), which implies that $\mathfrak{U}$ converges. ■

It follows that any closed hypercube in $\mathbb{R}^n$ is compact (with the usual topology).

COROLLARY 2.44.– *If* $\left\{X_i, \psi_i^j; I\right\}$ *is an inverse system of compact spaces, then* $\varprojlim X_i$ *is a compact space.*

PROOF.– This follows from Theorem 2.43 and Corollary 2.33. ■

2.3.8. *Connectedness*

Readers will already be familiar with the notion of arc-connectedness ([P1], section 3.3.8(**VII**)). Connectedness is somewhat similar, but weaker in general.

DEFINITION 2.45.– *A topological space X is said to be* connected *if it cannot be expressed as the union of two disjoint non-empty open sets.*

By taking complements, this is equivalent to saying that X is not the union of two disjoint non-empty closed sets. Intuitively, a connected space is formed from a single piece. For example, the union of two (open or closed) disjoint discs in $\mathbb{R}^2$ is not connected. A subset A of $\mathbb{R}$ or $\bar{\mathbb{R}}$ is connected if and only if A is an interval ([BKI 74], Chapter IV, section 4.2, Proposition 5). If X is the union of a family of connected spaces with non-empty intersection, then X is connected (**exercise***: cf. [BKI 74], Chapter I, section 11.1, Proposition 2). However, the intersection of two connected sets is *not* necessarily connected (for example, consider the intersection of a disk and a horseshoe).

THEOREM 2.46.– *If X, Y are two topological spaces, X is connected, and $f : X \to Y$ is continuous, then $f(X)$ is connected (**exercise**).*

It follows that every quotient of a connected space is connected. It can be shown that the product of non-empty connected spaces X_i is connected if and only if each of the X_i is connected ([BKI 74], Chapter I, section 11.4, Proposition 8). It is worth comparing this property and Theorem 2.46 with Theorems 2.43 and 2.39.

COROLLARY 2.47.– *Let X be a topological space. If X is arc-connected, then it is connected.*

PROOF.– If $X = \varnothing$, then X is connected. Suppose that $X \neq \varnothing$ and that X is arc-connected; then X is the union of the images of the paths to every point from one of its point. Each of these images is connected, since the interval $[0, 1]$ is connected in $\mathbb{R}$, and they have non-empty intersection; by Theorem 2.46 and the observation made immediately prior to this theorem, X is connected. ■

Let X be a topological space and pick $x \in X$. The union of the connected subsets of X that contain x is connected, and is called the *connected component* of x. A topological space X is said to be *locally connected* if every point has a fundamental system of connected neighborhoods. A connected space is locally connected, but the converse may fail.

2.3.9. *Locally compact spaces*

(I) A Hausdorff topological space X is said to be *locally compact* if every point of X has a compact neighborhood. It is clear that every compact space is locally compact, but the converse is not true (for example, $\mathbb{R}^n$ is locally compact but is not compact). In a locally compact space, every point has a fundamental system of compact neighborhoods, and more generally every compact set has a fundamental system of compact neighborhoods (**exercise**). Therefore, every locally compact space X satisfies the T_3 *separation axiom* : for every point $a \in X$ and every closed set F in X that does not contain a, there exist disjoint neighborhoods of a and F.

Any space that satisfies both the T_1 and T_3 separation axioms is necessarily T_2 (**exercise**).

DEFINITION 2.48.– *A topological space is said to be* regular *if it satisfies both the T_1 and T_3 separation axioms.*

Locally compact spaces are therefore regular.

(II) It is possible to show the following result ([BKI 74], Chapter I, section 9.8, Theorem 4):

THEOREM 2.49.– *(Alexandrov) Let X be a locally compact space with topology $\mathfrak{T}$.*

1) There exists a point $\infty \notin X$ and, writing $X^{\infty} := X \cup \{\infty\}$, a set $\mathfrak{T}^{\infty}$ of subsets of X^{∞} consisting of elements of $\mathfrak{T}$ and of the set $\left\{\complement_{X^{\infty}} K : K \in \mathfrak{K}\right\}$, where $\mathfrak{K}$ is the set of compact subsets of X, such that the following conditions hold:

i) $\mathfrak{T}^{\infty}$ is a topology on X^{∞}.

ii) $\mathfrak{T}^{\infty}$ induces the topology $\mathfrak{T}$ on X.

iii) The topological space $(X^{\infty}, \mathfrak{T}^{\infty})$ is compact.

2) The topological space $(X^{\infty}, \mathfrak{T}^{\infty})$ that satisfies these conditions is unique up to homeomorphism.

DEFINITION 2.50.– *The topological space $(X^{\infty}, \mathfrak{T}^{\infty})$ is called the* one-point compactification *of the locally compact space $(X, \mathfrak{T})$.*

Part (2) of Theorem 2.49 shows that the one-point compactification is "essentially unique"[7]. The point ∞ is called the "point at infinity" of X^{∞}. If X is compact, then $X^{\infty} = X \uplus \{\infty\}$.

DEFINITION 2.51.– *A locally compact space X is said to be* countable at infinity *if it is the union of* countably many *compact subsets.*

Compact spaces are countable at infinity. The space $\mathbb{R}^n$ is countable at infinity, since it is locally compact and may be written as the union of the $B^c(0; m)$ such that $B^c(0; m) = \{x \in \mathbb{R}^n : \|x\| \leq m\}$. Given two sets A, B in a topological space X, we write $A \Subset B$ if A is contained in the interior of B. Then the following result can be shown (**exercise***: cf. [BKI 74], Chapter I, section 9.9, Proposition 15 and Corollary 1):

LEMMA 2.52.– *Let X be a locally compact space that is countable at infinity. Then there exists a sequence $(K_n)_{n \geq 1}$ of compact subsets of X such that $K_n \Subset K_{n+1}$ and $\bigcup_{n \geq 1} K_n = X$.*

With the notation of Definition 2.50, it is possible to show the following result ([BKI 74], Chapter IX, section 2.9, Proposition 16 and its corollary):

THEOREM 2.53.–

7 Despite this property, the one-point compactification is not the solution of a universal problem.

1) For a compact space X to be metrizable, it is necessary and sufficient for its topology to have a countable base.

2) Let X be a locally compact space. The following conditions are equivalent.

i) The topology of X has a countable base.

ii) X^∞ is metrizable.

iii) X is metrizable and countable at infinity.

Example The one-point compactification of the real line $X = \mathbb{R}$ (up to homeomorphism) may be constructed as follows: consider the unit circle S^1 in the plane $\Delta = \mathbb{R}^2$ and write ∞ for the point with coordinates $(0,1)$ in Δ. Now send every point $x \in \mathbb{R}$ to the point $P(x)$ at the intersection of S^1 and the straight line between ∞ and $(x,0)$ (Figure 2.1). Then $\{P(x) : x \in \mathbb{R}\} = \complement_{S^1}\{\infty\}$ and $X^\infty = S^1$. The homeomorphism $\complement_{S^1}\{\infty\} \tilde{\rightarrow} \mathbb{R}$ (where S^1 is equipped with the topology induced by the topology of $\mathbb{R}^2$) is known as *stereographic projection*. Note that the extended real line $\bar{\mathbb{R}}$, which is metrizable and compact ([DIE 82], Volume I, (3.3.2)), is not the same as the one-point compactification S^1, since $\bar{\mathbb{R}}$ in constructed from $\mathbb{R}$ by adding the *two* points $-\infty$ and $+\infty$. The one-point compactification of the plane $\mathbb{R}^2$ can be constructed similarly (or, equivalently, the one-point compactification of the complex plane $\mathbb{C}$, which yields the topological space known as the Riemann *sphere*: cf. [CAR 61], section III.5.1).

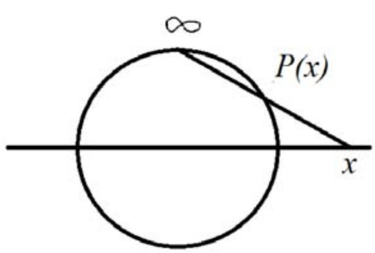

Figure 2.1. *Stereographic projection*

2.3.10. *Paracompact spaces*

A family $(A_i)_{i \in I}$ of subsets of a topological space X is said to be *locally finite* if, for all $x \in X$, there exists a neighborhood V of x such that $A_i \cap V = \varnothing$ except for finitely many indices $i \in I$. The notion of paracompact space, weaker than the notion of compact space, was introduced by J. Dieudonné (1944).

DEFINITION 2.54.– *A topological space X is said to be* paracompact *if it is Hausdorff and, for any open covering $\mathfrak{R}$ of X, there exists a* locally finite open covering $\mathfrak{R}'$ *that is* finer *than* $\mathfrak{R}$.

It is possible to show the following result ([BKI 74], Chapter I, section 9.10, Proposition 18 and Theorem 2):

THEOREM 2.55.– *The coproduct of a family of paracompact spaces is a paracompact space. For a space to be paracompact, it is necessary and sufficient for it to be the coproduct of a family of locally compact spaces each of which is countable at infinity.*

In particular, a locally compact space that is countable at infinity is paracompact, and, consequently (**exercise***: cf. [HOC 61], Theorem 2-66):

COROLLARY 2.56.– *Every locally compact space whose topology has a countable base is paracompact.*

Closed subspaces of a paracompact space are paracompact (**exercise**). The following result can also be shown ([BKI 74], Chapter IX, section 4.5, Theorem 4):

THEOREM 2.57.– *(Stone) Metrizable spaces are paracompact.*

A topological space X is said to be locally metrizable if each point of X admits a neighborhood which is homeomorphic to a metrizable space. The following result was obtained by Y. Smirnov in 1951:

THEOREM 2.58.– *(Smirnov's metrization theorem) A topological space is metrizable if and only if it is paracompact and locally metrizable.*

2.3.11. *Normal spaces*

We say that a topological space satisfies the T_4 or $\mathrm{O_V}$ separation axiom if, for any disjoint closed sets A and B in X, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

DEFINITION 2.59.– *A topological space X is said to be* normal *if it is T_1 and T_4.*

Every normal space is T_2 (**exercise**). Let X be a topological space; the following conditions are equivalent (**exercise**): (i) every subspace of X is T_4; (ii) for any subsets A, B of X such that $A \cap \overline{B} = \overline{A} \cap B = \varnothing$, there exist two disjoint open sets U, V such that $A \subset U$ and $B \subset V$. A topological space that satisfies these properties is said to be T_5. A space that is T_5 and T_1 is said to be *completely normal*. Metrizable spaces are completely normal (**exercise**).

THEOREM 2.60.– *Every paracompact space is normal.*

PROOF.– For the general case, cf. ([BKI 74], Chapter IX, section 4.4. Proposition 4). In the case of a metrizable space, let d be the metric that induces the topology of X and let A, B be two disjoint closed sets in X. Since the functions $x \mapsto d(x, A)$ and $x \mapsto d(x, B)$ are continuous (section 2.1.1), $U = \{x \in X : d(x, A) < d(x, B)\}$ and $V = \{x \in X : d(x, B) < d(x, A)\}$ are disjoint open sets such that $A \subset U$ and $B \subset V$. ■

2.3.12. *Continuous partitions of unity*

Let X be a topological space and let $f : X \to \mathbb{K}$ be a mapping. The *support* of f, written $\operatorname{supp}(f)$, is the largest closed set such that $f(x) \neq 0$ on $\complement_X S$ (this notion is analogous to the support of a family: cf. [P1], section 2.2.1(**IV**)).

DEFINITION 2.61.– *Let X be a topological space and let $(U_i)_{i \in I}$ be an open covering of X. A* partition of unity *on X subordinate to $(U_i)_{i \in I}$ is a family of functions $\psi_i : X \to \mathbb{R}$ such that:*

i) $\operatorname{supp}(\psi_i) \subset U_i$.

ii) $(\operatorname{supp}(\psi_i))_{i \in I}$ *is a locally finite family.*

iii) $\sum_{i \in I} \psi_i = 1$. *This partition of unity is said to be* continuous *if the ψ_i are continuous.*

The following result can be shown ([BKI 74], Chapter IX, section 4.3, Theorem 3, and section 4.4, Corollary 1):

THEOREM 2.62.– *Let X be a paracompact (resp. normal) space. Given any open (resp. locally finite open) covering $(U_i)_{i\in I}$ of X, there exists a continuous partition of unity subordinate to $(U_i)_{i\in I}$.*

2.3.13. *Germs of mappings*

DEFINITION 2.63.– *Let X be a topological space, Y a set, A a non-empty subset of X, and $(V_i)_{i\in I}$ a left directed family of open neighborhoods of A (i.e. $V_j \supset A$ and $V_i \subset V_j$ if $i \succeq j$) such that $\bigcap_{i\in I} V_i = A$. Let E be a set of mappings from X into Y and, for all $i \in E$, let F_i be the set of restrictions $f\,|_{V_i}$ of the mappings $f \in E$. For any pair $(i,j) \in I \times I$ such that $i \succeq j$, we have that $F_i = \rho_i^j(F_j)$, where ρ_i^j is the restriction $F_j \hookrightarrow F_i : f\,\big|_{V_j} \mapsto f\,|_{V_i}$. The inductive limit $\varinjlim_{i\in I} F_i$ (in $\mathbf{Set}$) of the direct system $\mathfrak{D} = \left(F_i, \rho_i^j; I\right)$ is called the set of* germs of the mappings $f \in E$ in a neighborhood of A.

Let $\rho_i : F_i \hookrightarrow \varinjlim_{i\in I} F_i$ be the canonical mapping. A representative of a germ $[f] \in \varinjlim_{i\in I} F_i$ is a mapping $f_i \in F_i$ such that $\rho_i(f_i) = [f]$ (the relation $\rho_i(f_i) = \rho_i(f_i')$ is an equivalence relation $\sim$, and the class of $f_i \pmod{\sim}$ may be identified with the germ $[f]$).

2.4. Uniform structures

2.4.1. *Entourages and uniformizable topologies*

(I) ENTOURAGES In a metric space (X,d), for $r > 0$, the subsets $U_d(r)$ and $U_d^c(r)$ of $X \times X$ defined by

$$U_d(r) \ := \left\{\left(x', x''\right) \in X \times X : d\left(x', x''\right) < r\right\},$$
$$U_d^c(r) \ := \left\{\left(x', x''\right) \in X \times X : d\left(x', x''\right) \leq r\right\},$$

may be used to define when the two points x, x' are considered to be close. This situation is generalized by the notion of *uniform structure* $\mathfrak{U}$ on a set X; this is specifically a *filter* on $X \times X$ (section 2.2.1**(I)**) whose elements are called *entourages* of X (section 2.1.3**(III)**). An entourage $V \in \mathfrak{U}$ is a subset of

$X \times X$ that satisfies the following conditions, which generalize the conditions (**D**$_1$), (**D**$_2$), (**D**$_3$) satisfied by metrics (section 2.1.1):

(**E**$_1$) $(\forall V \in \mathfrak{U}), (\forall x \in X) : (x, x) \in V$,

(**E**$_2$) $V \in \mathfrak{U} \Rightarrow V^{-1} \in \mathfrak{U}$, where $V^{-1} := \{(y, x) : (x, y) \in V\}$,

(**E**$_3$) $(\forall V \in \mathfrak{U}), (\exists W \in \mathfrak{U}) : W \circ W \subset V$, where $W \circ W' := \{(x, y) \in X \times X, \exists z \in X : (x, z) \in W' \;\&\; (z, y) \in W\}$.

In the case of a metric space (X, d), the subsets $\mathfrak{B} = \{U_d(r) : r > 0\}$ and $\mathfrak{B}^c = \{U_d^c(r) : r > 0\}$ of $\mathfrak{P}(X \times X)$ are both bases of the filter $\mathfrak{U}$ (**exercise**). In general, a base of $\mathfrak{U}$ is called a *fundamental system of entourages* of the uniform space $(X, \mathfrak{U})$ (often simply written X when doing so is not ambiguous). Condition (**E**$_3$) is similar to the triangle inequality (**D**$_3$). We inductively define $\overset{n}{W}$ $(n \geq 1)$ using the relations $\overset{n}{W} = \overset{n-1}{W} \circ W$, $\overset{1}{W} = W$, and we set $\overset{-1}{W} = \{(y, x) : (x, y) \in W\}$. An entourage W is said to be *symmetric* if $W = \overset{-1}{W}$.

(II) Uniformizable topologies For each entourage $V \in \mathfrak{U}$, let $V(x) := \{y \in X : (x, y) \in V\}$. There exists precisely one topology $\mathfrak{T}$ on X such that $\mathfrak{V}(x) := \{V(x) : V \in \mathfrak{U}\}$ is the filter of neighborhoods of x for $\mathfrak{T}$ (**exercise***: cf. [BKI 74], Chapter II, section 1.2, Proposition 1); a set O is an open set for $\mathfrak{T}$ if and only if, for each $x \in O$, there exists $V \in \mathfrak{U}$ such that $V(x) \subset O$. This topology $\mathfrak{T}$ is said to be *deduced from the uniform structure* $\mathfrak{U}$. We say that $\mathfrak{U}$ is Hausdorff if $\mathfrak{T}$ is Hausdorff. A topological space whose topology $\mathfrak{T}$ may be deduced from a uniform structure $\mathfrak{U}$ is said to be *uniformizable*. It is possible to show that ([BKI 74], Chapter II, section 1.2, Proposition 2 and Corollary 2):

Lemma 2.64.–

i) Let $\mathfrak{U}$ be a uniform structure on a set X. For any symmetric entourage $V \in \mathfrak{U}$ and any point $z = (x, y) \in X \times X$, $V \circ \{z\} \circ V$ is a neighborhood of z in the topological product space $X \times X$.

ii) Every uniform structure $\mathfrak{U}$ on a set X has a fundamental system of symmetric and open (resp. closed) entourages in the topological space $X \times X$, which are more precisely interiors (resp. closures) of entourages of this *structure. If* $V \in \mathfrak{U}$*, then, for any integer* $n \geq 1$*, there exists a symmetric entourage* $W \in \mathfrak{U}$ *such that* $\overset{n}{\overline{W}} \subset V$ *(resp.* $\overset{\overset{n}{\circ}}{W} \subset V$*).*

The next result gives a criterion that allows us to determine whether a space is uniformizable ([BKI 74], Chapter IX, section 1.5, Theorem 2):

THEOREM 2.65.– (A. *Weil*[8]*) A topological space* X *is uniformizable if and only if the condition* O_{IV} *below is satisfied:*

O_{IV} : *For every point* $x \in X$ *and every neighborhood* V *of* x*, there exists a continuous function* $f : X \to [0,1]$ *such that* $f(x) = 0$ *and* $f(y) = 1$ *for every* $y \in \complement_X V$*.*

A uniformizable topological space is T_0 if and only if it is T_2 (**exercise***: cf. [BKI 74], Chapter II, section 1, Exercise 4), in which case it is known as a *completely regular space* (or a Tychonov $\mathbf{T}$-space). The condition $\mathrm{O_{IV}}$ is known as the axiom of complete regularity. This implies:

LEMMA 2.66.– *Let* X *be a topological space whose topology* $\mathfrak{T}$ *is uniformizable (i.e. which satisfies* O_{IV}*). With this topology, the following equivalencies of separation axioms hold:* $T_0 \Leftrightarrow T_1 \Leftrightarrow T_2 \Leftrightarrow T_3 \Leftrightarrow \mathbf{T}$.

It can be shown ([BKI 74], Chapter IX, section 4.2, Theorem 2) that:

THEOREM 2.67.– *(Urysohn) In a topological space, the* T_4 *separation condition (section 2.3.11) is equivalent to:*

T_4' : *For any arbitrary disjoint closed sets* A *and* B*, there exists a continuous mapping from* X *into* $[0,1]$ *that is* 0 *at every point in* A *and* 1 *at every point in* B*.*

It is clear that (T_4' & T_1)$\Rightarrow \mathrm{O_{IV}}$, so normal spaces are uniformizable. Locally compact spaces are completely regular ([BKI 74], Chapter II, section 4.1, Corollary 2), and we saw in Theorem 2.60 that paracompact spaces (in particular, locally compact spaces that are countable at infinity) are normal.

8 [WEI 37], Theorem 1, p. 13, and p. 16.

2.4.2. *Uniform continuity*

DEFINITION 2.68.– *Let X, X' be two uniform spaces and let $f : X \to X'$. The mapping f is said to be* uniformly continuous *if, for any entourage V' of X', there exists an entourage V of X such that $(f(x), f(x')) \in V'$ whenever $(x, x') \in V$.*

In particular, let X, X' be two metric spaces with metrics d and d' respectively. Then $f : X \to X'$ is uniformly continuous if (and only if)

$$\forall \varepsilon > 0, \exists \eta > 0, \forall \left(x, x'\right) \in X \times X : d\left(x, x'\right) < \eta \Rightarrow d'\left(f(x), f\left(x'\right)\right) < \varepsilon.$$

Let A be a non-empty subset of a metric space (X, d); then the function $x \mapsto d(x, A)$ (section 2.1.1) is uniformly continuous on X (**exercise**). Any uniformly continuous mapping is continuous, and the composition of two uniformly continuous mappings is in turn uniformly continuous (**exercise**). The morphisms of the category $\mathbf{Topu}$ of uniform spaces are the uniformly continuous mappings. This is a concrete category, which enables us to define (as we did earlier for topological structures) the property of a uniform structure $\mathfrak{U}_2$ being *finer* than a uniform structure $\mathfrak{U}_1$ on a set X. Given a set X, a family of uniform spaces $(X_i)_{i \in I}$, and a family of mappings $(f_i : X \to X_i)_{i \in I}$, we define the *initial uniform structure* on X for the family $(f_i)_{i \in I}$ ([P1], section 1.3.2(**I**)) to be the *coarsest* uniform structure for which every f_i is uniformly continuous; the topology deduced from this initial uniform structure is the same as the initial topology for the family $(f_i)_{i \in I}$ ([BKI 74], Chapter II, section 2.3, Corollary). Given a space X, a uniform space $(X', \mathfrak{U}')$, and a mapping $f : X \to X'$, the initial uniform structure for f is called the *inverse image* of the uniform structure $\mathfrak{U}'$ under f; when $X \subset X'$ and $f : X \hookrightarrow X'$ is inclusion, the inverse image of the uniform structure $\mathfrak{U}'$ under f is the *uniform structure induced* by $\mathfrak{U}'$ on X; its entourages are the $V' \cap (X \times X)$, $V' \in \mathfrak{U}'$, and X is called a *uniform subspace* of $(X', \mathfrak{U}')$ when equipped with this induced uniform structure. The category $\mathbf{Topu}$, like $\mathbf{Top}$, admits concrete products and concrete projective limits, which are defined as in ([P1], section 1.3.3); see ([BKI 74], Chapter II, section 2) for more details on how to construct them. Projective limits commute with the two forgetful functors $\mathbf{Topu} \to \mathbf{Top}$ and $\mathbf{Top} \to \mathbf{Set}$ ([P1], section 1.3.1(**I**)).

However, quotient uniform structures (and especially final uniform structures) cannot be defined so easily [PLA 06], except in the case of topological groups (cf. below, Lemma 2.126), and consequently in the case of topological rings, topological vector spaces, etc.

2.4.3. *Families of pseudometrics and Lipschitz structures*

(I) A *pseudometric* on a set X is a mapping $d : X \times X \to \bar{\mathbb{R}}_+$ that satisfies the conditions ($\mathbf{D}_1$), ($\mathbf{D}_2$), ($\mathbf{D}_3$) of metrics, but which is not required to satisfy the separation condition ($\mathbf{D}_4$). Let $x \in X$ and $r > 0$; the set $B_d(x;r) = \{y \in X : d(x,y) < r\}$ (resp. $B_d^c(x;r) = \{y \in X : d(x,y) \leq r\}$) is called the open (resp. closed) *semi-ball* with center x and radius r with respect to the pseudometric d.

Given a pseudometric d on X and a real number $r > 0$, as above, set $U_d(r) = \{(x,y) \in X : d(x,y) < r\}$ and $U_d^c(r) = \{(x,y) \in X : d(x,y) \leq r\}$. Let $(d_i)_{i \in I}$ be a family of pseudometrics on X, and define $\mathfrak{U} \in \mathfrak{P}(X \times X)$ such that $V \in \mathfrak{U}$ if and only if there exists a *finite* set $J \subset I$ and real numbers $r_i > 0$ $(i \in J)$ such that $\bigcap_{i \in J} U_{d_i}(r_i) \subset V$. Then $\mathfrak{U}$ is a uniform structure on X and $\left\{U_{d_i}^c(r_i) : i \in I, r_i > 0\right\}$ is a fundamental system of entourages of $\mathfrak{U}$. With this notation, we conversely have that ([BKI 74], Chapter IX, section 1.2):

LEMMA 2.69.– *Given a uniform space $(X, \mathfrak{U})$, there exists a family of pseudometrics $(d_i)_{i \in I}$ on X such that $\{U_{d_i}(r_i) : i \in I, r_i > 0\}$, or alternatively $\left\{U_{d_i}^c(r_i) : i \in I, r_i > 0\right\}$, is a fundamental system of entourages of $\mathfrak{U}$.*

This family of pseudometrics may always be assumed to be *saturated* (namely, for any *finite* subset J of I, $\sup_{i \in J} d_i$ is an element of the family of pseudometrics $(d_i)_{i \in I}$) and may also be assumed to satisfy $d_i \leq 1, \forall i \in I$ (replacing d_i by $\inf(d_i, 1)$ if necessary) (**exercise**). The sets $U_{d_i}(r)$ (resp. $U_{d_i}^c(r_i)$) are open (resp. closed) with respect to the topology on $X \times X$ deduced from $\mathfrak{U}$, since the $d_i : X \times X \to \mathbb{R}_+$ are uniformly continuous by (2.1), (2.5). The uniform structure $\mathfrak{U}$ is Hausdorff if and only if, for any pair $(x,y) \in X \times X$ such that $x \neq y$, there exists $i \in I$ such that $d_i(x,y) > 0$.

(II) Let X, Y be two sets with the families of pseudometrics $d = (d_i)_{i \in I}$ and $\delta = (\delta_j)_{j \in J}$ respectively. A mapping $f : X \to Y$ is said to be locally Lipschitz if

$$\begin{gathered}(\forall j \in J), (\exists i \in I), \\ (\exists U \in \mathfrak{N}_X(x_0))(\exists k_{i,j,U} > 0) : (\forall (x', x'') \in U \times U), \\ \delta_j(f(x'), f(x'')) \leq k_{i,j,U} d_i(x', x'').\end{gathered} \qquad [2.7]$$

This mapping is said to be Lipschitz if $U = X$. In particular, if (X, d) and (Y, δ) are two metric spaces, then a mapping $f : X \to Y$ is said to be Lipschitz with Lipschitz constant $k > 0$ if

$$\left(\forall \left(x', x''\right) \in X \times X\right), \delta\left(f\left(x'\right), f\left(x''\right)\right) \leq kd\left(x', x''\right). \qquad [2.8]$$

Returning to the general case, a mapping $f : X \to Y$ is uniformly continuous if and only if

$$\begin{gathered}(\forall j \in J), (\forall \varepsilon > 0)(\exists i \in I), (\exists \eta > 0) : (\forall (x', x'') \in X \times X), \\ (d_i(x', x'') < \eta \Rightarrow \delta_j(f(x'), f(x'')) < \varepsilon),\end{gathered} \qquad [2.9]$$

and so Lipschitz mappings are uniformly continuous. However, the converse does not hold; for example, the function $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, 1]$ (cf. below, Theorem 2.86) but is not Lipschitz. Two families of pseudometrics $d = (d_i)_{i \in I}$ and $\delta = (\delta_j)_{j \in J}$ on X are said to be *topologically equivalent* (resp. *uniformly equivalent*, resp. Lipschitz *equivalent*) if the identity mapping from (X, d) onto (X, δ) and its inverse are continuous (resp. uniformly continuous, resp. Lipschitz). The pair $(X, [d])$, where $[d]$ is the Lipschitz equivalency class of d, is called a Lipschitz *space* (after the mathematician R. Lipschitz) ([SCW 67], Chapter VII, section 7, p. 431)[9]. The mapping f that satisfies (2.7) is locally Lipschitz (and Lipschitz if $U = X$) and this property depends solely on the Lipschitz equivalency classes $[d]$ and $[\delta]$ (and not the choice of representatives d and δ), and thus solely on the Lipschitz structures on X and Y. The composition of two Lipschitz mappings

9 Various other approaches to defining the concept of Lipschitz space can be found in the literature.

is Lipschitz (**exercise**). The morphisms of the category **Topl** of Lipschitz spaces are the Lipschitz mappings. This category is concrete with base **Topu**; the forgetful functors may be composed as follows: $\mathbf{Topl} \to \mathbf{Topu} \to \mathbf{Top} \to \mathbf{Set}$.

THEOREM 2.70.– *(*A. *Weil) Let $(X, \mathfrak{U})$ be a uniform Hausdorff space. The following conditions are equivalent:*

i) The uniform structure $\mathfrak{U}$ has a countable *fundamental system of entourages.*

ii) $\mathfrak{U}$ may be defined by a countable *family of pseudometrics $(d_n)_{n \geq 1}$.*

iii) $(X, \mathfrak{U})$ is metrizable.

PROOF.– (i)⇔(ii): **exercise**. We already know that (iii)⇒(ii), so we simply need to show that (ii)⇒(iii) in the case where the sequence (d_n) is infinite. We may assume that $d_n \leq 1$ for all n (cf. **(I)**, above). Therefore, the series $d(x,y) = \sum_{n=1}^{+\infty} \frac{1}{2^n} d_n(x,y)$ converges for all $x, y \in X$. We clearly have that $d(x,y) = d(y,x) \geq 0$; the triangle inequality holds; and, since $(X, \mathfrak{U})$ is Hausdorff, $d(x,y) = 0 \Leftrightarrow x = y$. For all $r > 0$, set $B_n(x;r) = \{x' \in X : d_n(x,x') < r\}$ and $B(x;r) = \{x' \in X : d(x,x') < r\}$. For all $r > 0$, we have that $B(x;r) \subset B_n(x;2^n r)$. Choosing n sufficiently large for the inequality $2^{n-1} \geq 1/r$ to hold, the relation $\sum_{k=1}^{\infty} 2^{-n-k}$ $d_{n+k}(x) \leq 2^{-n} \leq r/2$ also holds, which implies that

$$d(x,y) = \underbrace{\sum\nolimits_{k=1}^{n} 2^{-k} d_k(x,y)}_{<r/2} + \underbrace{\sum\nolimits_{k=1}^{\infty} 2^{-n-k} d_{n+k}(x)}_{<r/2} < r$$

and hence $\bigcap_{k=1}^{n} B_n(x;r) \subset B(x;r)$. ■

COROLLARY 2.71.– *(i) Any countable product of metrizable spaces is metrizable. (ii) The category of metrizable spaces (and continuous mappings) admits countable projective limits.*

PROOF.– (i) immediately follows from Theorem 2.70 and (ii) follows from (i) by (2.6). ■

2.4.4. *Completeness*

(I) CAUCHY FILTERS AND NETS Let X be a uniform space and let V be an entourage of X. We say that two elements x', x'' are V*-neighbors* if $(x', x'') \in V$; we say that a set $A \subset X$ is V*-small* if, for any pair (x', x'') of elements of A, x' and x'' are V-neighbors. For example, in a metric space (X, d), a set A is $U_d^c(\varepsilon)$-small $(\varepsilon > 0)$ if and only if $\delta(A) \leq \varepsilon$, where $\delta(A)$ is the diameter of A.

DEFINITION 2.72.– *We say that a filter $\mathfrak{F}$ on a uniform space X is a Cauchy* filter *if, for any entourage V of X, there exists a V-small set in $\mathfrak{F}$. We say that a net $\mathfrak{x} = (x_i)_{i \in I}$ is a Cauchy* net *in X if, for any entourage V of X, there exists $i_0 \in I$ such that the set $\{x_i : i \succeq i_0\}$ is V-small.*

Readers may wish to derive Lemmas 2.73 and 2.74 below as an **exercise** (for the second lemma, see [BKI 74], Chapter II, section 2.1, Proposition 4):

LEMMA 2.73.– *Let $\mathfrak{F}$ be a Cauchy filter on a uniform space X. Then every net associated with $\mathfrak{F}$ (section 2.2.2) is a Cauchy net. Conversely, if $\mathfrak{x}$ is a Cauchy net in X, then the elementary filter associated with $\mathfrak{x}$ is a Cauchy filter.*

LEMMA 2.74.– *Let X be a set, $(Y_i)_{i \in I}$ a family of uniform spaces, and, for each $i \in I$, f_i a mapping from X into Y_i. Let $\mathfrak{U}$ be the initial uniform structure for the family $(f_i)_{i \in I}$. For a filter base $\mathfrak{B}$ on X to be a Cauchy filter base, it is necessary and sufficient for $f_i(\mathfrak{B})$ to be a Cauchy filter base on Y_i for all $i \in I$.*

THEOREM 2.75.– *Let X be a uniform space. Then every Cauchy filter on X and every Cauchy net in X with a cluster point is convergent.*

PROOF.– It is sufficient to consider the case of a Cauchy net by Corollary 2.19 and Lemma 2.73. Let $\mathfrak{x} = (x_i)_{i \in I}$ be a Cauchy net in X and let V be an entourage of X. There exists an entourage W such that $W \circ W \subset V$ (by the condition ($\mathbf{E}_3$)). There also exists $i_0 \in I$ such that $\{x_i : i \succeq i_0\} \subset W$. If $a \in X$ is a cluster point of $\mathfrak{x}$, then there exists $j \geq i_0$ such that $(x_j, a) \in W$. For all $i \succeq i_0$, we therefore have that $(x_i, x_j) \in W$ and $(x_j, a) \in W$, so $(x_i, a) \in W \circ W \subset V$, and consequently $\mathfrak{x} \to a$. ■

The image of a Cauchy filter base (resp. net) under a uniformly continuous mapping is a Cauchy filter base (resp. net). Every convergent filter (resp. net) is Cauchy (**exercise**). The converse may fail. Consider for example a sequence of rational numbers converging to $\sqrt{2}$ in $\mathbb{R}$. This sequence is Cauchy in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$. This leads to the following notion:

(II) COMPLETE SPACES

DEFINITION 2.76.– *A* complete space *is a uniform space on which every Cauchy filter (or equivalently in which every Cauchy net) is convergent.*

LEMMA 2.77.– *Let X be a uniform space and $A \subset X$. If the uniform subspace A is complete, then it is closed in X; if X is complete and A is closed in X, then A is complete (**exercise**).*

THEOREM 2.78.–

1) (Cauchy's criterion*) Let $\mathfrak{F}$ be a filter on a set X and let f be a mapping from X into a complete uniform space X'. For f to have a cluster point with respect to $\mathfrak{F}$ (section 2.3.3), it is necessary and sufficient for the image of $\mathfrak{F}$ under f to be a Cauchy filter base.*

2) Let A be a dense subset of a topological space X and let f be a mapping from A into a Hausdorff and complete *uniform space X'. For there to exist a continuous mapping $\bar{f} : X \to X'$ extending f, it is necessary and sufficient for the image under f of the trace on A of the filter $\mathfrak{V}(x)$ of neighborhoods of x in X (written $\mathfrak{V}(x) \cap A$) to be a Cauchy filter base on X' for every $x \in \complement_X A$.*

3) If X is a uniform space, A is a dense subspace of X, and $f : A \to X'$ is uniformly continuous, then the continuous extension $\bar{f}$ of f into X is uniformly continuous.

PROOF.–

1) f has a cluster point with respect to $\mathfrak{F}$ if and only if the filter base $f(\mathfrak{F})$ is convergent. This implies that $f(\mathfrak{F})$ is a Cauchy filter base. Conversely, if $f(\mathfrak{F})$ is a Cauchy filter base, then this filter is convergent, since X' is complete.

2) If the continuous extension $\bar{f}$ exists, then it is unique by the principle of extension of identities (Lemma 2.30). The function $\bar{f}$ is continuous at the point $x \in X$ if and only if $\lim_{x' \to x, x' \in A} f(x) = f(a)$, which is equivalent to saying that $f(\mathfrak{V}(x) \cap A)$ is a Cauchy filter base.

3) Let V' be a closed symmetric entourage of X', and let V be an entourage of X such that, whenever $x', x'' \in A$ are V-neighbors, $f(x')$ and $f(x'')$ are

V'-neighbors. We may assume that V is the closure in $X \times X$ of an entourage W of A (Lemma 2.64). We have that $\left(\bar{f}(x'), \bar{f}(x'')\right) \in V'$ for all $(x', x'') \in W$, and, since $\bar{f}$ is continuous, $\left(\bar{f}(x'), \bar{f}(x'')\right) \in V'$ for all $(x', x'') \in V = \bar{W}$, since V' is closed (Lemma 2.28). ■

The next result follows from Lemmas 2.74, 2.77 and Corollary 2.33:

THEOREM 2.79.– *If $(X_i)_{i \in I}$ is a family of non-empty uniform spaces, then $\prod_{i \in I} X_i$ is complete if and only if each X_i is complete. If $\left\{X_i, \psi_i^j; I\right\}$ is an inverse system of Hausdorff and complete uniform spaces, then the projective limit $\varprojlim_{i \in I} X_i$ is a Hausdorff and complete uniform space.*

THEOREM 2.80.– *Let X be a uniform space.*

1) There exists a complete Hausdorff space $\hat{X}$, unique up to isomorphism, and a uniformly continuous mapping $\iota : X \to \hat{X}$ satisfying the following universal property ([P1], section 1.2.4, Definition 1.16):

***(SC)** For every uniformly continuous mapping f from X into a complete Hausdorff uniform space Y, there exists precisely one uniformly continuous mapping $g : \hat{X} \to Y$ such that $f = g \circ \iota$.*

2) The subspace $\iota(X)$ is dense in $\hat{X}$, and, if X is Hausdorff, then ι is an isomorphism (of uniform spaces) from X onto $\iota(X)$.

3) In particular, if Y is a complete Hausdorff space and the space X is dense in Y, then the inclusion $X \hookrightarrow Y$ may be extended to an isomorphism from $\hat{X}$ onto Y.

PROOF.–

a) *Uniqueness*: Let $\mathbf{Topuc}$ be the category of complete Hausdorff uniform spaces and let $|.| : \mathbf{Topuc} \to \mathbf{Topu}$ be the forgetful functor. Then $\left(\hat{X}, \iota\right)$ is a universal arrow from X to the functor $|.|$ ([P1], section 1.2.4, Definition 1.14), which implies the uniqueness of $\hat{X}$ up to isomorphism (*ibid.*, Lemma 1.15).

b) For the other claims, see ([BKI 74], Chapter II, section 3.7, Theorem 3) in the general case. We will show the following weaker result here in the form that was shown by A. Weil ([WEI 37], p. 19, Theorem II):

Every Hausdorff uniform space X may be associated with a complete space $\hat{X}$, unique up to isomorphism, such that X is isomorphic to a dense subspace of $\hat{X}$.

To simplify the notation, we will only consider the case where X is a metric space (with metric d), following a method that is analogous to the Cantor-Méray construction of the real numbers.

Given two Cauchy sequences (x_n), (y_n) in X, write $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. This is clearly an equivalence relation. Let X^* be the set of equivalence classes. First, let $\mathfrak{x} = (x_n)$, $\mathfrak{y} = (y_n)$ be Cauchy sequences in X. We will show that the limit $\delta(\mathfrak{x}, \mathfrak{y}) = \lim_{n\to+\infty} d(x_n, y_n)$ exists. By (2.1), we have that $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$, so $(d(x_n, y_n))$ is a Cauchy sequence of real numbers ≥ 0. This sequence has a limit $\delta(\mathfrak{x}, \mathfrak{y}) \in \mathbb{R}_+$ that depends solely on the equivalence classes x^* and y^* of $\mathfrak{x}$ and $\mathfrak{y}$ respectively (**exercise**), which we may therefore write as $d^*(x^*, y^*)$. Then d^* is a metric on X^* (**exercise**). If $x \in X$, consider the sequence $\mathfrak{x} = (x_n)$ such that $x_n = x$ for all n. Let x^* be the equivalence class of $\mathfrak{x}$. Then the mapping $\iota : X \to X^* : x \mapsto x^*$ is injective. If $x, y \in X$, then $d^*(x^*, y^*) = d(x, y)$, so $\iota : X \hookrightarrow \iota(X)$ is an *isometry*, i.e. a metric-preserving mapping (any such mapping is necessarily injective).

It simply remains to be shown that $\iota(X)$ is dense in (X^*, d^*). Let $x^* \in X^*$ and let $\varepsilon > 0$ be an arbitrary real number. Let (x_n) be a representative of x^*. This is a Cauchy sequence, so there exists an integer N such that, for all integers $n, m \geq N$, $d(x_n, x_m) \leq \varepsilon$. For all n, the mapping $x \mapsto d(x_n, x)$ is continuous in X, so, whenever $n \geq N$, $d^*(x_n, x^*) = \lim_{m\to+\infty} d(x_n, x_m) \leq \varepsilon$. ■

DEFINITION 2.81.– *We say that the complete Hausdorff uniform space $\hat{X}$ is the* Hausdorff completion *of X and that $\iota : X \to \hat{X}$ is the* canonical mapping.

We shall now explore a few properties of continuous images of complete metrizable spaces:

DEFINITION 2.82.– *Let X, Y be uniform spaces. A mapping $\varphi : X \to Y$ is said to be* uniformly open *if, for any entourage U of X, there exists an entourage V of Y such that, for all $x \in X$, $\varphi(U(x)) \supset V(\varphi(x))$ (where $U(x) = \{x' \in X : (x, x') \in U\}$, and $V(y)$ is defined similarly).*

THEOREM 2.83.– *(Kelley) Let X be a metrizable and complete space, Y a Hausdorff uniform space, and $\varphi : X \to Y$ a continuous and uniformly open surjection. Then Y is complete.*

PROOF.– The proof of this theorem is difficult in the general case ([KEL 55], Chapter 6, Corollary 37). We can capture the overall idea by considering the case where Y is a metric space and assuming that condition **(K)** below is satisfied, where d and δ denote the metrics on X and Y respectively:

(K) There exists some $\alpha > 0$ such that, $\forall (x'; x'') \in X \times X, d(x', x'') \leq \alpha\delta(\varphi(x'), \varphi(x''))$.

If condition **(K)** is satisfied, then the mapping φ (which is assumed to be continuous and surjective) is uniformly open; conversely, if X and Y are *normed vector spaces*, then the mapping φ (assumed to be linear, continuous, and surjective) satisfies **(K)** whenever it is uniformly open (**exercise**).

Assuming that **(K)** holds, let (y_n) be a Cauchy sequence in Y and, for all n, pick $x_n \in X$ such that $\varphi(x_n) = y_n$. We have that $d(x_n, x_m) \leq \alpha\delta(y_n, y_m)$, so (x_n) is a Cauchy sequence in X. This sequence converges to a point x, and (y_n) converges to $\varphi(x)$. ■

We also have the following result (**exercise***: cf. [BKI 81], Chapter I, section 3.3, Lemma 2)[10]:

LEMMA 2.84.– *Let (X, d) and (Y, δ) be two metric spaces and suppose that (X, d) is complete. Let $u : X \to Y$ be a continuous mapping with the following property: for any real number $r > 0$, there exists a real number $\rho(r) > 0$ such that, for $x \in X$, we have $B_\delta(u(x); \rho(r)) \subset \overline{u(B_d(x; r))}$. Then, for every $a > r$, $u(B_d(x; a))$ contains the ball $B_\delta(u(x); \rho(r))$.*

2.4.5. *Uniformity of compact spaces*

Every compact space X is normal, and so is uniformizable (Theorems 2.65 and 2.67). Let X be equipped with a uniform structure $\mathfrak{U}$ that determines the topology of X (section 2.4.1**(II)**).

10 The essential parts of this result were established by S. Banach: cf. [BAN 32], Chapter III, Part (1) of the proof of Theorem 3.

THEOREM 2.85.– *The compact space X is complete.*

PROOF.– Every Cauchy net in X has a cluster point (Lemma 2.36, Property (iv)), and therefore is convergent (Theorem 2.75). ■

Note that locally compact spaces are not complete in general; for example, the interval $]0,1[$ in $\mathbb{R}$ is locally compact but is not complete.

THEOREM 2.86.– *(Heine) Every continuous mapping f from a compact space X into a uniform space Y is uniformly continuous.*

PROOF.– The uniform structures of X and Y are defined by saturated families of pseudometrics $(d_i)_{i\in I}$ and $(\delta_j)_{j\in J}$ respectively (Lemma 2.69 sq.). Suppose that f is not uniformly continuous. Then (by (2.9)), there exist $j \in J$ and $\varepsilon > 0$ such that, $\forall i \in I, \forall \eta > 0, \exists (x', x'') \in U^c_{d_i}(\eta)$ and $\delta_j(f(x'), f(x'')) \geq \varepsilon$. Since the family of pseudometrics $(d_i)_{i\in I}$ is saturated, every finite intersection of the closed sets $U^c_{d_i}(\eta)$ contains another set $U^c_{d_{i'}}(\eta')$, and so is non-empty. Consequently, since X is compact, the intersection of the $U^c_{d_i}(\eta)$ $(i \in I, \eta > 0)$ is non-empty (Lemma 2.36, Property (i')). Pick (x', x'') in this intersection. Since X is Hausdorff, we have that $x' = x''$, but $\delta_j(f(x'), f(x'')) \geq \varepsilon$: contradiction. ■

COROLLARY 2.87.– *Let X be a compact topological space. There exists precisely one uniform structure that determines the topology of this space.*

PROOF.– The only thing that remains to be shown in uniqueness. Let $\mathfrak{U}$, $\mathfrak{U}'$ be uniform structures that determine the topology of X. By Theorem 2.86, $1_X : (X, \mathfrak{U}) \to (X, \mathfrak{U}')$ is uniformly continuous, and so is its inverse bijection. ■

LEMMA 2.88.– *Let X be a uniform space, $A \subset X$ a compact set, $B \subset X$ a closed set, and suppose that $A \cap B \neq \varnothing$. Then there exists an entourage V of X such that $V(A) \cap V(B) = \varnothing$ (with $V(A) := \{y \in X : \forall x \in A, (x,y) \in V\}$, and a similar definition for $V(B)$).*

PROOF.– Consider the case where X is a metric space. We know that the mapping $x \mapsto d(x,B)$ is continuous on X (section 2.1.1), so it attains its minimum value $\delta = d(A,B) > 0$ on A (Theorem 2.39). Thus $V(A) = \{x \in X : 0 < d(x,A) < \delta/2\}$ and $V(B) = \{x \in X : 0 < d(x,B) < \delta/2\}$ are disjoint. Extending this result to the case where X is an arbitrary uniform space is an **exercise***: cf. ([BKI 74], Chapter II, section 4.3, Proposition 4). ■

DEFINITION 2.89.– *We say that a uniform space X is* precompact *if its Hausdorff completion is compact.*

A Hausdorff uniform space is compact if and only if it is precompact and complete. It is possible to show the following result ([BKI 74], Chapter II, section 4.2, Theorem 3), which is analogous to the Borel-Lebesgue property (Lemma 2.36(i)):

THEOREM 2.90.– *A uniform space X is precompact if and only if, for any entourage V of X, there exists a* finite *covering of X consisting of V-small sets.*

If (X, d) is a metric space, then it is precompact if and only if, for all $\varepsilon > 0$, there exists a *finite* covering of X consisting of sets of diameter $< \varepsilon$. In general uniform spaces, the following results hold (**exercise***: cf. [BKI 74], Chapter II, section 4.2, Proposition 1):

LEMMA 2.91.– *Subsets of precompact spaces, finite unions of precompact spaces, closures of precompact spaces, and images of precompact spaces under* uniformly *continuous mappings are all precompact. A product of non-empty uniform spaces is precompact if and only if each space in the product is precompact.*

LEMMA 2.92.– *Let X be a uniform space and let (x_n) be a Cauchy sequence in X. Then $\{x_n : n \geq 0\}$ is a precompact subset of X.*

PROOF.– For each entourage U of X, there exists an integer $N > 0$ such that $\{x_n : n \geq N\} \subset U(x_N)$. Therefore, $\{x_n : n \geq N\}$ is precompact, and the same is true for the finite set $\{x_n : 0 \leq n \leq N\}$. ■

Note also the following result ([BKI 74], Chapter IX, section 2.10, Propositions 17 and 18):

LEMMA 2.93.– *Let X be a metrizable and complete space, $\sim$ an equivalence relation that is open and Hausdorff in X (section 2.3.4(**IV**)), and $\varphi : X \twoheadrightarrow X/\sim$ the canonical surjection. For every compact set K of $X/\sim$, there exists a compact subset K' of X such that $\varphi(K) = K'$, and K is a metrizable space.*

2.5. Bornologies

2.5.1. *Concept of bornology*

(I) Let X be a set. A *covering* $\mathfrak{S}$ of X is called a *bornology* if the following conditions are satisfied ([HOU 72], Chapter I, section 2): if $A \in \mathfrak{S}$ and $B \subset A$, then $B \in \mathfrak{S}$; if $A, B \in \mathfrak{S}$, then $A \cup B \in \mathfrak{S}$.

DEFINITION 2.94.– *The set X equipped with the bornology $\mathfrak{S}$ is called a* bornological set[11]*, and the elements of $\mathfrak{S}$ are called the $\mathfrak{S}$-bounded sets.*

If $\mathfrak{S}$ is a bornology on a non-empty set X and $x \in X$, then we clearly have that $\{x\} \in \mathfrak{S}$. Let $(X, \mathfrak{S})$ and $(Y, \mathfrak{T})$ be two bornological sets; the mapping $f : X \to Y$ is said to be $(\mathfrak{S}, \mathfrak{T})$-*bounded* (or bounded from $(X, \mathfrak{S})$ to, $(Y, \mathfrak{T})$) if, for all $A \in \mathfrak{S}$, $f(A) \in \mathfrak{T}$.

(II) CATEGORY OF BORNOLOGICAL SETS The morphisms of the category $\mathbf{Bor}$ of bornological sets are the bounded mappings. The category $\mathbf{Bor}$ is *concrete* with base $\mathbf{Set}$, and the general notions of initial and final structures ([P1], section 1.3.2) are realized in $\mathbf{Bor}$ by the notions of initial and final bornologies. The bornology $\mathfrak{S}_1$ on X is said to be *finer* than the bornology $\mathfrak{S}_2$ on X if $\mathfrak{S}_1 \subset \mathfrak{S}_2$, or in other words if 1_X is bounded from $(X, \mathfrak{S}_1)$ onto $(X, \mathfrak{S}_2)$. If $(X, \mathfrak{S})$ is a bornological space and $A \subset X$, then the *induced bornology* on A is the initial bornology for inclusion, given by the $A \cap B$ such that $B \in \mathfrak{S}$. The finest bornology on a set X, called the *discrete bornology*, consists of the finite subsets; this bornology is denoted s (or is said to be of type s). The coarsest bornology, called the *trivial bornology*, is $\mathfrak{P}(X)$, denoted u (or said to be of type u). For example, every mapping from a set X to a set Y is bounded when these sets are both equipped with bornologies of type s or type u.

Products, coproducts, equalizers, coequalizers, and projective and inductive limits all exist in $\mathbf{Bor}$ and are concrete. In particular, if $(X_i, \mathfrak{S}_i)_{i \in I}$ is a family of bornological sets, then the product $\prod_i X_i$ is equipped with the initial bornology for the family $(\mathrm{pr}_i : \prod_i X_i \to X_i)$; this bornology, called the *product bornology*, consists of the sets A such that $\mathrm{pr}_i(A) \in \mathfrak{S}_i$ for all $i \in I$. The coproduct (or sum) $\coprod_i X_i$ is equipped with the final bornology for

11 This must not be confused with the notion of a bornological space (Definition 3.61).

the family $\left(\mathrm{inj}_i : X_i \to \coprod_i X_i\right)$. If $(X, \mathfrak{S})$ is a bornological set and $\sim$ is an equivalence relation on X, then the *quotient bornology* on $X \twoheadrightarrow X/\sim$ consists of the sets $\pi(B)$, $B \in \mathfrak{S}$, where $\pi : X \twoheadrightarrow X/\sim$ is the canonical surjection.

(III) Let $\mathfrak{S}$ be a bornology on a set X. We say that a family $(B_i)_{i\in I}$ is a *fundamental system* of $\mathfrak{S}$ if, for all $A \in \mathfrak{S}$, there exists $i \in I$ such that $A \subset B_i$.

(IV) PRINCIPAL TYPES OF BORNOLOGY If X is a Hausdorff topological space, then the compact subsets of X form a bornology on X, denoted c (or said to be of type c). On a uniform space, we can also consider the bornology, denoted (or said to be of type) pc, consisting of the precompact subsets. A subset A of a Lipschitz space $(X, [d])$, is said to be a *bounded set* for the family of pseudometrics $d \in [d]$ if, for all $i \in I$, there exist a point x_i and a real number $r_i > 0$ such that $A \in B_{d_i}(x_i; r_i)$ ([SCW 67], Chapter VII, section 7, p. 434). Any two *Lipschitz equivalent* families of pseudometrics determine the same bounded sets, but this does not necessarily hold for two families of pseudometrics that are merely *uniformly continuous* (observe that replacing each d_i by $\inf\{d_i, 1\}$ does not change the uniform structure; however, it changes the Lipschitz structure and the bounded sets). The bounded sets form a bornology on $(X, [d])$, called the *canonical bornology* of this Lipschitz space; this bornology is denoted b, or is said to be of type b. A subset of a metric space is bounded if and only if it has finite diameter (section 2.1.1). In a Lipschitz space, if $A \in b$, then $\overline{A} \in b$ (**exercise**). By Theorem 2.39 and Lemma 2.91:

LEMMA 2.95.– *Every* continuous *(resp.* uniformly continuous*) mapping from a Hausdorff topological space (resp. from a uniform space) X into a Hausdorff topological space (resp. into a uniform space) Y is bounded whenever these spaces are both equipped with bornologies of type c or pc.*

LEMMA 2.96.– *Every precompact subset A of a Lipschitz space $(X, [d])$ is bounded (which implies that $s \subset pc \subset b$).*

PROOF.– Any such set A is contained in *finitely many* semi-balls $B_{d_i}(x_0; r_i)$. Each of these semi-balls is bounded, so A is bounded. ■

Therefore, on a Hausdorff Lipschitz space, the relations $s \subset c \subset pc \subset b \subset u$ hold.

2.5.2. *Compactness in* $\mathbb{K}^n$

Consider $\mathbb{K}^n$ equipped with one of the usual metrics d_p $(p \in [1, \infty])$ (section 2.1.1). A set A is bounded if and only if it is contained in some closed hypercube H. This hypercube H is compact (section 2.3.7); thus, if A is closed and bounded, it is compact. Conversely, if A is compact, then each of its projections $\mathrm{pr}_i(A)$ is compact by Tychonov's theorem (Theorem 2.43), and therefore is a closed and bounded set in $\mathbb{R}$, which implies:

THEOREM 2.97.– *(Heine-Borel-Lebesgue)*[12] *A subset A of $\mathbb{K}^n$ is compact if and only if it is closed and bounded.*

2.5.3. *Boundedness of filters and nets*

A net $(x_i)_{i \in I}$ in a bornological set $(X, \mathfrak{S})$ is said to be $\mathfrak{S}$-*bounded* (or *bounded* in the case where $(X, [d])$ is a Lipschitz space and $\mathfrak{S} = b$) if there exists an index $i_0 \in I$ such that $\{x_i : i \succeq i_0\} \in \mathfrak{S}$. If $(x_i)_{i \in I}$ is $\mathfrak{S}$-bounded, then every net equivalent to it is also $\mathfrak{S}$-bounded. Similarly, a filter on X is said to be $\mathfrak{S}$-*bounded* (or *bounded* in the case where $(X, [d])$ is a Lipschitz space and $\mathfrak{S} = b$) if it contains an element of $\mathfrak{S}$. The next theorem follows from Lemmas 2.92 and 2.96.

THEOREM 2.98.– *Let (x_n) be a Cauchy sequence in a Lipschitz space $(X, [d])$. Then $\{x_n : n \in \mathbb{N}\}$ is a bounded subset of X.*

DEFINITION 2.99.– *A Lipschitz space $(X, [d])$ is said to be* quasi-complete *if every bounded Cauchy net (or equivalently filter) converges, and* semi-complete *(or* sequentially complete*) if every Cauchy* sequence *converges.*

Since Cauchy sequences are bounded (Theorem 2.98), quasi-complete Lipschitz spaces are semi-complete.

THEOREM 2.100.– *Let $\mathfrak{I} = \left(X_i, \psi_i^j; I\right)$ be an inverse system of quasi-complete Hausdorff Lipschitz spaces. Then $\varprojlim X_i$ is a quasi-complete Hausdorff Lipschitz space.*

12 This theorem is sometimes called the Heine-Borel *theorem*, and sometimes the Borel-Lebesgue *theorem*.

PROOF.– This follows from Theorem 2.79, Lemma 2.77, and Corollary 2.33, as well as from the fact that a subset B is bounded in $\prod_{i\in I} X_i$ if and only if $B_i = \mathrm{pr}_i(B)$ is bounded in X_i for every $i \in I$ (section 2.5.1(**II**)). ∎

2.6. Baire spaces, Polish spaces, Suslin spaces, and Lindelöf spaces

2.6.1. *Baire spaces and Baire's category theorem*

DEFINITION 2.101.– *We say that a topological space X is a Baire space if condition (**EB**) below is satisfied:*

> *(**EB**) Every* countable *intersection of dense open sets in X is dense in X.*

THEOREM 2.102.– *(Baire) Locally compact spaces and complete metric spaces are all Baire spaces.*

PROOF.– These spaces are all regular (Definition 2.48 and Lemma 2.66). Let X be a regular topological space, (A_n) a sequence of dense open sets in X, and $O = O_1$ an arbitrary non-empty open set. Then $O_1 \cap A_1$ is non-empty. If $y \notin O_1 \cap A_1$, then there exists an open set O_2 containing y such that $O_2 \cap \complement_X (O_1 \cap A_1) = \varnothing$, so $\varnothing = \overline{O_2 \cap \complement_X (O_1 \cap A_1)} = \overline{O_2} \cap \complement_X (O_1 \cap A_1)$ (section 2.3.1(**I**)), and consequently $\overline{O_2} \subset O_1 \cap A_1$. By inductively repeating this construction, we obtain a sequence (O_n) of non-empty open sets such that $\overline{O_{n+1}} \subset O_n \cap A_n$ for every integer $n \geq 1$. We therefore have that $O \cap (\bigcap_{n=1}^{\infty} A_n) \supset \bigcap_{n=1}^{\infty} O_n = \bigcap_{n=1}^{\infty} \overline{O_n}$, and we simply need to show that this last intersection is non-empty. But if X is locally compact, we may assume that $\overline{O_n}$ is compact for all $n \geq 1$, which implies that $\bigcap_{n=1}^{\infty} \overline{O_n} \neq \varnothing$ by Lemma 2.36 (property (i')). If X is a complete metric space, we may choose the sequence (O_n) in such a way that $\delta\left(\overline{O_n}\right) \to 0$; then, by selecting a point $x_n \in \overline{O_n}$ for all n (by the axiom of choice[13]) (x_n) is a Cauchy sequence that converges to some point $x \in \bigcap_{n=1}^{\infty} \overline{O_n}$. ∎

By taking complements, condition (**EB**) is equivalent to saying:

13 Here, and in many other arguments on metrizable spaces, we are only using the "axiom of countable choice" (whose precise statement the reader will easily be able to guess), which is weaker than the usual axiom of choice.

(**EB'**) Every countable union of closed sets with empty interiors in X has empty interior in X.

DEFINITION 2.103.–

i) We say that a subset A of a topological space X is rare *if its closure $\overline{A}$ has empty interior.*

ii) We say that a subset A of a topological space is meager *(or* of first category*) if it a countable union of rare sets.*

iii) We say that a subset A of a topological spaces is of second category *if it is not of first category.*

COROLLARY 2.104.– *Let X be a topological space. The following conditions are equivalent: (i) X is a Baire space. (ii) Every non-empty open set in X is of second category. (iii) The complement of a meager set in X is dense in X (**exercise**).*

Every open set A in a Baire space X is a Baire space. If every point in a topological space X has a neighborhood that is a Baire space, then X is a Baire space (**exercise**).

2.6.2. *Polish spaces and Suslin spaces*

DEFINITION 2.105.– *We say that a topological space is* Polish *if it is metrizable, separable, and complete.*

Baire's theorem implies that every Polish space is a Baire space. Open subspaces of Polish spaces and products of countable families of Polish spaces are also Polish spaces (**exercise***: cf. [BKI 74], Chapter IX, section 6.1, Proposition 1).

THEOREM 2.106.– *Let X be a locally compact and metrizable space that is countable at infinity. Then X is Polish.*

PROOF.– The one-point compactification X^∞ of X is metrizable and separable by Theorem 2.53. Since it is compact, it is complete (Theorem 2.85), and is therefore a Polish space. They same is true for X, which is an open subspace of X^∞. ■

DEFINITION 2.107.– *We say that a topological space S is a Suslin space if it is Hausdorff and there exists a Polish space P together with a continuous surjection $P \twoheadrightarrow S$.*

Every Polish space is a Suslin space and every Suslin space is separable.

2.6.3. *Lindelöf spaces*

DEFINITION 2.108.– *We say that a topological space X is a Lindelöf space if a countable subcovering may be extracted from every open covering of X.*

Readers may wish to show the following result as an **exercise**:

THEOREM 2.109.–

*i) Every topological space with a countable base (*i.e. *satisfying the second axiom of countability) is a Lindelöf space. In particular, every Suslin space is a Lindelöf space.*

ii) Let $f : X \to Y$ be a continuous mapping from X into a topological space Y. If X is a Lindelöf space, then so is the subspace $f(X)$ of Y.

iii) Every regular Lindelöf space (Definition 2.48) is paracompact (section 2.3.10).

2.7. Uniform function spaces

2.7.1. $\mathfrak{S}$*-convergence*

Let X, Y be two sets. We write $\mathcal{F}(X;Y)$ for the set of mappings from X into Y. If $(X, \mathfrak{S})$ is a bornological set and $(Y, \mathfrak{U})$ is a uniform space, then, for all $A \in \mathfrak{S}$, $U \in \mathfrak{U}$, we define

$$\mathbf{W}(A, U) = \{(u, v) \in \mathcal{F}(X;Y) \times \mathcal{F}(X;Y) : (u(x), v(x)) \in U, \forall x \in A\}.$$

The $\mathbf{W}(A, U) \subset \mathcal{F}(X;Y) \times \mathcal{F}(X;Y)$ form a fundamental system of entourages of a uniform structure on $\mathcal{F}(X;Y)$, called the uniform structure of $\mathfrak{S}$-convergence. We write $\mathcal{F}_{\mathfrak{S}}(X;Y)$ for the set $\mathcal{F}(X;Y)$ equipped with this uniform structure. If $(Y, [d])$ is a Lipschitz space, let $d = (d_i)_{i \in I}$ be a representative family of pseudometrics of $[d]$ and, for all $f, g \in \mathcal{F}(X;Y)$, $i \in I$, $A \in \mathfrak{S}$, suppose that

$$\epsilon_{i,A}(f,g) = \sup_{x \in A} \{d_i(f(x), g(x))\}.$$

Then $(\epsilon_{i,A})_{i \in I, A \in \mathfrak{S}}$ is a family of pseudometrics on $\mathcal{F}(X;Y)$ that defines the uniform structure of $\mathcal{F}_{\mathfrak{S}}(X;Y)$.

In the following, X is a set equipped with suitable structure (trivial, topological, uniform, or Lipschitz), and Y is a uniform space. The uniform structure on $\mathcal{F}_s(X;Y)$ (resp. $\mathcal{F}_c(X;Y)$, resp. $\mathcal{F}_{pc}(X;Y)$, resp. $\mathcal{F}_b(X;Y)$, resp. $\mathcal{F}_u(X;Y)$) is called the *uniform structure of pointwise convergence* (resp. *compact convergence*, resp. *precompact convergence*, resp. *bounded convergence*, resp. *uniform convergence*). If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are bornologies on X, and $\mathfrak{S}_1$ is finer than $\mathfrak{S}_2$ (i.e. $\mathfrak{S}_1 \subset \mathfrak{S}_2$), then the uniform structure on $\mathcal{F}_{\mathfrak{S}_1}(X;Y)$ is coarser than the uniform structure on $\mathcal{F}_{\mathfrak{S}_2}(X;Y)$; if $\mathfrak{S}$ is a bornology on X and Y is Hausdorff, then $\mathcal{F}_{\mathfrak{S}}(X;Y)$ is Hausdorff, since $\mathfrak{S}$ is a covering of X (**exercise**).

LEMMA 2.110.– *Let $(X, \mathfrak{S})$ be a bornological set and Y a Hausdorff space. For a net $\mathfrak{f} = (f_i)_{i \in I}$ in $\mathcal{F}_{\mathfrak{S}}(X;Y)$ to converge to h (in $\mathcal{F}_{\mathfrak{S}}(X;Y)$), it is necessary and sufficient for $\mathfrak{f}$ to be a Cauchy net in $\mathcal{F}_{\mathfrak{S}}(X;Y)$ and for $\mathfrak{f}$ to converge pointwise to h (in other words, converge to h in $\mathcal{F}_s(X;Y)$).*

PROOF.– The necessary condition is obvious, so we will only show the sufficient condition. Let $\mathfrak{f} = (f_i)_{i \in I}$ be a Cauchy net in $\mathcal{F}_{\mathfrak{S}}(X;Y)$ and suppose that $\mathfrak{f}$ converges to h in $\mathcal{F}_s(X;Y)$. Let V, W be entourages of Y such that $\overset{2}{W} \subset V$. For every set $A \in \mathfrak{S}$, there exists $i_0 \in I$ such that $(f_i(x), f_{i'}(x)) \in W$ for all $i, i' \succeq i_0$ and all $x \in A$. For all $x \in A$, there exists $i'_0(x) \in I$ such that $(f_{i'}(x), h(x)) \in W$ for all $i' \succeq i'_0(x)$. Therefore, for all $x \in A$ and $i \succeq i_0$, $i' \succeq i''_0(x)$ with $i''_0(x) \succeq i_0, i'_0(x)$, we have that $(f_i(x), f_{i'}(x)) \in W$ and $(f_{i'}(x), h(x)) \in W$, which implies that $(f_i(x), h(x)) \in V$, and hence $\mathfrak{f}$ converges to h in $\mathcal{F}_{\mathfrak{S}}(X;Y)$. ■

The next result immediately follows from this lemma and Definition 2.76 (for the necessary condition, simply view Y as the subspace of constant mappings in $\mathcal{F}_{\mathfrak{S}}(X;Y)$).

THEOREM 2.111.– *Let* $(X, \mathfrak{S})$ *be a bornological set and* Y *a uniform space. Then* $\mathcal{F}_{\mathfrak{S}}(X; Y)$ *is Hausdorff (resp. complete) if and only if* Y *is Hausdorff (resp. complete).*

LEMMA 2.112.– *Let* X, Y *be sets with bornologies* $\mathfrak{S}$ *and* $\mathfrak{T}$ *respectively, and let* Z *be a uniform space. Consider the product bornology* $\mathfrak{S} \times \mathfrak{T}$ *on* $X \times Y$ *(section 2.5.1**(II)**). The two uniform spaces* $\mathcal{F}_{\mathfrak{S} \times \mathfrak{T}}(X \times Y; Z)$ *and* $\mathcal{F}_{\mathfrak{S}}(X; \mathcal{F}_{\mathfrak{T}}(Y; Z))$ *are isomorphic under the isomorphism* $[f : (x, y) \mapsto f(x, y)] \mapsto \{x \mapsto f_x : [y \mapsto f(x, y)]\}$ ***(exercise)**.*

2.7.2. Sets of continuous mappings

Let X be a topological space and Y a uniform space. We write $\mathcal{C}(X; Y)$ for the set of continuous mappings from X into Y. If $x_0 \in X$, we write $\mathcal{C}_{(x_0)}(X; Y)$ for the set of mappings from X into Y that are continuous at the point x_0.

THEOREM 2.113.– *The sets* $\mathcal{C}_{(x_0)}(X; Y)$ *and* $\mathcal{C}(X; Y)$ *are closed in* $\mathcal{F}_u(X; Y)$.

PROOF.– It is clear that $\mathcal{C}(X; Y) = \bigcap_{x_0 \in X} \mathcal{C}_{(x_0)}(X; Y)$; therefore, it suffices to show that $\mathcal{C}_{(x_0)}(X; Y)$ is closed in $\mathcal{F}_u(X; Y)$. Let h be a mapping in the closure of $\mathcal{C}_{(x_0)}(X; Y)$ in $\mathcal{F}_u(X; Y)$. There exists a net $(f_i)_{i \in I}$ in $\mathcal{C}_{(x_0)}(X; Y)$ that converges to h in $\mathcal{F}_u(X; Y)$ (Corollary 2.20). Let V be an entourage of Y and W a symmetric entourage of Y such that $\overset{3}{W} \subset V$ (Lemma 2.64). There exists $i_0 \in I$ such that $(h(x), f_i(x)) \in W, \forall i \succeq i_0$, $\forall x \in X$. For every neighborhood U of x_0 in X, there exists $i'_0 \in I$ such that $(f_i(x), f_i(x_0)) \in W, \forall i, i' \succeq i'_0, \forall x \in U$. Thus, let $i''_0 \succeq i_0, i'_0$; then, $\forall i, i' \succeq i''_0$, $\forall x \in U$, we have that $(h(x), f_i(x)) \in W$, $(f_i(x), f_i(x_0)) \in W$, $(h(x_0), f_i(x_0)) \in W$, which implies that, $\forall i \succeq i_0, \forall x \in U : (h(x), h(x_0)) \in \overset{3}{W} \subset V$. ■

We write $\mathcal{C}_{\mathfrak{S}}(X; Y)$ for the subset of $\mathcal{F}_{\mathfrak{S}}(X; Y)$ given by the continuous mappings. Whenever X is a Lipschitz space, we can similarly define $\mathcal{C}_b(X; Y)$.

COROLLARY 2.114.– *Suppose that the bornology* $\mathfrak{S}$ *satisfies the following condition: given a mapping* $f \in \mathcal{F}(X; Y)$*, if* $f|_A$ *is continuous for all*

$A \in \mathfrak{S}$, then f is continuous. Then the uniform space $\mathcal{C}_{\mathfrak{S}}(X;Y)$ is complete if and only if Y is complete.

PROOF.– This follows from Theorems 2.111, 2.113 and Lemma 2.77. ■

In particular:

COROLLARY 2.115.– *(i) $\mathcal{C}_u(X;Y)$ is complete if and only if Y is complete. (ii) If X is locally compact or metrizable, then $\mathcal{C}_c(X;Y)$ is complete if and only if Y is complete.*

PROOF.– (i) is clear, and so is (ii) in the case where X is locally compact. To show (ii) in the case where X is metrizable, we can use sequences: cf. section 2.1.3(**II**)). ■

2.7.3. *Equicontinuity*

DEFINITION 2.116.–

1) Let X be a topological space and Y a uniform space. We say that a subset H of $\mathcal{F}(X;Y)$ is equicontinuous *at the point $x_0 \in X$ if, for every entourage V of Y, there exists a neighborhood U of x_0 such that, for any choice of $f \in H$ and $x \in U$, we have that $(f(x), f(x_0)) \in V$.*

2) Let X, Y be uniform spaces. We say that a subset H of $\mathcal{F}(X;Y)$ is a uniformly equicontinuous *set if, for any entourage V of Y, there exists an entourage U of Y such that, for any choice of $f \in H$ and $(x', x'') \in U$, we have that $(f(x'), f(x'')) \in V$.*

Uniform equicontinuity clearly implies equicontinuity. Furthermore, these definitions imply that:

LEMMA 2.117.– *Let X be a topological space (resp. a uniform space), Y a uniform space, and H a subset of $\mathcal{F}(X;Y)$. For any $x \in X$, let $\tilde{x} : H \to Y$ be the Gelfand* transform *of x ([P1], section 3.2.2(**I**)), namely the mapping defined by $\tilde{x}(f) = f(x)$ $(f \in H)$. For H to be equicontinuous (resp. uniformly equicontinuous) at the point $x_0 \in X$, it is necessary and sufficient for the mapping $x \mapsto \tilde{x}$ from X into $\mathcal{F}_u(X;Y)$ to be continuous (resp. uniformly equicontinuous) at x_0.*

COROLLARY 2.118.– *Let X be a compact space and Y a uniform space. Every equicontinuous subset of $\mathcal{F}(X;Y)$ is uniformly equicontinuous.*

PROOF.– This is a consequence of Theorem 2.86 and Lemma 2.117. ■

The following result also holds (**exercise***, cf. [BKI 74], Chapter X, section 2.1, Corollary 3 and Proposition 2):

THEOREM 2.119.– *Let T and X be topological (resp. uniform) spaces, Y a uniform space, and f a mapping from $T \times X$ into Y. For each $x \in X$, write $f(.,x)$ for the partial mapping : $t \mapsto f(t,x)$ from T into Y and, for each $t \in T$, write $f(t,.)$ for the partial mapping $x \mapsto f(t,x)$ from X into Y. For f to be continuous (resp. uniformly continuous), it is necessary and sufficient for the following conditions to be satisfied:*

i) For every $x \in X$, $f(.,x)$ is continuous (resp. the set $\{f(.,x) : x \in X\}$ is equicontinuous in $\mathcal{F}(T;Y)$).

ii) The set $\{f(t,.) : t \in T\}$ is equicontinuous (resp. uniformly equicontinuous) in $\mathcal{F}(X;Y)$.

The first two of the three Ascoli-Arzelà theorems given below (which are not stated in their most general forms) may be shown as an **exercise*** (cf. [BKI 74], Chapter X, section 2.3, Proposition 6; section 2.4, Theorem 1); we will give a proof of the third using the first two.

THEOREM 2.120.– *(Ascoli-Arzelà, 1st* theorem*) Let X be a topological space, Y a uniform space, and H a subset of $\mathcal{F}(X;Y)$. For H to be equicontinuous, it is necessary and sufficient for the closure $\overline{H}$ of H in $\mathcal{F}_s(X;Y)$ to be equicontinuous.*

In particular, if H is equicontinuous, then every pointwise limit of a net in H is continuous.

THEOREM 2.121.– *(Ascoli-Arzelà, 2nd* theorem*) Let X be a topological (resp. uniform) space, Y a uniform space, and H an equicontinuous (resp. uniformly equicontinuous) subset of $\mathcal{C}(X;Y)$. Then, on H, the uniform structures of compact (resp. precompact) convergence and pointwise convergence on dense subsets D of X are identical.*

In other words, if $(f_i)_{i\in I}$ *is a net in* H *and* $(f_i(x)) \to f(x), \forall x \in D$, *then* $(f_i) \to f$ *uniformly on every compact (resp. precompact) set in* X.

THEOREM 2.122.– *(Ascoli-Arzelà, 3rd* theorem*) Let* X *be a topological (resp. uniform) space,* Y *a Hausdorff uniform space, and* $H \subset \mathcal{C}(X;Y)$.

1) H *is relatively compact in* $\mathcal{C}_c(X;Y)$ *(resp.* $\mathcal{C}_{pc}(X;Y)$*) whenever (i)* H *is equicontinuous (resp. uniformly equicontinuous) and (ii)* $H(x)$ *is relatively compact for all* $x \in X$, *where* $H(x) := \{f(x) : f \in H\}$,

2) If Y *is also* complete*, we can replace (ii) by (ii'):* $H(x)$ *is relatively compact in a* dense subset D *of* X.

3) Conversely, if X *is a locally compact topological space,* H *is always equicontinuous and condition (ii) is always satisfied whenever* H *is relatively compact in* $\mathcal{C}_c(X;Y)$.

PROOF.–

1) We will retrace the reasoning in the case where X is a topological space. Since $\overline{H(x)}$ is compact, $\prod_{x\in X} \overline{H(x)}$ is compact by Tychonov's theorem (Theorem 2.43), and therefore the closure $\overline{H}$ of H in $\mathcal{C}_s(X;Y)$ is compact. Since H is equicontinuous, $\overline{H}$ is also equicontinuous (Theorem 2.120), so $\overline{H}$ is compact in $\mathcal{C}_c(X;Y)$ and is the closure of H in $\mathcal{C}_c(X;Y)$ (Theorem 2.121).

2) It suffices to show that $H(x)$ is precompact for all $x \in X$. For any symmetric entourage V of Y, there exists a neighborhood U of x such that, for all $x' \in U \cap D$ and all $f \in H$, we have that $(f(x), f(x')) \in V$. But $H(x')$ is relatively compact in Y, so there exist finitely many points $y_k \in Y$ such that $H(x') \subset \bigcup_k V(y_k)$, and hence $H(x) \subset \bigcup_k \overset{2}{V}(y_k)$.

3) Since $\overline{H}$ is compact in $\mathcal{C}_c(X;Y)$, $\tilde{x}\left(\overline{H}\right) = \overline{H}(x)$ is compact for all $x \in X$ (Theorem 2.39), since the Gelfand transform (cf. Lemma 2.117) $\tilde{x} : \mathcal{C}_c(X;Y) \to Y$ is continuous. Let K be a compact set in X. The restriction mapping $r_K : f \mapsto f|_K$ from $\mathcal{C}_c(X;Y)$ into $\mathcal{C}_u(K;Y)$ is continuous, so $\overline{H}_K := r_K\left(\overline{H}\right)$ is compact. Let V be an arbitrary entourage of $\overline{H}_K$ and W a symmetric entourage of $\overline{H}_K$ such that $\overset{3}{W} \subset V$. There exist elements $f_1, ..., f_n \in \overline{H}$ such that the $W(f_i|_K)$ are a covering of $\overline{H}_K$ (Borel-Lebesgue property), hence such that $(f(x), f_i(x)) \in W, \forall i \in \{1, ..., n\}, \forall f \in H, \forall x \in K$. Given that the set of continuous mappings $\{f_i|_K : i = 1, ..., n\}$ is finite, it is equicontinuous at every point $a \in K$, so there exists a neighborhood

U of a in X such that $(f_i(x), f_i(a)) \in W, \forall i \in \{1, ..., n\}, \forall x \in U \cap K$. Hence, for every mapping $f \in H$ and every $x \in U \cap K$, we have that $(f(x), f(a)) \in \overset{3}{W} \subset V$. Since X is locally compact, we may choose the compact set K in such a way that a is in the interior of U, since every neighborhood U of a contains a compact set $K \ni a$; thus, H is equicontinuous at the point a, which was an arbitrary point of X. ■

2.7.4. *Equiboundedness*

Let E_1 and E_2 be two bornological sets, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ their respective bornologies, and $\mathfrak{b}(E_1; E_2)$ the set of bounded (or $(\mathfrak{S}_1, \mathfrak{S}_2)$-bounded) mappings from E_1 into E_2 (section 2.5.1(**I**)). A subset H of $\mathfrak{b}(E_1; E_2)$ is said to be *equibounded* (or $(\mathfrak{S}_1, \mathfrak{S}_2)$-bounded), if, for every bounded set A of E_1, $H(A) = \bigcup_{u \in H} u(A)$ is bounded in E_2. The equibounded sets of $\mathfrak{b}(E_1; E_2)$ form a bornology of this set, called the *equibornology*.

Let E_1, E_2 be two Lipschitz spaces. A subset H of $\mathcal{F}(E_1; E_2)$ is said to be pointwise bounded (resp. bounded with respect to the Lipschitz structure of bounded convergence) if it is $(\mathfrak{S}_1, \mathfrak{S}_2)$-bounded, where $\mathfrak{S}_2$ is the canonical bornology of E_2 and $\mathfrak{S}_1$ is the discrete (resp. canonical) bornology of E_1.

2.8. Topological algebra

2.8.1. *Topological groups*

(I) A *topological group* G is a group equipped with a topology $\mathfrak{T}$ for which the two mappings $(x, y) \mapsto xy$ from $G \times G$ into G and $x \mapsto x^{-1}$ from G into G are continuous; this is equivalent to saying that the mapping $(x, y) \mapsto xy^{-1}$ from $G \times G$ into G is continuous (**exercise**). The category of topological groups (whose morphisms are the continuous homomorphisms) is denoted $\mathbf{Topgrp}$. If $a \in G$, then the translations $a\circ : x \mapsto ax$ and $\circ a : x \mapsto xa$, both from G into G, are homeomorphisms, and hence the filter of neighborhoods of a is identical to both of the filters $a.\mathfrak{V}$ and $\mathfrak{V}.a$, where $\mathfrak{V}$ is the filter of neighborhoods of the neutral element e (**exercise**). The continuity of the mappings $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ at $x = y = e$, as well as the continuity of the mapping $x \mapsto axa^{-1}$, for all $a \in G$, implies the conditions (where $V.V := \{xy : (x, y) \in V \times V\}$)

(GT$_1$) $\forall U \in \mathfrak{V}, \exists V \in \mathfrak{V} : V.V \subset U,$

(GT$_2$) $(\forall U)\left(U \in \mathfrak{V} \Rightarrow U^{-1} \in \mathfrak{V}\right),$

(GT$_3$) $(\forall a)\,(\forall V)\left(a \in G, U \in \mathfrak{V} \Rightarrow a.V.a^{-1} \in \mathfrak{V}\right).$

Conversely, if $\mathfrak{V}$ is a filter on G that satisfies the conditions (**GT**$_1$), (**GT**$_2$), (**GT**$_3$), then there exists precisely one topology of topological groups for which $\mathfrak{V}$ is the filter of neighborhoods of e (**exercise***: cf. [BKI 74], Chapter III, section 1.2, Proposition 1).

As shown by A. Weil ([WEI 37], section 5), the topology on the topological group G determines two uniform structures: a left uniform structure, $\mathfrak{U}_g$, whose entourages V_g are the $(x, y) \in G \times G$ such that $x^{-1}y \in V$; and a right uniform structure, $\mathfrak{U}_d$, whose entourages V_d are the $(x, y) \in G$ such that $yx^{-1} \in V$, as V ranges over $\mathfrak{V}$; these structures are distinct in general, but G is complete with respect to $\mathfrak{U}_g$ if and only if it is complete with respect to $\mathfrak{U}_d$. Conversely, we have that (with the notation of section 2.4.1(**II**)) $\{V_g(e) : V_g \in \mathfrak{U}_g\} = \{V_d(e) : V_d \in \mathfrak{U}_d\} = \mathfrak{V}$ (**exercise**). Consequently, the topology on G may be deduced from either of the uniform structures $\mathfrak{U}_g$ and $\mathfrak{U}_d$, which implies:

LEMMA 2.123.– *The topology of a topological group is uniformizable.*

COROLLARY 2.124.– *Let G be a topological group and let e be its neutral element. The following conditions are equivalent:*

i) G is Hausdorff.

ii) G is completely regular.

iii) $\{e\}$ is closed in G.

iv) The intersection of the neighborhoods of e is equal to $\{e\}$.

PROOF.– We have that (i)⇔(ii) by Theorem 2.65, and (i)⇔(iii) by Lemma 2.66 (since $\{x\}$ is closed for every x if and only if the topology of G is T_1). (iii)⇔(iv): **exercise**. ■

We saw before that the categories $\mathbf{Grp}$ and $\mathbf{Top}$ admit projective limits and products: cf. ([P1], section 2.2.1(**III**)), and above, section 2.3.5). The same is true for $\mathbf{Topgrp}$; furthermore, if $\left\{X_i, \psi_i^j; I\right\}$ is an inverse system of topological groups, then $\varprojlim_{i\in I} X_i$ is given by (2.6). These products and projective limits commute with the two forgetful functors $\mathbf{Topgrp} \to \mathbf{Grp}$ and $\mathbf{Topgrp} \to \mathbf{Top}$ ([P1], section 1.3.3**(I)**). If the X_i are Hausdorff (resp. Hausdorff and complete, resp. compact), then $\varprojlim_{i\in I} X_i$ is a closed (resp. Hausdorff and complete, resp. compact) subgroup of the Hausdorff topological group $\prod_{i\in I} X_i$, and therefore is Hausdorff (resp. Hausdorff and complete, resp. compact). Moreover, the product isomorphism in $\mathbf{Grp}$ ([P1], section 2.2.3**(IV)**) is again an isomorphism in $\mathbf{Topgrp}$ (**exercise***: cf. [BKI 74], Chapter III, section 2.9, Proposition 26).

LEMMA 2.125.– *Let G and H be two topological groups, both equipped with their left or right uniform structures. Let $u : G \to H$ be a morphism of groups.*

1) The following conditions are equivalent: (i) u is continuous; (ii) u is continuous at some point; (iii) u is uniformly continuous.

*2) If u is a continuous and open epimorphism, then the mapping u is uniformly open (Definition 2.82) (**exercise**; for (2), cf. [JAM 90], p. 82).*

(II) If G is a Hausdorff topological group, then its uniform structures $\mathfrak{U}_g$ and $\mathfrak{U}_d$ are metrizable if and only if the neutral element e has a countable fundamental system of neighborhoods (Theorem 2.70); if so, $\mathfrak{U}_g$ and $\mathfrak{U}_d$ may be defined by metrics d_g and d_d that are left-invariant and right-invariant respectively, i.e. which satisfy $d_g(xy, xz) = d_g(y, z)$ and $d_d(yx, zx) = d_d(y, z)$ for any choice of $x, y, z \in G$ (Birkhoff-Kakutani theorem: cf. [BKI 74], Chapter IX, section 3.1, Proposition 2). Every *countable* product and similarly every *countable* projective limit of metrizable topological groups yields a metrizable topological group (Corollary 2.71).

(III) Let G be a topological group, H a normal subgroup of G ([P1], section 2.2.2), and $\varphi : G \twoheadrightarrow G/H$ the canonical surjection. The filter of neighborhoods of $\varphi(e)$ in the topological quotient space G/H has base $\varphi(\mathfrak{V})$, where $\mathfrak{V}$ is the filter of neighborhoods of e in G. It can be checked (**exercise**) that the filter generated by $\varphi(\mathfrak{V})$ satisfies the conditions ($\mathbf{GT}_1$), ($\mathbf{GT}_2$), ($\mathbf{GT}_3$). Hence, G/H, equipped with the quotient topology (section 2.3.4**(IV)**), is a topological group (called the *topological quotient group*). For G/H to be Hausdorff, it is necessary and sufficient for H to be closed in G (**exercise**); this necessary and sufficient condition also holds when $H \subseteq G$ is not normal, in which case G/H is a homogeneous set ([P1], section

2.2.8(**II**)), called a *homogeneous space* when equipped with the quotient topology (which is uniformizable: cf. [PON 66], Chapter 3, section 19, Theorem 10). The canonical surjection $\varphi : G \to G/H$ is uniformly open (**exercise**; cf. [JAM 90], p. 90).

LEMMA 2.126.– *Let G be a metrizable group and H a subgroup of G that is closed in G. Then the homogeneous space G/H is metrizable, and if G is complete, then G/H is complete.*

PROOF.– With the same notation as above, let (V_n) be a countable fundamental system of neighborhoods of e in G. Then $(\varphi(V_n))$ is a fundamental system of neighborhoods of $\varphi(e)$ in G/H, and, since $(\varphi(V_n))$ is countable, G/H is metrizable (Theorem 2.70). If G is also complete, then G/H is complete by Kelley's theorem (Theorem 2.83).[14] ■

THEOREM 2.127.– (open mapping) *Let G, G' be two metrizable topological groups and let $u : G \twoheadrightarrow G'$ be an epimorphism of* **Topgrp**. *Suppose that G is complete, and consider the three following conditions:*

i) The mapping u is open.

ii) G' is complete.

***(C)**: For every neighborhood U of the neutral element in G, $\overline{u(U)}$ is a neighborhood of the neutral element in G'.*

*1) (i) implies (ii) and **(C)** implies (i).*

*2) The condition **(C)** is satisfied whenever (ii) is satisfied and G is a Lindelöf space (Definition 2.6.3).*

*Consequently, if G is a Lindelöf space (and in particular if G is Polish (Theorem 2.109(i)), which is the most important case in practice), then (i)⇔(ii)⇔(**C**).*

PROOF.–

1) (a) (i)⇒(ii) by Kelley's theorem and Lemma 2.125(2). (b) Suppose that **(C)** is satisfied. Let e, e' be the neutral elements of G, G' and suppose that d, d' are metrics defining the topology of these two uniform spaces respectively.

14 A direct proof of this lemma can be found in ([DIE 82], Volume 2, (12.11.3)).

Then $\overline{u\left(B_d\left(e;r\right)\right)}$ is a neighborhood of e' in G' for all $r > 0$, and hence there exists $\rho\left(r\right) > 0$ such that $B_{d'}\left(e';\rho\left(r\right)\right) \subset \overline{u\left(B_d\left(e;r\right)\right)}$. By translation, $B_{d'}\left(u\left(x\right);\rho\left(r\right)\right) \subset \overline{u\left(B_d\left(x;r\right)\right)}$ for all $x \in G$. By Lemma 2.84, for any pair $\left(a,r\right)$ such that $a > r > 0$, we have that $B_{d'}\left(e';\rho\left(r\right)\right) \subset u\left(B_d\left(e;r\right)\right)$, which implies (i).

2) Let V be a neighborhood of e in G. By the conditions (**GT**$_1$), (**GT**$_2$), there exists a neighborhood U of e in G such that $U^{-1}U \subset V$. Since G is a Lindelöf space, it has a countable covering $\left\{Ux_n : n \in \mathbb{N}\right\}$. We have that $Ux_n \cong Ux_m$ for all $\left(n,m\right) \in \mathbb{N}\times\mathbb{N}$, so $\overline{u\left(Ux_n\right)} \cong \overline{u\left(Ux_m\right)}$; but $\bigcup_n \overline{u\left(Ux_n\right)} = G'$ and, by (ii), G' is not meager by Baire's (Theorem 2.102). Hence, $\overline{u\left(Ux_n\right)} \cong \overline{u\left(U\right)}$ has non-empty interior. Now, $\overline{u\left(U^{-1}U\right)} \supset \overline{u\left(U^{-1}\right)u\left(U\right)} \supset \overline{u\left(U^{-1}\right)}.\overline{u\left(U\right)} = \overline{u\left(U\right)}^{-1}.\overline{u\left(U\right)}$; if x' is in the interior of $\overline{u\left(U\right)}$, then x'^{-1} is in the interior of $\overline{u\left(U\right)}^{-1}$, so $e' = x'^{-1}x'$ is in the interior of $\overline{u\left(U\right)}^{-1}.\overline{u\left(U\right)} \supset \overline{u\left(V\right)}$. ■

THEOREM 2.128.– (closed graph) *Let G, H be two Polish topological groups and suppose that $u : G \to H$ is a group homomorphism. Then u is continuous if and only if its graph* $\mathrm{Gr}\left(u\right)$ *is closed in $G \times H$.*

PROOF.– This condition is necessary by Lemma 2.29. Conversely, if the group $\mathrm{Gr}\left(u\right)$ is closed in $G \times H$, then it is Polish (Theorem 2.79 and Lemma 2.77). Consider the two projections $p : \mathrm{Gr}\left(u\right) \to G : \left(g,u\left(g\right)\right) \mapsto g$ and $q : \mathrm{Gr}\left(u\right) \to H : \left(g,u\left(g\right)\right) \mapsto u\left(g\right)$. They are both continuous and $u = q \circ p^{-1}$. By Theorem 2.127, p is an open mapping, so p^{-1} is continuous, and therefore so is u. ■

(IV) Let $u : G \to G'$ be a morphism of $\mathbf{Topgrp}$; its kernel, image, cokernel, and coimage are defined in the same way as in $\mathbf{Ab}$, namely (writing e' for the neutral element of G')

$$\boxed{\ker\left(u\right) = u^{-1}\left(\left\{e'\right\}\right), \quad \mathrm{im}\left(u\right) = u\left(G\right), \quad \mathrm{coim}\left(u\right) = G/\ker\left(u\right),} \quad [2.10]$$

after equipping $u\left(G\right)$ with the topology induced by the topology on G', $u^{-1}\left(\left\{e'\right\}\right)$ with the topology induced by the topology on G, and $G/\ker\left(u\right)$ with the quotient topology of the topology on G. The homomorphism *of groups* u induces a bijective homomorphism $\breve{u} : \mathrm{coim}\left(u\right) \to \mathrm{im}\left(u\right)$ by Noether's first isomorphism ([P1], section 2.2.3(**I**), Theorem 2.12(1)), and $\breve{u}$

is continuous. Indeed, since u is continuous, if O' is open in G', then $u^{-1}(O') = u^{-1}(O' \cap \mathrm{im}(u))$ is open in G. Let $\pi : G \twoheadrightarrow G/\ker(u)$ be the canonical surjection. By the definition of the quotient topology (section 2.3.4(**IV**)), the open sets in $G/\ker(u)$ are the $\pi(O)$ such that O is an open set in G. Therefore, $\pi\left(u^{-1}(O' \cap \mathrm{im}(u))\right) = \breve{u}^{-1}(O' \cap \mathrm{im}(u))$ is open in $\mathrm{coim}(u)$.

Recall that a morphism u in a preabelian category is called a *strict morphism* if the induced morphism $\breve{u}$ is an isomorphism ([P1], section 3.3.7). *This definition may also be adopted in* $\mathbf{Topgrp}$*, even though the latter is not a preabelian category:*

DEFINITION 2.129.– *A morphism $u : G \to G'$ of* $\mathbf{Topgrp}$ *is said to be* strict *if the induced morphism $\breve{u} : \mathrm{coim}(u) \to \mathrm{im}(u)$ is an isomorphism.*

THEOREM 2.130.– *The following conditions are equivalent for a morphism of topological groups $u : G \to G'$:*

i) u is a strict morphism.

ii) For any open set O in G, $u(O)$ is open in $u(G)$.

iii) For any neighborhood U of 0 in G, $u(U)$ is a neighborhood of 0 in $u(G)$.

PROOF.– (ii)⇔(iii): **exercise**. (i)⇔(ii): (i) holds if and only if, for any open set Ω in $\mathrm{coim}(u)$, $\breve{u}(\Omega)$ is open in $\mathrm{im}(u)$. Moreover, Ω is open in $\mathrm{coim}(u)$ if and only if Ω is of the form $\pi(O)$, where O is open in G, in which case $\breve{u}(\Omega) = u(O)$. ■

COROLLARY 2.131.–

i) The composition of two strict epimorphisms (resp. monomorphisms) of topological groups is a strict epimorphism (resp. monomorphism).

*ii) If $v \circ u$ is a strict epimorphism (resp. monomorphism) of topological groups, then v is a strict epimorphism (resp. u is a strict monomorphism) of topological groups. (**Exercise**)*

COROLLARY 2.132.– *Let G, H be two metrizable and complete topological groups.*

i) Suppose that G is Polish. Every bimorphism $u : G \to H$ of $\mathbf{Topgrp}$ *([P1], section 1.1.1(**III**)) is an isomorphism; if $u : G \twoheadrightarrow H$ is an epimorphism* *of* $\mathbf{Topgrp}$*, then* u *induces an isomorphism from* $G/\ker(u)$ *onto* H*, so* H *is Polish.*

ii) Let $u : G \to H$ *be a morphism of* $\mathbf{Topgrp}$*. If* u *is a strict morphism, then* $u(G)$ *is closed in* H*. Conversely, if* G *and* H *are Polish and* $u(G)$ *is closed in* H*, then* u *is a strict morphism.*

PROOF.–

i) is an immediate consequence of Theorem 2.127.

ii) If u is a strict morphism, then $u(G) \cong G/\ker(u)$ is complete (Lemma 2.126), and so closed in H (Lemma 2.77). Conversely, supposing that G and H are Polish, if $u(G)$ is closed in H, then $u(G)$ is a Polish group and $\breve{u} : G/\ker(u) \to \mathrm{im}(u)$ is an isomorphism by (i). ■

2.8.2. *Topological abelian groups*

We write $\mathbf{Topab}$ for the category of topological abelian groups. Let $G \in \mathbf{Topab}$. Its left and right uniform structures, $\mathfrak{U}_g$ and $\mathfrak{U}_d$, are identical. The Birkhoff-Kakutani theorem (section 2.8.1**(II)**) implies that the topology on G may be defined by a metric that is both left-invariant *and* right-invariant. Given $G \in \mathbf{Topab}$, the Hausdorff completion $\hat{G}$ of G in $\mathbf{Topu}$ (Definition 2.81) is again a topological group[15]; this is defined as the Hausdorff completion of G in $\mathbf{Topab}$ (the reader may wish to show the case where G is metrizable as an **exercise**). In $\mathbf{Topab}$, every morphism $u : G \to G'$ has a cokernel

$$\boxed{\mathrm{coker}(u) = G'/\mathrm{im}(u)} \qquad [2.11]$$

equipped with the quotient topology of the topology on G', and the category $\mathbf{Topab}$ is preabelian ([P1], section 3.3.7**(III)**).

COROLLARY 2.133.– *A morphism of topological abelian groups* w *is strict if, and only if there exist a strict monomorphism of abelian groups* v *and a strict epimorphism of abelian groups* u *such that* $w = v \circ u$ *(**exercise**).*

15 This is not the case if G is not abelian ([WEI 37], p. 30-31).

2.8.3. *Topological rings and modules, etc.*

Let $\mathbf{A}$ be a ring on which a topology $\mathfrak{T}$ is defined. We say that $(\mathbf{A}, \mathfrak{T})$ is a *topological ring* if the two mappings $(x, y) \mapsto x - y$ and $(x, y) \mapsto xy$, from $\mathbf{A} \times \mathbf{A}$ into $\mathbf{A}$, are continuous. Let $\left(\hat{\mathbf{A}}, \hat{\mathfrak{T}}\right)$ be the Hausdorff completion of the underlying topological abelian group; then $\hat{\mathbf{A}}$ is a ring, and $\left(\hat{\mathbf{A}}, \hat{\mathfrak{T}}\right)$ is a topological ring, called the *Hausdorff completion* of the ring $\mathbf{A}$. The $\mathfrak{m}$-adic completion ([P1], section 3.1.8(**I**)) is one example of such a construction.

In a similar spirit, we can define the notions of *topological algebra*, *topological module* M over a topological ring $\mathbf{A}$ (where M is a topological abelian group and the action $(\lambda, x) \mapsto \lambda x$ from $\mathbf{A} \times M$ into M is continuous), *topological field* and *topological vector space* over a topological field. The reader is welcome to fill in the details. One example of the completion of a topological module was given in ([P1], section 3.1.8(**II**)) (Hausdorff completion of a topological module with respect to the $\mathfrak{m}$-adic topology). The category of topological rings and the category of topological modules over a topological ring (and continuous homomorphisms) both admit products and projective limits.

METATHEOREM 2.134.– *The properties stated above in the categories* $\mathbf{Topgrp}$ *and* $\mathbf{Topab}$ *also hold in the category of topological modules over a topological ring (and certainly hold in the category of topological vector spaces over a topological division ring).*

The field $\mathbb{R}$ is the completion of the valuation field $\mathbb{Q}$ equipped with its usual absolute value ([BKI 74], Chapter IV, section 1), and so $\mathbb{C} \cong \mathbb{R}^2$ is complete (Theorem 2.79). These two fields are furthermore locally compact by the Heine-Borel-Lebesgue theorem (Theorem 2.97).

3

Topological Vector Spaces

3.1. Introduction

In the 1930s and 1940s, after the publication of the book by S. Banach [BAN 32], the mathematical community might have been forgiven for thinking that functional analysis would henceforth essentially be set in stone, given the natural framework provided by normed vector spaces (and their completions: Banach spaces (Theorem 2.80)). But the holomorphic functions on an open set in the complex plane are just one example of a topogical vector space that is non-normable despite having the structure of a Fréchet-Montel space (section 4.3.2). The general theory of topological vector spaces was outlined by A. Kolmogorov and J. von Neumann in 1935, then completed in 1945-1946 by the fundamental contributions of G. Mackey on locally convex spaces [MAC 45], [MAC 46]. The significance of these generalizations became clear with the advent of distribution theory, developed by L. Schwartz between 1945 and 1950 following precursory work by S. Sobolev and S. Bochner[1]; this new theory propelled a previously unheard of type of topological vector space into the spotlight. Dieudonné and Schwartz showed in an article published in 1949 [DIE 49] that these spaces are in fact strict inductive limits of Fréchet spaces (Definitions 3.32 and 3.51). The groundwork for this article had been laid by a summary published in 1942 by Dieudonné on duality in locally convex spaces [DIE 42], as well as the

1 Schwartz was unaware of this work when he developed his distribution theory: cf. [SCW 97], pp. 234, 236. For an excellent presentation of the history leading up to distribution theory, cf. [LUT 82].

contributions by Mackey mentioned above. The article by Dieudonné-Schwartz was completed in the early 1950s by A. Grothendieck: generalization of the open mapping theorem and the closed graph theorem (Theorem 3.64)[2], notions of a Schwartz space [GRO 54] (section 3.4.6) and of a nuclear space (section 3.11), and general theory of tensor products of topological vector spaces [GRO 55] (which we sadly do not have the space to discuss here beyond a few basic ideas). Other generalizations of the open mapping theorem and the closed graph theorem can also be found in the literature, notably the ones derived by V. Pták. For an overview of these results, readers can refer to ([KOT 79], Volume II).

3.2. General topological vector spaces

3.2.1. *Topologies of topological vector spaces*

(I) Henceforth, the only topological vector spaces that we shall consider are defined over the field $\mathbb{K}$ of real or complex numbers[3]. In the following, $\mathbb{R}$ is embedded in $\mathbb{C}$ and, for $x \in \mathbb{K}^n$, $\Re(x), \Im(x), |x|$, and $\overline{x}$ respectively denote the real part of x, its imaginary part, any standard norm on x (cf. below, Theorem 3.17(2)), and the vector of complex conjugates of the components of x. In practice, there is little risk of confusing the notation $\overline{x}$ with that of the closure $\overline{A}$ of a subset in a topological space X. The category of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear mappings is written **Vec** (consistently with the notation of ([P1], section 1.1.1**(II)**) when $\mathbb{K}$ is implicit). As stated in section 2.8.3, a topological vector space $E[\mathfrak{T}]$ is $\mathbb{K}$-vector space E equipped with a topology $\mathfrak{T}$ that is compatible with its vector space structure, or in other words such that the mappings $E \times E \to E : (x, y) \mapsto x - y$ and $\mathbb{K} \times E \to E : (\lambda, x) \mapsto \lambda x$ are continuous (after equipping E and $\mathbb{K}$ with $\mathfrak{T}$ and the canonical topology (section 2.1.1) respectively). This topology $\mathfrak{T}$ (which we shall henceforth call a topology of $\mathbf{Tvs}$) determines a unique uniform structure (section 2.8.2). Any given topology of $\mathbf{Tvs}$ is invariant under translation in the sense that the filter of neighborhoods of any point $x \in E$ with respect to this topology is $\{U + x : U \in \mathfrak{U}\}$, where $\mathfrak{U}$ is the filter

2 The most important generalizations of these theorems and the relations between them are summarized and discussed in [BLS 14].

3 All of the results in this section, as well as Definition 3.27 and Theorem 3.30, still hold whenever $\mathbb{K}$ is a complete and locally compact non-discrete valued field.

of neighborhoods of 0. The latter is invariant under every homothety of non-zero ratio, which means that the set U belongs to $\mathfrak{U}$ if and only if λU belongs to $\mathfrak{U}$ for every $\lambda \in \mathbb{K}^{\times}$.

(II) A subset A of a $\mathbb{K}$-vector space E is said to be *balanced* if $\lambda A \subset A$ for all $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$. We say that A *absorbs* the set $B \subset E$ if there exists a real number $\lambda > 0$ such that $B \subset \mu A$ for every $\mu \in \mathbb{K}$ satisfying $|\mu| \geq \lambda$, and that it is *absorbing* if it absorbs every singleton subset. In a topological vector space, every neighborhood of 0 is absorbing (**exercise**). The following result holds (**exercise***: cf. [BKI 81], Chapter I, section 1.5, Proposition 4):

LEMMA 3.1.–

1) In a topological vector space, there exists a fundamental system $\mathfrak{V}$ of neighborhoods of 0 such that:

*(**TVS**$_1$) Every $V \in \mathfrak{V}$ is balanced and closed.*

*(**TVS**$_2$) For any $V \in \mathfrak{V}$, if $\lambda \neq 0$, then $\lambda V \in \mathfrak{V}$.*

*(**TVS**$_3$) For any $V \in \mathfrak{V}$, there exists $W \in \mathfrak{V}$ such that $W + W \subset V$.*

*2) Conversely, let E be a vector space over $\mathbf{K}$, and suppose that $\mathfrak{B}$ is a filter base on E satisfying the conditions (**TVS**$_1$), (**TVS**$_2$), (**TVS**$_3$) above. Then there exists a unique topology of $\mathbf{Tvs}$ for which $\mathfrak{B}$ is a fundamental system of neighborhoods of 0.*

(III) Let $(x_i)_{i \in I}$ be a family of elements in a topological vector space E. We write $\sum_{i \in I} x_i = x$, and say that the family $(x_i)_{i \in I}$ is *summable* in E with sum x, if, for any neighborhood V of 0 in E, there exists a *finite* set of indices $J \subset I$ such that the relation $\sum_{i \in K} x_i \in x + V$ holds for any *finite* set of indices K satisfying $J \subset K \subset I$. This implies that the set of $i \in I$ such that $x_i \neq 0$ is at most countable (**exercise**). If the set I is itself countable, then we may identify it with $\mathbb{N}$; if the sequence $(x_i)_{i \in \mathbb{N}}$ is summable in E, then $\sum_{i \in \mathbb{N}} x_i = \sum_{i=0}^{+\infty} x_i$ is said to be the sum of the *series* x_i, and this series is said to be *convergent* in E.

3.2.2. *Category of topological vector spaces*

(I) The morphisms of the category $\mathbf{Tvs}$ of topological vector spaces over $\mathbb{K}$ are the continuous linear mappings (also known as *continuous linear operators* in some contexts). Let $u : E \to F$ be a morphism of $\mathbf{Tvs}$; its image, kernel, coimage, and cokernel are defined in the same way as in $\mathbf{Topab}$ (cf. relations (2.10), (2.11) with $e' = 0$) and $\mathbf{Tvs}$ is a preabelian category. We write $\mathcal{L}(E;F)$ for the space of *continuous linear mappings* from E into F and $\mathrm{Hom}(E;F)$ for the space of *linear* mappings from E into F. When $F = \mathbb{K}$, $\mathrm{Hom}(E;F)$ is written E^* and is called the *algebraic dual* of E ([P1], section 3.1.2)); $\mathcal{L}(E;F)$ is written E' and is called the *dual* of E. An *operator in* E is defined to be a continuous endomorphism of E, or in other words an element of $\mathcal{L}(E) := \mathcal{L}(E;E)$; $\mathcal{L}(E)$ is a ring when equipped with the operations $+$ and $\circ$.

The definitions of the cokernel and image are different in the category $\mathbf{Tvsh}$ of Hausdorff topological vector spaces and continuous linear mappings. This is because, given a morphism $u : E \to F$ of $\mathbf{Tvsh}$, $F/u(E) \notin \mathbf{Tvsh}$ whenever $u(E)$ is not closed in F. From ([P1], section 3.3.7**(I)**), we deduce:

LEMMA 3.2.– *Let $u : E \to F$ be a morphism of* $\mathbf{Tvsh}$. *Then* $\ker(u) = u^{-1}(\{0\})$, $\mathrm{im}(u) = \overline{u(E)}$, $\mathrm{coker}(u) = F/\overline{u(E)}$, *and* $\mathrm{coim}(u) = E/\ker(u)$ *(where $\overline{u(E)}$ is the closure of $u(E)$ in F); the first two vector spaces are equipped with the induced topologies, and the latter two are equipped with the quotient topologies. The category* $\mathbf{Tvsh}$ *is preabelian.*

(II) Let $(E)_{i\in I}$ be a family of topological vector spaces; the product topology on $\prod_{i\in I} E_i$ is compatible with the $\mathbb{K}$-vector space structure of this space; consequently, the category $\mathbf{Tvsh}$ admits products and (since every morphism has a kernel and so every double arrow has an equalizer) arbitrary projective limits ([P1], section 1.2.8**(II)**, Proposition 1.22), which commute with the forgetful functors $\mathbf{Tvs} \to \mathbf{Vec}$ and $\mathbf{Tvs} \to \mathbf{Top}$ (cf. Remark 2.34). A topological vector space E over $\mathbb{C}$ is still a topological vector space E_0 over $\mathbb{R}$ by restriction of the field of scalars; conversely, a topological vector space F over $\mathbb{R}$ determines a topological vector space $\mathbb{C} \bigotimes_{\mathbb{R}} F = F \oplus iF$ $\left(i^2 = -1\right)$ over $\mathbb{C}$, called its *complexification*; this complex space is written $F_{(\mathbb{C})}$, and $\left(F_{(\mathbb{C})}\right)_0$ is isomorphic to $F \times F$. Every element $z \in F_{(\mathbb{C})}$ may therefore be uniquely expressed in the form $x + iy$, where $x, y \in F$; we write $x = \Re(z), y = \Im(z)$.

(III) Let E be a topological vector space, F a vector subspace of E, and $\varphi : E \to E/F$ the canonical surjection. The open sets of the *quotient topology* on E/F are the sets O such that $\varphi^{-1}(O)$ is open in E (section 2.3.4**(IV)**). When equipped with this topology, E/F is a topological vector space, which is Hausdorff if and only if F is closed in E (*ibid*). In particular, $E/\overline{\{0\}}$ is always Hausdorff. We say that a subset A of a topological vector space E is *total* in E if the $\mathbb{K}$-vector space $[A]$ generated by A is dense in E. The notion of a *total* family $(a_i)_{i \in I}$ in E may be defined similarly.

(IV) Let E be a topological vector space and suppose that E_1 is a closed subspace of E. We say that E_1 is a *split* subspace of E if there exists an isomorphism (of topological vector spaces) $E \to E_1 \times E_2$, where E_2 is another subspace of E, in which case E_2 is said to be a *topological complement* of E_1. The direct sum $E = E_1 \oplus E_2$ holds in **Vec**, as well as the isomorphism $E \cong E_1 \times E_2$ in **Tvs**, which is described by saying that $E_1 \oplus E_2$ is the *topological direct sum* of E_1 and E_2. More generally:

DEFINITION 3.3.– *Let E be a topological vector space that can be expressed as the direct sum $\bigoplus_{i=1}^{n} E_i$ in* **Vec**. *This direct sum is said to be a* topological direct sum *if $E \cong \prod_{i=1}^{n} E_i$ in* **Tvs**.

We know that any element $x \in \bigoplus_{i=1}^{n} E_i$ may be uniquely written in the form $x = \sum_{i=1}^{n} p_i(x), p_i(x) \in E_i$. We call $p_i : E \to E$ (such that $p_i(E) = E_i$) the *projector* of index i of E; each of these projectors is idempotent (i.e. $p_i \circ p_i = p_i$), and the following result holds:

LEMMA 3.4.–

1) The direct sum $E = \bigoplus_{i=1}^{n} E_i$ is a topological direct sum if and only if each projector p_i $(i = 1, ..., n)$ is continuous. If so, the projector p_i is a strict epimorphism from E onto E_i for each i.

2) In particular, a subspace E_1 of the topological vector space E is split if and only if there exists a continuous projector p_1 of E such that $p_1(E) = E_1$; if so, $E = E_1 \oplus \ker(p_1)$ is a topological direct sum and the projector p_2 of E such that $p_2(E) = \ker(p_1)$ is continuous.

PROOF.–

1) Each canonical projection $\pi_i : \prod_{i=1}^{n} E_i \twoheadrightarrow E : (x_i)_{1 \leq i \leq n} \mapsto x_i$ is continuous (section 2.3.5(**I**)), so the linear bijection $\theta : \prod_{i=1}^{n} E_i \to E : (x_i)_{1 \leq i \leq n} \mapsto \sum_{i=1}^{n} x_i$ is continuous and the mapping $E \to \prod_{i=1}^{n} E_i : x \mapsto (p_i(x))_{1 \leq i \leq n}$ is the inverse mapping of θ. Therefore, θ is an isomorphism of $\mathbf{Tvs}$ if and only if each p_i is continuous. We know that $\ker(p_i) = \bigoplus_{j=1, j \neq i}^{n} E_j$, so $E/\ker(p_i) \cong E_i$.

2) For all $x \in E$, we can write $x = p_1(x) + (x - p_1(x))$, where $p_1(x) \in E_1$ and $p_1(x - p_1(x)) = p_1(x) - p_1^2(x) = 0$, which implies that $x - p_1(x) \in \ker(p_1)$. Therefore, $E = E_1 + \ker(p_1)$, this sum is direct, and E_1 is split in E by (1). ■

THEOREM 3.5.– *Let E and F be topological vector spaces.*

1) A mapping $r \in \mathcal{L}(E; F)$ is right invertible in $\mathbf{Tvs}$*, or in other words admits a section $s \in \mathcal{L}(F; E) : r \circ s = 1_E$, if and only if it is a* split strict *epimorphism (i.e.* $\ker(r)$ *is a split subspace of E).*

2) A mapping $s \in \mathcal{L}(F; E)$ is left invertible in $\mathbf{Tvs}$*, or in other words admits a retraction $r \in \mathcal{L}(E; F) : r \circ s = 1_E$, if and only if it is a* split strict *monomorphism, or in other words an isomorphism of* $\mathbf{Tvs}$ *from F onto $s(F)$ such that $s(F)$ is split in E.*

3) Let $u \in \mathcal{L}(E; F)$. The following conditions are equivalent:

i) $u(E)$ is split in F and $\ker(u)$ *is split in E.*

ii) There exists a topological vector space G, an injection $\iota \in \mathcal{L}(G; F)$ such that $\iota(G)$ is split in F, and a surjection $\sigma \in \mathcal{L}(E; G)$ such that $\ker(\sigma)$ *is split in E, where $u = \iota \circ \sigma$.*

PROOF.– We will show the necessary condition of (1) (the sufficient condition of (1) and both directions of (2) are an **exercise**): if $r \circ s = 1_E$, then $r(s(E)) = E$, so $r(F) = E$, and $s(x) = 0 \Rightarrow r(s(x)) = 0 \Rightarrow x = 0$; consequently, r is surjective and s is injective. Let $p = s \circ r : E \to E$. Then $p^2 = s \circ (r \circ s) \circ r = s \circ r = p$, so p is idempotent. p is a continuous projector, so $p(E) = s(r(E)) = s(F)$ is split and admits the topological complement $\ker(p)$ (Lemma 3.4(2)). Furthermore, for all $x \in E$, $r(p(x)) = ((r \circ s) \circ r)(x) = r(x)$, and, since s is injective, $r(x) = 0 \Leftrightarrow (s \circ r)(x) = 0 \Leftrightarrow p(x) = 0$, which implies that $\ker(p) = \ker(r)$. We have that $\bar{r}^{-1} = \varphi \circ s$, where φ :

$E \twoheadrightarrow E/\ker(p)$ is the canonical epimorphism and $\bar{r}$ is the continuous linear bijection $E/\ker(p) \to F$ induced by r. Since $\varphi \circ s$ is continuous, $\bar{r}^{-1} : F \to E/\ker(p)$ is an isomorphism of **Evt**, and so r is a strict epimorphism.

(3): Conditions (i), (ii) are both equivalent to saying that u induces a mapping $\bar{u} \in \mathcal{L}(E;G)$, $G = u(E)$ for which the following diagram commutes, where η is an isomorphism of **Tvs** and $\sigma = \eta \circ \varphi$:

$$\begin{array}{ccccc} E & \xrightarrow{\bar{u}} & G & \xrightarrow{\iota} & F \\ \downarrow \varphi & \nearrow \eta & & & \\ E/\ker(u) & & & & \end{array}$$

■

If E is a one-dimensional Hausdorff topological vector space with basis $\{e\}$, then the mapping $\mathbb{K} \to E : \lambda \mapsto \lambda e$ is an isomorphism of **Tvs** (**exercise**); consequently, if E is a finite-dimensional Hausdorff topological vector space with basis $\{e_1, ..., e_n\}$, then the mapping $\psi : \mathbb{K}^n \to E : (\lambda_1, ..., \lambda_n) \mapsto \sum_{i=1}^n \lambda_i e_i$ is an isomorphism of **Tvs**. The next two lemmas follow:

LEMMA 3.6.– *On any finite-dimensional vector space E, there exists a unique Hausdorff topology of* **Tvs**.

PROOF.– The filter of neighborhoods of 0 in E has base $\{\psi(V) : V \in \mathfrak{V}\}$, where $\mathfrak{V}$ is the filter of neighborhoods of 0 in $\mathbb{K}^n$ with the above notation. ■

LEMMA 3.7.– *Let $u : E \to F$ be a linear mapping from a finite-dimensional Hausdorff topological vector space E into a topological vector space F. Then u is continuous.*

PROOF.– We have that $u = v \circ \psi^{-1}$, where ψ is the isomorphism given above, and $v : \mathbb{K}^n \to F$ is linear. The continuity of the mappings $F \times F \to E : (x, y) \mapsto x + y$ and $\mathbb{K} \times F \to F : (\lambda, x) \mapsto \lambda x$ implies the continuity of v (**exercise**), so u is continuous. ■

COROLLARY 3.8.– *In a Hausdorff topological vector space, every finite-dimensional vector subspace is complete, and hence closed.*

COROLLARY 3.9.– *Let E be a topological vector space and suppose that M is a closed vector subspace of finite codimension n (i.e. $\operatorname{codim}(M) := \dim(E/M) < \infty$). Every subspace N that is an algebraic complement of M is also a topological complement.*

PROOF.– The set $\{0\} = N \cap M$ is closed in N, so N is Hausdorff. Since M is closed in E, E/M is Hausdorff. The vector spaces N and E/M are isomorphic and the bijection $N \tilde{\to} E/M$ is an isomorphism of $\mathbf{Tvs}$ by Lemma 3.7. ■

Let E be a topological vector space and suppose that $f : E \to \mathbb{K}$ is a linear form. Suppose that $\alpha \in \mathbb{K}$; we say that the set $H = \{x \in E : f(x) = \alpha\}$ is a *hyperplane* in E. With this notation, the following result holds:

LEMMA 3.10.– *The hyperplane H is closed if and only if f is continuous.*

PROOF.– The sufficient condition is clear. For the converse, by performing translations, we can reduce to the case where $\alpha = 0$. Then E/H is a one-dimensional Hausdorff topological vector space, and $f = g \circ \varphi$, where the canonical surjection $\varphi : E \twoheadrightarrow E/H$ is continuous (by the definition of the quotient topology) and $g : E/H \to \mathbb{K}$ is continuous by Lemma 3.7. ■

3.2.3. *Riesz and Banach theorems*

THEOREM 3.11.– *(Riesz) Let E be a Hausdorff topological vector space E. Then E is locally compact if and only if E is finite-dimensional.*

PROOF.– (1) Suppose that E has dimension n. In $\mathbb{K}^n$, the hypercube $\mathbf{K} = \{x \in \mathbb{K}^n : |x_i| \leq 1, \forall i \in \{1, ..., n\}\}$ is compact by the Heine-Borel-Lebesgue theorem (Theorem 2.97). Therefore (with the notation defined before Lemma 3.6), $\psi(\mathbf{K})$ is a compact neighborhood of 0 in E (Theorem 2.39), and $\left\{m^{-1}\psi(\mathbf{K}) : m \geq 1\right\}$ is a fundamental system of compact neighborhoods of 0 in E. (2) Conversely, suppose that E has a compact neighborhood V of 0. Let $\lambda \in \mathbb{K}$ be such that $0 < |\lambda| < 1$. There exist finitely many points $a_i \in V$ such that $V \subset \bigcup_i (a_i + \lambda V)$. The subspace M generated by the a_i is isomorphic to $\mathbb{K}^p$, where $p = \dim_{\mathbb{K}}(M)$, by Lemma 3.6, and thus is complete by Theorem 2.79 (since $\mathbb{K}$ is complete), and so is closed in E (Lemma 2.77). In E/M, the canonical image of V is a compact neighborhood W of 0 such that $W \subset \lambda W$, which implies that $W \subset \lambda^n W$ for all $n \geq 1$, and, since W is absorbing (section 3.2.1(**II**)), $W = E/M$. If $E/M \neq \{0\}$, then it contains a straight line isomorphic to $\mathbb{K}$, which is complete and hence closed in E/M, and therefore compact, which is impossible. We conclude that $E = M$. ■

The definition of a *strict morphism* given above in **Topgrp** (Definition 2.129) remains valid, *mutatis mutandis*, in **Tvs**, as does the characterization of strict morphisms established by Theorem 2.130. If G and H are metrizable and complete topological vector spaces, then the condition **(C)** of Theorem 2.127 is satisfied, as can be seem by replacing the translations Ux_n by $\alpha^n U$ in part (2) of the proof of this theorem, where $\alpha \in \mathbb{K}$ satisfies $|\alpha| > 1$. This allows us to reformulate Theorems 2.127, 2.128, and Corollary 2.132 as follows:

THEOREM 3.12.– *(Banach* theorems*) Let E and F be two metrizable topological vector spaces and suppose that $u : E \to F$ is a linear mapping. Suppose further that E is complete.*

*1) (*open mapping *or Banach-Schauder* theorem*) Suppose that u is continuous. The following conditions are equivalent:*

i) u is a surjective strict morphism.

ii) F is complete and u is surjective.

2) Suppose that F is complete and u is continuous.

*i) (*inverse operator theorem*) If u is bijective, then it is an isomorphism.*

ii) u is a strict morphism if and only if $u(E)$ is closed in F.

*3) (*closed graph theorem*) Suppose that F is complete. Then u is continuous if and only if its graph* $\mathrm{Gr}(u)$ *is closed in $E \times F$.*

For more details, see section 3.8.1. There are many other classical results from the theory of locally convex spaces that may be extended to general topological vector spaces, some with more difficulty than others, but many of these generalizations currently seem to represent little more than curiosities, with few truly useful consequences in practice.

3.3. Locally convex spaces

3.3.1. *Convex sets and functions*

Let E be a vector space over $\mathbb{K}$. Let x and y be two points of E such that $x \neq y$. We say that $[x, y] = \{\lambda x + (1 - \lambda) y : \lambda \in [0, 1] \subset \mathbb{R}\}$ (resp. $]x, y[= \{\lambda x + (1 - \lambda) y : \lambda \in]0, 1[\subset \mathbb{R}\}$) is the *closed segment* (resp. *open segment*) with endpoints x and y. (In some cases, this terminology may be slightly misleading: when E is a Hausdorff topological vector space, the closed segment $[x, y]$ is compact as the image of the compact set $[0, 1]$ in $\mathbb{R}$ under the mapping $\lambda \mapsto \lambda x + (1 - \lambda) y$ (Theorem 2.39), and so is closed in E (section 2.3.7), but $]x, y[$ is not open in E if $E \neq \mathbb{R}$.) A subset A of E is said to be *convex* ([P1], section 3.3.3(**V**)) if, for any two points x, y in A, the whole closed segment $[x, y]$ is also contained in A. This set is said to be *strictly convex* if the open segment $]x, y[$ is contained in $\overset{\circ}{A}$ whenever x, y are in A and $x \neq y$. Let A be a convex subset of E. A function $f : E \to \bar{\mathbb{R}}$ is said to be *convex* (resp. *strictly convex*) if, given two points x, y in E such that $x \neq y$ and $\lambda \in]0, 1[\subset \mathbb{R}$, the relation $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$ (resp. $f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y)$) holds. In other words, a function $f : A \to \bar{\mathbb{R}}$ is convex (resp. strictly convex) if and only if its epigraph $\operatorname{epi}(f) := \left\{(x, y) \in A \times \bar{\mathbb{R}} : f(x) \leq y\right\}$ is convex (resp. strictly convex). Part (1) of the next result may be shown as an **exercise***: cf. ([BKI 81], Chapter II, section 2.6, Proposition 14 and Corollary 1 of Proposition 16). Part (2) is another very simple **exercise**.

LEMMA 3.13.–

1) In a topological vector space E, the interior $\overset{\circ}{A}$ and the closure $\overline{A}$ of a convex set A are also convex; furthermore, if $\overset{\circ}{A} \neq \varnothing$, then $\overline{A} = \overline{\overset{\circ}{A}}$ and $\overset{\circ}{A} = \overset{\circ}{\overline{A}}$.

2) Let u be a linear mapping from a vector space E into a vector space F; then, whenever A is a convex subset of E, $u(A)$ is also a convex subset of F.

DEFINITION 3.14.– *A subset A of a vector space E is called a* disk *(or said to be* disked*, or* circled*) if it is balanced and convex.*

Every disk in a vector space contains the origin (**exercise**).

DEFINITION 3.15.– *A topological vector space over* $\mathbb{K}$ *is said to be* locally convex *if the set of neighborhoods of* 0 *has a fundamental system of convex neighborhoods. The topology of any such space is said to be* locally convex.

LEMMA 3.16.– *Let* E *be a vector space over* $\mathbb{K}$.

i) Let $\mathfrak{T}$ *be a locally convex topology on* E. *There exists a fundamental system of neighborhoods of* 0 *for this topology consisting solely of closed disks.*

ii) Conversely, let $\mathfrak{B}$ *be a filter base on* E *consisting of absorbing disks. Then* $\mathfrak{V} = \{\lambda V : \lambda > 0, V \in \mathfrak{B}\}$ *is a fundamental system of neighborhoods of* 0 *for a locally convex topology on* E.

PROOF.– (i): Let V be a neighborhood of 0 in $\mathfrak{T}$. The continuity of the mapping $(\lambda, x) \mapsto \lambda x$ from $\mathbb{K} \times E$ into E implies the existence of a real number $\alpha > 0$ and a closed convex neighborhood $V' \subset V$ of 0 such that $\lambda x \in V'$ whenever $|\lambda| < \alpha$ and $x \in V'$. Then $V' \supset W$, where $W = \bigcup_{|\lambda|<\alpha} \lambda V'$, so $V' \supset \overline{W}$, and $\overline{W}$ is a closed disked neighborhood of 0. (ii) follows from Lemma 3.1. ■

3.3.2. *Semi-norms*

(I) A *semi-norm* on E is a mapping $p : E \to \mathbb{R}_+$ that satisfies the following conditions for all $x, y \in E$ and all $\lambda \in \mathbb{K}$:

(N$_1$) $p(0) = 0$.

(N$_2$) $p(\lambda x) = |\lambda| \, p(x)$.

(N$_3$) $p(x + y) \leq p(x) + p(y)$ (“triangle inequality”).

A semi-norm p is said to be a *norm* if it also satisfies the condition

(N$_4$) $p(x) = 0 \Rightarrow x = 0$.

Any semi-norm p is a convex mapping (**exercise**) and determines a pseudometric $d : (x, y) \mapsto p(x - y)$. If p is a norm, then d is a metric. Let p be a semi-norm. For all $\alpha > 0$, write

$$B_p(\alpha) = \{x \in E : p(x) < \alpha\}, \quad B_p^c(\alpha) = \{x \in E : p(x) \leq \alpha\}.$$

The sets $B_p(\alpha)$ and $B_p^c(\alpha)$ are both disked and absorbing, and the following conditions are equivalent (**exercise**): (i) p is continuous on E; (ii) p is continuous at 0; (iii) p is uniformly continuous; (iii) for all $\alpha > 0$, $B_p(\alpha)$ is open in E (so is an open neighborhood of 0); (iv) for all $\alpha > 0$, $B_p^c(\alpha)$ is a neighborhood of 0. Given a disked and absorbing set $A \subset E$, define $p_A(x) = \inf\{\rho : \rho > 0, x \in \rho A\}$ for all $x \in E$. Then p_A is a semi-norm on E (**exercise**), called the *gauge* (or Minkowski functional) of A; furthermore, $B_{p_A}(1) \subset A \subset B_{p_A}^c(1)$. If A is open (resp. closed) in E, then p_A is continuous (resp. lower semi-continuous, cf. Definition 2.31) and $A = B_{p_A}(1)$ (resp. $A = B_{p_A}^c(1)$) (**exercise***: cf. [BKI 81], Chapter II, section 2.10, Proposition 23). Conversely, suppose that p_A is lower semi-continuous; then $p_A^{-1}([0,1]) = p_A^{-1}\left(\complement_{\mathbb{R}_+}]1,\infty[\right)$ and $p_A^{-1}\left(\complement_{\mathbb{R}_+}]1,\infty[\right) = \complement_E\left(p_A^{-1}(]1,\infty[)\right)$ ([P1], section 1.1.2(**VI**)); but $p_A^{-1}(]1,\infty[)$ is open, so $p_A^{-1}([0,1])$ is closed.

(II) A topological vector space is said to be *semi-normed* if its topology $\mathfrak{T}$ is determined by a semi-norm p, that is, if $\{B_p(\alpha) : \alpha > 0\}$ is a fundamental system of neighborhoods of 0 for $\mathfrak{T}$ (any such space is Hausdorff if and only if p is a norm).

Let E be a vector space on $\mathbb{K}$ and suppose that p_1, p_2 are two semi-norms on E. These semi-norms are said to be *equivalent* if they determine the same topology on E. The proof of claim (1) of the next theorem is an **exercise**. Claim (2) follows from Lemma 3.6:

THEOREM 3.17.–

1) The two semi-norms p_1, p_2 on E are equivalent if and only if there exist real numbers $\alpha > 0, \beta > 0$ such that $\alpha p_1 \leq p_2 \leq \beta p_1$.

2) On $\mathbb{K}^n$, all norms are equivalent.

3.3.3. *Locally convex topologies*

Let E be a locally convex space; there exists a base $\mathfrak{B}$ of the filter of neighborhoods of 0 such that every $V \in \mathfrak{B}$ is a closed disk, which implies that $n^{-1}V \in \mathfrak{B}$ for every integer $n > 0$; if $(V_i)_{i \in J}$ is a *finite* family of elements of $\mathfrak{B}$, then $\bigcap_{i \in J} V_i \in \mathfrak{B}$, and $\bigcap_{i \in J} V_i = \bigcap_{i \in J} B_{p_i}^c(1)$, where p_j is the gauge of V_j. Conversely, let $(p_i)_{i \in I}$ be a family of semi-norms on a $\mathbb{K}$-vector space E. This family determines Lipschitz and uniform structures, as well as a topology $\mathfrak{T}$ (section 2.4.3). A subset V of E is a neighborhood of 0 in $\mathfrak{T}$ if and only if there exists a *finite* set of indices $J \subset I$ and a real number

$\lambda > 0$ such that $\bigcap_{i\in J} B^c_{p_i}(1) \subset \lambda V$. The topology $\mathfrak{T}$ is the coarsest of the topologies that are invariant under translation and for which all the semi-norms p_i are continuous; this topology is locally convex. Hence:

LEMMA 3.18.– *Let E be a vector space on $\mathbb{K}$. Any locally convex topology $\mathfrak{T}$ on E may be defined by a family of semi-norms that are continuous for $\mathfrak{T}$. Conversely, a family of semi-norms on E uniquely determines a locally convex topology $\mathfrak{T}$ on E for which the set of $\left\{n^{-1}\bigcap_{i\in J} B^c_{p_i}(1) : J \in \mathfrak{J}, n \in \mathbb{N}^*\right\}$ is a fundamental system of closed disked neighborhoods of* 0, *where $\mathfrak{J} := \mathfrak{P}_f(I)$ is the set of finite subsets of I.*

REMARK 3.19.– *If we replace the p_i by $p_J = \sup\{p_j : J \in \mathfrak{J}\}, \mathfrak{J} := \mathfrak{P}_f(I)$, the locally convex topology determined by the family $(p_i)_{i\in I}$ does not change. The family $(p_J)_{J\in\mathfrak{J}}$ is* right directed, *which means that $p_J \leq p_{J'}$ whenever $J \preceq J'$, where $\preceq$ is the order relation on $\mathfrak{J}$ defined by $J \preceq J' \Leftrightarrow J \subset J'$ $(J, J' \subset \mathfrak{J})$.*

It follows from Theorem 2.70 that:

COROLLARY 3.20.– *A Hausdorff locally convex space is metrizable if and only if its topology may be defined by a countable family of semi-norms.*

3.3.4. *Hahn-Banach theorem*

(I) CONES Let E be a vector space over $\mathbb{K}$ (the field of real *or* complex numbers). A subset M of E is said to be a *linear variety* if it is of the form $x_0 + F$, where F is a vector subspace of E. A *cone* with summit x_0 in a *real* vector space is a subset C that is invariant under every homothety of center x_0 and ratio $\lambda > 0$. A cone with summit x_0 is said to be *pointed* if it contains x_0, and *blunt* otherwise. A convex cone with summit x_0 is said to be *salient* if it does not contain any straight line through x_0. The image of a cone (resp. of a convex salient cone) with summit 0 under a linear mapping is a cone (resp. a convex salient cone) with summit 0 (**exercise**).

(II) Mazur's theorem and its consequences

THEOREM 3.21.– *(Mazur)*[4] *Let E be a topological vector space over $\mathbb{K}$, M a linear variety in E, and A a non-empty convex subset disjoint from M. Then there exists a closed hyperplane H in E containing M that is disjoint from A.*

PROOF.– By translation, we can reduce to the case where M contains 0, which implies that $M = F$. Let $\mathfrak{F}$ be the family of all subspaces L of E such that $F \subset L$ and $L \cap A = \varnothing$; $\mathfrak{F}$ may be ordered by inclusion. Let Φ be a chain in $\mathfrak{F}$ ([P1], section 1.1.2(**III**)). The union of the elements of Φ is a subspace L' of E such that $F \subset L'$ and $L' \cap A = \varnothing$, so is the largest element of Φ. By Zorn's lemma ([P1], section 1.1.2(**III**), Lemma 1.3), $\mathfrak{F}$ has a maximal element H. But its closure $\overline{H}$ must also be a maximal element of $\mathfrak{F}$, so $H = \overline{H}$ and H must be closed. Consequently, E/H is a Hausdorff topological vector space (section 2.8.1(**III**)). It simply remains to be shown that H is a hyperplane, or in other words that $\dim(E/H) = 1$. To do this, let us consider the following two cases separately: $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

(1) If $\mathbb{K} = \mathbb{R}$, let $\varphi : E \twoheadrightarrow E/H$ be the canonical surjection. Then $\varphi(A)$ is a convex open set that does not contain the origin of E/H. The set $C = \bigcup_{\lambda>0} \lambda\varphi(A)$ is a convex open cone that does not contain 0. Suppose that $\dim(E/H) \geq 2$; then C is not a half-line, as this would have empty interior, so the frontier $\mathrm{Fr}(C)$ must contain at least one point $x \neq 0$. But then $-x \notin C$, since otherwise there would exist points x' and $-x'$ in C arbitrarily close to x and $-x$ respectively, but distinct from 0; the segment $[x', -x']$ would be contained in C, which is impossible, because $0 \notin C$. Since x and $-x$ belong to $\complement_{E/H} C$, so does the line D passing through them; hence, $\varphi^{-1}(D)$ is a subspace of E disjoint from A that contains H, which contradicts the maximality of H. We conclude that H must be a hyperplane.

(2) If $\mathbb{K} = \mathbb{C}$, proceeding as in section 3.2.2(**II**), we first view E as a topological vector space over $\mathbb{R}$. By (1), there exists a *real* hyperplane H containing F and disjoint from A. We know that $F = iF$, so $F \subset H_1$, where $H_1 = H \cap iH$ is a closed *complex* hyperplane in E disjoint from A. ■

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4 This theorem is often objectionably (cf. [KOT 87]) called the *geometric* Hahn-Banach *theorem*. It was shown by S. Mazur in 1933 using the "analytic" Hahn-Banach theorem (Theorem 3.25), which itself was proven in 1929. The first of these forms is more intuitive than the second (readers may wish to draw a diagram), which is why we shall present it first.

In a topological vector space E, consider two non-empty sets A and B, as well as a closed *real* hyperplane H described by the equation $f(x) = \alpha$, where $f : E \to \mathbb{R}$ is a *real* continuous linear form (Lemma 3.10). We say that H *separates* (resp. *strictly separates*) A and B if $f(x) \leq \alpha$ (resp. $f(x) < \alpha$) for all $x \in A$ and $f(x) \geq \alpha$ (resp. $f(x) > \alpha$) for all $x \in B$ (this naturally still holds when the roles of A and B are switched).

THEOREM 3.22.– (convex set separation) *Let A and B be two non-empty open convex subsets of a topological vector space E such that A is open and $A \cap B = \varnothing$. Then there exists a closed real hyperplane that separates A and B. If B is also open, then there exists a closed real hyperplane that* strictly *separates A and B.*

PROOF.– The set $A - B := \{a - b : a \in A, b \in B\}$ is an open convex subset of E (Lemma 3.13(2)) that does not contain 0. By Theorem 3.21, there exists a closed hyperplane H containing 0 that is disjoint from $A - B$. But H is the kernel of a real continuous linear form f (Lemma 3.10). Therefore (redefining f as $-f$ if necessary), for all points $a \in A$ and $b \in B$, $f(a - b) > 0$, and so $f(a) > f(b)$. Since $B \neq \varnothing$, we have that $\alpha := \inf_{a \in A} f(a) > -\infty$, and so the real hyperplane $H = \{x \in E : f(x) = \alpha\}$ separates A and B. If B is also open, then $\overline{B}$ is convex (Lemma 3.13(1)) and disjoint from A; hence, there exists a closed real hyperplane H_1 that separates A and $\overline{B}$, which therefore strictly separates A and B. ■

From Lemma 3.13(1), we can deduce the next result:

COROLLARY 3.23.– *(Eidelheit's theorem) Let A and B be non-empty convex subsets of a topological vector space E, and suppose that A has non-empty interior $\overset{\circ}{A}$. If B is disjoint from $\overset{\circ}{A}$, then there exists a closed real hyperplane H that separates A and B.*

COROLLARY 3.24.– *Let A and K be non-empty disjoint convex subsets of a topological vector space E. Suppose that A is closed and K is compact. Then there exists a closed real hyperplane that strictly separates A and K.*

PROOF.– There exists a convex open neighborhood V of 0 in E such that $A + V$ and $K + V$ have no points in common (Lemma 2.88). These sets are open and convex in E, so, by Theorem 3.22, there exists a closed real hyperplane that strictly separates them, which must clearly also strictly separate A and K. ■

THEOREM 3.25.– *(Hahn-Banach) Let E be a vector space over $\mathbb{K}$. Let p be a semi-norm on E, V a vector subspace of E, and suppose that $f : V \to \mathbb{K}$ is a linear form on V such that $|f(y)| \leq p(y), \forall y \in V$. Then there exists a linear form $f_1 : E \to \mathbb{K}$ extending f that satisfies $|f(x)| \leq p(x), \forall x \in E$.*

PROOF.– Suppose that $f \neq 0$, since the case where $f = 0$ is trivial. The vector space E equipped with the semi-norm p is a semi-normed space for whose topology $\mathfrak{T}$ the set of convex disks $\left\{B_p\left(n^{-1}\right) : n > 0\right\}$ is a fundamental system of neighborhoods of 0 (section 3.3.2(**II**)). Let $H = \{x \in V : f(x) = 1\}$; H is a hyperplane in V and a linear variety in E. The set $A = B_p(1)$ is open and convex and disjoint from H, so, by Theorem 3.21, there exists a closed hyperplane H_1 in the topological vector space E that contains H and is disjoint from A. Therefore, $H_1 \cap V = H \cap V = H$. Since $0 \notin H_1$, there exists a continuous linear form (for the topology $\mathfrak{T}$) f_1 such that $H_1 = \{x \in E : f_1(x) = 1\}$; so $f_1|_H = f$. Furthermore, since $H_1 \cap A = \varnothing$, $|f_1(x)| \leq p(x)$ for all $x \in E$. ■

COROLLARY 3.26.– *Let E be a locally convex space and suppose that x_0 is a point that is not in the closure of $\{0\}$. Then there exists a continuous linear form $f_1 : E \to \mathbb{K}$ such that $f_1(x_0) \neq 0$.*

PROOF.– Let $M = \mathbb{K}x_0$. We know that $M \cong \mathbb{K}$, so M is Hausdorff, and the linear form $f : M \to \mathbb{K} : \xi \mapsto \xi x_0$ is continuous. Hence, there exists (Lemma 3.18) a continuous semi-norm p on E such that $|f(y)| \leq p(y), \forall y \in M$. By Theorem 3.25, there exists a linear form $f_1 : E \to \mathbb{K}$ extending f (which implies that $f_1(x_0) \neq 0$) such that $|f_1(x)| \leq p(x), \forall x \in E$, and which is therefore continuous. ■

3.3.5. *Category of locally complex spaces*

The category **Lcs** of locally convex spaces (over $\mathbb{K}$) and continuous linear mappings is a concrete category with base **Vec**. Let $\mathcal{S} = (f_i : E \to E_i)$ be a source of $(\mathbf{Lcs}, |.|, \mathbf{Vec})$ ([P1], section 1.3.2(**I**)); the coarsest topology on E for which the *linear mappings* $f_i : E \to E_i$ are continuous is locally convex ([BKI 81], Chapter II, section 4.3, Proposition 4)[5]. Hence, **Lcs**, like **Tvs**,

5 With the notation of ([P1], section 1.3.2(**I**)) (see also section 2.3.5 and footnote 5 on page 73 above), these linear mappings should strictly speaking be written $|f_i| : |E| \to |E_i|$, but we fear that such an overzealous level of precision might result in confusion. We will simplify the notation accordingly for the rest of the presentation.

admits arbitrary products and *projective limits* (cf. Remark 2.34), which commute with the forgetful functors $\mathbf{Lcs} \to \mathbf{Vec}$ and $\mathbf{Lcs} \to \mathbf{Top}$. Let E be a locally convex space, $(p_i)_{i \in I}$ a family of semi-norms that define the topology of E (Lemma 3.18), and $F \subset E$. Then F is a locally convex space when equipped with the family $(p_i\,|_E)_{i \in I}$ whose topology is induced by the topology of E (section 2.3.4**(II)**). Let $\varphi : E \to E/F$ be the canonical surjection. The quotient topology on E/F (section 2.3.4**(IV)**) is identical to the topology defined by the family of semi-norms $(\dot{p}_i)_{i \in I}$, where $\dot{p}_i(\bar{x}) = \inf \{p_i(x) : \varphi(x) = \bar{x}\}$ ([BKI 81], Chapter II, section 1.3), and is therefore locally convex. The image, kernel, coimage, and cokernel in $\mathbf{Lcs}$ of a morphism $u : E \to F$ are defined as in $\mathbf{Tvs}$ or $\mathbf{Topab}$ (cf. $(2.10), (2.11)$ with $e' = 0$). Moreoever, in the category $\mathbf{Lcsh}$ of Hausdorff locally convex spaces and continuous linear mappings, the kernel, cokernel, image, and coimage in $\mathbf{Lcsh}$ of a morphism $u : E \to F$ are defined in the same way as in $\mathbf{Tvsh}$ (Lemma 3.2). The categories $\mathbf{Lcs}$ and $\mathbf{Lcsh}$ are both preabelian.

3.3.6. *Bounded sets*

A subset A of a locally convex set E is *bounded* (section 2.5.1**(IV)**)[6] if and only if $\sup_{x \in A} p_i(x) < \infty$ for all $i \in I$, where $(p_i)_{i \in I}$ is a family of semi-norms that defines the topology on E; this is equivalent to the following condition (**exercise**), which we shall use as the definition of a bounded subset in a (possibly non-locally-convex) topological vector space:

DEFINITION 3.27.– *A subset A of a topological vector space E is said to be* bounded *if A is absorbed by every neighborhood of* 0.

REMARK 3.28.– *Let E be a $\mathbb{K}$-vector space, and let $\mathfrak{T}_1, \mathfrak{T}_2$ be topologies of* $\mathbf{Tvs}$ *on E such that $\mathfrak{T}_1$ is finer than $\mathfrak{T}_2$. It follows from Definition 3.27 that, if a set $A \subset E$ is bounded in $E[\mathfrak{T}_1]$, then it is also bounded in $E[\mathfrak{T}_2]$.*

LEMMA 3.29.– *Let E be a locally convex space and B a bounded subset of E. Then the disked envelope of B is bounded in E (**exercise**).*

THEOREM 3.30.– *Let A be a subset of a topological vector space E. Then A is bounded if and only if, for every sequence (λ_n) that converges to 0 in $\mathbb{K}$ and*

6 Here, we are working with the canonical bornology b (cf. section 2.5.1**(IV)**), also known as the von Neumann *bornology* in the context of topological vector spaces.

for every sequence (x_n) of points in A, the sequence $(\lambda_n x_n)$ converges to 0 in E. Hence, a subset A of E is bounded if and only if every countable subset of A is bounded.

PROOF.– (1) Suppose that A is bounded; let V be a balanced neighborhood of 0 and suppose that (λ_n) is a sequence that converges to 0 in $\mathbb{K}$. Since A is absorbed by V, there exists an integer $N > 0$ such that $\lambda_n x_n \in V$ for every integer $n \geq N$, so $(\lambda_n x_n)$ converges to 0 in E. (2) Conversely, let A be a subset of E satisfying the condition stated above, and suppose that A is not bounded. Then there exists a neighborhood U of 0 in E such that, for every sequence (λ_n) in $\mathbb{K}$, $A \nsubseteq \lambda_n U$. Pick λ_n such that $|\lambda_n| \geq n$ for all n and let $x_n \in A$ such that $x_n \notin \lambda_n U$. Then $\lambda_n^{-1} x_n \notin U$ for all n, but $\left(\lambda_n^{-1}\right) \to 0$: contradiction. ■

3.3.7. *Topological direct sums and inductive limits*

Definition 3.3 for *finite* topological sum can be generalized in **Lcs** to arbitrary sets of indices. Let $(E_i)_{i\in I}$ be a family of locally convex spaces, and consider the direct sum $\bigoplus_{i\in I} E_i$ ([P1], section 2.3.1(**III**)) in **Vec**, as well as the canonical injections $\mathrm{inj}_i : E_i \hookrightarrow E$. The *coproduct* $E = \coprod_{i\in I} E_i$ of the E_i in **Lcs** is called their *topological direct sum*. As a vector space, $E = \bigoplus_{i\in I} E_i$. This space is then equipped with the final locally convex structure for the family $(\mathrm{inj}_i)_{i\in I}$ ([P1], section 1.3.2(**II**)), that is, the finest *locally convex* topology for which the inj_i are continuous. We will reuse the same notation $\bigoplus_{i\in I} E_i$ to denote this locally convex space.

A disk D in $\bigoplus_{i\in I} E_i$ is a neighborhood of 0 if and only if, for any *finite* subset J of I, $D \bigcap E_J$ is a neighborhood of 0 in $E_J := \bigoplus_{i\in J} E_i$ (**exercise**). Since coproducts are also defined in **Lcs**, the existence of the cokernel of any morphism (thus of the coequalizer of any double arrow) implies the existence of *inductive limits* in **Lcs** ([P1], section 1.2.8(**II**), Proposition 1.22). More precisely, let $\mathfrak{D} = \left(E_i, \varphi_j^i; I\right)$ be a direct system of locally convex spaces ([P1], section 1.2.8(**I**)) such that each $\varphi_i^j : E_i \to E_j$ $(i \preceq j)$ is linear and continuous. By ([P1], section 3.3.7(**V**)), in **Lcs**, the canonical mapping $\varphi_i : E_i \to \varinjlim E_i$ is the composition

$$\varphi_i : E_i \xrightarrow{\mathrm{inj}_i} \bigoplus_{i\in I} E_i \xrightarrow{\mathrm{can}} \underbrace{\left(\bigoplus_{i\in I} E_i\right) / \left(\sum_{i\preceq j} \mathrm{im}\left(\mathrm{inj}_j \circ \varphi_i^j - \mathrm{inj}_i\right)\right)}_{\varinjlim E_i}.$$

If $i \preceq j$, then $\varphi_i = \varphi_j \circ \varphi_i^j$, so $\varphi_i(E_i) \subset \varphi_j(E_j)$, and $\varinjlim E_i = \bigcup_{i \in I} \varphi_i(E_i)$ when equipped with the final locally convex topology for the family $(\varphi_i)_{i \in I}$.

REMARK 3.31.–

*1) By the transitivity of final (or terminal) structures ([P1], section 1.3.2(**II**)), $\varinjlim E_i$ is equipped with the final locally convex topology for the family of canonical injections $\varphi_i(E_i) \hookrightarrow \varinjlim E_i$, where each $\varphi_i(E_i)$ is equipped with the final locally convex topology for φ_i.*

2) In general, the finest locally convex *topology for which the* inj_i *above are continuous is not necessarily the finest topology of* $\mathbf{Tvs}$ *for which the* inj_i *are continuous ([BKI 81], Chapter II, section 4, Exercise 15).*

3) Quotients and topological direct sums in $\mathbf{Lcs}$ *are special cases of inductive limits (dual statement to Remark 2.34(2)).*

DEFINITION 3.32.–

1) The inductive limit $E = \varinjlim E_i$ of a direct system $\mathfrak{D} = \left(E_i, \varphi_j^i; I\right)$ in $\mathbf{Lcs}$, *where $E_i = E_i[\mathfrak{T}_i]$, is said to be* increasing *(resp.* strict*) if, for any pair $(i,j) \in I \times I$ such that $i \preceq j$, we have that $E_i \subset E_j$, $\varphi_i^j : E_i \hookrightarrow E_j$ is the canonical injection, and φ_i^j is continuous (resp. $\mathfrak{T}_i$ is induced by $\mathfrak{T}_j$).*

2) The projective limit $E = \varprojlim E_i$ of an inverse system $\mathfrak{I} = \left(E_i, \psi_i^j; I\right)$ in $\mathbf{Lcs}$, *where $E_i = E_i[\mathfrak{T}_i]$, is said to be* decreasing *(resp.* strict*) if, for any pair $(i,j) \in I \times I$ such that $i \preceq j$, $E_i \subset E_j$, we have that $\psi_i^j : E_j \twoheadrightarrow E_i$ is the canonical surjection and ψ_i^j is continuous (resp. $\mathfrak{T}_i$ is induced by $\mathfrak{T}_j$).*

REMARK 3.33.–

1) Remark 3.31(1) shows that the inductive limit $\varinjlim E_i$ of the direct system $\mathfrak{D} = \left(E_i, \varphi_j^i; I\right)$ in $\mathbf{Lcs}$ *may be written as the* increasing inductive limit *of the $\varphi_i(E_i)$. This yields $\varinjlim E_i = \bigcup_{i \in I} E_i$ equipped with the finest locally convex topology for which the canonical injections $\varphi_i : E_i \hookrightarrow E$ are continuous.*

2) Dually, the projective limit $E := \varprojlim E_i$ of the inverse system $\mathfrak{I} = \left(E_i, \psi_i^j; I\right)$ in $\mathbf{Lcs}$ *may be written as the* decreasing projective limit *of the $\psi_i^j(E_j)$. Then $E \subset \prod_{i \in I} E_i$ is equipped with the coarsest topology for which the canonical mappings $\psi_i : E \hookrightarrow E_i$ are continuous; these mappings can be assumed to have a dense image ([SCF 99], p. 139).*

In the *countable* case, we have the following result ([BKI 81], Chapter II, section 4.6, Proposition 9; Chapter III, section 1.4, Proposition 5), parts (i), (ii) and (iv) of which were established by Dieudonné and Schwartz [DIE 49], and part (iii) of which was proven by G. Köthe in 1950.

THEOREM 3.34.– *Let* (E_n) *be a* sequence *of locally convex spaces,* $\mathfrak{T}_n$ *the topology of* E_n*,* $E = \varinjlim E_n$ *the* strict *inductive limit of the sequence* (E_n)*, and* $\mathfrak{T}$ *the topology of* E*. Then:*

i) For all n*, the topology induced by* $\mathfrak{T}$ *on* E_n *is identical to* $\mathfrak{T}_n$*; if the* $\mathfrak{T}_n$ *are Hausdorff, then* $\mathfrak{T}$ *is Hausdorff.*

ii) If E_n *is closed in* E_{n+1} *(for the topology* $\mathfrak{T}_{n+1}$*) for all* n*, then* E_n *is closed in* E *(for* $\mathfrak{T}$*).*

iii) If each E_n *is complete (for* $\mathfrak{T}_n$*), then* E *is complete for* $\mathfrak{T}$*.*

iv) A subset B *of* E *is bounded if and only if it is contained in one of the subspaces* E_n *and bounded in this space.*

COROLLARY 3.35.– *Let* (E_n) *be a sequence of quasi-complete locally convex spaces. Then their strict inductive limit is a quasi-complete space (**exercise**).*

3.3.8. $\mathfrak{S}$*-topology on* $\mathcal{L}(E;F)$

Let E and F be two locally convex spaces and let $\mathfrak{S}$ be a bornology on E[7]. The uniform structure of $\mathfrak{S}$-convergence in $\mathcal{L}(E;F)$ (section 2.7.1) determines a locally convex topology on $\mathcal{L}(E;F)$ if and only if $u(M)$ is bounded in F for all $u \in \mathcal{L}(E;F)$ and all $M \in \mathfrak{S}$ (**exercise***: cf. [BKI 81], Chapter III, section 3.1, Proposition 1). By Lemma 3.37, this condition is satisfied if every element $M \in \mathfrak{S}$ is bounded in E.

DEFINITION 3.36.– *Let* E *and* F *be two locally convex spaces and suppose that* $\mathfrak{S}$ *is a bornology on* E *consisting of bounded subsets in* E*. The topology determined by the uniform structure of* $\mathfrak{S}$*-convergence in* $\mathcal{L}(E;F)$ *is called the* $\mathfrak{S}$*-topology of* $\mathcal{L}(E;F)$*; when equipped with this topology, this space is denoted* $\mathcal{L}_{\mathfrak{S}}(E;F)$ *.*

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7 Recall that the bornologies on E that we consider in this book are all *covering* in the sense of Bourbaki [BKI 81]; in other words, they are coverings of E.

The most important bornologies on a locally convex space E include s, pc, and b, as well as c if E is Hausdorff (section 2.5.1(**IV**)). The bornology denoted (or said to be of type) c_c consisting of the compact convex subsets also has an important role to play (cf. below, Definition 3.92).

LEMMA 3.37.– *Let E_1 and E_2 be locally convex spaces respectively equipped with bornologies $\mathfrak{S}_1$, $\mathfrak{S}_2$ of same type, either s, c_c, c, pc, or b. If $u \in \mathcal{L}(E_1; E_2)$, then u is $(\mathfrak{S}_1, \mathfrak{S}_2)$-bounded (section 2.5.1(**I**)).*

PROOF.– The case of type s is trivial; for types c and pc, cf. Lemma 2.95. The case of type c_c then follows from Lemma 3.13(2). Consider now the case of type b. Let B be a bounded set in E_1 and $u \in \mathcal{L}(E_1; E_2)$. Let V be a neighborhood of 0 in E_2. Then $U = u^{-1}(V)$ is a neighborhood of 0 in E_1. By Definition 3.27 and section 3.2.1(**II**), there exists a real number $\lambda > 0$ such that, for all $\mu \in \mathbb{K}$ satisfying the condition $|\mu| \geq \lambda$, we have that $B \subset \mu U$. Hence, $u(B) \subset \mu u(U) = \mu u\left(u^{-1}(V)\right) \subset \mu V$, so $u(B)$ is bounded, and u is $(\mathfrak{S}_1, \mathfrak{S}_2)$-bounded. ■

It is possible to show the following result ([BKI 81], Chapter III, section 4.3, Corollary 1):

THEOREM 3.38.– *Let E be a semi-complete Hausdorff locally convex space (Definition 2.99), F a locally convex space, and $\mathfrak{S}$ a bornology on E consisting of bounded subsets. Then every bounded subset in $\mathcal{L}_s(E; F)$ is also bounded for the $\mathfrak{S}$-topology.*

The spaces $\mathcal{L}_s(\mathbb{K}; F) \cong \mathcal{L}_b(\mathbb{K}; F) \cong F$ are isomorphic under the isomorphism $u \mapsto u.1$ (**exercise**).

3.4. Important types of locally convex spaces

3.4.1. *Banach spaces*

(I) CATEGORY OF BANACH SPACES A Banach space is a complete normed vector space. Let $E_1, ..., E_n$ be Banach spaces, each equipped with a norm denoted $\|.\|$. Consider the mappings $\|.\|_p$ $(p \in [1, \infty])$ from $E = E_1 \times ... \times E_n$ into $\mathbb{R}_+$ defined below:

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} \|x_i\|^P} \text{ if } p \neq \infty, \quad \|x\|_\infty = \sup_{1 \leq i \leq n} \|x_i\|,$$

where $x = (x_1, ..., x_n)$. These mappings $\|.\|_p$ are norms on E by Minkowski's inequality (Lemma 2.2, (2.3)): they are all equivalent, and E, equipped with any one of them, is a Banach space. Let E be a normed vector space with norm p. Let F be a closed subspace of E. Then E/F is a normed vector space with norm

$$\dot{p}(\bar{x}) = \inf\{p(x) : \varphi(x) = \bar{x}\}, \qquad [3.1]$$

where $\varphi : E \twoheadrightarrow E/F$ is the canonical surjection (section 3.3.5).

THEOREM 3.39.– *If E is a Banach space, then the quotient space E/F is also a Banach space.*

PROOF.– This follows from Theorem 2.83. ■

The morphisms of every subcategory of $\mathbf{Tvs}$ or $\mathbf{Tvsh}$ considered below are the continuous linear mappings, unless otherwise stated. The category of Banach spaces is a preabelian full subcategory of **Tvsh**. Since it admits finite products, kernels, and cokernels, it also admits finite projective and inductive limits.

DEFINITION 3.40.– *A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a normed vector space is said to be* absolutely summable *if $\sum_n \|x_n\| < \infty$. If so, we say that the series $\sum_{i=0}^{+\infty} x_i$ is* absolutely convergent.

THEOREM 3.41.– *Any absolutely summable sequence of elements in a Banach space E is summable, and any absolutely convergent series in a Banach space is convergent.*

PROOF.– Let n, m be two integers such that $m \geq n \geq 0$. Then $\|\sum_{i=n}^{m} x_i\| \leq \sum_{i=n}^{m} \|x_i\| \to 0$ as $n, m \to +\infty$, so the sequence (s_n) defined by $s_n = \sum_{i=0}^{n} x_i$ is a Cauchy sequence in E, and therefore converges in E (Definition 2.76). Hence, $(x_n)_{n\in\mathbb{N}}$ is summable in E (section 3.2.1**(III)**). ■

(II) OPERATORS IN BANACH SPACES

THEOREM 3.42.– *Let E and F be two normed vector spaces with norms $|.|$. Let $u : E \to F$ be a linear mapping.*

1) The following conditions are equivalent:

i) $u \in \mathcal{L}(E; F)$.

ii) $\|u\| < \infty$, *where*

$$\boxed{\|u\| := \sup_{x \neq 0} \frac{|u(x)|}{|x|} = \sup_{|x| \leq 1} |u(x)| = \sup_{|x|=1} |u(x)|}. \qquad [3.2]$$

2) The mapping $\|.\|$ *is a norm on* $\mathcal{L}(E; F)$.

3) If F *is a Banach space, then* $\mathcal{L}(E; F)$, *equipped with the norm* $\|.\|$, *is a Banach space identical to* $\mathcal{L}_b(E; F)$.

4) Let G *be a normed vector space and pick* $v \in \mathcal{L}(F; G)$. *Then* $\|v \circ u\| \leq \|v\| \|u\|$.

PROOF.– The proof of the equalities (3.2) is a classic **exercise**, as is the proof of the equivalence (1)⇔(2), and the statement of (3) (which alternatively can be deduced from Theorem 3.98 below). (4) is an easy **exercise**. ■

DEFINITION 3.43.– *A normed* $\mathbb{K}$*-algebra* $\mathbf{A}$ *is an associative* $\mathbb{K}$*-algebra equipped with a norm* $\|.\|$ *that is* multiplicative, *meaning that, for all elements* u, v *of* $\mathbf{A}$, *the relation* $\|uv\| \leq \|u\| \|v\|$ *holds. If the underlying normed vector space* $\mathbf{A}$ *is complete, we say that* $\mathbf{A}$ *is a Banach* $\mathbb{K}$*-algebra.*

COROLLARY 3.44.– *Let* E *be a Banach space. Then* $\mathcal{L}(E) := \mathcal{L}(E; E)$ *is a Banach algebra.*

LEMMA-DEFINITION 3.45.– *Let* E *be a Banach space and suppose that* (u_n) *is a sequence of elements of* $\mathcal{L}(E)$. *The series of non-negative real numbers* $\sum_{n=0}^{+\infty} \|u_n\|$ *converges in* $\mathbb{R}$ *if and only if the series* $\sum_{n=0}^{+\infty} u_n$ *converges absolutely in* $\mathcal{L}(E)$, *in which case the sequence* $\sum_{n=0}^{+\infty} u_n$ *is said to be* normally convergent.

THEOREM 3.46.– *Let* E *be a Banach space and suppose that* $u \in \mathcal{L}(E)$ *satisfies* $\|u\| < 1$. *Then* $1_E - u$ *is invertible in* $\mathcal{L}(E)$ *and is the sum of the normally convergent series* $\sum_{n \geq 0} u^n$.

PROOF.– For every integer $N \geq 0$, we have that $(1_E - u) \sum_{n=0}^{N} u^n = 1_E - u^{N+1}$. ■

DEFINITION 3.47.– *Let* E *be a complex Banach space and let* $u \in \mathcal{L}(E)$.

1) We say that a complex number ζ is a regular value *of u if $\zeta.1_E - u$ has an inverse $R(\zeta; u)$ in $\mathcal{L}(E)$; the set $\rho(u)$ of regular values of u is called the* resolvent set *of u, and the mapping $\rho(u) \to \mathcal{L}(E) : \zeta \mapsto R(\zeta; u)$ is said to be its* resolvent.

2) The complex numbers that are not regular values of u are called the spectral values *of u, and the set of these values is called the* spectrum $\mathrm{Sp}(u)$ *of u (so $\mathrm{Sp}(u) = \complement_{\mathbb{C}} \rho(u)$).*

3) An eigenvalue *of u is a spectral value ζ such that* $\ker(\zeta.1_E - u) \neq \{0\}$; *an element $x \neq 0$ of* $\ker(\zeta.1_E - u)$ *is called an* eigenvector *of u with eigenvalue ζ, and* $\ker(\zeta.1_E - u)$ *is called the eigenspace of the eigenvalue ζ.*

This definition generalizes the usual finite-dimensional versions of these concepts.

COROLLARY 3.48.– *Let E be a complex Banach space and suppose that $u \in \mathcal{L}(E)$. (i) $\rho(u)$ is open in $\mathbb{C}$ and the resolvent $\zeta \mapsto R(\zeta; u)$ is analytic in $\rho(u)$ (cf. below, Definition 4.74). (ii)* $\mathrm{Sp}(u)$ *is a non-empty compact subset of $\mathbb{C}$ contained in the ball $|\zeta| \leq \|u\|$.*

PROOF.– (i): **exercise**. (ii): Since $\rho(u)$ is open in $\mathbb{C}$, $\mathrm{Sp}(u)$ is closed and contained in the ball $|\zeta| \leq \|u\|$; therefore, $\mathrm{Sp}(u)$ is compact by the Heine-Borel-Lebesgue theorem (Theorem 2.97). If $\mathrm{Sp}(u)$ were empty, we would have $\rho(u) = \mathbb{C}$, so the resolvent $\zeta \mapsto R(\zeta; u)$ would be an entire function. This function would also be bounded (**exercise***: cf. ([DIE 82], Volume 1, (11.1.3))), and hence constant by Liouville's theorem (cf. below, Theorem-Definition 4.81(3)); thus, its inverse $\zeta \mapsto \zeta.1_E - u$ would also be constant, which is impossible. ■

COROLLARY 3.49.– *Let E be a Banach space and suppose that $u \in \mathcal{L}(E)$ is an invertible operator in $\mathcal{L}(E)$. Then, for any $v \in \mathcal{L}(E)$ such that $\|v\| < \left\|u^{-1}\right\|^{-1}$, $u + v$ is invertible in $\mathcal{L}(E)$. Consequently, the set of invertible operators is open in $\mathcal{L}(E)$ (**exercise**).*

LEMMA 3.50.– *Let E and F be normed vector spaces with norms $\|.\|$. A mapping $u \in \mathcal{L}(E; F)$ is an injective strict morphism if and only if there exists a real number $k_1 > 0$ such that $\|u(x)\| \geq k_1 \|x\|$ for all $x \in E$. The set of continuous linear mappings satisfying this condition is open in $\mathcal{L}(E; F)$.*

PROOF.– An operator $u \in \operatorname{Hom}(E;F)$ is an injective strict morphism if and only if it is an isomorphism (of normed vector spaces) from u onto $u(E)$, or alternatively if there exist real numbers $k_1, k_2 > 0$ such that $k_1 \|x\| \leq \|u(x)\| \leq k_2 \|x\|$ for all $x \in E$. The rest is an **exercise**. ■

3.4.2. *Fréchet spaces*

(I) Fréchet spaces and their inductive limits are two of the most important types of locally convex space, together with Banach spaces.

DEFINITION 3.51.– *A Fréchet space (or $(\mathcal{F})$-space) is a metrizable and complete locally convex space. An $(\mathcal{LF})$-space is an inductive limit of a* sequence *of Fréchet spaces; an $(\mathcal{L}_s\mathcal{F})$-space is a* strict *inductive limit of a* sequence *of Fréchet spaces*[8].

The category of Fréchet spaces is preabelian, and, in particular, any Hausdorff quotient of a Fréchet space is a Fréchet space. By Theorem 3.34:

COROLLARY 3.52.– *Every $(\mathcal{L}_s\mathcal{F})$-space is Hausdorff and complete.*

(II) A locally convex space E is quasi-complete if and only if every closed and bounded subset of E is complete (Definition 2.99). Any closed subspace of a Fréchet space is a Fréchet space. If $\left(E_i, \psi_i^j; I\right)$ is an inverse system of *complete* Hausdorff (resp. *quasi-complete* Hausdorff) locally convex spaces, then $\varprojlim E_i$ is *complete* Hausdorff (resp. *quasi-complete* Hausdorff) by Theorems 2.79 and 2.100. We will see to what extent the converse holds below.

THEOREM 3.53.–

1) If a locally convex space is complete and Hausdorff, then it is the decreasing projective limit *(Definition 3.32(2)) of a family of Banach spaces.*

2) A locally convex space E is a Fréchet space if and only if it is the decreasing projective limit *of a* sequence *of Banach spaces.*

8 In [DIE 42], the concept of $(\mathcal{LF})$-space is defined as a *strict* inductive limit of a *sequence* of Fréchet spaces. We are instead adopting the terminology used by Grothendieck ([GRO 55], Chapter I, p. 18), which was also adopted by G. Köthe [KOT 79].

PROOF.–

1) Let E be a complete Hausdorff locally convex space, and let $\mathfrak{U}$ be a fundamental system of disked and closed neighborhoods of 0, ordered by inclusion. For all $V \in \mathfrak{U}$, let p_V be the gauge of V (section 3.3.2(**I**)). Then $\ker(p_V) := \{x \in E : p_V(x) = 0\}$ is a vector subspace of E and $E_V = E/\ker(p_V)$ is a normed vector space (said to be associated with V) when equipped with the norm $\|\bar{x}_V\|_V = p_V(x)$, where $\bar{x}_V := x + \ker(p_V)$. Let $\widehat{E_V}$ be the completion of E_V (Definition 2.81). Let $U, V \in \mathfrak{U}$ be such that $U \subset V$. Then $\ker(p_U) \subset \ker(p_V)$, so there exists a canonical mapping $\psi_V^U : E_U \to E_V$ induced by 1_E ([P1], section 2.2.3(**I**)); ψ_V^U is surjective (*ibid.*) and is continuous, since $\|\bar{x}_V\|_V \leq \|\bar{x}_U\|_U$. Thus, by Theorem 2.80, there exist canonical mappings $\widehat{\psi_V^U} : \widehat{E_U} \to \widehat{E_V}$ and $\widehat{\psi_U} : E \to \widehat{E_U}$; clearly, if $U \subset V \subset W$ $(U, V, W \in \mathfrak{U})$, then $\widehat{\psi_V} = \widehat{\psi_V^U} \circ \widehat{\psi_U}$ and $\widehat{\psi_W^U} = \widehat{\psi_W^V} \circ \widehat{\psi_V^U}$. We therefore have the inverse system $\mathfrak{I} = \left\{\widehat{E_V}, \widehat{\psi_V^U}; \mathfrak{U}\right\}$, and $F := \varprojlim \widehat{E_V} \subset \prod_{V \in \mathfrak{U}} \widehat{E_V}$ is equipped with the coarsest topology for which the canonical mappings $F \to \widehat{E_V}$ are continuous. Since E is Hausdorff, there exists a canonical injection $j : E \hookrightarrow F : x \mapsto (\bar{x}_V)_{V \in \mathfrak{U}}$; j is a strict monomorphism ([SCF 99], Chapter II, section 5.4), $j(E)$ is dense in F and $E \cong j(E)$ is complete, thus $j(E) = F$ and via j, which is an isomorphism of $\mathbf{Tvs}$, E and F are identified.

2) If E is a Fréchet space, then $\mathfrak{U}$ may be chosen to be countable (section 2.1.3(**II**)). ■

3.4.3. *Barreled spaces*

(I) Following the publication of the article by Dieudonné and Schwartz mentioned earlier, N. Bourbaki published another article introducing the fundamental concepts of barreled spaces and bornological spaces [BKI 50] (cf. below, section 3.4.4(**I**)). As well as the properties established in Corollary 3.52, $(\mathcal{L}_s\mathcal{F})$-spaces are both barreled and bornological. The category of barreled spaces admits arbitrary inductive limits and products, and the category of bornological spaces admits countable inductive limits and products, but neither admits projective limits. The category of Montel spaces and the category of Schwartz spaces, which we shall introduce below, both admit arbitrary projective limits and certain inductive limits (countable strict for the former, and Hausdorff countable for the latter) (Theorems 3.68 and

3.73). None of these four categories admit kernels, and so none of them are preabelian.

DEFINITION 3.54.– *Let E be a locally convex space.*

1) A subset A of E is said to be bornivorous *if it is balanced, convex, and absorbs the bounded subsets of E.*

2) A barrel *T in E is a closed and absorbing disk.*

DEFINITION 3.55.– *A locally convex space E is said to be* barreled *(resp.* infrabarreled[9]*) if every barrel (resp. every bornivorous barrel) is a neighborhood of* 0.

A locally convex space E is barreled if and only if every lower semi-continuous semi-norm on E is continuous (**exercise**).

THEOREM 3.56.– *Every locally convex space that is also a Baire space (Definition 2.101) is barreled.*

PROOF.– Let T be a barrel. Then $E = \bigcup_{n\in\mathbb{N}} nT$. If T had empty interior, then E would have empty interior (section 2.6.1, Condition (**EB'**)), which is impossible. Therefore, T has a point x in its interior. If $x \neq 0$, then $-x \in T$ and $0 \in [-x, x]$, so 0 is in the interior of T. ■

(II) The following results are given without proof. If $(F_i)_{i\in I}$ is a family of barreled (resp. infrabarreled) spaces, E is a $\mathbb{K}$-vector space, and, for all $i \in I$, f_i is a linear mapping from F_i into E, then E, equipped with the final locally convex topology for the family $(f_i)_{i\in I}$, is a barreled (resp. infrabarreled) space ([BKI 81], Chapter III, section 4.1, Proposition 3, and section 4, Exercise 7). Hence, every quotient and every inductive limit of barreled (resp. infrabarreled) spaces is a barreled (resp. infrabarreled) space. In particular, every $(\mathcal{LF})$-space is barreled, since every Fréchet space is barreled by Theorems 3.56 and 2.102. Every product of barreled (resp. infrabarreled) spaces is barreled (resp. infrabarreled) ([BKI 81], Chapter IV, section 1.5,

9 Readers should take care to distinguish between the concepts of infrabarreled spaces and the *semi-barreled spaces* defined by Bourbaki ([BKI 81], Chapter IV, section 3.1, Definition 1). Infrabarreled spaces are semi-barreled, and are studied in ([BKI 81], Chapter IV, section 3, Exercise 9), the claims of which are in fact incorrect (with the exception of those relating to $(\mathcal{DF})$-spaces) [BLS 13].

Corollary, and section 2, Exercise 2). The completion of either a Hausdorff infrabarreled space or a quasi-complete infrabarreled space (Definition 2.99) is barreled ([GRO 73], Chapter 3, section 3). Closed subspaces of a barreled space are not necessarily barreled ([BKI 50], p. 6), so the category of barreled spaces does not admit projective limits (even finite projective limits).

(III) Let E and F be two locally convex spaces and let $\mathfrak{S}$ be a bornology on E consisting of bounded subsets of E. Let H be a subset of $\mathcal{L}(E;F)$. By the definitions, the following conditions are equivalent: (i) H is bounded for the $\mathfrak{S}$-topology (Definition 3.36); (ii) H is $(\mathfrak{S},b)$-bounded, where b is the canonical bornology of F (section 2.5.1); (iii) $H(M) := \bigcup_{u \in H} u(M)$ is bounded in M for all $M \in \mathfrak{S}$; (iv) every neighborhood V of 0 in F absorbs $H(M)$ for all $M \in \mathfrak{S}$; (v) for every neighborhood V of 0 in F, $\bigcap_{u \in H} u^{-1}(V)$ absorbs every $M \in \mathfrak{S}$. Thus:

LEMMA 3.57.– *Let E and F be two locally convex spaces and let $\mathfrak{S}$ be a bornology on E consisting of bounded subsets of E. Every equicontinuous subset of $\mathcal{L}(E;F)$ is bounded for the $\mathfrak{S}$-topology.*

The following result gives two cases in which the converse also holds:

THEOREM 3.58.– *(Banach-Steinhaus) A locally convex space E is barreled (resp. infrabarreled) if and only if, given any locally convex space F, every pointwise bounded subset (resp. every subset bounded for the topology of bounded convergence[10]) H in $\mathcal{L}(E;F)$ is equicontinuous.*

PROOF.– (1) Suppose that E is barreled. Let V be a neighborhood of 0 in F, p a continuous semi-norm on F, and $\alpha > 0$ a real number such that $B_p^c(\alpha;F) \subset V$, where $B_p^c(\alpha;F) := \{y \in F : p(y) \leq \alpha\}$. Let $q = \sup_{u \in H}(p \circ u)$. Then $q(x) < \infty$ for all $x \in E$, so (section 3.3.2(**I**)) q is a lower semi-continuous semi-norm on E. The set $B_q^c(\alpha;E) := \{x \in E : q(x) \leq \alpha\}$ is a barrel, and thus a neighborhood U of 0. For all $x \in U$ and all $u \in H$, we have that $u(x) \in V$, which shows that H is equicontinuous. For the converse, simply pick $F = \mathbb{K}$ and observe that, if a set $T \subset E$ is a barrel, then it is bounded in the weak dual E'_s (cf. below, Theorem 3.86(1)).

10 See section 2.7.4, replacing the term “Lipschitz structure” with “topology” (section 3.3.3).

(2) The proof is similar in the case where E is infrabarreled: cf. ([GRO 73], Chapter III, section 3, Proposition 5). ■

LEMMA 3.59.– *Let E and F be two locally convex spaces and suppose that H is an equicontinuous subset of $\mathcal{L}(E;F)$. On H, the uniform structure of precompact convergence is identical to the uniform structure of pointwise convergence on a dense subset D of E. If a net $(u_i)_{i\in I}$ in H converges pointwise to a mapping $u : E \to F$, then $u \in \mathcal{L}(E;F)$ and $(u_i)_{i\in I}$ converges uniformly to u on any precompact subset of E.*

PROOF.– This follows from the 2nd Ascoli-Arzelà theorem (Theorem 2.121). The linearity of u follows from the principle of extension of identities (Lemma 2.30). ■

COROLLARY 3.60.– *Let E be a barreled space, F a Hausdorff locally convex space, and $(u_i)_{i\in I}$ a pointwise bounded net (for example a sequence) in $\mathcal{L}(E;F)$ that converges pointwise to a mapping $u : E \to F$. Then $u \in \mathcal{L}(E;F)$ and $(u_i)_{i\in I}$ converges uniformly to u on every precompact subset of E.*

PROOF.– If (u_i) is sequence converging pointwise, then it is pointwise bounded (i.e. for all $x \in E$, $\bigcup_{i\in\mathbb{N}} u_i(x)$ is bounded in F) by Theorem 2.98, and so is equicontinuous by Theorem 3.58. The same is true if $(u_i)_{i\in I}$ is a pointwise bounded net, so we can simply apply Lemma 3.59 to obtain the result. ■

3.4.4. *Bornological spaces and ultrabornological spaces*

(I) BORNOLOGICAL SPACES

DEFINITION 3.61.– *A locally convex space E is said to be a* bornological space *if every bornivorous subset of E is a neighborhood of 0.*[11]

Every bornological space is infrabarreled (**exercise**). A space E is bornological if and only if there exist a family $(E_i)_{i\in I}$ of semi-normed spaces (section 3.3.2(**II**)) and a linear mapping $u_i : E_i \to E$ for each $i \in I$ such that the topology on E is the final locally convex topology for the family $(u_i)_{i\in I}$

11 This should not be confused with the notion of bornological set (Definition 2.94).

([BKI 81], Chapter III, section 2, Proposition 1); in particular, every semi-normed space is bornological. By the transitivity of final structures ([P1], section 1.3.2(**II**)), inductive limits of bornological spaces are bornological spaces. Every metrizable locally convex space is bornological ([BKI 81], Chapter III, section 2, Proposition 2). Every bornological space that is Hausdorff and semi-complete (section 2.5.3) is the inductive limit of a family of Banach spaces. The importance of bornological spaces stems from the next result, which determines one case in which a partial converse of Lemma 3.37 holds:

THEOREM 3.62.– *Let E be a bornological space, F a locally convex space, and $u : E \rightarrow F$ a linear mapping. The following conditions are equivalent:*

i) u is continuous.

ii) $(u(x_n)) \rightarrow 0$ in F for every sequence (x_n) that tends to 0 in E.

iii) $u(B)$ is bounded in F for every subset B that is bounded in E.

PROOF.– We know that (i)⇒(ii)⇒(iii) when E is an arbitrary locally convex space (Lemma 3.37).

(iii)⇒(i): Let V be a disked neighborhood (Definition 3.14) of 0 in F and let B be a bounded subset of E. Then, since $u(B)$ is bounded in F, V absorbs $u(B)$, so $u^{-1}(V)$ absorbs B ([P1], section 1.1.2(**VI**)). Hence, $u^{-1}(V)$ is bornivorous. Since E is bornological, $u^{-1}(V)$ is a neighborhood of 0 in E, and u is continuous. ■

DEFINITION 3.63.– *Let E, F be two locally convex spaces. Any linear mapping $u : E \rightarrow F$ satisfying the property (iii) in Theorem 3.62 is said to be* bounded.

(II) ULTRABORNOLOGICAL SPACES A locally convex space E is said to be *ultrabornological* if it is Hausdorff and may be expressed as the inductive limit of a family of Banach spaces. Every Hausdorff and semi-complete bornological space is ultrabornological by **(I)** above (but the completion of a bornological space is neither ultrabornological nor even bornological in general). By the transitivity of inductive limits, any Hausdorff inductive limit of a family of ultrabornological spaces is ultrabornological. In particular, $(\mathcal{LF})$-spaces are ultrabornological and every Hausdorff quotient of an ultrabornological space is ultrabornological. Since Banach spaces are barreled, ultrabornological spaces are also barreled. Any *countable* product of bornological (resp. ultrabornological) spaces is bornological (resp. ultrabornological) ([KOT 79], section 28(4)(4); [BKI 81], Chapter III, section 4, Exercise 18, e)); however, since subspaces (even closed subspaces) of complete bornological spaces (and thus ultrabornological spaces) are not bornological in general ([KOT 79], section 28(4)(4)), neither the category of bornological spaces nor the category of ultrabornological spaces admits projective limits (even finite ones).

It is possible to show the following result ([GRO 55], Introduction, IV, Theorem B; [GRO 73], Chapter 4, section 5, Theorem 2), which generalizes the Banach-Schauder theorem and the closed graph theorem (Theorem 3.12):

THEOREM 3.64.– *(Grothendieck) Let E be an ultrabornological space (for example an $(\mathcal{LF})$ space) and suppose that F is an $(\mathcal{LF})$-space. Then:*

1) Every surjective continuous mapping $u : F \to E$ is open.

2) Every linear mapping $v : E \to F$ with a sequentially closed graph is continuous.

3.4.5. *Montel spaces*

DEFINITION 3.65.– *A Montel space (or $(\mathcal{M})$-space) is a barreled Hausdorff locally convex space in which every bounded subset is relatively compact.*

Every finite-dimensional Hausdorff space is a Montel space.

LEMMA 3.66.– *A normed space is a Montel space if and only if it is finite-dimensional.*

PROOF.– Let E be a normed $(\mathcal{M})$-space with norm p. The bounded subsets are those which are contained in some closed ball $B_p^c(\alpha)$, $\alpha > 0$. If this ball is compact, then it is a compact neighborhood of 0, so E is finite-dimensional by Riesz's theorem (Theorem 3.151). ■

LEMMA 3.67.– *Every Montel space E is quasi-complete (Definition 2.99).*

PROOF.– Let $(x_i)_{i\in I}$ be a bounded Cauchy net in E. The closure of $\{x : i \in I\}$ in E is a compact set, and so is complete (Theorem 2.85). Hence, E is quasi-complete. ■

We have the following result (**exercise***: cf. [SCF 99], p. 147; [BKI 81], Chapter IV, section 2.5, Example 3):

THEOREM 3.68.– *Projective limits and topological direct sums of a* family *of Montel spaces are Montel spaces; the* strict *inductive limit of a* sequence *of Montel spaces E_n such that E_n is closed in E_{n+1} for each n is also a Montel space.*

3.4.6. *Schwartz spaces*

(I) COMPACT MAPPINGS

DEFINITION 3.69.– *Let E and F be locally convex spaces (resp. locally convex Hausdorff spaces) and suppose that $u : E \to F$ is a linear mapping. We say that u is a* precompact mapping *(resp.* compact mapping*) if there exists a neighborhood V of 0 in E such that $u(V)$ is precompact (resp. relatively compact) in F.*

The reader can show the next results as an **exercise**. Let E and F be Hausdorff locally convex spaces. The set $\mathcal{K}(E;F)$ of compact operators from E into F is a $\mathbb{K}$-vector space. If $u \in \mathcal{L}(E;F)$ has finite rank (i.e. $\dim(u(E)) < \infty$), then $u \in \mathcal{K}(E;F)$. If E is a normed vector space and F is a quasi-complete locally convex space, then $\mathcal{K}(E;F)$ is closed in $\mathcal{L}_b(E;F)$; in particular, if (u_n) is a sequence of finite-rank elements of $\mathcal{L}(E;F)$ with limit u in $\mathcal{L}_b(E;F)$, then u is a compact operator (for the Hilbertian case, see Theorem 3.166 in section 3.10.2).

LEMMA 3.70.– *If E, F, G, and H are locally convex (resp. Hausdorff locally convex) spaces, $v : F \to G$ is precompact (resp. compact), and $u : E \to F$, $w : G \to H$ are linear and continuous, then $w \circ v \circ u$ is compact (resp. precompact) (**exercise**). In particular, in $\mathcal{L}(E)$, the set of precompact (resp. compact) operators is a two-sided ideal.*

(II) SCHWARZ SPACES This concept was introduced by Grothendieck ([GRO 54], section III.4). The terminology is motivated by the fact that every function space used by L. Schwartz to construct his distribution theory is a Schwartz space (cf. below, section 4.4.1). We will reuse the notation from the proof of Theorem 3.53:

LEMMA 3.71.– *Let E be a Hausdorff locally convex space. The following conditions are equivalent:*

i) For every disked neighborhood V of 0, there exists a disked neighborhood $U \subset V$ such that the canonical mapping $\widehat{\psi_V^U} : \widehat{E_U} \to \widehat{E_V}$ is compact.

ii) For every disked neighborhood V of 0, there exists a disked neighborhood $U \subset V$ such that the canonical mapping $\psi_V^U : E_U \to E_V$ is precompact.

iii) There exists a Banach space F such that every mapping $u \in \mathcal{L}(E; F)$ is compact.

PROOF.– (i)⇔(ii) follows from Definition 2.89. (iii)⇒(i): Pick $F = \widehat{E_V}$. (i)⇒(iii): Same again, choosing $U = u^{-1}(V)$. ∎

DEFINITION 3.72.– *Let E be a locally convex space. We say that E is a Schwartz* space *(or $(\mathcal{S})$-space) if it is Hausdorff and satisfies one of the equivalent condition in Lemma 3.71.*

Every finite-dimensional Hausdorff locally convex space is a Schwartz space by Riesz's theorem (Theorem 3.151) and Theorem 2.39. Conversely, any *normed* vector space that is also a Schwartz space is necessarily finite-dimensional.

THEOREM 3.73.– *The category of Schwartz spaces admits arbitrary projective limits and countable Hausdorff inductive limits.*

PROOF.– (1) Condition (ii) of Lemma 3.71 shows that every subspace of an $(\mathcal{S})$-space is also an $(\mathcal{S})$-space. Every finite product of $(\mathcal{S})$-spaces is an $(\mathcal{S})$-space (**exercise***: cf. [HOR 12], Chapter III, section 15, Proposition 6). If $(E_i)_{i\in I}$ is a family of $(\mathcal{S})$-spaces and F is a Banach space, then, by section 2.3.5**(I)** and Lemma 2.26, choosing a continuous linear mapping $\prod_{i\in I} E_i \to F$ is equivalent to choosing a continuous linear mapping $\prod_{i\in J} E_i \to F$, where $J \subset I$ is finite, so $\prod_{i\in I} E_i$ is an $(\mathcal{S})$-space. The category of $(\mathcal{S})$-spaces therefore admits arbitrary projective limits by ([P1], section 1.2.8**(III)**, Proposition 1.22).

(2) Every Hausdorff quotient of an $(\mathcal{S})$-space is an $(\mathcal{S})$-space by Lemma 3.70. The topological direct sum of a sequence of $(\mathcal{S})$-spaces is an $(\mathcal{S})$-space ([GRO 54], Proposition 18, p. 118), and hence the category of $(\mathcal{S})$-spaces

admits countable Hausdorff inductive limits by ([P1], section 1.2.8**(III),** Proposition 1.22). ■

THEOREM 3.74.– *Let E be a* complete *Hausdorff locally convex space. The following conditions are equivalent:*

i) E is a Schwartz space.

ii) E is the decreasing projective limit of a family of Banach spaces $(E_i)_{i \in I}$ satisfying the following condition: for every pair $(i,j) \in I \times I$ such that $i \preceq j$, the canonical surjection $\psi_i^j : E_j \twoheadrightarrow E_i$ is compact.

PROOF.– (ii)⇒(i): We have that $\psi_i = \psi_i^j \circ \psi_j$, so ψ_i is compact by Lemma 3.70. (i)⇒(ii): In the proof of Theorem 3.53(2), it suffices to choose $\mathfrak{U}$ in such a way that, given some $V \in \mathfrak{U}$, $\widehat{\psi_V^U}$ is compact for every $U \in \mathfrak{U}$ satisfying $U \subset V$. ■

(III) FRÉCHET-SCHWARTZ SPACES A Fréchet-Schwartz *space* or $(\mathcal{FS})$-space is a Fréchet space that is also a Schwartz space (similarly, a Fréchet-Montel *space* or $(\mathcal{FM})$-space is a Fréchet space that is also a Montel space, and a Montel-Schwartz *space* or $(\mathcal{MS})$-space is a Montel space that is also a Schwartz space). From Theorem 3.74, we deduce:

COROLLARY 3.75.– *Let E be a locally convex space. The following conditions are equivalent:*

i) E is a Fréchet-Schwartz space.

ii) E is the decreasing projective limit of a sequence of Banach spaces (E_n) such that the canonical surjection $\psi_n^{n+1} : E_{n+1} \twoheadrightarrow E_n$ is compact for each n.

THEOREM 3.76.– *Every Fréchet-Schwartz space is a Montel space.*

PROOF.– Let E be an $(\mathcal{FS})$-space. This space is barreled (section 3.4.3). It just remains to be shown that every closed subset B of E is compact, or alternatively that if (x_n) is a bounded sequence of elements of B, then it has a convergent subsequence. The set $\{\psi_j(x_n) : n \geq 0\}$ is relatively compact in E_j for all j, so there exists a subsequence $\left(x_n^{(1)}\right)$ of (x_n) such that $\left(\psi_1\left(x_n^{(1)}\right)\right)$ converges in E_1, a subsequence $\left(x_n^{(2)}\right)$ of $\left(x_n^{(1)}\right)$ such that $\left(\psi_2\left(x_n^{(2)}\right)\right)$ converges in E_2, ... , a subsequence $\left(x_n^{(j+1)}\right)$ of $\left(x_n^{(j)}\right)$ such that $\left(\psi_{j+1}\left(x_n^{(j+1)}\right)\right)$ converges in E_{j+1}. Therefore, $\left(\psi_j\left(x_n^{(j)}\right)\right)$ converges in E_j for all j, and hence $\left(x_n^{(n)}\right)$ converges in E (Remark 3.33(2)). ■

It is possible to find examples of $(\mathcal{FM})$-spaces that are not $(\mathcal{FS})$-spaces. By Theorems 3.34, 3.68, and 3.73:

COROLLARY 3.77.– *The product and projective limit of a* sequence *of* $(\mathcal{FS})$-*spaces and the Hausdorff quotient of an* $(\mathcal{FS})$-*space are also* $(\mathcal{FS})$-*spaces, and so are* $(\mathcal{M})$-*spaces.*

However, the Hausdorff quotient of an $(\mathcal{FM})$-space is not necessarily an $(\mathcal{M})$-space, or even reflexive in general ([GRO 54], p. 118).

3.5. Weak topologies

3.5.1. *Dual systems*

Let E and F be two vector spaces over $\mathbb{K}$ and suppose that $B : E \times F \to \mathbb{K}$ is a bilinear form. The triple (E, F, B) is called a *dual system*. We say that E and F are *in duality with respect to* B. The *weak topology* on E (resp. on F) *defined by the dual system* (E, F, B) is the coarsest topology for which every linear form $d_y : x \mapsto B(x, y)$ (resp. $s_x : y \mapsto B(x, y)$) is continuous. This topology is denoted $\sigma(E, F)$ (resp. $\sigma(F, E)$); the topology $\sigma(E, F)$ is defined by the family of semi-norms $p_y : x \mapsto |B(x, y)|$ $(y \in F)$, and E is therefore a locally convex space when equipped with the topology $\sigma(E, F)$ (the same is true for F equipped with the topology $\sigma(F, E)$).

The topology $\sigma(E, F)$ is Hausdorff if and only, for all $x \neq 0$, there exists $y \in F$ such that $B(x, y) \neq 0$, i.e. if B is not left-degenerate. If so, then for every $x \in E$, the mapping $\mathrm{s} : E \to F^* : x \mapsto s_x$ (where F^* is the algebraic dual of F : cf. section 3.2.2(**I**)) is injective, and E may be identified with the subspace $\mathrm{s}(E)$ of F^*. We say that the duality between E and F is *separating at* E. This duality is separating at both E and F if and only if B is not degenerate (on either side).

Suppose that $\sigma(E, F)$ is Hausdorff. By the above, $E \subset F^*$, and, for all $y \in F$, the canonical linear form $\langle ., y\rangle : F^* \to \mathbb{K} : y^* \mapsto \langle y^*, y\rangle$ ([P1], section 3.1.2)) extends $d_y = B(., y)$; we write $B(., y) = \langle ., y\rangle$, and $B = \langle ., .\rangle$ whenever this notation is not ambiguous.

3.5.2. *Bipolars*

Let $(E, F, \langle .,. \rangle)$ be a dual system and let M be a subset of E. The *polar set* of M in F is the set

$$\boxed{M^0 := \{y \in F : \Re(\langle x, y \rangle) \geq -1, \forall x \in M\}}.$$

This set contains 0 and is convex; it is closed for the topology $\sigma(E, F)$ (**exercise**). The correspondence between $\mathfrak{P}(E)$ and $\mathfrak{P}(F)$ (ordered by inclusion)

$$\mathfrak{P}(E) \ni M \mapsto M^0, \quad \mathfrak{P}(F) \ni N \mapsto N^0,$$

is a Galois connection ([P1], section 2.1.2(**II**)), and therefore $M \subset M^{00}$, $M^0 = M^{000}$. Furthermore, $M^0 = M_1^0$, where M_1 is the convex envelope of $M \cup \{0\}$ (**exercise**). Two sets $M \subset E$, $N \subset F$ are said to be *orthogonal* if $\langle x, y \rangle = 0, \forall x \in M, \forall y \in N$.

THEOREM 3.78.– *(*bipolar*) Let E, F be two vector spaces that are in duality. For any subset M of E, the polar set M^{00} of M^0 (called the* bipolar *of M) is the convex closed envelope of $M \cup \{0\}$ (for the topology $\sigma(E, F)$).*

PROOF.– By the above, we may assume without loss of generality that M is convex and contains 0. Let $\overline{M}$ be closure of M with respect to the topology $\sigma(E, F)$. We know that $\overline{M}$ is convex (Lemma 3.13(1)), and, since M^{00} is closed, we also have that $\overline{M} \subset M^{00}$. Given $a \in \complement_E M$, $\{a\}$ is a compact convex subspace of E (even if $\sigma(E, F)$ is not Hausdorff), so by Corollary 3.24 there exists a real closed hyperplane H that strictly separates $\{a\}$ and $\overline{M}$; furthermore, $0 \in M$, so H does not contain 0, and hence there exists $y \in F$ such that H has equation $\Re(\langle x, y \rangle) = -1$; consequently, $\Re(\langle x, y \rangle) > -1$ for all $x \in \overline{M}$ and $\Re(\langle a, y \rangle) < -1$. So $y \in M^0$ and $a \notin M^{00}$, which implies that $M^{00} = \overline{M}$. ■

Part (1) of the next result immediately follows; the reader may show part (2) as an **exercise**, and can refer to ([BKI 81], Chapter II, section 6.5, Corollary 1 of Proposition 7) for part (3):

COROLLARY 3.79.– *Let E, F be two vector spaces that are in duality with respect to the bilinear form B and suppose that M is a vector subspace of E.*

1) The closure $\overline{M}$ of M for the topology $\sigma(E,F)$ is identical to the bipolar M^{00} of M.

2) The bilinear form B induces a bilinear form $\bar{B} : M \times F/M^0 \to \mathbb{K} : (x,\bar{y}) \mapsto B(x,y)$, where $\bar{y}$ is the canonical image of $y \in F$ in F/M^0, and $\bar{B}$ is non-degenerate; thus, $\bar{B}$ puts M and F/M^0 in separating duality.

3) Suppose that the duality of E and F is separating at E. The quotient topology induced on F/M^0 by $\sigma(F,E)$ is identical to $\sigma\left(F/M^0, M\right)$ if and only if M is closed in $\sigma(E,F)$.

3.5.3. *Transpose of a linear mapping*

Consider two pairs (E_1, F_1), (E_2, F_2) of vector spaces in duality.

LEMMA 3.80.– *Let $u : E_1 \to E_2$ be a linear mapping.*

1) The following conditions are equivalent:

i) u is weakly continuous, *i.e. continuous for the weak topologies $\sigma(E_1, F_1)$, $\sigma(E_2, F_2)$.*

ii) There exists a mapping $v : F_2 \to F_1$ such that

$$\langle u(x_1), y_2\rangle = \langle x_1, v(y_2)\rangle, \quad \forall x_1 \in E_1, \forall y_2 \in F_2. \qquad [3.3]$$

2) If these conditions are satisfied and the duality between E_1 and F_1 is separating at F_1, then the mapping v satisfying (3.3) is unique and linear.

PROOF.– (1) (i)⇒(ii): If u is continuous for the weak topologies, then the linear form $x_1 \mapsto \langle u(x_1), y_2\rangle$ is continuous for all $y_2 \in F_2$, so is of the form $x_1 \mapsto \langle x_1, v(y_2)\rangle$, where $v(y_2) \in E_2$. This v is unique if the duality between E_1 and F_1 is separating at F_1 (**exercise**). (ii)⇒(i): clear. (2): **exercise**. ■

Suppose that the duality between E_1 and F_1 is separating at F_1 and the duality between E_2 and F_2 is separating at F_2. Then F_1 and F_2 may be identified with vector subspaces of E_1^* and E_2^* respectively (section 3.5.1) and the conditions (i), (ii) of Lemma 3.80 are also equivalent to ${}^t u(F_2) \subset F_1$, where ${}^t u : E_2^* \to E_1^*$ is the algebraic transpose of u ([P1], section 3.1.2(**I**)); hence, v is the restriction to F_2 of ${}^t u$, again written ${}^t u$. We shall adopt this convention even when the duality between E_2 and F_2 is not separating at F_2; given these conditions, we say that:

DEFINITION 3.81.– *The linear mapping* ${}^t u : F_2 \to F_1$ *is said to be the* transpose *of the weakly continuous mapping* $u : E_1 \to E_2$.

COROLLARY 3.82.– *Suppose that the duality between* E_1 *and* F_1 *is separating at* F_1. *If a linear mapping* $u : E_1 \to E_2$ *is weakly continuous, then its transpose* ${}^t u : F_2 \to F_1$ *is weakly continuous. Furthermore,* ${}^t\left({}^t u\right) = u$ *if the duality between* E_2 *and* F_2 *is separating at* E_2 *(**exercise**).*

THEOREM 3.83.– *Suppose that the duality between* E_1 *and* F_1 *is separating at* F_1. *Let* $u : E_1 \to E_2$ *be a weakly continuous linear mapping.*

1) The following relations hold:

$$\boxed{\ker\left({}^t u\right) = \left(\operatorname{im}(u)\right)^0, \quad \overline{\operatorname{im}\left({}^t u\right)} = \ker(u)^0}. \qquad [3.4]$$

2) If the duality E_1 *between* F_1 *and the duality between* E_2 *and* F_2 *are both separating, then* u *is a strict morphism (in* **Lcs**, *for the weak topologies) if and only if* $\operatorname{im}\left({}^t u\right)$ *is a closed subspace of* F_1 *for the topology* $\sigma\left(F_1, E_1\right)$.

PROOF.–

1) The relation ${}^t u\left(y_2\right) = 0$ $\left(y_2 \in F_2\right)$ holds if and only if, for all $x_1 \in E_1$, $B\left(u\left(x_1\right), y_2\right) = 0$, i.e. $B\left(\operatorname{im}(u), y_2\right) = 0$, or alternatively $y_2 \in \underline{\left(\operatorname{im}(u)\right)}^0$, which shows the first equality. For the next part, observe that $\overline{\operatorname{im}\left({}^t u\right)} = \operatorname{im}\left({}^t u\right)^{00}$ by Corollary 3.79(1); moreover, $\operatorname{im}\left({}^t u\right)^{00} = \ker\left({}^t u\right)^0$ by the first equality of (3.4); the second equality follows after replacing u by ${}^t u$.

2) Let $N_1 = \operatorname{im}\left({}^t u\right) \subset F_1$. Then $N_1^0 = \operatorname{im}\left({}^t u\right)^0 = \overline{\operatorname{im}\left({}^t u\right)}^0 = \ker(u)^{00} = \ker(u)$, so u induces a canonical linear bijection $\bar{u} : E_1/N_1^0 \to \operatorname{im}(u)$ (Noether's first isomorphism theorem: cf. [P1], section 2.2.3**(II)**, Theorem 2.12(1)). The vector spaces E_1/N_1^0 and N_1 are in separating duality (Corollary 3.79(2)) and the bilinear form $B_1 : E_1 \times F_1 \to \mathbb{K}$ induces a non-degenerate bilinear form $\bar{B}_1 : E_1/N_1^0 \times N_1 \to \mathbb{K}$; with the notation of Corollary 3.79(2), for all $x_1 \in E_1$ and all $y_2 \in F_2$, writing $\bar{x}_1 = x_1 + N_1^0$, we have that $B_2\left(\bar{u}\left(\bar{x}_1\right), y_2\right) = \bar{B}_1\left(\bar{x}_1, {}^t\bar{u}\left(y_2\right)\right)$, where B_2 is the bilinear form that puts E_2 and F_2 in duality. Consequently, $\bar{u}$ is continuous from E_1/N_1^0 to $\operatorname{im}(u)$ when E_1/N_1^0 is equipped with the topology $\sigma\left(E_1/N_1^0, N_1\right)$ and

$\operatorname{im}(u)$ is equipped with the topology induced by $\sigma(E_2, F_2)$. Moreover, for all $x_2 \in \operatorname{im}(u)$, let $x_1 \in E_1$ be such that $x_2 = u(x_1)$; then $x_2 = \bar{u}(\bar{x}_1)$, and, for all $y_2 \in F_2$, we have that $B_2(x_2, y_2) = \bar{B}_1\left(\bar{u}^{-1}(x_2), {}^t\bar{u}(y_2)\right)$, so $\bar{u}^{-1} : \operatorname{im}(u) \to E_1/N_1^0$ is continuous for the same topologies as above, and $\bar{u}$ is an isomorphism for these topologies. Hence, $\bar{u}$ is a strict morphism (Definition 2.129) if and only if $\sigma\left(E_1/N_1^0, N_1\right)$ is the topology induced by $\sigma(E_1, F_1)$, i.e. (since the duality is separating at E_1) N_1 is a closed subspace of F_1 for the topology $\sigma(F_1, E_1)$ (Corollary 3.79(3)). ■

Consider three pairs of vector spaces $(E_1, E_2), (F_1, F_2), (G_1, G_2)$ in separating duality, and consider the two following sequences of weakly continuous mappings:

$$E_1 \xrightarrow{u} F_1 \xrightarrow{v} G_1, \qquad [3.5]$$

$$E_2 \xleftarrow{{}^t u} F_2 \xleftarrow{{}^t v} G_2. \qquad [3.6]$$

The following result can be deduced from Theorem 3.83. The spaces E_i, F_i, G_i $(i = 1, 2)$ are equipped with the weak topologies:

COROLLARY 3.84.–

1) The sequence (3.5) *is algebraically exact (exact in the abelian category* **Vec***) if and only if the three following conditions are satisfied:*

i) ${}^t u \circ {}^t v = 0$*;*

ii) $\operatorname{im}\left({}^t v\right)$ *is dense in* $\ker\left({}^t u\right)$ *equipped with the topology induced by* $\sigma(F_2, F_1)$*;*

iii) $\operatorname{im}(u)$ *is closed in* F_1*, or, equivalently,* ${}^t u$ *is a strict morphism from* F_2 *into* E_2*.*

2) In particular, u *is surjective if and only if* ${}^t u$ *is an isomorphism from* F_2 *onto* ${}^t u(F_2)$ *equipped with the topology induced by* $\sigma(E_2, E_1)$*.*

3) The sequence (3.5) *is strictly coexact in the preabelian category* **Lcs** *([P1], section 3.3.7(**III**)) if and only if the sequence* (3.6) *is strictly coexact in* **Lcs***. This holds if and only if* v *is a strict morphism from* F_1 *into* G_1 *(or equivalently if and only if* $\operatorname{im}\left({}^t v\right)$ *is closed in* F_2*), provided that the conditions (i), (ii), and (iii) above are satisfied.*

PROOF.– (1), (2): **exercise**. (3) follows immediately, since, by definition, (3.5) is strictly exact (resp. coexact) in **Lcs** if and only if $\mathrm{im}\,(u) = \ker\,(v)$ and u (resp. v) is a strict morphism. ■

3.6. Dual of a locally convex space

3.6.1. *Notion of a dual space*

The *dual* E' of a locally convex space $E\,[\mathfrak{T}]$ is the vector subspace of continuous linear forms in its algebraic dual E^* (section 3.2.2(**I**)). The duality pairing $\langle .,. \rangle : E' \times E \to \mathbb{K}$ defined by $\langle x', x \rangle = x'\,(x)$ puts E and E' in duality. The next result follows from Corollary 3.26:

LEMMA 3.85.– *The duality between E and E' is separating at E'. Furthermore, if $E\,[\mathfrak{T}]$ is Hausdorff, then this duality is separating at E.*

THEOREM 3.86.–

1) Let M be a subset of E'. The following conditions are equivalent:

i) M is equicontinuous.

ii) There exists a disked neighborhood V of 0 in $E\,[\mathfrak{T}]$ such that $M \subset V^0$.

iii) M^0 is a neighborhood of 0 in $E\,[\mathfrak{T}]$.

2) Therefore, $\mathfrak{T}$ is identical to the topology of uniform convergence on equicontinuous subsets of E'.

PROOF.– Let V be a disked subset of E. For all $x' \in E'$,

$$\sup_{x \in V} \left| \langle x', x \rangle \right| = \sup_{|\xi|=1} \left(-\Re \left(\langle x', \xi x \rangle \right) \right) = - \inf_{|\xi|=1} \Re \left(\langle x', \xi x \rangle \right)$$

so $x' \in V^0$ if and only if $\sup_{x \in V} |\langle x', x \rangle| \leq 1$. ■

3.6.2. *Weak* topology on E'*

The dual E' of $E\,[\mathfrak{T}]$ may be equipped with various $\mathfrak{S}$-topologies (section 3.3.8). The s-topology on E' (Definition 3.36) is $\sigma\,(E', E)$; this is known as the *weak* topology*[12]; we will write E'_s for the space E' equipped with this topology, and E_s for the space E equipped with the topology $\sigma(E, E')$ (called the *weakened topology* on E, since this topology is coarser than $\mathfrak{T}$ (**exercise**); the latter is called the *initial topology*). Consider a second locally convex space $F[\mathfrak{U}]$ and a continuous linear mapping $u : E[\mathfrak{T}] \to F[\mathfrak{U}]$. Then u is *weakly continuous*, i.e. continuous from E_s into F_s (**exercise**). Since the topology $\sigma(E', E)$ is Hausdorff, the transpose ${}^t u : F'_s \to E'_s$ is continuous; furthermore, ${}^t\left({}^t u\right) = u$ if $F[\mathfrak{U}]$ is Hausdorff, since this implies that the topology $\sigma(F, F')$ is Hausdorff (Corollary 3.82 and Lemma 3.85).

DEFINITION 3.87.– *The "duality" functor* $(-)' : \mathbf{Lcsh} \to \mathbf{Lcsh}$ *is defined as*

$$\boxed{E[\mathfrak{T}] \mapsto E'_s, \quad (u : E[\mathfrak{T}] \to F[\mathfrak{U}]) \mapsto {}^t u : F'_s \to E'_s}.$$

See Theorem 3.133 below. The next result may be deduced from the 3rd Ascoli-Arzelà theorem (Theorem 2.122(1)) and Theorem 3.86:

THEOREM 3.88.– *(Alaoglu-Bourbaki) Let* $\mathcal{L}_s(E; F)$ *be the space* $\mathcal{L}(E; F)$ *equipped with the topology of pointwise convergence. Suppose that* F *is Hausdorff; then a subset* H *of* $\mathcal{L}(E; F)$ *is relatively compact in* $\mathcal{L}_s(E; F)$ *if and only if* $H(x) := \{u(x) : x \in E\}$ *is a relatively compact set in* F *for all* $x \in E$. *Hence, if* V *is a neighborhood of* 0 *in* $E[\mathfrak{T}]$, *then* V^0 *is relatively compact in* E'_s.

PROOF.– It suffices to show that whenever H is an equicontinuous subset of $\mathcal{L}(E; F)$ its closure $\overline{H}$ in $\mathcal{F}_s(E; F)$ is contained in $\mathcal{L}(E; F)$ and is equicontinuous. Readers may derive this property as an **exercise*** ([BKI 81], Chapter III, section 3.4, Proposition 4). ■

COROLLARY 3.89.– *(Banach) Let* E *be a Banach space. Then the unit ball of* E' *is compact in* E'_s.

A subset M of E'_s is bounded if and only if $\{\langle x', x\rangle : x' \in M\}$ is a bounded set in $\mathbb{K}$ for all $x \in E$. Consequently, every equicontinuous subset of E' is bounded in E'_s, and, therefore, in $E'[\mathfrak{T}']$, where $\mathfrak{T}'$ is any locally convex

12 Some authors call the topology $\sigma(E', E)$ the *weak topology* on E', but this might potentially be confused with the topology $\sigma(E', E'')$, where E'' is the bidual of $E[\mathfrak{T}]$, in the case where $E[\mathfrak{T}]$ is not semi-reflexive (cf. below, section 3.7.2).

topology on E' that is finer than $\sigma(E', E)$ (Remark 3.28). By Lemma 3.57 and the proof of Theorem 3.58:

COROLLARY 3.90.– *The following conditions are equivalent:*

i) $E[\mathfrak{T}]$ *is barreled.*

ii) The bounded subsets of E'_s *are equicontinuous.*

3.6.3. *Topologies compatible with duality*

Let E and F be two vector spaces in separating duality with respect to the bilinear form $\langle ., . \rangle$. For any $y \in F$, consider the bilinear form $d_y : E \to \mathbb{K} : x \mapsto \langle x, y \rangle$ (section 3.6.1). Given that the mapping $\mathbf{d} : F \to E^* : y \mapsto d_y$ is linear and injective, we may use it to identify F and $\mathbf{d}(F) \subset E^*$.

DEFINITION 3.91.– *Let* $\mathfrak{T}$ *be a locally convex topology on* E. *This topology is said to be* compatible with the duality *between* E *and* F *if* $\mathbf{d}$ *is a bijection from* F *onto* $E[\mathfrak{T}]'$.

DEFINITION 3.92.– *We define the Mackey* topology *on* E *to be the topology of uniform convergence on convex subsets of* F *that are compact for the topology* $\sigma(F, E)$ *(i.e. the topology of type* c_c *on the weak space* F*). This topology is denoted* $\tau(E, F)$.

In the following, we will write $E_\tau = E[\tau(E, E')]$ and $E'_\tau = E'[\tau(E', E)]$. It is possible to show the following result ([BKI 81], Chapter IV, section 1.1):

THEOREM 3.93.–

1) (*Mackey-Arens*) *Let* $\mathfrak{T}$ *be a locally convex topology on* E. *This topology is compatible with the duality between* E *and* F *if and only if it is finer than* $\sigma(E, F)$ *and coarser than* $\tau(E, F)$.

2) (*Mackey*) *The closed convex subsets of* E *are the same for every locally convex topology that is compatible with the duality between* E *and* F. *Similarly, the bounded subsets of* E *are the same for every locally convex topology that is compatible with the duality between* E *and* F.

Since the initial topology $\mathfrak{T}$ on a Hausdorff locally convex space $E[\mathfrak{T}]$ is compatible with the duality between E and $E[\mathfrak{T}]'$, $\mathfrak{T}$ is coarser than

$\tau(E, E')$. Whenever these two topologies are identical, $E[\mathfrak{T}]$ is said to be a Mackey *space*. The category $\mathbf{Mac}$ of Mackey spaces admits arbitrary products and inductive limits ([SCF 99], Chapter IV, section 4.3, Corollary 2). If E is a Mackey space, then its completion $\widehat{E}$ is also a Mackey space ([SCF 99], Chapter IV, section 3.5). Infrabarreled spaces (Definition 3.55), and therefore also bornological spaces (Definition 3.61), are Mackey spaces ([SCF 99], Chapter IV, section 5.2); in particular, every space considered in section 4.3 is a Mackey space[13].

We saw in section 3.6.2 that every continuous linear mapping from a locally convex space $E[\mathfrak{T}]$ into a locally convex space $F[\mathfrak{U}]$ is weakly continuous. Conversely ([BKI 81], Chapter IV, section 1.3, Proposition 7):

THEOREM 3.94.– *Let $E[\mathfrak{T}]$ and $F[\mathfrak{U}]$ be two Mackey spaces. A linear mapping $u : E[\mathfrak{T}] \to F[\mathfrak{U}]$ is continuous if and only if it is weakly continuous.*

3.6.4. *Other topologies on E'*

For any bornology $\mathfrak{S}$ on E, we write $E'_{\mathfrak{S}} := \mathcal{L}_{\mathfrak{S}}(E; \mathbb{K})$. We also write $\beta(E', E)$ for the topology of uniform convergence on the bounded subsets of $E[\mathfrak{T}]$ (i.e. on the elements of b), also known as the *strong topology* on E', as well as E'_b for the *strong dual* of E. Readers may show the following result as an **exercise*** (cf. [BKI 81], Chapter IV, section 1.3):

THEOREM 3.95.– *Let E and F be two locally convex spaces and suppose that $u \in \mathcal{L}(E; F)$.*

i) The transpose ${}^t u : F' \to E'$ of u is continuous when F' and E' are equipped with their respective strong topologies.

ii) ${}^t u$ is injective if and only if $\operatorname{im}(u)$ is dense in F. If E and F are Hausdorff, then u is injective if and only if $\operatorname{im}\left({}^t u\right)$ is dense in E'_s.

13 See the table on page 169.

Definition 3.54(2) implies that:

COROLLARY 3.96.– *A set* $T \subset E'$ *is a barrel in* E'_s *if and only if* T *is of the form* B^0*, where* B *is a bounded disk in* $E[\mathfrak{T}]$*. The set of barrels in* E'_s *is a fundamental system of neighborhoods of* 0 *in* E'_b*.*

The next result should be compared with Corollary 3.90. Its proof is similar.

COROLLARY 3.97.– *The following conditions are equivalent:*

i) $E[\mathfrak{T}]$ *is infrabarreled.*

ii) The bounded subsets of E'_b *are equicontinuous.*

We have that (**exercise***: cf. [KOT 79], section 39(6)(4)):

THEOREM 3.98.– *Let* E *be a bornological space,* F *a quasi-complete (resp. complete) locally convex Hausdorff space, and* $\mathfrak{S} \subset b$ *a bornology on* E *that contains the points of every sequence in* E *that converges to* 0 *(for example,* $\mathfrak{S} = c_c, c, pc,$ *or* b*). Then* $\mathcal{L}_{\mathfrak{S}}(E; F)$ *is quasi-complete (resp. complete).*

COROLLARY 3.99.– *The strong dual of a semi-normed space is a Banach space.*

DEFINITION 3.100.– *The strong dual of a locally convex metrizable space is said to be a* $(\mathcal{DF})$- space[14].

COROLLARY 3.101.– *The strong dual of a* $(\mathcal{DF})$*-space is a Fréchet space.*

PROOF.– If the locally convex space $E[\mathfrak{T}]$ is metrizable, then it is bornological, so its strong dual is complete (Theorem 3.98). Let $B_n = \{x \in E : d(x_n, 0) \leq n\}$, where d is a metric that determines the topology $\mathfrak{T}$. The sequence (B_n) is a fundamental system of bounded sets of E (section 2.5.1(**III**)). Furthermore, by Corollary 3.96, the set of B_n^0 is a fundamental system of neighborhoods of E'_b. This system is countable, so E'_b is metrizable (Theorem 2.70). ■

14 Grothendieck ([GRO 54], Definition 1, p. 63) gave a more general definition of the notion of $(\mathcal{DF})$-space (which encompasses normed spaces as a special case), but the two definitions are equivalent in any of the contexts that we shall encounter.

However, Theorem 3.98 doesn't say anything about E'_s. The following result may be deduced from the Banach-Steinhaus theorem (Theorem 3.58) (**exercise***: cf. [BKI 81], Chap. III, no. 4.2, Cor. 4):

THEOREM 3.102.– *Let E be a barreled space, F a quasi-complete Hausdorff locally convex space, and $\mathfrak{S} \subset b$ a bornology on E. Then $L_{\mathfrak{S}}(E;F)$ is quasi-complete (Definition 2.99).*

THEOREM 3.103.–

1) Every barreled $(\mathcal{DF})$-space is bornological.

2) The strong dual of a complete Schwartz space is ultrabornological.

PROOF.– (1): **exercise**: cf. [BKI 81], Chapter IV, section 3, Exercise 2c. (2): cf. [SCW 57], p. 43. ■

The following result complements Corollary 3.60 and so can be viewed as an extension of the Banach-Steinhaus theorem ([BKI 81], Chapter IV, section 3.2, Corollary):

THEOREM 3.104.– *Let E be a $(\mathcal{DF})$-space, F a locally convex Hausdorff space, and (u_n) a sequence in $\mathcal{L}(E;F)$ converging pointwise to a mapping u from E into F. Then $u \in \mathcal{L}(E;F)$ and (u_n) converges uniformly to u on every precompact subset of E.*

We also have the following result ([BKI 81], Chapter IV, section 1.5, Propositions 14 and 15; [MOR 93], Proposition A.3.7):

THEOREM 3.105.–

1) Let $(E)_i$ be a family of locally convex spaces. The following canonical isomorphisms exist in $\mathbf{Tvs}$:

$$\left(\bigoplus\nolimits_{i\in I} E_i\right)'_b \cong \prod\nolimits_{i\in I} (E'_i)_b\,, \quad \bigoplus\nolimits_{i\in I} (E'_i)_b \cong \left(\prod\nolimits_{i\in I} E_i\right)'_b.$$

2) Let $\mathfrak{D} = \left(E_i, \varphi^i_j; I\right)$ be an increasing direct system of locally convex spaces, where $\varphi^j_i : E_i \hookrightarrow E_j$ $(j \succeq i)$ is the canonical injection, and let $\mathfrak{I} = \left((E'_i)_b, {}^t\varphi^i_j; I\right)$ be the dual inverse system. There exist continuous linear bijections (which in general are not necessarily isomorphisms of $\mathbf{Tvs}$; cf.

below, Theorem 3.133)

$$\kappa : \varinjlim (E_i)'_b \tilde{\to} \left(\varprojlim E_i\right)'_b, \quad \lambda : \left(\varinjlim E_i\right)'_b \tilde{\to} \varprojlim \left(E'_i\right)_b.$$

3.6.5. *Riesz spaces*

(I) Let P be a closed salient convex cone with summit 0 in a *real* vector space E (section 3.3.4**(I)**). We may define an order relation on E by setting

$$x \geq y \text{ if } x - y \in P.$$

We say that (E, P) is a Riesz *space* and that P is its *positive cone*. If P has non-empty interior, we write $x > 0$ if x is in the interior of P.

Suppose that E is also a locally convex space. In its dual E', the polar set P^0 of P is also a closed convex salient cone, and $\left(E', P^0\right)$ is a Riesz space, called the *dual* of (E, P). Furthermore (**exercise**), $P^0 = \{x' \in E' : \langle x', x\rangle \geq 0, \forall x \in P\}$.

LEMMA 3.106.– *Let E be a Riesz space and P its positive cone, which we shall assume has non-empty interior.*

i) (Farkas' lemma*) $x \geq 0$ if and only if $\langle x', x\rangle \geq 0$ for all $x' \geq 0$.*

ii) Suppose that the strong dual E'_b of E is a Fréchet space. If $x < 0$ and $x' \geq 0, x' \neq 0$, then $\langle x', x\rangle < 0$.

PROOF.– (i): The sufficient condition is clear. We will show the necessary condition by contradiction. Let $x \notin P$. Then $\{x\}$ is compact and disjoint from P. Therefore, by Corollary 3.24, there exists an element $x' \in E'$ such that $\langle x', x\rangle < \langle x', p\rangle$ for all $p \in P$. It cannot be the case that $\langle x', p\rangle < 0$, since this would imply that $\langle x', \alpha p\rangle < 0$ for all $\alpha > 0$, so there would exist $\alpha > 0$ such that $\langle x', x\rangle > \langle x', \alpha p\rangle$, which is impossible, since $\alpha p \in P$. Therefore, $\langle x', p\rangle \geq 0$ for all $p \in P$, i.e. $x' \geq 0$. But $\inf_{p \in P} \langle x', p\rangle = 0$, so $\langle x', x\rangle < 0$: contradiction. (ii): We know that $\langle x', x\rangle \leq 0$. By the Banach-Schauder theorem (Theorem 3.12(1)), the image of the interior of P under x' is an open set in $\mathbb{R}$, so $\langle x', x\rangle < 0$. ■

A Riesz space is said to be a *lattice* if it is a lattice for the above ordering, and similarly a *complete lattice* if it is a complete lattice for this ordering ([P1], section 2.1.3**(I)**).

(II) CONVEX MAPPINGS We can now generalize the notion of a convex function. Consider the mapping $f : A \to F$, where A is a convex subset of a real vector space E and F is a Riesz space. The *epigraph* of f is the set $\mathrm{epi}\,(f) = \{(x, y) \in A \times E : f(x) \leq y\}$. The mapping f is said to be *convex* if its epigraph is convex.

3.7. Bidual and reflexivity

3.7.1. *Embedding of E within its bidual*

Let $E[\mathfrak{T}]$ be a Hausdorff locally convex space and let E'_b be its strong dual. By Mackey's theorem (Theorem 3.93(2)), the bornology b on $E[\mathfrak{T}]$ is identical to the canonical bornology on E_s (section 2.5.1**(IV)**), and is also written b. In E_s, $c_c \subset b$, so the topology $\beta(E', E)$ is finer than $\tau(E', E)$ (section 2.7.1). These topologies may be distinct, or in other words, by the Mackey-Arens theorem (Theorem 3.93(1)), the topology $\beta(E', E)$ is not necessarily compatible with the duality between E' and E, so the dual E'' of E'_b is not necessarily identical to E.

DEFINITION 3.107.– *We say that $E'' := (E'_b)'$ is the* bidual *of $E[\mathfrak{T}]$.*

We write $\mathbf{c}_E : E \to E''$ for the Gelfand transform ([P1], section 3.2.2) $x \mapsto \tilde{x}$, where $\tilde{x}$ is the bilinear form $x' \mapsto \langle x', x\rangle$ on E'; we say that $\mathbf{c}_E$ is the *canonical mapping*.

THEOREM 3.108.– *The canonical mapping $\mathbf{c}_E$ is injective.*

PROOF.– The kernel $\mathfrak{K}$ of $\mathbf{c}_E$ is the intersection of the kernels of all continuous linear forms on E. Therefore, $\mathfrak{K} \subset \overline{\{0\}}$ by Corollary 3.26, and, since E is Hausdorff, $\overline{\{0\}} = \{0\}$, so $\mathfrak{K} = \{0\}$. ■

The vector space E may now always be identified with $\mathbf{c}_E(E) \subset E''$.

3.7.2. *Semi-reflexive spaces*

DEFINITION 3.109.– *The locally convex space $E[\mathfrak{T}]$ is said to be* semi-reflexive *if the canonical mapping $\mathbf{c}_E$ is bijective, i.e. (after identification) if $E = E''$ (equality of vector spaces).*

The following conditions are equivalent by the above and Definition 3.92: (a) $E[\mathfrak{T}]$ is semi-reflexive; (b) $\beta(E', E) = \tau(E', E)$, i.e. E'_b is a Mackey space; (c) the convex bounded subsets of E_s are compact in E_s.

THEOREM 3.110.–

1) Let $E[\mathfrak{T}]$ be a Hausdorff locally convex space. The following conditions are equivalent:

i) $E[\mathfrak{T}]$ is semi-reflexive.

ii) Every bounded subset of $E[\mathfrak{T}]$ is relatively compact for the weakened topology $\sigma(E, E')$.

iii) E'_τ is barreled.

iii) E_s is quasi-complete.

2) If the locally convex space $E[\mathfrak{T}]$ is semi-reflexive, then it is Hausdorff and quasi-complete, and its strong dual E'_b is barreled.

PROOF.– The disked envelope of any bounded subset of E_s is bounded in E_s (Lemma 3.29), so (i)⇔(ii) by the equivalence (a)⇔(c) above. The other claims may be shown as an **exercise***: cf. ([SCF 99], Chapter IV, section 5.5) and ([BKI 81], Chapter IV, section 2.2, Theorem 1). ■

3.7.3. *Reflexive spaces*

Every equicontinuous subset of E' is strongly bounded (i.e. bounded in E'_b) (Lemma 3.57), so the topology on E induced by $\beta(E'', E')$ is finer than the initial topology $\mathfrak{T}$. By Corollary 3.97:

THEOREM 3.111.– *The initial topology $\mathfrak{T}$ of $E[\mathfrak{T}]$ is identical to the topology on E induced by $\beta(E'', E')$ if and only if $E[\mathfrak{T}]$ is infrabarreled.*

DEFINITION 3.112.– *A locally convex space $E[\mathfrak{T}]$ is said to be* reflexive *if $\mathbf{c}_E : E[\mathfrak{T}] \to (E'_b)_b$ is an isomorphism of* $\mathbf{Tvs}$.

This definition leads to the following result:

LEMMA 3.113.–

1) The strong dual of a reflexive space is reflexive.

2) If $E[\mathfrak{T}]$ is reflexive, then a subset of E (resp. E') is bounded in $E[\mathfrak{T}]$ (resp. E'_b) if and only if it is bounded in E_s (resp. E'_s); consequently, we may simply refer to the bounded subsets *of E (resp. E') without ambiguity.*

THEOREM 3.114.– *Let $E[\mathfrak{T}]$ be a locally convex space. The following conditions are equivalent:*

i) $E[\mathfrak{T}]$ is reflexive.

ii) $E[\mathfrak{T}]$ is semi-reflexive and barreled.

iii) $E[\mathfrak{T}]$ is semi-reflexive and infrabarreled.

iv) E_τ is reflexive.

v) E'_τ is reflexive.

PROOF.– (i)⇒(ii): If $E[\mathfrak{T}]$ is reflexive, then so is E'_b (Lemma 3.113(1)), and therefore $(E'_b)'_b = E[\mathfrak{T}]$ is barreled (Theorem 3.110(2)). The implications (ii)⇒(iii) and (iii)⇒(i) are obvious. The equivalence of (i) with (iv) and (v) is an **exercise*** (cf. [SCF 99], Chapter IV, section 5.7). ■

The product and topological direct sum of a family of semi-reflexive (resp. reflexive) spaces are again semi-reflexive (resp. reflexive) spaces; the projective limit of a family of semi-reflexive spaces is a semi-reflexive space; the strict inductive limit of a sequence of semi-reflexive (resp. reflexive) spaces is a semi-reflexive (resp. reflexive) space ([SCF 99], Chapter IV, section 5.8). Since every reflexive space is barreled, it is a semi-reflexive Mackey space, but it is possible to find semi-reflexive Mackey spaces that are not reflexive ([SCF 99], Chapter IV, section 5.6). The proof of (1) below is an **exercise***: cf. ([KOT 79], section 23(7)(1)).

LEMMA-DEFINITION 3.115.– *Let $E[\mathfrak{T}]$ be a locally convex space.*

1) The following conditions are equivalent: (i) Every bounded subset B_1 of $E''[\beta(E'', E')]$ is contained in the closure of some bounded subset of E for the topology $\beta(E'', E')$. (ii) For every bounded subset B_1 of $E''[\beta(E'', E')]$, there exists a bounded subset B of $E[\mathfrak{T}]$ such that $B_1 \subset B^{00}$. (iii) E'_b is barreled.

2) The space $E[\mathfrak{T}]$ is said to be distinguished *if one of these equivalent conditions is satisfied.*

Every normed vector space is distinguished, but there exist Fréchet spaces that are not distinguished (consequently, barreled spaces are not always distinguished). By Theorem 3.110(2), every semi-reflexive space is distinguished, and it is clear that every $(\mathcal{DF})$-space is distinguished. There exist $(\mathcal{DF})$-spaces that are not Mackey spaces ([GRO 54], p. 74, Remark 8), so distinguished spaces are not necessarily Mackey spaces.

COROLLARY 3.116.– *Every metrizable semi-reflexive locally convex space E is a reflexive Fréchet space.*

PROOF.– Since the space E is identical to its bidual, it is the strong dual of a $(\mathcal{DF})$-space, and is therefore a Fréchet space (Corollary 3.101). Therefore, it is barreled (section 3.4.3(**II**)), and thus reflexive (Theorem 3.114). ■

Let $E[\mathfrak{T}]$ be a locally convex space, F a closed subspace of $E[\mathfrak{T}]$, F^0 the polar set of F in E'_s, and $\mathfrak{S}$ a bornology on E.

THEOREM 3.117.–

1) There exist isomorphisms of vector spaces:

$$F' \tilde{\rightarrow} E'/F^0, \quad F^0 \cong (E/F)'. \qquad [3.7]$$

2) Let ψ be the first isomorphism and suppose that $\psi(\mathfrak{S}) := \{\psi(B) : B \in \mathfrak{S}\}$. Then the $\psi(\mathfrak{S})$-topology of E'/F^0 is identical to the topology induced by the topology of $E'_{\mathfrak{S}}$.

3) If $E[\mathfrak{T}]$ is semi-reflexive, then its closed subspace F is semi-reflexive, and $F'_b \cong E'_b/F^0$ is an isomorphism of $\mathbf{Tvs}$.

PROOF.–

1) Let $r_F : E' \to F' : x' \mapsto x'|_F$ be the restriction morphism. This morphism is clearly injective, and is also surjective by the Hahn-Banach theorem (Theorem 3.25), which yields the first isomorphism in (3.7). Any given element of F^0 is a linear form $x' \in E'$ that vanishes on F, which therefore induces the element $\overline{x'}$ in $(E/F)'$ ([P1], section 2.2.3(**I**)). The correspondence $x' \mapsto \overline{x'}$ from F^0 into $(E/F)'$ is bijective and $\mathbf{K}$-linear.

2) immediately follows from (1). (3): By (1), the topology on F_s is induced by the topology on E_s. Furthermore, F_s is closed in E_s by Mackey's theorem (Theorem 3.93(2)). Suppose that $E[\mathfrak{T}]$ is semi-reflexive. Any bounded subset

of F_s is bounded in E_s; thus, by Theorem 3.110, it is relatively compact in E_s, and therefore in F_s. Hence, F is semi-reflexive. ■

3.7.4. *Duality in Banach spaces*

(I) NORM ON THE BIDUAL AND NORM OF A TRANSPOSE MAPPING Let E be a normed vector space with norm $\|.\|$. The strong dual E'_b of E is a Banach space (Corollary 3.99); writing $\|.\|$ for the norm of this space, by Theorem 3.42, for all $x' \in E'$, it follows that

$$\left\|x'\right\| = \sup_{\|x\|\leq 1, x\in E} \left|\left\langle x', x\right\rangle\right|.$$

THEOREM 3.118.– *The strong bidual E''_b of E is a Banach space. For all $x \in E$,*

$$\|x\| = \sup_{\|x'\|\leq 1, x'\in E'} \left|\left\langle x', x\right\rangle\right|,$$

i.e. the canonical injection $\mathbf{c}_E$ from E into its strong bidual is isometric (meaning that it has norm 1).

PROOF.– For all $x \in E$ and all $x' \in E'$ such that $\|x'\| \leq 1$, we have that $|\langle x', x\rangle| \leq \|x'\| \|x\|$, so $\|\mathbf{c}_E(x)\| \leq \|x\|$ and $\|\mathbf{c}_E\| \leq 1$. Let $x_0 \in E$, $V = \mathbb{K}x_0$ and $y'_0 : V \to \mathbb{K} : \lambda x_0 \mapsto \lambda \|x_0\|$, so that $\|y'_0\| = 1$. By the Hahn-Banach theorem (Theorem 3.25), there exists an element $x'_0 \in E'$ such that $\|x'_0\| = 1$ and $\langle x'_0, x_0\rangle = \langle y'_0, x_0\rangle = \|x_0\|$, so $\sup_{\|x'\|\leq 1, x'\in E'} |\langle x', x_0\rangle| \geq \|x_0\|$ and $\|\mathbf{c}_E\| \geq 1$. ■

THEOREM 3.119.– *Let E, F be two normed vector spaces and suppose that $u \in \mathcal{L}(E; F)$. Then $\left\|{}^t u\right\| = \|u\|$.*

PROOF.– For all $y' \in F'$ and all $x \in E$, we have that $\left\langle {}^t u(y'), x\right\rangle = \langle y', u(x)\rangle$ and, writing $\|.\|$ for the norms in F' and E, $\left\|{}^t u(y')\right\| = \sup_{\|x\|\leq 1} \left|\left\langle {}^t u(y'), x\right\rangle\right|$. Therefore,

$$\left\|{}^t u\left(y'\right)\right\| = \sup_{\|x\|\leq 1} \left|\left\langle y', u(x)\right\rangle\right| \leq \left\|y'\right\| \sup_{\|x\|\leq 1} \|u(x)\| = \left\|y'\right\| \|u\|,$$

which implies that $\left\|{}^t u\right\| \leq \|u\|$. The reverse inequality may be shown using the Hahn-Banach theorem in a similar manner to Theorem 3.118. ■

(III) Reflexive Banach spaces A normed vector space is semi-reflexive if and only if it is reflexive, in which case it is a Banach space (Corollary 3.116). From Theorem 3.110, it follows that:

COROLLARY 3.120.– *(Banach) A Banach space is reflexive if and only if its unit ball is compact for the weakened topology* $\sigma(E, E')$ *.*

COROLLARY 3.121.– *In a reflexive Banach space* E*, every closed convex bounded subset is compact for the weakened topology* $\sigma(E, E')$ *.*

COROLLARY 3.122.– *Let* E *be a reflexive Banach space,* A *a closed convex subset of* E*, and* $f : A \to \bar{\mathbb{R}}$ *a lower semi-continuous convex function (Definition 2.31). Then* f *attains its minimum on* A *in both of the following two cases: (i)* A *is bounded; (ii)* A *is unbounded and* f *is* coercive*, i.e.* $\lim_{\|x\| \to +\infty, x \in A} f(x) = +\infty$.

PROOF.– Let $\bar{x} \in \bar{\mathbb{R}}$ and $B := \{x \in A : f(x) \leq f(\bar{x})\}$. The minimum of f exists on A if and only if it exists on B, which is a closed bounded convex subset of E, and hence is compact for the topology $\sigma(E, E')$. The function f is lower semi-continuous for the topology $\sigma(E, E')$, since, for all $y \in \bar{\mathbb{R}}$, $f^{-1}([-\infty, y[) = \complement_A \left(f^{-1}(]y, \infty])\right)$ is a closed convex set in E, and therefore in E_s by Mackey's theorem (Theorem 3.93(2)). This concludes the proof by Theorem 2.42. ■

3.7.5. *Strong dual of a Montel space*

THEOREM 3.123.– *Every Montel space is reflexive, and its strong dual is also a Montel space.*

PROOF.– Every $(\mathcal{M})$-space is reflexive by Definition 3.65 and Theorems 3.110, 3.114. Let $E[\mathfrak{T}]$ be an $(\mathcal{M})$-space. Its strong dual E'_b is barreled (Theorem 3.110(2)); furthermore, by Definition 3.65, every bounded subset of $E[\mathfrak{T}]$ is relatively compact, so $\beta(E', E)$ is identical to the topology of compact convergence $\gamma(E', E)$ of E' on E. Let B be a bounded subset of E'_b; it is bounded in E'_s (Remark 3.28), so is equicontinuous (Corollary 3.90), since $E[\mathfrak{T}]$ is barreled. By the 3rd Ascoli-Arzelà theorem (Theorem 2.122), the closure of B in E'_s (which is the same as the closure of B in E'_b by Mackey's theorem) is therefore compact for $\gamma(E', E) = \beta(E', E)$. Therefore, E'_b is an $(\mathcal{M})$-space. ■

COROLLARY 3.124.– *Let $E[\mathfrak{T}]$ be a Montel space. A bounded net of elements of E (resp. E') (Lemma 3.113(1)) converges in E_s (resp. E'_s) if and only if it converges in $E[\mathfrak{T}]$ (resp. E'_b). In particular, a sequence of elements of E (resp. E') converges in E_s (resp. E'_s) if and only if it converges in $E[\mathfrak{T}]$ (resp. E'_b).*

PROOF.– We will argue in E'. The necessary condition is clear when $E[\mathfrak{T}]$ is any arbitrary locally convex space, so we shall simply show the sufficient condition. Let $\mathfrak{x}' = (x'_i)_{i \in I}$ be a net such that the set $B = \{x'_i : i \in I\}$ is bounded in E'; the proof of Theorem 3.123 shows (i) that B is relatively compact in E'_b, and (ii) that the closure of B in E'_s is identical to the closure of B in E'_b. Let us write $\overline{B}$ for this closure, and $\overline{B}_s$ (resp. $\overline{B}_b$) for the set $\overline{B}$ equipped with the topology induced by the topology of E'_s (resp. of E'_b). Then the bijection $\iota : \overline{B}_b \to \overline{B}_s : x' \mapsto x'$ is continuous; it is therefore a homeomorphism (Corollary 2.41), since $\overline{B}_b$ is compact. Given that the net $\mathfrak{x}'$ converges in $\overline{B}_s$, it also converges in $\overline{B}_b$. If (x'_n) is a convergent sequence in E'_s, then it is Cauchy in E'_s and therefore bounded in E' (Theorem 2.98). ■

The table below summarizes the various implications between locally convex spaces:

$$\begin{array}{ccccccc}
\text{metrizable} & \Rightarrow & \text{bornological} & \Rightarrow & \text{infrabarreled} & \Rightarrow & \text{Mackey} \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
(\mathcal{F}) & \Rightarrow & \text{ultrabornological} & \Rightarrow & \text{barreled} & & \text{distinguished} \\
\Uparrow & & & & \Uparrow & & \Uparrow \\
(\mathcal{FS}) & \Rightarrow & (\mathcal{M}) & \Rightarrow & \text{reflexive} & \Rightarrow & \text{semi-reflexive}
\end{array}$$

3.8. Additional notes about $(\mathcal{F})$ and $(\mathcal{FS})$-spaces and their duals

3.8.1. *Strict morphisms of Fréchet spaces*

The following result was shown in ([DIE 49], Theorem 7, p. 92) (see also [BKI 81], Chapter IV, section 4.2, Theorem 1):

THEOREM 3.125.– *(Dieudonné-Schwartz) Let E and F be two Fréchet spaces and suppose that $u \in \mathcal{L}(E; F)$. The following conditions are equivalent:*

a) u is a strict morphism.

b) u is a strict morphism for the weakened topologies.

c) $u(E)$ is closed in F.

d) ${}^t u$ *is a strict morphism from* F' *into* E' *for the weakened topologies.*

e) ${}^t u\left(F'\right)$ *is closed in* E'_s*.*

f) ${}^t u\left(F'\right)$ *is closed in* E'_b*.*

We can deduce the following result, which extends Corollary 3.84:

COROLLARY 3.126.– *Let* E, F, G *be Fréchet spaces, suppose that* $u : E \to F$ *and* $v : F \to G$ *are continuous linear mappings, and consider the following sequences:*

$$E \overset{u}{\to} F \overset{v}{\to} G, \qquad [3.8]$$

$$E' \overset{{}^t u}{\leftarrow} F' \overset{{}^t v}{\leftarrow} G'. \qquad [3.9]$$

If the sequence (3.8) *is strictly exact and coexact*[15] *when* E, F*, and* G *are equipped with their respective initial topologies (or weakened topologies), then* (3.9) *is strictly exact and coexact when* E'*,* F'*, and* G' *are equipped with their respective weak* topologies.*

COROLLARY 3.127.– *Let* E *and* F *be Fréchet spaces and suppose that* $u \in \mathcal{L}\left(E; F\right)$*. Then* ${}^t u$ *(resp.* u*) is surjective if and only if* u *(resp.* ${}^t u$*) is an injective strict morphism.*

PROOF.– This follows from (3.4) and Theorem 3.125. ■

COROLLARY 3.128.– *Let* E *and* F *be Banach spaces. The subset of* $\mathcal{L}\left(E; F\right)$ *consisting of the surjective operators is open in* $\mathcal{L}\left(E; F\right)$*.*

PROOF.– This follows from Lemma 3.50, Theorem 3.119, and Corollary 3.127. ■

3.8.2. *Strict morphisms of* $(\mathcal{FS})$ *or Silva spaces*

(I) Theorem 3.117 may be extended as follows:

15 See the proof of Corollary 3.84.

LEMMA 3.129.– *Let E be an $(\mathcal{FS})$-space and suppose that F is a closed subspace of E. There exists an isomorphism of* $\mathbf{Tvs}$

$$\left(F^0\right)_b \cong (E/F)'_b\,,$$

where $\left(F^0\right)_b$ denotes the polar set of $F^0 \subset E'$ equipped with the topology induced by the strong topology of E'.

PROOF.– It suffices to show that every bounded subset B of E/F is contained in the closure of some bounded subset of E. But B is relatively compact, so we may simply apply Lemma 2.93. ■

THEOREM 3.130.– *(Palamodov) Let E and F be $(\mathcal{FS})$-spaces and suppose that $u \in \mathcal{L}(E;F)$. The equivalent conditions (a)-(f) in Theorem 3.125 are also equivalent to:*

(g) ${}^t u$ is a strict morphism from F' into E' for the strong topologies.

PROOF.– (a)⇒(g): Condition (a) means that $\check{u} : E/\ker(u) \to \operatorname{im}(u)$ is an isomorphism of $\mathbf{Tvs}$. Therefore, ${}^t\check{u} : \operatorname{im}(u)'_b \to (E/\ker(u))_b$ is an isomorphism of $\mathbf{Tvs}$. Theorem 3.117 and Lemma 3.129 respectively imply the isomorphisms of $\mathbf{Tvs}$ $\operatorname{im}(u)'_b \cong F'_b/(\operatorname{im}(u))^0_b$ and $(E/\ker(u))'_b \cong (\ker(u))^0_b$. But $(\operatorname{im}(u))^0 = \ker\left({}^t u\right)$ and $(\ker(u))^0 = \operatorname{im}\left({}^t u\right)$ by (3.4), since $\operatorname{im}\left({}^t u\right)$ is closed. Hence, the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{im}(u)'_b & \overset{{}^t\check{u}}{\longrightarrow} & (E/\ker(u))_b \\
\downarrow & & \downarrow \\
F'_b/\left(\ker\left({}^t u\right)\right)^0_b & \dashrightarrow & \operatorname{im}\left({}^t u\right)_b,
\end{array}$$

where the vertical arrows represent isomorphisms, and ${}^t\check{u}$ may be identified with the horizontal dotted arrow. This implies (g).

(g)⇒(f): Let ${}^t u : F'_b \to E'_b$ be a strict morphism. The commutative diagram above implies that there is an isomorphism $F'_b/\left(\ker\left({}^t u\right)\right)^0_b \cong \operatorname{im}\left({}^t u\right)_b$, and $\left(F'_b/\left(\ker\left({}^t u\right)\right)^0_b\right)'_b \cong \left(\ker\left({}^t u\right)\right)^0_b$ by Lemma 3.129; given that $\left(\ker\left({}^t u\right)\right)^0_b$ is a closed subspace of F, it is an $(\mathcal{FS})$-space, so is reflexive, and $\left(\left(\ker\left({}^t u\right)\right)^0_b\right)'_b \cong F_b/\left(\ker\left({}^t u\right)\right)_b$ by

Theorem 3.117(3); but $F_b / \left(\ker\left({}^t u\right)\right)_b$ is an $(\mathcal{FS})$-space, so its strong dual $\left(F'_b / \left(\ker\left({}^t u\right)\right)^0_b\right)'_b$ is complete. Since $\operatorname{im}\left({}^t u\right)_b$ is isomorphic to $\left(F'_b / \left(\ker\left({}^t u\right)\right)^0_b\right)'_b$, $\operatorname{im}\left({}^t u\right)_b$ is closed, and thus complete in E'_b. ■

DEFINITION 3.131.– *The strong dual of an* $(\mathcal{FS})$*-space is called a Silva space (or a* $(\mathcal{DFS})$*-space).*

Note that the strong dual of any metrizable Montel space is a Schwartz space ([GRO 54], p. 118), and so every Silva space is a Schwartz space. Every Silva space is ultrabornological by Theorem 3.103(2).

(II) The category **FS** of $(\mathcal{FS})$-spaces and the category **Sil** of Silva spaces are both preabelian (the images and cokernels of morphisms are given by Lemma 3.2). All $(\mathcal{FS})$ and Silva spaces are reflexive and furthermore complete Montel spaces by Theorems 3.76, 3.98, and 3.123; they are bornological and barreled by Theorem 3.103(1). The strong dual of a Silva space is an $(\mathcal{FS})$-space by Lemma 3.113(1). Closed subspaces and Hausdorff quotients of Silva spaces are Silva spaces; the inductive limit of a *sequence* of Silva spaces is a Silva space by Corollary 3.77. Therefore any finite product of Silva spaces is a Silva space.

By Theorem 3.130 (**exercise***: cf. [PAL 70], Chapter V, section 1, Proposition 8):

COROLLARY 3.132.– *(Palamodov) Let* E, F, G *be* $(\mathcal{FS})$*-spaces and suppose that* $u : E \to F$ *and* $v : F \to G$ *are continuous linear mappings. The sequence* (3.8) *is strictly exact and coexact*[16] *in* **FS** *if and only if the sequence* (3.9) *is strictly exact and coexact in* **Sil**.

The next theorem shows that the "duality" functor $(-)'$ (Definition 3.87) is "quasi-exact"[17] from $\mathbf{Sil}^{\mathrm{op}}$ into **FS** and from $\mathbf{FS}^{\mathrm{op}}$ into **Sil**:

16 See footnote 15.

17 We avoid using the term "exact" because the finite projective and injective limits used to define exactness (cf. [P1], section 1.2.9(**I**)) are replaced by countable *strict* projective and injective limits here.

THEOREM 3.133.–

1) Let $E = \varprojlim E_i$ *be the strict projective limit (Definition 3.32(2)) of a sequence of Silva spaces, and let* $\psi_i^j : E_j \twoheadrightarrow E_i$ $(j \geq i)$ *be the canonical surjection. Then* $\varinjlim (E_i)'_b$ *(with the canonical injection* $\varphi_i^j = {}^t\psi_i^j : (E_i)'_b \hookrightarrow (E_j)'_b$ $(j \geq i)$*) is a strict inductive limit of* $(\mathcal{FS})$*-spaces and the mapping* κ *in Theorem 3.105(2) is an isomorphism of* $\mathbf{Tvs}$.

2) Let $E = \varinjlim E_i$ *be the strict inductive limit of a sequence of* $(\mathcal{FS})$*-spaces and let* $\varphi_i^j : E_i \hookrightarrow E_j$ $(j \geq i)$ *be canonical injection. Then* $\varprojlim (E_i)'_b$ *(with the canonical surjection* $\psi_i^j = {}^t\varphi_i^j : (E_j)'_b \hookrightarrow (E_i)'_b$ $(j \geq i)$*) is a strict projective limit of Silva spaces and the mapping* λ *in Theorem 3.105(2) is an isomorphism of* $\mathbf{Tvs}$.

PROOF.–

1) E is the subspace $\bigcap_{i \leq j} \ker\left(u_i^j\right)$ of $F := \prod_{i \geq 0} E_i$ with $u_i^j = \psi_i^j \circ \mathrm{pr}_j - \mathrm{pr}_i$ (cf. (2.6)); but ψ_i^j is a strict epimorphism (since the projective limit is strict) and $\mathrm{pr}_j, \mathrm{pr}_i$ are strict epimorphisms (by the definition of the product topology), so u_i^j is a strict morphism by Corollary 2.131(ii). We know that F is reflexive, so $E'_b \cong F'_b/E^0$ by Theorem 3.117(3); furthermore, $F'_b \cong \bigoplus_{i \geq 0} (E_i)'_b$ by Theorem 3.105(1), $(E_i)'_b$ is an $(\mathcal{FS})$-space, and $E^0 = \overline{\sum_{i \leq j} \mathrm{im}\left({}^t u_i^j\right)} = \sum_{i \leq j} \mathrm{im}\left({}^t u_i^j\right)$ by Theorem 3.83(1) and the Dieudonné-Schwartz theorem (Theorem 3.125). Hence, $F'_b/E^0 = \varinjlim (E_i)'_b$. Finally, ${}^t u_i^j$ is a strict morphism of $(\mathcal{FS})$-spaces by Theorem 3.130, so the inductive limit $\varinjlim (E_i)'_b$ is strict.

2) $E = F/H$, where $F := \bigoplus_{i \geq 0} E_i$, $H = \sum_{i \leq j} \mathrm{im}\left(v_i^j\right)$, $v_i^j = \mathrm{inj}_j \circ \varphi_i^j - \mathrm{inj}_i$. An argument analogous to the above shows that H is closed in F, so, by Lemma 3.129, $(F/H)'_b \cong \left(H^0\right)_b = \bigcap_{i \leq j} \ker\left({}^t v_i^j\right)$, and ${}^t v_i^j = {}^t\varphi_i^j \circ \mathrm{pr}_j - \mathrm{pr}_i$. Hence, $(F/H)'_b \cong \varprojlim (E_i)'_b$, and this limit can be shown to be strict as above. ■

The “dual version” of Corollary 3.75 may be stated as follows:

COROLLARY 3.134.– *A locally convex space E is a Silva space if and only if E is the increasing inductive limit of a sequence of Banach spaces (E_n) such that the canonical injection $\varphi_{n+1}^n : E_n \hookrightarrow E_{n+1}$ is compact for each n. In particular, every Silva space is an $(\mathcal{LF})$-space.*

3.9. Continuous multilinear mappings

3.9.1. *Continuous bilinear mappings*

(I) SEPARATELY CONTINUOUS BILINEAR MAPPINGS Let E, F, and G be three topological vector spaces over $\mathbb{K}$ and suppose that $u : E \times F \to G$ is a $\mathbb{K}$-bilinear mapping. Let $(x_0, y_0) \in E \times F$. If the two partial linear mappings $u(x_0, .) : y \mapsto u(x_0, y)$ and $u(., y_0) : x \mapsto u(x, y_0)$ are continuous, then the mapping u is said to be *separately continuous* at the point (x_0, y_0). It is said to be *separately continuous* if it is separately continuous at every point $(x_0, y_0) \in E \times F$.

LEMMA 3.135.– *Let $\mathcal{L}_s(E; G)$ (resp. $\mathcal{L}_s(F; G)$) be the space of continuous linear mappings from E into G (resp. from F into G) equipped with the topology of pointwise convergence. The correspondences $u \mapsto [y \mapsto u(., y)]$, $u \mapsto [x \mapsto u(x, .)]$ are canonical isomorphisms between the space of separately continuous mappings from $E \times F$ onto G and the spaces $\mathcal{L}(F; \mathcal{L}_s(E; G))$ and $\mathcal{L}(E; \mathcal{L}_s(F; G))$ respectively, which allows us to identify these three spaces (**exercise**).*

(II) CONTINUOUS BILINEAR MAPPINGS With the same hypotheses as in **(I)**, we can define the notion of a continuous bilinear mapping at the point $(x_0, y_0) \in E \times F$ by equipping $E \times F$ with the product topology. Any bilinear mapping that is continuous at (x_0, y_0) is clearly separately continuous at (x_0, y_0). Furthermore:

LEMMA 3.136.– *The bilinear mapping $u : E \times F \to G$ is continuous at (x_0, y_0) if and only if it is continuous at $(0, 0)$.*

PROOF.– Simply observe that $u(x - x_0, y - y_0) = u(x, y) - u(x_0, y) - u(x, y_0) + u(x_0, y_0)$. ■

Suppose that E, F, and G are locally convex. With this notation, the following result also holds:

THEOREM 3.137.– *(Bourbaki [BKI 50], Grothendieck ([GRO 54], Corollary p. 66)) If* u *is separately continuous, then it is continuous in both of the following two cases:*

a) E *and* F *are metrizable and at least one of these spaces is barreled.*

b) E *and* F *are barreled* $(\mathcal{DF})$*-spaces and* G *is Hausdorff.*

PROOF.– (a): **exercise***: use the Banach-Steinhaus theorem (Theorem 3.58); cf. ([SCF 99], Chapter III, Theorem 5.1). (b): cf. *loc. cit.* or ([GRO 73], Chapter 4, Part 1, section 2, Corollary 1 of Theorem 2). ■

We also have the following result (**exercise***: cf. [BKI 81], Chapter III, section 5.5, Corollaries 1 and 2 of Proposition 9):

THEOREM 3.138.– *Let* R, S, T *be three locally convex spaces.*

1) Given any equicontinuous subset H *of* $\mathcal{L}(S;T)$, *the bilinear mapping* $(u,v) \mapsto v \circ u$ *from* $\mathcal{L}(R;S) \times H$ *into* $\mathcal{L}(R;T)$ *is continuous.*

2) Hence, if S *is barreled,* $(u_n) \to u$ *in* $\mathcal{L}(R,S)$*, and* $(v_n) \to v$ *in* $\mathcal{L}(S;T)$, *then* $(v_n \circ u_n) \to v \circ u$ *in* $\mathcal{L}(R;T)$.

3.9.2. *Hypocontinuous bilinear mappings*

The notion of hypocontinuous bilinear mapping was introduced by Bourbaki [BKI 50] as an intermediate stage between separately continuous bilinear mappings and continuous bilinear mappings. Let E, F, and G be three locally convex spaces and suppose that $\mathfrak{S}$ is a bornology on E (section 2.5.1).

DEFINITION 3.139.– *A bilinear mapping* $u : E \times F \to G$ *is said to be* $\mathfrak{S}$*-hypocontinuous if it is separately continuous and furthermore satisfies the property that, given any neighborhood* W *of* 0 *in* G *and any set* $M \in \mathfrak{S}$, *there exists a neighborhood* V *of* 0 *in* F *such that* $u(M \times V) \subset W$.

A bilinear mapping $u : E \times F \to G$ is $\mathfrak{S}$-hypocontinuous if and only if, for every set $M \in \mathfrak{S}$, the image of M under the mapping $x \mapsto u(x,.)$ is an equicontinuous subset of $\mathcal{L}(F;G)$ (**exercise**). We can similarly define the notion of $\mathfrak{T}$-hypocontinuous bilinear mapping from $E \times F$ into G when $\mathfrak{T}$ is a bornology on F; we say that a bilinear mapping from $E \times F$ into G is $(\mathfrak{S}, \mathfrak{T})$-hypocontinuous if it is both $\mathfrak{S}$-hypocontinuous and $\mathfrak{T}$-hypocontinuous. The next result (which may be proved as an **exercise**) generalizes Lemma 3.135:

THEOREM 3.140.– *The correspondences* $u \mapsto [y \mapsto u(., y)]$, $u \mapsto [x \mapsto u(x, .)]$ *are canonical isomorphisms between the space of* $(\mathfrak{S}, \mathfrak{T})$*-hypocontinuous mappings from* $E \times F$ *into* G *and the spaces* $\mathcal{L}_{\mathfrak{T}}(F; \mathcal{L}_{\mathfrak{S}}(E; G))$ *and* $\mathcal{L}_{\mathfrak{S}}(E; \mathcal{L}_{\mathfrak{T}}(F; G))$ *respectively, which allows us to identify these three spaces.*

Furthermore, the Banach–Steinhaus theorem (Theorem 3.58) implies the following result (**exercise***: cf. [BKI 81], Chapter III, section 5.3, Proposition 6):

THEOREM 3.141.– *If* E *is barreled, every separately continuous mapping from* $E \times F$ *into* G *is* $\mathfrak{S}$*-hypocontinuous for any bornology on* E*.*

The space $\mathcal{L}_b(F; \mathcal{L}_b(E; G)) \cong \mathcal{L}_b(E; \mathcal{L}_b(F; G))$ is called the space of hypocontinuous bilinear mappings.

3.9.3. *Bounded multilinear mappings*

(I) If E and F are two locally convex spaces, then every continuous linear mapping from E into F is bounded (Lemma 3.37). Moreover, if E is bornological, then conversely every bounded linear mapping is continuous (Theorem 3.62). In general, let $E_1, ..., E_n$, and F be locally convex spaces. The mapping u from $E_1 \times ... \times E_n$ into F is bounded if, for every bounded subset B of $E_1 \times ... \times E_n$, $u(B)$ is bounded in F (section 2.5.1). For every integer $k \in \{1, ..., n-i+1\}$, write $\mathcal{B}(E_i, ..., E_{i+k-1}; F)$ for the vector space of bounded k-linear mappings from $E_i \times ... \times E_{i+k-1}$ into F, and write $\mathcal{B}_b(E_i, ..., E_{i+k-1}; F)$ for this space equipped with its equibornology (section 2.7.4).

THEOREM 3.142.– *For all* $k \in \{1, ..., n-1\}$, *the canonical mapping*

$$\mathcal{B}_b(E_1, ..., E_n; F) \to \mathcal{B}_b(E_1, ..., E_{n-k}; \mathcal{B}_b(E_{n-k+1}, ..., E_n; F))$$

is an isomorphism of bornological sets.

PROOF.– Suppose that $n = 2$. We will show that $\mathcal{B}_b(E_1, E_2; F) \cong \mathcal{B}_b(E_1; \mathcal{B}_b(E_2; F))$ is an isomorphism of bornological sets. Let H be a set of bilinear mappings from $E_1 \times E_2$ into F. It suffices to show that the following conditions are equivalent:

i) For every bounded subset A_i of E_i $(i = 1, 2)$, $H(A_1 \times A_2)$ is bounded in F.

ii) For every bounded subset A_1 of E_1, $\{u(x_1, .) : u \in H, x_1 \in E_1\}$ is an equibounded set of $\mathcal{B}(E_2; F)$.

Now, observe that (ii) is equivalent to saying that, for every bounded set A_i of E_i $(i = 1, 2)$, $\bigcup_{u \in H, x_1 \in A_1}$ is bounded in F; we also know that $\bigcup_{u \in H, x_1 \in A_1} = H(A_1 \times A_2)$.

This result may be extended to any arbitrary integer $n \geq 2$ by induction. ■

Let $\mathcal{L}(E_1, ..., E_n; F)$ be the space of continuous n-linear mappings from $E_1 \times ... \times E_n$ into F. Then $\mathcal{B}(E_1, ..., E_n; F) \subset \mathcal{L}(E_1, ..., E_n; F)$ (cf. Lemma 3.37). The reader may wish to show the next result (which gives a partial generalization of Theorem 3.42) as an **exercise**:

THEOREM 3.143.– *Let $E_1, ..., E_n$, and F be normed vector spaces with norms $|.|$.*

1) Let u be an n-linear mapping from $E_1 \times ... \times E_n$ into F. The following conditions are equivalent:

i) $u \in \mathcal{B}(E_1, ..., E_n; F)$.

ii) $u \in \mathcal{L}(E_1, ..., E_n; F)$.

iii) $\|u\| < \infty$, where

$$\boxed{\|u\| := \sup_{x_1, ..., x_n \neq 0} \frac{|u(x_1, ..., x_n)|}{|x_1| ... |x_n|} = \sup_{|x_1|, ..., |x_n| \leq 1} |u(x_1, ..., x_n)|} \qquad [3.10]$$

2) $\|.\|$ is a norm on $\mathcal{L}(E_1, ..., E_n; F)$, and, whenever F is a Banach space, $\mathcal{L}(E_1, ..., E_n; F)$ is also a Banach space equipped with this norm.

(II) More generally, if $E_1, ..., E_n$ are normed vector spaces all equipped with the norm $|.|$ and F is a locally convex space whose topology is defined by a family of semi-norms $\left(|.|_\gamma\right)_{\gamma\in\Gamma}$, then the family $\left(\|.\|_\gamma\right)_{\gamma\in\Gamma}$ defined on $\mathcal{L}(E_1, ..., E_n; F)$ by

$$\|u\|_\gamma := \sup_{|x_1|,...,|x_n|\leq 1} |u(x_1, ..., x_n)|_\gamma$$

is a family of semi-norms. When equipped with this family of semi-norms, $\mathcal{L}(E_1, ..., E_n; F) = \mathcal{B}(E_1, ..., E_n; F)$ is a locally convex space that is Hausdorff whenever F is Hausdorff, and quasi-complete (resp. complete) whenever F is quasi-complete (resp. complete). With these conventions, the linear mapping $\mathcal{L}(E_1, E_2; F) \to \mathcal{L}(E_1, \mathcal{L}(E_2; F))$ defined by $u \mapsto \tilde{u} : x_1 \mapsto u(x_1, .)$ is an isomorphism of locally convex spaces, and is furthermore an isometry (see the proof of Theorem 2.80) whenever F is a normed vector space (**exercise**).

3.10. Hilbert spaces

3.10.1. *Pre-Hilbert spaces*

(I) HERMITIAN FORMS Let E and F be $\mathbb{K}$-vector spaces. A mapping $u : E \to F$ is said to be *antilinear* if it is additive (i.e. $\mathbb{Z}$-linear) and $u(\lambda x) = \bar{\lambda} u(x)$ for all $x \in E$ and all $\lambda \in \mathbb{K}$ (where $\bar{\lambda}$ is the complex conjugate of λ; thus $\bar{\lambda} = \lambda$ for all λ in the case where $\mathbb{K} = \mathbb{R}$). If E and F are normed vector spaces, then we define the norm of u as we did in (3.2) (cf. Theorem 3.42).

A *sesquilinear* form on E is a $\mathbb{Z}$-bilinear form ([P1], section 3.1.5) $f : E \times E \to \mathbb{K}$ such that $f(x, .)$ is linear and $f(., y)$ is antilinear for every $x, y \in E$ and all $\lambda, \mu \in \mathbb{K}$.[18] Any such form is said to be *Hermitian symmetric*, or is called a *Hermitian form* on E, if the relation $f(y, x) = \overline{f(x, y)}$ also holds for every $x, y \in E$. If $\mathbb{K} = \mathbb{R}$, then antilinear mappings are just linear mappings and Hermitian forms are just *symmetric forms*. A Hermitian form f is said to be *semi-positive definite* if $f(x, x) \geq 0$ for all $x \in E$. If so, we write $f(x, y) = \langle x|y\rangle$, and say that this form is a *scalar product* on E.

(II) PRE-HILBERT SPACES A $\mathbb{K}$-vector space E equipped with a scalar product $\langle x|y\rangle$ is said to be a *pre-Hilbert space*. The Cauchy-Schwarz *inequality*

18 Some authors adopt the reverse convention: $f(x, .)$ is antilinear and $f(., y)$ is linear.

holds in any such space[19]: for all $x, y \in E$, $|\langle x|y\rangle| \leq \sqrt{\langle x|x\rangle}.\sqrt{\langle y|y\rangle}$. The proof of this inequality is a classic **exercise**.

COROLLARY 3.144.– *(Minkowski*'s inequality*) For all* $x, y \in E$,

$$\sqrt{\langle x+y|x+y\rangle} \leq \sqrt{\langle x|x\rangle} + \sqrt{\langle y|y\rangle}$$

PROOF.– Observe that $\langle x+y|x+y\rangle = \langle x|x\rangle + \langle y|y\rangle + 2\Re\langle x|y\rangle$ and $|\Re\langle x|y\rangle| \leq |\langle x|y\rangle|$. Therefore, Minkowski's inequality follows from the Cauchy-Schwarz inequality (cf. also Lemma 2.2 in section 2.1.1). ■

Minkowski's inequality shows that the mapping $x \mapsto \sqrt{\langle x|x\rangle}$ is a semi-norm, called a *prehilbertian semi-norm*, which we shall denote $\|.\|$. Every pre-Hilbert space is equipped with this semi-norm; this space is Hausdorff (i.e. $\|.\|$ is a norm) if and only if the Hermitian form $\langle .|.\rangle$ is *positive definite*, or in other words $\langle x|x\rangle > 0$ for all $x \neq 0$. The Cauchy-Schwarz inequality may be rewritten as

$$\boxed{|\langle x|y\rangle| \leq \|x\|\,\|y\|\,, \quad \forall x, y \in E}. \qquad [3.11]$$

If $x = y$, then equality holds in (3.11), so the Hermitian form $\langle .|.\rangle$ has norm 1, provided that E is not the trivial space (the norm of the Hermitian form $\langle .|.\rangle$ is defined in the same way as the norm of a bilinear mapping: cf. Theorem 3.143); in particular, the Hermitian form $\langle .|.\rangle$ is continuous.

(III) ORTHOGONALITY For all $x, y \in E$, the following equation holds:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re\left(\langle x|y\rangle\right), \qquad [3.12]$$

and so $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if and only if $\langle x|y\rangle = 0$ (Pythagorus' *theorem*); these x and y are said to be *orthogonal*.

DEFINITION 3.145.– *Let A be a non-empty subset of a pre-Hilbert space E. The set $A^{\perp}$ of elements of E that are orthogonal to every element of A is called the orthogonal complement of A.*

19 See the Wikipedia article on the Cauchy-Schwarz *inequality*.

LEMMA 3.146.– *Let A be a non-empty subset of a pre-Hilbert space E. The orthogonal complement $A^{\perp}$ is a closed subspace of E. If E is Hausdorff, then $A \cap A^{\perp} = \{0\}$.*

PROOF.– Note that $A^{\perp} = \bigcap_{x \in A} \ker \langle x|.\rangle$, which shows that $A^{\perp}$ is closed in E. If E is Hausdorff and $x \in A \cap A^{\perp}$, then $\langle x|x\rangle = 0$, so $x = 0$. ■

(IV) RELATION BETWEEN SEMI-NORMS AND SCALAR PRODUCTS The expression (3.12) allows us to calculate the semi-norm of a pre-Hilbert space from its scalar product. The reader may show as an **exercise** that the reverse is also possible (by replacing y with iy, where $i^2 = -1$, an expression for $\Im(\langle x|y\rangle)$ can be derived after a few calculations).

3.10.2. *Hilbert spaces*

(I) A Hilbert *space* is a pre-Hilbert space that is Hausdorff and complete. Hilbert spaces are therefore Banach spaces. If E is a Hausdorff pre-Hilbert space with scalar product $\langle.|.\rangle$, then its completion $\hat{E}$ (Definition 2.81) is a Hilbert space whose scalar product extends $\langle.|.\rangle$ (**exercise**).

(II) PROJECTION THEOREM AND CONSEQUENCES

THEOREM 3.147.– *(*projection*) Let E be a Hausdorff pre-Hilbert space and suppose that H is a* complete *non-empty* convex *subset of E (one important special case of this situation is when E is a Hilbert space and H is a non-empty closed and convex subset of E).*

1) For all $x \in E$, there exists precisely one point $p_H(x)$ in H such that $\|x - p_H(x)\| = \inf_{y \in H} \|x - y\|$. This point $p_H(x)$ is called the projection *of x onto H, and is the unique element $a \in H$ such that*

$$\Re \langle x - a|y - a\rangle \leq 0. \qquad [3.13]$$

for all $y \in H$ (see Figure 3.1). The projection operator p_H is an idempotent and continuous linear mapping with norm 1 (cf. Theorem 3.42).

2) In particular, if H is a complete subspace of E, then $p_H(x)$ is the unique element $a \in H$ such that $\Re \langle x - a|y - a\rangle = 0$ for all $y \in H$, called the orthogonal projection *of x onto H. In this case, the projection operator p_H is linear with norm 1 (cf. Theorem 3.42); the topological direct sum $E = H \oplus H^{\perp}$ holds, so H is a* split subspace *of E (section 3.2.2**(IV)**).*

PROOF.–

1) Appolonius' median theorem[20] states that, if m is the midpoint of the segment $[b, c]$, then, for all $a \in E$, the following relation holds, writing d for the metric determined by the norm $\|.\|$ (section 3.3.2(**I**)):

$$d(b,a)^2 + d(c,a)^2 = 2d(m,a)^2 + \frac{1}{2}d(c,b)^2 . \qquad [3.14]$$

Let $x \in E$, $\delta = d(x, H)$. There exists a sequence (y_n) of elements of H such that $d(x, y_n) \to \delta$ as $n \to +\infty$. Therefore, by replacing a, b, c with x, y_n, y_m respectively in (3.14) , we see that (y_n) is a Cauchy sequence (**exercise**), which must therefore converge to some point $y \in H$. The mapping $d(x, .)$ is continuous (section 2.1.1), so $d(x, y) = \delta$, and the point $y \in H$ satisfying this property is unique; indeed, if $z \in H$ is a second such point, then the sequence (y_n) such that $y_n = y$ for n even and $y_n = z$ for n odd satisfies $d(x, y_n) \to \delta$ as $n \to +\infty$, so (y_n) is a Cauchy sequence, and so $z = y$.

By translation, we can reduce to the case where $a = p_H(x) = 0$. The inequality (3.13) may therefore be written as $\Re\langle x|y\rangle \leq 0$. We have that:

$$\|x - y\|^2 = \langle x - y|x - y\rangle = \|x\|^2 + \|y\|^2 - 2\Re\langle x|y\rangle$$

so $\|x - y\|^2 > \|x\|^2$ if (3.13) holds and $y \neq 0$, in which case 0 is the projection of x onto H. Conversely, suppose that 0 is the projection of x onto H; since H is convex, $tx \in H$ for all $t \in [0, 1]$ whenever $x \in H$; if so, the inequality $\|a - tx\|^2 \geq \|a\|^2$ also holds, which implies that $-2t\Re\langle a|x\rangle + t^2\|x\|^2 \geq 0$. But if $\Re\langle a|x\rangle > 0$, then this quantity would be < 0 for sufficiently small $t > 0$: contradiction.

The relation $p_H \circ p_H = p_H$ clearly holds. Moreover, let $a, b \in E$ and $\alpha = p_H(a), \beta = p_H(b)$. Then

$$\begin{aligned}\|\alpha - \beta\|^2 &= \Re\langle \alpha - \beta|\alpha - \beta\rangle \\ &= \Re\langle \alpha - a|\alpha - \beta\rangle + \Re\langle b - \beta|\alpha - \beta\rangle + \Re\langle a - b|\alpha - \beta\rangle ,\end{aligned}$$

20 See the Wikipedia article on Appolonius' *theorem*.

and the first two terms on the right-hand side are $\leq$ 0, so $\|\alpha - \beta\|^2 \leq \|a - b\| \, \|\alpha - \beta\|$, which implies that $\|\alpha - \beta\| \leq \|a - b\|$.

2) The reader may wish to show that p_H is linear with norm 1 as an **exercise**. Let $x \in E$ and $a = p_H(x)$; we know that $x = x - a + a$ and $\langle x - a, b\rangle = 0$ for all $b \in H$, so $x - a \in F^{\perp}$, $E = H^{\perp} + H$, and this is a direct sum by Lemma 3.146. It is furthermore a topological direct sum by Lemma 3.4. ■

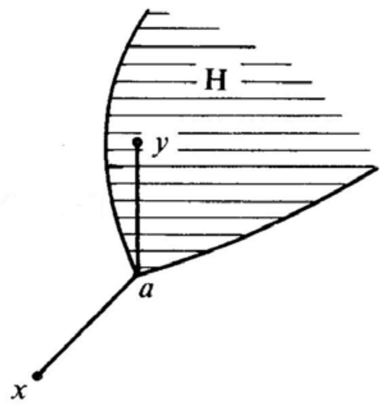

Figure 3.1. *Projection onto a closed convex set*

REMARK 3.148.– *Given a $\mathbb{K}$-vector space E and supplementary subspaces F_1, F_2 of E (i.e. which satisfy $F_1 \oplus F_2 = E$), every element x of E may be uniquely written in the form $x = x_1 + x_2$, where $x_1 \in F_1$ and $x_2 \in F_2$. The surjection $\pi_1 : x \mapsto x_1$ is called the projection of x onto E_1 parallel to E_2. Hence, if H is a complete subspace of a Hausdorff pre-Hilbert space E, then p_H is projection onto H parallel to $H^{\perp}$, and is called* orthogonal projection *onto H.*

COROLLARY 3.149.– *Let E be a Hilbert space, F a vector subspace of E, and $H = \overline{F}$ its closure in E.*

i) F is dense in E if and only if $F^{\perp} = \{0\}$.

ii) $H = F^{\perp\perp}$ and $F^{\perp\perp\perp} = F^{\perp}$.

PROOF.–

i) If F is dense in E, let $x \in E$. There exists a sequence (x_n) of points in F that converge to x. If $y \in F^{\perp}$, then $\langle x_n|y\rangle = 0$ for every integer n, so $\langle x, y\rangle = 0$, since the linear form $\langle ., y\rangle$ is continuous. Consequently, $F^{\perp} = E^{\perp} = \{0\}$. If F is not dense in E, then its closure H is a proper subspace of E. Let $x \in \complement_E H$ and $a = p_H(x)$; then $x - a \neq 0$ and $x - a \in F^{\perp}$.

ii) $F \subseteq F^{\perp\perp}$, so $\overline{F} \subseteq F^{\perp\perp}$, since $F^{\perp\perp}$ is closed in E by Lemma 3.146. Furthermore, $E = \overline{F} \oplus F^{\perp}$ by Theorem 3.147(2), and, replacing $\overline{F}$ by $F^{\perp}$, we find that $E = F^{\perp} \oplus F^{\perp\perp}$. Therefore, $\overline{F} = F^{\perp\perp}$. ■

Note that if $(F_i)_{i\in I}$ is a family of closed subspaces of a Hilbert space E, then

$$\left(\bigcup\nolimits_{i\in I} F_i\right)^{\perp} = \bigcap\nolimits_{i\in I} F_i^{\perp}, \quad \left(\bigcap\nolimits_{i\in I} F_i\right)^{\perp} = \overline{\left[\bigcup\nolimits_{i\in I} F_i\right]}.$$

COROLLARY 3.150.– *A family $(a_i)_{i\in I}$ of elements of a Hilbert space E is total (section 3.2.2(**III**)) if and only if $y = 0$ whenever $\langle a_i|y\rangle = 0$ for all $i \in I$.*

(II) QUOTIENTS OF HILBERT SPACES Let E be a Hilbert space with norm N, and suppose that F is a closed subspace of E. We know from Theorem 3.39 that E/F is a Banach space equipped with the norm (3.1) . Let $\varphi : E \twoheadrightarrow E/F$ be the canonical epimorphism, and, for all $x \in E$, suppose that $\bar{x} = \varphi(x)$. The direct topological sum $E = F \oplus F^{\perp}$ implies the existence of an isomorphism of **Tvs** $\psi : E/F \tilde{\rightarrow} F^{\perp} : \dot{x} \mapsto p_{F^{\perp}}(x)$, and $\dot{N}(\bar{x}) = N(p_{F^{\perp}}(x))$ by Theorem 3.147(2). If we define

$$\langle \bar{x}|\bar{y}\rangle := \langle p_{F^{\perp}}(x)\,|p_{F^{\perp}}(y)\rangle$$

for every $x, y \in E$, then ψ is an isomorphism of Hilbert spaces. Every quotient of a Hilbert space by a Hilbert subspace is therefore a Hilbert space.

Arguments similar to those given in section 3.4.1(**I**) show that the category of Hilbert spaces is preabelian and admits finite inductive and projective limits (**exercise**).

(III) Duality

THEOREM 3.151.–

1) (Riesz) Let E be a Hilbert space with scalar product $\langle .|. \rangle_E$, and suppose that $x' \in E'$ is a continuous linear form on E. There exists a unique element $x^* \in E$ such that $\langle x', . \rangle = \langle x^* | . \rangle_E$ and the mapping $\sigma_E : E' \to E : x' \mapsto x^*$ is an antilinear bijection from E' onto E with norm 1. Therefore, E is anti-isomorphic (as a Banach space) to its dual E'.

2) The dual E' of E is itself a Hilbert space with scalar product $\langle x' | y' \rangle_{E'} = \overline{\langle x^* | y^* \rangle_E}$, using the same notation as above.

PROOF.–

1) Suppose that $x' \neq 0$ and let $M = \ker(\langle x' | . \rangle)$. We know that the algebraic isomorphism $E/M \cong \operatorname{im}(\langle x' | . \rangle)$ holds (Noether's first isomorphism: cf. [P1], section 2.2.3(**II**), Theorem 2.12(1)). Moreover, $\operatorname{im}(\langle x' | . \rangle) = \mathbb{K}$, so $\operatorname{codim}(M) = 1$. Since $E = M \oplus M^\perp$ (Theorem 3.147(2)), $M^\perp \cong \mathbb{K}$. Let $v \in M^\perp$ be such that $M^\perp = \mathbb{K}v$; set $x^* = \bar{\lambda} v$ with $\lambda = \frac{1}{\|v\|^2} \langle x', v \rangle$. Then $\langle x', x \rangle = 0 = \langle x^* | x \rangle$ for all $x \in M$, and $\langle x^* | v \rangle = \bar{\lambda} \|v\|^2 = \langle x', v \rangle$; therefore, $\langle x^* | x \rangle = \langle x', x \rangle$ for all $x \in E$. The mapping $\sigma_E : x' \mapsto \frac{1}{\|v\|^2} \overline{\langle x', . \rangle}$ is clearly bijective and antilinear, and, since $\sup_{\|x\| \leq 1} |\langle x', x \rangle| = \sup_{\|x\| \leq 1} |\langle \sigma(x') | x \rangle|$, we also deduce that σ_E has norm 1.

2) It is easy to check that $\overline{\langle \sigma_E(.) | \sigma_E(.) \rangle_E}$ is indeed a positive definite Hermitian form on E', and that $\overline{\langle \sigma_E(x') | \sigma_E(x') \rangle_E} = \|x'\|^2$; we observed earlier that the norm of a pre-Hilbert space determines its scalar product (section 3.10.1(**IV**)). ■

We can immediately deduce the following result:

COROLLARY 3.152.– *Let E be a Hilbert space. Then σ^2 is a linear isometry from E'' onto E, so E is a reflexive Banach space (section 3.7.4).*

(IV) HILBERTIAN SUMS

THEOREM-DEFINITION 3.153.–

*1) Let $(E_i)_{i\in I}$ be a family of Hilbert spaces, each equipped with the scalar product $\langle .|.\rangle$ and norm $\|.\|$. Let $E \subset \prod_{i\in I} E_i$ be the set of families $(x_i)_{i\in I}$ such that $\sum_{i\in I} \|x_i\|^2 < +\infty$ (section 3.2.1(**III**)). Given any elements $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$, the family $\sum_{i\in I} \langle x_i|y_i\rangle$ is summable (*ibid.*) and E is a Hilbert space with scalar product*

$$\langle x|y\rangle = \sum_{i\in I} \langle x_i|y_i\rangle .$$

We write $E = \widetilde{\bigoplus}_{i\in I} E_i$, and say that E is the Hilbertian sum *of the family $(E_i)_{i\in I}$.*[21]

2) If F is a Hilbert space and $(E_i)_{i\in I}$ is a family of closed subspaces of E such that $E_i \cap E_j = \{0\}$ whenever $i \neq j$, then we similarly define the Hilbertian sum $E = \widetilde{\bigoplus}_{i\in I} E_i \subset F$. Writing x_i for the orthogonal projection of an arbitrary element $x \in F$, we have that

$$\boxed{\|x\|^2 \leq \|x\|^2 - \|x - p_E(x)\|^2 = \|p_E(x)\|^2 = \sum_{i\in I} \|x_i\|^2}. \qquad [3.15]$$

PROOF.– If $x, y \in E$, then $x + y \in E$, and, for all $\lambda \in \mathbb{K}$, $\lambda x \in E$ (**exercise**), so E is a $\mathbb{K}$-vector space. We have that $|\langle x_i|y_i\rangle| \leq \frac{1}{2}\left(\|x_i\|^2 + \|y_i\|^2\right)$ by the Cauchy-Schwarz inequality (3.11), and hence E is a Hausdorff pre-Hilbert space. The reader may show as an **exercise*** that this space is complete (cf. [BKI 81], Chapter V, section 2.1, Proposition 1). The rest of the result is clear. ■

The relations in (3.15) imply the following inequality, known as the Bessel-Parseval inequality:

$$\boxed{\|x\|^2 \leq \sum_{i\in I} \|x_i\|^2}.$$

21 The Hilbertian sum $\widetilde{\bigoplus}_{i\in I} E_i$ is often written $\bigoplus_{i\in I} E_i$; however, this notation can be confusing, since $\widetilde{\bigoplus}_{i\in I} E_i$ is not a direct sum except in the finite case.

We have that $E = F$ if and only if $\bigcup_{i \in I} E_i$ contains a total subset of F, in which case the Bessel-Parseval inequality becomes an equality.

(V) HILBERT BASES A family $(e_i)_{i \in I}$ in a Hausdorff pre-Hilbert space F is said to be *orthogonal* if e_i and e_j are orthogonal whenever $i \neq j$. If $\|e_i\| = 1$ also holds for every i, then this family is said to be *orthonormal*. A set $E \subset F$ is said to be *orthonormal* if $E = \{e_i : i \in I\}$, where the family $(e_i)_{i \in I}$ is orthonormal.

Let $(e_i)_{i \in I}$ be an orthonormal family in a Hilbert space F, let $E_i = \mathbb{K} e_i$, and suppose that $E = \widetilde{\bigoplus}_{i \in I} E_i$. Let $x_i = p_{E_i}(x)$ for all $x \in F$ and all $i \in I$; then there exists $\lambda_i \in \mathbb{K}$ such that $x_i = \lambda_i e_i$, so $\lambda_i = \langle e_i | x_i \rangle = \langle e_i | x \rangle$ and

$$p_E(x) = \sum_{i \in I} \langle e_i | x \rangle e_i, \quad \|p_E(x)\| = \sum_{i \in I} \|\langle e_i | x \rangle\|^2 .$$

COROLLARY-DEFINITION 3.154.– *The orthonormal family $(e_i)_{i \in I}$ is called a* Hilbert basis *of F if it is total in F. If so, the following relations hold for every $x \in E$:*

$$\boxed{x = \sum_{i \in I} \langle e_i | x \rangle e_i, \quad \|x\|^2 = \sum_{i \in I} |\langle e_i | x \rangle|^2},$$

the latter of which is also known as the Bessel-Parseval equality.

A Hilbert basis of a Hilbert space E is a basis of E if and only if E is finite-dimensional. The next result is analogous to the basis extension theorem ([P1], section 3.1.3**(IV)**, Theorem 3.10):

THEOREM 3.155.– *For every orthonormal set L in a Hilbert space E, there exists a Hilbert basis B containing L. In particular, every Hilbert space has a Hilbert basis.*

PROOF.– Let $\mathcal{D}$ be the set of orthonormal subsets of E containing L, ordered by inclusion. This set is of finite character ([P1], section 1.1.2(**III**)), and so has a maximum element B (*ibid.*, Lemma 1.4). It is also total, which can immediately be seen by arguing by contradiction. Therefore, B is a Hilbert basis of E. ■

(VI) $l^2(I)$-SPACES We now define

$$\boxed{l^2(I) := \left\{ (\lambda_i)_{i \in I} \in \mathbb{K}^I : \sum_{i \in I} |\lambda_i|^2 < +\infty \right\}}.$$

This is a Hilbert space with scalar product $\left\langle (\lambda_i)_{i\in I} \,|\, (\mu_i)_{i\in I} \right\rangle = \sum_{i\in I} \overline{\lambda_i}.\mu_i$.

THEOREM 3.156.– *Let E be a Hilbert space and suppose that $(e_i)_{i\in I}$ is a Hilbert basis of E. The following mapping is an isomorphism of Hilbert spaces.*

$$\boxed{E \cong l^2\left(I\right)}. \qquad [3.16]$$

PROOF.– We have that

$$E = \widetilde{\bigoplus_{i\in I}} \mathbb{K}e_i = \left\{ \sum\nolimits_{i\in I} \lambda_i e_i : \lambda_i \in \mathbb{K}, \quad \sum\nolimits_{i\in I} \left|\lambda_i\right|^2 < +\infty \right\},$$

from which we deduce the isomorphism $E \tilde{\rightarrow} l^2\left(I\right) : \sum_{i\in I} \lambda_i e_i \mapsto (\lambda_i)_{i\in I}$. ■

LEMMA-DEFINITION 3.157.–

1) The Hilbert spaces $l^2\left(I\right)$ and $l^2\left(J\right)$ are isomorphic if and only if $\mathrm{Card}\left(I\right) = \mathrm{Card}\left(J\right)$.

2) Let E be a Hilbert space for which the isomorphism (3.16) *holds. We say that* $\mathrm{Card}\left(I\right)$ *is the Hilbert* dimension *of E.*

3) A Hilbert space is separable if and only if it has Hilbert dimension $\leq \aleph_0$.

PROOF.–

1) Since every Hilbert basis of a finite-dimensional Hilbert space is a basis, by ([P1], section 3.1.7, Theorem 3.28(ii)), we simply need to consider the case of infinite-dimensional Hilbert spaces. Let B and C be two Hilbert bases of a given infinite-dimensional Hilbert space E. For each $b \in B$, let C_b be the subset of C consisting of the $c \in C$ such that $\langle b|c\rangle \neq 0$. Then $b = \sum_{c\in C_b} \langle b|c\rangle\, c$, so C_b is countable (section 3.2.1**(III)**). Furthermore, $C = \bigcup_{b\in B} C_b$, so $\mathrm{Card}\left(C\right) \leq \aleph_0 \,\mathrm{Card}\left(B\right) = \mathrm{Card}\left(B\right)$ ([P1], section 1.1.2**(IV)**, (1.4)). By symmetry, $\mathrm{Card}\left(C\right) = \mathrm{Card}\left(B\right)$. Conversely, if $\mathrm{Card}\left(I\right) = \mathrm{Card}\left(J\right)$, then $l^2\left(I\right) \cong l^2\left(J\right)$ clearly holds.

3) The Hilbert space E is separable if and only if it contains a total sequence from which a total free family can be extracted. By the Gram-Schmidt[22]

22 See the Wikipedia article on the Gram-Schmidt *algorithm*.

orthonormalization algorithm, we can construct a countable Hilbert basis from this family. ■

3.10.3. *Operators in Hilbert spaces*

(I) ADJOINT OF AN OPERATOR Let E, F be two Hilbert spaces, suppose that $u \in \mathcal{L}(E; F)$, and let ${}^t u : F' \to E'$ be its transpose (section 3.5.3).

LEMMA-DEFINITION 3.158.– *The operator*

$$u^* = \sigma_E \circ {}^t u \circ \sigma_F^{-1} : F \to E$$

(with the notation of Theorem 3.151(1)) is called the adjoint *of the operator* u. *It is continuous and linear, and the mapping* ${}^t u \mapsto u^*$ *from* $\mathcal{L}(F'; E')$ *into* $\mathcal{L}(F; E)$ *is antilinear with norm* 1.

DEFINITION 3.159.– *An endomorphism* $u \in \mathcal{L}(E)$ *is said to be a* normal *operator in* E *if* $u \circ u^* = u^* \circ u$, *a* Hermitian *operator if* $u^* = u$, *and a* unitary *operator if* $u^* \circ u = u \circ u^* = 1_E$. *If* $\mathbb{K} = \mathbb{R}$, *Hermitian operators are said to be* symmetric, *and unitary operators are said to be* orthogonal.

THEOREM 3.160.– *Let* E *be a Hilbert space. The set of normal operators in* E *is closed in* $\mathcal{L}(E)$. *The set* $\mathcal{H}(E)$ *of Hermitian operators and the set* $\mathrm{U}(E)$ *of unitary operators in* E *are closed subspaces of* $\mathcal{L}(E)$.

PROOF.– Let (u_n) be a sequence of normal operators in E that converges in $\mathcal{L}(E)$ to u. We know that $u_n \circ u_n^* = u_n^* \circ u_n$ for any integer n, so $u^* \circ u = u \circ u^*$ by Theorem 3.138, since every Hilbert space is barreled. A similar argument shows that the set of Hermitian operators and the set of unitary operators in E are also closed in $\mathcal{L}(E)$, and they are furthermore vector subspaces. ■

Moreover, $\mathrm{U}(E)$ is a closed subgroup of $\mathrm{GL}(E)$, as is the set $\mathrm{O}(E)$ of orthogonal endomorphisms of a real Hilbert space E (**exercise**).

(II) POSITIVE HERMITIAN OPERATORS Let E be a Hilbert space and suppose that $u \in \mathcal{H}(E)$. This operator is said to be *positive* if $\langle x | u(x) \rangle \geq 0$ for all $x \in E$. The set $\mathcal{H}_+(E)$ of positive operators in E is a closed cone with summit 0 in $\mathcal{H}(E)$ (section 3.3.4). We can define the following order relation on $\mathcal{H}(E)$: $u \geq v$ if $u - v \in \mathcal{H}_+(E)$.

Let E, F be Hilbert spaces and suppose that $u \in \mathcal{L}(E; F)$; then $u^* \circ u \in \mathcal{H}_+(E)$, $u \circ u^* \in \mathcal{H}_+(F)$, and $\|u^* \circ u\| = \|u \circ u^*\| = \|u\|^2$ (**exercise**).

We say that an operator $u \in \mathcal{H}(E)$ is *positive definite* if it is positive and $x = 0$ whenever the equality $\langle x | u(x) \rangle = 0$ holds. We say that this operator is *coercive* if there exists a real number $\varepsilon > 0$ such that $u \geq \varepsilon 1_E$.

LEMMA 3.161.– *If $u \in \mathcal{H}(E)$ and λ is an eigenvalue of u (Definition 3.47(3)), then $\lambda \in \mathbb{R}$; if $u \in \mathcal{H}_+(E)$, then $\lambda \geq 0$; if u is positive definite, then $\lambda > 0$ (**exercise**). For complements in the finite-dimensional case, see ([BLS 10], section 13.5).*

(III) SURJECTIVITY; LEFT AND RIGHT INVERTIBILITY

THEOREM 3.162.– *Let E, F be two Hilbert spaces and suppose that $r \in \mathcal{L}(E; F)$. The following conditions are equivalent:*

i) r is surjective.

ii) r is right invertible in $\mathbf{Tvs}$*, i.e. is a retraction in* $\mathbf{Tvs}$.

iii) $r \circ r^*$ is coercive.

iv) r^* is an injective strict morphism, i.e. there exists a real number $k > 0$ such that, for all $y \in F$, $\|r^*(y)\| \geq k \|y\|$.

PROOF.– (i)⇒(iv) by Corollary 3.127 and Lemma 3.50. (iv)⇒(iii): $\|r^*(y)\|^2 = (r^*(y) | r^*(y)) = (y | (r \circ r^*)(y))$. (iii)⇒(ii): $s = r^* \circ (r \circ r^*)^{-1}$ is a right inverse of r. (ii)⇒(i): clear. ■

COROLLARY 3.163.– *Let E, F be two Hilbert spaces and suppose that $s \in \mathcal{L}(E; F)$. The following conditions are equivalent:*

i) s^* is surjective.

ii) s is left invertible in $\mathbf{Tvs}$*, i.e. is a section in* $\mathbf{Tvs}$.

iii) $s^* \circ s$ is coercive.

iv) s is an injective strict morphism, i.e. there exists a real number $k > 0$ such that, for all $x \in E$, $\|s(x)\| \geq k \|x\|$. Therefore, for every left inverse r of s, it follows that $\|r\| \geq 1/k$.

PROOF.– $(r \circ s)^* = s^* \circ r^*$, so s is left invertible if and only if s^* is right invertible. Therefore, we can simply apply Theorem 3.162, replacing r with s^*. Let r be a left inverse of s; for all $x \in E$, we have that $\|x\| = \|r(s(x))\| \leq \|r\| \|s(x)\| \leq \|r\| k \|x\|$, so $\|r\| \geq 1/k$. ■

(IV) INJECTIVITY

THEOREM 3.164.– *Let E, F be two Hilbert spaces and suppose that $s \in \mathcal{L}(E; F)$. The following conditions are equivalent:*

i) s is injective.

ii) s is left invertible in **Vec**.

iii) $s^*(F)$ is dense in E.

iv) $s^* \circ s$ is positive definite.

PROOF.– (i)⇔(ii) by ([P1], section 3.1.4(**I**), Lemma-Definition 3.15). (ii)⇔(iii) by Theorem 3.83. (i)⇔(iv): for all $x \in E$, we have that $\langle s(x) | s(x) \rangle = \langle x, (s^* \circ s)(x) \rangle$, so $\ker(s) = \ker(s^* \circ s)$. ■

COROLLARY 3.165.– *Let E, F be two Hilbert spaces and suppose that $r \in \mathcal{L}(E; F)$. The following conditions are equivalent:*

i) r^* is injective.

ii) r^* is left invertible in **Vec**.

iii) $r(E)$ is dense in F.

iv) $r \circ r^*$ is positive definite.

(V) COMPACT OPERATORS

THEOREM 3.166.– *Let E be a normed vector space and suppose that F is a Hilbert space. Then the space $\mathcal{K}(E; F)$ of compact operators from E into F (section 3.4.6(**I**)) is the closure in $\mathcal{L}(E; F)$ of the subspace $\mathcal{FR}(E; F)$ of finite-rank operators.*

PROOF.– We know that $\overline{\mathcal{FR}(E; F)} \subset \mathcal{K}(E; F)$ by section 3.4.6(**I**), so we simply need to show the inclusion in the other direction. Let B be the unit ball of E, pick $u \in \mathcal{K}(E; F)$, and suppose that $\varepsilon > 0$. Since $u(B)$ is relatively compact in F, there exist $b_1, ..., b_n \in u(B)$ such that the balls of center b_i

and radius ε form a covering of $u(B)$. Let H be the vector subspace of F generated by the b_i. Since H is finite-dimensional, it is closed in F, and there exists an orthogonal projection operator $p_H : F \twoheadrightarrow H$ (Theorem 3.147(2)). The operator $p \circ u$ is of finite rank, and, for all $x \in B$, there exists an index $i \in \{1, ..., n\}$ for which $\|u(x) - b_i\| \leq \varepsilon$, so $\|u(x) - p \circ u(x)\| \leq \varepsilon$, and hence $\|u - p \circ u\| \leq \varepsilon$. ■

REMARK 3.167.– *S. Banach originally conjectured that Theorem 3.166 still holds when F is a Banach space that is not a Hilbert space. However, P. Enflo published a counterexample in 1972.*

COROLLARY 3.168.– *Let E and F be Hilbert spaces and suppose that $u \in \mathcal{L}(E; F)$. If u is a finite-rank (resp. compact) operator, then its adjoint u^* is also of finite rank and has the same rank as u (resp. is also compact).*

PROOF.– (1) If $\dim(\operatorname{im}(u)) = n$, then $\dim\left(\ker(u)^\perp\right) = n$. But $\ker(u)^\perp = \overline{\operatorname{im}(u^*)}$ by the relation (3.4) in Theorem (3.83), so $\dim\left(\overline{\operatorname{im}(u^*)}\right) = n$, and $\dim(\operatorname{im}(u^*)) \leq n$. Since $u^{**} = u$, $\dim(\operatorname{im}(u^*)) = n$.

(2) If u is compact, then it is the limit of a sequence of finite-rank operators (Theorem 3.166). If $(u_j) \to u$, where each u_j has finite rank, then $\left(u_j^*\right) \to u^*$, since the mapping $u \mapsto u^*$ is antilinear with norm 1 (Lemma-Definition 3.158). Each u_j^* has finite rank by (1), so u^* is compact. ■

REMARK 3.169.– *Let E and F be two Hausdorff locally convex spaces and suppose that $u \in \mathcal{L}(E; F)$. Part (1) of the above proof and Corollary 3.82 show that if u has finite rank, then ${}^t u$ also has finite rank. It can further be shown that ${}^t u$ is a compact operator whenever u is a compact operator ([GRO 73], Chapter 5, Part 2, section 1, Proposition 1).*

3.11. Nuclear spaces

We mentioned in the preface that we do not have the space to fully present the theory of nuclear spaces and its applications in this *Précis*. However, a brief overview will prove helpful later, so we shall outline a few of the key ideas below. Every result stated in this section was established by Grothendieck [GRO 55].

3.11.1. *Tensor product topologies*

(I) Let E and F be two vector spaces, and write $G = E \otimes F$ for their tensor product. There exists a canonical linear mapping $h : E \times F \to E \otimes F$ ([P1], section 3.1.5(**I**)).

If E and F are normed vector spaces, then we can equip G with the structure of a normed vector space as follows: if g is an arbitrary element of G, then there exist *finite and non-unique* families $(e)_{i\in I}$ and $(f_i)_{i\in I}$ of elements of E and F respectively such that

$$g = \sum_{i\in I} e_i \otimes f_i. \qquad [3.17]$$

It is easy to check that the mapping $\|.\|_1 : G \to \mathbb{R}_+$ defined by

$$\|g\|_1 = \inf\left\{\sum_{i\in I} \|e_i\| \, \|f_i\| : (3.17) \text{ holds}\right\}$$

is a norm on G. The norm $\|.\|_1$ determines the finest locally convex topology on G for which the canonical mapping h is continuous.

Suppose now that E and F are locally convex spaces.

DEFINITION 3.170.– *The finest locally convex topology for which the canonical mapping h is continuous is called the* projective topology *on $E \otimes F$. We write $E \otimes_\pi F$ for this space equipped with this topology.*

The projective topology on $E \otimes F$ is Hausdorff whenever E and F are Hausdorff. If so, we write $E \widehat{\otimes}_\pi F$ for the completion of $E \otimes_\pi F$.

(II) Write $\mathfrak{B}(E'_s, F'_s)$ for the space of separately continuous bilinear forms on $E'_s \times F'_s$ (section 3.9.1(**I**)). Then

$$E \otimes F \subset \mathfrak{B}(E'_s, F'_s) \subset (E' \otimes F')^*,$$

where $(.)^*$ denotes the algebraic dual, which follows from the relation ([P1], section 3.1.5(**I**))

$$(x \otimes y)(x' \otimes y') = \langle x', x\rangle \langle y', y\rangle$$

with the obvious notation.

DEFINITION 3.171.– *The* bi-equicontinuous topology *on* $\mathfrak{B}(E'_s, F'_s)$ *is the topology of uniform convergence on sets of the form* $S \otimes T \subset E' \otimes F'$, *where* S, T *are equicontinuous subsets (Definition 2.116, section 2.7.3) of* E', F' *respectively. We write* $\mathfrak{B}_e(E'_s, F'_s)$ *for the space* $\mathfrak{B}(E'_s, F'_s)$ *equipped with the bi-equicontinuous topology, and* $E \otimes_\varepsilon F$ *for the space* $E \otimes F$ *equipped with the induced topology.*

The bi-equicontinuous topology is always Hausdorff. We write $E \widehat{\otimes}_\varepsilon F$ for the completion of $E \otimes_\varepsilon F$.[23] If E and F are complete, then $\mathfrak{B}_e(E'_s, F'_s)$ is complete ([GRO 55], Introduction, III, 6), in which case $E \widehat{\otimes}_\varepsilon F$ is the closure of $E \otimes F$ in $\mathfrak{B}_e(E'_s, F'_s)$.

3.11.2. *Nuclear mappings*

(I) Let E, F be two locally convex spaces. Every element $v \in E' \otimes F$ determines a unique continuous linear mapping $u = \Phi(v) : E \to F$ such that

$$u(x) = \sum_{i=1}^{r} \langle x'_i, x \rangle y_i. \qquad [3.18]$$

The mapping $\Phi : E' \otimes F \to \mathcal{L}(E; F)$ is linear and injective; this mapping can be used to identify $E' \otimes F$ with a vector subspace of $\mathcal{L}(E; F)$: specifically, the subspace of finite-rank linear mappings (section 3.4.6**(I)**).

(II) Suppose that E and F are Banach spaces. By (3.18),

$$\|u\| = \sup_{\|x\| \le 1} \|u(x)\| \le \sum_{i=1}^{r} \|x'_i\| \, \|y_i\|,$$

so Φ is continuous. The vector space $E' \otimes F$ equipped with the norm $\|.\|_1$ (section 3.11.1**(I)**) therefore has a topology that is finer than the topology of $\mathcal{L}_b(E; F)$. The latter space is a Banach space (Theorem 3.42); since Φ is uniformly continuous (Lemma 2.125, section **(I)**), there exists a canonical monomorphism $\hat{\Phi}$ from the completion $E' \widehat{\otimes}_\pi F$ of $E' \otimes F$ into $\mathcal{L}_b(E; F)$ (Theorem 2.80) under which $E' \widehat{\otimes}_\pi F$ may be identified with a vector subspace of $\mathcal{L}(E; F)$.

23 The spaces $E \widehat{\otimes}_\pi F$ and $E \widehat{\otimes}_\varepsilon F$ are respectively denoted by $E \hat{\otimes} F$ and $E \hat{\hat{\otimes}} F$ in [GRO 55].

DEFINITION 3.172.– *The elements of* $E' \widehat{\otimes}_\pi F \subset \mathcal{L}(E; F)$ *are called* nuclear mappings.

(III) Let us now return to the general case: suppose that E, F are locally convex spaces.

DEFINITION 3.173.– *A linear mapping* u *from* E *into* F *is said to be* nuclear *if it may be expressed as the composition of three continuous linear mappings* α, β, γ

$$E \xrightarrow{\alpha} E_1 \xrightarrow{\beta} F_1 \xrightarrow{\gamma} F$$

where E_1, F_1 *are Banach spaces and* β *is nuclear.*

It can be shown ([GRO 55], Chapter I, section 3.2) that every nuclear mapping is compact (section 3.4.6(**I**)). If E, F, G, and H are locally convex spaces, $v : F \to G$ is nuclear, and $u : E \to F$, $w : G \to H$ are both continuous and linear, then $w \circ v \circ u$ is nuclear (compare this result with Lemma 3.70).

3.11.3. *Nuclear spaces*

(I) We have the following result ([GRO 55], Chapter II, section 2.1):

THEOREM-DEFINITION 3.174.– *Let* E *be a Hausdorff locally convex space.*

1) The following conditions are equivalent:

i) For any choice of Hausdorff locally convex space F*, the locally convex space* $E \otimes_\pi F$ *is isomorphic to a dense subspace of* $\mathfrak{B}_e(E'_s, F'_s)$.

ii) Every continuous linear mapping from E *into a Banach space* F *is nuclear.*

2) The space E *is said to be* nuclear *if one of the above conditions is satisfied.*

A Hausdorff locally convex space E is nuclear if and only if its completion $\widehat{E}$ is nuclear. If E is nuclear and complete, then, for any complete Hausdorff locally convex space F, the canonical mapping from $E \widehat{\otimes}_\pi F$ into $E \widehat{\otimes}_\varepsilon F$ is an isomorphism of $\mathbf{Lcs}$ and (after identification) the following equalities hold in $\mathbf{Lcs}$: $E \widehat{\otimes}_\pi F = E \widehat{\otimes}_\varepsilon F = \mathfrak{B}_e(E'_s, F'_s)$. Since every nuclear mapping is compact, every nuclear space is a Schwartz space (section 3.4.6(**II**)) (*ibid.*). Every finite-dimensional space is nuclear, and conversely every nuclear normed vector space is finite-dimensional. Every barreled quasi-complete locally convex space that is nuclear is a Montel space ([TRE 67], Corollary 3 of Proposition 50.2), and thus reflexive (Theorem 3.123). A nuclear Fréchet space is separable, since every metrizable Montel space is separable ([SCF 99], Chapter IV, Exercise 19(d)). Any nuclear $(\mathcal{DF})$-space (i.e. any nuclear Silva space: cf. Definition 3.131) is the strong dual of a nuclear Fréchet space (and hence of a separable Fréchet-Montel space); it is Suslin ([TRE 67], Appendix, Proposition A.9), and therefore separable (section 2.6.2). If E and F are nuclear, then $E \widehat{\otimes}_\pi F$ and $E \widehat{\otimes}_\varepsilon F$ are both nuclear.

(II) PERMANENCE PROPERTIES The following result can also be shown ([GRO 55], Chapter II, section 2.2):

Every vector subspace and every Hausdorff quotient of a nuclear space is nuclear. The projective limit (and in particular the product) of a family of nuclear spaces is nuclear. Every Hausdorff space that is the inductive limit of a sequence of nuclear spaces is nuclear.

If E is an $(\mathcal{F})$ or $(\mathcal{DF})$-space (Definitions 3.51 and 3.100)[24], then E is nuclear if and only if E'_b is nuclear. If so, whenever F is a nuclear space, then $\mathcal{L}_b(E; F)$ is also nuclear.

(III) RELATION WITH HILBERT SPACES It is possible to prove the following result ([SCF 99], Chapter III, section 7.3, Corollaries 2 and 3): Let E be a nuclear space. The topology on E is determined by a family of prehilbertian semi-norms (p_α) (section 3.10.1(**II**)).

Every complete nuclear space is the projective limit of a family of Hilbert spaces. This family is countable if E is a Fréchet space, in which case $E = \varprojlim \psi_m^n H_n$, where each H_n is a Hilbert space and ψ_m^n is nuclear for $m < n$.

COROLLARY 3.175.– *Let E be a countable product of nuclear Fréchet spaces and nuclear Silva spaces. Then E is a separable nuclear space (and thus a Lindelöf and paracompact nuclear space).*

24 The $(\mathcal{DF})$-spaces considered in this *Précis* are always complete.

PROOF.– Every product of nuclear spaces is nuclear and every countable product of separable spaces is separable (section 2.3.5(**I**)), hence Lindelöf (section 2.6.3). Furthermore, since E is Hausdorff, it is regular (Corollary 2.124), and therefore paracompact (Theorem 2.109(ii)). ■

4

Measure and Integration, Function Spaces

4.1. Measure and integration

The most essential groundwork for the notion of "abstract measure" and the theory of integration was laid by E. Borel and H. Lebesgue in the early 20th century. Abstract measures (section 4.1.1) are founded upon the notion of measurable sets (sets equipped with a σ-algebra $\mathcal{T}$) and play a key role in probability theory. Radon measures (which were formulated by F. Riesz and J. Radon between 1909 and 1913) and their properties are studied in section 4.1.5. These measures are based on the theory of duality in locally convex spaces. Abstract Borel measures (Example 4.4(1)) are equivalent to positive Radon measures on locally compact topological spaces (Theorem 4.31). Each type of measure has its own advantages and challenges: abstract measures may be defined in a more general context than Radon measures, and in particular calculating probabilities is extremely difficult with the latter. The theory of Radon measures is straightforward on locally compact topological spaces, but becomes more intricate when working with general topological spaces [SCW 73], ([BKI 69], Chapter IX). On the other hand, the images of abstract measures exhibit pathological behavior in ways that the images of Radon measures don't (section 4.1.5(**II**)); for more details, see ([SCW 93], Volume III, p. 366 sq.) and ([SCW 73], 1st Part, Chapter I, Theorem 8). Radon measures are distributions of order 0 (Example 4.94(1)).

The following presentation of measure and integration theory (which only includes a few proofs and therefore cannot substitute for a treatise on the topic) attempts to avoid redundancy between these two approaches by using the Riesz representation theorem (Theorem 4.31); we shall also take full advantage of the notion of absolutely continuous function (section 4.1.7). Radon measures are studied in the context of a locally compact space Ω that is countable at infinity (section 2.3.9**(II)**).

REMARK 4.1.– *Assuming that the space is countable at infinity is not necessary, but is almost never a restriction in practice. It will allow various simplifications: every Borel measure on Ω is "regular" (which simplifies the statement of Theorem 4.31); the space $\mathcal{M}(\Omega)$ of complex Radon measures is the dual of the space $\mathcal{K}(\Omega)$, which has the well-understood structure of an $\mathcal{L}_s\mathcal{F}$-space (section 4.1.4**(IV)**); the space $L^1_{loc}(\Omega,\mu)$ is a Fréchet space (section 4.1.5**(III)**). Furthermore, the tensor product of the Borel σ-algebras of two locally compact spaces Ω_1, Ω_2 that are countable at infinity is the Borel σ-algebra of $\Omega_1 \times \Omega_2$; every Radon measure μ is "moderated"; the essentially μ-integrable functions may be identified with the μ-integrable functions, which simplifies the Fubini-Tonelli theorem (Theorem 4.21) and allows us to apply it in its most immediate form when integrating with respect to the tensor product of two Radon measures (compare with [BKI 69], Chapter V, section 8.4, Theorem 1).*

Haar measures on Lie groups and harmonic analysis are discussed in [P3].

4.1.1. *"Abstract" measures*

(I) σ-ALGEBRAS Let X be a set. A σ-algebra $\mathcal{T}$ on X is defined as a subset of $\mathfrak{P}(X)$ containing $\varnothing$ that is closed under countable unions and complements. The pair $(X, \mathcal{T})$ is called a *measurable space*, and the elements of $\mathcal{T}$ are called the measurable sets in X. Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be two measurable spaces. The mapping $f : X \to Y$ is said to be $(\mathcal{S}$-$\mathcal{T})$*-measurable* if $f^{-1}(B) \in \mathcal{S}$ for every $B \in \mathcal{T}$ ([SCW 93], Volume III, Definition 5.2.16).

For example, if X is a topological space, then the σ-algebra generated by the opens sets of X (i.e. the smallest σ-algebra that contains every open set of X) is called the Borel σ*-algebra* on X. If X, Y are two topological spaces, $\mathcal{S}$ is the Borel σ-algebra of X, and $\mathcal{T}$ is the Borel σ-algebra on Y, then every continuous mapping from X into Y is $(\mathcal{S}$-$\mathcal{T})$-measurable.

If $(X, \mathcal{S})$ is a measurable space and $X_1 \subset X$, then the trace $\mathcal{T}_1 := \{A \cap X_1 : A \in \mathcal{T}\}$ of $\mathcal{T}$ on X_1 is a σ-algebra on X_1 (**exercise**). If

$\mathcal{E} \subset \mathfrak{P}(X)$, then the σ-algebra *generated* by $\mathcal{E}$ is defined as the smallest σ-algebra containing $\mathcal{E}$.

(II) POSITIVE, REAL, AND COMPLEX MEASURES Let $(X, \mathcal{T})$ be a measurable space. A function $\mu : \mathcal{T} \to \bar{\mathbb{R}}_+$ is said to be a *positive measure* if $\mu(\varnothing) = 0$ and μ is countably additive, i.e. every sequence (A_n) of pairwise disjoint elements of $\mathcal{T}$ satisfies

$$\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu(A_n).$$

The triple $(X, \mathcal{T}, \mu)$ (or, by abuse of language, the couple (X, μ)) is called a *measure space*. A set $A \in \mathcal{T}$ is said to be μ-negligible if $\mu(A) = 0$.

Every positive measure μ considered below will be σ-finite, which means that it satisfies the following condition: for every $A \in \mathcal{T}$, there exists a sequence (A_n) of elements of $\mathcal{T}$ such that $A \subset \bigcup_n A_n$ and $\mu(A_n) < \infty$ for all n. A positive measure is said to be *finite* if $\mu(X) < \infty$, and is said to be a *probability measure* if $\mu(X) = 1$.

A real (resp. complex) measure is a countably additive function $\mu : \mathcal{T} \to \mathbb{R}$ (resp. $\mu : \mathcal{T} \to \mathbb{C}$). The set of positive (resp. real, resp. complex) measures on $(X, \mathcal{T})$ is denoted $\mathfrak{M}_+(X, \mathcal{T})$ (resp. $\mathfrak{M}_{\mathbb{R}}(X, \mathcal{T})$, resp. $\mathfrak{M}(X, \mathcal{T})$). The difference $\mu - \nu$ of two measures $\mu, \nu \in \mathfrak{M}_+(X, \mathcal{T})$ is defined if and only if, for every $A \in \mathcal{T}$, the relation $\mu(A) = \nu(A) = +\infty$ does not hold, in which case it is called a *signed measure* ([HAL 50], section 28). The set $\mathfrak{M}_\pm(X, \mathcal{T})$ of signed measures strictly contains $\mathfrak{M}_{\mathbb{R}}(X, \mathcal{T})$ and is not a vector space.

DEFINITION 4.2.– *If X is a topological space, then any measure on the Borel σ-algebra of X (cf.* ***(I)****) is said to be a Borel measure.*

In the following, unless otherwise stated, "abstract" measures are always assumed to be positive.

DEFINITION 4.3.– *A positive measure $\mu : \mathcal{T} \to \bar{\mathbb{R}}_+$ is said to be* complete *if $B \in \mathcal{T}$ for every pair of subsets A, B of X such that $B \subset A$ and A is μ-negligible.*

If the above condition is satisfied, then clearly $\mu(B) = 0$.

EXAMPLE 4.4.–

1) The Dirac measure δ_a *at the point* $a \in X$ *(also described as the measure defined by the unit mass at the point* a*) is the positive measure defined by* $\delta_a(a) = 1$ *if* $a \in A$, $\delta_a(A) = 0$ *if* $a \notin A$.

2) The Lebesgue measure λ *on the real line is the Borel measure defined by the condition* $\lambda([a,b]) = b - a$ *for every pair of elements* $a, b \in \mathbb{R}$ *such that* $b \geq a$*. This measure is not complete ([HAL 50], section 15).*

3) Let I *be an interval of* $\mathbb{R}$ *and suppose that* $f : I \to \mathbb{R}$ *is a left-continuous non-decreasing function. The Stieltjes* measure μ_f *defined by* f *is the Borel measure on* I *uniquely determined by the condition* $\mu_f([a,b[) = f(b) - f(a)$ *for every interval* $[a,b[\subset I$*. In particular,* $\mu_f(\{a\}) = \lim_{x \to a^+} \mu_f([a,x[) = f(a+0) - f(a)$*. The Lebesgue measure on the real line is the Stieltjes measure defined by the function* $1_{\mathbb{R}} : x \mapsto x$.

4) The Dirac measure δ_0 *at the point* 0 *on the real line is the Stieltjes measure determined by the Heaviside* function $\Upsilon : \mathbb{R} \to \mathbb{R}$*, which is defined by* $\Upsilon(x) = 0$ *if* $x \leq 0$ *and* $\Upsilon(x) = 1$ *if* $x > 0$.

We have the following result ([HAL 50], section 13):

THEOREM 4.5.– *(Carathéodory) Let* $\mu : \mathcal{T} \to \bar{\mathbb{R}}_+$ *be a positive measure, where* $\mathcal{T}$ *is a* σ*-algebra on* X*. There exists a* σ*-algebra* $\hat{\mathcal{T}} \supset \mathcal{T}$ *and a positive complete measure* $\hat{\mu} : \hat{\mathcal{T}} \to \bar{\mathbb{R}}_+$ *that solve the following universal problem:*

i) $\hat{\mu}$ *extends* μ*, i.e.* $\hat{\mu}|_{\mathcal{T}} = \mu$.

ii) Every complete positive measure that extends μ *extends* $\hat{\mu}$.

DEFINITION 4.6.– *This measure* $\hat{\mu}$ *is called the* completion *of* μ.

Let $(X, \mathcal{T}, \mu)$ be a measurable set, $X_1 \subset X$, and suppose that $\mathcal{T}_1$ is the trace of $\mathcal{T}$ on X_1 (cf. **(I)**). Then, for all $A \in \mathcal{T}_1$, the value $\hat{\mu}(A)$ is well-defined ([SCW 93], Volume III, Proposition 5.10.1) and the function $\mu_1 : \mathcal{T}_1 \to \bar{\mathbb{R}}_+$ defined by $\mu_1(A) = \hat{\mu}(A)$ for all $A \in \mathcal{T}_1$ is a positive measure, said to be *induced* by μ on X_1.

(III) With the above notation, a set $A \subset X$ is said to be μ*-measurable* if it is $\hat{\mu}$-measurable, and μ*-negligible* if $\hat{\mu}(A) = 0$. Any subset of a μ-negligible set is therefore μ-negligible, and every countable union of μ-negligible sets is μ-negligible. A property P involving the elements x of X is said to be μ-almost everywhere true if the set of $x \in X$ for which $P(x)$ does not hold is μ-negligible.

Let $(X, \mathcal{T}, \mu)$ be a measure space and suppose that $(Y, \mathcal{S})$ is a measurable space. The function $f : X \to Y$ is said to be μ-measurable if it is $\left(\hat{\mathcal{T}}, \mathcal{S}\right)$-measurable. Any such function is μ-almost everywhere equal to a $(\mathcal{S}$-$\mathcal{T})$-measurable function. Suppose that $(Y, \mathcal{S}) = (E, \mathcal{B})$, where E is a Banach space or $\bar{\mathbb{R}}$, and $\mathcal{B}$ is the Borel σ-algebra on E; the function $f : X \to E$ is said to be *μ-negligible* if it is zero everywhere except on a μ-negligible set. With these conventions, we write $|x|$ for the absolute value of x when $E = \bar{\mathbb{R}}$, or for the norm of x when E is a Banach space. Let $\mathbf{f}, \mathbf{g}$ be functions from X into E. These functions are said to be *equivalent* for the measure μ if $\mathbf{f}(x) = \mathbf{g}(x)$ μ-almost everywhere. This is equivalent to saying that $|\mathbf{f} - \mathbf{g}|$ is μ-negligible, where $|\mathbf{f} - \mathbf{g}| : x \mapsto |\mathbf{f}(x) - \mathbf{g}(x)|$; this is clearly an equivalence relation $\sim_\mu$ (equivalence $\bmod .\mu$). We write $\mathring{\mathbf{f}}$ for the equivalence class $\bmod .\mu$ of the function $\mathbf{f} : X \to E$.

4.1.2. *Integration and L^p-spaces*

(I) Let $(X, \mathcal{T}, \mu)$ be a measure space, E a Banach space or $\bar{\mathbb{R}}$, and $\mathbf{f} : X \to E$. The function $\mathbf{f}$ is said to be a *simple $\mathcal{T}$-measurable* function if there exist finitely many disjoint $\mathcal{T}$-measurable sets A_i such that $\mathbf{f}$ is constant (with value $\mathbf{a}_i \in E$) on A_i and zero on $X - (\cup_i A_i)$. It is said to be μ-integrable if (with the above notation) $\sum_i |\mathbf{a}_i| \, \mu(A_i) < \infty$. If so, we define the integral of this simple function $\mathbf{f}$ by $\int \mathbf{f}.d\mu = \sum_i \mathbf{a}_i \mu(A_i)$.

The function $\mathbf{f} : X \to E$ is said to be μ-integrable if there exists a sequence of simple μ-integrable functions $(\mathbf{f}_n)$ such that $(\mathbf{f}_n(x))$ converges μ-almost everywhere to $\mathbf{f}(x)$ and $\lim_{n,m \to \infty} \int |\mathbf{f}_n - \mathbf{f}_m| \, .d\mu \to 0$. It can be shown that

$$\lim_{n \to \infty} \int \mathbf{f}_n.d\mu \qquad [4.1]$$

exists in E and is independent of the choice of approximating sequence $(\mathbf{f}_n)$.

DEFINITION 4.7.–

1) The limit (4.1) *is called the integral (or sometimes the "Bochner integral") of* $\mathbf{f}$ *with respect to* μ*, and is written*

$$\boxed{\int \mathbf{f}.d\mu \quad \text{or} \quad \int_X \mathbf{f}(x).d\mu(x)}.$$

2) If $X = \mathbb{R}$ *and* $\mu = \sigma_{\boldsymbol{F}}$ *(Stieltjes measure defined by* $\boldsymbol{F}$*, cf. Example 4.4(3)), then this integral (called the "Stieltjes integral") is written* $\int \mathbf{f}.d\boldsymbol{F}$*; in particular, in the case where* μ *is the Lebesgue measure (Example 4.4(2)), the integral is written* $\int_{\mathbb{R}} \mathbf{f}(x).dx$*.*[1]

The integral $\int_{]a,b[} \mathbf{f}(x).dx$ on the interval $]a,b[\subset \mathbb{R}$ is equal to the Riemann integral on this interval whenever the function $\mathbf{f}$ is Riemann-integrable.

The Dirac *comb* ϖ on $\mathbb{R}$ is defined by the conditions (i) $\varpi(\{n\}) = 1$ for every $n \in \mathbb{Z}$ and (ii) $\varpi(]n, n+1[) = 0$. Given a continuous function $\mathbf{f} : \mathbb{R} \to E$, the relation $\int \mathbf{f}.d\varpi = \sum_{n\in\mathbb{Z}} \mathbf{f}(n)$ holds whenever this sum is absolutely convergent.

LEMMA 4.8.–

1) The function $\mathbf{f} : X \to E$ *is* μ*-integrable if and only if* $|\mathbf{f}|$ *is* μ*-integrable, in which case* $\left|\int \mathbf{f}.d\mu\right| \leq \int |\mathbf{f}|.d\mu$*. If* $\mathbf{f}$ *is* μ*-integrable,* G *is another Banach space, and* $u \in \mathcal{L}(E;G)$*, then* $u(\mathbf{f}) : x \mapsto u(\mathbf{f}(x))$ *is* μ*-integrable,* $\int u(\mathbf{f}).d\mu = u\left(\int \mathbf{f}.d\mu\right)$*, and* $\left|\int u(\mathbf{f}).d\mu\right| \leq \|u\| \int |\mathbf{f}|.d\mu$*.*

2) The relation $\int |\mathbf{f}|.d\mu = 0$ *holds if and only if* $\mathbf{f}(x) = 0$ μ*-almost everywhere (**exercise**).*

(II) DOMINATED CONVERGENCE THEOREM AND ITS CONSEQUENCES

The dominated convergence theorem is one of the most important results of measure and integration theory ([SCW 93], Volume III, Theorem 5.7.25):

1 Although this notation has become entrenched by tradition, it is inconsistent. If we write $\int \mathbf{f}.d\sigma_F$ for the Stieltjes integral of $\mathbf{f}$, this expression becomes $\int \mathbf{f}.d(dF)$, since $\sigma_F = dF$. To write the Stieltjes integral of $\mathbf{f}$ consistently as $\int \mathbf{f}.dF$, we would have to write $\int \mathbf{f}.\mu$ for the integral of $\mathbf{f}$ with respect to the measure μ, which nobody ever does.

THEOREM 4.9.– *(Lebesgue's* dominated convergence theorem*) Let E be a Banach space or $\bar{\mathbb{R}}$ and suppose that $(\mathbf{f}_n)$ is a sequence of μ-integrable functions from X into E that converges μ-almost everywhere to a function $\mathbf{f} : X \to E$. If there exists a μ-integrable function $g : X \to \mathbb{R}_+$ such that $|\mathbf{f}_n(x)| \leq g(x)$ μ-almost everywhere, then $\mathbf{f}$ is μ-integrable and $\int \mathbf{f}_n.d\mu$ converges to $\int \mathbf{f}.d\mu$ in E as $n \to +\infty$.*

The following results can easily be deduced (**exercise**) from Theorem 4.9 (although they are often presented as preliminary results of this theorem)[2]:

THEOREM 4.10.–

1) (Beppo-Levi theorem*, also known as Lebesgue's* monotone convergence theorem*) Let (f_n) be an increasing sequence of μ-measurable functions from X into $\bar{\mathbb{R}}_+$ and suppose that f is a function from X into $\bar{\mathbb{R}}$ such that $f_n(x) \to f(x)$ μ-almost everywhere. Then f is μ-measurable and $\int f.d\mu = \lim \int f_n.d\mu$.*

2) (Fatou's lemma*) Let (f_n) be a sequence of μ-measurable functions from X into $\bar{\mathbb{R}}_+$. Then $\int \left(\liminf_{n\to+\infty} f_n \right) .d\mu \leq \liminf_{n\to+\infty} \int f_n.d\mu$.*

THEOREM 4.11.– *Let Ω be an open subset of $\mathbb{R}$, E a real Banach space, and $(x,t) \mapsto \mathbf{f}(x,t)$ a mapping from $X \times \Omega$ into E that satisfies the following conditions:*

i) For μ-almost every $x \in X$, the function $\mathbf{f}(x,.)$ has a derivative $\partial\mathbf{f}/\partial t$.

ii) There exists a μ-integrable function $g \geq 0$ such that $|\partial\mathbf{f}/\partial t(x,t)| \leq g(x)$ μ-almost everywhere in X for all $t \in \Omega$.

Then $\mathbf{h} : t \mapsto \int f(x,t)\, d\mu(x)$ is differentiable on Ω and

$$\dot{\mathbf{h}}(t) = \int \frac{\partial \mathbf{f}}{\partial t}(x,t)\, d\mu(x).$$

2 See the Wikipedia articles: *Monotone convergence theorem, Fatou's lemma*, and *Dominated convergence theorem.*

PROOF.– Let (h_n) be a sequence of non-zero real numbers that tends to 0. We need to show that, as $n \to +\infty$,

$$\int \frac{\mathbf{f}(x, t+h_n) - \mathbf{f}(x,t)}{h_n} d\mu(x) \to \int \frac{\partial \mathbf{f}}{\partial t}(x,t)\, d\mu(x).$$

By the theorem of averages and conditions (i) and (ii),

$$\left| \int \frac{\mathbf{f}(x, t+h_n) - \mathbf{f}(x,t)}{h_n} \right| \leq g(x)$$

μ-almost everywhere in X, so the result follows from Theorem 4.9. ■

(III) L^p-SPACES Let $(X, \mathcal{T}, \mu)$ be a measure space, and suppose that $\mathcal{L}^p(X, \mu; E)$ $(p \in [1, \infty[)$ is the space of functions $\mathbf{f} : X \to E$ such that $|\mathbf{f}|^p$ is μ-integrable. Furthermore, let $\mathcal{L}^\infty(X, \mu; E)$ be the space of functions $\mathbf{f} : X \to E$ such that $\operatorname{ess\,sup} |\mathbf{f}(x)| < \infty$, where, given any function $g : X \to \bar{\mathbb{R}}$,

$$\boxed{\operatorname{ess\,sup} g := \inf \left\{ a \in \bar{\mathbb{R}} : g(x) \leq a,\ \mu\text{-almost everywhere} \right\}}.$$

We say that $\operatorname{ess\,sup} g$ is the *essential supremum* of g; similarly, $\operatorname{ess\,inf} g := -\operatorname{ess\,sup}(-g)$ is the *essential infimum* of g.

Let $\mathcal{N}$ be the subspace of $\mathcal{L}^p(X, \mu; E)$ $(p \in [1, \infty])$ given by the μ-negligible functions, and suppose that $L^p(X, \mu; E) := \mathcal{L}^p(X, \mu; E) / \mathcal{N}$ (henceforth, $L^p(X, \mu) = L^p(X, \mu; \mathbb{K})$).

THEOREM 4.12.– *(Fischer-Riesz) For every $p \in [1, \infty]$, $L^p(X, \mu; E)$ is a Banach space equipped with the following norm:*

$$\boxed{\begin{aligned} N_\infty\left(\mathring{\mathbf{f}}\right) &:= \operatorname{ess\,sup} |\mathbf{f}(x)| \ \textit{if}\ p = \infty, \\ N_p\left(\mathring{\mathbf{f}}\right) &:= \sqrt[p]{\textstyle\int |\mathbf{f}|\,.d\mu}\ \textit{if}\ p < \infty. \end{aligned}}$$

In particular, if E is a Hilbert space, then $L^2(X, \mu; E)$ is also a Hilbert space.

PROOF.–

1) In the case $p = \infty$, N_∞ clearly defines a norm on $L^\infty(X, \mu; E)$. Furthermore, any sequence $\left(\mathring{\mathbf{f}}_n\right)$ converges to $\mathring{\mathbf{f}}$ in $L^\infty(X, \mu; E)$ if and only if $(\mathbf{f}_n)$ converges to $\mathbf{f}$ uniformly everywhere except on a μ-negligible set. It immediately follows that $L^\infty(X, \mu; E)$ is complete (cf. Theorem 2.111).

2) For any $p \in [1, \infty[$, N_p clearly satisfies the conditions ($\mathbf{N}_1$), ($\mathbf{N}_2$), as well as condition ($\mathbf{N}_4$) in section 3.3.2**(I)** by Lemma 4.8(2). Only the triangle inequality ($\mathbf{N}_3$) remains to be shown. Observe that $|\mathbf{f}|^p$ is μ-integrable if and only if there exists a sequence $(\mathbf{f}_n)$ of simple functions converging μ-almost everywhere to $\mathbf{f}$ such that $|\mathbf{f}_n|^p$ is μ-integrable for all n. The triangle inequality ($\mathbf{N}_3$) therefore follows from Theorem 4.9 and Minkowski's inequality (inequality (2.3) in Lemma 2.2). Hence, N_p is a norm on $L^p(X, \mu; E)$. A similar argument shows that every Cauchy sequence in $L^p(X, \mu; E)$ is convergent[3], so $L^p(X, \mu; E)$ is a Banach space.

In the case where $p = 2$ and E is a Hilbert space with scalar product $\langle | \rangle$, the scalar product $\langle . | . \rangle_2$ of $L^2(X, \mu; E)$ is given by

$$\boxed{\left\langle \mathring{\mathbf{f}} | \mathring{\mathbf{g}} \right\rangle_2 = \int_X \langle \mathbf{f}(x) | \mathbf{g}(x) \rangle . d\mu(x)}. \blacksquare$$

Each norm N_p ($p \in [1, \infty]$) determines a semi-norm $\tilde{N}_p$ on $\mathcal{L}^p(X, \mu; E)$ by setting $\tilde{N}_p(\mathbf{f}) = N_p\left(\mathring{\mathbf{f}}\right)$; we will simply write N_p for this semi-norm below. Replacing Hölder's inequality ((2.2), (2.4)) by Minkowski's inequality in the proof of Theorem 4.12 yields the following result:

LEMMA 4.13.– *Let E, F, G be three Banach spaces and suppose that $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}.\mathbf{y}$ is a continuous bilinear mapping from $E \times F$ into G with norm equal to 1 (Theorem 3.143). If $\mathbf{f} \in \mathcal{L}^p(X, \mu; E)$ and $\mathbf{g} \in \mathcal{L}^q(X, \mu; F)$ for the pair of conjugate exponents $p, q \in [1, \infty]$ (Definition 2.1), then $\mathbf{f}.\mathbf{g} \in \mathcal{L}^1(X, \mu; G)$ and $N_1(\mathbf{f}.\mathbf{g}) \leq N_p(\mathbf{f}) N_q(\mathbf{g})$, where $\mathbf{f}.\mathbf{g}$ is the function $x \mapsto \mathbf{f}(x).\mathbf{g}(x)$ (**exercise**).*

3 For more details, see the Wikipedia article on the Riesz-Fischer *theorem*.

This result can be generalized by considering $p_1, ..., p_n, r \in [1, \infty]$ such that $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$ and replacing $|\mathbf{f}|$ by $|\mathbf{f}|^r$ and $|\mathbf{g}|$ by $|\mathbf{g}|^r$ if $r < \infty$ in the above, then arguing by induction:

THEOREM 4.14.– *Let $E_1, ..., E_n$, and G be Banach spaces, suppose that $(\mathbf{x}_1, ..., \mathbf{x}_n) \mapsto \mathbf{x}_1 ... \mathbf{x}_n = \prod_{i=1}^{n} \mathbf{x}_i$ is a continuous n-linear mapping with norm ≤ 1 from $E_1 \times ... \times E_n$ into G (Theorem 3.143) and, for each $i \in \{1, ..., n\}$, pick $\mathbf{f}_i \in \mathcal{L}^{p_i}(X, \mu; E_i)$, where $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}$. Then $\prod_{i=1}^{n} \mathbf{f}_i \in \mathcal{L}^r(X, \mu; G)$, and the* generalized *Hölder* inequality *states that:*

$$\boxed{N_r\left(\prod_{i=1}^{n} \mathbf{f}_i\right) \leq \prod_{i=1}^{n} N_{p_i}(\mathbf{f}_i)}. \qquad [4.2]$$

Furthermore ([BKI 69], Chapter IV, section 6, Proposition 3):

THEOREM 4.15.– *Let E be a Banach space with strong dual E' (section 3.7.4(**I**)). Choose $p \in [1, \infty]$ and write q for its conjugate exponent. Let $(X, \mathcal{T}, \mu)$ be a measure space if $p \in [1, \infty[$ or consider a locally compact space X that is countable at infinity if $p = \infty$, and let μ be a Borel measure on X (Definition 4.2). Then, for every function $\mathbf{f} \in \mathcal{L}^p(X, \mu; E)$, $p \in [1, \infty]$,*

$$N_p(\mathbf{f}) = \sup\left\{|\langle \mathbf{f}, \mathbf{g}\rangle| : \mathbf{g} \in \mathcal{L}^q\left(X, \mu; E'\right), N_q(\mathbf{g}) = 1\right\}.$$

This result also holds when the roles of $\mathbf{f}$ and $\mathbf{g}$ are exchanged.

The following related result can also be shown ([YOS 80], Chapter IV, section 9, Example 3):

THEOREM 4.16.– *Let $p \in [1, \infty[$ and q be conjugate exponents. Then $L^q(X, \mu)$ is the dual of $L^p(X, \mu)$, and the duality pairing may be defined as*

$$\boxed{\langle f, g\rangle = \int_X f(x) g(x) .d\mu(x)}.$$

COROLLARY 4.17.– *For every $p \in]1, \infty[$, the Banach space $L^p(X, \mu)$ is reflexive.*

REMARK 4.18.– *In general, $L^1(X, \mu)$ is strictly contained in the dual of $L^\infty(X, \mu)$ ([BKI 69], Chapter V, p. 65), and therefore $L^p(X, \mu)$ is not reflexive in general for $p = 1$ or $p = \infty$ ([BKI 69], Chapter V, section 5, Exercise 10). If X is a locally compact topological space that is countable at* *infinity and μ is a Borel measure on X, then $L^p(X,\mu)$ is separable (section 2.3.1(**I**)) for all $p \in [1,\infty[$, but $L^\infty(\mathbb{R},\lambda)$ is not separable ([DIE 82], Volume II, Chapter XIII, section 11, Exercise 12, and section 12, Exercise 1).*

4.1.3. *Tensor product of measures*

(I) PRODUCT OF MEASURABLE SPACES Let $(X_1,\mathcal{T}_1)$, $(X_2,\mathcal{T}_2)$ be two measurable spaces. We define their Cartesian product to be $(X_1 \times X_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$, where $\mathcal{T}_1 \otimes \mathcal{T}_2$ is the σ-algebra generated by $\mathcal{E} = \{A_1 \times A_2 : A_1 \in \mathcal{T}_1, A_2 \in \mathcal{T}_2\}$, known as the *tensor product* of the σ-algebras $\mathcal{T}_1, \mathcal{T}_2$. Given $A \in X_1 \times X_2$, write $A_{x_1} = \{x_2 \in X_2 : (x_1,x_2) \in A\}$ for each $x_1 \in X_1$ and $A^{x_2} = \{x_1 \in X_1 : (x_1,x_2) \in A\}$ for each $x_2 \in X_2$. If A is $(\mathcal{T}_1 \otimes \mathcal{T}_2)$-measurable, then A_{x_1} is $\mathcal{T}_2$-measurable for every $x_1 \in X_1$ and A^{x_2} is $\mathcal{T}_1$-measurable for every $x_2 \in X_2$. Similarly, let $(Y,\mathcal{S})$ be a measurable space and suppose that $f : X_1 \times X_2 \to Y$; if f is $(\mathcal{T}_1 \otimes \mathcal{T}_2, \mathcal{S})$-measurable, then $f_{x_1} : x_2 \mapsto f(x_1,x_2)$ is $(\mathcal{T}_2,\mathcal{S})$-measurable for every $x_1 \in X_1$ and $f^{x_2} : x_1 \mapsto f(x_1,x_2)$ is $(\mathcal{T}_1,\mathcal{S})$-measurable for every $x_2 \in X_2$. If X_1, X_2 are topological spaces and $\mathcal{T}_i$ is the Borel σ-algebra on X_i $(i = 1,2)$ (section 4.1.1(**I**)), then $\mathcal{T}_1 \otimes \mathcal{T}_2$ is the Borel σ-algebra on $X_1 \times X_2$, provided that X_1, X_2 both have topologies with a countable base (section 2.3.2(**II**)) (cf, [SCW 93], Volume III, Theorem 5.12.2).

(II) EXISTENCE AND DEFINITION OF THE TENSOR PRODUCT OF MEASURES Let $(X_1,\mathcal{T}_1,\mu_1)$, $(X_2,\mathcal{T}_2,\mu_2)$ be two measure spaces. With the above notation, we have the following result ([HAL 50], section 35, Theorem B):

THEOREM 4.19.– *There exists a uniquely determined measure μ on the product $(X_1 \times X_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$ such that, for every set $A \in \mathcal{T}_1 \otimes \mathcal{T}_2$,*

$$\mu(A) = \int_{X_1} \mu_2(A_{x_1})\,.d\mu_1(x_1) = \int \mu_1(A^{x_2})\,.d\mu_2(x_2),$$

and which furthermore satisfies $\mu(A_1 \times A_2) = \mu_1(A_1)\,\mu_2(A_2)$ for every "rectangle" $A_1 \times A_2 \in \mathcal{T}_1 \otimes \mathcal{T}_2$.

DEFINITION 4.20.– *The measure μ above is said to be the* tensor product *of the measures μ_1 and μ_2, and is written $\mu_1 \otimes \mu_2$.*

Given n measure spaces $(X_i, \mathcal{T}_i, \mu_i)$ $(i = 1, ..., n)$, we can define their tensor product inductively:

$$\mu_1 \otimes \mu_2 ... \otimes \mu_n := (\mu_1 \otimes \mu_2 ... \otimes \mu_{n-1}) \otimes \mu_n.$$

The Lebesgue measure on $\mathbb{R}^n$ is the tensor product $\lambda^{n\otimes}$ of the measures μ_i $(i = 1, ..., n)$, where μ_i is the measure λ on $\mathbb{R}$ (Example 4.4(2)).

(III) FUBINI-TONELLI THEOREM This theorem exists in several forms. The following version will be most useful to us later ([SCW 93], Volume III, Theorems 5.12.14 and 5.12.15):

THEOREM 4.21.– *(Fubini-Tonelli)*[4] *Let* $(X_1, \mathcal{T}_1, \mu_1)$, $(X_2, \mathcal{T}_2, \mu_2)$ *be two measure spaces, suppose that* E *is a Banach space or* $\bar{\mathbb{R}}$, *and let* $\mathbf{f} : X_1 \times X_2 \to E$ *be a* $\mu_1 \otimes \mu_2$*-integrable function. The set* N *of* $x_1 \in X_1$ *such that the function* $x_2 \mapsto f(x_1, x_2)$ *is not* μ_2*-integrable is* μ_1*-negligible; for every* $x_1 \in \complement_{X_1} N$, *the partial mapping* $\mathbf{f}_{x_1}$ *is* μ_2*-integrable; the function* $x_1 \mapsto \int_{X_2} \mathbf{f}(x_1, x_2)\, d\mu_2(x_2)$, *defined on* $\complement_{X_1} N$, *is* μ_1*-integrable, and the following holds:*

$$\int_{X_1} \left[\int_{X_2} \mathbf{f}(x_1, x_2)\, d\mu_2(x_2) \right] d\mu_1(x_1) = \int_{X_1 \times X_2} \mathbf{f}.d(\mu_1 \otimes \mu_2),$$

which is sometimes also written as

$$\boxed{\int_{X_1} d\mu_1(x_1) \int_{X_2} \mathbf{f}(x_1, x_2)\, d\mu_2(x_2) = \iint_{X_1 \times X_2} \mathbf{f}(x_1, x_2)\, d\mu_1(x_1)\, d\mu_2(x_2)}.$$

The roles of $(X_1, \mathcal{T}_1, \mu_1)$ and $(X_2, \mathcal{T}_2, \mu_2)$ can of course be switched. This result can also be extended by induction to the case of an arbitrary finite product of measure spaces.

For each $p \in [1, \infty[$, this theorem determines isometries $\mathbf{f} \mapsto (x_1 \mapsto \mathbf{f}_{x_1})$ and $\mathbf{f} \mapsto (x_2 \mapsto \mathbf{f}_{x_2})$:

$$\begin{aligned} L^p(X_1 \times X_2, \mu_1 \otimes \mu_2; E) &\cong L^p(X_1, \mu_1; L^p(X_2, \mu_2; E)) \\ &\cong L^p(X_2, \mu_2; L^p(X_1, \mu_1; E)), \end{aligned}$$

4 This theorem is also known as the Lebesgue-Fubini *theorem*.

provided that E is assumed to be separable ([SCW 93], Volume III, Theorem 5.12.8).

4.1.4. *Spaces of continuous functions*

(I) Let Ω be a locally compact topological space that is countable at infinity (for example an open subset of $\mathbb{R}^n$), K a compact subset of Ω, $\mathcal{K}(\Omega; K)$ the $\mathbb{C}$-algebra of continuous complex functions with support in K (section 2.3.12), and $\mathcal{C}(K)$ the $\mathbb{C}$-algebra of complex functions that are defined and continuous on K. For any complex function $\varphi \in \mathcal{C}(K)$, let

$$p_K(\varphi) = \sup_{x \in K} |\varphi(x)|. \qquad [4.3]$$

It is easy to see that p_K is a norm on $\mathcal{C}(K)$. Furthermore, $\mathcal{C}(K)$ equipped with this norm is complete by Corollary 2.114, and so is a Banach $\mathbb{C}$-algebra (Definition 3.43). The restriction $\rho_K : \varphi \longmapsto \varphi|_K$ allows $\mathcal{K}(\Omega; K)$ to be embedded in $\mathcal{C}(K)$ (by identifying φ and $\rho_K(\varphi)$ when $\varphi \in \mathcal{K}_K(\Omega)$), and $\mathcal{K}(\Omega; K)$ is clearly closed in $\mathcal{C}(K)$, so $\mathcal{K}(\Omega; K)$ is a Banach $\mathbb{C}$-algebra.

(II) We write $\mathcal{C}_0(\Omega)$ for the $\mathbb{C}$-algebra of functions φ that are continuous on Ω and zero at infinity (in other words, zero at the point ∞ of the one-point compactification of Ω : cf. Definition 2.50). The function φ on Ω is zero at infinity if and only if, for all $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ such that $|\varphi(x)| \leq \varepsilon$ for all $x \in \complement_\Omega K$. We can define the norm $\|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)|$ on $\mathcal{C}_0(\Omega)$, which implies that (**exercise**):

THEOREM 4.22.– *$\mathcal{C}_0(\Omega)$ is a Banach $\mathbb{C}$-algebra.*

(III) Let $(K_j)_{j \geq 1}$ be a sequence of compact subsets of Ω such that $K_j \Subset K_{j+1}$ and $\bigcup_{j \geq 1} K_j = \Omega$ (Lemma 2.52). Then $\mathcal{K}(\Omega) = \bigcup_{j \geq 1} \mathcal{K}(\Omega; K_j)$. We equip the algebra $\mathcal{C}(\Omega)$ of continuous functions on Ω with the family of semi-norms $(p_{K_j})_{j \geq 1}$. The locally convex space thus obtained is Hausdorff, and, since the family $(p_{K_j})_{j \geq 1}$ is countable, it is metrizable (Corollary 3.20); furthermore, it is complete (**exercise**), which gives the following result:

THEOREM 4.23.– *$\mathcal{C}(\Omega)$ is a Fréchet space.*

(IV) Consider now the space $\mathcal{K}(\Omega)$ of continuous complex functions whose support is a compact subset of Ω. We can define a topology on $\mathcal{K}(\Omega)$ by specifying the convergent nets in this space:

DEFINITION 4.24.– *We say that a net $(\varphi_i)_{i\in I}$ converges to φ in $\mathcal{K}(\Omega)$ if, for every compact set $K \subset \Omega$, there exists $i_0 \in I$ such that $\varphi_i \in \mathcal{K}_K(\Omega)$ for all $i \succcurlyeq i_0$, and $(\varphi_i)_{i\succcurlyeq i_0}$ converges to φ in the Banach space $\mathcal{K}(\Omega; K)$.*

Thus, in **Lcs**, we have the strict inductive limit $\mathcal{K}(\Omega) = \varinjlim_{j\geq 1} \mathcal{K}(\Omega; K_j)$. The next result follows:

THEOREM 4.25.– *$\mathcal{K}(\Omega)$ is an $(\mathcal{L}_s\mathcal{F})$-space.*

Readers may wish to show the next result as an **exercise*** (cf. [BKI 69], Chapter III, section 1):

THEOREM 4.26.– *There exist* continuous *canonical injections $\mathcal{K}(\Omega) \hookrightarrow \mathcal{C}_0(\Omega) \hookrightarrow \mathcal{C}(\Omega)$. Furthermore, $\mathcal{K}(\Omega)$ is dense in $\mathcal{C}_0(\Omega)$, which is dense in $\mathcal{C}(\Omega)$.*

4.1.5. *Radon measures*

(I) NOTION OF RADON MEASURE Let Ω be a locally compact topological space that is countable at infinity. The strong dual $\mathcal{M}(\Omega) := \mathcal{K}'_b(\Omega)$ of $\mathcal{K}(\Omega)$ (section 4.1.4**(IV)**) is a complete Hausdorff locally convex space, since $\mathcal{K}(\Omega)$ is bornological (Theorem 3.98). An element of $\mathcal{M}(\Omega)$ is called a Radon measure on Ω. If $U \subset \Omega$ is open, then it is locally compact and countable at infinity; a function $\varphi \in \mathcal{K}(U)$ may be extended by continuity to a function in $\mathcal{K}(\Omega)$ by setting its value to 0 on $\complement_\Omega U$; therefore, $\mathcal{K}(U)$ may be identified with a subspace of $\mathcal{K}(\Omega)$. Let $\mu \in \mathcal{M}(\Omega)$; its restriction $\mu|_U$ to $\mathcal{K}(U)$ is called the *restriction* of μ to U (or the Radon *measure induced* by μ on U). Let $(U_i)_{i\in I}$ be an open covering of Ω. For all $i \in I$, let μ_i be a Radon measure on U_i, and suppose that, for any pair $(i,j) \in I \times I$, the restrictions of μ_i and μ_j to $U_i \cap U_j$ are identical. With these assumptions, we have the following result:

LEMMA 4.27.– *(*gluing principle*) There exists a unique Radon measure μ on Ω such that $\mu|_{U_i} = \mu_i$ for all $i \in I$.*

PROOF.– The space Ω is paracompact (section 2.3.10), so there exists a continuous partition of unity $(\psi_i)_{i\in I}$ subordinate to $(U_i)_{i\in I}$ (Theorem 2.62). For every function $\varphi \in \mathcal{K}(\Omega)$, define $\langle \mu, \varphi \rangle = \sum_{i\in I} \langle \mu_i, \psi_i \varphi \rangle$. Then μ is the required Radon measure. ■

Let U_μ be the union of all open sets $U \subset \Omega$ satisfying $\mu|_U = 0$. By Lemma 4.27, $\mu\left|_{U_\mu}\right. = 0$, and U_μ is therefore the largest open set on which μ vanishes.

DEFINITION 4.28.– *The* support *of a Radon measure μ on Ω, denoted* $\operatorname{supp}(\mu)$, *is the complement of U_μ in Ω* : $\operatorname{supp}(\mu) := \complement_\Omega U_\mu$.

(II) IMAGE OF A RADON MEASURE Let $\pi : \Omega \to \Omega'$ be a homeomorphism from Ω onto a locally compact space Ω' that is countable at infinity, and suppose that μ is a Radon measure on Ω. Given any function $\varphi \in \mathcal{K}(\Omega')$, the function $\varphi \circ \pi$ belongs to $\mathcal{K}(\Omega)$ and $\operatorname{supp}(\varphi \circ \pi) = \pi^{-1}(\operatorname{supp}(\varphi))$. Consequently, the mapping $\varphi \mapsto \mu(\varphi \circ \pi)$ is a Radon measure on Ω', called the *image* of μ under π, and is written $\pi(\mu)$.

(III) RADON MEASURES AND LOCALLY INTEGRABLE FUNCTIONS Let Ω be a locally compact space that is countable at infinity and suppose that μ is a Radon measure on Ω. We write $\mathcal{L}^1_{loc}(\Omega, \mu)$ for the space of locally μ-integrable functions from Ω into $\mathbb{C}$ (i.e. the functions that are μ-integrable on every compact set $K \subset \Omega$). The mapping $N_{1,K} : \mathcal{L}^1_{loc}(\Omega, \mu) \to \mathbb{R}_+$, defined for any given compact set $K \subset \Omega$ by $N_{1,K}(f) = \int_K |f(x)|\, d\mu(x)$, is a semi-norm on $\mathcal{L}^1_{loc}(\Omega, \mu)$. If $f(x) = 0$ μ-almost everywhere, then $N_{1,K}(f) = 0$, so $N_{1,K}(f)$ depends solely on the equivalence class $\mathring{f}$ of f $(\operatorname{mod}.\mu)$ (section 4.1.1(**III**)), and may be written $N_{1,K}\left(\mathring{f}\right)$. Let $\mathcal{N} \subset \mathcal{L}^1_{loc}(\Omega, \mu)$ be the subspace of μ-negligible functions and suppose that $L^1_{loc}(\Omega, \mu) = \mathcal{L}^1_{loc}(\Omega, \mu)/\mathcal{N}$. Then, for every compact set $K \subset \Omega$, $N_{1,K}$ is a semi-norm on $L^1_{loc}(\Omega, \mu)$. We can construct a covering of Ω by choosing an increasing sequence of compact sets (K_n); therefore, $L^1_{loc}(\Omega, \mu)$, equipped with the sequence of semi-norms (N_{1,K_n}), is a Fréchet space (**exercise**).

In particular, let Ω be an open set of $\mathbb{R}^n$ and suppose that $\mathcal{L}^1_{loc}(\Omega, dx)$ is the space of functions from Ω into $\mathbb{C}$ that are locally integrable for the measure dx induced on Ω by the Lebesgue measure $\lambda^{n\otimes}$ (section 4.1.1(**II**)).

THEOREM 4.29.– *There exist canonical injections* $L^1_{loc}(\Omega, dx) \hookrightarrow \mathcal{M}(\Omega)$ *and* $\mathcal{C}(\Omega) \hookrightarrow \mathcal{M}(\Omega)$, *the latter of which preserves support.*

PROOF.– Let $f \in \mathcal{L}^1_{loc}(\Omega, dx)$ and $\varphi \in \mathcal{K}(\Omega)$. The integral

$$\mu_f(\varphi) = \int_\Omega f(x)\,\varphi(x)\,dx \qquad [4.4]$$

converges and the mapping $\varphi \mapsto \mu_f(\varphi)$ is a linear form, i.e. an element of $\mathcal{K}(\Omega)^*$. Let $(\varphi_i)_{i\in I}$ be a net that converges to 0 in $\mathcal{K}(\Omega)$. By Definition 4.24, given any compact set $K \subset \Omega$ and any real number $\varepsilon > 0$, there exists $i_0 \in I$ such that, for any $i \succeq i_0$, $\operatorname{supp}(\varphi_i) \subset K$ and $|\varphi_i(x)| \leq \varepsilon$ for every $x \in K$. Therefore,

$$|\mu_f(\varphi_i)| = \left|\int_K f(x)\,\varphi_i(x)\,dx\right| \leq \int_K |f(x)\,\varphi_i(x)|\,dx \leq \varepsilon N_{1,K}\left(\mathring{f}\right),$$

so the linear form μ_f is continuous on $\mathcal{K}(\Omega)$. If $\mu_f(\varphi) = 0$ for all $\varphi \in \mathcal{K}(\Omega)$, then $\mathring{f} = 0$ ([SCW 65], Chapter II, Proposition 2), so $f = 0$ if $f \in \mathcal{C}(\Omega)$. The reader may show as an **exercise** that, in this case, $\operatorname{supp}(\mu_f) = \operatorname{supp}(f)$. ■

(IV) BOUNDED RADON MEASURES A *bounded* Radon *measure* on Ω is an element of the dual $\mathcal{C}_0'(\Omega)$. The strong dual $\mathcal{C}_0'(\Omega)_b$ is a Banach space; we write $\|\mu\|$ for the norm of a Radon measure μ on Ω (setting $\|\mu\| = +\infty$ if μ is not bounded). The Radon measure $\mu \in \mathcal{K}'(\Omega)$ has compact support if and only if it is continuous on $\mathcal{K}(\Omega)$ *equipped with the topology induced by* $\mathcal{C}(\Omega)$ (cf. Theorem 4.26); any such measure may therefore be identified with a continuous linear form on $\mathcal{C}(\Omega)$. The next result follows (where each of the considered measures is a Radon measure):

THEOREM 4.30.– *There exist canonical injections*

$$\underbrace{\mathcal{C}'(\Omega)}_{\text{space of measures with compact support}} \hookrightarrow \underbrace{\mathcal{C}_0'(\Omega)}_{\text{space of bounded measures}} \hookrightarrow \underbrace{\mathcal{K}'(\Omega)}_{\text{space of measures}}$$

The topology of the weak dual $\mathcal{K}_s'(\Omega)$ (which is quasi-complete by Theorem 3.102, since $\mathcal{K}(\Omega)$ is barreled) is known as the *vague topology*.

(V) REAL RADON MEASURES AND POSITIVE RADON MEASURES Write $\mathcal{K}_\mathbb{R}(\Omega)$ for the set of *real* continuous functions on Ω. The subset $\mathcal{K}_+(\Omega)$ formed by the $\varphi \in \mathcal{K}_\mathbb{R}(\Omega)$ such that $\varphi(x) \geq 0$ for all $x \in \Omega$ is a closed convex salient cone, and $(\mathcal{K}_\mathbb{R}(\Omega), \mathcal{K}_+(\Omega))$ is a Riesz space and a lattice (section 3.6.5). Similarly, the subset $\mathcal{K}_+(\Omega; K)$ of functions $\varphi \in \mathcal{K}_{\mathbb{R}}(\Omega; K)$ such that $\varphi(x) \geq 0$ for all $x \in \Omega$ is a closed salient cone. If $\varphi \in \mathcal{K}_{\mathbb{R}}(\Omega; K)$ satisfies $\varphi(x) > 0$ for all $x \in K$, then $\inf_{x \in K} \varphi(x) > 0$, since $\varphi(K)$ is compact; hence, φ is in the interior of $\mathcal{K}_+(\Omega; K)$, which must therefore be non-empty.

A Radon measure μ is said to be *real* if $\mu(\varphi) \in \mathbb{R}$ for every function $\varphi \in \mathcal{K}_{\mathbb{R}}(\Omega)$. If we also have that $\mu(\varphi) \geq 0$ for every function $\varphi \in \mathcal{K}_+(\Omega)$, then we write that $\mu \geq 0$ and say that the measure μ is *positive*. The set of real Radon measures and the set of positive Radon measures are respectively denoted $\mathcal{M}_{\mathbb{R}}(\Omega)$ and $\mathcal{M}_+(\Omega)$. The set $\mathcal{M}_{\mathbb{R}}(\Omega)$ is a real vector space and $(\mathcal{M}_{\mathbb{R}}(\Omega), \mathcal{M}_+(\Omega))$ is the Riesz space that is dual to $(\mathcal{K}_{\mathbb{R}}(\Omega), \mathcal{K}_+(\Omega))$. This space is a complete lattice ([SCW 93], Volume III, Theorem 5.4.28).

Let $\mu \in \mathcal{M}_{\mathbb{R}}(\Omega)$. We set $\mu^+ = \sup\{\mu, 0\}$ and $\mu^- = \sup\{-\mu, 0\}$, so that $\mu = \mu^+ - \mu^-$.

(VI) ABSOLUTE VALUE OF A RADON MEASURE If $\mu \in \mathcal{M}_{\mathbb{R}}(\Omega)$, define $|\mu| = \sup(\mu, -\mu) = \mu^+ + \mu^- \geq 0$; this measure is called the *absolute value* of μ. Clearly,

$$|\mu| = \sup_{|\zeta| \leq 1} |\zeta \mu|. \qquad [4.5]$$

If μ is a complex Radon measure, we can also use (4.5) to define its absolute value $|\mu|$ (where ζ now ranges over the complex values with modulus ≤ 1). Then $\|\mu\| = |\mu|(1)$ (cf. **(III)**).

(VII) EXAMPLES OF RADON MEASURES The Dirac measure δ_a at the point $a \in \Omega$ is defined by $\delta_a(\varphi) = \varphi(a)$.

The Dirac comb on $\mathbb{R}$ is defined by $\varpi(\varphi) = \sum_{n=-\infty}^{+\infty} \varphi(n)$.

The Lebesgue measure $\lambda_{]a,b[}$ on $]a,b[\subset \mathbb{R}$ is defined by $\lambda_{]a,b[}(\varphi) = \int_{]a,b[} \varphi(x)\, dx$ (usual Riemann integral).

These three Radon measures are all positive. The support of δ_a is $\{a\}$ and $|\delta_a| = 1$; $\left|\lambda_{]a,b[}\right| = b - a$; the support of ϖ is $\mathbb{Z}$ and $|\varpi| = \infty$ (**exercise**).

(VIII) VECTOR RADON MEASURES Let E be a finite-dimensional real vector space. A *vector measure* $\boldsymbol{\mu}$ with values in E' is an element of

$\mathcal{M}_{E'}(\Omega) := \mathcal{L}(\mathcal{K}_{\mathbb{R}}(\Omega); E')$. The space $\mathcal{K}_E(\Omega)$ of continuous functions on Ω taking values in E may be identified with the tensor product $\mathcal{K}_{\mathbb{R}}(\Omega) \otimes_{\mathbb{R}} E$, and its dual may be identified with $\mathcal{M}_{\mathbb{R}}(\Omega) \otimes_{\mathbb{R}} E'$, which in turn may be identified with $\mathcal{M}_{E'}(\Omega)$ as follows: let $\boldsymbol{\beta} = \{\mathbf{e}_1, ..., \mathbf{e}_p\}$ be a basis of E and suppose that $\boldsymbol{\beta}' = \{\mathbf{e}'_1, ..., \mathbf{e}'_p\}$ is the dual basis. Every element $\boldsymbol{\mu}$ of $\mathcal{M}_{E'}(\Omega)$ may be uniquely written in the form $\sum_{j=1}^{p} \mu_j.\mathbf{e}'_j$, $\mu_j \in \mathcal{M}_{\mathbb{R}}(\Omega)$, and every element $\boldsymbol{\varphi}$ of $\mathcal{K}_E(\Omega)$ may be uniquely written in the form $\sum_{j=1}^{p} \varphi_j.\mathbf{e}_j$, $\varphi_i \in \mathcal{K}_{\mathbb{R}}(\Omega)$. We set

$$\boxed{\langle \boldsymbol{\mu}, \boldsymbol{\varphi} \rangle = \sum_{j=1}^{p} \langle \mu_j, \varphi_j \rangle},$$

and so $\mu_j = \langle \boldsymbol{\mu}, \mathbf{e}_j \rangle : \varphi_j \mapsto \sum_{j=1}^{p} \langle \boldsymbol{\mu}, \varphi_j.\mathbf{e}_j \rangle$, $j = 1, ..., p$.

The set of positive measures $\{|\boldsymbol{\mu}.\mathbf{z}| : \mathbf{z} \in E, |\mathbf{z}| \leq 1\}$ is upper bounded, so has a supremum, since $\mathcal{M}_{\mathbb{R}}(\Omega)$ is a complete lattice (cf. **(V)**). Set $|\boldsymbol{\mu}| = \sup_{\mathbf{z} \in E, |\mathbf{z}| \leq 1} |\boldsymbol{\mu}.\mathbf{z}|$, which generalizes (4.5).

In particular, when E is the $\mathbb{R}$-vector space $\mathbb{C}$ and $\mu \in \mathcal{M}(\Omega) = \mathcal{M}_{\mathbb{R}}(\Omega) \oplus i\mathcal{M}_{\mathbb{R}}(\Omega)$, then we define

$$\begin{aligned} \Re(\mu) &: \mathcal{K}_{\mathbb{R}}(\Omega) \to \mathbb{R} : \varphi \mapsto \Re(\langle \mu, \varphi \rangle), \\ \Im(\mu) &: \mathcal{K}_{\mathbb{R}}(\Omega) \to \mathbb{R} : \varphi \mapsto \Im(\langle \mu, \varphi \rangle), \end{aligned}$$

which implies that $\mu = \Re(\mu) + i\Im(\mu)$. We also write $\bar{\mu} = \Re(\mu) - i\Im(\mu)$. Then $|\mu| \leq |\Re(\mu)| + |\Im(\mu)|$.

4.1.6. *Integral with respect to a Radon measure*

(I) Riesz representation theorem Let Ω be a locally compact space that is countable at infinity, $\mathcal{B}$ its Borel σ-algebra, and suppose that μ is a positive Radon measure on Ω. For every $A \in \mathcal{B}$ (writing χ_A for the characteristic function of A), define $\mu(\chi_A) := \sup\{\mu(g) : g \in \mathcal{K}_{\mathbb{R}}(\Omega), g \leq \chi_A\}$. The mapping

$$\mathfrak{b}(\mu) : \mathcal{B} \to \mathbb{R}_+ : A \mapsto \mu(\chi_A) \qquad [4.6]$$

is a Borel measure. Let $\mathfrak{M}(\Omega)$ be the space of complex Borel measures (section 4.1.1**(II)**). We have the following result ([RUD 87], Theorem 6.19):

THEOREM 4.31.– *(Riesz* representation*) Given any Radon measure* $\mu \in \mathcal{M}(\Omega)$, *there exists a unique Borel measure* $\mathfrak{b}(\mu) \in \mathfrak{M}(\Omega)$ *such that every function* $\varphi \in \mathcal{K}(\Omega)$ *satisfies* $\langle \mu, \varphi \rangle = \int_\Omega \varphi . d\mathfrak{b}(\mu)$. *Furthermore, the bijection* $\mathfrak{b}$ *(which extends the mapping defined in* (4.6)*) is an isomorphism of vector spaces from* $\mathcal{M}(\Omega)$ *onto* $\mathfrak{M}(\Omega)$ *whose inverse isomorphism* $\mathfrak{r}$*, defined for each measure* $\nu \in \mathfrak{M}(\Omega)$*, is given by*

$$\mathfrak{r}(\nu) : \mathcal{K}(\Omega) \to \mathbb{C} : \varphi \mapsto \int \varphi . d\nu. \qquad [4.7]$$

Under the isomorphism $\mathfrak{b}$ above, the order relation defined on $\mathcal{M}_{\mathbb{R}}(\Omega)$ can be carried over to $\mathfrak{M}_{\mathbb{R}}(\Omega)$, allowing the latter to be equipped with the structure of a complete lattice.

(II) INTEGRAL WITH RESPECT TO A RADON MEASURE Let μ be a positive Radon measure and suppose that $\mathfrak{b}(\mu)$ is the associated Borel measure (Theorem 4.31). Let E be a Banach space or $\bar{\mathbb{R}}$. We say that $\mathbf{f} : \Omega \to E$ is μ-measurable (resp. μ-integrable) if it is $\mathfrak{b}(\mu)$-measurable (resp. $\mathfrak{b}(\mu)$-integrable), and we define $\int \mathbf{f}.d\mu = \int \mathbf{f}.d\mathfrak{b}(\mu)$.

Let μ be a complex Radon measure. We say that $\mathbf{f} : \Omega \to E$ is μ-measurable (resp. μ-integrable) if it is $|\mu|$-measurable (resp. $|\mu|$-integrable). If μ is real, we define $\int \mathbf{f}.d\mu = \int \mathbf{f}.d\mu^+ - \int \mathbf{f}.d\mu^-$. If μ is complex, we define $\int \mathbf{f}.d\mu = \int \mathbf{f}.d(\Re(\mu)) + i \int \mathbf{f}.d(\Im(\mu))$.

It is sometimes useful to directly characterize measurability with respect to the Radon measure μ : this can be done with the concept of Lusin measurability. The function $\mathbf{f} : \Omega \to E$ is *μ-measurable* (in the sense defined above) if and only if it satisfies Lusin*'s property*: for every compact set $K \subset \Omega$ and every $\delta > 0$, there exists a compact set $K_\delta \subset K$ such that $|\mu|(K \backslash K_\delta) \leq \delta$ and the restriction $\mathbf{f}|_{K_\delta}$ is continuous. A set $A \subset \Omega$ is *μ-measurable* if and only if its characteristic function χ_A is μ-measurable (in the sense of Lusin's property).

(III) MEASURE WITH BASE μ Let $\mu \in \mathcal{M}(\Omega)$ and $g \in \mathcal{L}^1_{loc}(\Omega, \mu)$ (cf. section 4.1.5**(III)**). The mapping $\varphi \mapsto \int \varphi g . d\mu$ from $\mathcal{K}(\Omega)$ into $\mathbb{C}$ is a Radon measure, since $\left|\int \varphi g . d\mu\right| \leq \|\varphi\| \int |g\chi_K| . d\mu$ for every compact set $K \subset \Omega$ such that $\operatorname{supp}(\varphi) \subset K$. This Radon measure, written $g.\mu$, is said to have *density* g with respect to μ, and is said to have *base* μ. For example, the measure $g(a)\,\delta_a$ has density g (or $g(a)$) with respect to δ_a, and has base δ_a.

Suppose that $\mu \in \mathcal{M}_+(\Omega)$ and $\nu \in \mathcal{M}(\Omega)$. We have the following result ([BKI 69], Chapter V, section 5.2, Definition 2 and section 5.5, Theorem 2):

THEOREM 4.32.– *(Radon-Nikodym)*[5] *The following conditions are equivalent:*

i) ν has base μ (i.e. there exists $g \in \mathcal{L}^1_{loc}(\Omega, \mu)$ such that $\nu = g.\mu$).

ii) For every set $A \subset \Omega$, $\mu(A) = 0 \Longrightarrow |\nu(A)| = 0$. We write that $|\nu| \ll \mu$.

iii) For every compact set $K \subset \Omega$ and every $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $\mu(K) \leq \delta \Longrightarrow |\nu|(K) \leq \varepsilon$.

If ν has base μ, then its density g (which satisfies $\nu = g.\mu$) is uniquely determined μ-almost everywhere and is called the Radon-Nikodym *derivative* of ν with respect to μ; this is written $g = \frac{d\nu}{d\mu}$ ([HAL 50], section 32). If $|\nu| \ll \mu$ (in other words, if ν has base μ), we sometimes say that ν is *absolutely continuous* with respect to μ.

(IV) INTEGRAL WITH RESPECT TO A VECTOR RADON MEASURE Let E_1, E_2, E_3, E_4 be finite-dimensional real vector spaces, $\boldsymbol{\nu} \in \mathcal{M}_{\mathcal{L}(E_1;E_2)}(\Omega)$, $\boldsymbol{\mu} \in \mathcal{M}_{\mathcal{L}(E_3;E_4)}(\Omega)$. The function $\mathbf{f} : \Omega \to \mathcal{L}(E_2; E_3)$ is said to be $\boldsymbol{\mu}$-integrable (resp. $\boldsymbol{\nu}$-integrable) if it is $|\boldsymbol{\mu}|$-integrable (resp. $|\boldsymbol{\nu}|$-integrable). By choosing bases and identifying linear mappings with their matrices with respect to these bases, $\boldsymbol{\mu}$, $\mathbf{f}$, and $\boldsymbol{\nu}$ may be identified with the matrices $\left(\mu_i^j\right)$, $\left(f_j^k\right)$, and $\left(\nu_k^l\right)$ respectively; we define

$$\left(\int \mathbf{f}(x)\,.d\boldsymbol{\nu}(x)\right)_j^l = \sum_k \int f_j^k(x)\,.d\nu_k^l(x)$$

$$\left(\int d\boldsymbol{\mu}(x)\,.\mathbf{f}(x)\right)_i^k = \sum_j \int d\mu_i^j(x)\,.f_j^k(x)$$

5 This theorem is also known as the Lebesgue-Nikodym *theorem*.

which implies that

$$\boxed{\int \mathbf{f}(x)\,.d\boldsymbol{\nu}(x) \overset{\text{identification}}{\in} \mathcal{L}(E_1;E_2)\,,\ \int d\boldsymbol{\mu}(x)\,.\mathbf{f}(x) \overset{\text{identification}}{\in} \mathcal{L}(E_2;E_4)}.$$

Let Ω_1, Ω_2 be two locally compact topological spaces that are countable at infinity, $\boldsymbol{\nu} \in \mathcal{M}_{\mathcal{L}(E_1;E_2)}(\Omega_2)$, $\boldsymbol{\mu} \in \mathcal{M}_{\mathcal{L}(E_3;E_3)}(\Omega_1)$. A function $\mathbf{h} : \Omega_1 \times \Omega_2 \to \mathcal{L}(E_2;E_3)$ is said to be $\boldsymbol{\mu} \otimes \boldsymbol{\nu}$-integrable if it is $|\boldsymbol{\mu}| \otimes |\boldsymbol{\nu}|$-integrable. We define the double integral

$$\iint d\boldsymbol{\mu}(x_1) \circ \mathbf{h}(x_1,x_2) \circ d\boldsymbol{\nu}(x_2) \in \mathcal{L}(E_1;E_4)$$

in the obvious way. It can be useful to write this integral as

$$\boxed{\int d\boldsymbol{\mu}(x_1) \int \mathbf{h}(x_1,x_2)\, d\boldsymbol{\nu}(x_2)}. \qquad [4.8]$$

This expression generalizes the one in Theorem 4.21 (however, here, the order of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ matters). In particular, with the above hypotheses, we have the following result:

THEOREM 4.33.– *(Dirichlet*'s formula*) Let* $\Omega \subset \mathbb{R}^2$ *be the triangle with vertices* (a_1,a_2), (b_1,a_2), (b_1,b_2), *with a right-angle at* (b_1,a_2), *and suppose that* $a_1 < b_1, a_2 < b_2$ *(see Figure 4.1). Then :*

$$\boxed{\int_{a_1}^{b_1} d\boldsymbol{\mu}(x_1)\left\{\int_{a_2}^{x_2} \mathbf{h}(x_1,x_2)d\boldsymbol{\nu}(x_2)\right\} = \int_{a_2}^{b_2}\left\{\int_{x_1}^{b_1} d\boldsymbol{\mu}(x_1)\mathbf{h}(x_1,x_2)\right\} d\boldsymbol{\nu}(x_2)}.$$

PROOF.– Both integrals are equal to

$$\iint_{\mathbb{R}^2} d\boldsymbol{\mu}(x_1)\, \chi_\Omega(x_1,x_2)\, \mathbf{h}(x_1,x_2)\, d\boldsymbol{\nu}(x_2)\,. \blacksquare$$

4.1.7. *Absolutely continuous functions*

In the following, I is an interval of $\mathbb{R}$ that is assumed to have non-empty interior, $a = \inf(I) \geq -\infty$, and $b = \sup(I) \leq +\infty$. The Lebesgue measure is written dx or dt.

(I) LEBESGUE POINTS Let $\mathcal{L}^1_{loc}(I,dx)$ be the space of locally dx-integrable complex functions on I. If $g \in \mathcal{L}^1_{loc}(I,dx)$ is continuous, then it is a well-known fact that the function $f : x \mapsto \int_a^x g(t) .dt$ is differentiable, and its derivative $\dot{f}$ is g. In the following, we will show that this remains "almost true" when $g \in \mathcal{L}^1_{loc}(I, dx)$ is not continuous (Theorem 4.44).

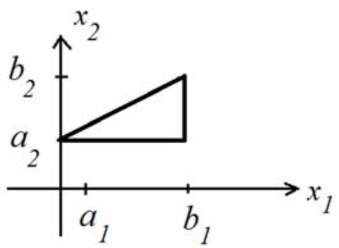

Figure 4.1. *Triangle*

DEFINITION 4.34.– *We say that a point c in the interior of I is a Lebesgue point of g if*

$$\lim_{h\downarrow 0, x\leq c\leq x+h} \frac{1}{h}\int_x^{x+h} |g(t) - g(c)|\, dt = 0$$

(where $h \downarrow 0$ means that $h \to 0^+$ from above).

LEMMA 4.35.– *Let $g \in \mathcal{L}^1_{loc}(I, dx)$. If c is a Lebesgue point of g, then f is differentiable at the point c and $\dot{f}(c) = g(c)$.*

PROOF.– Let $x, c, h \in]a, b[$ be such that $x \leq c \leq x + h$. Then, if c is a Lebesgue point for g,

$$\frac{1}{h}|f(x+h) - f(x)| = \left|\frac{1}{h}\int_x^{x+h} g(t)\, dt - g(c)\right| \leq \frac{1}{h}\int_x^{x+h} |g(t) - g(c)| dt$$
$$\Rightarrow g(c) = \lim_{h\downarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\downarrow 0} \frac{f(x) - f(x-h)}{h},$$

where the third expression follows from the second by changing x to $x - h$. This proves the lemma by choosing $x = c$. ■

The following result can also be shown ([SCW 93], Volume IV, Theorem 6.1.26):

THEOREM 4.36.– *(Lebesgue) Let $g \in \mathcal{L}^1_{loc}(I, dx)$. Then dx-almost every interior point of I is a Lebesgue point for g.*

(II) FUNCTIONS OF BOUNDED VARIATION Let $[a, b]$ be a compact interval of $\mathbb{R}$ and suppose that f is a real function defined on $[a, b]$. If f is increasing, the value $V_a^b(f) = f(b) - f(a)$ is said to be the *total variation* of f on $[a, b]$. In the general case, the total variation of a real function f specified on any interval I of $\mathbb{R}$ is defined as follows

$$V(f; I) = \sup_{[a_k, a_{k+1}] \in \sigma(I)} |f(a_{k+1}) - f(a_k)|,$$

where $\sigma(I)$ is the set of intervals $[a_k, a_{k+1}] \subset I$, $k \in \{0, ..., n-1\}$, and n is allowed to vary arbitrarily. We write $V(f; I) = V_a^b(f)$ if $I = [a, b]$ and $V(f; I) = V_{-\infty}^{+\infty}(f)$ if $I = \mathbb{R}$. This definition of $V(f; I)$ remains valid for the function $f : I \to E$ when E is a finite-dimensional real vector space (with norm $|.|$), and in particular when $E = \mathbb{C}$. We shall focus on the latter case below to avoid unnecessarily complicating the notation.

DEFINITION 4.37.– *A complex function f defined on a non-empty interval I of $\mathbb{R}$ is said to be* of bounded variation *if $V(f; I) < \infty$. A complex function f defined on I is said to be* of locally bounded variation *if its restriction to every compact interval $[a, b] \subset I$ is of bounded variation.*

LEMMA 4.38.– *Every* real *function of bounded variation on I may be expressed as the difference of two non-decreasing functions f^+ and f^-.*

PROOF.– For any point x in I such that $x > a$, let $I_x =]a, x[= \bigcup_k [a_k, x]$ (where $(a_k) \downarrow a$ is a sequence of real numbers such that $a < a_k < x$) and $v(x) = V(f; I_x)$. Clearly, v is a non-decreasing function f^+, and the same is true for $f^- := f^+ - f$ (**exercise**). ■

This lemma implies that, if f is of bounded variation on I, then the limits $f(x-0)$ and $f(x+0)$ exist at every point x in the interior of I, as well as $f(a+0)$ and $f(b-0)$. Write $NBV(I)$ for the vector space of complex functions defined and of bounded variation on I that are also left-continuous at every point in the interior of I and which satisfy $f(b-0) = 0$, extended to the point a if $a \notin I$ by setting $f(a) = f(a+0)$. The function $V(.; I) : f \mapsto V(f; I)$ gives a norm on the space $NBV(I)$; this space is called the space of "normalized" functions of bounded variation on I, and is a Banach space for the norm $V(.; I)$ (**exercise**).

Let $f \in NBV(I)$ be a real function. Then f^+ and f^- are non-decreasing and left-continuous at every point in the interior of I (**exercise***: cf. [SCW 93], Volume II, Theorem 3.2.32), and we can therefore associate Stietjes measures df^+ and df^- with these functions (Example 4.4(3) and Definition 4.7(2)). We say that $df = df^+ - df^-$ is the Stieltjes measure defined by f. We can proceed in a similar manner when f is complex by considering the real and imaginary parts. If $f \in NBV(I)$, then, for every point x in I such that $x > a$, $f(x) - f(a) = df([a, x[) = \int_{[a,x[} df$ (Example 4.4(4)). The Riesz representation theorem (Theorem 4.31) implies that:

COROLLARY 4.39.– *Let f be a complex left-continuous function of bounded variation on a compact interval $[a, b]$ of $\mathbb{R}$. The Stieltjes measure df may be identified with the Radon measure $\mu_f : \mathcal{K}([a, b]) \to \mathbb{C} : \varphi \mapsto \int_{[a,b]} \varphi.df$, where the integral is interpreted in the sense of Definition 4.7.*

Any non-decreasing real function on I is regulated ([DIE 82], Volume I, (7.6.2)), and therefore locally dx-integrable; in other words, given any compact interval $[a_1, b_1] \subset I$, we have that $NBV([a_1, b_1]) \subset \mathcal{L}^1([a_1, b_1], dx)$. Every left-continuous function f of bounded variation is therefore dx-almost everywhere differentiable by Theorem 4.36; the points at which this function is not differentiable are the points at which it is discontinuous; f may be uniquely written in the form $f_c + f_d$, where f_c is continuous and f_d is a step function. We have that $df_d = \sum_j \delta_{x_j}$ $(f(x_j + 0) - f(x_i))$, which implies that, for any function $\varphi \in \mathcal{K}([a_1, b_1])$,

$$\int_{[a_1,b_1]} \varphi.df_d = \sum_j \varphi(x_j)(f(x_j + 0) - f(x_i)),$$

where the $x_j \in [a_1, b_1[$ are the points at which f is discontinuous. The set of these points x_j is countable at most and $\sum_j |f(x_i + 0) - f(x_i)| \leq V_{a_1}^{b_1}(f)$. The Radon measure μ_{f_d} associated with df_d is said to be *atomic* and the Radon measure μ_{f_c} associated with df_c is said to be *diffuse*, since $|\mu_{f_c}|(\{x\}) = 0$ for every $x \in [a_1, b_1]$.

Let $f : [a, b] \to \mathbb{R}$ be a non-decreasing continuous function. This function is integrable on $[a, b]$, so is dx-almost everywhere differentiable on this interval by Lebesgue's theorem (Theorem 4.36). Write $\dot{f}$ for its derivative, defined dx-almost everywhere. For every $x \in [a, b]$, observe that $f(x) - f(a) \leq \int_a^x \dot{f}(t)\,dt$. Therefore, write $F(x)$ for the integral on the right-hand side of this inequality and $G(x) = f(x) - F(x) + f(b) - f(a)$. Now, $G \in NBV([a,b])$, this function is dx-almost everywhere differentiable, and $\dot{G}(x) = \dot{f}(x) - \frac{d}{dx}\int_a^x \dot{f}(t)\,dt = 0$ for dx-almost every x. Any such function G, i.e. which is continuous, of bounded variation, and whose derivative is dx-almost everywhere zero, is said to be *singular*. The Radon measure dG associated with this function is also said to be singular.

We can further refine the statement of Corollary 4.39 ([KOL 77], Chapter VI, section 6, Theorem 4), ([SCW 93], Volume IV, Corollary 6.1.9 and Theorem 6.1.10): Let $[a,b]$ be a compact interval of $\mathbb{R}$; then the norm $\|\mu_f\|$ of μ_f on the Banach space $\mathcal{M}([a,b])$ is identical to the total variation $V_a^b(f)$, and $NBV([a,b])$ may be identified with the Banach subspace of $\mathcal{M}([a,b])$ consisting of the Radon measures μ such that $\mu(\{b\}) = 0$ (this condition is required because the functions $f \in NBV([a,b])$ are not discontinuous at the point b).

DEFINITION 4.40.– *Let Ω be a locally compact space that is countable at infinity. Two Radon measures $\mu, \nu \in \mathcal{M}(\Omega)$ are said to be* disjoint *if* $\inf\{|\mu|, |\nu|\} = 0$*, in which case we write that $\mu \perp \nu$.*

If $\mu \perp \nu$, then $|\mu + \nu| = |\mu| + |\nu|$. It is possible to show the following result ([BKI 69], Chapter V, section 5.7, Theorem 3):

THEOREM 4.41.– *(Lebesgue* decomposition theorem*) Every measure $\theta \in \mathcal{M}(\Omega)$ may be uniquely written in the form $\theta = g.\mu + \theta'$, where $g \in \mathcal{L}^1_{loc}(\Omega, \mu)$, $\mu \in \mathcal{M}_+(\Omega)$, and $\theta' \perp \mu$. We then have that $|\theta| = |g|.\mu + |\theta'|$.*

COROLLARY 4.42.– *(Lebesgue) Every Radon measure μ on $[a,b]$ may be uniquely written in the form $g.dx + \mu_d + \mu_a$, where $g \in \mathcal{L}^1([a,b], dx)$, the measure $\mu_d \perp dx$ is diffuse and singular, and the measure μ_a is atomic.*

(III) ABSOLUTELY CONTINUOUS FUNCTIONS

DEFINITION 4.43.– *A complex function f on I is said to be* absolutely continuous *if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every finite family of pairwise disjoint intervals $[a_k, b_k] \subset I$ $(k = 1, ..., n)$ satisfying $\sum_{k=1}^n (b_k - a_k) < \delta$, we have that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$. A complex function f on I is said to be* locally absolutely continuous *if its restriction to every compact interval $[a,b]$ of $\mathbb{R}$ is absolutely continuous.*

Every Lipschitz function on I (cf. section 2.4.3(**II**), (2.8)) is absolutely continuous on I, and every absolutely continuous function on I is uniformly continuous and of bounded variation on I (**exercise**). The function $x \mapsto x^2$ is not uniformly continuous on $\mathbb{R}$ ([DIE 82], Volume I, Chapter III, section 11, Exercise 4), so is not absolutely continuous, but is locally absolutely continuous. The Radon-Nikodym theorem (Theorem 4.32) shows that a complex function f on an interval I of $\mathbb{R}$ is locally absolutely continuous if and only if $|df| \ll dx$. This yields the following result ([SCW 93], Volume IV, Proposition 6.1.18 and Theorem 6.1.32):

THEOREM 4.44.–

1) Let f be a complex function on an interval $I \subset \mathbb{R}$. The following conditions are equivalent:

*i) f is locally absolutely continuous and $g = \frac{df}{dx}$ is the Radon-Nikodym derivative of df with respect to dx (section 4.1.6(**III**)).*

ii) f is differentiable dx-almost everywhere, its derivative $g := \dot{f}$ belongs to $\mathcal{L}^1_{loc}(I, dx)$, and, for every $x \in I$, $f(x) - f(c) = \int_c^x g(t)\,.dt$ (where $c \in I$ is arbitrary).

2) Let $[a, b]$ be a compact interval of $\mathbb{R}$ and write $AC([a, b])$ for the set of complex absolutely continuous functions on $[a, b]$. Then $f \in AC([a, b])$ if and only if (i) f is of bounded variation on $[a, b]$ and (ii) its total variation $V_a^b(f)$ is equal to $\int_a^b \left|\dot{f}(x)\right| dx$.

Equipping the vector space $AC([a, b])$ with the norm $\|f\|_{[a,b]} = |f(a)| + V_a^b(f)$ gives the isomorphism of normed vector spaces $AC([a, b]) \cong L^1([a, b], dx) \oplus \mathbb{C}$; by the Fischer-Riesz theorem (Theorem 4.12) and section 3.4.1(**I**), $AC([a, b])$ is therefore a Banach space.

Readers may wish to show the following results as an **exercise** ([ROY 10], Chapter 6): If $f, g \in AC([a, b])$, then $fg \in AC([a, b])$, and $f/g \in AC([a, b])$ if g does not vanish on $[a, b]$. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous and $g : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then $g \circ f \in AC([a, b])$. If $f \in AC([a, b])$ and $\dot{f} \in \mathcal{L}^\infty([a, b]; dx)$, then f is Lipschitz with Lipschitz constant $N_\infty\left(\dot{f}\right)$ (cf. section 2.4.3(**II**))

4.1.8. *Continuous sums of Hilbert spaces*

Let Z be a topological space, π a Borel measure on Z (Definition 4.2), $(\mathbf{H}_z)_{z\in Z}$ a family of Hilbert spaces, and $\mathbf{V} = \prod_{z\in Z} \mathbf{H}_z$; the family $(\mathbf{H}_z)_{z\in Z}$ is called a field of Hilbert spaces; an element $f = (f_z)_{z\in Z}$, where $f_z \in \mathbf{H}_z, \forall z \in Z$, is called a vector field; an element $u = (u_z)_{z\in Z}$, where $u_z \in \operatorname{End}(\mathbf{H}_z), \forall z \in Z$, is called a field of operators. The field $(\mathbf{H}_z)_{z\in Z}$ is said to be π-measurable if there exists a subspace $\mathbf{S}$ of $\mathbf{V}$ such that (i) for every vector field $\varepsilon = (\varepsilon_z) \in \mathbf{S}$, the real-valued function $z \mapsto \|\varepsilon_z\|$ is π-measurable (section 4.1.1**(III)**); (ii) given $f \in \mathbf{V}$, then $f \in \mathbf{S}$ whenever the function $z \mapsto \langle f_z|\varepsilon_z\rangle_{\mathbf{H}_z}$ is π-measurable for every $\varepsilon \in \mathbf{S}$; (iii) there exists a sequence $\left(\varepsilon^i\right)$ of vector fields in $\mathbf{V}$ such that, for any $z \in Z$, the sequence $\left(\varepsilon_z^i\right)$ is total in $\mathbf{H}_z$ (section 3.2.2**(III)**). If these conditions hold, the vector field f is said to be π-measurable if the function $z \mapsto \left\langle f_z|\varepsilon_z^i\right\rangle_{\mathbf{H}_z}$ is π-measurable for every i; it is said to be square-integrable (with respect to π) if it is π-measurable and $\int_Z \|f_z\|^2 d\pi(z) < \infty$.

The square-integrable vector fields form a pre-Hilbert complex space $\mathbf{K}$; taking the quotient by the subspace of π-negligible vector fields yields a Hausdorff pre-Hilbert space $\mathbf{H}$. It can be shown that $\mathbf{H}$ is complete, and is therefore a Hilbert space. This space is said to be the *continuous sum* of the field of Hilbert spaces $(\mathbf{H}_z)_{z\in Z}$ ([GOD 03], section XI.6), denoted by:

$$\boxed{\mathbf{H} = \int_Z \mathbf{H}_z d\pi(z)}.$$

If Z is discrete, this continuous sum is a Hilbertian sum (Theorem-Definition 3.153) and the $\mathbf{H}_z$ may be canonically identified with Hilbert subspaces of $\mathbf{H}$ (this does not hold in general if Z is not discrete).

4.2. Functions in a single complex variable

4.2.1. *Holomorphic functions in a single complex variable*

THEOREM-DEFINITION 4.45.–

1) Let $\Omega \subset \mathbb{C}$ be an open set in the complex plane, suppose that E is a real Banach space, and write $E_{(\mathbb{C})}$ for the complexification of E (section 3.4.1).

A function $\mathbf{f} : \Omega \to E_{(\mathbb{C})} : \mathbf{f} \mapsto \mathbf{f}(z)$ *is said to be* holomorphic *at the point* $z_0 \in \Omega$ *if the "complex derivative"*

$$\frac{d\mathbf{f}}{dz}(z_0) := \lim_{h \to 0, h \neq 0} \frac{\mathbf{f}(z_0 + h) - \mathbf{f}(z_0)}{h}$$

(where h *is a small increment in* $\mathbb{C}$*) exists in* $E_{(\mathbb{C})}$.

2) By viewing Ω *as an open subset of* $\mathbb{R} \times \mathbb{R}$, *the function* $\mathbf{f} = U + iV$ *(*$U = \Re(\mathbf{f}), V = \Im(\mathbf{f})$*) defined above can be viewed as a function of the two real variables* x *and* y, *with* $x = \Re(z)$ *and* $y = \Im(z)$. *The following conditions are equivalent:*

i) $\mathbf{f}$ *is holomorphic.*

ii) $\mathbf{f} : (x, y) \mapsto \mathbf{f}(x, y)$ *is differentiable and satisfies the Cauchy-Riemann condition*

$$\frac{\partial \mathbf{f}}{\partial x} + i\frac{\partial \mathbf{f}}{\partial y} = 0, \text{ or equivalently } \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \qquad [4.9]$$

PROOF.– Let $z_0 = x_0 + iy_0 \in \Omega$. By setting $c = \frac{d\mathbf{f}}{dz}(z_0)$, condition (i) means that $\mathbf{f}(z_0 + h) = \mathbf{f}(z_0) + h.c + o(h)$, where h is a small complex increment. The function $\mathbf{f} : (x, y) \mapsto \mathbf{f}(x, y)$ is differentiable at the point (x_0, y_0) with partial derivatives $a = \frac{\partial \mathbf{f}}{\partial x}(x_0, y_0)$, $b = \frac{\partial \mathbf{f}}{\partial y}(x_0, y_0)$ if and only if

$$\mathbf{f}(x_0 + k, y_0 + l) = \mathbf{f}(x_0, y_0) + k.a + l.b + o(|(k, l)|),$$

where k, l are small real increments. Write $h = k + il$, so that $h.c = (k + il)c$. Then

$$h.c = k.a + l.b \iff (a = c, b = ic) \implies a + ib = 0,$$

so (i)⇔(ii). ■

In Theorem-Definition 4.45(2), $z = x + iy$ and $\bar{z} = x - iy$ are viewed as independent variables. Since $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$, we have that $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}.\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}.\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

COROLLARY 4.46.– *The Cauchy-Riemann equation* (4.9) *is equivalent to* $\frac{\partial \mathbf{f}}{\partial \bar{z}} = 0$. *If* $\mathbf{f}$ *is continuous on* Ω *and differentiable on* $\Omega - D$, *where* D *is* *finite, then this condition is satisfied on $\Omega - D$ if and only if the differential form $\omega = \mathbf{f}.dz$ is closed, i.e. $d\omega = 0$*[6].

PROOF.– We have that $\frac{\partial \mathbf{f}}{\partial x} + i\frac{\partial \mathbf{f}}{\partial y} = \frac{\partial \mathbf{f}}{\partial \bar{z}}$ and $d\omega = \frac{\partial \mathbf{f}}{\partial \bar{z}} d\bar{z} \wedge dz = 2i\frac{\partial \mathbf{f}}{\partial \bar{z}} dx \wedge dy$, so $d\omega = 0$ if and only if $\frac{\partial \mathbf{f}}{\partial \bar{z}} = 0$. ■

Clearly, the set of holomorphic functions on an open subset Ω of $\mathbb{C}$ with values in $E_{(\mathbb{C})}$ is a $\mathbb{C}$-vector space. This vector space is written $\mathcal{H}\left(\Omega; E_{(\mathbb{C})}\right)$, or simply $\mathcal{H}(\Omega)$ if $E_{(\mathbb{C})} = \mathbb{C}$.

4.2.2. *Homology. Cauchy's theorem*

DEFINITION 4.47.– *A path $\gamma : [0,1] \to \mathbb{C}$ ([P1], section 3.3.8(**VII**), p. 195) is said to be* rectifiable *if the mapping γ is continuous and of bounded variation (section 4.1.7(**II**)), in which case the total variation $V_0^1(\gamma)$ is said to be the* length *of γ.*

If γ is a rectifiable path, we saw in section 4.1.7(**II**) that its derivative $\dot{\gamma}$ is defined λ-almost everywhere, where λ is the Lebesgue measure on $[0,1]$. Let Ω be an open set in the complex plane, γ a rectifiable path in Ω, and $\mathbf{f} : \Omega \to E_{(\mathbb{C})}$ a function such that $(\mathbf{f} \circ \gamma)\dot{\gamma} \in \mathcal{L}^1\left([0,1], \lambda; E_{(\mathbb{C})}\right)$ (section 4.1.2(**III**)). Define $\int_\gamma \mathbf{f}(z).dz := \int_0^1 \mathbf{f}(\gamma(t))\dot{\gamma}(t).dt$. If γ is a closed path, the value of this integral is denoted $\oint_\gamma \mathbf{f}(z).dz$.

DEFINITION 4.48.– *The value $\int_\gamma \mathbf{f}(z).dz$ (or $\oint_\gamma \mathbf{f}(z).dz$ when γ is a closed path) is said to be the integral of $\mathbf{f}$ along the rectifiable path γ.*

LEMMA-DEFINITION 4.49.– *Let Ω be an open set in the complex plane, $\gamma : [0,1] \to \mathbb{C}$ a rectifiable closed path, and $z \in \complement_\Omega \gamma([0,1])$. If $n \neq -1$, then $\oint_\gamma (\xi - z)^n d\xi = 0$; in the case $n = -1$, the value $j(z;\gamma) := \frac{1}{2\pi i}\oint_\gamma \frac{d\xi}{\xi - z}$ is an integer in $\mathbb{Z}$, and represents the number of times N that $\gamma(t)$ winds around z in the counterclockwise direction as t ranges over the interval $[0,1]$, called the* winding number *of γ with respect to z.*

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6 This is equivalent to saying that ω is a cocycle: cf. [P1], section 3.3.8(**VII**) p. 195, where, for algebraic reasons, E is assumed to be finite-dimensional and ω is taken to be C^∞.

PROOF.– If $n \neq -1$, then $\oint_\gamma s^n ds = \left[\frac{s^{n+1}}{n+1}\right]_{\gamma(0)}^{\gamma(1)} = 0$. If $n = 1$, write $h(t) = \frac{1}{2\pi i}\int_0^t \frac{\dot{\gamma}(\tau)}{\gamma(\tau)-z}d\tau$; then h is an absolutely continuous function (section 4.1.7**(III)**), and, if $g(t) = e^{-h(t)}(\gamma(t) - z)$, then $\dot{g}(t) = 0$ dt-almost everywhere, so $e^{h(t)} = \frac{\gamma(t)-z}{\gamma(0)-z}$, and $e^{h(1)} = 1$, which implies that $h(1) = i.2\pi N$, where $N = \arg(\gamma(1) - z) - \arg(\gamma(0) - z)$,[7] and $j(z;\gamma) = N$. ■

DEFINITION 4.50.– *A rectifiable closed path γ in an open subset Ω of $\mathbb{C}$ is said to be* homologous to 0 *if $j(z;\gamma) = 0$ for every $z \in \complement_{\mathbb{C}}\Omega$. ([AHL 66], Chapter 4, section 4.4), ([LAN 93], Chapter IV, section 2).*

THEOREM 4.51.– *(Cauchy) If $\mathbf{f}$ is continuous on an open set $\Omega \subset \mathbb{C}$ and holomorphic on $\Omega - D$, where D is a set of isolated points, then $\oint_\gamma \mathbf{f}(z).dz = 0$ for any rectifiable closed path γ in Ω that is homologous to 0*

PROOF.–

1) Suppose first that $D = \varnothing$ and Ω is simply connected, i.e. every rectifiable path γ is homotopic to a point. By Corollary 4.46, the differential form $\omega = \mathbf{f}.dz$ is closed, so is exact by Poincaré's lemma ([P1], section 3.3.8**(VII)**, Theorem 3.184); in other words, there exists a function $\mathbf{F} : \Omega \to E_{(\mathbb{C})}$ such that $\omega = d\mathbf{F}$ (we say that $\mathbf{f}$ *has a primitive*). Consequently, for any rectifiable closed path γ in Ω, $\int_\gamma \mathbf{f}(z).dz = \mathbf{F}(\gamma(0)) - \mathbf{F}(\gamma(1)) = 0$. However, this property only depends on whether γ is homologous to 0; Ω being simply connected is just a *sufficient condition* for every rectifiable closed path γ in Ω (cycle) to be homologous to 0 (and therefore a boundary) ([P1], section 3.3.8**(II)**, Corollary-Definition 3.173.

2) If $D = \{z_0\}$, we need to do some surgery in the complex plane: let $C(r)$ be the closed path given by the circle of center z_0 and radius $r > 0$, traveled once in the counterclockwise direction. We can rewrite this as $\int_\gamma = \int_{\gamma - C(r)} + \int_{C(r)}$, where $\gamma - C(r)$ is the closed path obtained by juxtaposing γ and $-C(r)$ via a line segment AB, traveled first in one direction, then in the other (for an illustration, see Figure 4.2, where γ is the larger circle, traveled

7 The function arg (usually specified up to $\mathrm{mod}\ .2\pi$) is chosen at each point in such a way that $t \mapsto \arg(\gamma(t) - z)$ is a continuous function on $[0, 1]$.

once in the counterclockwise direction from A to A, $-C(r)$ is the smaller circle $C(r)$, traveled once in the clockwise direction from B to B, and $\gamma - C(r)$ is the juxtaposition of γ, AB, $-C(r)$, BA). Thus, $\oint_{\gamma-C(r)} \mathbf{f}(z).dz = 0$ by (1). Since $\mathbf{f}$ is continuous on $C(r)$, $|\mathbf{f}|$ is upper bounded on this compact set by some real number $M > 0$ (Theorem 2.42). Therefore, $\left|\int_{C(r)} \mathbf{f}(z).dz\right| \leq \int_{C(r)} |\mathbf{f}(z)|.dz \leq M.2\pi r$. Taking the limit as r tends to 0, $\oint_{\gamma} \mathbf{f}(z).dz = 0$.
(3) The result may be deduced from (2) by induction in the case where D is any arbitrary discrete set. ■

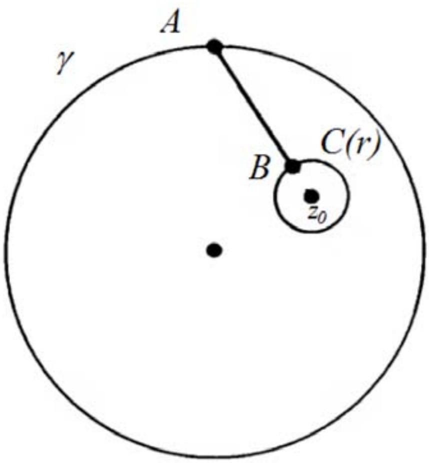

Figure 4.2. *Surgery in the complex plane*

THEOREM 4.52.– *(Morera) Let Ω be a simply connected open set in the complex plane. If $\oint_{\gamma} \mathbf{f}(z).dz = 0$ for every rectifiable closed path γ in Ω, then $\mathbf{f}$ is holomorphic on Ω.*

PROOF.– Simply note that $\mathbf{f}$ has a primitive $\mathbf{F}$ (see the proof of Theorem 4.51), and so $\mathbf{f} = \frac{d\mathbf{F}}{dz}$. ■

4.2.3. *Cauchy's integral formula*

THEOREM 4.53.– (*Cauchy*'s integral formula) *Let Ω be an open set in the complex plane, $\gamma : [0,1] \to \Omega$ a rectifiable closed path that is homologous to* 0, $z \in \complement_{\Omega}\gamma([0,1])$, *and $\mathbf{f} : \Omega \to E_{(\mathbb{C})}$ a function that is continuous on Ω and* *holomorphic on $\Omega - D$, where D is a set of isolated points. Cauchy's integral formula states that:*

$$j(z;\gamma).\mathbf{f}(z) = \frac{1}{2\pi i}\oint_{\gamma}\frac{\mathbf{f}(\xi)}{\xi - z}d\xi. \qquad [4.10]$$

PROOF.– Observe that $\frac{\mathbf{f}(\xi)}{\xi-z} = \mathbf{g}(\xi) + \frac{\mathbf{f}(z)}{\xi-z}$, where $\mathbf{g}$ is the function defined by $\mathbf{g}(\xi) = \frac{\mathbf{f}(\xi)-\mathbf{f}(z)}{\xi-z}$ for $\xi \in \Omega - \{z\}$, extended by continuity at $\xi = z$ by $\mathbf{g}(z) = \dot{\mathbf{f}}(z)$. By Theorem 4.51 and Lemma-Definition 4.49,

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{\mathbf{f}(\xi)}{\xi - z}d\xi = \underbrace{\frac{1}{2\pi i}\oint_{\gamma}\mathbf{g}(\xi)\,d\xi}_{0} + \mathbf{f}(z)\,.\underbrace{\frac{1}{2\pi i}\oint_{\gamma}\frac{d\xi}{\xi - z}}_{j(z;\gamma)}. \blacksquare$$

THEOREM 4.54.– (*Cauchy*'s generalized integral formula) *With the same assumptions as in Theorem 4.53, for every integer $k \geq 0$,*

$$j(z;\gamma).\mathbf{f}^{(k)}(z) = \frac{k!}{2\pi i}\oint_{\gamma}\frac{\mathbf{f}(\xi)}{(\xi - z)^{k+1}}d\xi. \qquad [4.11]$$

Hence, $\mathbf{f}$ is infinitely differentiable on Ω (in the sense of "complex differentiation").

PROOF.– This follows from Theorem 4.11 (allowing the sequence (h_n) in the proof to take complex values). $\blacksquare$

4.2.4. *Analytic functions in a single complex variable*

Let Ω be an open set in the complex plane and suppose that E is a complex Banach space.

DEFINITION 4.55.– *A function $\mathbf{f} : \Omega \to E$ is said to be analytic if, for every $z \in \Omega$, there exists a real number $r > 0$ such that the following equality holds on the disk $D(z;r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$ (where r is sufficiently small that $D(z;r) \subset \Omega$):*

$$\mathbf{f}(\zeta) = \sum_{k \geq 0}\mathbf{c}_k(\zeta - z)^k,$$

where the power series on the right-hand side is absolutely convergent (Definition 3.40).

PROPOSITION 4.56.– *The function* $\mathbf{f} : \Omega \to E$ *is analytic if and only if it is holomorphic.*

PROOF.– To see that this condition is sufficient, observe that the series $\sum_{k\geq 0} \frac{d}{dz}\mathbf{c}_k (\zeta - z)^k = \sum_{k\geq 0} k\mathbf{c}_k (\zeta - z)^{k-1}$ has the same radius of convergence as the series $\sum_{k\geq 0} \mathbf{c}_k (\zeta - z)^k$ ([CAR 61], Chapter I, Proposition 1.7), so $\mathbf{f}$ is differentiable with respect to ζ with derivative $\frac{d\mathbf{f}(\zeta)}{d\zeta} = \sum_{k\geq 0} k\mathbf{c}_k (\zeta - z)^k$. We will show the necessary condition later (Theorem 4.79) in a more general context. ■

4.2.5. *Residue theorem*

Let (a_n) be a (finite or infinite) sequence of isolated and distinct points in a non-empty open set Ω of the complex plane. Write S for the set of these points a_n, and suppose that E is a complex Banach space. Let $\mathbf{f} : \Omega - S \to E$ be an analytic function and pick $a \in S$. There exists a real number $r > 0$ such that $\mathbf{f}$ is analytic on the whole of the open punctured disk $\dot{D}(a; r) := \{z \in \Omega : 0 < |z - a| < r\}$, and it can be shown ([DIE 82], Volume I, (9.15.1)) that $\mathbf{f}$ has the following absolutely convergent Laurent series expansion (Definition 3.40) on $\dot{D}(a; r)$:

$$\mathbf{f}(z) = \sum_{k=-\infty}^{+\infty} \mathbf{c}_k (z - a)^k, \qquad [4.12]$$

where $\mathbf{c}_k = \frac{1}{2\pi i} \int_\gamma \frac{\mathbf{f}(\zeta)}{(\zeta - a)^{k+1}} d\zeta$, and γ denotes the circle of center a and radius $\rho \in]0, r[$ traveled once in the counterclockwise direction.

DEFINITION 4.57.– *We define the* residue *of* $\mathbf{f}$ *at the point* a *as the element* $\mathbf{c}_{-1} \in E$, *written* $\operatorname{res}(\mathbf{f}; a)$. *We also define* $\omega(a; \mathbf{f}) := \sup\{k \in \mathbb{Z} : \mathbf{c}_k \neq 0\} \in \overline{\mathbb{Z}}$, *and say that this number is the* order *of* $\mathbf{f}$ *at the point* a.

A point $a \in A$ is said to be a *singular point* of $\mathbf{f}$ if $\omega(a; \mathbf{f}) < 0$, and is said to be a *regular point* if $\omega(a; \mathbf{f}) \geq 0$. If a is a regular point, then we can extend $\mathbf{f}$ at a by continuity, in which case Morera's theorem (Theorem 4.52) implies that $\mathbf{f}$ is holomorphic at a. If $E = \mathbb{C}$ and $\omega(a;\mathbf{f}) > 0$, we say that $\mathbf{f}$ is a *zero* of order $\omega(a;\mathbf{f})$ at the point a. If $-\infty < \omega(a;\mathbf{f}) < 0$, we say that $\mathbf{f}$ has a *pole* of order $m = -\omega(a;\mathbf{f})$ at the point a. If so, then (**exercise**):

$$\operatorname{res}(\mathbf{f};a) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m \mathbf{f}(z)\right].$$

A singular point a that is not a pole is said to be an *essential singular point*. For example, 0 is an essential singular point for the function $z \mapsto e^{\frac{1}{z^2}}$ and $e^{\frac{1}{z^2}} = 1 + \frac{1}{1!}\frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \ldots$, so $\operatorname{res}\left(e^{\frac{1}{z^2}};0\right) = 0$.

Cauchy's integral formula (Theorem 4.53) implies the following result ([DIE 82], Volume I, (9.16.1)):

THEOREM 4.58.– (residue theorem) *Let Ω be an open set in the complex plane, (a_n) a sequence of isolated and distinct points of Ω without a cluster point in Ω (section 2.3.2(**III**)), S the set of the points a_n, $\mathbf{f} : \Omega - S \to E$ an analytic function, and $\gamma : [0,1] \to \Omega$ a rectifiable closed path homologous to 0 such that $\gamma([0,1]) \subset S$. Then, for every $z \in \Omega - S$,*

$$\boxed{\oint_\gamma \mathbf{f}(z)\,dz = 2\pi i \sum\nolimits_n j(a_n;\gamma)\operatorname{res}(\mathbf{f};a_n)},$$

and there are only finitely many terms on the right-hand side.

Readers can find a detailed presentation of how the residue theorem may be applied to calculate integrals along the real axis in ([CAR 61], Chapter III, section 6).

4.2.6. *Meromorphic functions*

Consider once again a (finite or infinite) sequence (a_n) of isolated and distinct points in a non-empty *connected* open subset Ω of the complex plane. Let S be the set of the points a_n, and suppose that E is a complex Banach space. Finally, let $\mathbf{f} : \Omega - S \to E$ be an analytic function.

LEMMA-DEFINITION 4.59.–

i) If $\mathbf{f} : \Omega - S \to E$ only has poles and Ω is connected, then f is said to be meromorphic *on Ω.*

ii) The function $\mathbf{f}$ *is meromorphic on* Ω *if and only if, for every* $z \in \Omega$*, there exists an open neighborhood* $U \subset \Omega$ *of* z *and holomorphic functions* $\mathbf{g} : U \to E$ *and* $h : U \to \mathbb{C}$ *such that* $\mathbf{f}$ *may be expressed as* $\frac{\mathbf{g}}{h}$ *on* U *(**exercise**).*

Let $a \in S$ be a pole of $\mathbf{f}$ with order m. There exists a neighborhood $U \subset \Omega$ of a and a holomorphic function $\mathbf{g} : U \to E$ such that, for every $z \in U - \{a\}$,

$$\mathbf{f}(z) = \mathbf{g}(z) + \mathbf{P}(z), \quad \mathbf{P}(z) := \sum_{k=1}^{m} \frac{\mathbf{c}_{-k}}{(z-a)^k}.$$

DEFINITION 4.60.– *The function* $\mathbf{P} : U - S \to E$ *is said to be the* principal part *of* $\mathbf{f}$ *at the point* a.

For a generalization, see sections 5.1 and 5.4.4.

Below, we shall prove the result stated in ([P1], section 2.3.8, Example 2.55):

THEOREM 4.61.– *(Weierstrass* factorization theorem*) Let* Ω *be a connected open set in the complex plane and suppose that* f *is a meromorphic function on* Ω.

1) If f *is holomorphic and does not vanish anywhere on* Ω*, then there exists a holomorphic function* g *on* Ω *such that* f *may be expressed as* $f(z) = e^{g(z)}$.

2) Let Z *be the set of zeroes of* f *and* P *the set of poles of* f *on* Ω*. There exists a holomorphic function* g *on* Ω *such that, for every* $z \in \Omega$*, we have with the notation in Definition 4.57*

$$\boxed{f(z) = e^{g(z)} \prod_{z \in Z \cup P} (z-a)^{\omega(a;f)}}. \qquad [4.13]$$

PROOF.– (1): The function $\dot{f}/f$ is holomorphic on Ω and therefore has a primitive g (cf. proof of Theorem 4.51). Therefore, f and e^g have the same logarithmic derivative $\dot{f}/f$. Since Ω is connected, there exists a constant $k > 0$ such that $f = ke^g$, which can be absorbed into the exponential. (2) clearly follows from (1) when Z and P are finite. When $P = \varnothing$ and Z is arbitrary, the equality (4.13) still holds, since the product on the right-hand side converges uniformly on every compact set ([RUD 87], Theorem 15.10), from which we deduce that (2) holds in the general case. ■

Conversely, it is possible to show the following results. The second is a "multiplicative version" of the first. Write $\mathcal{H}^*(U)$ (resp. $\mathcal{M}^*(U)$) for the multiplicative group of invertible elements of $\mathcal{H}(U)$ (resp. $\mathcal{M}(U)$); naturally, $\mathcal{M}^*(U) = \mathcal{M}^\times(U)$.

THEOREM 4.62.– *(Mittag-Leffler) Let U be a non-empty connected open subset of $\mathbb{C}$.*

*1) (*first form*) Let S be a discrete subset of U and, for each point $a \in S$, let P_a be a polynomial in $(z-a)^{-1}$ with no constant term. Then there exists a meromorphic function f on U whose principal part at each point $a \in S$ is P_a and which does not have any poles on $U - S$.*

*2) (*second form*) Let $(U_i)_{i\in I}$ be an open covering of U and, for each $i \in I$, let $h_i \in \mathcal{M}(U_i)$. If $h_i - h_j \in \mathcal{H}(U_i \cap U_j)$ for all $i, j \in I$, then there exists $h \in \mathcal{M}(U)$ such that $h - h_i \in \mathcal{H}(U_i)$ for every $i \in I$.*

*3) (*third form*) Let $(U_i)_{i\in I}$ be an open covering of U and, for all $j, k \in I$, let $g_{jk} \in \mathcal{H}(U_i \cap U_j)$ be functions satisfying*

$$g_{jk} = -g_{kj}, \quad g_{jk} + g_{kl} + g_{lj} = 0 \text{ on } U_j \cap U_k \cap U_l \text{ for all } j, k, l.$$

Then there exist $g_j \in \mathcal{H}(U_j)$ such that

$$g_{jk} = g_k - g_j \text{ on } U_j \cap U_k \text{ for all } j, k.$$

PROOF.– See ([TAY 02], 1.6.1) for the proof of (3). The proof of (1)⇔(2) is straightforward, and (2)⇒(3) and (3)⇒(1) are shown in ([HOR 90a], p. 13) and ([TAY 02], 1.6.2) respectively. ■

THEOREM 4.63.– *(Weierstrass) Let U be a non-empty connected open subset of $\mathbb{C}$.*

*1) (*first form*) Let S be a discrete subset of U and, for each point $a \in S$, let $\omega(a)$ be a non-zero integer. There exists a meromorphic function on U such that, at every point $a \in S$, this function has a zero of order $\omega(a)$ if $\omega(a) > 0$ and a pole of order $-\omega(a)$ if $\omega(a) < 0$, with no other zeroes or poles on $U - S$.*

*2) (*second form*) Let $(U_i)_{i\in I}$ be an open covering of U and, for all $i \in I$, let $h_i \in \mathcal{M}^*(U_i)$. If $h_i h_j^{-1} \in \mathcal{H}^*(U_i \cap U_j)$ for all $i, j \in I$, then there exists*

$h \in \mathcal{M}^*(U)$ such that $hh_i^{-1} \in \mathcal{H}^*(U_i)$ for all $i \in I$.

*3) (*third form*) Let $(U_i)_{i \in I}$ be an open covering of U and, for all $j, k \in I$, let $g_{jk} \in \mathcal{H}^*(U_i \cap U_j)$ be functions satisfying*

$$g_{jk} g_{kj} = 1, \quad g_{jk} g_{kl} g_{lj} = 1 \text{ on } U_j \cap U_k \cap U_l \text{ for all } j, k, l.$$

Then there exist $g_j \in \mathcal{H}^*(U_j)$ such that

$$g_{jk} = g_k g_j^{-1} \text{ on } U_j \cap U_k \text{ for all } j, k.$$

PROOF.– See ([HOR 90a], Theorem 1.5.1) for the proof of (1). The equivalences (1)⇔(2)⇔(3) may be shown in the same way as the Mittag-Leffler theorem. ■

Theorem 4.61 implies the following result:

COROLLARY 4.64.– *The set $\mathcal{M}(\Omega)$ of meromorphic complex functions on a connected open subset Ω of $\mathbb{C}$ is the field of fractions of the ring $\mathcal{H}(\Omega)$ of holomorphic complex functions on Ω.*

4.2.7. *Argument principle*

The argument principle plays an extremely important role, especially in the field of automation (Nyquist criterion): cf. ([BLS 10], section 4.1.4).

THEOREM 4.65.– *(*argument principle*) Let f be a meromorphic complex function on a connected subset Ω of the complex plane, Z the set of zeroes of f, P the set of poles of f, and suppose that $\gamma : [0,1] \to \Omega$ is a rectifiable closed path homologous to 0 such that $\gamma([0,1]) \subset \Omega - (Z \cup P)$. If Γ is the circuit $t \mapsto f(\gamma(t))$, then*

$$\boxed{j(0;\Gamma) = \textstyle\sum_{a \in Z \cup P} j(a;\gamma)\, \omega(a;f)},$$

and the sum in the right-hand side only has finitely many non-zero terms.

PROOF.– Setting $u = f(t)$ gives that $\oint_\Gamma \frac{du}{u} = \oint_\gamma \frac{\dot{f}(z)}{f(z)} dz$. The value on the left is $j(0;\Gamma)$ (Lemma-Definition 4.49). For the right-hand side, by Theorem 4.61, we can write $f(z) = e^{g(z)} \prod_{a \in Z \cup P} (z-a)^{\omega(a;f)}$, where g is holomorphic on

Ω. Therefore, by taking the logarithmic derivative of this expression, we find that $\frac{\dot{f}(z)}{f(z)} = \sum_{a \in Z \cup P} \omega(a; f) \frac{1}{z-a} + \dot{g}(z)$; furthermore, $\oint_\gamma \frac{dz}{z-a} = j(a; \gamma)$ (Lemma-Definition 4.49) and $\oint_\gamma \dot{g}(z)\, dz = 0$ by Cauchy's theorem (Theorem 4.51), so $\oint_\gamma \frac{\dot{f}(z)}{f(z)} dz = \sum_{a \in Z \cup P} \omega(a; f)\, j(a; \gamma)$. ■

Any function f of a complex variable defined on $|z| > r$, where $r > 0$, is said to be *holomorphic* (resp. *meromorphic*) *at infinity* if the function $\zeta \mapsto f\left(\frac{1}{\zeta}\right)$ is holomorphic (resp. meromorphic) at $\zeta = 0$. Any function that is holomorphic on the Riemann sphere (section 2.3.9(**II**)) $\mathbb{C} \cup \{\infty\}$ is in fact constant, and every function that is meromorphic on $\mathbb{C} \cup \{\infty\}$ is in fact rational (**exercise**).

4.3. Function spaces

4.3.1. *Spaces of infinitely differentiable functions*

(I) Let Ω be a non-empty subset of $\mathbb{R}^n$ and, for every integer $m \geq 0$, let $\mathcal{C}^m(\Omega)$ be the $\mathbb{C}$-algebra of complex functions of class C^m on Ω. For every integer q such that $0 \leq q \leq m$, every multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ such that $|\alpha| \leq q$, where $|\alpha| = \sum_{i=1}^n \alpha_i$ ([P1], section 2.3.9(**I**)), and every function $f \in \mathcal{C}^m(\Omega)$, define

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}. \qquad [4.14]$$

Furthermore, for every compact set $K \subset \Omega$, define $p_{q,K}(f) = \sup_{|\alpha| \leq q, x \in K} |\partial^\alpha f(x)|$. Clearly, $p_{q,K}$ is a semi-norm on $\mathcal{C}^m(\Omega)$. Let $(K_j)_{j \geq 1}$ be a sequence of compact subsets of Ω such that $K_j \Subset K_{j+1}$ and $\bigcup_{j \geq 1} K_j = \Omega$ (Lemma 2.52). Equipping the vector space $\mathcal{C}^m(\Omega)$ with the countable family of semi-norms $(p_{q,Kj})$ gives a locally convex space denoted $\mathcal{E}^{(m)}(\Omega)$; this space is Hausdorff, and is therefore metrizable (Corollary 3.20). Let (f_i) be a Cauchy sequence in $\mathcal{E}^{(m)}(\Omega)$. The space $\mathcal{C}(K)$ is complete (section 4.1.4(**I**)); therefore, for every multi-index α such that $|\alpha| \leq m$ and every compact set K, $(\partial^\alpha f_i\,|_K)$ converges in $\mathcal{C}(K)$ to a function g_α. It can also be shown that (**exercise**):

LEMMA 4.66.– *Let $(\varphi_i)_{i\geq 0}$ be a sequence of functions in $\mathcal{C}^1(\Omega)$ such that (φ_i) converges to a function g_0 in the space $\mathcal{C}_c(\Omega)$ of continuous functions on Ω equipped with the topology of compact convergence (section 2.7.1), with the additional assumption that, for every index $j \in \{1, ...n\}$, $\left(\partial\varphi_i/\partial x^j\right)$ converges to a function g_j in $\mathcal{C}_c(\Omega)$. Then $g_j = \left(\partial g/\partial x^j\right)$ $(j \geq 1)$ and $g_0 \in \mathcal{C}^1(\Omega)$.*

By induction, we deduce that (f_i) converges to g_0 in $\mathcal{E}^{(m)}(\Omega)$. Therefore, $\mathcal{E}^{(m)}(\Omega)$ is complete, and is hence a Fréchet space. The linear mapping $\partial/\partial x^i : \mathcal{E}^{(m)}(\Omega) \to \mathcal{E}^{(m+1)}(\Omega) : f \mapsto \partial f/\partial x^i$ is continuous (**exercise**).

The $\mathbb{C}$-algebra $\mathcal{E}(\Omega) = \bigcap_{m\geq 0} \mathcal{E}^{(m)}(\Omega)$ of infinitely differentiable functions on Ω is defined in the same way. The topology of this space is defined by the countable family of semi-norms $(p_{q,Kj})_{(q,j)\in\mathbb{N}\times\mathbb{N}}$. The same reasoning as above shows that $\mathcal{E}(\Omega)$ is a Fréchet space.

LEMMA 4.67.– *Let B be a bounded set in $\mathcal{E}^{(m+1)}(\Omega)$. Then B is relatively compact in $\mathcal{E}^{(m)}(\Omega)$.*

PROOF.– For every integer $q \leq m+1$ and every compact set $K \subset \Omega$, we have that $p_{q,K}(B) < \infty$. Therefore, for every multi-index α such that $|\alpha| \leq m+1$ and every $x \in \Omega$, the set $\partial^\alpha B(x) := \{\partial^\alpha f(x) : f \in B\}$ is bounded and thus relatively compact in $\mathbb{R}$. For the rest of the argument, we will assume $n = 1$ for simplicity (generalization to arbitrary n is left to the reader). The set B is bounded in $\mathcal{E}^{(m+1)}(\Omega)$, so, for every compact set $K \subset \Omega$, there exists a real number $\lambda > 0$ such that $|f'(x)| \leq \lambda, \forall x \in K, \forall f \in \mathcal{E}^{(m)}(\Omega)$. Let $x_0 \in \Omega$ and $K = \{x \in \mathbb{R} : |x - x_0| \leq \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small to ensure that $K \subset \Omega$. By the formula of averages, for all $x \in K$, we have that $|f(x) - f(x_0)| \leq \lambda\varepsilon$, $\forall f \in B$, so $\partial^\alpha B$ is equicontinuous on $\mathcal{C}(\Omega)$. Hence, by the 3rd Ascoli-Arzelà theorem (Theorem 2.122), $\partial^\alpha B$ is relatively compact in $\mathcal{C}(\Omega)$ for every multi-index α such that $|\alpha| \leq m$. Therefore, B is relatively compact in $\mathcal{E}^{(m)}(\Omega)$ (by the definition of the topology of this space). ■

THEOREM 4.68.– *The space $\mathcal{E}(\Omega)$ is an $(\mathcal{FS})$-space.*

PROOF.– This follows from Corollary 3.75 and Lemma 4.67. ■

(II) SPACE OF INFINITELY DIFFERENTIABLE FUNCTIONS WITH COMPACT SUPPORT Let Ω be a non-empty open subset of $\mathbb{R}^n$ and, for every compact set $K \subset \Omega$ and every integer $m \geq 0$, let $\mathcal{D}_K^{(m)}(\Omega)$ be the

$\mathbb{C}$-algebra of complex functions with class C^m on Ω and support contained in K. The reasoning presented above shows that $\mathcal{D}_K^{(m)}(\Omega)$ equipped with the countable family of semi-norms $(p_{m,K})$ is a Fréchet space and a closed subspace of $\mathcal{E}^{(m)}(\Omega)$. If $K_1 \subset K_2$, then $\mathcal{D}_{K_1}^{(m)}(\Omega) \subset \mathcal{D}_{K_2}^{(m)}(\Omega)$, and the topology of $\mathcal{D}_{K_1}^{(m)}(\Omega)$ is identical to the topology induced by the topology of $\mathcal{D}_{K_2}^{(m)}(\Omega)$.

Let $\mathcal{D}^{(m)}(\Omega)$ be the $\mathbb{C}$-algebra of complex functions with class C^m and compact support contained in Ω. Consider an increasing sequence $(K_j)_{j\geq 1}$ of compact subsets of Ω that satisfy the conditions stated in Lemma 2.52. The topology of $\mathcal{D}^{(m)}(\Omega)$ is defined in such a way that $\mathcal{D}^{(m)}(\Omega) = \varinjlim_{j\geq 1} \mathcal{D}_{K_j}^{(m)}(\Omega)$ is a strict inductive limit in **Lcs** (as in Definition 4.24, *mutatis mutandis*). Therefore, $\mathcal{D}^{(m)}(\Omega)$ is an $(\mathcal{L}_s\mathcal{F})$-space.

Consider the $\mathbb{C}$-algebra $\mathcal{D}(\Omega) = \bigcap_{m\geq 0} \mathcal{D}^{(m)}(\Omega)$ of infinitely differentiable complex functions with compact support contained in Ω and, for every compact set $K \subset \Omega$, define $\mathcal{D}_K(\Omega) = \bigcap_{m\geq 0} \mathcal{D}_K^{(m)}(\Omega)$. Then $\mathcal{D}_K(\Omega)$ is a vector subspace of $\mathcal{E}(\Omega)$ that is closed in $\mathcal{E}(\Omega)$ when equipped with the induced topology, and so is an $(\mathcal{FS})$-space (section 3.4.2(**I**) and Theorem 3.73). By defining the topology of $\mathcal{D}(\Omega)$ in such a way that $\mathcal{D}(\Omega) = \varinjlim_{j\geq 1} \mathcal{D}_{K_j}(\Omega)$ is a strict inductive limit in **Lcs** (cf. Definition 4.24, *mutatis mutandis*), Corollary 3.77 implies that:

THEOREM 4.69.– *The space $\mathcal{D}(\Omega)$ is an $(\mathcal{L}_s\mathcal{F})$-space and an $(\mathcal{MS})$-space.*

REMARK 4.70.– *$\mathcal{D}(\Omega)$ is not the trivial space $\{0\}$. For example, let x_1, x_2 be two real numbers such that $x_1 < x_2$. It is a classic exercise to show that the function φ defined by $\varphi(x) = 0$ if $x \in \complement_{\mathbb{R}} \left]x_1, x_2\right[$ and $\varphi(x) = e^{1/(x-x_1)(x-x_2)}$ if $x \in \left]x_1, x_2\right[$ is an element of $\mathcal{E}(\mathbb{R})$ and has support $[x_1, x_2]$.*

REMARK 4.71.– *The topologies on $\mathcal{D}(\Omega)$, $\mathcal{D}^{(m)}(\Omega)$, and $\mathcal{K}(\Omega)$ defined above do not depend on the particular choice of sequence $(K_j)_{j\geq 1}$ satisfying the conditions stated in Lemma 2.52 (**exercise**).*

(III) SPACE OF DECLINING FUNCTIONS Let $f \in \mathcal{E}(\mathbb{R}^n)$. For every index $\alpha \in \mathbb{N}$ and every multi-index $\beta \in \mathbb{N}^n$, set

$$q_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} (1+|x|)^\alpha \left|\partial^\beta f(x)\right|.$$

Write $\mathcal{S}(\mathbb{R}^n)$ for the vector subspace of $\mathcal{E}(\mathbb{R}^n)$ formed by the functions f such that $q_{\alpha,\beta}(f) < \infty$ for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$. A function $f \in \mathcal{E}(\mathbb{R}^n)$ such that $q_{\alpha,0}(f) < \infty$ is said to be a *rapidly decreasing infinitely differentiable* function, and any function $f \in \mathcal{S}(\mathbb{R}^n)$ (namely, a function whose derivatives of all orders are rapidly decreasing infinitely differentiable, including the function itself) is said to be *declining* ([DIE 82], Volume 6, section 22.16). The $q_{\alpha,\beta}$ are semi-norms on $\mathcal{S}(\mathbb{R}^n)$ that endow this vector space with a Hausdorff locally convex structure, again denoted $\mathcal{S}(\mathbb{R}^n)$. Since the family $(q_{\alpha,\beta})_{\alpha\in\mathbb{N},\beta\in\mathbb{N}^n}$ is countable, $\mathcal{S}(\mathbb{R}^n)$ is metrizable (Corollary 3.20). It is also complete (**exercise**), so is a Fréchet space. For every integer $\alpha \in \mathbb{N}$ and every function $f \in \mathcal{E}(\mathbb{R}^n)$, define $q_\alpha(f) := \sup_{|\beta|\leq\alpha} q_{\alpha,\beta}(f)$, and let $\mathcal{S}_\alpha(\mathbb{R}^n)$ be the subspace of $\mathcal{E}(\mathbb{R}^n)$ formed by the functions f such that $q_\alpha(f) < \infty$. Then $\mathcal{S}_\alpha(\mathbb{R}^n)$ is a Banach space; the sequence

$$\mathcal{S}_0(\mathbb{R}^n) \supset \mathcal{S}_1(\mathbb{R}^n) \supset \mathcal{S}_2(\mathbb{R}^n) \supset ...$$

is decreasing, and an argument similar to the one presented in Lemma 4.67 shows that, for every $\alpha \in \mathbb{N}$, the injection $\mathcal{S}_{\alpha+1}(\mathbb{R}^n) \hookrightarrow \mathcal{S}_\alpha(\mathbb{R}^n)$ is compact (**exercise***: cf. [GEL 68], Volume II, Chapter II, section 2) and $\mathcal{S}(\mathbb{R}^n) = \varprojlim \mathcal{S}_\alpha(\mathbb{R}^n)$. Therefore:

THEOREM 4.72.– *The space $\mathcal{S}(\mathbb{R}^n)$ is an $(\mathcal{FS})$-space.*

The injections $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n)$ are continuous, and $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

THEOREM 4.73.– *(L. Schwartz) For every multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, the differentiation operator ∂^α (cf. (4.14)) is a continuous linear mapping from $\mathcal{E}(\Omega)$ into $\mathcal{E}(\Omega)$, from $\mathcal{D}(\Omega)$ into $\mathcal{D}(\Omega)$, and from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. Furthermore, ∂^α is a surjective strict morphism from $\mathcal{E}$ onto $\mathcal{E}$, where $\mathcal{E} := \mathcal{E}(\mathbb{R}^n)$.*

PROOF.– The linearity of ∂^α is clear, and continuity follows immediately from the definitions of the topologies of $\mathcal{E}(\Omega), \mathcal{S}(\mathbb{R}^n)$, and $\mathcal{D}(\Omega)$. Let

$\mathcal{I}_1 : \mathcal{E} \to \mathcal{E}$ be the linear mapping defined for any given function $\varphi \in \mathcal{E}$ by $\mathcal{I}_1(\varphi)\left(x^1, ..., x^n\right) = \int_0^{x^1} \varphi\left(t, x^2, ..., x^n\right) dt$. Let $K \subset \mathbb{R}^n$ be a compact set, $a > 0$ a real number, and $K_2 \subset \mathbb{R}^{n-1}$ a compact set such that $K' \supset K$, where $K' := [-a, a] \times K_2$. Let (φ_k) be a sequence in $\mathcal{E}$ such that $(\varphi_k) \to 0$ in $\mathcal{E}$. Then $(\varphi_k) \to 0$ uniformly on K', and, for all $x^1 \in [-a, a]$,

$$\begin{aligned} |\mathcal{I}_1(\varphi_k)(x)| &= \left|\int_0^{x_1} \varphi_k\left(t, x^2, ..., x^n\right) dt\right| \leq \int_0^{x_1} \left|\varphi_k\left(t, x^2, ..., x^n\right)\right| dt \\ &\leq a \sup_{x \in K'} |\varphi_k(x)|, \end{aligned}$$

so $(\mathcal{I}_1(\varphi_k)) \to 0$ uniformly on K'. The same is true for $\partial_1 \mathcal{I}_1(\varphi_k) = \varphi_k$, where $\partial_1 := \frac{\partial}{\partial x^1}$, as well as for the partial derivatives of $\mathcal{I}_1(\varphi_k)$ of all orders, by induction. Therefore, $\mathcal{I}_1 : \mathcal{E} \to \mathcal{E}$ is a continuous linear mapping. Since $\partial_1 \circ \mathcal{I}_1 = 1_{\mathcal{E}}$, the mapping $\partial_1 : \mathcal{E} \to \mathcal{E}$ is a surjective strict morphism by Theorem 3.5(1), and it follows from Corollary 2.131 by induction that $\partial^\alpha : \mathcal{E} \to \mathcal{E}$ is a surjective strict morphism for every multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$. ■

4.3.2. *Spaces of analytic functions*

(I) ANALYTIC FUNCTIONS IN MULTIPLE VARIABLES Consider the formal power series

$$S_{\mathbf{c}}(X) = \sum_{0 \leq \alpha_1, ..., \alpha_n} \mathbf{c}_{\alpha_1, ..., \alpha_n} X_1^{\alpha_1} ... X_n^{\alpha_n} = \sum_{\alpha \in \mathbb{N}^n} \mathbf{c}_\alpha X^\alpha$$

([P1], section 2.3.9**(II)**), where the c_α belong to a Banach $\mathbb{K}$-space E with norm $|.|$. Let $z = (z_1, ..., z_n) \in \mathbb{K}^n$. The *open polydisk* of center z and radius $r = (r_1, ..., r_n)$ $(r_j > 0, j = 1, ..., n)$ in $\mathbb{K}^n$ is defined as the set

$$P(z; r) = \{\xi = (\xi_1, ..., \xi_n) = |\xi_j - z_j| < r_j, j = 1, ..., n\}.$$

We can similarly define the closed polydisk $\overline{P}(z; r)$ of center z and radius r by replacing the strict inequalities with non-strict inequalities. If there exists $\zeta \in \mathbb{K}^n$ such that the series of real numbers $\sum_{0 \leq \alpha_1, ..., \alpha_n} |\mathbf{c}_{\alpha_1, ..., \alpha_n}| \left|\zeta_1^{\alpha_1}\right| ... \left|\zeta_n^{\alpha_n}\right|$ converges, then the power series $S_{\mathbf{c}}(\zeta)$ is said to be *absolutely* *convergent* at the point ζ. If so, by Abel's lemma (**exercise**)[8], $S_{\mathbf{c}}(\xi)$ converges absolutely on the open polydisk $P(0;|\zeta|)$, where $|\zeta| := (|\zeta_1|, ..., |\zeta_n|)$, and *converges normally* on every closed polydisk $\overline{P}(0;r)$ such that $0 < r_i < |\zeta_i|$ $(i = 1, ..., n)$; this means that the series of terms $\sup_{\xi \in \overline{P}(0;r)} |\mathbf{c}_\alpha \xi^\alpha|$ (where $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$) converges.

DEFINITION 4.74.– *Let Ω be an open subset of $\mathbb{K}^n$ and suppose that E is a Banach $\mathbb{K}$-space. A function $\mathbf{f} : \Omega \to E$ is said to be* analytic *(or of class C^ω) on Ω if, for every point $z \in \Omega$, there exists a polydisk $P(z;r) \subset \Omega$ such that $\mathbf{f}(\xi)$ may be expressed as the sum of an absolutely convergent series*

$$\mathbf{f}(\xi) = \sum_{\alpha \in \mathbb{N}^n} \mathbf{c}_\alpha (\xi - z)^\alpha \qquad [4.15]$$

on $P(z;r)$. When $\mathbb{K} = \mathbb{R}$, we say that the function is real analytic *on Ω.*

LEMMA 4.75.– *Any function that is analytic on an open subset Ω of $\mathbb{K}^n$ has class C^∞.*

PROOF.– See the proof of Proposition 4.56 and argue by induction. ■

The space $\mathcal{A}_E(\Omega)$ of analytic functions on $\Omega \subset \mathbb{K}^n$ taking values in E is a $\mathbb{K}$-vector space, and $\mathcal{A}_{\mathbb{K}}(\Omega)$ is an associative, commutative, and unitary $\mathbb{K}$-algebra ([P1], section 2.3.10) that is furthermore entire as a subring of the entire ring $\mathbb{K}[[X_1, ..., X_n]]$ ([P1], section 2.3.9(**II**)).

THEOREM 4.76.– *(*principle of analytic extension*) Let $\Omega \subset \mathbb{K}^n$ be a* connected *open set, E a Banach $\mathbb{K}$-space, K a compact and infinite subset of Ω, and suppose that the functions $\mathbf{f}, \mathbf{g} : \Omega \to E$ are analytic. If $\mathbf{f}(\xi) = \mathbf{g}(\xi)$ for each $\xi \in K$, then $\mathbf{f}(\xi) = \mathbf{g}(\xi)$ on Ω (**exercise***: cf. [DIE 82], Volume 1, (9.4.3)).*

(II) ANALYTIC FUNCTIONS IN MULTIPLE COMPLEX VARIABLES

LEMMA-DEFINITION 4.77.–

1) A function $\mathbf{f} : \Omega \to E : z \mapsto \mathbf{f}(z)$ is said to be holomorphic *if all of its partial complex derivatives $\frac{\partial \mathbf{f}}{\partial z_j}$ $(j = 1, ..., n)$ are defined on Ω.*

2) The following conditions are equivalent:

8 See the Wikipedia article on *Power series*.

i) $\mathbf{f}$ *is holomorphic.*

ii) $\mathbf{f} : (x, y) \mapsto \mathbf{f}(x, y)$ *is differentiable on* Ω *viewed as an open subset of* $\mathbb{R}^n \times \mathbb{R}^n$, *and satisfies the Cauchy-Riemann conditions* $\frac{\partial \mathbf{f}}{\partial \bar{z}_j} = 0 \ (i = 1, ..., n)$.

PROOF.– (2) is an obvious generalization of Corollary 4.46. ∎

LEMMA-DEFINITION 4.78.– *Let* Ω *be a non-empty open subset of* $\mathbb{C}^n$, E *a real Banach space, and* $E_{(\mathbb{C})}$ *the complexification of* E. *Suppose that* $\mathbf{f}$ *is a holomorphic function on* Ω *taking values in* $E_{(\mathbb{C})}$, z *a point of* Ω, *and* $P(z; r)$ *an open polydisk whose closure* $\overline{P}(z; r)$ *is contained in* Ω. *For every multi-index* $\alpha = (\alpha_1, ..., \alpha_n)$, *define* $\alpha! := \alpha_1!...\alpha_n!$ *and* $r^\alpha = r_1^{\alpha_1}...r_n^{\alpha_n}$.

1) Let C_j *be the closed path* $[0, 1] \to \mathbb{C} : \theta_j \mapsto \xi_j = z_j + r_j e^{2\pi i \theta_j}$ $(j = 1, ..., n)$. *Then, with* $\mathbf{c}_\alpha := \frac{\partial^\alpha \mathbf{f}(z)}{\alpha!}$, *we have that*

$$\boxed{\mathbf{c}_\alpha = \frac{1}{(2\pi i)^n} \oint_{C_1} ... \oint_{C_n} \frac{\mathbf{f}(\xi) . d\xi_1 ... d\xi_n}{(\xi_1 - z_1)^{\alpha_1 + 1} ... (\xi_n - z_n)^{\alpha_n + 1}}}. \qquad [4.16]$$

2) Furthermore, $M := \sup_{\xi \in \overline{P}(z;r)} |\mathbf{f}(z)| < \infty$, *and "Cauchy's inequalities" hold:*

$$\boxed{|\mathbf{c}_\alpha| \leq \frac{M}{r^\alpha}}. \qquad [4.17]$$

PROOF.– (1): Every analytic function is holomorphic, and we can differentiate (4.15) under the $\sum$ sign (Proposition 4.56). Therefore, (4.16) may be deduced by induction from Cauchy's integral formula (4.11) with $j(z; \gamma) = 1$ ([DIE 82], Volume 1, (9.4.4)). (2): Since $\overline{P}(z; r)$ is compact, $|\mathbf{f}|$ is bounded on $\overline{P}(z; r)$ (Theorem 2.39), and (4.17) may be immediately deduced from (4.16). ∎

THEOREM 4.79.– *Let* Ω *be a non-empty open subset of* $\mathbb{C}^n$, E *a real Banach space, and* $E_{(\mathbb{C})}$ *the complexification of* E. *The function* $\mathbf{f} : \Omega \to E_{(\mathbb{C})}$ *is analytic on* Ω *if and only if it is holomorphic on* Ω.

PROOF.– We know that every analytic function on Ω is holomorphic (by the proof of Lemma-Definition 4.78), so we simply need to show the converse.

If $\mathbf{f}$ is a holomorphic function on Ω, then Cauchy's inequalities (4.17) imply that the power series on the right-hand side of (4.15) converges absolutely on $P(z;r)$, and is in fact precisely the Taylor series of $\mathbf{f}(\xi)$ about z. ■

COROLLARY 4.80.– *(Hartogs'* theorem*) Any function* $(\xi_1, ..., \xi_n) \mapsto \mathbf{f}(\xi_1, ..., \xi_n)$ *taking values in a complex Banach space* E *is analytic on an open set* $\Omega \subset \mathbb{C}^n$ *if and only if it is analytic in each of its variables when holding the others fixed.*

THEOREM-DEFINITION 4.81.–

1) We say that a function $\mathbf{f}$ *taking values in a complex Banach space* E *is an* entire function *if it is analytic on* $\mathbb{C}^n$.

2) If $\mathbf{f}$ *is an entire function, the power series in the right-hand side of* (4.15) *converges absolutely on* $\mathbb{C}^n$.

*3) (Liouville'*s theorem*) Every bounded entire function is constant.*

PROOF.– (2) and (3) follow from Cauchy's inequalities (4.17). ■

(III) SPACES OF ANALYTIC FUNCTIONS Let Ω be a non-empty open subset of $\mathbb{C}^n$ and suppose that $\mathcal{H}(\Omega)$ is the $\mathbb{C}$-algebra of holomorphic complex functions on Ω. This space may be canonically identified with the subspace of $\mathcal{E}(\Omega)$ (viewing Ω as an open subset of $\mathbb{R}^{2n}$) formed by the functions $\mathbf{f}$ that satisfy the Cauchy-Riemann conditions (Lemma-Definition 4.77). Therefore, $\mathcal{H}(\Omega)$ is a closed subspace of $\mathcal{E}(\Omega)$ (**exercise**), and:

THEOREM 4.82.– *The space* $\mathcal{H}(\Omega)$ *is an* $(\mathcal{FS})$*-space.*

Since every $(\mathcal{FS})$-space is a Montel space (Theorem 3.76), this result generalizes Montel's theorem:

COROLLARY 4.83.– *(Montel'*s theorem*) The space* $\mathcal{H}(\Omega)$ *(equipped with the topology of compact convergence) satisfies the following condition (known as Montel'*s property*): a subset of* $\mathcal{H}(\Omega)$ *is compact if and only if it is closed and bounded; equivalently, every sequence of elements in* $\mathcal{H}(\Omega)$ *that is bounded in this space has a convergent subsequence.*

REMARK 4.84.– *The spaces* $\mathcal{E}^{(m)}(\Omega), \mathcal{E}(\Omega), \mathcal{D}_K^{(m)}(\Omega), \mathcal{D}^{(m)}(\Omega), \mathcal{D}_K(\Omega)$, $\mathcal{D}(\Omega)$ *(resp.* $\mathcal{H}(\Omega)$*) may be defined in the same way (using charts) when* Ω *is a locally compact differential (resp. holomorphic) manifold that is countable* *at infinity (cf. [P3]). The spaces $\mathcal{E}(\Omega), \mathcal{D}(\Omega), \mathcal{H}(\Omega)$, and $\mathcal{S}(\mathbb{R}^n)$ are nuclear ([GRO 55], Chapter II, section 2.3).*

(IV) SPACES OF GERMS OF ANALYTIC FUNCTIONS Let $S \subset \mathbb{C}^n$ $(n \geq 1)$ be a set and suppose that $(\Omega_j)_{j\geq 0}$ is a decreasing sequence of open sets containing S such that $\bigcap_{j\geq 0} \Omega_j = S$. The restriction $\rho^{\Omega j}_{\Omega_{j+1}} : \mathcal{H}(\Omega_j) \hookrightarrow \mathcal{H}(\Omega_{j+1})$ is linear and continuous; the $\rho^{\Omega j}_{\Omega_{j+1}}$ inductively determine a (continuous linear) restriction $\rho^{\Omega_j}_{\Omega_k} : \mathcal{H}(\Omega_j) \hookrightarrow \mathcal{H}(\Omega_k)$ for every pair of integers j, k such that $k \geq j$. In **Lcs**, we therefore have the direct system $\mathfrak{D} = \left\{\mathcal{H}(\Omega_j), \rho^{\Omega_j}_{\Omega_k}; \mathbb{N}\right\}$, whose inductive limit $\mathcal{H}(S) = \varinjlim \mathcal{H}(\Omega_j)$ is the set of germs of holomorphic functions on a neighborhood of S (Definition 2.63). By Theorem 3.73:

COROLLARY 4.85.– *The space $\mathcal{H}(S)$ of germs of holomorphic functions on a neighborhood of the set $S \subset \mathbb{C}^n$ is a Silva space (and in particular an $(\mathcal{LF})$-space and an $(\mathcal{S})$-space).*

DEFINITION 4.86.– *Let S be a subset of $\mathbb{R}^n$ that is locally closed in $\mathbb{C}^n$. The set $\mathfrak{N}_{\mathbb{C}^n}(S)$ of open-neighborhoods of S in $\mathbb{C}^n$ (section 2.3.1(**I**)) is called the set of* complex neighborhoods *of S (this is a directed set).*

We write that

$$\mathcal{A}(S) = \varinjlim_{\Omega_j \in \mathfrak{N}_{\mathbb{C}^n}(S)} \mathcal{H}(\Omega_j)$$

and say that $\mathcal{A}(S)$ is the *space of germs of real analytic functions* on S. By the above, $\mathcal{A}(S)$ is a Silva space.

REMARK 4.87.– *The spaces $\mathcal{H}(S)$ and $\mathcal{A}(S)$ are nuclear (**exercise**: use Remark 4.84 and the permanence properties of nuclear spaces (section 3.11.3(**II**)).*

4.4. Generalized function spaces

4.4.1. *Distributions*

(I) DISTRIBUTIONS, DISTRIBUTIONS WITH COMPACT SUPPORT, TEMPERED DISTRIBUTIONS Distribution theory was developed by L.

Schwartz (cf. section 3.1). Let Ω be an open subset of $\mathbb{R}^n$. An element of the dual $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ (section 4.3.1**(III)**) is said to be a *distribution* on Ω. The notion of the *restriction* of a distribution on Ω to an open set $U \subset \Omega$ is defined in the same way as for a Radon measure. A gluing principle for distributions similar to Lemma 4.27 can be shown, since Ω admits infinitely differentiable partitions of unity ([SCW 66], Chapter I, section 2)[9]:

THEOREM 4.88.– *(Whitney) For any open covering $(U_i)_{i \in I}$ of Ω, there exists a partition of unity $(\psi_i)_{i \in I}$ subordinate to $(U_i)_{i \in I}$ (cf. Definition 2.61), where each ψ_i has class C^∞.*

The *support* of a distribution can therefore be defined as above. Clearly, every Radon measure on Ω uniquely determines a distribution on Ω, and so $\mathcal{K}'(\Omega) \subset \mathcal{D}'(\Omega)$ via an identification that preserves support. By Theorem 4.29, there exists a support-preserving canonical injection

$$\boxed{\mathcal{C}(\Omega) \hookrightarrow \mathcal{D}'(\Omega) : f \mapsto T_f},$$

specifically,

$$\boxed{\langle T_f, \varphi \rangle = \int_\Omega f(x)\, \varphi(x)\, dx \quad (\varphi \in \mathcal{D}(\Omega))}. \qquad [4.18]$$

Every element of the dual $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$ may be identified with a distribution with compact support. When $\Omega = \mathbb{R}^n$, any such element may further be identified with an element of the dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$; we say that $\mathcal{S}'(\mathbb{R}^n)$ is the space of *tempered distributions*. This yields canonical injections between the following spaces, all of which are continuous when the dual spaces are equipped with the strong topology (**exercise**):

$$\boxed{\underbrace{\mathcal{E}'}_{\substack{\text{space of} \\ \text{distributions with compact support}}} \hookrightarrow \underbrace{\mathcal{S}'}_{\substack{\text{space of} \\ \text{tempered distributions}}} \hookrightarrow \underbrace{\mathcal{D}'}_{\substack{\text{space of} \\ \text{distributions}}}}$$

Let $f \in \mathcal{C}(\mathbb{R}^n)$. Then T_f is a tempered distribution if and only if f is a so-called slowly increasing function, or in other words there exists an integer

9 A proof of this result is given in the Wikipedia article on *Partition of unity*.

$k \geq 0$ such that the function $x \mapsto (1 + |x|)^k |f(x)|$ is bounded (**exercise**). In the following, we will write $\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{D}', \mathcal{E}', \mathcal{S}'$ for $\mathcal{D}(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n), \mathcal{E}'(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)$ respectively.

THEOREM 4.89.– *The spaces $\mathcal{E}'$ and $\mathcal{S}'$ (which are the strong duals of $\mathcal{E}$ and $\mathcal{S}$ respectively) are both Silva spaces. The space $\mathcal{D}'$ is a complete ultrabornological Montel space.*

PROOF.– For $\mathcal{E}'$ and $\mathcal{S}'$, this follows from Theorems 4.68, 4.72, and section 3.8.2(**II**); for $\mathcal{D}'$, this follows from Theorems 3.98, 3.103(2), 3.123, and 4.69, since $\mathcal{D}$ is bornological. ■

REMARK 4.90.– *The spaces $\mathcal{D}'(\Omega), \mathcal{E}'(\Omega)$ may be defined in the same way (using charts) when Ω is a locally compact differential manifold that is countable at infinity (cf. [P3]). The spaces $\mathcal{D}', \mathcal{E}', \mathcal{S}'$ are nuclear. This follows from the fact that $\mathcal{E}$ and $\mathcal{S}$ are Fréchet nuclear spaces (Remark 4.84) together with the fact that $\mathcal{D}$ is both complete and the inductive limit of a sequence of nuclear Fréchet spaces, as well as the permanence properties of nuclear spaces (section 3.11.3(**II**)).*

DEFINITION 4.91.– *Let $\mathcal{T}$ be any one of the spaces $\mathcal{E}, \mathcal{S}, \mathcal{D}$, and suppose that $\mathcal{T}'$ is its strong dual. The space $\mathcal{T}$ is called the space of* test functions *of $\mathcal{T}'$.*

(II) DISTRIBUTIONAL DERIVATIVE Consider a function f of class C^1 on $\mathbb{R}^n$ and, to simplify the reasoning, suppose that $n = 1$. Let $\varphi \in \mathcal{D}$. The formula for integration by parts implies that

$$\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = -\int_{\mathbb{R}} f'(x)\,\varphi(x)\,dx,$$

which by (4.18) implies that

$$\langle T_f', \varphi \rangle = -\langle T_f, \varphi' \rangle .$$

with the notation $T_f' = T_{f'}$.

THEOREM-DEFINITION 4.92.– *With the notation of Definition 4.91, the linear operator $\partial^\alpha : \mathcal{T}' \to \mathcal{T}'$ defined for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, every distribution $T \in \mathcal{T}'$, and every test function $\varphi \in \mathcal{T}$ by*

$$\boxed{\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle}$$

(where $|\alpha| = \sum_{i=1}^{n} \alpha_i$) is a continuous linear mapping. If $\mathcal{T} = \mathcal{E}$, this operator is an injective strict morphism.

PROOF.– The mapping $u : \mathcal{T} \to \mathcal{T} : \varphi \mapsto \partial^\alpha \varphi$ is continuous and linear (Theorem 4.73). Furthermore, $\mathcal{T}$ is reflexive, and therefore a Mackey space. Hence, ${}^t u : \mathcal{T}' \to \mathcal{T}'$ is continuous (Theorem 3.94) and ${}^t u = (-1)^{|\alpha|} \partial^\alpha$, where $\partial^\alpha : \mathcal{T}' \to \mathcal{T}' : T \mapsto \partial^\alpha T$. If $\mathcal{T} = \mathcal{E}$, u is a surjective strict morphism, and $\mathcal{T}$ is an $(\mathcal{FS})$-space, so ${}^t u$ is an injective strict morphism by Corollary 3.132. ■

REMARK 4.93.– *It can be shown that $\partial^\alpha : \mathcal{D}' \to \mathcal{D}'$ is a strict morphism ([SCW 66], Chapter III, section 5, Theorem XVIII) (which is of course not injective).*

The distributional derivative ∂^α defined by Theorem-Definition 4.92 (for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$) extends the usual notion of derivative for C^∞ functions in the sense that, if $f \in \mathcal{C}^\infty(\Omega)$, then $\partial^\alpha T_f = T_{\partial^\alpha f}$ (**exercise**). By embedding $\mathcal{C}(\Omega)$ in $\mathcal{D}'(\Omega)$, and writing $\partial^\alpha f$ for $\partial^\alpha T_f$, the derivative $\partial^\alpha f$ of $f \in \mathcal{C}(\Omega)$ may be defined even when f is not differentiable in the usual sense: the *distribution* $\partial^\alpha f$ is called the *distributional derivative* of f.

This process can also be applied to piecewise continuous functions on the real line, assuming for example that they are right-continuous. Write $\mathcal{K}_m$ for the vector space of such functions. There exists a canonical injection $\mathcal{K}_m \hookrightarrow \mathcal{D}' : f \mapsto T_f$, where T_f is the distribution defined by (4.18) (**exercise**). We will reuse the same notation for f and T_f when $f \in \mathcal{K}_m$.

Radon measures are said to be distributions of order 0, and a distribution is said to have order $\leq m$ if it is the m-th derivative of a distribution of order 0.

EXAMPLE 4.94.–

1) Consider the Heaviside function Υ (Example 4.4(4)). This function satisfies $\langle \Upsilon, \varphi' \rangle = \int_0^{+\infty} \varphi'(x)\, dx = -\varphi(0)$, so

$$\frac{d\Upsilon}{dx} = \delta, \text{ where } \langle \delta, \varphi \rangle := \varphi(0)\,(\varphi \in \mathcal{D}).$$

We say that δ (also written δ_0) is the Dirac distribution. *It satisfies* $\operatorname{supp}(\delta) = \{0\}$. *The linear form $\mathcal{C}(\mathbb{R}) \to \mathbb{C} : \varphi \mapsto \varphi(0)$ is continuous, so δ is a Radon* *measure. The m-th derivative of the Dirac distribution is given by*

$$\boxed{\left\langle \delta^{(m)}, \varphi \right\rangle = (-1)^m \varphi^{(m)}(0)}.$$

2) Let $\Xi \in \mathcal{K}_m$ be the step function defined by $\Xi(x) = i$ for each $i \in \mathbb{Z}$ and all $x \in [i-1, i[$. Then

$$\frac{d\Xi}{dx} = \varpi := \sum_{i=-\infty}^{+\infty} \delta_i, \quad \delta_i : \varphi \mapsto \varphi(i) \ \ (\varphi \in \mathcal{D})$$

*and $\varpi \in \mathcal{S}'$ (**exercise**). The distribution ϖ is the Dirac* comb.

(III) Structure of distributions It is possible to show the following result ([SCW 66], Chapter III, Theorems XXVI and XXXII):

THEOREM 4.95.– *(L. Schwartz)*

1) Every distribution with compact support on $\mathbb{R}^n$ is the sum of finitely many derivatives of continuous functions on $\mathbb{R}^n$, and therefore has finite order.

2) Every distribution T on $\mathbb{R}^n$ is the sum of a locally finite series of distributions with compact support; more precisely, given any covering $(\Phi_i)_{i \in I}$ of $\mathbb{R}^n$ consisting of closed sets (which may be chosen to be compact), we have that $T = \sum_{i \in I} T_i$, where $\operatorname{supp}(T_i) \subset \Phi_i$.

(IV) Multiplication of a distribution by a function Let $g \in \mathcal{C}^\infty(\Omega)$ and suppose that $\varphi \in \mathcal{D}(\Omega)$. Then $g\varphi \in \mathcal{D}(\Omega)$, so, given any distribution $T \in \mathcal{D}'(\Omega)$, the value $\langle T, g\varphi \rangle$ is well-defined. The linear form $\varphi \mapsto \langle T, g\varphi \rangle$ is continuous on $\mathcal{D}(\Omega)$ (**exercise**), and is therefore a distribution, written gT :

$$\boxed{\langle gT, \varphi \rangle := \langle T, g\varphi \rangle}. \qquad [4.19]$$

It can be shown that $\operatorname{supp}(gT) \subset \operatorname{supp}(g) \cap \operatorname{supp}(T)$ (**exercise**). The same approach can be used, *mutatis mutandis*, to multiply a distribution T with compact support by a function $g \in \mathcal{C}^\infty(\Omega)$. For example, $g\delta_a = g(a)\,\delta_a$ if $g \in \mathcal{C}^\infty(\mathbb{R})$.

(V) Case of tempered distributions The situation is slightly different when we consider the product of a tempered distribution by an infinitely differentiable function. Write $\mathcal{O}_M$ for the subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$

given by the functions g such that, for every multi-index $\alpha \in \mathbb{N}^n$, there exists a polynomial $p_\alpha(X)$ for which the function $x \mapsto \frac{|g(x)|}{1+|p_\alpha(x)|}$ is bounded on $\mathbb{R}^n$. The space $\mathcal{O}_M$ is a vector subspace of $\mathcal{S}'(\mathbb{R}^n)$, and a subalgebra of the multiplicative algebra $\mathcal{C}^\infty(\Omega)$. We say that $\mathcal{O}_M$ is the space of *slowly increasing infinitely differentiable functions* on $\mathbb{R}^n$. For every pair of functions $g \in \mathcal{O}_M$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have that $g\varphi \in \mathcal{S}(\mathbb{R}^n)$, so the value $\langle T, g\varphi \rangle$ is well-defined for every tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$. The linear form $\varphi \mapsto \langle T, g\varphi \rangle$ is continuous on $\mathcal{S}(\mathbb{R}^n)$ (**exercise**), and is therefore a tempered distribution; this allows us to define the product gT using (4.19). For example, $g\varpi = \sum_{n\in\mathbb{Z}} g(n)\,\delta_n$ if $g \in \mathcal{O}_M(\mathbb{R})$.

(VI) MULTIPLICATION OF DISTRIBUTIONS In general, we cannot multiply two arbitrary Schwartz distributions; for example, we cannot simply define the product $\delta \times \delta$ ([SCW 66], Chapter V, section 1). However, an alternative approach is possible by introducing the notion of Colombeau distributions. Colombeau distributions generalize Schwartz distributions, but are beyond the scope of this book: see [COL 84].

DEFINITION 4.96.– *The* singular support *of a distribution $T \in \mathcal{D}'(\Omega)$ (where Ω is a non-empty open subset of $\mathbb{R}^n$) is defined as the subset* $\operatorname{sing\,supp}(T)$ *of Ω consisting of the points that do not have an open neighborhood $V \subset \Omega$ on which the restriction $T|_V$ is an infinitely differentiable function.*

The set $\operatorname{sing\,supp}(T)$ is a closed subset of $\operatorname{supp}(T)$. If $T, U \in \mathcal{D}'(\Omega)$ and $\operatorname{sing\,supp}(T) \cap \operatorname{sing\,supp}(U) = \varnothing$, let $\Omega_T = \complement_\Omega \operatorname{sing\,supp}(T)$ and $\Omega_U = \complement_\Omega \operatorname{sing\,supp}(U)$. Then $T|_{\Omega_T}$ and $U|_{\Omega_U}$ are C^∞ functions, and $\{\Omega_T, \Omega_U\}$ is an open covering of Ω. Let $\{\psi_T, \psi_U\}$ be a C^∞ partition of unity subordinate to $\{\Omega_T, \Omega_U\}$ (Theorem 4.88). Then, for every function $\varphi \in \mathcal{D}(\Omega)$, the values $\langle U, T|_{\Omega_T}\varphi \rangle$ and $\langle T, U|_{\Omega_U}\varphi \rangle$ are well-defined, and we can define $\langle TU, \varphi \rangle$ by

$$\langle TU, \varphi \rangle = \langle U, T|_{\Omega_T}\varphi \rangle + \langle T, U|_{\Omega_U}\varphi \rangle .$$

We can use this to define TU. For example, $\delta_a\delta_b = 0$ if $a \neq b$ (and is not defined if $a = b$).

4.4.2. *Hyperfunctions on the real line*

The theory of hyperfunctions was developed by M. Sato between 1958 and 1960 [SAT 60].

(I) ANALYTIC FUNCTIONALS Let S be a subset of $\mathbb{C}^n$. The elements of the dual space $\mathcal{H}'(S)$ (section 4.3.2(**IV**)) are called locally analytic functionals on S. When $\mathcal{H}'(S)$ is equipped with the strong topology, Corollary 4.85 implies that:

COROLLARY 4.97.– *The space $\mathcal{H}'_b(S)$ is an $(\mathcal{FS})$-space.*

Let K be a compact subset of $\mathbb{C}$ and suppose that $\Omega \in \mathfrak{N}_{\mathbb{C}}(K)$. There exists a canonical injection $\mathcal{H}(\Omega) \hookrightarrow \mathcal{H}(\Omega - K)$ (since every holomorphic function on Ω is also holomorphic on $\Omega - K := \complement_{\Omega} K$).

THEOREM 4.98.– *(Köthe-Silva-Grothendieck* duality theorem *[KOT 53], [GRO 53]) Let K be a compact subset of $\mathbb{C}$ and suppose that $\Omega \in \mathfrak{N}_{\mathbb{C}}(K)$. Then there exists an isomorphism of* **Tvs**

$$\frac{\mathcal{H}(\Omega - K)}{\mathcal{H}(\Omega)} \cong \mathcal{H}'_b(K)$$

(where $\mathcal{H}'_b(K)$ denotes the strong dual of $\mathcal{H}(K)$).

PROOF.– We will give a rough sketch of the proof: we may assume that K is a compact interval (since, in the general case, K is a countable union of disjoint compact sets). Suppose that $f \in \mathcal{H}(\Omega - K)$, $\mathring{\varphi} \in \mathcal{H}(K)$, and let $\varphi \in \mathcal{H}(\Omega)$ be a representative of the germ $\mathring{\varphi}$. Let $U \in \mathfrak{N}_{\mathbb{C}}(K)$ be an open set whose frontier ∂U is the image of a rectifiable closed path γ that winds around K once in the counterclockwise direction, and set

$$\boxed{\langle [f], \varphi \rangle := -\oint_{\gamma} f|_U(z)\, \varphi(z)\, dz}. \qquad [4.20]$$

Then $\langle [f], . \rangle : \varphi \mapsto \langle [f], \varphi \rangle$ is a linear form on $\mathcal{H}(\Omega)$. If $(\varphi_i)_{i \in I}$ is a net that converges to 0 in $\mathcal{H}(\Omega)$, then this net also converges to 0 in $\mathcal{C}(\Omega)$, and so converges uniformly to 0 on every compact set. Consequently, $(\langle [f], \varphi_i \rangle)_{i \in I} \to 0$ in $\mathbb{C}$, so the linear form $\langle [f], . \rangle$ is continuous, or in other words $\langle [f], . \rangle \in \mathcal{H}'(K)$. Every continuous linear form on $\mathcal{H}(K)$ is of this form (this result was shown by E. Silva and is presented in [GRO 53], p. 47-48). Finally, $\langle [f], \varphi \rangle = 0$ if and only if $f \in \mathcal{H}(\Omega)$, by Cauchy's theorem and Morera's theorem (Theorems 4.53 and 4.52). Hence, Noether's first isomorphism theorem implies that $\frac{\mathcal{H}(\Omega - K)}{\mathcal{H}(\Omega)} \tilde{\to} \mathcal{H}'(K) : \mathring{f} \mapsto \langle [f], . \rangle$ is an isomorphism of vector spaces, where $\mathring{f} := f + \mathcal{H}(\Omega)$. Since $\mathcal{H}(\Omega - K)$ and

$\mathcal{H}(\Omega)$ are $(\mathcal{FS})$-spaces (Theorem 4.82), the quotient $\frac{\mathcal{H}(\Omega-K)}{\mathcal{H}(\Omega)}$ is also an $(\mathcal{FS})$-space (section 3.4.2 and Theorem 3.73), and the same is true for $\mathcal{H}'_b(K)$ (Corollary 4.97). It is easy to see that this isomorphism is continuous. It is therefore an isomorphism of $\mathbf{Tvs}$ by the inverse operator theorem (Theorem 3.12(2)(i)). ■

(II) NOTION OF HYPERFUNCTION Let S be a locally closed subset of $\mathbb{R}$ and suppose that $\Omega \in \mathfrak{N}_{\mathbb{C}}(S)$ (Definition 4.86). The set S is closed in Ω, so $\Omega - S$ is open in Ω and therefore also in $\mathbb{C}$. We can write $\Omega - S = \Omega_+ \cup \Omega_-$, where $\Omega_\pm = \{z \in \Omega : \pm\Im(z) > 0\}$ (see Figure 4.3, where S is the open interval $]a, b[$ in $\mathbb{R}$).

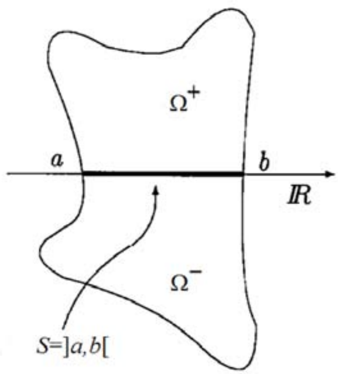

Figure 4.3. *Decomposition of* Ω

Since the open sets Ω_+, Ω_- are disjoint, every function $f \in \mathcal{H}(\Omega - S)$ may be uniquely written in the form $f_+ - f_-$, where $f_\pm \in \mathcal{H}(\Omega_\pm)$. If $f \in \mathcal{H}(\Omega)$, then $f\left|_{\Omega_\pm}\right. \in \mathcal{H}(\Omega_\pm)$, and, by the principle of analytic extension (Theorem 4.76(3)), f is the unique extension of $f\left|_{\Omega_\pm}\right.$ to Ω. Consequently, $\iota : f \mapsto f\left|_{\Omega_\pm}\right.$ is a canonical injection from $\mathcal{H}(\Omega)$ into $\mathcal{H}(\Omega_\pm)$.

LEMMA 4.99.– *Let S be a locally closed subset of $\mathbb{R}$ (section 2.3.1(**II**)), and thus a locally closed subset of $\mathbb{C}$. Then $\mathcal{B}[S] := \frac{\mathcal{H}(\Omega-S)}{\mathcal{H}(\Omega)}$ does not depend on* $\Omega \in \mathfrak{N}_{\mathbb{C}}(S)$ ("excision theorem")*, which implies that*

$$\boxed{\mathcal{B}[S] = \varinjlim_{\Omega \in \mathfrak{N}_{\mathbb{C}}(S)} \frac{\mathcal{H}(\Omega-S)}{\mathcal{H}(\Omega)}}. \qquad [4.21]$$

PROOF.– Let $\Omega_1, \Omega_2 \in \mathfrak{N}_{\mathbb{C}}(S)$, and suppose that $\Omega_2 \subset \Omega_1$ (readers should easily be able to convince themselves that this is without loss of generality). We will show that the induced homomorphism theorem ([P1], section 2.2.3(**I**)) implies that

$$\frac{\mathcal{H}(\Omega_2 - S)}{\mathcal{H}(\Omega_2)} \cong \frac{\mathcal{H}(\Omega_1 - S)}{\mathcal{H}(\Omega_1)}.$$

Let $\rho : \mathcal{H}(\Omega_1 - S) \hookrightarrow \mathcal{H}(\Omega_2 - S)$ be the canonical injection and write $\varphi_k : \mathcal{H}(\Omega_k - S) \twoheadrightarrow \frac{\mathcal{H}(\Omega_k - S)}{\mathcal{H}(\Omega_k)}$ $(k = 1, 2)$ for the canonical surjection. Since ρ is the restriction $f \mapsto f\,|_{\Omega_2 - S}$, we have that $\rho(\mathcal{H}(\Omega_1)) = \mathcal{H}(\Omega_2)$, which implies that there exists an induced homomorphism

$$\bar{\rho} : \frac{\mathcal{H}(\Omega_1 - S)}{\mathcal{H}(\Omega_1)} \to \frac{\mathcal{H}(\Omega_2 - S)}{\mathcal{H}(\Omega_2)}.$$

Let $f \in \rho^{-1}(\mathcal{H}(\Omega_2))$; then $f\,|_{\Omega_2 - S} \in \mathcal{H}(\Omega_2)$ and $\varphi_1(f) \in \mathcal{H}(\Omega_1)/\mathcal{H}(\Omega_2) = 0$ by the principle of analytic extension, so ρ is injective. It simply remains to be shown that $\bar{\rho}$ is surjective. To do this, we can apply the Mittag-Leffler theorem (Theorem 4.62(3)) with $I = 3, U_3 = \varnothing$, and $g_{12} = -g_{21} = g$, $g_{i3} = g_{3i} = 0$ $(i = 1, 2)$. For every pair of non-empty open subsets U_1, U_2 of $\mathbb{C}$ and every function $f \in \mathcal{H}(U_1 \cap U_2)$, this theorem implies that there exist functions $f_i \in \mathcal{H}(U_i)$ $(i = 1, 2)$ such that $f(z) = f_1(z) - f_2(z)$ on $U_1 \cap U_2$. Pick $U_1 = \Omega_1 - S$, $U_2 = \Omega_2$, so that $U_1 \cap U_2 = \Omega_2 - S$. Let $f \in \mathcal{H}(\Omega_2 - S)$; then there exists $f_1 \in \mathcal{H}(\Omega_1 - S), f_2 \in \mathcal{H}(\Omega_2)$ such that $f(z) = f_1(z) - f_2(z)$ on $\Omega_2 - S$. Then $\varphi_2 : f \mapsto f + \mathcal{H}(\Omega_2) = f_1 + \mathcal{H}(\Omega_2)$, and $f_1 + \mathcal{H}(\Omega_2) \mapsto f_1 + \mathcal{H}(\Omega_1)$ is an isomorphism by the principle of analytic extension. Therefore, $\operatorname{im}(\bar{\rho}) = \varphi_2(\rho(\mathcal{H}(\Omega_1 - S))) = \varphi_2(\mathcal{H}(\Omega_2 - S))$, which shows that $\bar{\rho}$ is surjective. ■

DEFINITION 4.100.– *An element* $[f]$ *of* $\mathcal{B}[S]$ *is said to be a* hyperfunction *on* S. *If* $f \in \mathcal{H}(\Omega - S)$, *then the equivalence class of* f $(\bmod .\mathcal{H}(\Omega))$ *is called the hyperfunction* $[f]$ *determined by* f.

The hyperfunction $[f]$ determined by f is often written $[f]_{z=x}$ (where x denotes a variable in S) or $f(x + i0) - f(x - i0)$, where the “boundary values" $f(x \pm i0) \in \mathcal{B}[S]$ are defined by

$$\begin{aligned} f(x+i0) &= [\epsilon(z) f(z)]_{z=x}, \quad f(x-i0) = -[\bar{\epsilon}(z) f(z)]_{z=x}, \\ \text{with } \epsilon(z) &= \begin{cases} 1 \\ 0 \end{cases}, \quad \bar{\epsilon}(z) = \epsilon(-z) = \begin{cases} 0 \\ 1 \end{cases} \text{ for } \begin{cases} \Im(z) > 0 \\ \Im(z) < 0 \end{cases}. \end{aligned}$$

If $f \in \mathcal{H}(\Omega)$, then it is clear that $[f] = 0$.

COROLLARY 4.101.– *Let K be a compact subset of $\mathbb{R}$. The set $\mathcal{B}[K] \cong \mathcal{H}'(K)$ (Theorem 4.98) of hyperfunctions on K is a nuclear Fréchet space.*

Let Ω, Ω' be open subsets of $\mathbb{C}$ such that $\Omega' \supset \Omega$, suppose that S is a subset of Ω that is closed in Ω, and let $S' \subset S$ be a subset of Ω' that is closed in Ω'. The following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}(\Omega') & \xrightarrow{\text{rest}} & \mathcal{H}(\Omega' - S') & \longrightarrow & \mathbf{H}^1_{S'}(\Omega'; \mathcal{H}) & \longrightarrow & 0 \\ & & \downarrow \text{rest} & & \downarrow \text{rest} & & \downarrow \text{can} & & \\ 0 & \longrightarrow & \mathcal{H}(\Omega) & \xrightarrow{\text{rest}} & \mathcal{H}(\Omega - U) & \longrightarrow & \mathbf{H}^1_S(\Omega; \mathcal{H}) & \longrightarrow & 0, \end{array}$$

whose rows are exact sequences (in **Vec**), and where

$$\mathbf{H}^1_S(\Omega; \mathcal{H}) := \frac{\mathcal{H}(\Omega - S)}{\mathcal{H}(\Omega)} \qquad [4.22]$$

($\mathbf{H}^1_{S'}(\Omega'; \mathcal{H})$ is defined similarly) and the arrows $\xrightarrow{\text{rest}}$ are the restriction morphisms. The third vertical arrow is a canonical morphism that induces a second canonical morphism

$$\varinjlim_{\Omega' \in \mathfrak{N}_{\mathbb{C}}(S')} \mathbf{H}^1_{S'}\left(\Omega'; \mathcal{H}\right) \xrightarrow{\text{rest}} \varinjlim_{\Omega \in \mathfrak{N}_{\mathbb{C}}(S)} \mathbf{H}^1_S(\Omega; \mathcal{H}),$$

which is also known as the restriction morphism. In particular, let S be a locally closed subset of $\mathbb{R}$ and suppose that S' is a subset of S that is open in S. From the above, it follows that:

COROLLARY 4.102.– *There exists a* restriction *morphism* $\rho_{S'}^{S} : \mathcal{B}[S] \to \mathcal{B}[S']$.

DEFINITION 4.103.– *Let* $[f] \in \mathcal{B}[S]$. *The complement in* S *of the largest open subset* S' *of* S *such that* $\rho_{S'}^{S}[f] = 0$ *is called the* support *of* $[f]$, *and is written* $\operatorname{supp}([f])$.

LEMMA 4.104.– *Let* $\mathfrak{u}$ *be an open subset of* $\mathbb{R}$. *The restriction mapping* $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathfrak{u})$ *is surjective. More precisely, if* $[f] \in \mathcal{B}(\mathfrak{u})$, *then there exists a hyperfunction* $[g] \in \mathcal{B}(\mathbb{R})$ *whose restriction to* $\mathfrak{u}$ *is equal to* $[f]$ *and which satisfies* $\operatorname{supp}([g]) \subset \overline{\mathfrak{u}}$, *where* $\overline{\mathfrak{u}}$ *is the closure of* $\mathfrak{u}$ *in* $\mathbb{R}$.

PROOF.– Let $\partial\mathfrak{u} := \overline{\mathfrak{u}} - \mathfrak{u}$ be the frontier of $\mathfrak{u}$. Suppose that $\Omega \in \mathfrak{N}_{\mathbb{C}}(\mathfrak{u})$ and $f \in \mathcal{H}((\mathbb{C} - \partial\mathfrak{u}) - \mathfrak{u}) = \mathcal{H}(\mathbb{C} - \overline{\mathfrak{u}})$, and let $[f] = f + \mathcal{H}(\Omega)$ be the hyperfunction determined by f. Every element of $\mathcal{B}(\mathfrak{u})$ is of this form. Let $g = f|_{\mathbb{C}-\mathbb{R}}$; then $[g] \in \mathcal{B}(\mathbb{R})$ and the restriction of $[g]$ to $\mathfrak{u}$ is equal to $[f]$; furthermore, $\operatorname{supp}([g]) \subset \overline{\mathfrak{u}}$. ■

(III) OPERATIONS ON HYPERFUNCTIONS The space $\mathcal{B}[S]$ is a $\mathbb{C}$-vector space. The n-th derivative of the hyperfunction $[f]_{z=x}$ is defined by the relation $\frac{d^n}{dx^n}[f]_{z=x} = \left[\frac{d}{dz^n} f\right]_{z=x}$. If $g \in \mathcal{H}(\Omega)$, then we define $g[f] := [gf]$. Thus, if

$$P\left(x, \frac{d}{dx}\right) = \sum\nolimits_{j=0}^{m} a_j(x) \frac{d^j}{dx^j}, \quad a_m \neq 0$$

is a linear differential operator with coefficients in $\mathcal{H}(\Omega)$, we define

$$\boxed{P\left(x, \tfrac{d}{dx}\right)[f] = \left[P\left(x, \tfrac{d}{dx}\right) f\right]}.$$

The *singular support* of a hyperfunction is defined in the same way as for a distribution (Definition 4.96), replacing the condition of *infinitely differentiable* by that of *analytic*. Hyperfunctions can be multiplied whenever the conditions for multiplying distributions stated in section 4.4.1**(VI)** are met.

(IV) HYPERFUNCTIONS WITH COMPACT SUPPORT Let $\mathfrak{u}$ be a non-empty open subset of $\mathbb{R}$ and suppose that $(K_j)_{j\geq 0}$ is a sequence of compact subsets of $\mathbb{R}$ such that $\bigcup_{j\geq 0} K_j = \mathfrak{u}$. Then, in **Vec**,

$$\mathcal{A}(\mathfrak{u}) = \varprojlim_j \mathcal{A}(K_j).$$

Equip $\mathcal{A}(\mathfrak{u})$ with the strict projective limit topology of the sequence of spaces $\mathcal{A}(K_j)$ (which are Silva spaces by Corollary 4.85). The space of hyperfunctions with compact support contained in $\mathfrak{u}$ is the countable strict inductive limit of the following $(\mathcal{FS})$-spaces:

$$\mathcal{B}_c[\mathfrak{u}] := \varinjlim_j \mathcal{B}[K_j].$$

COROLLARY 4.105.– *There exists a canonical isomorphism of* $\mathbf{Tvs}$ $\mathcal{B}_c[\mathfrak{u}] \cong \mathcal{A}'(\mathfrak{u})$.

PROOF.– This follows from Corollary 4.101 and Theorem 3.133. ■

THEOREM 4.106.– *There exists a linear injection* $\iota : \mathcal{E}'(\mathfrak{u}) \hookrightarrow \mathcal{B}_c[\mathfrak{u}]$ *that preserves support.*

PROOF.– Note that $\mathcal{A}(\mathfrak{u}) \subset \mathcal{E}(\mathfrak{u})$. Let $T \in \mathcal{E}'(\mathfrak{u}) : \varphi \mapsto \langle T, \varphi \rangle$ $(\varphi \in \mathcal{E}(\mathfrak{u}))$ and $\iota(T) : \varphi \mapsto \langle T, \varphi \rangle$ $(\varphi \in \mathcal{A}(\mathfrak{u}))$. We have that $\iota(T) \in \mathcal{B}_c[\mathfrak{u}]$, and, if $\iota(T) = 0$, or in other words if $\langle T, \varphi \rangle = 0, \forall \varphi \in \mathcal{A}(\mathfrak{u})$, then $\langle T, \varphi \rangle = 0, \forall \varphi \in \mathcal{E}(\mathfrak{u})$, since $\mathcal{A}(\mathfrak{u})$ is dense in $\mathcal{E}(\mathfrak{u})$ by a theorem shown by Whitney ([WHI 34], Lemma 5, p. 74). Therefore, the linear mapping ι is injective. For the property of preservation of support, cf. ([MOR 93], Proposition 9.2.1). ■

It can be shown that every hyperfunction on $\mathbb{R}$ is the sum of a locally finite series of hyperfunctions with compact support. Therefore, by Theorem 4.95(2):

THEOREM 4.107.– *There exists an injection* $\mathcal{D}'(\mathbb{R}) \hookrightarrow \mathcal{B}[\mathbb{R}]$ *that preserves support.*

(V) HYPERFUNCTIONS WITH SUPPORT AT THE ORIGIN The space $\mathcal{B}[\{0\}]$ of hyperfunctions with support contained in $\{0\}$ may be identified with $\mathcal{H}'_b(\{0\})$. The duality pairing $\langle -, - \rangle : \mathcal{B}[\{0\}] \times \mathcal{H}(\{0\})$ is defined in the same way as in (4.20), for example choosing U to be the disk $|z| < r$ of radius $r > 0$ in the complex plane.

We will now generalize our earlier remarks about distributions with support at the origin. The Dirac hyperfunction is defined as the linear form $\delta : \mathcal{H}(\{0\}) \to \mathbb{C} : \varphi \mapsto \varphi(0)$. Cauchy's integral formula (4.10) implies that

$$\langle \delta, \varphi \rangle = \varphi(0) = -\frac{1}{2\pi i} \oint_{c_r(0)} \frac{\varphi(z)}{z} dz,$$

so $\delta = \left[-\frac{1}{2\pi i z}\right]$. Similarly, by (4.11),

$$\left\langle \delta^{(n)}, \varphi \right\rangle = (-1)^n \varphi^{(n)}(0) = \frac{(-1)^{n+1} n!}{2\pi i} \oint_{c_r(0)} \frac{\varphi(z)}{z^{n+1}} dz,$$

so

$$\boxed{\delta^{(n)}(x) = \left[\frac{(-1)^{n+1} n!}{2\pi i} \frac{1}{z^{n+1}}\right]_{z=x}}.$$

Every function $f_1 \in \mathcal{H}(\mathbb{C} - \{0\})$ has a Laurent series expansion $f_1(z) = \sum_{n=-\infty}^{+\infty} a_n z^{-n}$ (section 4.3.2(**V**)) that belongs to the same equivalence class $(\mathrm{mod}\, .\mathcal{H}(\mathbb{C}))$ as $f(z) = \sum_{n=0}^{+\infty} a_n z^{-(n+1)}$. By the Cauchy-Hadamard theorem, the radius of convergence R of this power series in z^{-1} (in other words, the real number $R \geq 0$ for which $f(z)$ converges absolutely for every z such that $|z| \geq R$) is

$$R = \lim \sup_{n \to +\infty} \sqrt[n]{|a_n|}$$

(**exercise**)[10]. Moreover, $f \in \mathcal{H}(\mathbb{C} - \{0\})$ if and only if $R = 0$. We have that

$$[f] = \sum_{n=0}^{+\infty} b_n \left[(-1)^{n+1} n! \frac{1}{z^{n+1}}\right] = \sum_{n=0}^{+\infty} b_n \delta^{(n)},$$

where $a_n = b_n (-1)^{n+1} n!$, which implies that:

THEOREM 4.108.– *The vector space of hyperfunctions with support in $\{0\}$ is*

$$\boxed{\mathcal{B}[\{0\}] = \left\{ \sum_{n=0}^{+\infty} b_n \delta^{(n)} : \lim \sup_{n \to +\infty} \sqrt[n]{|b_n| \, n!} = 0 \right\}}.$$

REMARK 4.109.– *By Theorem 4.95(1), $\sum\limits_{n=0}^{+\infty} b_n \delta^{(n)}$ is a distribution if and only if all but finitely many of the b_n are zero. Therefore, $\mathcal{D}'(\mathbb{R}) \subsetneq \mathcal{B}[\mathbb{R}]$.*

10 See the Wikipedia article on the Cauchy-Hadamard *theorem*.

(VI) OTHER HYPERFUNCTIONS The Heaviside hyperfunction (cf. Example 4.4(4)) is defined by

$$\Upsilon(x) := \left[-\frac{1}{2\pi i}\ln(-z)\right],$$

which implies that $\frac{d}{dx}\Upsilon = \delta$.

The *sign* hyperfunction is given by

$$\operatorname{sgn}(x) := \left[\frac{1}{2\pi i}\left(\ln(z) + \ln(-z)\right)\right].$$

Other examples can be found in [GRA 10], [IMA 92].

REMARK 4.110.– *By definition,* $\ln(z) = \zeta \Leftrightarrow z = e^{\zeta} = e^{\zeta + i2\pi k}$ $(k \in \mathbb{Z})$, *so* $\ln(z)$ *is only determined* $\bmod .i2\pi$. *This problem of the multiplicity in the determination of the logarithm can only be properly resolved by introducing the notion of a Riemann surface (cf. [P3]).*

(VII) EMBEDDING OF THE SPACE OF ANALYTIC FUNCTIONS IN THE SPACE OF HYPERFUNCTIONS Given any function $f \in \mathcal{A}(\mathfrak{u})$, let $\tilde{f}$ be the extension of f to the complex neighborhood Ω of $\mathfrak{u}$. The mapping $f \mapsto \left[\varepsilon\tilde{f}\right]$ is well-defined and injective from $\mathcal{A}(\mathfrak{u})$ into $\mathcal{B}(\mathfrak{u})$, and therefore gives an embedding of the former space into the latter.

5

Sheaves

5.1. Introduction

Sheaf theory can be viewed as the area of geometry that studies the mathematical transition from local properties to global properties. The cohomology of a sheaf tells us about any obstacles that may be preventing us from making this transition. For example, let B be a connected open subset of $\mathbb{C}^n$. A *meromorphic* complex function on B is a complex function h defined on $B - S$, where S is a set of isolated points in B, satisfying the following local property for every $b \in B - S$: there exists an open neighborhood U of b in B and holomorphic complex functions f_U, g_U on U (section 4.3.2**(II)**) such that $h|_U = \frac{f_U}{g_U}$. This is a direct generalization of Lemma-Definition 4.59. Poincaré showed in 1883 that every meromorphic function h on B is of the form $h = \frac{f}{g}$, where f, g are two functions that are holomorphic on B (global property) when $B = \mathbb{C}^2$ [POI 83]. This result had previously been shown by Weierstrass in the case where $B = \mathbb{C}$ (Corollary 233). One of the key questions studied by the theory of functions in multiple complex variables is which types of open subset B of $\mathbb{C}^n$ still allow this transition from the local to the global property. Any such open sets B are known as *solutions of the* Poincaré *problem* (we shall denote this problem by **(P)** below).

Derived categories (developed by Grothendieck and J.-L. Verdier) ultimately proved to be the most suitable framework for sheaf cohomology, but we sadly do not have the space to present them here, as explained in the preface of [P1]. Readers can however find a comprehensive presentation in [GEL 03] (see also [KAS 90], [KAS 06]). Sheaf theory plays essential roles in algebraic geometry, the theory of functions in multiple complex variables (analytic geometry), differential geometry, the theory of differential equations (microlocal analysis), etc.

The notion of sheaf was introduced by J. Leray shortly after the end of the Second World War, contemporaneously with the notion of spectral sequence. It was later expanded by H. Cartan (between 1947 and 1953), who defined the notion of a cohomology taking values in a sheaf in order to study functions in multiple complex variables, and then by J.P. Serre, in his articles on coherent algebraic sheaves and analytic sheaves [SER 55], [SER 56][1]. A. Grothendieck made an important breakthrough in sheaf theory by generalizing the results established by Serre, then later by integrating sheaf cohomology into the more general context of the homological algebra of abelian categories ([P1], section 3.3.7) in a famous article published in 1957 [GRO 57]. These results were summarized and systemized in a book by R. Godement [GOD 58], the first ever to discuss this topic, and which remains an essential reference to this day; in particular, Godement introduced the notions of *flabby sheaves* and *soft sheaves*. Sheaf theory was widely used after 1959 by M. Sato in his theory of hyperfunctions (section 4.4.2) and, as mentioned earlier, in the field of microlocal analysis that he founded.

5.2. General results about sheaves

5.2.1. *Presheaves*

(I) Let B be a topological space and suppose that U is an open subset of B. Let $\mathfrak{C}(U)$ be the space of continuous complex functions on U. For every open set $V \subset U$, let $\rho_V^U(\mathfrak{C}(U)) := \mathfrak{C}(V)$ be the space of continuous complex functions on V, where $\rho_V^U : \mathfrak{C}(U) \to \mathfrak{C}(V)$ is the restriction morphism. If $W \subset V \subset U$, then $\rho_W^U = \rho_W^V \circ \rho_V^U$. Let $\mathfrak{Top}_B$ be the category whose objects are the open subsets of B and whose morphisms are defined by: $\mathrm{Hom}(V,U) = \varnothing$ if $V \nsubseteq U$, $\mathrm{Hom}(V,U) = V \hookrightarrow U$ (canonical injection) if $V \subset U$. Suppose further that $\mathfrak{C}_B$ is the category whose objects are the continuous complex functions on an open subset of B and whose morphisms

1 Interested readers can find a presentation of the early history of sheaf theory written by C. Houzel in the introduction of [KAS 90].

are the ρ_V^U defined as above when $V \subset U$ and by $\rho_V^U = \varnothing$ when $V \not\subseteq U$. Then the functor $\mathcal{F} : \mathfrak{Top}_B \to \mathfrak{C}_B$

$$U \mapsto \mathfrak{C}(U), \quad \mathrm{Hom}(V, U) \longmapsto \rho_V^U$$

is a contravariant functor, known as the presheaf of continuous functions on an open set of B.

(II) If B is a locally compact topological space, we can establish the same definitions for Radon measures on B by duality (section 4.1.5**(I)**), as well as for infinitely differentiable functions (with or without compact support), and, by duality, for distributions (with or without compact support) when B is an open subset of $\mathbb{R}^n$ (sections 4.3.1 and 4.4.1), for holomorphic functions when B is an open subset of $\mathbb{C}^n$ (section 4.3.2**(II)**), for germs of analytic functions when B is a subset of $\mathbb{C}^n$ (section 4.3.2**(IV)**), for hyperfunctions on the real line (with or without compact support) (section 4.4.2), and for regular functions taking values in an algebraically closed field $\mathbf{k}$ when B is an algebraic set over $\mathbf{k}$ ([P1], section 3.2.7**(III)**).

(III) FORMALIZATION

DEFINITION 5.1.– *Let B be a topological space and suppose that $\boldsymbol{C}$ is a category. A* presheaf *$\mathcal{F}$ with base B taking values in $\boldsymbol{C}$ is a rule which specifies*

– an object $\mathcal{F}(U) \in \boldsymbol{C}$ for every open subset U of B, called the object of sections *of $\mathcal{F}$ over U,*

– a morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ for every open subset $V \subset U$, called the restriction morphism *from U to V, such that, for every inclusion of open sets $W \subset V \subset U$, the relation $\rho_W^U = \rho_W^V \circ \rho_V^U$ holds.*

We say that $\mathcal{F}(B)$ is the object of global sections.

Any presheaf $\mathcal{F}$ on B taking values in $\boldsymbol{C}$ is a contravariant functor $\mathcal{F} : \mathfrak{Top}_B \to \boldsymbol{C} : U \mapsto \mathcal{F}(U), \mathrm{Hom}(V, U) \mapsto \rho_V^U$. We write this functor as $\left(\mathcal{F}(U), \rho_V^U\right)$.

Furthermore, we write $\Gamma(U, \mathcal{F})$ for the object $\mathcal{F}(U)$. If $\boldsymbol{C}$ is a concrete category with base $\mathbf{Set}$ ([P1], section 1.3.1), then $\Gamma(U, \mathcal{F})$ is a set, and every element s of $\Gamma(U,\mathcal{F})$ is called a section of $\mathcal{F}$ over U; if $V \subset U$, we write that $\rho_V^U(\mathbf{s}) = \mathbf{s}|_V$.

DEFINITION 5.2.– *Let $\mathcal{F},\mathcal{G}$ be two presheaves with base B taking values in a category $\boldsymbol{C}$. A* morphism of presheaves *$f : \mathcal{F} \to \mathcal{G}$ is a rule specifying a family of $\boldsymbol{C}$-morphisms $\{f_U : \Gamma(U,\mathcal{F}) \to \Gamma(U,\mathcal{G}) : U \in \mathfrak{Top}_B\}$ such that, for $V \subset U$, the following diagram commutes:*

$$\begin{array}{ccc} \Gamma(U,\mathcal{F}) & \xrightarrow{f_U} & \Gamma(U,\mathcal{G}) \\ \rho_V^U \downarrow & & \downarrow \rho_V^U \\ \Gamma(V,\mathcal{F}) & \xrightarrow{f_V} & \Gamma(V,\mathcal{G}) \end{array} .$$

This determines the category $\mathfrak{PSh}_{B,\boldsymbol{C}}$ of presheaves with base B taking values in $\boldsymbol{C}$. If $\boldsymbol{C}$ is a concrete category with base **Set***, then the commutativity of the above diagram is equivalent to the condition $f_V(\mathbf{s}|_V) = f_U(\mathbf{s})|_V$ for $V \subset U$ and every section $\mathbf{s} \in \Gamma(U,\mathcal{F})$. We write $\mathrm{Hom}_B(\mathcal{F},\mathcal{G})$ for the set of morphisms of presheaves $f : \mathcal{F} \to \mathcal{G}$ on B.*

Let $B' \subset B$ be an open set and suppose that $\mathcal{F}$ is a presheaf with base B taking values in a category $\boldsymbol{C}$. By replacing each open set U of B by $U \cap B'$, we obtain a presheaf $\mathcal{F}|B' : U \cap B' \mapsto \Gamma(U \cap B',\mathcal{F})$ with base B' taking values in $\boldsymbol{C}$, said to be *induced* by $\mathcal{F}$ on B' (or called the *restriction* of $\mathcal{F}$ to B'). If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, we write $f|B' : \mathcal{F}|B' \to \mathcal{G}|B'$ for the morphism of presheaves defined by the $f_U, U \subset B'$.

Let B, B' be topological spaces, $\mathcal{F}$ a presheaf on B taking values in a category $\boldsymbol{C}$, and suppose that $\psi : B \to B'$ is a continuous mapping. Then $\psi_*(\mathcal{F}) : U \mapsto \mathcal{F}(\psi^{-1}(U))$, $\rho_V^U : \mathcal{F}(\psi^{-1}(U)) \mapsto \mathcal{F}(\psi^{-1}(V))$ if $V \subset U$, is a presheaf on B' taking values in $\boldsymbol{C}$, called the *direct image* of $\mathcal{F}$ under ψ. It can easily be checked that

$$\psi_*(.) : \mathcal{F} \mapsto \psi_*(\mathcal{F}),\ \psi_*(f) = \left\{f_{\psi^{-1}(U)} : \mathcal{F}\left(\psi^{-1}(U)\right) \mapsto \mathcal{G}\left(\psi^{-1}(U)\right)\right\}$$

is a covariant functor from the category of presheaves with base B taking values in $\boldsymbol{C}$ into the category of presheaves with base B' taking values in $\boldsymbol{C}$. If $\mathcal{F}$ is a presheaf with base B and $\mathcal{G}$ is a presheaf with base B', both taking values in $\boldsymbol{C}$, then the morphism $u : \mathcal{G} \to \psi_*(\mathcal{F})$ is said to be a *ψ-morphism*.

The notion of the germ of a mapping (section 2.3.13) may be extended to presheaves:

DEFINITION 5.3.– *Let $\boldsymbol{C}$ be category that admits inductive limits ([P1], section 1.2.8). Let $b \in B$ and write $\mathfrak{V}(b)$ for the filter of neighborhoods of b; $\mathfrak{V}(b)$ is a directed set for the ordering relation $V \succeq U \Leftrightarrow V \subset U$, which leads us to consider the direct system $\mathfrak{D} = \left\{U, \iota_V^U; \mathfrak{V}(b)\right\}$.*

1) The stalk *of $\mathcal{F}$ at the point $b \in B$ is defined as the object $\mathcal{F}_b = \varinjlim_{U \in \mathfrak{V}(b)} \Gamma(U, \mathcal{F}) \in \boldsymbol{C}$. Given a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$, for all $b \in B$, we define the* morphism of stalks *$f_b : \mathcal{F}_b \to \mathcal{G}_b$ by $f_b = \varinjlim_{U \in \mathfrak{V}(b)} f_U$.*

2) If $\boldsymbol{C}$ is a concrete category with base $\mathbf{Set}$*, then the canonical image (under $\varphi_U : \Gamma(U, \mathcal{F}) \to \mathcal{F}_b$: cf. [P1], section 1.2.8(**I**), Definition 1.20) of the section $\mathrm{s} \in \Gamma(U, \mathcal{F})$ in $\mathcal{F}_b$ is written s_b and is called the* germ *of* s *at the point b.*

(IV) PRESHEAVES OF ABELIAN GROUPS AND RINGS A presheaf $\mathcal{F}$ of abelian groups with base B (where B is a topological space) is a presheaf that takes values in the category $\mathbf{Ab}$ of abelian groups. For every open set $U \in \mathfrak{Top}_B$, $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ is a group, and, for every $V \subset U$, ρ_V^U is a morphism of groups. We can similarly define presheaves of (not necessarily commutative) rings taking values in $\mathbf{Rng}$.

5.2.2. *Sheaves*

(I) Consider again the example from section 5.2.1(**I**). Let B be a topological space. The function $f \in \mathfrak{C}(B)$ is fully specified whenever we know its restriction $f|_U$ to every open subset U of B. Moreover, suppose that U and V are two open subsets of B, and that $f \in \mathfrak{C}(U)$ and $g \in \mathfrak{C}(V)$ are two functions that are identical on $U \cap V$. We can uniquely define a function h such that $h(b) = f(b)$ if $b \in U$ and $h = g(b)$ if $b \in V$, and $h \in \mathfrak{C}(U \cup V)$. We can "glue together" any two given continuous functions like this because continuity is a local property, which is why we say that the presheaf of continuous functions on B is a *sheaf*. The same property holds in every example listed in section 5.2.1(**II**), for the same reasons.

(II) SHEAVES OF SETS A presheaf of sets with base B is a presheaf on B taking values in $\mathbf{Set}$. We will now formalize the remarks made in **(I)** (**[GOD 58]**, Chapter II, section 1.1):

DEFINITION 5.4.– *Let $\mathcal{F}$ be a presheaf of sets with base B. We say that $\mathcal{F}$ is a* sheaf of sets *if the following conditions are satisfied, where $(U_i)_{i \in I}$ is an arbitrary family of open subsets of B, writing U for their union:*

$(\mathbf{F}_1)$ *Let $\mathrm{s}', \mathrm{s}''$ be two sections in $\Gamma(U, \mathcal{F})$; if $\mathrm{s}'|_{U_i} = \mathrm{s}''|_{U_i}$ for every $i \in I$, then $\mathrm{s}' = \mathrm{s}''$.*

$(\mathbf{F}_2)$ *Given $\mathrm{s}_i \in \Gamma(U_i, \mathcal{F})$, if $\mathrm{s}_i\big|_{U_i \cap U_j} = \mathrm{s}_j\big|_{U_i \cap U_j}$ for every pair of indices i, j, then there exists a section $\mathrm{s} \in \Gamma(U, \mathcal{F})$ such that $\mathrm{s}|_{U_i} = \mathrm{s}_i$ for all $i \in I$.*

Condition $(\mathbf{F}_1)$ implies that the section s of $(\mathbf{F}_2)$ is uniquely determined. Conditions $(\mathbf{F}_1), (\mathbf{F}_2)$ are equivalent to saying that the following diagram is exact ([P1], section 1.1.1**(III)**):

$$\Gamma(U, \mathcal{F}) \xrightarrow{\varepsilon} \prod\nolimits_{i \in I} \Gamma(U_i, \mathcal{F}) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod\nolimits_{(i,j) \in I \times I} \Gamma(U_i \cap U_j, \mathcal{F}),$$

where $\alpha((\mathrm{s}_i)_{i \in I}) = \left(\rho^{U_i}_{U_i \cap U_j}(\mathrm{s}_i)\right)_{(i,j) \in I \times I}$, $\beta((\mathrm{s}_i)_{i \in I}) = \left(\rho^{U_j}_{U_i \cap U_j}(\mathrm{s}_j)\right)_{(i,j) \in I \times I}$, and $\varepsilon = \mathrm{eq}(\alpha, \beta) = \left(\rho^{U}_{U_i}\right)_{i \in I}$ is the *equalizer* of (α, β).

If $\mathcal{F}$ is a presheaf with base B, note that $\rho^U_{\varnothing}$ is a mapping from $\Gamma(U, \mathcal{F})$ into $\Gamma(\varnothing, \mathcal{F})$, so $\Gamma(\varnothing, \mathcal{F})$ is non-empty except in the trivial case where $\Gamma(U, \mathcal{F}) = \varnothing$ for every open subset U of B. Given a sheaf $\mathcal{F}$, the condition $(\mathbf{F}_2)$ with $I = \varnothing$ implies that $\Gamma(\varnothing, \mathcal{F})$ is a singleton (except in the trivial case).

(III) SHEAVES TAKING VALUES IN A CATEGORY Let B be a topological space and suppose that $\boldsymbol{C}$ is a category. We set the following definition ([GRO 71], Chapter 0, (3.1.1)):

DEFINITION 5.5.– *We say that a presheaf $\mathcal{F}$ on B taking values in $\boldsymbol{C}$ is a sheaf if it satisfies the following condition:*

$(\mathbf{F})$ *For every object $\mathbf{T}$ of $\boldsymbol{C}$, $U \mapsto \mathrm{Hom}_{\boldsymbol{C}}(\mathbf{T}, \Gamma(U, \mathcal{F}))$ is a sheaf of sets.*

If $\boldsymbol{C}$ is a concrete category with base $\mathbf{Set}$ and $\mathfrak{O} : \boldsymbol{C} \to \mathbf{Set}$ is the forgetful functor ([P1], section 1.3.1), then the condition $(\mathbf{F})$ means that

$\mathfrak{O} \circ \mathcal{F}$ is a sheaf of sets with base B and, for every open subset U of B and every open covering (U_i) of U, the structure of $\mathcal{F}(U)$ is initial for the family of mappings $\left(\mathfrak{O}\left(\rho_{U_i}^U\right) : \Gamma(U, \mathfrak{O} \circ \mathcal{F}) \to \Gamma(U_i, \mathfrak{O} \circ \mathcal{F})\right)_{i \in I}$ ([P1], section 1.3.2). We write $\mathfrak{Sh}_{B,\boldsymbol{C}}$ for the category of sheaves on B taking values in $\boldsymbol{C}$, whose morphisms are the morphisms of presheaves (Definition 5.2). Specifying a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is equivalent to specifying restrictions $f_U : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ of f to U for every $U \in \mathfrak{B}$, where $\mathfrak{B}$ is a base of the topology of B. In the following, whenever it is not ambiguous to do so, we will write $\mathfrak{Psh}$ and $\mathfrak{Sh}$ for $\mathfrak{Psh}_{B,\boldsymbol{C}}$ and $\mathfrak{Sh}_{B,\boldsymbol{C}}$ respectively.

If $\mathcal{F}$ is a sheaf with base B taking values in $\boldsymbol{C}$, then, for every open subset B' of B, the presheaf $\mathcal{F}|B'$ induced by the presheaf $\mathcal{F}$ on B' is a sheaf. For each $B' \subset B$, the morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ induces a morphism of sheaves $f|B' : \mathcal{F}|B' \to \mathcal{G}|B'$, which must not be confused with f_U when $B' = U$ (if $\boldsymbol{C} = \mathbf{Set}$, f_U is a *mapping* frrom $\Gamma(U, \mathcal{F})$ into $\Gamma(U, \mathcal{G})$ whereas $f|U$ is the *family* of mappings $f_V, V \subset U$).

Let $\mathcal{F}, \mathcal{G}$ be sheaves on B taking values in a category $\boldsymbol{C}$ that admits inductive limits, and suppose that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. The general notions of monomorphism and epimorphism in a category ([P1], section 1.1.1(**III**)) imply that:

LEMMA 5.6.– *The morphism f is a monomorphism (resp. an epimorphism) if and only if $f_b : \mathcal{F}_b \to \mathcal{G}_b$ is a monomorphism (resp. an epimorphism) of $\boldsymbol{C}$ for every $b \in B$ (**exercise**).*

DEFINITION 5.7.– *The category $\boldsymbol{C}$ is said to be of type $\boldsymbol{CSh}_0$ if (i) it is concrete with base* $\mathbf{Set}$*; (ii) it admits inductive limits; (iii) every bijective morphism of $\boldsymbol{C}$ is an isomorphism.*

The category $\mathbf{Ab}$ (of abelian groups) and the category $\mathbf{Rng}$ (of rings) are both of type $\mathbf{CSh}_0$. The following result is immediate ([GRO 71], Chapter 0, (3.1.3)):

LEMMA 5.8.– *Let $\boldsymbol{C}$ be a category of type $\boldsymbol{CSh}_0$ and suppose that $\mathcal{F}$ is a presheaf with base B taking values in $\boldsymbol{C}$. If the presheaf $U \mapsto \Gamma(U, \mathcal{F})$ is a sheaf of* sets*, then $\mathcal{F}$ is a sheaf with base B taking values in $\boldsymbol{C}$.*

(IV) SHEAF ASSOCIATED WITH A PRESHEAF

LEMMA 5.9.– *Let $\mathcal{F}$ be a presheaf with base B taking values in a category $\boldsymbol{C}$ of type $\boldsymbol{CSh}_0$. There exists a sheaf $\tilde{\mathcal{F}}$ and a morphism of presheaves $f : \mathcal{F} \to \tilde{\mathcal{F}}$ with the following universal property ([P1], section 1.2.4): for every sheaf $\mathcal{G}$ on B taking values in $\boldsymbol{C}$ and every morphism of presheaves $g : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\tilde{g} : \tilde{\mathcal{F}} \to \mathcal{G}$ for which the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F} \arrow[d, "f"'] \arrow[dr, "g"] & \\ \tilde{\mathcal{F}} \arrow[r, dashed, "\tilde{g}"] & \mathcal{G} \end{tikzcd} .$$

PROOF.– By Lemma 5.8, we simply need to consider the case where $\boldsymbol{C} = \mathbf{Set}$. For every open subset U of B, let $\tilde{\mathcal{F}}(U)$ be the subset of the sum $\biguplus_{b\in U} \mathcal{F}_b$ ([P1], section 1.2.6(**II**)) formed by the sections $\mathrm{s} \in \Gamma(U, \mathcal{F})$ satisfying (i) $\mathrm{s}_b \in \mathcal{F}_b$ for every $b \in U$; (ii) there exists an open neighborhood $V \subset U$ of b together with a section $\mathrm{s}' \in \Gamma(V, \mathcal{F})$ such that $\mathrm{s}_{b'} = \mathrm{s}'_{b'}$ for every $b' \in V$. Then $U \mapsto \tilde{\mathcal{F}}(U)$ is the required sheaf. ■

By construction, $f_b : \mathcal{F}_b \to \tilde{\mathcal{F}}_b$ is a canonical $\boldsymbol{C}$-isomorphism. Therefore, after identification,

$$\boxed{\tilde{\mathcal{F}}_b = \varinjlim_{U\in\mathfrak{V}(b)} \Gamma(U, \mathcal{F}) = \mathcal{F}_b}. \qquad [5.1]$$

(Definition 5.3). For every sheaf $\mathcal{G}$ on B taking values in $\boldsymbol{C}$,

$$\boxed{\mathrm{Hom}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}\left(\tilde{\mathcal{F}}, \mathcal{G}\right) \cong \mathrm{Hom}_{\mathfrak{PSh}_{B,\boldsymbol{C}}}(\mathcal{F}, \mathcal{G})}, \qquad [5.2]$$

or in other words the “sheafification functor”

$$\tilde{\Gamma} : \mathfrak{PSh}_{B,\boldsymbol{C}} \to \mathfrak{Sh}_{B,\boldsymbol{C}} : \mathcal{F} \mapsto \tilde{\mathcal{F}}, g \mapsto \tilde{g}$$

is left adjoint to the covariant “section functor” $\Gamma : \mathfrak{Sh}_{B,\boldsymbol{C}} \hookrightarrow \mathfrak{PSh}_{B,\boldsymbol{C}}$ defined by

$$\Gamma : \mathcal{F} \mapsto [U \mapsto \Gamma(U, \mathcal{F})], [g : \mathcal{F}_1 \to \mathcal{F}_2] \mapsto \Gamma(g) = (g_U)_{U\in\mathfrak{Top}_B}.$$

REMARK 5.10.– *Let $\boldsymbol{C}$ be a category of type $\boldsymbol{CSh}_0$. The proof of Lemma 5.9 shows that specifying a sheaf $\mathcal{F}$ is equivalent to specifying its stalks $\mathcal{F}_b$ $(b \in B)$, and thus the germs s_b of its sections* s. *This explains why the sheaf of continuous (resp. infinitely differentiable, resp. holomorphic, etc.) functions is commonly called the sheaf of* germs *of continuous (resp. infinitely differentiable, resp. holomorphic, etc.) functions: cf. section 5.2.1(**I,II**). The same is true for the sheaf of distributions, of hyperfunctions, etc.*

More generally, let $\mathcal{F}$ be a presheaf on B taking values in a category $\boldsymbol{C}$. Then $\mathcal{G} \mapsto \mathrm{Hom}_{\mathfrak{PSh}}(\mathcal{F}, \mathcal{G})$ is a covariant functor of the category $\mathfrak{Sh}_{B,\boldsymbol{C}}$ in **Set**. Once again, (5.2) holds if and only if the functor $\mathcal{G} \mapsto \mathrm{Hom}_{\mathfrak{PSh}}(\mathcal{F}, \mathcal{G})$ is *representable* ([P1], section 1.2.5(**I**)) by the sheaf $\tilde{\mathcal{F}}$, in which case this sheaf is unique up to isomorphism.

DEFINITION 5.11.– *Let $\mathcal{F}$ be a presheaf with base B taking values in a category $\boldsymbol{C}$. If the functor $\mathcal{G} \mapsto \mathrm{Hom}_{\mathfrak{PSh}}(\mathcal{F}, \mathcal{G})$ is representable by a sheaf $\tilde{\mathcal{F}}$, this sheaf is said to be* associated *with the presheaf $\mathcal{F}$ and (since it is uniquely determined up to functor isomorphism) is written $\tilde{\Gamma}(\mathcal{F})$. A category $\boldsymbol{C}$ is said to be of type **CSh** if it admits inductive limits and, for every presheaf $\mathcal{F}$ taking values in $\boldsymbol{C}$, there exists a sheaf associated with $\tilde{\mathcal{F}}$.*

From Lemma 5.9, it follows that:

COROLLARY 5.12.– *Every category of type $\boldsymbol{CSh}_0$ is of type **CSh**. If $\boldsymbol{C}$ is a category of type **CSh**, then (5.1) holds.*

THEOREM 5.13.– *Let I be a set. If the category $\boldsymbol{C}$ is of type **CSh** and every inverse system indexed by any finite set of indices and by I admits a projective limit in $\boldsymbol{C}$, then the functor $\tilde{\Gamma}$ is exact and commutes with arbitrary inductive limits, and the functor Γ is left exact and commutes with projective limits indexed by I. If I is* finite, *given an inverse system $\mathfrak{I} = \left\{\mathcal{F}_i, \psi_i^j; I\right\}$ in $\mathfrak{Sh}_{B,\boldsymbol{C}}$, we have that*

$$\boxed{\left(\varprojlim_{i \in I} \mathcal{F}_i\right)_b = \varprojlim_{i \in I} (\mathcal{F}_i)_b, \quad \forall b \in B}. \tag{5.3}$$

PROOF.– Since the functor Γ is right adjoint to the functor $\tilde{\Gamma}$, it commutes with projective limits indexed by I and is therefore left exact, whereas $\tilde{\Gamma}$

commutes with arbitrary inductive limits and is therefore right exact ([P1], section 1.2.9). It remains to be shown that $\tilde{\Gamma}$ is left exact. For every $b \in B$, we have that $\left(\varprojlim_{i\in I} \mathcal{F}_i\right)_b = \varinjlim_{U\in\mathfrak{V}(b)} \left(\varprojlim_{i\in I} \mathcal{F}_i\right)$ and, *since I is finite*, this is equal to $\varprojlim_{i\in I} \left(\varinjlim_{U\in\mathfrak{V}(b)} \mathcal{F}_i\right)$ ([MAC 98], Chapter IX, section 2, Theorem 1). ■

COROLLARY 5.14.– *If $\boldsymbol{C}$ is a category of type* ***CSh*** *and $f : \mathcal{F} \to \mathcal{G}$ is a monomorphism, then $\Gamma(f)$ is a monomorphism. In other words, for every open subset U of B, $f|_U$ is a $\boldsymbol{C}$-monomorphism.*

REMARK 5.15.– *No equivalent result exists for epimorphisms ([GOD 58], p. 115), which is precisely why we need the theory of sheaf cohomology.*

Let B, B' be topological spaces, $\mathcal{F}$ a *presheaf* with base B' taking values in a category $\boldsymbol{C}$, and $\psi : B \to B'$ a continuous mapping. If the category $\boldsymbol{C}$ is of type **CSh**, then the sheaf associated with the presheaf $U \mapsto \varinjlim_{V\supset\psi(U)} \Gamma(V, \mathcal{F})$ is a sheaf with base B, said to be the *inverse image* of $\mathcal{F}$ under ψ, and denoted $\psi^{-1}(\mathcal{F})$. The following functor isomorphism holds in $\mathcal{G}$:

$$\mathrm{Hom}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}\left(\psi^{-1}(\mathcal{F}), \mathcal{G}\right) \tilde{\to} \mathrm{Hom}_{\mathfrak{PSh}_{B',\boldsymbol{C}}}(\mathcal{F}, \psi_*(\mathcal{G})),$$

which implies that $\mathcal{F} \mapsto \psi^{-1}(\mathcal{F})$ is the left adjoint functor of the functor $\mathcal{G} \mapsto \psi_*(\mathcal{G})$ ([P1], section 1.2.9(**II**)). If $\psi = 1_B$, then clearly $\psi^{-1}(\mathcal{F}) = \tilde{\mathcal{F}}$. If B' is a subspace of B, $\psi : B' \hookrightarrow B$ is inclusion, and $\mathcal{F}$ is a *sheaf* with base B, then $\psi^{-1}(\mathcal{F})$ is the sheaf $\mathcal{F}|B'$ induced by $\mathcal{F}$ on B' (also known as the *restriction* of $\mathcal{F}$ to B'); the functor $\mathcal{F} \mapsto \mathcal{F}|B'$ is exact (**exercise**).

Suppose that $\boldsymbol{C}$ admits finite projective limits. Since the functor $\mathcal{G} \mapsto \psi_*(\mathcal{G})$ has a left adjoint, it commutes with finite projective limits and is therefore left exact.

5.2.3. *Fibered spaces and étalé spaces*

(I) FIBERED SPACES The notion of fibered space was introduced by E. Cartan. Let X, B be two topological spaces and suppose that $p : X \to B$ is a continuous mapping. The triple $\lambda = (X, B, p)$ is called a *fibered space with* *base* B ([GRO 58], section 1.1) (or B-space ([BKI 16], Chapter. I, section 1.1), or a cleaved space with base B ([GOD 58], Chapter II, section 1.1)), and p is said to be the *projection* of λ. For every element $b \in B$, the subset $X_b := p^{-1}(\{b\})$ of X is called the *fiber* of λ over b.

Let U be an open subset of B. We know that the projection p admits a *section* if and only if p is a surjection ([P1], section 1.1.1**(III)**). Without assuming that this condition holds, we define a *continuous section* of p *over an open subset* U of B to be a continuous mapping $\mathbf{s} : U \to B$ such that $p \circ \mathbf{s} = \iota_U$, where $\iota_U : B \hookrightarrow U$ is the canonical injection, which therefore makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{p} & B \\ & \nwarrow \mathbf{s} & \downarrow \iota_U \\ & & U \end{array} .$$

We have that $\mathbf{s}(b) \in X_b$ for every $b \in B$. For every open subset U of B, let $\mathcal{S}(U;p)$ be the set of continuous sections of p over U. If $V \subset U$ is an open set, then we define the restriction $\rho_V^U : \mathcal{S}(U;p) \to \mathcal{S}(V;p) : \mathbf{s} \mapsto \mathbf{s}|_V$. It can easily be checked (**exercise**) that:

LEMMA-DEFINITION 5.16.– $\left(\mathcal{S}(U;p), \rho_V^U\right)$ *is a sheaf, called the* sheaf of continuous sections *of* p.

A *morphism of fibered spaces* $f : (X,B,p) \to (X',B,p')$ is by definition a continuous mapping $f : X \to X'$ for which the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & \overset{p}{\searrow} & \downarrow p' \\ & & B \end{array}$$

and so the fibered spaces form a category.

(II) ÉTALÉ SPACES Let X, B be topological spaces. A continuous mapping $p : X \to B$ is said to be (topologically) *étale* if p is a local homeomorphism, i.e. every point $x \in X$ has an open neighborhood W such that p induces a homeomorphism from W onto an open neighborhood U of $p(x)$ in B.

DEFINITION 5.17.– *A fibered space* $\lambda = (X,B,p)$ *whose projection* p *is étale is said to be an* étalé space *in* B.

The fibered space λ is sometimes written (X, p), or even X when the base and projection are clear from context. The category of étalé spaces is a full subcategory of the category of fibered spaces. The sheaf of continuous sections of p is $\left(\mathcal{S}(U; p), \rho_V^U\right) : U \mapsto \Gamma(U, \mathcal{S})$ (Lemma-Definition 5.16).

LEMMA 5.18.– *Let $\lambda = (X, B, p)$ be an étalé space.*

*i) The projection p is open (section 2.3.4(**III**)).*

ii) For every open subset U of B and every continuous section $\mathrm{s} \in \Gamma(U, \mathcal{S})$, $\mathrm{s}(U)$ is open in X.

iii) For every $x \in X$, there exists a continuous section $\mathrm{s} \in \Gamma(U, \mathcal{S})$, where U is an open neighborhood of $p(x)$, such that $\mathrm{s}(b) = x$; the family $\{\mathrm{s}(U) : \mathrm{s} \in \Gamma(U, \mathcal{S}), U \in \mathfrak{Top}_B\}$ is a base for the topology of X (section 2.1.2).

iv) If $\mathrm{s} \in \Gamma(U, \mathcal{S}), \mathrm{s}' \in \Gamma(U', \mathcal{S})$, where U, U' are neighborhoods of $b \in B$ and $\mathrm{s}(b) = \mathrm{s}'(b)$, then s and s' are identical on every neighborhood $U'' \subset U \cap U'$ of b.

PROOF.– (i): Let W be a non-empty open subset of X, $b \in p(W)$ and suppose that $x \in W$ satisfies $p(x) = b$. Then x has an open neighborhood $W' \subset W$ in X such that $p(W')$ is open in B. Consequently, b has an open neighborhood $p(W') \subset p(W)$. Therefore, $p(W)$ is a neighborhood for each of its points, so is open (section 2.1.2). (ii): Every $x \in \mathrm{s}(U)$ has an open neighborhood W in X such that $p|_W$ is a homeomorphism onto an open subset V of B. Therefore, $p|_W (W \cap \mathrm{s}(U)) = V \cap U$ is an open neighborhood of x that is contained in $\mathrm{s}(U)$; consequently, $\mathrm{s}(U)$ is a neighborhood for each of its points, and so is open in X. (iii) follows from (ii) and from the definition of an étale mapping. (iv): Let U, U' be neighborhoods of $b \in B$ and $W = \mathrm{s}(U) \cap \mathrm{s}'(U')$. Then $p|_W \circ \mathrm{s} = \iota_{U \cap U'} = p|_W \circ \mathrm{s}'$, and $p|_W$ is bijective from W onto $U \cap U'$, so $\mathrm{s}|_{U''} = \mathrm{s}'|_{U''}$ for every open neighborhood U'' of b such that $U'' \subset U \cap U'$. ■

THEOREM 5.19.– *Consider the direct system $\mathfrak{D} = \left\{\Gamma(U, \mathcal{S}), \rho_V^U; \mathfrak{V}(b)\right\}$ (cf. Definition 5.3). The relation $p^{-1}(\{b\}) = \varinjlim_{U \in \mathfrak{V}(b)} \Gamma(U, \mathcal{S})$ holds between stalks, and the topology of the subspace $p^{-1}(\{b\})$ of B is the discrete topology.*

PROOF.– For every $b \in B$ and every $U \in \mathfrak{V}(b)$, we have the mapping

$$\Gamma(U, \mathcal{S}) \overset{\rho_U}{\to} p^{-1}(\{b\}) : \mathbf{s} \mapsto \mathbf{s}(b)$$

and $\rho_V = \rho_U \circ \rho_V^U$ if $V \subset U$. For every set E, every $U \in \mathfrak{V}(b)$, and every mapping $f|_U : \Gamma(U, \mathcal{S}) \to E$, let $f : p^{-1}(\{b\}) \to E : \mathbf{s}(b) \mapsto f(\mathbf{s}(b))$. Then $f|_U = f \circ \rho_U$, which implies the expression for $p^{-1}(\{b\})$ by ([P1], section 1.2.8(**I**)). Every $x \in p^{-1}(\{b\})$ has an open neighborhood W in X such that $p|_W : W \to U$ is a homeomorphism, with $U \in \mathfrak{V}(b)$. Therefore, $p^{-1}(\{b\}) \cap W = \{x\}$ is an open neighborhood of x in $p^{-1}(\{b\})$, which is therefore a discrete topological space (section 2.3.1(**III**)). ■

(III) ÉTALÉ SPACE ATTACHED TO A SHEAF Given a presheaf of sets $\mathcal{F}$ with base B, consider the sheaf $\tilde{\mathcal{F}}$ associated with $\mathcal{F}$ (Definition 5.11). Let $b \in B$; then the equality (5.1) holds, and, by the definition of the inductive limit, for every open set $U \in \mathfrak{V}(b)$, there is a canonical mapping $\rho_U : \Gamma(U, \mathcal{F}) \to \tilde{\mathcal{F}}_b$. Write $\tilde{\mathbf{s}}(b) \in \tilde{\mathcal{F}}_b$ for the image of $\mathbf{s}$ under ρ_U; every section $\mathbf{s} \in \Gamma(U, \mathcal{F})$ uniquely determines a mapping $\tilde{\mathbf{s}} : U \to \tilde{\mathcal{F}} : b \mapsto \tilde{\mathbf{s}}(b)$. Let $p : \tilde{\mathcal{F}} \to B : \tilde{\mathcal{F}}_b \mapsto b$; then $(p \circ \tilde{\mathbf{s}})(b) = b$ for all $b \in U$. If V, U are nonempty open subsets of B such that $V \subset U$, then the restriction $\mathbf{s}|_V$ corresponds to the restriction $\tilde{\mathbf{s}}|_V$ via the above construction (**exercise**). The base of the finest topology $\tilde{\mathfrak{T}}$ on $\tilde{\mathcal{F}}$ for which all of the mappings $\tilde{\mathbf{s}}$ $(\mathbf{s} \in \mathcal{F}(U))$ are continuous is the base formed by the $\tilde{\mathbf{s}}(V)$ such that $\mathbf{s} \in \mathcal{F}(V)$ and $V \subset U$ (section 2.1.2). Given any open subset U of B,

$$p^{-1}(U) = \bigcup_{\mathbf{s} \in \mathcal{F}(V), V \subset U} \tilde{\mathbf{s}}(V).$$

This set is open when $\tilde{\mathcal{F}}$ is equipped with the topology $\tilde{\mathfrak{T}}$, in which case the mapping p is continuous (Lemma 2.28). Therefore, $\left(\tilde{\mathcal{F}}, B, p\right)$ (where $\mathcal{F}$ is equipped with the topology $\tilde{\mathfrak{T}}$) is an étalé space, and we have the following result ([GOD 58], Theorem 1.2.1):

THEOREM 5.20.– *Every sheaf of sets with base B is isomorphic to the sheaf of continuous sections of an étalé space in B, which itself is unique up to isomorphism (of fibered spaces).*

This result explains why some authors (see for example [SER 55], [SER 56]) view a sheaf of sets as a *topological space*. However, this perspective was abandoned in [GRO 71].

5.2.4. *Examples*

(I) Let B be a topological space and suppose that X is a set. For any open subset U of B, write $\mathfrak{F}(U;X)$ for the set of mappings from U into X as in section 5.2.1**(I)**, and, for any pair (U,V) of open subsets of B such that $V \subset U$, let $\rho_V^U : \mathfrak{F}(U;X) \to \mathfrak{F}(V;X)$ be the restriction mapping $f \longmapsto f|_V$. Then $\left(\mathfrak{F}(U;X), \rho_V^U\right)$ is a sheaf on B, called the sheaf on B of mappings with values in X, denoted $\underline{\mathfrak{F}}(B;X)$.

(II) Suppose now that X is a topological space; let $\mathfrak{C}(U;X)$ be the set of continuous mappings from $U \in \mathfrak{Top}_B$ into X. Then $\underline{\mathfrak{C}}(B;X) := \left(\mathfrak{C}(U;X), \rho_V^U\right)$ is a subsheaf of $\underline{\mathfrak{F}}(B;X)$, known as the sheaf on B of continuous mappings with values in X.

(III) Simple sheaf Let X be a set and suppose that B is a topological space. Consider the presheaf $\mathcal{F}$ defined by $\Gamma(U,\mathcal{F}) = X$ for every open subset U of B, whose restriction morphisms are taken to be $\rho_V^U = 1_X$ for any nonempty open subsets U,V of B such that $V \subset U$. For every $b \in B$, we have that $\mathcal{F}_b = X$, and $\mathcal{F}$ is called the *constant presheaf* with base B and stalk X. Every section $\mathrm{s} \in \Gamma(U,\mathcal{F})$ is a constant function $U \to X : b \mapsto x$. The proof of Lemma 5.9 shows that the sheaf $\tilde{\mathcal{F}}$ associated with $\mathcal{F}$ is $\biguplus_{b\in X} \mathcal{F}_b = B \times X$. Now, consider $\tilde{\mathcal{F}}$ as an étalé space on B (section 5.2.3**(III)**) and write p for its projection. By (5.1) and Theorem 5.19, the fiber of $\left(\tilde{\mathcal{F}}, p\right)$ at each point $b \in B$ is $p^{-1}(\{b\}) = X$, and p is the canonical projection $B \times X \twoheadrightarrow B$. The sections $\tilde{\mathfrak{s}} : U \to U \times X$ are of the form $b \mapsto (b,x)$. The base of the finest topology $\tilde{\mathfrak{T}}$ on $\tilde{\mathcal{F}}$ for which these sections are continuous is the set of $U \times \{x\}$ $(x \in X)$, and is therefore obtained by equipping X with the discrete topology. Therefore, $\tilde{\mathcal{F}} = \underline{\mathfrak{C}}(B;X)$, where X is equipped with the discrete topology. The sections $\tilde{\mathfrak{s}} \in \Gamma\left(U,\tilde{\mathcal{F}}\right)$ are the *locally constant* functions from U into X (in other words, constant on every *connected component* of U). In general:

Definition 5.21.– *We say that a presheaf $\mathcal{F}$ with base B taking values in a category $\boldsymbol{C}$ is* constant *if the restriction morphisms ρ_V^U $(V \subset U)$ are $\boldsymbol{C}$-isomorphisms. We say that a sheaf is a* simple sheaf *if it is isomorphic to the* *sheaf associated with a constant presheaf. We say that a sheaf $\mathcal{F}$ is* locally simple[2] *if every $b \in B$ has an open neighborhood B' such that $\mathcal{F}\,|_{B'}$ is simple.*

If B is irreducible ([P1], section 3.2.6**(I)**) and $\mathcal{F}$ is a presheaf with base B, then the following conditions are equivalent (**exercise***: cf. [GRO 71], Chapter 0, (3.6.2)): (a) the presheaf $\mathcal{F}$ is constant; (b) $\mathcal{F}$ is a simple sheaf; (c) $\mathcal{F}$ is a locally simple sheaf.

Let $\boldsymbol{C}$ be a category of type **CSh**; the simple sheaf $\mathcal{F}$ with base B and stalk $\mathbf{T} \in \boldsymbol{C}$ may be identified with $\mathbf{T}$.

5.2.5. *Sheaves of germs of homomorphisms*

Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves with base B taking values in a category $\boldsymbol{C}$. We already defined the morphisms from $\mathcal{F}$ into $\mathcal{G}$ earlier (section 5.2.2**(III)**), namely the elements $f : \mathcal{F} \to \mathcal{G}$ that are morphisms of presheaves; we write $f \in \mathrm{Hom}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}(\mathcal{F}, \mathcal{G})$, or $f \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ whenever this is not ambiguous.

For each $U \in \mathfrak{Top}_B$, consider the morphisms of sheaves $\mathcal{F}\,|U$ and $\mathcal{G}\,|U$ induced by $\mathcal{F}$ and $\mathcal{G}$ on U (section 5.2.2**(III)**,**(IV)**). Every morphism $f \in \mathrm{Hom}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}(\mathcal{F}, \mathcal{G})$ determines a morphism $f\,|U : \mathcal{F}\,|U \to \mathcal{G}\,|U$, $f\,|U \in \mathrm{Hom}_{\mathfrak{Sh}_{U,\boldsymbol{C}}}(\mathcal{F}\,|U, \mathcal{G}\,|U)$. If $V \subset U$, then $\mathrm{Hom}_{\mathfrak{Sh}_{U,\boldsymbol{C}}}(\mathcal{F}\,|U, \mathcal{G}\,|U) \to \mathrm{Hom}_{\mathfrak{Sh}_{V,\boldsymbol{C}}}(\mathcal{F}\,|V, \mathcal{G}\,|V)$ is a canonical mapping; this mapping satisfies the usual compatibility conditions, which implies that $U \mapsto \mathrm{Hom}_{\mathfrak{Sh}_{U,\boldsymbol{C}}}(\mathcal{F}\,|U, \mathcal{G}\,|U)$ is a presheaf, and furthermore a sheaf of sets with base B (**exercise**). This sheaf is called the *sheaf of germs of homomorphisms* from $\mathcal{F}$ into $\mathcal{G}$ and is written $\mathfrak{Hom}(\mathcal{F}, \mathcal{G})$. By definition, for every $U \in \mathfrak{Top}_B$,

$$\Gamma(U, \mathfrak{Hom}(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{\mathfrak{Sh}_{U,\boldsymbol{C}}}(\mathcal{F}\,|U, \mathcal{G}\,|U).$$

Let $b \in B$ and $f_b \in \mathfrak{Hom}(\mathcal{F}, \mathcal{G})_b$. Then f_b is the germ at b of a morphism $f\,|U : \mathcal{F}\,|U \to \mathcal{G}\,|U$ $(U \in \mathfrak{V}(b))$, which induces a $\boldsymbol{C}$-morphism $\mathcal{F}_b \to \mathcal{G}_b$; we therefore have the canonical mapping

$$\boxed{\mathfrak{Hom}(\mathcal{F}, \mathcal{G})_b \to \mathrm{Hom}_{\boldsymbol{C}}(\mathcal{F}_b, \mathcal{G}_b)}. \qquad [5.4]$$

2 Some authors use the term *constant sheaf* for what we have defined as a simple sheaf, and the term *locally constant sheaf* for what we have defined as a locally simple sheaf ([BKI 16], Chapter I, section 4.8).

In general, this mapping is neither injective nor surjective (cf. below, Theorem 5.29).

5.2.6. *Projective limits of sheaves*

(I) PROJECTIVE LIMIT Let $\mathfrak{I} = \left\{\mathcal{F}_i, \psi_i^j; I\right\}$ be an inverse system in $\mathfrak{Sh}_{B,\boldsymbol{C}}$, where the category $\boldsymbol{C}$ is of type **CSh** and admits projective limits indexed by I. We know that the presheaf $U \mapsto \varprojlim_{i\in I}\mathcal{F}_i(U)$ is a sheaf (Theorem 5.13), which we write $\varprojlim_{i\in I}\mathcal{F}_i$. Furthermore, if I is *finite*, then the relation (5.3) holds.

(II) PRODUCT OF SHEAVES By ([P1], section 1.2.8**(II),** Proposition 1.22), Theorem 5.13 implies the following result: Let $\boldsymbol{C}$ be a category that admits products and suppose that $(\mathcal{F}_i)_{i\in I}$ is a family of sheaves with base B taking values in $\boldsymbol{C}$. The presheaf $U \mapsto \prod_{i\in I}\Gamma(U, \mathcal{F}_i)$ is a sheaf, denoted $\mathcal{F} = \prod_{i\in I}\mathcal{F}_i$, and is the product of the family $(\mathcal{F}_i)_{i\in I}$ in the category $\mathfrak{Sh}_{B,\boldsymbol{C}}$ ([P1], section 1.2.6**(I)**).

For every $b \in B$, we have a *canonical monomorphism* $\mathcal{F}_b \hookrightarrow \prod_{i\in I}(\mathcal{F}_i)_b$ (**exercise**) that *is not* an epimorphism in general when I is infinite; indeed, if so, suppose that the $\mathcal{F}_i$ are sheaves of sets. Given a germ $(\mathcal{F}_i)_b$ for each $i \in I$, we cannot simultaneously extend every germ to a neighborhood of b in B in general. This shows that the hypothesis that I is finite in Theorem 5.13 is essential in general in order to ensure that (5.3) holds.

(III) SUBSHEAF AND EQUALIZER Let B be a topological space and suppose that $\mathcal{F} = \left(\mathcal{F}(U), \rho_V^U\right)$ is a presheaf (resp. sheaf) with base B taking values in a category $\boldsymbol{C}$. For every $U \in \mathfrak{Top}_B$, let $\mathcal{L}(U)$ be a subobject of $\mathcal{F}(U)$ in $\boldsymbol{C}$ ([P1], section 1.1.1**(III)**). If $\rho_V^U(\mathcal{L}(U))$ is a subobject of $\mathcal{L}(V)$ in $\boldsymbol{C}$ for every pair (U, V) of open subsets of B such that $V \subset U$, then $\mathcal{L} = \left(\mathcal{L}(U), \bar{\rho}_V^U\right)$, where $\bar{\rho}_V^U$ is the morphism induced by ρ_V^U, is a presheaf (resp. sheaf) taking values in $\boldsymbol{C}$, called a *subpresheaf* (resp. *subsheaf*) of $\mathcal{F}$, and is a subobject of $\mathcal{F}$ in $\mathfrak{Psh}_{B,\mathbf{C}}$ (resp. $\mathfrak{Sh}_{B,\boldsymbol{C}}$).

Suppose that $\boldsymbol{C}$ admits finite projective limits, and therefore equalizers by ([P1], section 1.2.8, Proposition 1.22). Let $\mathcal{F}, \mathcal{G}$ be sheaves with base B taking values in $\boldsymbol{C}$, and let $\mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G}$ be a double arrow in $\mathfrak{Sh}_{B,C}$. Suppose further that $(\mathcal{K}(U), \mathrm{eq}(f_U, g_U))$ is an equalizer of $\mathcal{F}(U) \underset{g_U}{\overset{f_U}{\rightrightarrows}} \mathcal{G}(U)$ in $\boldsymbol{C}$ for each $U \in \mathfrak{Top}_B$; then $U \to (\mathcal{K}(U), \mathrm{eq}(f_U, g_U))$ is an equalizer $\mathrm{eq}(f, g)$ of $\mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G}$ in $\mathfrak{Sh}_{B,C}$; in other words, the diagram $\mathcal{K} \overset{\mathrm{eq}(f,g)}{\longrightarrow} \mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G}$ is exact in this category and $\mathrm{eq}(f, g)_U = \mathrm{eq}(f_U, g_U)$. If $\boldsymbol{C}$ is also of type **CSh**, then Theorem 5.13 implies that:

COROLLARY 5.22.– *In* $\boldsymbol{C}$*, the diagram* $\mathcal{K}_b \overset{\mathrm{eq}(f,g)_b}{\to} \mathcal{F} \underset{g_b}{\overset{f_b}{\rightrightarrows}} \mathcal{G}_b$ *is exact for all* $b \in B$*, with* $\mathrm{eq}(f, g)_b = \mathrm{eq}(f_b, g_b)$.

PROOF.– Equalizers are finite projective limits. ■

5.2.7. *Inductive limits of sheaves*

(I) INDUCTIVE LIMIT Let $\mathfrak{D} = \left\{\mathcal{F}_i, \varphi_j^i; I\right\}$ be a direct system in $\mathfrak{PSh}_{B,\boldsymbol{C}}$, where the category $\boldsymbol{C}$ is of type **CSh**. Suppose that $\mathcal{F}_i$ is a sheaf for all i. The presheaf $U \mapsto \varinjlim \Gamma(U, \mathcal{F}_i)$ $(U \in \mathfrak{Top}_B)$ *is not a sheaf in general*. However, by the interchangeability property of inductive limits ([MAC 98], Chapter IX, section 2), for every $b \in B$, we have that

$$\boxed{\left(\varinjlim \mathcal{F}_i\right)_b = \varinjlim (\mathcal{F}_i)_b}.$$

With the hypotheses stated above, this implies:

THEOREM 5.23.– $\mathfrak{Sh}_{B,\boldsymbol{C}}$ *admits inductive limits and* $\varinjlim \mathcal{F}_i$ *is the sheaf associated with the presheaf* $U \mapsto \varinjlim \Gamma(U, \mathcal{F}_i)$ $(U \in \mathfrak{Top}_B)$.

(II) QUOTIENT SHEAF AND COEQUALIZER Let $\boldsymbol{C}$ be a category of type $\mathbf{CSh}_0$ and suppose that $\mathcal{F}, \mathcal{G}$ are sheaves with base B taking values in $\boldsymbol{C}$. Suppose that $\mathcal{G}_b$ is a quotient object of $\mathcal{F}_b$ for every $b \in B$ ([P1], section 1.1.1**(III)**). For every open subset U of B, let $\tilde{\mathcal{G}}(U)$ be the object of $\boldsymbol{C}$ constructed in the proof of Lemma 5.11. Then $\tilde{\mathcal{G}}$ is a quotient object of $\mathcal{F}$ in $\mathfrak{Sh}_{B,\boldsymbol{C}}$.

Now, let $\boldsymbol{C}$ be a category of type **CSh**. This category admits coequalizers by ([P1], section 1.2.8, Proposition 1.22). Let $\mathcal{F}, \mathcal{G}$ be sheaves with base B taking values in $\boldsymbol{C}$, and let $\mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G}$ be a double arrow in $\mathfrak{Sh}_{B,\boldsymbol{C}}$. Suppose further that, for each $U \in \mathfrak{Top}_B$, $(\mathcal{PC}(U), \pi\gamma_U = \mathrm{coeq}_{\boldsymbol{C}}(f_U, g_U))$ is a coequalizer of $\mathcal{F}(U) \underset{g_U}{\overset{f_U}{\rightrightarrows}} \mathcal{G}(U)$ in $\boldsymbol{C}$, and consider the presheaf $\mathcal{PC} : U \to (\mathcal{C}(U), \pi\gamma_U)$. Then the sheaf $\mathcal{C} = \widetilde{\mathcal{PC}}$ associated with $\mathcal{PC}$ is a coequalizer of $\mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G}$ in $\mathfrak{Sh}_{B,\boldsymbol{C}}$; in other words, the diagram $\mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G} \xrightarrow{\gamma} \mathcal{C}$, $\gamma = \mathrm{coeq}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}(f, g) = \widetilde{\pi\gamma}$, is exact in this category. In $\boldsymbol{C}$, this yields the exact diagram $\mathcal{F} \underset{g_b}{\overset{f_b}{\rightrightarrows}} \mathcal{G}_b \xrightarrow{\gamma_b} \mathcal{C}_b$ for every $b \in B$, where $\mathcal{C}_b = \mathcal{PC}_b$, $\gamma_b = \pi\gamma_b$. This and Corollary 5.22 imply that:

THEOREM 5.24.– *Let* $\boldsymbol{C}$ *be category of type* ***CSh*** *that admits finite projective limits,* $\mathcal{F}$ *and* $\mathcal{G}$ *sheaves with base* B *taking values in* $\boldsymbol{C}$*, and* $\mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G}$ *a double arrow in* $\mathfrak{Sh}_{B,C}$*. Then the following implications and equivalences hold:*

$$\boxed{\begin{array}{c}
\mathcal{K} \xrightarrow{\mathrm{eq}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}(f,g)} \mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G} \xrightarrow{\mathrm{coeq}_{\mathfrak{Sh}_{B,\boldsymbol{C}}}(f,g)} \mathcal{C} \\
\Updownarrow \\
\mathcal{K}_b \xrightarrow{\mathrm{eq}_{\boldsymbol{C}}(f_b,g_b)} \mathcal{F}_b \underset{g_b}{\overset{f_b}{\rightrightarrows}} \mathcal{G} \xrightarrow{\mathrm{coeq}_{\boldsymbol{C}}(f_b,g_b)} \mathcal{C}_b, \quad \forall b \in B \\
\Uparrow \\
\mathcal{K} \xrightarrow{\mathrm{eq}_{\mathfrak{PSh}_{B,\boldsymbol{C}}}(f,g)} \mathcal{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{G} \xrightarrow{\mathrm{coeq}_{\mathfrak{PSh}_{B,\boldsymbol{C}}}(f,g)} \mathcal{PC} \\
\Updownarrow \\
\mathcal{K}(U) \xrightarrow{\mathrm{eq}_{\boldsymbol{C}}(f_U,g_U)} \mathcal{F}(U) \underset{g_U}{\overset{f_U}{\rightrightarrows}} \mathcal{G}(U) \xrightarrow{\mathrm{coeq}_{\boldsymbol{C}}(f_U,g_U)} \mathcal{PC}(U), \quad \forall U \in \mathfrak{Top}_B.
\end{array}}$$

(IV) COPRODUCT OF SHEAVES Coproducts are a special case of inductive limits. If $(\mathcal{F}_i)_{i \in I}$ is a family of sheaves with base B taking values in a category $\boldsymbol{C}$ that admits coproducts, then the presheaf $U \mapsto \coprod_{i \in I} \Gamma(U, \mathcal{F}_i)$ is not a sheaf in general; the coproduct $\coprod_{i \in I} \mathcal{F}_i$ in $\mathfrak{Sh}_{B,\boldsymbol{C}}$ is the sheaf associated with this presheaf. If $\boldsymbol{C}$ is of type **CSh**, then $\left(\coprod_{i \in I} \mathcal{F}_i\right)_b = \coprod_{i \in I} (\mathcal{F}_i)_b$ for all $b \in B$.

5.2.8. *Sheaves taking values in an abelian category*

If B is a topological space and $\boldsymbol{C}$ is an abelian category of type **CSh**, then the results from sections 5.2.6 and 5.2.7 show that $\mathfrak{Sh}_{B,C}$ is an abelian category ([P1], section 3.3.7**(IV)**). If $f : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathfrak{Sh}_{B,C}$, then its kernel $\ker(f)$ is the sheaf $U \mapsto \ker(f_U)$, and its cokernel is the sheaf associated with the presheaf $U \mapsto \operatorname{coker}(f_U)$. By definition ([P1], section 3.3.7**(I)**), $\operatorname{im}(f) = \ker(\operatorname{coker}(f))$, so $\operatorname{im}(f)$ is the sheaf associated with the presheaf $U \mapsto \operatorname{im}(f_U)$. By Theorem 5.24, any short sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{\varphi} \mathcal{C} \to 0$$

is exact in $\mathfrak{Sh}_{B,C}$ if and only if, for every $b \in B$, the short sequence

$$0 \to \mathcal{F}_b \xrightarrow{f_b} \mathcal{G}_b \xrightarrow{\varphi_b} \mathcal{C}_b \to 0$$

is exact in $\boldsymbol{C}$. If so, the sheaf $\mathcal{C}$ is the quotient $\mathcal{F}/\mathcal{G}$, whose stalks are the $\mathcal{F}_b/\mathcal{G}_b$ $(b \in B)$.

These remarks naturally also apply in the case where $\boldsymbol{C}$ is the category of abelian groups, as well as in the case studied in the next section.

5.3. Sheaves of Modules

5.3.1. *Modules over a ringed space*

(I) Ringed space Following in the footsteps of H. Cartan, we define a *ringed space* (resp. *commutatively ringed space*) to be a pair $(B, \mathcal{A})$ formed by a topological space B and a sheaf of rings (resp. sheaf of commutative rings) $\mathcal{A}$ with base B. We say that $\mathcal{A}$ is the *structure sheaf* of the ringed space $(B, \mathcal{A})$. The stalk of $\mathcal{A}$ at the point $b \in B$ is denoted $\mathcal{A}_b$. We say that the sheaf $\mathcal{A}$ is entire (resp. reduced, resp. regular, resp. normal) at the point b if the stalk $\mathcal{A}_b$ is an entire (resp. reduced, resp. regular, resp. entire and integrally closed) ring ([P1], sections 2.3.1**(I)**, 2.3.6, 3.3.5**(II)**, 3.2.5**(I)**). We say that the ringed space $(B, \mathcal{A})$ is reduced (resp. regular, resp. normal) if $\mathcal{A}_b$ is reduced (resp. regular, resp. entire and integrally closed) for every $b \in B$. Given two ringed spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a morphism $(X, \mathcal{A}) \to (Y, \mathcal{B})$ is the pair (ψ, u) formed by a continuous mapping $\psi : X \to Y$ and a ψ-morphism $u : \mathcal{B} \to \mathcal{A}$ (section 5.2.1). Ringed spaces and their morphisms form a *category*.

(II) SHEAF OF $\mathcal{A}$-MODULES Let B be a topological space and suppose that $\mathcal{A}$ is a sheaf of rings with base B. We define a left $\mathcal{A}$-Module (always with an uppercase M) to be a sheaf of sets $\mathcal{G}$ with base B equipped with the following structure: for every open subset U of B, we specify a left module structure over the ring $\mathcal{A}(U) = \Gamma(U, \mathcal{A})$ on $\mathcal{G}(U) = \Gamma(U, \mathcal{G})$ such that the restriction mapping $\rho_V^U : \mathcal{G}(U) \to \mathcal{G}(V)$ $(V \subset U)$ is a morphism of modules that is compatible with the canonical morphism of rings $\mathcal{A}(U) \to \mathcal{A}(V)$.

Constructing the sheaf products $\mathcal{A} \times \mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$ (section 5.2.6**(II)**) on B is equivalent to specifying morphisms $\mathcal{A} \times \mathcal{G} \to \mathcal{G}$ and $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ of sheaves of *sets* such that the resulting mappings $\mathcal{A}_b \times \mathcal{G}_b \to \mathcal{G}_b$ and $\mathcal{G}_b \times \mathcal{G}_b \to \mathcal{G}_b$ define a left $\mathcal{A}_b$-module structure on $\mathcal{G}_b$ for every $b \in B$. The category of left (resp. right) $\mathcal{A}$-Modules is denoted ${}_{\mathcal{A}}\mathfrak{Mod}$ (resp. $\mathfrak{Mod}_{\mathcal{A}}$); a morphism of sheaves of $\mathcal{A}$-Modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that the mapping $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{A}(U)$-linear for every open subset U of B. We write $\bigoplus_{i \in I} \mathcal{G}_i$ for the coproduct of a family $\mathcal{G}_i$ of $\mathcal{A}$-Modules and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ for the abelian group of morphisms from $\mathcal{F}$ into $\mathcal{G}$.

LEMMA 5.25.– *Let U be an open subset of B, $\mathcal{G}$ a sheaf of $\mathcal{A}$-Modules on B, and $\mathcal{A}_U$ the extension of the restriction of $\mathcal{A}$ to U by 0. There exists a canonical isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_U, \mathcal{G}) \cong \Gamma(U, \mathcal{G})$, and in particular a canonical isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{G}) \cong \Gamma(B, \mathcal{G})$ (**exercise**).*

LEMMA 5.26.– *(Grothendieck ([GRO 57], p. 134, Example)): For every $b \in B$ and every $U \in \mathfrak{V}(b)$ (where $\mathfrak{V}(b)$ is the set of neighborhoods of B ordered by inclusion), let $\underline{\mathcal{A}_{b,U}}$ be the zero sheaf over $\complement_B U$ and identical to $\mathcal{A}_b$ over U. Then $\bigoplus_{b \in B, U \in \mathfrak{V}(b)} \underline{\mathcal{A}_{b,U}}$ is a generator ([P1], section 1.2.11) of ${}_{\mathcal{A}}\mathfrak{Mod}$.*

This lemma allows us to establish the following fundamental result:

THEOREM 5.27.– *(Grothendieck ([GRO 57], Proposition 3.1.1 and its Corollary)) The category ${}_{\mathcal{A}}\mathfrak{Mod}$ is abelian, admits arbitrary products, and has sufficiently many injectives ([P1], section 1.2.10**(II)**).*

The details of applying the general notions studied in sections 5.2.6 and 5.2.7 to the special case of the category ${}_{\mathcal{A}}\mathfrak{Mod}$ are left to the reader.

Let $Y \subset B$. We write $\mathcal{A}_Y$ (resp. $\mathcal{G}_Y$) for the sheaf whose stalks $\mathcal{A}_{x,Y}$ (resp. $\mathcal{G}_{x,Y}$) are equal to $\mathcal{A}_x$ (resp. $\mathcal{G}_x$) when $x \in Y$ and equal to 0 when $x \in \complement_B Y$.

(III) Tensor product of $\mathcal{A}$-Modules Let $\mathcal{A}$ be a sheaf of rings with base B, $\mathcal{G}$ a left $\mathcal{A}$-Module, and $\mathcal{D}$ a right $\mathcal{A}$-Module. Then the tensor product $\mathcal{G} \bigotimes_{\mathcal{A}} \mathcal{D}$ is the sheaf of abelian groups whose stalk at any arbitrary point $b \in B$ is the abelian group $\mathcal{G}_b \bigotimes_{\mathcal{A}_b} \mathcal{D}_b$. Formulating the notion of $\mathcal{A}$-multi-Module and its consequences for tensor products ([P1], section 3.1.5**(I)**) is left to the reader.

Let ${}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}}$ be a $(\mathcal{A}, \mathcal{B})$-bi-Module, ${}_{\mathcal{B}}\mathcal{G}$ a left $\mathcal{B}$-Module, and ${}_{\mathcal{A}}\mathcal{K}$ a left $\mathcal{A}$-Module.

THEOREM 5.28.– *There exists an isomorphism of sheaves of abelian groups*

$$\mathrm{Hom}_{\mathcal{A}}\left(\mathcal{F} \bigotimes\nolimits_{\mathcal{B}} \mathcal{G}, \mathcal{K}\right) \cong \mathrm{Hom}_{\mathcal{B}}\left(\mathcal{G}, \mathfrak{Hom}_{\mathcal{A}}\left(\mathcal{F}, \mathcal{K}\right)\right), \qquad [5.5]$$

which implies the functor isomorphism

$$\mathrm{Hom}_{\mathcal{A}}\left(\mathcal{F} \bigotimes\nolimits_{\mathcal{B}} -, -\right) \cong \mathrm{Hom}_{\mathcal{B}}\left(-, \mathfrak{Hom}_{\mathcal{A}}\left(\mathcal{F}, -\right)\right).$$

In other words, the following pairs of functors are adjoint: (a) $\left(\mathcal{F} \bigotimes_{\mathcal{B}} -, \mathfrak{Hom}_{\mathcal{A}}\left(\mathcal{F}, -\right)\right)$, (b) $\left(- \bigotimes_{\mathcal{A}} \mathcal{F}, \mathfrak{Hom}_{\mathcal{B}}\left(\mathcal{F}, -\right)\right)$. Consequently, the functors $\mathcal{F} \bigotimes_{\mathcal{B}} -$ and $- \bigotimes_{\mathcal{A}} \mathcal{F}$ are right exact (the former is covariant, the latter is contravariant), and the functors $\mathfrak{Hom}_{\mathcal{A}}\left(\mathcal{F}, -\right)$ and $\mathfrak{Hom}_{\mathcal{B}}\left(\mathcal{F}, -\right)$ are left exact (the former is covariant, the latter is contravariant).

PROOF.– Each element of the left-hand side of (5.5) maps every open subset U of B to an $\mathcal{A}\left(U\right)$-morphism

$$\mathcal{F}\left(U\right) \bigotimes\nolimits_{\mathcal{B}(U)} \mathcal{G}\left(U\right) \to \mathcal{K}\left(U\right)$$

that commutes with the restriction morphisms. By the adjoint isomorphism theorem for modules over a ring ([P1], section 3.1.5**(II)**, Theorem 3.19), this morphism induces a $\mathcal{B}\left(U\right)$-morphism

$$\mathcal{G}\left(U\right) \to \mathrm{Hom}_{\mathcal{A}(U)}\left(\mathcal{F}\left(U\right), \mathcal{K}\left(U\right)\right)$$

that holds for every open subset U of B, which implies that there exists a morphism $\mathcal{G} \to \mathfrak{Hom}_{\mathcal{A}}\left(\mathcal{F}, \mathcal{K}\right)$. A similar argument works in the other direction, which gives the isomorphism (5.5). ■

5.3.2. *Coherent sheaves*

The ideas studied in this section are "local versions" of notions classically applied to modules over a ring ([P1], Chapter 3), which may be recovered by considering simple sheaves. The notion of coherent sheaf was introduced by H. Cartan in 1950.

(I) FREE $\mathcal{A}$-MODULE, $\mathcal{A}$-MODULE GENERATED BY ITS SECTIONS Let $\mathcal{A}$ be a sheaf of rings with base B. An $\mathcal{A}$-Module $\mathcal{F}$ is said to be *free* if it is isomorphic to a copower $\mathcal{A}^{(I)}$ of $\mathcal{A}$, and is said to be *locally free* if, for every $b \in B$, there exists an open neighborhood U of b such that $\mathcal{F}\,|U \cong \mathcal{A}^{(I)}\,|U$, where $\mathcal{A}^{(I)} := \coprod_{i \in I} \mathcal{B}_i$ with $\mathcal{B}_i = \mathcal{A}$ for all i (and where I depends on U).

Let $(\mathbf{s}_i)_{i \in I}$ be a family of sections of $\mathcal{F}$; we say that $\mathcal{F}$ is generated by the family $(\mathbf{s}_i)_{i \in I}$ if, for all $b \in B$, the $\mathcal{A}_b$-module $\mathcal{F}_b$ is generated by the family $\left((\mathbf{s}_i)_b\right)$. If so, we have an epimorphism

$$u : \mathcal{A}^{(I)} \twoheadrightarrow \mathcal{F} : (a_i) \in \Gamma\left(U, \mathcal{A}\right)^{(I)} \mapsto \sum\nolimits_{i \in I} a_i \left(\mathbf{s}_i\right)_U,$$

and therefore an exact sequence

$$\mathcal{A}^{(I)} \to \mathcal{F} \to 0,$$

and $\mathcal{F}$ is a quotient of $\mathcal{A}^{(I)}$ in ${}_{\mathcal{A}}\mathfrak{Mod}$. In this case, we say that $\mathcal{F}$ is *generated by its sections* over B. Unlike modules over a ring, an $\mathcal{A}$-Module $\mathcal{F}$ can have a point $b \in B$ for which $\mathcal{F}\,|U$ is not generated by its sections over U for any open neighborhood U of b ([GRO 71], Chapter 0, (5.1.1)); therefore, ${}_{\mathcal{A}}\mathcal{A}$ is not a generator of ${}_{\mathcal{A}}\mathfrak{Mod}$ (compare with Lemma 5.26).

(II) QUASI-COHERENT $\mathcal{A}$-MODULE, FINITELY GENERATED $\mathcal{A}$-MODULE We say that the $\mathcal{A}$-Module $\mathcal{F}$ is *quasi-coherent* if, for every $b \in B$, there exists an open neighborhood U of b and an exact sequence in ${}_{\mathcal{A}|U}\mathfrak{Mod}$

$$\mathcal{A}^{(I)}\,|U \xrightarrow{f|U} \mathcal{A}^{(J)}\,|U \to \mathcal{F}\,|U \to 0,$$

or, in other words, $\mathcal{F}\,|U \cong \operatorname{coker}_{{}_{\mathcal{A}|U}\mathfrak{Mod}}\left(f\,|U\right)$.

We say that the $\mathcal{A}$-Module $\mathcal{F}$ is *finitely generated* if, for every $b \in B$, there exists an open neighborhood U of b and an exact sequence

$$\mathcal{A}^p\,|U \to \mathcal{F}\,|U \to 0,$$

where $p \in \mathbb{N}$ depends on U. If so, $\mathcal{F}_b$ is a finitely generated $\mathcal{A}_b$-module, but the converse does not hold.

(III) FINITELY PRESENTED $\mathcal{A}$-MODULE We say that the $\mathcal{A}$-Module $\mathcal{F}$ is *finitely presented* ([GRO 71], Chapter 0, (5.2.5)) (or *pseudo-coherent* [GRO 57], section 4.1)) if, for every $b \in B$, there exists an exact sequence

$$\mathcal{A}^q\,|U \longrightarrow \mathcal{A}^p\,|U \to \mathcal{F}\,|U \to 0,$$

where the integers $p, q \in \mathbb{N}$ depend on U. Clearly, every finitely presented $\mathcal{A}$-Module is finitely generated and quasi-coherent. The following result is an improvement by A. Grothendieck ([GRO 57], Proposition 4.1.1) of a result originally shown by J.-P. Serre ([SER 55], section 2, Proposition 5):

THEOREM 5.29.– *(Serre-Grothendieck) Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of $\mathcal{A}$-Modules. If $\mathcal{F}$ is finitely generated (resp. finitely presented), then the canonical homomorphism* (5.4) *is a monomorphism (resp. an isomorphism).*

PROOF.–

1) Suppose that $\mathcal{F}$ is finitely generated at b. Then there exists an open neighborhood U of b and an exact sequence $0 \longrightarrow \mathcal{A}^p\,|U \longrightarrow \mathcal{F}\,|U \longrightarrow 0$, which implies the exact sequences $\mathrm{Hom}_{\mathcal{A}|U}\left(\mathcal{A}^p\,|U\,, \mathcal{G}\,|U\right) \longleftarrow \mathrm{Hom}_{\mathcal{A}|U}\left(\mathcal{F}\,|U\,, \mathcal{G}\,|U\right) \longleftarrow 0$, $0 \longrightarrow \mathcal{A}_b^p \longrightarrow \mathcal{F}_b \longrightarrow 0$ and $\mathrm{Hom}_{\mathcal{A}_b}\left(\mathcal{A}_b^p, \mathcal{G}_b\right) \longleftarrow \mathrm{Hom}_{\mathcal{A}_b}\left(\mathcal{F}_b, \mathcal{G}_b\right) \longleftarrow 0$. Moreover, there is a canonical isomorphism $\mathrm{Hom}_{\mathcal{A}}\left(\mathcal{A}, \mathcal{G}\right) \cong \mathcal{G}$, and therefore a canonical isomorphism $\mathrm{Hom}_{\mathcal{A}}\left(\mathcal{A}^p, \mathcal{G}\right) \cong \mathcal{G}^p$. Hence, the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{G}_b^p & \xleftarrow{i} & \mathfrak{Hom}\left(\mathcal{F}, \mathcal{G}\right)_b & \longleftarrow & 0 \\
\downarrow\cong & & \downarrow f & & \\
\mathcal{G}_b^p & \xleftarrow{j} & \mathrm{Hom}_{\mathcal{A}_b}\left(\mathcal{F}_b, \mathcal{G}_b\right) & \longleftarrow & 0
\end{array}$$

and so f is a monomorphism. Indeed, if $f\left(x\right) = 0$, then $j\left(f\left(x\right)\right) \cong i\left(x\right) = 0$, so $x = 0$.

2) If $\mathcal{F}$ is finitely presented, then a similar reasoning yields the exact sequence

$$\begin{array}{ccccccc} \mathcal{G}_b^q & \xleftarrow{k} & \mathcal{G}_b^p & \longleftarrow & \mathfrak{Hom}\left(\mathcal{F},\mathcal{G}\right)_b & \longleftarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ \mathcal{G}_b^q & \xleftarrow{l} & \mathcal{G}_b^p & \longleftarrow & \mathrm{Hom}_{\mathcal{A}_b}\left(\mathcal{F}_b,\mathcal{G}_b\right) & \longleftarrow & 0 \end{array}$$

and $\mathfrak{Hom}\left(\mathcal{F},\mathcal{G}\right)_b = \ker\left(k\right) \cong \ker\left(l\right) = \mathrm{Hom}_{\mathcal{A}_b}\left(\mathcal{F}_b,\mathcal{G}_b\right)$. ■

(IV) COHERENT $\mathcal{A}$-MODULE

DEFINITION 5.30.–

1) We say that an $\mathcal{A}$-Module is coherent *if (a) it is finitely generated and (b) for every open set $U \in \mathfrak{Top}_B$, every integer $p \geq 0$, and every homomorphism $u : \mathcal{A}^p\left|U\right. \to \mathcal{F}\left|U\right.$, $\ker\left(u\right)$ is finitely generated.*

2) We say that the sheaf of rings $\mathcal{A}$ is left coherent if ${}_{\mathcal{A}}\mathcal{A}$ is a coherent $\mathcal{A}$-Module, or in other words if, for every open set $U \in \mathfrak{Top}_B$, every integer $p \geq 0$, and every homomorphism $u : \mathcal{A}^p\left|U\right. \to \mathcal{A}\left|U\right.$, $\ker\left(u\right)$ is finitely generated.

We immediately observe that:

LEMMA 5.31.– *Every finitely generated sub-$\mathcal{A}$-Module of a coherent $\mathcal{A}$-Module is coherent.*

PROPOSITION 5.32.– *Every coherent $\mathcal{A}$-Module is finitely presented. Conversely, if $\mathcal{A}$ is a coherent sheaf of rings, then every finitely presented $\mathcal{A}$-Module is coherent (**exercise**).*

The direct sum $\mathcal{F} \oplus \mathcal{G}$ of two $\mathcal{A}$-Modules $\mathcal{F}$ and $\mathcal{G}$ is a coherent $\mathcal{A}$-Module if and only if $\mathcal{F}$ and $\mathcal{G}$ are both coherent (**exercise**), which in particular implies that the category ${}_{\mathcal{A}}\mathfrak{Coh}$ of coherent sheaves of $\mathcal{A}$-Modules is additive ([P1], section 3.3.7(**II**)). J. P. Serre ([SER 55], section 2, Theorem 1) showed the following result (**exercise***: cf. *op. cit.* or [GRO 71], Chapter 0, Proposition 5.3.2):

THEOREM 5.33.– *(Serre) Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be a short exact sequence of $\mathcal{A}$-Modules. If any two of them are coherent, then so is the third.*

COROLLARY 5.34.– *Let $\mathcal{F}$, $\mathcal{G}$ be two coherent $\mathcal{A}$-Modules and suppose that $u : \mathcal{F} \to \mathcal{G}$ is a morphism; then* $\ker(u)$, $\mathrm{im}(u)$ *and* $\mathrm{coker}(u)$ *are coherent.*

PROOF.– The $\mathcal{A}$-Module $\mathrm{im}(u) \subseteq \mathcal{G}$ is finitely generated because $\mathcal{F}$ is finitely generated, and, since $\mathcal{G}$ is coherent, $\mathrm{im}(u)$ is coherent. Theorem 5.33 and the exact sequences

$$\begin{aligned} 0 &\longrightarrow \ker(u) \longrightarrow \mathcal{F} \longrightarrow \mathrm{im}(u) \longrightarrow 0 \\ 0 &\longrightarrow \mathrm{im}(u) \longrightarrow \mathcal{F} \longrightarrow \mathrm{coker}(u) \longrightarrow 0 \end{aligned}$$

therefore imply that $\ker(u)$ and $\mathrm{coker}(u)$ are coherent. ■

THEOREM 5.35.– *(1) The category ${}_{\mathcal{A}}\mathfrak{Coh}$ of coherent sheaves of $\mathcal{A}$-Modules is abelian. (2) If $\mathcal{A}$ is a coherent sheaf of rings, then the category ${}_{\mathcal{A}}\mathfrak{Mod}^{fp}$ of finitely presented sheaves of $\mathcal{A}$-Modules is abelian.*

PROOF.– (1) follows from the fact that ${}_{\mathcal{A}}\mathfrak{Mod}^{fp}$ is an additive category that admits kernels and cokernels, and so is preabelian ([P1], section 3.3.7**(III)**); furthermore, it is a full subcategory of the category ${}_{\mathcal{A}}\mathfrak{Mod}$, so every image in ${}_{\mathcal{A}}\mathfrak{Mod}^{fp}$ is isomorphic to a coimage, and ${}_{\mathcal{A}}\mathfrak{Mod}^{fp}$ is therefore an abelian category. (2) may be deduced from (1) by Proposition 5.32. ■

Let Y be a closed subspace of B and suppose that $\mathcal{A}$ is a sheaf of rings on Y. Let $\mathcal{A}^B$ be the sheaf of rings on B obtained by extending $\mathcal{A}$ by 0 outside of Y.

LEMMA 5.36.– *$\mathcal{A}$ is a coherent sheaf of rings if and only if $\mathcal{A}^B$ is a coherent sheaf of rings (**exercise**).*

5.3.3. *Examples of coherent sheaves*

(I) COHERENT ALGEBRAIC SHEAVES Let $Y = \mathcal{Z}(\mathfrak{a})$ be an algebraic set in the n-dimensional affine space $\mathbb{A}^n_{\mathbf{k}}$ over an algebraically closed field $\mathbf{k}$ ([P1], section 3.2.7), where $\mathfrak{a}$ is an ideal in $\mathbf{A} := \mathbf{k}[X_1, ..., X_n]$, and equip Y with the Zariski topology. Earlier, we defined regular functions in Y as the restrictions of polynomial functions to Y ([P1], section 3.2.1), and we will now "localize" this point of view. A regular function *in the neighborhood of a point* $x \in Y$ is a polynomial function on an open neighborhood $\Omega_Q \cap Y$ of x

in Y, where Ω_Q is a principal open set, since the principal open sets form a base of the Zariski topology on $\mathbb{A}^n_{\mathbf{k}}$ ([P1], section 3.2.7**(II)**). Moreover, we have that $\Omega_Q = \{y \in Y : Q(y) \neq 0\}$, where Q is an irreducible polynomial, so Ω_Q is the set of points of Y at which the polynomial function $y \mapsto Q(y)$ is invertible. Consequently, any function that is regular in the neighborhood of x may identified with a rational fraction P/Q such that $Q(x) \neq 0$. The sets of the form $\mathfrak{m}_x := \{F \in \mathbf{A} : F(x) = 0\}$ are the maximal ideals of $\mathbf{A}$ by the weak version of the *Nullstellensatz* ([P1], section 3.2.5**(III)**, Theorem 3.91). Therefore, functions that are regular in the neighborhood of x may be identified with elements of the local ring $\mathbf{A}_{\mathfrak{m}_x}$ of $\mathfrak{m}_x$ ([P1], section 3.2.1**(I)**).

Now let U be an open subset of $\mathbb{A}^n_{\mathbf{k}}$. We say that a rational function from U to $\mathbf{k}$ is *regular* on U if it is of the form $x \mapsto R(x)$, where R is an irreducible rational fraction P/Q, $P, Q \in \mathbf{A}$, and $Q(x) \neq 0$ on U. Given that $\mathbf{k}$ is algebraically closed, it is infinite ([P1], section 2.3.5**(II)**, Lemma 2.34), so we may identify the function $x \mapsto R(x)$ with the rational fraction R, since the mapping $\mathbf{A} \to \mathfrak{F}(\mathbf{k}^n; \mathbf{k}) : P \mapsto [x \mapsto P(x)]$ is injective (**exercise**). Saying that the function R is regular on U is equivalent to saying that it is continuous on U. If $\mathcal{O}(U)$ is the ring of regular functions on U, then we observe that $U \mapsto \mathcal{O}(U)$ is a sheaf of rings $\mathcal{O}$. If $U = \mathbb{A}^n_{\mathbf{k}}$, then $\mathcal{O}(U)$ is once again the ring of polynomial functions, which shows that we have correctly "localized" the original formulation of the notion of a regular function.

Let Y be an algebraic set, i.e. a closed subspace of $\mathbb{A}^n_{\mathbf{k}}$ (section 2.3.1**(II)**). A section of $\mathcal{O}_Y$ on an open subset U of Y is a mapping $f : U \to \mathbf{k}$ that is equal to the restriction of a function that is regular and rational at every x in U and zero at every point $x \in \mathbb{A}^n_{\mathbf{k}} - Y$ (section 5.3.1**(II)**).

Let X, Y be algebraic sets, $Y \subset \mathbb{A}^n_{\mathbf{k}}$, and suppose that $\iota : Y \hookrightarrow \mathbb{A}^n_{\mathbf{k}}$ is the canonical injection. From our current local perspective ("localized" from the global perspective in [P1], section 3.2.7**(III)**), a mapping $\varphi : X \to Y$ is a *morphism of algebraic sets* if the components of $\iota \circ \varphi$ are regular, which determines the *category of algebraic sets*. $\varphi : X \to Y$ is a morphism if and only if (a) φ is continuous, and (b) if $x \in X$ and $f \in \mathcal{O}_{\varphi(x),Y}$, then $f \circ \varphi \in \mathcal{O}_{x,X}$ (**exercise**).

DEFINITION 5.37.– *Let Y be an algebraic set. We say that $\mathcal{O}_Y$ is the sheaf of local rings of Y.*

For every $x \in Y$, the stalk $\mathcal{O}_{x,Y}$ of $\mathcal{O}_Y$ at the point x is a local ring. Let $\mathfrak{I}(Y)$ be the ideal of Y in $\mathbf{A}$ ([P1], section 3.2.7(**I**)).

LEMMA 5.38.– *Let $x \in Y$ and suppose that $\varepsilon_x : \mathcal{O}_x \twoheadrightarrow \mathcal{O}_{x,Y}$ is the canonical surjection. Then* $\ker(\varepsilon_x)$ *is the ideal* $\mathfrak{I}(Y).\mathcal{O}_x$ *of* $\mathcal{O}_x$*, and* $\mathcal{O}_{x,Y} \cong \mathcal{O}_x / (\mathfrak{I}(Y).\mathcal{O}_x) \cong \mathbf{A}/\mathfrak{I}(Y)$.

PROOF.– We have that $y \in Y \Rightarrow [f(y) = 0, \forall f \in \mathfrak{I}(Y)]$, so $\mathfrak{I}(Y).\mathcal{O}_x \subset \ker(\varepsilon_x)$. Conversely, let $R = P/Q$ be an element of ε_x, where $Q(x) \neq 0$. such that $P(y) = 0, \forall y \in W \cap Y$. Let $F = \complement_X W$; F is closed in X and $x \notin F$, so there exists a polynomial P_1 that vanishes on F such that $P_1(x) \neq 0$. Therefore, $P.P_1 \in \mathfrak{I}(Y)$, and $R = P.P_1/Q.P_1 \in \mathfrak{I}(Y).\mathcal{O}_x$. This yields the equality $\ker(\varepsilon_x) = \mathfrak{I}(Y).\mathcal{O}_x$, which, by Noether's first isomorphism theorem ([P1], section 2.2.3(**II**), Theorem 2.12(1)) and by ([P1], section 3.1.9, Theorem 3.34), implies that $\mathcal{O}_{x,Y} \cong \mathcal{O}_x / (\mathfrak{I}(Y).\mathcal{O}_x) \cong \mathbf{A}/\mathfrak{I}(Y)$. ■

LEMMA 5.39.– *The sheaf $\mathcal{O}$ of local rings of $\mathbb{A}^n_{\mathbf{k}}$ is a coherent sheaf of rings.*

PROOF.– Let $x \in \mathbb{A}^n_{\mathbf{k}}$ and suppose that U is a neighborhood of x. For every integer $p \geq 0$ and every homomorphism $u : \mathcal{O}^p|U \to \mathcal{O}|U$, we must show that $\ker(u)$ is finitely generated. Observe that u is of the form $(\mathbf{s}_1, ..., \mathbf{s}_p) \mapsto \sum_{i=1}^p f_i.\mathbf{s}_i$, where the $\mathbf{s}_i$ and f_i are sections of $\mathcal{O}$ over U for every i. By replacing U with a smaller open set if necessary, we may assume that the $\mathbf{s}_i$ can be written as P_i/Q, where the $P_i, Q \in \mathbf{A}$ and Q does not vanish on U. Let $y \in U$ and consider the relation $\sum f_{i,y}.\mathbf{s}_{i,y} = 0$. We can also write that $f_{i,y} = R_i/S$, where $R_i, S \in \mathbf{A}$ and S does not vanish on U. The relation "$\sum_{i=1}^p f_i.\mathbf{s}_i = 0$ in the neighborhood of y" is therefore equivalent to "$\sum_{i=1}^p R_i P_i = 0$"; this is the equation of $\ker(g)$, where $g : \mathbf{A}^n \to \mathbf{A} : (P_1, ..., P_n) \mapsto \sum_{i=1}^p R_i P_i$. By Hilbert's basis theorem, $\mathbf{A}$ is Noetherian ([P1], section 3.1.11, Corollary 3.49), so $\ker(g)$ is finitely generated, and the same is true for $\ker(u)$. ■

LEMMA 5.40.– *Let Y be an algebraic set. For each $x \in Y$, let $\mathcal{J}_{Y,x}$ be the ideal of $\mathcal{O}_x$ formed by the elements $f \in \mathcal{O}_x$ whose restriction to Y is zero in the neighborhood of x. The sheaf $\mathcal{J}_Y$ with stalk $\mathcal{J}_{Y,x}$ at the point x is a subsheaf of $\mathcal{O}$ and a coherent sheaf of $\mathcal{O}$-Modules.*

PROOF.– We have that $\mathcal{J}_{Y,x} = \mathfrak{I}(Y).\mathcal{O}_x$ if $x \in Y$ by Lemma 5.38, and, if $x \notin Y$, then $\mathcal{J}_{Y,x} = \mathcal{O}_x$, and $\mathfrak{I}(Y).\mathcal{O}_x = \mathcal{O}_x$. The ideal $\mathfrak{I}(Y)$ is finitely generated, so $\mathcal{J}_Y$ is also finitely generated, and $\mathcal{J}_Y$ is coherent by Lemmas 5.31 and 5.39. ■

We can now state the following result ([SER 55], section 2, Proposition 1):

THEOREM 5.41.– *(Serre)*[3] *If Y is an algebraic set, then $\mathcal{O}_Y$ is a coherent sheaf of rings.*

PROOF.– By Lemma 5.40, $\mathcal{J}_Y$ is a coherent sheaf of ideals, so $\mathcal{O}/\mathcal{J}_Y$ is a coherent sheaf of rings by Theorem 5.33. This sheaf is zero outside of Y, and its restriction to Y is identical to $\mathcal{O}_Y$, so $\mathcal{O}_Y$ is a coherent sheaf of rings by Lemma 5.36. ■

(II) COHERENT ANALYTIC SHEAVES A subset X of $\mathbb{C}^n$ is said to be analytic (we say that it is an *analytic set*[4]) if, for every $z \in X$, there exist holomorphic functions $f_1, ..., f_k$ defined on a neighborhood U of z such that

$$X \cap U = \{z \in U : f_j(z) = 0, i = 1, ..., k\}.$$

Any such set is locally closed in $\mathbb{C}^n$ (section 2.3.1**(II)**), so is locally compact (sections 2.3.9 and 2.5.2).

Consider once again the sheaf $\mathcal{H}$ of holomorphic functions on $\mathbb{C}^n$ and let $\mathcal{H}_X$ be the sheaf defined in section 5.3.1**(II)**. As in **(I)**, let $\varepsilon_z : \mathcal{H}_z \twoheadrightarrow \mathcal{H}_{z,X}$ be the canonical surjection and suppose that $\mathcal{J}_{z,X} = \ker(\varepsilon_z)$. Then, by Noether's first isomorphism theorem, $\mathcal{H}_{z,X}$ may be identified with the quotient ring $\mathcal{H}_z/\mathcal{J}_{z,X}$, and therefore $\mathcal{H}_X$ may be identified with $\mathcal{H}/\mathcal{J}$, where $\mathcal{J}$ is the sheaf on X whose fiber at any arbitrary given point $z \in X$ is $\mathcal{J}_{z,X}$. Let $Y \subset \mathbb{C}^m$ be another analytic set. We say that the mapping $\varphi : X \to Y$ is *holomorphic* if (a) it is continuous and (b) whenever $z \in X$ and $f \in \mathcal{H}_{\varphi(z),Y}$, then $f \circ \varphi \in \mathcal{H}_{z,X}$. The morphisms of the *category of analytic sets* are the holomorphic functions (compare this with the notion of morphism of algebraic sets in **(I)**). The two lemmas stated below are the

3 Serre showed this result more generally after introducing the notion of "algebraic varieties" (see [SER 55] or ([DIE 74], Vol. 2)), which we will avoid defining here. The proof given here is exactly the same as the proof given by Serre.

4 These analytic sets are "local versions" of the analytic spaces considered in [SER 56]. The transition to the notion of analytic space is achieved using the idea of a manifold, which is studied in [P3].

analytic versions of the Lemmas 5.39 and 5.40, but they are much more difficult to prove ([CAR 50], Theorems 1 and 2). The next theorem is an immediate consequence by the same reasoning as is used in the proof of Theorem 5.41, which is given in ([SER 56], section 1, Proposition 1):

LEMMA 5.42.– *(Oka*'s coherence theorem*) Let U be a non-empty open subset of $\mathbb{C}^n$. The sheaf of rings $\mathcal{H}_U$ of U is coherent.*

LEMMA 5.43.– *(Oka-Cartan* theorem*) If $X \subset \mathbb{C}^n$ is an analytic set and Y is a closed subset of X, then the sheaf $\mathcal{J}_Y$ is coherent.*

THEOREM 5.44.– *(Serre) If X is an analytic set and Y is a closed subset of X, then $\mathcal{H}_Y$ is a coherent sheaf of rings.*

5.4. Cohomology of sheaves

5.4.1. *Acyclic, fine, soft, flabby sheaves*

(I) DEFINITIONS Let $(X, \mathcal{A})$ be a ringed space, $\mathcal{F}$ a sheaf of $\mathcal{A}$-Modules, and Y a locally closed subset of B. Since ${}_{\mathcal{A}}\mathfrak{Mod}$ has sufficiently many injectives (Theorem 5.27), we can define the p-th right derived functor $\mathbf{H}^p(Y, \mathcal{F})$ of the functor $\Gamma : \mathcal{F} \to \mathcal{F}(Y)$ ([P1], section 3.3.9(**III**)). The canonical isomorphism $\mathcal{F}(X) \cong \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F})$ (Lemma 5.25) implies the canonical isomorphism $\mathbf{H}^p(X, \mathcal{F}) \cong \operatorname{Ext}^n_{\mathcal{A}}(\mathcal{A}, \mathcal{F})$.

DEFINITION 5.45.– *We say that $\mathcal{F}$ is* acyclic *over Y if $\mathbf{H}^p(Y, \mathcal{F}) = 0, \forall p \geq 1$.*

DEFINITION 5.46.–

i) We say that $\mathcal{F}$ is c-soft *(resp.* soft*, resp.* flabby*) if the restriction mapping $\Gamma(X, F) \to \Gamma(Y, F)$ is surjective for every compact (resp. closed, resp. open) subset Y of X.*

ii) We say that $\mathcal{F}$ is fine *if, for every open covering (U_i) of X, there exists a family of morphisms $(\phi_i : \mathcal{F} \to \mathcal{F})$ such that* $\operatorname{supp}(\phi_i) \subset U_i$ *and, for all $x \in X$, there exists an open neighborhood of x on which all but finitely many of the ϕ_i are zero, and $\sum_i \phi_i = 1_{\mathcal{F}}$.*

It is possible to show the following result (for (1) and (2) cf. [TAY 02], Theorems 7.5.2, 7.5.3, 7.5.4; (3) is an **exercise**):

THEOREM 5.47.–

1) If $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0$ *is an exact sequence of* $\mathcal{A}$*-Modules and* $\mathcal{G}, \mathcal{F}$ *are flabby, then* $\mathcal{H} \cong \mathcal{F}/\mathcal{G}$ *is flabby. Every flabby sheaf of* $\mathcal{A}$*-Modules is acyclic.*

2) Suppose that X *is paracompact (section 2.3.10). If* $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0$ *is an exact sequence of* $\mathcal{A}$*-Modules and* $\mathcal{G}, \mathcal{F}$ *are soft, then* $\mathcal{H} \cong \mathcal{F}/\mathcal{G}$ *is soft. If* Y *is a locally closed subspace of* X *and* $\mathcal{G}$ *is a soft sheaf of* $\mathcal{A}$*-Modules, then* $\mathcal{G}_Y$ *is soft. Every flabby sheaf and every fine sheaf of* $\mathcal{A}$*-Modules is soft, and every soft sheaf of* $\mathcal{A}$*-Modules is acyclic.*

3) Every soft sheaf of $\mathcal{A}$*-Modules is c-soft, and the converse holds whenever* X *is locally compact and countable at infinity (Definition 2.51).*

(II) EXAMPLES

a) The sheaves $\underline{\mathfrak{F}}(B;X)$ and $\underline{\mathfrak{C}}(B;X)$ (section 5.2.4) are flabby (**exercise**).

b) The sheaves $\underline{\mathcal{E}}(\Omega)$ and $\underline{\mathcal{D}}(\Omega)$, of infinitely differentiable functions on Ω and compactly supported infinitely differentiable functions on Ω respectively, where Ω is a non-empty open subset of $\mathbb{R}^n$ (section 4.3.1), are fine by Whitney's theorem (Theorem 4.88). Indeed, each C^∞ partition of unity (φ_i) determines a family of mappings $f_x \mapsto \varphi_i(x) f_x$, with $f_x \in \underline{\mathcal{E}}(\Omega)_x$ or $f_x \in \underline{\mathcal{D}}(\Omega)_x$.

c) The same reasoning shows that the sheaves $\underline{\mathcal{D}}'(\Omega)$ and $\underline{\mathcal{E}}'(\Omega)$, of distributions on Ω and compactly supported distributions on Ω respectively (section 4.4.1), are fine.

(III) FLABBY RESOLUTIONS In ${}_{\mathcal{A}}\mathfrak{Mod}$, consider a right resolution ([P1], section 3.3.5(**I**)) of the sheaf of $\mathcal{A}$-Modules $\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^0 \longrightarrow \mathcal{L}^1 \longrightarrow \mathcal{L}^2 ...$$

This resolution is said to be *flabby* if each of the sheaves of $\mathcal{A}$-Modules $\mathcal{L}^i$ $(i = 0, 1, 2, ...)$ is flabby.

Let $\mathcal{G}$ be a sheaf of $\mathcal{A}$-Modules, viewed as an étalé space (section 5.2.3(**II**)), and write $\mathcal{C}^0(U;\mathcal{G})$ for the abelian group of germs of (not necessarily continuous) sections from an open set $U \subset X$ into $\mathcal{G}$. Each element of $\mathcal{C}^0(U;\mathcal{G})$ is therefore a mapping s from U into the étalé space $\mathcal{G}$

satisfying the condition $p \circ \mathbf{s} = 1_U$. Therefore, $\mathcal{C}^0(\mathcal{G}) : U \mapsto \mathcal{C}^0(U;\mathcal{G})$ is a flabby sheaf, and $j : \mathcal{G} \hookrightarrow \mathcal{C}^0(\mathcal{G})$ is a canonical injection (**exercise**). We now set

$$\mathcal{C}^1(U;\mathcal{G}) = \mathcal{C}^0\left(U;\mathbf{Z}^1(U;\mathcal{G}(U))\right), \quad \mathbf{Z}^1(U;\mathcal{G}) := \mathcal{C}^0(U;\mathcal{G})/\mathcal{G}(U),$$
$$\mathcal{C}^2(U;\mathcal{G}) = \mathcal{C}^0\left(U;\mathbf{Z}^2(U;\mathcal{G}(U))\right), \quad \mathbf{Z}^2(U;\mathcal{G}) := \mathcal{C}^1(U;\mathcal{G})/\mathbf{Z}^1(U;\mathcal{G}(U)),$$

etc. Let $d^p : \mathcal{C}^p(\mathcal{G}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{G})$ be the homomorphism obtained by composing the surjection $\mathcal{C}^p(\mathcal{G}) \twoheadrightarrow \mathbf{Z}^{p+1}(\mathcal{G}) := \mathcal{C}^p(\mathcal{G})/\mathbf{Z}^p(\mathcal{G})$ with the injection $\mathbf{Z}^{p+1}(\mathcal{G}) \hookrightarrow \mathcal{C}^{p+1}(\mathcal{G}) := \mathcal{C}^0\left(U;\mathbf{Z}^{p+1}(\mathcal{G})\right)$. Then

$$0 \longrightarrow \mathcal{G} \xrightarrow{j} \mathcal{C}^0(\mathcal{G}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{G}) \xrightarrow{d^1} \mathcal{C}^2(\mathcal{G}) \longrightarrow \ldots$$

is an exact sequence, and is therefore a flabby resolution of $\mathcal{G}$.

DEFINITION 5.48.– *The above flabby resolution is written* $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{C}^*(\mathcal{G})$ *and is called the* canonical flabby resolution *of* $\mathcal{G}$.

5.4.2. *Cohomology of coherent sheaves*

(I) ALGEBRAIC CASE Proofs of the following results can be found in ([SER 55], section 2.45) (they are not difficult, but are somewhat lengthy).

THEOREM 5.49.– *(Serre)*

1) Let $\mathcal{F}$ be a coherent algebraic sheaf on an algebraic set Y. For every $x \in Y$, the $\mathcal{O}_{x,Y}$-module $\mathcal{F}_x$ is generated by the global sections of $\mathcal{F}$, i.e. the elements of $\Gamma(Y,\mathcal{F})$.

2) The restriction of the functor Γ to the category ${}_{\mathcal{A}}\mathfrak{Coh}$ of coherent sheaves of $\mathcal{A}$-Modules is exact.

Proposition (1) means that there exists a family of global sections $(\mathbf{s}_i)$ such that $\mathcal{F}_x = \sum_i \mathcal{O}_{x,Y}.(\mathbf{s}_i)_x$.

COROLLARY 5.50.– *If Y is an algebraic set, then every coherent algebraic sheaf on Y, and in particular $\mathcal{O}_Y$, is acyclic.*

PROOF.– This immediately follows from Theorems 5.41 and 5.49(2). ■

(II) Analytic case This case is more difficult and requires some preliminary work.

Definition 5.51.– *A non-empty connected open subset (or non-empty "domain") Ω of $\mathbb{C}^n$ is said to be a* domain of holomorphy *if there* do not exist *open subsets Ω_1 and Ω_2 of $\mathbb{C}^n$ with the following properties:*

i) $\varnothing \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

ii) Ω_2 is convex and $\Omega_2 \nsubseteq \Omega$.

iii) For every function $f \in \mathcal{H}(\Omega)$, there exists a function $f_2 \in \mathcal{H}(\Omega_2)$ such that $f|_{\Omega_1} = f_2|_{\Omega_1}$.

See Figure 5.1. In the case $n = 1$, every connected open set is a domain of holomorphy (if z_0 belongs to the frontier ∂B of B, consider the function $z \mapsto \frac{1}{z-z_0}$). By contrast ([HOR 90a], Theorem 2.3.2):

Theorem 5.52.– *(Hartogs'* extension theorem*) For $n \geq 2$, if Ω is a non-empty connected open subset of $\mathbb{C}^n$ and $K \subset \Omega$ is compact, then every holomorphic function on $\complement_\Omega K$ may be extended to a holomorphic function on Ω.*

Therefore, $\complement_\Omega K$ is not a domain of holomorphy.

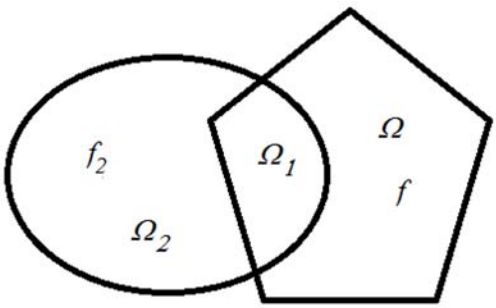

Figure 5.1. *Illustration of the properties of Definition 5.51*

The next theorem combines the Cartan-Thullen theorem [CAR 32], Cartan's theorems A and B [CAR 52b] ([CAR 79] is also especially relevant), and an observation made by J.-P. Serre.

THEOREM 5.53.–

1) Let X be an analytic set. The following conditions are equivalent:

i) For every coherent analytic sheaf $\mathcal{F}$ on X and for every $z \in X$, the $\mathcal{H}_{z,X}$-module $\mathcal{F}_z$ is generated by the global sections of $\mathcal{F}$.

ii) Every coherent analytic sheaf is acyclic over X.

2) If X is a domain in $\mathbb{C}^n$, then conditions (i), (ii) are equivalent to:

iii) X is a domain of holomorphy.

DEFINITION 5.54.– *Let X be an analytic set. We say that X is a Stein* set *if the condition (ii) of Theorem 5.53 is satisfied.*

REMARK 5.55.– *Originally, a Stein variety was defined as a holomorphic variety with "sufficiently many holomorphic functions" (see [CAR 52a], Definition 2 for more details). The Cartan-Thullen theorem states that any domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is a Stein set. Given this setup, (iii)⇒(i) and (iii)⇒(ii) are the statements of Cartan's theorems A and B. Conversely, J.-P. Serre [SER 53] showed that (i)⇒(ii)⇒(iii).*

EXAMPLE 5.56.– *Every open subset of a Stein set, every Cartesian product of finitely many Stein sets (and in particular every Cartesian product of finitely many subsets of $\mathbb{C}$), every union of a sequence (X_n) of Stein sets such that $X_n \Subset X_{n+1}$, and every intersection of finitely many Stein sets is a Stein set ([GEL 79], Chapter V, section 1).*

5.4.3. *Relative cohomology*

The theory of relative cohomology was developed independently by both M. Sato [SAT 60] and A. Grothendieck [GRO 67].

(I) SUPPORT Let $(X, \mathcal{A})$ be a ringed space and suppose that $\mathcal{F}$ is a sheaf of $\mathcal{A}$-Modules. The *support* of $\mathcal{F}$, denoted $\mathrm{supp}\,(\mathcal{F})$, is defined as the set of $x \in X$ such that $\mathcal{F}_x \neq \{0\}$. The set $\mathrm{supp}\,(\mathcal{F})$ is not necessarily closed in X ([GRO 71], Chapter 0, (3.1.5)). Given any section s of $\mathcal{F}$ over an open set U, the support of s is the set $\mathrm{supp}\,(\mathrm{s})$ of $x \in U$ such that $\mathrm{s}_x \neq 0_x$, and this set is closed in U (**exercise**).

Given a locally closed subset S of X, we define

$$\Gamma_S(X, \mathcal{F}) := \{s \in \Gamma(X, S) : \mathrm{supp}(\mathbf{s}) \subset S\},$$

which implies that, for all $U \in \mathfrak{N}(S)$ (section 2.3.1**(II)**), the sequence $0 \longrightarrow \Gamma_S(X, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U - S, \mathcal{F})$ is exact (**exercise**). Furthermore, we set

$$\boxed{\Gamma[S; \mathcal{F}] = \varinjlim_{U \in \mathfrak{N}(S)} \Gamma_S(U, \mathcal{F})}.$$

LEMMA 5.57.– *There exists a canonical isomorphism* $\Gamma[S; \mathcal{F}] \cong \Gamma_S(U, \mathcal{F})$ *for every* $U \in \mathfrak{N}(S)$.

PROOF.– If $U, V \in \mathfrak{N}(S)$, then $\Gamma_S(U, \mathcal{F}) \cong \Gamma_S(V, \mathcal{F})$. ■

(II) RELATIVE COHOMOLOGY AND ITS PROPERTIES The relative cohomology groups $\mathbf{H}_S^p(X, \mathcal{F})$, where $S \subset X$ is closed, are defined below ([SAT 60], II, section 4.1), ([MOR 93], Chapter 4, section 3).

Let $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{C}^*(\mathcal{F})$ be the canonical flabby resolution of $\mathcal{F}$ (section 5.4.1**(III)**). For $p = 0, 1, 2, ...,$ set (see [P1], section 3.3.8**(I)**)

$$\begin{aligned}
\mathbf{Z}_S^p(X; \mathcal{F}) &= \ker(d^p), \quad d^p : \Gamma_S(X, \mathcal{C}^p(\mathcal{F})) \longrightarrow \Gamma_S\left(X, \mathcal{C}^{p+1}(\mathcal{F})\right), \\
\mathbf{B}_S^p(X; \mathcal{F}) &= \mathrm{im}\left(d^{p-1}\right), \quad d^{p-1} : \Gamma_S\left(X, \mathcal{C}^{p-1}(\mathcal{G})\right) \longrightarrow \Gamma_S(X, \mathcal{C}^p(\mathcal{G})), \\
\mathbf{H}_S^p(X, \mathcal{F}) &= \mathbf{Z}_S^p(X; \mathcal{F}) / \mathbf{B}_S^p(X; \mathcal{F}), \quad \mathcal{C}^{-1}(\mathcal{F}) = 0.
\end{aligned}$$

DEFINITION 5.58.– *We say that* $\mathbf{H}_S^p(X, \mathcal{F})$ *is the* p*-th cohomology group of* X *with coefficients in* $\mathcal{F}$ *and support in* S.

As in ([P1], section 3.2.8**(III)**), the following properties hold (for additional results, see [MOR 93], Chapter 4, sections 3 and 4):

i) $\mathbf{H}_S^0(U, \mathcal{F}) = \Gamma_S(X, \mathcal{F})$.

ii) If $\mathcal{F}$ is a flabby sheaf, then $\mathbf{H}_S^n(X, \mathcal{F}) = 0, \forall n \geq 1$.

iii) The short exact sequence of $\mathcal{A}$-Modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \to \mathcal{K} \to 0$$

implies the long exact sequence of relative cohomologies

$$\begin{aligned} 0 \longrightarrow \mathbf{H}_S^0(X, \mathcal{F}) &\longrightarrow \mathbf{H}_S^0(X, \mathcal{G}) \longrightarrow \mathbf{H}_S^0(X, \mathcal{K}) \\ \xrightarrow{\delta^*} \mathbf{H}_S^1(X, \mathcal{F}) &\longrightarrow \mathbf{H}_S^1(X, \mathcal{G}) \longrightarrow \mathbf{H}_S^1(X, \mathcal{K}) \\ &... \\ \xrightarrow{\delta^*} \mathbf{H}_S^n(X, \mathcal{F}) &\longrightarrow \mathbf{H}_S^n(X, \mathcal{G}) \longrightarrow \mathbf{H}_S^n(X, \mathcal{K}) \xrightarrow{\delta^*} ... \end{aligned}$$

iv) $\mathbf{H}_X^n(X, \mathcal{F}) = \mathbf{H}^n(X, \mathcal{F})$.

v) If S is locally closed and $U_1, U_2 \in \mathfrak{N}(S)$, then there exists a canonical isomorphism $\mathbf{H}_S^n(U_1, \mathcal{F}) \cong \mathbf{H}_S^n(U_2, \mathcal{F})$ ("excision theorem").

vi) If S is locally closed, then, for every $U \in \mathfrak{N}(S)$, we have the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{H}_S^0(U, \mathcal{F}) &\longrightarrow \mathbf{H}^0(U, \mathcal{F}) \longrightarrow \mathbf{H}^0(U - S, \mathcal{F}) \xrightarrow{\delta'^*} \\ &... \\ \xrightarrow{\delta'^*} \mathbf{H}_S^n(U, \mathcal{F}) &\longrightarrow \mathbf{H}^n(U, \mathcal{F}) \longrightarrow \mathbf{H}^n(U - S, \mathcal{F}) \xrightarrow{\delta'^*} ... \end{aligned}$$

vii) If S, T are closed and $S \subset T$, then we have the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{H}_S^0(X, \mathcal{F}) &\longrightarrow \mathbf{H}_T^0(X, \mathcal{F}) \longrightarrow \mathbf{H}_{T-S}^0(X - S, \mathcal{F}) \xrightarrow{\delta''^*} \\ &... \\ \xrightarrow{\delta''^*} \mathbf{H}_S^n(X, \mathcal{F}) &\longrightarrow \mathbf{H}_T^n(X, \mathcal{F}) \longrightarrow \mathbf{H}_{S-T}^n(X - S, \mathcal{F}) \xrightarrow{\delta''^*} ... \end{aligned}$$

(III) DERIVED SHEAF OF RELATIVE COHOMOLOGY GROUPS Let S be a locally closed subset of X, suppose that $U, V \in \mathfrak{N}(S)$ are two open subsets such that $U \supset V$, and consider a flabby resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^*$ of $\mathcal{F}$. By Lemma 5.57, $\mathbf{H}^n[S \cap U; \mathcal{F}] \cong \mathbf{H}_S^n(U; \mathcal{F})$. There exists a canonical restriction mapping

$$\rho_V^U : \mathbf{H}^n[S \cap U; \mathcal{F}] \to \mathbf{H}^n[S \cap V; \mathcal{F}],$$

which determines a presheaf $U \mapsto \mathbf{H}^n[S \cap U; \mathcal{F}]$.

DEFINITION 5.59.– *We write $\mathfrak{H}_S^n(\mathcal{F})$ for the sheaf associated with the presheaf $U \mapsto \mathbf{H}^n[S \cap U; \mathcal{F}]$, and we call it the n-th derived sheaf of $\mathcal{F}$ with support in S.*

In particular, for every $x \in X$,

$$\mathfrak{H}_S^n(\mathcal{F})_x = \varinjlim_{U \in \mathfrak{N}(x)} \mathbf{H}^n[S \cap U; \mathcal{F}].$$

From this, we can deduce (**exercise**):

LEMMA 5.60.– *Let S be a locally closed subset of X.*

i) If $x \notin S$, then $\mathfrak{H}_S^n(\mathcal{F})_x = 0$, $n = 0, 1, 2, ...$

ii) If x is in the interior of S, then $\mathfrak{H}_S^n(\mathcal{F})_x = \mathcal{F}_x$ for $n = 0$ and $\mathfrak{H}_S^n(\mathcal{F})_x = 0$ for $n = 1, 2, ...$

5.4.4. *Meromorphic functions in multiple variables*

Let X be an analytic set and let $\mathcal{H}$ be the sheaf of germs of holomorphic functions on X. We can use the language of sheaves to reformulate the local definition of a meromorphic function in the sense imagined by Poincaré (section 5.1) as follows:

First, observe that the ring $\mathcal{H}_0$ of germs of functions that are holomorphic at the point 0 is the ring of convergent power series (sometimes written $\mathbb{C}\{Z_1, ..., Z_n\}$); this ring is entire and local, and the Weierstrass preparation theorem implies that it is a unique factorization domain ([GUN 65], Chapter II, section B, Theorem 7), as is the ring of formal power series $\mathbb{C}[[Z_1, ..., Z_n]]$ ([P1], section 2.3.9(**II**)); write $\mathcal{M}_0$ for its field of fractions. By translation, for every $z \in X$, the ring $\mathcal{H}_z$ of germs of functions that are holomorphic at the point z is entire, local, and a unique factorization domain; write $\mathcal{M}_z$ for the field of fractions of this new ring, so that each element h_z of $\mathcal{M}_z$ is the quotient f_z/g_z of two functions that are holomorphic at z.

DEFINITION 5.61.– *The sheaf $\mathcal{M}$ with stalks $\mathcal{M}_z$ is known as the sheaf of germs of meromorphic functions on X.*

Every global section of $\mathcal{M}$ (i.e. every element of $\mathcal{M}(X)$) is a meromorphic function on X; the converse holds if and only if X is a solution of the Poincaré problem **(P)**.

It immediately follows that $\mathcal{M}$ is a sheaf of $\mathcal{H}$-Modules. By applying the section functor Γ (section 5.2.2**(IV)**) to the exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{H} \to 0,$$

we obtain the long exact cohomology sequence ([P1], section 3.3.8**(III)**)

$$\begin{aligned}0 &\longrightarrow \mathcal{H}(X) \longrightarrow \mathcal{M}(X) \xrightarrow{\varphi} (\mathcal{M}/\mathcal{H})(X) \xrightarrow{\delta} \mathbf{H}^1(X, \mathcal{H}) \\ &\xrightarrow{\psi} \mathbf{H}^1(X, \mathcal{M}) \longrightarrow ...,\end{aligned}$$

since Γ is left exact (Theorem 5.13).

Every element of $(\mathcal{M}/\mathcal{H})_z$ is the principal part of an element $h_z \in \mathcal{M}_z$ (generalization of Definition 4.60), and so any section of $\mathcal{M}/\mathcal{H}$ is also known as a *system of principal parts*. Let $h = f/g \in \mathcal{M}(X)$, where f, g have no common non-unitary factors in $\mathcal{H}(X)$. The set $N_h = \{z \in X : f(z) = g(z) = 0\}$ is called the set of indeterminacy of h. If this set is non-empty, the meromorphic function h is improperly determined; we shall seek to circumvent this obstacle by strengthening the Poincaré problem **(P)** considered in section 5.1. The *strong form* of the Poincaré problem, written **(SP)**, is as follows:

(SP) Given a meromorphic function $h \in \mathcal{M}(X)$, do there exist two holomorphic functions $f, g \in \mathcal{H}(X)$ such that $h = f/g$ whose germs f_z, g_z in the unique factorization domain $\mathcal{H}_z$ are coprime for every $z \in X$?

The two Poincaré problems **(P)** and **(SP)** were tackled by Cousin in 1894 by setting two new problems, which we shall denote by **(C1)** and **(C2)** below. An analytic set X is a solution of the first Cousin problem **(C1)** (resp. the second Cousin problem **(C2)**) if and only if the generalization of the Mittag-Leffler theorem (Theorem 4.62(2)) (resp. the Weierstrass theorem (Theorem 4.63(2))) to multiple variables holds on X.

The *first* Cousin *problem* (**C1**) may be stated as follows ([HOR 90a], section 7.1):

(**C1**) Let $(U_i)_{i \in I}$ be an open covering of X and, for all $i \in I$, let $h_i \in \mathcal{M}(U_i)$. Suppose that $h_i - h_j \in \mathcal{H}(U_i \cap U_j)$ for every $i, j \in I$. Does there exist $h \in \mathcal{M}(X)$ such that $h - h_i \in \mathcal{H}(U_i)$ for every $i \in I$?

THEOREM 5.62.– *Let X be an analytic set.*

1) The implications and equivalences (i)⇒(ii)⇔(iii)⇒(iv) hold for the following conditions:

i) $\mathbf{H}^1(X, \mathcal{H}) = 0.$

ii) The canonical morphism $\psi : \mathbf{H}^1(X, \mathcal{H}) \longrightarrow \mathbf{H}^1(X, \mathcal{M})$ *is injective.*

*iii) X is a solution of **(C1)**.*

*iv) X is a solution of **(P)**.*

2) By Definition 5.54, condition (i) is equivalent to:

v) X is a Stein set.

PROOF.– The analytic set X is a solution of **(C1)** if and only if φ is surjective. Since $\operatorname{im}(\varphi) = \ker(\delta)$, this is equivalent to $\ker(\delta) = (\mathcal{M}/\mathcal{H})(X)$; but $\ker(\psi) = \operatorname{im}(\delta) \cong (\mathcal{M}/\mathcal{H})(X) / \ker(\delta)$, by Noether's first isomorphism theorem, and this quotient is zero if and only if $\ker(\delta) = (\mathcal{M}/\mathcal{H})(X)$. ■

Oka had already shown in 1937 that every domain of holomorphy is a solution of **(C1)**. There exist analytic sets that are not Stein sets but which are nonetheless solutions of **(C1)**, for example $\mathbb{C}^3 - \{0\}$: this set X is not a domain of holomorphy by Hartogs' extension theorem (Theorem 5.52), so is not a Stein set, but H. Cartan showed in 1937 that $\mathbf{H}^1(X, \mathcal{H}) = 0$ ([GEL 79], Chapter V, section 1.4); furthermore, $\mathbf{H}^2(X, \mathcal{H}) \neq 0$ (*ibid.*), which gives another way of showing that X is not a Stein set.

For every non-empty open subset of $\mathbb{C}^n$, let $\mathcal{M}^*(U)$ and $\mathcal{H}^*(U)$ be the multiplicative groups of the invertible elements of $\mathcal{M}(U)$ and $\mathcal{H}(U)$ respectively (section 4.2.6). The *second* Cousin *problem* **(C2)** may be stated as follows:

(C2) Let $(U_i)_{i \in I}$ be an open covering of X and, for every $i \in I$, let $h_i \in \mathcal{M}(U_i)^*$. Suppose that $h_i/h_j \in \mathcal{H}(U_i \cap U_j)$ for every $i, j \in I$. Does there exist $h \in \mathcal{M}^*(X)$ such that $h_i/h \in \mathcal{H}^*(U_i)$ for every $i \in I$?

The sheaves $\mathcal{M}^* : U \mapsto \mathcal{M}^*(U)$ and $\mathcal{H}^* : U \mapsto \mathcal{H}^*(U)$ are sheaves of abelian groups for which the following short sequence is exact:

$$1 \longrightarrow \mathcal{H}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{D} \longrightarrow 1, \qquad [5.6]$$

where $\mathcal{D} := \mathcal{M}^*/\mathcal{H}^*$ is called the *sheaf of germs of divisors* on X, and the sections of $\mathcal{D}$ are called the *divisors* (or the Cartier *divisors*) on X. Each element d of $\mathcal{D}_z$ is the principal part of an element $h_z \in \mathcal{M}_z^*$ (generalization of Definition 4.60); any such element is said to be ≥ 0 if $h_z \in \mathcal{H}_z$, and ≤ 0 if $1/h_z \in \mathcal{H}_z$. Each non-zero meromorphic function on X, i.e. each element of $\mathcal{M}^*(X)$, determines a divisor on X, specifically its canonical image in $\mathcal{D}(X)$, and any such divisor is said to be *principal*. By applying the functor Γ to the short exact sequence (5.6), we obtain the long exact sequence

$$1 \longrightarrow \mathcal{H}^*(X) \longrightarrow \mathcal{M}^*(X) \xrightarrow{\psi} \mathcal{D}(X) \xrightarrow{\eta} \mathbf{H}^1(X, \mathcal{H}^*) \longrightarrow ...$$

By the same reasoning as in the proof of Theorem 5.62, we deduce the following result:

THEOREM 5.63.–

i) The principal divisors of the analytic set X are the elements of $\operatorname{im}(\psi) = \ker(\eta)$.

ii) The analytic set X is a solution of **(C2)** *if and only if the canonical morphism* $\mathbf{H}^1(X, \mathcal{H}^\times) \to \mathbf{H}^1(X, \mathcal{M}^\times)$ *is injective.*

Therefore, for the analytic set X to be a solution of **(C2)**, it *suffices* that $\mathbf{H}^1(X, \mathcal{H}^*) = 0$.

Gronwall showed in 1917 that the open subset $\mathbb{C}^\times \times \mathbb{C}^\times$ of $\mathbb{C}^2$ is not a solution of **(SP)** ([GEL 79], Chapter V, section 2), despite being a Stein set (Example 5.56). The sheaf $\mathcal{H}^*$ is not coherent; this complicates the problem and ultimately entails the existence of Stein sets that are not solutions of **(C2)**. We shall write the abelian group $\mathcal{D}(X)$ additively, as well as the cohomology groups $\mathbf{H}^k(X, \mathcal{H}^*), \mathbf{H}^k(X, \mathcal{M}^*), \mathbf{H}^k(X, \mathcal{D})$ for $k \geq 1$. Let $\exp$ be the exponential mapping $\mathbf{s}_x \mapsto e^{i2\pi x}\mathbf{s}_x$ $(\mathbf{s}_x \in \mathcal{H}_x)$. We have the exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{H} \xrightarrow{\exp} \mathcal{H}^* \longrightarrow 1,$$

where $\underline{\mathbb{Z}}$ is the simple sheaf on X with stalk $\mathbb{Z}$. Applying the functor Γ yields the exact cohomology sequence

$$\begin{aligned} &\longrightarrow \quad \mathbf{H}^1(X,\underline{\mathbb{Z}}) \xrightarrow{\omega} \mathbf{H}^1(X,\mathcal{H}) \longrightarrow \mathbf{H}^1(X,\mathcal{H}^*) \xrightarrow{\delta} \\ \mathbf{H}^2(X,\underline{\mathbb{Z}}) &\longrightarrow \quad \mathbf{H}^2(X,\mathcal{H}) \longrightarrow ..., \end{aligned}$$

which implies ([KAU 83], Theorems 54.4 and 54A.3):

THEOREM 5.64.– *Let X be a Stein set.*

1) There exist canonical isomorphisms $\mathbf{H}^1(X,\underline{\mathbb{Z}}) \cong \mathcal{H}^*(X)/\exp(\mathcal{H}(X))$ and $\mathbf{H}^1(X,\mathcal{H}^*) \cong \mathbf{H}^2(X,\underline{\mathbb{Z}})$.

2) The following conditions are equivalent:

i) Every divisor of X is principal.

ii) $\mathbf{H}^2(X,\underline{\mathbb{Z}}) = 0$.

*iii) X is a solution of the strong Poincaré problem **(SP)**.*

*iv) X is a solution of the second Cousin problem **(C2)**.*

PROOF.– If X is a Stein set, then $\mathbf{H}^1(X,\mathcal{H}) = \mathbf{H}^2(X,\mathcal{H}) = 0$, so $\omega = 0$, and δ is an isomorphism. ■

There exist analytic sets X that are solutions of **(SP)** (and therefore of **(P)**) but which are solutions of neither **(C1)** nor **(C2)**, for example the notched cylinder (which is of course not a Stein set) [BEH 37]

$$\begin{aligned} \mathfrak{D}^* &= \left\{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\right\} \\ &\quad -\left\{(z_1,z_2) \in \mathbb{C}^2 : (|z_1|-1)^2 + \left(|z_2| - \tfrac{1}{2}\right)^2 < \tfrac{1}{16}\right\} \end{aligned}$$

(see Figure 5.2).

If X is a Stein set, then the canonical image $c(d)$ in $\mathbf{H}^1(X,\mathcal{H}^*)$ of a divisor $d \in \mathcal{D}(X)$ is called the Chern *class* of d. Any such divisor is principal if and only if its Chern class is zero[5]. The condition $\mathbf{H}^2(X,\mathbb{Z}) = 0$ dictates the topological nature of X. Gronwall showed that, in order for a product $X = X_1 \times ... \times X_n$ of analytic sets $X_k \subset \mathbb{C}$ to satisfy the condition $\mathbf{H}^2(X,\mathbb{Z}) = 0$, it is necessary for at least one of the factors to be simply connected. J.-P. Serre showed in 1953 that there exist simply connected analytic sets X such that $\mathbf{H}^2(X,\mathbb{Z}) \neq 0$ ([GEL 79], Chapter V, section 2.4).

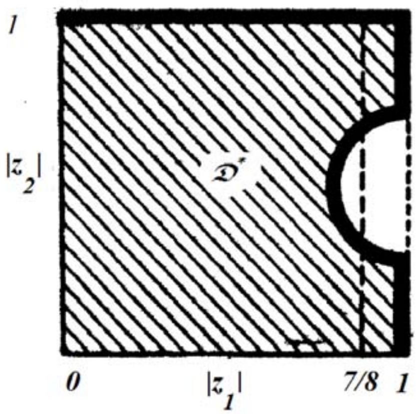

Figure 5.2. *Notched cylinder* $\mathfrak{D}^*$

Note that there is also a theory of divisors in algebraic geometry: see ([WEI 46], Chapter VIII) or, within a more general context, ([HAR 77], Chapter 2, section 6).

5.4.5. *Sheaf of hyperfunctions*

(I) Hyperfunction on an open subset of $\mathbb{R}^n$ Let $V = \mathbb{R}^n$; this space may be embedded in its complexification $E := \mathbb{C}^n$.

DEFINITION 5.65.–

*i) The sheaf $\mathcal{B} := \mathfrak{H}^n_V(\mathcal{H})$ (section 5.4.3**(III)**) is called the sheaf of germs of hyperfunctions on V.*

5 More precisely, $c(d)$ is the canonical image of d in the Picard group $\mathrm{Pic}(X)$ of X, which is canonically isomorphic to $\mathbf{H}^1(X,\mathcal{H}^*)$ ([GRO 71], Chapter 0, (5.6.3)).

ii) Given an open subset $\mathfrak{u}$ *of* V*, we write* $\mathcal{B}(\mathfrak{u})$ *for the space* $\Gamma(\mathfrak{u}, \mathcal{B})$ *of sections of* $\mathcal{B}$ *over* $\mathfrak{u}$*. The elements of* $\mathcal{B}(\mathfrak{u})$ *are called hyperfunctions on* $\mathfrak{u}$*.*

We have that $\mathfrak{H}_V^n(\mathcal{H})(\mathfrak{u}) = \mathbf{H}^n[\mathfrak{u} \cap \Omega; \mathcal{H}] \overset{\text{identification}}{=} \mathbf{H}_{\mathfrak{u}}^n(\Omega; \mathcal{H})$ for every $\Omega \in \mathfrak{N}_E(\mathfrak{u})$ (section 5.4.3(**III**)). Below, we shall write

$$\boxed{\mathcal{B}(\mathfrak{u}) = \mathbf{H}_{\mathfrak{u}}^n(\Omega; \mathcal{H}), \quad \Omega \in \mathfrak{N}_E(\mathfrak{u})}.$$

Consider the case $n = 1$. Here, $\mathbf{H}^1(\Omega, \mathcal{H}) = 0$ by Theorem 5.53, since Ω is a Stein set (Example 5.56). Therefore, by the long exact sequence (vi) from section 5.4.3(**II**), and the relations $\mathbf{H}^0(\Omega, \mathcal{H}) = \mathcal{H}(\Omega)$, $\mathbf{H}^0(\Omega - \mathfrak{u}, \mathcal{H}) = \mathcal{H}(\Omega - \mathfrak{u})$, we have that

$$\mathcal{B}(\mathfrak{u}) = \mathbf{H}_{\mathfrak{u}}^1(\Omega; \mathcal{H}) \cong \frac{\mathcal{H}(\Omega - \mathfrak{u})}{\mathcal{H}(\Omega)}, \quad \Omega \in \mathfrak{N}_E(\mathfrak{u}),$$

which yields the relations (4.21), (4.22).

In the case $n > 1$, Sato showed that $\mathbf{H}_{\mathfrak{u}}^p(\Omega; \mathcal{H}) = 0$ for $p = 1, ..., n-1$. Expressing hyperfunctions as the "boundary values of holomorphic functions", as in (4.21), is more complicated when $n > 1$; see for example ([KAT 99], Theorem 1.5.1)[6].

THEOREM 5.66.– *The sheaf* $\mathcal{B}$ *of hyperfunctions is flabby.*

PROOF.– See Lemma 4.104 in the case where $n = 1$. ■

REMARK 5.67.– *Hyperfunctions can be defined similarly on real analytic manifold that is locally compact countable at infinity* Ω *([MOR 93, Chapter 9).*

(II) HYPERFUNCTIONS WITH COMPACT SUPPORT The next theorem generalizes Theorem 4.98 and Corollary 4.101 ([MOR 93], Theorem 6.5.1):

THEOREM 5.68.– *(Martineau) Let* K *be a compact subset of* $\mathbb{R}^n$ *and suppose that* $\mathcal{A}(K)$ *is the nuclear Silva space of germs of real analytic functions on* K *(section 4.3.2(**IV**)). Then* $\mathcal{B}[K] \overset{\text{identification}}{=} \mathcal{A}_b'(K)$ *(where* $\mathcal{A}_b'(K)$ *is the nuclear Fréchet space defined to be the strong dual of* $\mathcal{A}(K)$*).*

6 See also the Wikipedia article on *Hyperfunctions*, where readers will find more useful information about hyperfunctions in multiple variables.

From this, we can deduce:

THEOREM 5.69.– *Let* $\mathfrak{u}$ *be a bounded open subset of* $\mathbb{R}^n$*. Then*

$$\boxed{\mathcal{B}(\mathfrak{u}) \cong \frac{\mathcal{A}'(\overline{\mathfrak{u}})}{\mathcal{A}'(\partial\mathfrak{u})}}. \qquad [5.7]$$

PROOF.– Since the sheaf $\mathcal{B}$ is flabby, we have the exact sequence (**exercise**)

$$0 \longrightarrow \mathcal{B}(\partial\mathfrak{u}) \longrightarrow \mathcal{B}(\overline{\mathfrak{u}}) \longrightarrow \mathcal{B}(u) \longrightarrow 0.$$

We can then simply apply Theorem 5.68. ■

REMARK 5.70.– *The quotient in the right-hand side of* (5.7) *was used to define the space of hyperfunctions on a* bounded *open subset of* $\mathbb{R}^n$ *in ([HOR 90b], Definition 9.2.1).*

5.4.6. *Application to systems of linear differential equations*

(I) The theory of systems of linear differential equations with constant coefficients was fully elucidated in ([P1], section 3.4.4). We shall now consider the case of systems of differential equations with variable coefficients of the form

$$\begin{aligned} R\left(t, \frac{d}{dt}\right) w &= 0 \\ R(t, X) &= \sum_{j=0}^{m} R_j(t)\, X^j \in \mathbf{A}^{q\times k}, \end{aligned}$$

where $\mathbf{A} = \mathbf{K}[X, d/dt]$ is the ring of linear differential operators. We will look for solutions in W^k, where W is the $\mathbf{A}$-module W, with X acting on $w \in W$ by $X : w \mapsto \frac{dw}{dt}$ ([P1], section 3.3.3(**I**)). We will avoid considering the algebraically pathological case where $\mathbf{A}$ is not entire, e.g. when $\mathbf{K} = \mathcal{C}^\infty(\mathbb{R})$, since then we might have $a, b \in \mathcal{C}^\infty(\mathbb{R})^\times$ such that $ab = 0$ (for example, pick a and b with disjoint support in $\mathcal{D}(\mathbb{R})^\times$). This leads us to the natural requirement that the coefficients should be analytic. It will be useful to embed $\mathbb{R}$ in $\mathbb{C}$, therefore assuming that $\mathbf{K}$ is a subring of $\mathcal{A}(\mathbb{R})$

(section 4.3.2**(IV)**)). However, we still need to determine more precisely which choice of ring $\mathbf{K}$ is most appropriate (an essentially algebraic question), as well as which function space W is most appropriate (a predominantly analytic question); to unite these two conditions, we must do some "algebraic analysis". Write ∂ for the indeterminate X, as in ([P1], section 3.4.4). In the following, $\mathfrak{u}$ is a non-empty open subset of the real line that contains the origin.

(II) DIFFERENTIAL EQUATIONS We will be able to fully solve the differential equations $p(\partial)\, w = 0, p(\partial) \in \mathbb{C}[\partial]$ if and only if W contains the exponential polynomials ([P1], section 3.4.4).

Equations of the form $t^m w = 0$ do not have non-zero solutions in $\mathcal{C}^\infty(\mathfrak{u})$, and so the latter is not a good choice for W. However, (i) this equation admits the solutions $\sum_{j=0}^{m-1} c_j \delta^{(j)}$, where the c_j are constants and $\delta^{(j)}$ denotes the j-th derivative of the Dirac distribution; (ii) this is in fact the complete set of solutions.

Claim (i) can easily be seen by applying the Laplace transform[7]: writing $\mathcal{L}$ for this transform and s for the Laplace variable, we have that $\mathcal{L} : w \mapsto \hat{w}(s)$, $tw \mapsto -\frac{d}{ds}(\hat{w})$, so $t^m w \mapsto (-1)^m \frac{d^m}{ds^m}(\hat{w})$. But $\frac{d^m}{ds^m}(\hat{w}) = 0 \Leftrightarrow \hat{w} = \sum_{j=0}^{m-1} c_j s^j \Leftrightarrow w = \sum_{j=0}^{m-1} c_j \delta^{(j)}$, since $\mathcal{L}\left(\delta^{(j)}\right) = s^j$. The complex vector space generated by these solutions has dimension m. Similarly, if we suppose that $\mathbb{C}[t] \subset \mathbf{K} \subset \mathcal{A}(\mathfrak{u})$, then $\frac{\mathbf{A}}{\mathbf{A}t^m}$ is an m-dimensional $\mathbb{C}$-vector space generated by $1, t, ..., t^{m-1}$. The fact that these two spaces have the same dimension means that there exists an isomorphism $\frac{\mathbf{A}}{\mathbf{A}t^m} \cong \mathrm{Hom}_{\mathbf{A}}\left(\frac{\mathbf{A}}{\mathbf{A}t^m}, \Delta\right)$, where $\Delta := \mathbb{C}[\partial]\,\delta$, which implies (ii). Therefore, the space $\mathcal{D}'(\mathfrak{u})$ of distributions on $\mathfrak{u}$ seems to represent a viable candidate for W.

However, this soon proves to be false after delving a little deeper. Laurent Schwartz ([SCW 66], Chapter V, section 6, (V,6; 15)) showed that the equation $\left(t^3\partial + 2\right) w = 0$ only admits the trivial solution $w = 0$ in $\mathcal{D}'$. But, given any reasonable choice of ring of coefficients $\mathbf{K}$, we would have $t^3 \in \mathbf{K}$, so $t^3\partial + 2 \in \mathbf{A}$, and $t^3\partial + 2$ would be non-invertible in $\mathbf{A}$, so $\frac{\mathbf{A}}{\mathbf{A}(t^3\partial+2)} \neq 0$. This observation, among others, motivates the use of hyperfunctions.

7 See the Wikipedia article on the *Bilateral Laplace transform*.

THEOREM 5.71.– *(Sato [SAT 60], Part I, section 26) Let* $\mathbf{K} \subset \mathcal{A}(\mathfrak{u})$, $\mathbf{K} \neq \{0\}$ *be a ring and suppose that* $\mathbf{A} = \mathbf{K}\left[\partial, \frac{d}{dt}\right]$ *is the ring of linear differential operators with coefficients in* $\mathbf{K}$. *The* $\mathbf{A}$*-module* $\mathcal{B}(\mathfrak{u})$ *is divisible ([P1], section 3.3.1(**II**)).*

PROOF.– We need to show that, for every hyperfunction $f \in \mathcal{B}(\mathfrak{u})$ and every differential operator $P(t, \partial) \in \mathbf{A}^{\times}$,

$$P(t, \partial) = \sum_{j=0}^{m} a_j(t)\, \partial^j, \qquad [5.8]$$

there exists a hyperfunction $u \in \mathcal{B}(\mathfrak{u})$ such that $P(t, \partial)\, u = f$. Let $\Omega \in \mathfrak{N}_{\mathbb{C}}(\mathfrak{u})$ be an open subset of $\mathbb{C}$ on which the coefficients a_j are holomorphic. Set $\Omega - \mathfrak{u} = \Omega^+ \cup \Omega^-$ (section 4.4.2(**II**), Figure 4.3); we may assume that Ω^+ and Ω^- are both simply connected and that every zero of a_m in Ω belongs to $\mathfrak{u}$. Let $F \in \mathcal{H}(\Omega - \mathfrak{u})$ be such that $[F] = f$. The classical theory of differential equations in the complex domain, which is well-understood ever since Riemann ([INC 56], Part II, Chapter XV), shows that there exists a solution $U \in \mathcal{H}(\Omega - \mathfrak{u})$ of the equation $P(t, \partial)\, U = F$. Therefore, setting $u = [U]$, we have that $P(t, \partial)\, u = f$. ■

The second key result of the theory of linear differential equations with analytic coefficients is stated below without proof (see [KOM 73]). Let $\mathrm{ord}_t(a_i) \geq 0$ $(j = 0, ..., m)$ be the order of the zero at the point $t \in \mathfrak{u}$ of the coefficient a_j (setting $\mathrm{ord}_t(a_j) = 0$ if t is not a zero of a_j). We say that $t \in \mathfrak{u}$ is a *singular point* for $P(t, \partial)$ if $\mathrm{ord}_t(a_m) > 0$, and a *regular point* otherwise. The Newton polygon of t for $P(t, \partial)$ is the highest convex polyhedron below the $m + 1$ points

$$(j, \mathrm{ord}_t(a_j)), \quad 0 \leq j \leq m.$$

To illustrate the idea, two Newton polygons of the point 0 are shown in Figure 5.3: the solid line shows the Newton polygon of 0 for $t^2\partial + t\partial + 1$, and the dashed line shows the Newton polygon of 0 for $t^3\partial + t\partial + 1$. A more general example is shown in Figure 5.4.

Write σ_t for the greatest slope of the Newton polygon (e.g. $\sigma_0 = 1$ in the first example, $\sigma_0 = 2$ in the second). With this notation, the following result holds, where $\ker_{\mathcal{B}(\mathfrak{u})}(P\bullet) := \{w \in \mathcal{B}(\mathfrak{u}) : P(t, \partial)\, w = 0\}$.

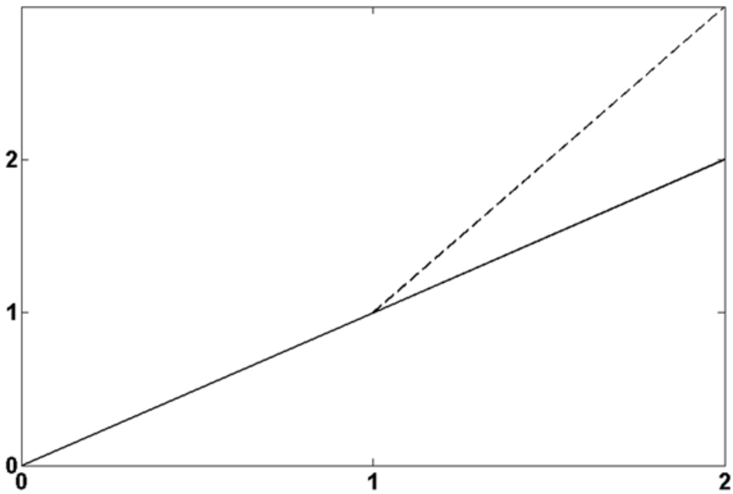

Figure 5.3. *Newton polygons*

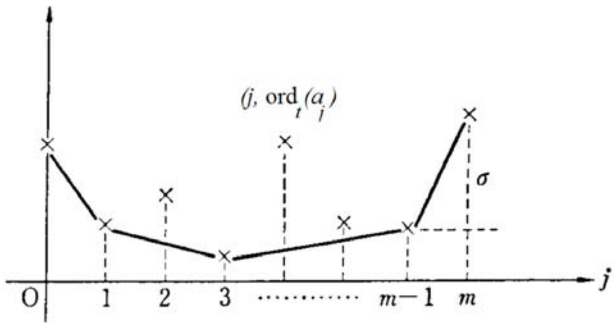

Figure 5.4. *Newton polygon (general form)*

THEOREM 5.72.– *(Komatsu [KOM 73])*

1) We have the relation

$$\dim_{\mathbb{C}}\left(\ker_{\mathcal{B}(\mathfrak{u})}(P\bullet)\right) = m + \sum\nolimits_{t\in\mathfrak{u}} \mathrm{ord}_t\,(a_m)\,. \quad [5.9]$$

2) The following conditions are equivalent:

a) $a_m(t) \neq 0$ *for every* $t \in \mathfrak{u}$.

b) $\ker_{\mathcal{B}(\mathfrak{u})}(P\bullet) \subset \mathcal{A}(\mathfrak{u})$.

c) $P(t,\partial)\, w \in \mathcal{A}(\mathfrak{u}) \Longrightarrow w \in \mathcal{A}(\mathfrak{u})$.

3) The following conditions are equivalent:

a') $\sigma_t \leq 1$ *for every* $t \in \mathfrak{u}$.

b') $\ker_{\mathcal{B}(\mathfrak{u})}(P\bullet) \subset \mathcal{D}'(\mathfrak{u})$.

c') $P(t,\partial)\, w \in \mathcal{D}'(\mathfrak{u}) \Longrightarrow w \in \mathcal{D}'(\mathfrak{u})$.

DEFINITION 5.73.– *A singular point* $t \in \mathfrak{u}$ *is said to be* regular singular *if* $\sigma_t \leq 1$, *and* irregular singular *otherwise.*

The point $t = 0$ is a regular singular point for the differential operator $t^2\partial + t\partial + 1$, and irregular singular for $t^3\partial + t\partial + 1$ and $t^3\partial + 2$.

REMARK 5.74.– *Komatsu's formalism is not the approach most commonly adopted by experts working in the field of $\mathcal{D}$-modules [MAI 93]. If we set $D = t\partial$, the commutation relation $Da - aD = \boldsymbol{\delta}(a)$ holds for every $a \in \mathbf{K}$ ([P1], section 3.1.11), where $\boldsymbol{\delta} : \mathbf{K} \to \mathbf{K}$ is the derivation $\boldsymbol{\delta}(a) = t\frac{da}{dt}$. Every linear differential equation $P(t,\partial)\, u = f$ can be expressed in the form $Q(t,D)\, u = tf$, where $Q(t,D) \in \mathbf{K}[D;\delta]$. The Newton polygon of t for $Q(t,D)$ can be defined in the same way as for $P(t,\partial)$, and, with this formalism, a singular point is regular if and only if its Newton polygon is contained in the quadrant $x \geq 0, y \leq 0$.*

EXAMPLE 5.75.–

1) Consider the differential equation $t(t\partial - 1)\, w = 0$. We have that $P(t,\partial) = t(t\partial - 1)$, which implies that $\sigma_0 = 1$, so $t = 0$ is a regular singular point for $P(t,\partial)$, and we should look for solutions in $\mathcal{D}'(\mathbb{R})$; by (5.9), it must be the case that $\dim_{\mathbb{C}}\left(\ker_{\mathcal{B}(\mathbb{R})}(P\bullet)\right) = 3$. By setting $v = (t\partial - 1)\, w$, we find that $tv = 0$, so $v = -c_1\delta$, and we need to solve $(t\partial - 1)\, w = c_1\delta$. The homogeneous equation yields $w = c_2 t$. The solutions with positive support can be found by using the unilateral Laplace transform[8].

8 See the Wikipedia article on the Laplace *transform*.

Denoting again s the Laplace variable, this gives

$$-\frac{d}{ds}(s\hat{w}) - \hat{w} = c_1 \Leftrightarrow -s\frac{d\hat{w}}{ds} - 2\hat{w} = c_1.$$

*The homogeneous equation of this second differential equation yields the solution $\hat{w} = \frac{c_2}{s^2}$. By applying the method of variation of constants, we obtain $\frac{dc_2}{ds} = c_1 s$, so $c_2 = c_1\frac{s^2}{2} + c_3$, and finally $\hat{w} = \frac{c_1}{2} + \frac{c_3}{s^2}$; by adding the solution of the homogeneous equation, we obtain the general solution $w(t) = \frac{c_1}{2} + c_3 t\Upsilon(t) + c_2 t$, where Υ is the Heaviside function (section 4.1.1(**II**), Example 4.4(4)). The $\mathbb{C}$-vector space generated by these solutions is indeed three-dimensional, and $\ker_{\mathcal{B}(\mathbb{R})}(P\bullet) = \ker_{\mathcal{D}'(\mathbb{R})}(P\bullet) = \mathbb{C}t \oplus \mathbb{C}t\Upsilon \oplus \mathbb{C}\delta \subset \mathcal{D}'(\mathbb{R})$.*

2) Consider the equation $(t^2\partial - 1)\, w = 0$. With $P(t,\partial) = t^2\partial - 1$, we have that $\sigma_0 = 2$ and, by (5.9), $\dim_{\mathbb{C}}\left(\ker_{\mathcal{B}(\mathbb{R})}(P\bullet)\right) = 3$. The point $t = 0$ is irregular singular, so $\ker_{\mathcal{B}(\mathbb{R})}(P\bullet) \nsubseteq \mathcal{D}'(\mathbb{R})$. By writing this solution in the form $\frac{\dot{w}}{w} = \frac{1}{t^2} = -\frac{d}{dt}\left(\frac{1}{t}\right)$, we obtain $w = ce^{-\frac{1}{t}}$ on $]0,+\infty[$ and $]-\infty,0[$. We therefore have the solution $w_1 \in \mathcal{C}^\infty(\mathbb{R}) : t \mapsto ce^{-\frac{1}{t}}$ for $t \in\ ]0,+\infty[$ and $t \mapsto 0$ for $t \in\ ]-\infty,0]$. Two other examples of linearly independent solutions are $e^{-1/(t+i0)}$ and $\left[e^{-\frac{1}{z}}\right] = e^{-1/(t+i0)} - e^{-1/(t-i0)}$.

(III) Systems of linear differential equations

THEOREM 5.76.– *Let $\mathbf{A} = \mathbf{K}\left[\partial; \frac{d}{dt}\right]$, where $\mathbb{C}[t] \subset \mathbf{K} \subset \mathcal{A}(\mathfrak{u})$. Then $\mathbf{A}$ is a simple non-commutative Dedekind domain and the $\mathbf{A}$-module $\mathcal{B}(\mathfrak{u})$ is injective.*

PROOF.– Since $\mathbf{K} \supset \mathbb{C}[t]$, $\mathbf{A}$ is simple, as shown in the proof of ([P1], section 3.1.11(**II**), Theorem 3.51(1)). Therefore, $\mathbf{A}$ is a non-commutative Dedekind domain by ([P1], section 3.3.4(**II**), Theorem 3.141). Since $\mathbf{K} \subset \mathcal{A}(\mathfrak{u})$, the $\mathbf{A}$-module $\mathcal{B}(\mathfrak{u})$ is divisible by Theorem 5.71, and is therefore injective by ([P1], section 3.3.4(**II**), Theorem 3.142). ■

From the perspective of algebraic analysis, by ([P1], section 3.3.3) (situation analogous to that of ([P1], section 3.4.4) in the case with constant coefficients), it would be favorable for $\mathcal{B}(\mathfrak{u})$ to be a cogenerator $\mathbf{A}$-module. For this to be possible, we must choose $\mathbf{K}$ appropriately. If $\mathbf{K} = \mathbb{C}[t]$, then $\mathbf{A}$ is the first Weyl algebra $A_1(\mathbb{C})$. However, this is not a suitable choice, since the equation $\left(t^2+1\right)w = 0$ only has the solution $w = 0$, whereas $\frac{A_1(\mathbb{C})}{A_1(\mathbb{C})(t^2+1)} \neq 0$, and $\mathcal{B}(\mathbb{R})$ is therefore not a cogenerator $\mathbf{A}$-module. This observation shows that coefficients such as t^2+1 that are not invertible in $\mathbb{C}[t]$ but which do not vanish in $\mathbb{R}$ need to be invertible in $\mathbf{K}$, which inspired U. Oberst [FRO 98] to choose $\mathbf{K} = \mathbb{C}(t) \cap \mathcal{A}(\mathfrak{u})$, i.e. the largest ring of analytic rational functions on $\mathfrak{u}$.

THEOREM 5.77.– *Let* $\mathbf{K} = \mathbb{C}(t) \cap \mathcal{A}(\mathfrak{u})$ *and suppose that* $\mathbf{A} = \mathbf{K}\left[\partial; \frac{d}{dt}\right]$. *The* $\mathbf{A}$*-module* $\mathcal{B}(\mathfrak{u})$ *is an injective cogenerator* $\mathbf{A}$*-module.*

PROOF.– By ([P1], section 3.3.3**(III)**, Corollary 3.127), since this module is injective, it suffices to show that $\operatorname{Hom}_{\mathbf{A}}(S, \mathcal{B}(\mathfrak{u})) \neq 0$ for every simple $\mathbf{A}$-module S. There exists a maximal left ideal $\mathfrak{m}$ in $\mathbf{A}$ such that $S \cong \mathbf{A}/\mathfrak{m}$ ([P1], section 2.3.5**(III)**, Theorem 2.38(1)), and Stafford's theorem states that $\mathfrak{m}$ is generated by two elements ([P1], section 3.3.4**(II)**). If the ideal $\mathfrak{m}$ is principal, then there exists $P \in \mathbf{A}^{\times}$ such that $\mathfrak{m} = \mathbf{A}P$, and $\operatorname{Hom}_{\mathbf{A}}(\mathbf{A}/\mathbf{A}P, \mathcal{B}(\mathfrak{u})) = \{w \in \mathcal{B}(\mathfrak{u}) : P(t,\partial)\, w = 0\} \neq 0$ by Theorem 5.71. Stafford's theorem can be made more precise as follows ([MCC 01], 5.7.12): if $\mathfrak{m}$ is an essential ideal (i.e. not a principal ideal), let $L \in \mathfrak{m}$ be a differential operator of minimum degree in $\mathfrak{m}$ with respect to the indeterminate ∂; there exists a differential operator P such that $\mathfrak{m} = \mathbf{A}L + \mathbf{A}P$. Thus, we have the exact sequence

$$0 \longleftarrow \mathbf{A}/\mathfrak{m} \longleftarrow \mathbf{A} \xleftarrow{\bullet \begin{bmatrix} L \\ P \end{bmatrix}} \mathbf{A}^{1\times 2}.$$

By performing right Euclidean division of P by L in $B_1 := \mathbb{C}(t)\left[\partial; \frac{d}{dt}\right]$, we find that $P = QL + R$, where $d°_{\partial}(R) < d°_{\partial}(L)$. Therefore, $R = 0$ and $P = QL$, where $Q \in B_1$; so $L(t,\partial)\, w = 0 \Rightarrow P(t,\partial)\, w = 0 \; (w \in \mathcal{B}(\mathfrak{u}))$ and $\operatorname{Hom}_{\mathbf{A}}(\mathbf{A}/\mathfrak{m}, \mathcal{B}(\mathfrak{u})) \supset \ker_{\mathcal{B}(\mathfrak{u})}(L(t,\partial)\,\bullet) \neq 0$; this inequality follows from the fact that L is not invertible in $\mathbf{A}$, as well as from (5.9). ■

Observe that the $\mathbf{A}$-module $\mathcal{M}(\Omega)$ of meromorphic functions on a complex neighborhood Ω of $\mathfrak{u}$ is neither injective nor a cogenerator module ([FRO 98], Example 6).

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The three volumes of this series of books, of which this is the second, put forward the mathematical elements that make up the foundations of a number of contemporary scientific methods: modern theory on systems, physics and engineering.

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