Iteration Theories

The Equational Logic of Iterative Processes

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To
Beatriz, Zsuzsa
and
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Preface

This monograph contains the results of our joint research over the last ten years on the logic of the fixed point operation. The intended audience consists of graduate students and research scientists interested in mathematical treatments of semantics. We assume the reader has a good mathematical background, although we provide some preliminary facts in Chapter 1.

Written both for graduate students and research scientists in theoretical computer science and mathematics, the book provides a detailed investigation of the properties of the fixed point or iteration operation. Iteration plays a fundamental role in the theory of computation: for example, in the theory of automata, in formal language theory, in the study of formal power series, in the semantics of flowchart algorithms and programming languages, and in circular data type definitions.

It is shown that in all structures that have been used as semantical models, the equational properties of the fixed point operation are captured by the axioms describing *iteration theories*. These structures include ordered algebras, partial functions, relations, finitary and infinitary regular languages, trees, synchronization trees, 2-categories, and others.

We begin with an introduction to the study of universal algebra in the framework of algebraic theories. A calculus is developed for manipulating algebraic theory terms. We proceed to develop the theory and investigate particular classes of examples. The emphasis is on equational proofs, as the title suggests.

Some History

The topics treated here originated in some investigations of Calvin C. Elgot. He was interested in finding a framework for the discussion and comparison of various approaches to the semantics of programming languages. His background in logic and algebra, combined with some prompting from Samuel Eilenberg, drew him to use the concepts and language of category theory. His primary subject was the semantics of what he called *monadic flowchart algorithms*. The word monadic refers to the simplification of the treatment of the external memory used by these algorithms. At any rate, he emphasized the fact that one could express all of the various standard semantics of such algorithms by a system of fixed point equations, or one vector fixed point equation (the *iteration equation*). In certain classes of semantic interpretations, such systems have unique fixed points, and he concentrated on these interpretations. Elgot called the algebraic systems that resulted from his investigations *iterative theories* [Elg75].

In the early 1970s, Elgot was teaching a course on mathematical semantics at the Stevens Institute, and interested Bloom in this topic. Their collaboration lasted until Elgot's death in 1980. During this period, some basic results were obtained, and the generalization, *iteration theory*, was defined. Several additional people were occasionally involved in the study of iterative and iteration theories: John Shepherdson in Bristol, and Jess Wright and Joe Rutledge at IBM, Yorktown Heights, Susanna Ginali, then a student at Chicago, Ralph Tindell and Douglas Troeger, from Stevens. Independent contributors inspired by Elgot's work include Jerzy Tiuryn, Evelyn Nelson, Ernie Manes and Michael Arbib. Of course, the work of the "ADJ" group at IBM, Jim Thatcher, Eric Wagner, and Jess Wright, provided additional stimulation.

In August 1981, Bloom gave a talk in Szeged on Elgot's work. There he learned of Ésik's results on an axiomatization of rational and iterative theories. Since that time, we have been collaborating on various aspects of iteration (see the references).

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Introduction

0.1 Mathematical Motivation

Iteration theories arose from an investigation of certain structures which occur in the syntax and semantics of the class of *flowchart* algorithms. These algorithms derive their name from the fact that they may be described by means of certain labeled directed graphs, the *flowcharts*. Descriptions by means of a list of labeled instructions are also possible. One example of a flowchart algorithm is the following familiar method to compute the n-th power of a positive integer m.

```
1.     x := m; y := n; z := 1
2.     if (y > 0) goto 3. else return z;
3.          if (y is even)
4.               {x := x*x; y := y/2; goto 2.};   else
5.               {z := z*x; y := y-1; goto 2.}
```

The list of instructions (or alternatively, the labeled directed graph) of a flowchart algorithm is a syntactic object, an *algorithm description*, or *presentation*, we might say. The meaning or semantics of the algorithm is an object of quite a distinct kind.

There are several aspects to the semantics of (flowchart) algorithms. There is the *input-output* behavior of the algorithm, a partial function from the set of all possible inputs to the set of all possible outputs. There is the stepwise behavior of the algorithm, which, on a given input is the finite or possibly infinite sequence of "states" assumed during the computation. These stepwise behaviors are modeled by the *sequacious functions* discussed below. Possibly one is interested not just in one algorithm, but all algorithms with the same "shape" that arise from a given one by reinterpreting the atomic components (in our example, the atomic components are: `y > 0; y is even; z := 1; x := x*x; y := y/2; z := z*x; y := y-1`). One may abstract from any particular interpretation of the atomic components of an algorithm by considering flowchart *schemes*, which capture only the *flow of control* of the algorithm.

In each of these cases, the corresponding semantics can be given by solving a system of fixed point equations in an appropriate domain. We illustrate this fact by computing the input-output behavior.

Let $X := \mathbf{N}^5$, the set of 5-tuples of nonnegative integers. We define several functions, corresponding to the lines of the algorithm above:

$$\begin{array}{rcl}
\varphi_1(x,y,z,m,n) & := & (m,n,1,m,n) \\
\varphi_2(x,y,z,m,n) & := & \left\{\begin{array}{ll} ((x,y,z,m,n),\ 1) & \text{if } y > 0 \\ ((x,y,z,m,n),\ 2) & \text{otherwise} \end{array}\right. \\
\varphi_3(x,y,z,m,n) & := & \left\{\begin{array}{ll} ((x,y,z,m,n),\ 1) & \text{if } y \text{ is even} \\ ((x,y,z,m,n),\ 2) & \text{otherwise} \end{array}\right. \\
\varphi_4(x,y,z,m,n) & := & (x^2, y/2, z, m, n) \\
\varphi_5(x,y,z,m,n) & := & (x, y-1, z*x, m, n) \\
\mathbf{1}_X(x,y,z,m,n) & := & (x,y,z,m,n).
\end{array}$$

Note that φ_2 and φ_3 are functions $X \to X \times \{1,2\}$; $\mathbf{1}_X$ is the identity function on X.

If $\psi : X \to X \times \{1,2\}$ and $\psi_i : X \to X$, for $i = 1,2$, then $\psi \cdot \langle \psi_1, \psi_2 \rangle$ is the function

$$x \quad \mapsto \quad \left\{\begin{array}{ll} y\psi_1 & \text{if } x\psi = (y,1) \\ y\psi_2 & \text{if } x\psi = (y,2). \end{array}\right.$$

Now consider the following system of equations in the variables $F_1, \ldots, F_5$, which range over the set of partial functions $X \to X$.

$$\begin{array}{lcl} F_1 & = & \varphi_1 \cdot F_2 \\ F_2 & = & \varphi_2 \cdot \langle F_3, \mathbf{1}_X \rangle \\ F_3 & = & \varphi_3 \cdot \langle F_4, F_5 \rangle \qquad (S) \\ F_4 & = & \varphi_4 \cdot F_2 \\ F_5 & = & \varphi_5 \cdot F_2. \end{array}$$

This system describes the partial functions computed by each line of this algorithm, as well as the flow of control. For example, the fourth equation for F_4 describes a partial function which is the composite of the function φ_4 with the partial function F_2; but F_2 is described in terms of F_3 and hence F_4 itself.

One method to solve such systems of equations is to regard the right-hand sides of the system as a function Φ which maps a 5-tuple of partial functions on X to another 5-tuple. Thus, if $G_1, \ldots, G_5$ are known partial functions, then define $(F_1, \ldots, F_5) := \Phi(G_1, \ldots, G_5)$ by replacing F_i by G_i on the right. Then one wants to find some 5-tuple which is kept fixed by Φ. Due to the properties of the partial ordering of partial functions ($f \leq g$ if $xf = xg$ whenever xf is defined), there is always a least fixed point for these systems. In fact, for most of the systems of equations arising from flowchart algorithms, there is a unique solution. But certain systems, for example,

$$F = \varphi_3 \cdot \langle F, \mathbf{1}_X \rangle$$

have infinitely many (with φ_3 as above). In these cases, it seems reasonable to say that the solution of interest is the least solution.

In this particular case, there is a unique solution of the system (S), say $G_1, \ldots, G_5$. The input-output semantics of this algorithm is then the composite:

$$\mathbf{N}^2 \xrightarrow{\alpha} \mathbf{N}^5 \xrightarrow{G_1} \mathbf{N}^5 \xrightarrow{\pi_3} \mathbf{N},$$

where $(m,n)\alpha$ is defined only if $n > 0$, and then

$$(m,n)\alpha \;:=\; (0,0,0,m,n) \quad \text{and} \quad (x,y,z,m,n)\pi_3 \;:=\; z.$$

0.2 Why Iteration Theories?

There are many other situations in theoretical computer science where one wants to find fixed points of various kinds: least fixed points in an order-theoretic context, unique fixed points, initial fixed points of endofunctors. Context free grammars describe fixed point equations, circular data type specifications describe initial algebras, recursive program schemes are themselves fixed point systems. Many of the fixed point questions may be posed as follows: for an appropriate category and endofunctor $F :\to$, find the least, unique, initial, or canonical x such that

$$x \quad = \quad F(x).$$

To give one more example, suppose that is the category of sets and suppose that F is the functor taking the set X to the set $A + X \times X$, where $+$ is disjoint union, and the function $f : X \to Y$ to the function $fF : XF \to YF$ defined by:

$$\begin{array}{rcl} a \in A & \mapsto & a \\ (x, x') \in X \times X & \mapsto & (xf, x'f). \end{array}$$

Then an "initial" fixed point of this functor can be regarded as the set of binary trees with leaves labeled by elements of A. Note that this fixed point depends on the parameter A. We can make this dependence apparent if we write $F(X) := G(X, A)$, where $G : \times \to$ is the functor whose value on the pair (X, A) is $A + X \times X$. The fixed point equation

$$X \quad = \quad G(X, A)$$

has an initial solution which we write $G^\dagger(A)$. It can be shown that $G^\dagger$ is also a functor. The fixed point functor $G^\dagger$ satisfies a number of identities, such as:

$$G^\dagger \quad \approx \quad \xrightarrow{\langle \mathbf{1}, G^\dagger \rangle} \times \xrightarrow{G}$$

where $\approx$ means "is naturally isomorphic to".

Thus, one wants not just to find fixed points, but to examine how these fixed points interact with other relevant operations, such as the composition operation. For example, what are all of the identities satisfied by the functors $G^\dagger$?

That is precisely what the study of iteration theories is about. Iteration theories describe the equational properties of the fixed point operation, in combination with other operations: composition, coproducts, and sometimes other operations (e.g. orderings and sums).

Since iteration theories occur with great frequency, our results about abstract iteration theories have as corollaries facts about:

- ordered theories, e.g. continuous functions on complete posets;
- continuous 2-theories, such as ω-functors and natural transformations on ω-categories;
- matrix theories, e.g. matrices of regular sets over some alphabet;
- synchronization trees, and other theories of trees;
- partial correctness logic.

As will be shown here, whether one is interested in least fixed points, in unique fixed points, or initial fixed points, the fixed point operation obeys all the laws that determine iteration theories. If the solution of fixed point equations has any intrinsic merit, then the study of iteration theories will be rewarding. We hope that this book will serve as an introduction to the study of iteration theories, as a reference for the technical results, and as a spur to further investigations.

0.3 Suggestions for the Impatient Reader

The mathematically sophisticated reader who wants to get to some new material as quickly as possible should glance through Chapters 1 and 2, slow down a bit for Chapters 3 and 4, and then begin reading in earnest. The first half of the results are contained in Chapters 5–8, with Chapter 6 being fundamental to everything that follows. In the remaining chapters, Chapter 8 is applied in Chapter 13, and Chapter 14 depends only slightly on the other chapters of the second half. Those interested in regular sets should read Chapters 9 and 10; those interested in structures for interpreting sequential and parallel algorithms should aim for Chapters 12 and 13. Floyd-Hoare logic is treated in Chapter 14.

0.4 A Disclaimer

Our bibliography is admittedly not a complete listing of papers related to the theory of iteration. It contains only papers directly related to the text.

0.5 Numbering

When reference is made within a chapter to Theorem 5.3, look in Section 5 of that chapter; item **7**.5.2 occurs in Chapter **7**, Section 5.

Chapter 1

Preliminary Facts

In this chapter, we establish our notation and review several concepts needed throughout the book. We assume that the reader has some familiarity with basic universal algebra and elementary category theory.

1.1 Sets and Functions

If $f : X \to Y$ is a function, the value of f on x in X will be written, in descending order of preference:

$$xf, \;\; f(x), \;\; f_x \text{ and, rarely, } \;\langle f, x\rangle.$$

The composite of the function $f : X \to Y$ with $g : Y \to Z$ will be written: $f \cdot g : X \to Z$ or just fg. The identity function $X \to X$ is denoted $\mathbf{1}_X$. For nonnegative integers n, we let $[n]$ denote the set consisting of the first n positive integers:

$$[n] \;=\; \{1, 2, ..., n\},$$

so that $[0]$ is another name for the empty set. We let $\mathbf{N}$ denote the set of nonnegative integers, and let $[\omega]$ denote the set of positive integers.

On occasion, we will need to name various sets of sequences. A *finite word* of length n, $n \geq 0$, on A is a function $u : [n] \to A$. We let A^* denote the set of finite words of elements of A, including the empty word ϵ; A^+ is $A^* \setminus \{\epsilon\}$. An *infinite word*, or ω-word on A is a function $[\omega] \to A$; A^ω denotes the set of all infinite words on A. Finally, we let A^∞ denote the union of A^ω with A^*. Subsets of A^∞ are sometimes called *languages*; subsets of A^* are finitary languages and subsets of A^ω are infinitary or ω-languages. The length of a word u in A^* is written $\mid u \mid$. *We identify A with the subset of A^* consisting of words of length* 1. If u and v are in A^* with lengths n and m respectively, we let both uv and $u \cdot v$ denote the word w of length $n+m$ with:

$$w_i \quad = \quad \begin{cases} u_i & \text{if } i \in [n]; \\ v_j & \text{if } i = n+j,\ j \in [m]. \end{cases}$$

We define $u \cdot v$ similarly if u and v are in A^∞. If $u \in A^\omega$, $u \cdot v = u$.

The word u is a *prefix* of the word v if $v = uw$, for some word w; u is a *proper prefix* of v if u is a prefix of v and $u \neq v$. If X and Y are subsets of A^∞, define the operation of set concatenation by:

$$X \cdot Y \ := \ \{x \cdot y : \ x \in X,\ y \in Y\}.$$

A *semiring* is a set S, containing two distinguished elements 0 and 1, equipped with two binary operations $+$ and $\cdot$ such that (S,+,0) is a commutative monoid, (S,$\cdot$,1) is a monoid, and $\cdot$ distributes over $+$ on the left and the right; 0 is both a right and left annihilator: $0 \cdot x = x \cdot 0 = 0$. A *semiring homomorphism* $h : S \to S'$ is a function which preserves 0 and 1 as well as the operations $+$ and $\cdot$:

$$\begin{aligned} 0h &= 0 \\ 1h &= 1 \\ (x+y)h &= xh + yh \\ (x \cdot y)h &= xh \cdot yh. \end{aligned}$$

(Usually we will write just xy and not $x \cdot y$ in semirings.)

Example 1.1.1 Some examples of semirings, other than rings, are:

- the Boolean semiring $\mathbf{B} = \{0, 1\}$ where $1 + 1 = 1$;
- the nonnegative integers N, with addition and multiplication;
- the semiring $L(A^*)$ whose elements are all subsets of A^*, with binary union and set concatenation; the multiplicative unit here is $\{\epsilon\}$.

We write an ordered pair as (a, b), and an write $(a_1, \ldots, a_n)$ for an ordered n-tuple. When $n = 1$, we identify (a) with a. The set $A \times B$ is the set of all ordered pairs (a, b), with a in A and b in B.

The disjoint union of a set A with itself can be represented by the set $A \times [2]$; similarly, the disjoint union of A with itself n times, for $n \geq 0$, can be represented by the set $A \times [n]$. For ease of notation, we usually identify the sets A and $A \times [1]$.

1.2 Posets

A partial ordering $\leq$ on the set A is a reflexive, antisymmetric and transitive binary relation on A. The pair $(A, \leq)$ is called a partially ordered set, or poset for short. The poset is ω-*complete* if there is a least element $\perp_A$ in A, and if every ω-*chain*, i.e. increasing ω-sequence

$$a_0 \ \leq \ a_1 \ \leq \ \ldots$$

has a least upper bound $\sup_n a_n$.

Note that if A_i is an ω-complete poset for each i in a set I, then $\prod_{i \in I} A_i$ is also ω-complete, with the pointwise ordering:

$$(a_i) \ \leq \ (b_i) \quad \Leftrightarrow \quad a_i \ \leq \ b_i, \qquad \text{for all } i \in I.$$

In particular, if $(A, \leq)$ is an ω-complete poset, then for any $n \geq 0$, A^n is also ω-complete.

If $(A, \leq)$ and $(B, \leq)$ are ω-complete posets, a function $f : A \to B$ is ω-*continuous* if for every ω-chain $a_n, \ n \geq 0$, in A, $\sup_n(a_n f)$ exists and $\sup_n(a_n f) = (\sup_n a_n)f$. Thus, any ω-continuous function is order preserving.

1.3 Categories

We recall some basic notions of category theory. A *category* $\mathcal{C}$ consists of a class $Ob()$ of *objects,* and for each pair a,b of objects, a (small) set $\mathcal{C}(a, b)$ of *morphisms* or *arrows* with source a and target b. If

$f \in \mathcal{C}(a, b)$, one writes $f : a \to b$. A category is equipped with an operation of *composition*

$$\begin{array}{rcl}(a,b) \times (b,c) & \to & (a,c) \\ (f,g) & \mapsto & f \cdot g,\end{array}$$

for all triples a, b, c of objects in $\mathcal{C}$. There is a distinguished morphism $\mathbf{1}_a : a \to a$ for each object a. The composition operation is required to be associative, when defined, and the morphisms $\mathbf{1}_a$ are neutral elements with respect to composition. The most familiar example of a category is , the category with each set as an object, and with (a, b) the collection of all functions from a to b. Other examples are mentioned below.

An $\mathbf{N}$-*category* is a category whose objects are the nonnegative integers. For example, the poset $\mathbf{N}$ is an $\mathbf{N}$-category having a morphism $n \to p$ exactly when $n \leq p$; thus in the poset category N, for any two objects n, p there is at most one morphism $n \to p$. More generally, any poset is a category where there is a morphism $a \to b$ when $a \leq b$.

We list some other examples of categories.

- The category of partially ordered sets and order preserving functions.
- The category *pSET* of sets and partial functions. The objects are all sets and the morphisms $X \to Y$ are all partial functions $X \to Y$.
- The category *GR* of groups. The objects are all groups; a morphism $f : a \to b$ is a group homomorphism.
- The $\mathbf{N}$-category of finite functions. The morphisms $n \to p$ are all functions $[n] \to [p]$.
- The category of real vector spaces. The objects are the vector spaces over the field of the real numbers and the morphisms are linear transformations.

The reader will probably be familiar with several classes of morphisms in any category . Suppose that $f : a \to b$ is a -morphism.

- f is an *isomorphism* if there is a morphism $g : b \to a$ such that $f \cdot g = \mathbf{1}_a$ and $g \cdot f = \mathbf{1}_b$.
- f is an *epimorphism* (epi or epic, for short) if $g = h : b \to c$ whenever $f \cdot g = f \cdot h$.
- f is a *monomorphism* (monic for short) if $g = h$ whenever $g \cdot f = h \cdot f$, for any $g, h : c \to a$.
- f is a *coequalizer* of $g, h : c \to a$ if $g \cdot f = h \cdot f$ and f is universal with this property; i.e. if $f' : a \to b'$ is any morphism such that $g \cdot f' = h \cdot f'$ then there is a unique morphism $k : b \to b'$ such that $f \cdot k = f'$.
- f is a *regular epi* if f is a coequalizer of some pair.
- f is an *equalizer* of $g, h : b \to c$ if $f \cdot g = f \cdot h$ and for any $f' : a' \to b$ such that $f' \cdot g = f' \cdot h$ there is a unique $k : a' \to a$ with $f' = k \cdot f$.
- f is called an *extremal epi* if f is epi and if whenever $f = g \cdot m$ with m a monic, it follows that m is an isomorphism.

An object t of the category is a *terminal object* if for each object x of , there is a unique morphism $x \to t$. Dually, an object i of is *initial* if for each object x of , there is a unique morphism $i \to x$. Note that if t and t' are both terminal objects, then there is a (unique) isomorphism $t \to t'$. In , the singleton sets are the terminal objects; the empty set is the only initial object. In the category of groups, a singleton group is both initial and terminal. In *pSET*, the empty set is both initial and terminal.

A category is *small* if the objects of form a small set. Thus, any $\mathbf{N}$-category is small.

If is a category, ^{op}, the *opposite category* of , is the category with the same objects and identities as satisfying $^{op}(a, b) := (b, a)$; further, $f \cdot g$ in ^{op} is $g \cdot f$ in .

Suppose that and are categories. is a *subcategory* of if each object in is an object in and each -morphism is a -morphism and if the composition operation and identities $\mathbf{1}_a$ in are those of ; is a *full subcategory* of if the -morphisms $a \to b$ between the -objects a, b are all -morphisms $a \to b$; thus a full subcategory is determined by its class of objects. is a *replete subcategory* of if whenever a is an object in and $f : a \to b$ is an isomorphism in , then b is also an object in .

A *functor*

$$U : \quad \to$$

is an assignment of an object aU in to each object a of and a morphism $fU : aU \to bU$ in to each $f : a \to b$ in . The assignment must satisfy the following two requirements:

$$\begin{aligned} (f \cdot g)U &= fU \cdot gU \\ \mathbf{1}_a U &= \mathbf{1}_{aU}. \end{aligned}$$

Examples of functors are:

- The identity functor $\mathbf{1}$ on a category .
- The inclusion functor from the category of abelian groups to GR.
- The underlying set functor $U : GR \to$; if $h : a \to b$ is a group homomorphism, $hU : aU \to bU$ is the function h on the underlying sets of a and b.
- The free group functor $F :\to GR$. If X is a set XF is the free group generated by X; if $f : X \to Y$ is a function, $fF : XF \to YF$ is the unique homomorphism with the property that $xfF = xf$, for each $x \in X$.

A functor $F :\to$ is *full* if, for each pair of objects a, b in , every -arrow $aF \to bF$ is fF, for some $f : a \to b$; the functor F is *faithful* if whenever $fF = f'F$, for $f, f' : a \to b$ in , it follows that $f = f'$. A functor $\to$ which is full, faithful and bijective on objects is an *isomorphism.*

Suppose that $U :\to$ and $F :\to$ are functors. We say that F *is a left adjoint of* U if, for each object x in , there is a morphism

$$\eta_x : x \quad \to \quad xFU,$$

called a *U-universal arrow for x,* with the following property. For any -morphism $f : x \to aU$ whose target is a U-image, there is a unique -morphism $f^\sharp : xF \to a$, such that the diagram [x‘xFU‘aU;η_x‘f‘$f^\sharp U$] commutes. (Left adjoints are essentially unique, as shown in Exercise 1.3.2, so one usually says F is *the* left adjoint.) For example, the free group functor is the left adjoint of the underlying set functor $U : GR \to$. In general, a left adjoint represents some kind of free construction.

Note that the functor F is completely determined by the collection of morphisms $\eta_x : x \to xFU$ and the objects xF, for x an object of . The value of F on -morphisms is forced by the universal property.

An *ω-diagram* in the category is a functor from the poset N of the natural numbers, considered to be a category as indicated above, to . Thus, the functor assigns to each natural number i an object a_i of , and a morphism $f_i : a_i \to a_{i+1}$. (The value of the functor on other arrows $i \leq j$ is forced.) We indicate ω-diagrams variously as follows:

$$\Gamma : \qquad a_0 \overset{f_0}{\to} a_1 \overset{f_1}{\to} a_2 \overset{f_2}{\to} \ldots$$

or

$$a_i \quad \overset{f_i}{\to} \quad a_{i+1}$$

or just (a_i, f_i). A *cone* (a, μ_i) *over the ω-diagram* Γ is an object a together with a morphism $\mu_i : a_i \to a$, for each $i \geq 0$, such that

[a$_i$‘a_{i+1}‘a; f_i‘μ_i‘μ_{i+1}] commutes for each $i \geq 0$. A cone over the ω-diagram Γ is a *colimit of* Γ if for any other cone (b, ν_i) over Γ, there is a unique morphism $\kappa : a \to b$ such that $\mu_i \cdot \kappa = \nu_i$, for all $i \geq 0$. It is clear that if both (a, μ_i) and (b, ν_i) are colimits of the same ω-diagram, then the objects a and b are isomorphic. A category is *ω-complete* if has an initial object and each ω-diagram in has a colimit in .

One last notion. Suppose that $F, G :\to$ are functors. A *natural transformation* $\tau : F \xrightarrow{\bullet} G$ assigns to each object c in an arrow, written τ_c or $c\tau$ from cF to cG in such that for each -arrow $f : c \to c'$ the square

[1‘1‘1‘1;800‘600] [cF‘cG‘c'F‘c'G;τ_c‘fF‘fG‘$\tau_{c'}$] commutes. The arrows τ_c are called the *components* of the natural transformation. When all the components of a natural transformation τ are isomorphisms, τ is called a *natural isomorphism.*

The above concepts, a small part of elementary category theory, will all be used at some time in the following chapters. For information on category theory not detailed here, two standard sources are [Mac71] and [HS73].

Exercise 1.3.1 Show that is a small category iff the disjoint union $\bigcup_{a,b}(a, b)$ is a small set, where a, b range over all objects of .

Exercise 1.3.2 Suppose that both $F, F' :\to$ are left adjoints to the functor $U :\to$. Show there is a natural isomorphism $\tau : F \stackrel{\bullet}{\to} F'$.

Exercise 1.3.3 Show that a functor $F :\to$ is an isomorphism iff there is a functor $G :\to$ such that $F \cdot G = \mathbf{1}$ and $G \cdot F = \mathbf{1}$.

Exercise 1.3.4 Let A be a poset. Show A is ω-complete iff it is ω-complete when considered to be a category.

1.4 2-Categories

We will have several occasions to make use of what we call 2-*theories*, which are a particular kind of 2-category. We review the concept of 2-category here.

Definition 1.4.1 *A* 2-**category** *consists of the following data:*

1. *A category, usually called also; the morphisms in are called* **horizontal morphisms***.*
2. *For each pair (a, b) of -objects, a category (a, b) whose objects are the horizontal morphisms $a \to b$ in . The morphisms in (a, b) are called* **vertical morphisms***; if $f : a \to b$ is a -morphism, the identity morphism $f \to f$ in (a, b) will be denoted $\mathbf{I}(f)$. The composite of vertical morphisms $u : f \to g$, $v : g \to h$ in (a, b) is written $u \circ v$, and the operation is called* **vertical composition***.*
3. *There is an operation, called* **horizontal composition** *on the vertical morphisms: if $u : f \to g$ in (a, b) and $v : f' \to g'$ in (b, c), then $u \cdot v : f \cdot f' \to g \cdot g'$ in (a, c).*

These data are subject to the following requirements:

$$
\begin{tikzcd}
a \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Downarrow u"] & b
\end{tikzcd}
\cdot
\begin{tikzcd}
b \arrow[r, bend left=60, "f'"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\Downarrow u'"] & c
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r, bend left=60, "f \cdot f'"] \arrow[r, bend right=60, "g \cdot g'"'] \arrow[r, phantom, "\Downarrow u \cdot u'"] & c
\end{tikzcd}
$$

Figure 1.1: Horizontal composition of cells

1. *Horizontal composition is associative, when defined.*

2. *The* **interchange law:** *If* $u : f \to g$ *and* $v : g \to h$ *in* (a, b), *and* $u' : f' \to g'$, $v' : g' \to h'$ *in* (b, c) *then*

$$(u \circ v) \cdot (u' \circ v') \quad = \quad (u \cdot u') \circ (v \cdot v').$$

Further,

$$
\begin{aligned}
\mathbf{I}(f) \cdot \mathbf{I}(f') &= \mathbf{I}(f \cdot f') \\
\mathbf{I}(\mathbf{1}_a) \cdot u &= u \\
u \cdot \mathbf{I}(\mathbf{1}_b) &= u
\end{aligned}
$$

for any $f, g : a \to b$, $f' : b \to c$ *in and* $u : f \to g$.

The *objects* of the 2-category are the objects of . The *underlying category* of a 2-category consists only of the objects and the horizontal morphisms of .

It is convenient to define a 2-*cell* or just *cell* $x : a \to b$ in the 2-category as a triple:

$$x \quad = \quad (u, f, g),$$

where $f, g : a \to b$ are horizontal morphisms and $u : f \to g$ is a vertical morphism. Then we may construe both horizontal and vertical composition as operations on cells:

$$(u, f, g) \cdot (v, f', g') \quad = \quad (u \cdot v,\ f \cdot f',\ g \cdot g'),$$

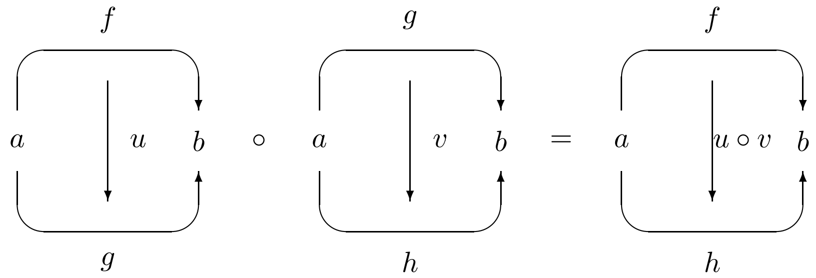

Figure 1.2: Vertical composition of cells

where $(u, f, g) : a \to b$ and $(v, f', g') : b \to c$; also

$$(u, f, g) \circ (v, g, h) \quad = \quad (u \circ v,\ f,\ h),$$

where (u, f, g) and $(v, g, h) : a \to b$. The collection of cells becomes a category whose objects are those of , and with the cells $a \to b$ as morphisms; composition in is horizontal composition of cells.

Thus, there are two important categories involved in a 2-category: the underlying category and the category of cells .

We will usually identify a horizontal morphism f with the corresponding identity vertical morphism $\mathbf{I}(f) : f \to f$. Thus, if $f : n \to p$, and $u : g \to h$ is a cell $p \to q$, then $f \cdot u$ is the vertical morphism

$$\mathbf{I}(f) \cdot u : f \cdot g \to f \cdot h.$$

The context should make clear when this convention is being used.

Example 1.4.2 Any category may be considered to be a 2-category whose only vertical morphisms are the identity morphisms $\mathbf{I}(f)$.

Example 1.4.3 Let be the category of posets and order preserving functions. is a 2-category, where for each pair of posets a, b, and each pair of order preserving maps $f, g : a \to b$, there is a vertical morphism $f \to g$ if $f \leq g$; i.e. if $xf \leq xg$, all $x \in a$.

Example 1.4.4 This is the example that motivated the concept of 2-category. Let be the category of all (small) categories; the horizontal morphisms $a \to b$

are all functors $a \to b$; the vertical morphisms $f \to g$ are all natural transformations $f \xrightarrow{\bullet} g$. If $u : f \xrightarrow{\bullet} g$ and $v : g \xrightarrow{\bullet} h$, where f, g, h are functors $a \to b$, then the vertical composite $u \circ v : f \xrightarrow{\bullet} h$ has components $u_c \cdot v_c$, for each object c of a. For natural transformations $u : f \xrightarrow{\bullet} f'$ and $v : g \xrightarrow{\bullet} g'$, where $f, f' : a \to a'$ and $g, g' : a' \to a''$, the horizontal composite $u \cdot v : f \cdot g \xrightarrow{\bullet} f' \cdot g'$ is defined as follows. For any a-object c, the square

[cfg'cf'g'cfg''cf'g';u$_c g$'v_{cf}'$v_{cf'}$'$u_c g'$] commutes, since v is a natural transformation. The diagonal is the c-component of $u \cdot v$.

Suppose that and are 2-categories.

Definition 1.4.5 *A* 2-**functor** $F :\to$ *is a functor on the underlying categories which is also a functor*

$$(a, b) \quad \to \quad (aF, bF)$$

for each pair of -objects a, b. In addition, F preserves horizontal composition of vertical morphisms.

Thus if $F :\to$ is a 2-functor, F maps a horizontal morphism $f : a \to b$ in to a horizontal morphism $fF : aF \to bF$ in , a vertical morphism $u : f \to f'$ in (a, b) to a vertical morphism $uF : fF \to f'F$ in (aF, bF), and

$$\begin{aligned}
(f \cdot g)F &= fF \cdot gF \\
(u \circ v)F &= uF \circ vF \\
(u \cdot v)F &= uF \cdot vF \\
\mathbf{1}_a F &= \mathbf{1}_{aF} \\
\mathbf{I}(f)F &= \mathbf{I}(fF),
\end{aligned}$$

for all appropriate horizontal morphisms f, g and vertical morphisms u, v.

The following proposition is quite useful.

Proposition 1.4.6 *Suppose that is a* 2*-category and that $u_i, v_i,\ i = 1, 2$, are vertical morphisms in (a, b). Let w_i be a vertical morphism in (b, c) and w_i' a vertical morphism in (d, a), $i = 1, 2$. If*

$$u_1 \circ u_2 \quad = \quad v_1 \circ v_2$$

then, when the following equations are meaningful,

$$\begin{aligned}(u_1 \cdot w_1) \circ (u_2 \cdot w_2) &= (v_1 \cdot w_1) \circ (v_2 \cdot w_2) \\ (w_1' \cdot u_1) \circ (w_2' \cdot u_2) &= (w_1' \cdot v_1) \circ (w_2' \cdot v_2).\end{aligned}$$

The easy proof uses the interchange law.

Exercise 1.4.7 Consider the map that takes $f : a \to b$ in the underlying category to the *identity cell* $(\mathbf{I}(f), f, f) : a \to b$. Show that this map determines an embedding of the underlying category of a 2-category into the category which is the identity on objects.

Exercise 1.4.8 Suppose that is a 2-category. Suppose that $u : g \to g'$ and $v : g' \to g''$ are vertical morphisms in (b, c) and that $f : a \to b$ is a horizontal morphism. Show that

$$f \cdot (u \circ v) = (f \cdot u) \circ (f \cdot v).$$

Exercise 1.4.9 Let be a 2-category. A vertical morphism $u : f \to g$ in (a, b) is a *retraction* if for some $v : g \to f$,

$$v \circ u = \mathbf{I}(g).$$

Show that if $u : f \to g$ is a retraction in (a, b), and if $u' : f' \to g'$ is a retraction in (b, c), then the horizontal composite $u \cdot u'$ is a retraction in (a, c).

1.4.1 is a 2-Category, Too

Suppose that is a 2-category. Then becomes a 2-category as follows. The objects of are those of ; the horizontal morphisms $a \to b$ in are the cells $a \to b$ in . We have already defined the horizontal composition $x \cdot y$ of cells $x : a \to b$ and $y : b \to c$. Suppose now that $x = (u, f, g)$ and $y = (u', f', g')$ are cells $a \to b$. We define a *vertical morphism* $x \to y$ as a pair (v, w) of vertical morphisms in ,

$$\begin{aligned} v : f &\to f' \\ w : g &\to g' \end{aligned}$$

such that

$$v \circ u' = u \circ w.$$

The vertical and horizontal composition of the vertical morphisms in is defined pointwise:

$$\begin{aligned}(v,w)\circ(v',w') &:= (v\circ v',\, w\circ w')\\ (v,w)\cdot(v',w') &:= (v\cdot v',\, w\cdot w').\end{aligned}$$

Exercise 1.4.10 Verify that the operations just defined do, in fact, produce vertical morphisms on cells.

Exercise 1.4.11 Verify that is a 2-category whenever is.

1.5 Σ-Trees

On several occasions, labeled trees will play a significant role. In the case of finite trees, the notion can be rephrased. But for infinite trees, we will use the definitions given here.

A *ranked set* or *signature* $\Sigma = \{\Sigma_n : n \geq 0\}$ is a family of pairwise disjoint sets indexed by the nonnegative integers. Suppose that Σ is a signature. Let $V_p = \{x_1, \ldots, x_p\}$ be a set disjoint from Σ. We occasionally call the elements of V_p *variables.* (As is customary, we identify Σ with the union $\bigcup \Sigma_n$.) The set $[\omega]^*$ is the set of all sequences of positive integers.

Definition 1.5.1 *A* **Σ-tree** $t : 1 \to p$ *is a partial function*

$$t : [\omega]^* \quad \to \quad \Sigma \cup V_p$$

which satisfies the following requirements:

- *the domain of t is a nonempty, prefix closed subset of $[\omega]^*$;*
- *if $ut \in \Sigma_n$, $n > 0$, then $(u \cdot i)t$ is defined iff $i \in [n]$;*
- *if $ut \in \Sigma_0 \cup V_p$, then u is a leaf; i.e. $(u \cdot i)t$ is not defined for any $i \in [\omega]$.*

The correspondence of this notion of tree with a more familiar one can be seen by defining the *vertices* of a tree $t : 1 \to p$ as those sequences in the domain of t; the *label* of the vertex u is the value of t on u, ut. The root of the tree is the empty sequence ϵ. The *successors* of a vertex u are the sequences $u \cdot 1, \ldots, u \cdot n$ if the label of u belongs to $\Sigma_n,\ n \geq 1$. A *leaf* of t is a vertex with no successor, i.e. a vertex whose label is either in Σ_0 or V_p. Thus, the leaves of a Σ-tree $1 \to p$ are labeled either by a letter in Σ_0 or by one of the variables $x_i,\ i \in [p]$.

A Σ-tree is *finite* if its domain is finite. The finite Σ-trees can be seen to be another representation of the Σ-terms discussed in the next chapter.

Each letter $\sigma \in \Sigma_n$ determines an *atomic tree*, also denoted $\sigma : 1 \to n$, as follows: assume first that $n = 0$. Then the atomic tree $\sigma : 1 \to 0$ has only one vertex, the root, which is also a leaf labeled σ. If $n \geq 1$ then the atomic tree $\sigma : 1 \to n$ has $1+n$ vertices; the root is labeled σ and the leaf $i \in [n]$ is labeled x_i. The tree $1 \to p$ with only one vertex labeled $x_i,\ i \in [p]$, will be denoted as x_i also.

Suppose that $t : 1 \to p$ is a Σ-tree and v is a vertex of t. The *subtree of t rooted at v* is the Σ-tree $t_v : 1 \to p$ defined as follows for $u \in [\omega]^*$:

$$ut_v \quad := \quad (v \cdot u)t.$$

A Σ-tree $t : 1 \to p$ is *regular* if t has only finitely many subtrees. We consider regular trees in Chapter 5.

Chapter 2

Varieties and Theories

We will review briefly some of the main concepts of standard universal algebra in order to motivate the definition of one of our central concepts: algebraic theories. We will show how theories are related to equational classes of algebras. Most of the facts mentioned in this chapter are well-known in some form.

2.1 Σ-Algebras

Suppose that Σ is a signature. A Σ-*algebra* $\mathbf{A}$ consists of a set A and for each letter σ in Σ_n, $n \geq 0$, a function

$$\sigma_{\mathbf{A}} : A^n \quad \to \quad A.$$

Sometimes we write σ_A rather than $\sigma_{\mathbf{A}}$. When $n = 0$, the function $\sigma_A : A^0 \to A$ may be identified with its value. The set A is called the *underlying set* of the algebra $\mathbf{A}$, and is sometimes written $|\mathbf{A}|$. (We will usually write A for the underlying set of $\mathbf{A}$, B for the underlying set of $\mathbf{B}$, etc.)

If $\mathbf{A}$ and $\mathbf{B}$ are Σ-algebras, a *homomorphism* $\mathbf{A} \to \mathbf{B}$ is a function

$$h : A \quad \to \quad B$$

such that for each $\sigma \in \Sigma_n,\ n \geq 0$, the following diagram commutes:

[A^{n}‘A‘B^{n}‘B; $\sigma_{\mathbf{A}}$‘h^{n}‘h‘$\sigma_{\mathbf{B}}$] The function $h^n : A^n \to B^n$ takes the n-tuple $(a_1, \ldots, a_n) \in A^n$ to the n-tuple $(a_1 h, \ldots, a_n h) \in B^n$.

The collection of all Σ-algebras and homomorphisms forms a category, denoted Σ-Alg. In this category, two algebras $\mathbf{A}, \mathbf{B}$ are isomorphic iff there is a bijective homomorphism $h : \mathbf{A} \to \mathbf{B}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be Σ-algebras. We say $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$ if $A \subseteq B$ and if the inclusion function $A \to B$ is a homomorphism. Equivalently, $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ if $A \subseteq B$ and for each $\sigma \in \Sigma_n,\ n \geq 0$, and each n-tuple $(a_1, \ldots, a_n)$ of elements in A,

$$\sigma_{\mathbf{A}}(a_1, \ldots, a_n) \quad = \quad \sigma_{\mathbf{B}}(a_1, \ldots, a_n).$$

We say that $\mathbf{B}$ is a *quotient* or *homomorphic image* of $\mathbf{A}$ if there is a surjective homomorphism $\mathbf{A} \to \mathbf{B}$. Finally, if for each i in a set I, $\mathbf{A}_i$ is a Σ-algebra, then the (direct) *product* $\prod_{i \in I} \mathbf{A}_i$ of these algebras has the Cartesian product $A = \prod_{i \in I} A_i$ as underlying set; the functions $\sigma_{\mathbf{A}}$ are defined pointwise. For example, if $\sigma \in \Sigma_2$, then

$$\sigma_{\mathbf{A}}((a_i), (b_i)) \quad = \quad (\sigma_{\mathbf{A}_i}(a_i, b_i)),$$

for any two I-indexed families $(a_i), (b_i) \in \prod_{i \in I} A_i$.

If K is a class of Σ-algebras, we write

- $\mathbf{S}(K)$ for the class of all Σ-algebras which are subalgebras of an algebra in K;
- $\mathbf{H}(K)$ for the class of all quotients of algebras in K;
- $\mathbf{P}(K)$ for the class of all products of algebras in K.

When $K = \{\mathbf{A}\}$ is a singleton, we write simply $\mathbf{S}(\mathbf{A})$, etc. The three operations **S, H**, and **P** are used to define the notion of a variety.

Definition 2.1.1 *Let K be a class of Σ-algebras. Then K is a* **variety** *if*

$$K \ = \ \mathbf{S}(K) \ = \ \mathbf{H}(K) \ = \ \mathbf{P}(K).$$

The first important theorem of universal algebra is that varieties are precisely those classes of Σ-algebras that can be defined by means of equations. We discuss equations in the next section.

We will have occasion to consider a class of $\mathbf{N} \times \mathbf{N}$-sorted algebras. The algebras we consider consist of sets $A(n,p)$ indexed by all pairs n, p of nonnegative integers, as well as functions

- composition: for all $n, p, q \geq 0$,

$$\begin{array}{rcl} \cdot : A(n,p) \times A(p,q) & \to & A(n,q) \\ (f,g) & \mapsto & f \cdot g \end{array}$$

- (source) tupling: for all $n, p \geq 0$,

$$\begin{array}{rcl} \langle\ \rangle : A(1,p)^n & \to & A(n,p), \\ (f_1, \ldots, f_n) & \mapsto & \langle f_1, \ldots, f_n \rangle \end{array}$$

- and constants: for $n \geq 1$,

$$i_n \ \in \ A(1,n).$$

We write 0_p for the value of the tupling operation $\langle\ \rangle$ in the case that $n = 0$. A *homomorphism* $h : A \to B$ of such algebras is an $\mathbf{N} \times \mathbf{N}$-indexed collection of functions $\varphi : A(n,p) \to B(n,p)$ such that

$$\begin{array}{rcl} (f \cdot g)\varphi & = & f\varphi \cdot g\varphi \\ \langle f_1, \ldots, f_n \rangle\varphi & = & \langle f_1\varphi, \ldots, f_n\varphi \rangle \\ i_n\varphi & = & i_n, \quad \text{for all } n \geq 1 \text{ and } i \in [n]. \end{array}$$

2.2 Terms and Equations

Let Σ be a fixed signature. Let X be any set, assumed disjoint from Σ, whose elements we call *variables*. We define a class of linguistic expressions in an inductive manner.

Definition 2.2.1 *A* **Σ-term on the variables** X *is either*

- *a letter in* Σ_0, *or*
- *a variable* $x \in X$, *or*

- *an expression* $\sigma(t_1, \ldots, t_p)$*, where* $\sigma \in \Sigma_p,\ p > 0,$ *and* $t_1, \ldots, t_p$ *are* Σ*-terms on* X*.*

When Σ is understood, we will say just *term*, instead of Σ-term.

It is convenient to assign an integer to each Σ-term t, called the *depth of* t; each term in $\Sigma_0 \cup X$ has depth 0; the depth of the term $\sigma(t_1, \ldots, t_p)$ is $1 + \max\{\text{depth } t_i :\ i \in [p]\}$.

Let $V = \{x_1, x_2, \ldots\}$ be a countably infinite set, and let V_n be the finite set $\{x_1, \ldots, x_n\}$, $n \geq 0$. Then we call a Σ-term on V_n an *n-ary term.* Note that any n-ary term is also an m-ary term, if $n \leq m$.

Exercise 2.2.2 Reformulate the notion of a Σ-term so that an n-ary term is a finite tree in the sense of Section 1.1.5.

The collection of all Σ-terms on X is itself a Σ-algebra, where for each $p \geq 0$, the operation on terms determined by the letter $\sigma \in \Sigma_p$ is the function

$$(t_1, \ldots, t_p) \quad \mapsto \quad \sigma(t_1, \ldots, t_p).$$

In particular, when $X = V_n$ we will denote the Σ-algebra of n-ary Σ-terms by

$$\Sigma T_n.$$

If $\mathbf{A}$ is a Σ-algebra, each n-ary term t induces a function

$$t_{\mathbf{A}} : A^n \ \rightarrow \ A$$

as follows. If $t = x_i$, then $t_{\mathbf{A}} : A^n \rightarrow A$ is the i-th projection function; if $t = \sigma \in \Sigma_0$, then $t_{\mathbf{A}}$ is the constant with value $\sigma_{\mathbf{A}}$. Lastly, if $t = \sigma(t_1, \ldots, t_p)$, for some $\sigma \in \Sigma_p,\ p > 0$, then $t_{\mathbf{A}}$ is the composite

The function

$$\langle (t_1)_{\mathbf{A}}, \ldots, (t_p)_{\mathbf{A}} \rangle : A^n \quad \rightarrow \quad A^p$$

is the target tupling of the p functions $(t_i)_{\mathbf{A}}, i \in [p]$;

$$(a_1, \ldots, a_n) \quad \mapsto \quad (b_1, \ldots, b_p)$$

where for each $i \in [p]$, $b_i = (t_i)_{\mathbf{A}}(a_1, \ldots, a_n)$. An *equation* between Σ-terms is a pair of n-ary terms (t, t'), for some $n \geq 0$, usually written $t = t'$. We say that the equation $t = t'$ *holds* or is *true* or *valid* in the algebra $\mathbf{A}$, or $\mathbf{A}$ *satisfies* the equation $t = t'$, when the functions $t_{\mathbf{A}}$ and $t'_{\mathbf{A}}$ are equal. It makes no difference whether we consider t and t' to be n-ary or m-ary, if both make sense. For any Σ-algebra $\mathbf{A}$, let $EQ(\mathbf{A})$ denote the set of all equations true in $\mathbf{A}$.

We indicate how the constructions **S, H, P** are related to equations.

Proposition 2.2.3 *If* $\mathbf{B}$ *is in* $\mathbf{S}(\mathbf{A})$ *or in* $\mathbf{H}(\mathbf{A})$ *then* $EQ(\mathbf{A}) \subseteq EQ(\mathbf{B})$. *If* $\mathbf{A}$ *is the product of the algebras* $\mathbf{B}_i$, $i \in I$, *then* $\cap_{i \in I} EQ(\mathbf{B}_i) \subseteq EQ(\mathbf{A})$, *and if none of the algebras* $\mathbf{B}_i$ *has an empty underlying set, then*

$$\cap_{i \in I} EQ(\mathbf{B}_i) \;=\; EQ(\mathbf{A}).$$

If E is a set of equations, a *model of* E is a Σ-algebra $\mathbf{A}$ such that $E \subseteq EQ(\mathbf{A})$. We let $Mod(E)$ denote the class of all models of E; also let $\overline{E}$ be the set of all equations which are true in each algebra in $Mod(E)$. Then $\overline{E}$ is the set of all equations which *follow logically* from E in that any model of E is a model of all equations in $\overline{E}$.

Corollary 2.2.4 *Let* E *be any set of equations between* Σ*-terms. Then* $Mod(E) = Mod(\overline{E})$ *and* $Mod(E)$ *is a variety.*

Any subclass K of Σ-Alg can be considered to be a full subcategory of Σ-Alg. As such, it is equipped with an *underlying set functor*

$$U_K : K \;\rightarrow\; .$$

The well-known Birkhoff theorem is the converse of Corollary 2.2.4. Its proof requires a preliminary result which is of independent interest.

Proposition 2.2.5 *Suppose that* K *is a class of* Σ*-algebras which is closed under* $\mathbf{S}$ *and* $\mathbf{P}$*. Then* K *has all free algebras; i.e. for any set* X *there is an algebra* XF *in* K *and a function*

$$\eta : X \;\; \rightarrow \;\; XFU_K$$

with the following universal property. For any algebra $\mathbf{A}$ *in* K *and any function* $f : X \to \mathbf{A}U_K$*, from* X *to the underlying set of* $\mathbf{A}$*, there is a unique homomorphism* $f^\sharp : XF \to \mathbf{A}$ *such that the diagram*

$[X`XFU_K`\mathbf{A}U_K; \eta`f`f^\sharp U_K]$ *commutes.*

The elements $x\eta$, $x \in X$, are called the *free generators* of the free algebra XF. We also say XF is freely generated by η, or by X, in K. When K contains an algebra with at least two elements, the function η is injective, and thus X is sometimes identified with a subset of XF.

The universal property determines this algebra up to isomorphism, since it may be rephrased as follows: F is the left adjoint of U_K. When X is finite with n elements, the corresponding free algebra is called the *n-generated free algebra.*

(Warning: sometimes we will adopt the sloppy but standard practice of not writing the underlying set functor, as in Example 2.2.7 below.)

Using Proposition 2.2.5, one can prove

Theorem 2.2.6 Birkhoff's Theorem. *Suppose that* K *is a variety of* Σ*-algebras. Then there is a set* E *of equations such that* K *is the class of all models of* E*.*

Proofs of Birkhoff's theorem and Proposition 2.2.5 may be found in most texts on universal algebra (e.g. in [Grä79, MMT87] or [Wec91]). We will prove these results in the context of "T-algebras", in Chapter 4.

Due to Birkhoff's theorem, varieties of algebras are sometimes called *equational classes.*

Example 2.2.7 Let K be the collection of *all* Σ-algebras. Since K is closed under any operations on Σ-algebras, K is a variety. The free algebras in K are term algebras over X. The Σ-algebra of terms over the set X is freely generated in K by the inclusion function of X into the set of Σ-terms. In particular, the n-generated free algebra in K is the algebra ΣT_n of n-ary Σ-terms, where the generating set is the set of variables V_n and where $\eta : V_n \to \Sigma T_n$ is the inclusion function. Further, if $f : V_n \to A$ is a function, where $A = |\mathbf{A}|$, for some Σ-algebra $\mathbf{A}$, the unique extension $f^\sharp : \Sigma T_n \to \mathbf{A}$ is the map $t \mapsto t_{\mathbf{A}}(x_1 f, ..., x_n f)$.

Example 2.2.8 A group G is torsion free if for any $x \in G$ and for any $n \geq 1$, $x^n = 1 \Rightarrow x = 1$. Let K be the collection of those groups that are homomorphic images of torsion free groups. It is easy to see that the class of torsion free groups is closed under **S** and **P**. Hence K is closed under **H, S** and **P**, showing that K can be defined by some set of equations. (In fact, K is the class of all groups, so that K is definable by any set of equations defining groups.)

2.3 Theories

To motivate the main definition, we use the following notation. If t is a p-ary Σ-term, write

$$t : 1 \quad \to \quad p.$$

More generally, we define a Σ-term $n \to p$ as an n-tuple $(t_1, \ldots, t_n)$ of p-ary terms. When $n = 0$, the empty sequence is the only term $0 \to p$, and when $n = 1$, $(t) = t$. The intuitive operation of substitution of terms for variables is exploited to define an operation of composition on terms. A rough description of the composite $s \cdot t$ of $s : 1 \to p$ and $t = (t_1, \ldots, t_p) : p \to q$ is this: the composite is the q-ary term obtained from s by substituting the term t_i for each occurrence of the variable x_i in s, $i \in [p]$. The official, pedantic definition of composition uses the universal property of the map

$$\eta : V_p \quad \to \quad \Sigma T_p.$$

Definition 2.3.1 *The* **composite** *$s \cdot t$ of $s : n \to p$ with $t : p \to q$ is defined as follows. When $n > 1$:*

$$(s_1, \ldots, s_n) \cdot t \quad := \quad (s_1 \cdot t, \ldots, s_n \cdot t).$$

When $n = 0$, $s \cdot t$ is the unique term $0 \to q$. Now suppose that $n = 1$. Given $t = (t_1, \ldots, t_p) : p \to q$, let $f_t : V_p \to \Sigma T_q$ be the function mapping x_i to t_i in ΣT_q, for each $i \in [p]$. Let $t^\sharp : \Sigma T_p \to \Sigma T_q$ be the unique homomorphism with $(x_i \eta) t^\sharp = x_i f_t$, $i \in [p]$. Then, for any $s : 1 \to p$, we define $s \cdot t$ as the value of $t^\sharp$ on the term s:

$$s \cdot t \quad := \quad s t^\sharp \;=\; s_{\Sigma T_q}(t_1, \ldots, t_p).$$

Remark 2.3.2 Note the following special case:

$$x_i \cdot (t_1, \ldots, t_n) \quad = \quad t_i,$$

for each $i \in [n]$, since the function $(t_1, \ldots, t_n)^\sharp$ maps x_i to t_i.

Occasionally, it pays to be pedantic. In this case, Definition 2.3.1 has the advantage that it allows one to give a rigorous proof that substitution is associative.

Proposition 2.3.3 *Suppose that* $s : k \to n,\ t : n \to m,\ u : m \to p$ *are terms. Then*

$$s \cdot (t \cdot u) \quad = \quad (s \cdot t) \cdot u.$$

Proof. We may as well assume that $k = 1$, since the general case follows from this one. Write $t = (t_1, \ldots, t_n)$ and $u = (u_1, \ldots, u_m)$. Using Definition 2.3.1,

$$\begin{aligned} t \cdot u \quad &= \quad (t_1 \cdot u, \ldots, t_n \cdot u) \\ &= \quad (t_1 u^\sharp, \ldots, t_n u^\sharp). \end{aligned}$$

Hence, $(t \cdot u)^\sharp$ is the unique homomorphism $\Sigma T_n \to \Sigma T_p$ such that for each $i \in [n]$,

$$x_i \eta \quad \mapsto \quad t_i u^\sharp.$$

But, by definition of the homomorphism $t^\sharp$,

$$x_i \eta t^\sharp \quad = \quad t_i.$$

Hence, for each $i \in [n]$,

$$x_i \eta t^\sharp u^\sharp \quad = \quad x_i \eta (t \cdot u)^\sharp,$$

proving

$$(t \cdot u)^\sharp \quad = \quad \Sigma T_n \xrightarrow{t^\sharp} \Sigma T_m \xrightarrow{u^\sharp} \Sigma T_p.$$

Thus,

$$\begin{aligned} s \cdot (t \cdot u) \quad &= \quad s (t \cdot u)^\sharp \\ &= \quad s t^\sharp u^\sharp \\ &= \quad (s \cdot t) u^\sharp \\ &= \quad (s \cdot t) \cdot u. \end{aligned}$$

Define the term $\mathbf{1}_n : n \to n$ as $(x_1, \ldots, x_n) : n \to n$. We let the reader verify that according to Definition 2.3.1,

$$\begin{aligned} \mathbf{1}_n \cdot t &= t \qquad (2.1) \\ t \cdot \mathbf{1}_p &= t, \qquad (2.2) \end{aligned}$$

for any term $t : n \to p$.

We now define the N-category .

Definition 2.3.4 *A morphism $n \to p$ in is a Σ-term $n \to p$. The composition operation has already been defined (Definition 2.3.1). The identity morphism $n \to n$ is the term $\mathbf{1}_n$.*

Since composition is associative, it follows from (2.1) and (2.2) that the structure is indeed a category. This category has the following property.

Proposition 2.3.5 *In the category , for each $n \geq 1$, the morphisms*

$$x_1 : 1 \to n, \ldots, x_n : 1 \to n$$

are coproduct injections making the object n the n-th copower of the object 1. Furthermore, when $n = 1$, the copower injection $x_1 : 1 \to 1$ is the identity $\mathbf{1}_1$.

Proof. We must show that for any n terms $t_i : 1 \to p$ there is a unique term $t : n \to p$ such that $x_i \cdot t = t_i$, for each $i \in [n]$. By Remark 2.3.2, the term $(t_1, \ldots, t_n)$ is one such term: $x_i \cdot (t_1, \ldots, t_n) = t_i$, for each $i \in [n]$. Now if t' is any term $n \to p$ with $x_i \cdot t' = t_i,\ i \in [n]$, we can write $t' = (t'_1, \ldots, t'_n)$ for some terms $t'_i : 1 \to p$. But again, since $x_i \cdot t' = t'_i$, it follows that $t' = t$.

The category is our first example of an algebraic theory.

Definition 2.3.6 *An* **algebraic theory**, *or* **theory** *for short, is an N-category T such that for each $n \geq 0$, there are n* **distinguished morphisms**

$$i_n : 1 \quad \to \quad n$$

with the following coproduct property. For any $p \geq 0$ and each family $f_1, \ldots, f_n$ of morphisms $1 \to p$ there is a unique morphism $f : n \to p$ such that

$$i_n \cdot f \quad = \quad f_i,$$

for each $i \in [n]$. Lastly, the distinguished morphism $1_1 : 1 \to 1$ is the identity $\mathbf{1}_1$, i.e. $1_1 = \mathbf{1}_1$.

The morphism f determined by the family $f_i, i \in [n]$, is called the *(source) tupling* of the family, and is denoted

$$\langle f_1, \ldots, f_n \rangle.$$

In the case $n = 0$, the source tupling of the empty family yields a necessarily unique morphism $0_p : 0 \to p$. It follows that for each $n \geq 0$,

$$\mathbf{1}_n \quad = \quad \langle 1_n, \ldots, n_n \rangle.$$

Thus, in an algebraic theory, the object n is the n-th copower of 1, so that 0 is the initial object. We insist moreover that distinguished copower injections be given in the specification of a theory. The condition that the coproduct injection 1_1 is the identity $\mathbf{1}_1$ is a normalizing requirement which simplifies many considerations.

When T is a theory, we write $T(n,p)$ for the hom-set of all morphisms $f : n \to p$ in T. The elements in $T(n,p)$ are called *T-morphisms*, or *morphisms in T*. A morphism in $T(1,p)$ is called a *scalar morphism*.

In the theory , the source tupling $\langle t_1, \ldots, t_n \rangle$ of the terms $t_i : 1 \to p,\ i \in [n]$, is just the n-tuple $(t_1, \ldots, t_n)$. The distinguished morphism $i_n : 1 \to n$ is the term x_i.

Remark 2.3.7 Suppose that T is an algebraic theory. Then $\langle f \rangle = f$, for any $f : 1 \to p$. Indeed, since $1_1 = \mathbf{1}_1$,

$$\begin{aligned} \langle f \rangle &= \mathbf{1}_1 \cdot \langle f \rangle \\ &= 1_1 \cdot \langle f \rangle \\ &= f. \end{aligned}$$

Definition 2.3.8 *Let T and T' be algebraic theories. A* **theory morphism**

$$\varphi : T \quad \rightarrow \quad T'$$

is a collection of functions $T(n,p) \rightarrow T'(n,p)$, for each $n, p \geq 0$, such that

$$(f \cdot g)\varphi \quad = \quad f\varphi \cdot g\varphi \tag{2.3}$$

$$i_n\varphi \quad = \quad i_n, \qquad \textit{for each} \;\; i \in [n], \;\; n \geq 1, \tag{2.4}$$

for all appropriate f, g in T.

Exercise 2.3.9 Show that for any theory morphism $\varphi : T \rightarrow T'$,

$$\langle f_1, \ldots, f_n \rangle \varphi \quad = \quad \langle f_1\varphi, \ldots, f_n\varphi \rangle, \tag{2.5}$$

all $f_i : 1 \rightarrow p$, $i = 1, \ldots, n$ in T.

Exercise 2.3.10 Show that $\varphi : T \rightarrow T'$ is a theory morphism iff φ satisfies equations (2.4), (2.5), as well as the equation (2.3) just for scalar f.

If $\varphi : T \rightarrow T'$ is a theory morphism, φ is (or determines) a functor $T \rightarrow T'$ which preserves objects and distinguished morphisms. Conversely, any such functor is a theory morphism. The collection of theories and theory morphisms determines a category .

Definition 2.3.11 *The objects of* **the category** *are all theories. A morphism $\varphi : T \rightarrow T'$ is a theory morphism.*

A theory morphism $\varphi : T \rightarrow T'$ is an *isomorphism* in iff $\varphi : T(n,p) \rightarrow T'(n,p)$ is a bijection, for each $n, p \geq 0$.

If T and T' are theories, T is a *subtheory* of T' if $T(n,p) \subseteq T'(n,p)$, for each $n, p \geq 0$, and the inclusion $T \rightarrow T'$ is a theory morphism. T is a *quotient* of T' if there is a surjective theory morphism $T' \rightarrow T$; lastly, if T_i, $i \in I$, are theories, then $T = \prod_{i \in I} T_i$ is a theory when $T(n,p)$ is defined as the direct product of the sets $T_i(n,p)$, $i \in I$, and if the theory operations are defined pointwise. Then each projection map

$$\prod_{i \in I} T_i \quad \rightarrow \quad T_j$$

becomes a theory morphism. The operations **S**, **H** and **P** on classes of Σ-algebras are now meaningful on classes of theories.

Exercise 2.3.12 Let T be an algebraic theory and let T' be a collection of morphisms in T. If T' contains each distinguished morphism, and contains $f \cdot g$, whenever it contains $f : 1 \to n$ and $g : n \to p$, and contains $\langle f_1, \dots, f_n \rangle : n \to p$ whenever it contains each $f_i : 1 \to p$, then T' is a subtheory of T.

Definition 2.3.13 *A* **theory congruence** $\approx$ *on an algebraic theory T is a family $\approx_{np}$ of equivalence relations on each set $T(n,p)$, $n, p \geq 0$, such that*

$$\begin{array}{rcl} f \ \approx_{np} \ g \text{ and } f' \ \approx_{pq} \ g' & \Rightarrow & f \cdot f' \ \approx_{nq} \ g \cdot g' \\ f_i \ \approx_{1p} \ g_i, \ i \in [n] & \Rightarrow & \langle f_1, \dots, f_n \rangle \ \approx_{np} \ \langle g_1, \dots, g_n \rangle. \end{array}$$

We will usually omit the subscripts and write just $f \approx g$. Alternatively, sometimes we write $f \equiv g \ (\theta)$ to indicate that the morphisms f and g are congruent modulo the congruence θ. If $\approx$ is a theory congruence on T, we let $[f]$, or $f/\approx$, denote the equivalence class of the morphism $f : n \to p$. Then the collection of equivalence classes becomes a theory $T/\approx$, where

$$\begin{array}{rcl} [f] \cdot [g] & := & [f \cdot g] \\ \langle [f_1], \dots, [f_n] \rangle & := & [\langle f_1, \dots, f_n \rangle]. \end{array}$$

The distinguished morphism i_n in $T/\approx$ is the equivalence class $[i_n]$. The *quotient map* (or *natural map*) taking a T-morphism f to $[f]$ is a theory morphism

$$T \ \to \ T/\approx .$$

The next result is obvious.

Proposition 2.3.14 *Suppose that θ_i, $i \in I$, are theory congruences on the theory T. Then the relation*

$$f \approx g \ \Leftrightarrow \ f \equiv g \ (\theta_i) \text{ for all } i \in I$$

is also a theory congruence. Hence, if $R = R_{np}$ is any family of relations on $T(n,p)$, $n, p \geq 0$, there is a smallest theory congruence θ on T such that $f \ R \ g \ \Rightarrow \ f \ \theta \ g$.

Theory congruences and theory morphisms are related to one another in the same way that homomorphisms of Σ-algebras are related to congruences on Σ-algebras, as we indicate next.

If $\varphi : T \to T'$ is a theory morphism, then the *kernel of* φ, $\ker\varphi$, is the family of equivalence relations on $T(n,p)$ defined by:

$$f \;\equiv\; g\ (\ker\varphi) \quad\Leftrightarrow\quad f\varphi = g\varphi.$$

We will usually write

$$f \;\sim_\varphi\; g$$

instead of $f \;\equiv\; g\ (\ker\varphi)$.

Proposition 2.3.15 *For any theory morphism $\varphi : T \to T'$, $\ker\varphi$ is a congruence on T. Conversely, if $\approx$ is any congruence on T there is a theory T' and a surjective theory morphism $\varphi : T \to T'$ such that $\approx$ is $\ker\varphi$.*

We now show that the term theories can be characterized by a universal property. For each signature Σ, let

$$\eta_\Sigma : \Sigma \quad \to$$

be the function which maps the letter $\sigma \in \Sigma_n,\ n \geq 0$, to the term

$$\sigma\eta_\Sigma \quad := \quad \sigma(x_1, \ldots, x_n)$$

in $(1,n)$.

We will usually write just η instead of η_Σ and identify σ with $\sigma\eta$.

Proposition 2.3.16 *Let Σ be any signature. For any theory T and any rank preserving function $\varphi : \Sigma \to T$, i.e. for each $n \geq 0$, φ maps letters in Σ_n to morphisms $1 \to n$ in T, there is a unique theory morphism $\varphi^\sharp :\to T$ such that the diagram*

$[\Sigma$“$T;\eta_\Sigma$‘φ‘$\varphi^\sharp]$ commutes. In short, is freely generated in by η_Σ : $\Sigma \to$.

Proof. In this argument we use the fact that by definition, has the *unique factorization property:* each morphism $1 \to p$ is either a variable or factors uniquely as $\sigma \cdot \langle t_1, \ldots, t_n \rangle$, for some letter $\sigma \in \Sigma_n$ and some terms $t_i,\ i \in [n]$. (Tree theories enjoy the same property, and we will exploit this fact in Chapter 5.) We use induction on the depth of terms in in order to define $\varphi^\sharp$. We need only define $\varphi^\sharp$ on scalar morphisms, since all theory morphisms preserve source tupling. If $t : 1 \to n$ has depth 0, then either $t \in \Sigma_0$ or t is one of the distinguished morphisms $x_i : 1 \to n$. In the first case, $t\varphi^\sharp$ is defined as the element $t\varphi : 1 \to n$ in T; in the second $x_i\varphi^\sharp := i_n : 1 \to n$ in T. Now assume that $\varphi^\sharp$ has been defined on all terms $1 \to n,\ n \geq 0$, of depth at most k. Let $t : 1 \to n$ have depth $k + 1$. Then, for some $p > 0$ and some letter $\sigma \in \Sigma_p$,

$$t \quad = \quad \sigma(t_1, \ldots, t_p),$$

where $t_i : 1 \to n$ are terms of depth at most k. Then, in the theory ,

$$t \quad = \quad (\sigma\eta) \cdot \langle t_1, \ldots, t_p \rangle.$$

Hence, if ψ is any theory morphism,

$$t\psi \quad = \quad (\sigma\eta\psi) \cdot \langle t_1\psi, \ldots, t_p\psi \rangle.$$

Since we want $\sigma\eta\varphi^\sharp = \sigma\varphi$, we *define*

$$t\varphi^\sharp \quad := \quad (\sigma\varphi) \cdot \langle t_1\varphi^\sharp, \ldots, t_p\varphi^\sharp \rangle.$$

For terms $t : n \to p$, for $n \neq 1$, if $\varphi^\sharp$ is any theory morphism,

$$t\varphi^\sharp \quad = \quad \langle (1_n \cdot t)\varphi^\sharp, \ldots, (n_n \cdot t)\varphi^\sharp \rangle.$$

Hence we make this equation the definition of $\varphi^\sharp$ for morphisms whose source is not 1. This completes the definition of $\varphi^\sharp$. We omit the routine verification that with this definition, $\varphi^\sharp$ is indeed a theory morphism. We note only that at every stage, the definition was forced. Hence $\varphi^\sharp$ is uniquely determined by φ.

Remark 2.3.17 Let $^{\mathbf{N}}$ denote the category of **N**-ranked sets. The objects in this category are families of sets (X_n) indexed by the nonnegative integers $n \geq 0$. A morphism $h : (X_n) \to (Y_n)$ is a family of functions $h_n : X_n \to$

$Y_n,\ n \geq 0$. There is a functor U from to $^{\mathbf{N}}$. On objects, U is defined as follows:

$$\begin{array}{rcl} U: & \to & ^{\mathbf{N}} \\ T & \mapsto & (X_n), \end{array}$$

where for each $n \geq 0,\ X_n = T(1,n)$. The theory morphism $\varphi : T \to T'$ maps under U to the family of functions $T(1,n) \to T'(1,n)$ determined by φ. Now Proposition 2.3.16 can be rephrased as follows: *the functor U has a left adjoint.*

Exercise 2.3.18 Let $^{\mathbf{N}\times\mathbf{N}}$ denote the category of $\mathbf{N} \times \mathbf{N}$-ranked sets. An object (X_{np}) in this category is a collection of sets indexed by pairs of non-negative integers. Morphisms are defined in the obvious way. There is a forgetful functor $U : \to\ ^{\mathbf{N}\times\mathbf{N}}$, which takes the theory T to the collection $(T(n,p))$. Show U has a left adjoint.

Exercise 2.3.19 Using the terminology of the previous exercise, suppose Δ is an $\mathbf{N} \times \mathbf{N}$-ranked set. Show the theory freely generated by Δ is also freely generated by an $\mathbf{N}$-ranked set Σ which is constructed from Δ in a canonical way.

Corollary 2.3.20 *For any theory T there is some signature Σ and a surjective theory morphism*

$$\to T.$$

Proof. Suppose that Σ is any signature such that there is a surjective rank preserving function $\varphi : \Sigma \to T$. For example, let $\Sigma_n := T(1,n)$. Then by the preceding proposition, φ extends to a theory morphism $\to T$.

We give some examples of algebraic theories.

Example 2.3.21 The smallest theory **Tot**. A morphism $n \to p$ in **Tot** is a function $[n] \to [p]$; the distinguished morphism $i_n : 1 \to n$ is the function with value i; composition is function composition. **Tot** is initial in the category of theories: if T is any theory, there is a unique theory morphism **Tot** $\to T$ which maps the function $f : [n] \to [p]$ to the morphism $\langle (1f)_p, \ldots, (nf)_p \rangle$, where jf is the value of f on the integer $j \in [n]$.

Example 2.3.22 Define the theory $[X]$ as follows. Let Σ be the signature with $\Sigma_0 = X$ and $\Sigma_n = \emptyset$ otherwise, then

$$[X] \quad := \quad .$$

Exercise 2.3.23 Show that the following theory T is isomorphic to $[X]$. A morphism $f : n \to p$ in T is a function $f : X \cup [n] \to X \cup [p]$ such that $xf = x$, for all $x \in X$. Theory composition is function composition. Note that a morphism $1 \to 0$ is determined by the value of the function on 1.

Every theory must contain the distinguished morphisms $i_n : 1 \to n$, for each $i \in [n]$, $n \geq 0$, as well as all morphisms obtainable from them by source-tupling. The *smallest theory* **Tot** contains only these morphisms. Note that in the case that Σ is the empty signature, is isomorphic to **Tot**. Thus, Proposition 2.3.16 has as a corollary the fact that **Tot** is initial in .

Example 2.3.24 The trivial theories. If T is a theory in which the two distinguished morphisms $1_2 : 1 \to 2$ and $2_2 : 1 \to 2$ are identical, then there is at most one morphism $1 \to p$, for each $p \geq 0$. (Hence, there is at most one morphism $n \to p$, for all $n \geq 0$.) Indeed, if $f, g : 1 \to p$, then

$$\begin{aligned} f &= 1_2 \cdot \langle f, g \rangle \\ &= 2_2 \cdot \langle f, g \rangle \\ &= g. \end{aligned}$$

Exercise 2.3.25 Show that up to isomorphism there are only two trivial theories. *Hint:* Consider $T(1, 0)$.

It is not difficult to see that T is nontrivial iff the unique theory morphism **Tot** $\to T$ is injective. In any theory, trivial or not, we will usually identify the function $\rho : [n] \to [p]$ with the morphism $\rho' : n \to p$ in T such that $i_n \cdot \rho' = j_p$ when $i\rho = j \in [p]$.

The morphisms in any theory corresponding to the functions in **Tot** are called the base morphisms, and will usually be denoted by Greek letters. We spell this out.

Definition 2.3.26 *In any algebraic theory, a* **base morphism** $\rho : n \to p$ *is either* 0_p *if* $n = 0$*, or a distinguished morphism if* $n = 1$ *or, is a source tupling of distinguished morphisms, when* $n \geq 2$.

We will call a base morphism injective, surjective, etc. when it corresponds to a function with this property. If ρ is a bijective base morphism, ρ^{-1} denotes its inverse.

If T is a nontrivial theory, the base morphisms in T form a subtheory isomorphic to **Tot**.

Example 2.3.27 Let A be a set. The theory A has as morphisms $n \to p$ all functions

$$A^p \ \to \ A^n.$$

Note the reversal of direction. The distinguished morphism $i_n : 1 \to n$ is the i-th projection function $A^n \to A$; the composite of $f : n \to p$ with $g : p \to q$ is the function composite

$$A^q \xrightarrow{g} A^p \xrightarrow{f} A^n.$$

The tupling $\langle f_1, \ldots, f_n \rangle$ of the morphisms $f_i : 1 \to p$ is the function

$$\begin{array}{rcl} A^p & \to & A^n \\ a & \mapsto & (af_1, \ldots, af_n). \end{array}$$

A morphism $1 \to 0$ in A can be identified with an element in the set A. Note that the two trivial theories are isomorphic to theories of the form A, namely the case that A is a singleton and the case that A is empty.

Example 2.3.28 For a fixed set A, the theory A has as morphisms $n \to p$ all partial functions

$$A \times [n] \quad \to \quad A \times [p].$$

Composition in the theory is composition of partial functions; identifying A with $A \times [1]$, the distinguished morphism $i_n : 1 \to n$, is the function

$$\begin{array}{rcl} A & \to & A \times [n] \\ a & \mapsto & (a, i). \end{array}$$

The tupling $\langle f_1, \ldots, f_n \rangle$ of the morphisms $f_i : 1 \to p$ is the partial function

$$\begin{array}{rcl} A \times [n] & \to & A \times [p] \\ (a, i) & \mapsto & af_i. \end{array}$$

Note that the theory A has the property that there is a unique morphism $1 \to 0$.

Example 2.3.29 Matrix theories. Let S be a semiring. The theory $\mathbf{Mat}_S$ has all n by p matrices over S as morphisms $n \to p$. The composition operation is matrix multiplication. The matrix $i_n : 1 \to n$ is the row matrix whose only nonzero entry is a 1 in the i-th column. Thus, if $A : n \to p$ in $\mathbf{Mat}_S$, $i_n \cdot A$ is the i-th row of A, and the source tupling a1n of the row matrices a_i is the matrix $n \to p$ whose i-th row is a_i.

In $\mathbf{Mat}_S$, one can define an addition operation on the set of morphisms $n \to p$, for each $n, p \geq 0$: if $f, g : n \to p$, then the (i, j)-th entry of $f + g$ is the sum in S of f_{ij} with g_{ij}. Composition distributes over addition on both sides:

$$\begin{aligned} f \cdot (g + h) &= f \cdot g + f \cdot h \\ (g + h) \cdot k &= g \cdot k + h \cdot k, \end{aligned}$$

for all appropriate f, g, h, k. We let 0_{np} denote the $n \times p$ matrix of all zero's. In matrix theories, 0_{10} is the unique morphism $1 \to 0$, the only 1×0 matrix. A base morphism in a nontrivial matrix theory is a matrix such that each row contains exactly one occurrence of 1; all other entries are 0.

Example 2.3.30 Theories of trees. In the theory $\Sigma\,\mathbf{TR}$, a morphism $1 \to p$ is a Σ-tree $t : 1 \to p$. (See Chapter 1, Section 1.5.) A morphism $n \to p$ in $\Sigma\,\mathbf{TR}$ is an n-tuple $(t_1, \ldots, t_n)$ of morphisms $1 \to p$. The composite of $t : 1 \to p$ with $s = (s_1, \ldots, s_p) : p \to q$ is the tree $t \cdot s : 1 \to q$ defined as follows: for any word u in $[\omega]^*$,

$$u(t \cdot s) \quad := \quad \begin{cases} ut & \text{if } ut \in \Sigma_0 \text{ or } u \text{ is a non-leaf vertex of } t; \\ vs_i & \text{if } u = wv,\ wt = x_i \text{ and } vs_i \text{ is defined;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

When $n \neq 1$, the composite of $t = (t_1, \ldots, t_n) : n \to p$ with $s : p \to q$ is defined as

$$t \cdot s \quad := \quad (t_1 \cdot s, \ldots, t_n \cdot s).$$

More intuitively, the tree $t \cdot s$ is obtained by attaching the tree s_i to each leaf of t labeled x_i, $i \in [p]$. The distinguished morphism i_n is the tree $x_i : 1 \to n$, for each $i \in [n]$. The collection of finite trees forms a subtheory of $\Sigma\,\mathbf{TR}$ isomorphic to .

Next, we give an example of an important class of theory morphisms.

Example 2.3.31 Consider the theory $T =$, for some signature Σ. Let $\mathbf{A}$ be a Σ-algebra. Then by Proposition 2.3.16, there is a unique theory morphism $\varphi_{\mathbf{A}} : T \to A$ such that $\varphi_{\mathbf{A}}$ maps the Σ-term $\sigma(x_1, \ldots, x_n) : 1 \to n$ in T to the corresponding function $\sigma_{\mathbf{A}} : A^n \to A$. In fact, $\varphi_{\mathbf{A}}$ maps the term $t = \langle t_1, \ldots, t_n \rangle : n \to p$ to the function $t_{\mathbf{A}} = \langle (t_1)_{\mathbf{A}}, \ldots, (t_n)_{\mathbf{A}} \rangle : A^p \to A^n$.

We will use the morphisms $\varphi_{\mathbf{A}}$ in the next section.

2.4 The Theory of a Variety

We now show how every variety of Σ-algebras determines an algebraic theory, and that every algebraic theory arises in this way. Suppose that $K = Mod(E)$ is a variety. When needed we may assume that $E = \overline{E}$, by Corollary 2.2.4. We will show that E determines a theory congruence $\sim_E$ on the free theory ; the resulting theory $/E$ is called the *theory of* K. Conversely, we show if T is any theory, T is isomorphic to the theory of some variety.

Now assume $K = Mod(E)$. Define the relation $\sim_E$ on (n, p) as follows:

Definition 2.4.1 *For Σ-terms $t, t' : 1 \to p$ in ,*

$$t \sim_E t' \quad \Leftrightarrow \quad \text{the equation } t = t' \text{ belongs to } \overline{E}.$$

For $t, t' : n \to p$, define

$$t \sim_E t' \quad \Leftrightarrow \quad i_n \cdot t \sim_E i_n \cdot t', \quad \text{for each } i = 1, \ldots, n.$$

An equation $t = t'$ belongs to $\overline{E}$ iff $t_{\mathbf{A}} = t'_{\mathbf{A}}$ in every algebra $\mathbf{A}$ in K. Thus, for $t, t' : n \to p$ in ,

$$\begin{aligned} t \sim_E t' \quad &\Leftrightarrow \quad t\varphi_{\mathbf{A}} = t'\varphi_{\mathbf{A}} \\ &\Leftrightarrow \quad t \sim_{\varphi_{\mathbf{A}}} t', \end{aligned}$$

for all $\mathbf{A}$ in K, where the theory morphisms $\varphi_{\mathbf{A}} :\to A$ were just defined in Example 2.3.31. It follows immediately from Proposition 2.3.15 that the family $\sim_E$ is a theory congruence.

We have proved

Proposition 2.4.2 *For each variety $K = Mod(E)$ of Σ-algebras, the relation $\sim_E$ is a theory congruence on .*

When the variety K is $Mod(E)$, we write $/E$ for the theory of K. When T is the theory of the variety K, then, as shown in the next exercise, $T(1, n)$ is the free n-generated algebra in K.

Exercise 2.4.3 Suppose that $K = Mod(E)$ is a variety of Σ-algebras and T is the resulting theory $/E$. Show that for each $n \geq 0$, the quotient map of the Σ-algebra of n-ary terms ΣT_n onto $T(1,n)$ induces a Σ-algebra structure on $T(1,n)$ such that $T(1,n)$ is the free algebra in K, freely generated by

$$\overline{\eta} \quad := \quad V_n \xrightarrow{\eta_n} \Sigma T_n \to T(1,n).$$

Example 2.4.4 Let K be the equational class of commutative monoids. We can axiomatize K by a set E of equations which say that $+$ is a commutative, associative binary operation with a neutral element 0. Thus, the corresponding signature Σ has one letter $+$ in Σ_2 and one letter 0 in Σ_0. It can be shown readily that any Σ-term on the variables x_1 and x_2 is provably equal to one of the form

$$nx_1 + mx_2,$$

for nonnegative integers n and m, where nx is an abbreviation for $x + \ldots + x$, n times. Thus, $\Sigma T_2/E$ is isomorphic to the monoid $\mathbf{N} \times \mathbf{N}$, which is freely generated in K by the generators $(1,0)$, $(0,1)$. Indeed, a concrete description of the theory T of commutative monoids is the following. $T(1,p) = \mathbf{N}^p$, the collection of all $1 \times p$ matrices of nonnegative integers; $T(n,p)$ is, of course, the collection of n-tuples of morphisms in $T(1,p)$, and hence the collection of $n \times p$ matrices over N. The composite of $t = [n_1, \ldots, n_p] : 1 \to p$ with $\langle s_1, \ldots, s_p \rangle : p \to q$ is

$$t \cdot s \quad = \quad \sum_{i=1}^{p} n_i \, s_i.$$

Recall that each morphism s_i is a $1 \times q$ matrix of nonnegative integers. For any such row matrix, we define

$$n[m_1, \ldots, m_q] \quad := \quad [nm_1, \ldots, nm_q].$$

Thus composition is given by matrix multiplication, showing that this theory is (isomorphic to) the matrix theory $\mathbf{Mat}_{\mathbf{N}}$.

Example 2.4.5 If K is the variety of commutative groups, the theory of K is the matrix theory $\mathbf{Mat}_Z$, where Z is the semiring of all integers.

Now we show that each theory T is determined by a variety of Σ-algebras, for at least one signature Σ.

Proposition 2.4.6 *If T is any theory, there is a signature Σ and a variety $K = Mod(E)$ of Σ-algebras such that T is isomorphic to $/E$.*

Proof sketch. By Corollary 2.3.20, there is a signature Σ and a surjective theory morphism $\varphi :\rightarrow T$. Define E as the set of all equations $s = t$ with

$$s\varphi \quad = \quad t\varphi,$$

where $s, t : 1 \rightarrow n$ in .

Remark 2.4.7 When the signature Σ is defined by $\Sigma_n := T(1, n)$, for each $n \geq 0$, and the map $\rightarrow T$ is induced by the function $\sigma \in \Sigma_n \mapsto \sigma : 1 \rightarrow n$ in T, then the corresponding variety K is the collection of all Σ-algebras $\mathbf{A}$ satisfying the following equations:

$$\begin{aligned} i_n(x_1, \ldots, x_n) &= x_i \\ h(x_1, \ldots, x_p) &= f(g_1(x_1, \ldots, x_p), \ldots, g_n(x_1, \ldots, x_p)), \end{aligned}$$

for each $i \in [n]$ and $h = f \cdot \langle g_1, \ldots, g_n \rangle$ in T, where $f : 1 \rightarrow n$, $g_j : 1 \rightarrow p$, $j \in [n]$.

Remark 2.4.8 We have shown in Proposition 2.4.2 and Proposition 2.4.6 that the map from varieties to theories is surjective. We might call two varieties *equivalent* when they determine the same theory, up to isomorphism. In fact, perhaps the most important reason for considering theories in connection with varieties is that there is something arbitrary in the standard signature dependent treatment. Consider the following example. Let Σ be the signature having one letter in Σ_0, Σ_1 and Σ_2, namely the letters 1, $^{-1}$ and $\cdot$, respectively; otherwise Σ_n is empty. Let E be the following set of equations between Σ-terms:

$$\begin{aligned} 1 \cdot x &= x \\ x \cdot 1 &= x \\ x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\ x \cdot x^{-1} &= 1 \\ x^{-1} \cdot x &= 1 \end{aligned}$$

where we are writing $x \cdot y$ rather than $\cdot(x, y)$, etc. Now let Σ' be the signature with one letter 1 in Σ'_0 and one letter $\odot$ in Σ'_2. Let E' be the following set of equations:

$$\begin{aligned} x \odot x &= 1 \\ x \odot 1 &= x \\ 1 \odot (1 \odot x) &= x \\ (x \odot (1 \odot y)) \odot (1 \odot z) &= x \odot (1 \odot (y \odot (1 \odot z))). \end{aligned}$$

Then, even though the signatures Σ and Σ' are distinct, a Σ-algebra $\mathbf{A}$ in $Mod\,(E)$ can be transformed into a Σ' algebra $\mathbf{A}'$ in $Mod\,(E')$ having the same underlying set. Indeed, given $\mathbf{A}$, define $x \odot y := x \cdot y^{-1}$. Conversely, any Σ'-algebra $\mathbf{B}$ in $Mod\,(E')$ can be transformed into a Σ-algebra in $Mod\,(E)$ with the same underlying set by defining

$$\begin{aligned} x^{-1} &:= 1 \odot x \\ x \cdot y &:= x \odot (1 \odot y). \end{aligned}$$

Homomorphisms are preserved under these transformations. Clearly, $Mod\,(E)$ is the usual version of the equational class of groups; $Mod\,(E')$ is a nonstandard version of the class of groups. What the two versions have in common is that, up to isomorphism, each determines the same algebraic theory. It is the algebraic theory which is intrinsic to groups. The use of theories avoids the necessity of making a somewhat arbitrary choice of signature.

Exercise 2.4.9 Find a theory morphism $\varphi : T \to T'$ which is an epi in but is not a surjection. *Hint:* Let $\Sigma_1 = \{f\}$ and $\Sigma_n = \emptyset$ otherwise. Let T be the theory of the variety of Σ-algebras determined by the equation

$$f(x_1) \quad = \quad f(x_2).$$

Let T' be the theory of Σ'-algebras, where $\Sigma'_0 = \{c\}$ and $\Sigma'_n = \emptyset$ otherwise.

Chapter 3

Theory Facts

In this chapter we collect some technical results about the category of theories, as well as document some observations which will be used frequently in later chapters. In the last section, we introduce the concept of a 2-theory.

3.1 Pairing and Separated Sum

It is useful to introduce two operations on theories, each of which is definable in terms of source tupling. First, we make the following observation.

Proposition 3.1.1 *In any theory, the object $n+m$ is the coproduct of the objects n and m. More precisely, the base morphisms $\kappa : n \to n+m$ and $\lambda : m \to n+m$ corresponding to the inclusion and translated inclusion function are coproduct injections.*

Proof. We may specify κ and λ

$$\begin{aligned} \kappa : n &\to n+m \\ \lambda : m &\to n+m \end{aligned}$$

as follows:

$$\begin{aligned} i_n \cdot \kappa &= i_{n+m}, & &\text{for each } i \in [n]; \\ i_m \cdot \lambda &= (n+i)_{n+m}, & &\text{for each } i \in [m]. \end{aligned}$$

We should write $\kappa_{n,n+m}$, $\lambda_{m,n+m}$, but we will omit the subscripts when the source and target of these morphisms can be deduced from the context. Now if $f : n \to p$ and $g : m \to p$, we must show that there is a unique morphism $\langle f, g\rangle : n + m \to p$ such that $\kappa \cdot \langle f, g\rangle = f$ and $\lambda \cdot \langle f, g\rangle = g$. But, the definitions of κ and λ require that

$$\langle f, g\rangle : n + m \quad \to \quad p$$

is the unique morphism $n + m \to p$ such that

$$i_{n+m} \cdot \langle f, g\rangle \quad = \quad \begin{cases} i_n \cdot f & \text{if } i \in [n]; \\ j_m \cdot g & \text{if } i = n + j,\ j \in [m]. \end{cases}$$

The operation taking $f : n \to p$ and $g : m \to p$ to $\langle f, g\rangle : n+m \to p$ is called *(source) pairing*. Note that the targets of f and g must be the same in order that $\langle f, g\rangle$ be meaningful.

If $f : n \to p$ and $g : m \to q$ are any two morphisms in a theory, we define the *separated sum*

$$f \oplus g : n + m \ \to \ p + q$$

as the source pairing

$$\langle f \cdot \kappa,\ g \cdot \lambda\rangle$$

where $\kappa : p \to p + q$ and $\lambda : q \to p + q$ are the coproduct injections, as above. Note that using separated sum, we can write the coproduct injections $\kappa : n \to n + p$ and $\lambda : p \to n + p$ as

$$\begin{aligned} \kappa &= \mathbf{1}_n \oplus 0_p \\ \lambda &= 0_n \oplus \mathbf{1}_p. \end{aligned}$$

The operations of source pairing and separated sum satisfy a number of identities, which we will use frequently in the remaining chapters.

Remark 3.1.2 In a category with binary coproducts $x + y$, one usually writes $f + g : x + x' \to y + y'$ for the unique morphism determined by the arrows $f : x \to y$ and $g : x' \to y'$. We use $\oplus$ since later we will study theories T with an additive structure on each hom-set $T(n, p)$ and we write $f + g$ for the addition operation on $T(n, p)$. See Section 3.5.

Proposition 3.1.3 *In any theory, the following identities hold for all morphisms with appropriate sources and targets:*

$$\begin{aligned}
\kappa \cdot \langle f, g \rangle &= f \\
\lambda \cdot \langle f, g \rangle &= g \\
\langle \kappa \cdot f,\, \lambda \cdot f \rangle &= f \\
\langle f,\, \langle g, h \rangle \rangle &= \langle \langle f, g \rangle,\, h \rangle \\
f \oplus (g \oplus h) &= (f \oplus g) \oplus h \\
\langle f, g \rangle \cdot h &= \langle f \cdot h,\, g \cdot h \rangle \\
(f \oplus g) \cdot \langle h, k \rangle &= \langle f \cdot h,\, g \cdot k \rangle \\
(f \oplus g) \cdot (h \oplus k) &= f \cdot h \,\oplus\, g \cdot k.
\end{aligned}$$

We emphasize the fact that pairing and separated sum are associative, so that we can use the notations

$$\langle f, g, h \rangle \quad \text{and} \quad f \oplus g \oplus h$$

unambiguously, when f, g and h have appropriate sources and targets.

Recall that in any theory, for each $p \geq 0$ there is a unique morphism

$$0_p : 0 \;\to\; p.$$

These *zero morphisms* play a very useful role, and we list some of their properties.

Proposition 3.1.4 *In any algebraic theory, the following equations hold whenever meaningful.*

$$\begin{aligned}
\langle 0_p, f \rangle &= \langle f, 0_p \rangle \;=\; f \\
\langle f, g \rangle \oplus 0_p &= \langle f \oplus 0_p,\, g \oplus 0_p \rangle \\
0_p \oplus \langle f, g \rangle &= \langle 0_p \oplus f,\, 0_p \oplus g \rangle \\
0_p \cdot f &= 0_q \\
0_0 \oplus f &= f \;=\; f \oplus 0_0 \\
f \oplus 0_q &= f \cdot (\mathbf{1}_p \oplus 0_q) \\
0_q \oplus f &= f \cdot (0_q \oplus \mathbf{1}_p).
\end{aligned}$$

The proofs of the two previous propositions are left as exercises. To get the reader started, we note that the first three identities in Proposition 3.1.3 are restatements of the coproduct property; the fact that $\langle f, g\rangle \cdot h = \langle f \cdot h,\, g \cdot h\rangle$ follows from these coproduct identities together with the facts that

$$\begin{aligned}
\kappa \cdot (\langle f, g\rangle \cdot h) &= \kappa \cdot \langle f \cdot h,\, g \cdot h\rangle \\
\lambda \cdot (\langle f, g\rangle \cdot h) &= \lambda \cdot \langle f \cdot h,\, g \cdot h\rangle.
\end{aligned}$$

We explain the meaning of the source pairing and separated sum operations in several examples.

- Let T be the theory A. For $f : n \to p,\ g : m \to p$ in T, the morphism $\langle f, g\rangle$ is the target tupling of the functions f, g:

  $$\begin{aligned}
  \langle f, g\rangle : A^p &\to A^{n+m} \\
  a &\mapsto (af,\, ag).
  \end{aligned}$$

  Similarly, for $f : n \to p$ and $g : m \to q$,

  $$\begin{aligned}
  f \oplus g : A^{p+q} &\to A^{n+m} \\
  (a, b) &\mapsto (af,\, bg).
  \end{aligned}$$

  Also,

  $$\begin{aligned}
  0_q \oplus f : A^{q+p} &\to A^n \\
  (a, b) &\mapsto bf.
  \end{aligned}$$

- If T is the theory A, and if $f : n \to p,\ g : m \to p$,

  $$\begin{aligned}
  \langle f, g\rangle : A \times [n+m] &\to A \times [p] \\
  (a, i) &\mapsto \begin{cases} (a, i)f & \text{if } i \in [n]; \\ (a, j)g & \text{if } i = n + j,\ j \in [m]. \end{cases}
  \end{aligned}$$

  Similarly, for $f : n \to p$ and $g : m \to q$,

  $$\begin{aligned}
  f \oplus g : A \times [n+m] &\to A \times [p+q] \\
  (a, i) &\mapsto \begin{cases} (a, i)f & \text{if } i \in [n]; \\ (b, p+k) & \text{if } i = n + j,\ j \in [m] \\ & \text{and } (a, j)g = (b, k). \end{cases}
  \end{aligned}$$

Also,

$$\begin{array}{rcll} 0_q \oplus f : A \times [n] & \to & A \times [q+p] & \\ (a,i) & \mapsto & (b, q+j) & \text{if } (a,i)f = (b,j). \end{array}$$

- Lastly, in a matrix theory $\mathbf{Mat}_S$, if $f : n \to p,\ g : m \to p$, then $\langle f, g \rangle$ is the $(n+m) \times p$ matrix whose first n rows are those of f, and whose last m rows are those of g. If $f : n \to p$ and $g : m \to q$, then the separated sum $f \oplus g : n+m \to p+q$ is the matrix indicated:

$$f \oplus g \quad = \quad \begin{bmatrix} f & 0_{nq} \\ 0_{mp} & g \end{bmatrix}.$$

Recall that 0_{np} is the n by p matrix all of whose entries are 0. Further, if $f : n \to p$,

$$0_q \oplus f \quad = \quad [0_{nq} \quad f].$$

Exercise 3.1.5 Suppose that T is an **N**-category. Show that T is an algebraic theory iff 0 is an initial object and for each $n, m \geq 0$, $n + m$ is the coproduct of n and m.

3.2 Elementary Properties of

We say that the theory morphism $\varphi : T \to T'$ is surjective or injective if each one of the maps $\varphi : T(n,p) \to T'(n,p)$ is surjective or injective, respectively.

Proposition 3.2.1 *Let $\varphi : T \to T'$ be a theory morphism. Then φ can be written as a composite*

$$\varphi \quad = \quad T \xrightarrow{\varphi_1} T'' \xrightarrow{\varphi_2} T',$$

where φ_1 is surjective and φ_2 is injective.

Proof. Let $T''(n,p)$ be the collection of all morphisms $f\varphi$, for $f : n \to p$ in T. Then it is easy to see that T'' is a subtheory of T'. The rest is routine. We usually write $T\varphi$ for the theory T''.

An interesting fact is that every theory is isomorphic to a theory of functions.

Proposition 3.2.2 *For any theory T, there is a set A such that T is isomorphic to a subtheory of A.*

Proof. Let $n_0 := 0$ if $T(1,0) \neq \emptyset$; otherwise let $n_0 := 1$. Let A be the direct product of the sets $T(1,k),\ k \geq n_0$. We denote an element of A as (g_k), where $g_k : 1 \to k$ in T. We observe that there is a bijection

$$\begin{array}{rcl} A^p & \to & \prod_{k \geq n_0} T(p,k) \\ ((g_k^1), \ldots, (g_k^p)) & \mapsto & (\langle g_k^1, \ldots, g_k^p \rangle). \end{array}$$

Thus we will regard an element in A^p as a sequence of morphisms with source p.

We define a theory morphism

$$\varphi : T \quad \to \quad A.$$

Given the T-morphism $f : 1 \to p$, define the function $f\varphi$ as follows:

$$\begin{array}{rcl} f\varphi : A^p & \to & A \\ (g_k) & \mapsto & (f \cdot g_k). \end{array}$$

When $f : n \to p,\ n \neq 1$, define $f\varphi$ so that φ preserves tupling. It is easy to check that φ is a theory morphism. In order to show that φ is injective, suppose that $f : 1 \to p$ in T. Then if (g_k) is any sequence in $\prod_k T(p,k)$ with $g_p = \mathbf{1}_p$, we see that the p-th element in the sequence $(g_k)f\varphi$ is f itself.

The construction in Proposition 3.2.2 will be referred to several times below.

Recall the definition of a theory congruence (Definition 2.2.3.13).

Corollary 3.2.3 *If θ is any theory congruence on the theory T, there is a set A and a theory morphism $\alpha : T \to A$ such that $\theta = \ker\, \alpha$.*

Exercise 3.2.4 Prove the corollary using Proposition 3.2.2.

Exercise 3.2.5 Let be a category with all finite copowers of some object A. Let $n \cdot A$ denote some chosen copower of A with itself n-times. Let $_A$ be the full subcategory of determined by the objects $n \cdot A$. Show that $_A$ determines an algebraic theory, in which the morphisms $n \to p$ are all -morphisms $n \cdot A \ \to\ p \cdot A$.

Exercise 3.2.6 Show for any algebraic theory, there is a category and an object A in such that T is isomorphic to the theory determined by $_A$ in the previous exercise.

Our next task will be to show that the category is complete and cocomplete. Both facts follow if we can show that has all products, all coproducts, equalizers and coequalizers. All of these results follow from the fact that as many-sorted algebras, theories form a many-sorted equational class, as we show next. Nevertheless, will give some indication of the constructions.

3.3 Theories as $\mathbf{N} \times \mathbf{N}$-Sorted Algebras

If T is a theory, T determines a many-sorted algebra in which the sorts are the pairs n, p of nonnegative integers; $T(n,p)$ is the underlying set of sort n, p. In this setting, one can give an equational axiomatization of the class of algebraic theories. An algebraic theory, as an algebra, is an $\mathbf{N} \times \mathbf{N}$-sorted algebra T having constants $i_n \in T(1,n),\ n \geq 0,\ i \in [n]$, operations of composition $T(n,p) \times T(p,q) \to T(n,q)$ and source tupling $T(1,p)^n \to T(n,p)$ satisfying the following identities:

$$
\begin{aligned}
f \cdot (g \cdot h) &= (f \cdot g) \cdot h \\
\langle 1_n, \ldots, n_n \rangle \cdot f &= f \\
f \cdot \langle 1_p, \ldots, p_p \rangle &= f \\
i_n \cdot \langle f_1, \ldots, f_n \rangle &= f_i \\
\langle 1_n \cdot f, \ldots, n_n \cdot f \rangle &= f \\
\langle 1_1 \rangle &= 1_1
\end{aligned}
$$

for all $f : n \to p$, $g : p \to q$, $h : q \to r$, and $f_j : 1 \to p$, for $j \in [n]$. The second equation above is redundant, but its inclusion makes obvious the connection between this notion and the categorical one.

We have just shown that the class of theories may be considered to be an equational class of $\mathbf{N} \times \mathbf{N}$-sorted algebras, and hence shares all properties common to such equational classes.

Corollary 3.3.1 *The category is complete, cocomplete, has surjective-injective factorizations, is well-powered and cowell-powered.*

We give a brief description of some of the categorical constructions.

- Products: Products of theories were defined in Section 2.2.3. Recall that the product $T = \prod_{i \in I} T_i$ of a collection of theories is formed at the level of sets; $T(n,p) = \prod_{i \in I} T_i(n,p)$, for each $n, p \geq 0$. The theory operations are defined pointwise. For example, the composite of $(f_i) : n \to p$ and $(g_i) : p \to q$ is the family $(f_i \cdot g_i) : n \to q$, for any two I-indexed families (f_i) and (g_i).

- Equalizers: If $\varphi_1, \varphi_2 : T \to T'$ are theory morphisms, the collection of all T-morphisms $f : n \to p$ such that $f\varphi_1 = f\varphi_2$ forms a subtheory T_0 of T; the inclusion $T_0 \to T$ is the equalizer of φ_1 and φ_2.

- Coequalizers: If $\varphi_1, \varphi_2 : T \to T'$ are theory morphisms, let θ be the smallest theory congruence on T' such that $f\varphi_1 \equiv f\varphi_2$ (θ), for all $f : n \to p$ in T. Such a congruence exists by Proposition 2.2.3.14. Then, if T'/θ is the resulting quotient theory, the natural map $T' \to T'/\theta$ is the coequalizer of φ_1 and φ_2 in .

Lastly, we consider coproducts of theories. It is simpler to describe binary coproducts, although the constructions involved in this special case generalize to arbitrary coproducts. First we describe the coproduct of two term theories and $\Delta\,\mathbf{Term}$, where we assume that no letter is common to Σ and Δ. Let $\Sigma + \Delta$ denote the union of the two signatures:

$$(\Sigma + \Delta)_n \quad := \quad \Sigma_n \cup \Delta_n, \quad n \geq 0.$$

Let $in_\Sigma : \Sigma\,\mathbf{Term} \to (\Sigma + \Delta)\,\mathbf{Term}$ be the inclusion of the theory of Σ-terms into the theory of $\Sigma + \Delta$-terms; it is the unique theory morphism induced by the inclusion

The morphism $in_\Delta : \Delta\,\mathbf{Term} \to (\Sigma + \Delta)\,\mathbf{Term}$ is defined similarly. We claim:

Proposition 3.3.2 *The two morphisms in_Σ, in_Δ are coproduct injections; i.e. for any theory T and theory morphisms $\varphi : \Sigma\,\mathbf{Term} \to T$,*

$\varphi' : \Delta\,\mathbf{Term} \to T$ *there is a unique theory morphism*

$$\psi : (\Sigma + \Delta)\,\mathbf{Term} \;\to\; T$$

such that

=3000 =1000

commutes.

The proof is obvious. On any letter σ in $(\Sigma + \Delta)_p$, if $\sigma \in \Sigma_p$, then $\sigma\psi := \sigma\varphi$. Otherwise $\sigma\psi := \sigma\varphi'$.

We can use Proposition 3.3.2 to show that any two theories have a coproduct. Indeed, suppose that $T_i,\ i = 1, 2$, are theories. Suppose that Σ and Δ are disjoint signatures such that there are surjective theory morphisms

$$\begin{aligned} \varphi_1 : \Sigma\,\mathbf{Term} &\;\to\; T_1 \\ \varphi_2 : \Delta\,\mathbf{Term} &\;\to\; T_2. \end{aligned}$$

Such morphisms exist by Corollary 2.2.3.20. Now let θ be the smallest theory congruence on the theory $(\Sigma + \Delta)\,\mathbf{Term}$ such that

- for any $f, g : n \to p$ in , $f\varphi_1 = g\varphi_1 \;\Rightarrow\; f\,in_\Sigma\;\theta\;g\,in_\Sigma$;
- and similarly, for any $f, g : n \to p$ in $\Delta\,\mathbf{Term}$, $f\varphi_2 = g\varphi_2 \;\Rightarrow\; f\,in_\Delta\;\theta\;g\,in_\Delta$.

Let T denote the quotient theory $(\Sigma + \Delta)\,\mathbf{Term}/\theta$, and let $\overline{\theta} : (\Sigma + \Delta)\,\mathbf{Term} \to T$ denote the corresponding theory morphism taking a morphism in $(\Sigma + \Delta)\,\mathbf{Term}$ to its θ-equivalence class. Then the maps

$$\begin{aligned} \kappa : T_1 &\;\to\; T \\ t\varphi_1 &\;\mapsto\; t\,(in_\Sigma \cdot \overline{\theta}) \qquad \text{for } t : n \to p \text{ in } \Sigma\,\mathbf{Term} \end{aligned}$$

$$\begin{aligned} \lambda : T_2 &\;\to\; T \\ t\varphi_2 &\;\mapsto\; t\,(in_\Delta \cdot \overline{\theta}) \qquad \text{for } t : n \to p \text{ in } \Delta\,\mathbf{Term} \end{aligned}$$

are well-defined theory morphisms. It follows easily that is a coproduct diagram.

The same construction works to yield the coproduct $\oplus_{i \in I} T_i$ of any set of theories.

3.4 Special Coproducts

Suppose that T is a theory and X is a set. In this section, we will outline an alternative construction of the theory $T[X]$ obtained by adding the elements of X freely to $T(1,0)$, i.e. the coproduct in the category of theories of T and the free theory $[X]$ generated by the set X (considered to be a set of morphisms $1 \to 0$). See Example 2.2.3.22. The theories $T[X]$ will be used several times in later chapters. In a series of exercises, we outline a proof of the following theorem.

Theorem 3.4.1 *For each theory T and set X, there is a theory $T[X]$ with the following properties.*

- *There is an injective theory morphism $\kappa : T \to T[X]$.*
- *There is a function $\lambda : X \to T[X](1,0)$, which determines a theory morphism $\lambda : [X] \to T[X]$. When T is nontrivial, λ is also injective.*
- *For any theory T', any theory morphism $\varphi : T \to T'$ and any function $h : X \to T'(1,0)$ there is a unique theory morphism $\varphi_h : T[X] \to T'$ such that both of the following equations hold:*

$$\begin{aligned} \kappa \cdot \varphi_h &= \varphi \\ \lambda \cdot \varphi_h &= h. \end{aligned}$$

- *For any $f : 1 \to 0$ in $T[X]$, there is a $g : 1 \to k$ in T and distinct $x_1, \ldots, x_k$ in X such that*

$$f = g\kappa \cdot \langle x_1\lambda, \ldots, x_k\lambda \rangle.$$

More generally, if f_1 and f_2 are any morphisms in $T[X](1,0)$, then there are T-morphisms $g_1, g_2 : 1 \to k$ and distinct elements $x_j \in X,\ j \in [k]$, such that

$$f_i \quad = \quad g_i\kappa \cdot \langle x_1\lambda, \ldots, x_k\lambda \rangle, \quad i = 1, 2.$$

Sometimes, when more than one theory is involved, we write κ_T and λ_T. When both κ and λ are injective, we may write

$$f \cdot \langle x_1, \ldots, x_n \rangle$$

instead of $f\kappa \cdot \langle x_1\lambda, \ldots, x_n\lambda \rangle$.

Remark 3.4.2 In a *trivial theory*, there is at most one morphism in each hom-set. Thus, up to isomorphism, there is one trivial theory having a morphism $1 \to 0$. Note that if T is trivial and $\varphi : T \to T'$ is a theory morphism, then T' is also trivial. We will assume below that T is a nontrivial theory, for otherwise the trivial theory with one morphism $1 \to 0$ will serve for $T[X]$ when $X \neq \emptyset$; $T[\emptyset] = T$.

The definition of $T[X]$ is by means of an equivalence on a category $=$ (X) of pairs. The objects of are the nonnegative integers. A morphism $n \to p$ in is a pair $(f; x)$ consisting of a T-morphism $f : n \to p + k$, for some $k \geq 0$, and a function $x : [k] \to X$. The *weight* of $(f; x)$ is k if x has domain $[k]$. If x has weight zero, we write just (f). The intuition here is that $x : [k] \to X$ becomes a morphism $x : k \to 0$ in and the pair $(f; x)$ represents the morphism $f \cdot (\mathbf{1}_p \oplus x)$. Thus, we will define a category equivalence on such that $(f; x) \approx (g; y)$ iff $(f; x)$ and $(g; y)$ represent the same morphism.

First, we will define a composition operation on and show that is a category.

Definition 3.4.3 *The* **composite** *of* $(f; x) \in (n, p)$ *and* $(g; y) \in (p, q)$ *is given by the formula*

$$(f; x) \cdot (g; y) \quad := \quad (f \cdot (g \oplus \mathbf{1}_k);\ y \oplus x),$$

where the weight of $(f; x)$ is k, and, if the weight of $(g; y)$ is k', then $y \oplus x$ is the function $[k' + k] \to X$ taking $i \in [k']$ to $iy \in X$ and $k' + j$ to $jx \in X$, for $j \in [k]$.

Exercise 3.4.4 Prove the composition operation is associative.

Exercise 3.4.5 Suppose that $(f;x) \in (n,p)$. Then prove

$$\begin{aligned}(\mathbf{1}_n)\cdot(f;x) &= (f;x)\\ (f;x)\cdot(\mathbf{1}_p) &= (f;x).\end{aligned}$$

Conclude that is a category containing a subcategory isomorphic to T.

From now on, we identify pairs of weight zero with morphisms in T, and identify the function $x : [k] \to X$ with the pair $(\mathbf{1}_k; x) : k \to 0$ in .

Since T is a subcategory of , we define a *base morphism* in to be a base morphism in T. We define a source pairing operation on .

Definition 3.4.6 *The* **source pairing** *of $(f;x) \in (n,p)$ and $(g;y) \in (m,q)$ is given by:*

$$\langle (f;x),\ (g;y)\rangle \quad := \quad (\,\langle f\cdot(\mathbf{1}_p \oplus \mathbf{1}_k \oplus 0_{k'}),\ g\cdot(\mathbf{1}_p \oplus 0_k \oplus \mathbf{1}_{k'})\rangle;\ x \oplus y\,),$$

where $(f;x)$ and $(g;y)$ have weights k and k', respectively.

Exercise 3.4.7 Prove that source pairing in is associative.

Exercise 3.4.8 Show that in general, the category is not a theory.

Source pairing determines separated sum in the usual way.

$$\begin{aligned}&(f;x)\oplus(g;y) \ =\\ &\quad := \ \langle (f;x)\cdot(\mathbf{1}_p \oplus 0_q),\ (g;y)\cdot(0_p \oplus \mathbf{1}_q)\rangle\\ &\quad = \ (\langle f\cdot(\mathbf{1}_p \oplus 0_q \oplus \mathbf{1}_k \oplus 0_j),\ g\cdot(0_p \oplus \mathbf{1}_q \oplus 0_k \oplus \mathbf{1}_j)\rangle;\ x \oplus y),\end{aligned}$$

where $(f;x) : n \to p$ has weight k and $(g;y) : m \to q$ has weight j.

Exercise 3.4.9 Show that for $x : [k] \to X$,

$$(\mathbf{1}_k; x) \quad = \quad (\mathbf{1}_1; x_1) \oplus \ldots \oplus (\mathbf{1}_1; x_k).$$

Exercise 3.4.10 Using the above-mentioned identifications, show that in ,

$$(f;x) \quad = \quad f\cdot(\mathbf{1}_p \oplus x),$$

for all $f : n \to p + k$, $x : [k] \to X$.

Remark 3.4.11 Source pairing determines a source tupling operation: if $n \geq 1$, and $(f_i; x_i) : 1 \to p$ has weight k_i, for $i \in [n]$, then by induction on n one defines $\langle (f_1; x_1), \ldots, (f_n; x_n) \rangle$ by

$$\langle (f_1; x_1), \ldots, (f_n; x_n) \rangle \quad := \quad \begin{cases} (f_1; x_1) \\ \qquad \text{if } n = 1; \\ \langle \langle (f_1; x_1), \ldots, (f_{n-1}; x_{n-1}) \rangle, (f_n; x_n) \rangle \\ \qquad \text{if } n > 1. \end{cases}$$

The weight of the source tupling is $k_1 + \ldots + k_n$, and one can show

$$\begin{aligned} &\langle (f_1; x_1), \ldots, (f_n; x_n) \rangle \\ &\quad = \quad (\langle f_1 \cdot (\mathbf{1}_p \oplus \tau_1), \ldots, f_n \cdot (\mathbf{1}_p \oplus \tau_n) \rangle;\ x_1 \oplus \ldots \oplus x_n), \end{aligned}$$

where $\tau_i : k_i \to k_1 + \ldots + k_n$ is the base morphism corresponding to the function

$$j \quad \mapsto \quad k_1 + \ldots + k_{i-1} + j.$$

Exercise 3.4.12 Prove the following special case of composition and separated sum with morphisms in T:

$$\begin{aligned} f \cdot (g; y) \quad &= \quad (f \cdot g;\ y) \\ f \oplus (g; y) \quad &= \quad (f \oplus g;\ y). \end{aligned}$$

Now suppose that $(f; x)$ and $(f'; x')$ are two pairs in (n, p), of weights k_1 and k_2 respectively. Let $\rho : [k_1] \to [k_2]$ be a function.

Definition 3.4.13 *Write*

$$(f; x) \quad \xrightarrow{\rho} \quad (f'; x') \tag{3.1}$$

if both of the following hold:

$$\begin{aligned} f \cdot (\mathbf{1}_p \oplus \rho) \quad &= \quad f' \\ x \quad &= \quad \rho \cdot x'. \end{aligned}$$

We write $(f; x) \to (f'; x')$ if for some base ρ, (3.1) holds.

Let $\approx$ be the least equivalence relation on (n, p) containing $\to$, so that $\approx$ is the reflexive, transitive closure of $(\to \cup \leftarrow)$. We want to show that $\approx$ is the relation $(\to \cdot \leftarrow)$, the relational composite of $\to$ with its converse $\leftarrow$.

Exercise 3.4.14 Suppose that $(f;x)$ and $(f';x')$ are pairs in (n,p) of weights k_1 and k_2, respectively. Show that $(f;x)\ (\rightarrow\cdot\leftarrow)\ (f';x')$ iff there is an integer k, a function $z:[k]\rightarrow X$ and base morphisms $\rho_i : k_i \rightarrow k,\ i=1,2$, such that

$$\begin{aligned} f\cdot(\mathbf{1}_p\oplus\rho_1) &= f'\cdot(\mathbf{1}_p\oplus\rho_2)\\ x &= \rho_1\cdot z\\ x' &= \rho_2\cdot z. \end{aligned}$$

The meaning of the equation $x=\rho_1\cdot z$, say, is that $x_j = z_{j\rho_1}$, for all $j\in[k_1]$.

Exercise 3.4.15 Show that the relation $(\rightarrow\cdot\leftarrow)$ is transitive on each set (n,p). *Hint:* Use Exercise 3.4.14 and a pushout diagram in . Conclude that

$$(f;x)\approx(f';x') \text{ iff } (f;x)\ (\rightarrow\cdot\leftarrow)\ (f';x').$$

Exercise 3.4.16 Suppose that $(f;x)\approx(f';x')$ in (n,p) and that $(g;y)\approx(g';y')$ in (p,q). Show that $(f;x)\cdot(g;y)\ \approx\ (f';x')\cdot(g';y')$ in (n,q). *Hint:* It is enough to consider the case that $(f;x)\rightarrow(f';x')$ and $(g;y)\rightarrow(g';y')$.

Exercise 3.4.17 Suppose that $(f;x)\approx(f';x')$ in (n,p) and that $(g;y)\approx(g';y')$ in (m,p). Show that

$$\langle (f;x),\ (g;y)\rangle\ \approx\ \langle (f';x'),\ (g';y')\rangle.$$

Exercise 3.4.18 Suppose that $f:n\rightarrow p+k$ in T. Show that for any $x:[k]\rightarrow X$ and $y:[j]\rightarrow X,\ \ (f;x)\ \approx\ (f\oplus 0_j;\ x\oplus y)$.

Definition 3.4.19 *The category $T[X]$ is an $\mathbf{N}$-category whose arrows $n\rightarrow p$ are the $\approx$-equivalence classes of the pairs in (n,p); composition is well-defined by Exercise 3.4.16. The identities are the $\approx$-equivalence classes of the morphisms $\mathbf{1}_n$.*

Exercise 3.4.20 Prove $T[X]$ is a theory, with source pairing given by Definition 3.4.6 (and Exercise 3.4.17). *Hint:* Use Exercise 3.1.5.

Define the following maps:

$$\begin{aligned} \iota_T : T &\rightarrow \\ t &\mapsto (t)\\ \iota_X : X &\rightarrow (1,0)\\ x &\mapsto (\mathbf{1}_1;\ x). \end{aligned}$$

Note that in the following exercise, in the category , the morphism $(\mathbf{1}_p \oplus x)$ is $(\mathbf{1}_{p+k}; x) \in (p+k, p)$; if $h : X \to T'(1,0)$ is a function, then for any $x : [k] \to X$, xh denotes the morphism $x_1h \oplus \ldots \oplus x_kh : k \to 0$ in T'.

Exercise 3.4.21 Show that the category has the following property: For any theory T', any theory morphism $\varphi : T \to T'$ and any function $h : X \to T'(1,0)$ there is a unique functor $\psi : \to T'$ which is the identity on objects such that

$$\begin{aligned} \iota_T \cdot \psi &= \varphi \\ \iota_X \cdot \psi &= h \\ (\mathbf{1}_p \oplus x)\psi &= \mathbf{1}_p \oplus xh, \end{aligned}$$

for all $x : [k] \to X$.

Exercise 3.4.22 Show that if $\psi : \to T'$ is any functor from to a theory T' which has the properties in Exercise 3.4.21, then whenever $(f;x) \approx (f';x')$ it follows that

$$(f;x)\psi \quad = \quad (f';x')\psi.$$

Remark 3.4.23 The previous Exercise is the justification for the definition of $\approx$. It shows that if $(f;x) \approx (f';x')$, then when mapped to theories, $(f;x)$ and $(f';x')$ will be identified.

Let $\Delta : \to T[X]$ be the functor which takes the pair $(f;x)$ in (n,p) to its equivalence class $(f;x)/\approx : n \to p$. Now define $\kappa : T \to T[X]$ as the composite of ι_T with the map of $\Delta : \to T[X]$; similarly, define $\lambda : X \to T[X](1,0)$ as the composite of ι_X with Δ.

Exercise 3.4.24 Show that if T is nontrivial, λ is injective.

Exercise 3.4.25 Show the functor $\kappa : T \to T[X]$ is an injective theory morphism.

The next exercise completes the proof Theorem 3.4.1.

Exercise 3.4.26 Show that for any theory T' and any theory morphism $\varphi : T \to T'$ and any function $h : X \to T'(1,0)$ there is a unique theory morphism $\varphi_h : T[X] \to T'$ such that

$$\begin{aligned} \kappa \cdot \varphi_h &= \varphi \\ \lambda \cdot \varphi_h &= h. \end{aligned}$$

Exercise 3.4.27 Show that for any $f : 1 \to 0$ in $T[X]$, there is a $g : 1 \to k$ in T and distinct $x_1, \ldots, x_k$ in X such that

$$f \quad = \quad g\kappa \cdot \langle x_1\lambda, \ldots, x_k\lambda \rangle.$$

More generally, if f_1 and f_2 are any morphisms in $T[X](1,0)$, then there are T-morphisms $g_1, g_2 : 1 \to k$ and distinct elements $x_j \in X,\ j \in [k]$, such that

$$f_i \quad = \quad g_i\kappa \cdot \langle x_1\lambda, \ldots, x_k\lambda \rangle, \quad i = 1, 2.$$

Corollary 3.4.28 *Suppose that* $f : n \to p + k_1$ *and* $g : n \to p + k_2$ *in* T *and that* $x_i, y_j \in X$*, for* $i \in [k_1]$ *and* $j \in [k_2]$*. Then*

$$f\kappa \cdot (\mathbf{1}_p \oplus \langle x_1\lambda, \ldots, x_{k_1}\lambda \rangle) \quad = \quad g\kappa \cdot (\mathbf{1}_p \oplus \langle x_1\lambda, \ldots, x_{k_2}\lambda \rangle)$$

in $T[X]$ *iff there is an integer* k*, base morphisms* $\rho_i : k_i \to k,\ i = 1, 2,$ *and* $z : [k] \to X$ *such that*

$$\begin{aligned} f \cdot (\mathbf{1}_p \oplus \rho_1) &= g \cdot (\mathbf{1}_p \oplus \rho_2) \\ x_i &= z_{i\rho_1} \\ y_j &= z_{j\rho_2} \end{aligned}$$

for all $i \in [k_1],\ j \in [k_2]$*.*

If T is trivial, so is the result. For nontrivial T, the proof is immediate from Exercise 3.4.14.

Exercise 3.4.29 Suppose that X is a singleton set and we write $\perp_m$ for the unique function $[m] \to X$, $m \geq 0$. Show that if $f : n \to p{+}k$ and $g : n \to p{+}j$ in T, then

$$f\kappa \cdot (\mathbf{1}_p \oplus \perp_k\lambda) \quad = \quad g\kappa \cdot (\mathbf{1}_p \oplus \perp_j\lambda)$$

in $T[X]$ iff

$$f \cdot (\mathbf{1}_p \oplus \tau_k) \quad = \quad g \cdot (\mathbf{1}_p \oplus \tau_j)$$

in T, where $\tau_m : m \to 1$ is the unique base morphism.

3.5 Matrix and Matricial Theories

In this section, we exhibit two equationally defined classes of theories, the matrix theories and the matricial theories.

3.5.1 Matrix Theories

A *pointed algebraic theory* is a theory with a distinguished morphism $0 : 1 \to 0$. In any pointed theory, define the morphisms

$$0_{np} \quad := \quad \overbrace{\langle 0, \ldots, 0 \rangle}^{n} \cdot 0_p : n \to p,$$

so that 0 is 0_{10}. A *morphism of pointed theories* is a theory morphism which preserves the point 0.

Definition 3.5.1 *A* **matrix theory** T *is a pointed algebraic theory such that each hom-set* $T(n, p)$ *is a commutative monoid under* $+$*, with* 0_{np} *as neutral element. Further, composition distributes over addition on both sides:*

$$\begin{aligned} f \cdot (g + h) &= (f \cdot g) \; + \; (f \cdot h) \\ (g + h) \cdot k &= (g \cdot k) \; + \; (h \cdot k), \end{aligned}$$

for all $f : m \to n,\ g, h : n \to p,\ k : p \to q$*. Lastly, the zeros are left and right annihilators:*

$$\begin{aligned} 0_{np} \cdot f &= 0_{nq} \\ f \cdot 0_{qr} &= 0_{pr} \end{aligned}$$

all $f : p \to q$*.*

We will sometimes omit the subscripts on 0 when they can be determined from the context.

The terminology is motivated by the fact that, as we will show below, if T is a matrix theory, the hom-set $S := T(1, 1)$ is a semiring in a natural way, and T is isomorphic to the theory $\mathbf{Mat}_S$ described in Example 2.2.3.29.

We do not need a new definition for morphisms of matrix theories.

Definition 3.5.2 *Let* T *and* T' *be matrix theories. A morphism* $\varphi : T \to T'$ *is a* **morphism of matrix theories** *if* φ *is just a theory morphism.*

It will be shown that theory morphisms between matrix theories necessarily preserve the additive structure on the hom-sets.

Since matrix theories are defined by means of equations involving the theory operations as well as the addition operation and 0, they form a variety of (enriched) theories.

Example 3.5.3 For each semiring S, the theory of all matrices over S, $\mathbf{Mat}_S$, is a matrix theory. See Example 2.3.29 in Chapter 2.

In matrix theories, one may form the *target tupling* of the morphisms $a_i : n \to 1$, $i \in [p]$,

$$\begin{aligned}[][a_1, \ \ldots, \ a_p] \ &:= \ a_1 \oplus 0_{p-1} + 0_1 \oplus a_2 \oplus 0_{p-2} + \ldots + 0_{p-1} \oplus a_p \\ &= \ a_1 \cdot 1_p + \ldots + a_p \cdot p_p.\end{aligned}$$

In $\mathbf{Mat}_S$, target tupling yields the n by p matrix whose i-th column is $a_i, \ i \in [p]$. If $p = 0$, the value is 0_{n0}.

It is clear that if $h : m \to n$ and $a_i : n \to 1$, for $i \in [k]$, then

$$h \cdot [a_1, \ \ldots, \ a_k] \quad = \quad [h \cdot a_1, \ \ldots, \ h \cdot a_k].$$

In any matrix theory, we can extend the target tupling operation and define the *target pairing*

$$[a \quad b] \ := \ a \oplus 0_q + 0_p \oplus b : n \to p + q$$

of two morphisms $a : n \to p, \ b : n \to q$. (Sometimes we use a comma and write $[a, \ b]$.) For each $n \geq 1$ and each $i \in [n]$, define $i_n^T : n \to 1$ as the morphism

$$i_n^T \quad := \quad \langle 0_{11}, \ldots, \mathbf{1}_1, \ldots, 0_{11} \rangle : n \to 1, \tag{3.2}$$

where the nonzero morphism occurs in the i-th position.

Lemma 3.5.4 *For each $i, j \in [n]$,*

$$i_n \cdot j_n^T \quad = \quad \begin{cases} \mathbf{1}_1 & \textit{if } i = j; \\ 0_{11} & \textit{otherwise.} \end{cases}$$

Furthermore,

$$1_n^T \cdot 1_n + \ldots + n_n^T \cdot n_n \quad = \quad \mathbf{1}_n.$$

Lastly, for any $f, g : n \to p$,

$$f + g \;=\; [f,\ g] \cdot \langle \mathbf{1}_p, \mathbf{1}_p \rangle \;=\; [\mathbf{1}_n\ \mathbf{1}_n] \cdot \langle f, g \rangle.$$

Proof. The first statement is obvious, and the second follows by showing that

$$i_n \cdot (1_n^T \cdot 1_n + \ldots + n_n^T \cdot n_n) \quad = \quad i_n.$$

Indeed,

$$\begin{aligned} i_n \cdot (1_n^T \cdot 1_n + \ldots + n_n^T \cdot n_n) &= \sum_{j \in [n]} i_n \cdot (j_n^T \cdot j_n) \\ &= \sum_{j \in [n]} (i_n \cdot j_n^T) \cdot j_n \end{aligned}$$

and all of the terms but one on the right-hand side are 0_{1n}; the remaining one is $(i_n \cdot i_n^T) \cdot i_n = i_n$.

Now we check one of the final assertions:

$$\begin{aligned} [f,\ g] \cdot \langle \mathbf{1}_p,\ \mathbf{1}_p \rangle &= (f \oplus 0_p + 0_p \oplus g) \cdot \langle \mathbf{1}_p,\ \mathbf{1}_p \rangle \\ &= f + g. \end{aligned}$$

We let the reader verify the last equation.

Lemma 3.5.5 *In any matrix theory, for each $n \geq 0$, there is a unique morphism $n \to 0$.*

Proof. It is enough to prove this statement for the case $n = 1$. Suppose that $f : 1 \to 0$. Then f is the composite

since $0_1 \cdot 0_{10}$ is the unique morphism $0 \to 0$, namely the identity morphism $\mathbf{1}_0$. But 0_{10} is a right annihilator, so

$$(f \cdot 0_1) \cdot 0_{10} \quad = \quad 0_{10}.$$

Thus, the "point" $0_{10} : 1 \to 0$ is the unique morphism $1 \to 0$ in a matrix theory.

We use the family of morphisms $i_n^T : n \to 1,\ i \in [n]$, to prove the following proposition.

Proposition 3.5.6 *In a matrix theory T, the object n is the n-th power of the object 1; the morphisms $i_n^T : n \to 1,\ i \in [n]$, are product projections.*

Proof. We must show that for any $n \geq 0$ and any family of morphisms $f_i : p \to 1,\ i \in [n]$, there is a unique morphism $f : p \to n$ such that

$$f \cdot i_n^T \quad = \quad f_i,$$

for each $i \in [n]$. By Lemma 3.5.5, the object 0 is terminal, and hence the product of zero copies of 1. Assume now $n \geq 1$. Define f as the target tupling of the morphisms f_i:

$$f \quad := \quad [f_1, \ldots, f_n] : p \to n.$$

Then,

$$\begin{aligned} f \cdot i_n^T &= (\sum_{j \in [n]} f_j \cdot j_n) \cdot i_n^T \\ &= \sum_{j \in [n]} (f_j \cdot j_n \cdot i_n^T) \\ &= f_i, \end{aligned}$$

by Lemma 3.5.4. This is the only choice for f, since if $f, g : p \to n$ and $f \cdot i_n^T = g \cdot i_n^T$, for all $i \in [n]$, then

$$\begin{aligned} f &= f \cdot \mathbf{1}_n \\ &= f \cdot (1_n^T \cdot 1_n + \ldots + n_n^T \cdot n_n) \\ &= \sum_{i \in [n]} (f \cdot i_n^T) \cdot i_n \\ &= \sum_{i \in [n]} (g \cdot i_n^T) \cdot i_n \\ &= g. \end{aligned}$$

It follows that any morphism $n \to p+q$ can be written as the target pairing $[a \;\; b]$, where $a : n \to p$ and $b : n \to q$. If $a : n \to p$ and $b : n \to q$, then $[a \;\; b]$ is the unique morphism $c : n \to p+q$ such that

$$\begin{aligned} c \cdot \langle \mathbf{1}_p, 0_{qp} \rangle &= a \\ c \cdot \langle 0_{pq}, \mathbf{1}_q \rangle &= b. \end{aligned}$$

Finally we can show the class of the matrix theories $\mathbf{Mat}_S$ is representative of all matrix theories, in the following sense.

Proposition 3.5.7 *Let T be a matrix theory. There is a semiring S such that T is isomorphic to $\mathbf{Mat}_S$.*

Proof. Let $S := (T(1,1), +, \cdot, 0_{11}, \mathbf{1}_1)$, where $s \cdot s'$ is the theory composite, and addition is the addition in the theory. It is immediate that S is a semiring. Further, for each morphism $f : n \to p$, define fM as the $n \times p$ matrix whose ij-th entry is the morphism

$$(fM)_{ij} \quad := \quad i_n \cdot f \cdot j_p^T .$$

We omit the easy proof that $f \mapsto fM$ is a bijective theory morphism $T \to \mathbf{Mat}_S$.

Example 3.5.8 Let $\mathbf{B}$ denote the Boolean semiring $\{0, 1\}$ with $1 + 1 = 1$. Then $\mathbf{Mat_B}$ is isomorphic to the theory whose morphisms $n \to p$ are all relations $[n] \to [p]$; the composite of $r : [n] \to [p]$ with $s : [p] \to [q]$ is the relation $r \cdot s$ defined by

$$i \; (r \cdot s) \; j \quad \Leftrightarrow \quad \exists k (i \, r \, k \text{ and } k \, s \, j).$$

Morphisms of matrix theories are in bijective correspondence with semiring homomorphisms.

Proposition 3.5.9 *Any morphism $\varphi : T \to T'$ of matrix theories preserves the additive structure and is totally determined by the map from $T(1,1)$ to $T'(1,1)$, which is necessarily a semiring homomorphism. Conversely, any semiring homomorphism $\varphi : T(1,1) \to T'(1,1)$ extends uniquely to a matrix theory morphism $T \to T'$.*

Proof. We show only that if $\varphi : T \to T'$ is a theory morphism between the matrix theories T and T', then necessarily φ preserves the additive structure; i.e. $0_{np}\varphi = 0_{np}$ and for any $f, g : n \to p$ in T,

$$(f+g)\varphi \quad = \quad f\varphi + g\varphi.$$

Indeed, $0_{10}\varphi = 0_{10}$, since 0_{10} is the unique morphism $1 \to 0$ in both theories. It follows that $0_{np}\varphi = 0_{np}$ and $i_n^T\varphi = i_n^T$, for each $i \in [n]$, $n \geq 1$, since φ preserves source tupling. If $h = [f \quad g] : n \to p+p$ in T, $f = h \cdot \langle \mathbf{1}_p, 0_{pp}\rangle$ and $g = h \cdot \langle 0_{pp}, \mathbf{1}_p\rangle$. Thus,

$$\begin{aligned} f\varphi &= (h \cdot \langle \mathbf{1}_p, 0_{pp}\rangle)\varphi \\ &= h\varphi \cdot \langle \mathbf{1}_p, 0_{pp}\rangle \qquad \text{and} \\ g\varphi &= h\varphi \cdot \langle 0_{pp}, \mathbf{1}_p\rangle. \end{aligned}$$

Thus,

$$[f \quad g]\varphi \quad = \quad [f\varphi \quad g\varphi].$$

But,

$$f + g \quad = \quad [f \quad g] \cdot \langle \mathbf{1}_p, \mathbf{1}_p\rangle.$$

So,

$$\begin{aligned} (f+g)\varphi &= ([f \quad g] \cdot \langle \mathbf{1}_p, \mathbf{1}_p\rangle)\varphi \\ &= [f \quad g]\varphi \cdot \langle \mathbf{1}_p, \mathbf{1}_p\rangle \\ &= [f\varphi \quad g\varphi] \cdot \langle \mathbf{1}_p, \mathbf{1}_p\rangle \\ &= f\varphi + g\varphi. \end{aligned}$$

Definition 3.5.10 *Let T be a matrix theory. A subtheory T' of T is a* **submatrix theory** *of T if T' contains the zero morphism 0_{10} and is closed under addition.*

It follows that a subtheory of a matrix theory contains all of the zero morphisms 0_{np}.

Proposition 3.5.11 *If T is the matrix theory $\mathbf{Mat}_S$, then a submatrix theory of T is a matrix theory $\mathbf{Mat}_{S'}$ where S' is a subsemiring of S.*

From now on, without any loss of generality, we may restrict our attention to only the matrix theories $\mathbf{Mat}_S$. Accordingly, we sometimes use the word matrix instead of morphism.

Each relation $\rho : [n] \to [p]$ determines a zero-one matrix $\rho' : n \to p$ in $\mathbf{Mat}_S$ as follows: $\rho'_{ij} = 1$ if $i \, \rho \, j$; $\rho'_{ij} = 0$ otherwise. Thus, we call a zero-one matrix over S a *relational matrix.* When the relation ρ is a function, the corresponding morphism ρ' in $\mathbf{Mat}_S$ is sometimes called a *base matrix.* Thus, if S is not trivial, i.e. if $0 \neq 1$, in a base matrix $n \to p$, each row contains exactly one occurrence of 1; all other entries are 0. We will usually use the same name for a function and its corresponding matrix.

Note that if $\rho : m \to n$ is a base matrix and if $a_i : 1 \to p, i \in [n]$, are any row matrices, and if $h : p \to q$ is any matrix, then

$$\begin{aligned} \rho \cdot \langle a_1, \ldots, a_n \rangle &= \langle a_{1\rho}, \ldots, a_{m\rho} \rangle \\ \langle a_1, \ldots, a_n \rangle \cdot h &= \langle a_1 \cdot h, \ldots, a_n \cdot h \rangle. \end{aligned}$$

(See Exercise 3.5.12.) In the case that $\pi : n \to n$ is a base matrix which corresponds to a permutation of $[n]$, then the inverse of π is given by π^T, the transpose matrix of π. Thus

$$\pi \cdot \pi^T = \pi^T \cdot \pi = \mathbf{1}_n.$$

If $A : n \to p$ is any morphism in the matrix theory $\mathbf{Mat}_S$, we write $A^T : p \to n$ for its transpose:

$$A^T{}_{ij} = A_{ji}.$$

Later, we will frequently use the facts that if ρ is a relational matrix and A is any matrix, then

$$\begin{aligned} (A \cdot \rho)^T &= \rho^T \cdot A^T \\ (\rho \cdot A)^T &= A^T \cdot \rho^T. \end{aligned}$$

The results of the next two exercises will be applied later.

Exercise 3.5.12 Prove that in any theory, if $f = \langle f_1, \ldots, f_n \rangle : n \to p$ and $\rho : m \to n$ is base, then

$$\rho \cdot f = \langle f_{1\rho}, \ldots, f_{m\rho} \rangle.$$

Exercise 3.5.13 Prove that in a matrix theory, if $A = [a_1, \ldots, a_p] : n \to p$ and $\rho : m \to p$ is a base matrix, then

$$A \cdot \rho^T \quad = \quad [a_{1\rho}, \ldots, a_{m\rho}].$$

Hint: Use the previous exercise after taking an appropriate transpose.

Exercise 3.5.14 This problem concerns theories with an addition operation on each hom-set. Suppose T is a pointed theory with point 0 which has a constant

$$+ : 1 \ \to \ 2.$$

For any $f, g : 1 \to p$, define

$$f + g \quad := \quad + \cdot \langle f, g \rangle.$$

When $f = \langle f_1, \ldots, f_n \rangle, \ g = \langle g_1, \ldots, g_n \rangle : n \to p$, define

$$f + g \quad := \quad \langle f_1 + g_1, \ldots, f_n + g_n \rangle.$$

Last, define

$$\begin{aligned} 0_{1p} \quad &:= \quad 0 \cdot 0_p \\ 0_{np} \quad &:= \quad \langle 0_{1p}, \ldots, 0_{1p} \rangle \end{aligned}$$

as before. Now show that

$$\begin{array}{rcll} (f+g) \cdot h & = & (f \cdot h) + (g \cdot h), & \text{all } f, g : n \to p, \ h : p \to q \\ 0_{np} \cdot f & = & 0_{nq}, & \text{all } f : p \to q \\ i_n \cdot (f+g) & = & i_n \cdot f + i_n \cdot g, & \text{all } f, g : n \to p, \ i \in [n] \\ i_n \cdot 0_{np} & = & 0_{1p}, & \text{all } i \in [n]. \end{array}$$

Exercise 3.5.15 Suppose that T has the morphism $+ : 1 \to 2$ and the constant $0 : 1 \to 0$ as in the previous exercise, so that $f + g$ is defined for all $f, g : n \to p$. Suppose that the following three equations hold:

$$\begin{array}{rcl} (1_3 + 2_3) + 3_3 & = & (1_3 + 2_3) + 3_3 \\ 1_2 + 2_2 & = & 2_2 + 1_2 \\ 1_1 + 0_{11} & = & 1_1. \end{array}$$

Show that all of the following identities also hold:

$$\begin{array}{rcll} (f+g) + h & = & f + (g + h), & f, g, h : n \to p \\ f + g & = & g + h, & f, g : n \to p \\ f + 0_{np} & = & f, & f : n \to p. \end{array}$$

Define n as the morphism $\overbrace{\mathbf{1}_1 + \ldots + \mathbf{1}_1}^{n} : 1 \to 1$. Show

$$n \cdot (f + g) \quad = \quad n \cdot f + n \cdot g.$$

Exercise 3.5.16 Suppose that T is a pointed theory. For each $n \geq 1$ and each $i \in [n]$, define the morphism $i_n^T : n \to 1$ as in (3.2). Show that if T is a matrix theory, then T^{op}, the opposite category of T, is also a theory with the morphisms i_n^T as distinguished morphisms.

Exercise 3.5.17 Suppose that T is a pointed theory and T^{op} is also a theory with the morphisms i_n^T as distinguished morphisms. Show that T is a matrix theory. *Hint:* Show that the operation of $+$ on $T(n,p)$ can be defined by the appropriate equation in Lemma 3.5.4.

3.5.2 Theories of Relations

For any fixed set A, let A be the theory whose morphisms $n \to p$ are all relations

$$r : A \times [n] \quad \to \quad A \times [p].$$

The composite of two relations is given by standard relational composition:

$$(a,i)\ r \cdot s\ (a',j) \quad \Leftrightarrow \quad \exists\, b,k\ [(a,i)\, r\, (b,k) \text{ and } (b,k)\, s\, (a',j)].$$

The *graph* of the relation $r : n \to p$ is the set of ordered pairs $((a,i),(a',j))$ such that $(a,i)\, r\, (a',j)$. If $r,s : n \to p$, the graph of their sum $r+s$ is the union of the graph of r with that of s, so that

$$(a,i)\ r+s\ (a',j) \quad \Leftrightarrow \quad (a,i)\ r\ (a',j) \text{ or } (a,i)\, s\, (a',j).$$

Exercise 3.5.18 Show that A is a matrix theory. Show that 0_{np} is the relation $n \to p$ whose graph is empty. What is the semiring S, where A is isomorphic to $\mathbf{Mat}_S$?

Exercise 3.5.19 Show that each hom-set in A has infinite, as well as finite sums, and that

$$\begin{aligned} r \cdot \sum_{i \in I} s_i &= \sum_{i \in I} r \cdot s_i \\ (\sum_{i \in I} s_i) \cdot r &= \sum_{i \in I} (s_i \cdot r) \end{aligned}$$

whenever the composites make sense.

3.5.3 Matricial Theories

Matricial theories are a generalization of matrix theories. These theories occur naturally in linear algebra, and subtheories of matricial theories have been used as semantic models of flowchart computation.

Definition 3.5.20 *A* **matricial theory** T *is a pointed algebraic theory such that each hom-set is equipped with a binary operation* $+$ *of addition and a unary operation* ¶ *with the following properties:*

1. *Each hom-set* $T(n,p)$ *is a commutative monoid under addition, with neutral element* 0_{np}.
2. *The function* $\P : T(n,p) \to T(n,p)$ *satisfies*

$$\begin{aligned} (f+g)\P &= f\P + g\P \\ (h \cdot g)\P &= h\P \cdot g\P \\ f\P \cdot 0_{pq} &= 0_{nq} \\ f\P\P &= f\P, \end{aligned}$$

for all $f, g : n \to p,\ h : m \to n$.

3. *Composition only partially distributes over addition:*

$$\begin{aligned} (f+g) \cdot h &= f \cdot h \ + \ g \cdot h \\ f\P \cdot (h+k) &= f\P \cdot h \ + \ f\P \cdot k, \end{aligned}$$

all $f, g : n \to p,\ h, k : p \to q$.

4. *The zeros are left (but not necessarily right) annihilators*

$$0_{np} \cdot f = 0_{nq},$$

all $f : p \to q$.

5. *Each* $f : n \to p$ *is expressible as a sum*

$$f = f\P + f \cdot 0_{pp}.$$

A **morphism of matricial theories** $\varphi : T \to T'$ *is a theory morphism which preserves the additive structure on each hom-set.*

Thus every matricial theory morphism is a pointed theory morphism. Suppose that T is a matrix theory. If we define the operation ¶ on each hom-set of T by

$$f\P \quad := \quad f,$$

then T is a matricial theory. Hence matrix theories are also matricial theories.

Exercise 3.5.21 Show that in any matricial theory, $f = f\P \Leftrightarrow f \cdot 0 = 0$. Conclude $0_{np}\P = 0_{np}$. Prove that whenever a morphism $f : n \to p$ is expressible as a sum $f = g\P + h \cdot 0$, then $g\P = f\P$ and $h \cdot 0 = f \cdot 0$. Conclude that every morphism in a matricial theory is *uniquely* expressible as

$$f \quad = \quad g + h,$$

where $g \cdot 0 = 0$ and $h \cdot 0 = h$.

Exercise 3.5.22 Show that if ρ is a base morphism in a matricial theory, then $\rho\P = \rho$ and $\langle f_1, \ldots, f_n\rangle\P = \langle f_1\P, \ldots, f_n\P\rangle$, for $f_i : 1 \to p,\ i \in [n]$. Conclude that

$$\begin{aligned} \langle f, g\rangle\P \quad &= \quad \langle f\P, g\P\rangle \\ (f \oplus g)\P \quad &= \quad f\P \oplus g\P, \end{aligned}$$

for all appropriate f and g.

We observe that if $\varphi : T \to T'$ is a morphism of matricial theories, then $(f\P)\varphi = (f\varphi)\P$. Indeed, note first that

$$f\varphi \quad = \quad (f\P)\varphi + f\varphi \cdot 0,$$

and necessarily

$$(f\P)\varphi \cdot 0 \quad = \quad 0.$$

Thus, by Exercise 3.5.21,

$$(f\P)\varphi \quad = \quad (f\varphi)\P.$$

One class of matricial theories involves the notion of a semiring module, defined as follows.

Definition 3.5.23 *Suppose that S is a semiring. A (left) S-***module** $V = (V, +, 0)$ *is a commutative monoid with an S-action*

$$\begin{array}{rcl} S \times V & \to & V \\ (s, v) & \mapsto & s \circ v \ (or\ sv) \end{array}$$

such that

$$\begin{array}{rcl} 0v & = & 0 \\ 1v & = & v \\ s0 & = & 0 \\ s(s'v) & = & (ss')v \\ (s + s')v & = & sv + s'v \\ s(v + v') & = & sv + sv'. \end{array}$$

Example 3.5.24 For any semiring S, both $\{0\}$ and S itself are S-modules.

We extend the action of S to matrices. If a is an $n \times p$ matrix over S and $v = (v_1, \ldots, v_p)$ is a p-tuple of elements of V, then $a \circ v$, or just av, is the n-tuple $w = (w_1, \ldots, w_n)$, where for each $i \in [n]$,

$$w_i = \sum_j a_{ij} v_j.$$

Example 3.5.25 If V is an S-module, define the theory $\mathbf{Matr(}S; V\mathbf{)}$ as the theory whose morphisms $n \to p$ consist of pairs $(a; x)$, sometimes written $[a; x]$, where a is an n by p matrix over S, and where x is an element in the monoid V^n. We call an n-tuple $v = (v_1, \ldots, v_n)$ with $v_i \in V$ an *n-vector.* The *sum* of $(a; v)$ and $(b; w)$ in $\mathbf{Matr(}S; V\mathbf{)}$ is $(a + b;\ v + w)$. The *composite* of $(a; x) : n \to p$ with $(b; y) : p \to q$ is given by the formula

$$(a; x) \cdot (b; y) \quad := \quad (ab;\ ay + x). \tag{3.3}$$

Here, ab is the standard matrix product of a and b; $ay + x$ is the n-vector in V^n obtained by adding x (componentwise) to the action of a on y.

The base morphisms $n \to p$ in $\mathbf{Matr(}S; V\mathbf{)}$ are those pairs $(a; 0)$ in which a is a base morphism $n \to p$ in the matrix theory $\mathbf{Mat}_S$ and where $0 = 0^n$ is the n-tuple of 0's in V. If $(a; x) : n \to p$ is a morphism in the matricial theory $\mathbf{Matr(}S; V\mathbf{)}$, $(a; x)\P : n \to p$ is the morphism $(a; 0) : n \to p$ and $(a; x) \cdot 0 = (0; x)$.

The class of examples in Example 3.5.25 of matricial theories is representative.

Proposition 3.5.26 *If T is a matricial theory, there is a semiring S and an S-module V such that T is isomorphic to* $\mathbf{Matr}(S;V)$.

Indeed, the set of all morphisms $f\P$, for $f : n \to p$ in T forms a subtheory $T\P$ of T which is itself a matrix theory, called the *underlying matrix theory* of T; thus T is isomorphic to $\mathbf{Matr}(S;V)$ and $T\P$ is isomorphic to $\mathbf{Mat}_S$, where $S := T(1,1)\P$ and $V := T(1,0)$.

Exercise 3.5.27 Give a complete proof of Proposition 3.5.26.

Without loss of generality, we may restrict attention to matricial theories only of the kind $\mathbf{Matr}(S;V)$. The underlying matrix theory $T\P$ of $T = \mathbf{Matr}(S;V)$ is isomorphic to $\mathbf{Mat}_S$. $\mathbf{Mat}_S$ is a subtheory of T when we identify a matrix $a : n \to p$ over S with the pair $(a;0)$, where 0 is the n-tuple of 0's in V. We will write either $(;v)$ or just v for a morphism $n \to 0$ in $\mathbf{Matr}(S;V)$. A *relational morphism* in a matricial theory $\mathbf{Matr}(S;V)$ is a pair $(\rho;0)$, where ρ is a relational matrix in the underlying matrix theory. By an abuse of terminology, we call such pairs relational matrices also.

Remark 3.5.28 If T is a matricial theory in which $f\P = f$, for all morphisms f, then for some semiring S, T is isomorphic to $\mathbf{Matr}(S;V)$ where $V = \{0\}$.

Definition 3.5.29 *If T is a matricial theory, a* **submatricial theory** T' *of T is a subtheory containing the distinguished zero* $0 : 1 \to 0$ *which is closed under $+$ and $\P$.*

Thus each morphism 0_{np} belongs to any submatricial theory.

Proposition 3.5.30 *If T is the matricial theory* $\mathbf{Matr}(S;V)$*, and if T' is a submatricial theory of T then T' is* $\mathbf{Matr}(S';V')$*, where S' is a subsemiring of S, V' is a submonoid of V and*

$$s \in S' \text{ and } v \in V' \quad \Rightarrow \quad sv \in V'.$$

Proposition 3.5.31 *A matricial theory morphism*

$$\varphi : \mathbf{Matr}(S; V) \quad \to \quad \mathbf{Matr}(S'; V')$$

is determined by a pair (φ_S, φ_V) *consisting of a semiring homomorphism* $\varphi_S : S \to S'$ *and a monoid homomorphism* $\varphi_V : V \to V'$*. The value of* φ *on pairs* $(a; x) : n \to p$ *is* $(a\varphi_S; x\varphi_V)$*, where* $(a\varphi_S)_{ij} = a_{ij}\varphi_S$ *and* $(x\varphi_V)_i = x_i\varphi_V$*, for all* $i \in [n],\ j \in [p]$*. The two maps must be compatible with the module action in the sense that the diagram*

$$[1`1`1`1; 1000`500][S \times V`S' \times V'`V`V'; \varphi_S \times \varphi_V` \circ ` \circ `\varphi_V] \tag{3.4}$$

commutes.

Exercise 3.5.32 Prove that the unrestricted left distributive law

$$f \cdot (g + h) \quad = \quad f \cdot g + f \cdot h$$

holds in $\mathbf{Matr}(S; V)$ iff V is idempotent, i.e. $v + v = v$, all $v \in V$.

Exercise 3.5.33 Suppose that T is the matricial theory determined by the semiring S and module V. Each morphism $(a; x) : n \to p$ determines an *affine transformation*

$$\begin{aligned} \overline{(a; x)} : V^p \quad &\to \quad V^n \\ v \quad &\mapsto \quad av + x. \end{aligned}$$

Show

$$\overline{(a; x) \cdot (b; y)} \quad = \quad \overline{(a; x)} \cdot \overline{(b; y)}.$$

Find an example of a semiring module pair $(S; V)$ such that the implication

$$\overline{(a; x)} \ = \ \overline{(b; y)} \quad \Rightarrow \quad (a; x) \ = \ (b; y)$$

fails. Show that if S is a ring and $V = S$, then the implication holds.

Exercise 3.5.34 [Elg76a] Suppose that T is a pointed theory, with point 0, and let $T¶$ consist of all morphisms f in T such that $f \cdot 0 = 0$. Show that $T¶$ is a subtheory of T.

Exercise 3.5.35 Suppose that T is a pointed theory. Define, for each $i \in [n],\ n \geq 1$, $i_n^T : n \to 1$ as in (3.2) above. Show that $T¶$ is a matrix theory if for each collection of morphisms $f_i : m \to 1$, $i \in [n]$, in $T¶$, there is a unique morphism $f : m \to n$ such that $f \cdot i_n^T = f_i$, for all $i \in [n]$.

Exercise 3.5.36 [Elg76a] With the terminology of the previous two Exercises, let T be a pointed theory such that $T\P$ is a matrix theory. Suppose further that every morphism in T is *uniquely* expressible as a sum

$$f \quad = \quad f\P + f \cdot 0,$$

where $f\P$ belongs to $T\P$. Show that T is a matricial theory in a natural way. *Hint:* See Exercises 3.5.16 and 3.5.17 above.

Exercise 3.5.37 [Elg76a] A *quemiring* is a set Q equipped with operations $+, \cdot$, constants $0, 1$ such that $(Q, +, 0)$ is a commutative monoid, $(Q, \cdot, 1)$ is a monoid, and the following hold:

$$\begin{array}{rcl} (a+b)c & = & ac + bc \\ a0 \ = \ 0 & \Rightarrow & a(b+c) \ = \ ab + ac \\ 0a & = & 0 \end{array}$$

and, finally, for each $a \in Q$, there is a unique decomposition

$$a \quad = \quad b + a0,$$

where $b0 = 0$. Show that if T is a matricial theory, $T(1, 1)$ is a quemiring.

Exercise 3.5.38 [Elg76a] Show that if Q is a quemiring, $Q\P = \{a \in Q : a0 = 0\}$ is a semiring and $Q \cdot 0 = \{a0 : a \in Q\}$ is a $Q\P$-module. Show, for every semiring S and S-module V there is a quemiring Q with $Q\P$ isomorphic to S and $Q \cdot 0$ isomorphic to V. *Hint:* Let $Q = S \times V$, with pointwise addition and multiplication given by $(s, v) \cdot (s', v') := (ss', \, sv' + v)$.

The theories A are an important class of matricial theories.

3.5.4 Sequacious Relations

Sequacious relations model the stepwise behaviors of nondeterministic flowchart algorithms. The theories of *sequacious functions* were introduced by Elgot in [Elg75] to model the stepwise behaviors of deterministic flowchart algorithms. We will describe these theories here. They will be studied further in Chapter 10.

Suppose that A is a nonempty set. In the theory A, a morphism $n \to p$ is a *sequacious relation*

$$r : A^+ \times [n] \cup A^\omega \ \to \ A^+ \times [p] \cup A^\omega.$$

By definition, such a relation satisfies the following conditions, where (x, i) ranges over $A \times [n]$:

1. for any $u, v \in A^+$, if $(u, i)\ r\ (v, j)$ then $v = uv'$, for some $v' \in A^*$;
2. for any $u \in A^+$, if $(u, i)\ r\ w,\ w \in A^\omega$, then $w = uw'$, for some $w' \in A^\omega$;
3. $(x, i)\ r\ (v, j)$ iff $(ux, i)\ r\ (uv, j)$, all $u \in A^*$;
4. $(x, i)\ r\ w,\ w \in A^\omega$, iff $(ux, i)\ r\ uw$, all $u \in A^*$;
5. for $w \in A^\omega,\ w\ r\ v\ \Leftrightarrow v = w$.

Thus a sequacious relation $n \to p$ is completely determined by its values on pairs of the form $(x, i) \in A \times [n]$. Note that if $r, s : n \to p$ are sequacious relations, then so is $r + s$, where the graph of the sum $r + s$ is the union of the graph of r with the graph of s; in fact the sum of any number of sequacious relations $n \to p$ is also sequacious. If $t : p \to q$ is sequacious, so is the relational composite $r \cdot t$.

The base morphism $n \to p$ in A determined by the function $\beta : [n] \to [p]$ is the sequacious relation r determined by the condition

$$(x, i)\ r\ v \quad \Leftrightarrow \quad v \ = \ (x, i\beta).$$

The theory A is a pointed theory, where $0 : 1 \to 0$ is the sequacious relation with the property that $(x, 1)\ 0\ w$ is always false, for any $x \in A$ and $w \in A^\omega$. If $r : n \to p$ is any sequacious relation, then $r \cdot 0_{pp}$ is a sequacious relation such that if $(x, i)\ (r \cdot 0_{pp})\ v$, then $v \in A^\omega$ and $(x, i)\ r\ v$.

A sequacious relation $r : n \to p$ is *finite* if no pair $(u, i) \in A^+ \times [n]$ is related to an infinite word, i.e. such that if $(u, i) \in A^+ \times [n]$ and $(u, i)\ r\ v$, then $v \in A^+ \times [p]$. Similarly, we call a sequacious relation $r \cdot 0_{pp} : n \to p$ which factors through 0 an *infinite* relation.

We can write a sequacious relation $r : n \to p$ as the sum of a finite sequacious relation $r¶$ and a sequacious relation $r \cdot 0$ which factors through 0. In detail, for $u, v \in A^+$ and $w \in A^\omega$,

$$\begin{aligned} (u, i)\ r¶\ (v, j) \quad &\Leftrightarrow \quad (u, i)\ r\ (v, j) \\ (u, i)\ (r \cdot 0)\ w \quad &\Leftrightarrow \quad (u, i)\ r\ w. \end{aligned}$$

Then

$$r \;=\; r\P + r \cdot 0.$$

We note the following. If $r, r' : n \to p,\ s, s' : p \to q$ are sequacious relations, then

$$\begin{aligned}
r\P\P &= r\P \\
r\P \cdot 0_{pp} &= 0_{np} \\
(r + r')\P &= r\P + r'\P \\
(r \cdot s)\P &= (r\P) \cdot (s\P) \\
0_{np} \cdot s &= 0_{nq} \\
(r + r') \cdot s &= (r \cdot s) \ + \ (r' \cdot s) \\
r \cdot (s + s') &= (r \cdot s) \ + \ (r \cdot s').
\end{aligned}$$

All of these assertions are easy to check.

It follows that A is a matricial theory, isomorphic to the theory $\mathbf{Matr}(S_A; V_A)$ where S_A is the semiring of finite sequacious relations

$$A^+ \cup A^\omega \to A^+ \cup A^\omega,$$

and where V_A is the S_A-module of sequacious relations

$$A^+ \cup A^\omega \to A^\omega.$$

Exercise 3.5.39 Let $S(A)$ consist of all relations $r : A^+ \to A^+$ satisfying the following conditions:

1. $u\ r\ v \ \Rightarrow\ v = uv'$, some $v' \in A^*$;
2. $a\ r\ u \ \Leftrightarrow\ va\ r\ vu$, all $u, v \in A^*$, all $a \in A$.

How are the relations in $S(A)$ related to those in S_A?

Exercise 3.5.40 Show that $S(A)$ is a semiring, with $+$ as union, $\cdot$ as relational product, 0 as the empty relation and 1 as the identity relation.

Exercise 3.5.41 Let S_0 consist of those relations r in $S(A)$ such that for all $a \in A$, $u \in A^+$,

$$a\ r\ u \quad\Rightarrow\quad u \ = \ a.$$

Show that S_0 is a subsemiring of $S(A)$.

Exercise 3.5.42 Let I consist of those relations r in $S(A)$ such that for all $a \in A$, $u \in A^+$,

$$a \ r \ u \quad \Rightarrow \quad |u| \ \geq \ 2.$$

Show that I is an ideal in $S(A)$ (i.e. $0 \in I$, $I + I = I$ and $S \cdot I = I \cdot S = I$) and that every relation in $S(A)$ can be written uniquely as a sum $r = x + y$ where $x \in S_0$ and $y \in I$.

Exercise 3.5.43 Is A a subtheory of X, for some set X?

3.5.5 Sequacious Functions

One subtheory A of A consists of the sequacious functions on A. A *sequacious function*

$$f : A^+ \times [n] \ \cup \ A^\omega \quad \rightarrow \quad A^+ \times [p] \ \cup \ A^\omega$$

is a sequacious relation such that for each $(x, i) \in A \times [n]$ there is some $v \in A^+ \times [p] \cup A^\omega$ with $(x, i) \ f \ v$; secondly, if $(x, i) \ f \ v$ and $(x, i) \ f \ v'$ then $v = v'$. Note that A is not a *submatricial* theory of A, but only a subtheory, since in general the sum of two sequacious functions is not a function.

3.6 Pullbacks and Pushouts of Base Morphisms

In this section, we give a number of facts, in the form of exercises, concerning pullbacks and pushouts of certain base morphisms in arbitrary theories. The results of these exercises will be applied in Chapter 11.

Recall that in any category a *pullback* of

[0‘0‘1‘1;500‘500] [‘B‘C‘D;“r‘b]

is a commuting square

[A‘B‘C‘D;t‘l‘r‘b]

of morphisms in (and t is a pullback of b, and l of r), if for any other commuting square

[A’‘B‘C‘D;t’‘l’‘r‘b]

there is a unique morphism $k : A' \to A$ such that

$$\begin{aligned} k \cdot t &= t' \\ k \cdot l &= l'. \end{aligned}$$

Of course, pullbacks are unique up to isomorphism. Dually, a commuting square

[A‘B‘C‘D;t‘l‘r‘b]

is a *pushout* of

[1‘1‘0‘0;500‘500] [A‘B‘C‘ ;t‘l‘ ‘]

and r and b are pushouts of l and t respectively, if for any other commuting square

[A‘B‘C‘D’;t‘l‘r’‘b’]

there is a unique morphism $k : D \to D'$ with

$$\begin{aligned} r \cdot k &= r' \\ b \cdot k &= b'. \end{aligned}$$

Exercise 3.6.1 Show that in , the pullback of

[0‘0‘1‘1;500‘500] [‘B‘C‘D;“r‘b]

can be given by $A := \{(x; y) \in B \times C : xr = yb\}$ with the maps $t : A \to B,\ l : A \to C$ as the first and second projection functions.

Exercise 3.6.2 Show that pullbacks in **Tot** can be constructed in a similar way.

Exercise 3.6.3 Show that in and **Tot**, pullbacks of surjections are surjections, and pushouts of injections are injections.

Exercise 3.6.4 Suppose that T is a theory containing at least one morphism $\perp : 1 \to 0$. In this exercise, it will be shown that the pullback of any two injective base morphisms exists in T.

1. Show that if $\rho : n \to p$ is an injective base morphism, and if we define $\rho^{-1} : p \to n$ by:

$$i_p \cdot \rho^{-1} \quad := \quad \begin{cases} j_n & \text{if } j_n \cdot \rho = i_p; \\ \perp \cdot 0_n & \text{otherwise,} \end{cases}$$

   then

$$\rho \cdot \rho^{-1} \quad = \quad \mathbf{1}_n.$$

2. If ρ and τ are composable injective base morphisms, show

$$(\rho \cdot \tau)^{-1} \quad = \quad \tau^{-1} \cdot \rho^{-1}.$$

3. Suppose that

   [n‘p‘m‘q;τ'‘ρ'‘ρ‘τ] is a pullback square in **Tot**. Show that if ρ and τ are injective, so are τ' and ρ'. *Hint:* This is a fact true in any category: pullbacks of monics are monics.

4. Show that the pullback square in the previous exercise is also a pullback in T. (More precisely, the image of this square in T under the unique theory morphism from **Tot** to T is a pullback in T.) *Hint:* Show that all ways to get from one corner of the square to the diagonally opposite corner are equivalent; in particular,

$$\begin{aligned} {\tau'}^{-1} \cdot \rho' \quad &= \quad \rho \cdot \tau^{-1} \\ {\rho'}^{-1} \cdot \tau' \quad &= \quad \tau \cdot \rho^{-1}. \end{aligned}$$

   Also show that if $u : r \to p$ and $v : r \to m$ are any T-morphisms with $u \cdot \rho = v \cdot \tau$,

$$u \cdot \tau'^{-1} \quad = \quad v \cdot \rho'^{-1}.$$

   Conclude that $k := u \cdot \tau'^{-1}$ is the unique morphism $r \to n$ such that

$$\begin{aligned} k \cdot \tau' \quad &= \quad u \\ k \cdot \rho' \quad &= \quad v. \end{aligned}$$

Exercise 3.6.5 Suppose that ρ and τ are injections and that [n‘p‘m‘q;τ'‘ρ'‘ρ‘τ] is a pullback square in **Tot** and $n \geq 1$. Show that the square is a pullback in any theory, even if there is no morphism $1 \to 0$ in T. *Hint:* Note first that $n \leq p, m \leq q$. Now show how to define the inverses so that all the equations used in the previous exercise hold.

Exercise 3.6.6 Find the pullback in **Tot** of

$$[0‘0‘1‘1; 500‘500][‘1‘1‘2; “\mathbf{1}_1 \oplus 0_1‘0_1 \oplus \mathbf{1}_1] \tag{3.5}$$

Exercise 3.6.7 Let T be the theory of the variety determined by the equation

$$f(x_1) \quad = \quad f(x_2).$$

Thus, T has no morphisms $1 \to 0$ and, regarding f as a morphism $1 \to 1$ in T, we have $f \cdot (\mathbf{1}_1 \oplus 0_1) = f \cdot (0_1 \oplus \mathbf{1}_1)$. Show the pullback in **Tot** of the pair (3.5) in the previous problem is not the pullback in T. What is the pullback in T?

Exercise 3.6.8 Let T be the theory of the variety determined by the equations

$$\begin{aligned} f(x_1) &= f(x_2) \\ g(x_1) &= h(x_2), \end{aligned}$$

where f, g, h are distinct unary function symbols. Show that the pullback of $\mathbf{1}_1 \oplus 0_1$ and $0_1 \oplus \mathbf{1}_1$ does not exist in T.

Exercise 3.6.9 Suppose that

[n‘p‘m‘q;ρ‘τ‘τ'‘ρ'] is a pushout square in **Tot** and suppose that ρ and τ are surjective base morphisms. In this exercise, we show the same square is a pushout in any theory T.

1. Show that both ρ' and τ' are surjective base morphisms. *Hint:* Show that in any category, pushouts of epis are epis.
2. Show that it is possible to choose base morphisms $\rho^{-1} : p \to n$, $\tau^{-1} : m \to n$, ${\tau'}^{-1} : q \to p$ and ${\rho'}^{-1} : q \to m$ which are left inverses of ρ, τ, τ' and ρ', respectively, and such that

$$\rho^{-1} \cdot \tau \quad = \quad \tau' \cdot {\rho'}^{-1} \tag{3.6}$$

$$\tau^{-1} \cdot \rho \quad = \quad \rho' \cdot {\tau'}^{-1}. \tag{3.7}$$

   Hint: First choose left inverses ρ^{-1}, τ^{-1} for ρ and τ, which exist since both are surjections. Then show it is possible to define left inverses for ρ' and τ' satisfying the equations (3.6) and (3.7).

3. Now show that if $\rho \cdot u = \tau \cdot v$, for any morphisms $u : p \to r$, $v : m \to r$, and if $k : q \to r$ is any morphism satisfying
$$\tau' \cdot k = u \quad (3.8)$$
$$\rho' \cdot k = v \quad (3.9)$$
then $k = {\rho'}^{-1} \cdot v = {\tau'}^{-1} \cdot u$.

4. Show
$${\rho'}^{-1} \cdot v = {\tau'}^{-1} \cdot u.$$
Now show that if k is defined as the morphism ${\rho'}^{-1} \cdot v = {\tau'}^{-1} \cdot u$, then the equations (3.8) and (3.9) hold. Thus, the pushout in **Tot** of ρ and τ is the pushout in T. *Hint:* Use the equations (3.6) and (3.7).

Exercise 3.6.10 Suppose that both ρ and τ are injective base morphisms in T. Show that their pushout in **Tot** is also a pushout in T. *Hint:* If $\pi : p \to p$ and $\psi : m \to m$ are isomorphisms, and if

[n‘p‘m‘q;ρ‘τ‘τ'‘ρ'] is a pushout square (in any category!), then so is

[n‘p‘m‘q;$\rho \cdot \pi$‘$\tau \cdot \psi$‘$\pi^{-1} \cdot \tau'$‘$\psi^{-1} \cdot \rho'$] Thus, in order to prove the claim, one can prove it for the case that $p = n + n'$, $m = n + n''$, $\rho = \mathbf{1}_n \oplus 0_{n'}$ and $\tau = \mathbf{1}_n \oplus 0_{n''}$.

Exercise 3.6.11 Using the same idea, show that the pushout in **Tot** of an injective base and a surjective base is also a pushout in T.

We have shown that the pushouts of any combination of injective or surjective base morphisms exist in any theory. We want to show that the pushout of any pair of base morphisms also exists.

Exercise 3.6.12 Show that if

[A‘B‘C‘D;t‘l‘r‘b]

and

[B‘E‘D‘F;t’‘r‘r’‘b’]

are pushout squares, so is

[A‘E‘C‘F;t$\cdot t'$‘l‘r'‘$b \cdot b'$] Use this fact twice, and the fact that any base morphism is the composite of a surjection followed by an injection, to show that in any theory T, the pushout of any two base morphisms exists, and is given by the pushout in **Tot**.

3.7 2-Theories

In this section, we slightly generalize the notion of theory in order to make some later applications easier to formulate. Theories are to 2-theories what categories are to 2-categories (defined in Section 1.1.4). Suppose that T is a 2-category whose underlying category is a theory. Thus, for each $n, p \geq 0$, the collection of horizontal morphisms $T(n,p)$ are the objects of a category, also denoted $T(n,p)$. Let T denote the category of cells with horizontal composition. Then roughly, if T is itself a theory, T is called a 2-*theory.*

Definition 3.7.1 *A* **2-theory** *T is a 2-category whose underlying category is an algebraic theory such that T is also a theory with distinguished morphisms the identity cells $(\mathbf{I}(i_n), i_n, i_n) : 1 \to n$, where $i_n : 1 \to n$, $i \in [n]$, $n \geq 1$, is the distinguished (horizontal) morphism in T.*

In more detail, in a 2-theory, there are two operations of tupling, one defined on the horizontal morphisms and one on the vertical morphisms. When $f_i, g_i : 1 \to p$ is a horizontal morphism and $u_i : f_i \to g_i$ is a vertical morphism, for each $i \in [n]$, then

$$\langle u_1, \ldots, u_n \rangle : \langle f_1, \ldots, f_n \rangle \quad \to \quad \langle g_1, \ldots, g_n \rangle$$

is the unique vertical morphism u with $\mathbf{I}(i_n) \cdot u = u_i$, for each $i \in [n]$. Similarly, $\langle f_1, \ldots, f_n \rangle$ is the unique horizontal morphism f with $i_n \cdot f = f_i$, for each $i \in [n]$. Thus every cell $x = (u, f, g) : n \to p$ in a 2-theory can be written uniquely as

$$x \quad = \quad (\langle u_1, \ldots, u_n \rangle, \, \langle f_1, \ldots, f_n \rangle, \, \langle g_1, \ldots, g_n \rangle),$$

where for each $i \in [n]$, $(u_i, f_i, g_i) : 1 \to p$. In particular, the unique cell $0 \to p$ is $(\mathbf{I}(0_p), 0_p, 0_p)$. Since $\langle 1_1 \rangle = \mathbf{1}_1$, for the distinguished horizontal morphism $1_1 : 1 \to 1$, it follows that in any 2-theory, $\langle \mathbf{I}(1_1) \rangle = \mathbf{I}(1_1)$.

For the next two definitions, suppose that T and T' are 2-theories.

Definition 3.7.2 *T is a* **sub-2-theory** *of T' if each cell in T is a cell in T', the distinguished morphisms and the horizontal and vertical identities in T are those of T', and the operations of horizontal and vertical composition in T are those in T'.*

It follows that if T is a sub-2-theory of T', then T is closed under tupling.

Definition 3.7.3 *A* **2-theory morphism** $\varphi : T \to T'$ *is a 2-functor which preserves all distinguished morphisms.*

It follows that a 2-theory morphism preserves tupling.

If $\varphi : T \to T'$ is a 2-theory morphism, then φ extends to a map of cells $x = (u, f, g) : n \to p$ in T to cells $x\varphi := (u\varphi, f\varphi, g\varphi) : n \to p$ in T'. As such, φ preserves both the horizontal and vertical composition on cells, as well as the identities. In particular, $\mathbf{I}(f)\varphi = \mathbf{I}(f\varphi)$.

Proposition 3.7.4 *Suppose that T is a 2-theory. Then*

- *tupling preserves identities: for any horizontal morphisms $f_i : 1 \to p,\ i \in [n]$,*

$$\langle \mathbf{I}(f_1), \ldots, \mathbf{I}(f_n) \rangle \quad = \quad \mathbf{I}(\langle f_1, \ldots, f_n \rangle); \tag{3.10}$$

- *for any cells $x_i : 1 \to p$ and cell $y : p \to q$,*

$$\langle x_1, \ldots, x_n \rangle \cdot y \quad = \quad \langle x_1 \cdot y, \ldots, x_n \cdot y \rangle.$$

Proof. The proof of (3.10) follows by precomposing each side with $\mathbf{I}(i_n)$ and using the fact that in any 2-category, $\mathbf{I}(f) \cdot \mathbf{I}(g) = \mathbf{I}(f \cdot g)$. The rest is easy.

We give several examples of 2-theories.

Example 3.7.5 Any theory is a 2-theory in which each cell is an identity cell $(\mathbf{I}(f), f, f)$.

Example 3.7.6 Suppose that T is a theory with the property that each hom-set $T(n, p)$ is a partially ordered set. If the ordering is preserved by both composition and tupling, i.e.

$$\begin{array}{rcl} f \le g,\ f' \le g' & \Rightarrow & f \cdot f' \le g \cdot g' \\ f_1 \le g_i,\ i \in [n] & \Rightarrow & \langle f_1, \ldots, f_n \rangle \le \langle g_1, \ldots, g_n \rangle, \end{array}$$

then T is an *ordered theory.* An ordered theory is a 2-theory in which there is a vertical morphism $f \to g$ if $f \le g$, for any $f, g : n \to p$.

Example 3.7.7 Suppose that is a small category. For each $n \geq 0$, let ^{n} be the n-fold product category of with itself. We define the 2-theory $Th()$ of functors over . A horizontal morphism $n \to p$ is a functor f $:^p \to ^n$. (Note the reversal of direction.) A vertical morphism $u : f \to g$ is a natural transformation $u : f \xrightarrow{\bullet} g$. Horizontal composition of functors is usual functor composition; horizontal and vertical composition of natural transformations is also standard. The distinguished horizontal morphism $i_n : 1 \to n$ is the i-th projection functor $^n \to$. We will be interested in this kind of 2-theory when the category has some additional properties, such as all ω-colimits.

This last example is representative.

Proposition 3.7.8 *Suppose that T is a 2-theory. Then there is a category such that T is isomorphic to a sub-2-theory of $Th()$.*

Proof. We modify the construction in Proposition 3.2.2. Define A as the product of all horizontal morphism $1 \to k$ in T:

$$A \quad := \quad \prod_{k \geq n_0} T(1, k).$$

(Recall that $n_0 = 0$ if $T(1,0) \neq \emptyset$ and $n_0 = 1$ otherwise.) Let be the following category. The objects of are the elements in A. A morphism

$$(f_k) \quad \to \quad (g_k)$$

is a sequence (u_k) of vertical morphisms, where for each $k \geq n_0$,

$$u_k : f_k \quad \to \quad g_k \text{ in } T(1, k).$$

Composition is pointwise vertical composition: if $(u_k) : (f_k) \to (g_k)$ and $(v_k) : (g_k) \to (h_k)$, then the composite $(u_k) \circ (v_k) : (f_k) \to (h_k)$ is $(u_k \circ v_k)$.

Now suppose that $t : 1 \to p$ is a horizontal morphism in T. Define the functor φ_t $:^p \to$ as follows. Given a p-tuple of objects in , or equivalently, given a sequence (f_k) of horizontal morphisms with $f_k : p \to k$, all $k \geq n_0$, define

$$(f_k)\varphi_t \quad := \quad (t \cdot f_k).$$

If $(u_k) : (f_k) \to (g_k)$ in ^{p}, so that for each $k \geq n_0$, $u_k : f_k \to g_k$ in $T(p, k)$, define

$$(u_k)\varphi_t \quad := \quad (t \cdot u_k),$$

where $t \cdot u_k$ is the horizontal composite of $\mathbf{I}(t) : t \to t$ with $u_k : f_k \to g_k$, $k \geq n_0$.

We check that φ_t is a functor. Suppose that for each $k \geq n_0$,

$$\begin{array}{rcl} u_k : f_k & \to & g_k \\ v_k : g_k & \to & h_k \end{array}$$

in $T(p,k)$. Then the i-th component of $((u_k) \circ (v_k))\varphi_t$ is $t \cdot (u_i \circ v_i)$. The i-th component of $(u_k)\varphi_t(v_k)\varphi_t$ is $(t \cdot u_i) \circ (t \cdot v_i)$. But using the interchange law,

$$\begin{array}{rcl} (t \cdot u_i) \circ (t \cdot v_i) & = & (t \circ t) \cdot (u_i \circ v_i) \\ & = & t \cdot (u_i \circ v_i). \end{array}$$

It is clear that φ_t preserves identities. Thus φ_t is a functor.

When $t = \langle t_1, \ldots, t_n \rangle : n \to p$, then

$$\begin{array}{rcl} \varphi_t & := & \langle \varphi_{t_1}, \ldots, \varphi_{t_n} \rangle \\ (f_k) & \mapsto & ((f_k)\varphi_{t_1}, \ldots, (f_k)\varphi_{t_n}). \end{array}$$

On morphisms $(u_k) : (f_k) \to (g_k)$, φ_t is defined in a similar way.

Now suppose that $v : t \to t'$ is a vertical morphism in $T(1,p)$. We show how v determines a natural transformation

$$\varphi_v \ : \ \varphi_t \ \to \ \varphi_{t'}.$$

On the object (f_k) in ^{p}, the component $(f_k)\varphi_v$ is defined as follows:

$$(f_k)\varphi_v \ = \ (v \cdot f_k \ : \ t \cdot f_k \to t' \cdot f_k).$$

To check that φ_v in indeed a natural transformation, we must show that for any morphism $(u_k) : (f_k) \to (g_k)$ in ^{p}, the square

[1‘1‘1‘1;800‘500] [(f$_k$)φ_t‘$(f_k)\varphi_{t'}$‘$(g_k)\varphi_t$‘$(g_k)\varphi_{t'}$;$(f_k)\varphi_v$‘$(u_k)\varphi_t$‘$(u_k)\varphi_{t'}$‘$(g_k)\varphi_v$] commutes. But for any fixed $k \geq n_0$,

$$\begin{array}{rcl} (v \cdot f_k) \circ (t' \cdot u_k) & = & (v \circ t') \cdot (f_k \circ u_k) \\ & = & (t \circ v) \cdot (u_k \circ g_k) \\ & = & (t \cdot u_k) \circ (v \cdot g_k). \end{array}$$

Thus φ_v is natural. We omit the easy proof that φ is a 2-theory morphism $T \to Th()$; one shows that φ is injective just as in the earlier proposition. The proof of this proposition is complete.

Exercise 3.7.9 Let T be a 2-theory. Recall that a vertical morphism $u : f \to g$ in $T(n, p)$ is a retraction if for some $v : g \to f$,

$$v \circ u \quad = \quad \mathbf{I}(g).$$

Show that if $u : f \to g$ is a retraction in $T(n, p)$, and if $u' : f' \to g'$ is a retraction in $T(m, p)$, then the pairing $\langle u, u' \rangle : \langle f, f' \rangle \to \langle g, g' \rangle$ is a retraction in $T(n + m,\, p)$.

Exercise 3.7.10 Suppose that Σ is a ranked set, and for each $\sigma, \sigma' \in \Sigma_n$, $V(\sigma, \sigma', n)$ is a set. Show that there is a 2-theory $T(\Sigma, V)$ with the following universal property. Suppose T is a 2-theory and F is a function which assigns to each $\sigma \in \Sigma_n$ a horizontal morphism

$$\sigma F : 1 \quad \to \quad n$$

in T; and to each letter u in $V(\sigma, \sigma', n)$ F assigns a vertical morphism

$$uF : \sigma F \quad \to \quad \sigma' F$$

in T. Then there is a unique 2-theory morphism

$$F^\sharp : T(\Sigma, V) \quad \to \quad T$$

extending F. Give a concrete description of $T(\Sigma, V)$.

Chapter 4

Algebras

In this chapter, we show how each theory T determines a category $T^\flat$ of T-algebras. Each such category is an abstraction of the notion of a variety of Σ-algebras. The category whose objects are the varieties $T^\flat$ and whose morphisms are functors which commute with the underlying set functors is shown to be dually isomorphic to the category of theories and theory morphisms.

4.1 T-algebras

Suppose that T is a fixed algebraic theory.

Definition 4.1.1 *A* T**-algebra** *is a pair* $\mathbf{A} = (A, \alpha)$ *where* A *is a set and* $\alpha : T \to A$ *is a theory morphism. We write the image under* α *of the* T*-morphism* $f : n \to p$ *as* $f_\alpha : A^p \to A^n$*. If* $\mathbf{A} = (A, \alpha)$ *and* $\mathbf{B} = (B, \beta)$ *are* T*-algebras, a* **homomorphism** $h : \mathbf{A} \to \mathbf{B}$ *is a function* $h : A \to B$ *such that for each* $f : n \to p \in T$*, the following diagram commutes:*

[A^p‘A^n‘B^p‘B^n; f_α‘h^p‘h^n‘f_β*]*

Remark 4.1.2 The following two statements are equivalent.

- $\alpha : T \to A$ is a theory morphism.

- α is a functor from the category T^{op} to such that the value of α on $i_n : 1 \to n$ is the i-th projection function $A^n \to A$, where $A = 1\alpha$.

Furthermore, the following two statements mean the same thing.

- $h : (A, \alpha) \to (B, \beta)$ is a homomorphism.
- h is a natural transformation between the two product preserving functors $\alpha, \beta : T^{op} \to$.

Definition 4.1.3 *For each theory T, the collection of all T-algebras and homomorphisms determines a category, denoted $T^\flat$.*

The set $A = |\mathbf{A}|$ is called the underlying set of the algebra $\mathbf{A} = (A, \alpha)$. In fact, there is an *underlying set functor*

$$U_T : T^\flat \ \to$$

mapping the homomorphism $h : (A, \alpha) \to (B, \beta)$ to the function $h : A \to B$.

Note that if $n \neq 1$ and $f : n \to p$ is a T-morphism, the function $f_\alpha : A^p \to A^n$ is determined by the n functions $g_i := (i_n \cdot f)_\alpha : A^p \to A,\ i \in [n]$, as follows:

$$f_\alpha(a_1, l, a_p) \ = \ (g_1(a_1, l, a_p), l, g_n(a_1, l, a_p)).$$

Hence, we may specify a T-algebra by defining only the functions f_α for $f : 1 \to p$.

Exercise 4.1.4 Show that a function $A \to B$ is a homomorphism $(A, \alpha) \to (B, \beta)$ of T-algebras iff $f_\alpha \cdot h = h^n \cdot f_\beta$, for all $f : 1 \to n$ in T.

Remark 4.1.5 To see that T-algebras are generalizations of Σ-algebras, suppose that T is the theory . If $\alpha : T \to A$ is a theory morphism, then α assigns to each term $t : 1 \to n$ in a function $t_\alpha : A^n \to A$ in such a way that the term x_i, $i \in [n]$, is assigned the i-th projection function and for all $\sigma \in \Sigma_n,\ g = \langle g_1, \ldots, g_n \rangle : n \to p$, the term $(\sigma \cdot g)_\alpha : A^p \to A$ is the following function:

Of course, a Σ-algebra $\mathbf{A}$ is just a set A together with an assignment of a function $\sigma_{\mathbf{A}} : A^n \to A$ to each letter in Σ_n, $n \geq 0$. Any such assignment of functions to the letters in Σ extends uniquely to an assignment of functions to all terms as above (see Example 2.2.3.31). Thus, in this case, there is only a minor difference between a T-algebra and a Σ-algebra.

If $\mathbf{A}$ and $\mathbf{B}$ are T-algebras, we say that $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$, in symbols, $\mathbf{A} \subseteq \mathbf{B}$, if $A \subseteq B$ and the inclusion is an injective homomorphism $h : \mathbf{A} \to \mathbf{B}$; we say that $\mathbf{B}$ is a *quotient* or *homomorphic image* of $\mathbf{A}$ if there is a surjective homomorphism $h : \mathbf{A} \to \mathbf{B}$. If $\mathbf{A}_i = (A_i, \alpha_i),\ i \in I$, are T-algebras, the product

$$\prod_{i\in I} \mathbf{A}_i \;=\; (\prod_{i\in I} A_i, \alpha)$$

is defined pointwise: for $f : 1 \to 2$, say, in T,

$$f_\alpha((a_i),(b_i)) \;=\; (f_{\alpha_i}(a_i, b_i)).$$

We will use the operators $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{P}$ in connection with T-algebras as usual.

It is a simple but important fact that morphisms between theories induce functors between the categories of their algebras. This observation will be used in later sections.

Proposition 4.1.6 *Suppose that $\varphi : T \to R$ is a theory morphism. Then φ induces a functor*

$$\begin{aligned} \varphi^\flat : R^\flat &\to T^\flat \\ h : (A,\alpha) \to (B,\beta) &\mapsto h : (A,\, \varphi\cdot\alpha) \to (B,\, \varphi\cdot\beta) \end{aligned}$$

which commutes with the underlying set functors.

Proof. Since φ is a theory morphism, $(A,\alpha)\varphi^\flat = (A,\, \varphi\cdot\alpha)$ is a T-algebra, for each R-algebra (A,α). We need to check that if $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism of R-algebras, then h is also a homomorphism $h : \mathbf{A}\varphi^\flat \to \mathbf{B}\varphi^\flat$. Thus, one must check that for all $t : 1 \to p$ in T, the diagram

[A^p'A'B^p'B; $t_{\varphi\cdot\alpha}$'h^p'h'$t_{\varphi\cdot\beta}$] commutes. But $t_{\varphi\cdot\alpha} = (t\varphi)_\alpha$ and $t_{\varphi\cdot\beta} = (t\varphi)_\beta$. Since h is an R-algebra homomorphism, the result follows.

Exercise 4.1.7 Show that if $\varphi : T \to R$ and $\psi : R \to S$ are theory morphisms, then $(\varphi \cdot \psi)^\flat = \psi^\flat \cdot \varphi^\flat$. Show also that if $\varphi : T \to T$ is the identity theory morphism, then $\varphi^\flat$ is the identity functor on $T^\flat$. Lastly, show that if $\varphi : T \to R$ is a surjective theory morphism, then $\varphi^\flat : R^\flat \to T^\flat$ is full, cf. Chapter 1.

We see that $T^\flat$ is closed under the operations of subalgebras, quotients and products. In fact, the following holds.

Theorem 4.1.8 *Let T be an algebraic theory. There is some signature Σ and set E of equations between Σ-terms such that the two categories $T^\flat$ and $K = Mod(E)$ are isomorphic over the category ; i.e. there is an isomorphism $\varphi : T^\flat \to K$ such that*

$[T^\flat$'K';φ'U_T'$U_K]$ commutes. Conversely, if Σ is any signature and $K = Mod(E)$ is any variety of Σ-algebras, then there is some theory T such that $T^\flat$ and K are isomorphic over .

Proof sketch. We know that any theory is isomorphic to one of the form $/E$, for some signature Σ and some set E of equations between Σ-terms. If $\psi : /E \to T$ is a theory isomorphism, then it follows from Exercise 4.1.7, $\psi^\flat$ is an isomorphism $T^\flat \to (/E)^\flat$. But a $(/E)$-algebra is, generalizing Remark 4.1.5 above, just a Σ-algebra which satisfies all the equations in E.

Remark 4.1.9 It follows that category $T^\flat$ enjoys all of the following properties: it is complete, cocomplete, well-powered, cowell-powered, and has surjective-injective factorizations. The monics in $T^\flat$ are just the injective homomorphisms. Lastly, the surjective homomorphisms are precisely the regular epis.

Remark 4.1.10 If T is a trivial theory, then all nonempty algebras in $T^\flat$ are singletons. Indeed, if (A, α) is an algebra of a trivial theory and $a, b \in A$, then

$$\begin{aligned} a &= (1_2)_\alpha(a,b) \\ &= (2_2)_\alpha(a,b) \\ &= b. \end{aligned}$$

If $T(1,0) = \emptyset$, then the empty algebra belongs to $T^\flat$.

Remark 4.1.11 A more interesting result is that each of the functors $\varphi^\flat$ induced by theory morphisms has a left adjoint (see [BW85] for one proof). For an example, consider the theories T and G of commutative monoids and commutative groups given in Examples 2.2.4.4 and 2.2.4.5. The inclusion $\varphi : T \to G$ determines the forgetful functor $G^\flat \to T^\flat$, which maps any commutative group to the underlying commutative monoid. This functor, in particular, has a left adjoint.

4.1.1 An Example: Algebras of Matrix Theories

In this section, we show that the algebras of a matrix theory $T = \mathbf{Mat}_S$ are essentially the same as S-modules, defined in Definition 3.3.5.23.

Now if V is an S-module, for any $n \times p$ matrix f over S, define the function $f_\alpha : V^p \to V^n$ using the matrix product:

$$f_\alpha(v_1, \ldots, v_p) \ := \ \begin{bmatrix} f_{11} & \ldots & f_{1p} \\ \vdots & & \vdots \\ f_{n1} & \ldots & f_{np} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix}.$$

Note that if $f = i_n$, then f_α is the i-th projection function. We let the reader verify that

$$\begin{aligned} (f \cdot g)_\alpha &= f_\alpha \cdot g_\alpha \\ (i_n)_\alpha(v_1, \ldots, v_n) &= v_i. \end{aligned}$$

Thus, S-modules determine T-algebras.

Conversely, suppose that $V = (V, \alpha)$ is a T-algebra. Define the operation $+ : V^2 \to V$ as the function determined by the morphism $f = [1\ 1] : 1 \to 2$ in T.

$$v + v' \ := \ [1\ 1]_\alpha(v, v') \ = \ \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix}.$$

Let $0 \in V$ be the element determined by the morphism $0 : 1 \to 0$ in T. Lastly, define the action of S on V as follows:

$$sv \ := \ s_\alpha(v),$$

where we identify S with the 1×1 matrices over S.

Claim 1. $(V, +, 0)$ is a commutative monoid. We prove only a few of the required facts, leaving the rest to the diligent reader.
Proof that $x + y = y + x$, for any $x, y \in V$.

$$x + y \ = \ \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} \\
&= y + x.
\end{aligned}
$$

Proof that $x + (y + z) = (x + y) + z$, for any $x, y, z \in V$.

$$
\begin{aligned}
x + (y + z) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= (x + y) + z.
\end{aligned}
$$

Proof that $0 + x = x$, for any $x \in V$. Note that if $0 : 1 \to 0$ in T, then the morphism $0 \oplus \mathbf{1}_1 : 2 \to 1$ determines the function $V \to V^2$ taking x to $(0, x)$. Hence

$$
\begin{aligned}
0 + x &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [x] \\
&= [1][x] \\
&= x.
\end{aligned}
$$

Claim 2. V is an S-module. First note that if $f = [s_1\ s_2] : 1 \to 2$, say, in T, then

$$
\begin{aligned}
(s_1 \oplus 0_1)_\alpha(v, v') &= [s_1\ 0_{11}] \begin{bmatrix} v \\ v' \end{bmatrix} \\
&= s_1 v, \qquad \text{and similarly} \\
(0_1 \oplus s_2)_\alpha(v, v') &= s_2 v'.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f_\alpha(v, v') &= ([1\ 1] \cdot \langle s_1 \oplus 0_1,\ 0_1 \oplus s_2 \rangle)_\alpha(v, v') \\
&= [1\ 1]_\alpha(s_1 v,\ s_2 v) \\
&= s_1 v + s_2 v'.
\end{aligned}
$$

Similarly, if $f = [s_1, \ldots, s_p] : 1 \to p$, then f_α is the function $V^p \to V$,

$$
f_\alpha(v_1, \ldots, v_p) = s_1 v_1 + \ldots + s_p v_p.
$$

It follows that for any $n \times p$ matrix $f = [s_{ij}]$ over S, the function $f_\alpha : V^p \to V^n$ is given by the matrix multiplication formula:

$$
\begin{aligned}
f_\alpha(v_1, \ldots, v_p) &= (w_1, \ldots, w_n), \qquad \text{where} \\
w_i &= s_{i1} v_1 + \ldots + s_{ip} v_p, \quad \text{for each } i \in [n].
\end{aligned}
$$

Proof that $s(v+v') = sv + sv'$.

$$\begin{aligned} s(v+v') &= [s][1\ 1]\begin{bmatrix} v \\ v' \end{bmatrix} \\ &= [1\ 1]\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} v \\ v' \end{bmatrix} \\ &= sv + sv'. \end{aligned}$$

Proof that $(s+s')v = sv + s'v$.

$$\begin{aligned} (s+s')v &= [s+s'][v] \\ &= [1\ 1]\begin{bmatrix} s \\ s' \end{bmatrix}[v] \\ &= [1\ 1]\begin{bmatrix} sv \\ s'v \end{bmatrix} \\ &= sv + s'v. \end{aligned}$$

We omit the easy proofs of the remaining facts.

4.2 Free Algebras in $T^\flat$

A T-algebra $\mathbf{A} = (A, \alpha)$ is *freely generated by the set* X in $T^\flat$, more precisely, freely generated by the function

$$\eta_X : X \rightarrow A,$$

if for any T-algebra $\mathbf{B} = (B, \beta)$ and any function $h : X \to B$ there is a unique T-algebra homomorphism $h^\sharp : \mathbf{A} \to \mathbf{B}$ such that

$$\eta_X \cdot h^\sharp = h.$$

Since the categories $T^\flat$ are varieties, we know all free algebras exist. In this section, we give more detailed descriptions of these algebras. When X is finite, there is a particularly simple description.

4.2.1 The T-algebras T_n

For any theory T and any $n \geq 0$, we will define a T-algebra T_n whose underlying set is the set $T(1, n)$ of all T-morphisms $1 \to n$.

If $f : 1 \to p$ is any T-morphism, f determines a function $f_{\kappa_n} : T(1,n)^p \to T(1,n)$, as follows:

$$f_{\kappa_n}(g_1, \ldots, g_p) \quad := \quad f \cdot \langle g_1, \ldots, g_p \rangle.$$

When $f : q \to p$, define

$$f_{\kappa_n}(g_1, , \ldots, g_p) \quad := \quad ((f_1)_{\kappa_n}(g_1, \ldots, g_p), \ldots, (f_q)_{\kappa_n}(g_1, \ldots, g_p)),$$

where $f_i := i_q \cdot f,\ i \in [q]$.

We show that κ_n is a theory morphism $T \to T_n$. Indeed, if $f : 1 \to p,\ f' : p \to q$, then writing f'_i for $i_p \cdot f'$,

$$\begin{aligned}
&(f \cdot f')_{\kappa_n}(g_1, \ldots, g_q) \ = \\
&\quad = \ (f \cdot f') \cdot \langle g_1, \ldots, g_q \rangle \\
&\quad = \ f \cdot \langle f'_1 \cdot \langle g_1, \ldots, g_q \rangle, \ldots, f'_p \cdot \langle g_1, \ldots, g_q \rangle \rangle \\
&\quad = \ f_{\kappa_n}((f'_1)_{\kappa_n}(g_1, \ldots, g_q), \ldots, (f'_p)_{\kappa_n}(g_1, \ldots, g_q)).
\end{aligned}$$

It follows that κ_n preserves composition. The image of the distinguished morphism $i_p : 1 \to p$ is the i-th projection function, since

$$\begin{aligned}
(i_p)_{\kappa_n}(g_1, \ldots, g_p) \quad &= \quad i_p \cdot \langle g_1, \ldots, g_p \rangle \\
&= \quad g_i.
\end{aligned}$$

Thus we have proved

Proposition 4.2.1 *For each theory T and each $n \geq 0$, $(T(1,n), \kappa_n)$ is a T-algebra, denoted T_n.*

Recall that $V_n = \{x_1, \ldots, x_n\}$. Let $\eta_n : V_n \to T(1,n)$ be the function mapping x_i to the distinguished morphism $i_n : 1 \to n$.

Theorem 4.2.2 *The T-algebra T_n is freely generated in $T^\flat$ by η_n.*

Proof. Let $\mathbf{A} = (A, \alpha)$ be any T-algebra, and let $h : V_n \to A$ be any function. Write the value $x_i h$ as $a_i \in A$. Then if $h^\sharp$ is any homomorphism with $\eta_n \cdot h^\sharp = h$, then for each $f : 1 \to n$ the following square commutes.

[1‘1‘1‘1;1000‘500] [T(1,n)^n‘$T(1,n)$‘A^n‘A; f_{κ_n}‘$(h^\sharp)^n$‘$h^\sharp$‘f_α] In particular,

$$\begin{aligned} h^\sharp(f) &= h^\sharp(f_{\kappa_n}(1_n,\ldots,n_n)) \\ &= f_\alpha(h^\sharp(1_n),\ldots,h^\sharp(n_n)) \\ &= f_\alpha(a_1,\ldots,a_n). \end{aligned}$$

Thus, we *define* $h^\sharp : T(1,n) \to A$ by

$$h^\sharp(f) \quad := \quad f_\alpha(a_1,\ldots,a_n). \tag{4.1}$$

Since α is a theory morphism, it follows that for each $p \geq 0$, $f : 1 \to p$, the square

[1‘1‘1‘1;1000‘500] [T(1,n)^p‘$T(1,n)$‘A^p‘A; f_{κ_n}‘$(h^\sharp)^n$‘$h^\sharp$‘f_α] commutes. Hence $h^\sharp$ is the unique T-algebra homomorphism extending h.

These T-algebras are closely related to one another. For example, we have the following fact.

Corollary 4.2.3 *Suppose that $f : n \to m$ in T. Then f determines a T-algebra homomorphism*

$$\begin{aligned} f^\sharp : T_n &\to T_m \\ g &\mapsto g \cdot f. \end{aligned}$$

Exercise 4.2.4 Prove that Corollary 4.2.3 follows directly from Proposition 4.2.2.

It follows not only that $T^\flat$ has all finitely generated free algebras, but that we have a reasonable description of these algebras. In particular, consider the T-algebra

$$T_0 \quad = \quad (T(1,0), \kappa_0). \tag{4.2}$$

Proposition 4.2.5 *T_0 is the initial algebra in $T^\flat$: i.e. for any T-algebra $\mathbf{A} = (A,\alpha)$ there is a unique homomorphism $h : T_0 \to \mathbf{A}$. In fact, the unique homomorphism $h : T_0 \to (A,\alpha)$ is the restriction of α to the set of morphisms $T(1,0)$.*

Proof. Indeed, the restriction is a homomorphism: the equation (4.1) shows that under any homomorphism,

$$f \in T(1,0) \;\mapsto\; f_\alpha.$$

Hence this function is the unique homomorphism.

Proposition 4.2.6 *Suppose that $f, f' : 1 \to n$ in the theory T. Then the following are equivalent:*

- $f = f'$;
- *for each T-algebra $\mathbf{A} = (A, \alpha)$, $f_\alpha \;=\; f'_\alpha$;*
- *in the T-algebra T_n, $f_{\kappa_n}(1_1, \ldots, n_n) \;=\; f'_{\kappa_n}(1_1, \ldots, n_n)$.*

Proof. Clearly, we need to show only that the third condition implies the first. But, in the T-algebra $T_n = (T(1,n), \kappa_n)$,

$$f_{\kappa_n}(1_n, \ldots, n_n) \;=\; f \cdot \mathbf{1}_n \;=\; f.$$

4.2.2 Infinitely Generated Free Algebras in $T^\flat$

For a given theory T and (infinite) set X, recall the coproduct theory $T[X]$ from Section 3.3.4. In particular, we assume familiarity with Theorem 3.4.1 from that section, and the notation used there. We will show here that the T-algebra obtained from the initial $T[X]$-algebra by applying the functor $\kappa^\flat$ of Proposition 4.1.6 is freely generated by the function $\lambda : X \to T[X](1,0)$.

Now $T[X]_0\kappa^\flat := (T[X](1,0),\, \kappa \cdot \kappa_0)$, where $\kappa_0 : T[X] \to T[X](1,0)$ is the theory morphism (4.2) above. We write

$$\tau_X \;\;:=\;\; \kappa \cdot \kappa_0.$$

Suppose that (A, α) is a T-algebra and that $f : X \to A$ is any function. Since A may be identified with $A(1,0)$, it follows from Theorem 3.3.4.1 that there is a unique theory morphism

$$\alpha_f : T[X] \;\;\to\;\; A$$

such that

$$\begin{aligned} \kappa \cdot \alpha_f &= \alpha && (4.3) \\ \lambda \cdot \alpha_f &= f. && (4.4) \end{aligned}$$

By Proposition 4.2.5, the restriction of α_f to $T[X](1,0)$ is the unique $T[X]$-algebra homomorphism with target (A, α_f). Applying the functor $\kappa^\flat$, we see that α_f is also a T-algebra homomorphism $T[X]_0\kappa^\flat \to (A, \alpha)$, by Proposition 4.1.6. It only remains to show that α_f is the unique T-algebra homomorphism with the properties (4.3) and (4.4). By the construction of $T[X]$, any morphism $1 \to 0$ in $T[X]$ can be written in the form

$$g\kappa \cdot \langle x_1\lambda, \ldots, x_n\lambda\rangle,$$

for some $g : 1 \to n$ in T, and some $x_i \in X,\ i \in [n]$. But, by definition,

$$g\kappa \cdot \langle x_1\lambda, \ldots, x_n\lambda\rangle \quad = \quad g_{\tau_X}(x_1\lambda, \ldots, x_n\lambda).$$

The uniqueness of α_f follows immediately.

We have thus proved the following fact.

Corollary 4.2.7 *For any set X, the T-algebra $(T[X](1,0), \tau_X)$ is freely generated in $T^\flat$ by the function $\lambda : X \to T[X](1,0)$.*

When needed, we write the X-generated free T-algebra as

$$\mathbf{F}_T[X] \quad := \quad (T[X](1,0), \tau_X).$$

When several theories are involved, we may also write τ_X^T.

Proposition 4.2.8 *Suppose that X has at least n elements. Then for any $f, f' : 1 \to n$ in T, if*

$$f_{\tau_X}(x_1\lambda, \ldots, x_n\lambda) \quad = \quad f'_{\tau_X}(x_1\lambda, \ldots, x_n\lambda),$$

where the x_i are distinct, then $f = f'$. Thus, if X is infinite, the kernel of τ_X is the identity congruence on T.

Proof. This fact follows from Proposition 4.2.6. If $f \neq f'$, there is some T-algebra $\mathbf{A} = (A, \alpha)$ and some n-tuple $(a_1, \ldots, a_n)$ in A with

$$f_\alpha(a_1, \ldots, a_n) \quad \neq \quad f'_\alpha(a_1, \ldots, a_n).$$

Let $h : X \to A$ be a function mapping x_i to a_i, for $i \in [n]$. Since X has at least n elements, there is such a function. Then, if $h^\sharp : \mathbf{F}_T[X] \to \mathbf{A}$ is the unique extension of h to a homomorphism,

$$\begin{aligned} f_{\tau_X}(x_1\lambda, \ldots, x_n\lambda)h^\sharp &= f_\alpha(a_1, \ldots, a_n) \\ f'_{\tau_X}(x_1\lambda, \ldots, x_n\lambda)h^\sharp &= f'_\alpha(a_1, \ldots, a_n) \end{aligned}$$

showing $f_{\tau_X} \neq f'_{\tau_X}$.

Remark 4.2.9 The fact that $\tau_X : T \to T[X](1, 0)$ is injective when X is infinite, gives another proof that every theory is isomorphic to a subtheory of A, for some set A. See Proposition 3.3.2.2.

Exercise 4.2.10 Recall that in any category, an object x is a retract of an object y if there are maps $x \to y$ and $y \to x$ such that $\mathbf{1}_x = x \to y \to x$. Suppose that the initial algebra T_0 in $T^\flat$ is nonempty. Show that for any nonempty set X, T_0 is a retract of the X-generated free algebra $\mathbf{F}_T[X]$ in $T^\flat$. Further, show that if $0 < n \leq m \leq \infty$, the n-generated free algebra in $T^\flat$ is a retract of the m-generated free algebra in $T^\flat$.

4.3 Subvarieties of $T^\flat$

Suppose that T is a fixed algebraic theory. In this section, we consider varieties of T-algebras.

Definition 4.3.1 *A* **variety of T-algebras** *is a full subcategory of $T^\flat$ closed under the operators* **S, H** *and* **P**.

We will show that any such subcategory is isomorphic to a category $R^\flat$, where R is a quotient theory of T. The argument mimics the standard proofs of Birkhoff's theorem.

Lemma 4.3.2 *Suppose that is a variety of T-algebras. Then has all free algebras, i.e. for each set X there is a T-algebra $\mathbf{F}[X]$ in and* *a function $\eta_X : X \to |\mathbf{F}[X]|$ with the following property. For any T-algebra $\mathbf{A} = (A, \alpha)$ in and any function $h : X \to A$ there is a unique homomorphism $h^\sharp : \mathbf{F}[X] \to \mathbf{A}$ such that*

$$\eta_X \cdot h^\sharp \quad = \quad h. \tag{4.5}$$

Proof. For each algebra $\mathbf{A} = (A, \alpha)$ in , the theory morphism $\alpha : T \to A$ determines a theory congruence on T. Let θ be the intersection of all of these congruences, and let the resulting quotient theory T/θ be denoted R, with the corresponding quotient map

$$\overline{\theta} : T \quad \to \quad R.$$

For any set X, consider the free R-algebra $\mathbf{F}_R[X]$ generated by $\lambda : X \to |\mathbf{F}_R[X]|$. Apply the functor $\overline{\theta}^\flat$ to this algebra, obtaining a T-algebra we denote

$$\mathbf{F}[X] \quad := \quad \mathbf{F}_R[X]\overline{\theta}^\flat.$$

Define the function η_X by

$$\eta_X \quad := \quad \lambda : X \to |\mathbf{F}_R[X]| \;=\; |\mathbf{F}[X]|.$$

Claim 1. For any algebra $\mathbf{A} = (A, \alpha)$ in and any function $h : X \to A$, there is a unique T-algebra homomorphism $h^\sharp : \mathbf{F}[X] \to \mathbf{A}$ such that (4.5) holds. Indeed, there is a unique theory morphism $\overline{\alpha} : R \to A$ such that $\overline{\theta} \cdot \overline{\alpha} = \alpha$, and a unique R-algebra homomorphism $h^\sharp : \mathcal{F}_R[X] \to (A, \overline{\alpha})$ with $\eta_X \cdot h^\sharp = h$. Applying $\overline{\theta}^\flat$ we obtain a T-algebra homomorphism $\mathbf{F}[X] \to \mathbf{A}$. Since $\overline{\theta}$ is surjective, $\overline{\theta}^\flat$ is full, by Exercise 4.1.7, showing this is the only such T-algebra homomorphism.

Claim 2. $\mathbf{F}[X]$ belongs to . Indeed, let s_1 and s_2 be any two elements in $|\mathbf{F}_R[X]|$. By Theorem 3.3.4.1, (and the fact that $\eta_X = \lambda$) we can find an n-tuple $(x_1, \ldots, x_n)$ of distinct elements in X and morphisms $t_i : 1 \to n,\ i = 1, 2$, in R such that

$$s_i \quad = \quad (t_i \kappa_R) \cdot \langle x_1\, \eta_X, \ldots, x_n\, \eta_X \rangle, \quad i = 1, 2.$$

Since $\overline{\theta}$ is surjective, there are morphisms t'_i in T with $t'_i\overline{\theta} = t_i,\ i = 1, 2$. If $s_1 \neq s_2$ then $t'_1\overline{\theta} \neq t'_2\overline{\theta}$. Hence, by definition of the congruence θ, there is an algebra $\mathbf{A} = (A, \alpha)$ in and an n-tuple $(a_1, \ldots, a_n)$ of elements in A such that $(t_1')_\alpha(a_1,\ldots,a_n) \neq (t_2')_\alpha(a_1,\ldots,a_n)$. Now let $h : X \to A$ be any function such that $x_i \mapsto a_i$, $i \in [n]$. Then by Claim 1, there is an extension of h to a T-algebra homomorphism $\mathbf{F}[X] \to \mathbf{A}$, which necessarily separates s_1 and s_2. Let I be the set of all pairs $i = (s_i, s_i')$ of distinct elements in $|\mathbf{F}_R[X]|$. For each $i \in I$ we have shown that there is an algebra $\mathbf{A}_i \in$ and a homomorphism $h_i : \mathbf{F}[X] \to \mathbf{A}_i$ which separates s_i from s_i'. Thus the target tupling of these homomorphisms gives an injective homomorphism $\mathbf{F}[X] \to \prod_{i\in I}\mathbf{A}_i$. Thus $\mathbf{F}[X]$ is isomorphic to a subalgebra of a product of algebras in , showing that $\mathbf{F}[X]$ itself is in .

As a corollary, we obtain the following version of Birkhoff's theorem.

Theorem 4.3.3 *Suppose that is a variety of T-algebras. Then there is a theory R and a surjective theory morphism*

$$\overline{\theta} : T \quad \to \quad R$$

such that a T-algebra $\mathbf{A} = (A, \alpha)$ belongs to iff α factors as

[T‘R‘A;$\overline{\theta}$‘α‘$\overline{\alpha}$] for some $\overline{\alpha} : R \to A$.

Proof. Let R and $\overline{\theta} : T \to R$ be defined from as in the proof of Lemma 4.3.2. By construction then, if (A, α) is an algebra in then $\alpha = \overline{\theta} \cdot \overline{\alpha}$, for some $\overline{\alpha} : R \to A$. Conversely, if $\alpha = \overline{\theta} \cdot \overline{\alpha}$, $(A, \overline{\alpha})$ is a quotient of some free R-algebra. Applying the functor $\overline{\theta}^\flat$, and the construction of Lemma 4.3.2, we see that the T-algebra (A, α) is a quotient of a T-algebra $\mathbf{F}[X]$, which belongs to , as shown above. Hence (A, α) itself belongs to .

Remark 4.3.4 The theory morphism $\overline{\theta} : T \to R$ induces the functor $\overline{\theta}^\flat : R^\flat \to T^\flat$ whose image is the variety of T-algebras; further, $\overline{\theta}^\flat$ is an isomorphism $R^\flat \to$ over .

4.4 The Categories and

The results of earlier sections show that one may identify a variety with a category of the form $T^\flat$, for some theory T. Let denote the category whose objects are the categories $T^\flat$. A morphism

$$G : T^\flat \quad \to \quad R^\flat$$

in is a functor which commutes with the underlying set functors, i.e. such that the diagram

[T$^\flat$'$R^\flat$'; G'U_T'U_R] commutes. In this section, we show that the opposite of the category of theories is isomorphic to the category .

Recall the map $^\flat$ from to in Proposition 4.1.6 which maps the theory morphism

$$\varphi : R \quad \to \quad T$$

to the functor

$$\varphi^\flat : T^\flat \quad \to \quad R^\flat .$$

According to Exercise 4.1.7, this map is indeed a functor $^{op} \to$. The main result of this section is that every morphism in is determined by a unique theory morphism. In other words, the functor $^\flat$ is full and faithful. It follows easily that this functor is an isomorphism.

We give some preliminary results as exercises.

Exercise 4.4.1 Suppose that $\varphi_i : T \to R,\ i = 1, 2$, are theory morphisms. Show that if $\varphi_1^\flat = \varphi_2^\flat$ then $\varphi_1 = \varphi_2$. *Hint:* Consider the finitely generated free algebras in $R^\flat$.

By definition of , the $^\flat$-functor is bijective on objects and the previous exercise shows that the $^\flat$-functor is faithful. The next theorem shows it is also full, showing it is an isomorphism.

Theorem 4.4.2 *Suppose that $G : R^\flat \to T^\flat$ is a morphism in . There is a unique theory morphism $\varphi : T \to R$ such that $G = \varphi^\flat$.*

Proof. We make use of the R-algebras $R_n = (R(1, n), \kappa_n),\ n \geq 0$. Recall from Proposition 4.2.3 that if $f : n \to m$ is any R-morphism, then $\overline{f} : R_n \to R_m$ is a homomorphism, where $g\overline{f} := g \cdot f$, for any $g : 1 \to n$ in R. Note that if $g : 1 \to n$, then in the R-algebra R_n,

$$\begin{aligned} g_{\kappa_n}(1_n, \ldots, n_n) &= g \cdot \langle 1_n, \ldots, n_n \rangle && (4.6) \\ &= g. && (4.7) \end{aligned}$$

The fact that the functor G commutes with the underlying set functors implies in particular that for each $n \geq 0$, there is a theory morphism $\alpha_n : T \to R(1, n)$ such that

$$(R(1,n), \kappa_n)G \quad = \quad (R(1,n), \alpha_n)$$

and for each R-morphism $f : n \to m$,

$$\overline{f} : (R(1,n), \alpha_n) \quad \to \quad (R(1,m), \alpha_m)$$

is a homomorphism of these T-algebras.

Now if $\varphi : T \to R$ is a theory morphism such that $G = \varphi^\flat$, then the theory morphism α_n is in fact the composite $\varphi \cdot \kappa_n$, and for $t : 1 \to n$ in T,

$$\begin{aligned} t\varphi &= t\varphi_{\kappa_n}(1_n, \ldots, n_n), \quad \text{by (4.6)}, \\ &= t_{\alpha_n}(1_n, \ldots, n_n). \end{aligned}$$

Hence, in order to prove that $G = \varphi^\flat$, we *define* $\varphi : T(1, n) \to R(1, n)$ for each $n \geq 0$ by the equation

$$t\varphi \quad := \quad t_{\alpha_n}(1_n, \ldots, n_n).$$

For $t = \langle t_1, \ldots, t_m \rangle : m \to n$ in T, define

$$t\varphi \quad := \quad \langle t_1\varphi, \ldots, t_m\varphi \rangle.$$

We now show that φ is indeed a theory morphism.

φ ***preserves distinguished morphisms.*** Indeed, since α_n is a theory morphism, $(i_n)_{\alpha_n}$ is the i-th projection function. So,

$$\begin{aligned} i_n\varphi &= (i_n)_{\alpha_n}(1_n, \ldots, n_n) \quad \text{by definition of } \varphi, \\ &= i_n. \end{aligned}$$

φ ***preserves source tupling.*** This holds by definition.

φ ***preserves composition.*** It is enough to show that

$$(t \cdot g)\varphi \quad = \quad t\varphi \cdot g\varphi$$

in the case that $1 \xrightarrow{t} n \xrightarrow{g} p$ in T. Write

$$g \quad = \quad \langle g_1, \ldots, g_n \rangle,$$

where $g_i = i_n \cdot g,\ i \in [n]$. Now, by definition of φ,

$$g\varphi \quad = \quad \langle (g_1)_{\alpha_p}(1_p, \ldots, p_p), \ldots, (g_n)_{\alpha_p}(1_p, \ldots, p_p) \rangle : n \to p.$$

Also, $\overline{g\varphi}$ is a homomorphism

$$\overline{g\varphi} : (R(1, n), \alpha_n) \quad \to \quad (R(1, p), \alpha_p).$$

So, in particular, the diagram

[1‘1‘1‘1;800‘600] [R(1,n)^{n}‘$R(1,n)$‘$R(1,p)^{n}$‘$R(1,p); t_{\alpha_n}$‘$\overline{g\varphi}^{n}$‘$\overline{g\varphi}$‘t_{α_p}]
commutes. Choose the n-tuple $\overline{x} = (1_n, \ldots, n_n)$ in $R(1, n)$. Applying the top and right-hand functions to $\overline{x}$ we obtain

$$\begin{aligned} \overline{x}(t_{\alpha_n} \cdot \overline{g\varphi}) &= t\varphi\overline{g\varphi} \\ &= t\varphi \cdot g\varphi, \end{aligned}$$

by the definition of $t\varphi$ and $\overline{g\varphi}$. Now applying the left-side and bottom functions to $\overline{x}$, we obtain

$$\begin{aligned} \overline{x}(\overline{g\varphi}^{n} \cdot t_{\alpha_p}) &= (1_n\overline{g\varphi}, \ldots, n_n\overline{g\varphi})t_{\alpha_p} \\ &= ((g_1)_{\alpha_p}(1_p, \ldots, p_p), \ldots, (g_n)_{\alpha_p}(1_p, \ldots, p_p))t_{\alpha_p} \\ &= (t \cdot g)_{\alpha_p}(1_p, \ldots, p_p), \end{aligned}$$

since α_p is a theory morphism and thus itself preserves composition. But

$$(t \cdot g)_{\alpha_p}(1_p, \ldots, p_p) \quad = \quad (t \cdot g)\varphi$$

by definition of φ, completing the proof that φ is a theory morphism $T \to R$. The uniqueness of φ follows from the fact that $^\flat$ is faithful.

4.5 Notes

Algebraic theories, and their algebras, were introduced by F.W. Lawvere [Law63] to give a uniform, signature free mathematical treatment to universal algebra. The version used here is due to Elgot [Elg75]. Infinitary versions of algebraic theories, sometimes called monads or triples, are treated in the text by Manes [Man76]. Algebraic theories were introduced into the study of automata by Eilenberg and Wright [EW67]. A nice discussion of the place of Lawvere theories in standard universal algebra can be found in Taylor [Tay79]. There is an alternative treatment of algebraic theories, called clones, or Maltsev algebras. For the first, see the book by P.M. Cohn [Coh81], and for the second, the paper by Ivo Rosenberg [Ros90]. Standard treatments of universal algebra can be found in [Grä79, MMT87, Wec91]. Matrix theories arise naturally in the study of (semi)additive categories [HS73, Fre64]. Matrix and matricial theories were studied in [Elg76a]. Theorem 4.4.2 is due to Lawvere [Law63]. The argument given here was outlined in [BW85].

Chapter 5

Iterative Theories

In this chapter, we define both ideal and iterative theories. The main definitions are in Sections 5.1 and 5.2. Section 5.3 has a more technical nature, and most details may be omitted at first reading. However, the results of Section 5.3 will be applied in later sections.

5.1 Ideal Theories

In this section we define ideal theories and give some examples of such theories.

Definition 5.1.1 *A* **scalar ideal** *I in an algebraic theory T is a collection of scalar morphisms closed under composition on the right, i.e. if $f : 1 \to p$ is in I then $f \cdot g$ is in I, for any $g : p \to q$ in T. An* **ideal** *I in T is a collection of morphisms closed under tupling, composition with arbitrary morphisms on the right, and composition with base morphisms on the left.*

Thus, if $f : n \to p$ is in I, then so are $\rho \cdot f$ and $f \cdot g$, for any base $\rho : m \to n$ and any morphism $g : p \to q$. The next proposition shows that an ideal is determined by its scalar members.

Proposition 5.1.2 *If I is a scalar ideal in T, the collection of all* *morphisms*

$$\langle f_1, \ldots, f_n \rangle : n \quad \rightarrow \quad p$$

such that each f_i is in I, is an ideal. If I is an ideal, the collection of all scalar morphisms in I is a scalar ideal. The correspondence between ideals and scalar ideals is bijective.

We give several examples of ideals.

Example 5.1.3 In any theory T, the collection of all scalar morphisms is the largest scalar ideal, and the smallest scalar ideal is empty. The largest ideal in T is T itself, and the smallest ideal consists of just the morphisms $0_p : 0 \rightarrow p$.

Example 5.1.4 If $T =$ and σ is a letter in Σ_k, for some k, then the collection of all scalar terms of the form $\sigma \cdot t$, for $t : k \rightarrow p$ in T, is a scalar ideal. This example generalizes. If T is any theory and T_0 is any fixed set of scalar morphisms in T, then the collection of all morphisms that factor through T_0, i.e. the collection of all morphisms of the form $f \cdot g$, for $f : 1 \rightarrow p$ in T_0 and $g : p \rightarrow q$ in T, is a scalar ideal.

Example 5.1.5 Suppose that T is an algebraic theory such that each set $T(n, p)$, $n, p \geq 0$, is a metric space. We suppose further that the metrics are related by the equation

$$d(f, g) \quad = \quad \max\{d(i_n \cdot f,\ i_n \cdot g) \ : \ i \in [n]\}, \tag{5.1}$$

when $f, g : n \rightarrow p$, $n > 1$. Assume that composition on the left is distance decreasing, so that

$$d(f \cdot g_1,\ f \cdot g_2) \quad \leq \quad d(g_1, g_2),$$

for each triple $f : n \rightarrow p$, $g_1, g_2 : p \rightarrow q$. Theories with these properties are called *contraction theories.* Let I be the collection of all morphisms which induce proper contractions, i.e. $f : n \rightarrow p$ is in I if there is a real number c with $0 \leq c < 1$ such that

$$d(f \cdot g_1,\ f \cdot g_2) \quad \leq \quad c \cdot d(g_1, g_2),$$

for all $g_1, g_2 : p \rightarrow q$ in T. Then I is an ideal in T, the ideal of *proper morphisms.*

Example 5.1.6 The theories and $\Sigma\,\mathbf{TR}$ are contraction theories with respect to the following (nonarchimedian) metric. For trees $t, t' : 1 \to p$, if $t = t'$ we define $d(t,t') := 0$, otherwise

$$d(t,t') \quad := \quad 2^{-\min\{|w|\,:\,wt \neq wt'\}}.$$

Finally, for $n \neq 1$ and trees $t, t' : n \to p$, we let

$$d(t,t') \quad := \quad \max\{d(i_n \cdot t,\ i_n \cdot t')\ :\ i \in [n]\}.$$

Thus a *tree* $t : n \to p$ is *proper* if and only if $i_n \cdot t$ is not distinguished, for all $i \in [n]$. Indeed, if none of the components $i_n \cdot t$ is distinguished, then

$$d(t \cdot g,\ t \cdot g') \quad \leq \quad \frac{1}{2}\, d(g, g'),$$

for all $g, g' : p \to q$.

For later use we note that, under the above metric, each set of trees $\Sigma\,\mathbf{TR}(n,p)$ is a complete metric space. Let $(t_k) : 1 \to p$ be a Cauchy sequence of scalar Σ-trees. Given a word $w \in [\omega]^*$, there is an integer k_0 such that, for all $k, m \geq k_0$,

$$d(t_k, t_m) \quad < \quad 2^{-|w|}.$$

Thus the sequence (wt_k) is either eventually constant or is eventually undefined. In either case, defining the function t by $wt := wt_{k_0}$, we obtain a partial function $t : [\omega]^* \to \Sigma \cup X_p$ which is easily seen to be a tree $1 \to p$. Also t is the limit of the sequence (t_k). Thus each set $\Sigma\,\mathbf{TR}(1,p)$ is a complete metric space, as are the sets $\Sigma\,\mathbf{TR}(n,p)$.

Example 5.1.7 Let M be a metric space with metric d. We assume that $d(x,y) \leq 1$, for all $x, y \in M$. For each $n \geq 0$, the product space M^n is a metric space with metric

$$d((a_1, \ldots, a_n),\ (b_1, \ldots, b_n)) \quad := \quad \max\{d(a_i, b_i)\ :\ i \in [n]\}.$$

Let $\mathbf{Contr}_M$ denote the subtheory of M consisting of all contractions. When $f, g : n \to p$ in $\mathbf{Contr}_M$, define

$$d'(f,g) \quad := \quad \sup\{d(xf, xg)\ :\ x \in M^p\}.$$

It follows that $\mathbf{Contr}_M$ is a contraction theory as defined in Example 5.1.5. The proper morphisms in $\mathbf{Contr}_M$ are exactly the proper contractions, i.e. the functions $f : M^p \to M^n$ such that for some $0 \leq c < 1$,

$$d(xf, yf) \quad \leq \quad c \cdot d(x,y), \quad \text{all } x, y \in M^p.$$

By Example 5.1.5, the proper contractions form an ideal in $\mathbf{Contr}_M$.

Definition 5.1.8 *A theory T is called an* **ideal theory** *if the collection of all nondistinguished scalar morphisms is a scalar ideal; i.e. if $f : 1 \to p$ is not distinguished, then $f \cdot g$ is not distinguished for any $g : p \to q$. An* **ideal theory morphism** $\varphi : T \to T'$ *between ideal theories is a theory morphism which satisfies the condition: if $f\varphi$ is distinguished in T', then f is distinguished in T. In any theory, the morphisms $f : n \to p$ such that $i_n \cdot f$ is not distinguished for each $i \in [n]$, are called* **ideal morphisms**.

The following proposition is an immediate consequence of the definitions.

Proposition 5.1.9 *A theory is an ideal theory if and only if the ideal morphisms form an ideal. A theory morphism $\varphi : T \to T'$ between ideal theories is an ideal theory morphism if and only if $f\varphi$ is ideal, for each ideal $f : n \to p$ in T.*

Suppose that T is a theory and I is an ideal (or scalar ideal) in T. Let $T(I)$ denote the smallest subtheory of T containing I. Thus, $f : n \to p$ is in $T(I)$ if and only if $i_n \cdot f$ is distinguished or is in I, for each $i \in [n]$.

Proposition 5.1.10 *If no morphism in I is distinguished, the subtheory $T(I)$ is an ideal theory whose ideal morphisms are the morphisms in I.*

Proof. A scalar morphism $f : 1 \to p$ in $T(I)$ is either a distinguished morphism i_p, which is not in I, or a morphism in I. Thus the ideal morphisms are those in I. Since I is an ideal in T, I is an ideal in $T(I)$ as well. Thus, by Proposition 5.1.9, $T(I)$ is an ideal theory.

We next give several examples of ideal theories.

Example 5.1.11 The initial theory **Tot** is an ideal theory; there is no nondistinguished scalar morphism. More generally, each theory or Σ **TR** is ideal, whose ideal morphisms are the proper trees. Indeed, if the root of t is labeled by a letter σ in Σ then so is the root of any tree $t \cdot g$, for all $g : p \to q$.

Example 5.1.12 A subtheory T of A is ideal if and only if $f \cdot g$ is not a projection $A^p \to A$ whenever $f : A^n \to A$ is not a projection, for all $f : A^n \to A$ and $g : A^p \to A^n$ in T.

Example 5.1.13 Let T be the theory of a variety K of Σ-algebras. Then T is ideal iff for every $\sigma \in \Sigma_k$, $k > 0$, either there exists an integer $i \in [k]$ such that

$$\sigma(x_1, \ldots, x_k) \ = \ x_i$$

holds in K, or for all terms $t_1, \ldots, t_k : 1 \to 1$, the equation

$$\sigma(t_1, \ldots, t_k) \ = \ x_1$$

does not hold in K.

Example 5.1.14 The theory A is not an ideal theory when A has at least two elements. Indeed, let $f : 1 \to 1$ be a nontrivial permutation of A; then f is nondistinguished, and $f \cdot f^{-1} = \mathbf{1}_1$. However, A is an ideal theory when A has at most one element. When A is a singleton set, we write just for A. A morphism $n \to p$ in may be identified with a partial function $[n] \to [p]$.

Example 5.1.15 Let T be a contraction theory and let I be the ideal of proper morphisms (defined in Example 5.1.5 above). Then $T(I)$ is an ideal theory. Thus, if $T = \mathbf{Contr}_M$, for a metric space M, then $T(I)$ is an ideal theory when I is the ideal of proper contractions. In both theories $T(I)$, the ideal morphisms are those in I unless the contraction theory is trivial or M has at most one element. We thus obtain another argument showing that and $\Sigma\,\mathbf{TR}$ are ideal theories whose ideal morphisms are the proper trees.

Exercise 5.1.16 Let S be a semiring and $R \subseteq S$ a right ideal of S, so that $0 \in R$, $R + R = R$ and $RS = R$. Then the collection I of all matrices $A : n \to p$ in $\mathbf{Mat}_S$ such that $A_{ij} \in R$, for all $i \in [n]$ and $j \in [p]$, is an ideal. Further, I is closed under sum and contains the matrices 0_{np}. Conversely, if I is an ideal in $\mathbf{Mat}_S$ which is closed under the additive structure, then $I(1,1)$ is a right ideal of S. This correspondence between right ideals of S and additively closed ideals of $\mathbf{Mat}_S$ is bijective.

Exercise 5.1.17 Let I be an ideal of the theory T. Show that I is a subtheory of T iff $\mathbf{1}_1$ is in I iff there is a morphism $f : n \to p$ in I such that some $i_n \cdot f$ is distinguished iff $I = T$.

Exercise 5.1.18 Suppose that T is a nontrivial theory. Show that T is ideal iff there is a theory morphism $T \to$ such that the only T-morphism mapped to a distinguished morphism i_p is the morphism i_p itself. (The theory was defined in Example 5.1.14.)

5.2 Iterative Theories Defined

In any theory, the *iteration equation* for a morphism $t : n \to n+p$ is the fixed point equation

$$\xi \;=\; t \cdot \langle \xi, \mathbf{1}_p \rangle$$

in the variable $\xi : n \to p$. The theory $\Sigma\,\mathbf{TR}$ and the subtheory Σtr defined below in Definition 5.2.10 are ideal theories with the following property: if $t : n \to n+p$ is ideal, the iteration equation for t has a unique solution. In this section we consider the collection of all ideal theories with this property.

Definition 5.2.1 *An* **iterative theory** T *is an ideal algebraic theory with the property that for each ideal morphism* $f : n \to n+p$ *there is a unique solution of the iteration equation for* f*. This solution is denoted* $f^\dagger$*. Thus* $f^\dagger$ *is the unique morphism* $n \to p$ *in* T *such that*

$$f^\dagger \;=\; f \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$$

A morphism $\varphi : T \to T'$ *between iterative theories is an ideal theory morphism.*

In iterative theories, we regard † as an operation on the ideal morphisms $n \to n+p$. Note that if $f : n \to n+p$ is ideal, so is $f^\dagger$, since $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$. The partial operation

$$f \;\mapsto\; f^\dagger$$

is called the *iteration* or *dagger* operation.

We prove first that only the scalar ideal morphisms require unique solutions of their fixed point equations.

Theorem 5.2.2 *An ideal theory* T *is an iterative theory if and only if the iteration equation has a unique solution for each scalar ideal morphism* $1 \to 1+p$*.*

Proof. Suppose that the iteration equation has a unique solution for each ideal scalar morphism $1 \to 1+p$. We prove by induction on n

that the same holds for ideal morphisms $n \to n+p$. The case $n=0$ is trivial and the case $n=1$ holds by assumption. To complete the induction we show that if for all ideal morphisms $m \to m+q$ and $k \to k+q$ the iteration equations have unique solutions then, for any ideal $f : m \to m+k+p$ and $g : k \to m+k+p$, there is a unique morphism $\xi : m+k \to p$ with

$$\xi \;=\; \langle f, g\rangle \cdot \langle \xi, \mathbf{1}_p\rangle. \tag{5.2}$$

Let $f^\dagger : m \to k+p$ be the unique solution of the iteration equation for f. Define the morphism h by

$$h \;:=\; g \cdot \langle f^\dagger, \mathbf{1}_{k+p}\rangle : k \to k+p.$$

We show that $\xi = \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\ h^\dagger\rangle$ satisfies (5.2), where $h^\dagger$ is the unique solution of the iteration equation for h. Indeed,

$$\begin{aligned}
\langle f, g\rangle \cdot \langle \xi, \mathbf{1}_p\rangle &= \langle f, g\rangle \cdot \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger, \mathbf{1}_p\rangle \\
&= \langle f, g\rangle \cdot \langle f^\dagger, \mathbf{1}_{k+p}\rangle \cdot \langle h^\dagger, \mathbf{1}_p\rangle \\
&= \langle f \cdot \langle f^\dagger, \mathbf{1}_{k+p}\rangle,\, g \cdot \langle f^\dagger, \mathbf{1}_{k+p}\rangle\rangle \cdot \langle h^\dagger, \mathbf{1}_p\rangle \\
&= \langle f^\dagger, h\rangle \cdot \langle h^\dagger, \mathbf{1}_p\rangle \\
&= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h \cdot \langle h^\dagger, \mathbf{1}_p\rangle\rangle \\
&= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger\rangle \\
&= \xi.
\end{aligned}$$

We now show that the solution is unique. We will make use of the following fact. Given any ideal morphism $f' : m \to m+q$ and any $g' : q \to s$ in T,

$$(f' \cdot (\mathbf{1}_m \oplus g'))^\dagger \;=\; f'^\dagger \cdot g', \tag{5.3}$$

where $(f' \cdot (\mathbf{1}_m \oplus g'))^\dagger$ and $f'^\dagger$ are the unique solutions of the iteration equations for $f' \cdot (\mathbf{1}_m \oplus g')$ and f', respectively. Indeed,

$$\begin{aligned}
f' \cdot (\mathbf{1}_m \oplus g') \cdot \langle f'^\dagger \cdot g',\, \mathbf{1}_s\rangle &= f' \cdot \langle f'^\dagger \cdot g',\, g'\rangle \\
&= f' \cdot \langle f'^\dagger, \mathbf{1}_q\rangle \cdot g' \\
&= f'^\dagger \cdot g',
\end{aligned}$$

so that (5.3) follows by the uniqueness of the solution of the iteration equation for $f' \cdot (\mathbf{1}_m \oplus g')$. Suppose now that $\xi = \langle \xi_1, \xi_2\rangle$ is a solution to (5.2), where $\xi_1 : m \to p$ and $\xi_2 : k \to p$. Thus

$$\begin{aligned} \xi_1 &= f \cdot \langle \xi_1, \xi_2, \mathbf{1}_p \rangle \\ \xi_2 &= g \cdot \langle \xi_1, \xi_2, \mathbf{1}_p \rangle. \end{aligned}$$

Since

$$\begin{aligned} \xi_1 &= f \cdot \langle \xi_1, \xi_2, \mathbf{1}_p \rangle \\ &= f \cdot (\mathbf{1}_m \oplus \langle \xi_2, \mathbf{1}_p \rangle) \cdot \langle \xi_1, \mathbf{1}_p \rangle, \end{aligned}$$

we have

$$\xi_1 = (f \cdot (\mathbf{1}_m \oplus \langle \xi_2, \mathbf{1}_p \rangle))^\dagger,$$

i.e. ξ_1 is the unique solution of the iteration equation for $f \cdot (\mathbf{1}_m \oplus \langle \xi_2, \mathbf{1}_p \rangle)$. By (5.3) we can also write ξ_1 as

$$\xi_1 = f^\dagger \cdot \langle \xi_2, \mathbf{1}_p \rangle. \tag{5.4}$$

But

$$\begin{aligned} \xi_2 &= g \cdot \langle \xi_1, \xi_2, \mathbf{1}_p \rangle \\ &= g \cdot \langle f^\dagger, \mathbf{1}_{k+p} \rangle \cdot \langle \xi_2, \mathbf{1}_p \rangle \\ &= h \cdot \langle \xi_2, \mathbf{1}_p \rangle. \end{aligned}$$

By the uniqueness of the solution of the iteration equation for h,

$$\xi_2 = h^\dagger,$$

so that

$$\xi_1 = f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle$$

from equation (5.4).

We note that ideal theory morphisms between iterative theories necessarily preserve the iteration operation.

Proposition 5.2.3 *Suppose that $\varphi : T \to T'$ is an ideal theory morphism between the iterative theories T and T'. Then, for each ideal morphism*
$f : n \to n + p$ in T,

$$f^\dagger \varphi = (f\varphi)^\dagger.$$

Proof. Since φ is an ideal theory morphism, $f\varphi : n \to n+p$ is ideal in T'. Since φ is a theory morphism and since $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$ in T,

$$f^\dagger \varphi \;=\; f\varphi \cdot \langle f^\dagger \varphi, \mathbf{1}_p \rangle \tag{5.5}$$

in T'. But the only morphism satisfying the iteration equation for $f\varphi$ is $(f\varphi)^\dagger$, since T' is an iterative theory. Thus, $f^\dagger \varphi = (f\varphi)^\dagger$.

In some iterative theories, $f^\dagger$ is obtained by a limiting process involving the powers of f.

Definition 5.2.4 *Suppose $f : n \to n+p$ is a morphism in the theory T. Then for each $k \geq 0$, we define the* **powers** *$f^k : n \to n+p$ by induction as follows:*

$$\begin{aligned} f^0 &:= \mathbf{1}_n \oplus 0_p \\ f^{k+1} &:= f \cdot \langle f^k,\, 0_n \oplus \mathbf{1}_p \rangle. \end{aligned}$$

Thus, $f^1 = f$.

Exercise 5.2.5 Show that

$$f^{k+m} \;=\; f^k \cdot \langle f^m,\, 0_n \oplus \mathbf{1}_p \rangle,$$

for all $k, m \geq 0$. In particular, $f^{k+1} = f^k \cdot \langle f,\, 0_n \oplus \mathbf{1}_p \rangle$.

We now give some examples of iterative theories.

Example 5.2.6 Let T be a nontrivial contraction theory such that each set $T(1,p)$, $p \geq 0$, is a complete metric space. Since the metrics are related by the equation (5.1), it follows that all sets $T(n,p)$, $n, p \geq 0$, are complete metric spaces. Let I be the ideal of proper morphisms. Suppose that either $I = \emptyset$, or there is at least one morphism $1 \to 0$ in T, so that none of the sets $T(n,p)$ is empty. If $f : n \to n+p$ is in I then, by the Banach fixed point theorem, the iteration equation for f has a unique solution denoted $f^\dagger$. Further,

$$f^\dagger \;=\; \lim_{k \to \infty} f^k \cdot \langle g_0, \mathbf{1}_p \rangle$$

where $g_0 : n \to p$ is any morphism and the powers f^k are defined in Definition 5.2.4. Since $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$, $f^\dagger$ is a proper morphism. Let $T(I)$ be the smallest subtheory of T containing I. By Example 5.1.15, $T(I)$ is an ideal theory whose ideal morphisms are the proper morphisms I. Thus, by the preceding argument, $T(I)$ is an iterative theory.

Example 5.2.7 Let M be a complete metric space with at least two elements. Thus, for each $n \geq 0$, the product space M^n is a complete metric space also. Let I be the ideal of proper contractions in the theory $T = \mathbf{Contr}_M$. By Example 5.1.15, $T(I)$, the subtheory of T generated by I, is an ideal theory with ideal morphisms those in I. Let $f : n \to n + p$ be a proper contraction. Given $y \in M^p$, we define the function

$$f_y : M^n \quad \to \quad M^n$$

by $xf_y := (x, y)f$, for all $x \in M^n$. It follows that f_y is a proper contraction. Thus, by the Banach fixed point theorem again, f_y has a unique fixed point that we denote $yf^\dagger$. The function $f^\dagger : M^p \to M^n$ is the unique solution of the iteration equation for f in M. Using the formula

$$yf^\dagger \quad = \quad \lim_{k \to \infty} x_0 f_y^k,$$

where f_y^k is the k-th power of f_y, $k \geq 0$, and where x_0 is any fixed element of the space M^n, it is not hard to see that $f^\dagger$ is a proper contraction also. Thus $T(I)$ is an iterative theory. (When M has at most one element, each morphism in the trivial theory M is a proper contraction, so that $T(I) = M$ is iterative.)

Example 5.2.8 Let Σ be a ranked set. We have seen, cf. Example 5.1.11, that $\Sigma\,\mathbf{TR}$ is an ideal theory. In Example 5.1.6 we imposed a metric on $\Sigma\,\mathbf{TR}$ such that $\Sigma\,\mathbf{TR}$ is a contraction theory where the proper morphisms are the ideal trees. Further, each set $\Sigma\,\mathbf{TR}(n, p)$ is a complete metric space. Thus, by Example 5.2.6, $\Sigma\,\mathbf{TR}$ is an iterative theory.

Example 5.2.9 The theory defined in Example 5.1.14 is iterative. In fact, when Σ_0 is a singleton and Σ_n is empty otherwise, the iterative theories $\Sigma\,\mathbf{TR}$ and are isomorphic.

We describe an important subtheory of the theory $\Sigma\,\mathbf{TR}$.

Definition 5.2.10 *Let Σ be a ranked set. A tree $t : 1 \to p$ in $\Sigma\,\mathbf{TR}$ is called* **regular** *if t has a finite number of subtrees. A regular tree $t : n \to p$, $n \neq 1$, is a Σ-tree such that each $i_n \cdot t$, for $i \in [n]$, is regular. We let Σtr denote the collection of regular Σ-trees.*

Clearly, all finite trees are regular, but there are also infinite regular trees. For one simple example, suppose that $\sigma \in \Sigma_1$. Then the Σ-tree

$t : 1 \to 0$ whose domain is the infinite set $\{\epsilon, 1, 11, 111, \ldots, 1^n, \ldots\}$ with $1^n t = \sigma$, $n \geq 0$, has only one subtree, since for any n, the subtree of t rooted at 1^n is t itself.

We show that regular Σ-trees form a subtheory of $\Sigma\,\mathbf{TR}$.

Proposition 5.2.11 *Let Σ be a ranked set. Then Σtr is a subtheory $\Sigma\,\mathbf{TR}$. Further, if $f : n \to n + p$ is a proper regular tree, then $f^\dagger$ is regular.*

Proof. Since each finite Σ-tree is regular, so are the distinguished trees i_n. Further, it is obvious that regular trees are closed under tupling. Suppose that $t : 1 \to p$ and $t' : p \to q$ are regular Σ-trees. Note that each subtree s of the composite $t \cdot t'$ is either a subtree of one of the trees $i_p \cdot t'$, $i \in [p]$, or $s = g \cdot t'$, where g is a subtree of t. Thus $t \cdot t'$ has a finite number of subtrees. This shows that regular trees are closed under composition.

We now show closure under iteration. Let $t : n \to n + p$ be a proper regular Σ-tree. Since

$$t^\dagger \;=\; t \cdot \langle t^\dagger, \mathbf{1}_p \rangle,$$

it follows that

$$t^\dagger \;=\; t^k \cdot \langle t^\dagger, \mathbf{1}_p \rangle,$$

for all $k \geq 0$. Thus each subtree of any $i_n \cdot t^\dagger$, $i \in [n]$, is of the form $g \cdot \langle t^\dagger, \mathbf{1}_p \rangle$, where g is a subtree of one of the components $j_n \cdot t$, for $j \in [n]$. Since t is regular, it follows that $t^\dagger$ is regular.

Corollary 5.2.12 *Σtr is an iterative theory. The ideal morphisms in Σtr are the proper regular trees.*

For later use we prove that every regular tree can be obtained from the atomic trees and the distinguished trees by a finite number of applications of the dagger operation and the theory operations of composition and tupling.

Definition 5.2.13 *Let Σ be a ranked set. We call a tree $f : n \to p$* **primitive** *if each component $i_n \cdot f$, $i \in [n]$, is of the form $\sigma \cdot \rho$, where σ is a letter in Σ (more precisely, $\sigma : 1 \to k$ is an atomic tree) and where ρ is a base morphism.*

Proposition 5.2.14 *Let Σ be a ranked set. For every regular tree $t : n \to p$, there is a primitive tree $f : s \to s+p$ and a base $\alpha : n \to s+p$ such that*

$$t \ = \ \alpha \cdot \langle f^{\dagger}, \mathbf{1}_p \rangle.$$

Further, for each proper subtree of one of the trees $i_n \cdot t$, $i \in [n]$, there is a unique integer $j \in [s]$ with $i_n \cdot t = j_s \cdot f^{\dagger}$; and each tree $j_s \cdot f^{\dagger}$ is a proper subtree of t.

Proof. Let $t_1, \ldots, t_s$ be all of the distinct proper subtrees of the trees $i_n \cdot t$, $i \in [n]$. Let g denote the tree $\langle t_1, \ldots, t_s \rangle : s \to p$. Each t_i, $i \in [s]$, can be written as

$$\sigma_i \cdot \rho_i \cdot \langle g, \mathbf{1}_p \rangle, \tag{5.6}$$

where $\sigma_i : 1 \to k_i$ is an atomic tree and $\rho_i : k_i \to s+p$ is base. We define

$$f \ := \ \langle \sigma_1 \cdot \rho_1, \ldots, \sigma_s \cdot \rho_s \rangle : s \to s+p.$$

By (5.6), g is a solution of the iteration equation for f, so that $g = f^{\dagger}$, since the solution is unique. To end the proof, define $\alpha : n \to s+p$ by $i\alpha = j$ if $i_n \cdot t = t_j$, $j \in [s]$; and $i\alpha = s+j$ if $i_n \cdot t$ is the distinguished tree j_p, $j \in [p]$.

Exercise 5.2.15 [Elg75] Let T be an iterative theory. A morphism $f : n \to n+p$ is called *power ideal*, if for some $k > 0$, f^k is an ideal morphism. Show that if f is power ideal, then there is a unique solution of the iteration equation for f. Characterize the set of morphisms $n \to n+p$ which have unique solution of their iteration equation.

5.3 Properties of Iteration in Iterative Theories

In iterative theories, iteration is an operation only on the set of ideal morphisms $f : n \to n+p$, $n, p \geq 0$, yielding a morphism $f^{\dagger} : n \to p$. We record some of the properties of this operation in the next theorem. The equations derived below will be used in the next section to prove that Σtr is the free iterative theory generated by the ranked alphabet Σ.

Theorem 5.3.1 *Let T be an iterative theory. The following equations are valid in T:*

[a] *the* **left zero identity**

$$(0_n \oplus f)^\dagger \;=\; f,$$

for all ideal $f : n \to p$; the case that $n = 1$ is called the scalar left zero identity*;*

[b] *the* **parameter identity**

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger \;=\; f^\dagger \cdot g,$$

for all ideal $f : n \to n + p$ and for all $g : p \to q$ in T. The particular case that g is a base morphism is called the base parameter identity*, or the* scalar base parameter identity *if g is base and $n = 1$;*

[c] *the* **(left) pairing identity**

$$\langle f, g\rangle^\dagger \;=\; \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\; h^\dagger\rangle,$$

for all ideal $f : n \to n + m + p$ and $g : m \to n + m + p$, where $h := g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle : m \to m + p$; the case that $m = 1$ is called the scalar (left) pairing identity.

Proof. Let $f : n \to p$ be an ideal morphism. In any theory, we have

$$f \;=\; (0_n \oplus f) \cdot \langle \xi, \mathbf{1}_p\rangle$$

for any morphism $\xi : n \to p$. This proves the left zero identity. The parameter identity and the pairing identity follow from the proof of Theorem 5.2.2.

Below we will often use the following notation.

Definition 5.3.2 *If $f = \langle f_1, \ldots, f_n\rangle : n \to m + p$ and $g_i : m \to k$, $i \in [n]$, are morphisms in a theory T, then*

$$f \parallel (g_1, \ldots, g_n)$$

denotes the morphism

$$\langle f_1 \cdot (g_1 \oplus \mathbf{1}_p), \ldots, f_n \cdot (g_n \oplus \mathbf{1}_p)\rangle : n \;\to\; k + p.$$

Note that if $g_1 = \ldots = g_n = g$, then

$$f \parallel (g_1, \ldots, g_n) \ = \ f \cdot (g \oplus \mathbf{1}_p).$$

Also,

$$(f \parallel (g_1, \ldots, g_n)) \cdot (h \oplus \mathbf{1}_p) \ = \ f \parallel (g_1 \cdot h, \ldots, g_n \cdot h),$$

for all $h : k \to q$. Further, when $\rho : q \to n$ is base,

$$\rho \cdot (f \parallel (g_1, \ldots, g_n)) \ = \ (\rho \cdot f) \parallel (g_{1\rho}, \ldots, g_{q\rho}).$$

Theorem 5.3.3 *The following equations hold in each iterative theory:*

[a] *the* **commutative identity**

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \ = \ \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \tag{5.7}$$

where $f : n \to m + p$ *is ideal,* $\rho : m \to n$ *is a surjective base morphism, and* $\rho_i : m \to m$ *are base with* $\rho_i \cdot \rho = \rho$, $i \in [m]$*;*

[b] *the* **functorial dagger implication**

$$f \cdot (h \oplus \mathbf{1}_p) \ = \ h \cdot g \quad \Rightarrow \quad f^\dagger \ = \ h \cdot g^\dagger, \tag{5.8}$$

for any ideal $f : m \to m + p$, $g : n \to n + p$, *any* $h : m \to n$.

Remark 5.3.4 The reason for the terminology "functorial dagger" is the following. Given the iterative theory T, let be the category whose objects are all *ideal* morphisms $f : m \to m + p$ in T; there is a morphism h from $f : m \to m + p$ to $g : n \to n + q$ only if $p = q$ and then h is a T-morphism $m \to n$ such that

$$f \cdot (h \oplus \mathbf{1}_p) \ = \ h \cdot g.$$

Let be the category whose objects are the ideal morphisms in T; there is a morphism from $f : m \to p$ to $g : n \to q$ only if $p = q$; and such a morphism is a T-morphism $h : m \to n$ such that

$$f \ = \ h \cdot g.$$

The functorial dagger implication states that † determines a functor $\to$.

We will prove that the functorial implication is true in iterative theories and that it implies the commutative identity. In fact we will prove this statement in the larger context of partial preiteration theories, defined below.

Definition 5.3.5 *A* **partial preiteration theory** *is a triple* $(T, I, ^\dagger)$ *such that* T *is an algebraic theory,* I *is an ideal in* T *and the iteration operation is defined for morphisms* $f : n \to n + p$ *in* I *yielding a morphism* $f^\dagger : n \to p$ *in* T*. A morphism* $(T, I, ^\dagger) \to (T', I', ^\dagger)$ *between partial preiteration theories is a theory morphism* $\varphi : T \to T'$ *which maps morphisms in* I *to morphisms in* I' *and which preserves the iteration operation.*

Lemma 5.3.6 *Let* $(T, I, ^\dagger)$ *be a partial preiteration theory. If the iteration operation satisfies the functorial dagger implication, i.e. if the functorial dagger implication (5.8) holds for all morphisms* f, g *in* I *and* h *in* T*, then the commutative identity (5.7) holds.*

Proof. Since $\rho_i \cdot \rho = \rho$ for each $i \in [m]$, we have

$$\begin{aligned} ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m)) \cdot (\rho \oplus \mathbf{1}_p) &= (\rho \cdot f) \parallel (\rho_1 \cdot \rho, \ldots, \rho_m \cdot \rho) \\ &= (\rho \cdot f) \parallel (\rho, \ldots, \rho) \\ &= \rho \cdot f \cdot (\rho \oplus \mathbf{1}_p). \end{aligned}$$

Thus, by the functorial dagger implication (5.8),

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger = \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger.$$

Remark 5.3.7 It is important to observe that only a weak form of the functorial dagger implication was used to prove the commutative identity, namely the case that $h = \rho$ is a surjective base morphism.

Now we show that the functorial dagger implication always holds in iterative theories.

Proof of Theorem 5.3.3.a. We claim that $h \cdot g^\dagger$ is a solution of the iteration equation for f, when $f \cdot (h \oplus \mathbf{1}_p) = h \cdot g$. Indeed,

$$\begin{aligned} f \cdot \langle h \cdot g^\dagger, \mathbf{1}_p \rangle &= f \cdot (h \oplus \mathbf{1}_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= h \cdot g \cdot \langle g^\dagger, \mathbf{1}_p \rangle, \end{aligned}$$

by assumption,

$$= h \cdot g^\dagger.$$

Since f is ideal, $h \cdot g^\dagger = f^\dagger$.

The previous results motivate the following definition.

Definition 5.3.8 *A* **partial iteration theory** *is a partial preiteration theory* $(T, I, ^\dagger)$ *which satisfies the left zero identity, the base parameter identity, the pairing identity and the commutative identity for morphisms in* I. *A morphism of partial iteration theories is a partial preiteration theory morphism.*

Let T be an iterative theory and I the collection of all ideal morphisms in T. By Theorems 5.3.1 and 5.3.3, the triple $(T, I, ^\dagger)$ is a partial iteration theory, where for any ideal $f : n \to n+p$, $f^\dagger : n \to p$ is the unique solution of the iteration equation for f. Below we will prove that, in any partial iteration theory $(T, I, ^\dagger)$, the equation

$$f^\dagger \;=\; f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$$

holds for all $f : n \to n+p$ in I, so that the morphism $f^\dagger$ is in I also. Thus each iterative theory gives rise to a unique partial iteration theory $(T, I, ^\dagger)$, where I is the collection of ideal morphisms. Also, by Proposition 5.2.3, a theory morphism $\varphi : T \to T'$ between iterative theories is an iterative theory morphism if and only if φ is a partial iteration theory morphism between the corresponding triples. If we want to treat an iterative theory T as a partial iteration theory $(T, I, ^\dagger)$, we always take I to be the collection of ideal morphisms and assume that the iteration operation provides unique solutions of the iteration equations for ideal morphisms $f : n \to n+p$.

Theorem 5.3.9 *The following equations are valid in any partial iteration theory* $(T, I, ^\dagger)$.

[a] *The* **permutation identity**

$$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger \;=\; \pi \cdot f^\dagger,$$

for all $f : n \to n+p$ *in* I *and all base permutations* $\pi : n \to n$. *In the case that* π *is a block transposition* $n \to n$, *i.e.* $\pi = \langle 0_k \oplus \mathbf{1}_m,\, \mathbf{1}_k \oplus 0_m \rangle$, *where* $m + k = n$, *the permutation identity is called the* **block transposition identity**; *when* $n = 2$ *and* π *is the nontrivial base permutation* $2 \to 2$, *this identity is called the* **transposition identity**.

[b] *The* **right zero identity**

$$(f \oplus 0_q)^\dagger \;=\; f^\dagger \oplus 0_q,$$

for all $f : n \to n + p$ *in* I. *The particular case that* $n = q = 1$ *will be referred to as the* scalar right zero identity.

[c] *The* **right pairing identity**

$$\langle f, g\rangle^\dagger \;=\; \langle k^\dagger,\ (g \cdot \rho)^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle\rangle,$$

for all $f : n \to n + m + p$ *and* $g : m \to m + n + p$ *in* I, *where*

$$\begin{aligned} \rho &:= \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n\rangle \oplus \mathbf{1}_p \qquad \textit{and} \\ k &:= f \cdot \langle \mathbf{1}_n \oplus 0_p, (g \cdot \rho)^\dagger, 0_n \oplus \mathbf{1}_p\rangle. \end{aligned}$$

When $m = 1$ *the right paring identity is called the* scalar right pairing identity.

[d] *The* **symmetric pairing identity**

$$\langle f, g\rangle^\dagger \;=\; \langle k^\dagger, h^\dagger\rangle,$$

for all $f : n \to n + m + p$ *and* $g : m \to n + m + p$ *in* I, *where* h *and* k *are defined as in the pairing identity and in the right pairing identity, respectively. When* $m = 1$ *this equation is called the* scalar symmetric pairing identity.

[e] *The* **fixed point identity**

$$f^\dagger \;=\; f \cdot \langle f^\dagger, \mathbf{1}_p\rangle,$$

for all $f : n \to n + p$ *in* I. *When* $n = 1$ *this equation is called the* scalar fixed point identity.

[f]

$$\alpha \cdot (f \cdot (\langle \mathbf{1}_n, \alpha\rangle \oplus \mathbf{1}_p))^\dagger \;=\; (\alpha \cdot f^\dagger)^\dagger,$$

for all $f : n \to n + m + p$ *in* I, *all base* $\alpha : m \to n$; *the particular case that* $n = m$ *and* $\alpha = \mathbf{1}_n$ *is called the* **double dagger identity**, *or the* scalar double dagger identity, *if* $n = m = 1$ *and* $\alpha = \mathbf{1}_1$.

[g]
$$\langle f,\ 0_n \oplus g\rangle^\dagger \ = \ \langle f^\dagger \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\ g^\dagger\rangle,$$

for all $f : n \to n + m + p$ *and* $g : m \to m + p$ *in* I.

[h]
$$\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p),\ g\rangle^\dagger \ = \ \langle f^\dagger,\ (g \cdot \rho)^\dagger \cdot \langle f^\dagger, \mathbf{1}_p\rangle\rangle,$$

for all $f : n \to n + p$ *and* $g : m \to n + m + p$ *in* I, *where* $\rho = \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n\rangle \oplus \mathbf{1}_p$.

[i]
$$\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p),\ 0_n \oplus g\rangle^\dagger \ = \ \langle f^\dagger, g^\dagger\rangle,$$

for all $f : n \to n + p$ *and* $g : m \to m + p$ *in* I.

[j] *The* **parameter identity**

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger \ = \ f^\dagger \cdot g$$

of Theorem 5.3.1.b, for all $f : n \to n + p$ *and* $g : p \to q$ *in* I.

[k] *The* **composition identity**

$$f \cdot \langle (g \cdot \langle f,\ 0_m \oplus \mathbf{1}_p\rangle)^\dagger,\ \mathbf{1}_p\rangle \ = \ (f \cdot \langle g,\ 0_n \oplus \mathbf{1}_p\rangle)^\dagger,$$

for all $f : n \to m + p$ *and* $g : m \to n + p$ *in* I. *When* $n = m = 1$ *this is called the* scalar composition identity.

[l] *The* **simplified composition identity**

$$(f \cdot g)^\dagger \ = \ f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger,$$

for all $f : n \to m$ *and* $g : m \to n + p$.

We will refer to the left zero identity and the right zero identity as the *zero identities*, or the *scalar zero identities.* Note that, unlike the case of iterative theories, in part [j] above the morphism g is in the ideal I. Theorem 5.3.9 will be proved in a number of propositions below. We suppose that a partial preiteration theory $(T, I,^\dagger)$ is given.

Proposition 5.3.10 *If the commutative identity holds in* $(T, I,^\dagger)$, *then the permutation identity is valid in* $(T, I,^\dagger)$ *also.*

Proof. When $m = n$ the commutative identity of Theorem 5.3.3.a simplifies to the permutation identity. Indeed, ρ must be a base permutation and we must have $\rho_1 = \ldots = \rho_n = \mathbf{1}_n$. Thus the commutative identity takes the particular form

$$(\rho \cdot f)^\dagger \;=\; \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger.$$

The result follows by substituting $f \cdot (\rho^{-1} \oplus \mathbf{1}_p)$ for f.

Proposition 5.3.11 *If the base parameter identity holds in $(T, I, ^\dagger\,)$, then so does the right zero identity.*

Proof. For $f : n \to n + p \in I$ we have

$$(f \oplus 0_q)^\dagger \;=\; (f \cdot (\mathbf{1}_n \oplus \mathbf{1}_p \oplus 0_q))^\dagger \;=\; f^\dagger \cdot (\mathbf{1}_p \oplus 0_q) \;=\; f^\dagger \oplus 0_q.$$

Proposition 5.3.12 *Suppose that the block transposition identity holds. Then the pairing identity is valid in $(T, I, ^\dagger\,)$ if and only if the right pairing identity is valid.*

Proof. We only prove that if the pairing identity and the block transposition identity hold, then the right pairing identity is valid in $(T, I, ^\dagger\,)$. Let $f : n \to n + m + p$ and $g : m \to n + m + p$ be in I, and let τ denote the block transposition $\langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n \rangle : n + m \to m + n$. We wish to prove that

$$\langle f, g\rangle^\dagger \;=\; \langle k^\dagger,\, (g \cdot \rho)^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle\rangle,$$

where $\rho = \tau \oplus \mathbf{1}_p$ and $k = f \cdot \langle \mathbf{1}_n \oplus 0_p,\, (g \cdot \rho)^\dagger,\, 0_n \oplus \mathbf{1}_p\rangle$. We have

$$\begin{aligned} \langle f, g\rangle^\dagger &= (\tau \cdot \tau^{-1} \cdot \langle f, g\rangle \cdot \rho \cdot \rho^{-1})^\dagger \\ &= \tau \cdot (\tau^{-1} \cdot \langle f, g\rangle \cdot \rho)^\dagger, \end{aligned}$$

by the permutation identity applied to the block transposition τ,

$$\begin{aligned} &= \tau \cdot \langle g \cdot \rho,\, f \cdot \rho\rangle^\dagger \\ &= \tau \cdot \langle (g\rho)^\dagger \cdot \langle k'^\dagger, \mathbf{1}_p\rangle,\, k'^\dagger\rangle \\ &= \langle k'^\dagger,\, (g\rho)^\dagger \cdot \langle k'^\dagger, \mathbf{1}_p\rangle\rangle, \end{aligned}$$

by the pairing identity, where

$$\begin{aligned} k' &= f \cdot \rho \cdot \langle (g \cdot \rho)^\dagger, \mathbf{1}_{n+p}\rangle \\ &= f \cdot \langle \mathbf{1}_n \oplus 0_p,\, (g \cdot \rho)^\dagger,\, 0_n \oplus \mathbf{1}_p\rangle \\ &= k. \end{aligned}$$

The proof is complete.

Corollary 5.3.13 *Suppose that either the pairing identity and the block transposition identity hold in $(T, I,^\dagger)$, or that the pairing identity and the right pairing identity hold in $(T, I,^\dagger)$. Then the symmetric pairing identity is valid in $(T, I,^\dagger)$.*

Remark 5.3.14 If the symmetric pairing identity holds in $(T, I,^\dagger)$ then so does the block transposition identity. Thus if both the pairing identity and the right pairing identity are valid in $(T, I,^\dagger)$, then the block transposition identity holds.

We now prove that the fixed point identity of Theorem 5.3.9.e and the identity given in Theorem 5.3.9.f hold in partial iteration theories.

Proposition 5.3.15 *If the left zero identity, the pairing identity and the commutative identity hold in $(T, I,^\dagger)$, then so does the fixed point identity and the following equation:*

$$\alpha \cdot (f \cdot (\langle \mathbf{1}_n, \alpha \rangle \oplus \mathbf{1}_p))^\dagger \;=\; (\alpha \cdot f^\dagger)^\dagger,$$

for all $f : n \to n + m + p$ *in* I, *and for all base* $\alpha : m \to n$.

Proof. Let $f : n \to n + p$ in I. Define $\overline{f} := 0_n \oplus f : n \to 2n + p$ and $g := \langle \overline{f}, \overline{f} \rangle : 2n \to 2n + p$. The commutative identity gives

$$(\mathbf{1}_n \oplus 0_n) \cdot g^\dagger \quad = \quad f^\dagger.$$

Indeed, if $\rho = \langle \mathbf{1}_n, \mathbf{1}_n \rangle$, then

$$g \quad = \quad (\rho \cdot \overline{f}) \parallel (\mathbf{1}_{2n}, \ldots, \mathbf{1}_{2n}),$$

so that

$$g^\dagger \;=\; \rho \cdot (\overline{f} \cdot (\rho \oplus \mathbf{1}_p))^\dagger \;=\; \rho \cdot f^\dagger \;=\; \langle f^\dagger, f^\dagger \rangle.$$

On the other hand, we have

$$\begin{aligned} (\mathbf{1}_n \oplus 0_n) \cdot g^\dagger &= \overline{f}^\dagger \cdot \langle (\overline{f} \cdot \langle \overline{f}^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger, \mathbf{1}_p \rangle \\ &= (0_n \oplus f)^\dagger \cdot \langle ((0_n \oplus f) \cdot \langle \overline{f}^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot \langle f^\dagger, \mathbf{1}_p \rangle, \end{aligned}$$

by the pairing identity and the left zero identity. This proves the fixed point identity. It follows that if $(T, I,^\dagger)$ is a partial iteration theory and $f : n \to n + p$ is a morphism in I, then $f^\dagger$ belongs to I.

Now let $f : n \to n+m+p$ in I and let $\alpha : m \to n$ be a base morphism. From the pairing identity and the fixed point identity,

$$\begin{aligned}(0_n \oplus \mathbf{1}_m) \cdot \langle f,\, \alpha \cdot f \rangle^\dagger &= (\alpha \cdot f \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle)^\dagger \\ &= (\alpha \cdot f^\dagger)^\dagger.\end{aligned}$$

The commutative identity yields

$$\begin{aligned}(0_n \oplus \mathbf{1}_m) \cdot \langle f,\, \alpha \cdot f \rangle^\dagger &= (0_n \oplus \mathbf{1}_m) \cdot (\langle \mathbf{1}_n, \alpha \rangle \cdot f)^\dagger \\ &= (0_n \oplus \mathbf{1}_m) \cdot \langle \mathbf{1}_n, \alpha \rangle \cdot (f \cdot (\langle \mathbf{1}_n, \alpha \rangle \oplus \mathbf{1}_p))^\dagger \\ &= \alpha \cdot (f \cdot (\langle \mathbf{1}_n, \alpha \rangle \oplus \mathbf{1}_p))^\dagger,\end{aligned}$$

proving the result.

Remark 5.3.16 Let $(T, I, ^\dagger)$ be a partial preiteration theory. If I is also a subtheory of T, so that $I = T$, the fixed point identity follows from the left zero identity, the pairing identity and the block transposition identity instead of the more complicated commutative identity.

Indeed, since the right pairing identity holds in $(T, I, ^\dagger)$ by Proposition 5.3.12, we have

$$\begin{aligned}&(\mathbf{1}_n \oplus 0_n) \cdot \langle 0_n \oplus f,\, \mathbf{1}_n \oplus 0_{n+p} \rangle^\dagger = \\ &\quad = ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, (0_n \oplus \mathbf{1}_n \oplus 0_p)^\dagger, 0_n \oplus \mathbf{1}_p \rangle)^\dagger \\ &\quad = ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p \rangle)^\dagger \\ &\quad = f^\dagger.\end{aligned}$$

Similarly, by the pairing identity and the left zero identity,

$$\begin{aligned}&(\mathbf{1}_n \oplus 0_n) \cdot \langle 0_n \oplus f,\, \mathbf{1}_n \oplus 0_{n+p} \rangle^\dagger = \\ &\quad = (0_n \oplus f)^\dagger \cdot \langle ((\mathbf{1}_n \oplus 0_{n+p}) \cdot \langle (0_n \oplus f)^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger, \mathbf{1}_p \rangle \\ &\quad = f \cdot \langle ((\mathbf{1}_n \oplus 0_{n+p}) \cdot \langle f,\, \mathbf{1}_{n+p} \rangle)^\dagger, \mathbf{1}_p \rangle \\ &\quad = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle.\end{aligned}$$

A similar fact is true for the second part of the proof of Proposition 5.3.15.

Remark 5.3.17 If $I = T$, and if the left zero identity, the pairing identity and the block transposition identity hold in $(T, I, ^\dagger)$, then the equation in Theorem 5.3.9.f holds as well, for all $\alpha : m \to n$ in T.

Indeed,

$$\begin{aligned}(0_n \oplus \mathbf{1}_m) \cdot \langle f,\, \alpha \oplus 0_{m+p} \rangle^\dagger &= ((\alpha \oplus 0_{m+p}) \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle)^\dagger \\ &= (\alpha \cdot f^\dagger)^\dagger,\end{aligned}$$

by the pairing identity. Further,

$$\begin{aligned}
&(0_n \oplus \mathbf{1}_m) \cdot \langle f,\, \alpha \oplus 0_{m+p}\rangle^\dagger \;= \\
&\quad = \;(0_m \oplus \alpha \oplus 0_p)^\dagger \cdot \langle (f \cdot \langle \mathbf{1}_n \oplus 0_p, (0_m \oplus \alpha \oplus 0_p)^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\
&\quad = \;(\alpha \oplus 0_p) \cdot \langle (f \cdot \langle \mathbf{1}_n \oplus 0_p, \alpha \oplus 0_p, 0_n \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\
&\quad = \;\alpha \cdot (f \cdot (\langle \mathbf{1}_n, \alpha\rangle \oplus \mathbf{1}_p))^\dagger,
\end{aligned}$$

by the right pairing and left zero identities.

We now prove Theorem 5.3.9.g and i.

Proposition 5.3.18 *If the pairing identity holds in* $(T, I, ^\dagger)$*, then*

$$\langle f,\, 0_n \oplus g\rangle^\dagger \;=\; \langle f^\dagger \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g^\dagger\rangle,$$

for all $f : n \to n+m+p$ *and* $g : m \to m+p$ *in* I*. If in addition the block transposition identity is valid in* $(T, I, ^\dagger)$*, then also*

$$\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p),\, 0_n \oplus g\rangle^\dagger \;=\; \langle f^\dagger, g^\dagger\rangle,$$

for all $f : n \to n+p$ *and* $g : m \to m+p$ *in* I*.*

Proof. Let $f : n \to n+m+p$ and $g : m \to m+p$ be in I. Since

$$(0_n \oplus g) \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \;=\; g,$$

the pairing identity immediately gives

$$\langle f, g\rangle^\dagger \;=\; \langle f^\dagger \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g^\dagger\rangle.$$

To prove the second part of the proposition, let $f : n \to n+p$ and $g : m \to m+p$ in I. We will apply the symmetric pairing identity, which holds by Corollary 5.3.13.

$$\begin{aligned}
&\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p),\, 0_n \oplus g\rangle^\dagger \;= \\
&\quad = \;\langle (f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_n \oplus 0_p, (g \cdot (\mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p))^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger, \\
&\qquad\qquad ((0_n \oplus g) \cdot \langle (f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p))^\dagger, \mathbf{1}_{m+p}\rangle)^\dagger\rangle \\
&\quad = \;\langle f^\dagger, g^\dagger\rangle.
\end{aligned}$$

Remark 5.3.19 We note that if the base parameter identity holds in $(T, I, ^\dagger)$, then the second part of Proposition 5.3.18 follows more easily from the pairing identity. Indeed,

$$\begin{aligned}
\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p),\, 0_n \oplus g\rangle^\dagger &= \langle (f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p))^\dagger \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g^\dagger\rangle \\
&= \langle (0_m \oplus f^\dagger) \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g^\dagger\rangle \\
&= \langle f^\dagger, g^\dagger\rangle,
\end{aligned}$$

by the first part of the proof and the base parameter identity.

Remark 5.3.20 A similar calculation proves that if the right pairing identity holds in $(T, I,^\dagger)$, then so does the equation in Theorem 5.3.9.h. Thus, if both the pairing identity and the block transposition identity hold in $(T, I,^\dagger)$, then so does the equation in Theorem 5.3.9.h.

Finally we prove that the parameter identity (in the form of Theorem 5.3.9.j) and the composition identity hold in any partial iteration theory.

Proposition 5.3.21 *If the two zero identities, the pairing identity and the block transposition identity hold in $(T, I,^\dagger)$, then the parameter identity is valid in $(T, I,^\dagger)$, i.e.*

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger \;=\; f^\dagger \cdot g,$$

for all $f : n \to n + p$ *and* $g : p \to q$ *in* I.

Proof. Indeed,

$$\begin{aligned}(\mathbf{1}_n \oplus 0_p) \cdot \langle f \oplus 0_q,\, 0_{n+p} \oplus g\rangle^\dagger \;&=\; (f \oplus 0_q)^\dagger \cdot \langle (0_p \oplus g)^\dagger, \mathbf{1}_q\rangle \\ &=\; f^\dagger \cdot g,\end{aligned}$$

by Proposition 5.3.18 and the zero identities. Further,

$$\begin{aligned}&(\mathbf{1}_n \oplus 0_p) \cdot \langle f \oplus 0_q,\, 0_{n+p} \oplus g\rangle^\dagger \;= \\ &\quad=\; ((f \oplus 0_q) \cdot \langle \mathbf{1}_n \oplus 0_q, (0_{p+n} \oplus g)^\dagger, 0_n \oplus \mathbf{1}_q\rangle)^\dagger \\ &\quad=\; (f \cdot \langle \mathbf{1}_n \oplus 0_q,\, 0_n \oplus g)^\dagger \\ &\quad=\; (f \cdot (\mathbf{1}_n \oplus g))^\dagger,\end{aligned}$$

by the right pairing identity and the left zero identity.

Exercise 5.3.22 Let $(T, I,^\dagger)$ be a partial preiteration theory. Let $T(I)$ denote the smallest subtheory of T containing I. Show that if the parameter identity

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger \;=\; f^\dagger \cdot g$$

holds in T when f is in I and g is in I or is a base morphism, then it holds for all f in I and g in $T(I)$.

Proposition 5.3.23 *If the left zero identity, the pairing identity and the block transposition identity hold in $(T, I,^\dagger)$, then the composition identity and the simplified composition identity are both valid in $(T, I,^\dagger)$, i.e.*

$$f \cdot \langle (g \cdot \langle f,\, 0_m \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \;=\; (f \cdot \langle g,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger,$$

for all $f : n \to m + p$ *and* $g : m \to n + p$ *in* I*; and*

$$(f \cdot g)^\dagger \;=\; f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger,$$

for all $f : n \to m$ *and* $g : m \to n + p$ *in* I*.*

Proof. First we prove the composition identity.

$$\begin{aligned}(\mathbf{1}_n \oplus 0_m) \cdot \langle 0_n \oplus f,\ g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p)\rangle^\dagger \;&= \\ =\;& (0_n \oplus f)^\dagger \cdot \langle (g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p) \cdot \langle (0_n \oplus f)^\dagger, \mathbf{1}_{m+p}\rangle)^\dagger,\ \mathbf{1}_p\rangle \\ =\;& f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle,\end{aligned}$$

by the pairing and left zero identities. Similarly,

$$\begin{aligned}(\mathbf{1}_n \oplus 0_m) \cdot \langle 0_n \oplus f,\ g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p)\rangle^\dagger \;&= \\ =\;& ((0_n \oplus f) \cdot \langle \mathbf{1}_n \oplus 0_p, (0_m \oplus g)^\dagger, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\ =\;& (f \cdot \langle g,\ 0_n \oplus \mathbf{1}_p\rangle)^\dagger,\end{aligned}$$

by the right pairing identity and the left zero identity. The simplified composition identity easily follows from the composition identity.

In the sequel, we will make use of the following special case of the functorial dagger implication.

Proposition 5.3.24 *Let* $(T, I, ^\dagger)$ *be a partial preiteration theory satisfying the pairing and permutation identities. Suppose that*

$$f \cdot (\alpha \oplus \mathbf{1}_p) \;=\; \alpha \cdot g, \tag{5.9}$$

where $f : n \to n + p$ *and* $g : m \to m + p$ *are in* I *and* $\alpha : n \to m$ *is an injective base morphism. Then*

$$f^\dagger \;=\; \alpha \cdot g^\dagger.$$

Proof. First suppose that α is the inclusion of $[n]$ into $[m]$, i.e. $\alpha = \mathbf{1}_n \oplus 0_{m-n}$. Then the equation (5.9) can be reformulated as

$$g \;=\; \langle f \cdot (\mathbf{1}_n \oplus 0_{m-n} \oplus \mathbf{1}_p),\ h\rangle,$$

where $h = (0_n \oplus \mathbf{1}_{m-n}) \cdot g$. Thus

$$\begin{aligned}\alpha \cdot g^\dagger \;&=\; (\mathbf{1}_n \oplus 0_{m-n}) \cdot \langle f \cdot (\mathbf{1}_n \oplus 0_{m-n} \oplus \mathbf{1}_p),\ h\rangle^\dagger \\ &=\; f^\dagger,\end{aligned}$$

by Remark 5.3.20. The general case is reducible to the previous one by the permutation identity. Let $\pi : [m] \to [m]$ be a permutation which maps the integers in the range of α onto $[n]$. Since

$$\begin{aligned} f \cdot (\alpha \cdot \pi \oplus \mathbf{1}_p) &= f \cdot (\alpha \oplus \mathbf{1}_p) \cdot (\pi \oplus \mathbf{1}_p) \\ &= \alpha \cdot g \cdot (\pi \oplus \mathbf{1}_p) \\ &= \alpha \cdot \pi \cdot \pi^{-1} \cdot g \cdot (\pi \oplus \mathbf{1}_p), \end{aligned}$$

we have

$$\begin{aligned} f^\dagger &= \alpha \cdot \pi \cdot (\pi^{-1} \cdot g \cdot (\pi \oplus \mathbf{1}_p))^\dagger \\ &= \alpha \cdot \pi \cdot \pi^{-1} \cdot g^\dagger \\ &= \alpha \cdot g^\dagger, \end{aligned}$$

by the preceding case and the permutation identity.

Exercise 5.3.25 Use the uniqueness of the solutions of the iteration equations to prove that the identities appearing in Theorem 5.3.9 hold in iterative theories.

Next we derive an equivalent form of the commutative identity.

Proposition 5.3.26 *Let $(T, I, ^\dagger)$ be a partial preiteration theory. The following three conditions are equivalent.*

[a] $(T, I, ^\dagger)$ *satisfies the commutative identity*

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger = \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \tag{5.10}$$

for all $f : n \to m + p$ in I, surjective base morphisms $\rho : m \to n$, and for all base morphisms $\rho_i : m \to m$ with $\rho_i \cdot \rho = \rho$, $i \in [m]$.

[b] $(T, I, ^\dagger)$ *satisfies the commutative identity (5.10) above either when each ρ_i is a base permutation or when each ρ_i is an "aperiodic" base morphism.*

[c] $(T, I, ^\dagger)$ *satisfies the* **generalized commutative identity**

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger = \rho \cdot (f \parallel (\tau_1, \ldots, \tau_n))^\dagger, \tag{5.11}$$

for all $f : n \to k + p$ in I, surjective base morphisms $\rho : m \to n$, and for all base morphisms $\rho_i : k \to m$, $\tau_j : k \to n$, $i \in [m]$, $j \in [n]$, such that

$$\rho_i \cdot \rho = \tau_{i\rho},$$

for all $i \in [m]$.

Proof. The commutative identity is derivable from the generalized commutative identity by taking $k = m$ and $\tau_j = \rho$, $j \in [n]$. Suppose that $(T, I,^\dagger)$ satisfies the commutative identity (5.10). The assumptions made in the generalized commutative identity (5.11) yield the equation

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m)) \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot f \parallel (\tau_1, \ldots, \tau_n). \tag{5.12}$$

Let

$$\delta \;:=\; \langle \rho_1, \ldots, \rho_m, \mathbf{1}_m \rangle,$$

so that δ is a surjective base morphism $mk + m \to m$. For each $i \in [m]$ define $\beta_i : k \to mk + m$ as the base inclusion of $[k]$ into the i-th block of $[k]$'s in $[mk + m]$, i.e.

$$\beta_i \;:=\; 0_{(i-1)k} \oplus \mathbf{1}_k \oplus 0_{(m-i)k} \oplus 0_m : k \to mk + m.$$

Thus $\beta_i \cdot \delta = \rho_i$. Further, let

$$g \;:=\; (\rho \cdot f) \parallel (\beta_1, \ldots, \beta_m) : m \to mk + m + p.$$

Thus

$$\begin{aligned} g \cdot (\delta \oplus \mathbf{1}_p) \;&=\; (\rho \cdot f) \parallel (\beta_1 \cdot \delta, \ldots, \beta_m \cdot \delta) \\ &=\; (\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m). \end{aligned} \tag{5.13}$$

Thus, by the commutative identity,

$$\begin{aligned} (\delta \cdot g)^\dagger \;&=\; ((\delta \cdot g) \parallel (\mathbf{1}_{mk+m}, \ldots, \mathbf{1}_{mk+m}))^\dagger \\ &=\; \delta \cdot (g \cdot (\delta \oplus \mathbf{1}_p))^\dagger \\ &=\; \delta \cdot ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger. \end{aligned} \tag{5.14}$$

Since ρ is surjective, there is some base morphism $\alpha : n \to m$ with $\alpha \cdot \rho = \mathbf{1}_n$. From (5.12) and (5.13) we immediately obtain

$$\alpha \cdot g \cdot (\delta \cdot \rho \oplus \mathbf{1}_p) \;=\; f \parallel (\tau_1, \ldots, \tau_n).$$

Thus, again by the commutative identity, for any collection of base morphisms

$$\kappa_j : mk + m \;\to\; mk + m, \quad j \in [mk + m],$$

if $\kappa_j \cdot \delta \cdot \rho = \delta \cdot \rho$ then

$$\begin{aligned} ((\delta \cdot \rho \cdot \alpha \cdot g) \parallel (\kappa_1, \ldots, \kappa_{mk+m}))^\dagger \;&=\; \delta \cdot \rho \cdot (\alpha \cdot g \cdot (\delta \cdot \rho \oplus \mathbf{1}_p))^\dagger \\ &=\; \delta \cdot \rho \cdot (f \parallel (\tau_1, \ldots, \tau_n))^\dagger. \end{aligned} \tag{5.15}$$

Comparing (5.14) and (5.15), we see that the the generalized commutative identity is established once we can find the κ_j's so that

$$j_{mk+m} \cdot \delta \cdot g \;=\; j_{mk+m} \cdot \delta \cdot \rho \cdot \alpha \cdot g \cdot (\kappa_j \oplus \mathbf{1}_p).$$

But for each $j \in [mk+m]$,

$$\begin{aligned} j_{mk+m} \cdot \delta \cdot g &= (j\delta)_m \cdot g \\ &= (j\delta)_m \cdot \rho \cdot f \cdot (\beta_{j\delta} \oplus \mathbf{1}_p) \\ &= (j\delta\rho)_n \cdot f \cdot (\beta_{j\delta} \oplus \mathbf{1}_p). \end{aligned}$$

Further,

$$\begin{aligned} j_{mk+m} \cdot \delta \cdot \rho \cdot \alpha \cdot g \cdot (\kappa_j \oplus \mathbf{1}_p) &= (j\delta\rho\alpha)_m \cdot g \cdot (\kappa_j \oplus \mathbf{1}_p) \\ &= (j\delta\rho\alpha\rho)_n \cdot f \cdot (\beta_{j\delta\rho\alpha} \oplus \mathbf{1}_p) \cdot (\kappa_j \oplus \mathbf{1}_p) \\ &= (j\delta\rho)_n \cdot f \cdot (\beta_{j\delta\rho\alpha} \cdot \kappa_j \oplus \mathbf{1}_p), \end{aligned}$$

and

$$\begin{aligned} \beta_{j\delta} &= 0_{(j\delta-1)k} \oplus \mathbf{1}_k \oplus 0_{(m-j\delta)k} \oplus 0_m \\ \beta_{j\delta\rho\alpha} &= 0_{(j\delta\rho\alpha-1)k} \oplus \mathbf{1}_k \oplus 0_{(m-j\delta\rho\alpha)k} \oplus 0_m. \end{aligned}$$

Thus let $\kappa_j : [mk+m] \to [mk+m]$ be the function with

$$\begin{aligned} (j\delta\rho\alpha - 1)k + i &\mapsto (j\delta - 1)k + i \\ (j\delta - 1)k + i &\mapsto (j\delta\rho\alpha - 1)k + i, \end{aligned}$$

$i = 1, \ldots, k$. For integers i' not of the form $(j\delta\rho\alpha - 1)k + i$ or $(j\delta - 1)k + i$, define $i'\kappa_j = i'$. It is then clear that

$$(\beta_{j\delta\rho\alpha} \oplus \mathbf{1}_p) \cdot (\kappa_j \oplus \mathbf{1}_p) \;=\; \beta_{j\delta} \oplus \mathbf{1}_p.$$

Since

$$(j\delta\rho\alpha - 1)k + i \overset{\delta}{\mapsto} i\rho_{j\delta\rho\alpha} \overset{\rho}{\mapsto} i\rho_{j\delta\rho\alpha}\rho \;=\; i\tau_{j\delta\rho\alpha\rho}$$

and

$$(j\delta - 1)k + i \overset{\delta}{\mapsto} i\rho_{j\delta} \overset{\rho}{\mapsto} i\rho_{j\delta}\rho \;=\; i\tau_{j\delta\rho},$$

by assumption, and since $\alpha \cdot \rho = \mathbf{1}_n$, also $\kappa_j \cdot \delta \cdot \rho = \delta \cdot \rho$.

We have proved that if the commutative identity holds in $(T, I, ^{\dagger})$, then the generalized commutative identity holds also. To complete the proof of Proposition 5.3.26, note that in the above argument we used only the particular subcase of the commutative identity (5.10) that each ρ_i is a base permutation. For a different choice of the κ_j's, the other subcase that each ρ_i is an aperiodic base morphism equally suffices. Indeed, we could take for κ_j the map

$$\begin{aligned} (j\delta\rho\alpha - 1)k + i &\mapsto (j\delta - 1)k + i \\ (j\delta - 1)k + i &\mapsto (j\delta - 1)k + i, \end{aligned}$$

for all $i = 1, \ldots, k$; $i'\kappa_j := i'$ otherwise.

Exercise 5.3.27 Show that in any partial iteration theory, the generalized commutative identity (5.11) holds for *all* base morphisms ρ.

5.4 Free Iterative Theories

Recall that Σtr is the theory of regular Σ-trees, i.e. those with a finite number of subtrees. We will show that each theory Σtr can be characterized as the free iterative theory, freely generated by the function

$$\eta : \Sigma \ \rightarrow \ \Sigma tr,$$

which takes the letter σ in Σ_n, $n \geq 0$, to the atomic tree $\sigma : 1 \rightarrow n$. In fact we will prove a stronger result that applies to partial iteration theories $(T, I, ^\dagger)$ which will be used in the next chapter. We need several preliminary facts.

Definition 5.4.1 *Let $(T, I, ^\dagger)$ be a partial iteration theory. An I-***normal description** $D : n \rightarrow p$ *of weight s is a pair $(\alpha; f)$ consisting of a morphism $f : s \rightarrow s + p$ in I and a base morphism $\alpha : n \rightarrow s + p$. The* **behavior** $|D|$ *of the normal description is the T-morphism*

$$|D| \ := \ \alpha \cdot \langle f^\dagger, \mathbf{1}_p \rangle : n \rightarrow p.$$

Recall that when T is an iterative theory, the ideal I is always taken to be the collection of ideal morphisms.

Definition 5.4.2 *Let $T = \Sigma tr$. A* **primitive normal description**

$$D \ = \ (\alpha; f) : n \rightarrow p$$

of weight s is a normal description such that f is a primitive tree $s \rightarrow s + p$. Thus each component $i_s \cdot f$ is of the form

$$i_s \cdot f \ = \ \sigma_i \cdot \rho_i$$

where σ_i is in Σ (more precisely, σ_i is an atomic tree of the form $\sigma_i : 1 \rightarrow k_i$) and where $\rho_i : k_i \rightarrow s + p$ is base.

A primitive normal description $D = (\alpha; f) : n \rightarrow p$ of weight s in Σtr is nothing but a *flowchart scheme.* As a flowchart scheme, D has vertex set $[s + p]$. The vertices $i \in [s]$ are called *internal.* The vertices $s + j$, $j \in [p]$, are the *exits* of the scheme. Suppose $i \in [s]$ is an internal vertex and $i_s \cdot f = \sigma \cdot \rho$, where $\sigma \in \Sigma_k$ and $\rho : [k] \to [s+p]$. Then vertex i is labeled σ and has k outedges, namely there is a directed edge from i to $j\rho$, for each $j \in [k]$. These edges are linearly ordered according to the order on the set $[k]$. When $j \in [p]$, the exit vertex $s+j$ is labeled `exit j`. The exits have outdegree 0. Finally, the base morphism α, considered to be a function $[n] \to [s+p]$, determines the *begin vertices*. The behavior of the D is the (regular) tree obtained by unfolding the scheme in usual way. Below, we will interpret the abstract notions in the language of flowchart schemes.

Let $D = (\alpha; f) : n \to p$ be a primitive normal description of weight s in Σtr, so that

$$f \;=\; \langle \sigma_1 \cdot \rho_1, \ldots, \sigma_s \cdot \rho_s \rangle : s \to s+p,$$

where the σ's are in Σ and the ρ's are base. We say that the integer $i \in [s]$ *directly depends* on the integer $j \in [s]$ if j is in the range of ρ_i. An equivalent condition is that the variable x_j is a leaf of the primitive tree $\sigma_i \cdot \rho_i$. Further, we say that i depends on j if there is a sequence $i = i_0, i_1, \ldots, i_m = j$, $m \geq 0$, such that each i_{k-1} directly depends on i_k, for all $k \in [m]$.

Definition 5.4.3 *Given a primitive normal description*

$$D \;=\; (\alpha; f) : n \to p$$

of weight s in Σtr, an integer $j \in [s]$ is called (D-)accessible if there exists $i \in [s]$ in the range of α such that i depends on j. D itself is called **accessible** *if each $j \in [s]$ is accessible. Further, D is* **reduced***, if for each $i, j \in [s]$, if $i \neq j$ then*

$$i_s \cdot f^\dagger \;\neq\; j_s \cdot f^\dagger.$$

As a flowchart scheme, D is accessible iff for each internal vertex $i \in [s]$ there is a directed path from some begin vertex to i. D is reduced iff for any two (internal) vertices i and j of D, the tree obtained by unfolding D at vertex i is different from that obtained by unfolding D at vertex j, unless $i = j$.

If the weight of D is 0, then D is trivially accessible and reduced. We note the following fact.

Proposition 5.4.4 *Let $D = (\alpha; f) : n \to p$ be a primitive normal description of weight s in Σtr.*

- *D is accessible if and only if for every integer $j \in [s]$ there exist $i \in [n]$ and $k \geq 0$ such that $i\alpha \in [s]$ and x_j is a leaf of the tree $(i\alpha)_s \cdot f^k$.*
- *Suppose that $m \leq s$ and that the accessible integers are those in $[m]$. Then there exist $g : m \to m + p$ and $h : (s - m) \to s + p$ and a base $\beta : n \to m + p$ such that*
$$\alpha \ = \ \beta \cdot (\mathbf{1}_m \oplus 0_{s-m} \oplus \mathbf{1}_p)$$
and
$$f \ = \ \langle g \cdot (\mathbf{1}_m \oplus 0_{s-m} \oplus \mathbf{1}_p),\, h\rangle.$$
Since g is clearly primitive, $E := (\beta; g) : n \to p$ is a primitive normal description of weight m.

Let $D = (\alpha; f) : n \to p$ be an accessible primitive normal description of weight s in Σtr. By the fixed point identity,
$$f^\dagger \ = \ f^k \cdot \langle f^\dagger, \mathbf{1}_p\rangle,$$
for all $k \geq 0$. By Proposition 5.4.4, for every $j \in [s]$ there exist $k \geq 0$ and $i \in [n]$ such that x_j is a leaf of $(i\alpha)_s \cdot f^k$. Thus the equations
$$i_n \cdot |D| \ = \ (i\alpha)_s \cdot f^\dagger \ = \ (i\alpha)_s \cdot f^k \cdot \langle f^\dagger, \mathbf{1}_p\rangle \qquad (5.16)$$
show that $j_s \cdot f^\dagger$ is a subtree of $i_n \cdot |D|$. Further, this subtree is clearly proper. Conversely, it follows from (5.16) that if t is a proper subtree of some $i_n \cdot |D|$, then $t = j_s \cdot f^\dagger$ for an integer $j \in [s]$. Indeed, let k be the length of the directed path from the root of $i_n \cdot |D|$ to a vertex v such that the subtree at vertex v is the tree t. We have thus proved the following Proposition.

Proposition 5.4.5 *Let $D = (\alpha; f) : n \to p$ be a primitive normal description and let*
$$J \ := \ \{j_s \cdot f^\dagger \ : \ j \in [s]\}, \quad K \ := \ \{i_n \cdot |D| \ : \ i \in [n]\}.$$
If D is accessible then a tree t is in J if and only if t is a proper subtree of some tree in K.

Exercise 5.4.6 With the notations of Proposition 5.4.5, suppose that D is reduced. Show that D is accessible iff J is the set of all proper subtrees of the trees in K.

We can now reformulate Proposition 5.2.14 as follows.

Corollary 5.4.7 *For every tree $t : n \to p$ in Σtr there exists a reduced and accessible primitive normal description $D = (\alpha; f) : n \to p$ with $t = |D|$.*

Next we consider the question of when two normal descriptions have the same behavior. In other terminology, when do two flowchart schemes unfold to the same tree?

Definition 5.4.8 *Let $D = (\alpha; f) : n \to p$ and $E = (\beta; g) : n \to p$ be primitive normal descriptions in Σtr of weight s and r, respectively. We write*

- *$D \xrightarrow{\tau} E$ for a base morphism $\tau : s \to r$ if*

$$\alpha \cdot (\tau \oplus \mathbf{1}_p) \;=\; \beta \quad \textit{and} \quad f \cdot (\tau \oplus \mathbf{1}_p) \;=\; \tau \cdot g.$$

- *$D \hookrightarrow E$ if and only if $D \xrightarrow{\iota} E$ for some injective base morphism ι.*
- *$D \to E$ if and only if $D \xrightarrow{\rho} E$ for some surjective base morphism ρ.*
- *$D \leftrightarrow E$ if and only if $D \xrightarrow{\tau} E$ for some bijective base morphism τ.*

It is immediate that $D \leftrightarrow E$ holds for two primitive normal descriptions if and only if both $D \hookrightarrow E$ and $D \to E$ hold. Further, if τ is bijective and $D \xrightarrow{\tau} E$, then $E \xrightarrow{\tau^{-1}} D$.

Suppose that $D \xrightarrow{\tau} E$ as above. Then, considering D and E as schemes, τ determines a "flowchart scheme homomorphism". Indeed, as a function $[s{+}p] \to [r{+}p]$, $\tau \oplus \mathbf{1}_p$ maps vertices of D to vertices of E in a way that preserves edges, labels, begins and exits, as well as the ordering of the outgoing edges of each vertex. Thus $D \leftrightarrow E$ iff E is an isomorphic image of D, $D \hookrightarrow E$ iff D is a subscheme of E, and $D \rightarrow E$ iff E is a homomorphic image of D. Proposition 5.4.11 below shows that D and E have the same behavior whenever there is a homomorphism $D \xrightarrow{\tau} E$.

Exercise 5.4.9 Let D, E and F be primitive normal descriptions. Show that if $D \xrightarrow{\tau} E$ and $E \xrightarrow{\rho} F$ for some base morphisms τ and ρ, then $D \xrightarrow{\tau \cdot \rho} F$.

Exercise 5.4.10 Suppose that $D \xrightarrow{\tau} E$ for the primitive normal descriptions D and E and for some base τ. Write τ as $\tau = \rho \cdot \iota$, where ρ is surjective and ι is injective. Prove that there exists a primitive normal description F such that $D \xrightarrow{\rho} F$ and $F \xrightarrow{\iota} E$.

Proposition 5.4.11 *Let D and E be primitive normal descriptions in Σtr. If $D \xrightarrow{\tau} E$ for any base τ, then $|D| = |E|$. In particular, $|D| = |E|$ whenever $D \hookrightarrow E$ or $D \rightarrow E$.*

Proof. We will make use of the functorial dagger implication, Theorem 5.3.3.b. Let $D = (\alpha; f)$ and $E = (\beta; g)$ where $f : s \rightarrow s + p$ and $g : r \rightarrow r + p$. Supposing

$$\alpha \cdot (\tau \oplus \mathbf{1}_p) \;=\; \beta$$

and

$$f \cdot (\tau \oplus \mathbf{1}_p) \;=\; \tau \cdot g,$$

we have

$$f^\dagger \;=\; \tau \cdot g^\dagger$$

by the functorial implication. Thus

$$\begin{aligned} |D| &= \alpha \cdot \langle f^\dagger, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle \tau \cdot g^\dagger, \mathbf{1}_p \rangle \\ &= \alpha \cdot (\tau \oplus \mathbf{1}_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= \beta \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= |E|. \end{aligned}$$

We now obtain a result in the converse direction. Assuming that D and E are schemes with the same behavior, we will show that D and E are related by the smallest equivalence relation including the relations $\hookrightarrow$ and $\to$. The details are in Corollary 5.4.15, below.

Proposition 5.4.12 *Let $D = (\alpha; f) : n \to p$ and $E = (\beta; g) : n \to p$ be accessible primitive normal descriptions of weight s and r, respectively. Suppose that E is reduced. If $|D| = |E|$ then $D \to E$.*

Proof. In our argument we will make use of the unique factorization property of Σ-trees, mentioned in the proof of Proposition 2.2.3.16. By Proposition 5.4.5, the function $\rho : [s] \to [r]$ with $j\rho = k$ if and only if

$$1 \xrightarrow{j_s} s \xrightarrow{f^\dagger} p \quad = \quad 1 \xrightarrow{k_r} r \xrightarrow{g^\dagger} p$$

is a well-defined surjection. We will show that $D \xrightarrow{\rho} E$. Note that we have $f^\dagger = \rho \cdot g^\dagger$. Let $i \in [n]$ and suppose $i\alpha = s + j$, $j \in [p]$. Then

$$i_n \cdot |D| \;=\; i_n \cdot \alpha \cdot \langle f^\dagger, \mathbf{1}_p \rangle \;=\; j_p \;=\; i_n \cdot \beta \cdot \langle g^\dagger, \mathbf{1}_p \rangle \;=\; i_n \cdot |E|.$$

Thus $i\beta = r + j$. Suppose now that $i\alpha = j \in [s]$. Then

$$i_n \cdot |D| \;=\; j_s \cdot f^\dagger$$

is a proper tree. Thus, by the unique factorization property, $i\beta = k \in [r]$, so that

$$i_n \cdot |E| \;=\; k_r \cdot g^\dagger.$$

But $i_n \cdot |D| = i_n \cdot |E|$ and since $|E|$ is reduced, $k = j\rho$. This proves that $\alpha \cdot (\rho \oplus \mathbf{1}_p) = \beta$.

Let us now write f and g in more detailed form as

$$f \;=\; \langle \sigma_1 \cdot \rho_1, \ldots, \sigma_s \cdot \rho_s \rangle$$

and

$$g \;=\; \langle \sigma'_1 \cdot \tau_1, \ldots, \sigma'_r \cdot \tau_r \rangle,$$

where the σ's are in Σ and the ρ's and τ's are base. To see that

$$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g,$$

we must show that

$$\sigma_i \cdot \rho_i \cdot (\rho \oplus \mathbf{1}_p) \;=\; \sigma'_{i\rho} \cdot \tau_{i\rho},$$

for all $i \in [s]$. But

$$i_s \cdot f^\dagger \;=\; i_s \cdot f \cdot \langle f^\dagger, \mathbf{1}_p\rangle \;=\; \sigma_i \cdot \rho_i \cdot \langle f^\dagger, \mathbf{1}_p\rangle$$

and

$$(i\rho)_r \cdot g^\dagger \;=\; (i\rho)_r \cdot g \cdot \langle g^\dagger, \mathbf{1}_p\rangle \;=\; \sigma'_{i\rho} \cdot \tau_{i\rho} \cdot \langle g^\dagger, \mathbf{1}_p\rangle.$$

Since also

$$i_s \cdot f^\dagger \;=\; (i\rho)_r \cdot g^\dagger,$$

we conclude by the unique factorization property of Σ-trees that $\sigma_i = \sigma'_{i\rho}$ and

$$\rho_i \cdot \langle f^\dagger, \mathbf{1}_p\rangle \;=\; \tau_{i\rho} \cdot \langle g^\dagger, \mathbf{1}_p\rangle.$$

Since $f^\dagger = \rho \cdot g^\dagger$, it follows that

$$\rho_i \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle g^\dagger, \mathbf{1}_p\rangle \;=\; \tau_{i\rho} \cdot \langle g^\dagger, \mathbf{1}_p\rangle.$$

Since the components of $g^\dagger$ are all different and proper, it follows that $\rho_i \cdot (\rho \oplus \mathbf{1}_p) \;=\; \tau_{i\rho}$.

Corollary 5.4.13 *Let $D = (\alpha; f) : n \to p$ and $E = (\beta; g) : n \to p$ be reduced accessible primitive normal descriptions $n \to p$ in Σtr of weights s and s', respectively. If $|D| = |E|$, then $s = s'$ and $D \leftrightarrow E$.*

Proposition 5.4.14 *Suppose that $D : n \to p$ is a primitive normal description in Σtr. There is an accessible primitive normal description $E : n \to p$ such that $E \hookrightarrow D$.*

Proof. If D is accessible let $E := D$. Suppose that D is not accessible. Let $D = (\alpha; f)$ with $f : s \to s + p$. For a bijective base morphism $\rho : s \to s$, define $D_\rho := (\alpha_\rho; f_\rho) : n \to p$ where

$$\alpha_\rho \;=\; \alpha \cdot (\rho \oplus \mathbf{1}_p) : n \to s + p$$

and

$$f_\rho \;=\; \rho^{-1} \cdot f \cdot (\rho \oplus \mathbf{1}_p) : s \to s + p.$$

Thus D_ρ is a primitive normal description with $D \xrightarrow{\rho} D_\rho$. If now ρ is any bijection which maps the D-accessible integers in $[s]$ onto the set $[m]$, for some $m \leq s$, then the D_ρ-accessible integers are exactly those in $[m]$. By Proposition 5.4.4 we can write

$$\alpha_\rho \;=\; \beta \cdot (\mathbf{1}_m \oplus 0_{s-m} \oplus \mathbf{1}_p)$$

and

$$f_\rho \;=\; \langle g \cdot (\mathbf{1}_m \oplus 0_{s-m} \oplus \mathbf{1}_p),\, h \rangle,$$

for a base $\beta : n \to m + p$ and some primitive $g : m \to m + p$ and $h : s - m \to s + p$. Let $E := (\beta; g)$ and $\iota := \mathbf{1}_m \oplus 0_{s-m}$. We obviously have $E \xrightarrow{\iota} D_\rho$. Since also $D_\rho \xrightarrow{\rho^{-1}} D$, it follows that $E \xrightarrow{\iota \cdot \rho^{-1}} D$. Thus $E \hookrightarrow D$.

Corollary 5.4.15 *Let D and E be primitive normal descriptions $n \to p$ in Σtr. $|D| = |E|$ if and only if there exist primitive normal descriptions D_0, E_0 and F such that $D_0 \hookrightarrow D$, $E_0 \hookrightarrow E$, $D_0 \to F$ and $E_0 \to F$, i.e.*

$$D \hookleftarrow D_0 \to F \leftarrow E_0 \hookrightarrow E.$$

Proof. Supposing $|D| = |E|$ let F be a reduced primitive normal description with $|F| = |D|$, which exists by Corollary 5.4.7. By Proposition 5.4.14, there exist accessible normal descriptions D_0 and E_0 with $D_0 \hookrightarrow D$ and $E_0 \hookrightarrow E$. By Proposition 5.4.11, we have $|D_0| = |D| = |E| = |E_0|$. Thus, by Proposition 5.4.12, $D_0 \to F$ and $E_0 \to F$. The converse direction follows from Proposition 5.4.11.

Thus, two schemes $D, E : n \to p$ have the same behavior iff D has a subscheme D_0 and E has a subscheme E_0 which map onto the same scheme F. In fact, D_0 may be chosen to be the "accessible part" of D, i.e. the subscheme spanned by the accessible internal vertices (and the exits). The scheme E_0 is defined similarly. The scheme F may be obtained from either scheme D_0 or E_0 by collapsing two vertices if they are the roots of identical trees of the behavior, obtained by unfolding the scheme.

Exercise 5.4.16 Suppose that D and E are primitive normal descriptions $n \to p$ in Σtr. Show that if $|D| = |E|$, then there is also a connecting sequence D_0, F, E_0 such that

$$D \hookleftarrow D_0 \leftarrow F \to E_0 \hookrightarrow E.$$

D_0 and E_0 may be chosen as above. Also

$$D \to D' \hookleftarrow F' \hookrightarrow E' \leftarrow E,$$

for some descriptions D', E' and F'.

Exercise 5.4.17 Show that an accessible primitive normal description D is characterized by the property that $D' \hookrightarrow D$ implies $D \leftrightarrow D'$. Similarly, D is reduced if and only if $D \to D'$ implies $D \leftrightarrow D'$.

Exercise 5.4.18 Let $D, E : n \to p$ be primitive normal descriptions. Show that $|D| = |E|$ if and only if there is a sequence $F_0, \ldots, F_m$ such that $F_0 = D$, $F_m = E$ and such that for some base morphisms τ_i, $i \in [m-1]$, we have $F_i \overset{\tau_i}{\to} F_{i+1}$ or $F_{i+1} \overset{\tau_i}{\to} F_i$.

Exercise 5.4.19 Suppose that $D : n \to p$ is a primitive normal description. Show that there is an accessible and reduced description F which has the same behavior as D. Prove that if E is any description such that $|D| = |E|$, there exist descriptions E_1 and E_2 with

$$E \hookleftarrow E_1 \to F \hookrightarrow E_2 \leftarrow E.$$

The following technical lemma will be of importance later.

Lemma 5.4.20 *Let $f : s \to s + p$ and $g : r \to r + p$ be primitive trees in Σtr. Suppose that*

$$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g$$

for a surjective base morphism $\rho : s \to r$. Then there is a primitive tree $h : r \to k + p$ such that for some base $\rho_1, \ldots, \rho_s : k \to s$ and $\tau_1, \ldots, \tau_r : k \to r$ we have

$$f \;=\; (\rho \cdot h) \parallel (\rho_1, \ldots, \rho_s)$$

and

$$g \;=\; h \parallel (\tau_1, \ldots, \tau_r).$$

Further, $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [s]$.

Proof. Suppose that

$$g \;=\; \langle \sigma_1 \cdot \beta_1, \ldots, \sigma_r \cdot \beta_r \rangle$$

where $\sigma_j \in \Sigma_{k_j}$ and $\beta_j : k_j \to r + p$ is base, for all $j \in [r]$. Since $f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g$, we conclude by the unique decomposition property of Σ-trees that

$$f \;=\; \langle \sigma_{1\rho} \cdot \alpha_1, \ldots, \sigma_{s\rho} \cdot \alpha_s \rangle,$$

where each $\alpha_i : k_i\rho \to s + p$ is base with

$$\alpha_i \cdot (\rho \oplus \mathbf{1}_p) \;=\; \beta_{i\rho}.$$

Thus, if $i\rho = j$ then, for all $x \in [k_j]$ and $y \in [p]$, $x\alpha_i = s + y$ if and only if $x\beta_j = r + y$.

Let $k := \max\{k_1, \ldots, k_r\}$. We can factor each β_j as

$$\beta_j \;=\; \delta_j \cdot (\tau_j \oplus \mathbf{1}_p),$$

for some base $\delta_j : k_j \to k + p$ and $\tau_j : k \to r$ subject to the following condition. For all $x, x' \in [k_j]$, if $x\delta_j = x'\delta_j$ is in $[k]$, then $x = x'$. We obviously have $x\beta_j = r + y$ if and only if $x\delta_j = k + y$, for all $x \in [k_j]$ and $y \in [p]$.

Now let $h := \langle \sigma_1 \cdot \delta_1, \ldots, \sigma_r \cdot \delta_r \rangle : r \to k + p$, so that

$$j_r \cdot h \cdot (\tau_j \oplus \mathbf{1}_p) \;=\; j_r \cdot g,$$

for all $j \in [r]$. To complete the proof we must find the base morphisms $\rho_i : k \to s$, $i \in [s]$, such that

$$\delta_j \cdot (\rho_i \oplus \mathbf{1}_p) \;=\; \alpha_i \quad \text{and} \quad \rho_i \cdot \rho \;=\; \tau_j,$$

where $j = i\rho$. If $x \in [k]$ is in the range of δ_j, say $x = x'\delta_j$, we define $x\rho_i := x'\alpha_i$; otherwise we define $x\rho_i$ to be any integer y in $[s]$ with $y\rho = x\tau_j$. The function ρ_i is well-defined by the properties of δ_j and the fact that ρ is surjective. The equation $\delta_j \cdot (\rho_i \oplus \mathbf{1}_p) = \alpha_i$ holds by definition and the fact that, for all $x \in [k_j]$ and $y \in [p]$,

$$x\delta_j \;=\; k + y \quad \Leftrightarrow \quad x\beta_j \;=\; r + y \quad \Leftrightarrow \quad x\alpha_i \;=\; s + y.$$

Also $x\rho_i\rho = x\tau_j$ is obvious if $x \in [k]$ is not in the range of δ_j. If $x = x'\delta_j$, then

$$x\rho_i\rho \;=\; x'\alpha_i\rho \;=\; x'\beta_j \;=\; x'\delta_j\tau_j \;=\; x\tau_j.$$

Thus $\rho_i \cdot \rho = \tau_j$.

Recall that when Σ is a signature, each Σ-term is identified with the corresponding finite tree. Thus is a subtheory of the theory Σtr of regular Σ-trees. Let $(T, I, ^\dagger)$ be a partial iteration theory. A theory morphism

$$\varphi : \to T$$

from the free theory to T is *admissible* (more precisely, I-admissible) if for each σ in Σ, $\sigma\varphi$ is in I. It then follows that $t\varphi$ is in I for any ideal (i.e. proper) t in . If φ is an admissible theory morphism, φ extends to a map, also denoted φ, from the primitive normal descriptions in Σtr to the I-normal descriptions in T, namely

$$(\alpha; f)\varphi \ := \ (\alpha; f\varphi).$$

Thus, if $D : n \to p$ is a primitive normal description, $D\varphi : n \to p$ is an I-normal description.

Proposition 5.4.21 *Let $D : n \to p$ and $E : n \to p$ be primitive normal descriptions in Σtr. If $(T, I, ^\dagger)$ is a partial iteration theory and $\varphi :\to T$ is admissible, then $|D\varphi| = |E\varphi|$ whenever $D \hookrightarrow E$ or $D \to E$.*

Proof. Let $D = (\alpha; f)$ and $E = (\beta; g)$, where $f : s \to s + p$ and $g : r \to r + p$. First suppose $D \xrightarrow{\iota} E$, where ι is an injective base morphism, i.e.

$$\alpha \cdot (\iota \oplus \mathbf{1}_p) \ = \ \beta \quad \text{and} \quad f \cdot (\iota \oplus \mathbf{1}_p) \ = \ \iota \cdot g.$$

Since φ is a theory morphism, also

$$f\varphi \cdot (\iota \oplus \mathbf{1}_p) \ = \ \iota \cdot g\varphi.$$

Since φ is admissible, $f\varphi$ and $g\varphi$ are in I. Thus,

$$\begin{aligned} |D\varphi| \ &= \ |(\alpha; f\varphi)| \\ &= \ \alpha \cdot \langle (f\varphi)^\dagger, \mathbf{1}_p \rangle \\ &= \ \alpha \cdot \langle \iota \cdot (g\varphi)^\dagger, \mathbf{1}_p \rangle, \end{aligned}$$

by Proposition 5.3.24,

$$\begin{aligned} &= \ \alpha \cdot (\iota \oplus \mathbf{1}_p) \cdot \langle (g\varphi)^\dagger, \mathbf{1}_p \rangle \\ &= \ \beta \cdot \langle (g\varphi)^\dagger, \mathbf{1}_p \rangle \\ &= \ |(\beta; g\varphi)| \\ &= \ |E\varphi|. \end{aligned}$$

Now suppose that $D \xrightarrow{\rho} E$ for a surjective base morphism $\rho : s \to r$. Then

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) \;=\; \beta \quad \text{and} \quad f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g.$$

By Lemma 5.4.20, there exists a primitive tree $h : r \to k + p$ and there exist base morphisms $\rho_1, \ldots, \rho_s : k \to s$ and $\tau_1, \ldots, \tau_r : k \to r$ with

$$\begin{aligned} f &= (\rho \cdot h) \parallel (\rho_1, \ldots, \rho_s) \\ g &= h \parallel (\tau_1, \ldots, \tau_r), \end{aligned}$$

and such that $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [s]$. Since φ is an admissible theory morphism, each of $h\varphi, f\varphi$ and $g\varphi$ is in I. Further,

$$f \;=\; (\rho \cdot h\varphi) \parallel (\rho_1, \ldots, \rho_s)$$

and

$$g \;=\; h\varphi \parallel (\tau_1, \ldots, \tau_r).$$

Thus, by Proposition 5.3.26,

$$(f\varphi)^\dagger \;=\; \rho \cdot (g\varphi)^\dagger.$$

The rest is routine.

Exercise 5.4.22 Suppose that D and E are primitive normal descriptions as above, and that φ is admissible. Prove that if $D \xrightarrow{\tau} E$ holds for a base morphism τ, then $|D\varphi| = |E\varphi|$.

Corollary 5.4.23 *Under the assumptions of Proposition 5.4.21, if* $|D| = |E|$ *holds in* Σtr*, then* $|D\varphi| = |E\varphi|$ *in* T*.*

Proof. Immediate from Proposition 5.4.21 and Corollary 5.4.15.

We now state the main result of this section. Let η' denote the inclusion of into Σtr.

Theorem 5.4.24 *Let* Σ *be a ranked set and let* $(T, I, ^\dagger)$ *be a partial iteration theory. For each admissible* $\varphi : \to T$ *there is a unique partial iteration theory morphism*

$$\varphi^\sharp : \Sigma tr \;\to\; (T, I, ^\dagger)$$

such that

$$\eta' \cdot \varphi^\sharp \;=\; \varphi.$$

Thus $f^\dagger \varphi^\sharp = (f\varphi^\sharp)^\dagger$, *for all ideal* $f : n \to n + p$ *in* Σtr.

Proof. We assume first that $\varphi^\sharp$ exists. By Corollary 5.4.7, for each tree $t : n \to p$ in Σtr there is a primitive normal description $D : n \to p$ such that $t = |D|$. It is clear that $f\varphi^\sharp = f\varphi$, for primitive trees f in Σtr, since $\varphi^\sharp$ extends φ. Hence if t has the primitive normal description $(\alpha; f)$, where f is primitive, then $t = \alpha \cdot \langle f^\dagger, \mathbf{1}_p\rangle$ and we must have

$$t\varphi^\sharp \;=\; \alpha \cdot \langle (f\varphi)^\dagger, \mathbf{1}_p\rangle \;=\; |D\varphi|. \tag{5.17}$$

Thus, there is at most one extension of φ to a partial iteration theory morphism $\varphi^\sharp$. The only problem is that the left-hand side of the equation (5.17) has not been defined. Thus we make (5.17) the official definition of the left-hand side. Thus, for any primitive normal description $D : n \to p$ in Σtr,

$$|D|\varphi^\sharp \;:=\; |D\varphi|.$$

By Corollary 5.4.23, $\varphi^\sharp$ is well-defined. We turn now to the task of showing $\varphi^\sharp$ is a theory morphism. First, suppose that $t = i_p : 1 \to p$ is base. Then

$$i_p \;=\; |D|,$$

where D is the primitive normal description $(i_p; 0_p)$ of weight 0. Hence

$$i_p\varphi^\sharp \;=\; |(i_p; 0_p)| \;=\; i_p.$$

We now show that $\varphi^\sharp$ preserves composition and iteration. The argument involves defining corresponding operations on descriptions and showing that these operations preserve behaviors.

Proposition 5.4.25 *The following facts hold in any partial iteration theory* $(T, I,^\dagger)$.

[a] *Given I-normal descriptions* $D = (\alpha; f) : n \to p$ *and* $E = (\beta; g) : p \to q$, *of weights* s *and* r *respectively, define the I-normal description*

$$D \cdot E \;:=\; (\gamma; h) : n \to q$$

of weight $s + r$ *by*

$$\begin{aligned} \gamma &:= \alpha \cdot (\mathbf{1}_s \oplus \beta) : n \to s + r + q \\ h &:= \langle f \cdot (\mathbf{1}_s \oplus \beta),\, 0_s \oplus g \rangle : s + r \to s + r + q. \end{aligned} \tag{5.18}$$

Then,

$$|D \cdot E| \;=\; |D| \cdot |E|.$$

[b] *Given* I*-normal descriptions* $D = (\alpha; f) : n \to p$ *and* $E = (\beta; g) : m \to p$*, of weights* s *and* r *respectively, define the* I*-normal description*

$$\langle D, E \rangle \;:=\; (\gamma; h) : n + m \to p$$

of weight $s + r$ *by*

$$\begin{aligned} \gamma &:= \langle \alpha \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p),\, 0_s \oplus \beta \rangle : n + m \to s + r + p \\ h &:= \langle f \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p),\, 0_s \oplus g \rangle : s + r \to s + r + p. \end{aligned} \tag{5.19}$$

Then,

$$|\langle D, E \rangle| \;=\; \langle |D|, |E| \rangle.$$

[c] *Given the* I*-normal description* $D = (\alpha; f) : n \to n + p$ *of weight* s *such that* $\alpha = \beta \oplus 0_{n+p}$*, for some base* $\beta : n \to s$*, define the* I*-normal description*

$$D^\dagger \;:=\; (\beta \oplus 0_p;\, g) : n \to p$$

of weight s*, where*

$$g \;:=\; f \cdot (\langle \mathbf{1}_s, \beta \rangle \oplus \mathbf{1}_p) : s \to s + p. \tag{5.20}$$

Then,

$$|D^\dagger| \;=\; |D|^\dagger.$$

The construction given in Proposition 5.4.25 can be described as follows. Given the schemes $D : n \to p$ and $E : p \to q$, the flowchart scheme $D \cdot E : n \to q$ is the scheme obtained by taking the disjoint union of D and E and identifying the exits of D with the corresponding begins of E. The begin vertices of the composite are those of D, the exits those of E. When $D : n \to p$ and $E : m \to p$, to obtain the pairing $\langle D, E \rangle$, take the disjoint union as before but identify the exits of D with the corresponding exits of E. The first n begins of $\langle D, E \rangle$

are those of D, the last m begins those of E. Finally, the flowchart scheme $D^\dagger$ is obtained from D by identifying the first n exits of D with the corresponding begins; the exits of $D^\dagger$ are the last p exits of D. The operation makes sense since no begin of D is an exit vertex.

Proof of [a]. We show first that $h^\dagger = \langle f^\dagger \cdot \beta \cdot \langle g^\dagger, \mathbf{1}_q\rangle,\, g^\dagger\rangle$, where h is defined in (5.18). We make use of some of the identities of partial iteration theories.

$$\begin{array}{rcl} h^\dagger & = & \langle f\cdot(\mathbf{1}_s \oplus \beta),\, 0_s \oplus g\rangle^\dagger \\ & = & \langle (f\cdot(\mathbf{1}_s\oplus\beta))^\dagger \cdot \langle g^\dagger, \mathbf{1}_q\rangle,\, g^\dagger\rangle \\ & = & \langle f^\dagger\cdot\beta\cdot\langle g^\dagger,\mathbf{1}_q\rangle,\, g^\dagger\rangle, \end{array}$$

by Theorem 5.3.9.g and the base parameter identity. Thus,

$$\begin{array}{rcl} |D\cdot E| & = & \gamma\cdot\langle h^\dagger, \mathbf{1}_q\rangle \\ & = & \alpha\cdot(\mathbf{1}_s\oplus\beta)\cdot\langle f^\dagger\cdot\beta\cdot\langle g^\dagger,\mathbf{1}_q\rangle, g^\dagger,\, \mathbf{1}_q\rangle \\ & = & \alpha\cdot\langle f^\dagger\cdot\beta\cdot\langle g^\dagger,\mathbf{1}_q\rangle,\, \beta\cdot\langle g^\dagger,\mathbf{1}_q\rangle\rangle \\ & = & \alpha\cdot\langle f^\dagger\cdot|E|,\, |E|\rangle \\ & = & \alpha\cdot\langle f^\dagger,\mathbf{1}_p\rangle\cdot|E| \\ & = & |D|\cdot|E|. \end{array}$$

Proof of [b]. We have already shown in Theorem 5.3.9.i that for the morphism h defined in (5.19),

$$h \;=\; \langle f^\dagger, g^\dagger\rangle.$$

Hence

$$\begin{array}{rcl} |\langle D,E\rangle| & = & \gamma\cdot\langle h^\dagger,\mathbf{1}_p\rangle \\ & = & \langle\alpha\cdot(\mathbf{1}_s\oplus 0_r\oplus\mathbf{1}_p),\, 0_s\oplus\beta\rangle\cdot\langle f^\dagger, g^\dagger,\mathbf{1}_p\rangle \\ & = & \langle\alpha\cdot\langle f^\dagger,\mathbf{1}_p\rangle,\, \beta\cdot\langle g^\dagger,\mathbf{1}_p\rangle\rangle \\ & = & \langle |D|, |E|\rangle. \end{array}$$

Proof of [c]. We have already shown in Theorem 5.3.9.f that for the morphism g defined in (5.20),

$$\beta\cdot g^\dagger \;=\; (\beta\cdot f^\dagger)^\dagger.$$

Hence,

$$\begin{array}{rcl} |D^\dagger| & = & (\beta \oplus 0_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ & = & (\beta \cdot f^\dagger)^\dagger \\ & = & ((\beta \oplus 0_{n+p}) \cdot \langle f^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger \\ & = & |D|^\dagger. \end{array}$$

Exercise 5.4.26 Let $D : n \to p$ and $E : m \to q$ be I-normal descriptions. Define a description $D \oplus E$ such that $|D \oplus E| = |D| \oplus |E|$.

Proof of Theorem 5.4.24, continued. It follows that $\varphi^\sharp$ preserves composition. Indeed, if $t = |D| : n \to p$ and $t' = |E| : p \to q$ in Σtr, then

$$\begin{array}{rcl} (t \cdot t')\varphi^\sharp & = & |D \cdot E|\varphi^\sharp \;=\; |(D \cdot E)\varphi| \\ & = & |D\varphi \cdot E\varphi| \;=\; |D\varphi| \cdot |E\varphi| \;=\; t\varphi^\sharp \cdot t'\varphi^\sharp. \end{array}$$

Thus $\varphi^\sharp$ is a theory morphism. We prove that $\varphi^\sharp$ preserves iteration. Suppose that $t = |D| : n \to n + p$ is an ideal (i.e. proper) tree in Σtr, where $D = (\alpha; f)$ is a primitive normal description of weight s. Then $t = \alpha \cdot \langle f^\dagger, \mathbf{1}_{n+p} \rangle$. Since none of the components of t is base, it follows that $\alpha = \beta \oplus 0_{n+p}$ for some base $\beta : n \to s$. Hence we calculate:

$$t^\dagger\varphi^\sharp \;=\; |D^\dagger|\varphi^\sharp \;=\; |D^\dagger\varphi| \;=\; |(D\varphi)^\dagger| \;=\; |D\varphi|^\dagger \;=\; (t\varphi^\sharp)^\dagger.$$

Lastly we show that $\eta' \cdot \varphi^\sharp = \varphi$. Note that each atomic tree $\sigma : 1 \to p$ has primitive normal description $D_\sigma \;:=\; (\mathbf{1}_1 \oplus 0_p;\, 0_1 \oplus \sigma)$, by the left zero identity, say. Hence

$$\sigma\eta'\varphi^\sharp \;=\; |D_\sigma\varphi| \;=\; |(\mathbf{1}_1 \oplus 0_p;\, 0_1 \oplus \sigma\varphi)| \;=\; \sigma\varphi,$$

again by the left zero identity.

The uniqueness of $\varphi^\sharp$ was already shown above. The proof is complete.

Exercise 5.4.27 Show that $\varphi^\sharp$ preserves pairing or tupling by using Proposition 5.4.25.

We obtain two immediate corollaries. Corollary 5.4.29 is an important characterization of the regular trees.

Let $\eta : \Sigma \to \Sigma tr$ be the function which maps any letter σ in Σ_n, $n \geq 0$, to the atomic tree $\sigma : 1 \to n$.

Corollary 5.4.28 *Let $(T, I, ^\dagger)$ be a partial iteration theory and let $\varphi : \Sigma \rightarrow T$ be an admissible rank preserving map (i.e. each $\sigma\varphi$ is in I). There is a unique partial iteration theory morphism*

$$\varphi^\sharp : \Sigma tr \;\; \rightarrow \;\; (T, I, ^\dagger)$$

such that

$$\eta \cdot \varphi^\sharp \;\; = \;\; \varphi.$$

Corollary 5.4.29 *For each ranked set Σ, the theory Σtr is freely generated in the class of iterative theories by the map $\eta : \Sigma \rightarrow \Sigma tr$. In more detail, if T is an iterative theory and $\varphi : \Sigma \rightarrow T$ is admissible (i.e. $\sigma\varphi$ is ideal for all σ in Σ), then there is a unique iterative theory morphism $\varphi^\sharp : \Sigma tr \rightarrow T$ such that $\eta \cdot \varphi^\sharp = \varphi$.*

In the next chapter, we will use the notation $\Sigma\mathbf{tr}$ to denote an *iteration theory* of regular trees obtained by adding a new symbol to Σ_0 and extending the dagger operation to all regular trees. See Section 6.6.5.

Exercise 5.4.30 Show that the theory defined in Example 5.1.14 is an initial iterative theory.

Exercise 5.4.31 Show that if $(T, I, ^\dagger)$ is a partial iteration theory, and if $f : n \rightarrow n + p \in I$, then $(f^k)^\dagger = f^\dagger$, for all $k \geq 1$.

5.5 Notes

Ideal theories and iterative theories were introduced by Calvin Elgot in [Elg75] to model aspects of both the syntax and semantics of flowchart algorithms. Theorem 5.2.2 was proved in [BGR77]. Most of the equations of Theorem 5.3.9 were shown to hold in iterative theories in [Elg75]. The proofs given here show these equations hold in (partial) iteration theories, and are taken from [Ési80, Ési83]. In fact, many of these equations already hold for flowchart schemes and can be easily understood by drawing the appropriate pictures. Flowcharts themselves were studied e.g. in [Elg76b, ES79, CSS88, BÉ85]. The commutative identity was introduced in [Ési80] and the equivalent forms which are considered in Proposition 5.3.26 were derived in [Ési80]. The general functorial dagger implication of Theorem 5.3.3.b appears in a different context in [AM80]. The weaker form that h is a surjective or injective base morphism was used in [Ési80]. Proposition 5.3.24 and the fact that the funtorial dagger implication for surjective base morphisms implies the commutative identity were noted therein. I-normal descriptions in iterative theories were defined in [Elg75] to model “monadic machines”. The composition, pairing and iteration operations given in Proposition 5.4.25 were shown to preserve behavior in all iterative theories in [Elg75]. Here, this fact is derived for all partial iteration theories. The existence of free iterative theories was pointed out in [BE76]. The characterization of the theories Σtr as the free iterative theories is from [EBT78]. Independently, this result was proved in [Gin79]. Metric (iterative) theories were first defined in [Blo82] and metric iteration theories were considered in [Tro82]. It was shown in [Ési80] that if T is an algebraic theory equipped with an iteration operation which satisfies the zero identities, the pairing identity, and the commutative identity, and if $\varphi : \Sigma \to T$ is any rank preserving map, where Σ is some signature, then φ uniquely extends to a theory morphism $\varphi^\sharp : \Sigma tr \to T$ such that $\eta \cdot \varphi^\sharp = \varphi$ and $f^\dagger \varphi^\sharp = (f\varphi^\sharp)^\dagger$, for all ideal $f : n \to n + p$ in Σtr. (Here η denotes the embedding of Σ in Σtr.) Theorem 5.4.24 is a slight extension of this result and its proof differs from the argument in [Ési80] only in a few technical details. The concept of partial iteration theory is a technical one. Its only purpose is that the present formalization of Theorem 5.4.24 allows us to describe the free iterative theories (Corollary 5.4.29) and the free iteration theories (Theorem 6.6.5.2) at the same time.

Chapter 6

Iteration Theories

Iteration theories are a generalization of iterative theories. In iteration theories, the dagger operation can be applied to all morphisms $f : n \to n + p$, producing a canonical solution of the iteration equation for f. The properties of iteration are captured equationally.

6.1 Iteration Theories Defined

In this section we introduce at last the class of iteration theories. Iteration theories have a total operation $f \mapsto f^\dagger$ which, roughly speaking, satisfies all of the equations involving the theory and dagger operations which are meaningful and valid in iterative theories. For example, the equation

$$f^\dagger \;=\; f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$$

is meaningful in an iterative theory if $f : n \to n + p$ is ideal; it is valid (when meaningful) in all iterative theories. In an iteration theory, this equation holds for all $f : n \to n + p$. We start with a technical definition.

Definition 6.1.1 *A* **preiteration theory** T *is an algebraic theory equipped with a (total) iteration (or dagger) operation taking each morphism $f : n \to n+p$ to a morphism $f^\dagger : n \to p$, $n, p \geq 0$. The operation is not required to satisfy any particular properties. A* **preiteration**

theory morphism $\varphi : T \to T'$ *is a theory morphism which preserves iteration, i.e.*

$$f^\dagger \varphi \;=\; (f\varphi)^\dagger,$$

for all $f : n \to n + p$ *in* T.

When T is a subtheory of T' and the inclusion $T \to T'$ is a preiteration theory morphism, T is a *subpreiteration theory* of T'. When $\varphi : T \to T'$ is a surjective preiteration theory morphism, T' is called a *quotient* of T. If φ is bijective, T and T' are said to be isomorphic and φ is called an isomorphism.

We prefer not to give a precise definition of an "identity valid in all iterative theories". Anyone who feels the need for such precision may regard parts [a] and [b] of the following Theorem as heuristic rephrasing of the remaining parts.

Recall from Section 5.5.4 the definition of the map $\eta : \Sigma \to \Sigma tr$.

Theorem 6.1.2 *The following properties of a preiteration theory* T *are equivalent.*

[a] T *satisfies every identity valid in all iterative theories when iteration is applied only to ideal morphisms.*

[b] T *satisfies every identity that holds in any iterative theory* Σtr *when iteration is applied only to proper trees.*

[c] *The triple* $(T, T,^\dagger\,)$ *is a partial iteration theory, i.e.* T *satisfies the left zero identity, the base parameter identity, the pairing identity and the commutative identity.*

[d] *For each ranked set* Σ *and rank preserving function* $\varphi : \Sigma \to T$, *there is a unique theory morphism* $\varphi^\sharp : \Sigma tr \to T$ *such that* $\eta \cdot \varphi^\sharp = \varphi$ *and*

$$f^\dagger \varphi^\sharp \;=\; (f\varphi^\sharp)^\dagger,$$

for all proper trees $f : n \to n + p$ *in* Σtr.

[e] *There is some iterative theory* J *and a theory morphism* $\psi : J \to T$ *such that every morphism in* T *is the image of an ideal morphism and such that*

$$f^\dagger \psi \;=\; (f\psi)^\dagger,$$

for all ideal $f : n \to n + p$.

Proof. It is obvious that [a] implies [b]. That [b] implies [c] follows from Theorems 5.5.3.1 and 5.5.3.3. That [c] implies [d] is immediate from Corollary 5.5.4.28. To prove that [d] implies [e] choose Σ large enough so that there is a surjective rank preserving function $\varphi : \Sigma \to T$. Then let J be Σtr and let ψ be $\varphi^\sharp$. Finally, [e] implies [a] because surjective morphisms preserve the validity of equations.

The above result motivates the following definition of iteration theories.

Definition 6.1.3 *An* **iteration theory** *is a preiteration theory such that any of the equivalent conditions of Theorem 6.1.2 holds. An* **iteration theory morphism** $\varphi : T \to T'$ *between iteration theories is a preiteration theory morphism.*

Thus, iteration theories form an equational class. One equational axiomatization is given in Theorem 6.1.2.c. The next proposition shows that the base parameter identity in this equational axiomatization can be replaced by the right zero identity, which is a particular subcase of the base parameter identity.

Corollary 6.1.4 *A preiteration theory* T *is an iteration theory if and only if* T *satisfies the zero identities, the pairing identity and the commutative identity.*

Proof. The right zero identity holds in any iteration theory by Proposition 5.5.3.11. Conversely, if T satisfies the zero identities, the pairing identity and the block transposition identity, then the parameter identity holds in T by Proposition 5.5.3.21. The proof is completed by noting that the commutative identity implies the permutation identity, see Proposition 5.5.3.10.

Since iteration theories are equationally defined, any quotient or subpreiteration theory of an iteration theory is an iteration theory. Hence a *subiteration theory* of T is just a subpreiteration theory.

The pairing identity and its variants are exploited in the following two statements whose straightforward proofs are omitted. For convenience in stating these properties, the term *scalar p-identity* refers to one of the scalar pairing, scalar right pairing, or scalar symmetric pairing identities.

Proposition 6.1.5 *Let T and T' be preiteration theories. Suppose that one of the scalar p-identities holds in both theories T and T'. Then a theory morphism $\varphi : T \to T'$ is a preiteration theory morphism if and only if φ preserves scalar iteration, i.e. when*

$$f^\dagger \varphi \;=\; (f\varphi)^\dagger,$$

for all $f : 1 \to 1 + p$.

Thus a theory morphism φ between iteration theories is an iteration theory morphism if and only if φ preserves scalar iteration.

Proposition 6.1.6 *Let T be a preiteration theory which satisfies one of the scalar p-identities. Then a subtheory T' of T is a subpreiteration theory if and only if $f^\dagger$ is in T', for all $f : 1 \to 1 + p$ in T'.*

Thus a subtheory T' of an iteration theory T is a subiteration theory if and only if T' is closed under scalar iteration.

6.2 Other Axiomatizations of Iteration Theories

By Corollary 6.1.4, the two zero identities, the pairing identity and the commutative identity form a complete set of equational axioms of iteration theories. The pairing identity may be replaced by the right pairing identity, by Proposition 5.5.3.12. Below we present several alternative axiomatizations which make use of the notion of a Conway theory.

Definition 6.2.1 *A* **Conway theory** *is a preiteration theory satisfying the zero identities, the pairing identity and the permutation identity. A morphism of Conway theories is just a preiteration theory morphism.*

Exercise 6.2.2 Every preiteration theory T is pointed with point $\perp := \mathbf{1}_1^\dagger : 1 \to 0$, and preiteration theory morphisms preserve the point. As in every pointed theory, define

$$\perp_{np} \; := \; \langle \perp \cdot 0_p, \ldots, \perp \cdot 0_p \rangle : n \to p.$$

Show that when T is a Conway theory, $\perp_{np} = (\mathbf{1}_n \oplus 0_p)^\dagger = \mathbf{1}_n^\dagger \cdot 0_p$.

Again, by Proposition 5.5.3.12, a preiteration theory is a Conway theory if and only if the two zero identities, the right pairing identity and the permutation identity hold in T. Also, by Proposition 5.5.3.10, every iteration theory is a Conway theory, and a Conway theory T is an iteration theory if and only if the commutative identity holds in T.

Exercise 6.2.3 Show that there exists a Conway theory which is not an iteration theory.

The following corollary is an immediate consequence of results proved in Chapter 5.

Corollary 6.2.4 *The fixed point, parameter, composition and double dagger identities hold in any Conway theory. In fact, each of the identities listed in Theorem* 5.5.3.9 *holds in any Conway theory.*

Corollary 6.2.5 *A preiteration theory T is a Conway theory if and only if T satisfies the base parameter identity, the composition identity and the double dagger identity.*

Proof. Suppose that T is a preiteration theory which satisfies the base parameter identity, the composition identity

$$f \cdot \langle (g \cdot \langle f,\, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle \;=\; (f \cdot \langle g,\, 0_n \oplus \mathbf{1}_p \rangle)^\dagger, \qquad (6.1)$$

for all $f : n \to m + p$ and $g : m \to n + p$ in T, and the double dagger identity

$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger \;=\; f^{\dagger\dagger},$$

for all $f : n \to n + n + p$ in T. We will show that the left zero identity, the pairing identity and the permutation identity hold in T. Taking $n = m$ and substituting $\mathbf{1}_n \oplus 0_p$ for g in the composition identity, it follows that the fixed point identity, and hence the left zero identity, holds in T. Further, when $f = h \oplus 0_p$ for a morphism $h : n \to m$, the composition identity reduces to the simplified composition identity

$$h \cdot (g \cdot (h \oplus \mathbf{1}_p))^\dagger \;=\; (h \cdot g)^\dagger. \tag{6.2}$$

The simplified composition identity in turn implies the permutation identity. Indeed, let $f : n \to n + p$ and let $\pi : n \to n$ be a base permutation. Then, by (6.2),

$$\begin{aligned}(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger &= \pi \cdot (f \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot (\pi \oplus \mathbf{1}_p))^\dagger \\ &= \pi \cdot f^\dagger.\end{aligned}$$

We will establish the pairing identity first in two special cases. We show that the following two equations hold in T:

$$\langle 0_n \oplus f,\, 0_n \oplus g\rangle^\dagger \;=\; \langle f \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g^\dagger\rangle, \tag{6.3}$$

for all $f : n \to m + p$ and $g : m \to m + p$; and

$$\langle f \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p),\, g \cdot (\mathbf{1}_n \oplus 0_m \oplus \mathbf{1}_p)\rangle^\dagger \;=\; \langle f^\dagger,\, g \cdot \langle f^\dagger, \mathbf{1}_p\rangle\rangle, \tag{6.4}$$

for all $f : n \to n + p$ and $g : m \to n + p$. Indeed, since

$$\langle 0_n \oplus f,\, 0_n \oplus g\rangle \;=\; \langle f, g\rangle \cdot (0_n \oplus \mathbf{1}_m \oplus \mathbf{1}_p),$$

then by the simplified composition identity,

$$\begin{aligned}(0_n \oplus \mathbf{1}_m) \cdot \langle 0_n \oplus f,\, 0_n \oplus g\rangle^\dagger &= (0_n \oplus \mathbf{1}_m) \cdot (\langle f, g\rangle \cdot (0_n \oplus \mathbf{1}_m \oplus \mathbf{1}_p))^\dagger \\ &= ((0_n \oplus \mathbf{1}_m) \cdot \langle f, g\rangle)^\dagger \\ &= g^\dagger.\end{aligned}$$

Thus $\langle 0_n \oplus f,\, 0_n \oplus g\rangle^\dagger = \langle h, g^\dagger\rangle$ for some $h : n \to p$, so that by the fixed point identity,

$$\begin{aligned}\langle 0_n \oplus f,\, 0_n \oplus g\rangle^\dagger &= \langle 0_n \oplus f,\, 0_n \oplus g\rangle \cdot \langle h,\, g^\dagger, \mathbf{1}_p\rangle \\ &= \langle f \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g \cdot \langle g^\dagger, \mathbf{1}_p\rangle\rangle \\ &= \langle f \cdot \langle g^\dagger, \mathbf{1}_p\rangle,\, g^\dagger\rangle.\end{aligned}$$

The equation (6.4) now follows from (6.3) by the block transposition identity.

We now complete the proof by showing that the pairing identity holds in T. Let $f : n \to n + m + p$ and $g : m \to n + m + p$ in T and define $h := g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle$ and $\tau := \langle \mathbf{1}_{n+m}, \mathbf{1}_{n+m}\rangle$. Then

$$\begin{aligned}
\langle f, g\rangle^\dagger &= \\
&= (\langle f \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p}),\, g \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p})\rangle \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\
&= \langle f \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p}),\, g \cdot (\mathbf{1}_n \oplus 0_{m+n} \oplus \mathbf{1}_{m+p})\rangle^{\dagger\dagger},
\end{aligned}$$

by the double dagger identity,

$$= \langle 0_n \oplus f^\dagger,\, g \cdot \langle 0_n \oplus f^\dagger,\, 0_n \oplus \mathbf{1}_{m+p}\rangle\rangle^\dagger,$$

by (6.4) and the base parameter identity,

$$\begin{aligned}
&= \langle 0_n \oplus f^\dagger,\, 0_n \oplus h\rangle^\dagger \\
&= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger\rangle,
\end{aligned}$$

by (6.3).

Corollary 6.2.6 *A preiteration theory T is Conway theory if and only if the base parameter identity, the fixed point identity, the simplified composition identity and the double dagger identity hold in T.*

Corollary 6.2.7 *Let T be a preiteration theory. The following conditions are equivalent.*

- *T is an iteration theory.*
- *The base parameter, composition, double dagger and commutative identities hold in T.*
- *The base parameter, fixed point, simplified composition, double dagger and commutative identities hold in T.*

6.2.1 Scalar Axiomatizations

We now set out to establish some further axiomatizations of iteration theories that involve scalar iteration as much as possible. The main results are summarized in Theorem 6.2.15, Corollary 6.2.19, Theorem 6.2.20 and Corollary 6.2.21.

Proposition 6.2.8 *If the scalar left zero identity and the scalar pairing identity hold in a preiteration theory T, then so does the left zero identity.*

Proof. Let $f : n \to p$ be a morphism in T. We prove by induction on n that

$$(0_n \oplus f)^\dagger \;=\; f.$$

When $n = 0, 1$ this is either obvious or holds by assumption. Supposing $n > 1$, let $f = \langle f_1, f_2\rangle$ where $f_1 : n-1 \to p$ and $f_2 : 1 \to p$. We have

$$\begin{aligned}
&((0_n \oplus f_2) \cdot \langle (0_n \oplus f_1)^\dagger,\, \mathbf{1}_{1+p}\rangle)^\dagger \;= \\
&\quad= \; ((0_n \oplus f_2) \cdot \langle (0_n \oplus f_1)^\dagger,\, \mathbf{1}_1 \oplus 0_p,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \\
&\quad= \; (f_2 \cdot (0_1 \oplus \mathbf{1}_p))^\dagger \\
&\quad= \; (0_1 \oplus f_2)^\dagger \\
&\quad= \; f_2.
\end{aligned}$$

Thus

$$\begin{aligned}
(0_n \oplus f)^\dagger &= \langle 0_n \oplus f_1,\, 0_n \oplus f_2\rangle^\dagger \\
&= \langle (0_n \oplus f_1)^\dagger \cdot \langle f_2, \mathbf{1}_p\rangle,\, f_2\rangle \\
&= \langle (0_1 \oplus f_1) \cdot \langle f_2, \mathbf{1}_p\rangle,\, f_2\rangle \\
&= \langle f_1, f_2\rangle \\
&= f,
\end{aligned}$$

by the scalar pairing identity and the induction assumption.

Proposition 6.2.9 *Let T be a preiteration theory satisfying the scalar base parameter and scalar pairing identities. Then the base parameter identity holds in T.*

Proof. Let $f : n \to n + p$ be in T and let $\rho : p \to q$ be a base morphism. We must show that

$$(f \cdot (\mathbf{1}_n \oplus \rho))^\dagger \;=\; f^\dagger \cdot \rho.$$

If $n = 0$ this is obvious and if $n = 1$ this holds by assumption. We proceed by induction on n. If $n > 1$ we can write f as $f = \langle f_1, f_2\rangle$ for some $f_1 : n-1 \to n+p$ and $f_2 : 1 \to n+p$. Define h and k as follows:

$$h \;:=\; f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{1+p}\rangle$$

and

$$k \;:=\; f_2 \cdot (\mathbf{1}_n \oplus \rho) \cdot \langle (f_1 \cdot (\mathbf{1}_n \oplus \rho))^\dagger,\, \mathbf{1}_{1+q}\rangle.$$

Then

$$\begin{aligned}
k^\dagger &= (f_2 \cdot (\mathbf{1}_{n-1} \oplus \mathbf{1}_1 \oplus \rho) \cdot \langle (f_1 \cdot (\mathbf{1}_{n-1} \oplus \mathbf{1}_1 \oplus \rho))^\dagger,\, \mathbf{1}_{1+q} \rangle)^\dagger \\
&= (f_2 \cdot \langle f_1^\dagger \cdot (\mathbf{1}_1 \oplus \rho),\, \mathbf{1}_1 \oplus \rho \rangle)^\dagger \\
&= (f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{1+p} \rangle \cdot (\mathbf{1}_1 \oplus \rho))^\dagger \\
&= (f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{1+p} \rangle)^\dagger \cdot \rho \\
&= h^\dagger \cdot \rho,
\end{aligned}$$

by the induction assumption. Thus,

$$\begin{aligned}
(f \cdot (\mathbf{1}_n \oplus \rho))^\dagger &= \langle f_1 \cdot (\mathbf{1}_n \oplus \rho),\, f_2 \cdot (\mathbf{1}_n \oplus \rho) \rangle^\dagger \\
&= \langle (f_1 \cdot (\mathbf{1}_n \oplus \rho))^\dagger \cdot \langle k^\dagger, \mathbf{1}_q \rangle,\, k^\dagger \rangle \\
&= \langle f_1^\dagger \cdot (\mathbf{1}_1 \oplus \rho) \cdot \langle h^\dagger \cdot \rho,\, \mathbf{1}_q \rangle,\, h^\dagger \cdot \rho \rangle \\
&= \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle \cdot \rho,\, h^\dagger \cdot \rho \rangle \\
&= \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle \cdot \rho \\
&= \langle f_1, f_2 \rangle^\dagger \cdot \rho \\
&= f^\dagger \cdot \rho,
\end{aligned}$$

by the scalar pairing identity and the induction hypothesis again.

Remark 6.2.10 In a similar way, if the scalar pairing and scalar parameter identities hold in T, then so does the parameter identity.

Proposition 6.2.11 *Let T be a preiteration theory. If the scalar pairing identity holds in T then the pairing identity holds.*

Proof. Let $f : n \to n + m + p$ and $g : m \to n + m + p$ and define

$$h \;:=\; g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m \to m + p.$$

We must show that

$$\langle f, g \rangle^\dagger \;=\; \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle.$$

When $m = 0$ this is obvious and when $m = 1$ this holds by assumption. Now for the induction step. Supposing $m > 1$, write $g = \langle g_1, g_2 \rangle$ with $g_1 : m - 1 \to n + m + p$ and $g_2 : 1 \to n + m + p$. Define

$$\begin{aligned}
a &:= g_1 \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m - 1 \to m + p \\
b &:= g_2 \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : 1 \to m + p
\end{aligned}$$

and

$$k \;:=\; b \cdot \langle a^\dagger, \mathbf{1}_{1+p} \rangle : 1 \to 1 + p.$$

Then

$$\begin{aligned} h &= \langle g_1, g_2\rangle \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \\ &= \langle a, b\rangle, \end{aligned}$$

so that

$$h^\dagger = \langle a^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger\rangle,$$

by the scalar pairing identity. Thus

$$\langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger\rangle = \langle f^\dagger \cdot \langle a^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger, \mathbf{1}_p\rangle,\, a^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger\rangle. \tag{6.5}$$

On the other hand, by the induction assumption,

$$\begin{aligned} g_2 \cdot \langle \langle f, g_1\rangle^\dagger, \mathbf{1}_{1+p}\rangle &= g_2 \cdot \langle f^\dagger \cdot \langle a^\dagger, \mathbf{1}_{1+p}\rangle,\, a^\dagger,\, \mathbf{1}_{1+p}\rangle \\ &= g_2 \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \cdot \langle a^\dagger, \mathbf{1}_{1+p}\rangle \\ &= b \cdot \langle a^\dagger, \mathbf{1}_{1+p}\rangle \\ &= k. \end{aligned}$$

Thus,

$$\begin{aligned} \langle f, g\rangle^\dagger &= \langle f, g_1, g_2\rangle^\dagger \\ &= \langle \langle f, g_1\rangle^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger\rangle \\ &= \langle \langle f^\dagger \cdot \langle a^\dagger, \mathbf{1}_{1+p}\rangle, a^\dagger\rangle \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger\rangle \\ &= \langle f^\dagger \cdot \langle a^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger, \mathbf{1}_p\rangle,\, a^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger\rangle \\ &= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger\rangle, \end{aligned}$$

by (6.5) and the induction assumption.

Lemma 6.2.12 *Let T be a preiteration theory. If the permutation identity holds in T for some base permutations $\rho : n \to n$ and $\tau : n \to n$, then the permutation identity holds in T for the composite $\rho \cdot \tau$.*

Proof. For any $f : n \to n + p$ we have

$$\begin{aligned} (\rho \cdot \tau \cdot f \cdot ((\rho \cdot \tau)^{-1} \oplus \mathbf{1}_p))^\dagger &= (\rho \cdot \tau \cdot f \cdot (\tau^{-1} \oplus \mathbf{1}_p) \cdot (\rho^{-1} \oplus \mathbf{1}_p))^\dagger \\ &= \rho \cdot (\tau \cdot f \cdot (\tau^{-1} \oplus \mathbf{1}_p))^\dagger \\ &= \rho \cdot \tau \cdot f^\dagger. \end{aligned}$$

Proposition 6.2.13 *Let T be a preiteration theory which satisfies the scalar pairing identity, the scalar base parameter identity and the transposition identity. Then the permutation identity holds in T.*

Proof. Since the symmetric group of degree n of all permutations $[n] \to [n]$ is generated by the transpositions $\rho_i = (i, i+1)$, $i \in [n-1]$, it suffices to show by Lemma 6.2.12, that

$$(\rho_i \cdot f \cdot (\rho_i^{-1} \oplus \mathbf{1}_p))^\dagger \;=\; \rho_i \cdot f^\dagger,$$

for all $f : n \to n+p$ and ρ_i. Our argument uses induction on n. The basis case $n = 2$ holds by assumption. If $n > 2$, two cases arise. If $i \in [n-2]$, then we can write $\rho_i = \tau \oplus \mathbf{1}_1$ and $f = \langle f_1, f_2 \rangle$ for some base transposition $\tau : n-1 \to n-1$ and morphisms $f_1 : n-1 \to n+p$ and $f_2 : 1 \to n+p$. Define

$$h \;:=\; f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{1+p} \rangle.$$

Since

$$(\tau \cdot f_1 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}))^\dagger \;=\; \tau \cdot f_1^\dagger,$$

by the induction assumption,

$$\begin{aligned}
&f_2 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}) \cdot \langle (\tau \cdot f_1 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}))^\dagger,\, \mathbf{1}_{1+p} \rangle \;= \\
&\quad= f_2 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}) \cdot \langle \tau \cdot f_1^\dagger,\, \mathbf{1}_{1+p} \rangle \\
&\quad= f_2 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}) \cdot (\tau \oplus \mathbf{1}_{1+p}) \cdot \langle f_1^\dagger, \mathbf{1}_{1+p} \rangle \\
&\quad= f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{1+p} \rangle \\
&\quad= h.
\end{aligned}$$

Thus

$$\begin{aligned}
(\rho_i \cdot f \cdot (\rho_i^{-1} \oplus \mathbf{1}_p))^\dagger &= ((\tau \oplus \mathbf{1}_1) \cdot \langle f_1, f_2 \rangle \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}))^\dagger \\
&= \langle \tau \cdot f_1 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}),\, f_2 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}) \rangle^\dagger \\
&= \langle (\tau \cdot f_1 \cdot (\tau^{-1} \oplus \mathbf{1}_{1+p}))^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle \\
&= \langle \tau \cdot f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle \\
&= (\tau \oplus \mathbf{1}_1) \cdot \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle \\
&= \rho_i \cdot \langle f_1, f_2 \rangle^\dagger \\
&= \rho_i \cdot f^\dagger,
\end{aligned}$$

by the scalar pairing identity.

If $i = n-1$, write $\rho_i = \mathbf{1}_{n-2} \oplus \tau$, where τ is the base transposition $2 \to 2$, so that $\tau^{-1} = \tau$. Further let $f = \langle f_1, f_2 \rangle$, where $f_1 : n-2 \to n+p$ and $f_2 : 2 \to n+p$. Define

$$h \;:=\; f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{2+p} \rangle.$$

By Proposition 6.2.9, the base parameter identity holds in T. Thus, by the transposition identity,

$$(\tau \cdot f_2 \cdot (\mathbf{1}_{n-2} \oplus \tau \oplus \mathbf{1}_p) \cdot \langle (f_1 \cdot (\mathbf{1}_{n-2} \oplus \tau \oplus \mathbf{1}_p))^\dagger,\, \mathbf{1}_{2+p} \rangle)^\dagger \;=$$

$$
\begin{aligned}
&= (\tau \cdot f_2 \cdot (\mathbf{1}_{n-2} \oplus \tau \oplus \mathbf{1}_p) \cdot \langle f_1^\dagger \cdot (\tau \oplus \mathbf{1}_p),\, \mathbf{1}_{2+p} \rangle)^\dagger \\
&= (\tau \cdot f_2 \cdot \langle f_1^\dagger \cdot (\tau \oplus \mathbf{1}_p),\, \tau \oplus \mathbf{1}_p \rangle)^\dagger \\
&= (\tau \cdot f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{2+p} \rangle \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\
&= \tau \cdot (f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{2+p} \rangle)^\dagger \\
&= \tau \cdot h^\dagger .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(\rho_{n-1} \cdot f \cdot (\rho_{n-1}^{-1} \oplus \mathbf{1}_p))^\dagger &= ((\mathbf{1}_{n-2} \oplus \tau) \cdot \langle f_1, f_2 \rangle \cdot (\mathbf{1}_{n-2} \oplus \tau \oplus \mathbf{1}_p))^\dagger \\
&= \langle f_1 \cdot (\mathbf{1}_{n-2} \oplus \tau \oplus \mathbf{1}_p),\, \tau \cdot f_2 \cdot (\mathbf{1}_{n-2} \oplus \tau \oplus \mathbf{1}_p) \rangle^\dagger \\
&= \langle f_1^\dagger \cdot (\tau \oplus \mathbf{1}_p) \cdot \langle \tau \cdot h^\dagger,\, \mathbf{1}_p \rangle,\, \tau \cdot h^\dagger \rangle \\
&= \langle f_1^\dagger \cdot (\tau \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p) \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, \tau \cdot h^\dagger \rangle \\
&= \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, \tau \cdot h^\dagger \rangle \\
&= (\mathbf{1}_{n-2} \oplus \tau) \cdot \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle \\
&= \rho_{n-1} \cdot f^\dagger ,
\end{aligned}
$$

by the base parameter identity and the pairing identity, which holds in T by Proposition 6.2.11.

Proposition 6.2.14 *Let T be a preiteration theory satisfying the scalar pairing identity. Then the transposition identity holds in T if and only if the following* **scalar transposition identity** *holds:*

$$
f^\dagger \cdot \langle (g \cdot \langle f^\dagger, \mathbf{1}_{1+p} \rangle)^\dagger,\, \mathbf{1}_p \rangle \;=\; (f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p \rangle)^\dagger ,
$$

where $f, g : 1 \to 1 + 1 + p$ and $\rho : 2 \to 2$ is the nontrivial base permutation.

Proof. First suppose that the scalar pairing identity and the transposition identity hold in T. Thus, for $f, g : 1 \to 1 + 1 + p$ in T,

$$
1_2 \cdot \langle f, g \rangle^\dagger \;=\; f^\dagger \cdot \langle (g \cdot \langle f^\dagger, \mathbf{1}_{1+p} \rangle)^\dagger,\, \mathbf{1}_p \rangle ,
$$

by the scalar pairing identity. By the proof of Proposition 5.5.3.12, the right pairing identity holds in T in the case that $n = m = 1$. Thus

$$
1_2 \cdot \langle f, g \rangle^\dagger \;=\; (f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p \rangle)^\dagger ,
$$

proving that the scalar transposition identity holds in T.

Suppose for the converse that T satisfies the scalar pairing identity and the scalar transposition identity. Given $f, g : 1 \to 1 + 1 + p$ in T, define

$$
\begin{aligned}
h &:= g \cdot \langle f^\dagger, \mathbf{1}_{1+p} \rangle \\
k &:= f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p \rangle .
\end{aligned}
$$

Substituting $g \cdot (\rho \oplus \mathbf{1}_p)$ for f and $f \cdot (\rho \oplus \mathbf{1}_p)$ for g in the scalar transposition identity, we obtain

$$(g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle k^\dagger, \mathbf{1}_p \rangle \;=\; h^\dagger .$$

Since

$$f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle \;=\; k^\dagger ,$$

it follows that

$$\begin{aligned} \langle f, g \rangle^\dagger &= \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle,\, h^\dagger \rangle \\ &= \rho \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle k^\dagger, \mathbf{1}_p \rangle,\, k^\dagger \rangle \\ &= \rho \cdot \langle g \cdot (\rho \oplus \mathbf{1}_p),\, f \cdot (\rho \oplus \mathbf{1}_p) \rangle^\dagger \\ &= \rho \cdot (\rho \cdot \langle f, g \rangle \cdot (\rho \oplus \mathbf{1}_p))^\dagger . \end{aligned}$$

Thus the transposition identity holds in T.

The five previous propositions are summarized in the following theorem.

Theorem 6.2.15 *A preiteration theory T is a Conway theory if and only if T satisfies the scalar left zero, the scalar base parameter, the scalar pairing and the transposition (or scalar transposition) identities.*

We note that a similar result holds if the scalar pairing identity is replaced by the scalar right pairing identity.

Corollary 6.2.16 *Suppose that T is a theory with an operation of scalar iteration $f \mapsto f^\dagger$ defined on scalar morphisms $f : 1 \to 1+p$. If this operation satisfies the scalar left zero, base parameter and transposition identities, then there is a unique way to extend the iteration operation to all morphisms $f : n \to n+p$ such that T becomes a Conway theory. Conversely, if such an extension exists, then the scalar iteration satisfies the scalar left zero, base parameter and transposition identities.*

In fact, the extension is forced by the scalar pairing identity. Next we derive a slightly simpler form of the commutative identity.

Proposition 6.2.17 *Let T be a preiteration theory. The commutative identity holds in T if and only if both the permutation identity and the following* **scalar commutative identity** *hold:*

$$1_m \cdot ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \;=\; 1_n \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger , \qquad (6.6)$$

for all $f : n \to m+p$ in T, $n, m \geq 1$, and for all monotone surjective base $\rho : m \to n$ and base $\rho_i : m \to m$, $i \in [m]$, with $\rho_i \cdot \rho = \rho$.

Proof. That the scalar commutative identity is implied by the commutative identity is immediate and we have already noted that the commutative identity implies the permutation identity. Suppose for the converse that the permutation identity and the scalar commutative identity hold in T. We must show that

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \;=\; \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \tag{6.7}$$

where $f : n \to m+p$ in T, $n, m \geq 1$, and where $\rho : m \to n$ is a surjective base morphism and each $\rho_i : m \to m$, $i \in [m]$, is base with $\rho_i \cdot \rho = \rho$. We prove (6.7) first in the case that ρ is monotone. Given an integer $j \in [m]$, there exist base permutations $\pi : m \to m$, $\kappa : n \to n$ and a monotone surjective base $\tau : m \to n$ with $1\pi = j$ and

$$\pi \cdot \rho \;=\; \tau \cdot \kappa.$$

We define

$$g \quad := \quad \kappa \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p)$$

and

$$\tau_i \quad := \quad \pi \cdot \rho_{i\pi} \cdot \pi^{-1},$$

for all $i \in [m]$. Note that

$$i_m \cdot \pi \cdot \rho \cdot f \cdot (\rho_{i\pi} \cdot \pi^{-1} \oplus \mathbf{1}_p) \;=\; i_m \cdot \tau \cdot g \cdot (\tau_i \oplus \mathbf{1}_p) \tag{6.8}$$

and

$$\begin{aligned} \tau_i \cdot \tau &= \pi \cdot \rho_{i\pi} \cdot \pi^{-1} \cdot \pi \cdot \rho \cdot \kappa^{-1} \\ &= \pi \cdot \rho_{i\pi} \cdot \rho \cdot \kappa^{-1} \\ &= \pi \cdot \rho \cdot \kappa^{-1} \\ &= \tau, \end{aligned}$$

for all $i \in [m]$. Thus,

$$\begin{aligned} j_m \cdot ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger &= 1_m \cdot \pi \cdot ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \\ &= 1_m \cdot (\pi \cdot (\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m) \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger \\ &= 1_m \cdot ((\pi \cdot \rho \cdot f) \parallel (\rho_{1\pi} \cdot \pi^{-1}, \ldots, \rho_{m\pi} \cdot \pi^{-1}))^\dagger \\ &= 1_m \cdot ((\tau \cdot g) \parallel (\tau_1, \ldots, \tau_m))^\dagger, \end{aligned}$$

by (6.8),

$$\begin{aligned} &= 1_m \cdot \tau \cdot (g \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\ &= 1_m \cdot \pi \cdot \rho \cdot \kappa^{-1} \cdot (\kappa \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot (\pi \oplus \mathbf{1}_p) \cdot (\rho \oplus \mathbf{1}_p) \cdot (\kappa^{-1} \oplus \mathbf{1}_p))^\dagger \\ &= 1_m \cdot \pi \cdot \rho \cdot \kappa^{-1} \cdot (\kappa \cdot f \cdot (\rho \oplus \mathbf{1}_p) \cdot (\kappa^{-1} \oplus \mathbf{1}_p))^\dagger \\ &= 1_m \cdot \pi \cdot \rho \cdot \kappa^{-1} \cdot \kappa \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger \\ &= j_m \cdot \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \end{aligned}$$

by the permutation and scalar commutative identities.

When ρ is not monotone we can write $\rho = \pi \cdot \tau$ where $\pi : m \to m$ is a base permutation and $\tau : m \to n$ is a monotone base surjection. Define

$$g \quad := \quad f \cdot (\pi \oplus \mathbf{1}_p)$$

and

$$\tau_i \quad := \quad \pi^{-1} \cdot \rho_{i\pi^{-1}} \cdot \pi,$$

for all $i \in [m]$. Thus

$$\begin{aligned} \tau_i \cdot \tau &= \pi^{-1} \cdot \rho_{i\pi^{-1}} \cdot \pi \cdot \tau \\ &= \pi^{-1} \cdot \rho_{i\pi^{-1}} \cdot \rho \\ &= \pi^{-1} \cdot \rho \\ &= \tau. \end{aligned}$$

By the previous case,

$$\begin{aligned} ((\tau \cdot g) \parallel (\tau_1, \ldots, \tau_m))^\dagger &= \tau \cdot (g \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\ &= \tau \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger. \end{aligned}$$

Thus,

$$\begin{aligned} ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger &= (\pi \cdot (\tau \cdot g) \parallel (\tau_1, \ldots, \tau_m) \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger \\ &= \pi \cdot ((\tau \cdot g) \parallel (\tau_1, \ldots, \tau_m))^\dagger \\ &= \pi \cdot \tau \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger \\ &= \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger. \end{aligned}$$

Remark 6.2.18 As in the proof of Proposition 5.5.3.26, if the permutation identity holds in T, the scalar commutative identity (6.6) reduces to either of the two subcases that each ρ_i is a base permutation or that each is an aperiodic base morphism.

By combining Proposition 6.2.17 with Theorem 6.2.15 we obtain the following result.

Corollary 6.2.19 *A preiteration theory T is an iteration theory if and only if the scalar left zero, scalar base parameter, scalar pairing, (scalar) transposition and scalar commutative identities hold in T.*

We give one more axiomatization of Conway theories based on scalar iteration.

Theorem 6.2.20 *Let T be a preiteration theory. Then T is a Conway theory if and only if T satisfies the scalar parameter, scalar composition, scalar double dagger and scalar pairing identities.*

Proof. Suppose that the scalar parameter, scalar composition, scalar double dagger and scalar pairing identities hold in T. We have noted in the proof of Corollary 6.2.5 that the composition identity implies the fixed point identity. In a similar way, it follows that the scalar fixed point identity and hence the scalar left zero identity holds in T. By Theorem 6.2.15, to complete the proof it suffices to show that the scalar transposition identity holds in T. Let $f, g : 1 \to 1 + 1 + p$ be morphisms in T. First we prove that

$$(g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger \;=\; ((g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle f^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger, \qquad (6.9)$$

where $\rho : 2 \to 2$ is the nontrivial base permutation. We use the scalar parameter identity and the scalar double dagger identity.

$$\begin{aligned}
&((g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle f^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \;= \\
&\quad= \;(g \cdot (\rho \oplus \mathbf{1}_p) \cdot (\mathbf{1}_1 \oplus \langle f^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle))^{\dagger\dagger} \\
&\quad= \;(g \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_1 \oplus 0_{1+p},\, 0_1 \oplus f^\dagger,\, 0_2 \oplus \mathbf{1}_p\rangle)^{\dagger\dagger} \\
&\quad= \;(g \cdot \langle 0_1 \oplus f^\dagger,\, \mathbf{1}_1 \oplus 0_{1+p},\, 0_2 \oplus \mathbf{1}_p\rangle)^{\dagger\dagger} \\
&\quad= \;(g \cdot \langle 0_1 \oplus f^\dagger,\, \mathbf{1}_1 \oplus 0_{1+p},\, 0_2 \oplus \mathbf{1}_p\rangle \cdot (\langle \mathbf{1}_1, \mathbf{1}_1\rangle \oplus \mathbf{1}_p))^\dagger \\
&\quad= \;(g \cdot \langle 0_1 \oplus f^\dagger,\, \mathbf{1}_1 \oplus 0_{1+p},\, 0_2 \oplus \mathbf{1}_p\rangle \cdot \langle \mathbf{1}_1 \oplus 0_p,\, \mathbf{1}_1 \oplus 0_p,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \\
&\quad= \;(g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger.
\end{aligned}$$

Next, by the scalar composition identity,

$$f^\dagger \cdot \langle ((g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle f^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger,\, \mathbf{1}_p\rangle \;=\; (f^\dagger \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger. \quad (6.10)$$

Finally, we observe that

$$(f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \;=\; (f^\dagger \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger. \quad (6.11)$$

Indeed, by the scalar parameter identity and the scalar double dagger identity,

$$\begin{aligned}
&(f^\dagger \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \;= \\
&\quad= \;(f \cdot (\mathbf{1}_1 \oplus \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle))^{\dagger\dagger} \\
&\quad= \;(f \cdot \langle \mathbf{1}_1 \oplus 0_{1+p},\, 0_1 \oplus (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_2 \oplus \mathbf{1}_p\rangle)^{\dagger\dagger} \\
&\quad= \;(f \cdot \langle \mathbf{1}_1 \oplus 0_{1+p},\, 0_1 \oplus (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_2 \oplus \mathbf{1}_p\rangle \cdot (\langle \mathbf{1}_1, \mathbf{1}_1\rangle \oplus \mathbf{1}_p))^\dagger \\
&\quad= \;(f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger.
\end{aligned}$$

The proof is now easily completed. By (6.9)-(6.11),

$$\begin{aligned}
&(f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \;= \\
&\quad= \;(f^\dagger \cdot \langle (g \cdot (\rho \oplus \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger \\
&\quad= \;f^\dagger \cdot \langle ((g \cdot (\rho \oplus \mathbf{1}_p))^\dagger \cdot \langle f^\dagger,\, 0_1 \oplus \mathbf{1}_p\rangle)^\dagger,\, \mathbf{1}_p\rangle \\
&\quad= \;f^\dagger \cdot \langle (g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger,\, \mathbf{1}_p\rangle.
\end{aligned}$$

As a corollary, we obtain a final scalar axiomatization of iteration theories.

Corollary 6.2.21 *A preiteration theory T is an iteration theory if and only if T satisfies the scalar parameter, scalar composition, scalar double dagger, scalar pairing and scalar commutative identities.*

Corollary 6.2.22 *Suppose that T is a theory with an operation of iteration defined on scalar morphisms. If the scalar parameter, scalar composition and scalar double dagger identities hold, then there is a unique way to define iteration on all of T such that T becomes a Conway theory. Conversely, when such an extension to a Conway theory exists, then the scalar parameter, composition and double dagger identities hold in T.*

6.3 Theories with a Functorial Dagger

In this section we consider the functorial dagger implication mentioned in Section 5.5.3. We use the following alternative terminology.

Definition 6.3.1 *Let T be a preiteration theory and let be a set of morphisms in T. We say that T* **satisfies the functorial dagger implication for** *, or that T* **has a functorial dagger with respect to** *, if*

$$f \cdot (h \oplus \mathbf{1}_p) \;=\; h \cdot g \quad \Rightarrow \quad f^\dagger \;=\; h \cdot g^\dagger,$$

for all $f : n \to n+p$, $g : m \to m+p$ in T and $h : n \to m$ in . If T satisfies the functorial dagger implication for the set of all morphisms in T, then we say that T **satisfies the strong functorial dagger implication** *or that T* **has a strong functorial dagger**.

We note a necessary condition that a preiteration theory has a strong functorial dagger.

Proposition 6.3.2 *If a preiteration theory T has a strong functorial dagger then there is a unique morphism $1 \to 0$ in T.*

Proof. For any morphism $h : 1 \to 0$ in T we have

$$h \;=\; \mathbf{1}_1 \cdot h \;=\; h \cdot \mathbf{1}_0.$$

Hence, if T has a strong functorial dagger,

$$\mathbf{1}_1^\dagger \;=\; h \cdot \mathbf{1}_0^\dagger \;=\; h,$$

since $\mathbf{1}_0^\dagger : 0 \to 0$ is necessarily $\mathbf{1}_0$, the identity. Thus $\mathbf{1}_1^\dagger$ is the only morphism $1 \to 0$.

Exercise 6.3.3 [Man92] A morphism $h : n \to m$ in a preiteration theory T is called *pure* if $h \cdot \perp_{m0} = \perp_{n0}$. (Recall that $\perp_{n0}$ is an abbreviation for $\mathbf{1}_n^\dagger$.) Show that if T satisfies the functorial dagger implication for h, then h is a pure morphism. Use this fact to prove Proposition 6.3.2.

From Proposition 5.5.3.24 and the proof of Lemma 5.5.3.6 we directly obtain the following results. Let T be a preiteration theory.

Proposition 6.3.4 *If the pairing identity and the permutation identity hold in T then T satisfies the functorial dagger implication for injective base morphisms. Thus any Conway theory satisfies the functorial dagger implication for injective base morphisms.*

Proposition 6.3.5 *If T satisfies the functorial dagger implication for surjective base morphisms, then the commutative identity holds in T.*

Proposition 6.3.6 *If T satisfies the functorial dagger implication for all injective base morphisms and for all surjective base morphisms, then T has a functorial dagger with respect to all base morphisms.*

Proof. Suppose that

$$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g, \tag{6.12}$$

for some $f : n \to n + p$ and $g : m \to m + p$ in T and some base $\rho : n \to m$. If $n = 0$ then $f^\dagger = \rho \cdot g^\dagger = 0_p$. Suppose that $n > 0$. First we treat the case that the range of ρ is the set $[k]$, for some $k \in [m]$. Thus

$$\rho \;=\; \tau \oplus 0_{m-k} \;=\; \tau \cdot (\mathbf{1}_k \oplus 0_{m-k})$$

for a surjective base morphism $\tau : n \to k$. It follows from (6.12) that

$$g \;=\; \langle g_1 \cdot (\mathbf{1}_k \oplus 0_{m-k} \oplus \mathbf{1}_p),\, g_2 \rangle, \tag{6.13}$$

where $g_1 : k \to k+p$ and $g_2 : m-k \to m+p$. Further,

$$f \cdot (\tau \oplus 0_{m-k} \oplus \mathbf{1}_p) \;=\; \tau \cdot g_1 \cdot (\mathbf{1}_k \oplus 0_{m-k} \oplus \mathbf{1}_p),$$

so that

$$f \cdot (\tau \oplus \mathbf{1}_p) \;=\; \tau \cdot g_1.$$

Since T satisfies the functorial dagger implication for surjective base morphisms,

$$f^\dagger \;=\; \tau \cdot g_1^\dagger. \tag{6.14}$$

By (6.13), also

$$g_1 \cdot (\mathbf{1}_k \oplus 0_{m-k} \oplus \mathbf{1}_p) \;=\; (\mathbf{1}_k \oplus 0_{m-k}) \cdot g.$$

Thus,

$$g_1^\dagger \;=\; (\mathbf{1}_k \oplus 0_{m-k}) \cdot g^\dagger, \tag{6.15}$$

since T satisfies the functorial dagger implication for injective base morphisms. Thus,

$$\begin{aligned} f^\dagger &= \tau \cdot g_1^\dagger \\ &= \tau \cdot (\mathbf{1}_k \oplus 0_{m-k}) \cdot g^\dagger \\ &= \rho \cdot g^\dagger, \end{aligned}$$

by (6.14) and (6.15).

To complete the proof note that if (6.12) holds then there is a base permutation $\pi : m \to m$ such that the range of $\rho \cdot \pi$ is $[k]$, for some $k \in [m]$. Thus,

$$\begin{aligned} f \cdot (\rho \cdot \pi \oplus \mathbf{1}_p) &= f \cdot (\rho \oplus \mathbf{1}_p) \cdot (\pi \oplus \mathbf{1}_p) \\ &= \rho \cdot g \cdot (\pi \oplus \mathbf{1}_p) \\ &= \rho \cdot \pi \cdot \pi^{-1} \cdot g \cdot (\pi \oplus \mathbf{1}_p), \end{aligned}$$

and by the previous case and the permutation identity we have

$$\begin{aligned} f^\dagger &= \rho \cdot \pi \cdot (\pi^{-1} \cdot g \cdot (\pi \oplus \mathbf{1}_p))^\dagger \\ &= \rho \cdot \pi \cdot \pi^{-1} \cdot g^\dagger \\ &= \rho \cdot g^\dagger. \end{aligned}$$

Since the permutation identity is implied by either the assumption that T has a functorial dagger with respect to all injective base morphisms or the assumption that T has a functorial dagger with respect to all surjective base morphisms, the proof is complete.

Proposition 6.3.7 *Let T be a preiteration theory such that the base parameter identity, the permutation identity and the pairing identity hold. Then T has a functorial dagger with respect to all surjective base morphisms if and only if T has a functorial dagger with respect to all base morphisms $n \to 1$, $n \geq 1$.*

Proof. Suppose that T has a functorial dagger with respect to the base morphisms $n \to 1$, $n \geq 1$. Assume that

$$f \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot g \tag{6.16}$$

for some $f : n \to n + p$ and $g : m \to m + p$ in T and for a *monotone* surjective base morphism $\rho : n \to m$. We use induction on m to show $f^\dagger = \rho \cdot g^\dagger$. When $m = 0$ this is obvious and the basis case $m = 1$ holds by assumption. When $m > 1$, we can write $n = n_1 + n_2$ and

$$\begin{aligned}
f &= \langle f_1, f_2 \rangle, && f_1 : n_1 \to n + p,\ f_2 : n_2 \to n + p\\
g &= \langle g_1, g_2 \rangle, && g_1 : m - 1 \to m + p,\ g_2 : 1 \to m + p\\
\rho &= \rho_1 \oplus \rho_2, && \rho_1 : n_1 \to m - 1,\ \rho_2 : n_2 \to 1,
\end{aligned}$$

where ρ_1 and ρ_2 are monotone surjective base morphisms with

$$f_i \cdot (\rho_1 \oplus \rho_2 \oplus \mathbf{1}_p) \;=\; \rho_i \cdot g_i, \quad i = 1, 2. \tag{6.17}$$

The induction hypothesis and the base parameter identity yield

$$f_1^\dagger \cdot (\rho_2 \oplus \mathbf{1}_p) \;=\; \rho_1 \cdot g_1^\dagger. \tag{6.18}$$

Now let $h := f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{n_2+p} \rangle : n_2 \to n_2 + p$ and $k := g_2 \cdot \langle g_1^\dagger, \mathbf{1}_{1+p} \rangle : 1 \to 1 + p$. We have

$$h \cdot (\rho_2 \oplus \mathbf{1}_p) \;=\; \rho_2 \cdot k. \tag{6.19}$$

Indeed,

$$\begin{aligned}
h \cdot (\rho_2 \oplus \mathbf{1}_p) &= f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{n_2+p} \rangle \cdot (\rho_2 \oplus \mathbf{1}_p)\\
&= f_2 \cdot \langle f_1^\dagger \cdot (\rho_2 \oplus \mathbf{1}_p),\, \rho_2 \oplus \mathbf{1}_p \rangle\\
&= f_2 \cdot \langle \rho_1 \cdot g_1^\dagger,\, \rho_2 \oplus \mathbf{1}_p \rangle,
\end{aligned}$$

by (6.18),

$$\begin{aligned}
&= f_2 \cdot (\rho_1 \oplus \rho_2 \oplus \mathbf{1}_p) \cdot \langle g_1^\dagger, \mathbf{1}_{1+p} \rangle\\
&= \rho_2 \cdot g_2 \cdot \langle g_1^\dagger, \mathbf{1}_{1+p} \rangle\\
&= \rho_2 \cdot k,
\end{aligned}$$

by (6.17). From (6.19), by the induction hypothesis, we obtain

$$h^\dagger \;=\; \rho_2 \cdot k^\dagger. \tag{6.20}$$

The proof is completed by using the pairing identity.

$$\begin{aligned} f^\dagger &= \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger\rangle \\ &= \langle f_1^\dagger \cdot \langle \rho_2 \cdot k^\dagger, \mathbf{1}_p\rangle,\, \rho_2 \cdot k^\dagger\rangle, \end{aligned}$$

by (6.20),

$$\begin{aligned} &= \langle f_1^\dagger \cdot (\rho_2 \oplus \mathbf{1}_p) \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, \rho_2 \cdot k^\dagger\rangle \\ &= \langle \rho_1 \cdot g_1^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, \rho_2 \cdot k^\dagger\rangle, \end{aligned}$$

by (6.18),

$$\begin{aligned} &= (\rho_1 \oplus \rho_2) \cdot \langle g_1^\dagger \cdot \langle k^\dagger, \mathbf{1}_p\rangle,\, k^\dagger\rangle \\ &= \rho \cdot g^\dagger. \end{aligned}$$

Suppose now that (6.16) holds for a surjective base morphism ρ which is not necessarily monotone. There is a base permutation $\pi : n \to n$ such that $\pi \cdot \rho$ is monotone. Define

$$f_\pi \;:=\; \pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p).$$

Since

$$\begin{aligned} f_\pi \cdot (\pi \cdot \rho \oplus \mathbf{1}_p) &= \pi \cdot f \cdot (\rho \oplus \mathbf{1}_p) \\ &= \pi \cdot \rho \cdot g, \end{aligned}$$

thus

$$\pi \cdot f^\dagger \;=\; f_\pi^\dagger \;=\; \pi \cdot \rho \cdot g^\dagger,$$

by the previous case and the permutation identity. Hence $f^\dagger = \rho \cdot g^\dagger$.

We note that in the previous proposition we may replace the pairing identity by either the right pairing identity or the symmetric pairing identity.

Corollary 6.3.8 *A preiteration theory T is an iteration theory satisfying the functorial dagger implication for all (surjective) base morphisms if and only if T is a Conway theory and T has a functorial dagger with respect to all base morphisms $\rho : n \to 1$, $n \geq 1$.*

We end this section by providing a simple sufficient condition that a Conway theory has a functorial dagger with respect to all (surjective) base morphisms.

Definition 6.3.9 *Let T be a preiteration theory. We say that T satisfies the* **GA-implication***, or that* **the GA-implication holds in** *T, if for all $f, g : 1 \to 2 + p$,*

$$f^{\dagger\dagger} \;=\; g^{\dagger\dagger} \quad\Rightarrow\quad (g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger \;=\; f^{\dagger\dagger}.$$

Remark 6.3.10 When the scalar double dagger identity holds in T, T satisfies the GA-implication iff

$$\begin{aligned} &(f \cdot (\tau \oplus \mathbf{1}_p))^\dagger \;=\; (g \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\ &\quad\Rightarrow\quad (g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger \;=\; (f \cdot (\tau \oplus \mathbf{1}_p))^\dagger, \end{aligned} \tag{6.21}$$

for all $f, g : 1 \to 2 + p$ in T, where τ is the base morphism $2 \to 1$.

Proposition 6.3.11 *Suppose that T is a Conway theory which satisfies the GA-implication. Then T is an iteration theory which has a functorial dagger with respect to all (surjective) base morphisms.*

Proof. For every $m \geq 1$, let τ_m denote the base morphism $m \to 1$, so that τ_2 is the morphism τ defined just above in Remark 6.3.10. First we prove the following:

CLAIM. If

$$(f \cdot (\tau_{1+m} \oplus \mathbf{1}_p))^\dagger \;=\; (g \cdot (\tau_{1+m} \oplus \mathbf{1}_p))^\dagger \;=\; h^\dagger,$$

for some morphisms $f, g : 1 \to 1 + m + p$ and $h : 1 \to 1 + p$, $m \geq 1$, then

$$(g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \cdot (\tau_m \oplus \mathbf{1}_p))^\dagger \;=\; h^\dagger.$$

Indeed, since by assumption

$$(f \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p))^\dagger \;=\; (f \cdot (\tau_{1+m} \oplus \mathbf{1}_p))^\dagger \;=\; h^\dagger$$

and

$$(g \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p))^\dagger \;=\; h^\dagger,$$

it follows from (6.21), by substituting $f \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p)$ for f and $g \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p)$ for g, and from the parameter identity that

$$\begin{aligned}
&(g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \cdot (\tau_m \oplus \mathbf{1}_p))^\dagger \;= \\
&\quad= \;(g \cdot \langle f^\dagger \cdot (\tau_m \oplus \mathbf{1}_p),\, \tau_m \oplus \mathbf{1}_p\rangle)^\dagger \\
&\quad= \;(g \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p) \cdot \langle (f \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p))^\dagger,\, \mathbf{1}_{1+p}\rangle)^\dagger \\
&\quad= \;(f \cdot (\mathbf{1}_1 \oplus \tau_m \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\
&\quad= \;h^\dagger.
\end{aligned}$$

The claim is proved.

Suppose now that $f : n \to n + p$, $n \geq 1$, and $g : 1 \to 1 + p$ are T-morphisms with $f \cdot (\tau_n \oplus \mathbf{1}_p) = \tau_n \cdot g$. We will show that $f^\dagger = \tau_n \cdot g^\dagger$. The case $n = 1$ is obvious, so we assume that $n > 1$. We define

$$\begin{aligned}
h_n \;&:=\; f : n \to n + p, \\
h_j \;&:=\; (0_1 \oplus \mathbf{1}_j) \cdot h_{1+j} \cdot \langle (1_{1+j} \cdot h_{1+j})^\dagger,\, \mathbf{1}_{j+p}\rangle : j \to j + p,
\end{aligned}$$

for all $1 \leq j < n$. Since by assumption $h_n \cdot (\tau_n \oplus \mathbf{1}_p) = \tau_n \cdot g$, also $(i_n \cdot h_n \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger = g^\dagger$, for all $i \in [n]$. It follows by a straightforward induction that

$$(i_j \cdot h_j \cdot (\tau_j \oplus \mathbf{1}_p))^\dagger \;=\; g^\dagger,$$

for all $i \in [j]$, $1 \leq j \leq n$. In particular, $h_1^\dagger = g^\dagger$. Indeed, if

$$(i_{1+j} \cdot h_{1+j} \cdot (\tau_{1+j} \oplus \mathbf{1}_p))^\dagger \;=\; g^\dagger,$$

for all $i \in [1 + j]$, where j is some integer $1 \leq j < n$, then for any $i \in [j]$, by the Claim,

$$\begin{aligned}
&(i_j \cdot h_j \cdot (\tau_j \oplus \mathbf{1}_p))^\dagger \;= \\
&\quad= \;((1 + i)_{1+j} \cdot h_{1+j} \cdot \langle (1_{1+j} \cdot h_{1+j})^\dagger, \mathbf{1}_{j+p}\rangle \cdot (\tau_j \oplus \mathbf{1}_p))^\dagger \\
&\quad= \;g^\dagger.
\end{aligned}$$

Also, by the pairing identity, it follows that

$$j_j \cdot h_j^\dagger \;=\; n_n \cdot f^\dagger,$$

for all $j = n, n-1, \ldots, 1$. Thus $n_n \cdot f^\dagger = h_1^\dagger = g^\dagger$, showing that the last component of $f^\dagger$ is $g^\dagger$. It follows from the permutation identity that $f^\dagger = \tau_n \cdot g^\dagger$, so that T has a functorial dagger with respect to all base morphisms $n \to 1$. The proof is completed by using Corollary 6.3.8.

Corollary 6.3.12 *Suppose that T is a preiteration theory. If the implication (6.21) as well as the scalar parameter, scalar composition and scalar pairing identities hold in T, then T is an iteration theory which satisfies the functorial dagger implication for all base morphisms.*

Corollary 6.3.12 follows from Proposition 6.3.11, Theorem 6.2.20 and the following lemma.

Lemma 6.3.13 *If the scalar left zero and scalar base parameter identities hold in a preiteration theory T, and if the implication (6.21) holds, then the scalar fixed point and scalar double dagger identities also hold.*

Proof. Suppose that $f : 1 \to 1 + p$ in T. We have, by (6.21),

$$\begin{aligned}(f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p) \cdot \langle (f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p))^\dagger,\, \mathbf{1}_{1+p} \rangle)^\dagger &= \\ = (f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p))^\dagger.\end{aligned}$$

But

$$\begin{aligned} f \cdot \langle f^\dagger, \mathbf{1}_p \rangle &= (0_1 \oplus f \cdot \langle f^\dagger, \mathbf{1}_p \rangle)^\dagger \\ &= (f \cdot \langle 0_1 \oplus f^\dagger,\, 0_1 \oplus \mathbf{1}_p \rangle)^\dagger \\ &= (f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p) \cdot \langle 0_1 \oplus f^\dagger,\, \mathbf{1}_{1+p} \rangle)^\dagger \\ &= (f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p) \cdot \langle (f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p))^\dagger,\, \mathbf{1}_{1+p} \rangle)^\dagger, \end{aligned}$$

and

$$f^\dagger = (f \cdot (\mathbf{1}_1 \oplus 0_1 \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p))^\dagger.$$

This proves the scalar fixed point identity. Now we prove the scalar double dagger identity. Suppose that $f : 1 \to 2 + p$ in T. Then,

$$f^{\dagger\dagger} = (f \cdot \langle f^\dagger, \mathbf{1}_{1+p} \rangle)^\dagger = (f \cdot (\tau \oplus \mathbf{1}_p))^\dagger,$$

by (6.21) and the scalar fixed point identity.

Exercise 6.3.14 [Ési88] Show that there is an iteration theory which does not satisfy the functorial dagger implication for the base morphism $2 \to 1$.

Exercise 6.3.15 [Ési88] Prove that for each prime number q, there is a Conway theory which has a functorial dagger with respect to all base morphisms $n \to 1$, $n < q$, but does not satisfy the functorial dagger implication for the base morphism $q \to 1$.

Exercise 6.3.16 Let Z be any infinite subset of the set $\mathbf{N}$ of natural numbers. Prove that if a Conway theory T has a functorial dagger with respect to the base morphisms $n \to 1$, for all $n \in Z$, then T has a functorial dagger with respect to all base morphisms. Conclude that in this case, T is an iteration theory.

Exercise 6.3.17 Show that there exists an iteration theory which has a functorial dagger with respect to all base morphisms but does not satisfy the GA-implication.

6.4 Pointed Iterative Theories

If T is an iterative theory, the dagger operation is only defined on ideal morphisms $n \to n + p$. We wish to extend the operation to all morphisms $n \to n + p$ in such a way that the resulting preiteration theory becomes an iteration theory. It turns out that we need only define the dagger operation on $\mathbf{1}_1$. The remaining values are forced, as will be seen.

Proposition 6.4.1 *Let $\perp$ be any fixed morphism $1 \to 0$ in an iterative theory T. There is a unique operation taking $f : n \to n + p$ to $f^\dagger : n \to p$, defined for all f, such that*

- $\mathbf{1}_1^\dagger = \perp$*;*
- *with the total dagger operation, T is a Conway theory.*

Proof. We show how to define $f^\dagger$ for each morphism $f : n \to n + p$ by induction on n. The case $n = 0$ is trivial, $0_p^\dagger := 0_p$, since there is a unique morphism $0 \to p$, for each p. Suppose now that $n = 1$.

If $f : 1 \to 1 + p$ is ideal, the value of $f^\dagger$ is forced by the fixed point identity. Otherwise, $f = i_{1+p}$, for some $i \in [1 + p]$. There are two cases: $i = 1$ and $i > 1$. If $i = 1$ then $f = \mathbf{1}_1 \oplus 0_p$, so that

$$(\mathbf{1}_1 \oplus 0_p)^\dagger \;=\; \mathbf{1}_1^\dagger \oplus 0_p \;=\; \perp \oplus 0_p,$$

by the right zero identity. In the case $i > 1$, the value of $f^\dagger$ is forced by the left zero identity. Indeed, for $j := i - 1$,

$$i_{1+p}^\dagger \;=\; (0_1 \oplus j_p)^\dagger \;=\; j_p.$$

Thus the Conway theory identities force the definition of $f^\dagger$ on all scalar morphisms. When $f : n \to n + p$ with $n > 1$, the definition of $f^\dagger$ is forced by the scalar pairing identity. The definition of the operation is complete. We still have to show that T is a Conway theory. Indeed, when f is ideal, the scalar parameter identity

$$(f \cdot (\mathbf{1}_1 \oplus g))^\dagger \;=\; f^\dagger \cdot g, \quad f : 1 \to 1 + p,\; g : p \to q \tag{6.22}$$

holds in T by Theorem 5.5.3.1. When $f = 1_{1+p}$, both sides of (6.22) evaluate to $\perp \oplus 0_q$, and when $f = (1 + j)_{1+p}$, for some $j \in [p]$, both sides of (6.22) evaluate to $j_p \cdot g$. Thus the scalar parameter identity holds in T. The scalar left zero identity

$$(0_1 \oplus f)^\dagger \;=\; f, \quad f : 1 \to p$$

also holds in T, either by Theorem 5.5.3.1 or by construction. Since the scalar pairing identity holds in T by definition, to complete the proof that T is a Conway theory it suffices to verify by Theorem 6.2.15 that the scalar transposition identity

$$\begin{aligned} & f^\dagger \cdot \langle (g \cdot \langle f^\dagger, \mathbf{1}_{1+p} \rangle)^\dagger,\, \mathbf{1}_p \rangle \;= \\ & \quad = \;(f \cdot \langle \mathbf{1}_1 \oplus 0_p,\, (g \cdot (\rho \oplus \pi \mathbf{1}_p))^\dagger,\, 0_1 \oplus \mathbf{1}_p \rangle)^\dagger \end{aligned} \tag{6.23}$$

holds in T, for all $f, g : 1 \to 1 + 1 + p$. There are several cases. When both f and g are ideal, (6.23) holds by Theorem 5.5.3.3 and the proof of Proposition 6.2.14. When $f = 1_{2+p}$ both sides of (6.23) evaluate to $\perp \oplus 0_p$, and when $f = 2_{2+p}$ both sides are equal to $g^{\dagger\dagger}$. When $f = (2 + j)_{2+p}$, for some $j \in [p]$, both sides of (6.23) evaluate to j_p. The case that g is a distinguished morphism is similar.

The fixed point identity holds in the resulting Conway theory. Thus, for any ideal morphism $f : n \to n + p$, $f^\dagger$ is the unique solution of the iteration equation for f.

Proposition 6.4.2 *Suppose that T is an iterative theory. Then the GA-implication holds in the Conway theory $(T,\, \mathbf{1}_1^\dagger = \perp)$, constructed above.*

Proof. Suppose that $f^{\dagger\dagger} = g^{\dagger\dagger}$, for some $f, g : 1 \to 2 + p$ in T. We must show that

$$(g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger \;=\; f^{\dagger\dagger}.$$

There are several cases.

Case 1. g is ideal. In this case we show that $f^{\dagger\dagger}$ is a solution of the iteration equation for $h := g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle$. Since h is ideal also, the result follows by the uniqueness of the solution. But, denoting the base morphism $2 \to 1$ by τ,

$$\begin{aligned}
g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle \cdot \langle f^{\dagger\dagger}, \mathbf{1}_p\rangle &= g \cdot \langle f^\dagger \cdot \langle f^{\dagger\dagger}, \mathbf{1}_p\rangle,\, f^{\dagger\dagger}, \mathbf{1}_p\rangle \\
&= g \cdot \langle f^{\dagger\dagger}, f^{\dagger\dagger}, \mathbf{1}_p\rangle \\
&= g \cdot \langle g^{\dagger\dagger}, g^{\dagger\dagger}, \mathbf{1}_p\rangle \\
&= g \cdot (\tau \oplus \mathbf{1}_p) \cdot \langle (g \cdot (\tau \oplus \mathbf{1}_p))^\dagger,\, \mathbf{1}_p\rangle \\
&= (g \cdot (\tau \oplus \mathbf{1}_p))^\dagger \\
&= g^{\dagger\dagger} \\
&= f^{\dagger\dagger},
\end{aligned}$$

by the fixed point and double dagger identities.

Case 2. $g = 0_2 \oplus i_p$, for some $i \in [p]$. In this case $f = g$, and

$$(g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger \;=\; i_p \;=\; f^{\dagger\dagger}.$$

Case 3. $g = 1_{2+p}$. Then $(g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger = f^{\dagger\dagger}$ holds obviously.

Case 4. $g = 2_{2+p}$. Now

$$(g \cdot \langle f^\dagger, \mathbf{1}_{1+p}\rangle)^\dagger \;=\; \perp \oplus\, 0_p \;=\; f^{\dagger\dagger}.$$

Corollary 6.4.3 *Let T be an iterative theory. The Conway theory $(T,\, \mathbf{1}_1^\dagger = \perp)$ has a functorial dagger with respect to all base morphisms.*

Proof. Immediate from Proposition 6.4.2 and Proposition 6.3.11.

Exercise 6.4.4 Give a direct proof of Corollary 6.4.3.

We summarize the previous results in the following statement.

Theorem 6.4.5 *Let T be an iterative theory and let $\perp : 1 \to 0$ be a fixed morphism in T. There is a unique way to extend the iteration operation to all morphisms $f : n \to n + p$ in T so that the resulting structure is a Conway theory with $\mathbf{1}_1^\dagger = \perp$. The unique extension $(T, \mathbf{1}_1^\dagger = \perp)$ is an iteration theory satisfying the GA-implication and thus the functorial dagger implication for all (surjective) base morphisms.*

Definition 6.4.6 *We call the iteration theories $(T, \mathbf{1}_1^\dagger = \perp)$ obtained from iterative theories* **pointed iterative theories**. *A morphism $\varphi : (T, \mathbf{1}_1^\dagger = \perp) \to (T', \mathbf{1}_1^\dagger = \perp')$ between pointed iterative theories is an iterative theory morphism $T \to T'$ which is a pointed theory morphism, i.e. $\perp\varphi = \perp'$.*

Exercise 6.4.7 Suppose that φ is a pointed iterative theory morphism

$$(T, \mathbf{1}_1^\dagger = \perp) \to (T', \mathbf{1}_1^\dagger = \perp').$$

Show that φ preserves the dagger operation, so that φ is an iteration theory morphism.

Example 6.4.8 As long as Σ is nonempty, the previous theorem applies to the tree theories $\Sigma\,\mathbf{TR}$ and Σtr, yielding pointed iterative theories $(\Sigma\,\mathbf{TR}, \mathbf{1}_1^\dagger = \perp)$ and $(\Sigma tr, \mathbf{1}_1^\dagger = \perp)$, for each tree $\perp : 1 \to 0$ (respectively, each regular tree $\perp : 1 \to 0$). If Σ is empty, there is no morphism $1 \to 0$ in $\Sigma\,\mathbf{TR}$.

Example 6.4.9 Recall from Example 5.5.1.14 the definition of the theory . By Example 5.5.2.9, is an iterative theory. Let $\perp$ denote the unique partial function $1 \to 0$. Then $(, \mathbf{1}_1^\dagger = \perp)$ is a pointed iterative theory, hence an iteration theory. In fact, the iteration theory determined by the pointed iterative theory $(, \mathbf{1}_1^\dagger = \perp)$ is the unique iteration (or Conway) theory structure on the theory . Hence we may denote the pointed iterative theory and the corresponding iteration theory just by .

Example 6.4.10 Let T be a trivial theory with a morphism $1 \to 0$. There is a unique way to turn T into a preiteration theory. The resulting preiteration theory is an iteration theory and a pointed iterative theory.

We end this section by showing that for every iterative theory T there is a pointed iterative theory freely generated by T. Notice that each pointed iterative theory morphism $(R, \mathbf{1}_1^\dagger = \perp) \to (R', \mathbf{1}_1^\dagger = \perp')$ is both an iterative and an iteration theory morphism $R \to R'$, by Exercise 6.4.7.

Theorem 6.4.11 *For every iterative theory T there exists a pointed iterative theory $(T', \mathbf{1}_1^\dagger = \perp)$ and an injective iterative theory morphism $\eta : T \to T'$ with the following universal property. For every iteration theory R and theory morphism $\varphi : T \to R$ such that $f^\dagger \varphi = (f\varphi)^\dagger$, for all ideal morphisms $f : n \to n + p$ in T, there is a unique iteration theory morphism $\varphi^\sharp : (T', \mathbf{1}_1^\dagger = \perp) \to R$ with $\eta \cdot \varphi^\sharp = \varphi$. In particular, when R is a pointed iterative theory and φ is an iterative theory morphism, $\varphi^\sharp$ is the unique pointed iterative theory morphism such that $\eta \cdot \varphi^\sharp = \varphi$.*

Proof. Suppose that X is a singleton set, say $X = \{\perp\}$. In Section 3.3.4, we showed that there is a theory $T[X]$, an injective theory morphism $\kappa : T \to T[X]$ and an assignment $\perp \mapsto \perp\lambda$ with the following property. For any pointed theory R with point $\perp' : 1 \to 0$ and for any theory morphism $\varphi : T \to R$, there exists a unique theory morphism $\varphi^\sharp : T[X] \to R$ with $\kappa \cdot \varphi^\sharp = \varphi$ and $\perp\lambda\varphi^\sharp = \perp'$. Further, we showed that each morphism $n \to p$ in $T[X]$ can be written in the form

$$f \cdot (\mathbf{1}_p \oplus \perp_k),$$

where $f : n \to p + k \in T$ and $\perp_k := \perp_{k0} = \langle \perp, \ldots, \perp \rangle : k \to 0$. (Here and below we write just $\perp$ for $\perp\lambda$, f, g for $f\kappa, g\kappa$, etc.) By Exercise 3.3.4.29,

$$f \cdot (\mathbf{1}_p \oplus \perp_k) \;=\; g \cdot (\mathbf{1}_p \oplus \perp_m)$$

iff

$$f \cdot (\mathbf{1}_p \oplus \tau_k) \;=\; g \cdot (\mathbf{1}_p \oplus \tau_m),$$

where for each $j \geq 0$, τ_j denotes the base morphism $j \to 1$.

Suppose for the rest of the proof that T is an iterative theory. We define $T' := T[X]$ and $\eta := \kappa$. We must prove that T' is an iterative theory also. That T' is ideal follows easily. Indeed, when T is nontrivial, the ideal morphisms $1 \to p$ in $T[X]$ are those of the form

$$f' \;=\; f \cdot (\mathbf{1}_p \oplus \perp_k) : 1 \to p,$$

where f is either an ideal morphism in T or a distinguished morphism $0_p \oplus j_k$, for some $j \in [k]$. Suppose that

$$g' = g \cdot (\mathbf{1}_q \oplus \perp_m) : p \to q.$$

Then

$$f' \cdot g' = f \cdot (g \oplus \mathbf{1}_k) \cdot (\mathbf{1}_q \oplus \perp_{m+k}).$$

Supposing that $f' \cdot g'$ is a distinguished morphism

$$i_q \cdot (\mathbf{1}_q \oplus \perp_0),$$

we have

$$i_q = f \cdot (g \oplus \tau_{m+k})$$

in T. But since T is ideal, this is possible only if f is a distinguished morphism. Now it follows easily that f' is distinguished. When T is trivial, T' is also trivial hence ideal.

We next show that when

$$f' = f \cdot (\mathbf{1}_{n+p} \oplus \perp_k) : n \to n+p$$

is an ideal morphism in T', the iteration equation for f' has a unique solution. We will restrict ourselves to the case that f is ideal in T. Then,

$$\begin{aligned}
f \cdot (\mathbf{1}_{n+p} \oplus \perp_k) \cdot \langle f^\dagger \cdot (\mathbf{1}_p \oplus \perp_k),\, \mathbf{1}_p\rangle &= f \cdot \langle f^\dagger \cdot (\mathbf{1}_p \oplus \perp_k),\, \mathbf{1}_p \oplus \perp_k\rangle \\
&= f \cdot \langle f^\dagger, \mathbf{1}_{p+k}\rangle \cdot (\mathbf{1}_p \oplus \perp_k) \\
&= f^\dagger \cdot (\mathbf{1}_p \oplus \perp_k),
\end{aligned}$$

showing that $f^\dagger \cdot (\mathbf{1}_p \oplus \perp_k)$ is a solution. Suppose that $g \cdot (\mathbf{1}_p \oplus \perp_m)$ is any other solution. Then,

$$\begin{aligned}
g \cdot (\mathbf{1}_p \oplus \perp_m) &= f \cdot (\mathbf{1}_{n+p} \oplus \perp_k) \cdot \langle g \cdot (\mathbf{1}_p \oplus \perp_m),\, \mathbf{1}_p\rangle \\
&= f \cdot \langle g \cdot (\mathbf{1}_p \oplus \perp_m),\, \mathbf{1}_p \oplus \perp_k\rangle \\
&= f \cdot \langle g \oplus 0_k,\, \mathbf{1}_p \oplus 0_m \oplus \mathbf{1}_k\rangle \cdot (\mathbf{1}_p \oplus \perp_{m+k}).
\end{aligned}$$

Thus,

$$f \cdot \langle g \oplus 0_k,\, \mathbf{1}_p \oplus 0_m \oplus \mathbf{1}_k\rangle \cdot (\mathbf{1}_p \oplus \tau_{m+k}) = g \cdot (\mathbf{1}_p \oplus \tau_m),$$

i.e.,

$$f \cdot (\mathbf{1}_n \oplus \mathbf{1}_p \oplus \tau_k) \cdot \langle g \cdot (\mathbf{1}_p \oplus \tau_m),\, \mathbf{1}_{p+1}\rangle = g \cdot (\mathbf{1}_p \oplus \tau_m).$$

This shows that $g \cdot (\mathbf{1}_p \oplus \tau_m)$ is the unique solution of the iteration equation for $f \cdot (\mathbf{1}_n \oplus \mathbf{1}_p \oplus \tau_k)$. Thus,

$$\begin{aligned} g \cdot (\mathbf{1}_p \oplus \tau_m) &= (f \cdot (\mathbf{1}_n \oplus \mathbf{1}_p \oplus \tau_k))^\dagger \\ &= f^\dagger \cdot (\mathbf{1}_p \oplus \tau_k). \end{aligned}$$

It follows that $g \cdot (\mathbf{1}_p \oplus \perp_m) = f^\dagger \cdot (\mathbf{1}_p \oplus \perp_k)$, proving the uniqueness of the solution.

Since η clearly preserves ideal morphisms, η is an iterative theory morphism. Suppose now that R is an iteration theory and $\varphi : T \to R$ is a theory morphism such that $f^\dagger\varphi = (f\varphi)^\dagger$, for ideal morphisms $f : n \to n + p$. Let $\perp'$ denote the morphism $\mathbf{1}_1^\dagger$ in R. Then, writing $\perp$ for $\perp_0$, there is a unique extension of φ to a pointed theory morphism $\varphi^\sharp : T' \to R$ which maps $\perp$ to $\perp'$. But for all $g = f \cdot (\mathbf{1}_{1+p} \oplus \perp_k) : 1 \to 1 + p$ in T such that f is ideal,

$$\begin{aligned} (f \cdot (\mathbf{1}_{1+p} \oplus \perp_k))^\dagger \varphi^\sharp &= (f^\dagger \cdot (\mathbf{1}_p \oplus \perp_k))\varphi^\sharp \\ &= f^\dagger\varphi \cdot (\mathbf{1}_p \oplus \perp_k') \\ &= (f\varphi)^\dagger \cdot (\mathbf{1}_p \oplus \perp_k') \\ &= (f\varphi \cdot (\mathbf{1}_{1+p} \oplus \perp_k'))^\dagger \\ &= ((f \cdot (\mathbf{1}_{1+p} \oplus \perp_k))\varphi^\sharp)^\dagger, \end{aligned}$$

showing that $g^\dagger\varphi^\sharp = (g\varphi)^\dagger$. When $f = 0_{1+p} \oplus i_k$, for some $i \in [k]$,

$$g^\dagger\varphi^\sharp \;=\; \perp_{1p}\varphi^\sharp \;=\; \perp_{1p}' \;=\; (g\varphi^\sharp)^\dagger.$$

We leave the verification of the fact that $g^\dagger\varphi^\sharp = (g\varphi)^\dagger$ to the reader for the case that $f = i_{1+p} \oplus 0_k$, for some $i \in [1 + p]$. The uniqueness of $\varphi^\sharp$ is obvious. When $R = (R,\, \mathbf{1}_1^\dagger = \perp')$ is a pointed iterative theory and φ is an iterative theory morphism, one proves that $\varphi^\sharp$ takes ideal morphisms to ideal morphisms. Thus $\varphi^\sharp$ is the unique pointed iterative theory morphism from $(T',\, \mathbf{1}_1^\dagger = \perp)$ to R such that $\eta \cdot \varphi^\sharp = \varphi$.

Corollary 6.4.12 *For every iterative theory T there is a pointed iterative theory $(T',\, \mathbf{1}_1^\dagger = \perp)$ and an injective iterative theory morphism $\varphi : T \to T'$. In particular, each iterative theory embeds in some iteration theory.*

Remark 6.4.13 The pointed iterative theory $(T',\, \mathbf{1}_1^\dagger = \perp)$ also has the following property. For every pointed theory R with point $\perp' : 1 \to 0$ and for every theory morphism $\varphi : T \to R$ there is a unique theory morphism $\varphi^\sharp : T' \to R$ such that $\eta \cdot \varphi^\sharp = \varphi$ and $\perp\varphi^\sharp = \perp'$.

6.5 Free Iteration Theories

Let Σ be a ranked set and suppose that $\perp$ is a symbol not in Σ. $\Sigma_\perp$ is the ranked set obtained from Σ by adding $\perp$ to Σ_0, so that

$$(\Sigma_\perp)_0 \;:=\; \Sigma_0 \cup \{\perp\}; \quad (\Sigma_\perp)_k \;:=\; \Sigma_k, \quad k \geq 1.$$

Recall the map $\eta : \Sigma \to \Sigma tr$ taking each letter in Σ to the corresponding atomic tree. We write

$$\eta_\perp : \Sigma \;\to\; \Sigma_\perp tr$$

for the composite of η with the inclusion $\Sigma tr \to \Sigma_\perp tr$.

Definition 6.5.1 *Since $\Sigma_\perp tr$ is an iterative theory, $(\Sigma_\perp tr, \mathbf{1}_1^\dagger = \perp)$ is a pointed iterative theory and an iteration theory. We denote this iteration theory by $\Sigma\mathbf{tr}$.*

In this section we will show that $\Sigma\mathbf{tr}$ is freely generated by $\eta_\perp$ in the class of all iteration theories.

Theorem 6.5.2 *Let T be an iteration theory and suppose that*

$$\varphi : \Sigma \;\to\; T$$

is a rank preserving function. Then there is a unique iteration theory morphism

$$\varphi^\sharp : \Sigma\mathbf{tr} \;\to\; T$$

such that $\eta_\perp \cdot \varphi^\sharp = \varphi$.

Proof. We use $\perp$ to denote the morphism $\mathbf{1}_1^\dagger$ in T (as well as a letter in Σ) and let $\varphi_\perp : \Sigma_\perp \to T$ be the function extending φ which takes $\perp$ to $\perp$. By Theorem 5.5.4.24, there is a unique partial iteration theory morphism $\varphi^\sharp : \Sigma_\perp tr \to T$ with $\eta_\perp \cdot \varphi^\sharp = \varphi_\perp$. By Proposition 6.1.5 we must only show that $\varphi^\sharp$ preserves the iteration operation on scalar base morphisms.

Suppose that $f : 1 \to 1 + p$ is a distinguished morphism in $\Sigma_\perp tr$. If $f = i_{1+p}$, with $i > 1$, then $f = 0_1 \oplus g$ for a distinguished morphism $g : 1 \to p$, and $f^\dagger = g$, by the left zero identity. But then

$$f^\dagger \varphi^\sharp \;=\; g\varphi^\sharp \;=\; g \;=\; (f\varphi^\sharp)^\dagger,$$

since $\varphi^\sharp$ is a theory morphism and T satisfies the left zero identity. When $f = 1_{1+p}$, $f^\dagger = \perp \oplus 0_p$ by the right zero identity; and since $\varphi^\sharp$ is a theory morphism preserving $\perp$,

$$f^\dagger \varphi^\sharp \;=\; \perp \oplus 0_p.$$

Similarly, using the right zero identity,

$$(f\varphi^\sharp)^\dagger \;=\; (\mathbf{1}_1 \oplus 0_p)^\dagger \;=\; \mathbf{1}_1^\dagger \oplus 0_p \;=\; \perp \oplus 0_p.$$

The next two corollaries exhibit two simple universal Horn theories whose equational part is the equational theory of the class of iteration theories.

Corollary 6.5.3 $\Sigma\mathbf{tr}$ *is freely generated by* $\eta_\perp$ *in either of the following two classes of iteration theories.*

- *The class of all iteration theories satisfying the functorial dagger implication for (surjective) base morphisms.*
- *The class of all iteration theories that satisfy the GA-implication.*

Corollary 6.5.4 *The variety of iteration theories generated by those which have a functorial dagger with respect to (surjective) base morphisms, or which satisfy the GA-implication, is the class of all iteration theories. In fact, for every iteration theory T there is an iteration theory R satisfying the GA-implication, and hence the functorial dagger implication for all base surjections, such that T is a quotient of R.*

Example 6.5.5 When Σ is empty, the free iteration theory $\Sigma\mathbf{tr}$ is isomorphic to the iteration theory described in Example 6.4.9. Thus the theory is initial iteration theory. Suppose that T is an iteration theory and that φ is the unique iteration theory morphism $\to T$. The morphisms of the form $f\varphi$, $f \in$, are called *partial base morphisms.* The partial base morphisms form a subiteration theory T_0 of T. In fact T_0 is the smallest subiteration theory of T. When T is nontrivial, T_0 is isomorphic to .

Exercise 6.5.6 Show that when T is the free iterative theory Σtr, the iteration theory obtained in the proof of Theorem 6.4.11 is isomorphic to the free iteration theory $\Sigma\mathbf{tr}$.

We can use the above representation of the free iteration theories to show that the equational theory of iteration theories is decidable. In other words, there is an algorithm that tells whether an equation between two iteration terms (composed from the theory operations, the dagger operation and some elements of a ranked set Σ that play the role of the variables) holds in all iteration theories. Indeed, any such term determines a flowchart scheme similar to that considered in Section 5.5.4 in connection with descriptions. However, some internal vertices of this scheme may be labeled $\perp$, a symbol of rank 0. Now, an equation $t_1 = t_2$ between two iteration terms holds in all iteration theories iff the schemes corresponding to the terms t_1 and t_2 unfold to the same regular $\Sigma_\perp$-tree, i.e. when the two schemes are *equivalent*. This latter condition can be checked in a number of ways. One possible method follows on the lines of Section 5.5.4. Suppose that D and E are the schemes that correspond to the terms t_1 and t_2. Then, construct the accessible parts of the two schemes by deleting all of the inaccessible internal vertices. Let D_0 and E_0 denote the resulting schemes. Then D and E are equivalent iff D_0 and E_0 have a common homomorphic image, i.e. when the reduced schemes equivalent to D_0 and E_0 are isomorphic. The whole process resembles the minimization of deterministic finite automata.

Exercise 6.5.7 Give a polynomial time algorithm to decide whether an equation between "iteration terms" holds in all iteration theories.

PROBLEM 6.5.1 Give a concrete description of the free Conway theories.

PROBLEM 6.5.2 Is the equational theory of Conway theories decidable?

6.6 Constructions on Iteration Theories

Preiteration theories determine an equational class of many-sorted algebras as do iteration theories or Conway theories. Thus the class of iteration theories shares all of the general properties of varieties. In particular, the category of iteration theories is complete, cocomplete, has surjective-injective factorizations, is well-powered and cowell-powered. The same facts are true for the category of preiteration theories and the category of Conway theories. In this section we will review some elementary properties of the category of (pre)iteration theories that will be useful in the sequel.

Proposition 6.6.1 *Suppose that T and T' are preiteration theories and that $\varphi : T \to T'$ a preiteration theory morphism. The T'-morphisms*

$$T\varphi \ := \ \{f\varphi : f \in T\}$$

form a subpreiteration theory of T'. Further, φ has a factorization $\varphi = \psi \cdot \iota$, where $\iota : T\varphi \to T'$ is the inclusion of $T\varphi$ into T' and $\psi : T \to T\varphi$ is a surjective preiteration theory morphism.

Thus each preiteration theory morphism has a factorization as the composite of a surjective preiteration theory morphism with an injective preiteration theory morphism. A similar fact is true in each full subcategory of preiteration theories determined by a subvariety, such as iteration theories or Conway theories, since any quotient or subpreiteration theory of an iteration theory (or Conway theory) is an iteration theory (or Conway theory, respectively).

Definition 6.6.2 *Let T be a preiteration theory. A* **dagger congruence** *on T is a theory congruence θ such that*

$$f\,\theta\, g \quad \Rightarrow \quad f^\dagger\,\theta\, g^\dagger,$$

for all $f, g : n \to n + p$ in T.

As for theories, we will often write $f \equiv g\ (\theta)$ for $f\,\theta\,g$. When θ is a dagger congruence on T, the quotient theory T/θ may be turned into a preiteration theory. The equivalence class containing a morphism

f is variously denoted $[f]$ or f/θ. The iteration operation on T/θ is defined by the equation

$$[f]^\dagger \;:=\; [f^\dagger],$$

for all $f : n \to n + p$ in T. The operation is well-defined since θ is a dagger congruence. The quotient map (or natural map) taking a T-morphism f to its equivalence class $[f]$ is a surjective preiteration theory morphism

$$T \;\to\; T/\theta.$$

Thus, if T is an iteration or Conway theory then so is the quotient theory T/θ.

The next result shows that the intersection of any family of dagger congruences on a preiteration theory is a dagger congruence.

Proposition 6.6.3 *If θ_i, $i \in I$, are dagger congruences on the preiteration theory T, then the relation*

$$f\,\theta\,g \;\Leftrightarrow\; \forall i \in I\,(f\,\theta_i\,g)$$

is also a dagger congruence.

Thus the dagger congruences on a preiteration theory T form a complete lattice with respect to inclusion. The smallest dagger congruence on T is the identity relation and the greatest dagger congruence is the trivial relation that identifies any two morphisms $f, g : n \to p$, for each $n, p \geq 0$. It follows that if $R = R_{np}$ is any family of relations on $T(n,p)$, $n, p \geq 0$, there is a smallest dagger congruence θ on T which contains R.

Recall from Section 3.3.2 that if $\varphi : T \to T'$ is a theory morphism between the theories T and T', $\ker\varphi$ is a theory congruence on T.

Proposition 6.6.4 *Let $\varphi : T \to T'$ be a preiteration theory morphism between preiteration theories T and T'. Then $\theta := \ker\varphi$ is a dagger congruence and the preiteration theories T/θ and $T\varphi$ are isomorphic via the map*

$$[f] \;\mapsto\; f\varphi, \quad f : n \to p.$$

Conversely, if θ is a dagger congruence on T there is a preiteration theory T' and a surjective preiteration theory morphism $\varphi : T \to T'$ such that $\theta = \ker\varphi$.

When $\varphi : T \to T'$ is a preiteration theory morphism, we will sometimes write $f \sim_\varphi g$ for $f \equiv g$ $(\ker \varphi)$.

Next we discuss in brief some categorical constructions. For any subvariety of preiteration theories, limits in the corresponding category can be constructed on the level of sets. If T_i, $i \in I$, are theories, then the product theory $T = \prod_{i \in I} T_i$ has hom-sets $T(n,p) = \prod_{i \in I} T_i(n,p)$ and the theory operations are defined pointwise (see Chapter 3). When the T_i's are preiteration theories, we may define the iterate of an I-indexed family $(f_i) : n \to n + p$ by

$$(f_i)^\dagger \;:=\; (f_i^\dagger).$$

Thus the projections $T \to T_i$ are preiteration theory morphisms and T is the product of the T_i's in the category of preiteration theories. When each T_i is an iteration or Conway theory, so is the product $\prod_{i \in I} T_i$. Further, $\prod_{i \in I} T_i$ is the categorical product in the category of iteration or Conway theories.

Suppose that $\varphi_1, \varphi_2 : T \to T'$ is a parallel pair of preiteration theory morphisms. The collection of all T-morphisms f such that $f\varphi_1 = f\varphi_2$ is a subpreiteration theory T_0 of T. The inclusion $\varphi : T_0 \to T$ is the equalizer of φ_1 and φ_2. The same construction produces the equalizer in the category of iteration or Conway theories. To construct the coequalizer of φ_1 and φ_2, take the smallest dagger congruence θ on T' such that $f\varphi_1 \equiv f\varphi_2$ (θ), for all f in T. The natural map $\varphi' : T' \to T'/\theta$ is the coequalizer of φ_1 and φ_2. If T' is an iteration or Conway theory, so is T'/θ. Thus, $\varphi' : T' \to T'/\theta$ is the coequalizer of φ_1 and φ_2 in both categories of iteration theories and Conway theories.

Lastly, we consider binary coproducts of *iteration theories.* Let Σ and Δ be disjoint signatures. The union of Σ and Δ is denoted $\Sigma + \Delta$. Let in_Σ be the iteration theory morphism $\Sigma\mathbf{tr} \to (\Sigma + \Delta)\mathbf{tr}$ determined by the map

where $\eta_{\Sigma+\Delta}$ is the free embedding. Thus, in_Σ is the inclusion of $\Sigma\mathbf{tr}$ to $(\Sigma + \Delta)\mathbf{tr}$. The iteration theory morphism $in_\Delta : \Delta\mathbf{tr} \to (\Sigma + \Delta)\mathbf{tr}$ is defined similarly. It follows that in_Σ and in_Δ are coproduct inclusions, cf. Proposition 3.3.3.2. Thus $(\Sigma + \Delta)\mathbf{tr}$ is the coproduct of the free iteration theories $\Sigma\mathbf{tr}$ and $\Delta\mathbf{tr}$. When T_1 and T_2 are arbitrary iteration theories, their coproduct can be constructed as follows. Let Σ and Δ be disjoint signatures such that there exist surjective iteration theory morphisms $\varphi_1 : \Sigma\mathbf{tr} \to T_1$ and $\varphi_2 : \Delta\mathbf{tr} \to T_2$. Let θ be the smallest dagger congruence on $(\Sigma + \Delta)\mathbf{tr}$ which identifies any two trees $f_1, g_1 : n \to p \in \Sigma\mathbf{tr}$ with $f_1 \sim_{\varphi_1} g_1$ as well as any two trees $f_2, g_2 : n \to p \in \Delta\mathbf{tr}$ with $f_2 \sim_{\varphi_2} g_2$. The coproduct of T_1 and T_2 can be represented as the quotient iteration theory

$$T_1 \oplus T_2 \ := \ (\Sigma + \Delta)\mathbf{tr}/\theta.$$

The coproduct map $\kappa_1 : T_1 \to T_1 \oplus T_2$ is given by $f\varphi_1 \mapsto (f\, in_\Sigma)/\theta$. The coproduct map $\kappa_2 : T_2 \to T_1 \oplus T_2$ is defined similarly.

Next we consider a special case of the coproduct construction, the enrichment of iteration theories with constants from a set. Let T be a theory and X a set. In Section 3.3.4, we have constructed a theory $T[X]$, a theory morphism $\kappa : T \to T[X]$ and a function $\lambda : X \to T[X](1, 0)$ with the following universal property. For any theory T', theory morphism $\varphi : T \to T'$ and any function $h : X \to T'(1, 0)$, there is a unique theory morphism $\varphi_h : T[X] \to T'$ such that the following diagram commutes:

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We have shown that κ is injective and when T is nontrivial, λ is injective also.

Proposition 6.6.5 *Let T be a preiteration theory satisfying the parameter identity and let X be a set. There is a unique way to turn $T[X]$ into a preiteration theory satisfying the parameter identity and such that κ becomes a preiteration theory morphism. When T is an iteration theory, $T[X]$ too is an iteration theory (and thus κ an iteration theory morphism).*

Proof. We will use the notation of Section 3.3.4. When T is trivial so is $T[X]$, and the Proposition holds obviously. Hence we will assume below that T is a nontrivial preiteration theory. Suppose that $T[X]$ has been turned into a preiteration theory satisfying the parameter identity and such that κ is a preiteration theory morphism. Then, for all $f : n \to n+p+s \in T$ and $x : [s] \to X$,

$$(f\kappa \cdot (\mathbf{1}_{n+p} \oplus x\lambda))^\dagger \quad = \quad f^\dagger\kappa \cdot (\mathbf{1}_p \oplus x\lambda). \tag{6.24}$$

Thus there is at most one possible definition of the dagger operation on $T[X]$.

Suppose for the rest of the proof that the iteration operation on the theory $T[X]$ is defined by (6.24). That the operation is well-defined follows from the fact that if

$$f \cdot (\mathbf{1}_{n+p} \oplus \rho) \;=\; g \quad \text{and} \quad \rho \cdot y \;=\; x,$$

for some $g : n \to n+p+r \in T$, $y : [r] \to X$ and $\rho : [s] \to [r]$, then

$$\begin{aligned}
f^\dagger\kappa \cdot (\mathbf{1}_p \oplus x\lambda) &= f^\dagger\kappa \cdot (\mathbf{1}_p \oplus \rho \cdot y\lambda) \\
&= f^\dagger\kappa \cdot (\mathbf{1}_p \oplus \rho) \cdot (\mathbf{1}_p \oplus y\lambda) \\
&= (f \cdot (\mathbf{1}_{n+p} \oplus \rho))^\dagger\kappa \cdot (\mathbf{1}_p \oplus y\lambda) \\
&= g^\dagger\kappa \cdot (\mathbf{1}_p \oplus y\lambda).
\end{aligned}$$

We let the reader provide the proof that the parameter identity holds in $T[X]$ and that κ is a preiteration theory morphism. We only show that $T[X]$ is an iteration theory when T is. Hence we assume that T is an iteration theory. In the rest of the proof, we identify T and X with their images under κ and λ, respectively.

We will use the axiomatization in Corollary 6.2.7.

Proof of the double dagger identity. Let

$$\overline{f} \;:=\; f \cdot (\mathbf{1}_{n+n+p} \oplus x) : n \to n+n+p$$

in $T[X]$. Then

$$\begin{aligned}
\overline{f}^{\dagger\dagger} &= f^{\dagger\dagger} \cdot (\mathbf{1}_p \oplus x) \\
&= (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_{p+s}))^\dagger \cdot (\mathbf{1}_p \oplus x) \\
&= (f \cdot (\mathbf{1}_{n+n+p} \oplus x) \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger \\
&= (\overline{f} \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger,
\end{aligned}$$

by the parameter identity and the fact that the double dagger identity holds in T.

Proof of the composition identity. Suppose that

$$\overline{f} \ := \ f \cdot (\mathbf{1}_{m+p} \oplus x) : n \to m+p$$

and

$$\overline{g} \ := \ g \cdot (\mathbf{1}_{n+p} \oplus y) : m \to n+p$$

are $T[X]$-morphisms, where $f : n \to m+p+s$, $g : m \to n+p+r$, $x : [s] \to X$ and $y : [r] \to X$. Using the composition identity, which holds in T, and the parameter identity, one proves that

$$(\overline{f} \cdot \langle \overline{g}, 0_n \oplus \mathbf{1}_p \rangle)^\dagger \ = \ f' \cdot \langle h^\dagger, \mathbf{1}_{p+r+s} \rangle \cdot (\mathbf{1}_p \oplus y \oplus x), \tag{6.25}$$

where

$$\begin{aligned} f' &:= f \cdot (\mathbf{1}_{m+p} \oplus 0_r \oplus \mathbf{1}_s) : n \to m+p+r+s \\ g' &:= g \oplus 0_s : m \to n+p+r+s \\ h &:= g' \cdot \langle f', 0_m \oplus \mathbf{1}_{p+r+s} \rangle : m \to m+p+r+s. \end{aligned}$$

Similarly, by the parameter identity,

$$\begin{aligned} &\overline{f} \cdot \langle (\overline{g} \cdot \langle \overline{f}, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle \ = \\ &\quad = \ f \cdot (\langle h^\dagger, \mathbf{1}_p \oplus 0_{r+s} \rangle \oplus \mathbf{1}_s) \cdot (\mathbf{1}_p \oplus y \oplus x \oplus x). \end{aligned} \tag{6.26}$$

But when $\rho = \mathbf{1}_r \oplus \langle \mathbf{1}_s, \mathbf{1}_s \rangle : [r+2s] \to [r+s]$,

$$\begin{aligned} f \cdot (\langle h^\dagger, \mathbf{1}_p \oplus 0_{r+s} \rangle \oplus \mathbf{1}_s) \cdot (\mathbf{1}_p \oplus \rho) &= f \cdot \langle h^\dagger, \mathbf{1}_p \oplus 0_r \oplus \mathbf{1}_s \rangle \\ &= f' \cdot \langle h^\dagger, \mathbf{1}_{p+r+s} \rangle. \end{aligned}$$

But since

$$y \oplus x \oplus x \ = \ \rho \cdot (y \oplus x),$$

the right-hand sides of (6.25) and (6.26) are equal, proving that the composition identity holds in $T[X]$.

Proof of the commutative identity. Suppose that

$$\overline{f} \ = \ f \cdot (\mathbf{1}_{m+p} \oplus x) : n \to m+p$$

in $T[X]$, where $x : [s] \to X$. Suppose further that $\rho : m \to n$ is a surjective base morphism. Let $\rho_i : m \to m$, $i \in [m]$, be base morphisms with $\rho_i \cdot \rho = \rho$. It is easy to see that

$$(\rho \cdot \overline{f}) \parallel (\rho_1, \ldots, \rho_m) \ = \ ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m)) \cdot (\mathbf{1}_{m+p} \oplus x).$$

Thus,

$$\begin{aligned}((\rho \cdot \overline{f}) \parallel (\rho_1, \ldots, \rho_m))^\dagger &= ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \cdot (\mathbf{1}_p \oplus x) \\ &= \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_{p+s}))^\dagger \cdot (\mathbf{1}_p \oplus x) \\ &= \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_{p+s}) \cdot (\mathbf{1}_{n+p} \oplus x))^\dagger \\ &= \rho \cdot (f \cdot (\mathbf{1}_{m+p} \oplus x) \cdot (\rho \oplus \mathbf{1}_p))^\dagger \\ &= \rho \cdot (\overline{f} \cdot (\rho \oplus \mathbf{1}_p))^\dagger.\end{aligned}$$

Exercise 6.6.6 Complete the details of the proof of Proposition 6.6.5.

Remark 6.6.7 By a similar argument it can be shown that $T[X]$ satisfies the permutation identity whenever T does. Thus if T is a Conway theory, then so is $T[X]$.

Proposition 6.6.8 *Let T and T' be preiteration theories satisfying the parameter identity and let X be a set. Suppose that $\varphi : T \to T'$ is a preiteration theory morphism and that $h : X \to T'(1, 0)$ is a function. Then the unique theory morphism $\varphi_h : T[X] \to T'$ determined by φ and h is a preiteration theory morphism.*

Proof. Suppose that

$$\overline{f} = f\kappa \cdot (\mathbf{1}_{n+p} \oplus x\lambda) : n \to n + p$$

is a $T[X]$-morphism, where $x : [s] \to X$. If

$$xh := x_1 h \oplus \ldots \oplus x_s h,$$

then

$$\begin{aligned}\overline{f}^\dagger \varphi_h &= (f^\dagger \kappa \cdot (\mathbf{1}_p \oplus x\lambda))\varphi_h \\ &= f^\dagger \varphi \cdot (\mathbf{1}_p \oplus xh) \\ &= (f\varphi \cdot (\mathbf{1}_{n+p} \oplus xh))^\dagger \\ &= ((f\kappa \cdot (\mathbf{1}_{n+p} \oplus x\lambda))\varphi_h)^\dagger \\ &= (\overline{f}\varphi_h)^\dagger.\end{aligned}$$

For later reference, in the following lemma we summarize the previous two propositions as well as some results proved in Section 3.3.4 for theories.

Lemma 6.6.9 *Let T be a preiteration theory satisfying the parameter identity. For any set X, there is a preiteration theory $T[X]$ satisfying the parameter identity with the following universal properties.*

[a] *There is an injective preiteration theory morphism $\kappa : T \to T[X]$ and a function $\lambda : X \to T[X](1,0)$ (which is also injective unless T is trivial).*

[b] *For any theory (not necessarily preiteration theory) T', and any theory morphism $\varphi : T \to T'$ and any function $h : X \to T'(1,0)$, there is a unique theory morphism $\varphi_h : T[X] \to T'$ such that the following diagram commutes.*

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[c] *If T' is a preiteration theory satisfying the parameter identity and $\varphi : T \to T'$ is a preiteration theory morphism, then so is $\varphi_h : T[X] \to T'$.*

[d] *When T is an iteration or Conway theory, then so is $T[X]$.*

We end this section with a useful result on the dagger congruences T induced by an equivalence relation θ_0 on the set of morphism $1 \to 0$ in a preiteration theory.

Any such relation θ_0 extends to morphisms $n \to 0$. Indeed, given $f, g : n \to 0$, we define $f \, \theta_0 \, g$ if and only if $i_n \cdot f \quad \theta_0 \quad i_n \cdot g$, for all $i \in [n]$. The relation θ_0 is called *left invariant* if for all $f, g : n \to 0$ and $h : m \to n$,

$$f \ \theta_0 \ g \quad \Rightarrow \quad h \cdot f \ \ \theta_0 \ \ h \cdot g.$$

Lemma 6.6.10 The Zero Congruence Lemma. *Suppose that T is a preiteration theory satisfying the parameter identity. Let θ_0 be an equivalence relation on the set of morphisms $1 \to 0$.*

[a] *The least theory congruence θ containing θ_0 is also a dagger congruence.*

[b] *The relation* θ *is the transitive closure of the following relation* R*: for all* $f, g : n \to p$,

$$\begin{aligned} f \, R \, g \quad \Leftrightarrow \quad & (\exists u : n \to p + s)(\exists f', g' : s \to 0) \\ & [f = u \cdot (\mathbf{1}_p \oplus f') \text{ and} \\ & g = u \cdot (\mathbf{1}_p \oplus g') \text{ and} \\ & f' \, \theta_0 \, g']. \end{aligned}$$

[c] *If* θ_0 *is left invariant, then for all* $f, g : n \to 0$,

$$f \, \theta \, g \;\Leftrightarrow\; f \, \theta_0 \, g.$$

Proof. Since R is reflexive and symmetric, its transitive closure R^+ is an equivalence relation. Thus the fact that R^+ is a dagger congruence follows once we prove that R is preserved by all operations.

Proof that for all $f, g : n \to p$ and $h : p \to q$,

$$f \, R \, g \;\Rightarrow\; f \cdot h \;R\; g \cdot h.$$

Let $u : n \to p + s$ and $f', g' : s \to 0$ be morphisms with

$$f \;=\; u \cdot (\mathbf{1}_p \oplus f'), \;\; g \;=\; u \cdot (\mathbf{1}_p \oplus g') \text{ and } f' \, \theta_0 \, g'.$$

Then

$$\begin{aligned} f \cdot h &= u \cdot (\mathbf{1}_p \oplus f') \cdot h \\ &= u \cdot (h \oplus \mathbf{1}_s) \cdot (\mathbf{1}_q \oplus f'), \end{aligned}$$

and similarly,

$$g \cdot h \;=\; u \cdot (h \oplus \mathbf{1}_s) \cdot (\mathbf{1}_q \oplus g').$$

Thus, $f \cdot h \;R\; g \cdot h$.

Proof that for all $f, g : n \to p$ and $h : m \to n$,

$$f \, R \, g \;\Rightarrow\; h \cdot f \;R\; h \cdot g.$$

Using the above notations we have

$$\begin{aligned} h \cdot f &= h \cdot u \cdot (\mathbf{1}_p \oplus f') \\ h \cdot g &= h \cdot u \cdot (\mathbf{1}_p \oplus g'). \end{aligned}$$

Thus, $h \cdot f \; R \; h \cdot g$.

Proof that for all $f_1, f_2 : n \to p$ and $g_1, g_2 : m \to p$,

$$f_1 \, R \, g_1 \text{ and } f_2 \, R \, g_2 \;\Rightarrow\; \langle f_1, f_2 \rangle \; R \; \langle g_1, g_2 \rangle.$$

Suppose that

$$\begin{array}{rclcrcl} f_1 & = & u \cdot (\mathbf{1}_p \oplus f_1') & \qquad & g_1 & = & u \cdot (\mathbf{1}_p \oplus g_1') \\ f_2 & = & v \cdot (\mathbf{1}_p \oplus f_2') & \qquad & g_2 & = & v \cdot (\mathbf{1}_p \oplus g_2'), \end{array}$$

where $u : n \to p + s$, $v : m \to p + r$, $f_1', g_1' : s \to 0$ and $f_2', g_2' : r \to 0$ with $f_1' \, \theta_0 \, g_1'$ and $f_2' \, \theta_0 \, g_2'$. Define

$$w \;:=\; \langle u \oplus 0_r, \; v \cdot (\mathbf{1}_p \oplus 0_s \oplus \mathbf{1}_r) \rangle : n + m \to p + s + r.$$

Since

$$\begin{array}{rcl} \langle f_1, f_2 \rangle & = & w \cdot (\mathbf{1}_p \oplus f_1' \oplus f_2') \\ \langle g_1, g_2 \rangle & = & w \cdot (\mathbf{1}_p \oplus g_1' \oplus g_2'), \end{array}$$

and

$$f_1' \oplus f_2' \; \theta_0 \; g_1' \oplus g_2',$$

it follows that $\langle f_1, f_2 \rangle \, R \, \langle g_1, g_2 \rangle$.

Proof that for all $f, g : n \to n + p$,

$$f \, R \, g \;\Rightarrow\; f^\dagger \, R \, g^\dagger.$$

Indeed, suppose that $f = u \cdot (\mathbf{1}_{n+p} \oplus f')$ and $g = u \cdot (\mathbf{1}_{n+p} \oplus g')$, for some $u : n \to n + p + s$ and $f', g' : s \to 0$ with $f' \, \theta_0 \, g'$. Then, by the parameter identity,

$$f^\dagger \;=\; u^\dagger \cdot (\mathbf{1}_p \oplus f') \; R \; u^\dagger \cdot (\mathbf{1}_p \oplus g') \;=\; g^\dagger.$$

Thus R^+ is a dagger congruence. Since R^+ includes θ_0 and is clearly included by any theory congruence containing θ_0, it follows that R^+ is both the least theory congruence and the least dagger congruence containing θ_0.

If θ_0 is left invariant, then $f \, R \, g$ implies $f \, \theta_0 \, g$, for all $f, g : n \to 0$. Thus if $f \, \theta \, g$ then $f \, \theta_0 \, g$.

6.7 Feedback Theories

In this section we consider the operation of feedback in algebraic theories. We show how to define feedback in any iteration theory in such a way that iteration is in turn expressible in terms of feedback. Iteration theory identities can then be translated to equations involving feedback.

Definition 6.7.1 *A* **prefeedback theory** *is a theory T equipped with a feedback operation mapping morphisms $f : n + p \to n + q$ to morphisms $\uparrow f : p \to q$, for all $n, p, q \geq 0$. A* **prefeedback theory morphism** $\varphi : T \to T'$ *between prefeedback theories is a theory morphism that preserves the feedback operation.*

When the context is ambiguous, we will write the feedback $\uparrow f$ of a morphism $f : n + p \to n + q$ as $\uparrow^n f$.

Proposition 6.7.2 [a] *Let T be a preiteration theory. For each morphism $f : n + p \to n + q$, define*

$$\uparrow f \;:=\; f_2 \cdot \langle f_1^\dagger, \mathbf{1}_q \rangle : p \to q, \tag{6.27}$$

where $f = \langle f_1, f_2 \rangle$ with $f_1 : n \to n+q$ and $f_2 : p \to n+q$. Then iteration is expressible in terms of feedback, namely

$$f^\dagger \;=\; \uparrow\langle f,\, \mathbf{1}_n \oplus 0_p \rangle : n \to p, \tag{6.28}$$

for all $f : n \to n + p$. Further, the feedback operation satisfies the equation

$$\uparrow\langle f, g \rangle \;=\; g \cdot \langle \uparrow\langle f,\, \mathbf{1}_n \oplus 0_q \rangle,\, \mathbf{1}_q \rangle, \tag{6.29}$$

for all $f : n \to n + q$ and $g : p \to n + q$.

[b] *Let T be a prefeedback theory. For each morphism $f : n \to n+p$, define $f^\dagger : n \to p$ as in (6.28). If the equation (6.29) holds in T, then the feedback operation can be expressed in terms of iteration as in (6.27).*

[c] *Suppose that both iteration and feedback are defined on the theories T and T'. Suppose further that iteration and feedback are related by the equations (6.27) and (6.28). Then a theory morphism $\varphi : T \to T'$ preserves iteration if and only if φ preserves feedback.*

The straightforward proof of Proposition 6.7.2 is omitted.

If the iteration and feedback operations are related as above, any equation involving iteration has a trivial translation involving feedback and vice versa. To obtain the feedback translation of an iteration identity one has to express each dagger operation with feedback as given by (6.28). In the following proposition we provide translations of some iteration theory identities, some of which are obtained in a less obvious way.

Proposition 6.7.3 *Let T be a theory which is both a preiteration theory and a prefeedback theory. Suppose that the iteration and feedback operations are related by the equations (6.27) and (6.28).*

[a] *The left zero identity holds in T if and only if, for all $f : n \to p$, the* **feedback left zero identity**

$$\uparrow\langle 0_n \oplus f,\, \mathbf{1}_n \oplus 0_p\rangle \;=\; f$$

holds.

[b] *The right zero identity holds in T if and only if, for all $f : n + m \to n + p$, the* **feedback right zero identity**

$$\uparrow(f \oplus 0_q) \;=\; \uparrow f \oplus 0_q$$

holds.

[c] *The pairing identity holds in T if and only if, for all $f : n + m + p \to n + m + q$, the* **feedback pairing identity**

$$\uparrow^{n+m} f \;=\; \uparrow^m \uparrow^n f$$

holds.

[d] *The permutation identity holds in T if and only if, for all $f : n{+}p \to n{+}q$, and all base permutations $\pi : n \to n$, the* **feedback permutation identity**

$$\uparrow((\pi \oplus \mathbf{1}_p) \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_q)) \;=\; \uparrow f$$

holds.

Proof. The feedback left zero identity is the trivial translation of the left zero identity. Indeed, by (6.28),

$$(0_n \oplus f)^\dagger \;=\; \uparrow\langle 0_n \oplus f,\, \mathbf{1}_n \oplus 0_p\rangle,$$

for all $f : n \to p$. For the feedback right zero identity, let $f = \langle f_1, f_2\rangle : n+m \to n+p$ with $f_1 : n \to n+p$ and $f_2 : m \to n+p$. Then, by (6.27),

$$\begin{aligned}
\uparrow(f \oplus 0_q) &= \uparrow(\langle f_1, f_2\rangle \oplus 0_q)\\
&= \uparrow\langle f_1 \oplus 0_q,\, f_2 \oplus 0_q\rangle\\
&= (f_2 \oplus 0_q) \cdot \langle (f_1 \oplus 0_q)^\dagger,\, \mathbf{1}_{p+q}\rangle\\
&= f_2 \cdot \langle (f_1 \oplus 0_q)^\dagger,\, \mathbf{1}_p \oplus 0_q\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\uparrow f \oplus 0_q &= f_2 \cdot \langle f_1^\dagger, \mathbf{1}_p\rangle \oplus 0_q\\
&= f_2 \cdot \langle f_1^\dagger \oplus 0_q,\, \mathbf{1}_p \oplus 0_q\rangle.
\end{aligned}$$

Thus, the right zero identity holds if and only if the feedback right zero identity holds.

For the feedback pairing identity, write $f : n+m+p \to n+m+q$ as $f = \langle f_1, f_2, f_3\rangle$, where $f_1 : n \to n+m+q$, $f_2 : m \to n+m+q$ and $f_3 : p \to n+m+q$. Define

$$h \;:=\; f_2 \cdot \langle f_1^\dagger, \mathbf{1}_{m+q}\rangle : m \to m+q.$$

Then, again by (6.27),

$$\uparrow^{n+m} f \;=\; f_3 \cdot \langle \langle f_1, f_2\rangle^\dagger,\, \mathbf{1}_q\rangle,$$

and

$$\begin{aligned}
\uparrow^m\uparrow^n f^\dagger &= \uparrow^m(\langle f_2, f_3\rangle \cdot \langle f_1^\dagger, \mathbf{1}_{m+q}\rangle) \\
&= \uparrow^m\langle h,\ f_3 \cdot \langle f_1^\dagger, \mathbf{1}_{m+q}\rangle\rangle \\
&= f_3 \cdot \langle f_1^\dagger, \mathbf{1}_{m+q}\rangle \cdot \langle h^\dagger, \mathbf{1}_q\rangle \\
&= f_3 \cdot \langle f_1^\dagger \cdot \langle h^\dagger, \mathbf{1}_q\rangle,\ h^\dagger, \mathbf{1}_q\rangle.
\end{aligned}$$

Thus, the feedback pairing identity holds if and only if the pairing identity holds.

Now let $f = \langle f_1, f_2\rangle : n+p \to n+q$ with $f_1 : n \to n+p$ and $f_2 : p \to n+q$. Note that for any base permutation $\pi : n \to n$,

$$\begin{aligned}
&\uparrow((\pi \oplus \mathbf{1}_p) \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_q)) = \\
&\quad = \uparrow\langle \pi \cdot f_1 \cdot (\pi^{-1} \oplus \mathbf{1}_q),\ f_2 \cdot (\pi^{-1} \oplus \mathbf{1}_q)\rangle \\
&\quad = f_2 \cdot (\pi^{-1} \oplus \mathbf{1}_q) \cdot \langle (\pi \cdot f_1 \cdot (\pi^{-1} \oplus \mathbf{1}_q))^\dagger,\ \mathbf{1}_q\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\uparrow f &= f_2 \cdot \langle f_1^\dagger, \mathbf{1}_q\rangle \\
&= f_2 \cdot (\pi^{-1} \oplus \mathbf{1}_q) \cdot (\pi \oplus \mathbf{1}_q) \cdot \langle f_1^\dagger, \mathbf{1}_q\rangle \\
&= f_2 \cdot (\pi^{-1} \oplus \mathbf{1}_q) \cdot \langle \pi \cdot f_1^\dagger,\ \mathbf{1}_q\rangle.
\end{aligned}$$

It follows that the feedback permutation identity holds if and only if the permutation identity holds.

Proposition 6.7.4 *Suppose that a theory T is both a preiteration theory and a prefeedback theory and that the two operations are related by (6.27) and (6.28).*

[a] *The commutative identity holds in T if and only if the following* **feedback commutative identity** *holds:*

$$\begin{aligned}
&\uparrow(f \cdot (\rho \oplus \mathbf{1}_q)) = \\
&\quad = \uparrow\langle 1_{m+p} \cdot (\rho \oplus \mathbf{1}_p) \cdot f \cdot (\rho_1 \oplus \mathbf{1}_q), \ldots \\
&\qquad \ldots, (m+p)_{m+p} \cdot (\rho \oplus \mathbf{1}_p) \cdot f \cdot (\rho_{m+p} \oplus \mathbf{1}_q)\rangle,
\end{aligned}$$

for all $f : n+p \to m+q$, surjective base morphism $\rho : m \to n$ and for base morphisms $\rho_i : m \to m$, $i \in [m+p]$, with $\rho_i \cdot \rho = \rho$.

[b] *T satisfies the functorial dagger implication for a set of morphisms if and only if T satisfies the* **functorial feedback implication** *for , i.e. if for all $f : n + p \to n + q$ and $g : m + p \to m + q$ in T and $h : n \to m$ in ,*

$$f \cdot (h \oplus \mathbf{1}_q) \;=\; (h \oplus \mathbf{1}_p) \cdot g \quad \Rightarrow \quad \uparrow f \;=\; \uparrow g.$$

The proof of Proposition 6.7.4 is left as an exercise.

Exercise 6.7.5 Prove Proposition 6.7.4.

Exercise 6.7.6 Suppose that the iteration and feedback operations are related by the equations (6.27) and (6.28). Find feedback forms of some other iteration theory identities such as the parameter, fixed point and double dagger identities.

Corollary 6.7.7 [a] *Let T be a Conway theory. If the operation of feedback is defined by (6.27), then the feedback left zero, feedback right zero, feedback pairing and feedback permutation identities as well as the equation (6.29) hold in T. Further, iteration can be expressed in terms of feedback by (6.28).*

[b] *Let T be a prefeedback theory satisfying the feedback left zero, feedback right zero, feedback pairing and feedback permutation identities as well as the equation (6.29). If iteration is defined by (6.28), then T is a Conway theory and the feedback operation can be written in terms of iteration as in (6.27).*

Corollary 6.7.8 [a] *Let T be an iteration theory. If the operation of feedback is defined by (6.27), then the feedback left zero, feedback right zero, feedback pairing and feedback commutative identities as well as the equation (6.29) hold in T. Further, iteration can be expressed in terms of feedback by (6.28).*

[b] *Let T be a prefeedback theory satisfying the feedback left zero, feedback right zero, feedback pairing and feedback commutative identities as well as the equation (6.29). If iteration is defined by (6.28), then T is an iteration theory and the feedback operation can be written in terms of iteration as in (6.27).*

The above corollaries motivate the following definition.

Definition 6.7.9 *A* **Conway feedback theory** *is a prefeedback theory T which satisfies the feedback left zero, feedback right zero, feedback pairing and feedback permutation identities as well as the equation (6.29). A* **feedback theory** *is a prefeedback theory T satisfying the feedback left zero, feedback right zero, feedback pairing and feedback commutative identities and the equation (6.29). A morphism of feedback theories or Conway feedback theories is a prefeedback theory morphism.*

Thus a feedback theory is a Conway feedback theory in which the feedback commutative identity holds. By Corollaries 6.7.7 and 6.7.8, Conway feedback theories are equivalent to Conway theories and feedback theories to iteration theories. In fact, the category of iteration (or Conway) theories is isomorphic to the category of feedback (or Conway feedback) theories over $^{\mathbf{N}\times\mathbf{N}}$.

Corollary 6.7.10 *Let T be a Conway theory and a set of morphisms in T. T satisfies the functorial dagger implication for if and only if as a Conway feedback theory T satisfies the functorial feedback implication for .*

Exercise 6.7.11 Suppose that T is a Conway feedback theory. Show that if T satisfies the functorial feedback implication for all base morphisms $n \to 1$, $n \geq 1$, then T satisfies the functorial feedback implication for all (surjective) base morphisms. Conclude that, in this case, T is a feedback theory.

Exercise 6.7.12 Find the feedback form of the GA-implication considered in Section 6.3. Show that if the GA-implication holds in a Conway feedback theory T, then T satisfies the functorial feedback implication for base morphisms.

Exercise 6.7.13 Translate Theorems 6.2.15 and 6.2.20 to equivalent forms using the feedback operation.

Exercise 6.7.14 When T is a preiteration theory, define

$$f^{\ddagger} \;:=\; (f \cdot \pi)^{\dagger},$$

for all $f : n \to p + n$, where π is the block transposition $p + n \to n + p$. The operation ‡ is called *right iteration.* Conversely, when the theory T is equipped with an operation ‡, define

$$g^\dagger \; := \; (g \cdot \pi^{-1})^\ddagger,$$

where $g : n \to n + p$ and π is the base permutation defined above. Show that the iteration and right iteration operations are mutually expressible from one another. Translate the axioms of iteration theories to axioms involving right iteration.

6.8 Summary of the Axioms

For easy reference, we give a summary of iteration theory axioms and axioms for Conway theories studied in earlier sections.

6.8.1 Axioms for Iteration Theories

In essence, we have investigated three axiom systems of iteration theories.

The **A** group:

- the *left zero identity*

$$(0_n \oplus f)^\dagger \; = \; f, \quad f : n \to p; \tag{6.30}$$

- the *right zero identity*

$$(f \oplus 0_q)^\dagger \; = \; f^\dagger \oplus 0_q, \quad f : n \to n + p; \tag{6.31}$$

- the *(left) pairing identity*

$$\langle f, g\rangle^\dagger \; = \; \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle, \, h^\dagger\rangle, \tag{6.32}$$

  where $f : n \to n{+}m{+}p$, $g : m \to n{+}m{+}p$ and $h := g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle$;

- the *commutative identity*

$$((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \; = \; \rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \tag{6.33}$$

  where $f : n \to m + p$, $\rho : m \to n$ is a surjective base morphism, and where each $\rho_i : m \to n$, $i \in [m]$, is base with $\rho_i \cdot \rho = \rho$.

The **B** group:

- the *parameter identity*
$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger \;=\; f^\dagger \cdot g, \quad f : n \to n+p,\; g : p \to q; \qquad (6.34)$$
- the *composition identity*
$$(f \cdot \langle g,\, 0_n \oplus \mathbf{1}_p \rangle)^\dagger \;=\; f \cdot \langle (g \cdot \langle f,\, 0_m \oplus \mathbf{1}_p \rangle)^\dagger,\, \mathbf{1}_p \rangle, \qquad (6.35)$$
for all $f : n \to m+p$ and $g : m \to n+p$;
- the *double dagger identity*
$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger \;=\; f^{\dagger\dagger}, \qquad (6.36)$$
for all $f : n \to n+n+p$;
- the commutative identity (6.33).

The **C** group:

- the *scalar parameter identity* (6.34) above with $n = 1$;
- the *scalar composition identity* (6.35) with $n = m = 1$;
- the *scalar double dagger identity* (6.36) with $n = 1$;
- the *scalar pairing identity* (6.32) with $m = 1$;
- the *scalar commutative identity*
$$1_m \cdot ((\rho \cdot f) \parallel (\rho_1, \ldots, \rho_m))^\dagger \;=\; 1_n \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger, \qquad (6.37)$$
where $f : n \to m+p$, $\rho : m \to n$ is a monotone surjective base morphism, and where each $\rho_i : m \to n$, $i \in [m]$, is base with $\rho_i \cdot \rho = \rho$.

In the **B** group, the composition identity may be replaced by the following two identities:

- the *fixed point identity*
$$f \cdot \langle f^\dagger,\, \mathbf{1}_p \rangle \;=\; f^\dagger, \quad f : n \to n+p; \qquad (6.38)$$

- the *simplified composition identity*

$$(f \cdot g)^\dagger \;=\; f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger, \tag{6.39}$$

for all $f : n \to m$ and $g : m \to n + p$.

Some simplifications of the commutative identity may be found in Proposition 5.5.3.26.

It would be interesting to obtain a solution of the following

PROBLEM 6.8.1 Find a simple set of equational axioms for iteration theories. In particular, find essential simplifications of the commutative identity.

6.8.2 Axioms for Conway Theories

For each of the above three groups, there is a corresponding axiom system for Conway theories. In fact, replacing the commutative identity in group **A** by the *permutation* identity

$$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger \;=\; \pi \cdot f^\dagger, \tag{6.40}$$

for all $f : n \to n + p$ and base permutations $\pi : n \to n$, or simply by removing the commutative identity from group **B** or the scalar commutative identity from group **C**, we obtain a complete set of axioms of Conway theories. In group **A**, the block transposition identity (or the transposition identity) suffices instead of the permutation identity.

6.9 Notes

Iteration theories were defined in [BEW80a, BEW80b]. Theorem 6.1.2 is a combination of results in [BEW80a, BEW80b] (namely, parts [a], [b], [d] and [e]) and [Ési80] (part [c]). The present proof, which is based on Corollary 5.5.4.28 follows the argument in [Ési80]. Corollary 6.1.4 is also from [Ési80]. Many of the most frequently used identities of iteration theories already hold in Conway theories, justifying to our mind the relevance of this notion. For example, the results of the last chapter on program correctness proofs need only the Conway identities.

We use the term Conway theory because of the form these identities take in matrix iteration theories. In his book [Con71], Conway investigates the consequences of a number of identities in matrix theories, including the zero, pairing and permutation identities. We will investigate matrix iteration theories in Chapter 9. Corollaries 6.2.5-6.2.7 are based on [Ste87a]. The scalar axiomatizations of Section 6.2.1 are taken from [Ési82, Ési90]. The relation between the commutative identity and the functorial dagger implication was already noted in [Ési80]. The strong functorial dagger implication was considered by Arbib and Manes in [AM80] in a different context. Proposition 6.3.7 was proved in [Ési90]. The GA-implication has its origin in [AG87], where it used in the axiomatization of regular sets (see also Chapter 9). The present more general formalization as well as Proposition 6.3.11 and Corollary 6.3.12 are new. Pointed iterative theories were defined in [BEW80a, BEW80b]. Proposition 6.4.2 is new. Corollary 6.4.3 was noted in [Ési88]. The fact that for every iterative theory T and morphism $\perp : 1 \to 0$ there is a unique iteration theory structure on T such that $\mathbf{1}_1^\dagger = \perp$ was proved in [BEW80b]. The present proof, based on Theorem 6.4.5, is new, see also [Ési82]. Theorem 6.4.11 was proved in [Ési82]. Theorem 6.5.2 is from [BEW80b] and [Ési80]. Our proof using Corollary 5.5.4.28 is based on [Ési80]. The coproduct construction described in Lemma 6.6.9 is from [BÉb]. (Further useful results about congruences on iteration theories can be found in [Tro81].) The Zero Congruence Lemma is an extension of a result in [BT87]. The feedback operation was introduced into the study of iteration theories by Gh. Stefanescu. The results of Section 6.7 are based on [CSS88].

Chapter 7

Iteration Algebras

7.1 Definitions

In Chapter 4 we discussed the algebras of an arbitrary theory. This chapter deals with the algebras of (pre)iteration theories. When T is a preiteration theory, not all T-algebras respect the iteration operation. We say that a T-algebra $\mathbf{A} = (A, \alpha)$ is a T-*iteration algebra* whenever the kernel of the theory morphism $\alpha : T \to A$ is a dagger congruence.

Definition 7.1.1 *Let T be a preiteration theory. A T-algebra $\mathbf{A} = (A, \alpha)$ is a T-***iteration algebra** *if $(f^\dagger)_\alpha = (g^\dagger)_\alpha$ whenever $f_\alpha = g_\alpha$, for all T-morphisms $f, g : n \to n + p$, $n, p \geq 0$. We let $T^\dagger$ denote the full subcategory of $T^\flat$ determined by the T-iteration algebras.*

In this chapter, we will investigate the properties of the categories $T^\dagger$.

When T is a preiteration theory, we will write $\perp$ for $\mathbf{1}_1^\dagger$, and $\perp_n : n \to 0$ for the morphism $\langle \perp, \ldots, \perp \rangle : n \to 0$. Note that when T is a preiteration theory, the underlying set A of each T-algebra (A, α) is nonempty, since A must contain at least the element $\perp_\alpha$.

Exercise 7.1.2 Suppose that T is a preiteration theory satisfying the pairing identity (or one of its variants). Show that a T-algebra (A, α) is a T-iteration algebra if for each pair of scalar morphisms $f, g : 1 \to 1+p$ in T, $(f^\dagger)_\alpha = (g^\dagger)_\alpha$ whenever $f_\alpha = g_\alpha$.

If T is an iteration theory, then a T-iteration algebra $\mathbf{A}$ has solutions for all parameterized systems of fixed point equations of the form

$$\begin{aligned} x_1 &= (t_1)_\alpha(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+p}) \\ &\vdots \\ x_n &= (t_n)_\alpha(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+p}), \end{aligned}$$

where $t_i : 1 \to n+p$ in T, $i \in [n]$. Indeed, for any selection of elements $a_{n+1}, \ldots, a_{n+p}$ in A, the elements

$$a_i \;:=\; (i_n \cdot t^\dagger)_\alpha(a_{n+1}, \ldots, a_{n+p}), \quad i \in [n],$$

satisfy the system, where $t := \langle t_1, \ldots, t_n \rangle : n \to n+p$ in T. Further, the solutions to such systems are natural, in a well-defined sense.

Remark 7.1.3 For any preiteration theory T, the collection of all T-iteration algebras is an axiomatizable class of Σ-algebras, where $\Sigma_n := T(1,n),\ n \geq 0$. A Σ-algebra $\mathbf{A}$ is a T-iteration algebra iff $\mathbf{A}$ is a T-algebra, i.e. satisfies the equations mentioned in Remark 2.2.4.7, and also satisfies the following first order sentences:

$$(\forall x)(\forall y)[\ f(x,y) \;=\; g(x,y)\ ] \quad \Rightarrow \quad (\forall z)[\ f^\dagger(z) \;=\; g^\dagger(z)\ ],$$

for all $f, g : n \to n+p$ in T. Here, the variables y and z range over A^p and x ranges over A^n, and when $f = \langle f_1, \ldots, f_n \rangle$ and $g = \langle g_1, \ldots, g_n \rangle$, $f(x,y) = g(x,y)$ is an abbreviation for

$$\bigwedge_{i=1}^{n} f_i(x,y) \;=\; g_i(x,y).$$

(When the pairing identity holds in T, it is sufficient to require these sentences only for $n = 1$.) Thus, the collection of all T-iteration algebras is first-order axiomatizable either by universal-existential or by existential-universal sentences.

We give the fundamental definition. Recall the operations $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ on T-algebras.

Definition 7.1.4 *A full subcategory of* $T^\dagger$ *is a* **variety** *of* T*-iteration algebras if*

$$= \mathbf{H}() \cap T^\dagger \;=\; \mathbf{S}() \cap T^\dagger \;=\; \mathbf{P}() \cap T^\dagger.$$

Example 7.1.5 Let T be a free iteration theory $\Sigma\mathbf{tr}$. A T-algebra (A, α) is *ω-continuous* if the set A is an ω-complete partially ordered set with least element $\perp_A$, and each function $f_\alpha : A^p \to A$ preserves least upper bounds of ω-chains. It is also required that for each $f : n \to n + p$ in T, and each $a \in A^p$, $(f^\dagger)_\alpha(a)$ is the least fixed point of the equation

$$x \;\; = \;\; f_\alpha(x, a)$$

in the variable $x \in A^n$. Thus all ω-continuous algebras are T-iteration algebras. It can be proved that each ω-continuous $\Sigma\mathbf{tr}$-algebra (A, α) is totally determined by the partial order on A and the functions σ_α, for the atomic trees $\sigma : 1 \to n,\ \sigma \in \Sigma_n$. We will consider ordered algebras and theories in more detail in Chapter 8.

Example 7.1.6 Another class of iteration algebras is determined by the *iterative algebras*. Suppose that T is an iterative theory. A T-algebra $\mathbf{A} = (A, \alpha)$ is a *T-iterative algebra* if two conditions hold. First, t_α is a projection function $A^n \to A$ iff t is a distinguished morphism. Second, for each ideal morphism $t : n \to n+p$ in T, there is a unique function $f : A^p \to A^n$ which is a solution of the iteration equation for t_α, in A, i.e. f is the unique function such that

$$f(a) \;\; = \;\; t_\alpha(f(a), a),$$

for all $a \in A^p$. Thus $f = (t^\dagger)_\alpha$. Equivalently, for any n-tuple of ideal morphisms $t_i : 1 \to n + p$, $i = 1, \ldots, n$, and any p-tuple $a = (a_1, \ldots, a_p) \in A^p$, there is a unique solution in A to the system of equations

$$\begin{aligned} x_1 \;\; &= \;\; (t_1)_\alpha(x_1, \ldots, x_n, a_1, \ldots, a_p) \\ &\;\;\vdots \\ x_n \;\; &= \;\; (t_n)_\alpha(x_1, \ldots, x_n, a_1, \ldots, a_p). \end{aligned}$$

By choosing a distinguished morphism $\perp : 1 \to 0$ in T, T becomes a pointed iterative theory $(T,\ \mathbf{1}_1^\dagger = \perp)$ and the theory congruence $\ker \alpha$ on the resulting iteration theory is a dagger congruence. Indeed, assume that $t, t' : 1 \to 1 + p$ in T and $t_\alpha = t'_\alpha$. If either t or t' is ideal, then so is the other and hence $(t^\dagger)_\alpha = (t'^\dagger)_\alpha$, since each is a solution of the iteration equation for t_α; if either is not ideal, then $t = t'$ by the first assumption on α. Thus, the iterative algebras of an iterative theory are iteration algebras. (In the case T has no morphism $1 \to 0$, we can adjoin a morphism $\perp : 1 \to 0$ freely (using the free pointed iterative theory, and then choosing an element $\perp_A$ in A as the value of $\perp_\alpha$. See Theorem 6.6.4.11 and Remark 6.6.4.13.)

7.2 Free Algebras in $T^\dagger$

First we dispose of the case of free algebras in $T^\dagger$ generated by an infinite set. We recall from Corollary 4.4.2.7 that the T-algebra

$$\mathbf{F}_T[X] \quad := \quad (T[X](1,0), \tau_X)$$

is freely generated in $T^\flat$ by the map $\eta_X := \lambda : X \to T[X](1,0)$, defined in Corollary 4.4.2.7. Also, it was shown in Proposition 4.4.2.8 that if X is infinite, the kernel of τ_X is the identity theory congruence. Hence, when X is infinite, $\mathbf{F}_T[X]$ belongs to $T^\dagger$, and is therefore the free algebra in $T^\dagger$ also.

Corollary 7.2.1 *For each infinite set X, the T-algebra $\mathbf{F}_T[X]$ is freely generated in $T^\dagger$ by η_X.*

Next, we consider the finitely generated free algebras in $T^\dagger$. As will be seen, their existence is not assured. We have shown in Theorem 4.4.2.2 that for any theory T, the n-generated free algebra in $T^\flat$ is the T-algebra $T_n = (T(1,n), \kappa_n)$, where, for $t : 1 \to p$ in T,

$$t_{\kappa_n}(g_1, \ldots, g_p) \quad := \quad t \cdot \langle g_1, \ldots, g_p \rangle.$$

More precisely, T_n is freely generated in $T^\flat$ by the function

$$\begin{aligned} \eta_n : V_n &\quad\to\quad T(1,n) \\ x_i &\quad\mapsto\quad i_n : 1 \to n. \end{aligned}$$

Recall that V_n is the set $\{x_1, \ldots, x_n\}$. Whenever the kernel of κ_n, denoted $\sim_n$, is a dagger congruence, it follows that $T_n \in T^\dagger$. Hence when $\sim_n$ is a dagger congruence, T_n is freely generated (by η_n) in $T^\dagger$, as well.

We show now that $T^\dagger$ has an n-generated free algebra only when $T_n \in T^\dagger$.

Proposition 7.2.2 *If $\eta'_n : V_n \to A$ freely generates $\mathbf{A}$ in $T^\dagger$, then $\mathbf{A}$ is isomorphic to T_n, so that T_n is also in $T^\dagger$.*

Proof. Let X be the countable set $\{x_1, x_2, \ldots\}$. Consider the X-generated free algebra $\mathbf{F}_T[X]$ in $T^\flat$. As just shown, this algebra belongs to $T^\dagger$. By the fact that all algebras involved are free, there are homomorphisms h, g, h', g' such that $x_i \eta'_n h = x_i \eta_n h' = x_i \eta_X,\ i \in [n]$, and such that

$$\begin{aligned} \mathbf{1_A} &= \mathbf{A} \xrightarrow{h} \mathbf{F}_T[X] \xrightarrow{g} \mathbf{A} \\ \mathbf{1}_{T_n} &= T_n \xrightarrow{h'} \mathbf{F}_T[X] \xrightarrow{g'} T_n. \end{aligned}$$

It follows that $h \cdot g' : \mathbf{A} \to T_n$ is an isomorphism in $T^\flat$ with inverse $h' \cdot g$, since the composites $(h \cdot g') \cdot (h' \cdot g)$ and $(h' \cdot g) \cdot (h \cdot g')$ are the identity on the images of the generating set V_n.

Corollary 7.2.3 *For each $n \geq 0$, $T^\dagger$ contains an n-generated free algebra iff T_n belongs to $T^\dagger$.*

To show that these finitely generated iteration algebras do not always exist, we give an example of an iteration theory T with no initial algebra in $T^\dagger$.

Example 7.2.4 Let Σ be the signature with one letter σ in Σ_1 and otherwise Σ_n is empty. Let $\sim$ be the least theory congruence on $\Sigma\mathbf{tr}$ such that

$$\sigma \cdot \perp \quad \sim \quad \perp,$$

where $\perp = \mathbf{1}_1^\dagger$, as usual. By the Zero Congruence Lemma, $\sim$ is also a dagger congruence, so that $T = \Sigma\mathbf{tr}/\!\sim$ is an iteration theory. Note that in T, there are exactly two morphisms $1 \to 0$; namely $[\perp]$ and $[\sigma^\dagger]$, where we are writing $[t]$ for the congruence class of the Σ-tree t. It follows that $T_0 = (T(1,0), \kappa_0)$ is not in $T^\dagger$. Indeed,

$$[\sigma] \quad \sim_0 \quad [\mathbf{1}_1],$$

since for each of the two morphisms $f : 1 \to 0$ in T,

$$\sigma \cdot f \quad \sim \quad f.$$

However, $[\sigma^\dagger] \ \not\sim_0\ [\perp]$ since $\sigma^\dagger \ \not\sim\ \perp$. Thus, T_0 is not a T-iteration algebra. Therefore, by Proposition 7.2.2, there can be no initial algebra in $T^\dagger$. Note that for any $n \geq 1$, the congruence $\sim_n$ is the identity on T. Hence, $T^\dagger$ has all free algebras generated by nonempty sets.

Example 7.2.5 A *matrix iteration theory* is a matrix theory which is also an iteration theory. These theories will be considered in detail in Chapter 9. Suppose that $T = \mathbf{Mat}_S$ is a matrix iteration theory. Then, for any $n \geq 1$, the T-algebra $T_n = (T(1,n), \kappa_n)$ is a T-iteration algebra, since it is easy to see that $f_{\kappa_n} = g_{\kappa_n}$ iff $f = g$, for any $f, g : 1 \to k$ in T. Thus, the congruence $\sim_n$ is the identity. In the case $n = 0$, since there is a unique morphism $1 \to 0$ in T, the congruence $\sim_0$ is trivial: $f_{\kappa_0} = g_{\kappa_0}$, for all $f, g : 1 \to k$ in T. Hence, $T^\dagger$ has all finitely generated free algebras.

In fact, it is not just initial algebras that may fail to exist. The situation is the worst possible.

Theorem 7.2.6 *For any set Z of nonnegative integers, there is an* **iteration** *theory T such that $T^\dagger$ has an n-generated free iteration algebra iff $n \in Z$.*

Exercise 7.2.7 Show that the following fact implies Theorem 7.2.6. For each $n \geq 0$, there is an iteration theory U_n such that ${U_n}^\dagger$ has a k-generated free algebra iff $k \neq n$.

We now obtain a simple, but important property of iteration algebras.

7.3 The Retraction Lemma

The following result will be a principal tool in the remainder of the chapter.

Lemma 7.3.1 The Retraction Lemma. *Suppose that T is a preiteration theory satisfying the parameter identity. Let (A, α) be any T-algebra. Then there is a T-iteration algebra (B, β) such that (A, α) is a retract of (B, β).*

Proof. Consider the unique theory morphism $\alpha_{id} : T[A] \to A$ determined by α and the function taking $a \in A$ to $a : 1 \to 0$ in A. Since T satisfies the parameter identity, $T[A]$ is also a preiteration theory satisfying the parameter identity, as shown in Lemma 6.6.6.9. Further, it is shown there that the theory morphism $\kappa : T \to T[A]$ is a preiteration theory morphism. (From now on, we identify $t\kappa$ and $a\lambda$ in $T[A]$ with $t \in T$ and $a \in A$, respectively, since both maps are injective.) Let θ_0 be the following relation on the morphisms with target 0 in $T[A]$:

$$t \;\theta_0\; t' \quad \Leftrightarrow \quad t_{\alpha_{id}} \;=\; {t'}_{\alpha_{id}}.$$

Then θ_0 is a left invariant equivalence relation, so the least theory congruence θ on $T[A]$ over θ_0 is also a dagger congruence, by the Zero Congruence Lemma. Let $T(\mathbf{A})$ denote the quotient of $T[A]$ by this theory congruence. Since θ is included in the kernel of α_{id}, the theory morphism α_{id} factors through the quotient map $T[A] \to T(\mathbf{A})$. Writing $[a]$ for the θ-congruence class of the element $a \in A$, we claim that the function

$$\begin{array}{rcl} h : A & \to & T(\mathbf{A})(1,0) \\ a & \mapsto & [a] \end{array}$$

is a bijection. Indeed, h is surjective, since any morphism $1 \to 0$ in $T(\mathbf{A})$ is of the form $[t \cdot \langle a_1, \ldots, a_n \rangle]$, and

$$t \cdot \langle a_1, \ldots, a_n \rangle \quad \equiv \quad t_\alpha(a_1, \ldots, a_n) \quad (\theta),$$

by definition of θ. The function h is injective, since the identity factors through α_{id}. It now follows that h is an isomorphism between $\mathbf{A}$ and the T-algebra

$$T \to T[A] \to T(\mathbf{A}) \to T(\mathbf{A})(1,0).$$

But if X is an infinite set, by Exercise 4.4.2.10, the initial $T(\mathbf{A})$-algebra

$$T(\mathbf{A}) \;\to\; T(\mathbf{A})(1,0)$$

is a retract of the X-generated $T(\mathbf{A})$-algebra, $\mathbf{F}_{T(\mathbf{A})}[X]$. But by Corollary 7.2.1, this algebra is also a $T(\mathbf{A})$-iteration algebra. By precomposition with the preiteration theory morphism $T \to T[A] \to T(\mathbf{A})$, we see that $\mathbf{A}$ is a retract of the T-iteration algebra

$$T \to T[A] \to T(\mathbf{A}) \to T(\mathbf{A})[X] \to T(\mathbf{A})[X](1,0).$$

In the next two exercises, we outline an alternative proof of Proposition 7.2.2 for preiteration theories satisfying the parameter identity.

Exercise 7.3.2 Suppose that is a full subcategory of such that each object in is a retract of some object in ; i.e. for each object c in , there is an object d of and morphisms $\mu_c : c \to d$, $\varepsilon_c : d \to c$ such that

$$\mathbf{1}_c \quad = \quad \mu_c \cdot \varepsilon_c. \tag{7.1}$$

Let $U :\to$ be a functor, and let $U' :\to$ be the restriction of U to . Suppose that for some object x in , $\eta : x \to dU'$ is a U'-universal arrow for x. Then η is also a U-universal arrow for x. *Hint:* Suppose that c is a -object and $f : x \to cU$ is any -arrow. Let

$$\mathbf{1}_c \quad = \quad c \stackrel{\mu_c}{\to} d_1 \stackrel{\varepsilon_c}{\to} c.$$

Then define g as the composite

$$g \quad := \quad f \cdot \mu_c U : x \to d_1 U'.$$

Since η is U'-universal, there is a unique -morphism $g^\sharp : d \to d_1$ with $\eta \cdot g^\sharp U' = g$. Consider $f^\sharp := g^\sharp \cdot \varepsilon_c$. Show that $f^\sharp$ is the unique morphism $d \to c$ such that $\eta \cdot f^\sharp U = f$.

Exercise 7.3.3 Suppose that T is a preiteration theory which satisfies the parameter identity, so that every algebra in $T^\flat$ is a retract of an algebra in $T^\dagger$. Use the previous exercise to prove Proposition 7.2.2.

7.4 Some Categorical Facts

In this section, motivated by the Retraction Lemma, we state and prove a theorem which applies immediately to the categories $T^\flat$ and $T^\dagger$ whenever T is a preiteration theory satisfying the parameter identity. First, we make the following observation.

Lemma 7.4.1 *Suppose that is a full subcategory of such that each object in is a retract of some object in . Then epis, monics, and coequalizers in are also epis, monics and coequalizers in .*

Proof. First assume that $e : d \to d'$ is an epi in . We show e is an epi in . Assume that $f, g : d' \to c$ is a parallel pair in such that $e \cdot f = e \cdot g$. Assume that $\mathbf{1}_c = \mu_c \cdot \varepsilon_c$, where $\mu_c : c \to d_1$, say, with d_1 an object in . Then,

$$e \cdot (f \cdot \mu_c) \quad = \quad e \cdot (g \cdot \mu_c),$$

so that $f \cdot \mu_c = g \cdot \mu_c$, since e is epi in . But since μ_c is monic, it follows that $f = g$.

Similarly it follows that if m is a monic in then m is monic in .

Now assume that $e : d \to d'$ is a coequalizer in of the parallel pair $f, g : d_2 \to d$. Let $e' : d \to c$ be any -morphism such that $f \cdot e' = g \cdot e'$. Let

$$\mathbf{1}_c \quad = \quad \mu_c \cdot \varepsilon_c$$

where $\mu_c : c \to d_1$, say, with d_1 an object in . Then, since e is a coequalizer of f, g in , there is a unique morphism $k : d' \to d_1$ with $e \cdot k = e' \cdot \mu_c$. But then,

$$\begin{aligned} e \cdot (k \cdot \varepsilon_c) &= e' \cdot \mu_c \cdot \varepsilon_c \\ &= e'. \end{aligned}$$

The uniqueness of $k \cdot \varepsilon_c$ follows from the fact that e is an epi in and hence in . Thus e is the coequalizer of f, g in .

Theorem 7.4.2 *Suppose that is complete, cocomplete, well-powered and has an -$\mathcal{M}$-factorization system, for some collection of epis and monics $\mathcal{M}$. Suppose also that is subcategory of the category which has the following properties: is full, replete, cowell-powered and is closed under products. Lastly, assume that each object in is a retract of some object in . Then, the following statements are equivalent.*

[a] $=$.

[b] *Every morphism in factors as an epi composed with a monic.*

[c] *has equalizers.*

[d] *is complete.*

[e] *has coequalizers.*

[f] *is cocomplete.*

[g] *The inclusion functor $\to$ has a left adjoint.*

Proof that [a] implies [b]. Trivial, since has an -$\mathcal{M}$ factorization system.

Proof that [b] implies [c]. Suppose that $f, g : a \to b$ is a parallel pair of morphisms in , and that $m : c \to a$ is an equalizer in . Suppose that $\mathbf{1}_c = \mu_c \cdot \varepsilon_c$, where $\varepsilon_c : d \to c$, and d is an object in . Factor the composite $\varepsilon_c \cdot m : d \to a$ as $e' \cdot m'$ in , where e' is an epi and m' is a monic. Then we claim m' is an equalizer of f, g in . Clearly, m' forks f and g, since e' is epi. But if h is any morphism which forks f, g, then $h = k \cdot m$, for some k, since m is an equalizer in . But then

$$\begin{aligned} h &= k \cdot \mu_c \cdot \varepsilon_c \cdot m \\ &= (k \cdot \mu_c \cdot e') \cdot m' \\ &= k' \cdot m', \end{aligned}$$

for $k' := k \cdot \mu_c \cdot e'$. If h is in , k' is a -morphism. Since m' is monic, k' is unique. This shows that m' is an equalizer in .

Proof that [c] implies [d]. Trivial, since is assumed to have products.

Proof that [d] implies [e]. Since is cowell-powered and complete, if we can show is well-powered, then by [HS73], Theorem 23.11, it follows that has coequalizers. But since is well-powered, so is . (The hypothesis of Theorem 23.11 in [HS73] is that the category is only extremally cowell-powered; but any cowell-powered category is also extremally cowell-powered.)

Proof that [e] implies [a]. Suppose that c is any object of and assume (7.1) holds. Then it is easy to see that ε_c is a coequalizer in of $\mathbf{1}_d$ and $\varepsilon_c \cdot \mu_c$. Let $e' : d \to d'$ be a coequalizer in of $\mathbf{1}_d$ and $\varepsilon_c \cdot \mu_c$. Then by Lemma 7.4.1, e' is also a coequalizer in , so that c and d' are isomorphic. Since is replete, c belongs to .

Proof that [a] iff [f]. If = then is cocomplete, since is. Conversely, if is cocomplete, has coequalizers, and thus =, by the previous argument.

Proof that [a] iff [g]. If = then the inclusion functor is the identity, which has itself as left adjoint. Conversely, suppose the inclusion has a left adjoint. For any -object c, let

$$\rho : c \quad \to \quad d$$

be a universal arrow. Then if $\mathbf{1}_c = c \xrightarrow{\mu_c} d_1 \xrightarrow{\varepsilon_c} c$, there is a unique arrow $f : d \to d_1$ with

$$\rho \cdot f \quad = \quad \mu_c.$$

We claim that ρ is an isomorphism with inverse $f \cdot \varepsilon_c$. Indeed, for one direction

$$\rho \cdot (f \cdot \varepsilon_c) \quad = \quad \mu_c \cdot \varepsilon_c$$

$$= \mathbf{1}_c.$$

For the other,

$$\begin{aligned}\rho \cdot (f \cdot \varepsilon_c \cdot \rho) &= \rho \\ &= \rho \cdot \mathbf{1}_d,\end{aligned}$$

showing that $f \cdot \varepsilon_c \cdot \rho = \mathbf{1}_d$. Since is replete, c belongs to . The proof of the theorem is complete.

As corollaries of Theorem 7.4.2, we obtain some results about varieties of iteration algebras.

7.5 Properties of $T^\dagger$

Proposition 7.5.1 *Let T be a preiteration theory satisfying the parameter identity. In the category $T^\dagger$,*

[a] *the monics are precisely the injective homomorphisms;*

[b] *the epis are precisely the epis in $T^\flat$;*

[c] *the surjective homomorphisms are precisely the regular epis.*

Proof. By Lemma 7.4.1 we need to show these characterizations of monics and regular epis hold for $T^\flat$, but these facts are well-known. See Remark 4.4.1.9.

One obvious corollary is:

Proposition 7.5.2 *For any preiteration theory T satisfying the parameter identity, $T^\dagger$ is well-powered.*

What is perhaps not so obvious is

Proposition 7.5.3 *For any preiteration theory T, $T^\dagger$ is cowell-powered.*

Proof. We will show that for any fixed T-iteration algebra $\mathbf{A}$ that there is a representative set of epis in $T^\dagger$ with source $\mathbf{A}$. First, recall that according to Remark 7.1.3, $T^\dagger$ is an axiomatizable class. In Freyd [Fre64], page 93, it is shown that if $f : \mathbf{A} \to \mathbf{B}$ is an epi in any category consisting of all models of some set of first order sentences written in a language L, then the cardinality of $\mathbf{B}$ is not greater that $2^{|\mathbf{A}|+|L|}$. It follows that such categories, including $T^\dagger$, are cowell-powered.

Proposition 7.5.4 *For any preiteration theory T, $T^\dagger$ has products, and products are those of $T^\flat$. Thus, if $\mathcal{V}$ is any collection of T-iteration algebras, $\mathbf{P}(\mathcal{V}) \cap T^\dagger = \mathbf{P}(\mathcal{V})$.*

Proof. Suppose that (A_i, α_i), $i \in I$, are T-iteration algebras. Let (A, α) be the product of these algebras in $T^\flat$. Write $\sim_i$ for the theory congruence determined by $\alpha_i : T \to A_i$. Then since α is the target tupling of the α_i, we have $\sim_\alpha = \bigcap_{i \in I} \sim_i$, so that $\sim_\alpha$ is a dagger congruence when each $\sim_i$ is.

The following result is a version of Birkhoff's theorem. Its proof a minor variation of the proof of Theorem 4.4.3.3.

Theorem 7.5.5 *Let T be a preiteration theory. Suppose that $\mathcal{V}$ is a full subcategory of $T^\dagger$. Then $\mathcal{V}$ is a variety of T-iteration algebras iff there is a dagger congruence θ on T such that a T-iteration algebra $\mathbf{A} = (A, \alpha)$ belongs to iff α factors through the canonical map $T \to T/\theta$:*

[T‘T/θ‘A; ‘α‘$\overline{\alpha}$] Hence is isomorphic to $(T/\theta)^\dagger$ over .

Exercise 7.5.6 Provide a proof for this theorem.

The next theorem is one of the main results in this section.

Theorem 7.5.7 *The following statements are equivalent for a preiteration theory T satisfying the parameter identity.*

[a] $T^\dagger = T^\flat$.

[b] *Every theory congruence on T is a dagger congruence.*

[c] *Every homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *in* $T^\dagger$ *factors as*

$$h \quad = \quad \mathbf{A} \xrightarrow{e} \mathbf{C} \xrightarrow{m} \mathbf{B},$$

where e *is a regular epi and* m *is a monic.*

[d] *Every homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *in* $T^\dagger$ *factors as* $h = e \cdot m$, *where* e *is an epi and* m *is a monic.*

[e] $T^\dagger$ *has equalizers.*

[f] $T^\dagger$ *is complete.*

[g] $T^\dagger$ *has coequalizers.*

[h] $T^\dagger$ *is cocomplete.*

[i] *The inclusion functor* $T^\dagger \to T^\flat$ *has a left adjoint.*

[j] *For each* $\mathbf{A} \in T^\flat$, *and* $\mathbf{B} \in T^\dagger$ *with* $\mathbf{A}$ *a subalgebra of* $\mathbf{B}$, *there is a least subalgebra* $\mathbf{C}$ *in* $T^\dagger$ *with* $|\mathbf{A}| \subseteq |\mathbf{C}| \subseteq |\mathbf{B}|$.

Proof. The equivalence of [a] and [b] follows from the definition of $T^\dagger$ together with Corollary 3.3.2.3. Otherwise, the only thing that doesn't follow immediately from Theorem 7.4.2 is the equivalence of [j] and [a]. But the fact that [a] implies [j] is obvious, and it is clear that [j] implies the existence of equalizers in $T^\dagger$.

Exercise 7.5.8 Give a direct proof that $T^\dagger$ is extremally cowell-powered.

We note the following relationship between varieties of T-algebras and varieties of T-iteration algebras.

Theorem 7.5.9 *Suppose that* T *is a preiteration theory.*

- *Let* $\mathcal{V}$ *be a variety of* T*-algebras. Then* $\mathcal{V} \cap T^\dagger$ *is a variety of* T*-iteration algebras.*
- *Let* $\mathcal{V}$ *be a variety of* T*-iteration algebras. Then* $\mathbf{H}(\mathcal{V})$, *the full subcategory of* $T^\flat$ *determined by all quotients of algebras in* $\mathcal{V}$, *is a variety of* T*-algebras. Moreover, if* T *satisfies the parameter identity,* $\mathbf{H}(\mathcal{V}) = \mathbf{S}(\mathcal{V})$.

- *The map from varieties of T-iteration algebras to varieties of T-algebras*

$$\mathcal{V} \quad \mapsto \quad \mathbf{H}(\mathcal{V}),$$

is injective, but not always surjective.

Proof. The first two statements are obvious, as well as the claim that $\mapsto \mathbf{H}()$ is injective, since $= \mathbf{H}() \cap T^{\dagger}$. When T satisfies the parameter identity, and $\mathcal{V}$ is a variety of T-iteration algebras, the fact that $\mathbf{H}() \subseteq \mathbf{S}()$ follows from the Retraction Lemma and the Birkhoff Theorem 7.5.5. The opposite inclusion is routine and uses Corollary 7.2.1.

Now we show the map $\mapsto \mathbf{H}()$ is not always surjective. Let T be the free iteration theory $\Sigma\mathbf{tr}$, where $\Sigma_1 = \{f, g\}$ and $\Sigma_n = \emptyset,\ n \neq 1$. Let $\mathcal{W}$ be the full subcategory of $T^{\flat}$ determined by those T-algebras (A, α) in which $f_\alpha = g_\alpha$. If $\mathbf{B} = (B, \beta)$ is in $\mathcal{W} \cap T^{\dagger}$, then $(f^{\dagger})_\beta = (g^{\dagger})_\beta$. Thus, if $(A, \alpha) \in \mathcal{W}$ and if $(f^{\dagger})_\alpha \neq (g^{\dagger})_\alpha$, then (A, α) is not a quotient of any T-iteration algebra in $\mathcal{W}$. A slight modification of Example 7.2.4 shows that such algebras exist. Hence if $\mathbf{H}(\mathcal{V}) \subseteq \mathcal{W}$, for some variety of T-iteration algebras, then $\mathbf{H}() \neq \mathcal{W}$.

Exercise 7.5.10 Modify Example 7.2.4 to find a $\Sigma\mathbf{tr}$-algebra $\mathbf{A}$, with $\Sigma_1 = \{f, g\}$ and Σ_n empty otherwise, such that $f_\alpha = g_\alpha$ but $(f^{\dagger})_\alpha \neq (g^{\dagger})_\alpha$. *Hint:* Consider the least theory congruence on $\Sigma\mathbf{tr}$ such that $f \sim g$.

Exercise 7.5.11 Prove or disprove the following statement. If T is an iteration theory such that $T^{\dagger}$ has all free algebras, then $T^{\flat} = T^{\dagger}$.

Each theory A will be shown to be an iteration theory in Chapter 8. We now show, by means of a number of exercises, that each of the theories $T = A$ has the property that $T^{\flat} = T^{\dagger}$. We find all of the theory congruences on A, and show that each one is also a dagger congruence.

Exercise 7.5.12 For each cardinal number $\kappa > 0$, define the relation

$$f \ \sim_\kappa \ g \quad \Leftrightarrow \quad |D(f, g)| \ < \ \kappa,$$

where $f, g : n \to p$ in A, and where

$$D(f, g) \quad := \quad \{(x, j) \in A \times [p] \ : \ (x, j)f^{-1} \ \neq \ (x, j)g^{-1}\}.$$

Here, $(x, j)f^{-1}$ is the set of all elements in $A \times [n]$ which map by f to the element (x, j). Show if $\kappa = 1$ or κ is infinite, then the relation $\sim_\kappa$ is a dagger congruence on A.

Exercise 7.5.13 Show that if A has at least two theory congruences, (i.e. if A is nonempty) then the smallest nonidentity theory congruence is $\sim_{\aleph_0}$.

Exercise 7.5.14 [Tro81] Prove that if A is finite and nonempty, there are only two theory congruences on A: the identity and the trivial congruence. When A is empty, there is only one congruence on A.

Exercise 7.5.15 Suppose that A is an infinite set. Prove that any theory congruence θ on A is determined by the relation $\theta_{1,1}$; i.e. for $f, g : 1 \to p$, $p \geq 2$, $f \ \theta \ g$ iff $(f \cdot \pi) \ \theta \ (g \cdot \pi)$, where $\pi : A \times [p] \to A$ is any bijection.

Exercise 7.5.16 Let $\sim$ be a theory congruence on A. Let $f, g : 1 \to 1$. Suppose that $D = D(f, g) = \{a : af^{-1} \neq ag^{-1}\}$. If the partial identity function p_D is congruent to the identity $\mathbf{1}_A$, show that $f \sim g$. The function p_D is defined as follows:

$$xp_D \quad := \quad \begin{cases} x & \text{if } x \notin D; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Exercise 7.5.17 Let $\sim$ be a theory congruence on A. Suppose that

$$f_0 \ \sim \ g_0 : 1 \ \to \ 1$$

and let

$$D \ := \ \{a \in A \ : \ af_0^{-1} \setminus ag_0^{-1} \ \neq \ \emptyset\}.$$

Prove there is a partial function $h_0 : 1 \to 1$ with

- $h_0 \ \sim \ \mathbf{1}_A$;
- $\{a : ah_0 \neq a\} \ = \ D$.

Exercise 7.5.18 Suppose that $h_0 : 1 \to 1$ in A and

$$D \ = \ \{a \in A \ : \ ah_0 \ \neq \ a\}.$$

If D is infinite, show there is a subset E of D with $|E| = |D|$ and $E \cap h_0(E) = \emptyset$.

Exercise 7.5.19 Suppose that $h_0 : 1 \to 1$ in A and $\sim$ is a theory congruence on A with $h_0 \sim \mathbf{1}_A$. If E is a subset of A with $E \cap h_0(E) = \emptyset$, then show $p_E \sim \mathbf{1}_A$.

Exercise 7.5.20 Suppose that $\sim$ is a theory congruence on A such that $p_E \sim \mathbf{1}_A$, for some subset E of A. Show that if $i : E' \to E$ is an injective function, then $p_{E'} \sim \mathbf{1}_A$.

Exercise 7.5.21 Suppose that $\sim$ is a theory congruence on A. Suppose that for some pair of partial functions $f_0, g_0 : 1 \to 1$ with $f_0 \sim g_0$, the set $\{a : af_0^{-1} \neq ag_0^{-1}\}$ is infinite with cardinality κ. Show that for any two partial functions $f, g : 1 \to 1$, if $|\{a : af^{-1} \neq ag^{-1}\}| \leq \kappa$, then $f \sim g$.

Exercise 7.5.22 Suppose A is infinite and $\sim$ is a nonidentity theory congruence on A. Let κ be the least infinite cardinal greater than $|\{a : af^{-1} \neq ag^{-1}\}|$, for all $f, g : 1 \to 1$ such that $f \sim g$. Prove that $\sim$ is $\sim_\kappa$.

Exercise 7.5.23 [BÉb] Use the previous exercises to prove that for infinite sets A, the lattice of theory congruences on A is a well-ordered chain

$$\sim_1, \sim_{\aleph_0}, l, \sim_{|A|}, \sim_{|A|^+}$$

with least element the identity congruence $\sim_1$, and greatest element the trivial congruence $\sim_{|A|^+}$. ($|A|^+$ is the least cardinal greater than $|A|$.)

7.6 A Characterization Theorem

Suppose that T and R are theories, not necessarily iteration theories. Suppose that $\varphi : T \to R$ is a theory morphism. Then, recalling Proposition 4.4.1.6, φ induces a functor

$$\varphi^\flat : R^\flat \quad \to \quad T^\flat$$

which takes

$$(A, \alpha) \quad \xrightarrow{h} \quad (B, \beta)$$

to

$$(A, \varphi \cdot \alpha) \quad \xrightarrow{h} \quad (B, \varphi \cdot \beta).$$

The functors $\varphi^\flat$ are called *algebraic functors*. The following facts are known about algebraic functors (see [Law63, HS73] and Section 4.4.4).

Theorem 7.6.1 [a] *A functor $R^\flat \xrightarrow{H} T^\flat$ is algebraic iff H commutes with the underlying set functors; i.e., iff*

$$U_R = H \cdot U_T.$$

[b] *Every algebraic functor $R^\flat \xrightarrow{H} T^\flat$ has a left adjoint.*

In this section, we consider functors

$$\varphi_\dagger : R^\dagger \rightarrow T^\dagger$$

induced by preiteration theory morphisms $\varphi : T \to R$. We call such functors *iteration functors.* In order to obtain a necessary condition that a functor be an iteration functor, we consider the category ICL, of *iteration clones.* An object in ICL is a pair $(A, \mathcal{F})$ where A is a set and $\mathcal{F}$ is a *concrete preiteration theory* of functions on A; i.e. a morphism $n \to p$ is a function $A^p \to A^n$; it is assumed that for each function $f : A^{n+p} \to A^n$ in $\mathcal{F}$ there is a function $f^\dagger : A^p \to A^n$ in $\mathcal{F}$. A morphism $(A, \mathcal{F}) \to (B, \mathcal{G})$ in ICL is a pair of functions:

$$\begin{aligned} h : A &\rightarrow B \\ \varphi : \mathcal{F} &\rightarrow \mathcal{G} \end{aligned}$$

such that φ is a preiteration theory morphism, and for all $f : A^p \to A^n$ in $\mathcal{F}$, the following diagram commutes:

[A^p'A^n'B^p'B^n; f'h^p'h^n'$f\varphi$] For objects $(A, \mathcal{F})$ and $(A, \mathcal{G})$ in ICL with the same underlying set, write $(A, \mathcal{F}) \sqsubseteq (A, \mathcal{G})$ if $\mathcal{F}$ is a subpreiteration theory of $\mathcal{G}$, i.e. the set of functions $\mathcal{F}$ is a subset of $\mathcal{G}$, and if the dagger operation on the functions in $\mathcal{F}$ is the same as that on the functions in $\mathcal{G}$.

For each preiteration theory T, there is a function mapping objects in $T^\dagger$ to objects in ICL.

$$\begin{aligned} C_T : Ob(T^\dagger) &\rightarrow Ob(ICL) \\ (A, \alpha) &\mapsto (A, T\alpha) \end{aligned}$$

where $T\alpha$ is the concrete preiteration theory consisting of all of the functions f_α, for $f : n \to p$ in T. For any *surjective* homomorphism $h : (A, \alpha) \to (B, \beta)$ of T-iteration algebras, we can define

$$hC_T : (A, T\alpha) \rightarrow (B, T\beta)$$

by

$$hC_T \quad := \quad (h, \varphi),$$

where $f_\alpha \varphi = f_\beta$. But unless h is surjective, there is no guarantee that if the functions f_α and g_α are the same in (A, α) then $f_\beta = g_\beta$ in (B, β).

We record two facts. Suppose that $\varphi : T \to R$ is a morphism of preiteration theories. Let $\varphi_\dagger : R^\dagger \to T^\dagger$ be the corresponding iteration functor.

Proposition 7.6.2 *The functor $\varphi_\dagger$ commutes with the underlying set functors $U_T : T^\dagger \to$ and $U_R : R^\dagger \to$. Further, if $(A, \alpha) \in R^\dagger$, then*

$$(A, \alpha)\varphi_\dagger C_T \quad \sqsubseteq \quad (A, \alpha)C_R.$$

Proof. The first fact is obvious, and the second follows from the observation that the functions in $(A, \alpha)\varphi_\dagger C_T$ are those of the form $f_{\varphi\cdot\alpha}$, for $f : n \to p$ in T. But any such function is g_α, for some $g : n \to p$ in R, namely $f\varphi$. Also, writing

$$\begin{aligned} (A, \mathcal{F}) \quad &= \quad (A, \alpha)\varphi_\dagger C_T \\ (A, \mathcal{G}) \quad &= \quad (A, \alpha)C_R, \end{aligned}$$

for any function $h : A^{n+p} \to A^n$ in $\mathcal{F}$, the dagger operation in $\mathcal{F}$ is the same as the dagger operation in $\mathcal{G}$. Indeed, if $h = f_{\varphi\cdot\alpha} = (f\varphi)_\alpha$, for $f : n \to n + p$ in T, then we must show

$$(f^\dagger)_{\varphi\cdot\alpha} \quad = \quad (f\varphi)^\dagger{}_\alpha.$$

But

$$\begin{aligned} (f^\dagger)_{\varphi\cdot\alpha} \quad &= \quad (f^\dagger\varphi)_\alpha \\ &= \quad (f\varphi)^\dagger{}_\alpha, \end{aligned}$$

since φ preserves †.

We sketch a proof of a converse.

Theorem 7.6.3 *Suppose that T and R are preiteration theories. Let $G : R^\dagger \to T^\dagger$ be a functor which commutes with the underlying set functors and which satisfies*

$$(A, \alpha)GC_T \ \sqsubseteq \ (A, \alpha)C_R,$$

for all R-iteration algebras (A,α). Then $G = \varphi_\dagger$, for some unique preiteration theory morphism $\varphi : T \to R$.

Proof. If either T or R is trivial, the result is obvious. Thus, we assume neither theory is trivial. Let X be the countably infinite set $X = \{x_1, x_2, l\}$. Consider the free algebra $\mathbf{F}_R[X] = (R[X](1,0), \tau_X^R)$ in $R^\dagger$ freely generated by $\eta : X \to |\mathbf{F}_R[X]|$. (Until the end of this argument, we write just τ instead of τ_X^R and identify elements in X with their η-image.) Then applying the functor G to this algebra, we obtain a T-iteration algebra with the same underlying set, say $(R[X](1,0), \alpha)$. Now for each $t : 1 \to p$ in T, consider the element $t_\alpha(x_1, l, x_p)$. By Theorem 3.3.4.1, there is some R-morphism $g : 1 \to k$ such that

$$t_\alpha(x_1, l, x_p) \quad = \quad g_\tau(x_1, l, x_k).$$

Unfortunately, we cannot say immediately that $k = p$. But, we claim that there is an R-morphism $t' : 1 \to p$ with the same target as t, namely p, such that

$$g_\tau(x_1, l, x_k) \quad = \quad t'_\tau(x_1, l, x_p).$$

Indeed, if $k < p$, let $t' = g \oplus 0_{p-k}$. If $k > p$, consider the endomorphism h of $\mathbf{F}_R[X]$ in $R^\dagger$ determined by the function

$$x_i \quad \mapsto \quad \begin{cases} x_i & \text{if } 1 \le i \le p; \\ \perp_\tau & \text{otherwise.} \end{cases}$$

Since G commutes with the underlying set functor, h is also an endomorphism in $T^\dagger$. Thus

$$\begin{aligned} t_\alpha(x_1, l, x_p) &= h(t_\alpha(x_1, l, x_p)) \\ &= h(g_\tau(x_1, l, x_p, x_{p+1}, l, x_k)) \\ &= g_\tau(x_1, l, x_p, \perp_\tau, l, \perp_\tau) \\ &= (g \cdot (\mathbf{1}_p \oplus \perp_{k-p}))_\tau(x_1, l, x_p). \end{aligned}$$

So, in this case, we can let $t' := g \cdot (\mathbf{1}_p \oplus \perp_{k-p})$.

Now if $t_\alpha(x_1, \ldots, x_p) = t'_\tau(x_1, \ldots, x_p)$, for some R-morphism $t' : 1 \to p$, we define $t\varphi := t'$. We extend φ to morphisms $n \to p$ by requiring that φ preserve tupling. The assignment $t \mapsto t'$ is a well-defined function, by Proposition 4.4.2.8. The argument that the map $t \mapsto t\varphi$ is a theory morphism is the same as in Theorem 4.4.4.2. We now show that $\varphi : T \to R$ also preserves dagger. Restating the definition of $\varphi : T \to R$, for any morphism t in T,

$$t_\alpha = (t\varphi)_\tau.$$

If $t : n \to n + p$, then

$$\begin{aligned} (t^\dagger)_\alpha &= (t_\alpha)^\dagger \\ &= ((t\varphi)_\tau)^\dagger \\ &= ((t\varphi)^\dagger)_\tau, \end{aligned}$$

since both α and τ preserve †. Thus,

$$t^\dagger \varphi = t\varphi^\dagger.$$

The uniqueness of φ also follows from Proposition 4.4.2.8. Since $t_\alpha = (t\varphi)_\tau$, for each T-morphism t, it follows that $G = \varphi_\dagger$.

Exercise 7.6.4 Find an example of an iteration functor which does not have a left adjoint. *Hint:* Consider Example 7.2.4.

7.7 Strong Iteration Algebras

In this section, we consider the class of strong T-iteration algebras for a given preiteration theory T. *Note: Everywhere below T is a preiteration theory satisfying the parameter identity.* We will make use of the theories $T[X]$ (see Section 3.3.4 and Lemma 6.6.6.9).

Definition 7.7.1 *A T-algebra $\mathbf{A} = (A, \alpha)$ is* **a strong T-iteration algebra** *if* $\mathbf{A}$ *is a model for each of the following sentences:*

$$\forall y \in A^p [\ \forall x \in A^n [f(x, y) = g(x, y)] \quad \Rightarrow \quad f^\dagger(y) = g^\dagger(y)],$$

for each pair $f, g : n \to n + p$ in T. We let

$$T_s^\dagger$$

denote the full subcategory of $T^\dagger$ determined by the strong T-iteration algebras.

Remark 7.7.2 We note that each strong T-iteration algebra $\mathbf{A} = (A, \alpha)$ has the following apparently stronger property. For any $f, g : n \to n+m+p$ in T, the following sentence is true in $\mathbf{A}$.

$$\forall z[\ \forall x \forall y[f(x,y,z) \ = \ g(x,y,z)] \quad \Rightarrow \quad \forall y[f^\dagger(y,z) \ = \ g^\dagger(y,z)].$$

Here, x ranges over A^n, y ranges over A^m and z ranges over A^p. Indeed, fix a p-tuple z in A^p such that for all x, y,

$$f_\alpha(x,y,z) \ = \ g_\alpha(x,y,z).$$

Hence, for any fixed m-tuple $y \in A^m$,

$$\forall x[f_\alpha(x,y,z) \ = \ g_\alpha(x,y,z)].$$

For this choice of y,

$$(f^\dagger)_\alpha(y,z) \ = \ (g^\dagger)_\alpha(y,z).$$

But since y is arbitrary,

$$\forall y[(f^\dagger)_\alpha(y,z) \ = \ (g^\dagger)_\alpha(y,z)].$$

Clearly, each strong T-iteration algebra is a T-iteration algebra. The converse is not true. See Example 7.7.8. In addition, when X is infinite, the free algebra $\mathbf{F}_T[X] = (T[X](1,0), \tau_X)$, freely generated by $\eta : X \to |\mathbf{F}_T[X]|$ in $T^\flat$, is a strong T-iteration algebra. Indeed, identifying elements in X with their η-images, fix the p-tuple y in $\mathbf{F}_T[X]$ and assume that the functions

$$f_{\tau_X}(-,y),\ g_{\tau_X}(-,y) : A^n \quad \to \quad A^n$$

are identical, where $A = |\mathbf{F}_T[X]|$. Choose distinct elements $z_1, l, z_q \in X$ and $h : p \to q$ in T with $y = h_{\tau_X}(z)$. Now choose distinct elements u_1, l, u_n in X such that the sets $\{u_1, l, u_n\}$ and $\{z_1, l, z_q\}$ are disjoint. Then

$$\begin{aligned} (f \cdot (\mathbf{1}_n \oplus h))_{\tau_X}(u,z) \quad &= \quad f_{\tau_X}(u,y) \\ &= \quad g_{\tau_X}(u,y) \\ &= \quad (g \cdot (\mathbf{1}_n \oplus h))_{\tau_X}(u,z). \end{aligned}$$

Thus, $f \cdot (\mathbf{1}_n \oplus h) = g \cdot (\mathbf{1}_n \oplus h)$ in T. Hence,

$$f^\dagger \cdot h \ = \ (f \cdot (\mathbf{1}_n \oplus h))^\dagger \ = \ (g \cdot (\mathbf{1}_n \oplus h))^\dagger \ = \ g^\dagger \cdot h,$$

by the parameter identity. It follows that

$$(f^\dagger)_{\tau_X}(y) \;=\; (f^\dagger \cdot h)_{\tau_X}(z) \;=\; (g^\dagger \cdot h)_{\tau_X}(z) \;=\; (g^\dagger)_{\tau_X}(y).$$

We may use the preceding result together with the fact that the theory morphism τ_X in $\mathbf{F}_T[X] = (T[X](1,0), \tau_X)$ is injective when X is infinite, to derive the following corollary.

Corollary 7.7.3 *For each preiteration theory T satisfying the parameter identity, there is a strong T-iteration algebra $\mathbf{A} = (A, \alpha)$ such that the concrete preiteration theory $T\alpha$ is isomorphic to T.*

We indicate several results concerning strong iteration algebras.

Proposition 7.7.4 *Let X be an infinite set. A T-algebra (A, α) is a strong T-iteration algebra iff each of the $T[X]$-algebras $\alpha_h : T[X] \to A$ is a $T[X]$-iteration algebra.*

Proof. We assume that T is nontrivial. First, assume that (A, α) is strong. Given $t, t' : n \to n+m$ in $T[X]$, there are $f, g : n \to n+m+p$ in T and $u : [p] \to X$ with

$$\begin{aligned} t &= f \cdot (\mathbf{1}_{n+m} \oplus u) \\ t' &= g \cdot (\mathbf{1}_{n+m} \oplus u). \end{aligned}$$

(Here, as usual, we identify T and X with their images in $T[X]$.) Assume that $t_{\alpha_h} = t'_{\alpha_h}$ in order to show that $(t^\dagger)_{\alpha_h} = (t'^\dagger)_{\alpha_h}$. It follows that

$$f_\alpha(x, y, a) \;=\; g_\alpha(x, y, a),$$

for all $x \in A^n$, all $y \in A^m$, where $uh = a \in A^p$. Since (A, α) is strong,

$$(f^\dagger)_\alpha(y, a) \;=\; (g^\dagger)_\alpha(y, a),$$

for all $y \in A^m$ by Remark 7.7.2, i.e., by the parameter identity,

$$(t^\dagger)_{\alpha_h} \;=\; (t'^\dagger)_{\alpha_h}.$$

Conversely, if $f_\alpha(x, a) = g_\alpha(x, a)$ for some $a \in A^p$ and all $x \in A^n$, then there is some $u : [p] \to X$ with $uh = a$ and

$$(f \cdot (\mathbf{1}_n \oplus u))_{\alpha_h} \;=\; (g \cdot (\mathbf{1}_n \oplus u))_{\alpha_h}.$$

Hence, if (A, α_h) is a $T[X]$-iteration algebra,

$$(f \cdot (\mathbf{1}_n \oplus u))^\dagger_{\alpha_h} \quad = \quad (g \cdot (\mathbf{1}_n \oplus u))^\dagger_{\alpha_h},$$

i.e.

$$(f^\dagger)_\alpha(a) \quad = \quad (g^\dagger)_\alpha(a),$$

showing that (A, α) is strong.

Note that the assumption that X is infinite is needed only to show that every algebraic function is in the image of some α_h. If (A, α) is a strong T-iteration algebra, then for every set X, each of the theory morphisms α_h is a preiteration theory morphism. Hence, Proposition 7.7.4 has the following corollaries.

Proposition 7.7.5 *The following are equivalent for a T-iteration algebra (A, α).*

- (A, α) *is a strong T-iteration algebra.*
- *For each set X and each function $h : X \to A$, the theory morphism $\alpha_h : T[X] \to A$ is a preiteration theory morphism.*
- *For some infinite set X and each function $h : X \to A$, the theory morphism $\alpha_h : T[X] \to A$ is a preiteration theory morphism.*
- *For each finite set X and each function $h : X \to A$, the theory morphism $\alpha_h : T[X] \to A$ is a preiteration theory morphism.*

In the same way, we get the following.

Proposition 7.7.6 *The following conditions are equivalent.*

- $T^\flat = T_s^\dagger$.
- *For every set X, every theory congruence on $T[X]$ is a dagger congruence.*
- *For some infinite set X, every theory congruence on $T[X]$ is a dagger congruence.*

- *For all finite sets X, every theory congruence on $T[X]$ is a dagger congruence.*

We note several other facts.

- If X is an infinite set, then the T-iteration algebra freely generated by X is a strong iteration algebra, and is thus same as the free strong iteration algebra.
- Any T-algebra is a retract of a strong T-iteration algebra.
- When X is finite, the strong T-iteration algebra freely generated by X exists iff the free T-algebra generated by X is a strong T-iteration algebra.
- In the category of strong T-iteration algebras, the monics are the injective homomorphisms, the epis are the same as the epis in $T^\flat$, and the regular epis are the surjective homomorphisms.

A version of Theorem 7.5.7 holds for strong iteration algebras. We give it below.

Theorem 7.7.7 *The following statements are equivalent.*

[a] $T_s^\dagger = T^\flat$.

[b] *For every set A, $T[A]^\flat = T[A]^\dagger$.*

[c] *Every homomorphism $h : \mathbf{A} \to \mathbf{B}$ in $T_s^\dagger$ factors as*

$$h \quad = \quad \mathbf{A} \xrightarrow{e} \mathbf{C} \xrightarrow{m} \mathbf{B},$$

where e is a coequalizer and m is a monic.

[d] *Every homomorphism $h : \mathbf{A} \to \mathbf{B}$ in $T_s^\dagger$ factors as*

$$h \quad = \quad \mathbf{A} \xrightarrow{e} \mathbf{C} \xrightarrow{m} \mathbf{B},$$

where e is an epi and m is a monic.

[e] $T_s^\dagger$ *has equalizers.*

[f] $T_s^\dagger$ *is complete.*

[g] $T_s^\dagger$ *has coequalizers.*

[h] $T_s^\dagger$ *is cocomplete.*

[i] *The inclusion functor* $T_s^\dagger \to T^\flat$ *has a left adjoint.*

[j] *For each* $\mathbf{A} \in T^\flat$, *and* $\mathbf{B} \in T_s^\dagger$ *with* $\mathbf{A}$ *a subalgebra of* $\mathbf{B}$, *there is a least subalgebra* $\mathbf{C} \in T_s^\dagger$ *with* $|\mathbf{A}| \subseteq |\mathbf{C}| \subseteq |\mathbf{B}|$.

Except for part [b], the proofs are the same as for Theorem 7.5.7, and we have already proved [b] in Proposition 7.7.6.

Lastly, we mention that analogues of Theorems 7.5.5 and 7.5.9 hold. One defines a variety $\mathcal{V}$ of strong T-iteration algebras as a class of strong T-iteration algebras closed under products, subalgebras and homomorphic images in $T_s^\dagger$; i.e.

$$= \mathbf{H}(\mathcal{V}) \cap T_s^\dagger \;=\; \mathbf{S}(\mathcal{V}) \cap T_s^\dagger \;=\; \mathbf{P}(\mathcal{V}) \cap T_s^\dagger.$$

Note that $\subseteq T_s^\dagger \Rightarrow \mathbf{P}(\mathcal{V}) \subseteq T_s^\dagger$.

Example 7.7.8 Suppose that $\Sigma_2 = \{\sigma\}$ and that $\Sigma_0 = \{a\}$; otherwise $\Sigma_n = \emptyset$. Let θ denote the least theory congruence (not necessarily a dagger congruence) on $\Sigma\mathbf{tr}$ such that

$$\sigma \cdot (\mathbf{1}_1 \oplus a) \;\theta\; \mathbf{1}_1.$$

Let

$$R \;:=\; \Sigma\mathbf{tr}/\theta$$

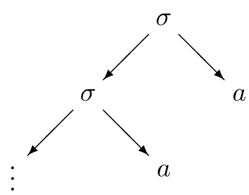
be the quotient theory. Let (A, α) be the $\Sigma\mathbf{tr}$-algebra determined by the initial R-algebra. Thus, the elements of A are the θ-congruence classes $[h]$ of the $\Sigma\mathbf{tr}$-morphisms $h : 1 \to 0$. If Δ is the signature which differs from Σ in that Δ does not contain the letter a, there is an obvious inclusion morphism $\iota : \Delta\mathbf{tr} \to \Sigma\mathbf{tr}$. Thus, $\mathbf{A} = (A, \beta = \iota \cdot \alpha)$ is a $\Delta\mathbf{tr}$-algebra. In fact, $\mathbf{A}$ is a $\Delta\mathbf{tr}$-iteration algebra. Indeed, if $t, t' : 1 \to p$ in $\Delta\mathbf{tr}$, and $t_\beta = t'_\beta$, then $t = t'$. However, $\mathbf{A}$ is not a strong $\Delta\mathbf{tr}$-iteration algebra. Consider the two Δ-trees $t, t' : 1 \to 2$:

$$\begin{aligned} t &= 1_2 : 1 \to 2 \\ t' &= \sigma \cdot \langle 1_2, 2_2 \rangle. \end{aligned}$$

Then, for all $[x] \in A$,

$$\begin{aligned} t_\beta([x],[a]) &= [x]; \\ t'_\beta([x],[a]) &= [\sigma(x,a)] \;=\; [x]. \end{aligned}$$

However, $t^\dagger_\beta([a]) = [\perp]$ and $t'^\dagger([a])$ is not $[\perp]$, but the θ-congruence class of the infinite tree indicated in the following figure.



Exercise 7.7.9 State and prove analogues of Theorems 7.5.5 and 7.5.9 for strong T-iteration algebras.

Exercise 7.7.10 Show that every ω-continuous ordered $\Sigma\mathbf{tr}$-algebra, defined in Example 7.1.5, is strong.

Exercise 7.7.11 Show every iterative algebra, defined in Example 7.1.6, determines a strong iteration algebra.

7.8 Notes

The entire chapter is based on [BÉb]. In particular, Theorem 7.2.6 is proved in [BÉb]. For more information about the ω-continuous algebras mentioned in Example 7.1.5, we refer to [Sco71, GTWW77, Niv75, CN76, Gue81]. A modification of ω-continuous algebras, called regular algebras, is investigated in [Tiu78, Tiu79]. The iterative algebras of Example 7.1.6 were studied in [Tiu80, Nel83]. Strong iteration algebras were introduced in [Ési83], called there simply iteration algebras. For iteration theories, Corollary 7.7.3 is proved in [Ési83]. Michael Barr observed that if and satisfy the hypotheses of the Lemma 7.4.1, and if has equalizers, then is equivalent to the "idempotent completion" of (see [Fre64]).

Chapter 8

Continuous Theories

In the first three sections of this chapter we consider theories whose hom-sets are equipped with a partial order which is compatible with the theory operations; iteration is defined using least fixed points. An ordered theory is a special kind of 2-theory, one in which there is a vertical morphism $f \to g$ iff $f \leq g$. In Section 8.4, the connection between initiality and the fixed point properties of iteration is examined in the context of 2-theories. We consider the properties of initial f-algebras, for horizontal morphisms f in a 2-theory. Many properties of iteration theories hold when all such initial algebras exist. In the last section, some of the constructions for ω-continuous ordered theories are generalized to ω-continuous 2-theories. In particular, it is shown how any ω-continuous 2-theory determines an iteration theory.

8.1 Ordered Algebraic Theories

We begin with some definitions.

Definition 8.1.1 • *An* **ordered algebraic theory** T *is an algebraic theory such that for each pair n, p of nonnegative integers, the set $T(n,p)$ is equipped with a partial order. The order on $T(n,p)$ will be written $f \leq g : n \to p$. The theory operations respect the ordering, i.e. if $f_1 \leq f_2 : n \to p$ and $g_1 \leq g_2 : p \to q$*

then

$$f_1 \cdot g_1 \ \leq \ f_2 \cdot g_2.$$

Further, if $f_i \leq g_i : 1 \to p$*, for each* $i \in [n]$*, then*

$$\langle f_1, \ldots, f_n \rangle \ \leq \ \langle g_1, \ldots, g_n \rangle.$$

- *A* **pointed ordered theory** *is an ordered theory which is pointed; i.e. there is a distinguished morphism* $\perp : 1 \to 0$*; as usual, we define* $\perp_{1p}$ *as* $\perp \cdot 0_p$*, for all* $p \geq 0$*, and* $\perp_{np}$ *as* $\langle \perp_{1p}, \ldots, \perp_{1p} \rangle$*, for* $n \neq 1$*. Furthermore, the morphisms* $\perp_{np}$ *are the* least *elements in* $T(n,p)$*. Note that composition in pointed theories is* **left strict***:*

$$\perp_{np} \cdot f \ = \ \perp_{nq},$$

for all $f : p \to q$*.*

- *A* **strict ordered theory** *is a pointed ordered theory in which composition is right strict, i.e.*

$$f \cdot \perp_{pq} \ = \ \perp_{nq},$$

for all $f : n \to p$*.*

- *An* **ordered theory morphism** $\varphi : T \to T'$ *is a theory morphism which preserves the ordering; i.e. if* $f \leq g : n \to p$ *in* T*, then* $f\varphi \leq g\varphi$ *in* T'*.*

- *A* **pointed ordered theory morphism** *is an ordered theory morphism which also preserves the distinguished elements; i.e. for all* n, p*,* $\perp_{np}\varphi = \perp_{np}$*.*

Exercise 8.1.2 Show that an ordered theory morphism φ between pointed ordered theories is a pointed ordered theory morphism iff $\perp_{10}\varphi = \perp_{10}$. Thus morphisms of pointed ordered theories are pointed theory morphisms as defined in Section 3.3.5.

Exercise 8.1.3 Show that in any ordered theory, the order relation is determined by the order on the scalar morphisms, in that

$$f \ \leq \ g : n \to p \quad \Leftrightarrow \quad i_n \cdot f \ \leq \ i_n \cdot g, \quad \text{all } i \in [n].$$

Exercise 8.1.4 Suppose that T is an ordered theory such that each set $T(n,p)$ has a least element λ_{np}. Suppose that composition in T is left strict, i.e. suppose that $\lambda_{np} \cdot f = \lambda_{nq}$, for all $f : p \to q$. Show that T is pointed, i.e. show that

$$\begin{aligned} \lambda_{1p} &= \lambda_{10} \cdot 0_p \qquad \text{and} \\ \lambda_{np} &= \langle \lambda_{1p}, \ldots, \lambda_{1p} \rangle : n \to p. \end{aligned}$$

We give some examples of ordered theories.

Example 8.1.5 Let $(A, \leq)$ be a poset. The set A^n is a poset with the pointwise ordering. Let T be the subtheory of A whose morphisms $n \to p$ are the order preserving functions $A^p \to A^n$. With the pointwise ordering,

$$f \ \leq \ g : n \to p \quad \Leftrightarrow \quad af \ \leq \ ag, \quad \text{all } a \in A^p,$$

T becomes an ordered theory. If A has a least element $\perp$, then T is a pointed ordered theory where $\perp_{np} : A^p \to A^n$ is the constant function returning the n-tuple of $\perp$'s. An order preserving function $A^p \to A^n$ is *strict* if $f(\perp, \ldots, \perp) = (\perp, \ldots, \perp)$. The collection of strict order preserving functions is a strict ordered theory.

Example 8.1.6 Let Σ be a ranked set. An *ordered* Σ*-algebra* is a Σ-algebra $\mathbf{A}$ whose underlying set A is a poset such that each operation $\sigma_{\mathbf{A}} : A^n \to A$ is order preserving. The theory T whose morphisms $1 \to n$ are all functions $A^n \to A$ determined by Σ-terms is an ordered theory. If A has a least element $\perp$ and there is a letter $\sigma \in \Sigma_0$ with $\sigma_{\mathbf{A}} = \perp$, T is a pointed ordered theory.

Example 8.1.7 Let $T = A$. Then T is a strict ordered theory with the order defined as follows:

$$f \ \leq \ g : A \times [n] \to A \times [p] \quad \Leftrightarrow \quad f\gamma \ \subseteq \ g\gamma,$$

where $f\gamma$ is the graph of the relation f. The element $\perp_{np} : A \times [n] \to A \times [p]$ is the empty relation. Note that A is a strict ordered subtheory of A.

Example 8.1.8 The theories L_Z. Suppose that Z is a set disjoint from the sets $V_n = \{x_1, \ldots, x_n\}$, $n \geq 1$. A morphism $1 \to p$ in L_Z is a subset of $(Z \cup V_p)^*$; a morphism $A = (A_1, \ldots, A_n) : n \to p$ is an n-tuple of morphisms $A_i : 1 \to p$, $i \in [n]$. Let $B = (B_1, \ldots, B_p)$ be a morphism $p \to q$ in L_Z. The composite $A \cdot B$ of $A : 1 \to p$ with B is defined by:

$$A \cdot B \ \ := \ \ \bigcup_{w \in A} w \cdot B,$$

where $w \cdot B$ will be defined by induction. Intuitively, $w \cdot B$ is the set of all words formed by replacing an occurrence of the letter x_i in w by a word in the set B_i.

$$\begin{array}{rcl} \varepsilon \cdot B & := & \emptyset \\ x_i \cdot B & := & B_i \\ z \cdot B & := & z, \quad \text{all } z \in Z \\ (aw) \cdot B & := & \{uv \; : \; u \in a \cdot B, \; v \in w \cdot B\}, \end{array}$$

for any $a \in Z \cup V_p$ and any word $w \in (Z \cup V_p)^*$. Finally,

$$(A_1, \ldots, A_n) \cdot B \quad := \quad (A_1 \cdot B, \ldots, A_n \cdot B).$$

The distinguished morphism $i_n : 1 \to n$ is the singleton $\{x_i\}$. The ordering is set inclusion:

$$A \; \leq \; B : 1 \to p \quad \Leftrightarrow \quad A \; \subseteq \; B.$$

Thus, the empty set is the least element $1 \to p$.

Two subtheories of L_Z will be mentioned below: REG_Z, whose morphisms $1 \to p$ are the regular subsets of $(Z \cup V_p)^*$, and CF_Z, whose morphisms are the context free sets.

Exercise 8.1.9 a) Show that every ordered theory is isomorphic to a subtheory of a theory of the kind in Example 8.1.5. b) Show that every ordered theory is isomorphic to a theory of the kind in Example 8.1.6. (Hint: use the construction in Proposition 3.3.2.2.)

Exercise 8.1.10 Show that every ordered theory T is a 2-theory, in which there is a vertical morphism $f \to g$ in $T(n,p)$ if $f \leq g$.

8.1.1 Free Ordered Theories

We show how free ordered theories are constructed. The case of non-pointed theories is easy. The theory , equipped with the trivial ordering (i.e. $f \leq g \Leftrightarrow f = g$) is freely generated by Σ in the class of ordered theories, since it is freely generated by Σ in the class of all theories (see Chapter 2, Section 2.3.)

There is an added complication in the case of pointed theories. Suppose that Σ is a ranked set and $\perp$ is a letter not in Σ. Let $\Sigma_\perp$ be the ranked set obtained by adding $\perp$ to Σ_0. We impose the following ordering on $\Sigma_\perp$**Term**.

Definition 8.1.11 *Suppose that* $t, t' : 1 \to p$ *in* $\Sigma_\perp\mathbf{Term}$. *Then* $t \leq t'$ *if*

- $t = \perp$, *or*
- $t = x_i$ *and* $t' = x_i$, *for some* $i \in [p]$, *or*
- *there is a letter* $\sigma \in \Sigma_n$, $n \geq 0$, *and terms* t_i, $t'_i : 1 \to p$ *such that*

$$\begin{array}{rcl} t & = & \sigma(t_1, \ldots, t_n) \\ t' & = & \sigma(t'_1, \ldots, t'_n) \end{array}$$

and $t_i \leq t'_i$, *for each* $i \in [n]$.

If $t, t' : n \to p$, *then* $t \leq t'$ *if for each* $i \in [n]$, $i_n \cdot t \ \leq \ i_n \cdot t'$.

We omit the easy verification of the following fact.

Proposition 8.1.12 *With the above order,* $\Sigma_\perp\mathbf{Term}$ *is a pointed ordered theory.*

We can express the definition of $t \leq t'$ in more intuitive terms: $t \leq t' : 1 \to p$ iff t' can be obtained from t by replacing some occurrences of $\perp$ in t by some terms $1 \to p$. Different occurrences of $\perp$ in t may be replaced by different terms.

Let

$$\eta_\perp : \Sigma \quad \to \quad \Sigma_\perp\mathbf{Term}$$

be the function mapping the letter $\sigma \in \Sigma_n$ to the term $\sigma(x_1, \ldots, x_n)$.

Theorem 8.1.13 *The theory* $\Sigma_\perp\mathbf{Term}$ *is freely generated in the class of pointed ordered theories by* $\eta_\perp$; *for any pointed ordered theory* T *and any rank preserving function* $\varphi : \Sigma \to T$ *there is a unique pointed ordered theory morphism* $\varphi^\sharp : \Sigma_\perp\mathbf{Term} \to T$ *such that* *commutes.*

Proof. Since $\Sigma_\perp$**Term** is freely generated in , there is a unique theory morphism

$$\varphi^\sharp : \Sigma_\perp\mathbf{Term} \quad \rightarrow \quad T$$

agreeing with $\varphi_\perp$ on $\Sigma_\perp$. By induction on the number of function symbols in the term t one shows that if $t \leq t' : 1 \rightarrow p$ in $\Sigma_\perp$**Term**, then $t\varphi^\sharp \leq t'\varphi^\sharp$.

Exercise 8.1.14 Show that free strict ordered theories exist, and find a concrete description of such theories as a theory of trees.

8.2 ω-Continuous Theories

Recall the definition of an ω-complete poset from Chapter 1.

Definition 8.2.1 *A pointed ordered theory T is* ω**-continuous** *if each hom-set $T(n,p)$ is an ω-complete poset and if composition is also ω-continuous; i.e.*

$$\begin{aligned}(\sup_n f_n) \cdot g \quad &= \quad \sup_n f_n \cdot g \\ f \cdot (\sup_n g_n) \quad &= \quad \sup_n f \cdot g_n,\end{aligned}$$

for ω-chains $(f_n), (g_n)$, where $f_n : m \rightarrow p$ and $g_n : p \rightarrow q,\ n \geq 0$, and for $f : m \rightarrow p,\ g : p \rightarrow q$. An ω**-continuous theory morphism** *$\varphi : T \rightarrow T'$ between ω-continuous theories is a pointed ordered theory morphism such that*

$$(\sup_n f_n)\varphi \quad = \quad \sup_n f_n\varphi,$$

for each ω-chain (f_n) in T.

Exercise 8.2.2 If T is an ω-continuous theory, then show

$$\sup_n \langle f_n,\ g_n\rangle \quad = \quad \langle \sup_n f_n,\ \sup_n g_n\rangle,$$

for any ω-chains $f_n : m \rightarrow p,\ g_n : q \rightarrow p$.

Some of the previous examples of ordered theories are in fact ω-continuous theories.

Example 8.2.3 The theories A and A are ω-continuous, for any set A.

Example 8.2.4 The theory L_Z is ω-continuous, for any set Z.

Example 8.2.5 If the poset A is ω-complete, then the subtheory of A consisting of the ω-continuous functions $A^p \to A^n$ is an ω-continuous theory.

8.2.1 Free ω-Continuous Theories

We will show that for each ranked set Σ there is an ω-continuous theory freely generated by Σ. Let $\Sigma_\perp$ be the ranked set above. We consider the theory $\Sigma_\perp\,\mathbf{TR}$ of *all* trees, equipped with the following ordering.

Definition 8.2.6 *Let $t, t' : 1 \to p$ in $\Sigma_\perp\,\mathbf{TR}$. Then $t \leq t'$ if for each word $u \in [\omega]^*$ either*

- *$ut = ut'$, or*
- *$ut = \perp$ and ut' is defined.*

For trees $t, t' : n \to p$, $n \neq 1$, define $t \leq t'$ iff $i_n \cdot t \leq i_n \cdot t'$, for all $i \in [n]$.

Briefly, $t \leq t'$ if the tree t' can be obtained by attaching some trees to some leaves of t which are labeled $\perp$.

Proposition 8.2.7 *With the above ordering, $\Sigma_\perp\,\mathbf{TR}$ is an ω-continuous theory. The subtheory of $\Sigma_\perp\,\mathbf{TR}$ consisting of the finite trees is isomorphic, as a pointed ordered theory, to the theory $\Sigma_\perp\mathbf{Term}$.*

Let

$$\eta_\perp : \Sigma \;\to\; \Sigma_\perp\,\mathbf{TR}$$

now denote the function identifying each letter in Σ with the corresponding atomic tree.

Theorem 8.2.8 *For each ω-continuous theory T and rank preserving function $\varphi : \Sigma \to T$ there is a unique ω-continuous theory morphism $\varphi^\sharp : \Sigma_\perp \mathbf{TR} \to T$ such that*

commutes.

Proof. We may assume that φ is immediately extended to the finite trees, using Theorem 8.1.13. Now each tree $t : 1 \to p$ in $\Sigma_\perp \mathbf{TR}$ can be written as the least upper bound of an ω-chain (t_n), where each tree $t_n : 1 \to p$ is finite and either the sequence is constant or the depth of t_{n+1} is just one greater than that of t_n. For example, choose t_n as the tree that results from deleting all vertices of t of depth greater than n, and labeling all the resulting leaves $\perp$. These requirements determine the sequence (t_n) sufficiently so that the following definition makes sense:

$$t\varphi^\sharp \quad := \quad \sup_n (t_n\varphi).$$

Indeed, if (t_n) and (s_n) are two ω-chains of finite trees with the same least upper bound, then the ω-chains $(t_n\varphi)$ and $(s_n\varphi)$ in T are cofinal, and have the same least upper bound. With this definition, it is easy to check that $\varphi^\sharp$ is an ω-continuous theory morphism. Clearly, this is the only possible definition, since continuous morphisms must preserve sups.

The ω-continuous theories are of interest because in such theories there is a well-known method to find least fixed points. In fact, these theories are iteration theories when $f^\dagger$ is defined as a least fixed point, as we show now.

Suppose that T is a pointed theory. For $f : n \to n + p$, recall from Definition 5.5.2.4 the *powers of* f: $f^{k+1} := f \cdot \langle f^k, 0_n \oplus \mathbf{1}_p\rangle$ and $f^0 := \mathbf{1}_n \oplus 0_p$. We define the *modified powers* of f as follows.

Definition 8.2.9 *The* **modified powers** *of $f : n \to n + p$ are the following morphisms $n \to p$:*

$$\begin{aligned} f^{(0)} &:= \perp_{np} \\ f^{(k+1)} &:= f^{k+1} \cdot \langle \perp_{np}, \mathbf{1}_p \rangle \\ &= f \cdot \langle f^{(k)}, \mathbf{1}_p \rangle. \end{aligned}$$

Then, when T is a pointed ordered theory, for each $k \geq 0$,

$$f^{(k)} \quad \leq \quad f^{(k+1)}.$$

Indeed, when $k = 0$, $f^{(0)} = \perp_{np}$ is the least morphism $n \to p$, so that $f^{(0)} \leq f^{(1)}$. Assuming that $f^{(k)} \leq f^{(k+1)}$, we obtain

$$\begin{aligned} f^{(k+1)} &= f \cdot \langle f^{(k)}, \mathbf{1}_p \rangle \\ &\leq f \cdot \langle f^{(k+1)}, \mathbf{1}_p \rangle \\ &= f^{(k+2)}. \end{aligned}$$

Thus, the morphisms $f^{(k)}$, $k \geq 0$, form an ω-chain of morphisms $n \to p$. When T is an ω-continuous theory, we *define*

$$f^\dagger \quad := \quad \sup_k f^{(k)}. \tag{8.1}$$

Proposition 8.2.10 *Suppose that in the ω-continuous theory T the dagger operation is defined on all morphisms $n \to n + p$ by (8.1). Then for any $f : n \to n + p$, $f^\dagger$ is the least solution of the iteration equation for f:*

$$\xi \quad = \quad f \cdot \langle \xi, \mathbf{1}_p \rangle, \quad \xi : n \to p.$$

Proof. First we show that $f^\dagger$ is a solution of the iteration equation for f. But this fact follows from the continuity of the composition operation:

$$\begin{aligned} f \cdot \langle f^\dagger, \mathbf{1}_p \rangle &= f \cdot \langle \sup_k f^{(k)}, \mathbf{1}_p \rangle \\ &= \sup_k f \cdot \langle f^{(k)}, \mathbf{1}_p \rangle \\ &= \sup_k f^{(k+1)} \\ &= f^\dagger. \end{aligned}$$

To see that $f^\dagger$ is the least solution, let $g = f \cdot \langle g, \mathbf{1}_p \rangle$ be any solution. More generally, assume only that $f \cdot \langle g, \mathbf{1}_p \rangle \leq g$. One shows by induction on k that $f^{(k)} \leq g$. Hence $f^\dagger = \sup_k f^{(k)} \leq g$.

Definition 8.2.11 *The ω-chain $(f^{(k)})$ determined by the morphism $f : n \to n + p$ will be denoted f^Δ.*

$$f^\Delta \quad := \quad (f^{(k)}), \quad k = 0, 1, \ldots.$$

These chains will be used in the next section. Note that in any ω-continuous theory, for any $n, p \geq 0$,

$$(\mathbf{1}_n \oplus 0_p)^\dagger \quad = \quad \perp_{np}.$$

In particular, $\mathbf{1}_1^\dagger = \perp$.

Proposition 8.2.12 *The operation $f \mapsto f^\dagger$ is itself ω-continuous; i.e. if $(f_k : n \to n+p)$ is an ω-chain in the ω-continuous theory T with least upper bound f, then*

$$f^\dagger \quad = \quad \sup_k f_k^\dagger.$$

Proposition 8.2.13 *Suppose that $\varphi : T \to T'$ is an ω-continuous morphism of the ω-continuous theories T and T'. Then for each $f : n \to n+p$ in T,*

$$f^\dagger \varphi \quad = \quad (f\varphi)^\dagger.$$

The previous three propositions show that each ω-continuous theory T determines a preiteration theory, called the *underlying preiteration theory* of T, and ω-continuous theory morphisms preserve the iteration operation.

Remark 8.2.14 Now apparently the tree theories $\Sigma_\perp \, \mathbf{TR}$ have two preiteration theory structures: one with dagger defined by the ordering and one with dagger determined by the structure of a pointed iterative theory. In fact, the two operations are necessarily the same, by Theorem 6.6.4.5.

Theorem 8.2.15 *The underlying preiteration theory of each ω-continuous theory is an iteration theory. Morphisms between ω-continuous theories preserve the †-operation. The variety of iteration theories generated by the ω-continuous theories is the variety of all iteration theories.*

Proof. For any ω-continuous theory T, there is a signature Σ and a surjective ω-continuous theory morphism $\varphi : \Sigma_\perp \, \mathbf{TR} \to T$, by Theorem 8.2.8. Hence any equation between "iteration terms" which is true in $\Sigma_\perp \, \mathbf{TR}$ is true in T, showing T is an iteration theory.

Now let be the variety of iteration theories generated by the ω-continuous theories. Since varieties are closed under subtheories, the subtheory $\Sigma\,\mathbf{tr}$ of $\Sigma_\perp\,\mathbf{TR}$ belongs to ; hence all free iteration theories belong to , showing is the variety of all iteration theories.

Exercise 8.2.16 Give an equational proof that the pairing identity holds in ω-continuous theories. *Hint:* Suppose first that $f : n \to n + m$ and $g : m \to n + m$. Show that for any $y : m \to m$, the least solution of the iteration equation for

$$f(y) \quad := \quad f \cdot (\mathbf{1}_n \oplus y) : n \to n + m$$

is $f^\dagger \cdot y$. Then, find the least solution $h^\dagger$ of the iteration equation for

$$h \quad := \quad g \cdot \langle f^\dagger, \mathbf{1}_m \rangle : m \to m.$$

Now let $y = h^\dagger$, and show that the pair

$$\langle f^\dagger \cdot h^\dagger,\ h^\dagger \rangle$$

is the least solution of the iteration equation for $\langle f, g \rangle$. Use the same idea to prove the pairing identity when $f : n \to n + m + p,\ g : m \to n + m + p$.

Exercise 8.2.17 Show that any ω-continuous theory satisfies the functorial dagger implication for all base morphisms, indeed, for all pure morphisms (see Exercise 6.6.3.3).

Exercise 8.2.18 Give a purely equational proof of Theorem 8.2.15 which uses the definition of $f^\dagger$ given in (8.1).

Exercise 8.2.19 For each set A, show the theory A is an ω-continuous theory, where the ordering on relations is set-inclusion of their graphs:

$$r \ \leq\ r' : n \to p \qquad \text{iff} \qquad u\, r\, v \ \Rightarrow\ u\, r'\, v,$$

all $u \in A \times [n],\ v \in A \times [p]$. Show that A is also an ω-continuous theory with the same ordering.

Exercise 8.2.20 For each set A, show the theory A is an ω-continuous theory, with the same ordering as in the previous problem.

Exercise 8.2.21 Prove the claim in Remark 8.2.14.

Exercise 8.2.22 Suppose that L is a complete lattice and that T is the subtheory of L consisting of the order preserving functions. Show that when $f : n \to n + p$ in T and $y \in L^p$, there is a least element $x_0 \in L^n$ with $f(x_0, y) = x_0$. Define $yf^\dagger := x_0$. Show that $f^\dagger$ is in T and that T is an iteration theory which has a functorial dagger with respect to all base morphisms.

8.3 Rational Theories

Some ordered theories fail to be ω-continuous because their hom-sets are not ω-complete. However, in many cases there are enough least upper bounds to find least solutions of iteration equations. This situation is formalized by the notion of a rational theory.

First, we explain some notation. Recall the ω-chains f^Δ in Definition 8.2.11. For $f : n \to n + p$, $g : m \to n$, and $h : p \to q$, we let $g \cdot f^\Delta$ and $f^\Delta \cdot h$ denote the following ω-chains.

$$\begin{array}{rcl} g \cdot f^\Delta & := & (g \cdot f^{(k)}) \\ & = & (g \cdot f^k \cdot \langle \perp_{np}, \mathbf{1}_p \rangle) \\ f^\Delta \cdot h & := & (f^{(k)} \cdot h) \\ & = & (f^k \cdot \langle \perp_{nq}, h \rangle) \end{array}$$

Definition 8.3.1 *A* **rational theory** T *is a pointed ordered theory with the following two properties.*

- *For any $f : n \to n + p$, the ω-chain f^Δ has a least upper bound $f^\dagger$.*
- *For $f : n \to n + p$, $g : m \to n$ and $h : p \to q$,*

$$\begin{array}{rcl} g \cdot f^\dagger & = & \sup(g \cdot f^\Delta) \\ f^\dagger \cdot h & = & \sup(f^\Delta \cdot h). \end{array}$$

A **morphism** *of rational theories is a pointed ordered theory morphism which preserves the sups of the chains f^Δ, and thus preserves the dagger operation.*

We note that if T is a rational theory and $f : n \to n + p$, then $f^\dagger$ is the least solution of the iteration equation for f.

We list some examples of rational theories.

Example 8.3.2 Every ω-continuous theory is a rational theory.

Example 8.3.3 The subtheories REG_Z and CF_Z of L_Z in Example 8.1.8 are rational but not ω-continuous, unless $Z = \emptyset$.

Example 8.3.4 The theory $\Sigma\,\mathbf{tr}$ with the order inherited from $\Sigma_\perp\,\mathbf{TR}$ is rational but not always ω-continuous.

In fact these last theories are the free theories in the class of rational theories.

Theorem 8.3.5 *For any rational theory T and any rank preserving function $\varphi : \Sigma \to T$ there is a unique rational theory morphism $\varphi^\sharp : \Sigma\,\mathbf{tr} \to T$ such that*

commutes.

We leave the proof of this theorem as an exercise. It can be proved by showing how to represent any infinite tree $t : 1 \to p$ in $\Sigma\,\mathbf{tr}$ as one component of the sup of the ω-chain $f^{(k)},\ k \geq 0$, of the modified powers of f, where $f : n \to n + p$ is a finite tree.

Since any rational theory is a quotient of an iteration theory $\Sigma\,\mathbf{tr}$, the following corollary follows immediately.

Corollary 8.3.6 *Suppose that T is a rational theory. Then the underlying preiteration theory of T is an iteration theory.*

The dagger operation in rational theories is nicely behaved.

Proposition 8.3.7 *Suppose that T is a rational theory. Then T satisfies the functorial dagger implication for all (surjective) base morphisms. T has a strong functorial dagger iff T has a unique morphism $1 \to 0$.*

Proof. First, suppose that $f : n \to n+p$ and $g : m \to m+p$ in T and $\rho : n \to m$ is a base surjection such that the diagram

[n‘n+p‘m‘m+p;f‘ρ‘$\rho \oplus \mathbf{1}_p$‘g] commutes. By induction on k, one shows that the square

[n‘n+p‘m‘m+p;f^k‘ρ‘$\rho \oplus \mathbf{1}_p$‘g^k] also commutes, for all $k \geq 0$. But then

$$\begin{aligned} \rho \cdot g^\dagger &= \sup_k \rho \cdot g^k \cdot \langle \perp_{mp}, \mathbf{1}_p \rangle \\ &= \sup_k f^k \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle \perp_{mp}, \mathbf{1}_p \rangle \\ &= \sup_k f^k \cdot \langle \perp_{np}, \mathbf{1}_p \rangle \ = \ f^\dagger. \end{aligned}$$

Note that $\rho \cdot \perp_{mp} = \perp_{np}$ because ρ is base.

Now, if there is a unique morphism $1 \to 0$, then if $h : n \to m$ is any morphism, $h \cdot \perp_{mp} = \perp_{np}$, since

$$\begin{aligned} h \cdot \perp_{mp} &= (h \cdot \perp_{m0}) \cdot 0_p \\ &= \perp_{n0} \cdot 0_p \\ &= \perp_{np}. \end{aligned}$$

Hence, the above argument also shows that in this case too, T has a strong functorial dagger.

The fact that any theory with a strong functorial dagger has a unique morphism $1 \to 0$ was established in Chapter 6, Section 6.3. The proof is complete.

Exercise 8.3.8 Suppose that T is an ω-continuous theory and T' is a subiteration theory of T. Then T' is a rational theory.

Exercise 8.3.9 Show each rational theory embeds in some ω-continuous theory.

Exercise 8.3.10 [Blo76] The collection of ordered theories and ordered theory morphisms forms a category, as does the collection of ω-continuous theories and ω-continuous theory morphisms. Show that the inclusion functor from the category of ω-continuous theories to the category of ordered theories has a left adjoint. What about the inclusion functor from the ω-continuous theories to the rational theories, and from the rational theories into ordered theories?

8.4 Initiality and Iteration in 2-Theories

In this section, we make extensive use of the notion of an F-algebra [AM74, Bar74], where $F :\to$ is an endofunctor on some category. We wish to investigate which identities are forced by the existence of initial F-algebras for functors determined by the horizontal morphisms $f : n \to n + p$ in a 2-theory. The major results are summarized in Theorem 8.4.16. These results will be used to show that in the case that the 2-theory is ω-continuous, the underlying theory is an iteration theory in which, roughly, $f^\dagger$ is an initial F_f-algebra, for a suitable functor F_f.

Definition 8.4.1 *Let $F :\to$ be a functor on the category . An F-* **algebra** *is a pair (x, u), where $u : xF \to x$. If (x, u) and (x', u') are F-algebras, an F-***homomorphism** $h : (x, u) \to (x', u')$ *is a morphism $h : x \to x'$ such that*

[xF‘x‘x’F‘x’;u‘hF‘h‘u’]

commutes.

The collection of F-algebras and F-homomorphisms forms a category, F-**ALG**. There is an obvious forgetful functor

$$U_F : \text{F-}\mathbf{ALG} \quad \to \quad .$$

Note that (x, u) is an *initial* object in F-**ALG** iff for any F-algebra (x', u') there is a unique F-homomorphism $h : (x, u) \to (x', u')$. Clearly, if (x, u) and (x', u') are both initial in F-**ALG**, they are isomorphic in F-**ALG** and hence the objects x and x' are isomorphic in .

Now suppose henceforth that T is a 2-theory. For any horizontal morphism $f : n \to n+p$ in T, we define a functor F_f on $T(n, p)$ as follows:

Definition 8.4.2 *The functor F_f is the endofunctor on $T(n, p)$ which takes $g : n \to p$ to $f \cdot \langle g, \mathbf{1}_p \rangle$, and the vertical morphism $u : g \to g'$ to*

$$f \cdot \langle u, \mathbf{1}_p \rangle : \; f \cdot \langle g, \mathbf{1}_p \rangle \quad \to \quad f \cdot \langle g', \mathbf{1}_p \rangle.$$

The fact that F_f is a functor follows from the following Exercise.

Exercise 8.4.3 Suppose that $h : n \to r + s$ is a horizontal morphism in the 2-theory T, and that u_i and v_i, $i = 1, 2$ are vertical morphisms, with u_i in $T(r, p)$ and v_i in $T(s, p)$. Show that

$$(h \cdot \langle u_1, v_1 \rangle) \circ (h \cdot \langle u_2, v_2 \rangle) \quad = \quad h \cdot \langle u_1 \circ u_2,\ v_1 \circ v_2 \rangle.$$

Sometimes we write just f-algebra, f-homomorphism, f-**ALG** instead of F_f-algebra, etc.

Remark 8.4.4 If $f : n \to n + p$ is a morphism in an ordered theory (considered as a 2-theory), an F_f-algebra is a morphism $k : n \to p$ such that

$$f \cdot \langle k, \mathbf{1}_p \rangle \quad \leq \quad k.$$

Thus, k is an initial F_f-algebra iff $k \leq k'$, for any F_f-algebra k'.

Let $f : n \to n + p$ be a horizontal morphism in T. If there is an initial f-algebra, we choose one and write it as follows:

$$\iota : f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \quad \to \quad f^\dagger.$$

When there is an initial f-algebra, we say that $f^\dagger$ *exists.*

Proposition 8.4.5 *Suppose that $F :\to$ is a functor and (x, u) is an initial F-algebra. Then u is an isomorphism.*

Proof. Note that $uF : xFF \to xF$ is an F-algebra. One shows that the unique F-homomorphism from (x, u) to (xF, uF) is the inverse of u.

Corollary 8.4.6 *If $f^\dagger$ exists, $\iota : f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \to f^\dagger$ is an isomorphism.*

Proposition 8.4.7 *For any $f : n \to p$, $(0_n \oplus f)^\dagger$ exists and is isomorphic to f; in fact,*

$$\mathbf{I}(f) : f \quad \to \quad f$$

is an initial $0_n \oplus f$-algebra.

Proof. Since $(0_n \oplus f) \cdot \langle g, \mathbf{1}_p \rangle = f$, a $(0_n \oplus f)$-algebra (g, u) is just a morphism

$$u : f \quad \rightarrow \quad g$$

and a $(0_n \oplus f)$-homomorphism $(g, u) \rightarrow (g', u')$ is a vertical morphism $v : g \rightarrow g'$ such that $u \circ v = I(f) \circ u'$. The result follows immediately.

Thus, the left zero identity will always hold in arbitrary 2-theories.

Proposition 8.4.8 *For any fixed n, p, the category $T(n, p)$ has an initial object iff $(\mathbf{1}_n \oplus 0_p)^\dagger$ exists.*

Proof. First, note that since

$$(\mathbf{1}_n \oplus 0_p) \cdot \langle g, \mathbf{1}_p \rangle \;=\; g,$$

a $(\mathbf{1}_n \oplus 0_p)$-algebra consists of a vertical morphism

$$u : g \quad \rightarrow \quad g.$$

Hence, let (g, u) be an initial $(\mathbf{1}_n \oplus 0_p)$-algebra. We show that $u = \mathbf{I}(g)$. Indeed, since (g, u) itself is a $(\mathbf{1}_n \oplus 0_p)$-algebra, there is a unique $v : g \rightarrow g$ such that

$$v \circ u \quad = \quad u \circ v.$$

But both $v = \mathbf{I}(g)$ and $v = u$ work. Thus, $(g, \mathbf{I}(g))$ is the initial $(\mathbf{1}_n \oplus 0_p)$-algebra. It follows immediately that g is initial in $T(n, p)$.

Conversely, if $\perp_{np}$ is initial in $T(n, p)$, $(\perp_{np}, \mathbf{I}(\perp_{np}))$ is an initial $(\mathbf{1}_n \oplus 0_p)$-algebra.

Now we consider the right zero identity.

Proposition 8.4.9 *Suppose that the 2-theory T has an initial object $\perp_{np}$ in each category $T(n, p)$ and that $\perp_{np} \cdot h$ and $\perp_{nq}$ are isomorphic, for all $h : p \rightarrow q$. Let $f : n \rightarrow n + p$ be a horizontal morphism in T. If $f^\dagger$ and $(f \oplus 0_q)^\dagger$ both exist, then $(f \oplus 0_q)^\dagger$ is isomorphic to $f^\dagger \oplus 0_q$.*

Proof. We may as well assume that $\perp_{np} \cdot h = \perp_{nq}$, for all $h : p \to q$. We give the proof for the case $p = 0$, leaving to the reader the task of proving the general case, which differs from this one only in notation. In our argument, we make heavy use of Proposition 1.1.4.6, which we refer to here as the *Preservation Proposition.* Note first that when $p = 0$, an $f \oplus 0_q$-algebra is a pair (j, β), where

$$\beta : \; f \cdot j \quad \to \quad j$$

is a vertical morphism in $T(n, q)$. So suppose that

$$\beta_0 : \; f \cdot j_0 \quad \to \quad j_0$$

is an initial $f \oplus 0_q$-algebra in $T(n, q)$ and that

$$\alpha_0 : \; f \cdot k_0 \quad \to \quad k_0$$

is an initial f-algebra in $T(n, 0)$. Lastly, let

$$i : \perp_{qq} \quad \to \quad \mathbf{1}_q$$

be the unique vertical morphism in $T(q, q)$. By the initiality of (k_0, α_0), there is a unique $a : k_0 \to j_0 \cdot \perp_{q0}$ such that

$$[1`1`1`1;1000`600][f \cdot k_0`k_0`f \cdot j_0 \cdot \perp_{q0}`j_0 \cdot \perp_{q0};\alpha_0`f \cdot a`a`\beta_0 \cdot \perp_{q0}]$$

commutes. Hence, by the Preservation Proposition,

$$[1`1`1`1;1000`600][f \cdot k_0 \cdot 0_q`k_0 \cdot 0_q`f \cdot j_0 \cdot \perp_{qq}`j_0 \cdot \perp_{qq};\alpha_0 \cdot 0_q`f \cdot a \cdot 0_q`a \cdot 0_q`\beta_0 \cdot \perp_{qq}] \tag{8.2}$$

commutes. It is not difficult to verify that the square

$$[1`1`1`1;1000`600][f \cdot j_0 \cdot \perp_{qq}`j_0 \cdot \perp_{qq}`f \cdot j_0`j_0;\beta_0 \cdot \perp_{qq}`f \cdot j_0 \cdot i`j_0 \cdot i`\beta_0] \tag{8.3}$$

commutes, using the interchange law. Lastly, by the initiality of (j_0, β_0), there is a unique vertical morphism b such that

$$[1`1`1`1;1000`600][f \cdot j_0`j_0`f \cdot k_0 \cdot 0_q`k_0 \cdot 0_q;\beta_0`f \cdot b`b`\alpha_0 \cdot 0_q] \tag{8.4}$$

commutes. Stacking square (8.4) on top of (8.2) on top of (8.3), we see that

$$b \circ (a \cdot 0_q) \circ (j_0 \cdot i) \quad = \quad \mathbf{I}(j_0), \tag{8.5}$$

by the initiality of (j_0, β_0). We will show that b is in fact an isomorphism with inverse

$$(a \cdot 0_q) \circ (j_0 \cdot i).$$

Now if we put square (8.2) on (8.3) on (8.4) and take the horizontal composite with $\perp_{q0}$ on the right, we see by the initiality of (k_0, α_0) and the Preservation Proposition,

$$a \circ (b \cdot \perp_{q0}) \quad = \quad \mathbf{I}(k_0). \tag{8.6}$$

We need to prove that

$$(a \cdot 0_q) \circ (j_0 \cdot i) \circ b \quad = \quad \mathbf{I}(k_0 \cdot 0_q). \tag{8.7}$$

But

$$\begin{aligned} (j_0 \cdot i) \circ b &= (j_0 \cdot i) \circ (b \cdot \mathbf{1}_q) \\ &= (j_0 \circ b) \cdot i \\ &= b \cdot i. \end{aligned}$$

By (8.6) and the Preservation Proposition,

$$(a \cdot 0_q) \circ (b \cdot \perp_{qq}) \quad = \quad \mathbf{I}(k_0 \cdot 0_q).$$

Now using i and the previous equation,

$$(a \cdot 0_q) \circ (b \cdot i) \quad = \quad \mathbf{I}(k_0 \cdot 0_q),$$

by the Preservation Proposition and the fact that $0_q \cdot i = 0_q$ and $\perp_{qq} \cdot i = i$. This proves (8.7) and completes the proof of the proposition.

When T is the 2-theory of an ordered theory, we can say more.

Proposition 8.4.10 *Suppose that T is a pointed ordered theory. For any $f : n \to n + p$ and any $q \geq 0$, $f^\dagger$ exists iff $(f \oplus 0_q)^\dagger$ exists. When both exist,*

$$(f \oplus 0_q)^\dagger \quad = \quad f^\dagger \oplus 0_q.$$

We let the reader prove this result. See Exercise 8.5.20.

Remark 8.4.11 The assumption on T in Proposition 8.4.9 is equivalent to the condition that $\bot_{10} \cdot 0_p$ is initial in $T(1,p)$, for all $p \geq 0$.

The permutation identity holds in the following form:

Proposition 8.4.12 *Suppose that $f : n \to n+p$ and that $\pi : n \to n$ is a base permutation. Then $f^\dagger$ exists iff $(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger$ exists, and when both exist, $\pi \cdot f^\dagger$ is isomorphic to $(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger$.*

Proof. We observe that

[1‘1‘1‘1;1000‘600] [f$\cdot\langle k, \mathbf{1}_p\rangle$‘$k$‘$f \cdot \langle k', \mathbf{1}_p\rangle$‘$k'$;$\alpha$‘$f \cdot \langle h, \mathbf{1}_p\rangle$‘$h$‘$\alpha'$] commutes iff the square

[1‘1‘1‘1;1500‘600] [$\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot \langle \pi \cdot k, \mathbf{1}_p\rangle$‘$\pi \cdot k$‘$\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot \langle \pi \cdot k', \mathbf{1}_p\rangle$‘$\pi \cdot k'$;$\pi \cdot \alpha$‘$\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p) \cdot \langle \pi \cdot h, \mathbf{1}_p\rangle$‘$\pi \cdot h$‘$\pi \cdot \alpha'$] does. Thus $h : (k, \alpha) \to (k', \alpha')$ is a homomorphism of f-algebras iff $\pi \cdot h : (\pi \cdot k,\ \pi \cdot \alpha) \to (\pi \cdot k',\ \pi \cdot \alpha')$ is a $\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p)$-algebra homomorphism.

Next, we will show that the parameter identity implies the pairing identity. Note that if $r : n \to n+p$ and $s : p \to q$ are horizontal morphisms, then an $r \cdot (\mathbf{1}_n \oplus s)$-algebra is a pair (y, v), where $y : n \to q$ is a horizontal morphism and

$$v : r \cdot (\mathbf{1}_n \oplus s) \cdot \langle y, \mathbf{1}_q\rangle \quad \to \quad y,$$

i.e.

$$v : r \cdot \langle y, s\rangle \quad \to \quad y.$$

Now if $f : n \to n+m+p$ and $g : m \to n+m+p$ are horizontal morphisms, an $\langle f, g\rangle$-algebra is determined by a vertical morphism

$$\langle f, g\rangle \cdot \langle r, \mathbf{1}_p\rangle \quad \to \quad r$$

in $T(n+m,\ p)$, or, writing $r = \langle r_1, r_2\rangle$ with $r_1 : n \to p$ and $r_2 : m \to p$, by a pair consisting of a vertical morphism

$$\alpha : \ f \cdot \langle r_1, r_2, \mathbf{1}_p\rangle \quad \to \quad r_1 \tag{8.8}$$

in $T(n,p)$, and a vertical morphism

$$\beta : \; g \cdot \langle r_1, r_2, \mathbf{1}_p \rangle \quad \to \quad r_2 \tag{8.9}$$

in $T(m,p)$. Let $\iota : \; f \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle \to f^\dagger$ be an initial F_f-algebra. Define the morphism h by

$$h \quad := \quad g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle.$$

Note that an h-algebra is determined by a vertical morphism $h \cdot \langle z, \mathbf{1}_p \rangle \to z$ in $T(m,p)$, i.e.

(8.10)

Let $\gamma : h \cdot \langle h^\dagger, \mathbf{1}_p \rangle \to h^\dagger$ be an initial h-algebra. Less succinctly,

Noting that

$$f \cdot \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, \, h^\dagger, \mathbf{1}_p \rangle \quad = \quad f \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle \cdot \langle h^\dagger, \mathbf{1}_p \rangle,$$

we will show that the pair consisting of

and

forms an initial $\langle f, g \rangle$-algebra, proving that $\langle f, g \rangle^\dagger$ is isomorphic to

$$\langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, \, h^\dagger \rangle.$$

In the present setting, the parameter identity means the following:

The Parameter Identity in 2-Theories. *For any horizontal morphisms $r : n \to n+p$ and $s : p \to q$, if $(r^\dagger, \iota)$ is an initial r-algebra, then $(r^\dagger \cdot s, \, \iota \cdot s)$ is an initial $r \cdot (\mathbf{1}_n \oplus s)$-algebra.*

Thus, if $\iota : r \cdot \langle r^\dagger, \mathbf{1}_p \rangle \to r^\dagger$ is an initial r-algebra, then for any

$$v : r \cdot \langle y, s \rangle \quad \to \quad y$$

there is a unique $v^\# : r^\dagger \cdot s \to y$ such that the following square commutes.

[1‘1‘1‘1;900‘700] [r$\cdot\langle r^\dagger \cdot s,\ s\rangle$‘$r^\dagger \cdot s$‘$r \cdot \langle y,\ s\rangle$‘$y$;$\iota \cdot s$‘$r \cdot \langle v^\#,\ s\rangle$‘$v^\#$‘$v$]

When this condition holds for some fixed integer n, and for all r and s as above, then we will say that *the parameter identity holds for* n. When $n = 1$, we will also say that the scalar parameter identity holds.

Proposition 8.4.13 *Suppose that the parameter identity holds in the* 2*-theory* T. *Then the pairing identity holds in the following form: Let* $f : n \to n + m + p$ *and* $g : m \to n + m + p$ *be horizontal morphisms. Then if* $f^\dagger$ *and* $h^\dagger$ *exist, where* h *is defined by* $h := g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle$, *then* $\langle f, g\rangle^\dagger$ *exists and is isomorphic to*

$$\langle f^\dagger \cdot \langle h^\dagger,\ \mathbf{1}_p\rangle,\ \ h^\dagger\rangle.$$

Restating the conclusion, if $(f^\dagger, \iota)$ is an initial f-algebra and $(h^\dagger, \gamma)$ is an initial h-algebra, then

$$(\langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\ h^\dagger\rangle,\ \langle \iota \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\ \gamma\rangle)$$

is an initial $\langle f, g\rangle$-algebra.

Proof. Assume that $(\langle r_1, r_2\rangle, \langle \alpha, \beta\rangle)$ is a fixed $\langle f, g\rangle$-algebra, as in (8.8) and (8.9). Since

$$f \cdot \langle r_1, r_2, \mathbf{1}_p\rangle \quad = \quad f \cdot (\mathbf{1}_n \oplus \langle r_2, \mathbf{1}_p\rangle) \cdot \langle r_1, \mathbf{1}_p\rangle,$$

it follows that

is an $f \cdot (\mathbf{1}_n \oplus \langle r_2, \mathbf{1}_p\rangle)$-algebra. Thus, by the parameter identity, there is a unique vertical morphism

such that the following square commutes:

[1‘1‘1‘1; 1600‘600][$f \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \cdot \langle r_2, \mathbf{1}_p\rangle$‘$f^\dagger \cdot \langle r_2, \mathbf{1}_p\rangle$‘$f \cdot \langle r_1, r_2, \mathbf{1}_p\rangle$‘$r_1$;$\iota \cdot \langle r_2, \mathbf{1}_p\rangle$‘$f \cdot \langle \alpha^\wedge,\ r_2, \mathbf{1}_p$

Recalling (8.10), define the vertical morphism $\overline{\beta}$ as the composite

(8.12)

Then $(r_2, \overline{\beta})$ is an h-algebra. Since $(h^\dagger, \gamma)$ is an initial h-algebra, there is a unique vertical morphism $\beta^\sharp : h^\dagger \to r_2$ such that the square

$[1‘1‘1‘1; 1000‘600][g \cdot \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger, \mathbf{1}_p\rangle ‘h^\dagger ‘g \cdot \langle f^\dagger \cdot \langle r_2, \mathbf{1}_p\rangle,\, r_2, \mathbf{1}_p\rangle ‘r_2; \gamma ‘g \cdot \langle f^\dagger \cdot \langle \beta^\sharp, \mathbf{1}_p\rangle,\, \beta^\sharp, \mathbf{1}_p\rangle ‘\beta^\sharp ‘\overline{\beta}](8.13$

commutes. It follows from the interchange law that the following square commutes also:

$[1‘1‘1‘1; 1300‘600][f \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \cdot \langle h^\dagger, \mathbf{1}_p\rangle ‘f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle ‘f \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle \cdot \langle r_2, \mathbf{1}_p\rangle ‘f^\dagger \cdot \langle r_2, \mathbf{1}_p\rangle; \iota \cdot \langle h^\dagger, \mathbf{1}_p\rangle ‘f \cdot \langle f$

Putting square (8.14) on top of square (8.11), we obtain the following commuting diagram.

Now define the vertical morphism $\alpha^\sharp$ as the composite of the vertical morphisms on the right side:

$$\alpha^{\#} \;\; := \;\; f^\dagger \cdot \langle \beta^\sharp, \mathbf{1}_p\rangle \circ \alpha^\wedge : \; f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle \; \to \; r_1. \qquad (8.15)$$

We will show that $\langle\, \alpha^\sharp,\, \beta^\sharp \,\rangle$ is the unique $\langle f, g\rangle$-homomorphism from

$$(\langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger\rangle,\; \langle \iota \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, \gamma\rangle)$$

to

$$(\langle r_1, r_2\rangle,\, \langle \alpha, \beta\rangle).$$

We first verify that the following two squares commute, showing that $\langle\, \alpha^\sharp,\, \beta^\sharp \,\rangle$ is an $\langle f, g\rangle$-homomorphism.

$[1‘1‘1‘1; 1645‘600][f \cdot \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger, \mathbf{1}_p\rangle = f \cdot \langle f^\dagger, \mathbf{1}_p\rangle \cdot \langle h^\dagger, \mathbf{1}_p\rangle ‘f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle ‘f \cdot \langle r_1, r_2, \mathbf{1}_p\rangle ‘r_1; \iota \cdot \langle h^\dagger, \mathbf{1}_p\rangle ‘$

and

$[1‘1‘1‘1; 1200‘600][g \cdot \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p\rangle,\, h^\dagger, \mathbf{1}_p\rangle ‘h^\dagger ‘g \cdot \langle r_1, r_2, \mathbf{1}_p\rangle ‘r_2; \gamma ‘g \cdot \langle \alpha^\sharp,\, \beta^\sharp, \mathbf{1}_p\rangle ‘\beta^\sharp ‘\beta](8.17)$

But the fact that (8.16) commutes follows from Exercise 8.4.3. Now, if we rearrange the information in (8.12) and (8.13), we obtain the commuting square

Now again using Exercise 8.4.3, it follows that (8.17) commutes also.

As for uniqueness, let u, v be any $\langle f, g\rangle$-homomorphism whose target is $(\langle r_1, r_2\rangle,\ \langle\alpha,\beta\rangle)$, so that the squares

[1‘1‘1‘1;1600‘700] [f$\cdot\langle f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle,\ h^\dagger,\mathbf{1}_p\rangle`f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle`f\cdot\langle r_1,r_2,\mathbf{1}_p\rangle`r_1;\iota\cdot\langle h^\dagger,\mathbf{1}_p\rangle`f\cdot\langle u,v,\mathbf{1}_p\rangle`u`\alpha$] and

[1‘1‘1‘1;1600‘700] [g$\cdot\langle f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle,\ h^\dagger,\mathbf{1}_p\rangle`h^\dagger`g\cdot\langle r_1,r_2,\mathbf{1}_p\rangle`r_2;\gamma`g\cdot\langle u,v,\mathbf{1}_p\rangle`v`\beta$] commute. Then, by the parameter identity, u is the unique vertical morphism such that the square

[1‘1‘1‘1; 1600‘700]$[f\cdot\langle f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle,\ h^\dagger,\mathbf{1}_p\rangle`f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle`f\cdot\langle r_1,\ h^\dagger,\mathbf{1}_p\rangle`r_1;\iota\cdot\langle h^\dagger,\mathbf{1}_p\rangle`f\cdot\langle u,\ h^\dagger,\mathbf{1}$

commutes, where $\overline{\alpha}$ is the composite

$$\overline{\alpha} \quad := \quad (f\cdot\langle r_1, v, \mathbf{1}_p\rangle)\circ\alpha.$$

Recall the definitions of $\alpha^\wedge$ in (8.11) and $\overline{\beta}$ in (8.12). By the interchange law again, for any vertical morphism $v : h^\dagger \to r_2$, the square

[1‘1‘1‘1; 1350‘700]$[f\cdot\langle f^\dagger,\mathbf{1}_{m+p}\rangle\cdot\langle h^\dagger,\mathbf{1}_p\rangle`f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle`f\cdot\langle f^\dagger,\mathbf{1}_{m+p}\rangle\cdot\langle r_2,\mathbf{1}_p\rangle`f^\dagger\cdot\langle r_2,\mathbf{1}_p\rangle;\iota\cdot\langle h^\dagger$

commutes. If we place the square (8.19) on top of square (8.11), we see that

$$(f^\dagger\cdot\langle v,\mathbf{1}_p\rangle)\circ\alpha^\wedge$$

is another homomorphism from $(f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle,\ \iota\cdot\langle h^\dagger,\mathbf{1}_p\rangle)$ to $(r_1,\overline{\alpha})$. Thus

$$u \quad = \quad (f^\dagger\cdot\langle v,\mathbf{1}_p\rangle)\circ\alpha^\wedge.$$

But now it follows that the square

[1‘1‘1‘1;1600‘700] [g$\cdot\langle f^\dagger\cdot\langle h^\dagger,\mathbf{1}_p\rangle,\ h^\dagger,\mathbf{1}_p\rangle`h^\dagger`g\cdot\langle f^\dagger\cdot\langle r_2,\mathbf{1}_p\rangle,\ r_2,\mathbf{1}_p\rangle`r_2;\gamma`g\cdot\langle f^\dagger\cdot\langle v,\mathbf{1}_p\rangle,\ v,\mathbf{1}_p\rangle`v`\overline{\beta}$] commutes, showing that $v = \beta^\sharp$, by (8.13). But now,

$$\begin{aligned} u &= (f^\dagger\cdot\langle v,\mathbf{1}_p\rangle)\circ\alpha^\wedge \\ &= (f^\dagger\cdot\langle\beta^\sharp,\mathbf{1}_p\rangle)\circ\alpha^\wedge \\ &= \alpha^\sharp, \end{aligned}$$

by (8.15), completing the proof of uniqueness.

Corollary 8.4.14 *Suppose that n and m are fixed integers such that $f^\dagger$ and $g^\dagger$ exist, for all horizontal morphisms $f : n \to n + p$ and $g : m \to m + p$, $p \geq 0$. If the parameter identity holds for n, then for any $h : n + m \to n + m + p$, $h^\dagger$ exists. If in addition the parameter identity holds for m, then the parameter identity also holds for $n + m$.*

Proof. The first fact follows by the proof of Proposition 8.4.13. The proof of the second fact is left as an exercise.

Example 8.4.15 This example shows that the parameter identity does not follow purely from considerations of initiality. Let $\mathbf{N}_\infty$ be the linearly ordered set consisting of the nonnegative integers, ordered as usual, and in addition, a point ∞ larger than all standard integers. Define a "successor function" σ as follows:

$$x\sigma \quad := \quad \begin{cases} x & \text{if } x = 0 \text{ or } x = \infty; \\ x + 1 & \text{otherwise.} \end{cases}$$

We define another function f on $\mathbf{N}_\infty$. For x, y in $\mathbf{N}_\infty$,

$$f(x, y) \quad := \quad \max\{x\sigma,\, y\sigma\}.$$

Let T be the least subtheory of $\mathbf{N}_\infty$ containing f and the constants $0, \infty : \mathbf{N}_\infty^0 \to \mathbf{N}_\infty$. T is a 2-theory where the vertical morphisms are given by the ordering. It can be seen that a function $k : \mathbf{N}_\infty^n \to \mathbf{N}$ belongs to T iff k is a projection or

$$k(x_1, \ldots, x_n) \quad = \quad \max\{x_{i_1}\sigma^{k_1}, \ldots, x_{i_m}\sigma^{k_m}, z\},$$

where $m \geq 0$, $1 \leq i_1 < \ldots < i_m \leq n$, $k_1, \ldots, k_m > 0$, and where $z = 0$ or $z = \infty$. Thus there are two constant functions in $T(1, n)$, the function g_n with value 0 and the function h_n with value ∞. Suppose that $k : 1 \to 1 + p \in T$ is not a projection. If k is of the form $k = 0_1 \oplus k'$, for some k' in T, then clearly k' is the only solution of the iteration equation for k. When

$$k(x_1, \ldots, x_{1+p}) \quad = \quad x_1\sigma^m,$$

for some $m > 0$, the iteration equation for k has two solutions in T, the functions g_p and h_p. Otherwise the only solution is h_p. Thus $k^\dagger$ exists, for all $k : 1 \to 1 + p$ in T. We also see that the scalar parameter identity fails in T, for

$$(f \cdot (\mathbf{1}_1 \oplus g_1))^\dagger \;=\; \sigma^\dagger \;=\; h_1,$$

but

$$f^\dagger \cdot g_1 \;=\; g_1.$$

We summarize and slightly simplify the results of this section in the following theorem.

Theorem 8.4.16 *Suppose that T is a 2-theory in which $f^\dagger$ exists, for all horizontal morphisms $f : n \to n + p,\ n, p \geq 0$. Then*

- *the fixed point identity holds;*
- *the permutation identity holds;*
- *the left zero identity holds;*
- *each category $T(n, p)$ has an initial object $\perp_{np}$;*
- *the right zero identity holds if $\perp_{np} \cdot h$ and $\perp_{nq}$ are isomorphic, for all $h : p \to q$.*

Lastly, if the parameter identity holds, so does the pairing identity, and, up to isomorphism, all of the identities true in all Conway theories hold in T.

Corollary 8.4.17 *Suppose that T is a 2-theory in which $f^\dagger$ exists, for all scalar horizontal morphisms $f : 1 \to 1 + p$. If the scalar parameter identity holds, then $g^\dagger$ exists, for all morphisms $g : n \to n+p$. Further, up to isomorphism, the underlying preiteration theory of T satisfies all of the Conway theory identities.*

Exercise 8.4.18 Under the hypotheses of Theorem 8.4.16, show that the simplified composition identity holds. Suppose that $f : n \to p$ and $g : p \to n + q$ are horizontal morphisms. Let (x, α) be an initial $f \cdot g$-algebra, and let (y, β) be an initial $g \cdot (f \oplus \mathbf{1}_q)$-algebra. Then show that (x, α) is isomorphic to $(f \cdot y,\ f \cdot \beta)$.

Exercise 8.4.19 Use Example 8.4.15 to show that the double dagger identity does not follow just from existence of initial algebras. *Hint.* Show $f^{\dagger\dagger} : 1 \to 0$ and $\sigma^\dagger$ exist and are distinct. But $\sigma = f \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle$, showing that the double dagger identity fails.

Exercise 8.4.20 Use Example 8.4.15 to show that the pairing identity does not follow from the existence of initial algebras. *Hint:* Consider $\langle f,\ 0_1 \oplus g_1 \rangle^\dagger$.

Exercise 8.4.21 Modify the example 8.4.15 to obtain an example of a *strict* ordered theory T such that for each morphism $f : 1 \to 1 + p$ there is a least solution $f^\dagger$ in T to the iteration equation for f; but for some $f : 1 \to 1 + p$ and $g : p \to q$, the parameter identity

$$f^\dagger \cdot g \quad = \quad (f \cdot (\mathbf{1}_1 \oplus g))^\dagger$$

fails in T. *Hint:* Let $A = \{0, 1, \ldots, \omega, \omega + 1, \ldots, \omega + \omega\}$. Define a successor function on A such that $x\sigma = x + 1$ unless $x = 0, \omega, \omega + \omega$. For these three cases, $x\sigma = x$. Let f be as before. Define $xg := \omega$ unless $x = 0$; similarly, $xh := \omega + \omega$ unless $x = 0$. Thus, the three functions f, g, h are strict order-preserving functions. Let T be the least subtheory of A containing f, g, h as well as the the constant $0 : A^0 \to A$.

Exercise 8.4.22 Complete the proof of Corollary 8.4.14.

8.5 ω-Continuous 2-Theories

In this section, we generalize some of the constructions on ω-continuous ordered theories to 2-theories. After defining ω-continuous 2-theories, we show how to use the vertical structure of such a 2-theory to define an initial f-algebra, for each horizontal morphism $f : n \to n + p$. In addition, the parameter identity holds in such theories, as well as the functorial implication for all base morphisms. It follows from Theorem 8.4.16 that up to isomorphism, the horizontal structure of an ω-continuous 2-theory is an iteration theory.

8.5.1 The Definition, with Examples

In this section, we provide some preliminary results.

Definition 8.5.1 *Suppose that T is a 2-theory. We say that T is* ω-**continuous** *if T has the following properties:*

1. *The underlying theory of T is pointed, with a distinguished horizontal morphism $\perp$ in $T(1, 0)$; as usual, for each $n, p \geq 0$, we define*

$$\begin{aligned} \perp_{1p} \quad &:= \quad \perp \cdot 0_p \\ \perp_{np} \quad &:= \quad \overbrace{\langle \perp_{1p}, \ldots, \perp_{1p} \rangle}^{n} . \end{aligned}$$

2. *Each morphism $\perp_{np}$ is initial in $T(n,p)$; i.e. for each horizontal morphism $f : n \to p$, there is a unique vertical morphism $\perp_{np} \to f$ in $T(n,p)$. Note:*

$$\perp_{np} \cdot f \quad = \quad \perp_{nq},$$

for each $f : p \to q$.

3. *For each $n, p \geq 0$, the category $T(n,p)$ has colimits of all ω-diagrams.*

4. *The colimits in each category $T(n,p)$ commute with horizontal composition on both sides.*

In detail, the last two conditions in the above definition mean the following: First, for each ω-diagram in $T(n,p)$,

(8.20)

there is a horizontal morphism $f : n \to p$, and a collection of vertical morphisms $\nu_i : f_i \to f,\ i \geq 0$, such that for each $i \geq 0$ the diagram

[f$_i$‘f_{i+1}‘f; u_i‘ν_i‘ν_{i+1}] commutes. Further, for any horizontal morphism $g : n \to p$ and any collection of vertical morphisms $\nu_i' : f_i \to g,\ i \geq 0$, such that $u_i \circ \nu_{i+1}' = \nu_i'$, all i, there is a unique vertical morphism $\kappa : f \to g$ such that the diagram

[f$_i$‘f‘g; ν_i‘ν_i'‘κ] commutes, for all $i \geq 0$. We write

$$(f, \nu_i) \;=\; \varinjlim\, (f_i, u_i)$$

or just $(f, \nu_i) = \operatorname{colim}(f_i, u_i)$, to mean that the cone (f, ν_i) is *some* colimit of the ω-diagram $u_i : f_i \to f_{i+1},\ i \geq 0$.

Second, if $(f, \nu_i) = \operatorname{colim}(f_i, u_i)$ in $T(n,p)$ then $(f \cdot g,\ \nu_i \cdot g)$ is a colimit cone of $(f_i \cdot g,\ u_i \cdot g)$; and $(h \cdot f,\ h \cdot \nu_i)$ is a colimit cone of $(h \cdot f_i,\ h \cdot u_i)$, where $g : p \to q$ and $h : m \to n$ are any horizontal morphisms.

Of course, if (f, ν_i) and (g, ν_i') are colimits of the same ω-diagram (f_i, u_i), then f and g are isomorphic, via the unique morphism $\kappa : f \to g$ such that

$$\nu_i \circ \kappa \quad = \quad \nu_i',$$

for all $i \geq 0$.

We list some immediate corollaries.

Proposition 8.5.2 *In an ω-continuous 2-theory, colimits commute with both source pairing and separated sum. In detail, suppose that*

$$\begin{array}{rcll} (f, \mu_k) & = & \mathrm{colim}\,(f_k, u_k) & \quad\text{and} \\ (g, \nu_k) & = & \mathrm{colim}\,(g_k, v_k). & \end{array}$$

in the ω-continuous 2-theory T. Then

$$\begin{array}{rcl} (\langle f, g\rangle,\ \langle \mu_k, \nu_k\rangle) & = & \mathrm{colim}\,(\langle f_k, g_k\rangle,\ \langle u_k, v_k\rangle) \\ (f \oplus g,\ \mu_k \oplus \nu_k) & = & \mathrm{colim}\,(f_k \oplus g_k,\ u_k \oplus v_k). \end{array}$$

Proof. The continuity of separated sum follows from the continuity of composition and source pairing. To prove the latter fact, assume $f_k : n \to p$ and $g_k : m \to p$, for all $k \geq 0$. Then since colimits commute with horizontal composition,

$$(\mathbf{1}_n \oplus 0_m) \cdot \mathrm{colim}\,(\langle f_k, g_k\rangle,\ \langle u_k, v_k\rangle)$$

is isomorphic to

$$\mathrm{colim}\,(\ (\mathbf{1}_n \oplus 0_m) \cdot \langle f_k, g_k\rangle,\ (\mathbf{1}_n \oplus 0_m) \cdot \langle u_k, v_k\rangle\ )$$

and thus is a colimit of (f_k, u_k). Similarly,

$$(0_n \oplus \mathbf{1}_m) \cdot \mathrm{colim}\,(\langle f_k, g_k\rangle,\ \langle u_k, v_k\rangle)$$

is isomorphic to $\mathrm{colim}\,(g_k, v_k)$.

Corollary 8.5.3 *The functor F_f preserves colimits of ω-diagrams, i.e. if*

is a colimit in $T(n, p)$ of the ω-diagram

then

is a colimit of the diagram

Proof. This follows from the fact that F_f is defined using the operations of horizontal composition and tupling:

$$(u : g \to g')F_f \quad = \quad f \cdot \langle u, \mathbf{1}_p \rangle : f \cdot \langle g, \mathbf{1}_p \rangle \to f \cdot \langle g', \mathbf{1}_p \rangle.$$

Both of these operations have just been shown to preserve ω-colimits.

Example 8.5.4 Suppose that T is an ω-continuous (ordered) theory. Then T is an ω-continuous 2-theory in which there is a vertical morphism $f \to g$ iff $f \leq g$. Thus, the sequence (8.20) is just an increasing sequence

$$f_1 \ \leq \ f_2 \ \leq \ldots$$

in $T(n,p)$, and if $f = \sup_k f_k$, then f determines the colimit cone of vertical morphisms.

Example 8.5.5 Suppose that is an ω-complete category, i.e. a category with an initial object $\perp$ and colimits of all ω-diagrams

$$a_0 \xrightarrow{f_0} a_1 \xrightarrow{f_1} a_2 \to \cdots \tag{8.21}$$

Consider the 2-theory $Th()$ (in Example 3.3.7.7). Let $Th_\omega()$ be the sub-2-theory of $Th()$ consisting of the ω-functors: i.e. those functors $F :^p \to^n$ which have the property that if

$$\Gamma : \qquad a_0 \xrightarrow{f_0} a_1 \xrightarrow{f_1} a_2 \to \cdots$$

is an ω-diagram in $\mathcal{C}^p$, with colimit (a, μ_i), then

$$\Gamma F : \qquad a_0F \xrightarrow{f_0F} a_1F \xrightarrow{f_1F} a_2 \to \cdots$$

has colimit (aF, μ_iF). A cell $x = (u, F, G)$ in $Th_\omega(\mathcal{C})$ consists of a natural transformation $u : F \stackrel{\bullet}{\to} G$, where $F, G : \mathcal{C}^p \to \mathcal{C}^n$ are ω-functors. We write $au : aF \to aG$ for the component of u determined by the object a. Now suppose that

$$F_0 \xrightarrow{u_0} F_1 \xrightarrow{u_1} \cdots$$

is an ω-diagram in $Th_\omega(\mathcal{C})$, where $F_i :^p \to^n,\ i \geq 0$. Then for each object a in ^{p},

$$\Gamma_a : \qquad aF_0 \overset{au_0}{\to} aF_1 \overset{au_1}{\to} \cdots$$

is an ω-diagram in ^{n}, with colimit $(aF, a\mu_i)$, say. The object aF is defined only up to isomorphism, but it is a routine exercise to check that for any choice of a colimit aF of the ω-diagram Γ_a, the result is the object map of an ω-functor F. The value of F on morphisms is forced by the choice of the colimit cones $(aF, a\mu_i)$. Suppose that $f : a \to b$ is a morphism in ^{p}. Then, since u_i is a natural transformation, each square

[aF$_i$'aF_{i+1}'bF_i'bF_{i+1}; au_i'fF_i'fF_{i+1}'bu_i] commutes. Thus, there is a unique morphism $fF : aF \to bF$ such that

[aF$_i$'aF'bF_i'bF; $a\mu_i$'fF_i'fF'$b\mu_i$] commutes, for all $i \geq 0$. It is well-known that the resulting functor is also an ω-functor whenever each F_i is. If, for each $i \geq 0$, F_i and G_i are ω-functors, then it is also well-known that colimits commute with horizontal composition (see Exercise 8.5.17 below). Further, the constant functors $^p \to^n$ whose value on $f : x \to y$ is the identity $\perp^n \to \perp^n$ are initial morphisms $n \to p$ in $Th_\omega()$. Thus $Th_\omega(\mathcal{C})$ is an ω-continuous 2-theory.

Exercise 8.5.6 Suppose that T is an ω-continuous 2-theory. Show that there is ω-complete category such that T is isomorphic to a sub-2-theory of $Th_\omega()$. *Hint:* Use a variation of the construction used to prove any theory is isomorphic to a subtheory of A, for some set A (see Proposition 3.3.2.2).

8.5.2 Initial Algebras Exist

In this section we show that the vertical structure of an ω-continuous 2-theory determines an iteration theory structure on the underlying theory. More precisely, we will show how to obtain an initial F_f-algebra, for any $f : n \to n + p$.

First, suppose that $f : n \to n + p$ is a horizontal morphism in T. Recalling the modified powers of f from Definition 8.2.9, we will define a sequence of vertical morphisms $u_k(f) : f^{(k)} \to f^{(k+1)}$.

Definition 8.5.7

$$\begin{aligned} u_0(f) &:= \text{the unique } f^{(0)} \to f^{(1)} \\ u_{k+1}(f) &:= f \cdot \langle u_k(f), \mathbf{1}_p \rangle. \end{aligned}$$

Note that $f^{(0)} = \perp_{np}$ and $u_{k+1}(f) = u_k(f)F_f$. As usual, in the right side of the last equation, we identify both f and $\mathbf{1}_p$ with the identity vertical morphisms whose source and target are f and $\mathbf{1}_p$, respectively.

Since T is ω-continuous, the diagram

$$f^{(0)} \overset{u_0(f)}{\longrightarrow} f^{(1)} \overset{u_1(f)}{\longrightarrow} \cdots$$

has a colimit, say

$$(f^\dagger,\, \mu_k(f)) \quad := \quad \varinjlim(f^{(k)},\, u_k(f)), \tag{8.22}$$

defining the horizontal morphism $f^\dagger$ only up to isomorphism.

Note: below, we usually write only u_k and μ_k, not $u_k(f)$ or $\mu_k(f)$.

We will show that there is a vertical isomorphism $\iota : f^\dagger F_f \to f^\dagger$ such that $(f^\dagger, \iota)$ is an initial F_f-algebra.

Proposition 8.5.8 *For any $f : n \to n + p$ in an ω-continuous 2-theory, there is a unique morphism*

$$\iota : f^\dagger F_f \quad \to \quad f^\dagger$$

such that for all $k \geq 0$, the following triangle commutes.

$$[1‘1‘1;700][f^{(k+1)}‘f^\dagger F_f‘f^\dagger;\mu_k F_f‘\mu_{k+1}‘\iota] \tag{8.23}$$

Furthermore, ι is an isomorphism.

Proof. For each $k \geq 0$, apply the functor F_f to the colimit diagram

[f$^{(k)}$‘$f^{(k+1)}$‘$f^\dagger; u_k$‘μ_k‘μ_{k+1}] obtaining another colimit diagram, by Corollary 8.5.3 above:

[1‘1‘1;700] [f$^{(k)}F_f$‘$f^{(k+1)}F_f$‘$f^\dagger F_f; u_k F_f$‘$\mu_k F_f$‘$\mu_{k+1} F_f$] But, for all $k \geq 0$,

$$\begin{aligned} f^{(k)}F_f &= f^{(k+1)} \qquad \text{and} \\ u_k F_f &= u_{k+1}. \end{aligned}$$

Hence, $(f^\dagger, \mu_k)$ and $(f^\dagger F_f,\, \mu_k F_f)$ are colimits of the same diagram. Thus there is a unique isomorphism

$$\iota : f^\dagger F_f \quad \to \quad f^\dagger$$

such that for all $k \geq 0$, the triangle (8.23) commutes.

Now we show $(f^\dagger, \iota)$ is an initial F_f-algebra.

Lemma 8.5.9 *Suppose that $f : n \to n + p$ is a horizontal morphism in the ω-continuous 2-theory T. If $w : f \cdot \langle g, \mathbf{1}_p \rangle \to g$ is a vertical morphism in $T(n,p)$ then there is a sequence of morphisms $w_k : f^{(k)} \to g,\ k \geq 0$, such that*

$$[f^{(k)}‘f^{(k+1)}‘g; u_k‘w_k‘w_{k+1}] \tag{8.24}$$

commutes, for all $k \geq 0$.

Corollary 8.5.10 *Under the hypothesis of Lemma 8.5.9, there is a unique morphism $w^\sharp : f^\dagger \to g$ such that for all $k \geq 0$,*

$$\mu_k \circ w^\sharp \quad = \quad w_k.$$

Proof of Lemma 8.5.9. Note that the square

[1‘1‘1‘-1;500‘500] [f$^{(0)}$‘$f^{(1)}$‘g‘$gF_f; u_0$‘!‘!F_f‘w] commutes, since $f^{(0)} = \perp_{np}$ is an initial object in $T(n,p)$; here $! : f^{(0)} \to g$ is the unique vertical morphism. Now by repeatedly applying the functor F_f to this diagram, we obtain the following commuting diagrams:

$< 1‘1‘1‘{-1}; 500‘500 > (0,0)[f^{(0)}‘f^{(1)}‘g‘gF_f; u_0‘!‘!F_f‘w] < 1‘0‘0‘{-1}; 500‘500 > (500,0)[\quad ‘\cdots‘ \quad ‘\cdots; u$

for each $k \geq 1$. Now define $w_0 :=\, !$ and, for $k \geq 1$,

$$w_k \quad := \quad !F_f^k \circ w F_f^{k-1} \circ \ldots \circ w F_f \circ w \ :\ f^{(k)} \to g.$$

Thus, w_k is the composite of the vertical morphisms on the path traced out by travelling from $f^{(k)}$ to g by going first down and then to the left around the outside of the above diagram. Then clearly, for $k \geq 0$,

$$w_{k+1} \quad = \quad w_k F_f \circ w, \tag{8.25}$$

and $w_k = u_k \circ w_{k+1}$.

In the next corollary, we use the notation and assumptions of the previous corollaries.

Corollary 8.5.11 *The morphism $w^\sharp$ is an F_f-algebra homomorphism*

$$(f^\dagger, \iota) \quad \to \quad (g, w).$$

Proof. Since $(f^\dagger F_f,\ \mu_k F_f)$ is a colimit of $(f^{(k+1)},\ u_{k+1})$, there is a unique vertical morphism $v : f^\dagger F_f \to g$ such that

$$\mu_k F_f \circ v \quad = \quad w_{k+1},$$

for all $k \geq 0$. We show that $v = w^\sharp F_f \circ w$ and $v = \iota \circ w^\sharp$ both work, proving the corollary. Indeed, for all $k \geq 0$,

$$\mu_k \circ w^\sharp \quad = \quad w_k,$$

by Corollary 8.5.10 above. Hence

$$\mu_k F_f \circ w^\sharp F_f \quad = \quad w_k F_f, \quad k \geq 0.$$

Thus

$$\begin{aligned} \mu_k F_f \circ w^\sharp F_f \circ w \quad &= \quad w_k F_f \circ w \\ &= \quad w_{k+1}, \end{aligned}$$

by (8.25), proving half the claim. Also,

$$\mu_k F_f \circ \iota \quad = \quad \mu_{k+1},$$

by (8.23), so that

$$\begin{aligned} \mu_k F_f \circ \iota \circ w^\sharp \quad &= \quad \mu_{k+1} \circ w^\sharp \\ &= \quad w_{k+1}, \end{aligned}$$

completing the proof.

In fact, there is a unique F_f-homomorphism $(f^\dagger, \iota) \to (g, w)$.

Proposition 8.5.12 *Using the previous assumptions, suppose that $v : f^\dagger \to g$ is any vertical morphism such that*

$$[f^\dagger F_f`f^\dagger`gF_f`g;\iota`vF_f`v`w] \tag{8.26}$$

commutes. Then, $v = w^\sharp$, showing that $(f^\dagger, \iota)$ is an initial F_f-algebra.

Proof. We show that for all $k \geq 0$,

$$\mu_k \circ v \quad = \quad w_k : f^{(k)} \to g.$$

Since $w^\sharp$ is the unique morphism with this property, by Corollary 8.5.10, it follows that $v = w^\sharp$. We use induction on k. The case that $k = 0$ holds by the initiality of $f^{(0)}$. Assume that

$$\mu_k \circ v \quad = \quad w_k.$$

Then

$$\mu_k F_f \circ v F_f \quad = \quad w_k F_f.$$

Thus,

$$\begin{aligned} \mu_{k+1} \circ v &= \mu_k F_f \circ \iota \circ v \\ &= \mu_k F_f \circ v F_f \circ w \\ &= w_k F_f \circ w \\ &= w_{k+1}. \end{aligned}$$

8.5.3 ω-Continuous 2-Theories are Iteration Theories

We turn now to the task of showing that with the above definition of $f^\dagger$ on the horizontal morphisms, the underlying preiteration theory of an ω-continuous 2-theory is an iteration theory.

Theorem 8.5.13 *If T is an ω-continuous 2-theory, then up to vertical isomorphism, all valid iteration theory identities and the functorial dagger implication for surjective base morphisms hold in the underlying theory of T.*

Note that all identities hold *up to isomorphism*; i.e. if $t = t'$ is a valid iteration theory identity, then for each valuation of the variables in T, the horizontal morphism denoted by t is isomorphic to the horizontal morphism denoted by t'. Regarding the functorial dagger implication, we prove that if $f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g$ then $f^\dagger$ is isomorphic to $\rho \cdot g^\dagger$.

We prove the theorem by verifying that the zero identities hold, that the functorial dagger implication for base morphisms holds, and lastly that the parameter identity holds.

Lemma 8.5.14 *The left and right zero identities hold.*

Proof. By Proposition 8.4.7 and Proposition 8.4.9 above.

Lemma 8.5.15 *The functorial dagger implication for all base morphisms holds in the underlying theory of T.*

Proof. Suppose that $\rho : m \to n$ is any base morphism, that $f : m \to m+p$ and $g : n \to n+p$ are horizontal morphisms and that the square

[1‘1‘1‘1;600‘400] [m‘m+p‘n‘n+p;f‘ρ‘$\rho \oplus \mathbf{1}_p$‘g] commutes. Let $(f^\dagger, \mu_k)$ be a colimit in $T(m,p)$ of the ω-diagram $f^{(k)} \stackrel{u_k}{\to} f^{(k+1)}$, and $(g^\dagger, \nu_k)$ be a colimit in $T(n,p)$ of the diagram $g^{(k)} \stackrel{v_k}{\to} g^{(k+1)}$. Thus, by the continuity of composition,

$$(\rho \cdot g^\dagger,\ \rho \cdot \nu_k)$$

is a colimit of the diagram

$$\rho \cdot v_k \ : \ \rho \cdot g^{(k)} \quad \longrightarrow \quad \rho \cdot g^{(k+1)}.$$

It is easy to check by induction that

$$\rho \cdot g^{(k)} \quad = \quad f^{(k)}, \quad k \geq 0. \tag{8.27}$$

We must verify that $\rho \cdot v_k = u_k$, for all $k \geq 0$. The basis case is $k = 0$. But $\rho \cdot v_0$ and u_0 are both vertical morphisms $\perp_{mp} \to f^{(1)}$, and hence they are equal. Assuming $\rho \cdot v_k = u_k$, we compute:

$$\begin{aligned}
\rho \cdot v_{k+1} \quad &= \quad \rho \cdot g \cdot \langle v_k, \mathbf{1}_p \rangle \\
&= \quad f \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle v_k, \mathbf{1}_p \rangle \\
&= \quad f \cdot \langle u_k, \mathbf{1}_p \rangle \\
&= \quad u_{k+1},
\end{aligned}$$

completing the induction.

We now show that the parameter identity holds, which implies, by Proposition 8.4.13 above, that the pairing identity holds as well.

Suppose that $f : n \to n + p$ in the ω-continuous 2-theory T. For each $g : p \to q$, let

$$f(g) \quad := \quad f \cdot (\mathbf{1}_n \oplus g) : n \to n + q. \tag{8.28}$$

Recall that an $F_{f(g)}$-algebra is a vertical morphism

$$f \cdot \langle k, g\rangle \quad \rightarrow \quad k$$

where $k : n \rightarrow q$. Indeed,

$$\begin{aligned} f(g) \cdot \langle k, \mathbf{1}_q\rangle &= f \cdot (\mathbf{1}_n \oplus g) \cdot \langle k, \mathbf{1}_q\rangle \\ &= f \cdot \langle k, g\rangle. \end{aligned}$$

Lemma 8.5.16 The parameter identity. *With the above notation, $f(g)^\dagger$ is isomorphic to $f^\dagger \cdot g$. In fact,*

$$\iota \cdot g : f^\dagger F_f \cdot g \quad \rightarrow \quad f^\dagger \cdot g$$

is an initial $F_{f(g)}$-algebra.

Proof. Let

be the ω-diagram for which $(f^\dagger, \mu_k)$ is a colimit. The lemma follows from the fact that if

is the ω-diagram determining the colimit $(f(g)^\dagger, \nu_k)$, then for each $k \geq 0$,

$$\begin{aligned} f(g)^{(k)} &= f^{(k)} \cdot g \qquad \text{and} \\ v_k &= u_k \cdot g. \end{aligned}$$

Thus $(f^\dagger \cdot g,\ \mu_k \cdot g)$ and $(f(g)^\dagger,\ \nu_k)$ are colimits of the same diagram. The proof of Theorem 8.5.13 is complete.

Exercise 8.5.17 Suppose that , and are ω-complete categories and that for each $i \geq 0$, $F_i :\rightarrow$ and $G_i :\rightarrow$ are ω-functors. Lastly suppose that

$$F_0 \overset{u_0}{\rightarrow} F_1 \overset{u_1}{\rightarrow} \cdots$$
$$G_0 \overset{v_0}{\rightarrow} G_1 \overset{v_1}{\rightarrow} \cdots$$

are two ω-chains of natural transformations. Show that if

$$\begin{aligned}(F, \mu_i) &= \lim_{\rightarrow}(F_i, u_i)\\ (G, \nu_i) &= \lim_{\rightarrow}(G_i, v_i)\end{aligned}$$

then

$$(F \cdot G,\, \mu_i \cdot \nu_i) \;=\; \lim_{\rightarrow}(F_i \cdot G_i,\, u_i \cdot v_i).$$

Hint: For each $i, j \geq 0$, the square $Sq(i, j)$

[1‘1‘1‘1;1000‘500] [F$_i \cdot G_j$‘$F_{i+1} \cdot G_j$‘$F_i \cdot G_{j+1}$‘$F_{i+1} \cdot G_{j+1}$; $u_i \cdot G_j$‘$F_i \cdot v_j$‘$F_{i+1} \cdot v_j$‘$u_i \cdot G_{j+1}$] commutes, and the diagonal transformation $F_i \cdot G_j \to F_{i+1} \cdot G_{j+1}$ is $u_i \cdot v_j$. Pasting these squares together with $Sq(i, j)$ immediately to the left of the square $Sq(i+1, j)$ and immediately above $Sq(i, j+1)$ yields a doubly infinite grid. Now show that the colimit of the j-th horizontal chain is $F \cdot G_j$, when G_j is an ω-functor; similarly, the colimit of the i-th column is $F_i \cdot G$. Use standard diagram chasing to show the colimit of the diagonal chain $u_i \cdot v_i : F_i \cdot G_i \to F_{i+1} \cdot G_{i+1}$ is also the colimit of both $F \cdot G_j$ and $F_i \cdot G$, namely $F \cdot G$.

Exercise 8.5.18 Suppose that T is a 2-theory and that $v : f \to g$ and $v' : f' \to g'$ are isomorphisms. Show that when the sources and targets are appropriate, $v \cdot v' : f \cdot f' \to g \cdot g'$ and $\langle v, v' \rangle : \langle f, f' \rangle \to \langle g, g' \rangle$ are isomorphisms. *Hint:* Use the interchange law and Proposition 3.3.7.4.

Exercise 8.5.19 Let T be the 2-theory of a pointed ordered theory. Show that for any $f : n \to n + p$, an f-algebra is a morphism $x : n \to p$ such that $f \cdot \langle x, \mathbf{1}_p \rangle \leq x$. Show that if $x : n \to p$ is an f-algebra then $x \oplus 0_q$ is an $(f \oplus 0_q)$-algebra. Show that if $y : n + q \to p$ is an $(f \oplus 0_q)$-algebra, then $y \cdot (\mathbf{1}_p \oplus \perp_q)$ is an f-algebra, where $\perp_q : q \to 0$ is the least morphism in $T(q, 0)$.

Exercise 8.5.20 Prove Proposition 8.4.10. *Hint:* Use the previous exercise. If $f^\dagger$ exists and if z is any $(f \oplus 0_q)$-algebra,

$$f^\dagger \;\leq\; z \cdot (\mathbf{1}_p \oplus \perp_q).$$

Thus,

$$\begin{aligned}f^\dagger \cdot (\mathbf{1}_p \oplus 0_q) &\leq z \cdot (\mathbf{1}_p \oplus \perp_q) \cdot (\mathbf{1}_p \oplus 0_q)\\ &= z \cdot (\mathbf{1}_p \oplus \perp_{qq})\\ &\leq z \cdot (\mathbf{1}_p \oplus \mathbf{1}_q) \;=\; z,\end{aligned}$$

showing $f^\dagger \oplus 0_q$ is the least $(f \oplus 0_q)$-algebra. The argument in the other direction is similar.

Exercise 8.5.21 Let $f, g : n \to n+p$ be horizontal morphisms in a 2-theory T. Assume that $(f^\dagger, \iota_f)$ and $(g^\dagger, \iota_g)$ are initial f- and g-algebras, respectively. Suppose that $u : f \to g$ is a vertical morphism. Show that there is a unique vertical $u^\dagger : f^\dagger \to g^\dagger$ such that

[f$\cdot\langle f^\dagger, \mathbf{1}_p\rangle$‘$f^\dagger$‘$g \cdot \langle g^\dagger, \mathbf{1}_p\rangle$‘$g^\dagger; \iota_f$‘$u \circ \langle u^\dagger, \mathbf{1}_p\rangle$‘$u^\dagger$‘$\iota_g$]

Exercise 8.5.22 Let T be a 2-theory such that $f^\dagger$ exists for all horizontal $f : n \to n+p$. Use the previous exercise to show how to extend † to all vertical morphisms so that † becomes a functor $T(n, n+p) \to T(n, p)$. Conclude that if f and g are isomorphic in $T(n, n+p)$, then $f^\dagger$ and $g^\dagger$ are isomorphic in $T(n, p)$. Similarly, show if $u : f \to g$ is a retraction in $T(n, n+p)$, then so is $u^\dagger : f^\dagger \to g^\dagger$.

Exercise 8.5.23 Suppose that $f, g : n \to n+p$ are horizontal morphisms in the 2-theory T, and that $u : f \to g$ is a vertical morphism. Then u determines a map

$$\begin{array}{rcl} u^\flat : F_g\text{-}\mathbf{ALG} & \to & F_f\text{-}\mathbf{ALG} \\ (x, v) & \mapsto & f \cdot \langle x, \mathbf{1}_p\rangle \stackrel{u\cdot\langle x, 1_p\rangle}{\to} g \cdot \langle x, \mathbf{1}_p\rangle \stackrel{v}{\to} x. \end{array}$$

Show that if $h : (x, v) \to (x', v')$ is a F_g-homomorphism, then h is also a F_f-homomorphism $(x, v)u^\flat \to (x', v')u^\flat$. Show $u^\flat$ determines a functor F_g-**ALG** $\to F_f$-**ALG** which commutes with the forgetful functors U_{F_f} and U_{F_g}. Conclude that if u is an isomorphism, then $f^\dagger$ exists iff $g^\dagger$ exists, and if both exist they are isomorphic. This gives an alternate argument to that outlined in Exercise 8.5.22

Exercise 8.5.24 For any category , in the 2-theory $Th()$, the horizontal morphisms $n \to p$ are the functors $C^p \to C^n$. Let T be a sub 2-theory of $Th()$. Suppose that for each $n, p \geq 0$, each functor $F :^{n+p} \to^n$ in T and each object y in ^{p} there is an initial F_y-algebra, where $F_y :^n \to^n$ takes x to $(x, y)F$. Show this fact allows one to define a functor $F^\dagger :^p \to^n$. Show that when $F^\dagger$ is in T, for all F in T, the resulting preiteration theory is a Conway theory, i.e. satisfies the zero, pairing and permutation identities, as well as the parameter identity.

Exercise 8.5.25 Show that if T is an ω-continuous 2-theory, so is the 2-theory of cells, T (see Section 1.4.1 in Chapter 1).

8.6 Notes

Partially orderings have been used in semantics for many purposes. Ordered sets make it meaningful to single out an element as the least or greatest member of a set with a certain property. Dana Scott [Sco72, Sco76] used lattices to great effect to find mathematical semantics for many languages. Ordered theories are a framework for the study of ordered algebras. The importance of ordered theories for semantics, particularly ω-continuous and rational ordered theories, was emphasized by the ADJ group (J. Goguen, J. Thatcher, E.G. Wagner and J. Wright) in a number of papers [WTGW76, GTWW77, WTW78]. They constructed the free continuous theories for various notions of continuity, in both the setting of one and many sorted theories. The stronger notions of 2-theories, and ω-complete 2-theories are useful in the study of the semantics of circular data types [Wan79, SP82, LS81, BÉ89]. ω-continuous 2-theories were introduced in [BÉT]. The fact that ω-functors form an iteration theory was proved in [BÉ89] by an extension of the method used to prove ω-continuous theories form iteration theories. The direct method used above in Section 8.5 gives more information. The essence of the proof of the pairing identity is contained in the paper by Lehmann and Smyth [LS81].

Chapter 9

Matrix Iteration Theories

Matrix theories were discussed in Section 3.3.5. We proved there that each matrix theory is isomorphic to a matrix theory $\mathbf{Mat}_S$, for some semiring S. Hence in this chapter we will consider only these theories. When the matrix theory $T = \mathbf{Mat}_S$ is a preiteration theory, then one can define a *star operation* on each hom-set $T(n, n)$ as follows. When A is $n \times n$, the matrix $[A \;\; \mathbf{1}_n]$ is a morphism $n \to n + n$ in T, and we define A^* by the equation

$$A^* \;=\; [A \;\; \mathbf{1}_n]^\dagger. \tag{9.1}$$

It turns out that when T is an iteration theory, the dagger operation is expressible using the star operation and the equational properties of the dagger operation are equivalent to certain equational properties of the star operation.

9.1 Notation

Recall that in any theory T, if $f = \langle f_1, \ldots, f_n \rangle : n \to m + p$ and $g_i : m \to k$, $i \in [n]$, then we denote by

$$f \parallel (g_1, \ldots, g_n)$$

the T-morphism

$$\langle f_1 \cdot (g_1 \oplus \mathbf{1}_p), \ldots, f_n \cdot (g_n \oplus \mathbf{1}_p) \rangle : n \;\to\; k + p.$$

Below, we will use this notation when T is a matrix theory and $p = 0$. Thus if $T = \mathbf{Mat}_S$ and if $A : n \to m$ is written as the source tupling $\langle A_1, \ldots, A_n \rangle$ of the n row matrices $A_i : 1 \to m$, $i \in [n]$, and if for each $i \in [n]$, B_i is an $m \times k$ matrix, then

$$A \parallel (B_1, \ldots, B_n)$$

stands for the $n \times k$ matrix

$$A \parallel (B_1, \ldots, B_n) \quad = \quad \langle A_1 B_1, \ldots, A_n B_n \rangle.$$

Similarly, if for each $i \in [p]$, $C_i : m \to n$, and D is the target tupling $D = [D_1, \ldots, D_p] : n \to p$, then

$$(C_1, \ldots, C_p) \parallel D \quad := \quad [C_1 D_1, \ldots, C_p D_p] : m \to p.$$

We call a *star theory* a matrix theory T such that each hom-set $T(n, n)$ is equipped with a star operation $^* : T(n, n) \to T(n, n)$ which need not satisfy any particular properties. A *star theory morphism* between star theories is a theory morphism which preserves the star operation.

9.2 Properties of the Star Operation

In Theorem 9.2.1 below, we suppose that T is a matrix theory and a preiteration theory. The operation $^* : T(n, n) \to T(n, n)$ was just defined in (9.1).

In this section we prove the following facts.

Theorem 9.2.1 *The parameter identity holds in T if and only if the dagger operation is determined by the star operation as follows. If $C : n \to n + p$, then*

$$C^\dagger \quad = \quad (C \cdot \langle \mathbf{1}_n, 0_{pn} \rangle)^* \cdot (C \cdot \langle 0_{np}, \mathbf{1}_p \rangle);$$

i.e. writing $C = [A \;\; B]$,

$$[A \;\; B]^\dagger \quad = \quad A^* \cdot B. \tag{9.2}$$

Suppose that the dagger and star operations are related by (9.1) and (9.2).

[a] *The* **star fixed point identity**

$$A^* \;=\; A \cdot A^* + \mathbf{1}_n, \quad A : n \to n, \tag{9.3}$$

holds in T if and only if the fixed point identity holds.

[b] *The* **star product identity**

$$\begin{aligned}(A \cdot B)^* \;&= \\ \;&= \mathbf{1}_n + A \cdot (B \cdot A)^* \cdot B, \quad A : n \to m,\; B : m \to n,\end{aligned} \tag{9.4}$$

holds in T if and only if the composition identity holds.

[c] *The* **star sum identity**

$$(A + B)^* \;=\; (A^* \cdot B)^* \cdot A^*, \quad A, B : n \to n, \tag{9.5}$$

holds in T if and only if the double dagger identity holds.

[d] *The* **star zero identity**

$$0_{nn}^* \;=\; \mathbf{1}_n \tag{9.6}$$

holds in T if and only if the left zero identity holds.

[e] *The* **star pairing identity** *(9.7) holds in T if and only if the pairing identity holds.*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \;=\; \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{9.7}$$

where

$$A : n \to n,\;\; B : n \to m,\;\; C : m \to n, \quad \text{and } D : m \to m$$

and where

$$\begin{aligned} \alpha \;&=\; A^*B\delta CA^* + A^*, \qquad \beta \;=\; A^*B\delta \\ \gamma \;&=\; \delta CA^*, \quad \delta \;=\; (D + CA^*B)^*. \end{aligned}$$

[f] *The* **star permutation identity** *(9.8) holds in T if and only if the permutation identity holds.*

$$(\pi \cdot A \cdot \pi^T)^* \;=\; \pi \cdot A^* \cdot \pi^T, \tag{9.8}$$

where $A : n \to n$ and where $\pi : n \to n$ is a base permutation.

[g] *The* **star commutative identity** *(9.9) holds in T if and only if the commutative identity holds.*

$$((\rho \cdot A) \parallel (\rho_1, \ldots, \rho_m))^* \cdot \rho \quad = \quad \rho \cdot (A \cdot \rho)^*, \tag{9.9}$$

where $A : n \to m$, $\rho : m \to n$ is a surjective base matrix, and where $\rho_i : m \to m$ is a base matrix such that $\rho_i \cdot \rho = \rho$, for each $i \in [m]$.

Proof. First we show that when the parameter identity holds in T, the dagger operation can be written in terms of the star operation as claimed. First, note that the matrix $[A, \ B] : n \to n+p$ can be written as the composite

$$[A \ \ B] \ = \ [A \ \ \mathbf{1}_n] \cdot \begin{bmatrix} \mathbf{1}_n & 0_{np} \\ 0_{nn} & B \end{bmatrix} \ = \ [A \ \ \mathbf{1}_n] \cdot (\mathbf{1}_n \oplus B).$$

Hence, by the parameter identity,

$$\begin{aligned} [A \ \ B]^\dagger \ &= \ [A \ \ \mathbf{1}_n]^\dagger \cdot B \\ &= \ A^* \cdot B, \end{aligned}$$

by definition of A^*. Conversely, if the dagger operation is determined by the star operation as in (9.2), then for any $C : p \to q$,

$$\begin{aligned} ([A \ \ B] \cdot (\mathbf{1}_n \oplus C))^\dagger \ &= \ [A, \quad B \cdot C]^\dagger \\ &= \ A^* \cdot B \cdot C \\ &= \ [A, \quad B]^\dagger \cdot C, \end{aligned}$$

showing that the parameter identity holds in T.

Proof of [a]. Let $f = [A \ \ B]$ be a matrix $n \to n + p$. Since

$$\begin{aligned} A \cdot A^* \cdot B \ + \ B \ &= \ [A \ \ B] \cdot \begin{bmatrix} A^* \cdot B \\ \mathbf{1}_p \end{bmatrix} \\ &= \ f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \end{aligned}$$

and

$$A^* \cdot B \ = \ f^\dagger,$$

the fixed point identity holds if and only if the star fixed point identity (9.3) holds.

Proof of [b]. Let $f = [A,\ C] : n \to m+p$ and $g = [B,\ D] : m \to n+p$. Then

$$\begin{aligned} f \cdot \langle (g \cdot \langle f,\, 0_m \oplus \mathbf{1}_p\rangle)^\dagger,\, \mathbf{1}_p\rangle \ &= \\ =\ & (A \cdot (B \cdot A)^* \cdot B + \mathbf{1}_n) \cdot C + A \cdot (B \cdot A)^* \cdot D \end{aligned}$$

and

$$(f \cdot \langle g,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \quad = \quad (A \cdot B)^* \cdot C + (A \cdot B)^* \cdot A \cdot D.$$

If the composition identity holds, then taking $p = n$ and substituting $\mathbf{1}_n$ for C and 0_{mn} for D, we see that the star product identity (9.4) holds. Conversely, if the star product identity holds, then so does

the *simplified star product identity*

$$A \cdot (B \cdot A)^* \quad = \quad (A \cdot B)^* \cdot A, \tag{9.10}$$

for all $A : n \to m$ and $B : m \to n$.

It then follows that the composition identity holds.

Proof of [c]. Let $f = [A,\, B,\, C] : n \to 2n + p$. Then

$$\begin{aligned} f^{\dagger\dagger} \quad &= \quad [A^* \cdot B,\ \ A^* \cdot C]^\dagger \\ &= \quad (A^* \cdot B)^* \cdot A^* \cdot C, \end{aligned}$$

and

$$\begin{aligned} (f \cdot (\langle \mathbf{1}_n,\ \ \mathbf{1}_n\rangle \oplus \mathbf{1}_p))^\dagger \quad &= \quad [A + B,\ \ C]^\dagger \\ &= \quad (A + B)^* \cdot C. \end{aligned}$$

Thus the double dagger identity holds in T if and only if so does the star sum identity (9.5).

Proof of [d]. Let A be an $n \times p$ matrix. Then $(0_n \oplus A)^\dagger = 0^*_{nn} \cdot A$. Thus the left zero identity holds if and only if the star zero identity (9.6) holds.

Proof of [e]. Suppose that

$$f = \langle f_1, f_2 \rangle : n+m \rightarrow n+m+p.$$

Write

$$f_1 = \begin{bmatrix} A & B & E \end{bmatrix}$$

and

$$f_2 = \begin{bmatrix} C & D & F \end{bmatrix}.$$

Thus

$$\begin{aligned} f_1^\dagger &= \begin{bmatrix} A^*B & A^*E \end{bmatrix} \\ h &= f_2 \cdot f_1^\dagger \mathbf{1}_{m+p} \\ &= \begin{bmatrix} C & D & F \end{bmatrix} \begin{bmatrix} A^*B & A^*E \\ \mathbf{1}_m & 0_{mp} \\ 0_{pm} & \mathbf{1}_p \end{bmatrix} \\ &= \begin{bmatrix} CA^*B + D & CA^*E + F \end{bmatrix}, \end{aligned}$$

so that

$$h^\dagger = (D + CA^*B)^*(CA^*E + F)$$

and

$$\begin{aligned} &f_1^\dagger \cdot h^\dagger \mathbf{1}_p h^\dagger = \\ &= \begin{bmatrix} A^*B(D + CA^*B)^*(CA^*E + F) + A^*E \\ (D + CA^*B)^*(CA^*E + F) \end{bmatrix} \\ &= \begin{bmatrix} A^*B(D + CA^*B)^*CA^* + A^* & A^*B(D + CA^*B)^* \\ (D + CA^*B)^*CA^* & (D + CA^*B)^* \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix}. \end{aligned}$$

Hence the pairing identity holds if and only if the star pairing identity does.

Proof of [g]. Write $f : n \rightarrow m + p$ as

$$f = [\ A \quad B\ ].$$

We identify the base morphism ρ with a function $[m] \to [n]$ as usual: $i\rho = j$ if $i_m \cdot \rho = j_n$. Then, if A is written as $\langle A_1, \ldots, A_n \rangle$ and B as $\langle B_1, \ldots, B_p \rangle$,

$$\begin{aligned}
&\langle 1_m \cdot \rho \cdot f \cdot (\rho_1 \oplus \mathbf{1}_p), \ldots, m_m \cdot \rho \cdot f \cdot (\rho_m \oplus \mathbf{1}_p) \rangle^\dagger \;= \\
&= \begin{bmatrix} \begin{bmatrix} A_{1\rho} & B_{1\rho} \end{bmatrix} \begin{bmatrix} \rho_1 & 0 \\ 0 & \mathbf{1}_p \end{bmatrix} \\ \vdots \\ \begin{bmatrix} A_{m\rho} & B_{m\rho} \end{bmatrix} \begin{bmatrix} \rho_m & 0 \\ 0 & \mathbf{1}_p \end{bmatrix} \end{bmatrix}^\dagger \\
&= \begin{bmatrix} A_{1\rho}\rho_1 & B_{1\rho} \\ \vdots & \vdots \\ A_{m\rho}\rho_m & B_{m\rho} \end{bmatrix}^\dagger \\
&= \begin{bmatrix} A_{1\rho}\rho_1 \\ \vdots \\ A_{m\rho}\rho_m \end{bmatrix}^* \begin{bmatrix} B_{1\rho} \\ \vdots \\ B_{m\rho} \end{bmatrix} \\
&= ((\rho \cdot A) \parallel (\rho_1, \ldots, \rho_m))^* \cdot \rho \cdot B.
\end{aligned}$$

Also,

$$\begin{aligned}
\rho \cdot (f \cdot (\rho \oplus \mathbf{1}_p))^\dagger &= \rho \cdot \left(\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & \mathbf{1}_p \end{bmatrix} \right)^\dagger \\
&= \rho \cdot \begin{bmatrix} A \cdot \rho & B \end{bmatrix}^\dagger \\
&= \rho \cdot (A \cdot \rho)^* \cdot B.
\end{aligned}$$

Thus the commutative identity holds if and only if the star commutative identity (9.9) holds. The proof that the permutation identity is equivalent to the star permutation identity is similar.

Proposition 9.2.2 *Suppose that T is a star theory. If the dagger operation is defined by the equation (9.2), then the star operation is expressible in terms of dagger as in (9.1) and the parameter identity holds. Further, all of the equivalences of the previous theorem hold.*

The following proposition is immediate.

Proposition 9.2.3 *Let T and T' be matrix theories. Suppose that the dagger and star operations are defined on both T and T' and that these operations are related by (9.1) and (9.2). Then a matrix theory morphism $\varphi : T \to T'$ preserves the dagger operation if and only if φ preserves the star operation.*

Exercise 9.2.4 Suppose that both dagger and star are defined on the theory $\mathbf{Mat}_S$ and the operations are related by the equations (9.1) and (9.2). Show that the simplified composition identity holds in $\mathbf{Mat}_S$ if and only if the simplified star product identity (9.10) holds.

Exercise 9.2.5 Show that the star fixed point and simplified star product identities are jointly equivalent to the star product identity.

Exercise 9.2.6 Suppose that T is a star theory and that the star operation satisfies the star product identity and the star sum identity. Then the star pairing identity can be written in either of the following three forms, where $A : n \to n$, $B : n \to m$, $C : m \to n$ and $D : m \to m$:

[a]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where

$$\begin{array}{ll} \alpha = (A + BD^*C)^* & \beta = \alpha BD^* \\ \gamma = \delta CA^* & \delta = (D + CA^*B)^*; \end{array}$$

[b]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix},$$

where

$$\begin{array}{ll} \alpha' = (A^*BD^*C)^*A^* & \beta' = \alpha' BD^* \\ \gamma' = \delta' CA^* & \delta' = (D^*CA^*B)^*D^*; \end{array}$$

[c]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{bmatrix},$$

where

$$\begin{array}{ll} \alpha'' = (A + BD^*C)^* & \beta'' = A^*B\delta'' \\ \gamma'' = D^*C\alpha'' & \delta'' = (D + CA^*B)^*. \end{array}$$

9.3 Matrix Iteration Theories Defined

Definition 9.3.1 *A* **matrix iteration theory** *is a matrix theory which is simultaneously an iteration theory. A* **Conway matrix theory** *is a matrix theory which is a Conway theory. A morphism of matrix iteration theories or Conway theories is a matrix theory morphism that preserves dagger, i.e. which is a preiteration theory morphism.*

The following corollaries are immediate from Theorem 9.2.1, Proposition 9.2.2, Proposition 9.2.3 and the axiomatization results of Chapter 6. See Section 6.6.8 for a summary.

Corollary 9.3.2 *Let T be a star theory. Then if dagger is defined by (9.2), T is a matrix iteration theory if and only if one of the following three groups of equations holds in T.*

The **A** *group:*

- *the star zero identity (9.6);*
- *the star pairing identity (9.7);*
- *the star commutative identity (9.9).*

The **B** *group:*

- *the star product identity (9.4);*
- *the star sum identity (9.5);*
- *the star commutative identity (9.9).*

The **C** *group:*

- *the scalar star product identity (9.4) with $n = m = 1$;*
- *the scalar star sum identity (9.5) with $n = 1$;*
- *the scalar star pairing identity (9.7) with $m = 1$;*

- *the scalar star commutative identity*

$$1_m \cdot ((\rho \cdot A) \parallel (\rho_1, \ldots, \rho_m))^* \cdot \rho \quad = \quad 1_n \cdot (A \cdot \rho)^*, \quad (9.11)$$

where $A : n \to m$, $\rho : m \to n$ is a monotone base surjection, and where $\rho_i : m \to m$ is base with $\rho_i \cdot \rho = \rho$, for each $i \in [m]$.

Conversely, if T is an iteration theory and if the star operation is defined by (9.1), then each of the above groups of equations holds in T. In either case, the iteration and star operations are related by the equations (9.1) and (9.2). Let T and T' be matrix iteration theories. A matrix theory morphism $\varphi : T \to T'$ is a matrix iteration theory morphism if and only if φ is a star theory morphism.

In the **B** group, the star commutative identity may be replaced by the scalar star commutative identity (9.11).

Corollary 9.3.3 *Let T be a star theory. If dagger is defined by (9.2), then T is a Conway matrix theory if and only if the star operation satisfies one of the following three groups of equational axioms.*

The **A** *group:*

- *the star zero identity (9.6);*
- *the star pairing identity (9.7);*
- *the star permutation identity (9.8).*

The **B** *group:*

- *the star product identity (9.4);*
- *the star sum identity (9.5);*

The **C** *group:*

- *the scalar star product identity (9.4) with $n = m = 1$;*
- *the scalar star sum identity (9.5) with $n = 1$;*

- *the scalar star pairing identity (9.7) with $m = 1$.*

Conversely, if T is a Conway theory and the star operation is defined by (9.1), then each of the above three groups of equations holds in T. In either case, the iteration and star operations are related by the equations (9.1) and (9.2). A matrix theory morphism between Conway matrix theories is a Conway matrix theory morphism if and only if it preserves the star operation, i.e. when it is a star theory morphism.

Below, we will always suppose that if T is a matrix preiteration theory, then the star operation is defined by equation (9.1), and if T is a star theory, then the dagger operation is defined by (9.2). When $T = \mathbf{Mat}_S$, we may identify the set of morphisms $1 \to 1$ in T with the elements of S, cf. Section 3.3.5.

Definition 9.3.4 *A* ***-semiring** *is a semiring S equipped with a star operation $^* : S \to S$ subject to no particular requirements. A morphism of *-semirings is a semiring homomorphism which preserves the star operation. A* **Conway semiring** *is a *-semiring S in which the scalar star product and scalar star sum identities hold, i.e.*

$$\begin{aligned} (ab)^* &= a(ba)^*b + 1 \\ (a+b)^* &= (a^*b)^*a^*, \end{aligned}$$

for all $a, b \in S$. An **iteration semiring** *S is a Conway semiring which satisfies the scalar star commutative identity (9.11), when the star of $n \times n$ matrices over S is defined by the scalar star pairing identity. A morphism of Conway or iteration semirings is just a *-semiring homomorphism.*

Thus, both Conway semirings and iteration semirings are equational classes and hence share all of the general properties of equational classes. In particular, any sub *-semiring or quotient of an iteration (or Conway) semiring is an iteration (Conway) semiring. Below a sub *-semiring of an iteration (or Conway) semiring S will sometimes be called a *subiteration semiring* (or *sub Conway semiring*) of S. Note that the scalar star commutative identity (9.11) determines n equations for *-semirings.

Exercise 9.3.5 Write down the equations determined by the scalar star commutative identity (9.11) when $m = 2$ and $n = 1$.

The following facts are immediate from the definitions.

Corollary 9.3.6 *If $T = \mathbf{Mat}_S$ is a matrix iteration (or Conway matrix) theory, then $S = T(1,1)$ is an iteration (or Conway, respectively) semiring. Conversely, if S is an iteration (or Conway) semiring, there is a unique way to extend the star operation on S to all $n \times n$ matrices such that $\mathbf{Mat}_S$ becomes a matrix iteration (or Conway matrix) theory.*

Thus each matrix iteration theory (or Conway matrix theory) can be represented as some theory $\mathbf{Mat}_S$ where S is an iteration (or Conway) semiring.

Let T be a matrix iteration theory. A *submatrix iteration theory* of T is a submatrix theory which is a subiteration theory. Since the dagger and star operations are related by the equations (9.1) and (9.2), a submatrix theory T' of T is a subiteration theory if and only if T' is closed under star. Similarly, a sub Conway matrix theory of a Conway matrix theory is a submatrix theory closed under the dagger or star operation. Any submatrix iteration theory of a matrix iteration theory $\mathbf{Mat}_S$ is uniquely determined by a sub *-semiring of S.

Proposition 9.3.7 *If S' is a sub *-semiring of the iteration (or Conway) semiring S then $\mathbf{Mat}_{S'}$ is a submatrix iteration theory (or sub Conway matrix theory) of $\mathbf{Mat}_S$. Conversely, if T' is a submatrix iteration theory (or sub Conway matrix theory) of $\mathbf{Mat}_S$, then $T' = \mathbf{Mat}_{S'}$ for some uniquely determined sub *-semiring S' of S.*

Proposition 9.3.8 *Let S_1 and S_2 be iteration (or Conway) semirings. The restriction of any matrix iteration theory morphism (or Conway matrix theory morphism) $\mathbf{Mat}_{S_1} \to \mathbf{Mat}_{S_2}$ to the set of morphisms $1 \to 1$ determines a *-semiring homomorphism $S_1 \to S_2$. Conversely, any *-semiring homomorphism $\varphi : S_1 \to S_2$ extends in a unique way to a matrix iteration theory morphism (or Conway matrix theory morphism) $\mathbf{Mat}_{S_1} \to \mathbf{Mat}_{S_2}$ by defining*

$$[a_{ij}] \;\mapsto\; [a_{ij}\varphi].$$

Next we treat the functorial implication in Conway matrix theories.

Definition 9.3.9 *Suppose T is a star theory. Let be a set of morphisms in T. We say T* **satisfies the functorial star implication for** *, or that T* **has a functorial star with respect to** *, if for each $A : m \to m,\ B : n \to n$ and for each $\rho : m \to n$ in , if*

[m‘m‘n‘n;A‘ρ‘ρ‘B] commutes, then so does

[m‘m‘n‘n;A^*‘ρ‘ρ‘B^*] When T satisfies the functorial star implication for the set of all morphisms, we say that T has a **strong functorial star***.*

Example 9.3.10 A semiring S is ω-*complete* if, for any countable family $(a_i)_{i\in I}$ of elements of S, there is an element $\sum_{i\in I} a_i$ such that

$$\begin{aligned}
\sum_{i\in[n]} a_i &= a_1 + \ldots + a_n \\
\sum_{(i,j)\in I\times J} a_i b_j &= (\sum_{i\in I} a_i)\cdot(\sum_{j\in J} b_j) \\
\sum_{i\in I} a_i &= \sum_{j\in J}(\sum_{i\in I_j} a_i),
\end{aligned}$$

where I is the union of the pairwise disjoint sets $I_j,\ j \in J$. It follows that $\sum\emptyset = 0$ and that summation is commutative, associative and completely distributive. Two examples of ω-complete semirings are the semiring $\mathbf{B}$ and the semiring

$$\mathbf{N}_\infty \ := \ \{0, 1, \ldots, \infty\}$$

with $n\cdot\infty = \infty\cdot n = \infty$, for all $n \neq 0$. Note that the definition of infinite sums is forced in $\mathbf{N}_\infty$ by the requirement that $\mathbf{N}$ be a subsemiring of $\mathbf{N}_\infty$. Similarly, if the infinite sum in $\mathbf{B}$ extends the finite sum, then $\alpha := 1 + 1 + \ldots = 1 + \alpha$, so that $\alpha = 1$. Another example of an ω-complete semiring is the semiring of binary relations on a set A. When S is this semiring, $\mathbf{Mat}_S$ may be identified with the theory A.

Let S be an ω-complete semiring, so that each semiring $S^{n\times n}$ is also ω-complete. We define

$$A^* \ := \ \sum_{k=0}^{\infty} A^k,$$

for any $n \times n$ matrix A. It is known that the star pairing identity holds in $\mathbf{Mat}_S$. It follows that $\mathbf{Mat}_S$ is a Conway matrix theory. We claim that

$\mathbf{Mat}_S$ has a strong functorial star, so that $\mathbf{Mat}_S$ is an iteration theory and S an iteration semiring (see Proposition 9.3.14 below). Indeed, if

$$A : m \to m,\ B : n \to n \ \text{ and } \ A \cdot \rho \ = \ \rho \cdot B,$$

then for all $k \geq 0$,

$$A^k \cdot \rho \quad = \quad \rho \cdot B^k,$$

as one shows by induction on k. But then,

$$\begin{aligned} A^* \cdot \rho &= (\sum_k A^k) \cdot \rho \\ &= \sum_k (A^k \cdot \rho) \\ &= \sum_k (\rho \cdot B^k) \\ &= \rho \cdot (\sum_k B^k) \\ &= \rho \cdot B^*. \end{aligned}$$

Thus, when S is $\mathbf{B}$ or the semiring $\mathbf{N}_\infty$, $\mathbf{Mat}_S$ is a matrix iteration theory and S is an iteration semiring. Similarly, since the semiring of binary relations on a set A is ω-complete, it follows that A is a matrix iteration theory. Note that the iteration theory structure so obtained on the theory A is the same as that derived from the fact that A is an ω-continuous theory, cf. Example 8.8.2.3.

Example 9.3.11 Let S be a semiring and X a set. A *formal power series* over S with (noncommuting) variables in X is a function $r : X^* \to S$. Denoting by $\langle r,\, w\rangle$ the value of r on the word w, r itself is written as the formal sum

$$r \ = \ \sum_{w \in X^*} \langle r,\, w\rangle w.$$

Each element $s \in S$ may be identified with the formal power series which takes the empty word ϵ to s and all other words to 0. The collection of all formal power series as defined above is denoted $S\langle\langle X^*\rangle\rangle$. Let $r_1, r_2 \in S\langle\langle X^*\rangle\rangle$. We define the *sum* $r_1 + r_2 \in S\langle\langle X^*\rangle\rangle$ to be the function

$$w \ \mapsto \ \langle r_1,\, w\rangle + \langle r_2,\, w\rangle, \quad w \in X^*.$$

Further, we define the (Cauchy) *product* $r_1 r_2 \in S\langle\langle X^*\rangle\rangle$ by

$$\langle r_1 r_2,\, w\rangle \ := \ \sum_{uv = w} \langle r_1,\, u\rangle\langle r_2,\, v\rangle.$$

It is well-known, see e.g. [Sal66] or [KS86], that $S\langle\langle X^*\rangle\rangle$ is a semiring with the above operations of sum and product and with the constants 0 and 1. Note that S is isomorphic to a subsemiring of $S\langle\langle X^*\rangle\rangle$.

When S is ω-complete, so is the semiring $S\langle\langle X^*\rangle\rangle$ with the pointwise definition of infinite sums. In particular, $\mathbf{N}_\infty\langle\langle X^*\rangle\rangle$ and $\mathbf{B}\langle\langle X^*\rangle\rangle$ are ω-complete semirings, hence iteration semirings by Example 9.3.10. Further, the theories $\mathbf{Mat}_{\mathbf{N}_\infty\langle\langle X^*\rangle\rangle}$ and $\mathbf{Mat}_{\mathbf{B}\langle\langle X^*\rangle\rangle}$ have a strong functorial star. For later use we note that $\mathbf{B}\langle\langle X^*\rangle\rangle$ is isomorphic to the ω-complete semiring $L(X^*)$ of languages over X. In the semiring $L(X^*)$, the sum operation is set union and the product of two languages is their concatenation. The constants 0 and 1 are the empty set and the set $\{\epsilon\}$. Below, the matrix iteration theory $\mathbf{Mat}_{L(X^*)}$ will be denoted $\text{Ł}(X^*)$.

Proposition 9.3.12 *Let $T = \mathbf{Mat}_S$ be a matrix theory in which both dagger and star are defined. Suppose that the two operations are related by (9.1) and (9.2). Let be a set of morphisms in T. Then T has a functorial dagger with respect to if and only if T has a functorial star with respect to .*

Proof. Suppose that $\rho : m \to n$ in . Let $f = [A,\ C] : m \to m + p$ and $g = [B,\ D] : n \to n + p$. If $f \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot g$, then $A \cdot \rho = \rho \cdot B$ and $C = \rho \cdot D$. Hence when T has a functorial star with respect to ,

$$\begin{aligned} f^\dagger &= A^* \cdot C \\ &= A^* \cdot \rho \cdot D \\ &= \rho \cdot B^* \cdot D \\ &= \rho \cdot g^\dagger, \end{aligned}$$

showing that T has a functorial dagger with respect to . Conversely, if $A \cdot \rho = \rho \cdot B$, for some $A : m \to m$, $B : n \to n$ and for some $\rho : m \to n$ in , then also $[A,\ \rho] \cdot (\rho \oplus \mathbf{1}_n) = \rho \cdot [B,\ \mathbf{1}_n]$. Thus if T has a functorial dagger with respect to , then

$$\begin{aligned} A^* \cdot \rho &= [A,\ \rho]^\dagger \\ &= \rho \cdot [B,\ \mathbf{1}_n]^\dagger \\ &= \rho \cdot B^*. \end{aligned}$$

In the next four propositions, let T be a star theory.

Proposition 9.3.13 *If the star pairing identity and the star permutation identity hold in T then T satisfies the functorial star implication for injective base matrices. Thus any Conway matrix theory satisfies the functorial star implication for injective base matrices.*

Proof. By Proposition 6.6.3.4, Proposition 9.3.12, Proposition 9.2.2 and Theorem 9.2.1.

Proposition 9.3.14 *If T satisfies the functorial star implication for surjective base matrices, then the star commutative identity holds in T.*

Proof. By Proposition 6.6.3.5, Proposition 9.3.12, Proposition 9.2.2 and Theorem 9.2.1.

Thus every Conway matrix theory with a functorial star with respect to all surjective base matrices is a matrix iteration theory.

Proposition 9.3.15 *If T satisfies the functorial star implication for all injective base matrices and for all surjective base matrices, then T has a functorial star with respect to all base matrices.*

Proof. By Proposition 6.6.3.6, Proposition 9.3.12, Proposition 9.2.2 and Theorem 9.2.1.

Proposition 9.3.16 *Suppose that the star permutation identity and the star pairing identity hold in T. Then T has a functorial star with respect to all surjective base matrices if and only if T has a functorial star with respect to all base matrices $n \to 1$, $n \geq 1$.*

Proof. By Proposition 6.6.3.7, Proposition 9.3.12, Proposition 9.2.2 and Theorem 9.2.1.

Corollary 9.3.17 *Let T be a star theory. T is a matrix iteration theory satisfying the functorial star implication for all (surjective) base matrices if and only if T is a Conway matrix theory and T has a functorial star with respect to all base matrices $\rho : n \to 1$, $n \geq 1$.*

A sufficient condition that a Conway theory have a functorial star with respect to base matrices will be given below.

Definition 9.3.18 *Suppose that $T = \mathbf{Mat}_S$ is a Conway matrix theory, so that S is a Conway semiring. We say that T (or S) satisfies the* **star GA-implication**, *or that the star GA-implication holds in T, if for all $a_i, b_i \in S$, $i = 1, 2, 3$, if*

$$(a_1 + a_2)^* a_3 \quad = \quad (b_1 + b_2)^* b_3$$

then

$$(b_2 + b_1 a_1^* a_2)^* (b_3 + b_1 a_1^* a_3) \quad = \quad (a_1 + a_2)^* a_3.$$

The semiring $L(X^*)$ satisfies the star GA-implication and its dual, see Example 9.7.24 and Exercise 9.7.25. However, there are ω-complete semirings in which the star GA-implication fails.

Example 9.3.19 In this example we show that when A is an infinite set, the star GA-implication fails in the iteration semiring of all binary relations on A. It follows that the GA-implication does not hold in the iteration theory A. Thus there exist ω-continuous theories in which the GA-implication fails.

First we define an infinite directed graph D whose edges are labeled by the letters a, b, c, or d. The graph D is the union of the directed graphs D_i, $i = 0, 1, \ldots$, defined by induction.

- D_0 is a two-point directed graph with a single edge $e_0 : u_0 \to v_0$ labeled a. We set $E_0 := \{e_0\}$.
- Suppose that D_i and E_i have been defined, where E_i is a subset of the set of edges of D_i. To obtain D_{i+1}, we enlarge D_i with a new point z and two new edges $e_1 : u \to z$ and $e_2 : z \to v$, for each edge $e : u \to v$ in E_i. The new edges are labeled as follows. When e is labeled a or b, we label e_1 by c and e_2 by d. When e is labeled c or d, we label e_1 by b and e_2 by a. The set E_{i+1} is the set of all new edges.

Claim. There is no directed path in D from u_0 to v_0 whose label belongs to the (regular) set

$$(d + ca^*b)^*(1 + ca^*) \;=\; (d^*ca^*b)^*(d^* + d^*ca^*).$$

Indeed, the label of any path from u_0 to v_0 belongs to the set generated by the rewriting system

$$a \to cd \quad b \to cd \quad c \to ba \quad d \to ba$$

with a being the start symbol. The intersection of this set with the above regular set is empty, for any word in the intersection has first letter c, and such that each occurrence of the letter c is followed by an occurrence of b, and each b is followed by c. There is no finite word with this property.

Now let A be the set of all points of D. Let a be the relation corresponding to the set of all edges labeled a. The relations b, c and d are defined in the same way. We have

$$(a+b)^* \;=\; (c+d)^*$$

by construction. Also

$$u_0 \ (a+b)^* \ v_0$$

but u_0 and v_0 are not related by the relation $(d+ca^*b)^*(1+ca^*)$.

Since the GA-implication fails in the semiring of binary relations on a countable set, it fails in the semiring of binary relations on any infinite set.

Exercise 9.3.20 Show that for infinite A, the dual star GA-implication (see below) does not hold in the iteration semiring of binary relations on A.

The proof of the following proposition is left to the reader.

Proposition 9.3.21 *Suppose that* $T = \mathbf{Mat}_S$ *is a Conway matrix theory. Then* T *satisfies the star GA-implication iff* T *satisfies the GA-implication. Hence* T *is a matrix iteration theory which has a functorial star for all (surjective) base matrices, whenever* T *satisfies the star GA-implication.*

Exercise 9.3.22 Prove Proposition 9.3.21.

When $S = (S, +, \cdot, 0, 1)$ is a semiring or a *-semiring, the multiplication in the *dual* semiring S^{op} is defined by

$$a \circ b \quad := \quad b \cdot a,$$

for all $a, b \in S$. The other operations and constants are those of S.

Proposition 9.3.23 *If S is a Conway semiring, then S^{op} is also.*

Proof. The Conway semiring axiom

$$(ab)^* \quad = \quad a(ba)^*b + 1 \tag{9.12}$$

is self-dual. Indeed, if (9.12) holds in a *-semiring S, then

$$(b \circ a)^* \quad = \quad b \circ (a \circ b)^* \circ a + 1$$

in S^{op}. Suppose now that both (9.12) and the equation

$$(a+b)^* \quad = \quad (a^*b)^*a^*$$

hold in S. Then also

$$(a+b)^* \quad = \quad a^*(ba^*)^*,$$

for all $a, b \in S$. Thus, in S^{op},

$$(a+b)^* \quad = \quad (a^* \circ b)^* \circ a^*.$$

Suppose that S is a Conway semiring. By Proposition 9.3.23, S^{op} is also a Conway semiring, so that both $T = \mathbf{Mat}_S$ and $T' = \mathbf{Mat}_{S^{\mathrm{op}}}$ are Conway matrix theories. Below we will write $^\otimes$ for the star operation on T' determined by the star pairing identity and the star operation on S^{op}, i.e. the star operation on S. Thus, when $A : n \to n$ in $\mathbf{Mat}_S$, A^* denotes the value of the star operation on A in the theory T, and $A^\otimes$ stands for the value of the star operation on A in the theory T'. The composition operation in T' will be denoted $\circ$.

Lemma 9.3.24 *The equation*

$$(A \cdot B)^T \quad = \quad B^T \circ A^T$$

holds for all $A : n \to p$ and $B : p \to q$. Equivalently, $(A \circ B)^T = B^T \cdot A^T$.

Lemma 9.3.25 *For all $a : n \to n$ in the Conway matrix theory $\mathbf{Mat}_S$,*

$$(A^T)^\otimes \quad = \quad (A^*)^T.$$

Hence also $(A^T)^* = (A^\otimes)^T$.

Proof. The argument uses induction on n. When $n = 0$ or $n = 1$, the claim is obvious. Suppose for the induction step that $A : n{+}1 \to n{+}1$, where $n \geq 1$. Let us partition A as

$$A \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a : n \to n$, $b : n \to 1$, $c : 1 \to n$ and $d : 1 \to 1$. Then

$$A^T \quad = \quad \begin{bmatrix} a^T & c^T \\ b^T & d^T \end{bmatrix}$$

and

$$(A^T)^{\otimes} \quad = \quad \begin{bmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{bmatrix}$$

where

$$\begin{aligned} \delta' &= (d^T + b^T \circ (a^T)^{\otimes} \circ c^T)^{\otimes} \\ \gamma' &= (a^T)^{\otimes} \circ c^T \circ \delta' \\ \beta' &= \delta' \circ b^T \circ (a^T)^{\otimes} \\ \alpha' &= (a^T)^{\otimes} \circ c^T \circ \delta' \circ b^T \circ (a^T)^{\otimes} + (a^T)^{\otimes}. \end{aligned}$$

Similarly,

$$A^* \quad = \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

where

$$\begin{aligned} \delta &= (d + ca^*b)^* \\ \gamma &= \delta c a^* \\ \beta &= a^* b \delta \\ \alpha &= a^* b \delta c a^* + a^*. \end{aligned}$$

It follows by the induction assumption and the previous lemma that $\alpha^T = \alpha'$, $\beta^T = \beta'$, $\gamma^T = \gamma'$ and $\delta^T = \delta'$.

Corollary 9.3.26 *Suppose that S is a Conway semiring, so that $T = \mathbf{Mat}_S$ and $T' = \mathbf{Mat}_{S^{\mathrm{op}}}$ are Conway matrix theories.*

[a] *The star commutative identity holds in T' iff the* **dual star commutative identity** *(9.13) holds in T:*

$$\rho^T \cdot ((\rho_1^T, \ldots, \rho_m^T) \parallel (A \cdot \rho^T))^* \quad = \quad (\rho^T \cdot A)^* \cdot \rho^T, \quad (9.13)$$

where $A = [A_1, \ldots, A_n] : m \to n$, $\rho : m \to n$ is a surjective base matrix with transpose $\rho^T : n \to m$, and $\rho_i : m \to m$, $i \in [m]$, are base, with $\rho_i \cdot \rho = \rho$, for each $i \in [m]$.

[b] *T' has a functorial star for a set   of matrices iff T has a functorial star for the set*

$$^T \;:=\; \{c^T \,:\, c \in\}.$$

[c] *The star GA-implication holds in T' iff the* **dual star GA-implication** *holds in T, i.e. when*

$$\begin{aligned} a_3(a_1 + a_2)^* \;&=\; b_3(b_1 + b_2)^* \\ \Rightarrow \quad (b_3 + a_3 a_1^* b_1)(b_2 + a_2 a_1^* b_1)^* \;&=\; a_3(a_1 + a_2)^*, \end{aligned}$$

for all $a_i, b_i \in S$, $i = 1, 2, 3$.

Corollary 9.3.27 *Suppose that $T = \mathbf{Mat}_S$ is a Conway matrix theory.*

[a] *T satisfies the functorial star implication for transposes of injective base morphisms.*

[b] *If T satisfies the functorial star implication for transposes of base surjections, then the dual star commutative identity holds in T.*

[c] *If T satisfies the functorial star implication for transposes of all base morphisms $n \to 1$, $n \geq 1$, then T has a functorial star with respect to transposes of base surjections.*

[d] *If the dual star GA-implication holds in T then T satisfies the functorial star implication for transposes of all base morphisms $n \to 1$, $n \geq 1$. Hence, in this case, T has a functorial star with respect to transposes of all (surjective) base morphisms.*

For later use, we introduce the following concept.

Definition 9.3.28 *A* **symmetric iteration semiring** *is a semiring S such that both S and S^{op} are iteration semirings. A morphism of symmetric iteration semirings is just a *-semiring homomorphism.*

Thus, a *-semiring S is a symmetric iteration semiring iff it is a Conway semiring such that both the star commutative identity and the dual star commutative identity hold in $\mathbf{Mat}_S$. Notice that symmetric iteration semirings form an equational class as do iteration semirings.

Example 9.3.29 When $\mathbf{Mat}_S$ is a Conway matrix theory which has a strong functorial star or which satisfies the star functorial implication for surjective base matrices and their transposes, S is a symmetric iteration semiring. In particular, any ω-complete semiring is a symmetric iteration semiring.

Exercise 9.3.30 Let $T = \mathbf{Mat}_S$ be a Conway matrix theory. When A is an $n \times n$ matrix, define

$$A^+ \ := \ A \cdot A^*.$$

Show that $A^* = A^+ + \mathbf{1}_n$ and $A^+ = A^* \cdot A$.

Exercise 9.3.31 Suppose that S is a Conway semiring. Show that for any a, b in S,

$$\begin{aligned} (a^*b)^* &= b^*(a^+b^+)^* \\ b^*(a^+b^+)^* &= a^*b^+(a^+b^+)^* + 1. \end{aligned}$$

Exercise 9.3.32 Suppose that S is a semiring or a *-semiring. When $a, b \in S$, we define

$$a \ \leq \ b \quad \text{iff} \quad \exists c \; a + c \ = \ b.$$

Show that the relation $\leq$ is a preordering of the set S. Show that if $a \leq b$ and $a' \leq b'$ then $a + a' \leq b + b'$, $ab \leq a'b'$, and when S is a Conway semiring, $a^* \leq b^*$. Show that if S is idempotent then $\leq$ is a partial order, and for all $a, b \in S$, $a \leq b$ iff $a + b = b$.

Exercise 9.3.33 Let S be a Conway semiring in which the equation $1^* = 1$ holds. Prove that

$$\begin{aligned} a + a &= a \\ a^* a^* &= a^* \\ a^{**} &= a^* \\ (a+b)^* &= (a^* b^*)^*, \end{aligned}$$

for all $a, b \in S$. What about the matrix versions of these equations?

In the following exercises let T be a star theory.

Exercise 9.3.34 Prove that if T has a functorial star with respect to transposes of base injections and transposes of base surjections, then T has a functorial star with respect to transposes of all base matrices.

Exercise 9.3.35 Suppose that the star pairing identity and the star permutation identity hold in T. Prove that T has a functorial star with respect to transposes of surjective base matrices if and only if T has a functorial star with respect to transposes of base matrices $n \to 1$, $n \geq 1$.

Exercise 9.3.36 The *generalized star commutative identity* is the equation

$$((\rho \cdot A) \parallel (\rho_1, \ldots, \rho_m))^* \cdot \rho \quad = \quad \rho \cdot (A \parallel (\tau_1, \ldots, \tau_n))^*, \qquad (9.14)$$

where $A : n \to k$, $\rho : m \to n$ is surjective base, and where $\rho_i : k \to m$, $\tau_j : k \to n$, $i \in [m]$, $j \in [n]$, are base with $\rho_i \cdot \rho = \tau_{i\rho}$.

Prove that the star commutative identity (9.9) holds in T if and only if the generalized star commutative identity (9.14) holds. Prove that the star commutative identity (9.9) reduces to the particular subcase that either each ρ_i is a base permutation or each ρ_i is aperiodic. A similar fact is true for the scalar star commutative identity (9.11). (See Proposition 5.5.3.26.)

Exercise 9.3.37 The *generalized dual star commutative identity* is the equation

$$((\tau_1^T, \ldots, \tau_n^T) \parallel B)^* \cdot \rho^T \quad = \quad \rho^T \cdot ((\rho_1^T, \ldots, \rho_m^T) \parallel (B \cdot \rho^T))^*, \quad (9.15)$$

$B : k \to n$, $\rho : m \to n$ is a surjective base matrix, and where $\rho_i : k \to m$, $i \in [m]$, and $\tau_j : k \to n$, $j \in [n]$, are base matrices satisfying $\rho_i \cdot \rho = \tau_{i\rho}$.

Prove the analogue of Exercise 9.3.36 for the generalized dual star commutative identity.

Exercise 9.3.38 Suppose that S is an iteration semiring. Write $n \in S$ for the sum of $1 \in S$ with itself n times. Show that all of the following equations hold in S:

$$\begin{aligned}
1^{**} &= (1^*1^*)^* \\
&= 1^{**}(1^*)^k, \quad k \geq 0 \\
&= (1^*)^k 1^{**}, \quad k \geq 0 \\
&= n + 1^{**} \\
&= (1^*)^k + 1^{**}, \quad k \geq 0 \\
&= 1^{***} \\
&= 1^{**} + 1^{**} \\
&= 1^{**}1^{**} \\
&= (2+n)^*, \quad n \geq 0.
\end{aligned}$$

Use the matrix $A = [1,\ 0] : 1 \to 2$, the base surjection $\rho : 2 \to 1$ and the two base matrices

$$\begin{aligned}
\rho_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\rho_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

in the star commutative identity to show

$$1^* + 1^* = 1^*.$$

Now prove that

$$\begin{aligned}
1^* + n &= 1^*, \quad n \geq 0 \\
(1^*)^k + n &= (1^*)^k, \quad k \geq 1,\ n \geq 0 \\
(1^*)^k + (1^*)^m &= (1^*)^k, \quad k \geq m \geq 0,\ \text{not both } 0. \\
1^{**} &= ((1^*)^k)^*, \quad k \geq 1.
\end{aligned}$$

9.4 Presentations in Matrix Iteration Theories

Let S be a Conway semiring. Suppose that S_0 is a sub *-semiring of S and that X is a subset of S. We define

$$V \ := \ \{\sum(s_i x_i \ : \ i \in [k],\ s_i \in S_0,\ x_i \in X)\},$$

so that V is the smallest S_0-submodule of S which contains the set X. We will give a description of the sub *-semiring generated by $S_0 \cup X$.

Since each iteration semiring is a Conway semiring, the result will also be applicable to iteration semirings.

Definition 9.4.1 *A* **presentation** *of weight $s \geq 0$ over the pair (S_0, X), or over (S_0, V), is a triple*

$$D \;=\; (\alpha; A; \gamma),$$

where $\alpha : 1 \to s \in \mathbf{Mat}_{S_0}$ and $\gamma : s \to 1 \in \mathbf{Mat}_{S_0}$ and where A is an $s \times s$ matrix all of whose entries are in V. The **behavior** *$|D|$ of D is defined by:*

$$|D| \;:=\; \alpha \cdot A^* \cdot \gamma.$$

Lemma 9.4.2 *Suppose that $D = (\alpha; A; \gamma)$ and $E = (\alpha'; B; \gamma')$ are presentations of weight s and r, respectively.*

[a] *Let $D + E$ be the following presentation of weight $s + r$:*

$$D + E \quad := \quad (\begin{bmatrix} \alpha & \alpha' \end{bmatrix}; \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}; \begin{bmatrix} \gamma \\ \gamma' \end{bmatrix}). \tag{9.16}$$

Then

$$|D + E| \quad = \quad |D| + |E|.$$

[b] *Let $D \cdot E$ be the following presentation of weight $s + r$:*

$$D \cdot E \quad := \quad (\begin{bmatrix} \alpha & 0 \end{bmatrix}; \begin{bmatrix} A & \gamma\alpha' B \\ 0 & B \end{bmatrix}; \begin{bmatrix} \gamma\alpha'\gamma' \\ \gamma' \end{bmatrix}). \tag{9.17}$$

Then

$$|D \cdot E| \quad = \quad |D| \cdot |E|.$$

[c] *Let D^* be the following presentation of weight $s + 1$:*

$$D^* \quad := \quad (\begin{bmatrix} \alpha & 1 \end{bmatrix}; \begin{bmatrix} (\gamma\alpha)^* A & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} (\gamma\alpha)^*\gamma \\ 1 \end{bmatrix}). \tag{9.18}$$

Then

$$|D^*| \quad = \quad |D|^*.$$

Proof of [a].

$$
\begin{aligned}
|D+E| &= \begin{bmatrix} \alpha & \alpha' \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^* \cdot \begin{bmatrix} \gamma \\ \gamma' \end{bmatrix} \\
&= \begin{bmatrix} \alpha & \alpha' \end{bmatrix} \cdot \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \cdot \begin{bmatrix} \gamma \\ \gamma' \end{bmatrix} \\
&= \alpha \cdot A^* \cdot \gamma + \alpha' \cdot B^* \cdot \gamma' \\
&= |D| + |E|
\end{aligned}
$$

Proof of [b].

$$
\begin{aligned}
|D \cdot E| &= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} A & \gamma\alpha' B \\ 0 & B \end{bmatrix}^* \cdot \begin{bmatrix} \gamma\alpha'\gamma' \\ \gamma' \end{bmatrix} \\
&= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} A^* & A^*\gamma\alpha' B^+ \\ 0 & B^* \end{bmatrix} \cdot \begin{bmatrix} \gamma\alpha'\gamma' \\ \gamma' \end{bmatrix} \\
&= \begin{bmatrix} \alpha A^* & \alpha A^*\gamma\alpha' B^+ \end{bmatrix} \cdot \begin{bmatrix} \gamma\alpha'\gamma' \\ \gamma' \end{bmatrix} \\
&= \alpha A^*\gamma \cdot \alpha'\gamma' + \alpha A^*\gamma \cdot \alpha' B^+\gamma' \\
&= \alpha A^*\gamma \cdot (\alpha'\gamma' + \alpha' B^+\gamma') \\
&= \alpha A^*\gamma \cdot \alpha' B^*\gamma' \\
&= |D| \cdot |E|
\end{aligned}
$$

Proof of [c].

$$
\begin{aligned}
|D^*| &= \begin{bmatrix} \alpha & 1 \end{bmatrix} \cdot \begin{bmatrix} (\gamma\alpha)^* A & 0 \\ 0 & 0 \end{bmatrix}^* \cdot \begin{bmatrix} (\gamma\alpha)^*\gamma \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha & 1 \end{bmatrix} \cdot \begin{bmatrix} ((\gamma\alpha)^* A)^* & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} (\gamma\alpha)^*\gamma \\ 1 \end{bmatrix} \\
&= \alpha((\gamma\alpha)^* A)^*(\gamma\alpha)^*\gamma + 1 \\
&= \alpha(\gamma\alpha + A)^*\gamma + 1 \\
&= \alpha(A^*\gamma\alpha)^* A^*\gamma + 1 \\
&= \alpha(A^*\gamma(\alpha A^*\gamma)^*\alpha + 1)A^*\gamma + 1 \\
&= (\alpha A^*\gamma(\alpha A^*\gamma)^* + 1)\alpha A^*\gamma + 1 \\
&= (\alpha A^*\gamma)^*\alpha A^*\gamma + 1 \\
&= |D|^*
\end{aligned}
$$

Corollary 9.4.3 Kleene's theorem for Conway semirings. *Let S be a Conway semiring, S_0 a sub *-semiring of S and X a subset of S. Let S' denote the smallest sub *-semiring of S containing $S_0 \cup X$. Then S' contains an element s if and only if $s = |D|$ for some presentation D over (S_0, X). In particular, S is generated by $S_0 \cup X$ if and only if each element of S is the behavior of some presentation over (S_0, X).*

Proof. Given $x \in X$, let D_x be the presentation of weight 2

$$D_x \ := \ (\begin{bmatrix} 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}).$$

We have $|D_x| = x$. When $\alpha \in S_0$, define

$$D_\alpha \ := \ (\alpha; 0_{11}; \mathbf{1}_1).$$

Clearly then, $|D_\alpha| = \alpha$. Thus each element of $S_0 \cup X$ is the behavior of some presentation. Thus, by Lemma 9.4.2, the elements that are behaviors of presentations over (S_0, X) form a sub *-semiring which contains both S_0 and X. By Proposition 9.3.7, this is the smallest such semiring.

Recall that when $T = \mathbf{Mat}_S$ is a matrix theory, we identify $T(1,1)$ with S.

Corollary 9.4.4 Kleene's theorem for Conway matrix theories. *Let $T = \mathbf{Mat}_S$ be a Conway matrix theory (or a matrix iteration theory). Let $\mathbf{Mat}_{S_0}$ be a sub Conway matrix theory (submatrix iteration theory, respectively) of $\mathbf{Mat}_S$ and $X \subseteq T(1,1)$. The smallest sub Conway matrix theory, (submatrix iteration theory, respectively) containing $\mathbf{Mat}_{S_0}$ and X consists of the matrices $[a_{ij}]$ where each entry a_{ij} is the behavior of some presentation over the pair (S_0, X). In particular, $\mathbf{Mat}_S$ is generated by $\mathbf{Mat}_{S_0}$ and X if and only if each $s : 1 \to 1$ in $\mathbf{Mat}_S$ is the behavior of some presentation over the pair (S_0, X).*

Example 9.4.5 Consider the matrix iteration theory Ł(X^*) of languages over the set X. We identify each letter $x \in X$ with the set $\{x\}$, so that we may consider X to be a subset of $L(X^*)$. Further, we may consider the Boolean semiring $\mathbf{B}$ to be a sub *-semiring of $L(X^*)$. The smallest sub $\mathbf{B}$-module of $L(X^*)$ which contains X is the set $P_f(X)$ of finite subsets of X.

A presentation $D = (\alpha; A; \gamma)$ of weight s over $(\mathbf{B}, X)$ may be considered to be a representation of the nondeterministic finite automaton (nfa)

$$_D \;:=\; (Q, X, \delta, Q_0, Q_f),$$

where the set of states Q is the set $[s]$, the initial states Q_0 are those integers $i \in [s]$ with $\alpha_i = 1$, and the final states Q_f are those $i \in [s]$ with $\gamma_i = 1$. Further,

$$j \in \delta(i, x) \;\Leftrightarrow\; x \in A_{ij},$$

for all $i, j \in Q$ and $x \in X$. The behavior $|D|$ of D is just the language accepted by the nfa $_D$. On the other hand, the smallest sub *-semiring of $L(X^*)$ is the semiring $RL(X^*)$ of *regular languages* over X. Thus Corollary 9.4.3 says that a language $a \in L(X^*)$ is regular if and only if a is accepted (or recognized) by an nfa. Thus $\mathbf{Mat}_{RL(X^*)}$, the smallest submatrix iteration theory of Ł(X^*) containing $\mathbf{Mat}_{\mathbf{B}}$ and the matrices $x : 1 \to 1$, for all $x \in X$, consists of the matrices $[a_{ij}]$ such that each a_{ij} is a language accepted by some nfa. In the sequel the matrix iteration theory $\mathbf{Mat}_{RL(X^*)}$ will be denoted Ł(X^*). Consider next the matrix iteration theory $\mathbf{Mat}_{\mathbf{N}_\infty\langle\langle X^*\rangle\rangle}$ which contains $\mathbf{Mat}_{\mathbf{N}_\infty}$ as a submatrix iteration theory. We may identify any letter $x \in X$ with the power series $x \mapsto 1$ and $w \mapsto 0$, for all $w \in X^*$, $w \neq x$. The sets X and $\mathbf{N}_\infty$ together generate the sub *-semiring $\mathbf{N}_\infty^{rat}\langle\langle X^*\rangle\rangle$. The formal power series in $\mathbf{N}_\infty^{rat}\langle\langle X^*\rangle\rangle$ are called *rational.* A presentation over $(\mathbf{N}_\infty, X)$ is a representation of a finite automaton with multiplicities in $\mathbf{N}_\infty$. The behavior of the presentation is the formal power series recognized by the corresponding automaton. Corollary 9.4.3 in this case asserts that a formal power series in $\mathbf{N}_\infty\langle\langle X^*\rangle\rangle$ is rational if and only if it is *recognizable.*

Exercise 9.4.6 The semirings $\mathbf{B}$ and $\mathbf{N}_\infty$ are linearly ordered with the usual ordering. Show that with the pointwise ordering of functions $X^* \to \mathbf{N}_\infty$ and $X^* \to \mathbf{B}$ and on matrices, the theories $\mathbf{Mat}_{\mathbf{N}_\infty\langle\langle X^*\rangle\rangle}$ and $\mathbf{Mat}_{\mathbf{B}\langle\langle X^*\rangle\rangle}$ are ω-continuous theories, as defined in Chapter 8. Conclude that the theories Ł(X^*) and $\mathbf{Mat}_{\mathbf{N}_\infty^{rat}\langle\langle X^*\rangle\rangle}$ are rational theories.

In the rest of this section, $\mathbf{Mat}_S$ is a Conway iteration theory, $\mathbf{Mat}_{S_0}$ is a sub Conway matrix theory of $\mathbf{Mat}_S$, and X is a subset of S.

Definition 9.4.7 *Let $D = (\alpha; A; \gamma)$ and $E = (\alpha'; B; \gamma')$ be two presentations over (S_0, X) of weight s and r, respectively. Let ρ be a relational matrix $s \to r$. We write*

$$D \xrightarrow{\rho} E$$

if the following diagram commutes:

Note that $D \overset{\rho^T}{\to} E$ if and only if $E^T \overset{\rho}{\to} D^T$, where the transpose of a presentation is defined below.

Definition 9.4.8 *If* $D = (\alpha; A; \gamma)$ *is a presentation, the* **transpose of** D *is the presentation*

$$D^T \ := \ (\gamma^T; A^T; \alpha^T).$$

Example 9.4.9 In this example we discuss the meaning of Definition 9.4.7 for the iteration semiring $RL(X^*)$. Suppose that $D = (\alpha; A; \gamma)$ and $E = (\alpha'; B; \gamma')$ are presentations of weight s and r, respectively. Let ρ be a base matrix $s \to r$. For later use it is worthwhile to express the condition that $D \overset{\rho}{\to} E$ in terms of the nfa ${}_D$ and ${}_E$, cf. Example 9.4.5. Let

$${}_D \ = \ (Q, X, \delta, Q_0, Q_f)$$

and

$${}_E \ = \ (Q', X, \delta', Q'_0, Q'_f).$$

Then $D \overset{\rho}{\to} E$ if and only if ρ, as a mapping from $Q = [s]$ to $Q' = [r]$, satisfies the following.

- For all $i \in Q_0$, $i\rho \in Q'_0$. Conversely, for every $i' \in Q'_0$ there exists $i \in Q_0$ with $i\rho = i'$.
- For all $i \in Q$, we have $i \in Q_f$ if and only if $i\rho \in Q'_f$.
- For all $i, j \in Q$ and $x \in X$, if $j \in \delta(i, x)$ then $j\rho \in \delta'(i\rho, x)$. Conversely, if $j' \in \delta'(i\rho, x)$, for some $i \in Q$, $j' \in Q'$ and $x \in X$, then there exists $j \in Q$ with $j\rho = j'$ and $j \in \delta(i, x)$.

Of course, these conditions have a simpler form when ρ is injective or surjective. If $E \overset{\rho^T}{\to} D$, i.e. $D^T \overset{\rho}{\to} E^T$ holds for the presentations D and E, we have analogues of the above conditions:

- For all $i \in Q$, we have $i \in Q_0$ if and only if $i\rho \in Q'_0$.
- For all $i \in Q_f$, $i\rho \in Q'_f$. Conversely, if $i' \in Q'_f$ then there exists $i \in Q_f$ with $i\rho = i'$.
- For all $i, j \in Q$ and $x \in X$, if $j \in \delta(i, x)$ then $j\rho \in \delta'(i\rho, x)$. Further, if $i' \in Q'$, $j \in Q$ and $x \in X$ with $j\rho \in \delta'(i', x)$, then there exists $i \in Q$ with $i\rho = i'$ and $j \in \delta(i, x)$.

Exercise 9.4.10 Suppose that $\approx$ is the smallest equivalence relation on the collection of presentations such that $D \approx E$ whenever $D \xrightarrow{\rho} E$ for some relational matrix ρ. When is $\approx$ is also the smallest equivalence relation on such that $D \approx E$ whenever $D \xrightarrow{\rho} E$ or $D \xrightarrow{\rho^T} E$, for some surjective or injective base matrix ρ? Define the operations of sum, product and star on by the equations (9.16)-(9.18). Show that when S_0 satisfies the functorial star implication for all relational matrices, the relation $\approx$ is preserved by all operations. Investigate the structure of the quotient $/\approx$.

Proposition 9.4.11 *Suppose that* $\mathbf{Mat}_S$ *has a functorial star with respect to all relational matrices. Let* $D = (\alpha; A; \gamma)$ *and* $E = (\alpha'; B; \gamma')$ *be presentations of weight* s *and* r*, respectively. If* $D \xrightarrow{\rho} E$ *for a relational matrix* $\rho : s \longrightarrow r$*, then* $|D| = |E|$*.*

Proof. Since $\mathbf{Mat}_S$ satisfies the functorial star implication for all relational matrices, we have

$$\begin{aligned} |D| &= \alpha \cdot A^* \cdot \gamma \\ &= \alpha \cdot A^* \cdot \rho \cdot \gamma' \\ &= \alpha \cdot \rho \cdot B^* \cdot \gamma' \\ &= \alpha' \cdot B^* \cdot \gamma' \\ &= |E|. \end{aligned}$$

Example 9.4.12 When $S = RL(X^*)$ (or $S = L(X^*)$), the matrix iteration theory $\mathbf{Mat}_S$ has a functorial star with respect to all matrices. Let D and E be presentations over the pair $(\mathbf{B}, X)$. The fact that $|D| = |E|$ when $D \xrightarrow{\rho} E$ for some base ρ corresponds to the fact that two nfa's accept the same language if there is a "homomorphism" between them.

In the next two exercises we assume that S is a Conway semiring, S_0 is a sub Conway semiring of S and that X is a subset of S.

Exercise 9.4.13 Show that for every presentation D over (S_0, X) there is a presentation $E = (\alpha; A; \gamma)$ of weight $s \geq 1$ such that $|D| = |E|$ and $\alpha = [1, 0, \ldots, 0]$.

Exercise 9.4.14 When $S_0 = \{0, 1\}$ and the equation $1^* = 1$ holds in S, there is an alternative definition of the star operation on presentations. Indeed, when $D = (\alpha; A; \gamma)$, define

$$D^* \quad := \quad (\begin{bmatrix} \alpha & 1 \end{bmatrix}; \begin{bmatrix} A + \gamma\alpha A & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} \gamma \\ 1 \end{bmatrix}).$$

Show that $|D^*| = |D|^*$.

9.5 The Initial Matrix Iteration Theory

Since iteration semirings form an equational class, for every set X there is an iteration semiring freely generated by X. Similarly, there is a free matrix iteration theory freely generated by any set. We do not have a concrete description of these theories, but we can give a concrete description of the initial such theory, i.e. a matrix iteration theory $T_0 = \mathbf{Mat}_{S_0}$ with the property that for any matrix iteration theory T there is a unique iteration theory morphism $T_0 \to T$.

Define a (linearly ordered) *-semiring S_0 as follows. The elements of S_0 are, in order:

$$0, 1, 2, \ldots, 1^*, (1^*)^2, (1^*)^3, \ldots, 1^{**}.$$

Sum and product on the integers are the standard operations; the sum and product on the remaining elements are forced by Exercise 9.3.38:

$$\begin{aligned} x + y &= \max\{x, y\}, \quad \text{if } x \geq 1^* \text{ or } y \geq 1^* \\ (1^*)^n (1^*)^p &= (1^*)^{n+p} \\ x1^{**} = 1^{**}x &= 1^{**}, \quad \text{if } x \neq 0. \end{aligned}$$

Lastly, the star operation is defined by:

$$x^* = \begin{cases} 1 & \text{if } x = 0; \\ 1^* & \text{if } x = 1; \\ 1^{**} & \text{otherwise.} \end{cases}$$

We define the matrix theory T_0 as $\mathbf{Mat}_{S_0}$. Note that the ordering on S_0 has the following property: For all $a, b \in S_0$,

$$a \leq b \Leftrightarrow \exists c \; a + c = b.$$

We will prove the following

Proposition 9.5.1 $T_0 = \mathbf{Mat}_{S_0}$ *is a matrix iteration theory which satisfies the star GA-implication.*

As a consequence we obtain the major result of this section.

Theorem 9.5.2 *If T is any matrix iteration theory, there is a unique matrix iteration theory morphism $\varphi : T_0 \to T$.*

Proof. Any matrix iteration theory morphism is determined by its restriction to the semiring homomorphism

$$T_0(1,1) \;\to\; T(1,1).$$

Thus, it is sufficient to show that S_0 is initial in the class of all iteration semirings. But any *-semiring homomorphism $h : S_0 \to S$ must take $n = \overbrace{(1 + \ldots + 1)}^{n \text{ times}}$ to n, xy to $h(x)h(y)$ and

$$h(x^*) \;=\; h(x)^*.$$

This map is well-defined, by Exercise 9.3.38.

Corollary 9.5.3 *The semiring S_0 is the initial iteration semiring.*

Proof of Proposition 9.5.1. By construction, S_0 is a Conway semiring, and thus T_0 is a Conway matrix theory. We will show that S_0 satisfies the star GA-implication, i.e.

$$(d + ca^*b)^*(ca^*e + f) \quad = \quad (a + b)^*e$$

whenever $(a + b)^*e = (c + d)^*f$. Thus, by Proposition 9.3.21, it will follow that T_0 is an iteration theory.

Case 1. $(a + b)^*e = (c + d)^*f = 1^{**}$. For any a, b, c, d, e, f, either

$$(d + ca^*b)^*(ca^*e + f) \quad \geq \quad (c + d)^*f$$

or

$$(d + ca^*b)^*(ca^*e + f) \quad \geq \quad (a + b)^*e.$$

Thus, in this case we have

$$1^{**} \ \geq \ (d + ca^*b)^*(ca^*e + f) \ \geq \ 1^{**}.$$

Case 2. $0 < (a + b)^*e = (c + d)^*f < 1^{**}$. In this case both $a + b$ and $c + d$ are either 0 or 1. We have several subcases.
Subcase $a + b = c + d = 1$. Then $e = f$ and one proves that

$$(d + ca^*b)^*(ca^*e + f) \ = \ 1^*e \ = \ (a + b)^*e.$$

Indeed, e.g. when $a = c = 1$ and $b = d = 0$,

$$(d + ca^*b)^*(ca^*e + f) \ = \ 1^*e + f \ = \ 1^*e.$$

Subcase $a + b = 1$ and $c + d = 0$, so that $c = d = 0$ and $1^*e = f$. In this case

$$(d + ca^*b)^*(ca^*e + f) \ = \ f \ = \ 1^*e \ = \ (a + b)^*e.$$

Subcase $a + b = 0$ and $c + d = 1$. In this case $a = b = 0$ and $e = 1^*f$. If $c = 1$ and $d = 0$ then

$$(d + ca^*b)^*(ca^*e + f) \ = \ e + f \ = \ e.$$

If $c = 0$ and $d = 1$ then

$$(d + ca^*b)^*(ca^*e + f) \ = \ 1^*f \ = \ e.$$

In either case,

$$(d + ca^*b)^*(ca^*e + f) \quad = \quad (a + b)^*e.$$

Subcase $a + b = c + d = 0$. Then $a = b = c = d = 0$ and one proves that

$$(d + ca^*b)^*(ca^*e + f) \ = \ f \ = \ e \ = \ (a + b)^*e.$$

Case 3. $(a + b)^*e = (c + d)^*f = 0$. Then $e = f = 0$ and

$$(d + ca^*b)^*(ca^*e + f) \ = \ 0 \ = \ (a + b)^*e.$$

Exercise 9.5.4 Use the fact that S_0 is commutative to show that S_0 satisfies the dual star GA-implication. Thus S_0 is initial symmetric iteration semiring.

Some computationally interesting quotients of S_0 are the two-element Boolean semiring $\mathbf{B} = \{0, 1\}$ and the infinite semiring

$$\mathbf{N}_\infty \;=\; \{0, 1, 2, \ldots, \infty = 1^* = (1^*)^2 = \ldots = 1^{**}\}$$

mentioned above. Another quotient of S_0 is the *-semiring $S_1 := \{0, 1, 1^*\}$ with the obvious *-operation. All of these semirings are (symmetric) iteration semirings.

Corollary 9.5.5 *The *-semiring $\mathbf{N}_\infty$ is the initial iteration semiring satisfying the equation $1^* = 1^{**}$; the semiring S_1 is the initial* **idempotent** *iteration semiring (i.e. $a + a = a$); finally, the semiring $\mathbf{B}$ is the initial* ω**-idempotent** *iteration semiring, i.e. $1^* = 1$.*

Problem 9.5.1 Find a concrete representation of the free matrix iteration theory and of the free Conway matrix theory on a set X.

Exercise 9.5.6 [Con71] Describe the initial Conway matrix theory.

Exercise 9.5.7 [Kui87] An ω-complete semiring is *L-complete* if

$$\sum_n a_n \;=\; \sum_n b_n$$

whenever

$$\sum_{n=0}^{k} a_n \;=\; \sum_{n=0}^{k} b_n,$$

for all $k \geq 0$. Show that the semiring S_1 in Corollary 9.5.5 is ω-complete but not L-complete.

9.6 An Extension Theorem

Suppose that the theory $T = \mathbf{Mat}_S$ is a matrix iteration theory. Let R be either the semiring $S\langle\langle X^* \rangle\rangle$ of all formal power series over S with variables in X defined in Example 9.3.11, or the semiring $S^{rat}\langle\langle X^*\rangle\rangle$ of all rational power series in $S\langle\langle X^*\rangle\rangle$ defined in Example 9.6.17 below. In this section we prove a theorem which implies that the theory $\mathbf{Mat}_R$ is an iteration theory whenever $\mathbf{Mat}_S$ is.

An *ideal* J in a semiring S is a subset that contains 0, is closed under the addition operation, and lastly contains both as and sa, for any a in J and s in S.

Notice that if R is either of the above power series semirings, the collection of all functions

$$r : X^* \;\rightarrow\; S$$

which are 0 on X^+, the set of all words on X of positive length, forms a subsemiring of R isomorphic to S. From now on, we identify S with this subsemiring of R. Let P denote the set of functions $a \in R$ such that $\langle a,\, \epsilon\rangle = 0$, where ϵ is the empty word; i.e. the support of the function a is a subset of X^+. Then P is an ideal in R and each function in R is uniquely expressible as a sum $x + a$ where x is in S and a is in P. Similarly, if $f : n \rightarrow m$ is an $n \times m$ matrix over R, f can be expressed uniquely as a sum

$$f \;=\; x + a$$

where $x : n \rightarrow m$ is in $\mathbf{Mat}_S$ and $a : n \rightarrow m$ has entries in P.

In Chapter 5, we defined the notion of an ideal in any theory. However, we will use a more restricted definition in connection with matrix theories, closely related to the additive structure of such theories.

Definition 9.6.1 *Suppose that T is a matrix theory. A collection I of morphisms in T is an* **ideal** *if*

- *each zero matrix 0_{np} is in I;*
- *if $a, b : n \rightarrow p$ are in I, so is $a + b$;*
- *if $a : n \rightarrow p$ is in I, so are $c \cdot a$ and $a \cdot b$, for all b and c composable with a;*
- *if $a_i : n_i \rightarrow p$ are in I, for $i \in [2]$, then $a_1 a_2 is in I$.*

We note some additional closure conditions on ideals in matrix theories. First, if $a : n \to p$ and $b : m \to q$ are in the ideal I, so is $a \oplus b$, since $a \oplus b = a \cdot \kappa b \cdot \lambda$, for appropriate base κ and λ. Thus I is also closed also under target tupling, since if $a : n \to p$ and $b : n \to q$,

$$[a \quad b] \quad = \quad a \oplus 0_q + 0_p \oplus b.$$

If I is an ideal in $\mathbf{Mat}_S$, then the collection $J := \{f : f \in I \text{ and } f : 1 \to 1\}$ is an ideal in S, and a matrix a is in I if and only if each of its components a_{ij} is in J. Conversely, given an ideal J in S, the collection of all matrices all of whose entries are in J forms an ideal in $\mathbf{Mat}_S$. In fact, this is the only ideal I such that set of elements a with $a : 1 \to 1 \in I$ is the set J.

We repeat a well-known fact. Let P be the ideal mentioned above in the semiring $S\langle\langle X^* \rangle\rangle$ of power series.

Proposition 9.6.2 *Suppose that* $A : n \to n$ *is a matrix with entries in* P*. Then for any matrix* $B : n \to p$ *there is a unique solution to the equation in the variable* $\xi : n \to p$

$$\xi \quad = \quad A\xi + B.$$

We are now able to state the extension theorem.

Theorem 9.6.3 The Matrix Extension Theorem. *Let* $T = \mathbf{Mat}_S$ *be a matrix theory having a submatrix theory* $T_0 = \mathbf{Mat}_{S_0}$*. Let* I *be an ideal in* S*. Suppose that the following three conditions hold.*

1. T_0 *is an iteration theory;*
2. *For each* a *in* I *and each* $b \in S$*, there is a unique element* ξ *in* S *satisfying*

$$\xi \quad = \quad a\xi + b;$$

3. *Each element in* S *can be written uniquely as a sum*

$$f \quad = \quad x + a$$

where $x \in S_0$ *and* a *is in* I*.*

Then there is a unique extension of the iteration operation in T_0 to all morphisms $n \to n+p$ of T such that T becomes an iteration theory.

Proof. The proof will occupy the rest of this section. Since T_0 is an iteration theory, there is a family of star operations $^* : T_0(n,n) \to T_0(n,n)$. We will show how to extend these operations to all of T so that the star sum, star product and star commutative identities hold. Thus, by Corollary 9.3.2, T is a matrix iteration theory.

For an $n \times n$ matrix, we will write $a \in I$ to mean that each entry of a belongs to the ideal I.

The first observation is that the hypotheses imply the following facts.

Lemma 9.6.4 *Each morphism $f : n \to p$ can be written uniquely as a sum*

$$f \quad = \quad x + a,$$

where $x : n \to p$ is in T_0 and $a : n \to p$ is in I.

Lemma 9.6.5 *For each $n \geq 1$ and $a : n \to n$ in I and each $b : n \to p$, there is a unique solution to the equation in the variable $\xi : n \to p$:*

$$\xi \quad = \quad a\xi + b. \tag{9.19}$$

Proof of Lemma 9.6.5. We prove the following assertion by induction on n: For $a \in I$, the equation

$$\xi \quad = \quad a\xi + \mathbf{1}_n$$

has a unique solution, denoted $a^{\ominus}$; further, the unique solution to (9.19) is $a^{\ominus}b$, for any $p \geq 0$ and any $b : n \to p$. The basis case $n = 1$ is proved in two steps: $p = 1$ and otherwise. When $p = 1$, there is a unique $a^{\ominus}$ such that $a^{\ominus} = a \cdot a^{\ominus} + 1$, since $a \in I$. The second statement is proved by showing that $a^{\ominus}b$ satisfies the equation. When $p = 0$, there is a unique morphism $1 \to 0$ in any matrix theory, and $a^{\ominus}b$ is such a morphism. When $p > 1$, write

$$b \quad = \quad [b_1, \ldots, b_p] : 1 \to p,$$

and

$$\xi \quad = \quad [\xi_1, \ldots, \xi_p] : 1 \rightarrow p.$$

Then the equation (9.19) becomes a system of p equations

$$\xi_i \quad = \quad a\xi_i + b_i, \quad i \in [p].$$

Since, by assumption, each one has a unique solution, namely $a^{\ominus}b_i$, the unique solution to (9.19) is $a^{\ominus}[b_1, \ldots, b_p]$.

For the induction step, assume that $a : n+1 \rightarrow n+1$ is written

$$a \quad = \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\xi \quad = \quad \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

and

$$b \quad = \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $a_{11} : n \rightarrow n$, etc. First, using the induction assumption, if ξ is a solution to (9.19), then

$$\xi_1 \quad = \quad a_{11}\xi_1 + a_{12}\xi_2 + b_1.$$

Hence,

$$\xi_1 \quad = \quad a_{11}^{\ominus}(a_{12}\xi_2 + b_1).$$

It follows that

$$\xi_2 \quad = \quad (a_{21}a_{11}^{\ominus}a_{12} + a_{22})\xi_2 + a_{21}a_{11}^{\ominus}b_1 + b_2.$$

Using the induction hypothesis to solve for ξ_2 and then for ξ_1, we can write

$$\xi \quad = \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where

$$\begin{array}{rcl}\alpha & = & a_{11}^{\ominus}a_{12}(a_{22}+a_{21}a_{11}^{\ominus}a_{12})^{\ominus}a_{21}a_{11}^{\ominus}+a_{11}^{\ominus}\\ \beta & = & a_{11}^{\ominus}a_{12}(a_{22}+a_{21}a_{11}^{\ominus}a_{12})^{\ominus}\\ \gamma & = & (a_{22}+a_{21}a_{11}^{\ominus}a_{12})^{\ominus}a_{21}a_{11}^{\ominus}\\ \delta & = & (a_{22}+a_{21}a_{11}^{\ominus}a_{12})^{\ominus}.\end{array}$$

This proves the uniqueness of the solution. We then verify that the array $\xi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is the solution to the equation $\xi = a\xi + \mathbf{1}_{n+1}$, and that $\xi \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is a solution to (9.19), completing the induction step.

We now begin the main part of the proof. First, since 0_{nn} is in I,

$$0_{nn}^{\ominus} \quad = \quad \mathbf{1}_n.$$

Since T_0 is already an iteration theory, each hom-set $T_0(n,n)$ is a *-semiring and

$$0_{nn}^{*} \quad = \quad \mathbf{1}_n.$$

Thus, the two operations * and $^{\ominus}$ agree on the zero matrices.

We now define the extension of the operation * to all of T.

Definition 9.6.6 *For $f = x + a : n \to n$ in T, with $x \in T_0$, $a \in I$, the morphism $f^{\otimes}$ is defined by:*

$$f^{\otimes} \quad := \quad (x^*a)^{\ominus}x^*.$$

Note that x^*a is in I, so that $(x^*a)^{\ominus}$ is meaningful. For $f = x \in T_0(n,n)$,

$$f^{\otimes} \;=\; (x + 0_{nn})^{\otimes} \;=\; (x^* \cdot 0_{nn})^{\ominus} \cdot x^* \;=\; \mathbf{1}_n \cdot x^* \;=\; x^*,$$

so that the new operation extends the operation *. Similarly, for $a \in I$,

$$a^{\otimes} \;=\; (0_{nn} + a)^{\otimes} \;=\; (0_{nn}^* a)^{\ominus} 0_{nn}^* \;=\; a^{\ominus},$$

so that $^{\otimes}$ also extends $^{\ominus}$. From now on, the reader will be relieved that we write just * for all three operations. Note that the definition of the extended operation is forced by the star sum identity. Thus there is at most one possible extension. We must show that with this extension T is an iteration theory.

We list a number of facts.

Lemma 9.6.7 *If $a : n \to n$ is in I, and $b : n \to p$ is in T, the unique solution to the equation*

$$\xi \quad = \quad a\xi + b$$

is a^*b. Further,

$$aa^* \quad = \quad a^*a.$$

For any $c : n \to n$ in either T_0 or in I, we write

$$c^+ \quad := \quad cc^* \;=\; c^*c.$$

Lemma 9.6.8 *If either $a : n \to m$ or $b : m \to n$ is in I, so that ab and ba are in I,*

$$(ab)^* \quad = \quad \mathbf{1}_n + a(ba)^*b.$$

Proof. Let

$$c \quad := \quad \mathbf{1}_n + a(ba)^*b.$$

Then

$$\begin{aligned} abc + \mathbf{1}_n &= ab + aba(ba)^*b + \mathbf{1}_n \\ &= a(\mathbf{1}_n + ba(ba)^*)b + \mathbf{1}_n \\ &= a(ba)^*b + \mathbf{1}_n \;=\; c. \end{aligned}$$

Thus $c = (ab)^*$.

Corollary 9.6.9 *For either $a : n \to m$ or $b : m \to n$ in I,*

$$a(ba)^* \quad = \quad (ab)^*a.$$

Lemma 9.6.10 *For* $a, b : n \to n$ *in* I,

$$(a+b)^* \quad = \quad (a^*b)^*a^*.$$

Proof. One shows that the right hand side is a solution to the equation

$$\xi \quad = \quad (a+b)\xi + \mathbf{1}_n.$$

Now we prove that T is a Conway theory. We will use the letters x and y to denote morphisms in T_0, and the letters a and b for morphisms in I.

Lemma 9.6.11 *For* $f = (x+a) : n \to n$ *in* T,

$$f^* \quad = \quad ff^* + \mathbf{1}_n.$$

Proof.

$$f^* \;=\; (x^*a)^*x^* \;=\; x^*(ax^*)^*,$$

so that

$$\begin{aligned}
\mathbf{1}_n + ff^* &= \mathbf{1}_n + xx^*(ax^*)^* + ax^*(ax^*)^* \\
&= \mathbf{1}_n + xx^*(\mathbf{1}_n + ax^*(ax^*)^*) + ax^*(ax^*)^* \\
&= xx^* + \mathbf{1}_n + (xx^* + \mathbf{1}_n)ax^*(ax^*)^* \\
&= x^*(\mathbf{1}_n + ax^*(ax^*)^*) \\
&= x^*(ax^*)^* = f^*.
\end{aligned}$$

It is more difficult to prove the star sum identity.

Lemma 9.6.12 *For* $f, g : n \to n$ *in* T,

$$(f+g)^* \quad = \quad (f^*g)^*f^*.$$

Proof. This lemma is proved in several steps.

Case 1. $f = a$ in I and $g = y$ in T_0.
Proof.

$$\begin{aligned}
(f+g)^* &= (a+y)^* \\
&= (y+a)^* \\
&= (y^*a)^*y^* \\
&= y^*(ay^*)^*,
\end{aligned}$$

by Corollary 9.6.9,

$$\begin{aligned} &= y^*(a + ay^+)^* \\ &= y^*(a^*ay^+)^*a^*, \end{aligned}$$

by Lemma 9.6.10,

$$\begin{aligned} &= y^*(a^+yy^*)^*a^* \\ &= (y^*a^+y)^*y^*a^*, \end{aligned}$$

by Corollary 9.6.9,

$$= (y + a^+y)^*a^*,$$

by definition,

$$\begin{aligned} &= (a^+y + y)^*a^* \\ &= (a^*y)^*a^* \\ &= (f^*g)^*f^*. \end{aligned}$$

CASE 2. $f = x + a$ and $g = b$ in I.
Proof.

$$\begin{aligned} (f+g)^* &= (x + a + b)^* \\ &= (x + (a + b))^* \\ &= (x^*(a + b))^*x^* \\ &= (x^*a + x^*b)^*x^* \\ &= ((x^*a)^*x^*b)^*(x^*a)^*x^*, \end{aligned}$$

by Lemma 9.6.10,

$$\begin{aligned} &= ((x + a)^*b)^*(x + a)^* \\ &= (f^*g)^*f^*. \end{aligned}$$

CASE 3. $f = x + a$ and $g = y \in T_0$.
Proof.

$$\begin{aligned} (f+g)^* &= ((x + a) + y)^* \\ &= ((x + y) + a)^* \\ &= ((x + y)^*a)^*(x + y)^* \\ &= ((x^*y)^*x^*a)^*(x^*y)^*x^*, \end{aligned}$$

since T_0 is an iteration theory,

$$\begin{aligned} &= (x^*y + x^*a)^*x^* \\ &= (x^*a + x^*y)^*x^* \\ &= ((x^*a)^*x^*y)^*(x^*a)^*x^*, \end{aligned}$$

by Case 1,

$$\begin{aligned} &= ((x+a)^*y)^*(x+a)^* \\ &= (f^*g)^*f^*. \end{aligned}$$

Case 4. The general case: $f = x + a$ and $g = y + b$.

$$\begin{aligned} (f+g)^* &= ((x+a+y)+b)^* \\ &= ((x+a+y)^*b)^*(x+a+y)^*, \end{aligned}$$

by Case 2,

$$= (((x+a)^*y)^*(x+a)^*b)^*((x+a)^*y)^*(x+a)^*$$

by Case 3,

$$= ((x+a)^*y + (x+a)^*b)^*(x+a)^*$$

by Case 2,

$$= (f^*g)^*f^*.$$

The proof of this lemma is complete.

The proof that T satisfies the identity

$$(fg)^*f \quad = \quad f(gf)^*$$

for all $f : n \to m$ and $g : m \to n$ is similar and is omitted. Since the star fixed point identity holds, it follows that the general star product identity holds.

It remains to show that T satisfies the star commutative identity. We will make use of the following observation, which shows that the morphisms in I satisfy the strong functorial star implication.

Proposition 9.6.13 *Suppose that $a : n \to n$ is in I and that $g : m \to m$ is in T. Let $\rho : n \to m$ be any morphism such that*

$$a \cdot \rho \quad = \quad \rho \cdot g.$$

Then

$$a^* \cdot \rho \quad = \quad \rho \cdot g^*.$$

Proof. We need only show that $\rho \cdot g^*$ is a solution to the equation

$$\xi \quad = \quad a \cdot \xi + \rho.$$

This is easy to do, using Lemma 9.6.11.

Now suppose that $f = x + a : n \to m$ and that $\rho : m \to n$, and $\rho_i : m \to m$ are base such that for each $i \in [m]$, $\rho_i \cdot \rho = \rho$. Abbreviate the m-tuple $(\rho_1, \ldots, \rho_m)$ by just R.

Lemma 9.6.14 *The star commutative identity holds, i.e. for f, ρ and R as above,*

$$((\rho f) \parallel R)^* \cdot \rho \quad = \quad \rho \cdot (f\rho)^*.$$

Proof.

$$\begin{aligned} (\rho f) \parallel R \quad &= \quad (\rho x + \rho a) \parallel R \\ &= \quad (\rho x \parallel R) + (\rho a \parallel R). \end{aligned}$$

Hence, since $(\rho x \parallel R) \in T_0$ and $(\rho a \parallel R) \in I$,

$$(\rho f \parallel R)^* \cdot \rho \quad = \quad ((\rho x \parallel R)^* \cdot (\rho a \parallel R))^* \cdot (\rho x \parallel R)^* \cdot \rho. \tag{9.20}$$

Define the morphism $b \in I$ by:

$$b \quad := \quad (\rho x \parallel R)^* \cdot (\rho a \parallel R).$$

We claim that

$$b \cdot \rho \quad = \quad \rho \cdot (x\rho)^*(a\rho). \tag{9.21}$$

Indeed,

$$b \cdot \rho \quad = \quad (\rho x \parallel R)^* \cdot (\rho a \parallel R) \cdot \rho,$$

but

$$(\rho a \parallel R) \cdot \rho \quad = \quad \rho \cdot a\rho,$$

since $\rho_i \cdot \rho = \rho$, and

$$(\rho x \parallel R)^* \cdot \rho \quad = \quad \rho \cdot (x\rho)^*, \tag{9.22}$$

since T_0 is an iteration theory. This proves the claim (9.21). Using Proposition 9.6.13 and the fact that the morphism b is in I, it follows that

$$b^* \cdot \rho \quad = \quad \rho \cdot ((x\rho)^* a\rho)^*. \tag{9.23}$$

Thus, using (9.20), (9.22) and (9.23),

$$\begin{aligned}(\rho f \parallel R)^* \cdot \rho \quad &= \quad b^* \cdot \rho \cdot (x\rho)^* \\ &= \quad \rho \cdot ((x\rho)^* a\rho)^* (x\rho)^*,\end{aligned}$$

by (9.23),

$$= \quad \rho \cdot (x\rho + a\rho)^*$$

by definition,

$$= \quad \rho \cdot (f\rho)^*.$$

The proof of the last lemma and the theorem is complete.

Remark 9.6.15 Using the notation of Theorem 9.6.3, we observe that, if the iteration theory T_0 has a functorial star with respect to base surjections, then so does the full theory T. Indeed suppose that

$$f \cdot \rho \quad = \quad \rho \cdot g$$

where $f : n \to n$, $\rho : n \to m$ is a surjective base morphism and $g : m \to m$. We show that

$$f^* \cdot \rho \quad = \quad \rho \cdot g^*.$$

Let $f = x + a$, $g = y + b$, with $x, y \in T_0$ and $a, b \in I$. Then

$$x \cdot \rho + a \cdot \rho \quad = \quad \rho \cdot y + \rho \cdot b.$$

Since $x \cdot \rho,\ \rho \cdot y$ are in T_0, and $a \cdot \rho,\ \rho \cdot b$ are in I, it follows by the uniqueness of the representation that

$$\begin{aligned} x \cdot \rho &= \rho \cdot y \\ a \cdot \rho &= \rho \cdot b. \end{aligned}$$

Hence

$$x^* \cdot \rho = \rho \cdot y^*,$$

since T_0 has a functorial star with respect to base surjections, and

$$a^* \cdot \rho = \rho \cdot b^*,$$

by Proposition 9.6.13. Thus,

$$f^* \cdot \rho = (x + a)^* \cdot \rho = (x^* \cdot a)^* \cdot x^* \cdot \rho = (x^* \cdot a)^* \cdot \rho \cdot y^*. \tag{9.24}$$

But

$$(x^* \cdot a) \cdot \rho = x^* \cdot (a \cdot \rho) = (x^* \cdot \rho) \cdot b = \rho \cdot y^* \cdot b,$$

and $x^* \cdot a$ is in I. Thus, again by Proposition 9.6.13,

$$(x^* \cdot a)^* \cdot \rho = \rho \cdot (y^* \cdot b)^*.$$

Using (9.24),

$$\begin{aligned} f^* \cdot \rho &= \rho \cdot (y^* \cdot b)^* \cdot y^* \\ &= \rho \cdot g^*, \end{aligned}$$

completing the proof.

Note that the above argument shows that if the iteration theory T_0 satisfies the implication

$$f \cdot \rho = \rho \cdot g \quad \Rightarrow \quad f^* \cdot \rho = \rho \cdot g^*,$$

for all $\rho \in F \subseteq T_0$, for some family F of morphisms in T_0, then T will satisfy the same implication for the morphisms in F. One place where this observation can be applied is the case that $T = \mathbf{Mat}_{\mathbf{B}\langle\langle X^* \rangle\rangle}$ and $F = T_0 = \mathbf{Mat}_{\mathbf{B}}$, where $\mathbf{B}$ is the Boolean semiring.

There are several corollaries.

Corollary 9.6.16 *Suppose that the theory $T = \mathbf{Mat}_S$ is a matrix iteration theory. Let R be the semiring $S\langle\langle X^* \rangle\rangle$ of all formal power series over S with variables in the set X. Then $\mathbf{Mat}_R$ is an iteration theory.*

Proof. As already noted, if we take the ideal P consisting of those power series in R which are zero on the empty word, all of the hypotheses of the Matrix Extension Theorem apply.

Example 9.6.17 Suppose that $\mathbf{Mat}_S$ is a matrix iteration theory, so that the matrix theory $\mathbf{Mat}_{S\langle\langle X^*\rangle\rangle}$ is an iteration theory as well. As in Example 9.4.5, we may consider S to be a sub *-semiring of $S\langle\langle X^*\rangle\rangle$ and X to be a subset of $S\langle\langle X^*\rangle\rangle$. The smallest submatrix iteration theory of $S\langle\langle X^*\rangle\rangle$ containing $\mathbf{Mat}_S$ and X is the matrix iteration theory $\mathbf{Mat}_{S^{rat}\langle\langle X^*\rangle\rangle}$, where $S^{rat}\langle\langle X^*\rangle\rangle$ is the sub *-semiring of $S\langle\langle X^*\rangle\rangle$ consisting of the *rational* power series. By Corollary 9.4.4, the rational power series in $S\langle\langle X^*\rangle\rangle$ are exactly the behaviors of presentations over (S, X).

The next result follows from the proof of the Matrix Extension Theorem.

Corollary 9.6.18 *Let S_0 be a subsemiring of the semiring S, and let I be an ideal in S. Suppose that*

1. *S_0 is a Conway semiring;*
2. *For each a in I and each b in S, there is a unique solution to the equation*
$$\xi \quad = \quad a\xi + b;$$
3. *Each element in S can be expressed uniquely as $x + a$, with $x \in S_0$ and $a \in I$.*

Then there is a unique way to define the star operations on $\mathbf{Mat}_S$ so that $\mathbf{Mat}_S$ is a Conway matrix theory and the star operation on $\mathbf{Mat}_S(1,1)$ extends the star operation on S_0.

Exercise 9.6.19 Consider again Exercises 3.5.40–3.5.42 in Chapter 3. Show that S_0, I and $S(A)$ satisfy the hypotheses of the Matrix Extension Theorem.

Problem 9.6.1 Find a base of identities for the equational class generated by all matrix iteration theories $\mathbf{Mat}_R$ where R is a semiring $\mathbf{N}_\infty\langle\langle X^*\rangle\rangle$.

The theories $\mathbf{Mat}_S$, where $S = \mathbf{B}\langle\langle X^*\rangle\rangle$, are considered below.

9.7 Matrix Iteration Theories of Regular Sets

In Sections 9.3 and 9.4 we described the matrix iteration theory $Ł(X^*)$ of languages over X. The matrix iteration theory $Ł(X^*)$ is a submatrix iteration theory of $Ł(X^*)$. The Matrix Extension Theorem provides another proof that $Ł(X^*)$ and $Ł(X^*)$ are matrix iteration theories. We noted that $Ł(X^*)$, and hence $Ł(X^*)$, has a strong functorial star. Thus, by Corollary 9.3.27, the dual star commutative identity holds in both theories. Further, it is immediate that the equation $1^* = 1$ holds as well. In this section we prove that $Ł(X^*)$ can be characterized as the matrix iteration theory freely generated by X in the class of matrix iteration theories satisfying these two additional identities. In other words, the semiring $RL(X^*)$ of regular sets is the ω-idempotent symmetric iteration semiring freely generated by the set X.

We need a number of preliminary facts. For the remainder of this section we suppose that a set X is fixed. We identify each letter $x \in X$ with the singleton $\{x\}$. Thus we may consider a letter either as a matrix $1 \to 1$ in $Ł(X^*)$ or as an element of the semiring $RL(X^*)$. Similarly, we identify the semiring $\mathbf{B}$ with the subsemiring $\{\emptyset, \{\epsilon\}\}$. By a presentation we shall always mean a presentation over $(\mathbf{B}, X)$. Recall that with each presentation D we have associated the nondeterministic finite automaton (nfa) $_D$, cf. Example 9.4.5. We may thus use automata theoretic concepts in connection with presentations.

Definition 9.7.1 *Let D be a presentation with corresponding nfa*

$$_D \;=\; (Q, X, \delta, Q_0, Q_f).$$

We call D **accessible** *if $_D$ is accessible, i.e. if for every $j \in Q$ there exist $i \in Q_0$ and $u \in X^*$ with $j \in \delta(i,u)$. Similarly, D is* **coaccessible** *if $_D$ is, i.e. if for every $i \in Q$ there exist $j \in Q_f$ and $u \in X^*$ with $j \in \delta(i,u)$. Note that D is coaccessible if and only if D^T is accessible. If D is both accessible and coaccessible, we call D* **biaccessible**. *Finally, we say that D is* **deterministic** *if $_D$ is a deterministic finite automaton (dfa), i.e. if Q_0 is either empty or a singleton set, and if for all $q \in Q$ and $x \in X$, $\delta(q,x)$ has at most one element.*

The following result is a translation of well-known facts.

Proposition 9.7.2 *For every presentation D there exist a coaccessible presentation D' and a biaccessible presentation E such that*

$$D \stackrel{\kappa^T}{\to} D' \quad \textit{and} \quad E \stackrel{\iota}{\to} D'$$

hold for some injective base matrices ι and κ. If D is deterministic, then so is E.

Proposition 9.7.3 *Let D be a presentation. There exist presentations E and F such that F is deterministic and $E \stackrel{\rho}{\to} D$ and $F \stackrel{\tau^T}{\to} E$ hold for some surjective base matrices ρ and τ.*

Proof. Let $= (Q, X, \delta, Q_0, Q_f)$. We define $'$ to be the well-known powerset automaton, i.e.

$$' \;:=\; (Q', X, \delta', Q'_0, Q'_f)$$

with

$$\begin{array}{rcl} Q' & := & \{U \subseteq Q : U \neq \emptyset\} \\ Q'_0 & := & \left\{ \begin{array}{ll} Q_0 & \text{if } Q_0 \neq \emptyset \\ \emptyset & \text{if } Q_0 = \emptyset \end{array} \right. \\ Q'_f & := & \{U \subseteq Q : U \cap Q_f \neq \emptyset\}, \end{array}$$

and for all $U \in Q'$ and $x \in X$,

$$\delta'(U, x) \;:=\; \bigcup(\delta(q, x) : q \in U),$$

if this set is nonempty. Otherwise $\delta'(U, x)$ is not defined. (In the case of $'$, just as for deterministic automata, we consider δ' to be a partial function $Q' \times X \to Q'$.) We then define the nfa

$$'' \;:=\; (Q'', X, \delta'', Q''_0, Q''_f)$$

to be a subautomaton of the direct product of with $'$. In detail,

$$\begin{array}{rcl} Q'' & := & \{(q, U) : q \in U,\ U \in Q'\} \\ Q''_0 & := & \{(q, Q_0) : q \in Q_0\} \\ Q''_f & := & \{(q, U) : q \in U \cap Q_f,\ U \in Q'\}, \end{array}$$

and for $(q,U),(p,V) \in Q''$ and $x \in X$,

$$(p,V) \in \delta''((q,U),x) \;\Leftrightarrow\; p \in \delta(q,x) \text{ and } V = \delta'(U,x).$$

Let $h_1 : Q'' \to Q$ and $h_2 : Q'' \to Q'$ be the mappings defined as follows:

$$h_1((q,U)) \;:=\; q$$

and

$$h_2((q,U)) \;:=\; U,$$

for all $(q,U) \in Q''$. It is straightforward to show that the mappings h_1 and h_2 satisfy analogues of the conditions of Example 9.4.9. Thus, if E and F are presentations whose automata are isomorphic to $''$ and $'$, and if ρ and τ are surjective base matrices corresponding to the mappings h_1 and h_2, respectively, then we have $E \xrightarrow{\rho} D$ and $F \xrightarrow{\tau^T} E$, as claimed.

As in Exercise 9.4.10, we define $\approx$ to be the smallest equivalence relation on the set of presentations such that $D \approx E$ whenever $D \xrightarrow{\rho} E$, for some relational matrix ρ. It will follow that $\approx$ is also the smallest equivalence relation such that $D \approx E$ whenever $D \xrightarrow{\rho} E$ or $D \xrightarrow{\rho^T} E$ holds for some injective or surjective base matric ρ.

Proposition 9.7.4 *Let D and E be presentations. If $|D| = |E|$ then $D \approx E$. In fact, if $|D| = |E|$, then there is a sequence*

$$D_0, D_1, \ldots, D_n, \quad n \le 10,$$

of presentations with $D_0 = D$, $D_n = E$, and such that for all $i \in [n]$, one of the relations

$$D_{i-1} \xrightarrow{\rho_i} D_i \quad \textit{and} \quad D_i \xrightarrow{\rho_i} D_{i-1}$$

holds, where each ρ_i is a surjective base matrix, or an injective base matrix, or the transpose of such.

Proof. By Propositions 9.7.2 and 9.7.3, it suffices to consider the case that both D and E are deterministic biaccessible presentations, so that $_D$ and $_E$ are equivalent biaccessible dfa. It is well-known that in this case $_D$ and $_E$ have a common homomorphic image , the reduced dfa equivalent to $_D$ and $_E$. This amounts to saying that there is a deterministic biaccessible presentation F such that

$$D \xrightarrow{\rho} F \quad \text{and} \quad E \xrightarrow{\tau} F$$

hold for some surjective base matrices ρ and τ.

Remark 9.7.5 The converse of Proposition 9.7.4 follows by Exercise 9.4.10.

The two lemmas below allow us to deduce more information from the assumption that $D \xrightarrow{\rho} E$ or $D \xrightarrow{\rho^T} E$ for a surjective base matrix ρ. Although these lemmas hold in a more general setting, in the sequel we will only make use of the present restricted form. (See also Lemma 5.4.20 in Chapter 5.)

Lemma 9.7.6 *Let $A : m \to m$ and $B : n \to n$ be matrices with entries in $P_f(X)$. Suppose that*

$$A \cdot \rho \;=\; \rho \cdot B$$

for a surjective base matrix $\rho : m \to n$. Then there is a matrix $C : n \to k$ with entries in $P_f(X)$ such that for some base matrices $\rho_1, \ldots, \rho_m : k \to m$ and $\tau_1, \ldots, \tau_n : k \to n$,

$$\begin{aligned} A &= (\rho \cdot C) \parallel (\rho_1, \ldots, \rho_m) \\ B &= C \parallel (\tau_1, \ldots, \tau_n), \end{aligned}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [m]$.

Lemma 9.7.7 *Let $A : n \to n$ and $B : m \to m$ be matrices with entries in $P_f(X)$ and let $\rho : m \to n$ be a surjective base matrix. Suppose that*

$$A \cdot \rho^T \;=\; \rho^T \cdot B.$$

Then there is a matrix $C : k \to n$ with entries in $P_f(X)$ such that for some base matrices $\rho_1, \ldots, \rho_m : k \to m$ and $\tau_1, \ldots, \tau_n : k \to n$,

$$\begin{aligned} B &= (\rho_1^T, \ldots, \rho_m^T) \parallel (C \cdot \rho^T) \\ A &= (\tau_1^T, \ldots, \tau_n^T) \parallel C, \end{aligned}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [m]$.

We omit the proof of the above technical lemmas, but note only that Lemma 9.7.6 implies Lemma 9.7.7. Indeed, $A \cdot \rho^T = \rho^T \cdot B$ holds if and only if $B^T \cdot \rho = \rho \cdot A^T$. Further we mention that Lemma 9.7.6, once proved in the subcase that $n = 1$, extends to arbitrary n by an inductive argument which does not use any specific properties of the matrix iteration theory $\mathrm{Ł}(X^*)$.

Remark 9.7.8 To illustrate the proof of Lemma 9.7.6, consider the matrices

$$A \ := \ \begin{bmatrix} \{x,y\} & \{x\} & \emptyset \\ \{x\} & \emptyset & \{y\} \\ \{y\} & \{x,y\} & \{y\} \end{bmatrix} \ \text{and} \ B \ := \ [\{x,y\}],$$

where x and y are letters in X. We have $A \cdot \rho = \rho \cdot B$ where ρ is the base matrix $3 \to 1$. In this case we may define the matrix C by

$$C \ := \ \begin{bmatrix} \{x\} & \{x\} & \{y\} & \{y\} & \{y\} \end{bmatrix}$$

and the base matrices $\rho_i : 5 \to 3$, $i = 1, 2, 3$, by

$$\begin{array}{lllll} 1\rho_1 = 1 & 2\rho_1 = 2 & 3\rho_1 = 1 & 4\rho_1 = 1 & 5\rho_1 = 1 \\ 1\rho_2 = 1 & 2\rho_2 = 1 & 3\rho_2 = 3 & 4\rho_2 = 3 & 5\rho_2 = 3 \\ 1\rho_3 = 2 & 2\rho_3 = 2 & 3\rho_3 = 1 & 4\rho_3 = 2 & 5\rho_3 = 3. \end{array}$$

Further, let τ be the base matrix $5 \to 1$. We have

$$A \ = \ (\rho \cdot C) \parallel (\rho_1, \rho_2, \rho_3)$$

and

$$B \ = \ C \parallel \tau.$$

Suppose now that $\mathbf{Mat}_S$ is a nontrivial matrix iteration theory which satisfies the dual star commutative identity and the equation $1^* = 1$, so that S is idempotent, by Exercise 9.3.33. Since $\mathbf{Mat}_S$ is nontrivial, $0 \neq 1$ in S, so that the elements $0, 1 \in S$ form a subiteration semiring isomorphic to $\mathbf{B}$. We will denote this subsemiring by $\mathbf{B}$ as well. Let

$$\varphi : X \ \to \ \mathbf{Mat}_S$$

be a mapping such that $x\varphi$ is a matrix $1 \to 1$, i.e. an element in S, for all $x \in X$. The mapping φ extends to matrices with entries in $P_f(X)$ in an obvious way. Given a presentation $D = (\alpha; A; \gamma)$, we then define

$$D\varphi \ := \ (\alpha; A\varphi; \gamma),$$

so that $D\varphi$ is a presentation over $(\mathbf{B}, X\varphi)$ and

$$|D\varphi| \;=\; \alpha \cdot (A\varphi)^* \cdot \gamma$$

is a matrix $1 \to 1$ in $\mathbf{Mat}_S$, i.e. an element of the semiring S.

Proposition 9.7.9 *Let D and E be presentations. If $|D| = |E|$ then $|D\varphi| = |E\varphi|$.*

Proof. By Proposition 9.7.4, it suffices to prove that $|D\varphi| = |E\varphi|$ whenever $D \xrightarrow{\rho} E$ or $D \xrightarrow{\rho^T} E$, where ρ is a surjective or injective base matrix.

Let $D = (\alpha, A, \gamma)$ and $E = (\alpha', B, \gamma')$. Supposing $D \xrightarrow{\rho} E$ for an injective base matrix ρ, we have $A \cdot \rho = \rho \cdot B$, so that also $(A\varphi) \cdot \rho = \rho \cdot (B\varphi)$. Since the functorial star implication holds for injective base matrices in any matrix iteration theory, cf. Proposition 9.3.13, it follows that

$$(A\varphi)^* \cdot \rho \;=\; \rho \cdot (B\varphi)^*.$$

Thus

$$\begin{aligned} |D\varphi| &= \alpha \cdot (A\varphi)^* \cdot \gamma \\ &= \alpha \cdot (A\varphi)^* \cdot \rho \cdot \gamma' \\ &= \alpha \cdot \rho \cdot (B\varphi)^* \cdot \gamma' \\ &= \alpha' \cdot (B\varphi)^* \cdot \gamma' \\ &= |E\varphi|. \end{aligned}$$

The proof that $|D\varphi| = |E\varphi|$ if $D \xrightarrow{\rho^T} E$ for an injective base matrix ρ is similar. One uses the fact that the functorial star implication holds in matrix iteration theories for transposes of injective base matrices.

Suppose now that $D \xrightarrow{\rho} E$ and that ρ is a surjective base matrix. Let s and r denote the weights of D and E, respectively. Since $A \cdot \rho = \rho \cdot B$, by Lemma 9.7.6 there is a matrix $C : r \to k$ all of whose entries are in $P_f(X)$, and there exist base matrices $\rho_1, \ldots, \rho_s : k \to s$ and $\tau_1, \ldots, \tau_r : k \to r$ with

$$\begin{aligned} A &= (\rho \cdot C) \parallel (\rho_1, \ldots, \rho_s) \\ B &= C \parallel (\tau_1, \ldots, \tau_r), \end{aligned}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [s]$. Then also

$$\begin{array}{rcl} A\varphi & = & (\rho \cdot C\varphi) \parallel (\rho_1, \dots, \rho_s) \\ B\varphi & = & C\varphi \parallel (\tau_1, \dots, \tau_r), \end{array}$$

so that

$$(A\varphi)^* \cdot \rho \;=\; \rho \cdot (B\varphi)^*,$$

by the generalized star commutative identity which holds in any matrix iteration theory. The proof can be completed as in the previous case. When $D \stackrel{\rho^T}{\to} E$ for a surjective base matrix ρ, a similar argument applies. One uses Lemma 9.7.7 and the generalized dual commutative identity which holds in $\mathbf{Mat}_S$ by our assumptions.

We now prove the main result of this section.

Theorem 9.7.10 *Let* $\mathbf{Mat}_S$ *be a matrix iteration theory satisfying both the dual commutative identity and the equation* $1^* = 1$*. Let* $\eta : X \to L(X^*)$ *be the function* $x\eta := [\{x\}] : 1 \to 1$*, for all* $x \in X$*. Let*

$$\varphi : X \;\to\; \mathbf{Mat}_S$$

be a mapping such that $x\varphi$ *is matrix* $1 \to 1$ *in* $\mathbf{Mat}_S$*, for each* $x \in X$*. Then there is a unique matrix iteration theory morphism*

$$\varphi^\sharp : L(X^*) \;\to\; \mathbf{Mat}_S$$

with $\eta \cdot \varphi^\sharp = \varphi$*.*

Proof. We only need to specify $\varphi^\sharp$ on the matrices $1 \to 1$ in $RL(X^*)$, i.e. on regular languages over X. Since any regular language is the behavior of a description D, the assignment

$$|D|\varphi^\sharp \;:=\; |D\varphi|$$

associates a matrix $1 \to 1$ in $\mathbf{Mat}_S$ with each regular language. By Proposition 9.7.9, $\varphi^\sharp$ is well-defined. That $\varphi^\sharp$ is a matrix iteration theory morphism follows from Lemma 9.4.2. Given $x \in X$, let D_x be the presentation of weight 2

$$D_x \;:=\; (\begin{bmatrix} 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & \{x\} \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}).$$

We have $|D_x| = x\eta$ and $|D_x\varphi| = x\varphi$. Thus $\eta \cdot \varphi^\sharp = \varphi$. Finally, $\varphi^\sharp$ is unique since each regular language is the behavior of a presentation.

The result contained in Theorem 9.7.10 can be expressed using conventional algebraic terms. Recall that when $\mathbf{Mat}_S$ is a matrix iteration theory, the *-semiring S is an iteration semiring. The dual commutative identity has been defined in Corollary 9.3.26.

Definition 9.7.11 *An* ω**-idempotent symmetric iteration semiring** *is a symmetric iteration semiring in which the equation* $1^* = 1$ *holds. A morphism of* ω*-idempotent symmetric iteration semirings is a* **-semiring homomorphism.*

Thus ω-idempotent symmetric iteration semirings form an equational class.

Corollary 9.7.12 *The semiring* $RL(X^*)$ *is the* ω*-idempotent symmetric iteration semiring freely generated by the set* X*. In more detail, let* $\eta : X \to RL(X^*)$ *be the inclusion and let* $\varphi : X \to S$ *be a function where* S *is a symmetric iteration semiring which satisfies the equation* $1^* = 1$*. Then there is a unique* **-semiring homomorphism* $\varphi^\sharp : RL(X^*) \to S$ *such that* $\eta \cdot \varphi^\sharp = \varphi$*.*

The above corollary characterizes the semirings $RL(X^*)$ as the free *-semirings in the variety of ω-idempotent symmetric iteration semirings. This class is the largest one in which these semirings are free, and it follows that the semirings $RL(X^*)$ are free in any subclass of ω-idempotent iteration semirings which contains these semirings. Three such classes are exhibited below.

Definition 9.7.13 *A* **Kozen semiring** S *is an idempotent Conway semiring which satisfies the following two implications for all* a, b *and* x *in* S*:*

$$ax + b \leq x \quad \Rightarrow \quad a^*b \leq x \tag{9.25}$$

$$xa + b \leq x \quad \Rightarrow \quad ba^* \leq x. \tag{9.26}$$

Note that since each Kozen semiring S is idempotent, S is partially ordered: $a \leq b$ if and only if $a + b = b$, for all $a, b \in S$.

Proposition 9.7.14 *Every Kozen semiring is an ω-idempotent symmetric iteration semiring.*

Proof. Suppose that S is a Kozen semiring, so that $T = \mathbf{Mat}_S$ is a Conway matrix theory. The partial order on S extends to a partial order on each hom-set $T(n,p)$ by defining, for $A, B : n \to p$, $A \leq B$ if and only if $A_{ij} \leq B_{ij}$, for all $i \in [n]$ and $j \in [p]$. It is easy to see that T is a (pointed) ordered theory as defined in Section 8.8.1. Let $f = [A \;\; B] : n \to n+p$ in T, where A is an $n \times n$ matrix and B is $n \times p$. When $n = 1$, the axiom (9.25) and the star fixed point identity imply that A^*B is an initial f-algebra. It follows from Corollary 8.8.4.17 that the same holds for all n, i.e.

$$AX + B \;\leq\; X \quad \Rightarrow \quad A^*B \;\leq\; X \tag{9.27}$$

for any $X : n \to n$. Dually, the equation $a^*a + 1 = a^*$ and (9.26) imply that

$$XA + B \;\leq\; X \quad \Rightarrow \quad B^*A \;\leq\; X \tag{9.28}$$

for all A, B and X as above.

That S is a symmetric iteration semiring follows from the fact that T has a strong functorial star, cf. Proposition 9.3.14 and Corollary 9.3.27. Indeed, suppose that

$$AC \;=\; CB, \tag{9.29}$$

where $A : n \to n$, $B : m \to m$ and $C : n \to m$. Then

$$ACB^* + C \;=\; CBB^* + C \;=\; CB^*,$$

so that

$$A^*C \;\leq\; CB^*$$

by (9.27). The proof that $CB^* \leq A^*C$ is symmetric, using (9.28). Thus if $AC = CB$ then $A^*C = CB^*$, showing that T has a strong functorial star. Finally we prove that the equation $1^* = 1$ holds in S. Indeed, since $1 + 1 = 1$, $1^* \leq 1$. On the other hand, $1 \leq 1^* + 1 = 1^*$ by the star fixed point identity.

Corollary 9.7.15 *The semiring $RL(X^*)$ is the free Kozen semiring on the set X.*

Proof. The semiring $RL(X^*)$ is a Kozen semiring. The rest follows by Corollary 9.7.12 and Proposition 9.7.14.

A simplified version of the Kozen semiring axioms is treated in the following exercise.

Exercise 9.7.16 [Koz90, Pra90] Show that a *-semiring S is a Kozen semiring if and only if the following hold for all a and x in S:

$$\begin{array}{rcl} a+a & = & a \\ aa^*+1 & = & a^* \\ a^*a+1 & = & a^* \\ ax \ \leq \ x & \Rightarrow & a^*x \ \leq \ x \\ xa \ \leq \ x & \Rightarrow & xa^* \ \leq \ x. \end{array}$$

Recall the star GA-implication and the dual star GA-implication defined in Definition 9.3.18 and Corollary 9.3.26.

Definition 9.7.17 *A* **Gorshkov-Arkhangelskii semiring** *is an ω-idempotent Conway semiring which satisfies the star GA-implication and the dual star GA-implication.*

Corollary 9.7.18 *The semiring $RL(X^*)$ is the free Gorshkov-Arkhangelskii semiring on the set X.*

Proof. Each Gorshkov-Arkhangelskii semiring is an ω-idempotent symmetric iteration semiring. Further, for any set X, the semiring $RL(X^*)$ is a Gorshkov-Arkhangelskii semiring. Indeed, $RL(X^*)$ satisfies both the star GA-implication and the dual star GA-implication.

A simplified version of the Gorshkov-Arkhangelskii semiring axioms is given in the following exercise.

Exercise 9.7.19 [AG87] Suppose that S is a *-semiring. Show that S is a Gorshkov-Arkhangelskii semiring iff S satisfies the GA-implications and the following equations:

$$\begin{aligned}(ab)^*a &= a(ba)^* \\ 1^* &= 1.\end{aligned}$$

Recall the definition of an ideal in a semiring.

Definition 9.7.20 *Let S be a nontrivial idempotent semiring, so that $0 \neq 1$. We call S an* **idempotent iterative semiring***, or a* **Salomaa semiring***, if the following conditions hold:*

- *The elements $a \in S$ with $1 + a \neq a$ form an ideal, denoted $I(S)$.*

- *Each $s \in S \setminus I(S)$ can be written uniquely as*

$$s = 1 + a,$$

  where $a \in I(S)$.

- *For all $a \in I(S)$ and $s \in S$, the equation*

$$x = ax + s$$

  has a unique solution in S.

We note that, in virtue of the second condition, $I(S)$ is an ideal if and only if $I(S)$ is an additive submonoid of S which is closed under product. We note that if S is a Salomaa semiring then 1 is not in $I(S)$. Examples of Salomaa semirings are the semirings $L(X^*)$ and $RL(X^*)$, for any set X. In fact, both $L(X^*)$ and $RL(X^*)$ are symmetric idempotent iterative semirings, see below. The following result follows from the Matrix Extension Theorem, cf. Theorem 9.6.3.

Corollary 9.7.21 *Let S be an idempotent iterative semiring. There is a unique way to define the star operation on $\mathbf{Mat}_S$ such that $\mathbf{Mat}_S$ becomes a matrix iteration theory with $1^* = 1$. Further $\mathbf{Mat}_S$ satisfies the functorial star implication for all relational matrices.*

Thus, if S is a Salomaa semiring, then S is an ω-idempotent symmetric iteration semiring.

Corollary 9.7.22 *The semiring $RL(X^*)$ is the free Salomaa semiring on the set X. More precisely, for any Salomaa semiring S and for any function $\varphi : X \to I(S)$, there is a unique semiring homomorphism $\varphi^\sharp : RL(X^*) \to S$ extending φ. The semiring homomorphism $\varphi^\sharp$ preserves the star operation.*

Exercise 9.7.23 Suppose that S is a Salomaa semiring, so that S is partially ordered by the relation $a \leq b$ iff $a + b = b$. Show that for all $a, b \in S$, a^*b is the least solution of the equation

$$x \;=\; ax + b.$$

In fact, if $ax + b \leq x$, for some a, b and x in S, then $a^*b \leq x$.

Example 9.7.24 In this example, we show that if S is a Salomaa semiring, then S satisfies the star GA-implication.

Suppose that $a, b, c, d, e, f \in S$ with

$$(a+b)^*e \;=\; (c+d)^*f.$$

We must show that

$$(d + ca^*b)^*(f + ca^*e) \;=\; (a+b)^*e. \tag{9.30}$$

First note that $(a+b)^*e$ is a solution of the equation

$$x \;=\; (d + ca^*b)x + f + ca^*e. \tag{9.31}$$

Indeed,

$$\begin{aligned}
(d + ca^*b)(a+b)^*e + f + ca^*e &= d(a+b)^*e + c(a^*b(a+b)^* + a^*)e + f \\
&= d(a+b)^*e + c(a+b)^*e + f \\
&= (c+d)(a+b)^*e + f \\
&= (c+d)(c+d)^*f + f \\
&= (c+d)^*f \\
&= (a+b)^*e,
\end{aligned}$$

for

$$\begin{aligned}
a^*b(a+b)^* + a^* &= a^*b(a^*b)^*a^* + a^* \\
&= (a^*b)^+a^* + a^* \\
&= (a^*b)^*a^* \\
&= (a+b)^*.
\end{aligned}$$

Since $(1+x)^* = (1^*x)^*1^* = x^*$ holds for all $x \in S$, it is sufficient to prove (9.30) for the case that $a, d \in I(S)$. If in addition either $b \in I(S)$ or $c \in I(S)$, then (9.30) follows. Indeed, by the previous argument, $(d + ca^*b)^*(f + ca^*e)$ is the unique solution of (9.31). When $b, c \notin I(S)$, we may write $b = 1 + b_1$ and $c = 1 + c_1$, for some $b_1, c_1 \in I(S)$. We prove that $(a + b_1)^*e$ is the unique solution of the equation

$$x \quad = \quad (d + a^+ + a^*b_1 + c_1a^* + c_1a^*b_1)x + f + ca^*e. \tag{9.32}$$

Indeed,

$$\begin{aligned}
(a+b_1)^* &\leq (a^* + a^*b_1)(a+b_1)^* \\
&\leq (a+b_1)^*(a+b_1)^* \\
&= (a+b_1)^*,
\end{aligned}$$

so that

$$(a^* + a^*b_1)(a+b_1)^* \quad = \quad (a+b_1)^*$$

and

$$(a^+ + a^*b_1)(a+b_1)^* \quad \leq \quad (a+b_1)^*.$$

Thus, using the assumptions

$$(a+b_1)^*e \;=\; (a+b)^*e \;=\; (c+d)^*f \;=\; (c_1+d)^*f,$$

we have

$$\begin{aligned}
&(d + a^+ + a^*b_1 + c_1a^* + c_1a^*b_1)(a+b_1)^*e + f + ca^*e = \\
&\quad = (d + a^+ + a^*b_1 + c_1a^* + c_1a^*b_1)(a+b_1)^*e + f + a^*e \\
&\quad = (d + a^+ + a^*b_1)(a+b_1)^*e + (c_1a^* + c_1a^*b_1)(a+b_1)^*e + f + a^*e \\
&\quad = (d + a^+ + a^*b_1)(a+b_1)^*e + c_1(a+b_1)^*e + f + a^*e \\
&\quad = (c_1+d)(a+b_1)^*e + f + (a^+ + a^*b_1)(a+b_1)^*e + a^*e \\
&\quad = (c_1+d)(c_1+d)^*f + f + (a^+ + a^*b_1)(a+b_1)^*e + a^*e \\
&\quad = (c_1+d)^*f + (a^+ + a^*b_1)(a+b_1)^*e + a^*e \\
&\quad = (a+b_1)^*e + (a^+ + a^*b_1)(a+b_1)^*e + a^*e \\
&\quad = (a+b_1)^*e.
\end{aligned}$$

Thus,

$$\begin{aligned}
(d + ca^*b)^*(f + ca^*e) &= (d + a^+ + a^*b_1 + c_1a^* + c_1a^*b_1)^*(f + ca^*e) \\
&= (a+b_1)^*e \\
&= (a+b)^*e,
\end{aligned}$$

since the equation (9.32) has a unique solution.

Exercise 9.7.25 When both S and S^{op} are idempotent iterative semirings, we call S a *symmetric idempotent iterative semiring.* Thus, if S is a symmetric idempotent iterative semiring, the equation

$$x = xa + b \tag{9.33}$$

has a unique solution, for all $a \in I(S)$ and $b \in S$. Conclude that the following are true in S.

- For all a, b and x in S, if $xa + b \leq x$ then $ba^* \leq x$. In particular, the equation (9.33) has a least solution, namely ba^*.
- The dual star GA-implication holds in S.

Thus every symmetric idempotent iterative semiring is both a Kozen semiring and a Gorshkov-Arkhangelskii semiring.

Exercise 9.7.26 Find an example of an idempotent iterative semiring which is not symmetric. *Hint:* Consider subsets of X^∞.

Exercise 9.7.27 An ω-complete semiring is ω-idempotent if $1^* = 1$, i.e. when it is ω-idempotent as an iteration semiring. Show that when S is ω-idempotent, $\sum_{i \in I} a_i = a$, whenever $a_i = a$ for all i in a countable nonempty set I. Show that every ω-complete ω-idempotent semiring is a Kozen semiring.

Exercise 9.7.28 Show that for any set X, there is an injective ω-additive semiring homomorphism from $L(X^*)$ to the ω-complete semiring of all binary relations on the set X^*. (A semiring homomorphism between ω-complete semirings is ω-additive if it preserves all countable sums.) It follows that the variety generated by the *-semirings of binary relations on a set is the class of ω-idempotent symmetric iteration semirings.

Problem 9.7.1 Give an explicit description of the free symmetric iteration semiring on a set X.

Problem 9.7.2 Does there exist an iteration semiring which is not a symmetric iteration semiring?

9.8 Notes

The star forms of most of the iteration theory identities considered in Theorem 9.2.1 were derived in [Ste87b]. The **C** groups of Corollaries 9.3.2 and 9.3.3 appear in [BÉc]. In his book [Con71], J.C. Conway already studied some consequences of the star sum and star product identities. Conway semirings were explicitly introduced in [BÉc]. The fact that ω-complete semirings give rise to iteration semirings having a strong functorial star should be considered well-known, but perhaps not in this form (see e.g. [Con71, Heb90].) The observation that Kleene's theorem holds for all Conway semirings in the form given by Corollary 9.4.3 is new (but perhaps was already known to Conway.) Particular instances of the result are well-known in the theory of automata. The initial iteration semiring was described in [BÉc]. The present proof using the star GA-implication is new. The material of Section 9.6 is taken from [BÉc]. Section 9.7 is based on [BÉa]. A complete equational axiomatization of regular *tree* languages based on the minimization of finite tree automata was outlined in [Ési81]. A simplified system, which applies to regular word languages, appears in [BÉ84]. The present set of equational axioms in Theorem 9.7.10 is different from that given in [BÉ84] in that the "power set axiom" is replaced by the dual star commutative identity. The proof of Theorem 9.7.10 explains the role of the commutative identities. While the Conway identities are sufficient for a nondeterministic version of Kleene's theorem, the commutative identities are used to capture in an equational way the process of obtaining a minimal deterministic automaton from an nfa. The concept of Salomaa semiring is abstracted from [Sal66]. Indeed, Corollary 9.7.22 is an algebraic counterpart of Salomma's axiomatization of regular sets. Corollary 9.7.15 is due to Dexter Kozen, cf. [Koz90]. The results of Corollary 9.7.18 and Exercise 9.7.19 were announced in [AG87].

Recently, Daniel Krob [Kro91] has published a result which is apparently stronger than our Theorem 9.7.10. Verifying a conjecture of Conway, he has shown that a complete equational axiomatization of the equational class generated by all iteration semirings $RL(X^*)$ consists of the Conway axioms, the equation $1^* = 1$, as well as some particular instances of the commutative identity, called *group identities*. Suppose that $G = \{g_1, \ldots, g_n\}$ is a finite (simple) group with identity element

g_1. Let $a_1, \ldots, a_n$ be variables corresponding to the group elements. The *group identity* corresponding to G is the equation

$$1_n \cdot {M_G}^* \cdot \tau \quad = \quad (a_1 + \ldots + a_n)^*,$$

where τ is the base matrix $n \to 1$, and where M_G is the Cayley matrix of G, i.e. the (i, j)-th entry of M_G is a_k iff $g_i g_k = g_j$. He also improved Kozen's result by showing that the Conway axioms, the equation $1^* = 1$ (or the equation $1 + 1 = 1$), together with the implication

$$a^2 \; = \; a \quad \Rightarrow \quad a^* \; = \; 1 + a$$

form another complete system of axioms.

Chapter 10

Matricial Iteration Theories

Recall the definition of matricial theories from Section 3.3.5.3. Since each matricial theory can be represented as a theory $\mathbf{Matr}(S;V)$, for some semiring module pair $(S;V)$, we will be considering only matricial theories of this sort. A matricial iteration theory is a matricial theory which is simultaneously an iteration theory. We show that when $T = \mathbf{Matr}(S;V)$ is a matricial iteration theory, the dagger operation determines and is determined by a star operation on the semiring S and an *omega operation* $^{\omega} : S \to V$ from the semiring to the module.

Notation: when $T = \mathbf{Matr}(S;V)$, we write $T\P$ for the underlying matrix theory $\mathbf{Mat}_S$.

10.1 From Dagger to Star and Omega, and Back

Suppose that $T = \mathbf{Matr}(S;V)$ is simultaneously a matricial theory and a preiteration theory. If f is any morphism $n \to n+p$, we write

$$f \quad = \quad ([a \ \ b];\ x),$$

where a is an $n \times n$ matrix over S, b is $n \times p$ and where x is an n-vector over V, which one may identify with an $n \times 1$ matrix over V. Hence the dagger operation applied to f produces $f^\dagger = (c; z)$ where c is an $n \times p$ matrix over S and where z is an n-vector over V. Two operations are implicitly defined by the operation $f \mapsto f^\dagger$. For each square matrix $a : n \to n$ in $\mathbf{Mat}_S$, let $f_a : n \to n+n$ in T be defined as follows:

$$f_a \quad := \quad ([a \ \ \mathbf{1}_n];\ 0).$$

Then,

$$f_a{}^\dagger \quad := \quad (a^*; a^\omega). \tag{10.1}$$

Thus, $a \mapsto a^*$ is a map $\mathbf{Mat}_S(n,n) \to \mathbf{Mat}_S(n,n)$, and $a \mapsto a^\omega$ is a map from $\mathbf{Mat}_S(n,n)$ to V^n, for each $n \geq 0$.

By using the same methods as those in Chapter 9, we can prove the following theorem.

Theorem 10.1.1 *Let $T = \mathbf{Matr}(S; V)$ be a matricial preiteration theory. Suppose that the star and omega operations are defined by (10.1). Then the parameter identity holds in T if and only if the dagger operation is determined by the star and omega operations:*

$$f^\dagger \quad = \quad (a^*b;\ a^*x + a^\omega), \tag{10.2}$$

for all $f = ([a \ \ b];\ x) : n \to n+p$ with $a : n \to n$ and $b : n \to p$ in T¶.

Suppose that the dagger, star and omega operations are related by the equations (10.1) and (10.2).

[a] *The star fixed point identity and the* **omega fixed point identity**

$$a \cdot a^\omega \ = \ a^\omega, \qquad a : n \to n \in T¶, \tag{10.3}$$

hold in T if and only if the fixed point identity holds.

[b] *The star product identity and the* **omega product identity**

$$(ab)^\omega \ = \ a \cdot (ba)^\omega, \tag{10.4}$$

$a : n \to m$, $b : m \to n \in T$¶, hold in T if and only if the composition identity holds.

[c] *The star sum identity and the* **omega sum identity**

$$(a+b)^\omega \;=\; (a^*b)^\omega + (a^*b)^* \cdot a^\omega, \tag{10.5}$$

$a, b : n \to n \in T\P$, hold in T if and only if the double dagger identity holds.

[d] *The star zero identity and the* **omega zero identity**

$$0^\omega_{nn} \;=\; 0^n \in V^n \tag{10.6}$$

hold in T if and only if the left zero identity holds.

[e] *The star pairing identity and the* **omega pairing identity** *(10.7) hold in T if and only if the pairing identity holds.*

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\omega &= \\ &= \begin{bmatrix} a^*b \cdot (d+ca^*b)^\omega + a^*b(d+ca^*b)^*c \cdot a^\omega + a^\omega \\ (d+ca^*b)^\omega + (d+ca^*b)^*c \cdot a^\omega \end{bmatrix} \end{aligned} \tag{10.7}$$

all $a:n \to n$, $b:n \to m$, $c:m \to n$ and $d:m \to m$ in $T\P$.

[f] *The star permutation identity and the* **omega permutation identity** *(10.8) hold in T if and only if the permutation identity holds.*

$$(\pi \cdot a \cdot \pi^T)^\omega \;=\; \pi \cdot a^\omega, \tag{10.8}$$

where $\pi : n \to n$ is any base permutation and $a : n \to n$ in $T\P$.

[g] *The star commutative identity and the* **omega commutative identity** *(10.9) hold in T if and only if the commutative identity holds.*

$$((\rho \cdot a) \parallel (\rho_1, \ldots, \rho_m))^\omega \;=\; \rho \cdot (a \cdot \rho)^\omega, \tag{10.9}$$

where $a : n \to m \in T\P$, and where ρ and ρ_i are as in the star commutative identity.

Conversely, if T is a matricial theory equipped with star and omega operations defined for all square matrices in $T\P$, and if the dagger operation is defined by the equation (10.2), then the preiteration theory T satisfies the parameter identity and all of the above equivalences hold.

The notation $(\rho \cdot a) \parallel (\rho_1, \ldots, \rho_m)$ was introduced in Section 9.9.1. The following fact is immediately seen.

Proposition 10.1.2 *Let T and T' be matricial theories. Suppose that each of the dagger, star and omega operations is defined in both theories T and T' and that these operations are related by the equations (10.1) and (10.2). Then a matricial theory morphism $\varphi : T \to T'$ preserves dagger if and only if φ preserves star and omega.*

For later use we define an *omega theory* to be a matricial theory T such that both star and omega are defined on the morphisms in T¶. A morphism of omega theories is a matricial theory morphism which preserves the star and omega operations. Thus, the underlying matrix theory of an omega theory is a star theory and any morphism of omega theories is also a morphism of the underlying star theories.

Remark 10.1.3 If the star and omega sum and product identities hold, then the omega pairing identity can be expressed in either of the following two forms:

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\omega &= \begin{bmatrix} (a+bd^*c)^\omega + (a+bd^*c)^*b \cdot d^\omega \\ (d+ca^*b)^\omega + (d+ca^*b)^*c \cdot a^\omega \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\omega &= \begin{bmatrix} (a^*bd^*c)^* \cdot a^\omega + (a^*bd^*c)^*a^*b \cdot d^\omega + (a^*bd^*c)^\omega \\ (d^*ca^*b)^*d^*c \cdot a^\omega + (d^*ca^*b)^* \cdot d^\omega + (d^*ca^*b)^\omega \end{bmatrix}.
\end{aligned}$$

Exercise 10.1.4 Prove Theorem 10.1.1.

10.2 Matricial Iteration Theories Defined

Definition 10.2.1 *A* **matricial iteration theory** *is a matricial theory which is also an iteration theory. A* **Conway matricial theory** *is a matricial theory which is a Conway theory. A morphism of matricial iteration theories or Conway matricial theories is a matricial theory morphism which preserves the dagger operation.*

The following results are immediate from Theorem 10.1.1, Proposition 10.1.2, Theorem 9.9.2.1 and the axiomatization results of Chapter 6.

Corollary 10.2.2 *Let T be a matricial theory. Suppose that either T is an omega theory in which the dagger operation is defined by (10.2), or that T is a preiteration theory satisfying the parameter identity in which the star and omega operations are defined by (10.1). Then T is a matricial iteration theory if and only if T, as an omega theory, satisfies either of the following groups of equational axioms:*

The **A** *group:*

- *the star zero identity and the omega zero identity (10.6);*
- *the star pairing identity and the omega pairing identity (10.7);*
- *the star commutative identity and the omega commutative identity (10.9).*

The **B** *group:*

- *the star product identity and the omega product identity (10.4);*
- *the star sum identity and the omega sum identity (10.5);*
- *the star commutative identity and the omega commutative identity (10.9).*

The **C** *group:*

- *the scalar star product identity and the scalar omega product identity (10.4) with $n = m = 1$;*
- *the scalar star sum identity and the scalar omega sum identity (10.5) with $n = 1$;*
- *the scalar star pairing identity and the scalar omega pairing identity (10.7) with $m = 1$;*
- *the scalar star commutative identity and the scalar omega commutative identity*

  $$1_m \cdot ((\rho \cdot a) \parallel (\rho_1, \ldots, \rho_m))^\omega \quad = \quad 1_n \cdot (a \cdot \rho)^\omega, \quad (10.10)$$

  where $a : n \to m \in T\P$, $\rho : m \to n$ is a monotone base surjection, and where $\rho_i : m \to m$ is a base morphism with $\rho_i \cdot \rho = \rho$, for each $i \in [m]$.

In either case, the dagger, star and omega operations are related by equations (10.1) and (10.2). A matricial theory morphism between matricial iteration theories is a matricial iteration theory morphism if and only if it is an omega theory morphism.

We note that in group **B**, the omega commutative identity may be replaced by the scalar omega commutative identity.

Corollary 10.2.3 *Let T be a matricial theory. Suppose that either T is an omega theory in which the dagger operation is defined by (10.2), or that T is a preiteration theory satisfying the parameter identity in which the star and omega operations are defined by (10.1). Then T is a Conway matricial theory if and only if either of the following three groups of equations holds in T:*

The **A** *group:*

- *the star zero identity and the omega zero identity (10.6);*
- *the star pairing identity and the omega pairing identity (10.7);*
- *the star permutation identity and the omega permutation identity (10.8).*

The **B** *group:*

- *the star product identity and the omega product identity (10.4);*
- *the star sum identity and the omega sum identity (10.5);*

The **C** *group:*

- *the scalar star product identity and the scalar omega product identity (10.4) with $n = m = 1$;*
- *the scalar star sum identity and the scalar omega sum identity (10.5) with $n = 1$;*
- *the scalar star pairing identity and the scalar omega pairing identity (10.7) with $n = 1$;*

The dagger, star and omega operations are related by equations (10.1) and (10.2). A matricial theory morphism between Conway matricial theories is a Conway matricial theory morphism if and only if it is an omega theory morphism.

Clearly, any matricial iteration theory is a Conway matricial theory, and a Conway matricial theory is an iteration theory iff the (scalar) commutative identities hold. Note that the star and omega fixed point identities hold in any Conway matricial theory.

The following corollary follows from the results of Chapter 9.

Corollary 10.2.4 *If T is a matricial iteration theory, then $T\P$ is a matrix iteration theory. Similarly, $T\P$ is a Conway matrix theory, when $T\P$ is a Conway matricial theory.*

Example 10.2.5 If $\mathbf{Matr}(S;V)$ is a matricial theory such that $\mathbf{Mat}_S$ is a star theory, we may turn $\mathbf{Matr}(S;V)$ to an omega theory by defining $a^\omega := 0^n$, for all $a : n \to n$ in $\mathbf{Mat}_S$. When $\mathbf{Mat}_S$ is a matrix iteration theory, $\mathbf{Matr}(S;V)$ is a matricial iteration theory. Similarly, if $\mathbf{Mat}_S$ is a Conway matrix theory, $\mathbf{Matr}(S;V)$ is a Conway matricial theory. In particular, any matrix iteration theory $\mathbf{Mat}_S$ may be viewed as the matricial iteration theory $\mathbf{Matr}(S;V)$ where $V = \{0\}$ is the trivial S-module.

When $T = \mathbf{Matr}(S;V)$ is a matricial theory, any morphism $1 \to 0$ may be identified with an element of V. Similarly, each morphism $1 \to 1$ in the underlying matrix theory $T\P$ may be considered to be an element of the semiring S. Thus, when the pairing identities hold, the star and omega operations are determined by operations $^* : S \to S$ and $^\omega : S \to V$.

Recall that a Conway semiring was defined in Chapter 9 as a *-semiring S in which the equations

$$
\begin{aligned}
(ab)^* &= 1 + a(ba)^*b && (10.11)\\
(a+b)^* &= (a^*b)^*a^*, && (10.12)
\end{aligned}
$$

are valid.

Definition 10.2.6 *A* **Conway semiring module pair** *consists of a Conway semiring S, an S-module V and an operation ${}^{\omega} : S \to V$ which satisfies the two equations*

$$(ab)^{\omega} = a \circ (ba)^{\omega} \tag{10.13}$$

$$(a+b)^{\omega} = (a^*b)^* \circ a^{\omega} + (a^*b)^{\omega}, \tag{10.14}$$

for all a,b in S. An **iteration semiring module pair** *consists of an iteration semiring S, an S-module V and an operation ${}^{\omega} : S \to V$ which satisfies (10.13) and (10.14) and the scalar omega commutative identity, where the omega operation on $n \times n$ matrices over S is defined by the scalar omega pairing identity. A morphism $\varphi : (S; V) \to (S'; V')$ of iteration or Conway semiring module pairs is determined by a *-semiring semiring homomorphism $\varphi_S : S \to S'$ and a monoid homomorphism $\varphi_V : V \to V'$ such that*

$$\begin{aligned} (s \circ v)\varphi_V &= s\varphi_S \circ v\varphi_V \\ (s^{\omega})\varphi_V &= (s\varphi_S)^{\omega}. \end{aligned}$$

The proof of the following facts is left as an exercise.

Proposition 10.2.7 [a] *When $T = \mathbf{Matr}(S; V)$ is a matricial iteration theory, $(S; V)$ is an iteration semiring module pair, and when T is a Conway matricial theory, $(S; V)$ is a Conway semiring module pair.*

[b] *Let $(S; V)$ be an iteration (or Conway) semiring module pair. There is a unique way to extend the * and ${}^{\omega}$ operations on S to all square matrices in $\mathbf{Mat}_S$ so that $\mathbf{Matr}(S; V)$ becomes a matricial iteration theory (or Conway matricial theory, respectively).*

[c] *Suppose that $T = \mathbf{Matr}(S; V)$ and $T' = \mathbf{Matr}(S'; V')$ are matricial iteration theories and that $\varphi : T \to T'$ is a matricial iteration theory morphism. Let φ_S be the restriction of φ to S and φ_V the restriction of φ to V. Then the pair (φ_S, φ_V) is a morphism between the iteration semiring module pairs $(S; V)$ and $(S'; V')$. Conversely, if (φ_S, φ_V) is a morphism $(S; V) \to (S'; V')$, then there is a unique matricial iteration theory morphism $\varphi : T \to T'$*

whose restriction to S and V determines the pair (φ_S, φ_V). This unique matricial iteration theory morphism is defined by

$$([a_{ij}];\, v_i) \quad \mapsto \quad ([a_{ij}\varphi_S];\, v_i\varphi_V),$$

for all $([a_{ij}];\, v_i) : n \to p$ in T with $a_{ij} \in S$ and $v_i \in V$, $i \in [n]$, $j \in [p]$. A similar fact holds for Conway matricial theories.

Definition 10.2.8 *A* **submatricial iteration theory** *of a matricial iteration theory T is a submatricial theory of T which is a subpreiteration theory. Similarly, a sub Conway matricial theory of a Conway matricial theory is just a submatricial theory which is closed under the dagger operation.*

Proposition 10.2.9 *Suppose that T' is a submatricial theory of the matricial iteration theory $T = \mathbf{Matr}(S;V)$. Then T' is a submatricial iteration theory of T iff T' is closed under the star and omega operations. When T' is a submatricial iteration theory of T, the pair $(S';V')$, where $S' := T'(1,1)$ and $V' := T'(1,0)$, is an iteration semiring module pair, and T' is the theory $\mathbf{Matr}(S';V')$. Conversely, if S' is a sub *-semiring of S, V' is a submonoid of V closed under the action of elements of S', and if a^ω is in V', for all a in S', then $\mathbf{Matr}(S';V')$ is a submatricial iteration theory of T. Similar facts are true for Conway matricial theories.*

In any Conway matricial theory, the commutative identities follow from the functorial star and omega conditions defined below.

Definition 10.2.10 *Let $T = \mathbf{Matr}(S;V)$ be an omega theory and let be a set of morphisms in T¶. We say that T* **satisfies the functorial star implication for** *, or that T* **has a functorial star with respect to** *, if for all $a : n \to n$, $b : m \to m$ in T¶ and all $c : n \to m$ in ,*

$$a \cdot c \;=\; c \cdot b \quad \Rightarrow \quad a^* \cdot c \;=\; c \cdot b^*.$$

We say that T **satisfies the functorial omega implication for** *, or that T* **has a functorial omega with respect to** *, if for all a, b and c as above,*

$$a \cdot c \;=\; c \cdot b \quad \Rightarrow \quad a^\omega \;=\; c \cdot b^\omega.$$

When T has a functorial star and omega with respect to the set of all morphisms in $T¶$, T is said to have a **strong functorial star and omega**.

The proof of Proposition 10.2.11 is the same as that of Propositions 9.9.3.12-9.9.3.16.

Proposition 10.2.11 *Suppose that T is a Conway matricial theory.*

[a] *For any set $\subseteq T¶$, T has a functorial dagger with respect to if and only if T has a functorial star and omega with respect to .*

[b] *T has a functorial star and omega with respect to all base injections.*

[c] *If T has a functorial star and omega with respect to all base surjections, then the star and omega commutative identities hold in T.*

[d] *T has a functorial star and omega with respect to all base surjections if and only if T has a functorial star and omega with respect to all base morphisms $n \to 1$, $n \geq 1$.*

Corollary 10.2.12 *Any Conway theory which satisfies the functorial star and omega implications with respect to all base surjections, or all base morphisms $n \to 1$, $n \geq 1$, is a matricial iteration theory.*

Exercise 10.2.13 Prove Proposition 10.2.7 and Proposition 10.2.11.

Definition 10.2.14 *Suppose that $T = \mathbf{Matr}(S;V)$ is a Conway matricial theory, so that $(S;V)$ is a Conway semiring module pair. We say that T or $(S;V)$ satisfies the* **star GA-implication** *if $T¶$ does, i.e. when*

$$\begin{aligned} (a_1+a_2)^*a_3 \;&=\; (b_1+b_2)^*b_3 \;=\; c \\ \Rightarrow\quad & (b_2+b_1a_1^*a_2)^*(b_3+b_1a_1^*a_3) \;=\; c, \end{aligned}$$

for all $a_i, b_i, c \in S$, $i = 1,2,3$. Similarly, we say that T or $(S;V)$ satisfies the **omega GA-implication** *if for all $a_1, a_2, b_1, b_2 \in S$ and $u, v, w \in V$ such that*

$$(a_1+a_2)^\omega + (a_1+a_2)^*u \;=\; (b_1+b_2)^\omega + (b_1+b_2)^*v \;=\; w,$$

it follows that

$$(b_2 + b_1a_1^*a_2)^\omega + (b_2 + b_1a_1^*a_2)^*(b_1a_1^\omega + b_1a_1^*u + v) \quad = \quad w.$$

We will sometimes say that the star or omega GA-implication holds in T.

Proposition 10.2.15 *The GA-implication holds in a Conway matricial theory T iff both the star and omega GA-implications hold.*

Proof. It suffices to rephrase the form of the GA-implication for morphisms

$$f_1 \;=\; ([a_1, a_2, a_3]; u) \quad \text{and} \quad f_2 \;=\; ([b_1, b_2, b_3]; v)$$

in terms of the star and omega operations.

Corollary 10.2.16 *If the star and omega GA-implications hold in a Conway matricial theory T, then T is a matricial iteration theory which has a functorial star and omega with respect to all (surjective) base morphisms.*

Thus, when $(S; V)$ is a Conway semiring module pair which satisfies the star and omega GA-implications, $(S; V)$ is an iteration semiring module pair.

In the following exercises, T denotes an omega theory $\mathbf{Matr}(S; V)$. In Exercises 10.2.19-10.2.21, we assume that T is a Conway matricial theory.

Exercise 10.2.17 Prove that if T has a functorial star and omega with respect to base injections and base surjections, then T has a functorial star and omega with respect to all base morphisms.

Exercise 10.2.18 Recall the dual star commutative identity in Corollary 9.9.3.26. The *dual omega commutative identity* is the equation

$$\rho^T \cdot ((\rho_1^T, \ldots, \rho_m^T) \parallel (a \cdot \rho^T))^\omega \quad = \quad (\rho^T \cdot a)^\omega, \tag{10.15}$$

where $a = [a_1, \ldots, a_n] : m \to n$, $\rho : m \to n$ is a surjective base matrix with transpose $\rho^T : n \to m$, and $\rho_i : m \to m$, $i \in [m]$, are base, with $\rho_i \cdot \rho = \rho$, for each $i \in [m]$. Show that if T has a functorial star and omega with respect to transposes of base surjections, then the dual star and omega commutative identities hold in T.

Exercise 10.2.19 Prove that if T has a functorial star and omega with respect to transposes of all base morphisms $n \to 1$, $n \geq 1$, then T has a functorial star and omega with respect to transposes of all base surjections.

Exercise 10.2.20 Show that if T has a functorial omega with respect to transposes of injective base morphisms, then the omega operation is trivial, i.e. $a^\omega = 0_{n0}$, for all $a : n \to n$ in $T\P$.

Exercise 10.2.21 Recall the generalized star commutative identity and the generalized dual star commutative identity from Exercises 9.9.3.36 and 9.9.3.37. The *generalized omega commutative identity* is the equation

$$((\rho \cdot a) \parallel (\rho_1, \ldots, \rho_m))^\omega \quad = \quad \rho \cdot (a \parallel (\tau_1, \ldots, \tau_n))^\omega, \tag{10.16}$$

where $a : n \to k$ in $T\P$, $\rho : m \to n$ is surjective base, and where $\rho_i : k \to m$, $\tau_j : k \to n$, $i \in [m]$, $j \in [n]$, are base with $\rho_i \cdot \rho = \tau_{i\rho}$.

The *generalized dual omega commutative identity* is the equation

$$\rho^T \cdot ((\rho_1^T, \ldots, \rho_m^T) \parallel (b \cdot \rho^T))^\omega \quad = \quad ((\tau_1^T, \ldots, \tau_n^T) \parallel b)^\omega, \tag{10.17}$$

where $b : k \to n$ in $T\P$ and where ρ, ρ_i and τ_j are as in (10.16).

Show that the star and omega commutative identities hold in T if and only if the generalized star and omega commutative identities hold. Prove the corresponding statement for the (generalized) dual commutative identities.

Exercise 10.2.22 Let S be an ω-complete semiring and V an S-module. Then $\mathbf{Matr(}S;V\mathbf{)}$ is a matricial iteration theory where the star and omega operations are defined as follows:

$$a^* \;:=\; \sum_{k=0}^{\infty} a^k \quad \text{and} \quad a^\omega \;:=\; 0_{n0},$$

for all $a : n \to n$ in $\mathbf{Mat}_S$. The theory $\mathbf{Matr(}S;V\mathbf{)}$ has a strong functorial star and omega but when V is nontrivial, $\mathbf{Matr(}S;V\mathbf{)}$ does not have a strong functorial dagger.

Exercise 10.2.23 Let $T = \mathbf{Matr(}S;V\mathbf{)}$ be a Conway matricial theory. We say that T or $(S;V)$ satisfies the *dual star GA-implication* if $T\P$ does, i.e.

$$a_3(a_1 + a_2)^* \;=\; b_3(b_1 + b_2)^* \;=\; c \quad \Rightarrow \quad (b_3 + a_3 a_1^* b_1)(b_2 + a_2 a_1^* b_1)^* \;=\; c,$$

for all $a_i, b_i, c \in S$. Further, we say that T or $(S; V)$ satisfies the *dual omega GA-implication* if for all $a_i, b_i \in S$ as above and for all $u, v \in V$, if

$$a_3(a_1 + a_2)^* \quad = \quad b_3(b_1 + b_2)^*$$

and

$$a_3(a_1 + a_2)^\omega + u \;=\; b_3(b_1 + b_2)^\omega + u \;=\; v$$

then

$$(b_3 + a_3 a_1^* b_1)((b_2 + a_2 a_1^* b_1)^\omega + (b_2 + a_2 a_1^* b_1)^* a_2 a_1^\omega) + a_3 a_1^\omega + u \quad = \quad v.$$

Again, we will sometimes say that the dual star GA-implication or the dual omega GA-implication holds in T or in $(S; V)$.

Show that when the dual star GA-implication holds in $T = \mathbf{Matr}(S; V)$, then T satisfies the dual omega GA-implication iff for all $a_i, b_i \in S$, $i = 1, 2, 3$, and for all $x, y, z, v \in V$, if

$$a_3(a_1 + a_2)^* \quad = \quad b_3(b_1 + b_2)^*$$

and

$$\begin{aligned} & a_3((a_1 + a_2)^\omega + (a_1 + a_2)^*(x + y)) + z \;= \\ = \;\; & b_3((b_1 + b_2)^\omega + (b_1 + b_2)^*(x + y)) + z \;=\; v \end{aligned}$$

then

$$\begin{aligned} & (b_3 + a_3 a_1^* b_1)((b_2 + a_2 a_1^* b_1)^\omega \\ + \;\; & (b_2 + a_2 a_1^* b_1)^*(a_2 a_1^\omega + a_2 a_1^* x + y)) + a_3 a_1^\omega + a_3 a_1^* x + z \\ = \;\; & v. \end{aligned}$$

Exercise 10.2.24 Suppose that $T = \mathbf{Matr}(S; V)$ is a Conway matricial theory which satisfies the dual star and omega GA-implications. Prove that T has a functorial star and omega with respect to all transposes of base morphisms $n \to 1$, $n \geq 1$. Hence T has a functorial star and omega with respect to the transposes of surjective base morphisms.

Hint: Suppose that $A : n \to n$ and $a : 1 \to 1$ in $\mathbf{Mat}_S$ with

$$a \cdot \tau^T \quad = \quad \tau^T \cdot A,$$

where τ denotes the base morphism $n \to 1$. Thus the sum of the entries of each column of A is a. Define a sequence of morphisms

$$f_i \,:\, n \to n - i + 1, \quad i = 1, \ldots, n + 1,$$

by induction on i. Let

$$f_1 \quad := \quad (A; 0^n),$$

and for all $i \in [n]$,

$$j_n \cdot f_{i+1} \quad := \quad \begin{cases} (i_n \cdot f_i)^\dagger & \text{if } i = j; \\ j_n \cdot f_i \cdot \langle (i_n \cdot f_i)^\dagger, \mathbf{1}_{n-i} \rangle & \text{if } j \neq i. \end{cases}$$

Then show

$$A^\omega \;=\; g_1^\dagger \;=\; g_2^\dagger \;=\; \ldots \;=\; g_{n+1}^\dagger \;=\; f_{n+1},$$

where

$$g_i \quad := \quad 0_{i-1} \oplus f_i : n \to n, \quad i = 1, \ldots, n+1.$$

Then prove the following. Assume that $i \in [n]$ is any fixed integer and write

$$f_i \quad = \quad (B; x)$$

where B is an $n \times (n - i + 1)$ matrix in $\mathbf{Mat}_S$ and x is an n-vector with components $x_1, \ldots, x_n$. Suppose that $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is some column of the matrix B. Show by induction on i that

$$a^* \quad = \quad (1 + a_1 + \ldots + a_{i-1})(a_i + \ldots + a_n)^*$$

and

$$\begin{aligned} a^\omega \;&= \\ &= \;(1 + a_1 + \ldots + a_{i-1})((a_i + \ldots + a_n)^\omega \\ &\quad + (a_i + \ldots + a_n)^*(x_i + \ldots + x_n)) + \; x_1 + \ldots + x_{i-1}. \end{aligned}$$

Use the previous exercise in the induction step. Thus, when $i = n$,

$$f_n \quad = \quad \begin{pmatrix} d_1; & w_1 \\ \vdots & \vdots \\ d_n; & w_n \end{pmatrix}$$

with

$$a^* \quad = \quad (1 + d_1 + \ldots + d_{n-1}) d_n^*$$

and

$$a^\omega \quad = \quad (1 + d_1 + \ldots + d_{n-1})(d_n^\omega + d_n^* w_n) + w_1 + \ldots + w_{n-1}.$$

Thus, writing

$$f_{n+1} = \begin{pmatrix} & z_1 \\ ; & \vdots \\ & z_n \end{pmatrix},$$

we have

$$\begin{aligned} z_j &= d_j d_n^\omega + d_j d_n^* w_n + w_j, \quad j \neq n \\ z_n &= d_n^\omega + d_n^* w_n. \end{aligned}$$

Thus

$$\begin{aligned} \tau^T \cdot A^\omega &= \tau^T \cdot f_{n+1} \\ &= z_1 + \ldots + z_n \\ &= a^\omega. \end{aligned}$$

Exercise 10.2.25 Show that if T is a matricial iteration theory and if $a : n \to n$ in $T\P$, then $(a^k)^\omega = a^\omega$, for all $k \geq 1$.

10.3 Examples

We will describe several matricial iteration theories, usually by giving the semiring module pairs $(S; V)$ together with the corresponding star and omega operations. All of these matricial iteration theories are idempotent.

10.3.1 A

The first example in this section involves theories of sequacious relations, defined in Section 3.3.5.4. The theory A is a pointed theory, where $0 : 1 \to 0$ is the sequacious relation with the property that $(x, 1)\ 0\ w$ is always false, for any $x \in A$ and $w \in A^\omega$. We have shown that A is a matricial theory, isomorphic to $\mathbf{Matr}(S_A; V_A)$, where S_A is the semiring of finite sequacious relations $A^+ \cup A^\omega \to A^+ \cup A^\omega$, and where V_A is the S_A-module of sequacious relations $A^+ \cup A^\omega \to A^\omega$. From now on, we identify A with $\mathbf{Matr}(S_A; V_A)$.

In this matricial theory, as in all idempotent matricial theories, composition distributes over sum on both the left and the right. Note that S_A is an ω-complete semiring, where the addition operation is the union of relations. We will turn $\mathbf{Matr}(S_A;V_A)$ into a matricial preiteration theory by defining an omega operation, since the star operation is defined using the standard infinite sum:

$$a^* \quad := \quad \mathbf{1}_n + a + a^2 + \dots$$

for each $a : n \to n$ in the underlying matrix theory.

Choose an element $\perp \in A$.

Definition 10.3.1 *If $a : n \to n$ in $\mathbf{Mat}_{S_A}$, then $a^\omega \in (V_A)^n$ is the following n-tuple of infinite sequacious relations: for $u \in A^+$, $u\ (a^\omega)_i\ w$ iff there is an infinite sequence $(u_k, i_k) \in A^+ \times [n]$ such that $i_0 = i,\ u_0 = u$ and*

$$u_k\ a_{i_k,i_{k+1}}\ u_{k+1},$$

for all $k \geq 0$; and lastly,

$$w \;=\; (\lim_{k\to\infty} u_k) \cdot \perp^\omega.$$

Here $\perp^\omega$ denotes the infinite word each of whose letters is $\perp$, and the expression $(\lim_{k\to\infty} u_k) \cdot \perp^\omega$ denotes either the unique infinite word $\lim_{k\to\infty} u_k$ having each word u_k as a prefix, if the lengths of the words u_k are unbounded, or the infinite word $v \cdot \perp^\omega$ if $u_k = v$, for all but finitely many values of k.

Note that in $\mathbf{Matr}(S_A;V_A)$, the relation 1^ω, for $1 \in S_A$, is the infinite sequacious relation $1 \to 0$ such that

$$u\ 1^\omega\ (u\perp^\omega),$$

for all $u \in A^+$.

It is a tedious but straightforward matter to verify that the omega operation satisfies the omega pairing identity.

Exercise 10.3.2 Prove that the omega pairing identity holds in $\mathbf{Matr}(S_A;V_A)$.

Next, we show that $\mathbf{Matr}(S_A;V_A)$ is a matricial iteration theory. Since S_A is an ω-complete semiring, the star zero, pairing and commutative identities hold in $\mathbf{Matr}(S_A; V_A)$. The omega zero identity is easy to verify, and it remains to prove that the omega commutative identity holds by showing that $\mathbf{Matr}(S_A; V_A)$ satisfies the functorial omega implication for all surjective base matrices. In order to do this, we will apply Corollary 10.2.12. But first, we must show that this theory satisfies the omega permutation identity.

Proposition 10.3.3 *If $r : n \to n$ is any matrix over S_A and $\pi : n \to n$ is a base permutation, then*

$$(\pi \cdot r \cdot \pi^T)^\omega \quad = \quad \pi \cdot r^\omega.$$

The proof is straightforward, using the fact that for $u, v \in A^+$ and any base matrix π,

$$u\ (\pi \cdot r \cdot \pi^T)_{ij}\ v \quad \Leftrightarrow \quad u\ r_{i\pi, j\pi}\ v.$$

It now follows from Corollary 10.2.12 that in order to verify that the theory $\mathbf{Matr}(S_A; V_A)$ satisfies the functorial omega implication for surjective base matrices $n \to m$, it suffices to show that the appropriate implication holds only when $m = 1$.

Lemma 10.3.4 *Suppose that r is an $n \times n$ matrix over S_A and $s \in S_A$. Suppose that $\rho : n \to 1$ is the unique base matrix, and that*

$$r \cdot \rho \quad = \quad \rho \cdot s,$$

i.e. for each $i \in [n]$, the sum $\sum_j r_{ij}$ of the entries in row i of r is the relation s. Then

$$r^\omega \quad = \quad \rho \cdot s^\omega.$$

We omit the proof, but point out only that the hypothesis concerning r and s is this: for all $u, v \in A^+$,

$$\begin{aligned} &\forall i, j \in [n]\ \ (u\ r_{ij}\ v \quad \Rightarrow \quad u\ s\ v) \\ &\forall i \in [n]\ \ (u\ s\ v \quad \Rightarrow \quad \exists j \in [n]\ u\ r_{ij}\ v)\,. \end{aligned}$$

Corollary 10.3.5 *For each nonempty set A, $\mathbf{Matr}(S_A; V_A)$ is a matricial iteration theory satisfying the functorial star and omega implications for surjective base matrices.*

Remark 10.3.6 Any infinite sequence in A^ω can play the role of $\perp^\omega$. The resulting theory is also an iteration theory.

10.3.2 Ł$(X^*; X^\omega)$

Let X be a set and let $S = L(X^*)$ be the semiring in Example 1.1.1.1 whose elements are all subsets of X^*. Let $V := L(X^\omega)$ be the S-module whose elements are all subsets of infinite words on X. The action of S on V is given by set concatenation: for $a \subseteq X^*,\ y \subseteq X^\omega$,

$$a \circ y \quad := \quad \{uv \ : \ u \in a,\ v \in y\}.$$

The star operation on square matrices over S is given by the infinite sum, as always; the omega operation on 1×1 matrices over S yields the vector

$$a^\omega \quad := \quad \{u_0 u_1 \ldots \ : \ u_j \neq \epsilon,\ u_j \in a\} \quad = \quad \{u_0 u_1 \ldots \in X^\omega \ : \ u_j \in a\}.$$

When a is $n \times n$, a^ω is the n-vector of sets of infinite words $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, where

$$y_i \quad := \quad \{u_0 u_1 \ldots \in X^\omega \ : \ u_j \in a_{i_j, i_{j+1}}\}$$

for each infinite sequence (i_j) of elements of $[n]$ such that $i_0 = i$. Note that

$$1^\omega \quad = \quad 0,$$

where 1 is the multiplicative identity in S. The matricial theory determined by this pair will be denoted Ł$(X^*; X^\omega)$.

We will use the result that $\mathbf{Matr}(S_A; V_A)$ is a matricial iteration theory, for all nonempty sets A, to derive the fact that the theory Ł$(X^*; X^\omega)$ is also. Given the set X, we will show that when $A := X_\perp = X \cup \{\perp\}$, where $\perp \notin X$, then Ł$(X^*; X^\omega)$ is a quotient of a submatricial iteration theory of $\mathbf{Matr}(S_A; V_A)$.

First, for any nonempty set A, we define a submatricial iteration theory of $\mathbf{Matr}(S_A; V_A)$.

Definition 10.3.7 *S_0 consists of all sequacious relations in S_A such that*

$$u \; r \; uv \quad \Rightarrow \quad u' \; r \; u'v,$$

for all $u, u' \in A^+, v \in A^*$. V_0 consists of all sequacious relations r in V_A such that

$$u \; r \; uw \quad \Rightarrow \quad u' \; r \; u'w,$$

for all $u, u' \in A^+$, $w \in A^\omega$.

Note that both the empty relation and the identity relation in S_A are in S_0; the empty relation in V_A is in V_0. In fact, the following holds.

Proposition 10.3.8 *S_0 is a subsemiring of S_A closed under infinite sums, and hence closed under star. V_0 is a submonoid of V_A and if $r \in S_0$, $v \in V_0$, then $r \circ v \in V_0$. If $r \in S_0$, then $r^\omega \in V_0$.*

Now define T as $\mathbf{Matr}(S_0; V_0)$. Then T is a matricial iteration theory by Proposition 10.2.9.

Given the set X, recall that $A = X_\perp$, where $\perp$ is a new letter not in X. We assume that $\perp$ is used to define the omega operation in $\mathbf{Matr}(S_A; V_A)$. In order to give a surjective theory morphism $T \to \text{Ł}(X^*; X^\omega)$, we will define a pair of maps

$$\begin{aligned} \varphi_S : S_0 \quad &\to \quad L(X^*) \\ \varphi_V : V_0 \quad &\to \quad L(X^\omega) \end{aligned}$$

satisfying the conditions of Definition 10.2.6.

First, for any word $u \in A^+$, let $\widehat{u}$ denote the word in X^* obtained by deleting all occurrences of the letter $\perp$ from u; similarly, if $w \in A^\omega$, let $\widehat{w} \in X^* \cup X^\omega$ be defined in the same way. Note that

$$\widehat{uv} \quad = \quad \widehat{u}\widehat{v}. \tag{10.18}$$

Now suppose that r is a finite sequacious relation in S_0. Let

$$r\varphi_S \quad := \quad \{\widehat{v} \; : \; \exists u \in A^+ \, (u \; r \; uv)\}.$$

Note that if r is the zero in S_0, i.e. the empty relation, then $r\varphi_S$ is the empty set; if r is the identity relation, then $r\varphi_S$ consists of the singleton set consisting of the empty word.

Proposition 10.3.9 *The function $\varphi_S : S_0 \to L(X^*)$ is a surjective semiring homomorphism which preserves infinite sums, and hence the star operation.*

Proof. We will check only that φ_S preserves products. For $r, s \in S_0$,

$$\begin{aligned}
&v \in (r \cdot s)\varphi_S \\
&\quad \Leftrightarrow \quad \exists u_1, u_2[u_1 \ (r \cdot s) \ u_1 u_2 \text{ and } \widehat{u}_2 = v] \\
&\quad \Leftrightarrow \quad \exists u_1, u_3, u_4[u_1 \ r \ u_1 u_3 \text{ and } u_1 u_3 \ s \ u_1 u_3 u_4 \text{ and } \widehat{u}_3 \widehat{u}_4 = v] \\
&\quad \Rightarrow \quad \exists u_3, u_4[u_3 \in (r\varphi_S) \text{ and } u_4 \in (s\varphi_S) \text{ and } v = u_3 u_4] \\
&\quad \Rightarrow \quad v \in (r\varphi_S) \cdot (s\varphi_S).
\end{aligned}$$

Also,

$$\begin{aligned}
&v \in (r\varphi_S) \cdot (s\varphi_S) \\
&\quad \Rightarrow \quad \exists u_1, u_2, v_1, v_2[u_1 \ r \ u_1 v_1 \text{ and } u_2 \ s \ u_2 v_2 \text{ and } v = \widehat{v}_1 \widehat{v}_2] \\
&\quad \Rightarrow \quad \exists u_1, v_1, v_2[u_1 \ r \ u_1 v_1 \text{ and } u_1 v_1 \ s \ u_1 v_1 v_2 \text{ and } v = \widehat{v}_1 \widehat{v}_2],
\end{aligned}$$

since $s \in S_0$, showing $v \in (r \cdot s)\varphi_S$.

We define φ_V similarly: if r is a relation in V_0,

$$r\varphi_V \quad := \quad \{\widehat{w} \ : \ \exists u \in A^+(u \ r \ uw)\} \cap X^\omega.$$

In particular, if $r = 1^\omega$, then $r\varphi_V$ is the empty set.

Proposition 10.3.10 *$\varphi_V : V_0 \to L(X^\omega)$ is a surjective monoid homomorphism which preserves infinite sums.*

One argues exactly as in the proof of Proposition 10.3.9 that jointly φ_S and φ_V preserve the action; i.e. the diagram

[1‘1‘1‘1;1000‘500] [$S_0 \times V_0$‘V_0‘$L(X^*) \times L(X^\omega)$‘$L(X^\omega)$; $\circ$ ‘$\varphi_S \times \varphi_V$‘$\varphi_V$‘ $\circ$] commutes.

The last thing to verify is that for each $r \in S_0$,

$$(r\varphi_S)^\omega \quad = \quad (r^\omega)\varphi_V.$$

First, suppose that $w \in (r\varphi_S)^\omega$. Then for each $i \geq 0$, there are words u_i, v_i with $u_i \ r \ u_i v_i$ and such that $\widehat{v}_i \neq \epsilon$ and $w = \widehat{v}_0 \widehat{v}_1 \ldots$. Since $r \in S_0$,

$$u_0 v_0 \ldots v_{i-1} \ r \ u_0 v_0 \ldots v_{i-1} v_i,$$

for each $i \geq 1$. Thus,

$$u_0 \ r^\omega \ u_0 w_1,$$

where $w_1 = \lim_{i\to\infty} \ v_0 \dots v_{i-1} v_i$, showing that $\widehat{w}_1 = w \in (r^\omega)\varphi_V$.

Conversely, suppose that $\widehat{w} \in (r^\omega)\varphi_V$. Then, there is some $u_0 \in A^+$ with $u_0 \ r^\omega \ u_0 w$. Hence, $w = (\lim \ u_i) \cdot \perp^\omega$ for some $u_i \in A^+, i \geq 1$, such that $u_i \ r \ u_{i+1}$ for all $i \geq 0$. Note that the sequence $(u_i), \ i \geq 1$, cannot be eventually constant, since $\widehat{w}$ is an infinite word. But then there are nonempty words v_i with $u_{i+1} = u_i v_{i+1}$, for each $i \geq 0$. Hence $\widehat{v}_i \in r\varphi_S$ for all $i \geq 1$, and

$$u_n \quad = \quad u_0 v_1 v_2 \dots v_n,$$

for all $n \geq 1$, showing that $\widehat{w} = \widehat{v}_1 \widehat{v}_2 \dots \in (r\varphi_S)^\omega$.

We have proved that the pair (φ_S, φ_V) determines a surjective matricial theory morphism from $\mathbf{Matr}(S_0; V_0)$ onto $\text{Ł}(X^*; X^\omega)$ which preserves the star and omega operations.

Corollary 10.3.11 *$\text{Ł}(X^*; X^\omega)$ is a matricial iteration theory.*

Exercise 10.3.12 Show that the theory $\text{Ł}(X^*; X^\omega)$ satisfies the functorial star and omega implications for surjective base matrices.

Exercise 10.3.13 Show that if X and Y are sets each having at least two elements, then the theories $\text{Ł}(X^*; X^\omega)$ and $\text{Ł}(Y^*; Y^\omega)$ satisfy the same equations involving the star, omega and matricial theory operations.

10.3.3 $\mathcal{RL}(X^*; X^\omega)$

The least submatricial iteration theory of $\text{Ł}(X^*; X^\omega)$ containing the pairs $(\{x\}; \emptyset) : 1 \to 1$, for each $x \in X$, is denoted $\mathcal{RL}(X^*; X^\omega)$. Thus, the theory $\mathcal{RL}(X^*; X^\omega)$ is the matricial iteration theory $\mathbf{Matr}(S; V)$, where S is the semiring $RL(X^*)$ of all regular subsets of X^* and V is the S-module $RL(X^\omega)$ of all regular subsets of X^ω. The regular subsets of X^ω are those subsets which can be written as finite unions of sets of the form $R_1 R_2^\omega$, where R_1 and R_2 are regular subsets of X^*.

10.3.4 $\text{Ł}(X^*; X^\omega)$

We mention a quotient of the matricial theory $\text{Ł}(X^*; X^\omega)$. We recall (from [BN80] for example) that for a subset A of X^ω,

$$\begin{aligned} \mathrm{pre}(A) &:= \{u \in X^* \ : \ \exists v \in X^\omega (uv \in A)\}; && (10.19) \\ \mathrm{adh}(A) &:= \{v \in X^\omega \ : \ \mathrm{pre}(\{v\}) \subseteq \mathrm{pre}(A)\}. && (10.20) \end{aligned}$$

A subset A of X^ω is *closed* if $\mathrm{adh}(A) \subseteq A$, or equivalently, $A = \mathrm{adh}(A)$. Let $CL(X^\omega)$ denote the submonoid of $L(X^\omega)$ consisting of the closed subsets of X^ω. We define an action of $L(X^*)$ on $CL(X^\omega)$: for $A \subseteq X^*,\ c \in CL(X^\omega)$,

$$A \circ c \ := \ \mathrm{adh}(\{uv \ : \ u \in A,\ v \in c\}).$$

We define an omega operation $L(X^*) \to CL(X^\omega)$ as follows:

$$A^{\overline{\omega}} \ := \ \mathrm{adh}(A^\omega),$$

where $s \mapsto s^\omega$ is the omega operation in $\text{Ł}(X^*; X^\omega)$. Thus, we have defined a matricial preiteration theory $\text{Ł}(X^*; X^\omega)$ whose morphisms $(A; v)$ are matrices over $L(X^*)$ and vectors over $CL(X^\omega)$. There is a natural theory morphism

$$\varphi : \text{Ł}(X^*; X^\omega) \ \to \ \text{Ł}(X^\omega; X^\omega)$$

determined by the pair of maps

$$\begin{aligned} \varphi_S : L(X^*) &\to L(X^*) \\ s &\mapsto s \\ \varphi_V : L(X^\omega) &\to CL(X^\omega) \\ v &\mapsto \mathrm{adh}(v). \end{aligned}$$

The definition of the action and omega operation in $\text{Ł}(X^*; X^\omega)$ ensures that these maps determine a matricial theory morphism that preserves the star and omega operations. By Corollary 10.3.11, we obtain

Corollary 10.3.14 *$\text{Ł}(X^*; X^\omega)$ is a matricial iteration theory.*

10.3.5 $\mathcal{RCL}(X^*; X^\omega)$

An important submatricial iteration theory $\mathcal{RCL}(X^*; X^\omega)$ of the theory $\mathcal{CL}(X^*; X^\omega)$ is

$$\mathbf{Matr}(RL(X^*);\ RCL(X^\omega)),$$

where $RL(X^*)$ is the subsemiring of $L(X^*)$ consisting of the regular subsets of X^*, and where $RCL(X^\omega)$ is the submodule of $L(X^\omega)$ consisting of the closed and regular subsets of X^ω. This theory is considered in Section 10.9 below. It is the least submatricial iteration theory of $\mathrm{Ł}(X^*; X^\omega)$ which contains the pairs $(\{x\}; \emptyset) : 1 \to 1$, for each $x \in X$.

Exercise 10.3.15 Show that $\mathcal{RCL}(X^*; X^\omega)$ is in fact a submatricial iteration theory of $\mathcal{CL}(X^*; X^\omega)$. *Hint:* Use Lemma 10.4.13 below.

10.3.6 More Commutative Identities

Below, we will need to know that certain subtheories of $\mathrm{Ł}(X^*; X^\omega)$ satisfy the dual omega commutative identity (10.15). We show that in fact each of the matricial theories $\mathbf{Matr}(S_A; V_A)$, $\mathrm{Ł}(X^*; X^\omega)$, $\mathrm{Ł}(X^*; X^\omega)$, $\mathcal{CL}(X^*; X^\omega)$, $\mathcal{RCL}(X^*; X^\omega)$ satisfies this identity. It is enough to prove this fact for the theories $\mathbf{Matr}(S_A; V_A)$, since as shown above, the other theories are in the variety of matricial iteration theories they generate.

Proposition 10.3.16 *For each nonempty set A,* $\mathbf{Matr}(S_A; V_A)$ *satisfies the dual omega commutative identity.*

Proof. By Exercise 10.2.18, the dual omega commutative identity follows whenever one can show that a matricial theory satisfies the functorial omega implication for transposes of surjective base matrices, i.e.

$$s \cdot \rho^T \ = \ \rho^T \cdot r \ \ \Rightarrow \ \ s^\omega \ = \ \rho^T \cdot r^\omega, \tag{10.21}$$

for each surjective base $\rho : m \to n$, each pair of matrices $r : m \to m$ and $s : n \to n$ over S_A. From Exercise 10.2.19, it follows that the functorial omega implication holds for all transposes of surjective base matrices ρ if it holds when ρ has target 1. In this case the assumption on $r : m \to m$ and $s : 1 \to 1$ is that the sum of the relations in each column of r is the relation s. Thus, for each $u, v \in A^+$,

$$\forall i, j \in [m]\,(u\ r_{ij}\ v \ \Rightarrow\ u\ s\ v)\,; \tag{10.22}$$

$$u\ s\ v \ \Rightarrow\ \forall j \in [m]\,\exists i \in [m]\,(u\ r_{ij}\ v)\,. \tag{10.23}$$

We must prove that if $u \in A^+,\ w \in A^\omega$,

$$\forall i \in [m]\,(u\ (r^\omega)_i\ w \ \Rightarrow\ u\ s^\omega\ w) \tag{10.24}$$

$$u\ s^\omega\ w \ \Rightarrow\ \exists i \in [m]\ u\ (r^\omega)_i\ w. \tag{10.25}$$

Assume first that $u\ (r^\omega)_i\ w$. Then there is an infinite sequence $u_0 = u, i_0 = i, u_1, i_1, \ldots$ such that for each $j \geq 0$,

$$u_j\ r_{i_j,i_{j+1}}\ u_{j+1}$$

and

$$w \quad = \quad (\lim_{k\to\infty}\ u_k) \cdot \perp^\omega.$$

By the assumption (10.22),

$$u_j\ s\ u_{j+1},$$

for all $j \geq 0$, so that $u\ s^\omega\ w$. This proves (10.24).

We use König's lemma to prove (10.25). Assume that $u\ s^\omega\ w$. Then, there is an infinite sequence $u = u_0, u_1, \ldots$ of words in A^+ such that for each $j \geq 0$,

$$u_j\ s\ u_{j+1}$$

and $w = (\lim_{k\to\infty} u_k)\cdot\perp^\omega$. The complete m-ary tree C_m has the words on $[m]$ as vertices; the root is the empty word and the successors of the word z are the words $z1, z2, \ldots, zm$. Now consider the subtree C' of C_m which has an edge from the root to each length one sequence, and an edge from $i_0 i_1 \ldots i_j$ to $i_0 i_1 \ldots i_j i_{j+1}$ iff $u_j\ r_{i_j,i_{j+1}}\ u_{j+1}$. Using (10.23), it is easy to show that for each $j \in [m]$ and each $k \geq 1$ there is path in C' from the root to a word of the form wj, where w has length k. Thus, by König's lemma, there is an infinite path in C', which shows that for some $i \in [m]$, $u\ (r^\omega)_i\ w$.

Remark 10.3.17 A similar argument shows that $\mathrm{Ł}(X^*; X^\omega)$ satisfies the functorial star and omega implications for transposes of surjective base matrices.

10.4 Additively Closed Subiteration Theories

In this section we will consider subtheories of matricial (iteration) theories $\mathbf{Matr}(S;V)$ closed under the additive structure. Since most of our results apply only when V is an idempotent S-module, i.e. V is a semilattice with zero, we will restrict ourselves to this case. We note however that V is idempotent whenever S is idempotent, since if $1+1=1$, then

$$v+v \;=\; 1\circ v + 1\circ v \;=\; (1+1)\circ v \;=\; 1\circ v \;=\; v,$$

for all $v \in V$. In this section, all matricial theories $\mathbf{Matr}(S;V)$ are assumed to have an idempotent S-module V.

Definition 10.4.1 *Let T be a subtheory of a matricial theory $\mathbf{Matr}(S;V)$, with an idempotent S-module V. We call T* **additively closed**, *a.c. for short, if $0_{10} \in T(1,0)$, and for all $p \geq 0$, $f+g \in T(1,p)$ whenever $f,g \in T(1,p)$.*

It then follows that $0_{np} \in T(n,p)$ and $f+g \in T(n,p)$, for all $f,g \in T(n,p)$, $n,p \geq 0$. It is clear that every submatricial theory of the theory $\mathbf{Matr}(S;V)$ is an a.c. subtheory and that an a.c. subtheory of $\mathbf{Matr}(S;V)$ is a submatricial theory iff $f\P \in T$ for all $f \in T$. Thus, when $V=\{0\}$, so that $\mathbf{Matr}(S;V)$ is conveniently identified with the matrix theory $\mathbf{Mat}_S$, an additively closed subtheory of $\mathbf{Matr}(S;V)$ is just a submatrix theory of $\mathbf{Mat}_S$.

Let T be an a.c. subtheory of $\mathbf{Matr}(S;V)$. If

$$f_1 \;=\; (A_1;v_1),\; \ldots,\; f_p \;=\; (A_p;v_p) : n \to 1$$

are in T, so is

$$[f_1,\ldots,f_p] \;\;:=\;\; ([A_1,\ldots,A_p];\, v_1+\ldots+v_p) : n \to p,$$

since

$$[f_1,\ldots,f_p] \;\;=\;\; f_1 \cdot 1_p + \ldots + f_p \cdot p_p.$$

Because the operation taking the p-tuple $f_i : n \to 1$, $i \in [p]$, to $[f_1,\ldots,f_p]$ agrees with target tupling when restricted to morphisms in the underlying matrix theory $\mathbf{Mat}_S$, we call it by the same name. When $p = 0$, target tupling returns the constant 0_{n0}. Similarly, we can define an operation of target pairing. For $f = (A; u) : n \to p$ and $g = (B; v) : n \to q$,

$$\begin{aligned} [f, g] &:= ([A, B];\ u + v) \\ &= (f \oplus 0_q) + (0_p \oplus g) : n \to p + q. \end{aligned}$$

Again, if f and g are in T, so is the target pairing $[f, g]$.

Exercise 10.4.2 Show that target pairing is associative and that

$$[f, 0_{n0}] \ = \ f \ = \ [0_{n0}, f]$$

for all $f : n \to p$. Show that target tupling can be expressed in terms of target pairing and the constants 0_{n0}.

Thus every additively closed subtheory of a matricial theory is closed under the operations of target tupling and target pairing. In particular, all relational matrices belong to any a.c. subtheory. Conversely, if a subtheory T of $\mathbf{Matr}(S; V)$ contains the constant 0_{10} and is closed under the target tupling of any two morphisms $1 \to p$ in T, then T is an a.c. subtheory of $\mathbf{Matr}(S; V)$, for

$$f + g \ = \ [f, g] \cdot \langle \mathbf{1}_p, \mathbf{1}_p \rangle$$

for all $f, g : 1 \to p$ in T. We have thus proved

Proposition 10.4.3 *A subtheory T of a matricial theory is additively closed if and only if T is closed under target tupling.*

We show next that the set of morphisms $n \to p$ in an a.c. subtheory of a matricial theory is uniquely determined by the set of morphisms $1 \to 1$. We write $0 = 0_{11} = (0; 0)$ and $1 = \mathbf{1}_1 = (1; 0)$.

Proposition 10.4.4 *Let T be an a.c. subtheory of the theory $\mathbf{Matr}(S; V)$. Then 0 and 1 are in $T(1, 1)$, and for all $f, g \in T(1, 1)$,*

$$f \cdot g \quad \textit{and} \quad f + g$$

are in $T(1,1)$*. Further, a morphism* $f = (A; v) : n \to p$ *in* $\mathbf{Matr}(S; V)$ *is in* T *if and only if each morphism*

$$i_n \cdot f \cdot j_p^T \;=\; (A_{ij}; v_i)$$

is in $T(1,1)$*, for* $i \in [n]$ *and* $j \in [p]$*. In particular, when* $p = 0$*,* f *is in* T *if and only if each morphism*

$$i_n \cdot f \cdot 0_1 \;=\; (0; v_i)$$

is in $T(1,1)$*, for* $i \in [n]$*.*

Conversely, let Q *be a set of morphisms* $1 \to 1$ *in* $\mathbf{Matr}(S; V)$ *containing the constants* 0 *and* 1 *and closed under sum and composition. There is a unique a.c. subtheory* T *of* $\mathbf{Matr}(S; V)$ *with* $T(1,1) = Q$*, namely that described above.*

Proof. Suppose that T is an additively closed subtheory of $\mathbf{Matr}(S; V)$. It is immediate from Definition 10.4.1 that $T(1,1)$ is closed under composition and sum. Further, for $j \in [p]$, the morphism

$$j_p^T \;=\; \langle 0_{11}, \ldots, \mathbf{1}_1, \ldots, 0_{11} \rangle \;=\; \left(\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} ; \begin{matrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{matrix} \right),$$

with 1 in the j-th position, is in T. Thus, if $f = (A; v) : n \to p$ is in T, then each

$$i_n \cdot f \cdot j_p^T \;=\; (A_{ij}; v_i)$$

is in $T(1,1)$. When $p = 0$, we have

$$i_n \cdot f \cdot 0_1 \;=\; (0; v_i) \in T(1,1),$$

for all $i \in [n]$. Conversely, if $f = (A; v) : n \to p$ in $\mathbf{Matr}(S; V)$ and each $(A_{ij}; v_i)$ is in $T(1,1)$, then it follows that f is in T, since

$$f \;=\; \langle [(A_{11}; v_1), \ldots, (A_{1p}; v_1)], \ldots, [(A_{n1}; v_n), \ldots, (A_{np}; v_n)] \rangle$$

and T is closed under tupling and target tupling. Note however that the fact that V is idempotent is needed in the above derivation. In particular, when $p = 0$ and each $(0; v_i)$ is in $T(1,1)$, then it follows that

$$f \;=\; \langle (0; v_1), \ldots, (0; v_n) \rangle \cdot 0_{10}$$

is in T.

We now turn to proving the second part of Proposition 10.4.4. Assume that Q is a set of morphisms $1 \to 1$ in $\mathbf{Matr}(S;V)$ containing 0 and 1 and closed under the sum and composition operations. We define an a.c. subtheory T with $T(1,1) = Q$. Let $T(n,p)$ consist of all

$$f \;=\; (A; v) : n \to p$$

in $\mathbf{Matr}(S;V)$ with $(A_{ij}; v_i) \in Q$, for each $i \in [n]$ and $j \in [p]$. When $p = 0$, let $T(n,0)$ consist of the morphisms

$$(;v) : n \to 0$$

with $(0; v_i) \in Q$, for all $i \in [n]$. Thus the morphisms $1 \to 1$ in T are exactly those in Q. It is then clear that T contains all the relational matrices and that T is closed under tupling and target tupling. Thus, to prove that T is an a.c. subtheory, it suffices to show that the composite $f \cdot g : 1 \to q$ is in T, for all $f : 1 \to p$ and $g : p \to q$ in T. Let

$$f \;=\; ([a_1, \ldots, a_p];\, v)$$

and

$$g \;=\; \left(\begin{bmatrix} b_{11} & \ldots & b_{1q} \\ \vdots & & \vdots \\ b_{p1} & \ldots & b_{pq} \end{bmatrix} ; \begin{matrix} v_1 \\ \vdots \\ v_p \end{matrix} \right).$$

Then, for all $j \in [q]$,

$$\begin{aligned} (f \cdot g) \cdot j_q^T \;&=\; (a_1 b_{1j} + \ldots + a_p b_{pj};\, v + a_1 v_1 + \ldots + a_p v_p) \\ &=\; (a_1; v) \cdot (b_{1j}; v_1) + \ldots + (a_p; v) \cdot (b_{pj}; v_p), \end{aligned}$$

proving that $f \cdot g$ is in T. Note that we are again using the fact that $v + v = v$. When $q = 0$, so that

$$g \;=\; \left(; \begin{matrix} v_1 \\ \vdots \\ v_p \end{matrix} \right),$$

we have $(0; v_i) \in Q$, for all $i \in [p]$. Also $(0; v) \in Q$ if $p = 0$. Here too it follows that $f \cdot g$ is in T.

We have proved that T is an additively closed subtheory with $T(1,1) = Q$. The uniqueness of T follows from the first part of the proposition.

Definition 10.4.5 *Let T be an a.c. subtheory of the theory $\mathbf{Matr}(S;V)$. An* **additive theory morphism** *$\varphi : T \to \mathbf{Matr}(S';V')$ is a theory morphism which preserves the constant 0_{10} and the sum operation, for morphisms in $T(1,p)$, $p \geq 0$; i.e.*

$$0_{10}\varphi \;=\; 0_{10}$$

and

$$(f+g)\varphi \;=\; f\varphi + g\varphi,$$

for all $f, g : 1 \to p$ in T.

It follows that any additive theory morphism $\varphi : T \to \mathbf{Matr}(S';V')$ preserves the constants 0_{np} and that

$$(f+g)\varphi \;=\; f\varphi + g\varphi$$

holds for all $f, g : n \to p$ in T. Further φ preserves the operation of target tupling as well as all of the relational morphisms. If $\varphi : T \to \mathbf{Matr}(S';V')$ is an additive theory morphism from an a.c. subtheory of $\mathbf{Matr}(S;V)$ into $\mathbf{Matr}(S';V')$, then the morphisms $f\varphi : n \to p$ in T' with $f : n \to p$ in T form an a.c. subtheory of $\mathbf{Matr}(S';V')$ denoted $T\varphi$.

Just as a.c. subtheories are determined by the set of morphisms $1 \to 1$, any additive theory morphism $\varphi : T \to \mathbf{Matr}(S';V')$ from an a.c. subtheory T of a matricial theory $\mathbf{Matr}(S;V)$ into a matricial theory $\mathbf{Matr}(S';V')$ is determined by its restriction to the set of morphisms $T(1,1)$.

Proposition 10.4.6 *Let T be an a.c. subtheory of the theory $\mathbf{Matr}(S;V)$ and let $\mathbf{Matr}(S';V')$ be a matricial theory. If $\varphi : T \to \mathbf{Matr}(S';V')$ is an additive theory morphism, then its restriction to the set of morphisms $T(1,1)$ is a surjective mapping*

$$T(1,1) \;\to\; (T\varphi)(1,1)$$

which preserves the constants 0 *and* 1 *as well as the sum and composition operations, i.e.*

$$(f+g)\varphi \;=\; f\varphi + g\varphi$$

and

$$(f\cdot g)\varphi \;=\; f\varphi\cdot g\varphi,$$

for all $f, g : 1 \to 1$ *in* T. *Given any morphism*

$$f \;=\; (A; v) : n \to p$$

in T, *we have*

$$i_n \cdot f\varphi \cdot j_p^T \;=\; (i_n \cdot f \cdot j_p^T)\varphi \;=\; (A_{ij}; v_i)\varphi,$$

for all $i \in [n]$ *and* $j \in [p]$. *When* $p = 0$, *i.e.*

$$f \;=\; (;v),$$

we have $i_n \cdot f \cdot 0_1 = (0; v_i) \in T(1,1)$, *and*

$$f\varphi \;=\; (;u),$$

where $(0; v_i)\varphi = (0; u_i)$, *for each* $i \in [n]$, *and where* u *is the* n*-vector of the* u_i.

Conversely, let Q *be the set of morphisms* $1 \to 1$ *of an a.c. subtheory* T *of* $\mathbf{Matr}(S; V)$. *Let* $\varphi : Q \to \mathbf{Matr}(S'; V')$ *be a mapping such that* $f\varphi$ *is a morphism* $1 \to 1$ *in* $\mathbf{Matr}(S'; V')$, *for all* f *in* Q. *Suppose that* φ *preserves the constants* 0 *and* 1 *and the operations of sum and composition on morphisms in* Q. *Then there is a unique additive theory morphism* $\overline{\varphi} : T \to \mathbf{Matr}(S'; V')$ *such that* $f\overline{\varphi} = f\varphi$ *for all* f *in* Q.

We omit the straightforward proof of Proposition 10.4.6. We now turn to additively closed subtheories of matricial iteration theories.

Definition 10.4.7 *Let* $\mathbf{Matr}(S; V)$ *be a Conway matricial theory. An* **a.i.c. subtheory** *of* $\mathbf{Matr}(S; V)$ *is an a.c. subtheory* T *which is closed under scalar iteration, i.e. for all* $f : 1 \to 1 + p$ *in* T, $f^\dagger$ *is in* T. *When* $\mathbf{Matr}(S; V)$ *is a matricial iteration theory and* T *is an a.i.c. subtheory of* $\mathbf{Matr}(S; V)$, T *is also called an* **additively closed subiteration theory**.

By the pairing identity, every a.i.c. subtheory of a Conway matricial theory is closed under iteration. More interestingly, the iteration operation in an a.i.c. subtheory is determined by an operation on the morphisms $1 \to 1$.

Definition 10.4.8 *In any omega theory* $\mathbf{Matr}(S;V)$*, we define*

$$f^{\otimes} \ := \ (A^*;\, A^{\omega} + A^* v) : n \to n,$$

for all $f = (A;v) : n \to n$.

Proposition 10.4.9 *The following hold in any Conway matricial theory* $\mathbf{Matr}(S;V)$.

[a]
$$f^{\otimes} \ = \ [f, \mathbf{1}_n]^{\dagger},$$
for all $f : n \to n$.

[b]
$$f^{\dagger} \ = \ g^{\otimes} \cdot h,$$
for all $f = [g,h] : n \to n+p$ *with* $g : n \to n$ *and* $h : n \to p$.

[c]
$$f^{\dagger} \ = \ f_1^{\otimes} \cdot [f_2, \ldots, f_{1+p}],$$
for all $f = [f_1, \ldots, f_{1+p}] : 1 \to 1+p$ *with* $f_i : 1 \to 1$, $i \in [1+p]$. *In particular, when* $p = 0$, $f^{\dagger} = f^{\otimes} \cdot 0_{10}$.

[d]
$$f^{\dagger} \ = \ (a_1;v)^{\otimes} \cdot ([a_2, \ldots, a_{1+p}];\, v),$$
for all $f = ([a_1, \ldots, a_{1+p}];\, v) : 1 \to 1+p$. *In particular, when* $p = 0$, $f^{\dagger} = (a_1;v)^{\otimes} \cdot (;v)$.

Proposition 10.4.10 *Let* T *be an a.i.c. subtheory of a Conway matricial theory* $\mathbf{Matr}(S;V)$. *The following conditions are equivalent.*

- T *is a subiteration theory.*
- $f^{\otimes}$ *is in* T, *for all* $f : n \to n$ *in* T.

- *$f^{\otimes}$ is in T, for all $f : 1 \to 1$ in T.*

Proof. By Proposition 10.4.9.a, if T is closed under iteration, then T is closed under the $^{\otimes}$ operation. Suppose that $f^{\otimes}$ is in T, for all $f : 1 \to 1$. Then, by Proposition 10.4.9.d, T is closed under scalar iteration. Indeed, if say $f = ([a_1, \ldots, a_{1+p}];\, v) : 1 \to 1 + p$ is in T, then

$$f^{\dagger} \;=\; (a_1; v)^{\otimes} \cdot ([a_2, \ldots, a_{1+p}];\, v)$$

is in T, for $(a_1; v)$ and $([a_2, \ldots, a_{1+p}];\, v)$ are in T and T is closed under composition and the "biscalar" $^{\otimes}$ operation.

Definition 10.4.11 *Let* $\mathbf{Matr}(S; V)$ *and* $\mathbf{Matr}(S'; V')$ *be Conway matricial theories and let T be an a.i.c. subtheory of* $\mathbf{Matr}(S; V)$*. An* **additive preiteration theory morphism** $\varphi : T \to \mathbf{Matr}(S'; V')$ *is an additive theory morphism such that*

$$(f\varphi)^{\dagger} \;=\; f^{\dagger}\varphi,$$

for all $f : 1 \to 1 + p$ in T.

Thus, by the pairing identity, $(f\varphi)^{\dagger} = f^{\dagger}\varphi$, for all $f : n \to n + p$ in T.

Supposing that $\varphi : T \to \mathbf{Matr}(S'; V')$ is an additive preiteration theory morphism, it follows that $T\varphi$ is an a.i.c. subtheory of $\mathbf{Matr}(S; V)$.

Proposition 10.4.12 *Let* $\mathbf{Matr}(S; V)$ *and* $\mathbf{Matr}(S'; V')$ *be Conway matricial theories. Let T be an a.i.c. subiteration theory of* $\mathbf{Matr}(S; V)$ *and let $\varphi : T \to$* $\mathbf{Matr}(S'; V')$ *be an additive theory morphism. If φ preserves iteration then*

$$(f\varphi)^{\otimes} \;=\; f^{\otimes}\varphi,$$

for all $f : n \to n$ in T. Conversely, if $(f\varphi)^{\otimes} = f^{\otimes}\varphi$ holds for all $f : 1 \to 1$ in T, then φ preserves iteration, so that φ is an additive preiteration theory morphism.

The easy proof of Proposition 10.4.12 is omitted. By Propositions 10.4.4 and 10.4.10, an a.i.c. subtheory T of a Conway matricial theory $\mathbf{Matr}(S; V)$ is completely determined by the set $T(1, 1)$, which must contain the constants 0 and 1 and must be closed under the sum, composition and $^{\otimes}$ operations. Supposing that T is an a.i.c. subiteration theory of a Conway matricial theory, by Propositions 10.4.6 and 10.4.12 it follows that any additive preiteration theory morphism $T \to \mathbf{Matr}(S;V)$ to a Conway matricial theory is uniquely determined by its restriction to the set $T(1,1)$. The restriction must preserve the constants 0 and 1 as well as the sum, composition and $^{\otimes}$ operations on the morphisms in $T(1,1)$.

10.4.1 An A.C. Subiteration Theory of Ł$(X^*;X^\omega)$

We end this section with an example. Recall that for a set X, Ł$(X^*;X^\omega)$ is an idempotent matricial iteration theory. We will describe an additively closed subiteration theory of Ł$(X^*;X^\omega)$.

Let $X^\infty = X^* \cup X^\omega$. We extend the pre and adh operators defined in Section 10.3 to subsets of X^∞. When $A \subseteq X^\infty$, we define

$$\mathrm{pre}(A) \quad := \quad \{u \in X^* : \exists v \in X^\infty \ (uv \in A)\}$$

and

$$\mathrm{adh}(A) \quad := \quad \{u \in X^\omega : \mathrm{pre}(\{u\}) \subseteq \mathrm{pre}(A)\}.$$

Recall that a set $A \subseteq X^\omega$ is closed if $\mathrm{adh}(A) \subseteq A$, or equivalently, $\mathrm{adh}(A) = A$. Since

$$\mathrm{adh}(A) \quad = \quad \mathrm{adh}(\mathrm{pre}(A)),$$

it follows that the closed sets $A \subseteq X^\omega$ are exactly those of the form $\mathrm{adh}(B)$ for some (prefix closed) $B \subseteq X^*$.

For $A \subseteq X^\infty$ we define

$$A_{\mathrm{fin}} \ := \ A \cap X^* \quad \text{and} \quad A_{\mathrm{inf}} \ := \ A \cap X^\omega.$$

We may extend the product, star and omega operations on subsets of X^* to subsets of X^∞. Given $A, B \subseteq X^\infty$, let

$$\begin{aligned} A \cdot B \quad &:= \quad A_{\mathrm{fin}} B + A_{\mathrm{inf}} \\ A^* \quad &:= \quad (A_{\mathrm{fin}})^* + (A_{\mathrm{fin}})^* A_{\mathrm{inf}} \end{aligned}$$

and

$$A^\omega \quad := \quad (A_{\mathrm{fin}})^\omega + (A_{\mathrm{fin}})^* A_{\mathrm{inf}}.$$

Thus, $A \cdot B$ is the set concatenation defined in Section 1.1.1.

Lemma 10.4.13 *For all* $A, B \subseteq X^\infty$,

$$\begin{aligned} \mathrm{adh}(A+B) &= \mathrm{adh}(A) + \mathrm{adh}(B) \\ \mathrm{adh}(AB) &= \mathrm{adh}(A) + A\,\mathrm{adh}(B) \\ \mathrm{adh}(A^*) &= A^\omega + A^*\,\mathrm{adh}(A) \\ \mathrm{adh}(A^\omega) &= \mathrm{adh}(A^*). \end{aligned}$$

Exercise 10.4.14 Prove Lemma 10.4.13.

Let Ł$(X^*; X^\omega)$ consist of the morphisms

$$(A; v) : n \ \rightarrow \ p$$

in Ł$(X^*; X^\omega)$ such that each v_i is a closed subset of X^ω and such that

$$\mathrm{adh}(A_{ij}) \ \subseteq \ v_i,$$

for all $i \in [n]$ and $j \in [p]$.

Proposition 10.4.15 *Ł$(X^*; X^\omega)$ is an a.c. subiteration theory of the theory Ł$(X^*; X^\omega)$.*

Proof. By Propositions 10.4.4 and 10.4.10, it suffices to show that the set of morphisms $1 \to 1$ in Ł$(X^*; X^\omega)$ contains the constants 0 and 1 and is closed under the sum, composition and $^\otimes$ operations. As for the constants, both

$$0 \ = \ (\emptyset; \emptyset) \quad \text{and} \quad 1 \ = \ (\{\epsilon\}; \emptyset)$$

are in Ł$(X^*; X^\omega)$. Suppose that $f = (A; u)$ and $g = (B; v)$ are morphisms $1 \to 1$ in Ł$(X^*; X^\omega)$. We use Lemma 10.4.13 to show that $f + g = (A+B;\ u+v)$, $f \cdot g = (AB;\ u + Av)$ and $f^\otimes = (A^*;\ A^\omega + A^* u)$ are all in Ł$(X^*; X^\omega)$.

Proof that $f + g = (A+B;\ u+v)$ is in Ł$(X^*; X^\omega)$. Since

$$\mathrm{adh}(u+v) \ = \ \mathrm{adh}(u) + \mathrm{adh}(v) \ = \ u + v,$$

$u + v$ is a closed subset of X^ω. Also

$$\mathrm{adh}(A+B) \ = \ \mathrm{adh}(A) + \mathrm{adh}(B) \ \subseteq \ u + v.$$

Proof that $f \cdot g = (AB;\ u + Av)$ is in Ł$(X^*; X^\omega)$. Since $\mathrm{adh}(A) \subseteq u$,

$$\begin{aligned} \mathrm{adh}(u + Av) &= \mathrm{adh}(u) + \mathrm{adh}(Av) \\ &= u + \mathrm{adh}(A) + A\,\mathrm{adh}(v) \\ &= u + Av, \end{aligned}$$

showing that $u + Av$ is closed. Further,

$$\begin{array}{rcl} \mathrm{adh}(AB) & = & \mathrm{adh}(A) + A\,\mathrm{adh}(B) \\ & \subseteq & u + Av, \end{array}$$

so that $f \cdot g$ is in Ł$(X^*; X^\omega)$.

Proof that $f^\otimes = (A^*;\ A^\omega + A^* u)$ is in Ł$(X^*; X^\omega)$. Since

$$\begin{array}{rcl} \mathrm{adh}(A^\omega + A^* u) & = & \mathrm{adh}(A^\omega) + \mathrm{adh}(A^* u) \\ & = & \mathrm{adh}(A^*) + \mathrm{adh}(A^*) + A^*\,\mathrm{adh}(u) \\ & = & \mathrm{adh}(A^*) + A^* u \\ & = & A^\omega + A^*\,\mathrm{adh}(A) + A^* u \\ & = & A^\omega + A^* u, \end{array}$$

$A^\omega + A^* u$ is a closed subet of X^ω. Further,

$$\begin{array}{rcl} \mathrm{adh}(A^*) & = & A^\omega + A^*\,\mathrm{adh}(A) \\ & \subseteq & A^\omega + A^* u. \end{array}$$

A subset of X^ω which is both closed and regular is called a closed regular ω-language.

Exercise 10.4.16 Prove that closed regular ω-languages are exactly those of the form $\mathrm{adh}(A)$, where $A \subseteq X^*$ is any regular set.

Let Ł$(X^*; X^\omega)$ consist of those morphisms

$$(A; v) : n \to p$$

in Ł$(X^*; X^\omega)$ such that each A_{ij} is regular, each $v_i \subseteq X^\omega$ is a closed regular set, and lastly,

$$\mathrm{adh}(A_{ij}) \ \subseteq \ v_i,$$

for all $i \in [n]$ and $j \in [p]$.

Proposition 10.4.17 *Ł$(X^*; X^\omega)$ is an a.c. subiteration theory of the matricial iteration theory Ł$(X^*; X^\omega)$, in fact it is the smallest a.c. subiteration theory containing the morphisms $(\{x\}; \emptyset)$, for $x \in X$.*

Proof. The proof that Ł$(X^*; X^\omega)$ is an a.c. subiteration theory of the theory Ł$(X^*; X^\omega)$ is similar to the proof of Proposition 10.4.15. However, one also uses the following facts. Let $A, B \subseteq X^*$ be regular sets and let $u, v \subseteq X^\omega$ be closed regular sets, say

$$u = \mathrm{adh}(C) \quad \text{and} \quad v = \mathrm{adh}(D),$$

for some regular $C, D \subseteq X^*$. In addition, we assume $\mathrm{adh}(A) \subseteq u$. Then

$$\begin{aligned} u + v &= \mathrm{adh}(C + D) \\ u + Av &= \mathrm{adh}(C + AD) \end{aligned}$$

and

$$A^\omega + A^* u = \mathrm{adh}(A^* C).$$

Indeed,

$$\begin{aligned} \mathrm{adh}(C + AD) &= \mathrm{adh}(C) + \mathrm{adh}(A) + A\,\mathrm{adh}(D) \\ &= u + Av, \end{aligned}$$

and

$$\begin{aligned} \mathrm{adh}(A^* C) &= \mathrm{adh}(A^*) + A^*\,\mathrm{adh}(C) \\ &= A^\omega + A^*\,\mathrm{adh}(A) + A^* u \\ &= A^\omega + A^* u. \end{aligned}$$

Thus, $u + v$, $u + Av$ and $A^\omega + A^* u$ are all closed regular ω-languages.

Now let T be an additively closed subiteration theory of $\text{Ł}(X^*; X^\omega)$ which contains the morphisms $(\{x\}; \emptyset)$, $x \in X$. We must show that any morphism $(A; u)$ in $\text{Ł}(X^*; X^\omega)$ is in T. First we prove that for regular $A \subseteq X^*$,

$$(A;\, \mathrm{adh}(A)) : 1 \to 1$$

is in T. Indeed, this is true by assumption for $A = \{x\}$, $x \in X$. The two constants $0 = (\emptyset; \emptyset)$ and $1 = (\{\epsilon\}; \emptyset)$ are clearly in T. We proceed by induction on the number of operations in $\{+, \cdot, ^*\}$ needed to obtain A. Suppose that

$$(B;\, \mathrm{adh}(B)) : 1 \to 1 \quad \text{and} \quad (C;\, \mathrm{adh}(C)) : 1 \to 1$$

are in T, for regular $B, C \subseteq X^*$. Then

$$\begin{aligned} (B;\, \mathrm{adh}(B)) + (C;\, \mathrm{adh}(C)) &= (B + C;\, \mathrm{adh}(B + C)) \\ (B;\, \mathrm{adh}(B)) \cdot (C;\, \mathrm{adh}(C)) &= (BC;\, \mathrm{adh}(B) + B\,\mathrm{adh}(C)) \\ &= (BC;\, \mathrm{adh}(BC)), \end{aligned}$$

and

$$\begin{aligned} (B;\, \mathrm{adh}(B))^\otimes &= (B^*;\, B^\omega + B^*\,\mathrm{adh}(B)) \\ &= (B^*;\, \mathrm{adh}(B^*)) \end{aligned}$$

are all in T, completing the induction.

Finally, suppose that $(A;u) : 1 \to 1$ is in $\text{Ł}(X^*;X^\omega)$, so that $A \subseteq X^*$ is regular, $u \subseteq X^\omega$ is a closed regular set and $\mathrm{adh}(A) \subseteq u$. Let $B \subseteq X^*$ be regular with $u = \mathrm{adh}(B)$, so that $(B;u)$ is in T. Thus,

$$(0;u) \;=\; (B;u) \cdot 0$$

and

$$(A;u) \;=\; (A;\, \mathrm{adh}(A)) + (0;u)$$

are both in T.

Let $T = \mathbf{Matr}(S;V)$ be a matricial theory with an idempotent module V and let T_0 be an a.c. subtheory of T. Define

$$\begin{aligned} S_0 \;&:=\; \{a \in S \,:\, \exists v \in V \;(a;v) \in T_0(1,1)\} \\ &=\; \{f\P \,:\, f \in T_0(1,1)\} \end{aligned}$$

and

$$V_0 \;:=\; \{\sum_{i=1}^{n} a_i \circ v_i : n \geq 0,\; a_i \in S_0,\; v_i \in T_0(1,0)\}.$$

Thus S_0 is a subsemiring of S and V_0 is a submonoid of V which is closed under the action of elements in S_0. It is easy to see that the smallest submatricial theory of T containing T_0 can be described as the theory $\mathbf{Matr}(S_0;V_0)$. When T is a Conway matricial theory and T_0 is an a.i.c. subtheory, then the theory $\mathbf{Matr}(S_0;V_0)$ is not necessarily a sub Conway matricial theory. The smallest sub Conway matricial theory containing T_0 is the theory $\mathbf{Matr}(S_0;V_1)$, where S_0 is defined above and

$$V_1 \;:=\; \{\sum_{i=1}^{n} a_i \circ v_i + \sum_{j=1}^{m} b_j \circ c_j^\omega \;:\; n,m \geq 0,\; a_i, b_j, c_j \in S_0,\; v \in V_0\}.$$

When T_0 is generated by the elements in S_0, i.e. by the pairs $(a;0)$, $a \in S$, it is possible to give a simpler description of V_1, namely

$$V_1 \;:=\; \{\sum_{i=1}^{n} a_i \circ s_i^\omega \;:\; n \geq 0,\; a_i, s_i \in S_0\}.$$

As an example, consider the a.c. subiteration theory $\text{Ł}(X^*;X^\omega)$ of the theory $\text{Ł}(X^*;X^\omega)$. The morphisms $1 \to 1$ in the smallest submatricial theory of $\text{Ł}(X^*;X^\omega)$ containing $\text{Ł}(X^*;X^\omega)$ are all ordered pairs $(A;v)$ consisting of a finitary regular language A in X^* and a regular ω-language v in X^ω of the form

$$v \;=\; B_1 u_1 + \ldots + B_n u_n,$$

$n \geq 0$, where each B_i is a finitary regular language and each u_i is a closed regular ω-language. The smallest submatricial iteration theory containing Ł$(X^*; X^\omega)$ is the theory Ł$(X^*; X^\omega)$ of finitary regular languages and all regular ω-languages.

We note that both theories Ł$(X^*; X^\omega)$ and Ł$(X^*; X^\omega)$ can be considered as a.c. subiteration theories of the matricial iteration theory Ł$(X^*; X^\omega)$ of finitary languages and closed ω-languages. Indeed, any morphism in the theory Ł$(X^*; X^\omega)$ is a morphism in Ł$(X^*; X^\omega)$, and the sum, composition and $^\otimes$ operations on morphisms $1 \to 1$ in Ł$(X^*; X^\omega)$ give the same result regardless whether these operations are evaluated in Ł$(X^*; X^\omega)$ or Ł$(X^*; X^\omega)$. The theory Ł$(X^*; X^\omega)$ is also an a.c. subiteration theory of the matricial iteration theory Ł$(X^*; X^\omega)$.

10.5 Presentations in Matricial Iteration Theories

Suppose that $T = \mathbf{Matr}(S; V)$ is a Conway matricial theory such that V is idempotent. Let S_0 be a sub *-semiring of S with $a^\omega = 0$, for all $a \in S_0$. Further, let X be a subset of S. In the first part of this section we will give a description of the a.i.c. subtheory of T generated by $S_0 \cup X$. We set

$$W \quad := \quad \{\sum_{i=1}^{k} a_i x_i \ : \ a_i \in S_0, \ x_i \in X\}. \tag{10.26}$$

Recall from Section 9.9.4 that a presentation of weight s over the pair (S_0, X) is a triple $D = (\alpha; A; \gamma)$ consisting of matrices $\alpha : 1 \to s$ and $\gamma : s \to 1$ in $\mathbf{Mat}_{S_0}$ and the $s \times s$ matrix A whose entries are in W. Below we redefine the notion of the *behavior* of a presentation in order to obtain a Kleene type result in Proposition 10.5.3 below. These behaviors will be used also in Section 10.8.

Definition 10.5.1 *Let $D = (\alpha; A; \gamma)$ be a presentation. We define $|D| := (|D|_1; |D|_0)$, where*

$$|D|_1 \quad := \quad \alpha \cdot A^* \cdot \gamma$$

and

$$|D|_0 \quad := \quad \alpha \cdot A^\omega.$$

Thus $|D|$ is a morphism $1 \to 1$ in T.

Let $D = (\alpha; A; \gamma)$ and $E = (\alpha'; B; \gamma')$ be presentations of weight s and r, respectively. The presentations $D + E$, $D \cdot E$ and D^* were defined in Lemma 9.9.4.2. Below we will use the notation $D^{\otimes}$ for D^*.

Lemma 10.5.2 [a] $|D + E| = |D| + |E|$;

[b] $|D \cdot E| = |D| \cdot |E|$;

[c] $|D^{\otimes}| = |D|^{\otimes}$.

Proof.
Proof of [a]. In Lemma 9.9.4.2, we have proved that $|D + E|_1 = |D|_1 + |E|_1$. A similar calculation shows that $|D + E|_0 = |D|_0 + |E|_0$. It follows that $|D + E| = |D| + |E|$.

Proof of [b]. For the presentation $D \cdot E$ we have $|D \cdot E|_1 = |D|_1 \cdot |E|_1$, cf. Lemma 9.9.4.2. Also

$$\begin{aligned}
|D \cdot E|_0 &= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} A & \gamma\alpha' B \\ 0 & B \end{bmatrix}^{\omega} \\
&= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} A^{\omega} + A^*\gamma\alpha' B B^{\omega} \\ B^{\omega} \end{bmatrix} \\
&= \alpha A^{\omega} + \alpha A^* \gamma \alpha' B^{\omega} \\
&= |D|_0 + |D|_1 \cdot |E|_0 \\
&= |D| \cdot |E| \cdot 0_{10}.
\end{aligned}$$

Thus, $|D \cdot E| = |D| \cdot |E|$.

Proof of [c]. First note that since $a^{\omega} = 0$, for all $a \in S_0$, it follows that $C^{\omega} = 0_{n0}$, for all $C : n \to n$ in $\mathbf{Mat}_{S_0}$. In particular, $(\gamma\alpha)^{\omega} = 0_{s0}$. We have

$$\begin{aligned}
|D^{\otimes}|_0 &= \begin{bmatrix} \alpha & 1 \end{bmatrix} \cdot \begin{bmatrix} (\gamma\alpha)^* A & 0 \\ 0 & 0 \end{bmatrix}^{\omega} \\
&= \alpha((\gamma\alpha)^* A)^{\omega} \\
&= \alpha((\gamma\alpha)^* A)^{\omega} + \alpha((\gamma\alpha)^* A)^* (\gamma\alpha)^{\omega} \\
&= \alpha(\gamma\alpha + A)^{\omega} \\
&= \alpha(A^*\gamma\alpha)^{\omega} + \alpha(A^*\gamma\alpha)^* A^{\omega}
\end{aligned}$$

$$
\begin{aligned}
&= (\alpha A^*\gamma)^\omega + (\alpha A^*\gamma)^* \cdot \alpha A^\omega \\
&= |D|_1^\omega + |D|_1^* \cdot |D|_0 \\
&= |D|^\otimes \cdot 0_{10}.
\end{aligned}
$$

It follows by Lemma 9.9.4.2 that $|D^\otimes| = |D|^\otimes$.

Proposition 10.5.3 *Let* $T = \mathbf{Matr}(S;V)$*,* S_0 *and* X *be as before. If* T' *is the smallest a.i.c. subtheory of* T *which contains the morphisms* $(a;0) : 1 \to 1$*,* $a \in S_0 \cup X$*, then a morphism* $f : 1 \to 1$ *belongs to* T' *if and only if* $f = |D|$*, for some presentation* D*.*

Exercise 10.5.4 Suppose that $S_0 = \{0,1\}$ and $1^* = 1$ in S_0. Show that for any presentation $D = (\alpha; A; \gamma)$, $|D|^\otimes = |D^\otimes|$, where

$$
D^\otimes \quad := \quad (\begin{bmatrix} \alpha & 1 \end{bmatrix}; \begin{bmatrix} A + \gamma\alpha A & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} \gamma \\ 1 \end{bmatrix}).
$$

See also Exercise 9.9.4.14.

Since each matricial iteration theory is a Conway matricial theory, the previous corollary applies to matricial iteration theories as well.

Example 10.5.5 In this example we again consider the matricial iteration theory $\mathrm{L}(X^*; X^\omega)$ studied in Section 10.3.2. Recall that the Boolean semiring $\mathbf{B}$ may be identified with a sub *-semiring S_0 of $L(X^*)$ and that we may identify each letter $x \in X$ with the singleton set $\{x\}$. Note that $0^\omega = 1^\omega = 0$. By definition, the smallest a.i.c. subtheory of T containing $S_0 \cup X$ is the theory $\mathcal{RACL}(X^*; X^\omega)$ whose morphisms $1 \to 1$ are the ordered pairs $(A; v)$ such that $A \subseteq X^*$ is regular, $v \subseteq X^\omega$ is closed and regular, and, moreover, $\mathrm{adh}(A) \subseteq v$. By Proposition 10.5.3, we may obtain an alternative characterization of $\mathcal{RACL}(X^*; X^\omega)$. Let $= (Q, X, \delta, Q_0, Q_f)$ be an nfa. If $u = u_0 u_1 \ldots$ is in X^ω, a *run* of u starting in state q_0 is an infinite word

$$
q_0 q_1 \ldots \in Q^\omega
$$

such that $q_{i+1} \in \delta(q_i, u_i)$, for all $i = 0, 1, \ldots$. The *ω-language accepted by* is the set $||_{\mathrm{inf}}$ of those words in X^ω which have at least one run starting in an initial state. Let $||_{\mathrm{fin}} \subseteq X^*$ denote the finitary language accepted by . We define the behavior of to be the ordered pair $|| := (||_{\mathrm{fin}}; ||_{\mathrm{inf}})$. Since any presentation D may be identified with the corresponding nfa $_D$, cf. Example 9.9.4.5, by Proposition 10.5.3 a morphism $f : 1 \to 1$ in T belongs to $\mathcal{RACL}(X^*; X^\omega)$ if and only if f is the behavior of some nfa.

In the rest of this section, just as above, let $T = \mathbf{Matr}(S; V)$ be a Conway matricial theory, S_0 a sub *-semiring of S and X a subset of S. However, we no longer require V to be idempotent or that $a^\omega = 0$ for the elements $a \in S_0$. The set W was defined by (10.26).

Definition 10.5.6 *A* **type** 1 **presentation** *of weight s over (S_0, X) is a triple $(\alpha; A; \gamma)$, where $\alpha : 1 \to s$ and $\gamma : s \to 1$ are matrices over S_0 and where A is an $s \times s$ matrix all of whose entries are in W. A* **type** 0 **presentation** *of weight s over (S_0, X) is a triple $(\alpha; A; \kappa)$, where α and A are as before and $\kappa : s \to s \in \mathbf{Mat}_{S_0}$. When $D = (\alpha; A; \gamma)$ is a type 1 presentation, the behavior of D is defined to be the $T\P$-morphism*

$$|D| \quad := \quad \alpha \cdot A^* \cdot \gamma : 1 \to 1.$$

When $E = (\alpha; A; \kappa)$ is a type 0 presentation, we define

$$|E| \quad := \quad \alpha \cdot (A^* \cdot \kappa)^\omega : 1 \to 0.$$

Thus a type 1 presentation in T is just a presentation in $T\P$ as defined in Section 9.9.4.

Lemma 10.5.7 [a] *Let $D = (\alpha; A; \kappa)$ and $E = (\alpha'; B; \kappa')$ be type 0 presentations of weight s and r, respectively. Define the type 0 presentation $D + E$ of weight $s + r$ by*

$$D + E \quad := \quad (\begin{bmatrix} \alpha & \alpha' \end{bmatrix}; \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}; \begin{bmatrix} \kappa & 0 \\ 0 & \kappa' \end{bmatrix}).$$

Then

$$|D + E| \quad = \quad |D| + |E|.$$

[b] *Let $D = (\alpha; A; \gamma)$ be a type 1 presentation of weight s and $E = (\alpha'; B; \kappa)$ a type 0 presentation of weight r. Let $D \cdot E$ be the type 0 presentation of weight $s + r$*

$$D \cdot E \quad := \quad (\begin{bmatrix} \alpha & 0 \end{bmatrix}; \begin{bmatrix} A & \gamma\alpha' B \\ 0 & B \end{bmatrix}; \begin{bmatrix} 0 & \gamma\alpha'\kappa \\ 0 & \kappa \end{bmatrix}).$$

Then

$$|D \cdot E| \quad = \quad |D| \cdot |E|.$$

[c] *Let $D = (\alpha; A; \gamma)$ be a type 1 presentation of weight s. Define D^ω to be the type 0 presentation of weight s*

$$D^\omega \quad := \quad (\alpha; A; \gamma\alpha).$$

Then

$$|D^\omega| \quad = \quad |D|^\omega.$$

We prove only [b] and [c].
Proof of [b].

$$\begin{aligned}
|D \cdot E| &= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} A & \gamma\alpha' B \\ 0 & B \end{bmatrix}^* \cdot \begin{bmatrix} 0 & \gamma\alpha'\kappa \\ 0 & \kappa \end{bmatrix} \right)^\omega \\
&= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} A^* & A^*\gamma\alpha' B^+ \\ 0 & B^* \end{bmatrix} \cdot \begin{bmatrix} 0 & \gamma\alpha'\kappa \\ 0 & \kappa \end{bmatrix} \right)^\omega \\
&= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & A^*\gamma\alpha' B^*\kappa \\ 0 & B^*\kappa \end{bmatrix}^\omega \\
&= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} A^*\gamma\alpha' B^*\kappa(B^*\kappa)^\omega \\ (B^*\kappa)^\omega \end{bmatrix} \\
&= \alpha A^*\gamma\alpha' B^*\kappa(B^*\kappa)^\omega \\
&= \alpha A^*\gamma \cdot \alpha'(B^*\kappa)^\omega \\
&= |D| \cdot |E|
\end{aligned}$$

Proof of [c].

$$|D^\omega| \;=\; \alpha(A^*\gamma\alpha)^\omega \;=\; (\alpha A^*\gamma)^\omega \;=\; |D|^\omega$$

Proposition 10.5.8 *Let $T = \mathbf{Matr}(S; V)$ be a Conway matricial theory or a matricial iteration theory. Suppose that S_0 is a sub *-semiring of S and that X is a subset of S. Let $T' = \mathbf{Matr}(S'; V')$ be the smallest sub Conway matricial theory (or submatricial iteration theory) of T which contains S_0 and X, so that S' is the smallest sub *-semiring of S containing $S_0 \cup X$. Let $a \in S$ and $v \in V$. Then a is in S' if and only if a is the behavior of some type 1 presentation over (S_0, X). Similarly, v is in V' if and only if $v = |D|$, for some type 0 presentation D.*

Exercise 10.5.9 The ω-language accepted by a nondeterministic finite automaton was defined in Example 10.5.5. In order that nondeterministic finite automata be able to accept ω-languages that are not necessarily closed, in this exercise we redefine the notion of acceptance. Suppose $\mathcal{A} = (Q, X, \delta, Q_0, Q_f)$ is an nfa. We define the ω-language accepted by $\mathcal{A}$ to be the set of all words in X^ω which have a run starting in an initial state which goes through some final state in Q_f infinitely often. Use Proposition 10.5.8 to show that an ω-language is regular iff it is accepted by some nfa.

10.6 The Initial Matricial Iteration Theory

In this section, we give a description of a matricial iteration theory $\mathbf{Matr}(S_0; V_0)$ with the property that for any matricial iteration theory $\mathbf{Matr}(S; V)$, there is a unique iteration theory morphism from $\mathbf{Matr}(S_0; V_0)$ to $\mathbf{Matr}(S; V)$.

We need some facts about elements in the S-module V, whenever the theory $\mathbf{Matr}(S; V)$ is a matricial iteration theory.

Proposition 10.6.1 *Assume that* $\mathbf{Matr}(S; V)$ *is an iteration theory. Define three elements in* V *as follows.*

$$\alpha \ := \ 1^\omega; \quad \beta \ := \ (1^*)^\omega; \quad \gamma \ := \ (1^{**})^\omega.$$

Then the following identities hold in V*.*

$$\begin{aligned}
2^\omega &= 1^{**}\alpha + \beta \\
\beta &= \gamma \\
n^\omega &= 1^{**}\alpha + \gamma, \ n \geq 2 \\
\gamma &= 1^{**}\alpha + \gamma \\
\gamma &= \alpha + \gamma \\
((1^*)^k)^\omega &= \gamma, \quad k \geq 1.
\end{aligned}$$

The proofs are straightforward and are left as exercises. For example, to show that $2^\omega = 1^{**}\alpha + \beta$, we use the omega sum identity:

$$\begin{aligned}
(1+1)^\omega &= 1^{**}1^\omega + (1^*)^\omega \\
&= 1^{**}\alpha + \beta.
\end{aligned}$$

Exercise 10.6.2 Prove the remaining facts in Proposition 10.6.1.

Let $\mathbf{Mat}_{S_0}$ be the initial matrix iteration theory. Recall from Chapter 9 that the elements of S_0 are

$$0, 1, 2, \ldots, n, \ldots, 1^*, (1^*)^2, \ldots, (1^*)^n, \ldots, 1^{**}.$$

It follows from the previous proposition that for nonzero $s \in S_0$,

$$s \circ \gamma \;=\; s \circ (1^{**} \circ \gamma) \;=\; (s1^{**}) \circ \gamma \;=\; \gamma,$$

since $s1^{**} = 1^{**}$.

If $\mathbf{Matr(}S; V\mathbf{)}$ is an iteration theory, the S-module V must contain at least the elements $\alpha = 1^\omega$ and $\gamma = (1^{**})^\omega$. Let $V_0 := (V_0, +, 0)$ be the following linearly ordered S_0-module, where $\mathbf{Mat}_{S_0}$ is the initial matrix iteration theory, whose elements, listed in increasing order are:

$$0,\ \alpha,\ 2 \circ \alpha, \ldots, 1^* \circ \alpha,\ (1^*)^2 \circ \alpha, \ldots, 1^{**} \circ \alpha,\ \gamma.$$

By Proposition 10.6.1, the addition operation in V_0 is forced to be defined by:

$$\begin{array}{rcl} 0 + x & = & x, \quad\ x \in V_0; \\ s \circ \alpha + s' \circ \alpha & = & (s + s') \circ \alpha, \quad s, s' \in S_0; \\ s \circ \alpha + \gamma & = & \gamma \;=\; \gamma + \gamma. \end{array}$$

Indeed, if $s \in S_0,\ s \neq 0,\ (s \circ \alpha) + \gamma = s \circ \alpha + s \circ \gamma = s \circ (\alpha + \gamma) = s \circ \gamma = \gamma$.

The action of S_0 on V_0 is also forced:

$$\begin{array}{rcl} 0 \circ x & = & 0; \\ s \circ (s' \circ \alpha) & = & (ss') \circ \alpha; \\ s \circ \gamma & = & \gamma, \quad s \neq 0 \in S_0. \end{array}$$

Note that the ordering on V_0 has the property that

$$x \;\leq\; y \;\;\Leftrightarrow\;\; \exists z (x + z \;=\; y).$$

We are forced to define the operation ${}^{\omega} : S_0 \to V_0$ by:

$$s^{\omega} \quad := \quad \begin{cases} 0 & \text{if } s = 0; \\ \alpha & \text{if } s = 1; \\ \gamma & \text{otherwise.} \end{cases}$$

For matrices $a : n \to n$ in $\mathbf{Mat}_{S_0}$, $n \geq 2$, a^{ω} is defined using the omega pairing identity.

We will prove that $M_0 := \mathbf{Matr}(S_0; V_0)$ is an iteration theory. It will follow that M_0 is the initial matricial iteration theory. Indeed, suppose that $T = \mathbf{Matr}(S; V)$ is any matricial iteration theory. Since $\mathbf{Mat}_S$ is a matrix iteration theory, there is a unique *-semiring homomorphism $\varphi_{S_0} : S_0 \to S$. The definition of the monoid homomorphism $\varphi_{V_0} : V_0 \to V$ is forced by the condition that it commute with the omega map and with scalar multiplication:

$$\begin{aligned} 0\varphi_{V_0} \quad &:= \quad 0; \\ (s \circ \alpha)\varphi_{V_0} \quad &:= \quad (s\varphi_{S_0}) \circ (1^{\omega}); \\ (\gamma)\varphi_{V_0} \quad &:= \quad (1^{**})^{\omega}. \end{aligned}$$

It is clear that φ_{S_0} and φ_{V_0} are compatible with the action:

$$s\varphi_{S_0} \circ v\varphi_{V_0} \quad = \quad (s \circ v)\varphi_{V_0}, \quad \text{all } s \in S_0,\ v \in V_0.$$

Also, for $s \in S_0$, $(s^{\omega})\varphi_{V_0} = (s\varphi_{S_0})^{\omega}$. Thus, if M_0 is an iteration theory, it is the initial one.

The argument that M_0 is indeed a matricial iteration theory follows the same outline as the proof that $\mathbf{Mat}_{S_0}$ is a matrix iteration theory; one shows that M_0 satisfies the omega GA-implication. (The fact that M_0 satisfies the star GA-implication was proved in Chapter 9.)

Indeed, suppose that

$$(a+b)^{\omega} + (a+b)^* u \quad = \quad (c+d)^{\omega} + (c+d)^* v,$$

for some $a, b, c, d \in S_0$ and $u, v \in V_0$. Since either

$$(d + ca^*b)^{\omega} + (d + ca^*b)^*(ca^{\omega} + ca^*u + v) \quad \geq \quad (a+b)^{\omega} + (a+b)^* u$$

or

$$(d + ca^*b)^{\omega} + (d + ca^*b)^*(ca^{\omega} + ca^*u + v) \quad \geq \quad (c+d)^{\omega} + (c+d)^* v,$$

if $(a+b)^\omega + (a+b)^* u = \gamma$ then also

$$(d + ca^*b)^\omega + (d + ca^*b)^*(ca^\omega + ca^*u + v) \quad = \quad \gamma.$$

When $(a+b)^\omega + (a+b)^*u$ is not γ there are four subcases: $a+b = c+d = 0$; $a+b = 1$ and $c+d = 0$; $a+b = 0$ and $c+d = 1$; and $a+b = c+d = 1$. The details are similar to the proof of Proposition 9.9.5.1.

We have proved the following result:

Theorem 10.6.3 *M_0 is the initial matricial iteration theory.*

We omit the proof of the following fact.

Corollary 10.6.4 *Let $S_1 = \{0, 1, 1^*\}$ be the idempotent semiring of Corollary 9.9.5.5. Let*

$$V_1 \quad := \quad \{0,\, \alpha = 1^\omega,\, 1^* \circ \alpha,\, \gamma = (1^{**})^\omega\}$$

be the indicated S_1-module. Then $\mathbf{Matr}(S_1; V_1)$ *is the initial idempotent matricial iteration theory.*

Corollary 10.6.5 *Let $S_2 = \{0, 1\}$ be the idempotent *-semiring with $1^* = 1$ and let $V_2 := \{0, \alpha = 1^\omega\}$. Then $M_2 :=$* $\mathbf{Matr}(S_2; V_2)$ *is the initial ω-idempotent matricial iteration theory.*

Problem 10.6.1 Find a "pleasant" equational axiomatization of the class of iteration semiring module pairs.

Problem 10.6.2 Give an explicit description of all free matricial iteration theories and all free Conway matricial theories.

Exercise 10.6.6 We know that for ω-complete semirings S, $\mathbf{Mat}_S$ is an iteration theory. Find that ω-complete semiring S such that $\mathbf{Mat}_S$ is initial in the full subcategory of matrix iteration theories determined by ω-complete semirings. Now do the same thing for matricial theories determined by ω-complete semirings.

Exercise 10.6.7 Describe the initial Conway matricial theory.

There is an extension theorem for matricial theories, analogous to the Extension Theorem in Chapter 9 for matrix theories. We will discuss this theorem in the next section.

10.7 The Extension Theorem

Suppose that T_0 is a submatricial theory of the matricial theory T. If T_0 is an iteration theory, when can one extend the star and omega operations in T_0 to all of T so that T becomes an iteration theory? This section contains one answer.

Positive semirings have been considered in several places (e.g. in [KS86]). We extend this notion to matricial theories. We suppose that S is a semiring and V is an S-module.

Definition 10.7.1 *V is a* **positive S-module** *and* **Matr$(S;V)$** *is a* **positive matricial theory** *if V has* at least two elements *and*

1. *for any $v, w \in V$, if $v + w = 0$ then both v and w are* 0;
2. *for any $s \in S$, $v \in V$, if $s \circ v = 0$ then either $s = 0 \in S$ or $v = 0 \in V$.*

The following observation is easy to prove.

Proposition 10.7.2 *If V is a positive S-module then S is a positive semiring, i.e. for $a, b \in S$, if $a + b = 0$, then $a = 0$ and $b = 0$; if $ab = 0$, then either $a = 0$ or $b = 0$.*

Example 10.7.3 The matricial theories $\mathcal{L}(X^*; X^\omega)$ and $\mathcal{CL}(X^*; X^\omega)$ defined in Section 10.3 are positive.

The extension theorem concerns matricial theories $T = \mathbf{Matr}(S;V)$ which are positive. We make several further assumptions on T. First, we assume that $T_0 = \mathrm{Matr}(S_0;V_0)$ is a submatricial theory of T and that T_0 is an iteration theory. Hence, there are functions

$$\begin{aligned} {}^* : S_0 &\rightarrow S_0 \\ {}^\omega : S_0 &\rightarrow V_0 \end{aligned}$$

which satisfy the conditions of Definition 10.2.6.

Second, we assume that I is an ideal in S which satisfies the following three hypotheses.

H1: Each element $s \in S$ can be written *uniquely* as a sum

$$s = x + a,$$

for $x \in S_0$ and $a \in I$. (It follows that any matrix $n \to p$ over S can be written uniquely as a sum of a matrix over S_0 and a matrix over I.)

H2: For any element a in I and for any b in S there is a unique ξ in S such that

$$\xi = a \cdot \xi + b.$$

H3: For any a in I and any $v \in V$ the equation

$$\xi = a \circ \xi + v$$

either has $0 \in V$ as the only solution (in the case that both v and a are 0); otherwise if a or v is not 0 the equation has a unique nonzero solution. In particular, for $a \neq 0$ in I, there is a unique nonzero solution in V of the equation

$$\xi = a \circ \xi.$$

Theorem 10.7.4 The Matricial Extension Theorem. *Suppose that $T = \mathbf{Matr}(S; V)$ is a positive matricial theory and $T_0 = \mathbf{Matr}(S_0, V_0)$ is a matricial iteration theory which is a submatricial theory of T. Suppose lastly that I is an ideal in the semiring S which satisfies the above hypotheses H1-H3. Then there is a unique extension of the star and omega operations on T_0 to T so that T becomes a matricial iteration theory with the iteration operation defined by*

$$f^\dagger = (a^*b;\, a^*x + a^\omega)$$

for $f = ([a \;\; b]; x)$, with $a : n \to n$, $b : n \to p$ in T¶. Further, if T_0 has a functorial star and omega operation with respect to surjective base morphisms, then so does T.

The proof will occupy the remainder of this section. It is easy to define the extended operations. For elements x in S_0, both x^* and x^ω are already defined. In particular, $0^* = 1$ and $0^\omega = 0 \in V_0$. Now we define the star and omega operations on elements of the ideal I. If $a \neq 0,\ a \in I$, the element $a^* \in S$ is defined as the unique solution to the equation

$$\xi = a\xi + 1,$$

and $a^\omega \in V$ is defined as the unique *nonzero* solution to

$$\xi = a\xi.$$

For all other elements s, write s uniquely as a sum $s = x + a$ with $x \in S_0$ and a in I. We then use the star and omega sum identities to define s^* and s^ω respectively.

$$(x+a)^* := (x^*a)^*x^* \tag{10.27}$$

$$(x+a)^\omega := (x^*a)^*x^\omega + (x^*a)^\omega. \tag{10.28}$$

We use the scalar pairing identities to define the star and omega operations on $n \times n$ matrices over S, for $n \geq 2$.

Note that the definition of the extended operations is forced, so that the uniqueness of the extensions is immediate. Also note that operations defined by (10.27) and (10.28) are indeed extensions of the operations in T_0 as well as those just defined on elements of I.

In the Matrix Extension Theorem in Chapter 9, we proved that under our current hypotheses, the matrix theory $\mathbf{Mat}_S$ is an iteration theory. Thus, according to Corollary 10.2.2 it remains to prove that the omega operation satisfies the scalar omega sum identity, the scalar omega product identity and the (scalar) omega commutative identity.

We first prove two lemmas concerning the omega operation applied to elements of I.

Lemma 10.7.5 *If either a or b is in I, then*

$$(ab)^\omega = a(ba)^\omega.$$

Proof. We need show only that the right-hand side is not zero if $ab \neq 0$ and that it is a solution to the equation

$$\xi = (ab)\xi.$$

Lemma 10.7.6 *If both a and b are in I, then*

$$(a+b)^\omega = (a^*b)^*a^\omega + (a^*b)^\omega. \tag{10.29}$$

Proof. The outline of the proof is the same as for the preceding lemma. First, if $a \neq 0$ then

$$(a^*b)^*a^\omega = a^\omega + b(a^*b)^*a^*a^\omega$$

is also nonzero. Also, if $b \neq 0$,

$$(a^*b)^\omega = (b + aa^*b)^\omega$$

is nonzero. Thus, if $a+b$ is nonzero, the right-hand side of (10.29) is nonzero.

In order to show that the right-hand side of (10.29) is a solution to the equation $\xi = (a+b)\xi$, we prove

$$(a+b)(a^*b)^*a^\omega = (a^*b)^*a^\omega \tag{10.30}$$

$$(a+b)(a^*b)^\omega = (a^*b)^\omega. \tag{10.31}$$

Hence, if neither a nor b is zero,

$$(a+b)^\omega = (a^*b)^*a^\omega = (a^*b)^\omega.$$

Proof of (10.30). We use the scalar star product identity in the second and last lines.

$$\begin{aligned}
a(a^*b)^*a^\omega + b(a^*b)^*a^\omega &= a(a^*b)^*a^\omega + (ba^*)^*ba^\omega \\
&= (a + aa^*(ba^*)^*b)a^\omega + (ba^*)^*ba^\omega \\
&= a^\omega + (aa^* + 1)(ba^*)^*ba^\omega \\
&= a^\omega + a^*(ba^*)^*ba^\omega \\
&= (1 + a^*(ba^*)^*b)a^\omega \\
&= (a^*b)^*a^\omega.
\end{aligned}$$

Proof of (10.31).

$$\begin{aligned}
(a+b)(a^*b)^\omega &= a(a^*b)^\omega + b(a^*b)^\omega \\
&= aa^*(ba^*)^\omega + (ba^*)^\omega \\
&= (aa^* + 1)(ba^*)^\omega \\
&= a^*(ba^*)^\omega \\
&= (a^*b)^\omega,
\end{aligned}$$

using Lemma 10.7.5 twice.

We now prove the fixed point identity holds for all elements in S. This fact will be used to prove the scalar omega product identity holds (Lemma 10.7.9 below).

Lemma 10.7.7 *For any $s \in S$,*

$$ss^\omega = s^\omega.$$

Proof. We write $s = x + a$, with $x \in S_0$ and $a \in I$.

$$\begin{aligned} ss^\omega &= (x+a)((x^*a)^*x^\omega + (x^*a)^\omega) \\ &= (x+a)(x^*a)^*x^\omega + (x+a)(x^*a)^\omega. \end{aligned}$$

We show

$$(x+a)(x^*a)^*x^\omega = (x^*a)^*x^\omega$$

and

$$(x+a)(x^*a)^\omega = (x^*a)^\omega.$$

Indeed,

$$\begin{aligned} x(x^*a)^*x^\omega + a(x^*a)^*x^\omega &= x((x^*a)(x^*a)^* + 1)x^\omega + a(x^*a)^*x^\omega \\ &= (x^*a(x^*a)^* + 1)x^\omega \\ &= (x^*a)^*x^\omega. \end{aligned}$$

Also

$$\begin{aligned} x(x^*a)^\omega + a(x^*a)^\omega &= xx^*(ax^*)^\omega + (ax^*)^\omega \\ &= x^*(ax^*)^\omega \\ &= (x^*a)^\omega. \end{aligned}$$

Thus,

$$\begin{aligned} (x+a)(x^*a)^*x^\omega + (x+a)(x^*a)^\omega &= (x^*a)^*x^\omega + (x^*a)^\omega \\ &= (x+a)^\omega. \end{aligned}$$

Lemma 10.7.8 *The scalar omega sum identity holds in* **Matr**$(S;V)$ *; i.e. for any* $s,t \in S$,

$$(s+t)^\omega = (s^*t)^*s^\omega + (s^*t)^\omega. \tag{10.32}$$

Proof. During the course of this argument we will use some facts about the star operation in any matrix iteration theory. Recall that for $s \in S$, $s^+ = ss^* = s^*s$.

By Exercise 9.9.3.31, we have

$$\begin{aligned} (s^*t)^* &= t^*(s^+t^+)^* && (10.33)\\ t^*(s^+t^+)^* &= s^*t^+(s^+t^+)^* + 1 && (10.34)\\ (s^*t)^* &= s^*t^+(s^+t^+)^* + 1, && (10.35) \end{aligned}$$

for all $s,t \in S$.

The proof of Lemma 10.7.8 will be divided into several cases. We always write $s = x + a,\ t = y + b$, with $x,y \in S_0$ and $a,b \in I$.

Case 1. $x = 0$ and $b = 0$. So $s = a$ and $t = y$.

$$\begin{aligned} (s+t)^\omega &= (a+y)^\omega \\ &= (y+a)^\omega \\ &= (y^*a)^*y^\omega + (y^*a)^\omega \\ &= (y^*a)^*y^\omega + (a+y^+a)^\omega \\ &= (y^*a)^*y^\omega + (a^*y^+a)^*a^\omega + (a^*y^+a)^\omega, \end{aligned}$$

by Lemma 10.7.6. We now compute the right-hand side of (10.32).

$$\begin{aligned} (s^*t)^*s^\omega + (s^*t)^\omega &= (a^*y)^*a^\omega + (a^*y)^\omega \\ &= (a^+y + y)^*a^\omega + (a^+y + y)^\omega \\ &= (y^*a^+y)^*y^*a^\omega + (y^*a^+y)^*y^\omega + (y^*a^+y)^\omega. \end{aligned}$$

The proof of this case will be completed once we prove the following identities.

$$\begin{aligned} (y^*a^+y)^*y^*a^\omega &= (a^*y^+a)^*a^\omega && (10.36)\\ (y^*a^+y)^*y^\omega &= (y^*a)^*y^\omega && (10.37)\\ (y^*a^+y)^\omega &= (a^*y^+a)^\omega. && (10.38) \end{aligned}$$

Proof of (10.36).

$$(y^*a^+y)^*y^*a^\omega = (1 + a^*y^+(a^+y^+)^*)a^\omega$$

by (10.34),

$$\begin{aligned}
&= a^\omega + a^*y^+(a^+y^+)^*a^\omega \\
&= a^\omega + a^*y^+(a^+y^+)^*aa^\omega \\
&= (1 + a^*y^+(a^+y^+)^*a)a^\omega \\
&= (1 + (a^*y^+a)(a^*y^+a)^*)a^\omega \\
&= (a^*y^+a)^*a^\omega.
\end{aligned}$$

Proof of (10.37).

$$\begin{aligned}
(y^*a^+y)^*y^\omega &= y^*a^+(yy^*a^+)^*yy^\omega + y^\omega \\
&= (y^*a^+(y^+a^+)^* + 1)y^\omega \\
&= (y^*a)^*y^\omega,
\end{aligned}$$

by the (10.35) above.

Proof of (10.38). This argument makes use of Lemma 10.7.5.

$$\begin{aligned}
(y^*a^+y)^\omega &= y^*(a^+y^+)^\omega \\
&= y^+(a^+y^+)^\omega + (a^+y^+)^\omega \\
&= (y^+a^+)^\omega + a^+(y^+a^+)^\omega \\
&= a^*(y^+a^+)^\omega \\
&= (a^*y^+a)^\omega.
\end{aligned}$$

The proof of this case is complete.

Case 2. $y = 0$. In the passage from third to the fourth line, we use Lemma 10.7.6.

$$\begin{aligned}
&(s^*t)^*s^\omega + (s^*t)^\omega \\
&= ((x+a)^*b)^*(x+a)^\omega + ((x+a)^*b)^\omega \\
&= ((x^*a)^*x^*b)^*[(x^*a)^*x^\omega + (x^*a)^\omega] + ((x^*a)^*x^*b)^\omega \\
&= ((x^*a)^*x^*b)^*(x^*a)^*x^\omega + ((x^*a)^*x^*b)^*(x^*a)^\omega + ((x^*a)^*x^*b)^\omega \\
&= (x^*a + x^*b)^*x^\omega + (x^*a + x^*b)^\omega \\
&= (x^*(a+b))^*x^\omega + (x^*(a+b))^\omega \\
&= (x + (a+b))^\omega \\
&= ((x+a)+b)^\omega \\
&= (s+t)^\omega.
\end{aligned}$$

Case 3. $a = 0$.

$$
\begin{aligned}
&(s+t)^\omega \\
&\quad = (x+(y+b))^\omega \\
&\quad = ((x+y)+b)^\omega \\
&\quad = ((x+y)^*b)^*(x+y)^\omega + ((x+y)^*b)^\omega \\
&\quad = ((x^*y)^*x^*b)^*((x^*y)^*x^\omega + (x^*y)^\omega) + ((x^*y)^*x^*b)^\omega \\
&\quad = ((x^*y)^*x^*b)^*(x^*y)^*x^\omega + ((x^*y)^*x^*b)^*(x^*y)^\omega + ((x^*y)^*x^*b)^\omega \\
&\quad = (x^*y+x^*b)^*x^\omega + (x^*y+x^*b)^\omega \\
&\quad = (s^*t)^*s^\omega + (s^*t)^\omega.
\end{aligned}
$$

Case 4. $b = 0$. In the third line we use Case 3, and we use Case 1 in the fifth line.

$$
\begin{aligned}
&(s+t)^\omega \\
&\quad = ((x+a)+y)^\omega \\
&\quad = (x+(a+y))^\omega \\
&\quad = (x^*(a+y))^*x^\omega + (x^*(a+y))^\omega \\
&\quad = (x^*a+x^*y)^*x^\omega + (x^*a+x^*y)^\omega \\
&\quad = ((x^*a)^*x^*y)^*(x^*a)^*x^\omega + ((x^*a)^*x^*y)^*(x^*a)^\omega + ((x^*a)^*x^*y)^\omega \\
&\quad = ((x^*a)^*x^*y)^*((x^*a)^*x^\omega + (x^*a)^\omega) + ((x^*a)^*x^*y)^\omega \\
&\quad = ((x+a)^*y)^*(x+a)^\omega + ((x+a)^*y)^\omega \\
&\quad = (s^*t)^*s^\omega + (s^*t)^\omega.
\end{aligned}
$$

Case 5. The general case.

$$
\begin{aligned}
(s+t)^\omega &= ((s+y)+b)^\omega \\
&= ((s+y)^*b)^*(s+y)^\omega + ((s+y)^*b)^\omega
\end{aligned}
$$

by Case 2,

$$
\begin{aligned}
&= ((s^*y)^*s^*b)^*(s+y)^\omega + ((s^*y)^*s^*b)^\omega \\
&= ((s^*y)^*s^*b)^*((s^*y)^*s^\omega + (s^*y)^\omega) + ((s^*y)^*s^*b)^\omega
\end{aligned}
$$

by Case 4,

$$
\begin{aligned}
&= ((s^*y)^*s^*b)^*(s^*y)^*s^\omega + ((s^*y)^*s^*b)^*(s^*y)^\omega + ((s^*y)^*s^*b)^\omega \\
&= (s^*y+s^*b)^*s^\omega + (s^*y+s^*b)^\omega,
\end{aligned}
$$

by Case 2,

$$= (s^*t)^*s^\omega + (s^*t)^\omega.$$

The proof of this lemma is complete.

Lemma 10.7.9 *The scalar omega product identity holds in* $\mathbf{Matr}(S;V)$*; i.e. for* $s = x + a$ *and* $t = y + b$*, with* $x, y \in S_0$ *and* $a, b \in I$*,*

$$(st)^\omega \quad = \quad s(ts)^\omega.$$

Proof. There are only three cases to consider.
Case 1. $a = 0$.

$$\begin{aligned}
(st)^\omega &= (x(y+b))^\omega \\
&= (xy + xb)^\omega \\
&= ((xy)^*xb)^*(xy)^\omega + ((xy)^*xb)^\omega \\
&= (x(yx)^*b)^*(xy)^\omega + (x(yx)^*b)^\omega \\
&= (x(yx)^*b)^*x(yx)^\omega + x((yx)^*bx)^\omega
\end{aligned}$$

by Lemma 10.7.5,

$$\begin{aligned}
&= x((yx)^*bx)^*(yx)^\omega + x((yx)^*bx)^\omega \\
&= x(yx + bx)^\omega \\
&= t(st)^\omega.
\end{aligned}$$

Case 2. $b = 0$. In the first line we use Lemma 10.7.7 and in the second line we use Case 1.

$$\begin{aligned}
((x+a)y)^\omega &= (x+a)y((x+a)y)^\omega \\
&= (x+a)(y(x+a))^\omega \\
&= s(ts)^\omega.
\end{aligned}$$

Case 3. The general case.

$$\begin{aligned}
(st)^\omega &= (sy + sb)^\omega \\
&= ((sy)^*sb)^*(sy)^\omega + ((sy)^*sb)^\omega,
\end{aligned}$$

by Lemma 10.7.8,

$$= \quad (s(ys)^*b)^*s(ys)^\omega + s((ys)^*bs)^\omega,$$

by Case 2 and Lemma 10.7.5,

$$\begin{aligned} &= s((ys)^*bs)^*(ys)^\omega + s((ys)^*bs)^\omega \\ &= s[((ys)^*bs)^*(ys)^\omega + ((ys)^*bs)^\omega] \\ &= s(ys+bs)^\omega \;=\; s(ts)^\omega, \end{aligned}$$

by Lemma 10.7.8. The proof of Lemma 10.7.9 is complete, showing that T is a Conway matricial theory.

Before considering the omega commutative identity, we characterize the value of a^ω in the case that a is an $n \times n$ matrix over I. Recall that when $n \geq 2$, a^ω is defined by the scalar omega pairing identity.

We define the *support* of the n-vector $v = \langle v_1, \ldots, v_n \rangle$, $\mathrm{supp}(v)$, as the set $\{i \in [n] : v_i \neq 0\}$.

Definition 10.7.10 *Suppose that v is an n-vector and $a \in \mathbf{Mat}_S(n,n)$. Then an n-vector u is called a* **maximal solution** *of the equation*

$$\xi \;=\; a\xi + v \tag{10.39}$$

in the variable $\xi : n \to 0$ if u is a solution of the equation (10.39) and $u = u'$ whenever u' is a solution with $\mathrm{supp}(u) \subseteq \mathrm{supp}(u')$.

It will follow from Proposition 10.7.12 below that if a is an $n \times n$ matrix over I, then a^ω is the unique maximal solution of $\xi = a\xi$.

Lemma 10.7.11 *Suppose that $a \in \mathbf{Mat}_S(n,n)$ and that v is an n-vector. Let $\pi : n \to n$ be a base permutation. Then an n-vector u is a solution of the equation (10.39) iff the n-vector πu is a solution of the equation*

$$\xi \;=\; (\pi a \pi^T)\xi + \pi v. \tag{10.40}$$

Thus u is a maximal solution to equation (10.39) iff πu is a maximal solution to equation (10.40).

Proof. The first statement is obvious, and the second follows from the fact that if u and u' are n-vectors with the support of u contained in the support of u', then the support of πu is a subset of the support of $\pi u'$.

Proposition 10.7.12 *Suppose that A is an $n \times n$ matrix over I, and that v is an n-vector. The equation*

$$\xi \quad = \quad A\xi + v \tag{10.41}$$

has a unique maximal solution. (ξ is a variable over n-vectors.) Further, if $A^{\otimes}$ denotes the unique maximal solution of

$$\xi \quad = \quad A\xi,$$

then $A^{\otimes} = A^{\omega}$ and the unique maximal solution to (10.41) is $A^{\omega} + A^{*}v$.

Proof. The proof is by induction on n. The case $n = 0$ is trivial. The case $n = 1$ holds by the hypothesis H3 on elements in the ideal I. Note that in this case, if $A = 0$ there is exactly one solution to (10.41), and otherwise there is exactly one nonzero solution, which is thus the maximal solution. Also, since $A^{\omega} + A^{*}v$ is a nonzero solution, it is the unique maximal solution.

Now assume that $n \geq 1$ and that $A : n+1 \to n+1$. Write

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ v &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \xi &= \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \end{aligned}$$

with $a : n \to n$, $b : n \to 1$, $c : 1 \to n$ and $d : 1 \to 1$. Similarly, v_1 and ξ_1 are n-vectors and v_2 and ξ_2 are elements in V. We can rewrite equation (10.41) as the system of equations

$$\xi_1 \quad = \quad a\xi_1 + b\xi_2 + v_1 \tag{10.42}$$
$$\xi_2 \quad = \quad c\xi_1 + d\xi_2 + v_2. \tag{10.43}$$

Suppose that $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a maximal solution to the system. If $u = 0$, it is then the unique solution. Also $v_1 = v_2 = 0$, so that $A^{\omega} = A^{\otimes}$.

Otherwise, if $u \neq 0$, by Lemma 10.7.11 we may as well assume that $u_2 \neq 0$. But then u_2 is a maximal solution of equation (10.43) when

ξ_1 has the value u_1. Hence by the basis step,

$$\begin{aligned} u_2 &= d^\omega + d^*(cu_1 + v_2) \\ &= d^* cu_1 + (d^\omega + d^* v_2). \end{aligned}$$

Consider the equation obtained from (10.42) by substituting this value of u_2 for ξ_2:

$$\xi_1 = a\xi_1 + bd^* cu_1 + b(d^\omega + d^* v_2) + v_1. \tag{10.44}$$

Since u_1 is a solution of (10.44) it is also a solution of

$$\xi_1 = (a + bd^* c)\xi_1 + b(d^\omega + d^* v_2) + v_1. \tag{10.45}$$

Conversely, if u'_1 is any solution of (10.45), then u' is a solution of the system of equations (10.42) and (10.43), where

$$u' = \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix},$$

with u'_2 defined by the equation

$$u'_2 = d^\omega + d^*(cu'_1 + v_2).$$

Thus, comparing the formulas for u_2 and u'_2, if the support of u_1 is contained in the support of u'_1 then the support of u_2 is contained in the support of u'_2. Thus, if u'_1 is the unique maximal solution of (10.45) (which exists by the induction hypothesis), then $u_1 = u'_1$ and $u_2 = u'_2$ by the maximality of u. Hence by the induction assumption we can write

$$\begin{aligned} u_1 &= (a + bd^* c)^*(b(d^\omega + d^* v_2) + v_1) + (a + bd^* c)^\omega \\ u_2 &= d^\omega + d^*(cu_1 + v_2). \end{aligned}$$

Applying the star and omega sum and product identities, we can rewrite these equations as the matrix equation

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} (a + bd^* c)^* & a^* b(d + ca^* b)^* \\ d^* c(a + bd^* c)^* & (d + ca^* b)^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} (a^* bd^* c)^* a^\omega + (a^* bd^* c)^* a^* bd^\omega + (a^* bd^* c)^\omega \\ (d^* ca^* b)^* d^* ca^\omega + (d^* ca^* b)^* d^\omega + (d^* ca^* ab)^\omega \end{bmatrix}. \end{aligned}$$

Thus we have shown that the maximal solution to (10.41) is obtained by the scalar pairing identities, and thus can be written A^*v+A^ω. The proposition is proved.

There are several corollaries. The first fact may have some independent interest, but will be used here only to prove Corollary 10.7.14.

Corollary 10.7.13 *Suppose that A is an $n \times n$ matrix over I having no zero rows (i.e. for each $i \in [n]$, there is some j such that $A_{ij} \neq 0$). Then the n-vector A^ω has no zero components.*

Proof. We use induction on n. When $n = 1$, the result follows by the hypothesis H3. Now assume that $n \geq 1$ and $A : n+1 \to n+1$. We wish to show that no component of A^ω can be the zero vector. We argue by contradiction. There are two cases.
CASE 1. A^ω is the zero $n+1$-vector 0^{n+1}. We write A as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a : n \to n$, $b : n \to 1$, $c : 1 \to n$ and $d : 1 \to 1$. Write $A^\omega = \begin{bmatrix} 0^n \\ 0 \end{bmatrix}$, with 0^n the zero n-vector and $0 \in V$.
Subcase 1. $d = 0$. Then, by the proof of the preceding proposition, 0^n is the maximal solution of the equation

$$\xi = (a+bc)\xi,$$

i.e. $0^n = (a+bc)^\omega$. However we show that no row of the matrix $a+bc$ is zero. Indeed, if row i of a is zero, then b_i is not zero, since the i-th row of A is not zero. Here, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. The i-th row of bc is

$$\begin{bmatrix} b_ic_1 & b_ic_2 & \dots & b_ic_n \end{bmatrix},$$

where

$$c = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}.$$

If each of the entries b_ic_j is zero, then c is the zero row matrix and the last row of A is zero. Thus, by the induction assumption, no

component of $(a+bc)^\omega$ is zero, a contradiction.
Subcase 2. $d \neq 0$. Then $d^\omega \neq 0$. But by the previous proposition, so that $0 = d^\omega + d^* c u_1$. Another contradiction. Thus, A^ω cannot be the zero $n+1$-vector.
Case 2. A^ω has some zero and some nonzero components. We may as well assume that

$$A^\omega \quad = \quad \begin{bmatrix} v \\ 0^p \end{bmatrix},$$

where v is a k-vector, none of whose components are zero, and 0^p is the zero p-vector, where $k+p = n+1$. The argument of Case 1 shows that $k \geq 1$. Now write

$$A \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a : k \to k$, $b : k \to p$, $c : p \to k$ and $d : p \to p$. But since $AA^\omega = A^\omega$, it follows that

$$\begin{aligned} av + b0^p &= v \\ cv + d0^p &= 0^p. \end{aligned}$$

Thus $cv = 0^p$ and $av = v$. Since $(S;V)$ is positive, and since no component of v is zero, it follows that each row of c is zero. Hence no row of d can be zero, and d^ω has no nonzero component, by the induction hypothesis. Also, $av = v$, so that $v = a^\omega$. Hence, if $x = a^* b d^\omega + a^\omega$ and $y = d^\omega$, then by the previous proposition,

$$\begin{aligned} ax + by &= x \\ cx + dy &= y, \end{aligned}$$

and the support of $\begin{bmatrix} x \\ y \end{bmatrix}$ is $[n+1]$, contradicting the assumption that A^ω contained some zero components.

Now let A be an arbitrary $n \times n$ matrix over I. Suppose that $Z \subseteq [n]$ is the set of indices of zero rows of A. Then if the cardinality of Z is p and $k+p = n$, there is a base permutation π of $[n]$ such that

$$\pi A \pi^T \quad = \quad \begin{bmatrix} a & a' \\ 0_{pk} & 0_{pp} \end{bmatrix},$$

where $a : k \to k$ and $a' : k \to p$. Thus, by the omega permutation identity,

$$A^\omega = \pi^T \begin{bmatrix} a & a' \\ 0_{pk} & 0_{pp} \end{bmatrix}^\omega .$$

Further, an n-vector v is a solution of the equation

$$\xi = \begin{bmatrix} a & a' \\ 0_{pk} & 0_{pp} \end{bmatrix} \xi$$

iff v is a solution of the equation

$$\xi = \begin{bmatrix} a & 0_{kp} \\ 0_{pk} & 0_{pp} \end{bmatrix} \xi.$$

Hence,

$$\begin{bmatrix} a & a' \\ 0_{pk} & 0_{pp} \end{bmatrix}^\omega = \begin{bmatrix} a & 0_{kp} \\ 0_{pk} & 0_{pp} \end{bmatrix}^\omega .$$

Corollary 10.7.14 *Suppose that A is an $n \times n$ matrix over I and that B is an $m \times m$ matrix over S. If $\rho : n \to m$ is a surjective base matrix such that $A\rho = \rho B$, then*

$$A^\omega = \rho B^\omega .$$

Proof. We use induction on n. Note that $n \geq m$ since ρ is a surjection. If $n = 1$ then ρ is the identity $1_1 \mathit{andthestatementistrivial.If}$n ≥ 2, assume that A has k nonzero rows and p zero rows (with $k + p = n$). If $p = 0$, then B^ω has no zero components, by the previous corollary. Indeed, B can have no zero rows (since ρ is surjective). Also $\xi = \rho B^\omega$ is a solution to the equation $\xi = A\xi$, since $A\rho = \rho B$. Hence ρB^ω is the unique maximal solution.

Thus we suppose that $p \geq 1$. There is a permutation π of $[n]$ such that

$$\pi A \pi^T = \begin{bmatrix} a & a' \\ 0_{pk} & 0_{pp} \end{bmatrix}$$

and a permutation ψ of $[m]$ such that

$$\psi B \psi^T = \begin{bmatrix} b & b' \\ 0_{rj} & 0_{rr} \end{bmatrix},$$

where B has j nonzero rows and r zero rows. (From now on we omit the subscripts on the 0-matrices.) Thus,

$$\begin{bmatrix} a & a' \\ 0 & 0 \end{bmatrix} \cdot (\pi\rho\psi^T) \quad = \quad (\pi\rho\psi^T) \cdot \begin{bmatrix} b & b' \\ 0 & 0 \end{bmatrix}.$$

Since row i of A is nonzero iff row $i\rho$ of B is nonzero, the surjection $\pi\rho\psi^T$ can be written

$$\pi\rho\psi^T \quad = \quad \rho_1 \oplus \rho_2 \quad = \quad \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}$$

for some surjective base morphisms $\rho_1 : k \to j$ and $\rho_2 : p \to r$. It follows that

$$a\rho_1 \quad = \quad \rho_1 b.$$

Since $k < n$, the induction hypothesis implies that

$$a^\omega \quad = \quad \rho_1 b^\omega.$$

Thus,

$$\begin{aligned}
\begin{bmatrix} a & a' \\ 0 & 0 \end{bmatrix}^\omega \quad &= \quad \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}^\omega \\
&= \quad \begin{bmatrix} a^\omega \\ 0 \end{bmatrix} \\
&= \quad \begin{bmatrix} \rho_1 b^\omega \\ 0 \end{bmatrix} \\
&= \quad (\rho_1 \oplus \rho_2) \begin{bmatrix} b^\omega \\ 0 \end{bmatrix} \\
&= \quad (\pi\rho\psi^T) \begin{bmatrix} b & b' \\ 0 & 0 \end{bmatrix}^\omega.
\end{aligned}$$

Thus, using the omega permutation identity again, we obtain

$$\begin{aligned}
A^\omega \quad &= \quad \pi^T \begin{bmatrix} a & a' \\ 0 & 0 \end{bmatrix}^\omega \\
&= \quad (\rho\psi^T) \begin{bmatrix} b & b' \\ 0 & 0 \end{bmatrix}^\omega \\
&= \quad \rho B^\omega,
\end{aligned}$$

as claimed.

We use the previous Corollary and the omega sum identity (10.5) to prove the last axiom.

Lemma 10.7.15 *The omega commutative identity holds in T, i.e. for any $n \times m$ matrix f over S, any surjective base $\rho : m \to n$, and base $\rho_i : m \to m$, such that $\rho_i \cdot \rho = \rho$ for each $i \in [m]$,*

$$(\rho f \parallel (\rho_1, \ldots, \rho_m))^\omega \quad = \quad \rho \cdot (f \cdot \rho)^\omega.$$

Proof. Write $f = x + a$, where x is an $n \times m$ matrix over S_0 and a is an $n \times m$ matrix over I. Abbreviate the m-tuple $(\rho_1, \ldots, \rho_m)$ by just R. Then

$$\begin{aligned}(\rho f \parallel R)^\omega &= (\rho x \parallel R + \rho a \parallel R)^\omega \qquad (10.46)\\ &= ((\rho x \parallel R)^*(\rho a \parallel R))^*(\rho x \parallel R)^\omega + ((\rho x \parallel R)^*(\rho a \parallel R))^\omega\end{aligned}$$

by the omega sum identity,

$$= \quad ((\rho x \parallel R)^*(\rho a \parallel R))^*\rho(x\rho)^\omega + ((\rho x \parallel R)^*(\rho a \parallel R))^\omega, \qquad (10.47)$$

since $\mathbf{Matr}(S_0, V_0)$ is an iteration theory.

Let b be the matrix defined by

$$b \quad := \quad (\rho x \parallel R)^*(\rho a \parallel R).$$

Then all of the entries of b belong to the ideal, and

$$b\rho \quad = \quad \rho(x\rho)^*(a\rho).$$

Thus, by Corollary 10.7.14,

$$b^\omega \quad = \quad \rho((x\rho)^*(a\rho))^\omega$$

and as shown in Chapter 9,

$$b^*\rho \quad = \quad \rho((x\rho)^*(a\rho))^*.$$

Substituting in (10.47), we obtain

$$\begin{aligned}(\rho f \parallel R)^\omega &= \rho((x\rho)^*(a\rho))^*(x\rho)^\omega + \rho((x\rho)^*(a\rho))^\omega\\ &= \rho(x\rho + a\rho)^\omega\\ &= \rho(f\rho)^\omega.\end{aligned}$$

We have finished the proof that T is an iteration theory. The fact that the extension of the operations in T_0 is unique is obvious, since the two sum identities must hold in any iteration theory.

Lastly we prove that if T_0 has a functorial star and omega with respect to base surjections, then so does T. It then follows that T satisfies the star and omega functorial implications for all base morphisms. So suppose that

$$f \cdot \rho \quad = \quad \rho \cdot g,$$

where $f : n \to n$ and $g : m \to m$ are in T¶ and $\rho : n \to m$ is a surjective base morphism. We show that

$$f^\omega \quad = \quad \rho \cdot g^\omega.$$

Let $f = x + a, \; g = y + b$, with $x, y \in T_0$ and $a, b \in I$. Then

$$x\rho + a\rho \quad = \quad \rho y + \rho b.$$

Since $x\rho, \; \rho y$ are in T_0, and $a\rho, \; \rho b$ are in I, it follows by the uniqueness of the representation that

$$\begin{aligned} x\rho &= \rho y \\ a\rho &= \rho b. \end{aligned}$$

Also,

$$(x^* a)\rho \quad = \quad \rho(y^* b),$$

and all entries of $x^* a$ belong to I. Hence

$$\begin{aligned} x^* \rho &= \rho y^* \\ x^\omega &= \rho y^\omega \\ (x^* a)^\omega &= \rho (y^* b)^\omega \\ (x^* a)^* \rho &= \rho (y^* b)^*, \end{aligned}$$

by Corollary 10.7.14 and the fact that T_0 has a functorial star and omega with respect to base surjections. Thus,

$$\begin{aligned} (x + a)^\omega &= (x^* a)^* x^\omega + (x^* a)^\omega \\ &= (x^* a)^* \rho y^\omega + \rho (y^* b)^\omega \\ &= \rho ((y^* b)^* y^\omega + (y^* b)^\omega) \\ &= \rho g^\omega. \end{aligned}$$

As a corollary of the proof the theorem, we obtain the following result.

Corollary 10.7.16 *Suppose that T, T_0 and I satisfy all of the hypotheses of the Matricial Extension Theorem except that we require only that T_0 is a Conway matricial theory, not necessarily a matricial iteration theory. Then there is a unique extension of the star and omega operations on T_0 to T so that T becomes a Conway matricial theory.*

Remark 10.7.17 It is easy to show that if T, T_0 and I satisfy all of the hypotheses of the Matricial Extension Theorem, and further if T_0 satisfies the implication

$$f \cdot \rho^T \;=\; \rho^T \cdot g \quad\Rightarrow\quad \left(f^* \cdot \rho^T \;=\; \rho^T \cdot g^* \text{ and } f^\omega \;=\; \rho^T \cdot g^\omega\right),$$

for all $f : m \to m,\ g : n \to n$ in $T¶$ and surjective base $\rho : n \to m$, where ρ^T is the transpose of ρ, then so will T when the star and omega operations are extended to T.

Example 10.7.18 The Matricial Extension Theorem provides us with yet another proof that the theory $\mathcal{CL}(X^*; X^\omega)$ defined in Section 10.3 above is a matricial iteration theory. Suppose that S is the semiring $L(X^*)$ and V is the S-module $CL(X^\omega)$ of closed subsets of X^ω. The matricial theory $T = \mathbf{Matr}(S; V)$ is clearly positive. Also the matrix iteration theory $\mathbf{Mat_B}$, considered to be a matricial iteration theory, is a subtheory of T. This subtheory plays the role of T_0 in Theorem 10.7.4. The ideal I in S is the collection of subsets of X^* which do not contain the empty word. Clearly, each set in S can be written uniquely as a sum

$$s \;=\; x + a$$

where x is in $\{0, 1\}$ and a is in the ideal I. We verify the remaining hypotheses.

If a is in I and b in S, it is well-known that a^*b is the unique s in S with

$$s \;=\; as + b.$$

If $v \in V$, and not both $a, v = 0$, there is a unique nonzero w in V with $w = aw + v$ for the following reason. It is a well-known fact that the collection of all nonempty sets in $V = CL(X^\omega)$ is a complete metric space $\widehat{V}$ when the distance between the closed sets $v_1 \neq v_2$ is defined by:

$$d(v_1, v_2) \;=\; 2^{-n},$$

where n is the least integer such that the symmetric difference of the sets $\mathrm{pre}(v_1)$ and $\mathrm{pre}(v_2)$ contains a word of length n. When a is in I, the map

$\xi \mapsto a \circ \xi + v$ is a proper contraction of $\widehat{V}$ to itself. Thus, by the Banach fixed point theorem, this map has a unique fixed point. (Of course, if both a and v are 0 the only solution is $0 \in V$.) Thus, according the Theorem 10.7.4, T is a matricial iteration theory. Also, T has a functorial star and omega with respect to base surjections and their transposes.

Exercise 10.7.19 Let S_A be the semiring of finite sequacious relations on the set A, and let V_A be the S_A-module of infinite sequacious relations. Determine which, if any, of the three hypotheses H1-H3 of Theorem 10.7.4 are satisfied by S_A, V_A.

10.8 Additively Closed Theories of Regular Languages

We have proved that the matricial iteration theory $Ł(X^*; X^\omega)$ satisfies the dual commutative identity. Also the equations $1^* = 1$ and $1^\omega = 0$ hold in $Ł(X^*; X^\omega)$. We have further described the a.c. subiteration theory $Ł(X^*; X^\omega)$ in which a morphism $(A; v) : 1 \to 1$ consists of a (finitary) regular language $A \subseteq X^*$ and a closed regular ω-language $v \subseteq X^\omega$ with $\mathrm{adh}(A) \subseteq v$. Consider the map

$$\eta : X \quad \to \quad Ł(X^*; X^\omega)$$

defined by

$$x \quad \mapsto \quad (\{x\}; \emptyset).$$

Let $\mathbf{Matr}(S; V)$ be a matricial iteration theory which also satisfies the dual commutative identity and the above two equations, so that the theory $\mathbf{Matr}(S; V)$ is idempotent. Given any function

$$\varphi : X \quad \to \quad \mathbf{Matr}(S; V)$$

such that $x\varphi$ is a morphism $1 \to 1$ in $\mathbf{Mat}_S$, the underlying matrix iteration theory of $\mathbf{Matr}(S; V)$, we will show that there is a unique additive iteration theory morphism

$$\varphi^\sharp : Ł(X^*; X^\omega) \quad \to \quad \mathbf{Matr}(S; V)$$

with $\eta \cdot \varphi^\sharp = \varphi$.

The restriction that each $x\varphi$ be a morphism in the underlying matrix iteration theory $\mathbf{Mat}_S$ is due to the fact that the morphisms $x\eta$ are in the underlying matrix iteration theory $\text{Ł}(X^*)$ of $\text{Ł}(X^*; X^\omega)$; further any additive theory morphism maps a morphism in the underlying matrix iteration theory of $\text{Ł}(X^*; X^\omega)$ to a morphism in $\mathbf{Mat}_S$. Thus, if there is any extension, then each $x\varphi$ must be in $\mathbf{Mat}_S$.

We need several preliminary facts. First we note a consequence of the equations $1^* = 1$ and $1^\omega = 0$.

Lemma 10.8.1 *Suppose that* $1^* = 1$ *and* $1^\omega = 0$ *hold in a Conway matricial theory* $T = \mathbf{Matr}(S; V)$. *Then*

$$(1 + f)^\omega \;=\; f^\omega,$$

for all $f : 1 \to 1$ *in* T¶.

Proof. We have

$$(1 + f)^\omega \;=\; (1^* f)^\omega + (1^* f)^* 1^\omega \;=\; f^\omega,$$

by the omega sum identity.

Given $A : n \to n$ in $\mathbf{Mat}_S$, we say that A *contains a cycle* if there is a sequence

$$A_{i_1 i_2}, A_{i_2 i_3}, \ldots, A_{i_m i_1}$$

$(m \geq 1)$ of nonzero entries of A. Otherwise A is called *cycle-free*.

Lemma 10.8.2 *Let* $\mathbf{Matr}(S; V)$ *be a Conway matricial theory and suppose that* $A : n \to n$ *is a matrix in the underlying matrix theory. If* A *is cycle-free, then* $A^\omega = 0_{n0}$.

Proof. Since A is cycle-free, $A^n = 0_{nn}$. Thus

$$A^\omega \;=\; A^n \cdot A^\omega \;=\; 0_{nn} \cdot A^\omega \;=\; 0_{n0}.$$

For the remainder of this section, we will be concerned with the a.c. subiteration theory $\text{Ł}(X^*; X^\omega)$ of the matricial iteration theory $\text{Ł}(X^*; X^\omega)$. By a presentation in $\text{Ł}(X^*; X^\omega)$ we shall mean a presentation over the pair $(\mathbf{B}, X)$ in the sense of Definition 9.9.4.1, where the Boolean semiring $\mathbf{B}$ and the set X are considered to be included in Ł(X^*) as usual. We use Definition 10.5.1 for the behavior $|D|$ of a presentation D. Two presentations D and E are called *equivalent* if $|D| = |E|$.

Recall that associated with each presentation D there is an nfa $_D$, cf. Example 10.5.5, where it was shown that the morphisms $1 \to 1$ in $\mathcal{RACL}(X^*; X^\omega)$ are the ordered pairs of the form $|| = (||_{\text{fin}}; ||_{\text{inf}})$. In fact, using Proposition 10.5.3 we obtain the following result:

Proposition 10.8.3 *Given* $(A; v) : 1 \to 1$ *in* Ł$(X^*; X^\omega)$*, there is a presentation* D *with* $|D| = (A; v)$*.*

We now study the relationship between two equivalent presentations. Recall Definition 9.9.4.7.

Proposition 10.8.4 *Let* D *and* E *be presentations of weight* s *and* r*, respectively. Suppose that*

$$D \xrightarrow{\rho} E,$$

where $\rho : s \to r$ *is either a surjective base matrix or the transpose of a surjective base matrix, or an injective base matrix. Then* $|D| = |E|$*, i.e.* D *and* E *are equivalent.*

Proof. The proof makes use of the fact, established in Section 10.3, that Ł$(X^*; X^\omega)$ satisfies the functorial star and omega implications for surjective base matrices and their transposes. Further, any matricial iteration theory satisfies the functorial star and omega implications for injective base matrices. Otherwise the proof is similar to that of Proposition 9.9.4.11.

If $D \xrightarrow{\iota^T} E$ holds for the transpose of an injective base matrix ι, we only have $|E|_0 \subseteq |D|_0$. Nevertheless we can deduce $|D| = |E|$ whenever ι satisfies certain additional conditions. For use in this section, we redefine the notion of a coaccessible state in an nfa. Recall Example 10.5.5.

Definition 10.8.5 *Let* $= (Q, X, \delta, Q_0, Q_f)$ *be an nfa. A state* $q \in Q$ *is called* **coaccessible** *if either*

$$\delta(q, u) \cap Q_f \neq \emptyset$$

for some word $u \in X^*$, or there is an infinite word in X^ω which has a run starting in state q. Further, is **biaccessible** *if is both accessible (cf. Definition 9.9.7.1) and coaccessible.*

Let $D = (\alpha; A; \gamma)$ and $E = (\alpha'; B; \gamma')$ be presentations of weight s and r, respectively. Suppose that $r \leq s$ and let $\iota : r \to s$ be an injective base matrix. We say that ι is **admissible** *if all coaccessible states of the nfa $_D$ belong to the range of ι.*

Proposition 10.8.6 *Let $D = (\alpha; A; \gamma)$ and $E = (\alpha'; B; \gamma')$ be presentations as above. If ι is admissible and $D \xrightarrow{\iota^T} E$, then $|D| = |E|$.*

Proof. By Proposition 9.9.4.11, the finitary behaviors are the same, i.e. $|D|_1 = |E|_1$. We must prove that $|D|_0 = |E|_0$. By the star and omega permutation identities it is enough to consider the case that $\iota = \mathbf{1}_r \oplus 0_{s-r}$. Since $D \xrightarrow{\iota^T} E$, we have

$$A = \begin{bmatrix} B & U \\ 0 & V \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \alpha' & \alpha'' \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} \gamma' \\ 0 \end{bmatrix},$$

for some $U : r \to s - r$, $V : s - r \to s - r$ and $\alpha'' : 1 \to s - r$. Since ι is admissible, V is cycle-free. Thus,

$$A^\omega = \begin{bmatrix} B^\omega \\ 0 \end{bmatrix},$$

by Lemma 10.8.2 and the omega pairing identity. Thus

$$|D|_0 = \alpha \cdot A^\omega = \alpha' \cdot B^\omega = |E|_0.$$

We have spelled out the above formal proof because it uses only some equations valid in matricial iteration theories $\mathrm{Ł}(X^*; X^\omega)$ and thus allows us to derive a similar fact in a more general situation below.

Definition 10.8.7 *We define a relation $\approx$ on presentations as the smallest equivalence relation such that $D \approx E$ whenever $D \xrightarrow{\rho} E$ holds, where ρ is a morphism of one of the following four types: a surjective base matrix, a transpose of a surjective base matrix, an injective base matrix, a transpose of an admissible injective base matrix.*

From our preceding results we immediately have the following.

Corollary 10.8.8 *Let D and E be two presentations. If $D \approx E$ then $|D| = |E|$.*

We now set out to prove the converse of Corollary 10.8.8. Our argument is a modification of that used in Section 9.9.7. Recall that a presentation D is deterministic if the corresponding automaton $_D$ is deterministic, dfa, for short.

Proposition 10.8.9 *Let D be a presentation. There exist presentations D_0 and D_1 and injective base matrices ι and κ with the following properties.*

- *D_0 is biaccessible and D_1 is coaccessible.*
- *$D_0 \xrightarrow{\iota} D_1$ and $D \xrightarrow{\kappa^T} D_1$.*
- *κ is admissible.*

If D is deterministic, then so are D_0 and D_1.

Proof. Let $_D = (Q, X, \delta, Q_0, Q_f)$ and let $_1 = (Q', X, \delta', Q'_0, Q'_f)$ be the coaccessible part of $_D$, i.e. the nfa obtained from $_D$ by removing all of the states which are not coaccessible. Let $_0 = (Q'', X, \delta'', Q''_0, Q''_f)$ be the accessible part of $_1$ which is the nfa obtained from $_1$ by removing those states which are not accessible. If D_i, $i = 0, 1$, is a presentation whose nfa is (isomorphic to) $_i$, and if ι and κ are the injective base matrices corresponding to the embedding of Q'' into Q' and the embedding of Q' into Q, respectively, then we have all of the required properties.

Definition 10.8.10 $=$ *(Q, X, δ, Q_0, Q_f) be a biaccessible dfa. For each $q \in Q$ let $_q$ be the same dfa except that the only initial state is q, i.e.*

$$_q \; := \; (Q, X, \delta, \{q\}, Q_f).$$

We define an equivalence relation θ on the state set Q. For any two states q_1 and q_2 in Q,

$$q_1 \, \theta q_2 \;\;\Leftrightarrow\;\; |_{q_1}| \;=\; |_{q_2}|.$$

We call the dfa **reduced** *if*

$$q_1 \, \theta q_2 \;\;\Rightarrow\;\; q_1 \;=\; q_2,$$

for all $q_1, q_2 \in Q$.

Proposition 10.8.11 *For every biaccessible dfa $= (Q, X, \delta, Q_0, Q_f)$ there is a reduced biaccessible dfa*

$$' \;=\; (Q', X, \delta', Q'_0, Q'_f)$$

with $|| = |'|$ such that $'$ is a homomorphic image of . In great detail, there is a surjective mapping $h : Q \to Q'$ with the following properties.

- $Q_0 h = Q'_0$;
- $Q_f h = Q'_f$ *and* $Q'_f h^{-1} = Q_f$;
- *for all $q_1, q_2 \in Q$ and $x \in X$,*

$$\delta(q_1, x) \;=\; q_2 \quad\Rightarrow\quad \delta'(q_1 h, x) \;=\; q_2 h;$$

- *for all $q_1 \in Q$, $q'_2 \in Q'$ and $x \in X$,*

$$\delta'(q_1 h, x) \;=\; q'_2 \quad\Rightarrow\quad \exists q_2 \in Q[\delta(q_1, x) \;=\; q_2 \text{ and } q_2 h \;=\; q'_2].$$

Let and $'$ be reduced biaccessible dfa's. If $|| = |'|$, then and $'$ are isomorphic, i.e. there is a bijective mapping $h : Q \to Q'$ with the above properties.

Proof. Given , we define $'$ to be the quotient of mod θ. The surjective homomorphism is then the quotient map $q \mapsto q/\theta$.

Corollary 10.8.12 *Let D and D' be equivalent deterministic biaccessible presentations. There exists a deterministic biaccessible presentation E such that*

$$D \xrightarrow{\rho} E \quad \text{and} \quad D' \xrightarrow{\tau} E,$$

for some surjective base matrices ρ and τ.

Proof. By Proposition 10.8.11, the dfa's $_D$ and $_{D'}$ have a common homomorphic image . Let E be a presentation corresponding to , i.e. such that $_E$ is isomorphic to . The base matrices ρ and τ are provided by the surjective homomorphisms $_D \to$ and $_{D'} \to$.

Corollary 10.8.13 *Let D and E be two presentations. If D and E are equivalent then $D \approx E$.*

Proof. Immediate from Proposition 9.9.7.3, Proposition 10.8.9 and Corollary 10.8.12.

Suppose now that $\mathbf{Matr}(S;V)$ is a nontrivial matricial iteration theory which satisfies the dual star and omega commutative identities as well as the equations $1^* = 1$ and $1^\omega = 0$. Let

$$\varphi : X \quad \to \quad \mathbf{Matr}(S;V)$$

be a mapping with $x\varphi : 1 \to 1$ in $\mathbf{Mat}_S$, for all $x \in X$. The mapping φ extends to matrices with entries in $P_f(X)$, the set of all finite subsets of X, in a natural way. For a presentation $D = (\alpha; A; \gamma)$ we define

$$D\varphi \ := \ (\alpha; A\varphi; \gamma),$$

so that $D\varphi$ is a presentation in $\mathbf{Matr}(S;V)$ over $(\mathbf{B}; X\varphi)$, and $|D\varphi|$ is a morphism $1 \to 1$ in $\mathbf{Matr}(S;V)$. (Note that since $\mathbf{Matr}(S;V)$ is nontrivial, $\mathbf{B}$ may be considered to be a sub *-semiring of S.)

Proposition 10.8.14 *Let D and E be presentations. If $|D| = |E|$, then $|D\varphi| = |E\varphi|$.*

Proof. By Corollary 10.8.13, it suffices to prove that $|D\varphi| = |E\varphi|$ whenever one of the following four cases applies.

- $D \xrightarrow{\rho} E$ for a surjective base matrix ρ.
- $D \xrightarrow{\rho^T} E$ for a surjective base matrix ρ.
- $D \xrightarrow{\iota} E$ for an injective base matrix ι.
- $D \xrightarrow{\iota^T} E$ for an admissible injective base matrix ι.

In the first two cases the proof uses Lemmas 9.9.7.6 and 9.9.7.7, the assumption that $\mathbf{Matr}(S;V)$ satisfies the (dual) star and omega commutative identities, and is otherwise similar to the argument used in the proof of Proposition 9.9.7.9. In the third case one simply uses that the functorial star and omega implications hold for injective base matrices in any matricial iteration theory. Finally, in the last case, the proof follows along the lines of the proof of Proposition 10.8.6.

We are now ready to prove the main result of this section.

Theorem 10.8.15 *Let* $\mathbf{Matr}(S;V)$ *be a matricial iteration theory which satisfies the dual star and omega commutative identities as well as the equations* $1^* = 1$ *and* $1^\omega = 0$. *Let* φ *be a mapping*

$$X \quad \to \quad \mathbf{Matr}(S;V)$$

such that $x\varphi$ *is a morphism* $1 \to 1$ *in* $\mathbf{Mat}_S$, *for all* x *in* X. *There is a unique additive iteration theory morphism*

$$\varphi^\sharp : \L(X^*;X^\omega) \quad \to \quad \mathbf{Matr}(S;V)$$

with $\eta \cdot \varphi^\sharp = \varphi$, *where*

$$\eta : X \quad \to \quad \L(X^*;X^\omega)$$

is the mapping $x \mapsto (\{x\};\emptyset)$.

Proof. Since the statement of the theorem is obvious when $\mathbf{Matr}(S;V)$ is trivial, we may assume that $\mathbf{Matr}(S;V)$, and hence S, is nontrivial. By results proved in Section 10.4, it suffices to define $\varphi^\sharp$ on the set of morphisms $1 \to 1$. Given $(A;v) : 1 \to 1$ in $\L(X^*;X^\omega)$, there is a presentation D with $|D| = (A;v)$. We define

$$(A;v)\varphi^\sharp \;:=\; |D\varphi|.$$

The definition of $\varphi^\sharp$ is unique by Proposition 10.8.14. That $\varphi^\sharp$ preserves the constants 0 and 1 is obvious. That $\varphi^\sharp$ preserves the sum, composition and $^\otimes$ operations follows from Lemma 10.5.2. Finally, the argument that $\varphi^\sharp$ extends φ is the same as that given in the proof of Theorem 9.9.7.10.

Exercise 10.8.16 Show that each free iteration theory $\Sigma\mathbf{tr}$ embeds in a theory $\mathcal{RACL}(X^*;X^\omega)$, for some set X. Hence $\Sigma\mathbf{tr}$ embeds in both $\text{Ł}(X^*;X^\omega)$ and $\mathcal{CL}(X^*,X^\omega)$. *Hint:* Define X to be the set

$$X \quad := \quad \Sigma \cup \bigcup_{n=1}^{\infty} \Sigma_n \times [n].$$

Then let $\varphi : \Sigma\mathbf{tr} \to \mathcal{RACL}(X^*;X^\omega)$ be the iteration theory morphism determined by

$$\sigma \quad \mapsto \quad ([(\sigma,1),\ldots,(\sigma,n)];\, \sigma^\omega), \quad \sigma \in \Sigma_n,\ n \geq 0.$$

10.9 Closed Regular ω-Languages

Let X be a set and consider the embedding

$$\begin{aligned} \eta : X \quad &\to \quad \text{Ł}(X^*;X^\omega) \\ x \quad &\mapsto \quad (\{x\};\emptyset) \end{aligned}$$

of X into the matricial iteration theory $\text{Ł}(X^*;X^\omega)$ of finitary regular languages and closed regular ω-languages over X. As usual, we may write just x for $x\eta$. When $|X| > 1$, the mapping η fails to be a free embedding of X into $\text{Ł}(X^*;X^\omega)$. Indeed, let x and y be different letters in X. In the theory $\text{Ł}(X^*;X^\omega)$ we have

$$x^*y^\omega + x^\omega \quad = \quad x^*y^\omega, \tag{10.48}$$

since both sides of the equation represent the closed regular ω-language consisting of the the ω-word x^ω and all ω-words x^ny^ω, $n \geq 0$. However, substituting $0 = (\emptyset;\emptyset)$ or $1 = (\{\epsilon\};\emptyset)$ for y, the equation (10.48) no longer holds, for the right-hand side evaluates to 0. Thus not all mappings

$$\varphi : X \quad \to \quad \text{Ł}(X^*;X^\omega)$$

extend to a matricial iteration morphism

$$\varphi^\sharp : \text{Ł}(X^*;X^\omega) \quad \to \quad \text{Ł}(X^*;X^\omega)$$

with $\eta \cdot \varphi^\sharp = \varphi$, not even when each $x\varphi$ is a finitary regular language, i.e. a matrix $1 \to 1$ in $\text{Ł}(X^*)$, the underlying matrix iteration theory of the theory $\text{Ł}(X^*;X^\omega)$.

In this section we show that the above discrepancy caused by the constants 0 and 1 is the only one preventing the theories Ł$(X^*; X^\omega)$ from being free in an equational class of matricial iteration theories. We will describe the appropriate subcategory of matricial iteration theories, by restricting both the class of objects and the class of *morphisms*, and identify the theories Ł$(X^*; X^\omega)$ as the free objects in this subcategory.

Let $_0$ denote the class of all matricial iteration theories satisfying the dual star and omega commutative identities as well as the two equations $1^* = 1$ and $1^\omega = 0$. Thus each matricial iteration theory in $_0$ is idempotent. To avoid trivial situations, we assume that when $\mathbf{Matr}(S; V)$ is in $_0$ then V is a nontrivial S-module. In this case, it follows that S is nontrivial also, i.e. $0 \neq 1$. We describe a subclass of $_0$. We let consist of all matricial iteration theories $\mathbf{Matr}(S; V)$ in $_0$ such that the following implications hold, for all $a, b \in S$ and $v \in V$:

$$\begin{array}{rlcl}
\text{(i)} & av = 0 & \Rightarrow & a = 0 \text{ or } v = 0 \\
\text{(ii)} & ab = 1 & \Rightarrow & a = b = 1 \\
\text{(iii)} & a^\omega = 0 & \Rightarrow & a = 0 \text{ or } a = 1 \\
\text{(iv)} & v \neq 0 & \Rightarrow & a^*v + a^\omega = a^*v.
\end{array}$$

It then follows that if $ab = 0$ then $a = 0$ or $b = 0$. Indeed, supposing $ab = 0$, also $a(bv) = (ab)v = 0$, where $v \in V$, $v \neq 0$. Thus either $a = 0$ or $bv = 0$, in which case $b = 0$. Another observation is that if $a + b = 0$ then $a = b = 0$. This follows at once, since if $a + b = 0$ then $a = a + a + b = a + b = 0$. Similarly, if for $u, v \in V$, $u + v = 0$, then $u = v = 0$. Thus if $\mathbf{Matr}(S; V)$ is in , then S is a positive semiring and V is a positive S-module as defined in Definition 10.7.1.

By (ii), if $\mathbf{Matr}(S; V)$ is in , then the constant 1 has only the trivial representation as the product of two elements in S. A similar fact is true for the representation of 1 as the sum of two elements of S: if $a + b = 1$ then either $a = 1$ and $b \in \{0, 1\}$ or $a = 0$ and $b = 1$. Indeed, if $a + b = 1$, then

$$(a + b)^\omega = (a^*b)^\omega + (a^*b)^*a^\omega = 0,$$

so that

$$(a^*b)^*a^\omega = 0.$$

But $(a^*b)^* = 1 + (a^*b)^+$ is not 0, thus $a^\omega = 0$. By (iii), it follows that a is 0 or 1. Now it follows easily that if $a^* = 1$ then $a = 0$ or $a = 1$.

Let $\mathbf{Matr}(S;V)$ and $\mathbf{Matr}(S';V')$ be matricial iteration theories in . We say that a matricial iteration theory morphism

$$\varphi : \mathbf{Matr}(S;V) \ \rightarrow \ \mathbf{Matr}(S';V')$$

is *strict* if for all $a \in S$ and $v \in V$,

$$\begin{array}{lcl} a\varphi = 1 & \Rightarrow & a \ = \ 1, \quad \text{and} \\ v\varphi \ = \ 0 & \Rightarrow & v \ = \ 0. \end{array}$$

It then follows that if $a\varphi = 0$ then $a = 0$, for all $a \in S$, and that any matrix mapped to a relational matrix ρ is the relational matrix ρ itself.

Next we exhibit a subclass of . Note that if V is a nontrivial S-module then S is nontrivial, and if S is idempotent then so is V.

Definition 10.9.1 *Let $(S;V)$ be a semiring module pair such that S is idempotent and V is nontrivial. We call $(S;V)$ an* **idempotent iterative semiring module pair** *if the following hold.*

- *The elements $a \in S$ with $1 + a \neq a$ form an ideal $I(S)$.*
- *Each $s \in S \setminus I(S)$ can be uniquely written in the form*

$$s \ = \ 1 + a,$$

  for some $a \in I(S)$.
- *For all $a \in I(S)$ and $s \in S$ the equation in the variable x*

$$x \ = \ ax + s$$

  has a unique solution in S.
- *For all $a \in I(S)$ and $v \in V$ the equation*

$$\xi \ = \ a\xi + v$$

  has either $0 \in V$ as its unique solution, or if $a \neq 0$ or $v \neq 0$ then the equation has a unique nonzero solution in V.

Thus if $(S;V)$ is an idempotent iterative semiring module pair then S is an idempotent iterative semiring (see Definition 9.9.7.20). If X is a nonempty set and if $S = L(X^*)$ and $V = CL(X^\omega)$, then $(S;V)$ is an idempotent iterative semiring module pair, as is $(RL(X^*);\ RCL(X^\omega))$, cf. Example 10.7.18. The following result is an immediate consequence of the Matricial Extension Theorem 10.7.4, see also Remark 10.7.17.

Theorem 10.9.2 *Let $(S;V)$ be an idempotent iterative semiring module pair. There is a unique way to define the * and $^\omega$ operations on $\mathbf{Mat}_S$ such that $\mathbf{Matr}(S;V)$ becomes a matricial iteration theory satisfying the equations $1^* = 1$ and $1^\omega = 0$ and such that for all $a \in I(S)$, if $a \neq 0$ then $a^\omega \neq 0$. Further $\mathbf{Matr}(S;V)$ satisfies the functorial star and omega implications for surjective base matrices and transposes of surjective base matrices.*

Below we show that the resulting matricial iteration theory is in . Thus Theorem 10.9.2 provides yet another proof that Ł$(X^*;X^\omega)$ and Ł$(X^*;X^\omega)$ are matricial iteration theories. Further, both satisfy the functorial star and omega implications for surjective base matrices and transposes of surjective base matrices, and thus the dual star and omega commutative identities.

Proposition 10.9.3 *Let $(S;V)$ be an idempotent iterative semiring module pair. If $\mathbf{Matr}(S;V)$ is turned into a matricial iteration theory as in Theorem 10.9.2, then $\mathbf{Matr}(S;V)$ is in .*

Proof. Since V is nontrivial and since the the equations $1^* = 1$ and $1^\omega = 0$ as well as the dual commutative identities hold in $\mathbf{Matr}(S;V)$, the theory $\mathbf{Matr}(S;V)$ is in ${}_0$. We note that if $a \in I(S)$ and $v \in V$ are not both zero, then the unique nonzero solution of the equation

$$\xi \;=\; a\xi + v$$

is $a^*v + a^\omega$. Indeed, if $v \neq 0$ then $a^*v = a^+v + v \neq 0$, and if $a \neq 0$ then $a^\omega \neq 0$, so that $a^*v + a^\omega$ is not zero. Also

$$\begin{aligned} a(a^*v + a^\omega) + v &= aa^*v + v + aa^\omega \\ &= (aa^* + 1)v + a^\omega \\ &= a^*v + a^\omega. \end{aligned}$$

Another observation is that

$$s^* \ = \ a^* \quad \text{and} \quad s^\omega \ = \ a^\omega,$$

for all $s = 1 + a$ with $a \in I(S)$, see Lemma 10.8.1.

It is now clear that if $s^\omega = 0$, for some $s \in S$, then either $s = 0$ or $s = 1$. Indeed, if $s \in I(S)$ then $s = 0$. Otherwise, $s = 1 + a$ with $a \in I(S)$, and $s^\omega = a^\omega = 0$, so that $a = 0$ and $s = 1$.

Assume now that $sv = 0$, for some $s \in S$ and $v \in V$, $v \neq 0$. Since for all $a \in I(S)$, $(1 + a)v = v + av \neq 0$, s must be in $I(S)$. But then v is the unique nonzero solution of the equation

$$\xi \ = \ s\xi + v,$$

so that

$$s^*v + s^\omega \ = \ v.$$

Thus

$$s^+v + s^\omega \ = \ s(s^*v + s^\omega) \ = \ sv \ = \ 0,$$

so that $s^\omega = 0$. Since s is in $I(S)$, we conclude that $s = 0$.

Suppose now that $s_1, s_2 \in S$ with $s_1 s_2 = 1$. We must show that $s_1 = s_2 = 1$. Indeed, since $I(S)$ is an ideal and $1 \notin I(S)$, the elements s_1 and s_2 are not in $I(S)$. Thus, $s_1 = 1 + a$ and $s_2 = 1 + b$, for some $a, b \in I(S)$. But then

$$1 \ = \ (1 + a)(1 + b) \ = \ 1 + a + b + ab,$$

and since 1 is uniquely written as the sum of 1 itself with an element of $I(S)$, it follows that each of a, b and ab is 0. Thus $s_1 = s_2 = 1$.

We still must prove that for all $s \in S$ and $v \in V$, if $v \neq 0$ then

$$s^*v + s^\omega \ = \ s^*v.$$

First let $s = a \in I(S)$. Since $v \neq 0$, both $a^*v + a^\omega$ and a^*v are nonzero solutions of the equation $\xi = a\xi + v$. Thus $a^*v + a^\omega = a^*v$. Now, supposing $s \notin I(S)$ we can write s as $s = 1 + a$ for some $a \in I(S)$. Thus,

$$s^*v + s^\omega \ = \ a^*v + a^\omega \ = \ a^*v \ = \ s^*v.$$

Recall that if S is an idempotent semiring, then the relation $\leq$ on S defined by

$$a \leq b \Leftrightarrow a + b = b$$

is a partial order. Further, if V is a S-module, then the relation on V,

$$u \leq v \Leftrightarrow u + v = v,$$

is a partial order on V. The partial orders on S and V extend pointwise to morphisms in $\mathbf{Matr}(S;V)$. Thus, if $f = (A;u),\ g = (B;v) : n \to p$ are in $\mathbf{Matr}(S;V)$, then we write $f \leq g$ if $A_{ij} \leq B_{ij}$ and $u_i \leq v_i$, for all $i \in [n]$ and $j \in [p]$. It is straightforward to see that the matrix theory operations are monotone, as are the * and $^\omega$ operations when $\mathbf{Matr}(S;V)$ is a matricial iteration theory. Indeed, if $A, B : n \to n$ in $\mathbf{Mat}_S$, then

$$(A+B)^\omega = (A^*B)^\omega + (A^*B)^*A^\omega \geq A^\omega.$$

Note that if $\mathbf{Matr}(S;V)$ is in and if $0 \leq a \leq 1$ in S, then either $a = 0$ or $a = 1$. Thus a matrix $A : n \to p$ in $\mathbf{Mat}_S$ is a relational matrix if and only if $A \leq B$, where B is the $n \to p$ matrix, all of whose entries are 1.

The following two lemmas will be useful in the sequel.

Lemma 10.9.4 *Let* $\mathbf{Matr}(S;V)$ *be a Conway matricial theory. If the equations* $1^* = 1$ *and* $1^\omega = 0$ *hold in* $\mathbf{Matr}(S;V)$*, then so do the following for all* $A : n \to n$ *in* $\mathbf{Mat}_S$*.*

$$\begin{array}{rlcl}
\text{(i)} & A^*A^* & = & A^* \\
\text{(ii)} & A^{**} & = & A^* \\
\text{(iii)} & (\mathbf{1}_n + A)^\omega & = & A^\omega \\
\text{(iv)} & A^{*\omega} & = & A^\omega \\
\text{(v)} & A^*A^\omega & = & A^\omega
\end{array}$$

Proof. The proof of the first three equations is left to the reader. For $n = 1$, (ii) was established in Lemma 10.8.1. We prove (iv). Since $A + A^* = A^*$, we have

$$\begin{aligned}
A^{*\omega} &= (A^*A^*)^\omega + (A^*A^*)^*A^\omega \\
&= A^{*\omega} + A^{**}A^\omega.
\end{aligned}$$

Also

$$(\mathbf{1}_n + A)^\omega \;=\; A^{*\omega} + A^{**}A^\omega,$$

so that

$$A^\omega \;=\; (\mathbf{1}_n + A)^\omega \;=\; A^{*\omega}.$$

We use (iv) to prove (v):

$$A^*A^\omega \;=\; A^*A^{*\omega} \;=\; A^{*\omega} \;=\; A^\omega.$$

Lemma 10.9.5 *Let* $\mathbf{Matr}(S;V)$ *be a Conway matricial theory in which the equations* $1^* = 1$ *and* $1^\omega = 0$ *hold. If* $A : n \to n$ *is a relational matrix then* $A^\omega = 0_{n0}$.

Proof. Let $B : n \to n$ be the relational matrix all of whose entries are 1. Since $A^\omega \leq B^\omega$, it suffices to show that $B^\omega = 0_{n0}$. This can be seen by induction on n together with the fact that $B^* = B$.

We establish a few more properties of the matricial iteration theories in .

Proposition 10.9.6 *Let* $\mathbf{Matr}(S;V)$ *be a matricial iteration theory in . Let* $A : n \to n$ *be a matrix in* $\mathbf{Mat}_S$ *and* $v : n \to 0$ *an* n*-vector. If no component of* v *is* 0 *then* $A^*v \geq A^\omega$.

Proof. If $n = 0$ this is obvious. If $n = 1$, then $A^*v \geq A^\omega$, by assumption. We proceed by induction on n. If $n > 1$ we can write

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad v \;=\; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where $a : n-1 \to n-1$, $b : n-1 \to 1$, $c : 1 \to n-1$ and $d : 1 \to 1$ in $\mathbf{Mat}_S$, and where $v_1 : n-1 \to 0$ and $v_2 : 1 \to 0$. Since $v_2 \neq 0$,

$$(d + ca^*b)^*v_2 \;\geq\; (d + ca^*b)^\omega \quad \text{and} \quad d^*v_2 \;\geq\; d^\omega.$$

Also

$$a^*v_1 \;\geq\; a^\omega \quad \text{and} \quad (a + bd^*c)^*v_1 \;\geq\; (a + bd^*c)^\omega,$$

by the induction hypothesis. Thus

$$\begin{array}{rcl} A^*v & = & \begin{pmatrix} (a+bd^*c)^*v_1 + (a+bd^*c)^*bd^*v_2 \\ (d+ca^*b)^*ca^*v_1 + (d+ca^*b)^*v_2 \end{pmatrix} \\ & \geq & \begin{pmatrix} (a+bd^*c)^\omega + (a+bd^*c)^*bd^\omega \\ (d+ca^*b)^*ca^\omega + (d+ca^*b)^\omega \end{pmatrix} \\ & = & A^\omega. \end{array}$$

Proposition 10.9.7 *Let* $\mathbf{Matr}(S;V)$ *be a matricial iteration theory in . Suppose that* $a : n \to n$, $b : n \to m$, $c : m \to n$ *and* $d : m \to m$ *are matrices in* $\mathbf{Mat}_S$. *Let* $v : n \to 0$ *be an* n*-vector. Suppose that each row of* d^*c *contains a nonzero entry and that no component of* v *is* 0. *Then*

$$(a+bd^*c)^*v \ \geq \ (a+bd^*c)^*bd^\omega$$

and

$$(d+ca^*b)^*ca^*v \ \geq \ (d+ca^*b)^\omega.$$

Proof. First, note that our assumptions imply that no component of d^*cv is 0. Since by Proposition 10.9.6 and Lemma 10.9.4

$$\begin{array}{rcl} (bd^*c)^*v & \geq & bd^*cv \\ & = & bd^*d^*cv \\ & \geq & bd^\omega, \end{array}$$

we have

$$\begin{array}{rcl} (a+bd^*c)^*v & = & ((bd^*c)^*a)^*(bd^*c)^*v \\ & = & ((bd^*c)^*a)^*(bd^*c)^*(bd^*c)^*v \\ & \geq & (a+bd^*c)^*bd^\omega. \end{array}$$

This proves the first inequality. For the second, since

$$(d+ca^*b)^\omega \ = \ (d^*ca^*b)^\omega + (d^*ca^*b)^*d^\omega,$$

we must show that

$$(d+ca^*b)^*ca^*v \ \geq \ (d^*ca^*b)^\omega$$

and

$$(d + ca^*b)^*ca^*v \ \geq \ (d^*ca^*b)^*d^\omega.$$

Since $d^*ca^*v \geq d^*cv$, no component of d^*ca^*v is 0. Thus

$$\begin{aligned} (d + ca^*b)^*ca^*v &= (d^*ca^*b)^*d^*ca^*v \\ &\geq (d^*ca^*b)^\omega \end{aligned}$$

and

$$\begin{aligned} (d + ca^*b)^*ca^*v &= (d^*ca^*b)^*d^*ca^*v \\ &= (d^*ca^*b)^*d^*d^*ca^*v \\ &\geq (d^*ca^*b)^*d^\omega, \end{aligned}$$

by Proposition 10.9.6.

Proposition 10.9.8 *Let* $\mathbf{Matr}(S;V)$ *be a matricial iteration theory in and let* $a : m \to n$, $b : n \to n$ *and* $c : n \to m$ *be matrices in* $\mathbf{Mat}_S$*. If no entry of* c *is* 0*, then* $(ab^*c)^\omega \geq ab^\omega$*.*

Proof. This is obvious if $m = 0$ or $n = 0$, so that we may assume $m > 0$ and $n > 0$. Further it suffices to prove the statement only in the case that exactly one entry of a is nonzero. Indeed, we can write a as the sum $a = a_1 + \ldots + a_k$ where each matrix a_i has exactly one nonzero entry. If

$$(a_ib^*c)^\omega \ \geq \ a_ib^\omega$$

holds for all $i \in [k]$, then

$$\begin{aligned} (ab^*c)^\omega &= ((a_1 + \ldots + a_k)b^*c)^\omega \\ &= (a_1b^*c + \ldots + a_kb^*c)^\omega \\ &\geq (a_1b^*c)^\omega + \ldots + (a_kb^*c)^\omega \\ &\geq a_1b^\omega + \ldots + a_kb^\omega \\ &= (a_1 + \ldots + a_k)b^\omega \\ &= ab^\omega. \end{aligned}$$

Another possible reduction is that by the star and omega permutation identities, it suffices to prove that

$$1_m \cdot (ab^*c)^\omega \ \geq \ 1_m \cdot ab^\omega,$$

and even then, just for the case that the only nonzero entry of a is in the left upper corner. We will prove this latter inequality by induction on n.

If $n = 1$ we can write

$$a \;=\; \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad c \;=\; \begin{bmatrix} c_1 & \dots & c_m \end{bmatrix}.$$

Thus

$$\begin{aligned} 1_m \cdot (ab^*c)^\omega &= 1_m \cdot \begin{bmatrix} a_1b^*c_1 & \dots & a_1b^*c_m \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}^\omega \\ &= (a_1b^*c_1)^\omega \\ &= a_1b^*(c_1a_1b^*)^\omega. \end{aligned}$$

Since by assumption, $a_1 \neq 0$ and $c_1 \neq 0$, $c_1a_1b^* \neq 0$. If $c_1a_1b^* = 1$, we have $a_1 = c_1 = 1$ and $b \in \{0, 1\}$. Thus

$$1_m \cdot (ab^*c)^\omega \;=\; 1_m \cdot ab^\omega \;=\; 0.$$

Otherwise $(c_1a_1b^*)^\omega \neq 0$ so that

$$\begin{aligned} 1_m \cdot (ab^*c)^\omega &= a_1b^*(c_1a_1b^*)^\omega \\ &\geq a_1b^\omega \\ &= 1_m \cdot ab^\omega, \end{aligned}$$

by the assumptions on $\mathbf{Matr}(S; V)$. We have thus proved the basis step of the induction.

If $n > 1$ we can write the matrices a, b and c as

$$a \;=\; \begin{bmatrix} a_1 & 0 \end{bmatrix}, \quad b \;=\; \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \text{and} \quad c \;=\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where $a_1 : m \to n-1$, $b_{11} : n-1 \to n-1$, $b_{12} : n-1 \to 1$, $b_{21} : 1 \to n-1$, $b_{22} : 1 \to 1$ and $c_1 : n-1 \to m$ and $c_2 : 1 \to m$. Thus

$1_m \cdot (ab^*c)^\omega =$

$$\begin{aligned}
&= 1_m \cdot (a_1(b_{11} + b_{12}b_{22}^*b_{21})^*c_1 + a_1(b_{11} + b_{12}b_{22}^*b_{21})^*b_{12}b_{22}^*c_2)^\omega \\
&\geq 1_m \cdot ((a_1(b_{11} + b_{12}b_{22}^*b_{21})^*c_1)^\omega + (a_1(b_{11} + b_{12}b_{22}^*b_{21})^*b_{12}b_{22}^*c_2)^\omega) \\
&\geq 1_m \cdot (a_1(b_{11} + b_{12}b_{22}^*b_{21})^\omega + a_1(b_{11} + b_{12}b_{22}^*b_{21})^*b_{12}b_{22}^\omega),
\end{aligned}$$

by the induction hypothesis,

$$\begin{aligned}
&= 1_m \cdot \left[\begin{array}{cc} a_1 & 0 \end{array}\right] \cdot \left[\begin{array}{c} (b_{11} + b_{12}b_{22}^*b_{21})^\omega + (b_{11} + b_{12}b_{22}^*b_{21})^*b_{12}b_{22}^\omega \\ (b_{22} + b_{21}b_{11}^*b_{12})^\omega + (b_{22} + b_{21}b_{11}^*b_{12})^*b_{21}b_{11}^\omega \end{array}\right] \\
&= 1_m \cdot ab^\omega.
\end{aligned}$$

Let $\mathbf{Matr}(S;V)$ be a matricial iteration theory in and let $A : n \to n$ be a matrix in $\mathbf{Mat}_S$. Below we will characterize those integers $i \in [n]$ for which $i_n \cdot A^\omega \neq 0$. We say that an integer $j_0 \in [n]$ is *eventually nonnullable* in A if there is a sequence of nonzero entries

$$A_{j_0j_1}, \ldots, A_{j_kj_{k+1}}, \ldots, A_{j_{k+m-1}j_k}$$

$(k \geq 0,\ m \geq 1)$ such that at least one of $A_{j_kj_{k+1}}, \ldots, A_{j_{k+m-1}j_k}$ is not 1. Otherwise, we say that j_0 is eventually nullable in A.

Proposition 10.9.9 *Suppose that* $\mathbf{Matr}(S;V)$ *is in and that* $A : n \to n$ *is a matrix in* $\mathbf{Mat}_S$*. Let* $i \in [n]$*. Then* $i_n \cdot A^\omega \neq 0$ *if and only if* i *is eventually nonnullable in* A*.*

Proof. Suppose that $i \in [n]$ is eventually nonnullable in A. Let

$$A_{j_0j_1}, \ldots, A_{j_kj_{k+1}}, \ldots, A_{j_{k+m-1}j_k}$$

be a sequence of entries as above with $j_0 = i$. Define

$$\begin{aligned}
s_0 &:= A_{j_0j_1} \cdot \ldots \cdot A_{j_{k-1}j_k} \\
s_1 &:= A_{j_kj_{k+1}} \cdot \ldots \cdot A_{j_{k+m-1}j_k},
\end{aligned}$$

so that $s_0 \neq 0$ and $s_1 \notin \{0, 1\}$ by our assumptions. By Exercise 10.2.25,

$$\begin{aligned}
i_n \cdot A^\omega &= i_n \cdot A^kA^\omega \\
&= i_n \cdot A^k(A^m)^\omega \\
&\geq s_0s_1^\omega,
\end{aligned}$$

and since $s_1^\omega \neq 0$, it follows that $i_n \cdot A^\omega \neq 0$.

Now let I be the set of all integers $i \in [n]$ which are eventually nullable in A. We show that if $i \in I$ then $i_n \cdot A^\omega = 0$. By the omega permutation identity we may assume that $I = [m]$ for some $m \leq n$. It is easy to see that A is of the form

$$A \;=\; \begin{bmatrix} a & 0 \\ c & d \end{bmatrix},$$

where $a : m \to m$, $c : n - m \to m$ and $d : n - m \to n - m$. Further, a is a relational matrix. Thus, for $i \in [m]$,

$$i_n \cdot A^\omega \;=\; i_m \cdot a^\omega \;=\; 0$$

by Lemma 10.9.5.

For the remainder of this section we suppose that a nonempty set X is fixed. We work with two types of presentations. A *presentation of type* 1 is what we called a presentation in Section 9.7 of Chapter 9 or in Section 10.8 above. (See also Definition 10.5.6.) If

$$D \;=\; (\alpha; A; \gamma)$$

is a presentation of type 1, its behavior is the finitary regular language

$$|D| \;:=\; \alpha \cdot A^* \cdot \gamma.$$

which was denoted $|D|$ in Section 9.9.7 and $|D|_1$ in Section 10.8. Below we redefine the notion of a type 0 presentation.

Definition 10.9.10 *A* **type** 0 **presentation** *of weight s is an ordered pair*

$$E \;:=\; (\beta; B),$$

where $\beta : 1 \to s$ is a relational matrix and $B : s \to s$ is a matrix with entries in $P_f(X)$, considered as a matrix in the underlying matrix iteration theory $Ł(X^*)$ of the matricial iteration theory $Ł(X^*; X^\omega)$. The **behavior** *of E is the closed regular ω-language*

$$|E| \;:=\; \beta \cdot B^\omega$$

in $Ł(X^*; X^\omega)$. Two type 0 presentations E and E' are **equivalent** *if $|E| = |E'|$.*

Let $D = (\alpha; A)$ be a type 0 presentation of weight s. There is a corresponding type 1 presentation

$$\overline{D} \;:=\; (\alpha; A; \gamma)$$

where γ is the zero matrix $s \to 1$. We may thus apply properties of (type 1) presentations such as accessibility, coaccessibility, etc. to type 0 presentations. Concerning coaccessibility of type 0 presentations, we use Definition 10.8.5 in Section 10.8, so that a type 0 presentation D is coaccessible if there is an infinite run starting at each state of the nfa corresponding to $\overline{D}$. Also we write

$$D \xrightarrow{\rho} E,$$

for two type 0 presentations $D = (\alpha; A)$ and $E = (\beta; B)$ and a relational matrix ρ of appropriate dimensions, if $\overline{D} \xrightarrow{\rho} \overline{E}$, for the corresponding type 1 presentations, i.e. when

$$\alpha \cdot \rho \;=\; \beta \quad \text{and} \quad A \cdot \rho \;=\; \rho \cdot B.$$

Given a type 0 presentation D, its behavior $|D|$ is related to the behavior of $\overline{D}$ as defined in Section 10.8. Indeed, if $D = (\alpha; A)$, then

$$|\overline{D}|_0 \;=\; \alpha \cdot A^\omega$$

is given by the same formula as $|D|$ except that the operations are evaluated in the theory Ł$(X^*; X^\omega)$ rather than in the theory Ł$(X^*; X^\omega)$ or in Ł$(X^*; X^\omega)$. However, $|\overline{D}|_0$ is closed, so that

$$|D| \;=\; \mathrm{adh}(|\overline{D}|_0) \;=\; |\overline{D}|_0.$$

Thus two type 0 presentations D and E are equivalent in the sense of Definition 10.9.10 above if and only if the type 1 presentations $\overline{D}$ and $\overline{E}$ are equivalent as defined in Section 10.8.

Corollary 10.9.11 *Let D and E be two type 0 presentations. Then $|D| = |E|$ if and only if there is a sequence*

$$D = D_0, \ldots, D_n = E$$

of type 0 presentations such that for all $i \in [n]$ either $D_{i-1} \xrightarrow{\rho_i} D_i$ or $D_i \xrightarrow{\rho_i} D_{i-1}$, where each ρ_i is one of the following: a surjective base matrix, an injective base matrix, the transpose of a surjective base matrix, or the transpose of an admissible injective base matrix.

Proof. For the definition of an admissible injective base matrix we refer to Definition 10.8.5. If $|D| = |E|$ then by Corollary 10.8.13 there exists a sequence $E_0, \ldots, E_n$ of type 1 presentations $E_i = (\alpha_i; A_i; \gamma_i)$ such that $\overline{D} = E_0$, $\overline{E} = E_n$ and for all $i \in [n]$, either $E_{i-1} \xrightarrow{\rho_i} E_i$ or $E_i \xrightarrow{\rho_i} E_{i-1}$ holds, where each ρ_i is a matrix as above. In fact each E_i may be given such that γ_i is zero. Thus the sequence of type 0 presentations $D_i := (\alpha_i; A_i)$, $i \in [n]$, will do. The converse direction follows from Corollary 10.8.8.

Now let $\mathbf{Matr}(S; V)$ be a matricial iteration theory in and let

$$\varphi : X \quad \to \quad \mathbf{Matr}(S; V)$$

be a mapping such that each $x\varphi$ is a matrix $1 \to 1$ in $\mathbf{Mat}_S$, i.e. an element of the semiring S. The mapping φ extends to matrices with entries in $P_f(X)$ as before. For a type 1 presentation $D = (\alpha; A; \gamma)$ define

$$D\varphi \quad := \quad (\alpha; A\varphi; \gamma)$$

and

$$|D\varphi| \quad := \quad \alpha \cdot (A\varphi)^* \cdot \gamma.$$

For a type 0 presentation $E = (\beta; B)$ define

$$E\varphi \quad := \quad (\beta; B\varphi)$$

and

$$|E\varphi| \quad := \quad \beta \cdot (B\varphi)^\omega.$$

Proposition 10.9.12 *Let D and D' be type 1 presentations and let E and E' be type 0 presentations. If $|D| = |D'|$ then $|D\varphi| = |D'\varphi|$. If $|E| = |E'|$ then $|E\varphi| = |E'\varphi|$.*

Proof. The proof that $|D\varphi| = |D'\varphi|$ whenever $|D| = |D'|$ is the same as that of Proposition 9.9.7.9. Supposing that $|E| = |E'|$, the presentations E and E' are related as in Corollary 10.9.11. Thus it suffices to prove that if $E \xrightarrow{\rho} E'$, where ρ is a surjective base matrix, the transpose of a surjective base matrix, an injective base matrix, or the transpose of an admissible injective base matrix, then $|E\varphi| = |E'\varphi|$. The argument which makes use of Lemmas 9.9.7.6 and 9.9.7.7 is similar to the proof of Proposition 10.8.14.

Recall that X is a fixed nonempty set and that

$$\eta : X \quad \to \quad Ł(X^*; X^\omega)$$

is the mapping which takes a letter x to the morphism

$$x\eta \quad = \quad (\{x\}; \emptyset) : 1 \to 1$$

in $Ł(X^*; X^\omega)$.

Theorem 10.9.13 *Let* $\mathbf{Matr}(S; V)$ *be a matricial iteration theory in and let*

$$\varphi : X \quad \to \quad \mathbf{Matr}(S; V)$$

be a mapping such that each $x\varphi$ *is a matrix* $1 \to 1$ *in* $\mathbf{Mat}_S$*, i.e. an element of the semiring* S*. Suppose further that each* $x\varphi$ *is different from the constants* 0 *and* 1*. There is a unique matricial iteration theory morphism*

$$\varphi^\sharp : Ł(X^*; X^\omega) \quad \to \quad \mathbf{Matr}(S; V)$$

such that $\eta \cdot \varphi^\sharp = \varphi$*; in addition* $\varphi^\sharp$ *is strict.*

Proof. We need to define $\varphi^\sharp$ only on finitary regular languages in X^* and on closed regular ω-languages in X^ω, i.e. on the matrices $1 \to 1$ in the underlying matrix iteration theory $Ł(X^*)$ of the matricial iteration theory $Ł(X^*; X^\omega)$ and on the set of morphisms $1 \to 0$ in $Ł(X^*; X^\omega)$. Given a finitary regular language a there is a type 1 presentation D with $|D| = a$. We define

$$a\varphi^\sharp \quad := \quad |D\varphi|.$$

For a closed regular ω-language v we define

$$v\varphi^\sharp \quad := \quad |E\varphi|,$$

where E is any type 0 presentation with $|E| = v$. By Proposition 10.9.12, $\varphi^\sharp$ is uniquely defined. Further $\eta \cdot \varphi^\sharp = \varphi$. By Theorem 9.9.7.10, the restriction of $\varphi^\sharp$ to the underlying matrix iteration theory $Ł(X^*)$ of $Ł(X^*; X^\omega)$ is a matrix iteration theory morphism $Ł(X^*) \to \mathbf{Mat}_S$. We will use this fact several times without explicit mention below.

We show that $\varphi^\sharp$ is strict. First we treat the constant $0 : 1 \to 1$. Let a be a nonempty regular language in X^* and let $x_1 \ldots x_n$ be a word in a. Since

$$\{x_1 \ldots x_n\} \subseteq a$$

also

$$(x_1\varphi) \ldots (x_n\varphi) = (\{x_1 \ldots x_n\})\varphi^\sharp \leq a\varphi^\sharp,$$

and since no $x\varphi$ is 0, it follows that $a\varphi^\sharp \neq 0$. The proof that $a\varphi^\sharp \neq 1$ whenever $a \neq \{\epsilon\}$ is similar. One uses the fact that no $x\varphi$, $x \in X$, is 1, as well as some conditions defining the class .

We now turn to the constant 0_{10}. It is obvious that $0_{10}\varphi^\sharp = 0_{10}$. Suppose that $v \subseteq X^\omega$ is a nonempty closed regular ω-language. Let $E = (\beta; B)$ be a type 0 presentation of weight s with $|E| = v$. Since v is nonempty, by Proposition 10.9.9 and the positivity of the matricial iteration theory $Ł(X^*; X^\omega)$ there is an integer $i \in [s]$ which is eventually nonnullable in B and such that $\beta_i = 1$. But then i is eventually nonnullable in $B\varphi$, so that $v\varphi^\sharp \neq 0_{10}$.

Since $\varphi^\sharp$ is clearly unique, to complete the proof we must show that if $a \subseteq X^*$ is a finitary regular language and if $u, v \subseteq X^\omega$ are closed regular ω-languages, then

$$(a \cdot v)\varphi^\sharp = a\varphi^\sharp \cdot v\varphi^\sharp$$

$$(a^\omega)\varphi^\sharp = (a\varphi^\sharp)^\omega$$

and

$$(u + v)\varphi^\sharp = u\varphi^\sharp + v\varphi^\sharp.$$

These facts follow from the propositions below.

Proposition 10.9.14 *Let $D = (\alpha; A; \gamma)$ be a type 1 coaccessible presentation of weight s and let $E = (\beta; B)$ be a type 0 presentation of weight r. Suppose that $|E| \neq 0_{10}$. Define the type 0 presentation of weight $s + r$ by:*

$$D \cdot E := \left(\begin{bmatrix} \alpha & 0 \end{bmatrix} ; \begin{bmatrix} A & \gamma\beta B \\ 0 & B \end{bmatrix} \right).$$

Then

$$|D \cdot E| \;\; = \;\; |D| \cdot |E|$$

and

$$|(D \cdot E)\varphi| \;\; = \;\; |D\varphi| \cdot |E\varphi|.$$

Proof. We have

$$\begin{aligned} |D \cdot E| &= \begin{bmatrix} \alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} A & \gamma\beta B \\ 0 & B \end{bmatrix}^{\omega} \\ &= \alpha A^{\omega} + \alpha A^{*}\gamma \cdot \beta B^{\omega}, \end{aligned}$$

and

$$|D| \cdot |E| \;\; = \;\; \alpha A^{*}\gamma \cdot \beta B^{\omega}.$$

Thus we must show that

$$\alpha A^{\omega} \;\; \leq \;\; \alpha A^{*}\gamma \cdot \beta B^{\omega}. \tag{10.49}$$

By the star permutation identity it suffices to consider the case that the first n entries of γ are 1 and the last m entries 0, for some integers n and m with $n + m = s$. Thus

$$\gamma \;\; = \;\; \begin{bmatrix} \gamma' \\ 0 \end{bmatrix},$$

where γ' is the $n \to 1$ base matrix. Let us partition A as

$$A \;\; = \;\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a : n \to n$, $b : n \to m$, $c : m \to n$ and $d : m \to m$. Then

$$A^{\omega} \;\; = \;\; \begin{bmatrix} (a + bd^{*}c)^{\omega} + (a + bd^{*}c)^{*}bd^{\omega} \\ (d + ca^{*}b)^{\omega} + (d + ca^{*}b)^{*}ca^{\omega} \end{bmatrix}$$

and

$$A^{*}\gamma \cdot \beta B^{\omega} \;\; = \;\; \begin{bmatrix} (a + bd^{*}c)^{*}\gamma'\beta B^{\omega} \\ (d + ca^{*}b)^{*}ca^{*}\gamma'\beta B^{\omega} \end{bmatrix}.$$

Since no component of $\gamma'\beta B^\omega$ is 0, we have

$$(a+bd^*c)^*\gamma'\beta B^\omega \quad \geq \quad (a+bd^*c)^\omega$$

and

$$(d+ca^*b)^*ca^*\gamma'\beta B^\omega \quad \geq \quad (d+ca^*b)^*ca^\omega,$$

by Proposition 10.9.6. Since D is coaccessible, each row of d^*c contains a nonzero entry. Also no entry of $\gamma'\beta B^\omega$ is 0. Thus

$$(a+bd^*c)^*\gamma'\beta B^\omega \quad \geq \quad (a+bd^*c)^*bd^\omega$$

and

$$(d+ca^*b)^*ca^*\gamma'\beta B^\omega \quad \geq \quad (d+ca^*b)^\omega,$$

by Proposition 10.9.7, proving (10.49).

The proof that $|(D\cdot E)\varphi| = |D\varphi|\cdot|E\varphi|$ is similar. One uses the fact, established above, that $\varphi^\sharp$ is strict in the constants.

Proposition 10.9.15 *Let $D=(\alpha;A;\gamma)$ be a deterministic biaccessible type 1 presentation of weight s. Define the type 0 presentation of weight s by:*

$$D^\omega \quad := \quad (\alpha;\, A+\gamma\alpha A).$$

Then

$$|D^\omega| \quad = \quad |D|^\omega$$

and

$$|D^\omega\varphi| \quad = \quad |D\varphi|^\omega.$$

Proof. We have

$$\begin{aligned}
|D^\omega| &= \alpha(A+\gamma\alpha A)^\omega\\
&= \alpha(A^*\gamma\alpha A)^\omega + \alpha(A^*\gamma\alpha A)^*A^\omega\\
&= \alpha A^*\gamma(\alpha A^+\gamma)^\omega + \alpha(A^*\gamma(\alpha A^+\gamma)^*\alpha A + \mathbf{1}_s)A^\omega\\
&= \alpha A^*\gamma(\alpha A^*\gamma)^\omega + \alpha A^*\gamma(\alpha A^+\gamma)^*\alpha A^\omega + \alpha A^\omega\\
&= (\alpha A^*\gamma)^\omega + \alpha A^*\gamma(\alpha A^+\gamma)^*\alpha A^\omega + \alpha A^\omega\\
&= (\alpha A^*\gamma)^\omega + ((\alpha A^*\gamma)^+ + \mathbf{1}_s)\alpha A^\omega\\
&= (\alpha A^*\gamma)^\omega + (\alpha A^*\gamma)^*\alpha A^\omega.
\end{aligned}$$

Note that we have used the fact that $\gamma\alpha \in \{0, 1\}$ and the fact, proved above, that

$$(1 + a)^* \;=\; a^* \quad \text{and} \quad (1 + a)^\omega \;=\; a^\omega$$

in any matricial iteration theory satisfying $1^* = 1$ and $1^\omega = 0$. Also

$$\begin{aligned} |D|^\omega &= (\alpha A^* \gamma)^\omega \\ &= \alpha A^* \gamma (\alpha A^* \gamma)^\omega \\ &= (\alpha A^* \gamma)^* (\alpha A^* \gamma)^\omega, \end{aligned}$$

by Lemma 10.9.4. Thus we need to show that

$$(\alpha A^* \gamma)^\omega \;\geq\; \alpha A^\omega,$$

which is obvious when $\alpha A^\omega = 0_{10}$. Let $\alpha A^\omega \neq 0_{10}$. By the star and omega permutation identities we may assume that γ is of the form

$$\gamma \;=\; \begin{bmatrix} \gamma' \\ 0 \end{bmatrix},$$

where all entries of $\gamma' : n \to 1$, $0 \leq n \leq s$, are 1, i.e. γ' is the unique base matrix $n \to 1$. Let $m := s - n$ and write A as

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a : n \to n$, $b : n \to m$, $c : m \to n$ and $d : m \to n$. Since D is deterministic and $\alpha A^\omega \neq 0_{10}$, exactly one entry of α is 1. We will work out the proof only in the case that $\alpha_i = 1$ for some $i \in [n]$. Thus we can write

$$\alpha \;=\; \begin{bmatrix} \alpha' & 0 \end{bmatrix}$$

for some $\alpha' : 1 \to n$. Since

$$\begin{aligned} (\alpha A^* \gamma)^\omega &= \left(\begin{bmatrix} \alpha' & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \cdot \begin{bmatrix} \gamma' \\ 0 \end{bmatrix} \right)^\omega \\ &= (\alpha'(a + bd^*c)^* \gamma')^\omega \end{aligned}$$

and

$$\alpha A^\omega \;=\; \alpha'(a + bd^*c)^\omega + \alpha'(a + bd^*c)^* bd^\omega,$$

it will follow that $(\alpha A^*\gamma)^\omega \geq \alpha A^\omega$ if we can show

$$(\alpha'(a+bd^*c)^*\gamma')^\omega \quad \geq \quad \alpha'(a+bd^*c)^\omega$$

and

$$(\alpha'(a+bd^*c)^*\gamma')^\omega \quad \geq \quad \alpha'(a+bd^*c)^*bd^\omega.$$

The first inequality is immediate from Proposition 10.9.8, since no entry of γ' is 0. The second inequality is derived as follows, making use of Lemma 10.9.4.

$$\begin{aligned}
&(\alpha'(a+bd^*c)^*\gamma')^\omega = \\
&= \alpha'(a+bd^*c)^*\gamma' \cdot (\alpha'(a+bd^*c)^*\gamma')^\omega \\
&= \alpha'((bd^*c)^*a)^*(bd^*c)^*\gamma' \cdot (\alpha'((bd^*c)^*a)^*(bd^*c)^*\gamma')^\omega \\
&= \alpha'((bd^*c)^*a)^*(bd^*c)^* \cdot (bd^*c)^*\gamma' \cdot (\alpha'((bd^*c)^*a)^*(bd^*c)^*(bd^*c)^*\gamma')^\omega \\
&= \alpha'(a+bd^*c)^* \cdot ((bd^*c)^*\gamma'\alpha'((bd^*c)^*a)^*(bd^*c)^*)^\omega \\
&= \alpha'(a+bd^*c)^* \cdot ((bd^*c)^*\gamma'\alpha'(a+bd^*c)^*)^\omega \\
&\geq \alpha'(a+bd^*c)^* \cdot (bd^*c\gamma'\alpha'(a+bd^*c)^*)^\omega \\
&= \alpha'(a+bd^*c)^* \cdot (bd^*d^*c\gamma'\alpha'(a+bd^*c)^*)^\omega \\
&\geq \alpha'(a+bd^*c)^*bd^\omega.
\end{aligned}$$

The last step follows by Proposition 10.9.8 and the fact that, since D is biaccessible, no entry of

$$U \quad := \quad d^*c\gamma'\alpha'(a+bd^*c)^*$$

is 0.

We have thus proved that $|D^\omega| = |D|^\omega$. Again, the proof that $|D^\omega\varphi| = |D\varphi|^\omega$ follows the same lines. In addition one uses some strictness properties of $\varphi^\sharp$. In particular, in the last step one uses the fact that no entry of

$$U\varphi^\sharp \;=\; (d\varphi)^*(c\varphi)\gamma'\alpha'((a\varphi)+(b\varphi)(d\varphi)^*(c\varphi))^*$$

is 0.

Proposition 10.9.16 *Let $E_1 = (\beta_1; B_1)$ and $E_2 = (\beta_2; B_2)$ be type 0 presentations of weight s and r, respectively. Define the type 0 presentation $E_1 + E_2$ of weight $s + r$*

$$E_1 + E_2 \quad := \quad \left(\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}; \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right).$$

Then

$$|E_1 + E_2| \quad = \quad |E_1| + |E_2|$$

and

$$|(E_1 + E_2)\varphi| \quad = \quad |E_1\varphi| + |E_2\varphi|.$$

We omit the proof of Proposition 10.9.16. One obvious corollary of Theorem 10.9.13 concerns matricial iteration theories over idempotent iterative semiring module pairs.

Corollary 10.9.17 *Let $(S;V)$ be an idempotent iterative semiring module pair, so that $\mathbf{Matr}(S;V)$ is a matricial iteration theory as in Theorem 10.9.2. Let*

$$\varphi : X \quad \to \quad \mathbf{Matr}(S;V)$$

be a mapping such that each $x\varphi$ is a matrix $1 \to 1$ in $\mathbf{Mat}_S$, i.e. an element of the semiring S. Suppose further that each $x\varphi$ is different from the constants 0 and 1. There is a unique matricial iteration theory morphism

$$\varphi^\sharp : L(X^*; X^\omega) \quad \to \quad \mathbf{Matr}(S;V)$$

such that $\eta \cdot \varphi^\sharp = \varphi$.

10.10 Notes

Sections 10.1, 10.2, 10.6 and 10.7 are based on [BÉd], and Sections 10.4, 10.8, 10.9 on [BÉ90]. The examples of Section 10.3 treat material from [BÉd, BÉ90]. The Kleene type results formulated in Propositions 10.5.3 and 10.5.8 of Section 10.5 are new. For a good survey of regular ω-languages, see [HR86].

Chapter 11

Presentations

In this chapter, we define a general notion of presentation, applicable to all Conway or iteration theories. We state and prove a general version of Kleene's theorem. We then apply presentations to give a necessary and sufficient condition that an iteration theory is the coproduct of an iteration theory and a free iteration theory. This technical result will be used in the axiomatization results of Chapters 12 and 13.

The general concept of a presentation is similar to that of a description used in Chapter 5. It also extends the notion of a presentation in a matrix or matricial iteration theory, defined in Chapters 9 and 10.

11.1 Presentations in Iteration Theories

In this section, we will prove that each iteration theory T can be represented as an iteration theory of presentations modulo behavioral equivalence.

Let T be a preiteration theory and T_0 a subpreiteration theory of T. We suppose that U is a collection of morphisms in T with the following properties:

1. $a \cdot \alpha$ is in U, for all $a : n \to p \in U$ and $\alpha : p \to q \in T_0$;

2. $\rho \cdot a$ is in U, for all $a : n \to p \in U$ and base morphisms $\rho : m \to n$;

3. $\langle a_1, \ldots, a_n \rangle$ is in U, for all $a_i : 1 \to p \in U$, $i \in [n]$.

Thus U is closed under right composition with morphisms in T_0, left composition with base morphisms, and tupling. In particular, all zero morphisms 0_p are in U. We call the pair (T_0, U) a *compatible pair*. We note that for any set Σ of scalar morphisms in T there is a least set $\Sigma(T_0)$ of morphisms which contains Σ and such that $(T_0, \Sigma(T_0))$ is a compatible pair. A morphism $f : n \to p \in T$ belongs to $\Sigma(T_0)$ if and only if each $i_n \cdot f$ can be written as the composite of a morphism in Σ with a T_0-morphism. For the rest of this section we assume that a compatible pair $(T_0, \Sigma(T_0))$ is given in a preiteration theory T.

Definition 11.1.1 *A* **presentation** $n \to p$ *of weight s over the pair (T_0, Σ), or over the compatible pair $(T_0, \Sigma(T_0))$ is an ordered pair*

$$D \quad := \quad (\alpha; a) : n \xrightarrow{s} p$$

consisting of a T_0-morphism $\alpha : n \to s{+}p$ and a morphism $a : s \to s{+}p$ in $\Sigma(T_0)$. The **behavior** *of D is the T-morphism*

$$|D| \quad := \quad \alpha \cdot \langle a^\dagger, \mathbf{1}_p \rangle : n \to p.$$

If we do not want to indicate the weight of a presentation D we will simply write $D : n \to p$.

Example 11.1.2 Let T be the free iteration theory $(\Delta + \Sigma)\mathbf{tr}$, where Δ and Σ are disjoint signatures. We may view $T_0 := \Delta\mathbf{tr}$ as a subiteration theory of T and identify each letter in Σ with the corresponding atomic tree. A regular tree $t : n \to p$ is in $\Sigma(T_0)$ if and only if each $i_n \cdot t$ has a factorization $t = \sigma_i \cdot v_i$, where σ_i is a letter in Σ and v_i is in $\Delta\mathbf{tr}$. A presentation over (T_0, Σ) is an ordered pair $(\alpha; a)$ consisting of a tree $\alpha : n \to s + p$ in $\Delta\mathbf{tr}$ and a tree $a : s \to s + p$ in $\Sigma(T_0)$ as described above. In particular, when Δ is empty, α is a partial base morphism (cf. Example 6.6.5.5), i.e. each component of α is either base or the tree $\perp \cdot 0_{s+p}$. Further, each component of a is a finite Σ-tree of depth at most 1 whose root is labeled by a letter in Σ and whose nonroot vertices are either labeled by a variable or by the symbol $\perp$.

Lemma 11.1.3 [a] *Let $D = (\alpha; a) : n \xrightarrow{s} p$ and $E = (\beta; b) : m \xrightarrow{r} p$ be presentations. Define the presentation*

$$\langle D, E \rangle \quad := \quad (\gamma; c) : n + m \xrightarrow{s+r} p$$

by

$$\begin{array}{rcl}\gamma & := & \langle \alpha \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p),\, 0_s \oplus \beta \rangle \\ c & := & \langle a \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p),\, 0_s \oplus b \rangle.\end{array}$$

Then $|\langle D, E\rangle| = \langle |D|, |E|\rangle$.

[b] *Let* $D = (\alpha; a) : n \xrightarrow{s} p$ *and* $E = (\beta; b) : p \xrightarrow{r} q$ *be presentations. Define the presentation*

$$D \cdot E \quad := \quad (\gamma; c) : n \xrightarrow{s+r} q$$

by

$$\begin{array}{rcl}\gamma & := & \alpha \cdot (\mathbf{1}_s \oplus \beta) \\ c & := & \langle a \cdot (\mathbf{1}_s \oplus \beta),\, 0_s \oplus b \rangle.\end{array}$$

Then $|D \cdot E| = |D| \cdot |E|$.

[c] *Let* $D = (\alpha; a) : n \xrightarrow{s} n + p$ *be a presentation. Define*

$$D^\dagger \quad := \quad (\beta; b) : n \xrightarrow{s} p$$

by

$$\begin{array}{rcl}\beta & := & (\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s,\, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^\dagger \\ b & := & a \cdot \langle \mathbf{1}_s \oplus 0_p,\, \beta,\, 0_s \oplus \mathbf{1}_p \rangle.\end{array}$$

Then $|D^\dagger| = |D|^\dagger$.

Proof of [a].

$$\begin{array}{rcl}|\langle D, E\rangle| & = & \gamma \cdot \langle c^\dagger, \mathbf{1}_p \rangle \\ & = & \gamma \cdot \langle a^\dagger,\, b^\dagger, \mathbf{1}_p \rangle \\ & = & \langle \alpha \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p) \cdot \langle a^\dagger,\, b^\dagger, \mathbf{1}_p \rangle,\, (0_s \oplus \beta) \cdot \langle a^\dagger,\, b^\dagger, \mathbf{1}_p \rangle \rangle \\ & = & \langle \alpha \cdot \langle a^\dagger,\, \mathbf{1}_p \rangle,\, \beta \cdot \langle b^\dagger, \mathbf{1}_p \rangle \rangle \\ & = & \langle |D|, |E| \rangle,\end{array}$$

using the identity in Theorem 5.5.3.9.i, which holds in any iteration theory.

Proof of [b]. We will make use of the parameter identity and the equation in Theorem 5.5.3.9.g.

$$\begin{array}{rcl} |D \cdot E| &=& \gamma \cdot \langle c^\dagger, \mathbf{1}_q\rangle \\ &=& \gamma \cdot \langle (a \cdot (\mathbf{1}_s \oplus \beta))^\dagger \cdot \langle b^\dagger, \mathbf{1}_q\rangle,\, b^\dagger,\, \mathbf{1}_q\rangle \\ &=& \alpha \cdot (\mathbf{1}_s \oplus \beta) \cdot \langle a^\dagger \cdot \beta \cdot \langle b^\dagger, \mathbf{1}_q\rangle,\, b^\dagger,\, \mathbf{1}_q\rangle \\ &=& \alpha \cdot \langle a^\dagger \cdot \beta \cdot \langle b^\dagger, \mathbf{1}_q\rangle,\, \beta \cdot \langle b^\dagger, \mathbf{1}_q\rangle\rangle \\ &=& \alpha \cdot \langle a^\dagger, \mathbf{1}_p\rangle \cdot \beta \cdot \langle b^\dagger, \mathbf{1}_q\rangle \\ &=& |D| \cdot |E|. \end{array}$$

Proof of [c]. First note that the following equation holds in any iteration theory:

$$(f \cdot \langle \mathbf{1}_n \oplus 0_p,\, g,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \;=\; (f^\dagger \cdot \langle g,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger, \qquad (11.1)$$

for all $f : n \to n+m+p$ and $g : m \to n+p$. Indeed, by the parameter identity and the double dagger identity, we have

$$\begin{array}{rcl} (f^\dagger \cdot \langle g,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger &=& (f \cdot (\mathbf{1}_n \oplus \langle g,\, 0_n \oplus \mathbf{1}_p\rangle))^{\dagger\dagger} \\ &=& (f \cdot \langle \mathbf{1}_n \oplus 0_{n+p},\, 0_n \oplus g,\, 0_{2n} \oplus \mathbf{1}_p\rangle)^{\dagger\dagger} \\ &=& (f \cdot \langle \mathbf{1}_n \oplus 0_p,\, g,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger. \end{array}$$

Thus,

$$\begin{array}{rcl} |D^\dagger| &=& \beta \cdot \langle b^\dagger, \mathbf{1}_p\rangle \\ &=& \beta \cdot \langle (a \cdot \langle \mathbf{1}_s \oplus 0_p,\, \beta,\, 0_s \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\ &=& \beta \cdot \langle (a^\dagger \cdot \langle \beta,\, 0_s \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\ &=& (\beta \cdot \langle a^\dagger,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\ &=& ((\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s\rangle \oplus \mathbf{1}_p))^\dagger \cdot \langle a^\dagger,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\ &=& (\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s\rangle \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_n \oplus 0_p,\, a^\dagger,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\ &=& (\alpha \cdot \langle a^\dagger,\, \mathbf{1}_n \oplus 0_p,\, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\ &=& (\alpha \cdot \langle a^\dagger, \mathbf{1}_{n+p}\rangle)^\dagger \\ &=& |D|^\dagger, \end{array}$$

using (11.1) and the composition identity.

Theorem 11.1.4 Kleene's theorem for iteration theories. *Suppose that T is an iteration theory and that $(T_0,\, \Sigma(T_0))$ is a compatible pair* *in T. Then a morphism $f : n \to p$ belongs to the smallest subiteration theory of T containing T_0 and Σ if and only if $f = |D|$ for some presentation $D : n \to p$ over $(T_0, \Sigma(T_0))$. In particular, the following conditions are equivalent:*

[a] $T_0 \cup \Sigma$ *(or $T_0 \cup \Sigma(T_0)$) generates T.*

[b] *For each $f : n \to p \in T$ there is a presentation $D : n \to p$ such that $|D| = f$.*

[c] *For each scalar morphism $f : 1 \to p \in T$ there is a presentation $D : 1 \to p$ such that $|D| = f$.*

Proof. If $\alpha : n \to p$ is in T_0, let

$$D_\alpha \quad := \quad (\alpha; 0_p) : n \xrightarrow{0} p.$$

We have

$$|D_\alpha| \;=\; \alpha \cdot \langle 0_p^\dagger, \mathbf{1}_p \rangle \;=\; \alpha.$$

Similarly, if $\sigma : 1 \to p \in \Sigma$, then $|D_\sigma| = \sigma$ for the presentation

$$D_\sigma \quad := \quad (\mathbf{1}_1 \oplus 0_p;\, 0_1 \oplus \sigma) : 1 \xrightarrow{1} p.$$

Thus any morphism in $T_0 \cup \Sigma$ is the behavior of some presentation. It follows now by Lemma 11.1.3 that the same holds for the morphisms belonging to the smallest subiteration theory containing $\Sigma \cup T_0$. The rest of the proof is obvious.

Remark 11.1.5 Since only Conway theory identities were used in the above proof and in the proof of Lemma 11.1.3, Theorem 11.1.4 holds for Conway theories.

Exercise 11.1.6 The following argument gives another proof of Theorem 11.1.4. First consider the free iteration theory $(\Delta + \Sigma)\mathbf{tr}$ treated in Example 11.1.2. Show directly that each tree in $(\Delta + \Sigma)\mathbf{tr}$ is the behavior of a presentation over $(\Delta\mathbf{tr}, \Sigma(\Delta\mathbf{tr}))$. If T is any iteration theory with a compatible pair $(T_0, \Sigma(T_0))$ such that $T_0 \cup \Sigma$ generates T, let Δ be large enough so that there exists a surjective iteration theory morphism

$$\varphi : (\Delta + \Sigma)\mathbf{tr} \;\to\; T$$

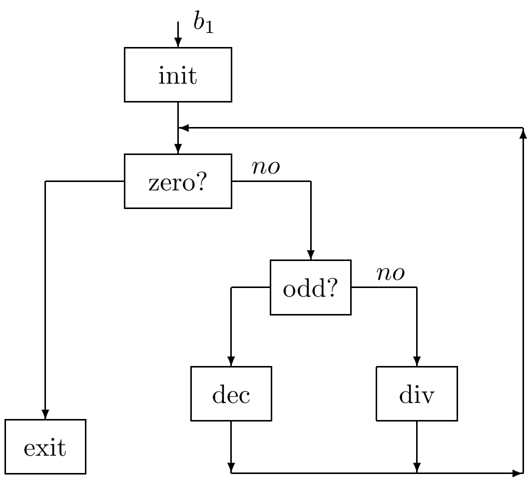
with $(\Delta\mathbf{tr})\varphi = T_0$ and $\Sigma\varphi = \Sigma$. If $D = (\alpha; a) : n \to p$ is a presentation over $(\Delta\mathbf{tr},\ \Sigma(\Delta\mathbf{tr}))$ in $(\Delta + \Sigma)\mathbf{tr}$, define

$$D\varphi \quad := \quad (\alpha\varphi; a\varphi).$$

Thus $D\varphi$ is a presentation $n \to p$ over $(T_0,\ \Sigma(T_0))$ in T. Show that $|D\varphi| = |D|\varphi$. Thus if $t = |D|$ then $t\varphi = |D\varphi|$. Since φ is surjective, it follows that each T-morphism is the behavior of a presentation over (T_0, Σ).

Example 11.1.7 Presentations are generalizations of flowchart schemes (which were discussed in Section 5.5.4). Suppose that T is the theory $\Sigma_\perp\,\mathbf{TR}$ or the free iteration theory $\Sigma\mathbf{tr}$ and T_0 is the initial iteration theory consisting only of partial base morphisms. Let $U = \Sigma(T_0)$ be the collection of all trees $a = \langle a_1, \ldots, a_n\rangle$ such that for each $i \in [s]$, $a_i = \sigma_i \cdot \rho_i$ is the composite of an atomic tree with a partial base morphism. Then a presentation $D = (\alpha; a) : n \to p$ of weight s over $(T_0, \Sigma(T_0))$ is a description of a *flowchart scheme* with n begin vertices and p exit vertices. Indeed, the nodes of the flowchart scheme are the integers $\{1, \ldots, s, s+1, \ldots, s+p, s+p+1\}$; the node $s+i,\ i \in [p]$, is exit i; the label of node $s+p+1$ is $\perp$; the label of node $j,\ j \in [s]$, is σ_j if $a_j := \sigma_j \cdot \rho_j$; the k-th edge of node j is connected to node $k\rho_j$, if $k\rho_j$ is defined and to the node labeled $\perp$ otherwise. The node $b_i := i\alpha$ is the i-th begin vertex, $i \in [n]$; the behavior of D is the n-tuple of labeled trees in $\Sigma\,\mathbf{tr}$ obtained by unfolding the scheme from each begin in turn. (When all of the partial base morphisms ρ_j are total, and if α is total, we will not include a node labeled $\perp$, as in the example below).

Consider the following flowchart scheme:



The corresponding presentation $D = (\alpha; a) : 1 \to 1$ has weight 5. We may let

$$\begin{array}{rcl} a_1 & := & \texttt{init} \cdot 2_6 \\ a_2 & := & \texttt{zero?} \cdot \langle 6_6, 3_6 \rangle \\ a_3 & := & \texttt{odd?} \cdot \langle 4_6, 5_6 \rangle \\ a_4 & := & \texttt{dec} \cdot 2_6 \\ a_5 & := & \texttt{div} \cdot 2_6 \\ \alpha & := & 1_6. \end{array}$$

Here, `zero?, odd?` are in Σ_2 and `init, dec` and `div` are in Σ_1.

Definition 11.1.8 *Let $D, E : n \to p$ be presentations over (T_0, Σ). We call D and E* **behaviorally equivalent***, denoted $D \approx_T E$, or $D \approx E$ for short, if and only if $|D| = |E|$.*

Remark 11.1.9 It follows that when T is generated by $T_0 \cup \Sigma$, then T can be represented as an iteration theory $\mathcal{B}(T_0, \Sigma)$ of behavioral equivalence classes $[D]$ of presentations D. The distinguished morphism i_n in $\mathcal{B}(T_0, \Sigma)$ is the class containing the presentation $(i_n; 0_n) : 1 \xrightarrow{0} n$, $i \in [n]$, $n \geq 0$. Similarly, the zero morphism 0_n in $\mathcal{B}(T_0, \Sigma)$ is the behavioral equivalence class of the presentation $(0_n; 0_n) : 0 \xrightarrow{0} n$. The composition, pairing and iteration operations are defined using Lemma 11.1.3, i.e.

$$\begin{array}{rcl} [D] \cdot [E] & := & [D \cdot E] \\ \langle [D], [E] \rangle & := & [\langle D, E \rangle] \\ [D]^\dagger & := & [D^\dagger], \end{array}$$

whenever D and E have appropriate source and target. The iteration theories T and $\mathcal{B}(T_0, \Sigma)$ are isomorphic via the map $|D| \mapsto [D]$.

11.2 Simulations of Presentations

For some iteration theories T, behavioral equivalence has an alternative 'syntactic' description. In Definition 11.2.1 below we will define two different notions of *simulations* of presentations, each of which gives rise to a syntactic equivalence. Suppose that $(T_0, \Sigma(T_0))$ is a compatible pair in the iteration theory T.

Definition 11.2.1 *Let $D = (\alpha; a) : n \xrightarrow{s} p$ and $E = (\beta; b) : n \xrightarrow{r} p$ be presentations and let $\rho : s \to r$ be a base morphism. We define*

[a] $D \xrightarrow{\rho} E$ *if*

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) \;=\; \beta \quad \textit{and} \quad a \cdot (\rho \oplus \mathbf{1}_p) \;=\; \rho \cdot b;$$

[b] $D \stackrel{\rho}{\rightharpoonup} E$ *if*

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) \;=\; \beta$$

as above and there exist $c : r \to k + p \in \Sigma(T_0)$ *and base morphisms* $\rho_i : k \to s$, $\tau_j : k \to r$ $(i \in [s],\ j \in [r])$ *such that*

$$\begin{aligned} a &= (\rho \cdot c) \parallel (\rho_1, \ldots, \rho_s) \\ b &= c \parallel (\tau_1, \ldots, \tau_r), \end{aligned}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, *for all* $i \in [s]$.

Note the use of two kinds of arrowheads.

Proposition 11.2.2 *Let* $D, E : n \to p$ *be presentations of weight* s *and* r, *respectively.*

[a] *If* $\rho : s \to r$ *is a base morphism and if* $D \stackrel{\rho}{\rightharpoonup} E$, *then* $D \xrightarrow{\rho} E$.

[b] *If* $\rho : s \to r$ *is an injective base morphism, then* $D \stackrel{\rho}{\rightharpoonup} E$ *if and only if* $D \xrightarrow{\rho} E$.

Proof. We only prove that if $D \stackrel{\rho}{\rightharpoonup} E$ then $D \xrightarrow{\rho} E$. Using the notation of Definition 11.2.1.b, if $D \stackrel{\rho}{\rightharpoonup} E$ we have

$$\begin{aligned} a \cdot (\rho \oplus \mathbf{1}_p) &= ((\rho \cdot c) \parallel (\rho_1, \ldots, \rho_s)) \cdot (\rho \oplus \mathbf{1}_p) \\ &= (\rho \cdot c) \parallel (\rho_1 \cdot \rho, \ldots, \rho_s \cdot \rho) \\ &= (\rho \cdot c) \parallel (\tau_{1\rho}, \ldots, \tau_{s\rho}) \\ &= \rho \cdot (c \parallel (\tau_1, \ldots, \tau_r)) \\ &= \rho \cdot b. \end{aligned}$$

Proposition 11.2.3 *Let* $D, E : n \to p$ *be presentations. If* $D \stackrel{\rho}{\rightharpoonup} E$ *holds for a base morphism* ρ *then* $D \approx E$. *Further, if* T *has a functorial dagger with respect to all base surjections, then* $D \approx E$ *whenever* $D \xrightarrow{\rho} E$.

Proof. The proof of this proposition uses the fact that each iteration theory T satisfies the generalized commutative identity (cf. Proposition 5.5.3.26 and Exercise 5.5.3.27) and has a functorial dagger with respect to injective base morphisms. When T has a functorial dagger with respect to base surjections, then T has a functorial dagger with respect to all base morphisms.

Definition 11.2.4 *We let $\overset{*}{\rightleftharpoons}_T$, or just $\overset{*}{\rightleftharpoons}$, denote the smallest equivalence relation on the set of presentations such that $D \overset{*}{\rightleftharpoons} E$ whenever $D, E : n \to p$ are presentations with $D \overset{\rho}{\rightharpoonup} E$, for some base morphism ρ. Similarly, $\overset{*}{\leftrightarrow}_T$ or $\overset{*}{\leftrightarrow}$ denotes the smallest equivalence relation such that $D \overset{*}{\leftrightarrow} E$ whenever $D \overset{\rho}{\to} E$, for some ρ.*

Corollary 11.2.5 *Let $D, E : n \to p$ be presentations.*

- [a] *If $D \overset{*}{\rightleftharpoons} E$ then $D \overset{*}{\leftrightarrow} E$ and $D \approx E$.*
- [b] *If T has a functorial dagger with respect to all base surjections and if $D \overset{*}{\leftrightarrow} E$, then $D \approx E$.*

Remark 11.2.6 We have seen that if T is generated by $T_0 \cup \Sigma$, T is isomorphic to the iteration theory $\mathcal{B}(T_0, \Sigma)$ of behavioral equivalence classes of presentations over (T_0, Σ). Of course, when behavioral equivalence $\approx$ coincides with the relation $\overset{*}{\rightleftharpoons}$ or with the relation $\overset{*}{\leftrightarrow}$, then $\mathcal{B}(T_0, \Sigma)$ and hence T can be represented as an iteration theory of $\overset{*}{\rightleftharpoons}$-equivalence classes or $\overset{*}{\leftrightarrow}$-equivalence classes of presentations. We note that whether or not the relation $\overset{*}{\rightleftharpoons}$ coincides with behavioral equivalence, the $\overset{*}{\rightleftharpoons}$-equivalence classes of presentations do form an iteration theory with operations and constants defined as in $\mathcal{B}(T_0, \Sigma)$, cf. Remark 11.1.9. The theory T is a quotient of this iteration theory. A similar fact is true for $\overset{*}{\leftrightarrow}$-equivalence classes of presentations. The details are left for exercises, cf. Exercise 11.3.4 and Exercise 11.3.5.

Below we will provide a simpler description of the relation $\overset{*}{\leftrightarrow}$. We will consider presentations $n \to p$ over (T_0, Σ) for a fixed pair (n, p) of integers.

Lemma 11.2.7 *Suppose that D, E, F are presentations such that $D \overset{\rho}{\to} E$ and $E \overset{\rho'}{\to} F$, for some base ρ and ρ'. If τ denotes the composite $\rho \cdot \rho'$, then $D \overset{\tau}{\to} E$.*

The proof of Lemma 11.2.7 is obvious. It follows that presentations $n \to p$ form a category with morphisms the simulations $D \xrightarrow{\rho} E$.

Lemma 11.2.8 *Let $D = (\alpha; a)$, $E = (\beta; b)$ be presentations of weight s and r, respectively. Suppose that*

$$D \xrightarrow{\varphi} E,$$

where $\varphi = \rho \cdot \tau$ with $\rho : s \to k$ a base surjection and $\tau : k \to r$ a base injection. Then there exists a presentation $F = (\gamma; c)$ of weight k such that

$$D \xrightarrow{\rho} F \quad \textit{and} \quad F \xrightarrow{\tau} E.$$

Proof. When $k = 0$ also $s = 0$, so that we may define $F := D$. Supposing $k \neq 0$, let

$$\begin{aligned} \gamma &:= \alpha \cdot (\rho \oplus \mathbf{1}_p) : n \to k + p \\ c &:= \rho' \cdot a \cdot (\rho \oplus \mathbf{1}_p) : k \to k + p, \end{aligned}$$

where $\rho' : k \to s$ is base with $\rho' \cdot \rho = \mathbf{1}_k$. Thus γ is a T_0-morphism and c is in $\Sigma(T_0)$. Since

$$\begin{aligned} \gamma \cdot (\tau \oplus \mathbf{1}_p) &= \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p) \\ &= \alpha \cdot (\rho \cdot \tau \oplus \mathbf{1}_p) \\ &= \beta \end{aligned}$$

and

$$\begin{aligned} c \cdot (\tau \oplus \mathbf{1}_p) &= \rho' \cdot a \cdot (\rho \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p) \\ &= \rho' \cdot a \cdot (\rho \cdot \tau \oplus \mathbf{1}_p) \\ &= \rho' \cdot \rho \cdot \tau \cdot b \\ &= \tau \cdot b, \end{aligned}$$

we have $F \xrightarrow{\tau} E$. Also

$$\begin{aligned} \rho \cdot c \cdot (\tau \oplus \mathbf{1}_p) &= \rho \cdot \tau \cdot b \\ &= a \cdot (\rho \cdot \tau \oplus \mathbf{1}_p) \\ &= a \cdot (\rho \oplus \mathbf{1}_p) \cdot (\tau \oplus \mathbf{1}_p), \end{aligned}$$

so that when $\tau' : r \to k$ is a morphism with $\tau \cdot \tau' = \mathbf{1}_k$,

$$\begin{aligned} \rho \cdot c &= \rho \cdot c \cdot (\tau \oplus \mathbf{1}_p) \cdot (\tau' \oplus \mathbf{1}_p) \\ &= a \cdot (\rho \oplus \mathbf{1}_p) \cdot (\tau \cdot \tau' \oplus \mathbf{1}_p) \\ &= a \cdot (\rho \oplus \mathbf{1}_p). \end{aligned}$$

(Note that since $k \neq 0$, there is some base τ' with $\tau \cdot \tau' = \mathbf{1}_k$.) Since $\alpha \cdot (\rho \oplus \mathbf{1}_p) = \gamma$ holds by definition, we have $D \xrightarrow{\rho} F$.

Corollary 11.2.9 *Let D, E, F be presentations. Suppose that $D \xrightarrow{\rho} E$ and $E \xrightarrow{\tau} F$ hold for a base injection ρ and base surjection τ. Then there is a presentation G such that for some surjective base morphism τ' and some injective base morphism ρ' the following square commutes:*

[1‘1‘1‘1;500‘500] [D‘E‘G‘F;ρ‘τ'‘τ‘ρ']

Proof. This is immediate from Lemma 11.2.7 and Lemma 11.2.8.

Lemma 11.2.10 *Let $D = (\alpha; a)$, $E = (\beta; b)$ and $F = (\gamma; c)$ be presentations of weight s, r and m, respectively. If $D \xrightarrow{\rho} F$ and $E \xrightarrow{\tau} F$ for some base injections ρ and τ, then there is a presentation G such that for some base injections τ' and ρ' the following square commutes:*

[1‘1‘1‘1;500‘500] [G‘D‘E‘F;τ'‘ρ'‘ρ‘τ]

Proof. We will use Exercise 3.3.6.4. Let

[k‘s‘r‘m;τ'‘ρ'‘ρ‘τ] be a pullback diagram in **Tot**, so that τ' and ρ' are injective base morphisms and the diagram

[1‘1‘1‘1;700‘700] [k+p‘s+p‘r+p‘m+p;$\tau' \oplus \mathbf{1}_p$‘$\rho' \oplus \mathbf{1}_p$‘$\rho \oplus \mathbf{1}_p$‘$\tau \oplus \mathbf{1}_p$] is a pullback square in **Tot** and hence a pullback square in both theories T_0 and T.

Thus, since

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) \;=\; \beta \cdot (\tau \oplus \mathbf{1}_p) \;=\; \gamma,$$

there is a morphism $\delta : n \to k+p$ in T_0 with $\delta \cdot (\tau' \oplus \mathbf{1}_p) = \alpha$ and $\delta \cdot (\rho' \oplus \mathbf{1}_p) = \beta$. Also, since

$$\begin{aligned} \tau' \cdot a \cdot (\rho \oplus \mathbf{1}_p) &= \tau' \cdot \rho \cdot c \\ &= \rho' \cdot \tau \cdot c \\ &= \rho' \cdot b \cdot (\tau \oplus \mathbf{1}_p), \end{aligned}$$

there is a morphism $d : k \to k+p \in T$ with

$$\tau' \cdot a \quad = \quad d \cdot (\tau' \oplus \mathbf{1}_p)$$

and

$$\rho' \cdot b \quad = \quad d \cdot (\rho' \oplus \mathbf{1}_p).$$

By Exercise 3.3.6.4, it follows that d is in $\Sigma(T_0)$. Thus $G := (\delta; d)$ is a presentation $n \to p$ of weight k and $G \xrightarrow{\tau'} D$ and $G \xrightarrow{\rho'} E$ as claimed.

Exercise 11.2.11 Show that the construction given in the above proof gives rise to pullbacks in the category of presentations.

Lemma 11.2.12 *Suppose that $D = (\alpha; a)$, $E = (\beta; b)$ and $F = (\gamma; c)$ are presentations of weight s, r and k such that*

$$F \xrightarrow{\rho} D \quad \textit{and} \quad F \xrightarrow{\tau} E,$$

for some base morphisms ρ and τ. Then there exist base morphisms τ' and ρ' and a presentation $G = (\delta; d)$ such that the following diagram commutes:

[1‘1‘1‘1;500‘500] [F‘D‘E‘G;ρ‘τ‘τ'‘ρ']

Proof. Recall that by Exercise 3.3.6.12 the pushout of any two base morphisms with a common source exists in any theory. Further, pushouts can be constructed in the initial theory **Tot**. Thus there exist base morphisms $\tau' : s \to m$ and $\rho' : r \to m$ such that the square [k‘s‘r‘m;ρ‘τ‘τ'‘ρ'] is a pushout diagram in both theories T and T_0. Define the T_0-morphism δ by

$$\delta \quad := \quad \gamma \cdot (\rho \cdot \tau' \oplus \mathbf{1}_p) : n \to m + p.$$

We have

$$\begin{aligned} \alpha \cdot (\tau' \oplus \mathbf{1}_p) &= \gamma \cdot (\rho \oplus \mathbf{1}_p) \cdot (\tau' \oplus \mathbf{1}_p) \\ &= \gamma \cdot (\rho \cdot \tau' \oplus \mathbf{1}_p) \\ &= \delta, \end{aligned}$$

and similarly,

$$\beta \cdot (\rho' \oplus \mathbf{1}_p) \quad = \quad \delta.$$

Also

$$\begin{aligned} \rho \cdot a \cdot (\tau' \oplus \mathbf{1}_p) &= c \cdot (\rho \oplus \mathbf{1}_p) \cdot (\tau' \oplus \mathbf{1}_p) \\ &= c \cdot (\tau \oplus \mathbf{1}_p) \cdot (\rho' \oplus \mathbf{1}_p) \\ &= \tau \cdot b \cdot (\rho' \oplus \mathbf{1}_p), \end{aligned}$$

so that there is a (unique) T-morphism $d : m \to m + p$ such that

$$\begin{aligned} a \cdot (\tau' \oplus \mathbf{1}_p) &= \tau' \cdot d \\ b \cdot (\rho' \oplus \mathbf{1}_p) &= \rho' \cdot d. \end{aligned}$$

Thus,

$$\langle \tau', \rho' \rangle \cdot d \quad = \quad \langle a \cdot (\tau' \oplus \mathbf{1}_p), b \cdot (\rho' \oplus \mathbf{1}_p) \rangle.$$

Since $\langle \tau', \rho' \rangle$ is a surjective base morphism and $\langle \tau', \rho' \rangle \cdot d$ is in $\Sigma(T_0)$, it follows that d is in $\Sigma(T_0)$ also. Thus $G := (\delta; d)$ is a presentation with $D \xrightarrow{\tau'} G$ and $E \xrightarrow{\rho'} G$.

Remark 11.2.13 If ρ is injective (respectively, surjective), then so is ρ'. A similar fact is true for the base morphisms τ and τ'.

Exercise 11.2.14 Show that the above construction provides the pushout of the diagram $E \xleftarrow{\rho} F \xrightarrow{\tau} D$ in the category of presentations $n \to p$.

We can now prove the following result concerning the class of presentations $n \to p$, for fixed n and p.

Proposition 11.2.15 *The following conditions are equivalent for any two presentations D and E:*

[a] $D \overset{*}{\leftrightarrow} E$.

[b] *There exists a presentation F such that*

$$D \xrightarrow{\rho} F \xleftarrow{\tau} E,$$

for some base morphisms ρ and τ.

[c] *There exist presentations D_1, F, E_1 such that*

$$D \xrightarrow{\rho_1} D_1 \xrightarrow{\tau_1} F \xleftarrow{\tau_2} E_1 \xleftarrow{\rho_2} E,$$

where ρ_i is a base injection and τ_i is a base surjection, for $i = 1, 2$.

[d] *There exist presentations D_1, F, E_1 such that*

$$D \xrightarrow{\rho_1} D_1 \xleftarrow{\tau_1} F \xrightarrow{\tau_2} E_1 \xleftarrow{\rho_2} E,$$

where ρ_i is a base surjection and τ_i is a base injection, for $i = 1, 2$.

Proof. That [a] implies [b] follows by Lemma 11.2.12. That [b] implies [c] is immediate by Lemma 11.2.8. That [c] implies [d] follows by Lemma 11.2.10. Finally, the last condition obviously implies the first one.

Exercise 11.2.16 Show that in each $\overset{*}{\leftrightarrow}$-equivalence class $\mathcal{D}$ of presentations $n \to p$ there is a minimal presentation F with the following property: For any presentation D in $\mathcal{D}$ there exists some presentation E in $\mathcal{D}$ such that

$$D \xrightarrow{\rho} E \xleftarrow{\tau} F,$$

where ρ is a base surjection and τ is a base injection.

Show that any two minimal presentations D and E in $\mathcal{D}$ are isomorphic, in the sense that their weights are equal and there is a base permutation π with $D \xrightarrow{\pi} E$. Show that a presentation is minimal if and only if it is accessible and reduced. A presentation D is *accessible* if for any presentation E, if $E \xrightarrow{\rho} D$ for an injective base morphism ρ, then ρ is a base permutation. Dually, D is *reduced* if whenever $D \xrightarrow{\tau} F$, where τ is surjective, it follows that τ is a base permutation.

Exercise 11.2.17 $(\Delta\mathbf{tr}, \Sigma(\Delta\mathbf{tr}))$ is a compatible pair in the free iteration theory $T = (\Delta + \Sigma)\mathbf{tr}$. Describe accessible and reduced presentations using graph theoretic terms. Show that if $D, E : n \to p$ are behaviorally equivalent presentations, then there exist accessible presentations D_1, E_1 and a reduced accessible (hence minimal) presentation F such that

$$D \overset{\rho_1}{\leftarrow} D_1 \overset{\tau_1}{\rightarrow} F \overset{\tau_2}{\leftarrow} E_1 \overset{\rho_2}{\rightarrow} E,$$

where ρ_i is a base injection and τ_i is a base surjection, for $i = 1, 2$. (Note that this connecting chain is different from those appearing in Proposition 11.2.15 above.) Thus $\approx = \overset{*}{\leftrightarrow}$.

In a number of iteration theories, the relations $\overset{*}{\rightleftharpoons}$ and $\overset{*}{\leftrightarrow}$ coincide.

Proposition 11.2.18 *Let $(T_0, \Sigma(T_0))$ be a compatible pair in an iteration theory T. Suppose that for any family of morphisms $a_1, \ldots, a_m : 1 \to s + p$ $(m \geq 1)$ in $\Sigma(T_0)$, if*

$$a_1 \cdot (\tau \oplus \mathbf{1}_p) \;=\; \ldots \;=\; a_m \cdot (\tau \oplus \mathbf{1}_p),$$

for a monotone surjective base morphism $\tau : s \to r$, then there is a morphism $c : 1 \to k + p$ in $\Sigma(T_0)$ and there exist base morphisms $\rho_1, \ldots, \rho_m : k \to s$ with

$$a_i \;=\; c \cdot (\rho_i \oplus \mathbf{1}_p), \quad i \in [m],$$

and

$$\rho_1 \cdot \tau \;=\; \ldots \;=\; \rho_m \cdot \tau.$$

Then for each pair of presentations D, E we have $D \overset{\rho}{\rightharpoonup} E$ for a base morphism ρ if and only if $D \overset{\rho}{\rightarrow} E$. Thus $D \overset{*}{\rightleftharpoons} E$ if and only if $D \overset{*}{\leftrightarrow} E$.

Proof. The proof consists of two steps. First let $a : m \to s + p$ and $b : n \to r + p$ in $\Sigma(T_0)$, and suppose that

$$a \cdot (\tau \oplus \mathbf{1}_p) \;=\; \rho \cdot b \tag{11.2}$$

holds for some monotone surjective base morphisms $\rho : m \to n$ and $\tau : s \to r$. We will show by induction on n that there exists a morphism $c : n \to k + p$ in $\Sigma(T_0)$ and there exist base morphisms $\rho_i : k \to s$, $\tau_j : k \to r$ $(i \in [m],\ j \in [n])$ such that

$$a \;=\; (\rho \cdot c) \parallel (\rho_1, \ldots, \rho_m) \tag{11.3}$$

and

$$\rho_i \cdot \tau \quad = \quad \tau_{i\rho}, \tag{11.4}$$

for all $i \in [m]$. It then follows that

$$b \quad = \quad c \parallel (\tau_1, \ldots, \tau_n).$$

When $n = m = 0$, this is obvious and when $n = 1$ this holds by assumption. If $n > 1$ we may write $m = m_1 + m_2$, $n = n_1 + 1$ and $\rho = \rho' \oplus \rho''$, where $\rho' : m_1 \to n_1$ and $\rho'' : m_2 \to 1$ are monotone surjective base morphisms. Let

$$\begin{aligned} a &= \langle a_1, a_2 \rangle, \quad a_1 : m_1 \to s + p,\ a_2 : m_2 \to s + p \\ b &= \langle b_1, b_2 \rangle, \quad b_1 : n_1 \to r + p,\ b_2 : 1 \to r + p, \end{aligned}$$

so that $a_i, b_i \in \Sigma(T_0)$, $i = 1, 2$. We have

$$\begin{aligned} a_1 \cdot (\tau \oplus \mathbf{1}_p) &= \rho' \cdot b_1 \\ a_2 \cdot (\tau \oplus \mathbf{1}_p) &= \rho'' \cdot b_2. \end{aligned}$$

Thus, by the induction hypothesis there exist $c_1 : n_1 \to k_1 + p$, $c_2 : 1 \to k_2 + p$ in $\Sigma(T_0)$ and base morphisms

$$\begin{aligned} \rho'_1, \ldots, \rho'_{m_1} &: k_1 \to s \\ \tau'_1, \ldots, \tau'_{n_1} &: k_1 \to r \\ \rho''_1, \ldots, \rho''_{m_2} &: k_2 \to s \end{aligned}$$

and $\tau'' : k_2 \to r$ such that

$$\begin{aligned} a_1 &= (\rho' \cdot c_1) \parallel (\rho'_1, \ldots, \rho'_{m_1}) \\ a_2 &= (\rho'' \cdot c_2) \parallel (\rho''_1, \ldots, \rho''_{m_2}) \end{aligned}$$

and $\rho'_i \cdot \tau = \tau'_{i\rho'}$, $i \in [m_1]$, $\rho''_i \cdot \tau = \tau''$, $i \in [m_2]$. Let $\gamma' : k_2 \to s$ and $\gamma'' : k_1 \to s$ be arbitrary base morphisms. (These morphisms exist since if $s = 0$ then $k_1 = k_2 = 0$.) Define $\delta' := \gamma' \cdot \tau : k_2 \to r$ and $\delta'' := \gamma'' \cdot \tau : k_1 \to r$. Then $k := k_1 + k_2$,

$$c \quad := \quad \langle c_1 \cdot (\mathbf{1}_{k_1} \oplus 0_{k_2} \oplus \mathbf{1}_p),\ 0_{k_1} \oplus c_2 \rangle : n \to k + p$$

and

$$\begin{aligned} \rho_i &:= \langle \rho'_i, \gamma' \rangle : k \to s, \quad i \in [m_1] \\ \rho_{m_1+i} &:= \langle \gamma'', \rho''_i \rangle : k \to s, \quad i \in [m_2] \\ \tau_i &:= \langle \tau'_i, \delta' \rangle : k \to r, \quad i \in [n_1] \end{aligned}$$

and $\tau_n := \langle \delta'', \tau'' \rangle : k \to r$. We have (11.3) and (11.4) as claimed.

To finish the proof let $D = (\alpha; a) : n \xrightarrow{s} p$ and $E = (\beta; b) : n \xrightarrow{r} p$ be presentations such that $D \xrightarrow{\rho} E$, for some base morphism ρ. The base morphism ρ has a factorization $\rho = \pi \cdot \rho' \cdot \tau$, where $\pi : s \to s$ is a base permutation, $\rho' : s \to r'$ is a monotone base surjection and $\tau : r' \to r$ is a base injection. By Lemma 11.2.8, there exist presentations F_1 and F_2 of weights s and r' such that

$$D \xrightarrow{\pi} F_1, \quad F_1 \xrightarrow{\rho'} F_2 \quad \text{and} \quad F_2 \xrightarrow{\tau} E.$$

Since ρ' is a monotone base surjection, we have $F_1 \overset{\rho'}{\rightharpoonup} F_2$. Since π and τ are base injections, also $D \overset{\pi}{\rightharpoonup} F_1$ and $F_2 \overset{\tau}{\rightharpoonup} E$. It follows now that $D \overset{\rho}{\rightharpoonup} E$.

Corollary 11.2.19 *Suppose that each scalar morphism a in $\Sigma(T_0)$ has a unique factorization $a = \sigma \cdot \alpha$ where $\sigma \in \Sigma$ and $\alpha \in T_0$. Suppose that for any family of morphisms $\alpha_1, \ldots, \alpha_m : 1 \to s + p$ $(m \geq 1)$ in T_0, if*

$$\alpha_1 \cdot (\tau \oplus \mathbf{1}_p) \;=\; \ldots \;=\; \alpha_m \cdot (\tau \oplus \mathbf{1}_p),$$

for a monotone surjective base morphism $\tau : s \to r$, then there is a morphism $\gamma : 1 \to k + p$ in T_0 and there exist base morphisms $\rho_1, \ldots, \rho_m : k \to s$ with

$$\alpha_i \;=\; \gamma \cdot (\rho_i \oplus \mathbf{1}_p), \quad i \in [m],$$

and

$$\rho_1 \cdot \tau \;=\; \ldots \;=\; \rho_m \cdot \tau.$$

Then for each pair of presentations D, E we have $D \overset{\rho}{\rightharpoonup} E$ for a base morphism ρ if and only if $D \xrightarrow{\rho} E$. Thus $D \overset{*}{\rightleftharpoons} E$ if and only if $D \overset{*}{\leftrightarrow} E$.

Proof. One shows that the assumptions imply those of Proposition 11.2.18. The details are similar to the first part of the proof of Proposition 11.2.18.

Exercise 11.2.20 This exercise concerns the theory $(\Delta + \Sigma)\mathbf{tr}$ again. Show that the assumptions of Corollary 11.2.19 hold for the compatible pair $(\Delta\mathbf{tr}, \Sigma(\Delta\mathbf{tr}))$. Thus for any two presentations D and E we have $D \overset{*}{\rightleftharpoons} E$ if and only if $D \overset{*}{\leftrightarrow} E$. Thus by Exercise 11.2.17, all of the three relations $\approx$, $\overset{*}{\rightleftharpoons}$ and $\overset{*}{\leftrightarrow}$ are the same.

We end this section by presenting an extension of the notion of simulation.

Definition 11.2.21 *Let $(T_0, \Sigma(T_0))$ be a compatible pair in an iteration theory T and let θ_0 be an equivalence relation on T_0 and θ_1 an equivalence relation on $\Sigma(T_0)$. We call (θ_0, θ_1) a* **compatible pair of equivalences** *if the following are valid:*

1. *θ_0 is a dagger congruence on T_0;*
2. *If $a\ \theta_1\ b$ and $\alpha\ \theta_0\ \beta$, then $a{\cdot}\alpha\ \theta_1\ b{\cdot}\beta$, for all $a, b : n \to p \in \Sigma(T_0)$ and $\alpha, \beta : p \to q \in T_0$;*
3. *If $a\ \theta_1\ b$ then $\rho \cdot a\ \ \theta_1\ \rho \cdot b$, for all $a, b : n \to p \in \Sigma(T_0)$ and base $\rho : m \to n$;*
4. *If $a_i\ \theta_1\ b_i$, $i \in [n]$, then $\langle a_1, \ldots, a_n\rangle\ \theta_1\ \langle b_1, \ldots, b_n\rangle$, for all $a_i, b_i : 1 \to p \in \Sigma(T_0)$, $i \in [n]$.*

For each dagger congruence θ_0 on T_0, there is a smallest equivalence θ_1 on $\Sigma(T_0)$ such that (θ_0, θ_1) is a compatible pair of equivalences. Suppose that each scalar morphism in $\Sigma(T_0)$ is uniquely written as the composite of a morphism from Σ with a T_0-morphism. Then this smallest equivalence relation θ_1 has the following direct description: For each pair of morphisms $a, b : n \to p \in \Sigma(T_0)$, $a\ \theta_1\ b$ if and only if for each $i \in [n]$ there is $\sigma_i \in \Sigma$ and T_0-morphisms α_i and β_i such that

$$\begin{aligned} i_n \cdot a &= \sigma_i \cdot \alpha_i \\ i_n \cdot b &= \sigma_i \cdot \beta_i \end{aligned}$$

and

$$\alpha_i\ \theta_0\ \beta_i.$$

The following definition is an extension of Definitions 11.2.1 and 11.2.4.

Definition 11.2.22 *Let (θ_0, θ_1) be a compatible pair of equivalences and let $D = (\alpha; a) : n \xrightarrow{s} p$ and $E = (\beta; b) : n \xrightarrow{r} p$ be presentations. We define the following relations.*

[a] $D \stackrel{\rho}{\rightarrow} E \bmod (\theta_0, \theta_1)$, *for a base morphism* $\rho : s \rightarrow r$, *if and only if*

$$\alpha \cdot (\rho \oplus \mathbf{1}_p)\ \theta_0\ \beta \quad \text{and} \quad a \cdot (\rho \oplus \mathbf{1}_p)\ \theta_1\ \rho \cdot b;$$

[b] $D \stackrel{\rho}{\rightharpoondown} E \bmod (\theta_0, \theta_1)$, *for a base morphism* $\rho : s \rightarrow r$, *if and only if*

$$\alpha \cdot (\rho \oplus \mathbf{1}_p)\ \theta_0\ \beta$$

as above and there exist $c : r \rightarrow k + p \in \Sigma(T_0)$ *and base morphisms* $\rho_i : k \rightarrow s$, $\tau_j : k \rightarrow r$ $(i \in [s], j \in [r])$ *such that*

$$\begin{array}{lll} a & \theta_1 & (\rho \cdot c) \parallel (\rho_1, \ldots, \rho_s) \\ b & \theta_1 & c \parallel (\tau_1, \ldots, \tau_r) \end{array}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, *for all* $i \in [s]$;

[c] *The relation* $\stackrel{*}{\rightleftharpoons} \bmod (\theta_0, \theta_1)$ $(\stackrel{*}{\leftrightarrow} \bmod (\theta_0, \theta_1))$ *is the smallest equivalence relation on presentations such that if* $D \stackrel{\rho}{\rightharpoondown} E$ $(D \stackrel{\rho}{\rightarrow} E) \bmod (\theta_0, \theta_1)$ *holds for some base morphism* ρ *then* $D \stackrel{*}{\rightleftharpoons} E \bmod (\theta_0, \theta_1)$ $(D \stackrel{*}{\leftrightarrow} E \bmod (\theta_0, \theta_1))$.

Exercise 11.2.23 Let $D, E : n \rightarrow p$ be presentations of weight s and r, respectively. Show that the following hold true:

[a] If $\rho : s \rightarrow r$ is a base morphism and if $D \stackrel{\rho}{\rightharpoondown} E \bmod (\theta_0, \theta_1)$, then also $D \stackrel{\rho}{\rightarrow} E \bmod (\theta_0, \theta_1)$.

[b] If $\rho : s \rightarrow r$ is an injective base morphism, then $D \stackrel{\rho}{\rightharpoondown} E \bmod (\theta_0, \theta_1)$ if and only if $D \stackrel{\rho}{\rightarrow} E \bmod (\theta_0, \theta_1)$.

Thus if $D \stackrel{*}{\rightleftharpoons} E \bmod (\theta_0, \theta_1)$ then $D \stackrel{*}{\leftrightarrow} E \bmod (\theta_0, \theta_1)$.

Exercise 11.2.24 The following conditions are equivalent, for any two presentations $D, E : n \rightarrow p$:

[a] $D \stackrel{*}{\leftrightarrow} E \bmod (\theta_0, \theta_1)$.

[b] There exists a presentation $F : n \rightarrow p$ such that

$$D \stackrel{\rho}{\rightarrow} F \stackrel{\tau}{\leftarrow} E \bmod (\theta_0, \theta_1),$$

for some base morphisms ρ and τ.

[c] There exist presentations $D_1, F, E_1 : n \to p$ such that

$$D \stackrel{\rho_1}{\to} D_1 \stackrel{\tau_1}{\to} F \stackrel{\tau_2}{\leftarrow} E_1 \stackrel{\rho_2}{\leftarrow} E \text{ mod } (\theta_0, \theta_1),$$

where ρ_i is a base surjection and τ_i is a base injection, for $i = 1, 2$.

[d] There exist presentations $D_1, F, E_1 : n \to p$ such that

$$D \stackrel{\rho_1}{\to} D_1 \stackrel{\tau_1}{\leftarrow} F \stackrel{\tau_2}{\to} E_1 \stackrel{\rho_2}{\leftarrow} E \text{ mod } (\theta_0, \theta_1),$$

where ρ_i is a base surjection and τ_i is a base injection, for $i = 1, 2$.

11.3 Coproducts Revisited

In this section we will give a necessary and sufficient condition that an iteration theory is the coproduct of an iteration theory and a free iteration theory. We start with a lemma which follows from the generalized commutative identity, cf. Proposition 5.5.3.26 and Exercise 5.5.3.27.

Lemma 11.3.1 *Let T be an iteration theory and θ a dagger congruence on T. Suppose that*

$$\begin{array}{lll} f & \theta & (\rho \cdot h) \parallel (\rho_1, \ldots, \rho_m) \\ g & \theta & h \parallel (\tau_1, \ldots, \tau_n), \end{array}$$

where $h : n \to k + p$, $f : m \to m + p$ and $g : n \to n + p$ in T and where $\rho : m \to n$ is a (surjective) base morphism and $\rho_i : k \to m$, $\tau_j : k \to n$ $(i \in [m],\ j \in [n])$ are base, with $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [m]$. Then

$$f^\dagger \ \theta \ \rho \cdot g^\dagger.$$

Corollary 11.3.2 *Let T be an iteration theory, $(T_0, \Sigma(T_0))$ a compatible pair in T and (θ_0, θ_1) a compatible pair of equivalences. Let θ be a dagger congruence on T which contains θ_0 and θ_1. If $D, E : n \to p$ are presentations with*

$$D \stackrel{*}{\rightleftharpoons} E \text{ mod } (\theta_0, \theta_1),$$

then $|D| \ \theta \ |E|$.

Proof. We show that if

$$D \xrightharpoonup{\rho} E \bmod (\theta_0, \theta_1),$$

for some presentations $D = (\alpha; a) : n \xrightarrow{s} p$ and $E = (\beta; b) : n \xrightarrow{r} p$ and a base morphism $\rho : s \to r$, then $|D|\ \theta\ |E|$. Indeed, there exists a morphism $c : r \to k + p \in \Sigma(T_0)$ and there exist base morphisms $\rho_i : k \to s$, $\tau_j : k \to r$ ($i \in [s]$, $j \in [r]$) such that

$$\begin{array}{lll} a & \theta_1 & (\rho \cdot c) \parallel (\rho_1, \ldots, \rho_s) \\ b & \theta_1 & c \parallel (\tau_1, \ldots, \tau_r) \end{array}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [s]$. Thus

$$a^\dagger\ \theta\ \rho \cdot b^\dagger,$$

by Lemma 11.3.1. Also

$$\alpha \cdot (\rho \oplus \mathbf{1}_p)\ \theta_0\ \beta.$$

Thus

$$\begin{array}{rcl} |D| & = & \alpha \cdot \langle a^\dagger, \mathbf{1}_p \rangle \\ & \theta & \alpha \cdot \langle \rho \cdot b^\dagger, \mathbf{1}_p \rangle \\ & = & \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle b^\dagger, \mathbf{1}_p \rangle \\ & \theta & \beta \cdot \langle b^\dagger, \mathbf{1}_p \rangle \\ & = & |E|. \end{array}$$

Theorem 11.3.3 *Let T be an iteration theory, $(T_0, \Sigma(T_0))$ a compatible pair in T and let (θ_0, θ_1) be a compatible pair of equivalences. Suppose that T is jointly generated by T_0 and Σ and that $\approx_T$ coincides with the relation $\overset{*}{\rightleftharpoons}_T$. Then the smallest dagger congruence on T containing θ_0 and θ_1 can be described as the following relation θ: For all $f, g : n \to p \in T$,*

$$\begin{array}{rl} f\ \theta\ g \ \Leftrightarrow & (\exists D, E : n \to p)\,[f = |D|,\ g = |E| \\ & \text{and } D \overset{*}{\rightleftharpoons} E \bmod (\theta_0, \theta_1)]. \end{array} \tag{11.5}$$

If $\alpha\ \theta\ \beta$ for some morphisms $\alpha, \beta : n \to p \in T_0$, then $\alpha\ \theta_0\ \beta$. Similarly, if $a, b : n \to p \in \Sigma(T_0)$ with $a\ \theta\ b$, then $a\ \theta_1\ b$. Thus θ_0 is the restriction of θ to T_0 and θ_1 is the restriction of θ to $\Sigma(T_0)$. Let

$$\begin{array}{lllll} T_0' & := & T_0/\theta & = & \{\alpha/\theta : \alpha \in T_0\} \\ \Sigma' & := & \Sigma/\theta & = & \{\sigma/\theta : \sigma \in \Sigma\}, \end{array}$$

so that

$$\Sigma'(T_0') \;=\; \Sigma(T_0)/\theta \;=\; \{a/\theta : a \in \Sigma(T_0)\}.$$

Then $(T_0', \Sigma'(T_0'))$ *is a compatible pair in the quotient iteration theory* T/θ*, and* T/θ *is jointly generated by* T_0' *and* Σ'*. Further, in* T/θ*, we have* $\approx_{T/\theta} \;=\; \overset{*}{\rightleftharpoons}_{T/\theta}$*, i.e. behavioral equivalence of presentations over* (T_0', Σ') *in* T/θ *coincides with the syntactic equivalence* $\overset{*}{\rightleftharpoons}_{T/\theta}$*.*

Proof. Let θ be defined by (11.5) above. First we prove that for any pair of morphisms $f, g : n \to p$,

$$\begin{aligned} f \;\theta\; g \quad \Leftrightarrow \quad & (\forall D, E : n \to p)\,[f = |D|,\ g = |E| \quad \Rightarrow \\ & D \overset{*}{\rightleftharpoons} E \bmod (\theta_0, \theta_1)]. \end{aligned}$$

Indeed, suppose that $f \;\theta\; g$, so that there exist $D', E' : n \to p$ with $f = |D'|$, $g = |E'|$ and $D' \overset{*}{\rightleftharpoons} E' \bmod (\theta_0, \theta_1)$. If $D, E : n \to p$ are any other presentations with $f = |D|$ and $g = |E|$, then $D \overset{*}{\rightleftharpoons} D'$ and $E \overset{*}{\rightleftharpoons} E'$. It follows that $D \overset{*}{\rightleftharpoons} E \bmod (\theta_0, \theta_1)$. The converse implication is valid, since by Theorem 11.1.4 each T-morphism is the behavior of some presentation.

It follows now that θ is an equivalence relation. To see that θ is a dagger congruence, we must prove the following facts for presentations with appropriate source and target.

[a] If

$$D \overset{\rho}{\rightharpoonup} D' \bmod (\theta_0, \theta_1),$$

for a base morphism ρ, then there exists a base morphism τ with

$$D \cdot E \overset{\tau}{\rightharpoonup} D' \cdot E \;\bmod (\theta_0, \theta_1).$$

[b] If

$$D \overset{\rho}{\rightharpoonup} D' \bmod (\theta_0, \theta_1),$$

for a base morphism ρ, then there is a base morphism τ with

$$E \cdot D \overset{\tau}{\rightharpoonup} E \cdot D' \;\bmod (\theta_0, \theta_1).$$

[c] If

$$D \overset{\rho_1}{\rightharpoonup} D' \bmod (\theta_0, \theta_1) \quad \text{and} \quad E \overset{\rho_2}{\rightharpoonup} E' \bmod (\theta_0, \theta_1),$$

for some base morphisms ρ_1, ρ_2, then there is a base τ with

$$\langle D, E\rangle \overset{\tau}{\rightharpoonup} \langle D', E'\rangle \ \bmod (\theta_0, \theta_1).$$

[d] If

$$D \overset{\rho}{\rightharpoonup} E \bmod (\theta_0, \theta_1),$$

for a base morphism ρ, then also

$$D^\dagger \overset{\rho}{\rightharpoonup} E^\dagger \bmod (\theta_0, \theta_1).$$

We will only prove [a] and [d].

Proof of [a]. Let $D = (\alpha; a) : n \xrightarrow{s} p$, $D' = (\alpha'; a') : n \xrightarrow{s'} p$ and $E = (\beta; b) : p \xrightarrow{r} q$. Suppose that $D \overset{\rho}{\rightharpoonup} D'$, for a base morphism $\rho : s \to s'$. Thus there exist $c : s' \to k+p \in \Sigma(T_0)$ and base morphisms $\rho_i : k \to s$, $\tau_j : k \to s'$ ($i \in [s]$, $j \in [s']$) such that

$$\begin{array}{rcl} a & \theta_1 & (\rho \cdot c) \parallel (\rho_1, \ldots, \rho_s) \\ a' & \theta_1 & c \parallel (\tau_1, \ldots, \tau_{s'}), \end{array}$$

and $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [s]$. Further,

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) \ \theta_0 \ \alpha'.$$

Recall that

$$D \cdot E \quad = \quad (\alpha \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p); \langle a \cdot (\mathbf{1}_s \oplus \beta),\, 0_s \oplus b\rangle) : n \xrightarrow{s+r} q$$

and

$$D' \cdot E \quad = \quad (\alpha' \cdot (\mathbf{1}_{s'} \oplus 0_r \oplus \mathbf{1}_p); \langle a' \cdot (\mathbf{1}_{s'} \oplus \beta),\, 0_{s'} \oplus b\rangle) : n \xrightarrow{s'+r} q.$$

We have

$$\begin{array}{rcl} \alpha \cdot (\mathbf{1}_s \oplus 0_r \oplus \mathbf{1}_p) \cdot (\rho \oplus \mathbf{1}_{r+p}) & = & \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot (\mathbf{1}_{s'} \oplus 0_r \oplus \mathbf{1}_p) \\ & \theta_0 & \alpha' \cdot (\mathbf{1}_{s'} \oplus 0_r \oplus \mathbf{1}_p). \end{array}$$

Let

$$d \quad := \quad \langle c \cdot (\mathbf{1}_k \oplus 0_r \oplus \mathbf{1}_p),\, 0_k \oplus b\rangle : s' + r \to k + r + p,$$

so that d is in $\Sigma(T_0)$. Further, let

$$\begin{array}{rcl} \rho' & := & \rho \oplus \mathbf{1}_r \\ \rho'_i & := & \begin{cases} \rho_i \oplus \mathbf{1}_r & \text{if } i \leq s \\ \rho_1 \oplus \mathbf{1}_r & \text{if } i > s \end{cases} \\ \tau'_j & := & \begin{cases} \tau_j \oplus \mathbf{1}_r & \text{if } j \leq s' \\ \tau_{1\rho} \oplus \mathbf{1}_r & \text{if } j > s', \end{cases} \end{array}$$

for all $i \in [s+r]$ and $j \in [s'+r]$. When both s and s' are 0, then $k = 0$, and then $\rho = \rho'_i = \tau_j = \mathbf{1}_r$, for all $i, j \in [r]$. It follows that

$$\begin{array}{rcl} \langle a \cdot (\mathbf{1}_s \oplus \beta),\, 0_s \oplus b \rangle & \theta_1 & (\rho' \cdot d) \parallel (\rho'_1, \ldots, \rho'_{s+r}) \\ \langle a' \cdot (\mathbf{1}_{s'} \oplus \beta),\, 0'_s \oplus b \rangle & \theta_1 & d \parallel (\tau'_1, \ldots, \tau'_{s'+r}). \end{array}$$

Since also $\rho'_i \cdot \rho' = \tau'_{i\rho'}$, for all $i \in [s+r]$, we have

$$D \cdot E \overset{\rho'}{\rightharpoondown} D' \cdot E.$$

Proof of [d]. Let $D = (\alpha; a) : n \xrightarrow{s} n + p$ and $E = (\beta; b) : n \xrightarrow{r} n + p$. Suppose that $D \overset{\rho}{\rightharpoondown} E$ mod (θ_0, θ_1), for a base morphism $\rho : s \to r$. Thus

$$\alpha \cdot (\rho \oplus \mathbf{1}_p)\ \theta_0\ \beta$$

and

$$\begin{array}{rcl} a & \theta_1 & (\rho \cdot c) \parallel (\rho_1, \ldots, \rho_s) \\ b & \theta_1 & c \parallel (\tau_1, \ldots, \tau_r), \end{array}$$

for some $c : r \to k{+}n{+}p \in \Sigma(T_0)$ and some base morphisms $\rho_i : k \to s$, $\tau_j : k \to r$ $(i \in [s],\ j \in [r])$ such that $\rho_i \cdot \rho = \tau_{i\rho}$, for all $i \in [s]$. By definition

$$\begin{array}{rcl} D^\dagger & = & (\gamma;\, a \cdot \langle \mathbf{1}_s \oplus 0_p,\, \gamma,\, 0_s \oplus \mathbf{1}_p \rangle) \\ E^\dagger & = & (\delta;\, b \cdot \langle \mathbf{1}_r \oplus 0_p,\, \delta,\, 0_r \oplus \mathbf{1}_p \rangle), \end{array}$$

where

$$\begin{array}{rcl} \gamma & = & (\alpha \cdot (\langle 0_n \oplus \mathbf{1}_s,\, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^\dagger : n \to s + p \\ \delta & = & (\beta \cdot (\langle 0_n \oplus \mathbf{1}_r,\, \mathbf{1}_n \oplus 0_r \rangle \oplus \mathbf{1}_p))^\dagger : n \to r + p. \end{array}$$

We have

$$\begin{array}{rcl}
\gamma\cdot(\rho\oplus\mathbf{1}_p) & = & (\alpha\cdot(\langle 0_n\oplus\mathbf{1}_s,\, \mathbf{1}_n\oplus 0_s\rangle\oplus\mathbf{1}_p))^\dagger\cdot(\rho\oplus\mathbf{1}_p)\\
& = & (\alpha\cdot(\langle 0_n\oplus\mathbf{1}_s,\, \mathbf{1}_n\oplus 0_s\rangle\oplus\mathbf{1}_p)\cdot(\mathbf{1}_n\oplus\rho\oplus\mathbf{1}_p))^\dagger\\
& = & (\alpha\cdot(\rho\oplus\mathbf{1}_{n+p})\cdot(\langle 0_n\oplus\mathbf{1}_r,\, \mathbf{1}_n\oplus 0_r\rangle\oplus\mathbf{1}_p))^\dagger\\
& \theta_0 & (\beta\cdot(\langle 0_n\oplus\mathbf{1}_r,\, \mathbf{1}_n\oplus 0_r\rangle\oplus\mathbf{1}_p))^\dagger\\
& = & \delta.
\end{array}$$

Let

$$d \;\;:=\;\; c\cdot\langle \mathbf{1}_k\oplus 0_{s+p},\, 0_k\oplus\gamma,\, 0_{k+s}\oplus\mathbf{1}_p\rangle : r\to k+s+p\in\Sigma(T_0).$$

Define

$$\begin{array}{rcl}
\rho_i' & := & \langle\rho_i,\mathbf{1}_s\rangle : k+s\to s \quad (i\in[s])\\
\tau_j' & := & \langle\tau_j,\,\rho\rangle : k+s\to r \quad (j\in[r]).
\end{array}$$

We have

$$\begin{array}{rl}
\multicolumn{2}{l}{(\rho\cdot d)\parallel(\rho_1',\ldots,\rho_s') =}\\
= & \langle 1_s\cdot\rho\cdot c\cdot\langle\rho_1\oplus 0_p,\,\gamma,\,0_s\oplus\mathbf{1}_p\rangle,\ldots\\
& \quad\ldots, s_s\cdot\rho\cdot c\cdot\langle\rho_s\oplus 0_p,\,\gamma,\,0_s\oplus\mathbf{1}_p\rangle\rangle\\
= & \langle 1_s\cdot\rho\cdot c\cdot(\rho_1\oplus\mathbf{1}_{n+p}),\ldots\\
& \quad\ldots, s_s\cdot\rho\cdot c\cdot(\rho_s\oplus\mathbf{1}_{n+p})\rangle\cdot\langle\mathbf{1}_s\oplus 0_p,\,\gamma,\,0_s\oplus\mathbf{1}_p\rangle\\
= & ((\rho\cdot c)\parallel(\rho_1,\ldots,\rho_s))\cdot\langle\mathbf{1}_s\oplus 0_p,\,\gamma,\,0_s\oplus\mathbf{1}_p\rangle\\
\theta_1 & a\cdot\langle\mathbf{1}_s\oplus 0_p,\,\gamma,\,0_s\oplus\mathbf{1}_p\rangle.
\end{array}$$

Similarly,

$$\begin{array}{rl}
\multicolumn{2}{l}{d\parallel(\tau_1',\ldots,\tau_r') =}\\
= & \langle 1_r\cdot c\cdot\langle\tau_1\oplus 0_p,\,\gamma\cdot(\rho\oplus\mathbf{1}_p),\,0_r\oplus\mathbf{1}_p\rangle,\ldots\\
& \quad\ldots, r_r\cdot c\cdot\langle\tau_r\oplus 0_p,\,\gamma\cdot(\rho\oplus\mathbf{1}_p),\,0_r\oplus\mathbf{1}_p\rangle\rangle\\
= & \langle 1_r\cdot c\cdot(\tau_1\oplus\mathbf{1}_{n+p}),\ldots\\
& \quad\ldots, r_r\cdot c\cdot(\tau_r\oplus\mathbf{1}_{n+p})\rangle\cdot\langle\mathbf{1}_r\oplus 0_p,\,\gamma\cdot(\rho\oplus\mathbf{1}_p),\,0_r\oplus\mathbf{1}_p\rangle\\
\theta_1 & (c\parallel(\tau_1,\ldots,\tau_r))\cdot\langle\mathbf{1}_r\oplus 0_p,\,\delta,\,0_r\oplus\mathbf{1}_p\rangle\\
\theta_1 & b\cdot\langle\mathbf{1}_r\oplus 0_p,\,\delta,\,0_r\oplus\mathbf{1}_p\rangle.
\end{array}$$

Also

$$\rho_i'\cdot\rho \;=\; \langle\rho_i\cdot\rho,\;\rho\rangle \;=\; \langle\tau_{i\rho},\,\rho\rangle \;=\; \tau_{i\rho}',$$

for all $i \in [s]$. Thus $D^\dagger \xrightharpoonup{\rho} E^\dagger$ mod (θ_0, θ_1).

Suppose that $a, b : n \to b \in \Sigma(T_0)$ with $a\ \theta_1\ b$. Suppose also that $D := (\mathbf{1}_n \oplus 0_p; 0_n \oplus a)$ and $E := (\mathbf{1}_n \oplus 0_p; 0_n \oplus b)$. Since $|D| = a$ and $|E| = b$ and since

$$D \xrightharpoonup{\mathbf{1}_n} E \text{ mod } (\theta_0, \theta_1),$$

it follows that $a\ \theta\ b$. Similarly, if $\alpha\ \theta_0\ \beta$ holds for some T_0-morphisms $\alpha, \beta : n \to p$ then $\alpha\ \theta\ \beta$. Thus θ contains θ_0 and θ_1. By Corollary 11.3.2, θ is the smallest such dagger congruence.

It is a routine matter to show that if $\alpha, \beta : 1 \to p \in T_0$ with

$$(\alpha; 0_p) \overset{*}{\leftrightarrow} (\beta; 0_p) \text{ mod } (\theta_0, \theta_1),$$

then $\alpha\ \theta_0\ \beta$. Thus $\alpha\ \theta_0\ \beta$ whenever $\alpha\ \theta\ \beta$, for all $\alpha, \beta : 1 \to p \in T_0$, and hence for all $\alpha, \beta : n \to p \in T_0$.

Assume next that $a, b : 1 \to p \in \Sigma(T_0)$ and that

$$(\mathbf{1}_1 \oplus 0_p;\ 0_1 \oplus a) \overset{*}{\leftrightarrow} (\mathbf{1}_1 \oplus 0_p;\ 0_1 \oplus b) \ \text{ mod } (\theta_0, \theta_1).$$

It follows by Exercise 11.2.24 that there is a presentation D such that for some base injections ρ and τ we have

$$(\mathbf{1}_1 \oplus 0_p;\ 0_1 \oplus a) \xleftarrow{\rho} D \xrightarrow{\tau} (\mathbf{1}_1 \oplus 0_p;\ 0_1 \oplus b) \ \text{ mod } (\theta_0, \theta_1).$$

One proves that $a\ \theta_1\ b$. Thus if $a\ \theta\ b$, then $a\ \theta_1\ b$, for all $a, b : 1 \to p$ in $\Sigma(T_0)$, and hence for all $a, b : n \to p$ in $\Sigma(T_0)$.

To complete the proof, note that if $D \xrightharpoonup{\rho} E$ mod (θ_0, θ_1) in T, for some presentations $D = (\alpha; a) : n \to p$ and $E = (\beta; b) : n \to p$ over (T_0, Σ) and some base morphism ρ, then also $D' \xrightharpoonup{\rho} E'$ in T/θ, for the presentations $D' := (\alpha/\theta; a/\theta)$ and $E' := (\beta/\theta; b/\theta)$ over (T_0', Σ'). Thus if $|D'| = |E'|$ in T/θ, then $D \overset{*}{\rightleftharpoons} E$ mod (θ_0, θ_1) in T and $D' \overset{*}{\rightleftharpoons} E'$ in T/θ. Thus behavioral equivalence in T/θ coincides with the relation $\overset{*}{\rightleftharpoons}_{T/\theta}$.

Exercise 11.3.4 Let $(T_0, \Sigma(T_0))$ be a compatible pair in an iteration theory T. By the proof of Theorem 11.3.3, $\overset{*}{\rightleftharpoons}$-equivalence of presentations is preserved by the pairing, composition and iteration operations as defined in Lemma 11.1.3. Show that $\overset{*}{\rightleftharpoons}$-equivalence classes form an iteration theory $\mathcal{S}(T_0, \Sigma)$. (See Remark 11.1.9 for the definition of the constants i_n and 0_n.) Prove that if T is generated by $T_0 \cup \Sigma$, then T is a quotient of $\mathcal{S}(T_0, \Sigma)$.

Exercise 11.3.5 Let $(T_0, \Sigma(T_0))$ be a compatible pair in T. It follows from the proof of Theorem 11.3.3 that $\stackrel{*}{\leftrightarrow}$-equivalence of presentations is preserved by all of the operations. Since $\stackrel{*}{\rightleftharpoons}$-equivalence is included in $\stackrel{*}{\leftrightarrow}$-equivalence, it follows from the preceding exercise that $\stackrel{*}{\leftrightarrow}$-equivalence classes also form an iteration theory $\mathcal{S}'(T_0, \Sigma)$ which is a quotient of $\mathcal{S}(T_0, \Sigma)$. Show that if T is generated by $T_0 \cup \Sigma$ and if T has a functorial dagger with respect to base surjections, then T is a quotient of $\mathcal{S}'(T_0, \Sigma)$. When is it true that $\mathcal{S}'(T_0, \Sigma)$ has a functorial dagger with respect to all base surjections?

Exercise 11.3.6 Suppose that $D \stackrel{*}{\rightleftharpoons} E$ whenever $D, E : 1 \to p$ are scalar presentations with the same behavior. It follows by Exercise 11.3.4 that in this case, behavioral equivalence coincides with $\stackrel{*}{\rightleftharpoons}$-equivalence. Similarly, by Exercise 11.3.5, if $D \approx E$ implies $D \stackrel{*}{\leftrightarrow} E$, for all scalar D and E, then $\approx = \stackrel{*}{\leftrightarrow}$. Give a direct proof of these facts.

In addition to the iteration theory T, we will now consider another iteration theory R and a surjective iteration theory morphism $\varphi : R \to T$. We assume that $(R_0, \Delta(R_0))$ is a compatible pair in R and define

$$\begin{aligned} T_0 &:= R_0\varphi \\ \Sigma &:= \Delta\varphi, \end{aligned}$$

so that $(T_0, \Sigma(T_0))$ is a compatible pair in T and $\Sigma(T_0) = \Delta(R_0)$. Note that if $R_0 \cup \Delta$ generates R then T is in turn generated by $T_0 \cup \Sigma$. The restriction of $\ker\varphi$ to R_0 is a dagger congruence θ_0 on R_0 and the restriction of $\ker\varphi$ to $\Delta(R_0)$ is an equivalence relation θ_1 on $\Delta(R_0)$. Further, (θ_0, θ_1) is a compatible pair of equivalences.

Proposition 11.3.7 *Suppose that $R_0 \cup \Delta$ generates R and that $\approx_T = \stackrel{*}{\rightleftharpoons}_T$ in T. Then $\ker\varphi$ is the smallest dagger congruence containing the relations θ_0 and θ_1.*

Proof. Let θ denote the smallest dagger congruence on R containing θ_0 and θ_1. We must show that if $f\varphi = g\varphi$ holds for $f, g : n \to p \in R$ then $f \; \theta \; g$. Since $R_0 \cup \Delta$ generates R, by Theorem 11.1.4 there exist presentations $D = (\alpha; a) : n \to p$ and $E = (\beta; b) : n \to p$ (over (R_0, Δ)) such that $f = |D|$ and $g = |E|$. Define

$$D\varphi \;:=\; (\alpha\varphi; a\varphi) \quad \text{and} \quad E\varphi \;:=\; (\beta\varphi; b\varphi),$$

so that $D\varphi$ and $E\varphi$ are presentations over (T_0, Σ) with

$$|D\varphi| \;=\; f\varphi \;=\; g\varphi \;=\; |E\varphi|.$$

Since $\approx_T = \overset{*}{\rightleftharpoons}_T$, by assumption, we have

$$D\varphi \overset{*}{\rightleftharpoons} E\varphi$$

in T. It follows that

$$D \overset{*}{\rightleftharpoons} E \text{ mod } (\theta_0, \theta_1)$$

in R. Thus $f\ \theta\ g$ by Corollary 11.3.2.

As an application of our previous results we now give a necessary and sufficient condition that an iteration theory is the coproduct of a nontrivial iteration theory and a free iteration theory. When Σ is a signature, we will identify a symbol in Σ with the corresponding atomic tree in the iteration theory $\Sigma\mathbf{tr}$. Recall Section 6.6.6.

Theorem 11.3.8 *Suppose that T_0 is a nontrivial iteration theory and $\kappa : T_0 \to T$ and $\lambda : \Sigma\mathbf{tr} \to T$ are iteration theory morphisms. Then κ and λ are coproduct injections if and only if the following conditions hold.*

[a] *κ is injective;*

[b] *$T_0\kappa \cup \Sigma\lambda$ generates T;*

[c] *For any two presentations $D, E : 1 \to p$ over $(T_0\kappa, \Sigma\lambda)$, if $|D| = |E|$ then $D \overset{*}{\rightleftharpoons} E$, i.e. $\approx_T = \overset{*}{\rightleftharpoons}_T$;*

[d] *For all $\sigma, \sigma' \in \Sigma$ and $\alpha, \alpha' \in T_0$ with appropriate source and target, if $\sigma\lambda \cdot \alpha\kappa = \sigma'\lambda \cdot \alpha'\kappa$ then $\sigma = \sigma'$ and $\alpha = \alpha'$.*

Proof. Suppose that Δ is large enough so that there is a surjective iteration theory morphism $\varphi_0 : \Delta\mathbf{tr} \to T_0$. Let

$$\kappa' : \Delta\mathbf{tr} \to (\Delta + \Sigma)\mathbf{tr} \quad \text{and} \quad \lambda' : \Sigma\mathbf{tr} \to (\Delta + \Sigma)\mathbf{tr}$$

be the inclusions of $\Delta\mathbf{tr}$ and $\Sigma\mathbf{tr}$ in $(\Delta + \Sigma)\mathbf{tr}$, respectively. Since κ' and λ' are coproduct injections, there is a unique iteration theory morphism

$$\varphi : (\Delta + \Sigma)\mathbf{tr} \to T$$

such that the following diagram commutes:

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We have seen in Section 6.6.6 that κ and λ are coproduct injections if and only if φ is surjective and $\ker\varphi$ is the smallest dagger congruence containing $\ker\varphi_0$. Clearly, φ is surjective if and only if T is generated by $T_0\kappa \cup \Sigma\lambda$.

Let $R := (\Delta + \Sigma)\mathbf{tr}$ and $R_0 := (\Delta\mathbf{tr})\kappa'$, so that $R_0 \cup \Sigma\lambda'$ generates R and (R_0, V) is a compatible pair in R, where V is the collection of all trees $a : n \to p$ in R such that each $i_n \cdot a$ is of the form $\sigma \cdot t$ for $\sigma \in \Sigma$ and $t \in \Delta\mathbf{tr}$. In fact V is the smallest set of morphisms containing $\Sigma\lambda'$ such that (R_0, V) is a compatible pair in R, i.e. $V = \Sigma\lambda'(R_0)$. Further, let U be the collection of morphisms $a : n \to p \in T$ such that each $i_n \cdot a$ is the composite of a morphism from $\Sigma\lambda$ with a T_0-morphism. Thus U is the smallest set of morphisms containing $\Sigma\lambda$ and such that $(T_0\kappa, U)$ is a compatible pair in T, in notation, $U = \Sigma\lambda(T_0)$. We have $R_0\varphi = T_0\kappa$ and $V\varphi = U$.

Suppose that κ and λ are coproduct injections so that $\ker\varphi$ is the smallest dagger congruence on R containing $\ker\varphi_0$. Since by Exercise 11.2.20 we have $\approx_R \;=\; \stackrel{*}{\rightleftharpoons}_R$, and since $R_0 \cup \Sigma\lambda'$ generates R, it follows by Theorem 11.3.3 that $\approx_T \;=\; \stackrel{*}{\rightleftharpoons}_T$. Further, κ is obviously injective. Let θ_0 denote the restriction of $\ker\varphi$ to R_0 and θ_1 the restriction of $\ker\varphi$ to V. Since κ is injective, $\theta_0 = \ker\varphi_0$. Thus, by Theorem 11.3.3 again, θ_1 is the smallest equivalence relation on V such that (θ_0, θ_1) is a compatible pair of equivalences. Thus for all $a, b : 1 \to p \in V$, $a \;\theta_1\; b$ if and only if there exist $\sigma \in \Sigma$ and $t, t' \in \Delta\mathbf{tr}$ with $a = \sigma \cdot t$, $b = \sigma \cdot t'$ and $t\varphi_0 = t'\varphi_0$. It follows that λ is injective and each scalar morphism in U can be written uniquely as the composite of a morphism in $\Sigma\lambda$ with a morphism in $T_0\kappa$.

For the converse first note that if κ is injective then the restriction θ_0 of $\ker\varphi$ to R_0 is the dagger congruence $\ker\varphi_0$. Also, since each scalar morphism in U can be written in a unique way as the composite of a morphism in $\sigma\lambda$ with a morphism in $T_0\kappa$, where $\sigma \in \Sigma$, the restriction of $\ker\varphi$ to V is the smallest equivalence relation θ_1 on V such that (θ_0, θ_1) is a compatible pair of equivalences. Thus, by Proposition 11.3.7, if $\approx_T \;=\; \stackrel{*}{\rightleftharpoons}_T$ then $\ker\varphi$ is the smallest dagger congruence on R which contains $\ker\varphi_0$.

Exercise 11.3.9 Suppose that T is an iteration theory and X is a set. Use Theorem 11.3.8 to show that the iteration theory $T[X]$ constructed in Section 6.6.6 is the coproduct of T with the free iteration theory $\Sigma\mathbf{tr}$, where $\Sigma_0 = X$ and Σ_n is empty, for all $n > 0$.

Exercise 11.3.10 One representation of the coproduct of a nontrivial iteration theory T_0 with a free iteration theory $\Sigma\mathbf{tr}$ can be obtained as follows. Define a presentation $n \to p$ to be a pair $(\alpha; a)$ where α is a morphism $n \to s + p$ in T_0 and $a : s \to s + p$ is a "formal expression"

$$a \;=\; \langle \sigma_1 \cdot \alpha_1, \ldots, \sigma_s \cdot \alpha_s \rangle,$$

where the σ's are in Σ and the α's are in T_0. Define the theory operations and $\overset{*}{\rightleftharpoons}$-equivalence of presentations as in Lemma 11.1.3 and Definitions 11.2.1 and 11.2.4. Then the $\overset{*}{\rightleftharpoons}$-equivalence classes of these presentations form an iteration theory T which is the coproduct of T_0 with the free iteration theory $\Sigma\mathbf{tr}$.

11.4 Notes

A concept analogous to that of presentation was introduced into the study of iterative theories by Calvin Elgot in [Elg75]. The explicit formalization of Theorem 11.1.4 is new. A corresponding result for iterative theories is given in the Main Theorem of [Elg75]. Theorem 11.3.8 is also new, although some particular insatances of this result are implicit in the proofs of the axiomatization results of [Ési80, Ési81, Ési85]. In fact, the present formalization is a rather starightforward generalization of the proof method used therein. Related results can be found in [CSS90].

Chapter 12

Flowchart Behaviors

This chapter deals with the structures which serve as the standard models for (functorial) flowchart semantics. Recall from Example 11.11.1.7 that one can identify a flowchart scheme $n \to p$ with a presentation $D = (\alpha; a)$ in a free tree theory $\Sigma\mathbf{tr}$. (Here α is a partial base morphism and each component of a is the composite of an atomic tree with a partial base morphism.) Thus, if T is any iteration theory and $\varphi : \Sigma \to T$ is any function mapping letters in Σ_n to morphisms $1 \to n$ in T, $n \geq 0$, φ extends uniquely to both a theory morphism $\Sigma\mathbf{tr} \to T$, and also to a morphism of presentations:

$$D\varphi \quad := \quad (\alpha; a\varphi).$$

The pair $(\alpha; a\varphi)$ might be called a flowchart *algorithm description*, or a *flowchart interpreted in the theory* T, and the theory morphism φ might be called a *semantics functor*. The behavior of the algorithm description is the T-morphism

$$|D\varphi| \quad = \quad \alpha \cdot \langle a\varphi^{\dagger}, \mathbf{1}_p \rangle.$$

Thus, a presentation D gives a syntactical description of a class of flowchart algorithms. Each instance of this class is determined by a particular iteration theory T and a semantics functor $\varphi : \Sigma\mathbf{tr} \to T$. In this chapter, we find equational axiomatizations for the varieties of iteration theories which form the standard interpretations of flowchart scheme models: the variety generated by the theories A and the variety generated by the theories A, with and without a set of distinguished predicates.

The theories A serve as targets of *input-output* semantics functors; if D is a flowchart scheme $1 \to 1$, say, and $\varphi : \Sigma\mathbf{tr} \to A$ is a theory morphism, then the behavior of the flowchart $D\varphi$ interpreted in A is the partial function $A \to A$ whose value on the input x in A is the value of the output, if any, of $D\varphi$ on x.

The theories A serve as targets of *stepwise-behavior* semantics functors. If D is a flowchart scheme $1 \to 1$ say and $\varphi : \Sigma\mathbf{tr} \to A$ is a theory morphism, then the behavior of $D\varphi$ is a sequacious function $A^+ \cup A^\omega \to A^+ \cup A^\omega$ whose value on input x in A is the finite or infinite sequence of "states" assumed by the algorithm $D\varphi$ running on input x.

We will also consider the case that certain letters in Σ are required to be interpreted as *predicates*. In the case there are no distinguished predicates, the variety generated by the class of theories A, as A ranges over all sets, is the variety of all iteration theories; the variety generated by the theories A is the variety consisting of those iteration theories with a unique morphism $1 \to 0$.

12.1 Axiomatizing Sequacious Functions

In Section 10.10.3, it was shown that A, the theory of sequacious relations on a nonempty set A, is an iteration theory. When $f : n \to n + p$ is a sequacious function, so is $f^\dagger : n \to p$. This shows that the theory $\mathbf{Seq}_A$ of sequacious functions is an iteration theory also. More precisely, the theory $\mathbf{Seq}_A$ is not determined uniquely by the the set A, for in order to define $\mathbf{1}_1^\dagger$ we used a fixed element a_0 of the set A. In this section, when we want to indicate this element, we will write $(\mathbf{Seq}_A, a_0)$. Let *SEQ* denote the variety of iteration theories generated by all theories of the form $\mathbf{Seq}_A$. We will show that *SEQ* is the variety of all iteration theories. To prove this result, we only need to prove that the free iteration theories $\Sigma\mathbf{tr}$ belong to *SEQ*. This fact will follow immediately from the first proposition below.

In the proof of Proposition 12.1.1, we will make use of sets B and $A_0 := B^*$, where

$$B \;\; := \;\; \cup(\Sigma_n \times [n] : n \geq 1) \cup \Sigma \cup \{\perp\}.$$

Recall that $\perp$ is a letter not in Σ used to define $\mathbf{1}_1^\dagger$ in $\Sigma\mathbf{tr}$. The use of A_0

causes a notational problem. The elements of A_0 are finite sequences of elements of B, and sequacious functions on A_0 involve sequences of elements of A_0. We will denote concatenation of words in B^* by juxtaposition, and will use $(u_1, \ldots, u_k)$ to denote a finite sequence in A_0^* and $(u_1, u_2, \ldots)$ an infinite sequence in A_0^ω.

Proposition 12.1.1 *For A_0 as above, there is an injective iteration theory morphism*

$$\varphi_0 : \Sigma\mathbf{tr} \quad \to \quad (\mathbf{Seq}_A, \bot).$$

Proof. Because this argument is a model for several to come, we will give a detailed proof. Since $\Sigma\mathbf{tr}$ is freely generated by Σ, to define φ_0 we need specify the value of φ_0 only on each letter in Σ.

We define $\sigma\varphi_0$, when $\sigma \in \Sigma_n$, $n \geq 0$, as the sequacious function determined by the following conditions. Let u be an element of A_0, i.e. a word in B^*, say $u = u_1 \ldots u_k$, $k \geq 0$, where $u_i \in B$, $i \in [k]$. Then, if $k > 0$ and $u_1 = (\sigma, i) \in \Sigma_n \times [n]$, $n > 0$,

$$u(\sigma\varphi_0) \quad := \quad ((u, u_2 \ldots u_k),\ i);$$

otherwise, i.e., if k or n is 0 or $u_1 \notin \Sigma_n \times [n]$,

$$u(\sigma\varphi_0) \quad := \quad (u, \sigma^\omega).$$

Thus, for all $u \in A_0$, $\sigma \in \Sigma_n$, $n \geq 0$, the value of $\sigma\varphi_0$ on u is an element of $A_0^2 \times [2] \ \cup \ A_0^\omega$. The point of this construction is that the sequacious function $F_t := t\varphi_0$ determined by the tree t determines the tree t itself in the following sense. If u codes the label of a path in t to a vertex v, F_t on u will produce the sequence of words obtained from u by deleting, from left to right, the letters in u perhaps followed, at the end, by the infinite sequence $(vt)^\omega$. On words that do not code a path in t, F_t will abort.

Example 12.1.2 Let σ_1 be a letter in Σ_1 and suppose that σ_2, σ_2' are in Σ_2. Let t be the tree $\sigma_1 \cdot \sigma_2$. Then

$$\begin{aligned}
(\sigma_1, 1)(\sigma_2, 2)F_t \quad &= \quad (((\sigma_1, 1)(\sigma_2, 2),\ (\sigma_2, 2),\ \epsilon),\ 2) \in A_0^3 \times [2] \\
(\sigma_1, 1)(\sigma_2', 1)F_t \quad &= \quad ((\sigma_1, 1)(\sigma_2', 1),\ (\sigma_2', 1),\ \sigma_2, \sigma_2, \ldots) \in A_0^\omega.
\end{aligned}$$

In order to prove the resulting iteration theory morphism φ_0 is injective, we restate some of the preceding definitions in the form of a lemma. When $u \in A_0$ and $(v, j) \in A_0^+ \times [p]$, we will understand $(u, (v, j))$ to mean $((u, v), j) \in A_0^+ \times [p]$.

Lemma 12.1.3 *Suppose that $t : 1 \to p$ is a tree in $\Sigma\mathbf{tr}$. Write F_t for $t\varphi_0$, and let u be an element of A_0.*

- [a] *If $t = j_p$, for some $j \in [p]$, then $uF_t = (u, j)$.*
- [b] *If $t = \sigma \cdot 0_p$, for some $\sigma \in \Sigma_0 \cup \{\perp\}$, then $uF_t = (u, \sigma^\omega)$.*
- [c] *Suppose $t = \sigma \cdot \langle t_1, \ldots, t_n \rangle$, for some $\sigma \in \Sigma_n$, $n \geq 1$. If $u = (\sigma, i)u'$, then $uF_t = (u, u'F_{t_i})$, otherwise $uF_t = (u, \sigma^\omega)$.*

We now show that $\varphi_0 : \Sigma\mathbf{tr} \to (\mathbf{Seq}_{A_0}, \perp)$ is injective. Assume that t and t' are distinct trees $1 \to p$ in $\Sigma\mathbf{tr}$. We use induction on the length of the shortest word $v \in [\omega]^*$ such that $vt \neq vt'$ to prove $uF_t \neq uF_{t'}$, for some element $u \in A_0$.

Case 1. $v = \epsilon$. In this case the labels of the roots of the two trees are distinct. Suppose, say, $t = \sigma_1 \cdot \langle t_1, \ldots, t_n \rangle$ and $t' = \sigma_2 \cdot \langle t'_1, \ldots, t'_m \rangle$. One of the letters σ_1 and σ_2 may be $\perp$. Then, by Lemma 12.1.3.b and c,

$$\epsilon F_t \;=\; (\epsilon, \sigma_1^\omega) \;\neq\; (\epsilon, \sigma_2^\omega) \;=\; \epsilon F_{t'}.$$

If $t' = j_p : 1 \to p$, then

$$\epsilon F_t \;=\; (\epsilon, \sigma_1^\omega) \;\neq\; (\epsilon, j) \;=\; \epsilon F_{t'}.$$

If t is also a trivial tree, say $t = k_p$, $\epsilon F_t = (\epsilon, k)$. The basis step is complete.

Case 2. $v = iw$, $t = \sigma \cdot \langle t_1, \ldots, t_n \rangle$ and $t' = \sigma \cdot \langle t'_1, \ldots, t'_n \rangle$, for some $i \leq n$. Then $wt_i \neq wt'_i$, so that by the induction hypothesis there is some element $u' \in A_0$ with

$$u'F_{t_i} \;\neq\; u'F_{t'_i}.$$

If we let $u := (\sigma, i)u'$, then

$$uF_t \;=\; (u, u'F_{t_i}) \;\neq\; (u, u'F_{t'_i}) \;=\; uF_{t'}.$$

We have completed the proof of the following theorem.

Theorem 12.1.4 *For any signature Σ, the iteration theory $\Sigma\mathbf{tr}$ is in the variety SEQ, since it is isomorphic to a subiteration theory of the form $\mathbf{Seq}_A$. Hence the variety SEQ is the variety of all iteration theories.*

Exercise 12.1.5 Show that $\Sigma\mathbf{tr}$ can be embedded in a theory $\prod_{i\in I}\mathbf{Seq}_{A_i}$, where the sets A_i are finite. Hence the variety of all iteration theories is generated by iteration theories of sequacious functions on finite sets.

12.2 Axiomatizing Partial Functions

In this section, an axiomatization is given for the variety generated by the class of all iteration theories of the form $\mathbf{Pfn}_A$. In the course of the proof, the free iteration theories in this variety are explicitly exhibited.

The property that an iteration theory has a unique morphism $1 \to 0$ is clearly an equational one, namely,

$$f \;=\; g, \quad \text{for all } f, g : 1 \to 0,$$

or

$$f \;=\; \mathbf{1}_1^\dagger, \quad \text{for all } f : 1 \to 0.$$

For the moment we adopt the definition that *PFN* is the variety of iteration theories with a unique morphism $1 \to 0$. Thus each iteration theory $\mathbf{Pfn}_A$ belongs to *PFN*. One of our results will show that *PFN* is generated by the theories $\mathbf{Pfn}_A$.

We will need the following terminology in the proof of the next propositions.

Definition 12.2.1 *Let Σ be a signature, $t : 1 \to p$ a tree in $\Sigma\mathbf{tr}$. We call a vertex v of t* **coaccessible**, *if there is a word $u \in [\omega]^*$ such that $(vu)t = x_i$, for some $i \in [p]$. A vertex v which is not coaccessible is referred to a* **0-vertex**. *A* **minimal 0-vertex** *is a 0-vertex v such that no proper prefix u of v is a 0-vertex. The tree t itself is coaccessible, if each vertex v of t is either coaccessible or labeled $\perp$, in notation $vt = \perp$. A tree $t : n \to p$ in $\Sigma\mathbf{tr}$, $n \neq 1$, is coaccessible if $j_n \cdot t$ is coaccessible, for all $j \in [n]$.*

In particular, no leaf of a coaccessible tree is labeled by a letter in Σ_0, and the only coaccessible tree $1 \to 0$ in $\Sigma\mathbf{tr}$ is the tree $\perp$. Further, all of the nonleaves of a coaccesible tree $t : 1 \to p$ are coaccessible.

Exercise 12.2.2 In any theory, we say that a morphism $f : n \to p$ factors through 0, if there is a morphism $g : n \to 0$ with $f = g \cdot 0_p$. Show that in the theory $\Sigma\mathbf{tr}$, if v is a vertex of a tree t, then v is a 0-vertex if and only if t_v, the subtree rooted at v, factors through 0.

Definition 12.2.3 *For a tree $t : 1 \to p$ in $\Sigma\mathbf{tr}$, let $\overline{t} : 1 \to p$ be the tree defined as follows:*

$$w\overline{t} \quad := \quad \begin{cases} wt & \text{if } w \text{ is coaccessible;} \\ \perp & \text{if } w \text{ is a minimal } 0\text{-vertex,} \end{cases}$$

otherwise wt is not defined. For a tree $t = \langle t_1, \ldots, t_n \rangle : n \to p$ in $\Sigma\mathbf{tr}$, $n \neq 1$, we define $\overline{t} := \langle \overline{t}_1, \ldots, \overline{t}_n \rangle$. When $f, g : n \to p$, we write $f \approx g$ iff $\overline{f} = \overline{g}$.

Thus, the tree $\overline{t}$ is obtained from $t : 1 \to p$ by relabeling all minimal 0-vertices v as $\perp$ and deleting the remaining vertices from each subtree t_v.

Exercise 12.2.4 Prove that for $f : 1 \to p \in \Sigma\mathbf{tr}$, $\overline{f}$ has a finite number of subtrees, so that $\overline{f}$ is in $\Sigma\mathbf{tr}$ also. In fact, there exists a coaccessible tree $f' : 1 \to p + k \in \Sigma\mathbf{tr}$ such that $\overline{f} = f' \cdot \langle \mathbf{1}_p, \perp_{kp} \rangle$ and $f = f' \cdot \langle \mathbf{1}_p, g \rangle$, for some $g : k \to p$ in $\Sigma\mathbf{tr}$ which factors through 0. Show that $\overline{f}$ is coaccessible. Show that $f = \overline{f}$ if and only if f is coaccessible.

Exercise 12.2.5 Prove that for $f, g : 1 \to p$ in $\Sigma\mathbf{tr}$, $\overline{f} = \overline{g}$ if and only if $f = h \cdot \langle \mathbf{1}_p, f_1 \rangle$ and $g = h \cdot \langle \mathbf{1}_p, g_1 \rangle$, for some trees $h : 1 \to p + k$ and $f_1, g_1 : k \to p$ in $\Sigma\mathbf{tr}$ such that f_1 and g_1 factor through 0.

The following proposition is immediate from the Zero Congruence Lemma (Lemma 6.6.6.10) and Exercise 12.2.5.

Proposition 12.2.6 *The relation $\approx$ is the smallest (dagger) congruence on $\Sigma\mathbf{tr}$ that identifies any two trees $t, t' : 1 \to 0$.*

Let

$$\eta_{\approx} \quad := \quad \Sigma \xrightarrow{\eta} \Sigma\mathbf{tr} \xrightarrow{\kappa_{\approx}} \Sigma\mathbf{tr}/{\approx},$$

where η is the free embedding of Σ to $\Sigma\mathbf{tr}$ and $\kappa_{\approx}$ is the quotient map.

Corollary 12.2.7 $\Sigma\mathbf{tr}/\approx$ *is freely generated by* $\eta_{\approx}$ *in the variety PFN.*

Proof. The theory $\Sigma\mathbf{tr}/{\approx}$ is in *PFN*, since it has a unique morphism $1 \to 0$. If T is in *PFN* and $\varphi : \Sigma \to T$ is a rank preserving map, φ extends to a unique iteration theory morphism $\varphi' : \Sigma\mathbf{tr} \to T$ such that $\eta \cdot \varphi' = \varphi$. By Proposition 12.2.6, φ' uniquely factors through $\kappa_{\approx}$.

Each $\approx$-equivalence class contains a unique tree t whose components are coaccessible, i.e. such that $\overline{t} = t$. Thus the iteration theory freely generated by Σ in the variety *PFN* can be alternatively represented as an iteration theory of regular coaccessible trees. We now set out to prove that *PFN* is generated by the theories $\mathbf{Pfn}_A$. We will find the following definition useful.

Definition 12.2.8 *Let* $t : 1 \to p$ *be a tree in* $\Sigma\mathbf{tr}$*. If* v *is a vertex of* t*, then the label of* v *in* t*,* $\operatorname{lab} v$ *is the word on the set* $\cup(\Sigma_n \times [n] : n > 0)$*, defined by induction on the length of* v *as follows:*

1. *If* $v = \epsilon$ *then* $\operatorname{lab} v = \epsilon$*;*
2. *If* $v = iw$ *and* $t = \sigma \cdot \langle t_1, \ldots, t_n \rangle$*, then* $\operatorname{lab} v = (\sigma, i) \cdot \operatorname{lab} w$*, where* $\operatorname{lab} w$ *is the label of* w *in* t_i*.*

Proposition 12.2.9 *If* t *and* t' *are distinct coaccessible trees* $1 \to p$ *in* $\Sigma\mathbf{tr}$*, then there is a leaf* v *of* t*, say, labeled* x_i*, for some* $i \in [p]$*, such that* $\operatorname{lab} v$ *is not the label of a path to a leaf labeled* x_i *in* t'*.*

Proof. Since t and t' are distinct, there is some vertex w in the domain of both trees such that $wt \neq wt'$. We may assume that wt is not $\perp$. If w is not a leaf of t labeled x_i, then some extension $v = wu$ is a leaf of t labeled x_i. But then $\operatorname{lab} v = \operatorname{lab} w \cdot \operatorname{lab} u$, where $\operatorname{lab} u$ denotes the label of u in the subtree t_w, and $\operatorname{lab} v$ is the required path label.

Proposition 12.2.10 *There is a set A_0 and a theory morphism $\varphi_0 : \Sigma\mathbf{tr} \to \mathbf{Pfn}_{A_0}$ such that if t and t' are distinct coaccessible trees $1 \to p$, then $t\varphi_0 \neq t'\varphi_0$.*

Proof. Let $B := \cup(\Sigma_n \times [n] : n > 0)$ and let $A_0 := B^*$, the set of all words on B. Again, to define φ_0, we need only to define the image $\sigma\varphi_0$ of each letter in Σ. On letters in Σ_0 the value of $\sigma\varphi_0$ is forced to be the empty partial function. Let $w = (\sigma_1, i_1) \ldots (\sigma_k, i_k)$, $k \geq 0$. If σ is in Σ_n, $n > 0$, and if $k \geq 1$ and $\sigma_1 = \sigma$, then $\sigma\varphi_0$ is defined by

$$w(\sigma\varphi_0) \quad := \quad ((\sigma_2, i_2) \ldots (\sigma_k, i_k),\, i_1);$$

otherwise, if $k = 0$ or $\sigma_1 \neq \sigma$, $w(\sigma\varphi_0)$ is undefined. In particular, $\epsilon(\sigma\varphi_0)$ is undefined.

Claim.

- If w is lab v, where v is a leaf of t labeled x_i, then $w(t\varphi_0) = (\epsilon, i)$.
- If w is not the label of a path in t to a leaf labeled by some x_i, $w(t\varphi_0)$ is undefined.

The claim is easily proved by induction on the length of w. The proof is completed by using Proposition 12.2.9.

Since $\approx$ is the smallest dagger congruence on $\Sigma\mathbf{tr}$ that collapses all trees $1 \to 0$, there is a unique iteration theory morphism $\psi_0 : (\Sigma\mathbf{tr}/\approx) \to \mathbf{Pfn}_{A_0}$ such that $\kappa_\approx \cdot \psi_0 = \varphi_0$, where $\kappa_\approx : \Sigma\mathbf{tr} \to \Sigma\mathbf{tr}/\approx$ is the quotient map. By Propositions 12.2.6, 12.2.9 and 12.2.10, ψ_0 is injective. We have thus completed the proof of the following theorem.

Theorem 12.2.11 *The variety of iteration theories generated by all theories $\mathbf{Pfn}_A$ is the class PFN of iteration theories with a unique morphism $1 \to 0$. For each signature Σ, $\Sigma\mathbf{tr}/\approx$ is the free iteration theory, freely generated by Σ, in PFN. Further, $\Sigma\mathbf{tr}/\approx$ is isomorphic to a subiteration theory of some $\mathbf{Pfn}_A$.*

Exercise 12.2.12 Prove that $\Sigma\mathbf{tr}/\approx$ embeds in a direct product of iteration theories $\mathbf{Pfn}_{A_i}$, $i \in I$, where each set A_i is finite.

We state without proof the following theorem.

Theorem 12.2.13 *There is a unique way to define the iteration operation † so that the theory of partial functions $\mathbf{Pfn}_A$ on a set A is an iteration theory.*

12.3 Diagonal Theories

Definition 12.3.1 *In any theory T, we define three collections of base morphisms: τ_{np}, δ_p, β_n, $n, p \geq 0$.*

$$\begin{array}{rcl} \tau_{np} : np & \to & pn \\ (j-1)n + i & \mapsto & (i-1)p + j, \quad i \in [n],\ j \in [p] \end{array}$$

and

$$\begin{array}{rcl} \delta_p : p & \to & p^2 \\ i & \mapsto & (i-1)p + i, \quad i \in [p]. \end{array}$$

We denote by β_n the unique base morphism $n \to 1$.

Some properties of the base morphisms τ_{np} and δ_p are given in the next three lemmas.

Lemma 12.3.2 *For all $f_{ij} : 1 \to p$, $i \in [n]$, $j \in [q]$, $n, p, q \geq 0$,*

$$\begin{array}{rl} & \tau_{nq} \cdot \langle f_{11} \oplus \ldots \oplus f_{1q}, \ldots, f_{n1} \oplus \ldots \oplus f_{nq} \rangle \\ = & \langle f_{11}, \ldots, f_{n1} \rangle \oplus \ldots \oplus \langle f_{1q}, \ldots, f_{nq} \rangle. \end{array}$$

Proof. For any $i \in [n]$, $j \in [q]$, the $((j-1)n+i)$-th component of both sides of the equation is the morphism

$$0_{(j-1)p} \oplus f_{ij} \oplus 0_{(q-j)p}.$$

Lemma 12.3.3 *For all $f_{ij} : 1 \to p$, $i, j \in [n]$, $n \geq 0$,*

$$\delta_n \cdot \langle f_{11} \oplus \ldots \oplus f_{1n}, \ldots, f_{n1} \oplus \ldots \oplus f_{nn} \rangle \quad = \quad f_{11} \oplus \ldots \oplus f_{nn}.$$

Lemma 12.3.4 *For all $n, p \geq 0$,*

$$\delta_n \cdot \tau_{nn} \ = \ \delta_n \quad \textit{and} \quad \tau_{np} \cdot \tau_{pn} \ = \ \mathbf{1}_{np}$$

Definition 12.3.5 *Let T be a theory, not necessarily an iteration theory. Suppose that $f : 1 \to n$ and $g : 1 \to p$ are in T. We say that*

- *f is* **idempotent***, if*

$$f \cdot \beta_n \quad = \quad \mathbf{1}_1; \tag{12.1}$$

- *f is* **diagonal***, if*

$$f \cdot (f \oplus \ldots \oplus f) \quad = \quad f \cdot \delta_n; \tag{12.2}$$

- *f* **commutes with** *g, if*

$$f \cdot (\overbrace{g \oplus \ldots \oplus g}^{n}) \quad = \quad g \cdot (\overbrace{f \oplus \ldots \oplus f}^{p}) \cdot \tau_{np}. \tag{12.3}$$

Remark 12.3.6 If (12.3) holds, then also

$$\begin{aligned} g \cdot (f \oplus \ldots \oplus f) &= f \cdot (g \oplus \ldots \oplus g) \cdot \tau_{np}^{-1} \\ &= f \cdot (g \oplus \ldots \oplus g) \cdot \tau_{pn}. \end{aligned}$$

Thus if f commutes with g, then g commutes with f, so that we may say that f and g commute. Each distinguished morphism i_n is idempotent and diagonal and commutes with any scalar morphism in T. If f is diagonal, then f commutes with itself. Indeed, by Lemma 12.3.4,

$$\begin{aligned} f \cdot (f \oplus \ldots \oplus f) &= f \cdot \delta_n \\ &= f \cdot \delta_n \cdot \tau_{nn} \\ &= f \cdot (f \oplus \ldots \oplus f) \cdot \tau_{nn}. \end{aligned}$$

Proposition 12.3.7 *Suppose that the theory T is generated by idempotent scalar morphisms. Then each scalar morphism in T is idempotent.*

Proof. Since the distinguished morphisms are idempotent, the proof can be easily completed once we prove that if $f : 1 \to n$ and $g_i : 1 \to p$, $i \in [n]$, are idempotent, then so is the composite $f \cdot \langle g_1, \ldots, g_n \rangle : 1 \to p$. But

$$\begin{aligned} f \cdot \langle g_1, \ldots, g_n \rangle \cdot \beta_p &= f \cdot \langle g_1 \cdot \beta_p, \ldots, g_n \cdot \beta_p \rangle \\ &= f \cdot \beta_n \\ &= \mathbf{1}_1. \end{aligned}$$

Proposition 12.3.8 *Suppose that the theory T is generated by a set Γ of scalar morphisms, any two of which commute. Then the same holds for any pair of scalar morphisms in T.*

Proof. For any scalar morphism h in T, let $\#(h)$ denote the minimum number of applications of the composition operation needed to obtain h from the morphisms in Γ and the distinguished morphisms. Let $f : 1 \to p$ and $g : 1 \to q$ in T. We prove that f and g commute by induction on $\#(f) + \#(g)$. When this sum is 0, f and g commute either by assumption or since each distinguished morphism commutes with any scalar morphism in T. Suppose that f and g commute when $\#(f) + \#(g)$ is at most k, for some integer $k \geq 0$. If $\#(f) + \#(g) = k + 1$, either $\#(f) > 0$ or $\#(g) > 0$. Assuming $\#(f) > 0$, we can write

$$f \quad = \quad h \cdot \langle f_1, \ldots, f_n \rangle,$$

for some $h : 1 \to n$ and some $f_i : 1 \to p$, $i \in [n]$, $n \geq 0$ with $\#(h) + \#(g) \leq k$ and $\#(f_i) + \#(g) \leq k$, for all $i \in [n]$. Applying the induction hypothesis twice together with Lemma 12.3.2, we have

$$\begin{aligned}
&h \cdot \langle f_1, \ldots, f_n \rangle \cdot (g \oplus \ldots \oplus g) = \\
&\quad = \quad h \cdot \langle f_1 \cdot (g \oplus \ldots \oplus g), \ldots, f_n \cdot (g \oplus \ldots \oplus g) \rangle \\
&\quad = \quad h \cdot \langle g \cdot (f_1 \oplus \ldots \oplus f_1), \ldots, g \cdot (f_n \oplus \ldots \oplus f_n) \rangle \cdot \tau_{pq} \\
&\quad = \quad h \cdot (g \oplus \ldots \oplus g) \cdot \langle f_1 \oplus \ldots \oplus f_1, \ldots, f_n \oplus \ldots \oplus f_n \rangle \cdot \tau_{pq} \\
&\quad = \quad g \cdot (h \oplus \ldots \oplus h) \cdot \tau_{nq} \cdot \langle f_1 \oplus \ldots \oplus f_1, \ldots, f_n \oplus \ldots \oplus f_n \rangle \cdot \tau_{pq} \\
&\quad = \quad g \cdot (h \oplus \ldots \oplus h) \cdot (\langle f_1, \ldots, f_n \rangle \oplus \ldots \oplus \langle f_1, \ldots, f_n \rangle) \cdot \tau_{pq} \\
&\quad = \quad g \cdot (h \cdot \langle f_1, \ldots, f_n \rangle \oplus \ldots \oplus h \cdot \langle f_1, \ldots, f_n \rangle) \cdot \tau_{pq},
\end{aligned}$$

so that $f \cdot (g \oplus \ldots \oplus g) = g \cdot (f \oplus \ldots \oplus f) \cdot \tau_{pq}$. The argument in the case that $\#(g) > 0$ is similar, and is omitted.

Proposition 12.3.9 *Suppose that the theory T is generated by a set Γ of diagonal scalar morphisms such that any two morphisms in Γ commute. Then all scalar morphisms in T are diagonal.*

Proof. We use the notation $\#(f)$ of the previous proposition. Suppose $f : 1 \to p$ is in T. We show f is diagonal by induction on $\#(f)$. When

$\#(f)$ is zero, $\#(f)$ is either a distinguished morphism i_p or a morphism in Γ. In either case, f is diagonal. Assuming $\#(f) > 0$, we can write

$$f \quad = \quad h \cdot \langle f_1, \ldots, f_n \rangle,$$

for some $h : 1 \to n$ and $f_i : 1 \to p$, $i \in [p]$, such that $\#(h) < \#(f)$ and $\#(f_i) < \#(f)$, for all i. Thus,

$$\begin{aligned}
&h \cdot \langle f_1, \ldots, f_n \rangle \cdot (h \oplus \ldots \oplus h) = \\
&\quad = \quad h \cdot \langle f_1 \cdot (h \oplus \ldots \oplus h), \ldots, f_n \cdot (h \oplus \ldots \oplus h) \rangle \\
&\quad = \quad h \cdot \langle h \cdot (f_1 \oplus \ldots \oplus f_1), \ldots, h \cdot (f_n \oplus \ldots \oplus f_n) \rangle \cdot \tau_{pn} \\
&\quad = \quad h \cdot (h \oplus \ldots \oplus h) \cdot \langle f_1 \oplus \ldots \oplus f_1, \ldots, f_n \oplus \ldots \oplus f_n \rangle \cdot \tau_{pn} \\
&\quad = \quad h \cdot \delta_n \cdot \langle f_1 \oplus \ldots \oplus f_1, \ldots, f_n \oplus \ldots \oplus f_n \rangle \cdot \tau_{pn} \\
&\quad = \quad h \cdot (f_1 \oplus \ldots \oplus f_n) \cdot \tau_{pn},
\end{aligned}$$

by Proposition 12.3.8, Lemma 12.3.3 and the induction hypothesis. Thus,

$$\begin{aligned}
&f \cdot (f \oplus \ldots \oplus f) = \\
&\quad = \quad h \cdot \langle f_1, \ldots, f_n \rangle \cdot (h \oplus \ldots \oplus h) \cdot (\langle f_1, \ldots, f_n \rangle \oplus \ldots \oplus \langle f_1, \ldots, f_n \rangle) \\
&\quad = \quad h \cdot (f_1 \oplus \ldots \oplus f_n) \cdot \tau_{pn} \cdot (\langle f_1, \ldots, f_n \rangle \oplus \ldots \oplus \langle f_1, \ldots, f_n \rangle) \\
&\quad = \quad h \cdot (f_1 \oplus \ldots \oplus f_n) \cdot \langle f_1 \oplus \ldots \oplus f_1, \ldots, f_n \oplus \ldots \oplus f_n \rangle \\
&\quad = \quad h \cdot \langle f_1 \cdot (f_1 \oplus \ldots \oplus f_1), \ldots, f_n \cdot (f_n \oplus \ldots \oplus f_n) \rangle \\
&\quad = \quad h \cdot \langle f_1, \ldots, f_n \rangle \cdot \delta_p \\
&\quad = \quad f \cdot \delta_p,
\end{aligned}$$

by Lemma 12.3.2 and the induction hypothesis.

The previous three propositions give the following result.

Theorem 12.3.10 *Suppose that the theory T is generated by a set Γ of idempotent and diagonal scalar morphisms such that any two morphisms in Γ commute. Then the same holds for the set of all scalar morphisms in T, i.e. each scalar morphism in T is idempotent and diagonal and any two scalar morphisms commute.*

Exercise 12.3.11 Show that if a theory T is generated by a single diagonal morphism $\Delta : 1 \to s$, $s > 0$, then each scalar morphism in T can be written in form $\Delta \cdot \rho$, for some base ρ.

Below we will make use of some additional base morphisms. Recall that β_m denotes the base morphism $m \to 1$. Let $r \geq 1$ be an integer.

Definition 12.3.12 *For all $1 \leq i \leq k \leq r$, the base morphism ν_i^k : $2^k \to 2$ is defined by induction as follows:*

$$\nu_1^1 \ := \ \mathbf{1}_1 \oplus \mathbf{1}_1 \ = \ \mathbf{1}_2.$$

For $i \leq k$, define

$$\nu_i^{k+1} \ := \ \langle \nu_i^k, \nu_i^k \rangle;$$

when $i = k+1$,

$$\nu_{k+1}^{k+1} \ := \ \beta_k \oplus \beta_k.$$

As an abbreviation, we omit the superscript and write $\nu_i := \nu_i^r$, for $1 \leq i \leq r$.

Exercise 12.3.13 Show that ν_i^k corresponds to the function $[2^k] \to [2]$, $j \mapsto 1$ iff $j = 2^{i-1}x + y$ for some even x and some $y \in [2^{i-1}]$.

Definition 12.3.14 *Let T be a theory, $r \geq 1$ and $s = 2^r$. When $\pi_1, \ldots, \pi_r$ are morphisms $1 \to 2$, we define the morphisms $\Delta_i : 1 \to 2^i$, $i \in [r]$, as follows:*

$$\Delta_1 \ := \ \pi_1 \quad \textit{and} \quad \Delta_{i+1} \ := \ \pi_{i+1} \cdot (\Delta_i \oplus \Delta_i). \tag{12.4}$$

In particular, the target of Δ_r is s. Conversely, if $\Delta : 1 \to s$, we define

$$\pi_i \ := \ \Delta \cdot \nu_i, \tag{12.5}$$

for all $i \in [r]$.

Proposition 12.3.15 *Suppose that the morphisms π_i, $i \in [r]$, are idempotent. If $\Delta := \Delta_r$ is defined as in (12.4), then the π_i's and Δ are related by (12.5).*

Proof. We prove that

$$\Delta_k \cdot \nu_i^k \ = \ \pi_i, \tag{12.6}$$

for each pair i, k with $0 \leq i \leq k \leq r$. When $k = 1$, $\Delta_1 = \pi_1$ and ν_1^1 is the identity morphism $2 \to 2$. We proceed by induction on k. Assume that (12.6) is true for k. If $i \leq k$,

$$\begin{aligned} \Delta_{k+1} \cdot \nu_i^{k+1} &= \pi_{k+1} \cdot (\Delta_k \oplus \Delta_k) \cdot \langle \nu_i^k, \nu_i^k \rangle \\ &= \pi_{k+1} \cdot \langle \pi_i, \pi_i \rangle \\ &= \pi_i, \end{aligned}$$

by the induction assumption and the fact that π_{k+1} is idempotent. If $i = k+1$ then

$$\begin{aligned} \Delta_{k+1}^{k+1} \cdot \nu_{k+1}^{k+1} &= \pi_{k+1} \cdot (\Delta_k \oplus \Delta_k) \cdot (\beta_k \oplus \beta_k) \\ &= \pi_{k+1} \cdot (\mathbf{1}_1 \oplus \mathbf{1}_1) \\ &= \pi_{k+1}, \end{aligned}$$

since Δ_k is idempotent, by Proposition 12.3.7.

A certain converse is also valid.

Proposition 12.3.16 *Suppose $\Delta : 1 \to s$ is idempotent and diagonal. If the morphisms π_i are defined by (12.5), and if the morphisms Δ_i are defined as in (12.4), then $\Delta = \Delta_r$.*

Proof. One can verify by induction on $i \in [r]$ that

$$\Delta_i = \Delta \cdot \langle \mathbf{1}_{2^i}, \ldots, \mathbf{1}_{2^i} \rangle.$$

Thus, in particular, $\Delta = \Delta_r$.

12.4 Sequacious Functions with Predicates

12.4.1 The Theory of One Predicate

Let T be the iteration theory $(\mathbf{Seq}_A, a_0)$. Let s be a fixed integer greater than 1. A sequacious function $\Delta : 1 \to s$ in T is a *predicate* of rank s if, for all $a \in A$, there is some $i \in [s]$ with

$$a\Delta = (a, i).$$

Thus any distinguished morphism $i_s : 1 \to s$ is a predicate. If we wish to emphasize that Δ has a fixed interpretation as a predicate, we will write $(\mathbf{Seq}_A, a_0, \Delta)$, or $(\mathbf{Seq}_A, \Delta)$.

Proposition 12.4.1 *Any predicate $\Delta : 1 \to s$ in $\mathbf{Seq}_A$ is idempotent and diagonal, i.e.*

$$\Delta \cdot \beta_s \quad = \quad \mathbf{1}_1 \tag{12.7}$$
$$\Delta \cdot (\Delta \oplus \ldots \oplus \Delta) \quad = \quad \Delta \cdot \delta_s, \tag{12.8}$$

where β_s is the base morphism $s \to 1$ and δ_s is the diagonal base morphism $s \to s^2$. In addition,

$$(\Delta \cdot \rho)^\dagger \quad = \quad \Delta \cdot \rho \cdot (\perp \oplus \mathbf{1}_p), \tag{12.9}$$

where $\rho : s \to 1 + p$ is any base morphism and $\perp$ is $\mathbf{1}_1^\dagger$. Further, any two predicates commute.

Conversely, any idempotent sequacious function $1 \to s$ is a predicate in T.

The equations (12.7)–(12.9) will be referred to as the Δ-*axioms*. Proposition 12.4.1 motivates the following definition.

Definition 12.4.2 *Let T be an iteration theory, not necessarily a theory of sequacious functions. A* **generalized predicate***, or predicate for short, of rank $s > 1$ in T is a morphism $\Delta : 1 \to s \in T$ which satisfies the Δ-axioms.*

A version of (12.9) holds for generalized predicates. Partial base morphisms in iteration theories were defined in Example 6.6.5.5. Recall that $\rho : n \to p$ is a partial base morphism if $i_n \cdot f$ is either a distinguished morphism or the morphism $\perp_{1p}$, for each $i \in [n]$.

Proposition 12.4.3 *In any iteration theory T, if $\Delta : 1 \to s$ is a generalized predicate, then*

$$(\Delta \cdot \tau)^\dagger \quad = \quad \Delta \cdot \tau \cdot (\perp \oplus \mathbf{1}_p),$$

where $\tau : s \to 1 + p$ is any partial base morphism.

Proof. Each partial base morphism $\tau : s \to 1 + p$ has a factorization

$$\tau \quad = \quad \rho \cdot (\mathbf{1}_1 \oplus \tau'),$$

where $\rho : s \to 1 + q$ is a base morphism and $\tau' : q \to p$ is partial base. Thus, by (12.9) and the parameter identity,

$$\begin{aligned}
(\Delta \cdot \tau)^\dagger &= (\Delta \cdot \rho)^\dagger \cdot \tau' \\
&= \Delta \cdot \rho \cdot (\perp \oplus \mathbf{1}_q) \cdot \tau' \\
&= \Delta \cdot \rho \cdot (\mathbf{1}_1 \oplus \tau') \cdot (\perp \oplus \mathbf{1}_p) \\
&= \Delta \cdot \tau \cdot (\perp \oplus \mathbf{1}_p).
\end{aligned}$$

Exercise 12.4.4 Show that when Δ is a predicate in the iteration theory T, each scalar morphism f in the subiteration theory generated by Δ can be written as $f = \Delta \cdot \tau$, where τ is partial base.

Again, when we want to indicate that the letter Δ has a fixed meaning in an iteration theory as a predicate, we will write (T, Δ). We let *SEQ*(Δ) denote the class of all iteration theories (T, Δ) with a generalized predicate. Thus *SEQ*(Δ) is a variety of (enriched) iteration theories. A morphism between two enriched theories in *SEQ*(Δ) is an iteration theory morphism which additionally preserves Δ. In the rest of this section we will show that *SEQ*(Δ) is the variety generated by all iteration theories $(\mathbf{Seq}_A, \Delta)$. In addition, we will provide an explicit description of the free theories in *SEQ*(Δ).

If Σ is any signature, let Σ_Δ be the signature obtained by adding the new symbol Δ of rank s to Σ. Recall that the free ω-continuous theory on Σ_Δ is the theory $\Sigma_\Delta\mathbf{TR}$ of all $(\Sigma_\Delta)_\perp$-trees. Nevertheless we will refer to these trees as Σ_Δ-trees.

Definition 12.4.5 *We call a scalar tree $t : 1 \to p$ in $\Sigma_\Delta\mathbf{TR}$ an* **alternating** *Σ_Δ-tree if for any vertex w of t, $wt = \Delta$ if and only if $|w|$ is even. A Σ_Δ-tree $t : n \to p$, $n \neq 1$, is alternating if $i_n \cdot t$ is alternating, for each $i \in [n]$.*

Thus any tupling of alternating trees is alternating. However, the composite of two alternating trees is never alternating, and the distinguished trees are not alternating either. Nevertheless we can redefine the composition operation and the distinguished morphisms so that we obtain a theory of alternating Σ_Δ-trees.

Definition 12.4.6 *Let $f : 1 \to p$ and $g = \langle g_1, \ldots, g_p \rangle : p \to q$ be arbitrary Σ_Δ-trees such that the depth of f is at least one. We define the composite $f \circ g : 1 \to q$ to be the tree such that for all w in $[\omega]^*$,*

$$w(f \circ g) \quad := \quad \begin{cases} wf & \text{if } wf \text{ is defined and } wf \notin X_p; \\ (jv)g_i & \text{if } w = ujv \text{ with } j \in [s] \text{ and } (uj)f = x_i. \end{cases}$$

The value of the function $f \circ g$ on w is not defined otherwise. The composite $f \circ g$ of trees $f : n \to p$ and $g : p \to q$, where $n \neq 1$ and each component of f has depth at least one, is determined by the formula

$$f \circ g \quad := \quad \langle (1_n \cdot f) \circ g, \ldots, (n_n \cdot f) \circ g \rangle.$$

The morphism $\widehat{i_p} : 1 \to p$, $i \in [p]$, is defined to be the alternating tree

$$\widehat{i_p} \quad := \quad \Delta \cdot \beta_s \cdot i_p \;=\; \Delta \cdot \langle i_p, \ldots, i_p \rangle.$$

Exercise 12.4.7 Show that the composite $f \circ g$ of alternating trees is alternating. Thus, since $\circ$ is associative and since $\widehat{i_n} \circ f = i_n \cdot f$, for alternating trees $f : n \to p$, it follows that the alternating trees form a theory.

Exercise 12.4.8 Show that for all Σ_Δ-trees f, g and h, if each component of f and g has depth at least one,

$$\begin{aligned} (f \circ g) \cdot h &= f \circ (g \cdot h) \\ (f \cdot g) \circ h &= f \cdot (g \circ h). \end{aligned}$$

We will denote by $\Sigma_\Delta \mathbf{ATR}$ the theory of alternating Σ_Δ-trees. The tupling operation need not be redefined in $\Sigma_\Delta \mathbf{ATR}$, for any tupling of alternating trees is alternating. Similarly, if f and g are alternating trees with appropriate source and target, their pairing $\langle f, g \rangle$ and separated sum $f \oplus g$ denote the same tree regardless whether these operations are evaluated in $\Sigma_\Delta \mathbf{TR}$ or in $\Sigma_\Delta \mathbf{ATR}$. However, in connection with base morphisms (or partial base morphisms, see below) we will follow the convention that if $\rho : [n] \to [p]$ is a function, then the corresponding base morphism in $\Sigma_\Delta \mathbf{ATR}$ is denoted $\widehat{\rho}$. In particular, the identity $n \to n$ is denoted $\widehat{\mathbf{1}}_n$. Note that

$$\widehat{\mathbf{1}}_n \quad = \quad \Delta \cdot \beta_s \oplus \ldots \oplus \Delta \cdot \beta_s,$$

and for $\rho : n \to p$,

$$\widehat{\rho} \;=\; \widehat{\mathbf{1}}_n \cdot \rho \;=\; \rho \cdot \widehat{\mathbf{1}}_p$$

is the alternating tree

$$(\Delta \oplus \ldots \oplus \Delta) \cdot \tau,$$

where $\tau : [ns] \to [p]$ is the function mapping $(i-1)s + j$ to $i\rho$, for all $i \in [n]$ and $j \in [s]$. The only exception to this convention is the case of zero morphisms, for which we keep the notation 0_p.

Each hom-set $\Sigma_\Delta\mathbf{ATR}(n,p)$ is partially ordered by the ordering inherited from the partial ordering of Σ_Δ-trees. Thus, for alternating trees $f, g : n \to p$, we have $f \le g$ if and only if the graph of $i_n \cdot f$ is included in the graph of $i_n \cdot g$, for all $i \in [n]$. In particular, there is a least alternating tree $\widehat{\perp}_{np} : n \to p$. Let $\widehat{\perp}$ denote the alternating tree

$$\widehat{\perp} \;\; := \;\; \Delta \cdot \beta_s \cdot \perp \; = \; \Delta \cdot \langle \perp, \ldots, \perp \rangle : 1 \to 0.$$

Then the least alternating tree $n \to p$ is the tree

$$\widehat{\perp}_{np} \;\; := \;\; \langle \widehat{\perp} \cdot 0_p, \ldots, \widehat{\perp} \cdot 0_p \rangle \; = \; \langle \widehat{\perp} \circ 0_p, \ldots, \widehat{\perp} \circ 0_p \rangle,$$

so that $\widehat{\perp}_{10} = \widehat{\perp}$. It is easy to see that the least upper bound of any ω-chain of alternating trees is alternating, the least upper bound taken in the theory $\Sigma_\Delta\mathbf{TR}$. Thus each hom-set $\Sigma_\Delta\mathbf{ATR}(n,p)$ is an ω-complete poset. We delegate to the reader the task of showing that composition in $\Sigma_\Delta\mathbf{ATR}(n,p)$ is ω-continuous. It follows that $\Sigma_\Delta\mathbf{ATR}$ is an ω-continuous theory, and hence an iteration theory, by Theorem 8.8.2.15. Although the iterate $f^\dagger$ of an alternating tree $f : n \to n+p$ denotes different trees in the theories $\Sigma_\Delta\mathbf{TR}$ and $\Sigma_\Delta\mathbf{ATR}$, e.g. in $\Sigma_\Delta\mathbf{ATR}$ we have $\widehat{\mathbf{1}}_1^\dagger = \widehat{\perp}$, we think that the context will resolve any ambiguity.

Exercise 12.4.9 Show that if $f_k : n \to p$ and $g_k : p \to q$ are ω-chains of alternating trees, then $\sup f_k \circ \sup g_k = \sup(f_k \circ g_k)$.

As usual, we will identify the letter Δ with the corresponding atomic tree, which is clearly alternating.

Proposition 12.4.10 *The atomic tree Δ satisfies the Δ-axioms, i.e. the equations*

$$\begin{aligned}
\Delta \circ \widehat{\beta}_s &= \widehat{\mathbf{1}}_1 \\
\Delta \circ (\Delta \oplus \ldots \oplus \Delta) &= \Delta \circ \widehat{\delta}_s \\
(\Delta \circ \widehat{\rho})^\dagger &= \Delta \circ \widehat{\rho} \circ (\widehat{\perp} \oplus \widehat{\mathbf{1}}_p)
\end{aligned}$$

hold in Σ_Δ**ATR**, *where* $\widehat{\rho} : s \to 1 + p$ *is any partial base morphism.*

The least subiteration theory T_0 of the iteration theory Σ_Δ**ATR** which contains Δ is of particular interest. T_0 is isomorphic to, in fact identical with the iteration theory Σ'_Δ**ATR**, where Σ' is the signature with $\Sigma'_n = \emptyset$, for all $n \geq 0$. The only trees $1 \to p$ in T_0 are those of the form $\Delta \cdot \tau$, where τ is a partial base morphism, see also Exercise 12.4.4. (These trees can be equally written as $\Delta \circ \widehat{\tau}$.) The trees $n \to p$ in T_0 are tuplings of such trees.

Exercise 12.4.11 Show that if $\Delta \cdot \rho_1 = \Delta \cdot \rho_2$ in T_0, where $\rho_1, \rho_2 : s \to p$ are partial base morphisms, then $\rho_1 = \rho_2$.

Exercise 12.4.12 This exercise concerns another representation of the theory T_0. Define a morphism $\alpha : n \to p$ in T'_0 to be a partial function

$$[n] \times [s] \;\; \to \;\; [p].$$

For $\beta : p \to q$ in T'_0, define $\alpha \cdot \beta : n \to q$ to be the partial function $[n] \times [s] \to [q]$,

$$(i, j) \;\; \mapsto \;\; ((i, j)\alpha, j)\beta.$$

Further, for any integer $i \in [p]$, let $i_p : [s] \to [p]$ be the constant function with value i. Thus i_p is a morphism $1 \to p$ in T'_0 if we identify $[1] \times [s]$ with $[s]$. Finally, define the tupling of $\alpha_1, \ldots, \alpha_p : 1 \to p$ to be the partial function $[n] \times [s] \to [p]$,

$$(i, j) \;\; \mapsto \;\; j\alpha_i.$$

Show that T'_0 is a theory, in fact an ω-continuous theory with the partial ordering $\alpha \leq \beta$ iff the graph of α is contained in the graph of β. Show that T_0 and T'_0 are isomorphic as ω-continuous theories and hence as iteration theories.

Associated with each letter $\sigma \in \Sigma_n$, $n \geq 0$, there is an atomic tree, also denoted σ. This atomic tree is not alternating. However, we can assign to σ an alternating tree

$$\widehat{\sigma} \;\; := \;\; \widehat{\mathbf{1}}_1 \cdot \sigma \cdot \widehat{\mathbf{1}}_n : 1 \to n.$$

Since Σ_Δ**TR** is the free ω-continuous theory on the signature Σ_Δ, there is a unique ω-continuous theory morphism

$$\widehat{\varphi} : \Sigma_\Delta\mathbf{TR} \;\; \to \;\; \Sigma_\Delta\mathbf{ATR}$$

with $\Delta\widehat{\varphi} = \Delta$ and $\sigma\widehat{\varphi} = \widehat{\sigma}$, for all $\sigma \in \Sigma$. Note that $\rho\widehat{\varphi} = \widehat{\rho}$, for any partial base morphism ρ. Below we will denote the value of $\widehat{\varphi}$ on a tree t alternatively as $\widehat{t}$. In the next proposition we establish a property of the theory morphism $\widehat{\varphi}$.

Lemma 12.4.13 *If t is alternating, then $\widehat{t} = t$.*

Proof. First suppose that t is of the form $t = \Delta \cdot \rho : 1 \to p$, for some partial base $\rho : s \to p$. Then

$$\widehat{t} \;=\; \Delta \circ \widehat{\rho} \;=\; \Delta \circ (\widehat{\mathbf{1}}_s \cdot \rho) \;=\; (\Delta \circ \widehat{\mathbf{1}}_s) \cdot \rho \;=\; \Delta \cdot \rho.$$

It follows that $\widehat{t} = t$, for all $t \in T_0$, i.e. for alternating trees of depth 1. Suppose that

$$t \;=\; \alpha \cdot \langle \sigma_1 \cdot t_1, \ldots, \sigma_m \cdot t_m,\, \rho\rangle,$$

where $\alpha : 1 \to m + r$ is an alternating tree of depth 1, $\sigma_i \in \Sigma_{k_i}$, $i \in [m]$, $m > 0$, and where ρ is a partial base morphism $r \to p$ and each t_i is a finite alternating tree $k_i \to p$ with $\widehat{t_i} = t_i$. Then

$$\begin{aligned}
\widehat{t} \;&=\; \alpha \circ \langle \widehat{\sigma}_1 \circ \widehat{t}_1, \ldots, \widehat{\sigma}_m \circ \widehat{t}_m,\, \widehat{\rho}\rangle \\
&=\; \alpha \circ \langle (\widehat{\mathbf{1}}_1 \cdot \sigma_1 \cdot \widehat{\mathbf{1}}_{k_1}) \circ t_1, \ldots, (\widehat{\mathbf{1}}_1 \cdot \sigma_m \cdot \widehat{\mathbf{1}}_{k_m}) \circ t_m,\, \widehat{\rho}\rangle \\
&=\; \alpha \circ \langle \widehat{\mathbf{1}}_1 \cdot \sigma_1 \cdot t_1, \ldots, \widehat{\mathbf{1}}_1 \cdot \sigma_m \cdot t_m,\, \widehat{\mathbf{1}}_r \cdot \rho\rangle \\
&=\; \alpha \circ ((\widehat{\mathbf{1}}_1 \oplus \ldots \oplus \widehat{\mathbf{1}}_1 \oplus \widehat{\mathbf{1}}_r) \cdot \langle \sigma_1 \cdot t_1, \ldots, \sigma_m \cdot t_m,\, \rho\rangle) \\
&=\; (\alpha \circ \widehat{\mathbf{1}}_{m+r}) \cdot \langle \sigma_1 \cdot t_1, \ldots, \sigma_m \cdot t_m,\, \rho\rangle \\
&=\; \alpha \cdot \langle \sigma_1 \cdot t_1, \ldots, \sigma_m \cdot t_m,\, \rho\rangle \\
&=\; t.
\end{aligned}$$

Thus $\widehat{t} = t$, for all finite alternating trees. If t is an infinite alternating tree, $t = \sup t_k$ for some ω-chain (t_k) of finite alternating trees. Since this least upper bound is the same in both theories $\Sigma_\Delta\mathbf{TR}$ and $\Sigma_\Delta\mathbf{ATR}$, and since $\widehat{\varphi}$ is ω-continuous, it follows that $\widehat{t} = t$.

We set $\widehat{\Sigma} := \{\widehat{\sigma} : \sigma \in \Sigma\}$. The least subiteration theory of $\Sigma_\Delta\mathbf{TR}$ which contains the atomic trees σ, for $\sigma \in \Sigma$, as well as the atomic tree Δ, is the free iteration theory $\Sigma_\Delta\mathbf{tr}$ of regular Σ_Δ-trees. $\Sigma_\Delta\mathbf{tr}$ contains the free iteration theory on the signature whose only letter is Δ. In the sequel, we will denote this theory by $\Delta\mathbf{tr}$. Let $\Sigma_\Delta\mathbf{Atr}$

be the smallest subiteration theory of $\Sigma_\Delta\mathbf{ATR}$ containing $\widehat{\Sigma}$ and the atomic tree Δ. Since the restriction of $\widehat{\varphi}$ to $\Sigma_\Delta\mathbf{tr}$ is clearly a surjective iteration theory morphism $\Sigma_\Delta\mathbf{tr} \to \Sigma_\Delta\mathbf{Atr}$, a Σ_Δ-tree is in $\Sigma_\Delta\mathbf{Atr}$ if and only if it is of the form $\widehat{t}$, for some regular Σ_Δ-tree t.

Proposition 12.4.14 *An alternating tree t is in $\Sigma_\Delta\mathbf{Atr}$ if and only if t is regular.*

Proof. If t is a regular alternating tree, then $\widehat{t}$ is in $\Sigma_\Delta\mathbf{Atr}$. But by Lemma 12.4.13, $t = \widehat{t}$. Suppose now that $t : n \to p$ is in $\Sigma_\Delta\mathbf{Atr}$. Since $T_0 \cup \widehat{\Sigma}$ generates $\Sigma_\Delta\mathbf{Atr}$, by Theorem 11.11.1.4, there is a presentation

$$D \quad = \quad (\alpha; a) : n \xrightarrow{k} p$$

over $(T_0, \widehat{\Sigma})$ with $t = |D| = \alpha \circ \langle a^\dagger, \widehat{\mathbf{1}}_p \rangle$. The tree $a : k \to k + p$ can be written as

$$a \quad = \quad \langle \widehat{\sigma}_1 \circ \alpha_1, \ldots, \widehat{\sigma}_k \circ \alpha_k \rangle,$$

where $\sigma_i \in \Sigma$ and $\alpha_i \in T_0$, for all $i \in [k]$. Define

$$\overline{a} \quad := \quad \langle \sigma_1 \cdot \alpha_1, \ldots, \sigma_k \cdot \alpha_k \rangle$$

and $\overline{D} := (\alpha; \overline{a})$, so that $\overline{D}$ is a presentation over $(\Delta\mathbf{tr}, \Sigma(\Delta\mathbf{tr}))$ in $\Sigma_\Delta\mathbf{tr}$. (Note that we regard Σ as a collection of scalar morphisms in $\Sigma_\Delta\mathbf{tr}$.) Since $\overline{a}\widehat{\varphi} = a$ and $\alpha\widehat{\varphi} = \alpha$, we have

$$\begin{aligned} |\overline{D}| &= \alpha \cdot \langle \overline{a}^\dagger, \mathbf{1}_p \rangle \\ &= (\alpha \cdot \langle \overline{a}^\dagger, \mathbf{1}_p \rangle)\widehat{\varphi} \\ &= \alpha \circ \langle a^\dagger, \widehat{\mathbf{1}}_p \rangle \\ &= |D|, \end{aligned}$$

by Lemma 12.4.13. Thus $t = |\overline{D}|$ and hence t is regular.

Recall from Chapter 11 that two presentations D and E are behaviorally equivalent, denoted $D \approx E$, if $|D| = |E|$. Recall Definition 11.11.2.2 and Definition 11.11.2.4.

Proposition 12.4.15 *If $D, E : n \to p$ are presentations over $(T_0, \widehat{\Sigma})$ in $\Sigma_\Delta\mathbf{Atr}$, then $D \approx E$ if and only if $D \overset{*}{\rightleftharpoons} E$.*

Proof. If $D = (\alpha; a)$, let $\overline{D} = (\alpha; \overline{a})$ be the presentation over the compatible pair $(\mathbf{\Delta tr}, \Sigma(\mathbf{\Delta tr}))$ defined in the proof of Proposition 12.4.14. Similarly, if $E = (\beta; b)$, let $\overline{E} = (\beta; \overline{b})$. If $|D| = |E|$ then $|\overline{D}| = |D| = |E| = |\overline{E}|$. As in the proof of Proposition 5.5.4.15 (see also Exercise 11.11.2.17), it can be seen that there exist presentations $\overline{D}_1 = (\alpha_1; a_1)$, $\overline{E}_1 = (\beta_1; b_1)$ and $\overline{F} = (\gamma; c)$ over $(\mathbf{\Delta tr}, \Sigma(\mathbf{\Delta tr}))$ in $\Sigma_\Delta \mathbf{tr}$ such that the simulations

$$\overline{D} \overset{\rho_1}{\leftarrow} \overline{D}_1 \overset{\tau_1}{\rightarrow} \overline{F} \overset{\tau_2}{\leftarrow} \overline{E}_1 \overset{\rho_1}{\rightarrow} \overline{E}$$

hold for some base injections ρ_i and some base surjections τ_i, $i = 1, 2$. Of course, the trees α_1, β_1 and γ are all in T_0. Define $D_1 := (\alpha_1; \widehat{a}_1)$, $E_1 := (\beta_1, \widehat{b}_1)$ and $F := (\gamma; \widehat{c})$. It follows that the corresponding simulations hold in $\Sigma_\Delta \mathbf{Atr}$, i.e.

$$D \overset{\widehat{\rho}_1}{\leftarrow} D_1 \overset{\widehat{\tau}_1}{\rightarrow} F \overset{\widehat{\tau}_2}{\leftarrow} E_1 \overset{\widehat{\rho}_1}{\rightarrow} E.$$

Indeed, e.g.

$$\alpha_1 \circ (\widehat{\tau}_1 \oplus \widehat{\mathbf{1}}_p) \;=\; (\alpha_1 \cdot (\tau_1 \oplus \mathbf{1}_p))\widehat{\varphi} \;=\; \gamma\widehat{\varphi} \;=\; \widehat{\gamma}$$

and

$$\widehat{a}_1 \circ (\widehat{\tau}_1 \oplus \widehat{\mathbf{1}}_p) \;=\; (a_1 \cdot (\tau_1 \oplus \mathbf{1}_p))\widehat{\varphi} \;=\; (\tau_1 \cdot c)\widehat{\varphi} \;=\; \widehat{\tau}_1 \circ \widehat{c}.$$

We have proved that if $D \approx E$ then $D \overset{*}{\leftrightarrow} E$.

It is easy to see that the compatible pair $(T_0, \widehat{\Sigma}(T_0))$ satisfies the assumptions of Corollary 11.11.2.19. Thus, for any two presentations $D, E : n \to p$ over $(T_0, \widehat{\Sigma}(T_0))$, $D \overset{*}{\leftrightarrow} E$ if and only if $D \overset{*}{\rightleftharpoons} E$. Thus if $D \approx E$ then $D \overset{*}{\leftrightarrow} E$ and hence $D \overset{*}{\rightleftharpoons} E$. The converse was shown in Corollary 11.11.2.5.

Now we show that $\Sigma_\Delta \mathbf{Atr}$ is the coproduct of T_0 and the free iteration theory $\Sigma\mathbf{tr}$. Let $\kappa : T_0 \to \Sigma_\Delta \mathbf{Atr}$ be the inclusion and $\lambda : \Sigma\mathbf{tr} \to \Sigma_\Delta \mathbf{Atr}$ the iteration theory morphism taking each atomic tree σ, $\sigma \in \Sigma$, to the alternating tree $\widehat{\sigma}$.

Proposition 12.4.16 *The diagram*

$$T_0 \xrightarrow{\kappa} \Sigma_\Delta \mathbf{Atr} \xleftarrow{\lambda} \Sigma\mathbf{tr}$$

is a coproduct diagram in the category of iteration theories.

Proof. The iteration theory $\Sigma_\Delta\mathbf{Atr}$ is generated by $T_0\kappa \cup \Sigma\lambda = T_0 \cup \widehat{\Sigma}$, by definition. Further, κ is clearly injective and if $\widehat{\sigma}_1 \circ \alpha = \widehat{\sigma}_2 \circ \alpha_2$, for some $\sigma_1, \sigma_2 \in \Sigma$ and $\alpha_1, \alpha_2 \in T_0$, then $\sigma_1 = \sigma_2$ and $\alpha_1 = \alpha_2$. By Proposition 12.4.15, for any two presentations D, E over $(T_0, \widehat{\Sigma}) = (T_0\kappa, \Sigma\lambda)$, $D \approx E$ if and only if $D \overset{*}{\rightleftharpoons} E$. Thus, by Theorem 11.11.3.8, κ and λ are coproduct injections.

By Proposition 12.4.10, the atomic tree Δ is a predicate in $\Sigma_\Delta\mathbf{Atr}$. Thus $(\Sigma_\Delta\mathbf{Atr}, \Delta)$ is in *SEQ*(Δ). Similarly, (T_0, Δ) is in *SEQ*(Δ).

Proposition 12.4.17 (T_0, Δ) *is initial in SEQ*(Δ)*.*

Proof. Let (T, Δ') be in *SEQ*(Δ). By Exercise 12.4.11, each morphism $1 \to p$ in T_0 can be written in the form $\Delta \circ \widehat{\tau}$ in a unique way, where $\widehat{\tau} : s \to p$ is a partial base morphism. Thus the function $\varphi : \Delta \circ \widehat{\tau} \mapsto \Delta' \cdot \tau$ is well-defined on scalar morphisms. For $t = \langle t_1, \ldots, t_n\rangle : n \to p \in T_0$, $n \neq 1$, define $t\varphi := \langle t_1\varphi, \ldots, t_n\varphi\rangle$. It is easy to see that φ is an iteration theory morphism $T_0 \to T$. Further, $\Delta\varphi = \Delta'$. The uniqueness of φ is obvious.

Theorem 12.4.18 *The iteration theory* $(\Sigma_\Delta\mathbf{Atr}, \Delta)$ *is freely generated by the mapping* $\sigma \mapsto \widehat{\sigma}$*,* $\sigma \in \Sigma$*, in the variety SEQ*(Δ)*.*

Proof. Suppose that (T, Δ') is in *SEQ*(Δ), and that $\varphi : \Sigma \to T$ is a rank preserving map. Then denote by $\overline{\varphi}$ the resulting iteration theory morphism

$$\overline{\varphi} : \Sigma\mathbf{tr} \quad\to\quad T.$$

Let $\eta : \Sigma \to \Sigma\mathbf{tr}$ be the free embedding of Σ to $\Sigma\mathbf{tr}$. Let $\kappa : T_0 \to \Sigma_\Delta\mathbf{Atr}$ and $\lambda : \Sigma\mathbf{tr} \to \Sigma_\Delta\mathbf{Atr}$ be the coproduct injections. Suppose that T is an iteration theory in *SEQ*(Δ), so that T has a distinguished constant Δ which satisfies the Δ-axioms. Since (T_0, Δ) is initial in *SEQ*(Δ), the diagram

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commutes for an iteration theory morphism $\varphi^\sharp$ if and only if the diagram

[1‘1‘1;800] [Σ‘Σ_Δ**Atr**‘T; $\eta \cdot \lambda$‘φ‘$\varphi^\sharp$]

commutes and $\varphi^\sharp$ preserves the constant Δ. But by Proposition 12.4.16, there is a unique iteration theory morphism with this property.

We now prove that the Δ-axioms capture the equational properties of predicates in iteration theories of the form $(\mathbf{Seq}_A, \Delta)$.

Proposition 12.4.19 *There is an iteration theory morphism*

$$\varphi_0 : (\Sigma_\Delta \mathbf{tr}, \Delta) \quad \rightarrow \quad (\mathbf{Seq}_{A_0}, \Delta)$$

which preserves the constant Δ such that if t and t' are distinct alternating trees in $\Sigma_\Delta \mathbf{tr}$, then $t\varphi_0 \neq t'\varphi_0$.

Proof. We only sketch the proof, since it is similar to the previous arguments. Let V denote $[s]^*$, the set of all words on $[s]$, which we think of as ‘truth values’. Let B denote the union of the sets $\Sigma_n \times [n]$, $n \geq 1$, with the set $\Sigma \cup \{\perp\}$. Let $C := B^*$. Finally, let $A_0 := V \times C$. We consider the theory $(\mathbf{Seq}_{A_0}, \tilde{\perp}, \Delta)$, where $\tilde{\perp}$ is the pair $(\epsilon, \perp)$ and in which the predicate Δ is defined below. In the following definitions, let

$$a \;=\; (b, c) \;=\; (b_1 \dots b_k,\, c_1 \dots c_m) \in A_0.$$

The predicate Δ in $(\mathbf{Seq}_{A_0}, \tilde{\perp}, \Delta)$ is defined as follows:

$$a\Delta \;:=\; \begin{cases} (a, b_1) & \text{if } k \geq 1; \\ (a, 1) & \text{otherwise.} \end{cases}$$

For $\sigma \in \Sigma_n$, $n \geq 0$, let $\sigma\varphi_0$ be the sequacious function determined by the condition

$$a(\sigma\varphi_0) \;:=\; \begin{cases} ((a, (b_2 \dots b_k, c_2 \dots c_m), i) & \text{if } k > 0,\ j > 0 \text{ and} \\ & c_1 = (\sigma, i); \\ (a, (\epsilon, \sigma)) & \text{otherwise.} \end{cases}$$

It is not difficult to show that if t and t' are distinct alternating trees $1 \rightarrow p$ in $\Sigma_\Delta \mathbf{tr}$, then $t\varphi_0 \neq t'\varphi_0$.

It follows that $(\Sigma_\Delta \mathbf{Atr}, \Delta)$ embeds in $(\mathbf{Seq}_{A_0}, \tilde{\perp}, \Delta)$. The results of this subsection are summarized in the following theorem.

Theorem 12.4.20 *The variety of all theories* $(\mathbf{Seq}_A, \Delta)$ *of sequacious functions with a generalized predicate is the variety* $SEQ(\Delta)$ *axiomatized by the* Δ*-axioms. The free iteration theory in* $SEQ(\Delta)$*, freely generated by a signature* Σ*, is the theory* $(\Sigma_\Delta \mathbf{Atr}, \Delta)$ *of regular alternating trees, which embeds in a theory* $(\mathbf{Seq}_{A_0}, \Delta)$*, for some set* A_0*.*

12.4.2 Several Binary Predicates

In this subsection, we show that the variety generated by theories of sequacious functions equipped with a finite nonempty set of predicates is equivalent to the variety $SEQ(\Delta)$, for a generalized predicate Δ of appropriate rank.

Let $\Pi = \{\pi_1, \ldots, \pi_r\}$, $r > 0$, be a fixed finite set. We will write (T, Π) to indicate that an iteration theory T has a distinguished constant $\pi_i : 1 \to 2$, for each $\pi_i \in \Pi$. Morphisms between two such pairs are iteration theory morphisms that preserve these additional constants. In (T, Π), we may define the morphisms $\Delta_i : 1 \to 2^i$, $i \in [r]$, as in Definition 12.3.14, i.e.

$$\begin{aligned} \Delta_1 &:= \pi_1 \\ \Delta_{j+1} &:= \pi_{j+1} \cdot (\Delta_j \oplus \Delta_j). \end{aligned}$$

We let $s := 2^r$.

Definition 12.4.21 *Let* T *be an iteration theory with a distinguished constant* $1 \to 2$ *for each letter in* Π*. We say that the pair* (T, Π) *belongs to the class* $SEQ(\Pi)$ *if the following equations hold:*

$$\begin{aligned} \pi_i \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle &= \mathbf{1}_1 && (12.10) \\ \pi_i \cdot (\pi_i \oplus \pi_i) &= \pi_i \cdot (\mathbf{1}_1 \oplus 0_2 \oplus \mathbf{1}_1) && (12.11) \\ \pi_i \cdot (\pi_j \oplus \pi_j) &= \pi_j \cdot (\pi_i \oplus \pi_i) \cdot (\mathbf{1}_1 \oplus \tau \oplus \mathbf{1}_1) && (12.12) \\ (\Delta_r \cdot \rho)^\dagger &= \Delta_r \cdot \rho \cdot (\perp \oplus \mathbf{1}_p), && (12.13) \end{aligned}$$

where $i, j \in [r]$*,* $\tau : 2 \to 2$ *is the nontrivial base permutation, and* $\rho : s \to 1 + p$ *is a base morphism.*

Thus each π_i is idempotent and diagonal and each π_i commutes with any π_j. The equations (12.10)-(12.13) will be referred to as the Π-*axioms*. By Proposition 12.3.15, the Π-axioms imply that

$$(\pi_i \cdot \rho)^\dagger \quad = \quad \pi_i \cdot \rho \cdot (\perp \oplus \mathbf{1}_p),$$

for all $i \in [r]$, where ρ is any base morphism $2 \to 1 + p$. Thus if (T, Π) is in $SEQ(\Pi)$, each constant π_i is a (binary) predicate in T.

Example 12.4.22 If T is the theory $\mathbf{Seq}_A$, for some set A, and if each π_i is a binary predicate on A, then $(\mathbf{Seq}_A, \Pi)$ is in $SEQ(\Pi)$.

In order to prove that the varieties $SEQ(\Pi)$ and $SEQ(\Delta)$ are equivalent, where $\Delta : 1 \to s$, we must exhibit a canonical way of defining a generalized predicate in any (T, Π) in $SEQ(\Pi)$, and conversely, we must prove that each (T, Δ) in $SEQ(\Delta)$ gives rise to a an iteration theory in $SEQ(\Pi)$. However, this has been, in essence, done in Section 12.3. Recall Definition 12.3.12.

Theorem 12.4.23 *Suppose that* (T, Π) *is in* $SEQ(\Pi)$*. Then*

$$(T, \Pi)\mathcal{D} \quad := \quad (T, \Delta_r)$$

is in $SEQ(\Delta)$*. Conversely, if* (T, Δ) *is in* $SEQ(\Delta)$ *and if* $\pi_i := \Delta \cdot \nu_i$*, for all* $i \in [r]$*, then*

$$(T, \Delta)\mathcal{P} \quad := \quad (T, \Pi)$$

is in $SEQ(\Pi)$*. Further,* $\mathcal{D}$ *and* $\mathcal{P}$ *are inverse assignments. An iteration theory morphism* $\varphi : T \to T'$ *is a morphism* $(T, \Pi) \to (T', \Pi)$ *in* $SEQ(\Pi)$ *if and only if* φ *is a morphism* $(T, \Pi)\mathcal{D} \to (T', \Pi)\mathcal{D}$ *in* $SEQ(\Delta)$*.*

Proof. Suppose (T, Π) is in $SEQ(\Pi)$. By Theorem 12.3.10, Δ_r is an idempotent diagonal morphism in T. Thus Δ_r is a predicate in T, by the Π-axiom (12.13). Thus (T, Δ_r) is in $SEQ(\Delta)$. Conversely, if (T, Δ) is in $SEQ(\Delta)$ and if $\pi_i := \Delta \cdot \nu_i$, for all $i \in [r]$, then by Theorem 12.3.10, each π_i is idempotent and diagonal and each π_i commutes with any π_j. By Proposition 12.3.16, $\Delta = \Delta_r$, where Δ_r is the morphism appearing in the last Π-axiom. Thus all of the Π-axioms hold for the morphisms π_i, so that $(T, \Delta)\mathcal{P}$ is in $SEQ(\Pi)$. The facts $\mathcal{D}$ and $\mathcal{P}$ are inverse assignments and that morphisms are preserved by these assignments follow from Propositions 12.3.15 and 12.3.16.

Corollary 12.4.24 *The variety SEQ*(Π) *is the variety generated by all theories of the form* $(\mathbf{Seq}_A, \Pi)$*, where* $\Pi = \{\pi_1, \ldots, \pi_r\}$ *is any collection of binary predicates on in* $\mathbf{Seq}_A$*.*

Proof. Since $SEQ(\Delta)$ is generated by the theories $(\mathbf{Seq}_A, \Delta)$, where $\Delta : 1 \to s$ is a predicate, $SEQ(\Pi)$ is generated by the theories $(\mathbf{Seq}_A, \Delta)\mathcal{P}$. However, these are exactly the theories of the form $(\mathbf{Seq}_A, \Pi)$, where the morphisms in Π are binary predicates.

Remark 12.4.25 We gave a description of the free theory $(\Sigma_\Delta \mathbf{Atr}, \Delta)$ in $SEQ(\Delta)$, freely generated by the signature Σ. It follows by Theorem 12.4.23 that the theory $(\Sigma_\Delta \mathbf{Atr}, \Delta)\mathcal{P}$ is the free theory on Σ in $SEQ(\Pi)$. Another description of this theory can be obtained as a theory of regular alternating Σ_Π-trees, where Σ_Π is the signature obtained by adding the letter in Π as symbols of rank two to Σ. An alternating Σ_Π-tree is obtained from an alternating Σ_Δ-tree by 'replacing' each node labeled Δ with the tree Δ_r whose nonleaf vertices are labeled by letters in Π.

Remark 12.4.26 When Π is infinite, it follows from the compactness of (many sorted) equational logic that the Π-axioms still form a complete set of axioms for $SEQ(\Pi)$.

12.5 Partial Functions with Predicates

Let $\Delta : A \to A \times [s]$ be a partial function, $s > 1$. Δ is a predicate on A, if for each $a \in A$, there is some $i \in [s]$ such that $a\Delta = (a, i)$. If Δ is a predicate on A, then Δ is a generalized predicate in the iteration theory $\mathbf{Pfn}_A$ in the sense of Definition 12.4.2. Conversely, every generalized predicate in $\mathbf{Pfn}_A$ is a predicate on the set A.

We let $(\mathbf{Pfn}_A, \Delta)$ denote the iteration theory of all partial functions $A \times [n] \to A \times [p]$, in which Δ denotes some fixed predicate. In this section we give an axiomatization of the variety $PFN(\Delta)$ of (enriched) iteration theories generated by all theories of the form $(\mathbf{Pfn}_A, \Delta)$. By our previous discussion it is clear that $PFN(\Delta)$ is a subvariety of $SEQ(\Delta)$, defined in Section 12.4. We will show that an iteration theory $(T, \Delta) \in SEQ(\Delta)$ belongs to $PFN(\Delta)$ if and only if T has a unique morphism $1 \to 0$. Thus an axiomatization of the variety $PFN(\Delta)$ can be obtained by adding to the Δ-axioms the axiom

$$f \;=\; g, \quad \text{for all } f, g : 1 \to 0. \tag{12.14}$$

Our argument is a slight modification of those applied in preceding sections. We omit most details.

Let Σ be a signature. Recall the definition of an alternating Σ_Δ-tree from Section 12.4 and recall Definition 12.2.1.

Suppose that $t : 1 \to p$ is an alternating tree in $\Sigma_\Delta\mathbf{Atr}$. We call a vertex v of t a *Σ-vertex* if $vt \in \Sigma$. A *$(\Sigma, 0)$-vertex* is Σ-vertex which is a 0-vertex. A *minimal $(\Sigma, 0)$-vertex* is a $(\Sigma, 0)$-vertex v such that all Σ-vertices which are a proper prefixes of v, are coaccessible.

Definition 12.5.1 *Suppose that $t : 1 \to p$ is in $\Sigma_\Delta\mathbf{Atr}$, the iteration theory of regular alternating trees. A vertex v is* **almost coaccessible** *if each Σ-vertex u such that u is a prefix of v is coaccessible. The tree t itself is almost coaccessible if its vertices are almost coaccessible. A regular alternating tree $t : n \to p$, $n \neq 1$, is almost coaccessible if and only if each $i_n \cdot t$ is almost coaccessible.*

Thus the tree $\widehat{\perp} = \Delta \cdot \perp_{s0}$ is the only almost coaccessible regular alternating tree $1 \to 0$. Further, if each subtree of the form $\widehat{\perp}_{1p}$ is replaced by the tree $\perp_{1p}$ in an almost coaccessible alternating tree $1 \to p$, the resulting tree is coaccessible.

Definition 12.5.2 *For a tree $t : 1 \to p$ in $\Sigma_\Delta\mathbf{Atr}$, let $\overline{t} : 1 \to p$ be the tree*

$$w\overline{t} \;:=\; \begin{cases} wt & \text{if } w \text{ is almost coaccessible;} \\ \perp & \text{if } w \text{ is a minimal } (\Sigma, 0)\text{-vertex,} \end{cases}$$

otherwise wt is undefined. For a tree $t = \langle t_1, \ldots, t_n \rangle : n \to p$ in $\Sigma\mathbf{tr}$, $n \neq 1$, we define $\overline{t} := \langle \overline{t}_1, \ldots, \overline{t}_n \rangle$.

It follows that $\overline{f} \in \Sigma_\Delta\mathbf{Atr}$. Further, a tree $f : 1 \to p \in \Sigma_\Delta\mathbf{Atr}$ is almost coaccessible if and only if $f = \overline{f}$.

Let $\approx$ be the smallest dagger congruence on $\Sigma_\Delta\mathbf{Atr}$ which collapses any two morphisms $1 \to 0$.

Proposition 12.5.3 *Let $f, g : n \to p$ in $\Sigma_\Delta\mathbf{Atr}$. Then $f \approx g$ if and only if $\overline{f} = \overline{g}$.*

Recall that $(\Sigma_\Delta\mathbf{Atr}, \Delta)$ is the free iteration theory in $SEQ(\Delta)$, freely generated by the signature Σ. Let

$$\eta_\approx \quad := \quad \Sigma \xrightarrow{\eta} \Sigma_\Delta\mathbf{Atr} \xrightarrow{\kappa_\approx} \Sigma_\Delta\mathbf{Atr}/\approx,$$

where η is the free embedding of Σ in $\Sigma_\Delta\mathbf{Atr}$ and $\kappa_\approx$ is the quotient map.

Corollary 12.5.4 *Let T be an iteration theory with a unique morphism $1 \to 0$. Let $\Delta' : 1 \to s$ be a generalized predicate in T. If $\varphi : \Sigma \to T$ is any rank preserving function, there is a unique iteration theory morphism*

$$\Sigma_\Delta\mathbf{Atr}/\approx \quad \to \quad T$$

such that $\eta_\approx \cdot \varphi^\# \ = \ \varphi$ and $\Delta/\approx \stackrel{\varphi^\#}{\longmapsto} \Delta'$.

Thus to complete the proof that $PFN(\Delta)$ is axiomatized by the Δ-axioms and the equation (12.14), it suffices prove the following statement.

Proposition 12.5.5 *There is a set A_0 and a predicate Δ_0 on A_0 and an iteration theory morphism $\varphi_0 : \Sigma_\Delta\mathbf{Atr} \to \mathbf{Pfn}_{A_0}$ such that $\Delta\varphi_0 = \Delta_0$ and $t\varphi_0 \neq t'\varphi_0$ whenever $t, t' : 1 \to p$ are distinct almost coaccessible alternating trees in $\Sigma_\Delta\mathbf{Atr}$.*

Proof. Let $B := \cup(\Sigma_n \times [n] : n \geq 1)$, let $C := \{\Delta\} \times [s]$ and define

$$A_0 \quad := \quad (C \cdot B)^* \cdot C,$$

the set of alternating words which begin and end with a letter of the form (Δ, i) and every other letter has the form (σ, j). We let $\Delta\varphi_0 := \Delta_0$ defined below:

$$w\Delta_0 \quad := \quad (w, i),$$

where $w = (\Delta, i)v$. Now define the partial function $\sigma\varphi_0$, for $\sigma \in \Sigma_n$, $n \geq 1$, by:

$$w(\sigma\varphi_0) \quad := \quad \begin{cases} (u, j) & \text{if } v = (\sigma, j)u \text{ for some } u \in A_0'; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The results of this section are summarized by the following theorem.

Theorem 12.5.6 *The variety PFN*(Δ) *is axiomatized by the* Δ*-axioms and the equation (12.14). The free iteration theory on a signature* Σ *in PFN*(Δ) *can be described as the theory* $(\Sigma_\Delta\mathbf{Atr}/{\approx}, \Delta/{\approx})$*, where* $\approx$ *is the dagger congruence identifying two regular alternating trees* $f, g : 1 \to p$ *if and only if they yield the same almost coaccessible tree.*

Just as in Section 12.4, we may alternatively consider the variety *PFN*(Π) of iteration theories generated by all theories of the form $(\mathbf{Pfn}_A, \Pi)$, where $\Pi = \{\pi_1, \ldots, \pi_r\}$ is a nonempty collection of binary predicates on the set A. Using the same methods as in Section 12.4, it is possible to show that this variety is equivalent to *PFN*(Δ), where $\Delta : 1 \to 2^r$.

12.6 Notes

This chapter is based on [Ési85] (Sections 12.2 and 12.5), and [BÉ88] (Sections 12.1 and 12.4). The present proofs of the main results which use Theorem 11.11.3.8 are analogous of the argument in [Ési85]. Theorem 12.2.13 was proved in [BÉ88]. We regard the material contained in Section 12.3 as well-known, but see Chapter 4.4 of the book [MMT87], and [Plo66, Urb65].

Chapter 13

Synchronization Trees

Synchronization trees were used by Milner as models of the computation of communicating processes. More accurately, it was the equivalence classes of trees under the relation of bisimulation which were his primary focus. He conjectured that with a different treatment of variables, his work might fit into the framework of *iterative* theories.

The current chapter shows that Milner's conjecture was essentially correct. A slightly different definition of synchronization tree is used here (reflecting a different treatment of variables), and the structure of these trees is characterized. In fact, both the structure of the trees themselves and the bisimulation equivalence classes of trees are characterized. The characterizations take the following form: the structure in question is free in a particular variety of iteration theories.

The structure of the synchronization trees (with edges labeled by letters in a fixed set A) is shown to be an ω-continuous 2-theory. It is shown that the category whose morphisms $n \to p$ are n-tuples of synchronization trees $1 \to p$ is an iteration theory. Further, each hom-set has the structure of a commutative monoid (where $f+g$ is the coproduct of f and g); composition distributes over the monoid operation on the right: $(f+g) \cdot h = f \cdot h + g \cdot h$, and $0 \cdot f = 0$. Such structures are called *grove iteration theories.* The categories of synchronization trees satisfy one further identity involving the † operation which we use to define the variety of *synchronization theories.*

The regular synchronization trees are those trees $1 \to 1$ in the least subsynchronization theory containing each *atomic* tree corresponding to an action symbol $a \in A$, which consists of a two edge path, the first edge labeled by a, and the second labeled 1. One of the main results shows that the regular trees are freely generated in the variety of synchronization theories.

The relation of bisimulation on synchronization trees is shown to be an iteration theory congruence. The iteration theory of bisimulation classes of the regular trees is shown to be freely generated in the variety of ω-*idempotent* synchronization theories.

13.1 Theories of Synchronization Trees

We let *pSET* denote the category whose objects are pointed sets (X, x) consisting of a set X and an element x in X. A morphism $f : (X, x) \to (Y, y)$ is a function $f : X \to Y$ such that $xf = y$. Thus the category *pSET* is essentially the same as the category of sets and partial functions considered in Chapter 1. The category $pSET_\omega$ is the full subcategory determined by the collection of objects (X, x) such that X is finite or countable.

Throughout this section A denotes a fixed set of *action symbols*. We assume that A is disjoint from the set of integers.

Definition 13.1.1 *A* **synchronization tree** $t : 1 \to p$, $p \in \mathbf{N}$, *over* A *is a* 4*-tuple* (V, v_0, E, l) *consisting of*

1. *an object* (V, v_0) *in* $pSET_\omega$*; the elements of* V *are* vertices *and the point* v_0 *is the* root*;*
2. *a set* $E \subseteq V \times V$ *of (directed)* edges*;*
3. *a function* $l : E \to A \cup [p]$*, the* labeling function.

These data obey the following restrictions. The directed graph (V, v_0, E) *is a rooted tree, i.e. there is a unique path from* v_0 *to* u*, for each vertex* u*. Further, if* $(u, v) \in E$ *and* $l(u, v) \in [p]$*, then the vertex* v *must be a leaf. An edge whose label is in* $[p]$ *is called an* **exit edge***. The target* v *of an exit edge* (u, v) *is called an* **exit vertex***.*

Note that since (V, v_0) is an object in $pSET_\omega$, a vertex has at most countably many successors.

If $t = (V, v_0, E, l)$ is a synchronization tree $1 \to p$, we introduce the following notation for certain subsets of vertices:

$$\begin{array}{rcl} V_{ex_i} & := & \{v \in V \ : \ \exists u \in V\,(\,(u,v) \in E \ \text{ and } \ l(u,v) = i\,)\} \\ V_{ex} & := & \bigcup (V_{ex_i} \ : \ i \in [p]) \\ \widehat{V} & := & V \setminus V_{ex} \\ \overline{V} & := & V \setminus \{v_0\}. \end{array}$$

Definition 13.1.2 *Let $t_i = (V_i, v_i, E_i, l_i)$, $i = 1, 2$, be synchronization trees $1 \to p$. A* **synchronization tree morphism** *$\varphi : t_1 \to t_2$ is a morphism $(V_1, v_1) \to (V_2, v_2)$ in $pSET_\omega$ such that*

1. *φ preserves edges, i.e. if $(u, v) \in E_1$ then $(u\varphi, v\varphi) \in E_2$;*
2. *φ preserves the labeling, i.e. $l_1(u, v) = l_2(u\varphi, v\varphi)$, for all edges $(u, v) \in E_1$.*

The composite $\varphi \circ \psi : t_1 \to t_3$ of two synchronization tree morphisms $\varphi : t_1 \to t_2$ and $\psi : t_2 \to t_3$ is the composite in $pSET_\omega$. We call this operation on tree morphisms *vertical* composition. Corresponding to any synchronization tree $t : 1 \to p$ there is an identity morphism $t \to t$ which is the identity mapping on the vertex set of t. Note that if $\varphi : t_1 \to t_2$ is a synchronization tree morphism, then for each vertex v of t_1, the depth of $v\varphi$ is the depth of v, where the depth is the length of the unique path from the root.

Thus the synchronization trees $1 \to p$ form a category $ST(A)(1, p)$. Two such trees are isomorphic if there is a bijection of their vertices which preserves the root, the edges and the labeling. For $n, p \geq 0$, we let $ST(A)(n, p)$ be the category which is the n-th direct power of $ST(A)(1, p)$. An object in $ST(A)(n, p)$ is an n-tuple $(t_1, \ldots, t_n)$, where each t_i, $i \in [n]$, is an object in $ST(A)(1, p)$. Morphisms in $ST(A)(n, p)$ are n-tuples of morphisms in $ST(A)(1, p)$. An object in $ST(A)(n, p)$ is called a synchronization tree $n \to p$.

We wish to prove that each category $ST(A)(n, p)$ has all countable colimits. It will be sufficient to prove this only for the case that $n = 1$.

In our argument we make use of the underlying $pSET_\omega$ functor

$$\begin{array}{rcl} U : ST(A)(1,p) & \rightarrow & pSET_\omega \\ (V,v,E,l) & \mapsto & (V,v). \end{array}$$

Proposition 13.1.3 *The functor U creates coequalizers.*

Proof. Suppose that $t_i = (V_i, v_i, E_i, l_i)$, $i = 1,2$, are synchronization trees $1 \rightarrow p$ and that $\varphi_1, \varphi_2 : t_1 \rightarrow t_2$ are synchronization tree morphisms. The coequalizer of φ_1 and φ_2 in $pSET_\omega$ can be obtained by taking the least equivalence relation $\theta = \theta(\varphi_1, \varphi_2)$ on V_2 such that for all $x \in V_1$,

$$x\varphi_1 \; \theta \; x\varphi_2.$$

We make three preliminary remarks about the equivalence relation θ.

[a] If $u \, \theta \, v$, then the depth of u equals that of v.

[b] If $(u,v),(u',v') \in E_2$ and $v \, \theta \, v'$, then $u \, \theta \, u'$.

[c] If $(u,v),(u',v') \in E_2$, $u \, \theta \, u'$ and $v \, \theta \, v'$, then $l_2(u,v) = l_2(u',v')$.

The θ-equivalence class of a vertex $u \in V_2$ is written $[u]$. Note that the equivalence class of the root of t_2 is $\{v_2\}$.

Let $\varphi : (V_2, v_2) \rightarrow (V_2/\theta, [v_2])$ be the coequalizer of φ_1 and φ_2 in $pSET_\omega$, so that $v\varphi = [v]$, for all $v \in V$. We must show that there is a unique synchronization tree $t = (V_2/\theta, [v_2], E, l)$ such that φ becomes a morphism $t_2 \rightarrow t$. In order that φ be a synchronization tree morphism $t_2 \rightarrow t$, E must contain at least the set

$$E_0 \quad := \quad \{([u],[v]) \, : \, (u,v) \in E_2\}.$$

Letting $E := E_0$, from the remarks [a] and [b] we see that $(V_2/\theta, [v_2], E)$ is a rooted tree. It is also clear that the labeling function must satisfy the condition $l([u],[v]) = l_2(u,v)$, for all $(u,v) \in E_2$. By [c], this definition makes sense, so that $\varphi : t_2 \rightarrow t$ is a synchronization tree morphism.

We claim that φ is the coequalizer of φ_1 and φ_2 in $ST(A)(1,p)$. Indeed, $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$, by construction. If $\psi : t_2 \to s$ is any synchronization tree morphism such that

$$\varphi_1 \circ \psi \;=\; \varphi_2 \circ \psi,$$

then for all $u, v \in V_2$,

$$u \;\theta\; v \quad\Rightarrow\quad u\psi \;=\; v\psi.$$

Thus, ψ factors through the morphism φ, completing the proof.

Proposition 13.1.4 *The functor U creates countable coproducts.*

Proof. This is easy. The coproduct $\Sigma_{i\in I} X_i$ in $pSET_\omega$ of the pointed sets $X_i = (V_i, v_i)$, for i in a finite or countable set I, can be obtained by taking the disjoint union $V = \bigcup_{i\in I} \overline{V}_i \times \{i\} \cup \{v\}$, where the element v is not a member of any of the sets V_i. The coproduct injection $\iota_i : X_i \to (V, v)$ takes $x \in \overline{V}_i$ to (x, i) and the point v_i to v.

Now assume that t_i, $i \in I$, is a finite or countable collection of synchronization trees $1 \to p$. The pointed set $\Sigma_{i\in I} t_i U$ is the underlying pointed set of a unique synchronization tree $\Sigma_{i\in I} t_i$ such that the functions ι_i become synchronization tree morphisms. This synchronization tree is the coproduct of the t_i.

Corollary 13.1.5 *The functor U creates all countable colimits.*

Corollary 13.1.6 *Each category $ST(A)(n,p)$ is countably cocomplete.*

We can now consider the *horizontal structure* of synchronization trees. Suppose that $t : 1 \to p$ is a synchronization tree in $ST(A)(1,p)$ and $s = (s_1, \ldots, s_p)$ is a synchronization tree in $ST(A)(p,q)$. We will define the *horizontal* composite $t \cdot s$ in $ST(A)(1,q)$. Intuitively, $t \cdot s$ is obtained from t by deleting each exit edge (u, v) of t labeled $i \in [p]$ and attaching a copy of the tree s_i with root merged with the vertex u.

Definition 13.1.7 *Using the above notations, let* $t = (V, v, E, l)$*, and write* $s_i = (V_i, v_i, E_i, l_i)$*, for* $i \in [p]$*. For each exit vertex* u *in* V_{ex_i} *we make a copy* s_i^u *of the tree* s_i*,*

$$s_i^u \quad := \quad (V_i \times \{u\},\, (v_i, u),\, E_i \times \{u\},\, l_i^u).$$

Of course, $E_i \times \{u\} = \{((u_1, u), (u_2, u)) : (u_1, u_2) \in E_i\}$*. The labeling function is defined by* $l_i^u((u_1, u), (u_2, u)) := l_i(u_1, u_2)$*. We define the set of vertices of* $t \cdot s$ *as the union of the nonexit vertices of* t *with the nonroot vertices of the synchronization trees* s_i^u*:*

$$\mathrm{vert}(t \cdot s) \quad := \quad \widehat{V} \cup \bigcup (\overline{V}_i \times \{u\} \,:\, i \in [p],\, u \in V_{ex_i}).$$

The edges of $t \cdot s$ *are all nonexit edges in* E*, all edges in the synchronization trees* s_i^u *whose source is not the root, and lastly an edge* $(u_1, (u_2, u))$ *for each* $i \in [p]$*,* $u_1 \in V$*,* $u \in V_{ex_i}$ *and* $u_2 \in V_i$*, if* $(u_1, u) \in E$ *and* $(v_i, u_2) \in E_i$*. If* $t = (t_1, \ldots, t_n) : n \to p$ *and* $s : p \to q$ *are synchronization trees, where* $n \neq 1$*, then we define*

$$t \cdot s \quad := \quad (t_1 \cdot s, \ldots, t_n \cdot s).$$

Next we define horizontal composition of synchronization tree morphisms. Suppose that $\varphi : t \to t'$ and $\psi : s \to s'$ are synchronization tree morphisms, where $t, t' : 1 \to p$ and $s, s' : p \to q$ are synchronization trees. Write

$$\begin{aligned} s &:= (s_1, \ldots, s_p) \\ s' &:= (s'_1, \ldots, s'_p) \\ \psi &:= (\psi_1, \ldots, \psi_p). \end{aligned}$$

We will define the horizontal composite $\varphi \cdot \psi : t \cdot s \to t' \cdot s'$.

Definition 13.1.8 *Let* $t = (V, v, E, l)$ *and* $s_i = (V_i, v_i, E_i, l_i)$*,* $i \in [p]$*. Using the notation of Definition 13.1.7, we define* $\varphi \cdot \psi$ *on the vertices of* $t \cdot s$ *as follows:*

$$\begin{aligned} w(\varphi \cdot \psi) &:= w\varphi, \quad \text{if } w \in \widehat{V}; \\ (w, u)(\varphi \cdot \psi) &:= (w\psi_i, u\varphi), \quad \text{if } w \in \overline{V}_i,\ u \in V_{ex_i} \text{ and } i \in [p]. \end{aligned}$$

If t *and* t' *are synchronization trees* $n \to p$ *and* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *is a synchronization tree morphism* $t \to t'$*,* $n \neq 1$*, then for any* $\psi : s \to s'$ *as above we define* $\varphi \cdot \psi := (\varphi_1 \cdot \psi, \ldots, \varphi_n \cdot \psi)$*.*

It is straightforward to show that $\varphi \cdot \psi$ is a synchronization tree morphism $t \cdot s \to t' \cdot s'$.

Proposition 13.1.9 *Horizontal composition is a bifunctor*

$$ST(A)(n,p) \times ST(A)(p,q) \quad \to \quad ST(A)(n,q).$$

The proof is omitted. The content of Proposition 13.1.9 is that the horizontal composite of identity morphisms is an identity morphism and moreover, horizontal and vertical compositions are related by the interchange law:

$$(\varphi \circ \varphi') \cdot (\psi \circ \psi') \quad = \quad (\varphi \cdot \psi) \circ (\varphi' \cdot \psi'),$$

for all composable $\varphi : t_1 \to t_2$, $\varphi' : t_2 \to t_3$ and $\psi : s_1 \to s_2$, $\psi' : s_2 \to s_3$ with $t_i \in ST(A)(n,p)$ and $s_i \in ST(A)(p,q)$, $i = 1, 2, 3$.

Next we indicate why horizontal composition preserves all countable colimits in its first argument and all ω-colimits in its second. Fix a synchronization tree $t : p \to q$. As usual, we will write t also for the vertical identity morphism $t \to t$. We define two functors R_t and L_t by right and left composition.

$$\begin{aligned} R_t : ST(A)(n,p) &\quad \to \quad ST(A)(n,q) \\ s \xrightarrow{\varphi} s' &\quad \mapsto \quad s \cdot t \xrightarrow{\varphi \cdot t} s' \cdot t \end{aligned}$$

$$\begin{aligned} L_t : ST(A)(q,r) &\quad \to \quad ST(A)(p,r) \\ s \xrightarrow{\varphi} s' &\quad \mapsto \quad t \cdot s \xrightarrow{t \cdot \varphi} t \cdot s'. \end{aligned}$$

Proposition 13.1.10 *R_t preserves countable colimits.*

Proof sketch. Although we omit most details, we note that it is sufficient to prove this statement for the case that $n = 1$. Secondly, we can argue at the level of pointed sets, since R_t preserves countable colimits iff $R_t \cdot U$ does. As for coequalizers, suppose that

$$\varphi_1, \varphi_2 : s_1 \quad \to \quad s_2$$

are a parallel pair of morphisms in $ST(A)(1,p)$. Recall that their coequalizer is determined by the equivalence relation $\theta = \theta(\varphi_1, \varphi_2)$ in Proposition 13.1.3. Thus it is enough to check that $\theta' = \theta(\varphi_1 \cdot t,\, \varphi_2 \cdot t)$ has the following property. If v_1 and v_2 are nonexit vertices of s_2 then $v_1 \,\theta\, v_2$ iff $v_1 \,\theta'\, v_2$. Moreover, if (v,u) is an exit edge in s_2 labeled $i \in [p]$, and if $v_1 \,\theta'\, v_2$, where v_1 and v_2 are vertices in the tree t_i^u, then $v_1 = v_2$. Of course, t_i denotes the i-th component of t, so that $t = (t_1, \ldots, t_p)$. In other words, no vertices inside the trees t_i^u are collapsed by $\theta(\varphi_1 \cdot t,\, \varphi_2 \cdot t)$. But t_i^u gets identified with $t_i^{u'}$ whenever $u \,\theta\, u'$.

Proposition 13.1.11 *L_t preserves ω-colimits.*

Proof sketch. Again, it is enough to consider the meaning of this statement for pointed sets. It is helpful to know that a concrete description of the colimit in $pSET_\omega$ of the diagram

$$(X_0, x_0) \overset{f_0}{\to} (X_1, x_1) \overset{f_1}{\to} \cdots$$

is obtained by placing the following equivalence on the disjoint union $\bigcup_n X_n$:

$$x \;\equiv\; y \qquad \text{iff} \qquad (\exists k)\,(x f_{n,k} \;=\; y f_{m,k}\,).$$

Here, $x \in X_n$, $y \in Y_m$ and, for $n < k$, $f_{n,k}$ is the composite of the functions $f_n, \ldots, f_{k-1}$, so that $f_{n,n+1} = f_n$.

The trees we have been considering up to now might be called concrete synchronization trees, since isomorphic trees with distinct sets of vertices are distinct. From now on, we will identify all isomorphic concrete synchronization trees, and define a synchronization tree as a suitably selected representative of an isomorphism class of concrete synchronization trees. This is spelled out below.

For each integer $p \in \mathbf{N}$, let $ST'(A)(1,p)$ be a skeletal subcategory of $ST(A)(1,p)$. Further, for $n \in \mathbf{N}$, $n \neq 1$, let $ST'(A)(n,p)$ be the n-th direct power of $ST'(A)(1,p)$. Thus, for each synchronization tree t in $ST(A)(n,p)$ there is a unique synchronization tree $\bar{t}$ in $ST'(A)(n,p)$ isomorphic to t. Let x_t denote a fixed isomorphism $t \to \bar{t}$.

The synchronization trees in $ST'(A)$ determine a 2-theory. Indeed, the vertical structure is provided by the structure of the component

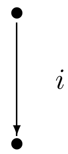


Figure 13.1: The synchronization tree i_n

categories $ST'(A)(n,p)$. To define horizontal composition, let $t, t' : n \to p$ and $s, s' : p \to q$ be synchronization trees in $ST'(A)$ with morphisms $\varphi : t \to t'$ and $\psi : s \to s'$. We define the horizontal composite of t by t' in $ST'(A)$ to be the synchronization tree $\overline{t \cdot t'}$. Similarly, we define the horizontal composite of φ and ψ in $ST'(A)$ as the morphism

$$x_{t \cdot t'}^{-1} \circ (\varphi \cdot \psi) \circ x_{s \cdot s'}.$$

The tupling of synchronization trees $t_i : 1 \to p$, $i \in [n]$, in $ST'(A)$ is just the n-tuple $(t_1, \ldots, t_n)$. Finally, for $i \in [n]$, we define i_n to be the synchronization tree in $ST'(A)(1,n)$ which has one edge labeled i, see Figure 13.1.

Proposition 13.1.12 *$ST'(A)$ is an ω-continuous 2-theory.*

The proof is immediate from the definitions and Propositions 13.1.9, 13.1.10 and 13.1.11.

13.2 Grove Iteration Theories

In this section we consider iteration theories with additional constants $+ : 1 \to 2$ and $\# : 1 \to 0$ which satisfy certain identities. Following [BT89], any right distributive abelian seminear ring with a left absorptive zero is called a *grove*. Therefore the algebraic theories arising in connection with groves will be called grove theories below.

Definition 13.2.1 *Let T be an algebraic theory with additional constants $+ : 1 \to 2$ and $\# : 1 \to 0$. We define an operation of sum. For*

$f, g : 1 \to p$, $p \in \mathbf{N}$, let

$$f + g \quad := \quad + \cdot \langle f, g \rangle, \tag{13.1}$$

and for $f = \langle f_1, \ldots, f_n \rangle$, $g = \langle g_1, \ldots, g_n \rangle : n \to p$, $n, p \in \mathbf{N}$, $n \neq 1$, let

$$f + g \quad := \quad \langle f_1 + g_1, \ldots, f_n + g_n \rangle. \tag{13.2}$$

Clearly, we have $+ = 1_2 + 2_2$.

Moreover, for each $n, p \in \mathbf{N}$, we define a morphism $0_{np} : n \to p$ by letting

$$0_{1p} \quad := \quad \# \cdot 0_p \tag{13.3}$$

and

$$0_{np} \quad := \quad \langle 0_{1p}, \ldots, 0_{1p} \rangle, \tag{13.4}$$

for $n \neq 1$, so that $0_{0p} = 0_p$.

Proposition 13.2.2 *The following equations hold in T:*

$$(f+g) \cdot h \quad = \quad f \cdot h + g \cdot h, \quad f, g : n \to p, \ h : p \to q \tag{13.5}$$
$$0_{np} \cdot f \quad = \quad 0_{nq}, \quad f : p \to q \tag{13.6}$$
$$i_n \cdot (f+g) \quad = \quad i_n \cdot f + i_n \cdot g, \quad f, g : n \to p, \ i \in [n] \tag{13.7}$$
$$i_n \cdot 0_{np} \quad = \quad 0_{1p}, \quad i \in [n]. \tag{13.8}$$

Proof. (13.7) and (13.8) are obvious. By (13.7), it is enough to prove (13.5) for $n = 1$:

$$(f+g) \cdot h \;=\; + \cdot \langle f,\, g \rangle \cdot h \;=\; + \cdot \langle f \cdot h,\, g \cdot h \rangle \;=\; f \cdot h + g \cdot h.$$

Similarly, by (13.8), it is enough to show that $0_{1p} \cdot f = 0_{1q}$, $f : p \to q$. However, this is immediate, for

$$0_{1p} \cdot f \;=\; \# \cdot 0_p \cdot f \;=\; \# \cdot 0_q \;=\; 0_{1q}.$$

The converse of Proposition 13.2.2 also holds.

Proposition 13.2.3 *Let T be an algebraic theory with an operation of sum defined on each set $T(n,p)$. Suppose that we are given constants $0_{np} : n \to p$. Define the morphism $+ : 1 \to 2$ by $1_2 + 2_2$ and let $\# := 0_{10}$. If the equations (13.5)–(13.8) hold, then so do the equations (13.1)–(13.4). Thus, sum can be defined in terms of the constant $+ : 1 \to 2$, and the constants 0_{np} are related to $\#$ by the equations (13.3) and (13.4).*

Proof. Let $f, g : 1 \to p$. Then

$$+ \cdot \langle f, g \rangle \;=\; (1_2 + 2_2) \cdot \langle f, g \rangle \;=\; 1_2 \cdot \langle f, g \rangle + 2_2 \cdot \langle f, g \rangle \;=\; f + g,$$

proving (13.1). (13.2) is now clear by (13.7). Now for (13.3):

$$\# \cdot 0_p \;=\; 0_{10} \cdot 0_p \;=\; 0_{1p},$$

by (13.6). The proof is easily completed by making use of (13.8).

Roughly speaking, a grove theory is an algebraic theory T such that each set $T(n,p)$ is an additive abelian monoid and composition distributes over finite sums on the right. Moreover, it is required that tupling creates an isomorphism between the n-th direct power of $T(1,p)$ and the monoid $T(n,p)$, for all $n, p \in \mathbf{N}$. An equivalent definition, motivated by the previous propositions, is given below.

Definition 13.2.4 *A* **grove theory** *is a theory T with additional constants $+ : 1 \to 2$ and $\# : 1 \to 0$ subject to the following conditions:*

$$(1_3 + 2_3) + 3_3 \;=\; 1_3 + (2_3 + 3_3) \qquad (13.9)$$

$$1_2 + 2_2 \;=\; 2_2 + 1_2 \qquad (13.10)$$

$$1_1 + 0_{11} \;=\; 1_1. \qquad (13.11)$$

A morphism of grove theories is a theory morphism that preserves the constants $+$ and $\#$.

Thus every matrix or matricial theory is a grove theory.

Proposition 13.2.5 *Each grove theory satisfies the following identities:*

$$(f + g) + h \;=\; f + (g + h), \quad f, g, h : n \to p \qquad (13.12)$$

$$f + g \;=\; g + f, \quad f, g : n \to p \tag{13.13}$$
$$f + 0_{np} \;=\; f, \quad f : n \to p \tag{13.14}$$
$$(f + g) \cdot h \;=\; f \cdot h + g \cdot h, \quad f, g : n \to p,\ h : p \to q \tag{13.15}$$
$$0_{np} \cdot f \;=\; 0_{nq}, \quad f : p \to q. \tag{13.16}$$

Proof. The last two conditions are just restatements of (13.5) and (13.6). By (13.7), it is enough to prove the first three statements for $n = 1$.

Proof of (13.12). Composing both sides of (13.9) with $\langle f, g, h\rangle$ on the right, we get

$$((1_3 + 2_3) + 3_3) \cdot \langle f, g, h\rangle \;=\; (1_3 + (2_3 + 3_3)) \cdot \langle f, g, h\rangle. \tag{13.17}$$

Now the left side reduces to $(f + g) + h$, by the right distributivity law (13.15):

$$\begin{aligned} ((1_3 + 2_3) + 3_3) \cdot \langle f, g, h\rangle &= (1_3 + 2_3) \cdot \langle f, g, h\rangle + 3_3 \cdot \langle f, g, h\rangle \\ &= (1_3 \cdot \langle f, g, h\rangle + 2_3 \cdot \langle f, g, h\rangle) + h \\ &= (f + g) + h. \end{aligned}$$

Similarly, the right side reduces to $f + (g + h)$.

Proof of (13.14). By (13.15), (13.16) and (13.11),

$$f + 0_{1p} \;=\; \mathbf{1}_1 \cdot f + 0_{11} \cdot f \;=\; (\mathbf{1}_1 + 0_{11}) \cdot f \;=\; \mathbf{1}_1 \cdot f \;=\; f.$$

We omit the proof of (13.13).

We further refine our notation. Let T be a grove theory and let $n \in \mathbf{N}$. We define the morphism $n : 1 \to 1$ by

$$n \;:=\; \mathbf{1}_1 + \ldots + \mathbf{1}_1 \qquad (n\text{-times}). \tag{13.18}$$

When $n = 0$ we take $0 = 0_{11}$. Thus we have a constant in T for each $n \in \mathbf{N}$.

Proposition 13.2.6 *In any grove theory, $n \cdot (f + g) = n \cdot f + n \cdot g$ and $n \cdot 0_{1p} = 0_{1p}$, for all $f, g : 1 \to p$, $n, p \in \mathbf{N}$.*

The proof uses a simple induction on n. As a consequence of this fact and Proposition 13.2.5, we obtain the fact that the morphisms $n : 1 \to 1 \in T$ form a semiring with respect to sum and composition which is a homomorphic image of the semiring $\mathbf{N}$ of nonnegative integers. It then follows that the matrix theory over $\mathbf{N}$ is initial in the class of grove theories.

We now turn to grove iteration theories.

Definition 13.2.7 *Let T be an iteration theory with additional constants $+ : 1 \to 2$ and $\# : 1 \to 0$, so that we obtain an operation of sum according to (13.1) and (13.2). We define the operation $^\otimes$ on morphisms $f : n \to n + p$ as follows:*

$$f^\otimes \;:=\; (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^\dagger : n \to n + p \tag{13.19}$$

On morphisms $n \to n$ in matrix iteration theories, the $^\otimes$-operation is the same as Kleene star. In matricial iteration theories, we have

$$f^\otimes \;=\; ([a^*, a^*b];\, a^\omega + a^*v),$$

for all $f = ([a, b]; v) : n \to n + p$, so that on morphisms $n \to n$, it is the operation defined in Section 10.10.4, cf. Definition 10.10.4.8. We know that iteration and Kleene star are equivalent operations in matrix iteration theories. This fact holds under weaker assumptions.

Proposition 13.2.8 *Suppose that $\mathbf{1}_1 + 0_{11} = \mathbf{1}_1$ holds in T. Then iteration can be expressed by $^\otimes$, namely*

$$f^\dagger \;=\; f^\otimes \cdot \langle 0_{np}, \mathbf{1}_p \rangle,$$

for all $f : n \to n + p$.

Proof. By the proof of Proposition 13.2.5, $\mathbf{1}_1 + 0_{11} = \mathbf{1}_1$ implies $g + 0_{np} = g$, for all $g : n \to p$. We make use of the parameter identity and the right distributivity law.

$$\begin{aligned}
f^\otimes \cdot \langle 0_{np}, \mathbf{1}_p \rangle \;&= \\
&= (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^\dagger \cdot \langle 0_{np}, \mathbf{1}_p \rangle \\
&= ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_n \oplus \langle 0_{np}, \mathbf{1}_p \rangle))^\dagger \\
&= (f \cdot \mathbf{1}_{n+p} + (0_n \oplus 0_{np}))^\dagger \\
&= (f + 0_{nn+p})^\dagger \\
&= f^\dagger.
\end{aligned}$$

Thus, whenever $\mathbf{1}_1 + 0_{11} = \mathbf{1}_1$ holds in T, identities involving iteration can be expressed by $^{\otimes}$ and conversely. It would be interesting to see what form some identities of iteration theories take when expressed by $^{\otimes}$. However, we do not consider this question here. Instead we provide some identities that will be useful in the sequel.

Proposition 13.2.9 *Let T be an iteration theory with additional constants $+ : 1 \to 2$ and $\# : 1 \to 0$. Then*

$$f \cdot \langle f^{\otimes}, 0_n \oplus \mathbf{1}_p \rangle \; + \; (\mathbf{1}_n \oplus 0_p) \;\; = \;\; f^{\otimes}, \quad f : n \to n + p \tag{13.20}$$

holds, when $f^{\otimes}$ is defined by (13.19). Further,

$$\begin{aligned}
(f \cdot (\mathbf{1}_n \oplus g))^{\otimes} &= f^{\otimes} \cdot (\mathbf{1}_n \oplus g), && (13.21)\\
&\qquad f : n \to n + p, \; g : p \to q &&\\
(f + g)^{\dagger} &= (f^{\otimes} \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^{\dagger}, && (13.22)\\
&\qquad f, g : n \to n + p &&\\
\mathbf{1}_1^{\otimes} + \mathbf{1}_1 &= \mathbf{1}_1^{\otimes} && (13.23)\\
(1_{1+p} + g)^{\dagger} &= (\mathbf{1}_1^{\otimes} \cdot g)^{\dagger}, \quad g : 1 \to 1 + p && (13.24)\\
(\mathbf{1}_1^{\otimes} \cdot 1_{1+p} + g)^{\dagger} &= (\mathbf{1}_1^{\otimes\otimes} \cdot g)^{\dagger}, \quad g : 1 \to 1 + p. && (13.25)
\end{aligned}$$

Proof of (13.20). We use the fixed point identity and the right distributivity law.

$$\begin{aligned}
f^{\otimes} &= (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot \langle f^{\otimes}, \mathbf{1}_{n+p} \rangle\\
&= f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) \cdot \langle f^{\otimes}, \mathbf{1}_{n+p} \rangle + (0_n \oplus \mathbf{1}_n \oplus 0_p) \cdot \langle f^{\otimes}, \mathbf{1}_{n+p} \rangle\\
&= f \cdot \langle f^{\otimes}, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p).
\end{aligned}$$

Proof of (13.21). We use the parameter identity and the right distributivity law.

$$\begin{aligned}
f^{\otimes} \cdot (\mathbf{1}_n \oplus g) &= (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \cdot (\mathbf{1}_n \oplus g)\\
&= ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_{2n} \oplus g))^{\dagger}\\
&= (f \cdot (\mathbf{1}_n \oplus 0_n \oplus g) + (0_n \oplus \mathbf{1}_n \oplus 0_q))^{\dagger}\\
&= (f \cdot (\mathbf{1}_n \oplus g) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_q) + (0_n \oplus \mathbf{1}_n \oplus 0_q))^{\dagger}\\
&= (f \cdot (\mathbf{1}_n \oplus g))^{\otimes}.
\end{aligned}$$

Proof of (13.22). We make use of the parameter identity, the double dagger identity and the right distributivity law.

$$
\begin{aligned}
&(f^{\otimes} \cdot \langle g,\, 0_n \oplus \mathbf{1}_p \rangle)^{\dagger} = \\
&= \; ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^{\dagger} \cdot \langle g,\, 0_n \oplus \mathbf{1}_p \rangle)^{\dagger} \\
&= \; ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)) \cdot (\mathbf{1}_n \oplus \langle g,\, 0_n \oplus \mathbf{1}_p \rangle))^{\dagger\dagger} \\
&= \; (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus g))^{\dagger\dagger} \\
&= \; ((f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus g)) \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^{\dagger} \\
&= \; (f + g)^{\dagger}.
\end{aligned}
$$

Proof of (13.23). Substitute $\mathbf{1}_1$ for f in (13.20).

Proof of (13.24). First observe that, by (13.21),

$$1^{\otimes}_{1+p} \;=\; (\mathbf{1}_1 \cdot (\mathbf{1}_1 \oplus 0_p))^{\otimes} \;=\; \mathbf{1}^{\otimes}_1 \cdot (\mathbf{1}_1 \oplus 0_p) \;=\; \mathbf{1}^{\otimes}_1 \cdot 1_{1+p}.$$

Thus, by (13.22),

$$
\begin{aligned}
(1_{1+p} + g)^{\otimes} \;&=\; (1^{\otimes}_{1+p} \cdot \langle g,\, 0_1 \oplus \mathbf{1}_p \rangle)^{\dagger} \\
&=\; (\mathbf{1}^{\otimes}_1 \cdot 1_{1+p} \cdot \langle g,\, 0_1 \oplus \mathbf{1}_p \rangle)^{\dagger} \\
&=\; (\mathbf{1}^{\otimes}_1 \cdot g)^{\dagger}.
\end{aligned}
$$

The proof of (13.25) is similar.

Remark 13.2.10 By letting $p = 0$ in (13.20) we obtain

$$f \cdot f^{\otimes} + \mathbf{1}_n \;=\; f^{\otimes},$$

for all $f : n \to n$.

Definition 13.2.11 *A theory T which is simultaneously a grove theory and an iteration theory is a* **grove iteration theory**. *Thus, T has constants $+$, $\#$ and an iteration operation †. A* **synchronization theory** *is a grove iteration theory which, in addition, satisfies the following conditions, where $^{\otimes}$ is defined by Definition 13.2.7:*

$$
\begin{aligned}
1^{\otimes} \cdot (1_2 + 2_2) \;&=\; 1^{\otimes} \cdot 1_2 + 1^{\otimes} \cdot 2_2 && (13.26)\\
1^{\otimes} \;&=\; 1^{\otimes\otimes} && (13.27)\\
\# \;&=\; \perp. && (13.28)
\end{aligned}
$$

A **grove iteration theory morphism** *is an iteration theory morphism which preserves the constants $+$ and $\#$. A* **synchronization theory morphism** *is a grove iteration theory morphism.*

Of course, $\perp$ denotes the morphism $\mathbf{1}_1^\dagger$. Since $\# = \perp$ holds in a synchronization theory, also $0_{np} = \perp_{np}$, for all $n, p \geq 0$, so that we can write $\perp_{np}$ instead of 0_{np} and $\perp$ instead of $\#$. Observe that an iteration theory morphism between synchronization theories is a synchronization theory morphism iff it preserves the constant $+$, since $\perp$ is preserved by any iteration theory morphism.

Proposition 13.2.12 *The following identities hold in each synchronization theory, where* $n \in \mathbf{N}$.

$$1^\otimes \cdot (f+g) = 1^\otimes \cdot f + 1^\otimes \cdot g, \quad f, g : 1 \to p \tag{13.29}$$
$$1^\otimes \cdot \perp_{1p} = \perp_{1p} \tag{13.30}$$
$$1^\otimes + n = 1^\otimes \tag{13.31}$$
$$1^\otimes + 1^\otimes = 1^\otimes \tag{13.32}$$
$$n \cdot 1^\otimes = 1^\otimes \cdot n = 1^\otimes, \quad n \neq 0 \tag{13.33}$$
$$1^\otimes \cdot 1^\otimes = 1^\otimes \tag{13.34}$$
$$n^\otimes = 1^\otimes, \quad n \neq 0 \text{ and } 0^\otimes = 1 \tag{13.35}$$
$$(n \cdot 1_{1+p} + f)^\dagger = (1^\otimes \cdot f)^\dagger, \quad n \neq 0, \ f : 1 \to 1+p \tag{13.36}$$
$$(1^\otimes \cdot 1_{1+p} + f)^\dagger = (1^\otimes \cdot f)^\dagger, \quad f : 1 \to 1+p. \tag{13.37}$$

Proof. Below we use some of the identities established for grove theories without explicit mention.

Proof of (13.29). We make use of (13.26).

$$\begin{aligned} 1^\otimes \cdot (f+g) &= 1^\otimes \cdot (1_2 + 2_2) \cdot \langle f, g \rangle = (1^\otimes \cdot 1_2 + 1^\otimes \cdot 2_2) \cdot \langle f, g \rangle \\ &= 1^\otimes \cdot f + 1^\otimes \cdot g. \end{aligned}$$

Proof of (13.30). Since $\perp_{1p} = \perp \cdot 0_p$, it suffices to show that $1^\otimes \cdot \perp = \perp$.

$$\begin{aligned} 1^\otimes \cdot \perp &= (1_2 + 2_2)^\dagger \cdot \perp \\ &= ((1_2 + 2_2) \cdot (1 \oplus \perp))^\dagger, \end{aligned}$$

by the parameter identity,

$$\begin{aligned} &= (1 + \perp_{11})^\dagger \\ &= 1^\dagger = \perp. \end{aligned}$$

Proof of (13.31). Simple induction on n. For $n = 0$ and $n = 1$, one uses (13.14) and (13.23).

Proof of (13.32).

$$\begin{aligned} 1^{\otimes} + 1^{\otimes} &= 1^{\otimes} \cdot 1 + 1^{\otimes} \cdot 1 \\ &= 1^{\otimes} \cdot 2 \\ &= (1_2 + 2_2)^{\dagger} \cdot 2 \\ &= ((1_2 + 2_2) \cdot (1 \oplus 2))^{\dagger}, \end{aligned}$$

by the parameter identity,

$$\begin{aligned} &= (1_2 \cdot (1 \oplus 2) + 2_2 \cdot (1 \oplus 2))^{\dagger} \\ &= (1_2 + 2 \cdot 2_2)^{\dagger} \\ &= (1_2 + 2_2 + 2_2)^{\dagger} \\ &= (1_2 + 2_2)^{\dagger} \\ &= 1^{\otimes}, \end{aligned}$$

by the identity $(f^2)^{\dagger} = f^{\dagger}$, $f : n \to n + p$, which holds in all iteration theories. (See Exercise 5.5.4.31.)

Proof of (13.33). Simple induction on n. One uses (13.29) and (13.32).

Proof of (13.34). First we observe that, by (13.27) and (13.20),

$$1^{\otimes} \cdot 1^{\otimes} + 1 \ = \ 1^{\otimes} \cdot 1^{\otimes\otimes} + 1 \ = \ 1^{\otimes\otimes} \ = \ 1^{\otimes}.$$

Thus,

$$\begin{aligned} 1^{\otimes} \cdot 1^{\otimes} &= (1^{\otimes} + 1) \cdot 1^{\otimes} \ = \ 1^{\otimes} \cdot 1^{\otimes} + 1^{\otimes} \\ &= 1^{\otimes} \cdot 1^{\otimes} + 1^{\otimes} + 1 \ = \ 1^{\otimes} + 1^{\otimes} \ = \ 1^{\otimes}, \end{aligned}$$

by making use of (13.31) and (13.32).

Proof of (13.35). By induction on n. The basis $n = 1$ is obvious. Suppose that the equation holds for some $n \geq 1$. Then,

$$\begin{aligned} (n+1)^{\otimes} &= ((n+1) \cdot 1_2 + 2_2)^{\dagger} \\ &= (1_2 + (n \cdot 1_2 + 2_2))^{\dagger} \\ &= (1^{\otimes} \cdot (n \cdot 1_2 + 2_2))^{\dagger}, \end{aligned}$$

by (13.24),

$$\begin{aligned} &= (1^{\otimes} \cdot n \cdot 1_2 + 1^{\otimes} \cdot 2_2)^{\dagger} \\ &= (1^{\otimes} \cdot 1_2 + 1^{\otimes} \cdot 2_2)^{\dagger}, \end{aligned}$$

by (13.29) and (13.33),

$$= (1^{\otimes\otimes} \cdot 1^{\otimes} \cdot 2_2)^{\dagger},$$

by (13.25),

$$\begin{aligned} &= (1^{\otimes} \cdot 1^{\otimes} \cdot 2_2)^{\dagger} \\ &= (1^{\otimes} \cdot 2_2)^{\dagger}, \end{aligned}$$

by (13.27) and (13.34),

$$\begin{aligned} &= (0_1 \oplus 1^{\otimes})^{\dagger} \\ &= 1^{\otimes}, \end{aligned}$$

by the left zero identity.

The identity $0^{\otimes} = 1$ is established as follows:

$$0^{\otimes} = 0 \cdot 0^{\otimes} + 1 = 1.$$

Proof of (13.36).

$$(n \cdot 1_{1+p} + f)^{\dagger} = ((n \cdot 1_{1+p})^{\otimes} \cdot \langle f,\, 0_1 \oplus \mathbf{1}_p \rangle)^{\dagger},$$

by (13.22),

$$\begin{aligned} &= ((n \cdot (\mathbf{1}_1 \oplus 0_p))^{\otimes} \cdot \langle f,\, 0_1 \oplus \mathbf{1}_p \rangle)^{\dagger} \\ &= (n^{\otimes} \cdot (\mathbf{1}_1 \oplus 0_p) \cdot \langle f,\, 0_1 \oplus \mathbf{1}_p \rangle)^{\dagger}, \end{aligned}$$

by (13.21),

$$\begin{aligned} &= (n^{\otimes} \cdot f)^{\dagger} \\ &= (1^{\otimes} \cdot f)^{\dagger}, \end{aligned}$$

by (13.35) above.

Proof of (13.37). This follows from (13.25) and (13.27).

The identities proved thus far readily imply that the morphisms $n : 1 \to 1,\ n \in \mathbf{N}$, and $1^{\otimes} : 1 \to 1$ form a semiring which is a homomorphic image of the semiring $\mathbf{N}_\infty = \mathbf{N} \cup \{\infty\}$ with $\infty = \infty + n = \infty \cdot n = n \cdot \infty$, for all $n \in \mathbf{N}_\infty, n \neq 0$. Since the matrix iteration theory over $\mathbf{N}_\infty$ is a synchronization theory, we have the following corollary.

Corollary 13.2.13 *The matrix iteration theory over* $\mathbf{N}_\infty$ *is initial in the category of synchronization theories.*

This fact allows us to introduce the notation ∞ for $1^{\otimes}$.

We continue by presenting a slightly different base of identities of synchronization theories.

Proposition 13.2.14 *Let T be a grove iteration theory. Then T is a synchronization theory if and only if (13.26), (13.28) and (13.38) hold, where*

$$2^{\otimes} \quad = \quad 1^{\otimes}. \tag{13.38}$$

The proof uses the following observation.

Lemma 13.2.15 *Let T be an iteration theory with additional constants $+ : 1 \to 2$ and $\# : 1 \to 0$. Then T satisfies*

$$(1_2 + (1_2 + 2_2))^{\dagger} \quad = \quad \mathbf{1}_1^{\otimes} \cdot \mathbf{1}_1^{\otimes\otimes}. \tag{13.39}$$

Proof.

$$(1_2 + (1_2 + 2_2))^{\dagger} \quad = \quad (\mathbf{1}_1^{\otimes} \cdot (1_2 + 2_2))^{\dagger},$$

by (13.24),

$$= \quad \mathbf{1}_1^{\otimes} \cdot ((1_2 + 2_2) \cdot (\mathbf{1}_1^{\otimes} \oplus \mathbf{1}_1))^{\dagger},$$

by the composition identity,

$$\begin{aligned} &= \quad \mathbf{1}_1^{\otimes} \cdot (\mathbf{1}_1^{\otimes} \cdot 1_2 + 2_2)^{\dagger} \\ &= \quad \mathbf{1}_1^{\otimes} \cdot \mathbf{1}_1^{\otimes\otimes}. \end{aligned}$$

Proof of Proposition 13.2.14. By (13.35), $2^{\otimes} = 1^{\otimes}$ holds in any synchronization theory. Conversely, if (13.38) holds, then

$$1^{\otimes\otimes} \;=\; 1^{\otimes} \cdot 1^{\otimes\otimes} + 1 \;=\; 2^{\otimes} + 1 \;=\; 1^{\otimes} + 1 \;=\; 1^{\otimes},$$

where the second equation follows from (13.39).

Next we treat an important subclass of synchronization theories.

Definition 13.2.16 *A grove theory is an* **idempotent grove theory** *if it satisfies the identity*

$$1 + 1 \;=\; 1. \tag{13.40}$$

*Let T be a grove iteration theory. T is an ω-***idempotent grove iteration theory** *if the following identity holds in T:*

$$1^{\otimes} \;=\; 1. \tag{13.41}$$

Proposition 13.2.17 [a] *The following law (13.42) holds in any idempotent grove theory:*

$$f + f \;=\; f, \quad f : n \to p. \tag{13.42}$$

[b] *Every ω-idempotent grove iteration theory with $\# = \perp$ is a synchronization theory.*

[c] *Every ω-idempotent grove iteration theory is an idempotent grove theory.*

Proof. We omit the proof of [a].
Proof of [b]. Let T be an ω-idempotent grove iteration theory with $\# = \perp$. Then,

$$1^{\otimes} \cdot (1_2 + 2_2) \;=\; 1 \cdot (1_2 + 2_2) \;=\; 1_2 + 2_2 \;=\; 1^{\otimes} \cdot 1_2 + 1^{\otimes} \cdot 2_2.$$

Moreover, $1^{\otimes} = 1^{\otimes\otimes} = 1$.

Proof of [c].

$$1 + 1 \;=\; 1^{\otimes} + 1 \;=\; 1^{\otimes} \;=\; 1.$$

By virtue of Proposition 13.2.17, an ω-idempotent grove iteration theory with $\# = \perp$ will be called *ω-idempotent synchronization theory.*

13.3 Axiomatizing Synchronization Trees

In this section we give an algebraic characterization of regular synchronization trees. We show that these trees form free synchronization theories. Thus the axioms of synchronization theories are an axiomatization of regular synchronization trees as well as the variety generated by all theories of synchronization trees. For this and the next section by a *synchronization tree* we shall mean a synchronization tree in a skeletal ω-continuous 2-theory $ST'(A)$. By Theorem 8.8.5.13, we may impose an iteration theory structure on $ST'(A)$. The resulting iteration theory is denoted $\mathcal{ST}(A)$. We observe that $\mathcal{ST}(A)$ contains a synchronization tree shown in Figure 13.2. Denoting this tree by $+$, the tree

$$t_1 + t_2 \;=\; + \cdot \langle t_1, t_2 \rangle$$

is just the coproduct of synchronization trees $t_1, t_2 : 1 \to p$. In $\mathcal{ST}(A)$, we interpret the constant $\# : 1 \to 0$ as the trivial synchronization tree consisting of only one vertex, so that $\# = \perp$.

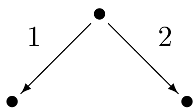


Figure 13.2: The synchronization tree $+$

Theorem 13.3.1 $\mathcal{ST}(A)$ *is a synchronization theory.*

Proof. Indeed, equations (13.26) and (13.27) are checked in a straightforward way. The rest of the properties derive from the fact that sum is defined via coproducts and the initiality of $\perp$. The synchronization tree $1^{\otimes}$ is shown in Figure 13.3.

Definition 13.3.2 *Let t be a synchronization tree $1 \to p$ and let v be a vertex of t. The vertices that are accessible from v by a directed path form the vertex set of a tree $1 \to p$ with root v, isomorphic to a synchronization tree t' in $\mathcal{ST}(A)$. We call t' the* **subtree** *of t at vertex*

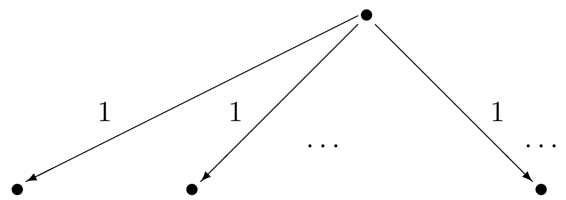


Figure 13.3: The synchronization tree $1^{\otimes}$

v. A synchronization tree $1 \to p$ is **regular** *if it has a finite number of subtrees, and it is* **finite** *if its vertex set is finite. A synchronization tree $t : n \to p$ is regular (finite, respectively) if $i_n \cdot t$ is regular (finite), for all $i \in [n]$.*

For any action symbol $a \in A$, let $a\eta$ denote the tree shown in Figure 13.4. We may identify the letter a with the atomic tree $a\eta$. Let $\mathcal{RST}(A)$ denote the subsynchronization theory of $\mathcal{ST}(A)$ generated by all the atomic synchronization trees a, for all $a \in A$. Note that $\mathcal{RST}(A)$ is the subiteration theory of $\mathcal{ST}(A)$ generated by the atomic trees a and the tree $+ : 1 \to 2$. Below we will show that a synchronization tree t is regular if and only if t belongs to $\mathcal{RST}(A)$.

When A is empty, $\mathcal{ST}(A) = \mathcal{RST}(A)$ is the theory of synchronization trees that do not have any edge labeled by an action symbol. We will denote this theory by T_0. Note that T_0 is isomorphic to the matrix iteration theory over the semiring $\mathbf{N}_\infty$. Indeed, any morphism $1 \to p$ in T_0 can be written uniquely in the form $n_1 \cdot 1_p + \ldots + n_p \cdot p_p$, where $n_i \in \mathbf{N}_\infty$, for all $i \in [p]$, which may be identified with the row matrix $[n_1, \ldots, n_p]$. By Corollary 13.2.13 we have the following fact:

Proposition 13.3.3 *The synchronization theory T_0 is initial in the class of synchronization theories.*

Definition 13.3.4 *A synchronization tree $t : 1 \to p$ is* **guarded** *if no edge starting at the root has its label in $[p]$. An equivalent condition is that t cannot be written as $t = i_p + t'$, for any $i \in [p]$ and any*

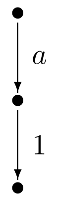


Figure 13.4: The synchronization tree $a\eta$

synchronization tree $t' : 1 \to p$. A synchronization tree $t : n \to p$ is guarded if $i_n \cdot t$ is guarded, for all $i \in [n]$.

Proposition 13.3.5 *Let $t : n \to n + p$ be guarded. Then $t^\dagger$ is the unique solution to the iteration equation for t:*

$$\xi \;=\; t \cdot \langle \xi, \mathbf{1}_p \rangle.$$

Proof. Supposing $f = t \cdot \langle f, \mathbf{1}_p \rangle$ we obtain

$$f \;=\; t^m \cdot \langle f, \mathbf{1}_p \rangle, \tag{13.43}$$

for all $m \geq 0$. The equation (13.43) uniquely defines f up to depth $m - 1$. Since this holds for every m, f is uniquely determined.

Definition 13.3.6 *Let $t : 1 \to p$ be a synchronization tree. If t' is the subtree of t at vertex v and if v is the target of an edge labeled $a \in A$, then $a \cdot t'$ is called an A-**subtree** of t.*

The following proposition is obvious.

Proposition 13.3.7 *A synchronization tree $t : 1 \to p$ is regular if and only if t has a finite number of A-subtrees.*

Proposition 13.3.8 *Suppose that $t : 1 \to p$ is regular. Then t is in $\mathcal{RST}(A)$.*

Proof. In our proof we will make use of Corollary 11.11.1.4. In fact, we will show that there exists a presentation D_t over (T_0, A) with $t = |D_t|$. (Recall that any action symbol $a \in A$ is identified with the corresponding atomic tree.)

Let

$$a_1 \cdot t_1, \ldots, a_s \cdot t_s$$

be all of the A-subtrees of t, in some fixed ordering. For each $i \in [s]$ there exists $\alpha_i : 1 \to s + p$ in T_0 with

$$a_i \cdot t_i \quad = \quad a_i \cdot \alpha_i \cdot \langle a_1 \cdot t_1, \ldots, a_s \cdot t_s,\, \mathbf{1}_p \rangle. \tag{13.44}$$

Similarly, there exists $\alpha : 1 \to s$ in T_0 such that

$$t \quad = \quad \alpha \cdot \langle a_1 \cdot t_1, \ldots, a_s \cdot t_s,\, \mathbf{1}_p \rangle. \tag{13.45}$$

Define

$$f \quad := \quad \langle a_1 \cdot t_1, \ldots, a_s \cdot t_s \rangle$$

and

$$u \quad := \quad \langle a_1 \cdot \alpha_1, \ldots, a_s \cdot \alpha_s \rangle.$$

By (13.44) above, f is a solution to the iteration equation for u. Since u is guarded, by Proposition 13.3.5 we have $f = u^\dagger$. Thus, by (13.45), also $t = \alpha \cdot \langle u^\dagger, \mathbf{1}_p \rangle$. This shows that $t = |D_t|$, where $D_t := (\alpha, u) : 1 \xrightarrow{s} p$.

Definition 13.3.9 *Let $D = (\alpha; u) : 1 \xrightarrow{s} p$ be a presentation over (T_0, A). Let $i, j \in [s]$. We say that the i-th component of u directly depends on the j-th component if some edge of the tree $i_s \cdot u$ is labeled j. This defines the* **direct dependency** *relation. The reflexive and transitive closure of it is called the* **dependency** *relation. We say that the j-th component of u is accessible, if for some $i \in [s]$, α has an edge labeled $i \in [s]$ and the i-th component of u depends on the j-th component of u.*

Definition 13.3.10 *Let $D = (\alpha; u) : 1 \xrightarrow{s} p$ be a presentation. We say that D is* **accessible** *if each component of u is accessible. We say that D is* **reduced**, *if $i_s \cdot u^\dagger \neq j_s \cdot u^\dagger$ whenever $i, j \in [s]$ with $i \neq j$.*

Example 13.3.11 The presentation D_t given in the proof of Proposition 13.3.8 is both accessible and reduced.

Exercise 13.3.12 Show that a presentation is accessible (or reduced) iff it is accessible (reduced, respectively) in the sense of Exercise 11.11.2.16.

Proposition 13.3.13 *Let $D = (\alpha; u) : 1 \xrightarrow{s} p$ be an accessible presentation over (T_0, A). Then the set*

$$\{i_s \cdot u^\dagger \,:\, i \in [s]\}$$

is the set of all A-subtrees of $|D|$.

Proof. That each $i_s \cdot u^\dagger$ is an A-subtree of $|D|$ can be seen by induction on the length of the shortest dependency chain which connects the j-th component of u to the i-th component of u, for some $j \in [s]$ such that α has an edge labeled j. Conversely, if $a \cdot t$ is an A-subtree of $|D|$, then $|D|$ has vertices v_1, v_2 and an edge $e : v_1 \to v_2$ labeled a such that t is the subtree at vertex v_2. One shows by induction on the length of the path from the root of $|D|$ to v_1 that there exists an integer $i \in [s]$ with $i_s \cdot u^\dagger = a \cdot t$. The assumption that D is accessible is not needed for the last step.

Corollary 13.3.14 *A synchronization tree t is in $\mathcal{RST}(A)$ if and only if t is regular.*

Proof. If t is regular then t is in $\mathcal{RST}(A)$ by Proposition 13.3.8. Suppose that $t : 1 \to p$ is in $\mathcal{RST}(A)$. By Corollary 11.11.1.4 there is a presentation D over (T_0, A) with $t = |D|$. It follows from the proof of Proposition 13.3.13 that t has a finite number of A-subtrees. Thus, by Proposition 13.3.7, t is regular.

Using the notation introduced in Chapter 11, a synchronization tree $t : n \to p$ is in $A(T_0)$ iff the components of t are of the form $a \cdot \alpha$, where $a \in A$ and $\alpha \in T_0$. We will make use of the following two lemmas.

Lemma 13.3.15 *Let $\alpha, \beta : 1 \to s + p$ be in T_0 and let*

$$t \;=\; \langle a_1 \cdot t_1, \ldots, a_s \cdot t_s \rangle,$$

where $a_i \in A$ and $t_i : 1 \to p$, for all $i \in [s]$. Suppose that $a_i \cdot t_i \neq a_j \cdot t_j$, for any pair of integers $i, j \in [s]$ with $i \neq j$, so that either $a_i \neq a_j$ or $t_i \neq t_j$. If

$$\alpha \cdot \langle t, \mathbf{1}_p \rangle \;=\; \beta \cdot \langle t, \mathbf{1}_p \rangle,$$

then $\alpha = \beta$.

Lemma 13.3.16 *Suppose that $u, v : n \to p$ are synchronization trees in $A(T_0)$. If $t : p \to q$ is any synchronization tree such that $i_p \cdot t \neq j_p \cdot t$, for all $i, j \in [p]$ with $i \neq j$, and if $u \cdot t = v \cdot t$, then $u = v$.*

Proposition 13.3.17 *Let $D = (\alpha; u) : 1 \xrightarrow{s} p$ and $E = (\beta; v) : 1 \xrightarrow{r} p$ be accessible presentations over (T_0, A). Suppose that $|D| = |E|$ and that E is reduced. Then there is a surjective base morphism $\rho : s \to r$ with $D \xrightarrow{\rho} E$, i.e. such that $\alpha \cdot (\rho \oplus \mathbf{1}_p) = \beta$ and $u \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot v$.*

Proof. Suppose $u = \langle u_1, \ldots, u_s \rangle$, $v = \langle v_1, \ldots, v_r \rangle$, $u^\dagger = \langle \overline{u}_1, \ldots, \overline{u}_s \rangle$ and $v^\dagger = \langle \overline{v}_1, \ldots, \overline{v}_r \rangle$. By Proposition 13.3.13, for every $i \in [s]$ there is a $j \in [r]$ with $\overline{u}_i = \overline{v}_j$ and vice versa. Since E is reduced, the integer j is uniquely determined by $i \in [s]$. Define $\rho : s \to r$ by the condition $i\rho = j$ if and only if $\overline{u}_i = \overline{v}_j$. We have

$$\begin{aligned} |D| &= \alpha \cdot \langle u^\dagger, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle \rho \cdot v^\dagger, \mathbf{1}_p \rangle \\ &= \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle v^\dagger, \mathbf{1}_p \rangle \end{aligned}$$

and

$$|E| \;=\; \beta \cdot \langle v^\dagger, \mathbf{1}_p \rangle.$$

Since $|D| = |E|$, by Lemma 13.3.15 we have $\alpha \cdot (\rho \oplus \mathbf{1}_p) = \beta$.

Let $i \in [s]$ and $j \in [r]$ with $\overline{u}_i = \overline{v}_j$. It remains to show that

$$u_i \cdot (\rho \oplus \mathbf{1}_p) \;=\; v_j. \tag{13.46}$$

But

$$\begin{aligned} u_i \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle v^\dagger, \mathbf{1}_p \rangle &= u_i \cdot \langle \rho \cdot v^\dagger, \mathbf{1}_p \rangle \\ &= u_i \cdot \langle u^\dagger, \mathbf{1}_p \rangle \\ &= \overline{u}_i \\ &= \overline{v}_j \\ &= v_j \cdot \langle v^\dagger, \mathbf{1}_p \rangle. \end{aligned}$$

Thus, (13.46) follows from Lemma 13.3.16.

Corollary 13.3.18 *Let* $D = (\alpha; u) : 1 \xrightarrow{s} p$ *and* $E = (\beta; v) : 1 \xrightarrow{r} p$ *be reduced accessible presentations over* (T_0, A)*. If* $|D| = |E|$ *then* $s = r$*, and there is a base permutation* $\pi : s \to s$ *with* $D \xrightarrow{\pi} E$*.*

Proposition 13.3.19 *For every presentation* $D = (\alpha; u) : 1 \xrightarrow{s} p$ *over* (T_0, A)*, there is an accessible presentation* $E = (\beta; v) : 1 \xrightarrow{r} p$ *over* (T_0, A) *with* $r \leq s$ *such that for some base injection* $\rho : r \to s$ *we have* $D \xrightarrow{\rho} E$*.*

Corollary 13.3.20 *Let* $D, E : 1 \to p$ *be presentations. If* $|D| = |E|$*, then* $D \overset{*}{\leftrightarrow} E$*. More exactly, there exist accessible presentations* D_1, E_1 *and a reduced accessible presentation* F *such that*

$$D \xleftarrow{\rho_1} D_1 \xrightarrow{\tau_1} F \xleftarrow{\tau_2} E_1 \xrightarrow{\rho_2} E,$$

for some base injections ρ_i *and base surjections* τ_i*, with appropriate source and target,* $i = 1, 2$*.*

The pair $(T_0, A(T_0))$ satisfies the assumptions of Corollary 11.11.2.19. In more detail, we have the following fact.

Lemma 13.3.21 *Each scalar tree* t *in* $A(T_0)$ *has a unique factorization* $t = a \cdot \alpha$*, where* $a \in A$ *and* $\alpha \in T_0$*. Further, for any family of morphisms* $\alpha_1, \ldots, \alpha_m : 1 \to s + p$ $(m \geq 1)$ *in* T_0*, if*

$$\alpha_1 \cdot (\tau \oplus \mathbf{1}_p) \;=\; \ldots \;=\; \alpha_m \cdot (\tau \oplus \mathbf{1}_p),$$

for some surjective base morphism $\tau : s \to r$*, then there is a morphism* $\gamma : 1 \to k + p$ *in* T_0 *and there exist base morphisms* $\rho_1, \ldots, \rho_m : k \to s$ *with*

$$\alpha_i \;=\; \gamma \cdot (\rho_i \oplus \mathbf{1}_p) \quad (i \in [m])$$

and

$$\rho_1 \cdot \tau \;=\; \ldots \;=\; \rho_m \cdot \tau.$$

By the previous lemma, Corollary 11.11.2.19 and Corollary 13.3.20, we obtain the following corollary.

Corollary 13.3.22 *If $D, E : 1 \to p$ are presentations with $|D| = |E|$ then $D \overset{*}{\rightleftharpoons} E$.*

We are now ready to prove the main result of this section.

Theorem 13.3.23 *The theory $\mathcal{RST}(A)$ is freely generated by the map $\eta : A \to \mathcal{RST}(A)$ in the class of synchronization theories.*

Proof. As an iteration theory, $\mathcal{RST}(A)$ is the coproduct of the initial synchronization theory T_0 and the free iteration theory on A, where each letter of A is a symbol of rank 1. This follows by Theorem 11.11.3.8 and Corollary 13.3.22.

Exercise 13.3.24 [BT89] Finite synchronization trees on a set A of action symbols form a subgrove theory $\mathcal{FST}(A)$ of $\mathcal{ST}(A)$. Show that $\mathcal{FST}(A)$ is the free grove theory on A.

Exercise 13.3.25 Describe the free synchronization theory on Σ, where Σ is any signature, i.e. when the letters in Σ may have any rank. *Hint:* Generalize the concept of synchronization trees to trees with hyper edges labeled by symbols in Σ or by integers.

13.4 Bisimilarity

There are many equivalence notions for behaviors of concurrent systems. One of the best known is the notion of bisimilarity. In this section, we show that the relation is an iteration congruence on the theory of synchronization trees, and characterize the quotient theory of the regular trees. For the sake of brevity some proofs are just sketched.

Definition 13.4.1 *Let $f, g : 1 \to p$ be synchronization trees in $\mathcal{ST}(A)$, say $f = (V, E, \{v_0\}, l)$ and $g = (V', E', \{v_0'\}, l')$. A* **bisimulation** *between f and g is a relation*

$$R \subseteq V \times V'$$

satisfying the following conditions.

1. *The roots are related, i.e.* $v_0 \, R \, v'_0$ *holds.*

2. *If* $u, v \in V$ *and* $u' \in V'$ *are vertices with* $u \, R \, u'$ *and* $(u, v) \in E$, *then there exists* $v' \in V'$ *with* $v \, R \, v'$, $(u', v') \in E'$ *and* $l(u, v) = l'(u', v')$.

3. *Symmetrically, if* $u', v' \in V'$ *and* $u \in V$ *are vertices with* $u \, R \, u'$ *and* $(u', v') \in E'$, *then there exists* $v \in V$ *with* $v \, R \, v'$, $(u, v) \in E$ *and* $l(u, v) = l'(u', v')$.

Definition 13.4.2 *Let* $f, g : 1 \to p$ *be synchronization trees in* $\mathcal{ST}(A)$. *We say that* f *is* **bisimilar** *to* g *if there exists a bisimulation between* f *and* g. *Two synchronization trees* $f, g : n \to p$ *are bisimilar, denoted* $f \sim g$, *if* $i_n \cdot f$ *is bisimilar to* $i_n \cdot g$, *for all* $i \in [n]$. *We call* $\sim$ *the relation of* **bisimilarity**.

Proposition 13.4.3 *Bisimilarity is an equivalence relation.*

Proof. That $\sim$ is reflexive and symmetric is obvious. Transitivity follows from the fact that if $f, g, h : 1 \to p$ are synchronization trees and if R and R' are bisimulations between f and g and between g and h, respectively, then the relational composite of R and R' is a bisimulation between f and h.

We now give an equivalent characterization of bisimilarity. We observe that that two synchronization trees $f, g : 1 \to p$ are bisimilar if and only if there is a synchronization tree $h : 1 \to p$ which is a "locally surjective" image of both f and g.

Definition 13.4.4 *Let* $\varphi : f \to g$ *be a synchronization tree morphism, where* $f, g : 1 \to p$. *We say that* φ *is* **locally surjective** *if for each vertex* v *of* f, *and each edge* $(v\varphi, u)$ *in* g, *there is an edge* (v, u') *in* f *with* $u'\varphi = u$. *A synchronization tree morphism between trees* $n \to p$, $n \neq 1$, *is locally surjective if its components are locally surjective.*

Note that any locally surjective tree morphism is necessarily surjective.

Proposition 13.4.5 *A tree morphism* $\varphi : f \to g$ *is locally surjective iff* φ, *as a relation, is a bisimulation.*

Exercise 13.4.6 Prove Proposition 13.4.5.

Let $f : 1 \to p$ be a synchronization tree, say $f = (V, E, v_0, l)$. We will define a synchronization tree $\tilde{f} = (\tilde{V}, \tilde{E}, \tilde{v}_0, \tilde{l}) : 1 \to p$. First, let $U_0 := \{v_0\}$. Suppose that we have already defined the set $U_i \subseteq V$, where $i \geq 0$ is an integer. We define U_{i+1} as follows. For each $u \in U_i$ and $a \in A \cup [p]$, let $u_1, u_2, \ldots$ be the finite or infinite sequence of the a-successors of u in some fixed ordering, i.e. the collection of all vertices which are the targets of edges labeled a originating in u. Then let $u_{i_1}, u_{i_2}, \ldots$ be the subsequence

$$\begin{aligned} u_{i_1} &:= u_1 \\ u_{i_{j+1}} &:= \text{the first } u_k \text{ such that } t_{u_k} \text{ is not bisimilar} \\ & \text{ to any of the trees } t_{u_{i_1}}, t_{u_{i_2}}, \ldots,\ t_{u_{i_j}}. \end{aligned}$$

Let U_{i+1} be the union of the sets $\{u_{i_1}, u_{i_2}, \ldots\}$, for all $u \in U_i$ and $a \in A \cup [p]$.

Definition 13.4.7 *Let f be the synchronization tree above. The synchronization tree $\tilde{f} = (\tilde{V}, \tilde{E}, \tilde{v}_0, \tilde{l}) : 1 \to p$ is defined as follows:*

$$\begin{aligned} \tilde{V} &:= \bigcup_{i=0}^{\infty} U_i \\ \tilde{E} &:= E \cap (\tilde{V} \times \tilde{V}) \\ \tilde{v}_0 &:= v_0 \\ \tilde{l} &:= l|_{\tilde{E}}. \end{aligned}$$

Proposition 13.4.8 *There is a locally surjective synchronization tree morphism $f \to \tilde{f}$.*

Proof. Suppose $u \in V$. If $u \in \tilde{V}$, let $u\varphi := u$. If $u \notin \tilde{V}$, there is a directed path

$$u_0, u_1, \ldots, u_k = u$$

from a vertex $u_0 \in \tilde{V}$ which contains no vertex in $\tilde{V}$ other than u_0. Define $w_0 := u_0$. Supposing that w_i $(1 \leq i < k)$ has been defined and that u_{i+1} is an a-successor of u_i, for some $a \in A \cup [p]$, let $w_{i+1} \in \tilde{V}$ be that a-successor of w_i for which the trees $t_{u_{i+1}}$ and $t_{w_{i+1}}$ are bisimilar. We define $u\varphi := w_k$. The function φ is easily shown to be a locally surjective synchronization tree morphism $t \to \tilde{t}$.

Exercise 13.4.9 Show that the morphism φ constructed in the above proof is the only locally surjective synchronization tree morphism $t \to \tilde{t}$. It is also the smallest bisimilation between t and $\tilde{t}$.

Next we list some properties of the previous construction.

Proposition 13.4.10 *Let t be a synchronization tree $1 \to p$.*

- [a] $t = \tilde{t}$ *if and only if whenever u is a vertex of t and v_1 and v_2 are a-successors of u, for some $a \in A \cup [p]$, $v_1 = v_2$ whenever the subtrees t_{v_1} and t_{v_2} are bisimilar.*
- [b] $t = \tilde{t}$ *if and only if* $t = \tilde{f}$, *for some f;*
- [c] *If t is regular, then so is $\tilde{t}$.*

Proof. We only prove [c]. Let $\varphi : t \to \tilde{t}$ be the locally surjective synchronization tree morphism. Let v_1 and v_2 be vertices of $\tilde{t}$ and let u_1 and u_2 be vertices of t with $u_i\varphi = v_i$, $i = 1, 2$. If $\tilde{t}_{v_1} \neq \tilde{t}_{v_2}$ then $t_{u_1} \neq t_{u_2}$. Thus if t has a finite number of subtrees, then $\tilde{t}$ has a finite number of subtrees also.

Below we will call a synchronization tree $t : 1 \to p$ with $t = \tilde{t}$ a $\sim$-*reduced* synchronization tree.

Lemma 13.4.11 *Suppose that $f, g : 1 \to p$ are synchronization trees and that g is $\sim$-reduced. If $f \sim g$ then there is a locally surjective synchronization tree morphism $f \to g$.*

Proof. Let, say, $f = (V, E, v_0, l)$ and $g = (V', E', v_0', l')$. Suppose that R is a bisimulation between f and g. We define a function $\varphi_R : V \to V'$ with the property that $(v, v\varphi) \in R$, by induction on the depth of the vertex $v \in V$. When the depth of v is 0, $v = v_0$. We define $v\varphi_R := v_0'$, the root of g. Suppose that the depth of v is $i + 1$, for some integer $i \geq 0$, and that the value of φ_R has been defined on all vertices of depth i. Suppose further that φ_R maps the vertices of f with depth i onto the vertices of g of depth i. Let u be the ancestor of v, so that u has depth i and (u, v) is an edge of f, labeled a, for some a in $A \cup [p]$. Let $u' := u\varphi_R$, so that $u\,R\,u'$ by the induction assumption. Since R is a bisimulation, vertex u' has an a-successor v' with $v\,R\,v'$. Also f_v is bisimilar to $g_{v'}$. Since g is $\sim$-reduced, v' is the only a-successor of u' with $v\,R\,v'$. We define $v\varphi_R := v'$. It follows from the facts that R is a bisimulation and that g is reduced, that each vertex of g of depth $i+1$ is the image of a vertex of f with depth $i+1$. Thus, by definition, φ_R is a surjective synchronization tree morphism $f \to g$. In fact, φ_R is locally surjective.

Corollary 13.4.12 *Let f and g be synchronization trees $1 \to p$.*

[a] *If both f and g are $\sim$-reduced, then f and g are bisimilar if and only if $f = g$;*

[b] *f and g are bisimilar if and only if $\tilde{f} = \tilde{g}$.*

[c] *f and g are bisimilar iff $f \xrightarrow{\varphi} h$ and $g \xrightarrow{\psi} h$, for some ($\sim$-reduced) tree h and locally surjective morphisms φ and ψ.*

Lemma 13.4.13 *Let $f, g : 1 \to p$ be synchronization trees and $\varphi : f \to g$ a locally surjective synchronization tree morphism. Then there is an injective synchronization tree morphism $\psi : g \to f$ such that $\psi \circ \varphi$ is the identity morphism $g \to g$. In other words, every locally surjective synchronization tree morphism is a retraction.*

Proof. Let, say, $f = (V, E, v_0, l)$ and $g = (V', E', v_0', l')$. Suppose that $\varphi : V \to V'$ is a locally surjective synchronization tree morphism $f \to g$. Define $\psi : V' \to V$ as follows. Let v be a vertex in V'. When v is the root v_0', define $v\psi := v_0$. Otherwise v is the successor of a vertex u. Supposing that $u\psi$ has been defined and that $u\psi\varphi = v$, since φ is locally surjective, there is a successor of $u\psi$ which maps to v under φ. Let $v\psi$ be one of these.

Exercise 13.4.14 Find an example of a surjection, even a retraction, which is not a bisimulation.

Exercise 13.4.15 Show that if $\varphi_i : f_i \to g_i$, $i = 1, 2$, are locally surjective, then so is

$$\varphi_1 \cdot \varphi_2 : f_1 \cdot f_2 \;\to\; g_1 \cdot g_2,$$

where we assume that the trees have appropriate sources and targets.

Exercise 13.4.16 Show that if $\varphi_i : f_i \to g_i$, $i = 1, 2$, are locally surjective, then so is

$$\langle \varphi_1, \varphi_2 \rangle : \langle f_1, f_2 \rangle \;\to\; \langle g_1, g_2 \rangle,$$

where we assume that the trees have appropriate sources and targets.

Exercise 13.4.17 Show that if $f, g : n \;\to\; n+p$ and if there is a locally surjective map $f \to g$, then there is a locally surjective map $f^\dagger \to g^\dagger$. Compare this fact with that given in Exercise 8.8.5.22.

Proposition 13.4.18 *Bisimilarity is an iteration theory congruence on $\mathcal{ST}(A)$.*

Proof. The proof follows from Proposition 13.4.5 and Corollary 13.4.12, together with Exercises 13.4.15, 13.4.16 and 13.4.17.

We continue with some technical results. For the definition of the presentation D_t, where t is a synchronization tree $1 \to p$, see the proof of Proposition 13.3.8.

Proposition 13.4.19 *Any subtree or A-subtree of a $\sim$-reduced tree is $\sim$-reduced.*

Proof. This is immediate from Proposition 13.4.10.

Proposition 13.4.20 *Let t be a regular $\sim$-reduced synchronization tree and let $D_t = (\alpha; u)$. Then D_t is $\sim$-reduced, i.e. α and u are $\sim$-reduced and the components of $u^\dagger$ are pairwise not bisimilar.*

Proof. We only prove that the components of $u^\dagger$ are pairwise not bisimilar. Suppose $i_s \cdot u^\dagger \sim j_s \cdot u^\dagger$, for some $i, j \in [s]$. Since both trees $i_s \cdot u^\dagger$ and $j_s \cdot u^\dagger$ are A-subtrees of the $\sim$-reduced tree t, they are $\sim$-reduced also. Thus, by Corollary 13.4.12, $i_s \cdot u^\dagger = j_s \cdot u^\dagger$. Since D_t is a reduced presentation, we have $i = j$, as claimed.

Lemma 13.4.21 *Let $\alpha, \beta : 1 \to s + p$ be synchronization trees in T_0 and let*

$$t \;=\; \langle a_1 \cdot t_1, \ldots, a_s \cdot t_s \rangle : s \to p,$$

be a synchronization tree, where $a_i \in A$ and $t_i : 1 \to p$, for all $i \in [s]$. Suppose that the trees $a_i \cdot t_i$ are pairwise not bisimilar. If

$$\alpha \cdot \langle t, \mathbf{1}_p \rangle \;\sim\; \beta \cdot \langle t, \mathbf{1}_p \rangle,$$

then $\alpha \sim \beta$.

Lemma 13.4.22 *Suppose that $u, v : n \to p$ are in $A(T_0)$, so that the components of u and v are of the form $a \cdot \alpha$, where $a \in A$ and $\alpha \in T_0$. If $t : p \to q$ is some synchronization tree whose components are pairwise not bisimilar, and if $u \cdot t \sim v \cdot t$, then $u \sim v$.*

Lemma 13.4.23 *Let $f : 1 \to p$ be a regular sychronization tree so that $\tilde{f} : 1 \to p$ is regular also. Let $D_f = (\alpha, u)$ and $D_{\tilde{f}} = (\beta; v)$, say. Then there exists a surjective base morphism $\rho : s \to r$ with*

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) \;\sim\; \beta \tag{13.47}$$

and

$$u \cdot (\rho \oplus \mathbf{1}_p) \;\sim\; \rho \cdot v. \tag{13.48}$$

Proof. By Proposition 13.4.8, there is a locally surjective synchronization tree morphism $\varphi : f \to \tilde{f}$. If t is an A-subtree of f, the images of the vertices of t determine an A-subtree of $\tilde{f}$ which is bisimilar to t. It follows that each A-subtree of f is bisimilar to an A-subtree of $\tilde{f}$ and vice versa.

The trees $i_s \cdot u^\dagger$, $i \in [s]$, are all of the A-subtrees of f. Similarly, the trees $j_r \cdot v^\dagger$ are all of the A-subtrees of $\tilde{f}$. By Proposition 13.4.19 and Corollary 13.4.12, the trees $j_r \cdot v^\dagger$, $j \in [r]$, are pairwise not bisimilar. Thus, for each $i \in [s]$ there is a unique $j \in [r]$ such that $i_s \cdot u^\dagger \sim j_r \cdot v^\dagger$. Define $i_s \cdot \rho := j_r$. It follows that ρ is a surjective base morphism $s \to r$.

In order to prove equation (13.47), note that $\beta \cdot \langle v^\dagger, \mathbf{1}_p \rangle = \tilde{f}$ and

$$\begin{aligned} \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle v^\dagger, \mathbf{1}_p \rangle &= \alpha \cdot \langle \rho \cdot v^\dagger, \mathbf{1}_p \rangle \\ &\sim \alpha \cdot \langle u^\dagger, \mathbf{1}_p \rangle \\ &= f. \end{aligned}$$

Since $f \sim \tilde{f}$, equation (13.47) follows by Lemma 13.4.21. Also, $\rho \cdot v^\dagger = \rho \cdot v \cdot \langle v^\dagger, \mathbf{1}_p \rangle$, and

$$\begin{aligned} u \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle v^\dagger, \mathbf{1}_p \rangle &= u \cdot \langle \rho \cdot v^\dagger, \mathbf{1}_p \rangle \\ &\sim u \cdot \langle u^\dagger, \mathbf{1}_p \rangle \\ &= u^\dagger. \end{aligned}$$

Since $u^\dagger \sim \rho \cdot v^\dagger$,

$$u \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle v^\dagger, \mathbf{1}_p \rangle \sim \rho \cdot v \cdot \langle v^\dagger, \mathbf{1}_p \rangle.$$

Thus (13.48) follows by Lemma 13.4.22.

We let $\mathcal{BST}(A)$ denote the quotient synchronization theory

$$\mathcal{BST}(A) := \mathcal{ST}(A)/\!\sim .$$

Similarly, $\mathcal{BRST}(A)$ denotes the synchronization theory $\mathcal{RST}(A)/\!\sim$.

Proposition 13.4.24 *$\mathcal{BST}(A)$ and $\mathcal{BRST}(A)$ are ω-idempotent synchronization theories.*

Proof. The synchronization trees $1^\otimes$ and 1 are bisimilar.

In the rest of this section we prove that $\mathcal{BRST}(A)$ is the free ω-idempotent synchronization theory on the set A. When $A = \emptyset$, $\mathcal{BRST}(A)$ is the synchronization theory $T_0' := T_0/\!\sim$. Recall that T_0 is the theory of all synchronization trees all of whose edges are exit edges.

Proposition 13.4.25 *T_0' is initial ω-idempotent synchronization theory.*

Proof. T_0' is isomorphic to $\mathbf{Mat_B}$, the matrix iteration theory over the Boolean semiring $\mathbf{B}$.

We will write $[f]$ for the $\sim$-equivalence class of a regular synchronization tree. In particular, $[a]$, or sometimes just a, will stand for the $\sim$-equivalence class of the atomic tree a, for each $a \in A$. Thus we may regard A as a subset of the set of morphisms $1 \to 1$ in $\mathcal{BRST}(A)$. Similarly, we may regard T_0' as a subsynchronization theory of $\mathcal{BRST}(A)$.

Since T_0' and A jointly generate $\mathcal{BRST}(A)$, each morphism $1 \to p$ in $\mathcal{BRST}(A)$ is the behavior of a presentation over T_0' and A. A presentation over (T_0', A) can be given in the form $[D] = ([\alpha]; [u])$, where $D = (\alpha; u)$ is a presentation over (T_0, A) in $\mathcal{RST}(A)$. The behavior of $[D]$ is then the $\sim$-equivalence class of the behavior of D, i.e.

$$|[D]| \;=\; [|D|] \;=\; [\alpha \cdot \langle u^\dagger . \mathbf{1}_p \rangle] \;=\; [\alpha] \cdot \langle [u]^\dagger, \mathbf{1}_p \rangle.$$

Lemma 13.4.26 *Let $f, g : 1 \to p$ be regular synchronization trees. If $f \sim g$ then there is a regular $\sim$-reduced synchronization tree $h : 1 \to p$ such that for some base surjections ρ and τ we have*

$$[D_f] \xrightarrow{\rho} [D_h] \xleftarrow{\tau} [D_g].$$

Proof. Let $h = \tilde{f} = \tilde{g}$. Then ρ and τ exist by Corollary 13.4.12 and Lemma 13.4.23.

Corollary 13.4.27 *Let D and E be presentations over (T_0', A) in the theory $\mathcal{BRST}(A)$. If $|D| = |E|$ then $D \overset{*}{\leftrightarrow} E$.*

Lemma 13.4.28 *Each scalar morphism t in $A(T_0')$ has a unique factorization $t = a \cdot \alpha$ where $a \in A$ and $\alpha \in T_0'$. Further, for any family of morphisms $\alpha_1, \ldots, \alpha_m : 1 \to s + p$ $(m \geq 1)$ in T_0', if*

$$\alpha_1 \cdot (\tau \oplus \mathbf{1}_p) \;=\; \ldots \;=\; \alpha_m \cdot (\tau \oplus \mathbf{1}_p),$$

for some surjective base morphism $\tau : s \to r$, then there is a morphism $\gamma : 1 \to k + p$ in T_0 and there exist base morphisms $\rho_1, \ldots, \rho_m : k \to s$ with

$$\alpha_i \;=\; \gamma \cdot (\rho_i \oplus \mathbf{1}_p), \quad i \in [m],$$

and

$$\rho_1 \cdot \tau \;=\; \ldots \;=\; \rho_m \cdot \tau.$$

By the previous lemma, Corollary 11.11.2.19 and Corollary 13.4.27 we have the following fact.

Corollary 13.4.29 *Let $D, E : 1 \to p$ be presentations over (T_0', A) with $|D| = |E|$. Then $D \overset{*}{\rightleftharpoons} E$.*

We are now ready to prove the main result of this section.

Theorem 13.4.30 *The theory $\mathcal{BRST}(A)$ is freely generated by the map $\eta_\sim : A \to \mathcal{BRST}(A)$, $a \mapsto [a]$, in the class of synchronization theories.*

Proof. As an iteration theory, $\mathcal{BRST}(A)$ is the coproduct of the initial synchronization theory T_0' and the free iteration theory on A, where each letter of A is a symbol of rank 1. This follows by Theorem 11.11.3.8 and Corollary 13.4.29.

Exercise 13.4.31 The $\sim$-equivalence classes of finite synchronization trees on a set A of action symbols form a subgrove theory $\mathcal{BFST}(A)$ of $\mathcal{BST}(A)$. Show that $\mathcal{BFST}(A)$ is the free idempotent grove theory on A.

13.5 Notes

The notion of bisimiliarity was introduced in [Par81], for finite automata. The same idea occurs in [Mil80, Theorem 5.6]. This notion has been called a "branching time semantics". It is a finer classification of the behavior of processes than the so-called linear time semantics or trace semantics. In [Mil84], Milner gave an axiomatization of bisimilarity equivalence classes of regular synchronization trees, similar to Salomaa's axiomatization of regular languages [Sal66]. Our treatment of synchronization trees is based on [BÉT].

There are many papers studying bisimulation. See [Mil89, Tau89] for a brief overview and further references.

Chapter 14

Floyd-Hoare Logic

Suppose that $f : Q \to Q$ is a partial function on a set Q, where we might think of Q as a set of "states" of a machine. If α and β are predicates on Q, i.e. total maps $Q \to \{\text{TRUE}, \text{FALSE}\}$, the *partial correctness assertion* $\{\alpha\}\ f\ \{\beta\}$, pca for short, means

$$\forall q \in Q\,[\alpha(q) \;=\; \text{TRUE} \;\wedge\; f(q) \text{ defined} \quad\Rightarrow\quad \beta(f(q)) \;=\; \text{TRUE}].$$

In his often-cited paper, Hoare [Hoa69] refined Floyd's [Flo67] methods to reason about pca's, where the partial function f was determined by a "while-program", and systems for reasoning about flowchart programs are usually called Floyd-Hoare logic. What is special about the rules of this logic? We show that partial correctness logic can be viewed as a special case of the equational logic of iteration theories. We show how to formulate a partial correctness assertion $\{\alpha\}\ f\ \{\beta\}$ as an equation between iteration theory terms. The *guards* α, β that appear in partial correctness assertions are equationally axiomatized, and a new representation theorem for Boolean algebras is derived. The familiar rules for the structured programming constructs of composition, if-then-else and while-do, are shown valid in all guarded iteration theories. A system of partial correctness logic is described which applies to all flowchart programs. The invariant guard condition is found to be both necessary and sufficient for the completeness of these rules. The role played by weakest liberal preconditions in connection with completeness is examined. The Cook completeness theorem [Coo78] follows as an easy corollary.

14.1 Guards

We will abbreviate the two base morphisms $1 \to 2$ as follows:

$$\begin{array}{rcl} \textsc{true} & := & 1_2 \;=\; \mathbf{1}_1 \oplus 0_1 \\ \textsc{false} & := & 2_2 \;=\; 0_1 \oplus \mathbf{1}_1. \end{array}$$

In the theories A, TRUE is the function

$$\begin{array}{rcl} A & \to & A \times [2] \\ a & \mapsto & (a, 1). \end{array}$$

Similarly, FALSE is the function taking a to $(a, 2)$. In A, TRUE is the first projection function $A^2 \to A$:

$$(x_1, x_2) \quad \mapsto \quad x_1,$$

and FALSE is the second.

Definition 14.1.1 *Let Γ be a set of morphisms $1 \to 2$ in any theory T (not necessarily an iteration theory). We say that Γ is a* **Boolean class** *in T if*

- *Γ contains the two base morphisms* TRUE *and* FALSE*;*
- *each $\alpha \in \Gamma$ is* **idempotent***:*
$$\alpha \cdot \langle \mathbf{1}_1, \mathbf{1}_1 \rangle \quad = \quad \mathbf{1}_1; \tag{14.1}$$
- *each $\alpha \in \Gamma$ is* **diagonal***:*
$$\alpha \cdot (\alpha \oplus \alpha) \quad = \quad \alpha \cdot (\mathbf{1}_1 \oplus 0_2 \oplus \mathbf{1}_1); \tag{14.2}$$
- *any two morphisms α, β in Γ* **commute***:*
$$\alpha \cdot (\beta \oplus \beta) \quad = \quad \beta \cdot (\alpha \oplus \alpha) \cdot (\mathbf{1}_1 \oplus \rho \oplus \mathbf{1}_1), \tag{14.3}$$
where $\rho = \langle \textsc{false}, \textsc{true} \rangle : 2 \to 2$ is the nontrivial base permutation.
- *If $\alpha, \beta, \gamma \in \Gamma$, then so is*
$$\alpha \cdot \langle \beta, \gamma \rangle.$$

Note that these properties were used in Chapter 12 to axiomatize certain varieties with distinguished predicates. See Definition 12.12.3.5.

Remark 14.1.2 If α_i, β_i, γ_i, $i = 1, 2$, are morphisms $1 \to 2$, let

$$\delta_i \ := \ \alpha_i \cdot \langle \beta_i, \gamma_i \rangle.$$

If α_i, β_i, γ_i are each idempotent and diagonal, so is δ_i; if any two commute then so do δ_1 and δ_2. Further, both TRUE and FALSE are idempotent and diagonal and both commute with any morphism $\alpha : 1 \to 2$. Hence in any theory, any set Δ of morphisms $1 \to 2$ each of which is idempotent and diagonal, and such that any two commute, generates a least Boolean class Γ containing Δ. See Theorem 12.12.3.10.

Definition 14.1.3 *A pair* (T, Γ) *consisting of an (iteration) theory* T *and a Boolean class* Γ *in* T *is a* **guarded (iteration) theory**. *The elements of* Γ *will be called* **guards.**

We will be considering triples

$$(T, \Gamma, T_0)$$

in which (T, Γ) is a guarded iteration theory and T_0 is a particular subiteration theory of T.

Remark 14.1.4 It is interesting to reformulate the meaning of the equations (14.1), (14.2), (14.3) in a subtheory of A. In particular, a morphism $\alpha : 1 \to 2$ is a function $A^2 \to A$. A function α is idempotent if for all x in A,

$$\alpha(x, x) \ = \ x;$$

α is diagonal if for all x, y, u, v in A,

$$\alpha(\alpha(x, y), \alpha(u, v)) \ = \ \alpha(x, v);$$

lastly, α and β commute if for all x, y, u, v in A,

$$\alpha(\beta(x, y), \beta(u, v)) \ = \ \beta(\alpha(x, u), \alpha(y, v)).$$

In this form, the motivation for the terminology should be clearer. For example, see the paper [Plo66] on diagonal algebras.

Remark 14.1.5 In any theory, the set consisting of just the two base morphisms TRUE and FALSE forms a Boolean class. But sometimes this class can be enlarged. Since the collection of all Boolean classes is partially ordered by set inclusion, by Zorn's lemma there are maximal Boolean classes. A maximal Boolean class Γ is characterized by the following property: any idempotent and diagonal morphism $\alpha : 1 \to 2$ which commutes with each morphism in Γ in fact belongs to Γ. Some theories have several maximal sets of guards (see Exercises 14.1.24 and 14.1.25 below).

The reason for the terminology "Boolean class" will be explained by the next theorem. First we define three operations on the morphisms $1 \to 2$ in any theory. Let α, $\beta : 1 \to 2$ be any two morphisms.

Definition 14.1.6

$$\begin{array}{rcl} \alpha \wedge \beta & := & \alpha \cdot \langle \beta, \text{ FALSE} \rangle \\ \alpha \vee \beta & := & \alpha \cdot \langle \text{TRUE}, \ \beta \rangle \\ \neg \alpha & := & \alpha \cdot \langle \text{FALSE}, \text{TRUE} \rangle. \end{array}$$

Example 14.1.7 If $\alpha(x, y)$ and $\beta(x, y)$ are morphisms $1 \to 2$ in A, then

$$\begin{array}{rcl} \neg\alpha(x, y) & = & \alpha(y, x) \\ (\alpha \wedge \beta)(x, y) & = & \alpha(\beta(x, y), y) \\ (\alpha \vee \beta)(x, y) & = & \alpha(x, \beta(x, y)) \\ \text{TRUE}(x, y) & = & x \\ \text{FALSE}(x, y) & = & y. \end{array}$$

Example 14.1.8 Suppose that $\mathbf{B} = (B, +, \cdot, \ ', 1, 0)$ is a Boolean algebra, where we use $+$, $\cdot$ and $'$ for the Boolean operations. For each element b in B, the function

$$if_b : B^2 \ \to \ B$$

is idempotent and diagonal, where

$$if_b(u, v) \ \ := \ \ (b \cdot u) + (b' \cdot v),$$

and any two such functions commute. It should be clear that these functions are closely related to the *if-then-else* operation (see [Man87]).

Theorem 14.1.9 *Let T be any algebraic theory, not necessarily an iteration theory.*

[a] *If* Γ *is a Boolean class in* T, *then* Γ *is closed under the operations* $\wedge$, $\vee$ *and* $\neg$ *just defined, and the structure*

$$(\Gamma, \vee, \wedge, \neg, \text{TRUE}, \text{FALSE})$$

is a Boolean algebra (with $\vee$ *as join,* $\wedge$ *as meet, etc.).*

[b] *Conversely, if* $\mathbf{B} = (B, +, \cdot, ', 1, 0)$ *is any Boolean algebra, let* T *be the theory* B. *Then* $\mathbf{B}$ *is isomorphic to the Boolean class of functions* if_b *in Example 14.1.8:*

$$\Gamma_B \quad := \quad \{\mathit{if}_b \,:\, b \in B\}. \tag{14.4}$$

[c] *The class* Γ_B *in (14.4) is maximal: any idempotent and diagonal morphism* $\alpha : B^2 \to B$ *which commutes with each function of the form* if_b *is itself in* Γ_B.

Proof of [a]. This is straightforward and is omitted.
Proof of [b]. To see that the map

$$\begin{array}{rcl} \varphi : B & \to & \Gamma_B \\ b & \mapsto & \mathit{if}_b \end{array}$$

is injective, we note that the value of if_b on $(1, 0)$ is b. We show that φ preserves negation. Indeed, $\varphi(b') = \mathit{if}_{b'}$, and

$$\mathit{if}_{b'}(x, y) \quad = \quad b' \cdot x + b \cdot y,$$

by definition,

$$\begin{array}{rl} = & \mathit{if}_b(y, x) \\ = & \neg \mathit{if}_b(x, y). \end{array}$$

Last, we show that φ preserves meet. We use the fact that in any Boolean algebra,

$$x + y \quad = \quad x + x' \cdot y. \tag{14.5}$$

$$\begin{array}{rcl} \mathit{if}_{b \cdot c}(x, y) & = & (b \cdot c) \cdot x + (b \cdot c)' \cdot y \\ & = & b \cdot c \cdot x + b' \cdot y + c' \cdot y \\ & = & b \cdot c \cdot x + b \cdot c' \cdot y + b' \cdot y, \end{array}$$

by (14.5),

$$\begin{aligned} &= if_b(if_c(x,y),y) \\ &= (if_b \wedge if_c)(x,y). \end{aligned}$$

The proof of [b] is complete.

Proof of [c]. Suppose that α is an idempotent and diagonal function $B^2 \to B$ which commutes with each function of the form if_b. We will establish three properties of α, namely (14.6), (14.7) and (14.8) below, which enable us to show that $\alpha = if_a$, for $a = \alpha(1,0)$.

$$\alpha(b \cdot x,\, b \cdot y) = b \cdot \alpha(x,y) \tag{14.6}$$

$$\alpha(b + x,\, b + y) = b + \alpha(x,y) \quad \text{and} \tag{14.7}$$

$$\alpha(1,0) = \alpha(0,1)', \tag{14.8}$$

for any b, x, y in B.
Proof of (14.6). Since $b \cdot x = if_b(x,b)$, for any x,

$$\begin{aligned} \alpha(b \cdot x,\, b \cdot y) &= \alpha(if_b(x,b),\, if_b(y,b)) \\ &= if_b(\alpha(x,y),\, \alpha(b,b)), \end{aligned}$$

since α commutes with if_b,

$$= if_b(\alpha(x,y),\, b),$$

since α is idempotent,

$$= b \cdot \alpha(x,y).$$

Similarly, we prove (14.7).

$$\alpha(b + x,\, b + y) = \alpha(b + b' \cdot x,\, b + b' \cdot y),$$

by (14.5),

$$\begin{aligned} &= \alpha(if_b(b,x),\, if_b(b,y)) \\ &= if_b(\alpha(b,b),\, \alpha(x,y)) \\ &= if_b(b,\, \alpha(x,y)) \\ &= b + b' \cdot \alpha(x,y) \\ &= b + \alpha(x,y), \end{aligned}$$

by (14.5) again, completing the proof of (14.7).
Proof of (14.8). We prove that $\alpha(1,0) = \alpha(0,1)'$ by showing the meet of $\alpha(1,0)$ and $\alpha(0,1)$ is 0 and their join is 1.

$$\alpha(1,0)\cdot\alpha(0,1) \quad = \quad \alpha(0,\, \alpha(1,0)),$$

by (14.6), with $x=0$, $y=1$ and $b=\alpha(1,0)$,

$$\begin{aligned} &= \alpha(\alpha(0,0),\, \alpha(1,0)) \\ &= \alpha(0,0) \;=\; 0, \end{aligned}$$

since α is diagonal and idempotent. Similarly,

$$\alpha(1,0)+\alpha(0,1) \quad = \quad \alpha(\alpha(1,0),\, 1),$$

by (14.7), with $y=1$, $b=\alpha(1,0)$, $x=0$,

$$\begin{aligned} &= \alpha(\alpha(1,0),\, \alpha(1,1)) \\ &= \alpha(1,1) \;=\; 1, \end{aligned}$$

completing the proof.

We will show that

$$\alpha(x,y) \quad = \quad x\cdot\alpha(1,0) + y\cdot\alpha(1,0)',$$

showing that $\alpha = \mathit{if}_a$, where $a := \alpha(1,0)$.

Indeed,

$$\alpha(x,y) \quad = \quad \alpha(\alpha(x, x+y),\, \alpha(0,y)),$$

since α is diagonal,

$$\begin{aligned} &= \alpha(x+\alpha(0,y),\, \alpha(0,y)) \\ &= \alpha(x,0)+\alpha(0,y), \end{aligned}$$

using (14.7) twice. Also,

$$\begin{aligned} \alpha(x,0)+\alpha(0,y) &= x\cdot\alpha(1,0)+y\cdot\alpha(0,1) \\ &= x\cdot\alpha(1,0)+y\cdot\alpha(1,0)', \end{aligned}$$

by (14.6) and (14.8), completing the proof.

Definition 14.1.10 *A* **guard algebra** *is a Boolean class* Γ *in a theory* A *of functions.*

We call the next result the *Cayley Representation Theorem* for Boolean algebras.

Corollary 14.1.11 *Every Boolean algebra is isomorphic to a guard algebra.*

Proof. This is a restatement of part [b] of the previous theorem.

The Stone representation theorem for Boolean algebras is that every Boolean algebra is isomorphic to a Boolean algebra of subsets of some set. The next few exercises show that this result is equivalent to the Cayley Representation Theorem.

Exercise 14.1.12 Show that for every Boolean algebra **B** of subsets of the set A, there is a Boolean class $\Gamma(B)$ of guards in A isomorphic to **B**. (*Hint:* For $X \subseteq A$, let $p_X : 1 \to 2$ be the partial function

$$ap_X \quad := \quad \begin{cases} (a,1) & \text{if } a \in X; \\ (a,2) & \text{otherwise.} \end{cases}$$

Consider the family of morphisms $p_X : X \in B$.)

Exercise 14.1.13 Let Γ be a Boolean class of guards in the theory T. Use the representation theorem for theories, Theorem 2.3.2.2, to show that Γ is isomorphic to a Boolean class of guards in a theory A, for some A.

Exercise 14.1.14 Use the two previous exercises to show that the Stone representation theorem implies the Cayley Representation Theorem.

Exercise 14.1.15 Show that if Γ is a Boolean class in A, and $\alpha \in \Gamma$, then the following relation is a congruence on the algebra (A, Γ):

$$a \ \sim_\alpha \ b \quad \Leftrightarrow \quad \forall c \in A[\alpha(a,c) \ = \ \alpha(b,c)].$$

Show that

$$a \ \sim_\alpha \ b \quad \Leftrightarrow \quad \exists c[\alpha(a,c) \ = \ \alpha(b,c)].$$

Exercise 14.1.16 Show that the following is a congruence on the Boolean algebra Γ:

$$\beta \ \approx_\alpha \ \gamma \quad \Leftrightarrow \quad \forall a,b[\beta(a,b) \ \sim_\alpha \ \gamma(a,b)].$$

Exercise 14.1.17 Show that if A has at least two elements, then there are exactly two congruences on (A,Γ) iff Γ is the two element Boolean algebra. Conclude that the only subdirectly irreducible guard algebra is the two element Boolean algebra.

Exercise 14.1.18 Use the previous exercise to show that the Cayley Representation Theorem implies the Stone representation theorem.

We now introduce one of the fundamental definitions: that of n-guards.

Definition 14.1.19 *Suppose that (T,Γ) is a guarded theory. The set of n-**guards** Γ_n is the set of all morphisms $n \to 2n$ in T which can be written in the form*

$$[\alpha_1, l, \alpha_n] \quad := \quad \langle \alpha_1 \cdot \rho_1, l, \alpha_n \cdot \rho_n \rangle,$$

where each α_i is in Γ and where

$$\rho_i \quad := \quad \langle i_{2n}, \, (n+i)_{2n} \rangle : 2 \to 2n$$

is the base morphism corresponding the function

$$\begin{array}{rcl} 1 & \mapsto & i \\ 2 & \mapsto & n+i. \end{array}$$

In A, an n-guard is a total function $\alpha : A \times [n] \to A \times [2n]$ such that $(a,i)\alpha = (a,i)$ or $(a,\, n+i)$, for each $(a,i) \in A \times [n]$. In A, if $\pi_i : A^2 \to A$ are guards, the function

$$[\pi_1, l, \pi_n] : A^{2n} \ \to \ A^n$$

is the following:

$$(a_1, a_2, l, a_n, b_1, lb_n) \ \mapsto \ (\pi_1(a_1,b_1), l, \pi_n(a_n,b_n)).$$

We define Boolean operations on Γ_n as follows. For $\alpha = [\alpha_1, l, \alpha_n]$, and $\beta = [\beta_1, l, \beta_n]$ in Γ_n,

$$\begin{aligned}
\text{TRUE}_n &:= [\text{TRUE}, l, \text{TRUE}] \\
\text{FALSE}_n &:= [\text{FALSE}, l, \text{FALSE}] \\
\alpha \wedge \beta &:= [\alpha_1 \wedge \beta_1, l, \alpha_n \wedge \beta_n] \\
\alpha \vee \beta &:= [\alpha_1 \vee \beta_1, l, \alpha_n \vee \beta_n] \\
\neg\alpha &:= [\neg\alpha_1, l, \neg\alpha_n].
\end{aligned}$$

The following equations are consequences of these definitions.

$$\begin{aligned}
\text{TRUE}_n &= \mathbf{1}_n \oplus 0_n \\
\text{FALSE}_n &= 0_n \oplus \mathbf{1}_n \\
\alpha \wedge \beta &= \alpha \cdot \langle \beta,\ \text{FALSE}_n \rangle \\
\alpha \vee \beta &= \alpha \cdot \langle \text{TRUE}_n,\ \beta \rangle \\
\neg\alpha &= \alpha \cdot \langle \text{FALSE}_n, \text{TRUE}_n \rangle.
\end{aligned}$$

It follows that Γ_1 is Γ and for $n \geq 2$, Γ_n equipped with the operations $\wedge, \vee, \neg$ defined above is also a Boolean algebra isomorphic to the direct power $\Gamma \times l \times \Gamma$ (n times). The set Γ_0 consists of the identity $0 \to 0$ only. The n-guards satisfy analogues of the guard axioms (14.1), (14.2) and (14.3) above, and we will refer to the three equations (14.9), (14.10) and (14.11) as the *Boolean axioms.*

Proposition 14.1.20 *For any $\alpha, \beta, \gamma \in \Gamma_n$, the morphism $\alpha \cdot \langle \beta, \gamma \rangle$ is also in Γ_n and*

$$\alpha \cdot \langle \mathbf{1}_n, \mathbf{1}_n \rangle = \mathbf{1}_n \tag{14.9}$$

$$\alpha \cdot (\alpha \oplus \alpha) = \alpha \cdot (\mathbf{1}_n \oplus 0_{2n} \oplus \mathbf{1}_n) \tag{14.10}$$

$$\alpha \cdot (\beta \oplus \beta) = \beta \cdot (\alpha \oplus \alpha) \cdot (\mathbf{1}_n \oplus \rho_n \oplus \mathbf{1}_n), \tag{14.11}$$

where $\rho_n = \langle \text{FALSE}_n, \text{TRUE}_n \rangle$.

Remark 14.1.21 An easy corollary of the Boolean axioms is that if α is in Γ_n for any $n \geq 1$, then α has the following *deletion properties:*

$$\begin{aligned}
\alpha \cdot \langle \alpha \cdot \langle u, v \rangle,\ w \rangle &= \alpha \cdot \langle u, w \rangle \\
\alpha \cdot \langle u,\ \alpha \cdot \langle v, w \rangle \rangle &= \alpha \cdot \langle u, w \rangle,
\end{aligned}$$

for all $u, v, w : n \to p$.

Exercise 14.1.22 Prove that for any nonempty set Δ of morphisms $1 \to 2$ each of which satisfy (14.1), (14.2) and (14.3), the least class of morphisms containing Δ, which is closed under the Boolean operations $\wedge, \vee$ and $\neg$ is also the least class of morphisms containing Δ, TRUE and FALSE which is closed under composition in the form $\alpha, \beta, \gamma \mapsto \alpha \cdot \langle \beta, \gamma \rangle$. See Section 12.12.3.

It is important to be able to "pair" the guards.

Definition 14.1.23 *For* $\alpha = [\alpha_1, l, \alpha_n]$ *in* Γ_n *and* $\beta = [\beta_1, l, \beta_p]$ *in* Γ_p, *we write*

$$[\alpha, \beta]$$

for

$$[\alpha_1, l, \alpha_n, \beta_1, l\beta_p]$$

in Γ_{n+p}.

Observe that

$$\begin{aligned} [\alpha, \beta] &= \langle \alpha \cdot (\mathbf{1}_n \oplus 0_p \oplus \mathbf{1}_n \oplus 0_p),\ \beta \cdot (0_n \oplus \mathbf{1}_p \oplus 0_n \oplus \mathbf{1}_p) \rangle \\ &= (\alpha \oplus \beta) \cdot (\mathbf{1}_n \oplus \tau \oplus \mathbf{1}_p), \end{aligned}$$

where $\tau : n + p \to p + n$ is the block transposition $\langle 0_p \oplus \mathbf{1}_n,\ \mathbf{1}_p \oplus 0_n \rangle$.

Exercise 14.1.24 [Blo90] This exercise provides examples of theories having several maximal Boolean classes, all of which are isomorphic as Boolean algebras.

Let V be a vector space and let S be the ring of linear transformations $V \to V$; if $u, v \in S$ and $x \in V$,

$$\begin{aligned} x(u \cdot v) &:= (xu)v \\ x(u + v) &:= xu + xv \\ x0 &:= 0 \\ x\mathbf{1} &:= x. \end{aligned}$$

Consider the matrix theory $T = \mathbf{Mat}_S$.

1. Show that if $\alpha = [u\ v]$ is a guard in T, then

$$\begin{aligned} u \cdot u &= u \\ u \cdot v &= v \cdot u = 0 \\ v \cdot v &= v. \end{aligned}$$

2. Let B be a basis for the vector space V. For each subset X of B, let $p_X : V \to V$ be the projection onto the subspace spanned by X. Let $\alpha_X := [p_X\ p_Y]$, where $Y := B \setminus X$. Show that the collection of morphisms $p_X,\ X \subseteq B$ forms a Boolean class in T, isomorphic to the powerset Boolean algebra of subsets of B.

3. Show that if V is finite dimensional, then any maximal Boolean family Γ is determined by some basis of V, as in the previous problem. Conclude that any two maximal Boolean classes are isomorphic as Boolean algebras.

Exercise 14.1.25 Find an example of a theory T having at least two nonisomorphic maximal Boolean classes.

Exercise 14.1.26 Suppose that K is a familiar equational class, say the class of all monoids, or all groups, or all lattices. Let T be the theory determined by K, so that $T(1, n)$ is the free K-algebra, freely generated by n-elements. Show that T contains only one maximal Boolean class, the two element Boolean algebra.

Exercise 14.1.27 For each $n \geq 1$, find an equational class K_n of Σ-algebras such that $T(1, 2)$ itself is a Boolean algebra with 2^n elements. *Hint:* let $\Sigma_n = \{f\}$, $\Sigma_k = \emptyset$ if $k \neq n$. Define K_n as the class of all models of the two equations

$$\begin{aligned} f(x, \ldots, x) &= x \\ f(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})) &= f(x_{11}, x_{22}, \ldots, x_{nn}). \end{aligned}$$

14.2 Partial Correctness Assertions

To motivate Definition 14.2.1 below, consider the theory A of partial functions. An idempotent morphism $1 \to 2$ in this theory is a total function $\alpha : A = A \times [1] \to A \times [2]$ such that if $a\alpha = (b, j)$ for some $j \in [2]$, then $b = a$. We usually identify such a function α with the subset of those a in A such that "$\alpha(a)$ holds", or "α is true of a", meaning $a\alpha = (a, 1)$. The collection Γ of all idempotent morphisms is the unique maximal Boolean class in A. Suppose that α and β are guards in A. When $f : 1 \to 1$, we can express the condition

$$\textit{if } \alpha(a) \textit{ holds and if } f(a) = b, \textit{ then } \beta(b) \textit{ holds}$$

by the following equation:

$$\alpha \cdot (f \oplus \mathbf{1}_1) \cdot (\beta \oplus \mathbf{1}_1) \quad = \quad \alpha \cdot (f \oplus \mathbf{1}_1) \cdot (\textsc{true} \oplus \mathbf{1}_1). \qquad (14.12)$$

Indeed, if α is true of a, and $af = (b, 1)$,

$$a\alpha \cdot (f \oplus \mathbf{1}_1) \cdot (\beta \oplus \mathbf{1}_1) \quad = \quad b\beta.$$

Also,

$$\begin{aligned} a\alpha \cdot (f \oplus \mathbf{1}_1) \cdot (\textsc{true} \oplus \mathbf{1}_1) &= b\textsc{true} \\ &= (b, 1). \end{aligned}$$

Hence, in this case equation (14.12) holds iff $b\beta = (b, 1)$. Similarly, if $a\alpha = (a, 2)$, (14.12) holds since the value of both sides of the equation is $(a, 3)$.

We now give the general definition of a partial correctness assertion, which applies to a morphism $n \to p$, any $n, p \geq 0$.

Definition 14.2.1 *Suppose that (T, Γ) is a guarded theory. For α in Γ_n, β in Γ_p and $f : n \to p$ in T, the* **partial correctness assertion**

$$\alpha f \beta$$

is the assertion that in the theory T,

$$\alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \oplus \mathbf{1}_n) \quad = \quad \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\textsc{true}_p \oplus \mathbf{1}_n). \qquad (14.13)$$

We may express equation (14.13) also by the equation

$$\alpha \cdot (f\beta \ \oplus \mathbf{1}_n) \quad = \quad \alpha \cdot (f \oplus 0_p \oplus \mathbf{1}_n). \qquad (14.14)$$

(Sometimes we omit the symbol $\cdot$ for composition, as with $f\beta$ just above.)

Every Boolean algebra is partially ordered by the relation

$$\alpha \ \leq \ \beta \quad \Leftrightarrow \quad \alpha \vee \beta \ = \ \beta.$$

In the Boolean algebra Γ_n, this ordering is definable by the following equation:

$$\alpha \leq \beta \ \Leftrightarrow \ [\alpha \cdot (\beta \oplus \mathbf{1}_n) = \alpha \cdot (\textsc{true}_n \oplus \mathbf{1}_n)].$$

We will frequently require that in the equations $\alpha f \beta$, the morphism f must belong to a certain subtheory T_0 of T. Thus, the object of interest will be the triple (T, Γ, T_0), as in the *standard example* below in Section 14.3.

14.2.1 An Alternative Definition of n-Guards

There is another possibility for defining the n-guards in a guarded theory (T, Γ). Let $\Delta_1 = \Gamma = \Gamma_1$, and for $n \neq 1$, let Δ_n be the collection of all morphisms of the form $[\![\alpha_1, l, \alpha_n]\!]$, where

$$[\![\alpha_1, l, \alpha_n]\!] \quad := \quad \langle \alpha_1 \cdot \tau_1, l, \alpha_n \cdot \tau_n \rangle : n \to n+1,$$

and where $\tau_i : 2 \to n+1$ is the base morphism corresponding to the function

$$\begin{array}{rcl} 1 & \mapsto & i \\ 2 & \mapsto & n+1. \end{array}$$

A Boolean algebra structure on Δ_n is determined as follows: for the guards $\alpha = [\![\alpha_1, l, \alpha_n]\!]$ and $\beta = [\![\beta_1, l, \beta_n]\!]$, define:

$$\begin{array}{rcl} \alpha \wedge \beta & := & [\![\alpha_1 \wedge \beta_1, l, \alpha_n \wedge \beta_n]\!] \\ \alpha \vee \beta & := & [\![\alpha_1 \vee \beta_1, l, \alpha_n \vee \beta_n]\!] \\ \neg\alpha & := & [\![\neg\alpha_1, l, \neg\alpha_n]\!] \\ True_n & := & [\![\text{TRUE}, l, \text{TRUE}]\!] \; = \; \mathbf{1}_n \oplus 0_1 \\ False_n & := & [\![\text{FALSE}, l, \text{FALSE}]\!] \; = \; 0_n \oplus \mathbf{1}_1. \end{array}$$

Clearly, the Boolean algebras Δ_n and Γ_n are isomorphic.

Definition 14.2.2 *For $f : n \to p$, α in Δ_n, β in Δ_p, we define the partial correctness assertion*

$$\{\{\alpha\}\} \; f \; \{\{\beta\}\}$$

as an abbreviation for the equation

$$\alpha \cdot \langle f \cdot \beta, \, 0_p \oplus \mathbf{1}_1 \rangle \; = \; \alpha \cdot \langle f \oplus 0_1, \, 0_p \oplus \mathbf{1}_1 \rangle.$$

The difference between the two possibilities is mostly aesthetic, in view of the following theorem (whose proof we omit).

Theorem 14.2.3 *In any guarded theory,*

$$[\alpha_1, l, \alpha_n] f [\beta_1, l, \beta_p] \quad \Leftrightarrow \quad \{\{[\![\alpha_1, l, \alpha_n]\!]\}\} \; f \; \{\{[\![\beta_1, l, \beta_p]\!]\}\}.$$

14.3 The Standard Example

Let $\mathbf{B} = (B, l)$ be a first order structure of finitary type, so that $\mathbf{B}$ may have both finitary relations and functions. We will consider a subiteration theory T_0 of the theory A, where A is the set B^ω, the set of functions $a : [\omega] \to B$. We write the value of $a \in A$ on $i \in [\omega]$ as a_i. If $\alpha(x_1, l, x_n)$ is a first order formula with free variables among $\{x_1, l, x_n\}$, then we identify α with the function

$$\begin{aligned} g_\alpha : A &\rightarrow A \times [2] \\ a &\mapsto \begin{cases} (a,1) & \text{if } \alpha(a_1, l, a_n) \text{ is true;} \\ (a,2) & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, if $\tau(x_1, l, x_n)$ is a term containing at most the variables x_1, l, x_n, we let the expression

$$x_i \;:=\; \tau(x_1, l, x_n)$$

denote the function $f_{i,\tau} : A \to A$ where the components of $f_{i,\tau}(a)$ are determined as follows:

$$f_{i,\tau}(a)_k \;=\; \begin{cases} \tau(a_1, l, a_n) & \text{if } k = i; \\ a_k & \text{otherwise.} \end{cases}$$

Let Γ denote the collection of all functions g_α determined by first order formulas. Let T_0 be the least subiteration theory of A which contains the functions $f_{i,\tau}$ and those functions g_α *where α is a quantifier-free formula*. In fact, T_0 contains all functions computable by some flowchart algorithm whose atomic operations and tests are specified in the structure $\mathbf{B}$. It is clear that Γ is a Boolean class in A. We say (A, Γ, T_0) is the triple *corresponding* to the structure $\mathbf{B}$. If $f : 1 \to 1$ is any morphism in this theory, and α and β are any formulas, then the equation

$$g_\alpha f g_\beta$$

holds iff whenever α is true of a and $f(a)$ is defined, then β is true of $f(a)$.

Note that the guards are not necessarily morphisms in T_0.

14.4 Rules for Partial Correctness

Now we will give some rules for deriving new partial correctness assertions from old ones. The rules generalize the standard Hoare logic rules for while-programs. A rule has the form: from X infer y, in symbols $X \vdash y$, where y is some partial correctness expression and X is a finite set of zero or more partial correctness expressions or inequalities between guard expressions. There is a rule for each of the fundamental iteration theory operations: composition, tupling and iteration. Since all flowchart algorithms can be constructed from atomic actions using the operations of composition, tupling and iteration (proved in [Elg75]; see also [Elg76b]), these rules will enable us to give a calculus for proving the correctness of any flowchart algorithm.

Before giving our rules, we need to discuss the precise meaning of the terms involved in an equation.

14.4.1 Formalization

In order to express our rules in a formal system, we introduce a system of "iteration terms" used to name morphisms and guards in a given (pre)iteration theory. Let Σ be a ranked set and Π be any set, not necessarily disjoint from Σ_2. Let Σ_Π denote the ranked set with

$$\begin{array}{rcl}(\Sigma_\Pi)_2 & := & \Sigma_2 \cup \Pi \\ (\Sigma_\Pi)_n & := & \Sigma_n,\ n \neq 2.\end{array}$$

The collection of Σ_Π-iteration terms is built in the standard way from the above letters using the function symbols for composition, tupling and iteration. The *constants* consist of the letters i_n, for $i \in [n]$, each $n \geq 1$, and the letters 0_n, for $n \geq 0$. Thus, for example, if f is a Σ_Π-iteration term $n \to n + p$, then $f^\dagger$ is a Σ_Π-iteration term $n \to p$; if f_i are Σ_Π-iteration terms $1 \to p$, for $i \in [n]$, then

$$\langle f_1, l, f_n \rangle$$

is a Σ_Π-iteration term $n \to p$. The functions of separated sum and pairing are to be thought of as the appropriate abbreviations, as are the expressions $\textsc{true}_n$ and $\textsc{false}_n$.

Remark 14.4.1 Note the difference between Σ_Π-iteration terms and the Σ-terms defined in Chapter 2. The Σ_Π-iteration terms here are used to denote morphisms in (pre)iteration theories.

We let $\Sigma_\Pi \mathbf{TM}$ denote the collection of all Σ_Π-iteration terms. $\Sigma_\Pi \mathbf{TM}$ becomes an $\mathbf{N} \times \mathbf{N}$-sorted algebra equipped with the constants i_n and 0_n, $n \geq 0$, and the operations of composition, tupling and iteration. In fact, $\Sigma_\Pi \mathbf{TM}$ is a *free* $\mathbf{N} \times \mathbf{N}$-sorted algebra, equipped with the above operations and constants, freely generated by the ranked set Σ_Π. A Σ-iteration term is a term having no occurrences of letters in Π. We let $\Sigma\,\mathbf{TM}$ denote the subalgebra of $\Sigma_\Pi \mathbf{TM}$ consisting of the Σ-iteration terms. A Σ_Π-iteration term is called *finite* if it contains no occurrence of the operation †.

The class of *guard expressions* is defined inductively. Each letter in Π is a guard expression, as are the two constants 1_2 and 2_2; if γ and γ' are guard expressions and $\alpha \in \Pi$ then $\alpha \cdot \langle \gamma, \gamma' \rangle$ is a guard expression. Nothing else is a guard expression.

A 0-*guard expression* is the iteration term 0_0; a 1-*guard expression* is just a guard expression; and, for $n \geq 2$, an n-*guard expression* is a iteration term of the form

$$[\alpha_1, l, \alpha_n] \quad := \quad \langle \alpha_1 \cdot \rho_1, l, \alpha_n \cdot \rho_n \rangle,$$

where for each $i \in [n]$, α_i is a guard expression and ρ_i is the base iteration term

$$\langle i_{2n},\, (n+i)_{2n} \rangle : 2 \quad \rightarrow \quad 2n.$$

Definition 14.4.2 *A* **partial correctness expression** *is an expression of the form*

$$\alpha f \beta,$$

where f is a Σ-iteration term $n \rightarrow p$, α is an n-guard expressions and β is a p-guard expression. A **guard inequality** *is an expression $\pi \leq \pi'$, where π and π' are n-guard expressions, for some $n \geq 0$.*

Note that we are using exactly the same notation

$$\{\alpha\}\ f\ \{\beta\}$$

to denote both a partial correctness **expression** and a partial correctness **assertion.** The difference between an expression and an assertion is that an expression is a string on a certain alphabet containing the letters in Σ_Π as well as $^\dagger, \}, \{, \rangle, \langle, \cdot$, etc.; an assertion is a statement that two morphisms in some guarded theory are identical.

A Σ_Π-iteration term can be interpreted in many ways: as a flowchart scheme, as a partial function, as a tree, or more generally, as a morphism in some iteration theory. The interpretation of an iteration term is determined by giving a meaning to each atomic letter; the operation symbols are interpreted by the operations in the theory. This process is made precise by means of homomorphisms.

Let T be an iteration theory. A *homomorphism*

$$\varphi : \Sigma_\Pi \mathbf{TM} \quad \to \quad T$$

is a homomorphism of $\mathbf{N} \times \mathbf{N}$-sorted algebras. Thus φ is a function which assigns to each iteration term $t : n \to p$ in $\Sigma_\Pi \mathbf{TM}$ a morphism $t\varphi : n \to p$ in T such that φ preserves the constants and the operations of composition, tupling and iteration. A homomorphism is an *admissible homomorphism* if the collection of morphisms

$$\{\pi\varphi : \pi \text{ is a guard expression}\}$$

is a Boolean class in T, i.e. the morphisms $\pi\varphi$ satisfy the equations (14.1), (14.2) and (14.3) in Section 14.1.

Definition 14.4.3 *Let (T, Γ, T_0) be a triple consisting of an iteration theory T, a Boolean class of guards Γ in T and a subiteration theory T_0 of T. A homomorphism $\varphi : \Sigma_\Pi \mathbf{TM} \to T$ is called* **appropriate for the triple** (T, Γ, T_0) *if Γ is the φ-image of the guard expressions and T_0 is the φ-image of $\Sigma\,\mathbf{TM}$.*

Maybe a better term than *appropriate* is *appropriately surjective.* An appropriate homomorphism provides names for each guard in Γ and each T_0-morphism. An admissible homomorphism φ is appropriate for the triple (T, Γ, T_0), where Γ is the φ-image of the guard expressions, and where T_0 is the φ-image of the iteration terms in $\Sigma\,\mathbf{TM}$. Conversely, if $\varphi : \Sigma_\Pi \mathbf{TM} \to T$ is appropriate for (T, Γ, T_0), φ is clearly admissible.

Remark 14.4.4 If $\psi : T \to T'$ is an iteration theory morphism and $\varphi : \Sigma_\Pi \mathbf{TM} \to T$ is an admissible homomorphism, then $\varphi \cdot \psi : \Sigma_\Pi \mathbf{TM} \to T'$ is also admissible. Indeed, if Γ is a Boolean class in T, then $\Gamma\psi$ is a Boolean class in T', because theory morphisms preserve equations.

Definition 14.4.5 *Let T be an iteration theory. Suppose that*

$$\varphi : \Sigma_\Pi \mathbf{TM} \ \to \ T$$

is an admissible homomorphism. We let $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$ denote the collection of all partial correctness expressions $\{\alpha\}\ f\ \{\beta\}$ such that

$$\alpha\varphi f\varphi\beta\varphi$$

is true in T; i.e. such that the equation

$$\alpha\varphi \cdot (f\varphi \oplus 0_p \oplus \mathbf{1}_n) \quad = \quad \alpha\varphi \cdot (f\varphi \cdot \beta\varphi \oplus \mathbf{1}_n)$$

holds in T. We will sometimes write

$$(T, \varphi, \Sigma_\Pi) \ \models \ \{\alpha\}\ f\ \{\beta\}$$

instead of $\{\alpha\}\ f\ \{\beta\} \ \in \ \mathbf{Val}(T, \varphi, \Sigma_\Pi)$.

Proposition 14.4.6 *If $\psi : T \to T'$ is an iteration theory morphism,*

$$\mathbf{Val}(T, \varphi, \Sigma_\Pi) \quad \subseteq \quad \mathbf{Val}(T', \, \varphi \cdot \psi, \, \Sigma_\Pi).$$

Proof. By the previous remark, $\varphi \cdot \psi$ is admissible if φ is. Also, if

$$(T, \varphi, \Sigma_\Pi) \ \models \ \{\alpha\}\ f\ \{\beta\},$$

then

$$(T', \, \varphi \cdot \psi, \, \Sigma_\Pi) \ \models \ \{\alpha\}\ f\ \{\beta\},$$

since ψ preserves equations.

Let T be an iteration theory. For any Σ_Π as above, and any set X of partial correctness expressions and guard inequalities, we write

$$X \quad \models_T \quad \{\alpha\}\ f\ \{\beta\}$$

if for *any* admissible homomorphism $\varphi : \Sigma_\Pi \mathbf{TM} \to T$, $\{\alpha\}\ f\ \{\beta\}$ is in $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$ whenever each partial correctness expression in X is in $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$ and $\pi\varphi \leq \pi'\varphi$ holds in T, for each guard inequality $\pi \leq \pi'$ in X.

We write

$$X \quad \models \quad \{\alpha\}\ f\ \{\beta\}$$

and say $X \models \{\alpha\}\ f\ \{\beta\}$ is *valid*, if $X \models_T \{\alpha\}\ f\ \{\beta\}$ for all iteration theories T.

Similarly, we write

$$X \quad \models_{(T,\Gamma,T_0)} \quad \{\alpha\}\ f\ \{\beta\}$$

if for any appropriate homomorphism φ, $\{\alpha\}\ f\ \{\beta\}$ is in $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$ whenever each partial correctness expression in X is in $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$ and $\pi\varphi \leq \pi'\varphi$ holds in T, for each guard inequality $\pi \leq \pi'$ in X.

14.4.2 Formal Rules

We now consider four formal rules to derive new partial correctness expressions from old. Each rule has the form

$$X \quad \vdash \quad \{\alpha\}\ f\ \{\beta\},$$

where X is a finite set of partial correctness expressions or inequalities between guard expressions on some fixed ranked alphabet Σ_Π as above.

1. The **composition rule**.

$$\alpha f \beta,\ \beta g \gamma \quad \vdash \quad \alpha f \cdot g \gamma,$$

for all n-guard expressions α, all p-guard expressions β, and all Σ-iteration terms $f : n \to p$, $g : p \to q$.

2. The **tupling rule**.

$$\begin{aligned}&\alpha_1 f_1 \beta, l, \alpha_n f_n \beta \\ &\quad \vdash \quad [\alpha_1, l, \alpha_n]\langle f_1, l, f_n\rangle \beta,\end{aligned}$$

for any Σ-iteration terms $f_i : 1 \to p$, and guard expressions α_i, for $i \in [n]$, and p-guard expression β.

3. The **iteration rule**.

$$\alpha' \ \leq \ \alpha,\ \alpha f[\alpha, \beta] \quad \vdash \quad \alpha' f^\dagger \beta,$$

where α, α' are n-guard expressions, $f : n \to n + p$ is in $\Sigma\,\mathbf{TM}$ and β is a p-guard expression.

4. **Rule for distinguished morphisms**.

$$\alpha \ \leq \ \beta_i \ \ \vdash \ \ \alpha i_n[\beta_1, l, \beta_n],$$

for any $i \in [n], \ n \geq 0$, and any guard expressions $\alpha, \beta_j, \ j \in [n]$.

Note that when $n = 0$, the tupling rule becomes a rule for the morphisms 0_p:

$$0_0 0_p \beta,$$

for any p-guard expression β.

In the next section, we will prove that each of the above rules $X \vdash \{\alpha\} \ f \ \{\beta\}$ is *sound*, i.e.

$$X \ \ \models \ \ \{\alpha\} \ f \ \{\beta\}.$$

14.5 Soundness

Theorem 14.5.1 *The iteration rule is valid in every iteration theory and the other three rules are valid in any theory.*

Proof. In the proof of the soundness of these rules, we do not mention any admissible homomorphism $\varphi : \Sigma_\Pi \mathbf{TM} \to T$, and simply identify a Σ-iteration term $f : n \to p$ with its image $f\varphi : n \to p$ in T, and an n-guard expression π with the n-guard $\pi\varphi$ in T. We let Γ_n denote the set of n-guards in T determined by the guards $\pi\varphi$, for $\pi \in \Pi$. In the arguments below, we will make use of the theory identities in Section 2.3.1 and the dagger identities in Section 5.5.3.

Proof of the composition rule. Assume that $f : n \to p, \ g : p \to q$ are any morphisms in T and that $\alpha \in \Gamma_n, \ \beta \in \Gamma_p$ and $\gamma \in \Gamma_q$ are guards. Further, we assume that

$$\alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\text{TRUE}_p \oplus \mathbf{1}_n) \ = \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \oplus \mathbf{1}_n) \quad (14.15)$$

$$\beta \cdot (g \oplus \mathbf{1}_p) \cdot (\text{TRUE}_q \oplus \mathbf{1}_p) \ = \ \beta \cdot (g \oplus \mathbf{1}_p) \cdot (\gamma \oplus \mathbf{1}_p). \quad (14.16)$$

We must show that

$$\alpha \cdot ((f \cdot g \cdot \gamma) \oplus \mathbf{1}_n) \ = \ \alpha \cdot ((f \cdot g) \oplus 0_q \oplus \mathbf{1}_n).$$

$$
\begin{aligned}
\alpha \cdot (f \cdot g \cdot \gamma \oplus \mathbf{1}_n) \ =& \\
=& \ \ \alpha \cdot (f \oplus 0_p \oplus \mathbf{1}_n) \cdot (\langle g \cdot \gamma,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\text{TRUE}_p \oplus \mathbf{1}_n) \cdot (\langle g \cdot \gamma,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \oplus \mathbf{1}_n) \cdot (\langle g \cdot \gamma,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \cdot (g \cdot \gamma \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{2q},\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \cdot (g \oplus \mathbf{1}_p) \cdot (\gamma \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{2q},\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \cdot (g \oplus \mathbf{1}_p) \cdot (\text{TRUE}_q \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{2q},\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \cdot (g \oplus 0_q \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{2q},\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \cdot \langle g,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\beta \oplus \mathbf{1}_n) \cdot (\langle g,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus \mathbf{1}_n) \cdot (\text{TRUE}_p \oplus \mathbf{1}_n) \cdot (\langle g,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \oplus 0_p \oplus \mathbf{1}_n) \cdot (\langle g,\ g \cdot \gamma \rangle \oplus \mathbf{1}_n) \\
=& \ \ \alpha \cdot (f \cdot g \oplus 0_q \oplus \mathbf{1}_n).
\end{aligned}
$$

The composition rule is proved. It is an interesting fact that in this argument, no particular properties of the guards were used. The validity of this rule is a consequence only of the equational logic of algebraic theories.

Proof of the tupling rule. The statement is obviously true if $n = 0$, so we assume that n is positive. For each $i \in [n]$, we have

$$\alpha_i \cdot ((f_i\beta) \oplus \mathbf{1}_1) \quad = \quad \alpha_i \cdot (f_i \oplus 0_p \oplus \mathbf{1}_1).$$

We must prove that

$$
\begin{aligned}
[\alpha_1, l, \alpha_n] \cdot (\langle f_1, l, f_n \rangle \beta \ \oplus \mathbf{1}_n) \ =& \\
=& \ \ [\alpha_1, l, \alpha_n] \cdot (\langle f_1, l, f_n \rangle \oplus 0_p \oplus \mathbf{1}_n).
\end{aligned}
\tag{14.17}
$$

We compose i_n with the left side (LHS) of equation (14.17). The result is

$$i_n \cdot \text{LHS} \quad = \quad \alpha_i \cdot \rho_i \cdot (\langle f_1, l, f_n \rangle \beta \ \oplus \mathbf{1}_n),$$

where $\rho_i = \langle i_{2n},\ (n+i)_{2n} \rangle$. By applying some of the theory identities in Section 2.3.1, we see

$$
\begin{aligned}
i_n \cdot \text{LHS} \quad &= \quad \alpha_i \cdot \langle (f_i\beta) \oplus 0_n,\ 0_{2p} \oplus i_n \rangle \\
&= \quad \alpha_i \cdot ((f_i\beta) \ \oplus \mathbf{1}_1) \cdot (\mathbf{1}_{2p} \oplus i_n) \\
&= \quad \alpha_i \cdot (f_i \oplus 0_p \oplus \mathbf{1}_1) \cdot (\mathbf{1}_{2p} \oplus i_n),
\end{aligned}
$$

by the assumption,

$$
\begin{aligned}
&= \alpha_i \cdot \rho_i \cdot (\langle f_1, l, f_n\rangle \oplus 0_p \oplus \mathbf{1}_n) \\
&= i_n \cdot \mathrm{RHS},
\end{aligned}
$$

where RHS is the right side of (14.17). Thus, the right and left sides of (14.17) are equal, completing the proof. Again, no special properties of guards were used.

Proof of the iteration rule. This argument does use the guard properties. We will first prove the validity of the iteration rule in the case that $\alpha = \alpha'$. The general rule will follow easily. The assumptions for this case are that α is an n-guard, β is a p-guard, $f : n \to n + p$ is a morphism in T and that

$$
\alpha \cdot (f \oplus 0_{n+p} \oplus \mathbf{1}_n) \;=\; \alpha \cdot ((f[\alpha, \beta]) \;\oplus \mathbf{1}_n). \tag{14.18}
$$

We must prove

$$
\alpha \cdot (f^\dagger \oplus 0_p \oplus \mathbf{1}_n) \;=\; \alpha \cdot ((f^\dagger \beta) \;\oplus \mathbf{1}_n).
$$

Of course we use the iteration theory identities in Section 5.5.3. First observe that

$$
\begin{aligned}
&\alpha \cdot (f \cdot (\alpha \oplus \beta) \oplus \mathbf{1}_n) \;= \\
&\quad= \alpha \cdot (f \cdot [\alpha, \beta] \oplus \mathbf{1}_n) \cdot (\mathbf{1}_n \oplus \langle 0_n \oplus \mathbf{1}_p,\, \mathbf{1}_n \oplus 0_p\rangle \oplus \mathbf{1}_{p+n}) \\
&\quad= \alpha \cdot (f \oplus 0_{n+p} \oplus \mathbf{1}_n) \cdot (\mathbf{1}_n \oplus \langle 0_n \oplus \mathbf{1}_p,\, \mathbf{1}_n \oplus 0_p\rangle \oplus \mathbf{1}_{p+n}),
\end{aligned}
$$

by (14.18),

$$
= \alpha \cdot (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p) \oplus \mathbf{1}_n).
$$

Now let $g : p \to 2p$ be any morphism. Since α is idempotent,

$$
\begin{aligned}
&f \cdot (\mathbf{1}_n \oplus 0_n \oplus g) \;= \\
&\quad= \alpha \cdot \langle f \cdot (\mathbf{1}_n \oplus 0_n \oplus g),\, f \cdot (\mathbf{1}_n \oplus 0_n \oplus g)\rangle \\
&\quad= \alpha \cdot (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p) \oplus \mathbf{1}_n) \cdot \\
&\qquad\qquad \cdot \langle \mathbf{1}_{2n} \oplus \langle g,\, 0_p \oplus \mathbf{1}_p\rangle,\, f \cdot (\mathbf{1}_n \oplus 0_n \oplus g)\rangle \\
&\quad= \alpha \cdot (f \cdot (\alpha \oplus \beta) \oplus \mathbf{1}_n) \cdot \\
&\qquad\qquad \cdot \langle \mathbf{1}_{2n} \oplus \langle g,\, 0_p \oplus \mathbf{1}_p\rangle,\, f \cdot (\mathbf{1}_n \oplus 0_n \oplus g)\rangle \\
&\quad= \alpha \cdot \langle f \cdot (\alpha \oplus \beta \cdot \langle g,\, 0_p \oplus \mathbf{1}_p\rangle),\, f \cdot (\mathbf{1}_n \oplus 0_n \oplus g)\rangle.
\end{aligned}
$$

We have established the fact that for any morphism $g : p \to 2p$,

$$\begin{aligned} f \cdot (\mathbf{1}_n \oplus 0_n \oplus g) &= \\ &= \alpha \cdot \langle f \cdot (\alpha \oplus \beta \cdot \langle g,\, 0_p \oplus \mathbf{1}_p\rangle),\ f \cdot (\mathbf{1}_n \oplus 0_n \oplus g)\rangle. \end{aligned} \tag{14.19}$$

Now, taking $g = \mathbf{1}_p \oplus 0_p$ in (14.19), we obtain

$$\begin{aligned} f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p) &= \\ &= \alpha \cdot \langle f \cdot (\alpha \oplus \beta),\ f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p)\rangle \qquad (14.20) \\ &= \alpha \cdot \langle f \cdot (\alpha \oplus \beta),\ a\rangle, \qquad (14.21) \end{aligned}$$

where $a = f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p) : n \to 2n + 2p$.

Similarly, if we take $g = \beta$ in (14.19), we obtain

$$\begin{aligned} f \cdot (\mathbf{1}_n \oplus 0_n \oplus \beta) &= \\ &= \alpha \cdot \langle f \cdot (\alpha \oplus \beta \cdot \langle \beta,\ 0_p \oplus \mathbf{1}_p\rangle),\ f \cdot (\mathbf{1}_n \oplus 0_n \oplus \beta)\rangle \qquad (14.22) \\ &= \alpha \cdot \langle f \cdot (\alpha \oplus \beta),\ b\rangle, \qquad (14.23) \end{aligned}$$

where $b = f \cdot (\mathbf{1}_n \oplus 0_n \oplus \beta) : n \to 2n + 2p$, since β has the deletion property. Applying the iteration operation to both sides of (14.20), it follows that

$$\begin{aligned} (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p))^\dagger &= (\alpha \cdot \langle f \cdot (\alpha \oplus \beta),\ a\rangle)^\dagger \\ &= \alpha \cdot (\langle f \cdot (\alpha \oplus \beta),\ a\rangle \cdot (\alpha \oplus \mathbf{1}_{n+2p}))^\dagger, \end{aligned}$$

by the simplified composition identity,

$$\begin{aligned} &= \alpha \cdot (\langle f \cdot (\alpha \oplus \beta) \cdot (\alpha \oplus \mathbf{1}_{n+2p}),\ a \cdot (\alpha \oplus \mathbf{1}_{n+2p})\rangle)^\dagger \\ &= \alpha \cdot (\langle f \cdot (\alpha \oplus \beta) \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_{n+2p}),\ a \cdot (\alpha \oplus \mathbf{1}_{n+2p})\rangle)^\dagger \\ &= \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger,\ a'\rangle, \qquad (14.24) \end{aligned}$$

for some $a' : n \to n + 2p$, using the iteration theory identities and the Boolean axioms.

Similarly, by applying the iteration operation to both sides of (14.22), we obtain

$$(f \cdot (\mathbf{1}_n \oplus 0_n \oplus \beta))^\dagger = \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger,\ b'\rangle, \tag{14.25}$$

for some $b' : n \to n + 2p$.

Define the base morphism τ by:

$$\tau \;:=\; \langle 0_{2p} \oplus \mathbf{1}_n,\ \mathbf{1}_{2p} \oplus 0_n \rangle.$$

Then

$$\begin{aligned}
(f \oplus 0_{p+n})^\dagger &= (f \cdot (\mathbf{1}_n \oplus \mathbf{1}_p \oplus 0_p \oplus 0_n))^\dagger \\
&= (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p) \cdot (\mathbf{1}_n \oplus \tau))^\dagger \\
&= (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_p))^\dagger \cdot \tau \\
&= \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger,\ a' \rangle \cdot \tau,
\end{aligned}$$

by (14.24), so that

$$(f \oplus 0_{p+n})^\dagger \;=\; \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger \cdot \tau,\ a' \cdot \tau \rangle. \tag{14.26}$$

A similar computation using (14.25) yields

$$(f \cdot (\mathbf{1}_n \oplus \beta \oplus 0_n))^\dagger \;=\; \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger \cdot \tau,\ b' \cdot \tau \rangle. \tag{14.27}$$

We can now complete the proof.

$$\alpha \cdot (f^\dagger \oplus 0_p \oplus \mathbf{1}_n) \;=\; \alpha \cdot \langle f^\dagger \oplus 0_{p+n},\ 0_{2p} \oplus \mathbf{1}_n \rangle,$$

by definition of $\oplus$,

$$\begin{aligned}
&= \alpha \cdot \langle (f \oplus 0_{p+n})^\dagger,\ 0_{2p} \oplus \mathbf{1}_n \rangle \\
&= \alpha \cdot \langle \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger \cdot \tau,\ a' \cdot \tau \rangle,\ 0_{2p} \oplus \mathbf{1}_n \rangle,
\end{aligned}$$

by (14.26),

$$= \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger \cdot \tau,\ 0_{2p} \oplus \mathbf{1}_n \rangle, \tag{14.28}$$

since α has the deletion property.

By the same kind of calculation,

$$\begin{aligned}
\alpha \cdot (f^\dagger \cdot \beta \oplus \mathbf{1}_n) &= \alpha \cdot \langle f^\dagger \cdot \beta \oplus 0_n,\ 0_{2p} \oplus \mathbf{1}_n \rangle \\
&= \alpha \cdot \langle (f \cdot (\mathbf{1}_n \oplus \beta \oplus 0_n))^\dagger,\ 0_{2p} \oplus \mathbf{1}_n \rangle \\
&= \alpha \cdot \langle \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger \cdot \tau,\ b' \cdot \tau \rangle,\ 0_{2p} \oplus \mathbf{1}_n \rangle,
\end{aligned}$$

by (14.27),

$$= \alpha \cdot \langle (f \cdot (\alpha \oplus \beta))^\dagger \cdot \tau,\ 0_{2p} \oplus \mathbf{1}_n \rangle.$$

The special case of the iteration rule is proved.

The general case follows from the composition rule and the fact that

$$\alpha' \mathbf{1}_n \alpha$$

is valid when $\alpha' \leq \alpha$.

We omit the easy proof of the validity of the distinguished morphism rule. The proof of the theorem is complete.

Remark 14.5.2 Since only Conway theory identities were used in the above proof, the iteration rule is valid for all Conway theories.

The validity of other rules can be established with the aid of the four rules we have just considered. One important example is the monotone rule.

Proposition 14.5.3 *The following* **monotone rule***, or* **rule of consequence** *holds in every theory:*

$$\alpha f \beta,\ \delta \ \leq \ \alpha,\ \beta \ \leq \ \gamma \quad \vdash \quad \delta f \gamma. \tag{14.29}$$

Here, $f : n \to p$ is in $\Sigma\,\mathbf{TM}$, $\alpha = [\alpha_1, l, \alpha_n]$, $\delta = [\delta_1, l, \delta_n]$ are n-guard expressions, and β and γ are p-guard expressions.

Proof. We prove the rule in two steps. First assume that $\delta \leq \alpha$ in T and that the equation $\alpha f \beta istrue.Thenforeach$i $\in [n]$,

$$\delta_i i_n \alpha,$$

by the rule for distinguished morphisms. By the tupling rule, it follows that

$$\delta \mathbf{1}_n \alpha$$

holds. Applying the composition rule, we derive the truth of the equation

$$\delta \mathbf{1}_n \cdot f \beta,$$

from which follows the truth of $\delta f \beta. Usingasimilarargument, itfollowsthat \delta f \gamma holdsin$T$also.$

For later use, we state some additional valid rules. We sometimes omit the statement of obvious syntactical restrictions. The easy proofs of the validity of these rules is left to the reader.

Theorem 14.5.4 *Each of the following rules is valid.*

1. **True-false rules**.

$$\begin{aligned} &\vdash \ \ \mathrm{FALSE}_n f \beta \\ &\vdash \ \ \alpha f \mathrm{TRUE}_p, \end{aligned}$$

for $f : n \to p$ in $\Sigma\,\mathbf{TM}$, n-guard expression α and p-guard expression β.

2. *The* **pairing rule**.

$$\alpha f \gamma,\ \beta g \gamma \ \ \vdash \ \ [\alpha, \beta]\langle f, g\rangle \gamma,$$

for any n-guard expression α, m-guard expression β, p-guard expression γ and Σ-iteration terms $f : n \to p$, $g : m \to p$.

3. **Converse pairing rule**.

$$\begin{aligned} [\alpha, \beta]\langle f, g\rangle \gamma \ \ &\vdash \ \ \alpha f \gamma \\ [\alpha, \beta]\langle f, g\rangle \gamma \ \ &\vdash \ \ \beta g \gamma, \end{aligned}$$

for any n-guard expression α, m-guard expression β, p-guard expression γ and Σ-iteration terms $f : n \to p$, $g : m \to p$.

4. **Untupling rule**.

$$[\alpha_1, l, \alpha_n]\langle f_1, l, f_n\rangle \beta \ \ \vdash \ \ \alpha_i f_i \beta, \quad \text{each } i \in [n],$$

for any p-guard expression β, any Σ-iteration terms $f_i : 1 \to p$ and guard expressions α_i, $i \in [n]$.

5. **Conjunction rule**.

$$\alpha_1 f \beta_1,\ \alpha_2 f \beta_2 \ \ \vdash \ \ \alpha_1 \wedge \alpha_2 f \beta_1 \wedge \beta_2.$$

6. **Disjunction rule**.

$$\alpha_1 f \beta_1,\ \alpha_2 f \beta_2 \ \ \vdash \ \ \alpha_1 \vee \alpha_2 f \beta_1 \vee \beta_2.$$

7. **Base morphism rule**.
Suppose that $\rho = \langle \rho_1, l, \rho_n\rangle : n \to m$ is a base morphism. Then

$$\alpha_i \ \leq \ \beta_{\rho_i}, \text{ for all } i \in [n] \ \ \vdash \ \ [\alpha_1, l, \alpha_n]\rho[\beta_1, l, \beta_m],$$

for all guard expressions α_i, β_j. A special case of this rule is the identity rule.

8. *The* **identity rule**.

$$\alpha \ \leq \ \beta \ \ \vdash \ \ \alpha \mathbf{1}_n \beta,$$

for n-guard expressions α, β.

9. **Converse of distinguished morphism rule**.

$$\alpha i_n[\beta_1, l, \beta_n] \ \ \vdash \ \ \alpha \ \leq \ \beta_i.$$

10. **Guard rules**.

$$\begin{aligned} \alpha\beta[\gamma,\delta] \ &\vdash \ \ \alpha \wedge \beta \ \leq \ \gamma \\ \alpha\beta[\gamma,\delta] \ &\vdash \ \ \alpha \wedge \neg\beta \ \leq \ \delta. \\ \alpha \wedge \beta \ \leq \ \gamma, \ \alpha \wedge \neg\beta \ \leq \ \delta \ &\vdash \ \ \alpha\beta[\gamma,\delta], \end{aligned}$$

for n-guard expressions $\alpha, \beta, \gamma, \delta$.

11. **Sum rule**.

$$\alpha_1 f_1 \beta_1, \ \alpha_2 f_2 \beta_2 \ \ \vdash \ \ [\alpha_1, \alpha_2] f_1 \oplus f_2 [\beta_1, \beta_2].$$

12. **Converse sum rule**.

$$[\alpha_1, \alpha_2] f_1 \oplus f_2 [\beta_1, \beta_2] \ \ \vdash \ \ \alpha_i f_i \beta_i, \ i \in [2].$$

13.

$$\begin{aligned} &\alpha \wedge \beta f \gamma, \ \alpha \wedge \neg\beta g \delta \\ &\quad \vdash \ \ \alpha\beta \cdot (f \oplus g)[\gamma, \delta]. \end{aligned}$$

14.

$$\begin{aligned} &\alpha \wedge \beta f \gamma, \ \alpha \wedge \neg\beta g \gamma \\ &\quad \vdash \ \ \alpha\beta \cdot \langle f, \ g\rangle \gamma. \end{aligned} \tag{14.30}$$

15.

$$\alpha \wedge \beta f \alpha \ \ \vdash \ \ \alpha(\beta \cdot (f \oplus \mathbf{1}_n))^{\dagger} \alpha \wedge \neg\beta,$$

for $f : n \to n$ in $\Sigma\,\mathbf{TM}$.

Some of these rules generalize those for the usual structured programming constructs. For example, in an iteration theory A, if β is a guard $A \to A \times [2]$, and f and g are partial functions $A \to A$, the program fragment

$$\textit{if } \beta \textit{ then do } f \textit{ else do } g$$

can be written as the morphism

$$\beta \cdot \langle f,\, g\rangle,$$

and the standard rule of the conditional becomes rule (14.30) above:

$$\alpha \wedge \beta f \gamma,\ \alpha \wedge \neg\beta g \gamma \quad \vdash \quad \alpha\beta \cdot \langle f,\, g\rangle\gamma.$$

Similarly the fragment

$$\textit{while } \beta \textit{ do } f$$

can be written as the morphism

$$(\beta(f \oplus \mathbf{1}_1))^\dagger,$$

and the familiar rule for *while-do* becomes

$$\alpha \wedge \beta f \alpha \quad \vdash \quad \alpha(\beta(f \oplus \mathbf{1}_1))^\dagger \alpha \wedge \neg\beta.$$

Remark 14.5.5 The soundness of the iteration rule required the longest argument. Its length is justified by the fact that it shows that the iteration rule is valid in all Conway theories. We outline in this section an easier argument which uses the commutative identity.

Lemma 14.5.6 *If α is an n-guard and $f : n \to n + p$, then*

$$f^\dagger \quad = \quad (\alpha \cdot \langle f,\, 0_n \oplus f^\dagger\rangle)^\dagger.$$

Proof. First, we assume $n = p = 1$. Consider the system of four fixed-point equations

$$\begin{aligned}
x_1 &= \alpha \cdot \langle x_2, x_3\rangle \\
x_2 &= f \cdot \langle x_1, x_5\rangle \\
x_3 &= f \cdot \langle x_4, x_5\rangle \\
x_4 &= \alpha \cdot \langle x_3, x_3\rangle.
\end{aligned}$$

Since α is idempotent, it is clear that the fourth equation is redundant, and the first component of the solution of these equations is $(\alpha \cdot \langle f, 0_1 \oplus f^\dagger \rangle)^\dagger$. Let $\rho : 4 \to 2$ be the base surjection taking $1, 4$ to 1 and $2, 3$ to 2. Define $F = \langle F_1, F_2 \rangle : 2 \to 4 + 1$ as follows:

$$\begin{aligned} F_1 &= \alpha \cdot \langle x_2, x_3 \rangle \\ F_2 &= f \cdot \langle x_1, x_5 \rangle. \end{aligned}$$

Now it is easy to find $\rho_1, \ldots, \rho_4 : 4 \to 4$ such that $\rho_i \cdot \rho = \rho$ and

$$\begin{aligned} F_1 \cdot (\rho_1 \oplus \mathbf{1}_1) &= \alpha \cdot \langle x_2, x_3 \rangle \\ F_2 \cdot (\rho_2 \oplus \mathbf{1}_1) &= f \cdot \langle x_1, x_5 \rangle \\ F_2 \cdot (\rho_3 \oplus \mathbf{1}_1) &= f \cdot \langle x_4, x_5 \rangle \\ F_1 \cdot (\rho_4 \oplus \mathbf{1}_1) &= \alpha \cdot \langle x_3, x_3 \rangle. \end{aligned}$$

Note that

$$\begin{aligned} F_1 \cdot (\rho \oplus \mathbf{1}_1) &= \alpha \cdot \langle x_2, x_2 \rangle \\ F_2 \cdot (\rho \oplus \mathbf{1}_1) &= f \cdot \langle x_1, x_3 \rangle. \end{aligned}$$

Hence $(F \cdot (\rho \oplus \mathbf{1}_1))^\dagger = \langle f^\dagger, f^\dagger \rangle$, and applying the commutative identity

$$\langle 1\rho \cdot F \cdot (\rho_1 \oplus \mathbf{1}_1), \ldots, 4\rho \cdot F \cdot (\rho_4 \oplus \mathbf{1}_1) \rangle^\dagger \;=\; \rho \cdot (F \cdot (\rho \oplus \mathbf{1}_1))^\dagger,$$

the lemma is proved for the case $n = p = 1$. The general case follows by a slight change in notation.

Now assume that $f : n \to n + p$, that α is an n-guard, β is a p-guard and that

$$\alpha f[\alpha, \beta]. \tag{14.31}$$

In order to make life at least notationally easier, we make the following abbreviations:

$$\begin{aligned} x &= \mathbf{1}_n \oplus 0_{n+2p+n} \\ x' &= 0_n \oplus \mathbf{1}_n \oplus 0_{2p+n} \\ y &= 0_n \oplus 0_n \oplus \mathbf{1}_p \oplus 0_{p+n} \\ y' &= 0_n \oplus 0_{n+p} \oplus \mathbf{1}_p \oplus 0_n \\ z &= 0_n \oplus 0_{n+2p} \oplus \mathbf{1}_n. \end{aligned}$$

Now we can express the equation (14.31) as follows:

$$\alpha \cdot \langle f \cdot \langle x, y \rangle,\, z \rangle \;=\; \alpha \cdot \langle f \cdot \langle \alpha \cdot \langle x, x' \rangle,\, \beta \cdot \langle y, y' \rangle \rangle,\, z \rangle,$$

or, omitting the dot, as

$$\alpha \langle f \langle x, y \rangle,\, z \rangle \;=\; \alpha \langle f \langle \alpha \langle x, x' \rangle,\; \beta \langle y, y' \rangle \rangle,\; z \rangle.$$

Similarly, Lemma 14.5.6 can be written

$$f^\dagger y \quad = \quad (\alpha\langle f\langle x, y\rangle, f^\dagger y\rangle)^\dagger.$$

We continue with this dot free notation.

We are trying to prove $L = R$, where, by definition,

$$\begin{array}{rcl} L & := & \alpha\langle f^\dagger y,\ z\rangle \\ R & := & \alpha\langle f^\dagger \beta\langle y, y'\rangle,\ z\rangle. \end{array}$$

(The reader who draws the following terms as flowchart schemes will follow the next argument easily.) Using the Lemma,

$$L \quad = \quad \alpha\langle \alpha\langle f\langle x, y\rangle, f^\dagger y\rangle^\dagger,\ z\rangle.$$

Applying the assumption (14.31), we see that

$$L \quad = \quad \alpha\langle \alpha\langle f\langle \alpha\langle x, x'\rangle, \beta\langle y, y'\rangle\rangle, f^\dagger y\rangle^\dagger,\ z\rangle. \tag{14.32}$$

Applying the Lemma now to R, we see

$$R \quad = \quad \alpha\langle \alpha\langle f\langle x, \beta\langle y, y'\rangle\rangle, f^\dagger \beta\langle y, y'\rangle\rangle^\dagger,\ z\rangle,$$

which, by the assumption (14.31) is

$$= \quad \alpha\langle \alpha\langle f\langle \alpha\langle x, x'\rangle, \beta\langle y, y'\rangle\rangle, f^\dagger \beta\langle y, y'\rangle\rangle^\dagger,\ z\rangle. \tag{14.33}$$

But this last expression can be transformed into the expression in (14.32) as follows. The right-hand side of (14.32) can be written

$$\alpha\langle \alpha\langle f\langle \alpha\langle \alpha\langle x, f^\dagger y\rangle, x'\rangle, \beta\langle y, y'\rangle\rangle^\dagger, f^\dagger y\rangle,\ z\rangle,$$

using the composition identity. The term in (14.33) can be written

$$\alpha\langle \alpha\langle f\langle \alpha\langle \alpha\langle x, f^\dagger \beta\langle y, y'\rangle\rangle, x'\rangle, \beta\langle y, y'\rangle\rangle^\dagger, f^\dagger \beta\langle y, y'\rangle\rangle,\ z\rangle.$$

These latter two expressions are equal, due to the deletion property of α.

14.6 The Standard Example, Continued

Recall the notation of Section 14.3. Let $\mathbf{B}$ be a first order structure, and let L be the corresponding first order language. We let Π denote the set of all L-formulas, and let Σ_2 denote the set of all quantifier free L-formulas. Lastly, let Σ_1 denote the set of all expressions of the form

$$x_i \ := \ \tau(x_1, l, x_n),$$

for an L-term τ. Otherwise $\Sigma_n = \emptyset$. We define the function $\varphi_B : \Sigma_\Pi \to B^\omega$ by

$$y\varphi_B \quad := \quad \begin{cases} g_\alpha : B^\omega \to B^\omega \times [2] & \text{if } y \text{ is } \alpha,\ \alpha \in \Pi; \\ f_{i,\tau} : B^\omega \to B^\omega & \text{if } y \text{ is } (x_i := \tau). \end{cases}$$

The unique extension of φ_B to a homomorphism $\Sigma_\Pi\mathbf{TM} \to B^\omega$ is appropriate for the triple (B^ω, Γ, T_0) corresponding to $\mathbf{B}$ of Section 14.3. This choice of φ_B is the *standard homomorphism* for the structure $\mathbf{B}$. Note that Σ and Π are the same for *all* L-structures.

14.7 A Floyd-Hoare Calculus for Iteration Theories

The *axioms* for the Floyd-Hoare type calculus we will describe are parameterized by the triples (T, φ, Σ_Π), where $\varphi : \Sigma_\Pi\mathbf{TM} \to T$ is an admissible homomorphism. The axioms for a given triple (T, φ, Σ_Π) consist of all partial correctness assertions of two kinds. The axioms of the first kind have the form

$$\alpha\sigma\beta \tag{14.34}$$

such that $\alpha\varphi\sigma\varphi\beta\varphi istrueinT.Here\sigma$ is a letter in Σ_n, for some n, α is a guard expression and β is an n-guard expression.

Axioms of the second kind are all partial correctness expressions of the form

$$\alpha i_n[\beta_1, l, \beta_n], \tag{14.35}$$

where α and the β_j are guard expressions and $\alpha\varphi \leq \beta_i\varphi$ in T; or expressions of the form

$$\alpha 0_n[\beta_1, l, \beta_n], \tag{14.36}$$

where $\alpha = 0_0$ is the unique 0-guard expression and the β_j are guard expressions.

Definition 14.7.1 *A partial correctness expression* $\{\alpha\}\ f\ \{\beta\}$ *is* **derivable***, written*

$$\vdash \quad \{\alpha\}\ f\ \{\beta\},$$

if there is a finite sequence

$$\alpha_1 f_1 \beta_1, l, \alpha_n f_n \beta_n$$

such that $\alpha_n f_n \beta_n is \{\alpha\}\ f\ \{\beta\}, and for each k \in [n],$ *either* $\alpha_k f_k \beta_k is an axiom or$

there are $i, j < k$ *such that* $\alpha_k f_k \beta_k follows from the two expressions \alpha_i f_i \beta_i and \alpha_j f_j \beta_j by the composition rule$

$$\alpha_k = \alpha_i;\ \beta_k = \beta_j;\ f_k = f_i \cdot f_j \quad and \quad \beta_i = \alpha_j; \text{or}$$

there are $i_1, l, i_m < k$ such that $\alpha_k f_k \beta_k follows from \alpha_{i_1} f_{i_1} \beta_{i_1}, l, \alpha_{i_m} f_{i_m} \beta_{i_m}, by the tupling rule : i.e. \mathrm{f}_k :$ $m \to p,\ m \geq 1,\ f_{i_j} : 1 \to p$ for each $j \in [m]$ and

$$\begin{aligned} f_k &= \langle f_{i_1}, l, f_{i_m} \rangle : m \to p \\ \alpha_k &= [\alpha_{i_1}, l, \alpha_{i_m}] \quad \text{and} \\ \beta_k &= \beta_{i_j}, \quad \text{for all } j \in [m]; \end{aligned}$$

or

there is an $i < k$ such that $\alpha_k f_k \beta_k follows from the expression \alpha_i f_i \beta_i by the iteration rule :$ $i.e. \mathrm{f}_i : m \to m + p$ and

$$\begin{aligned} f_k &= f_i^\dagger : m \to p \\ \beta_i &= [\alpha_i,\ \beta_k] \quad \text{and} \\ \alpha_k \varphi &\leq \alpha_i \varphi. \end{aligned}$$

We note that the axioms (and iteration rule) depend on the triple (T, φ, Σ_Π).

Remark 14.7.2 If one wished, one could reformulate the iteration rule slightly to eliminate the explicit dependence on the homomorphism φ as follows:

$$\alpha \mathbf{1}_n \gamma,\ \gamma f[\gamma,\ \beta] \quad \vdash \quad \alpha f^\dagger \beta.$$

In other words, the condition $\alpha_k \varphi \leq \alpha_i \varphi$ in the iteration rule may be replaced by the conditions that for some $j < k$, $\alpha_j = \alpha_k$ and $f_j = \mathbf{1}_n$ and $\beta_j = \alpha_i$.

In Section 14.10 we use this logic to prove the correctness of a particular flowchart algorithm.

Notation. Let $\varphi : \Sigma_\Pi \mathbf{TM} \to T$ be an admissible homomorphism. We let $\mathbf{FH}(T, \varphi, \Sigma_\Pi)$ denote the set of all partial correctness expressions which are derivable from the axioms for (T, φ, Σ_Π).

Definition 14.7.3 *The system of axioms for* (T, φ, Σ_Π) *together with our four rules is* **complete for** (T, φ, Σ_Π) *if*

$$\mathbf{FH}(T, \varphi, \Sigma_\Pi) = \mathbf{Val}(T, \varphi, \Sigma_\Pi).$$

Corollary 14.7.4 Soundness Theorem. *For any triple* (T, φ, Σ_Π),

$$\mathbf{FH}(T, \varphi, \Sigma_\Pi) \quad \subseteq \quad \mathbf{Val}(T, \varphi, \Sigma_\Pi).$$

Proof. By induction on the length of a derivation, using the fact that by Theorem 14.5.1, the rules preserve validity. The details are routine.

In the next section, we will find conditions on (T, Γ, T_0) which ensure

$$\mathbf{Val}(T, \varphi, \Sigma_\Pi) \quad = \quad \mathbf{FH}(T, \varphi, \Sigma_\Pi)$$

for any appropriate $\varphi : \Sigma_\Pi \mathbf{TM} \to T$.

14.8 The Standard Example, Again

If $\mathbf{B}$ is a first order structure and $\varphi_B : \Sigma_\Pi \mathbf{TM} \to B^\omega$ is the standard homomorphism, at the expense of adding the monotone rule (14.29), we can simplify the class of axioms of the kind (14.34). If σ is in Σ_1, then σ is an assignment statement

$$x_i \;:=\; \tau(x_1, l, x_n).$$

We can take as axioms for σ the standard Hoare axioms:

$$\beta(x_i/\tau)\sigma\beta$$

for all formulas β (these formulas are the guard expressions here). In the case that σ is in Σ_2, σ is an open formula, say θ. As axioms for σ we take all expressions $\alpha\sigma[\beta,\gamma] such that to$ Use "frak only in math modeB$\models (\alpha \wedge \theta \to \beta) \;\wedge\; (\alpha \wedge \neg\theta \to \gamma)$.

14.9 Completeness

In the first part of this section we give a necessary and sufficient condition that Floyd-Hoare logic for (T, φ, Σ_Π) is complete. The condition will be formulated for a triple (T, Γ, T_0), where φ is any homomorphism appropriate for (T, Γ, T_0). We will assume that (T, Γ, T_0) and an appropriate $\varphi : \Sigma_\Pi \to T$ are fixed for the remainder of this section. In the second part, we investigate the role of weakest liberal preconditions for the completeness of Floyd-Hoare logic. The sufficient condition for completeness given in Corollary 14.9.17 is one of the main results.

14.9.1 The Invariant Guard Property

We begin with a definition of one of the more important notions in this chapter.

Definition 14.9.1 *We say that* (T, Γ, T_0) *has* invariant guards for f, *where* $f : n \to n + p$ *is in* T_0*, if whenever*

$$\alpha f^\dagger \beta,$$

(i.e., whenever $\alpha f^\dagger \beta$ *holds in* T*) for* α *in* Γ_n*,* $\beta \in \Gamma_p$*, then there is some* n*-guard* γ *in* Γ_n *such that*

$$\alpha \quad \leq \quad \gamma$$

and

$$\gamma f[\gamma, \beta].$$

We will say that (T, Γ, T_0) *has* **invariant guards** *(respectively,* **scalar invariant guards***) if* (T, Γ, T_0) *has invariant guards for all* $f : n \to n + p$*, all* $n, p \geq 0$*, (respectively, for all* $f : 1 \to 1 + p$*,* $p \geq 0$*) in* T_0*.*

In the theory $T = A$, if $f : 1 \to 2$ say, and if

$$\gamma f[\gamma, \beta],$$

then γ is an invariant for f in the following sense. If $\gamma(a)$ holds and $af = (a', 1)$ then $\gamma(a')$ holds also.

Definition 14.9.2 *We say that* (T, Γ, T_0) *has the* interpolation property for f and g, *where* $f : n \to p,\ g : p \to q$ *are in* T_0, *if whenever*

$$\alpha f \cdot g\beta,$$

for $\alpha \in \Gamma_n$ *and* $\beta \in \Gamma_q$, *there is some* p*-guard* γ *such that*

$$\begin{aligned} &\alpha f \gamma \text{ and} \\ &\qquad \gamma g \beta. \end{aligned}$$

The triple (T, Γ, T_0) *has the* **interpolation property** *(respectively, the* **scalar interpolation property***) if* (T, Γ, T_0) *has the interpolation property for all* $f : n \to p$, *(respectively, for all* $f : 1 \to p$*) and all* $g : p \to q$ *in* T_0.

The following observation concerning these concepts is easy to prove using the form of the iteration and composition rules.

Proposition 14.9.3 *Let the homomorphism* $\varphi : \Sigma_\Pi \mathbf{TM} \to T$ *be appropriate for* (T, Γ, T_0). *Suppose that* $\mathbf{Val}(T, \varphi, \Sigma_\Pi) = \mathbf{FH}(T, \varphi, \Sigma_\Pi)$. *Then* (T, Γ, T_0) *has invariant guards and the interpolation property.*

The more important of these two conditions is the invariant guard property.

Proposition 14.9.4 *The triple* (T, Γ, T_0) *has the interpolation property whenever it has invariant guards.*

Proof. Let $f : n \to m,\ g : m \to p$ be T_0-morphisms and assume that

$$\alpha f \cdot g\beta,$$

where α is an n-guard and β is a p-guard. Define the morphism $h : n + m \to n + m + p$ as the pairing

$$h \quad = \quad \langle 0_n \oplus f \oplus 0_p,\ 0_{n+m} \oplus g \rangle.$$

Then using the identities in Section 5.5.3, it is easy to see that

$$h^\dagger = \langle f \cdot g,\ g \rangle.$$

Further, by the pairing rule,

$$[\alpha, \text{FALSE}_m]h^\dagger\beta,$$

since

$$\{\alpha\}\, f \cdot g\, \{\beta\}$$

and

$$\{\text{FALSE}_n\}\, g\, \{\beta\}.$$

Thus, by the hypothesis, there is some $n+m$-guard $[\gamma, \delta]$ with

$$[\alpha, \text{FALSE}_n] \;\leq\; [\gamma, \delta]$$

and

$$[\gamma, \delta]h[\gamma, \delta, \beta].$$

By the converse pairing rule, both

$$\gamma 0_n \oplus f \oplus 0_p[\gamma, \delta, \beta]$$

and

$$\delta 0_{n+m} \oplus g[\gamma, \delta, \beta].$$

By the converse sum rule it follows that

$$\gamma f \delta,$$

and

$$\delta g \beta.$$

Thus, by the monotone rule,

$$\{\alpha\}\, f\, \{\delta\} \quad \text{and} \quad \{\delta\}\, g\, \{\beta\},$$

showing that (T, Γ, T_0) has the interpolation property.

In the next section, we give an example to show that the interpolation property does not imply the invariant guard property.

14.9.2 Completeness of Floyd-Hoare Rules

Recall that the Floyd-Hoare rules were parameterized by the admissible homomorphisms $\varphi : \Sigma_\Pi\mathbf{TM} \to T$. When we are interested in triples (T, Γ, T_0), we must assume that φ is appropriate for (T, Γ, T_0).

Theorem 14.9.5 Completeness Theorem. *Suppose that φ is appropriate for (T, Γ, T_0). Then*

$$\mathbf{Val}(T, \varphi, \Sigma_\Pi) \;=\; \mathbf{FH}(T, \varphi, \Sigma_\Pi)$$

if and only if (T, Γ, T_0) has invariant guards.

Proof. By Proposition 14.9.3, we need prove only one direction. Thus, suppose (T, Γ, T_0) has invariant guards. We will prove by induction on the number of operation symbols that occur in the Σ-iteration term f that if

$$(T, \varphi, \Sigma_\Pi) \quad \models \quad \{\alpha\}\ f\ \{\beta\},$$

then

$$\vdash \quad \{\alpha\}\ f\ \{\beta\}.$$

The assumption on φ means that there are enough guard expressions to denote each guard, and enough iteration terms in $\Sigma\,\mathbf{TM}$ to denote each T_0-morphism.

The basis case is divided into two subcases.

CASE 1. f is 0_n or is in Σ_n, for some n. In this case, $\{\alpha\}\ f\ \{\beta\}$ is an axiom for (T, φ, Σ_Π), by definition.

CASE 2. f is one of the constants i_n, for some $n > 0$ and some $i \in [n]$. In this case, since β is in Π_n, $\beta = [\beta_1, l, \beta_n]$, for some guard expressions β_1, l, β_n. Since $\alpha i_n \beta$ is in $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$, it follows that

$$\alpha \;\leq\; \beta_i,$$

by the converse of the distinguished morphism rule. Hence again, the expression $\{\alpha\}\ f\ \{\beta\}$ is an axiom.

The induction step is divided into three parts.

Case 1. $f = \langle f_1, l, f_n \rangle$, for some Σ-iteration terms $f_i : 1 \to p,\ i \in [n],\ n \geq 1$. In this case, α must have the form $[\alpha_1, l, \alpha_n]$, for some guard expressions α_1, l, α_n. Since $\{\alpha\}\ f\ \{\beta\} \in \mathbf{Val}(T, \varphi, \Sigma_\Pi)$ it follows by the untupling rule that $\alpha_i f_i \beta \in \mathbf{Val}(T, \varphi, \Sigma_\Pi)$, for each i in $[n]$. But then, by the induction hypothesis, $\alpha_i f_i \beta \in \mathbf{FH}(T, \varphi, \Sigma_\Pi)$. Hence $\{\alpha\}\ f\ \{\beta\}$ is derivable by the tupling rule.
Case 2. $f = g \cdot h$, for some Σ-iteration terms $g : n \to p,\ h : p \to q$. But, (T, Γ, T_0) has the interpolation property by Proposition 14.9.4, and since φ is appropriate, there is some p-guard expression γ such that both $\alpha g \gamma$ and $\gamma h \beta$ are in $\mathbf{Val}(T, \varphi, \Sigma_\Pi)$. Thus, by the induction hypothesis and the composition rule, $\{\alpha\}\ f\ \{\beta\}$ is derivable.
Case 3. $f = g^\dagger$, for some Σ-iteration term $g : n \to n + p$. Using the hypothesis on (T, φ, Σ_Π), there is some n-guard expression γ with $\alpha \leq \gamma$ in T such that

$$\gamma g[\gamma, \beta] \ \in\ \mathbf{Val}(T, \varphi, \Sigma_\Pi).$$

Hence, $\gamma f \beta$ is derivable, by the induction hypothesis and the iteration rule.

The proof of the theorem is complete.

Remark 14.9.6 In [Art85], a necessary and sufficient condition for Hoare logic to be complete for WHILE-programs is given. The condition amounts to assuming weaker forms of the scalar invariant guard and the scalar interpolation properties. Since we have the power of vector iteration, we can get away with only the invariant guard property. See Proposition 14.9.9 below.

How crucial is the choice of the alphabets Σ, Π and an appropriate homomorphism $\varphi : \Sigma_\Pi\mathbf{TM} \to T$ for (T, Γ, T_0)?

Corollary 14.9.7 *Let*

$$\begin{aligned} \varphi : \Sigma_\Pi\mathbf{TM} &\ \to\ T \\ \varphi' : \Sigma'_{\Pi'}\mathbf{TM} &\ \to\ T \end{aligned}$$

be homomorphisms appropriate for (T, Γ, T_0). *Then*

$$\mathbf{FH}(T, \varphi, \Sigma_\Pi) \ =\ \mathbf{Val}(T, \varphi, \Sigma_\Pi)$$

if and only if

$$\mathbf{FH}(T, \varphi', \Sigma'_{\Pi'}) \ =\ \mathbf{Val}(T, \varphi', \Sigma'_{\Pi'}).$$

14.9.3 Guarantees of Completeness

The completeness theorem above shows that we may now forget any syntactic or linguistic concerns and concentrate on the following question:

What conditions will ensure that (T, Γ, T_0) *has invariant guards?*

We first make two technical observations.

Proposition 14.9.8 *The triple* (T, Γ, T_0) *has the interpolation property iff* (T, Γ, T_0) *has the scalar interpolation property.*

Proof. Suppose that $f = \langle f_1, l, f_m \rangle : m \to p$, $\alpha = [\alpha_1, l, \alpha_m]$ and that

$$\alpha f \cdot g\beta.$$

If $m = 0$, then $\alpha = 0_0$ and $f = 0_p$, say. In this case, we can use FALSE_p as an interpolant. If $m > 0$, then by the untupling rule,

$$\alpha_i f_i \cdot g\beta,$$

for each $i \in [m]$. Hence, by the scalar interpolation property, there is an n-guard γ_i with

$$\alpha_i f_i \gamma_i$$

and

$$\gamma_i g\beta,$$

for each $i \in [m]$. Let γ be $\gamma_1 \vee l \vee \gamma_m$. Then, by the monotone rule,

$$\alpha_i f_i \gamma$$

and, by the disjunction rule

$$\gamma g\beta.$$

Lastly, $\alpha f\gamma, by the tupling rule. The proof is complete.$

Proposition 14.9.9 *The triple* (T, Γ, T_0) *has invariant guards iff it has the interpolation property and scalar invariant guards.*

Proof. By Proposition 14.9.4 we need to prove only one direction. We show by induction on n that if $f : n \to n + p$ is in T_0 then (T, Γ, T_0) has invariant guards for f. When $n = 0$, there is nothing to prove, and the case $n = 1$ holds by assumption. In the induction step repeated use is made of the following lemma, whose easy proof we omit.

Lemma 14.9.10 *Suppose that* (T, Γ, T_0) *has the interpolation property and that*

$$\alpha g \cdot \langle h, \mathbf{1}_p \rangle \beta,$$

where $g : n \to k + p$ *and* $h : k \to p$ *are in* T_0*. Then there is a guard* γ *in* Γ_k *such that*

$$\alpha g[\gamma, \beta]$$

and

$$\gamma h \beta.$$

Proof of Proposition 14.9.9, *continued.* Suppose that $f : n \to n+m+p$ and $g : m \to n+m+p$ are in T_0, and for some $\alpha \in \Gamma_n$, $\beta \in \Gamma_m$, $\gamma \in \Gamma_p$,

$$[\alpha, \beta]\langle f, g \rangle^\dagger \gamma.$$

We must show that there are some guards δ_1, δ_2 such that

$$\alpha \ \leq \ \delta_1, \quad \beta \ \leq \ \delta_2 \tag{14.37}$$

with

$$[\delta_1, \delta_2]\langle f, g \rangle [\delta_1, \delta_2, \gamma]. \tag{14.38}$$

By the pairing identity,

$$\langle f, \ g \rangle^\dagger \quad = \quad \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, \ h^\dagger \rangle,$$

where $h = g \cdot \langle f^\dagger, \ \mathbf{1}_{m+p} \rangle$. Using this fact, together with Lemma 14.9.10, the interpolation property and the induction hypothesis, we will show that there are guards $\alpha_i, \ \beta_j, \ 1 \leq i \leq 5, \ 1 \leq j \leq 3$, satisfying the inequalities

$$\alpha \ \leq \ \alpha_5, \ \alpha_1 \ \leq \ \alpha_2, \ \alpha_3 \ \leq \ \alpha_4, \quad \text{and} \quad \beta \ \leq \ \beta_1, \ \beta_2 \ \leq \ \beta_3$$

such that

$$\begin{array}{c}\{\alpha_2\}\, g\, \{[\alpha_3, \alpha_2, \gamma]\}\\ \{\beta_1\}\, g\, \{[\beta_2, \beta_1, \gamma]\}\\ \{\alpha_5\}\, f\, \{[\alpha_5, \alpha_1, \gamma]\}\\ \{\alpha_4\}\, f\, \{[\alpha_4, \alpha_2, \gamma]\}\end{array}$$

and

$$\{\beta_3\}\, f\, \{[\beta_3, \beta_1, \gamma]\}.$$

We show how to get the β's. By the pairing identity above, and the converse pairing rule,

$$\{\beta\}\, h^\dagger\, \{\gamma\}.$$

Thus, by induction, there is a β_1 with $\beta \leq \beta_1$ such that

$$\beta_1 g \cdot \langle f^\dagger, \mathbf{1}_{m+p}\rangle [\beta_1, \gamma].$$

Using Lemma 14.9.10, there is a β_2 with

$$\{\beta_1\}\, g\, \{[\beta_2, \beta_1, \gamma]\}$$

and

$$\{\beta_2\}\, f^\dagger\, \{[\beta_1, \gamma]\}.$$

Again, by induction, there is a β_3 with $\beta_2 \leq \beta_3$ and

$$\{\beta_3\}\, f\, \{[\beta_3, \beta_1, \gamma]\}.$$

The guards $\alpha_j,\ 1 \leq j \leq 5$, are obtained in the same way. Now, using the disjunction and monotone rules, it follows that equations (14.37) and (14.38) will hold when

$$\begin{array}{rcl}\delta_1 & := & \beta_3 \vee \alpha_4 \vee \alpha_5;\\ \delta_2 & := & \beta_1 \vee \alpha_2.\end{array}$$

Thus, Proposition 14.9.9 is proved.

Some other conditions that ensure (T, Γ, T_0) has invariant guards involve weakest liberal preconditions.

14.9.4 Weakest Liberal Preconditions and Completeness

It is useful to make the following abbreviation, relative to (T, Γ, T_0). We write $\{\alpha\}\ f\ \{\beta\} \in \mathbf{PC}(T, \Gamma, T_0)$ if $f : n \to p$ is a morphism in T_0, $\alpha \in \Gamma_n$ and $\beta \in \Gamma_p$ and $\{\alpha\}\ f\ \{\beta\}$ holds in T.

Definition 14.9.11 *The triple* (T, Γ, T_0) **has weakest liberal preconditions for** f **and** β*, where* $f : n \to p$ *in* T_0 *and* $\beta \in \Gamma_p$*, if there is some* α_0 *in* Γ_n *such that for all* $\alpha \in \Gamma_n$*,*

$$\{\alpha\}\ f\ \{\beta\} \quad \Leftrightarrow \quad \alpha \ \leq \ \alpha_0.$$

One can express this condition by saying that there is some n-guard which is maximum among those n-guards α such that $\{\alpha\}\ f\ \{\beta\}$ is in $\mathbf{PC}(T, \Gamma, T_0)$. We will say that α_0 is the *weakest liberal precondition* of (f, β), and write $\alpha_0 = \mathrm{wlp}(f, \beta)$. (T, Γ, T_0) *has weakest liberal preconditions* (respectively, *weakest scalar preconditions*) if (T,Γ, T_0) has weakest liberal preconditions for every f (respectively, every scalar f) in T_0 and every β. Dually, (T, Γ, T_0) has *strongest liberal postconditions* if for any T_0-morphism $f : n \to p$ and any α in Γ_n, there is an element β in Γ_p which is minimum among those p-guards β' such that $\alpha f \beta' \in \mathbf{PC}(T, \Gamma, T_0)$. We will say that β is a strongest liberal postcondition of (α, f) and write $\beta = \mathrm{slp}(\alpha, f)$. Many of the facts about weakest liberal preconditions mentioned below have similar versions for strongest liberal postconditions. We will not make any use of strongest liberal postconditions.

We list some facts about weakest liberal preconditions in (T, Γ, T_0) which show that the notion involves essentially only scalar morphisms in T_0 and scalar guards. There are corresponding statements which hold for strongest liberal postconditions.

Proposition 14.9.12 *Let* $f = \langle f_1, l, f_n\rangle : n \to p$ *be a* T_0*-morphism and let* β *be a* p*-guard. Then* $\mathrm{wlp}(f, \beta)$ *exists in* (T, Γ, T_0) *iff* $\mathrm{wlp}(f_i, \beta)$ *exists for each* i *in* $[n]$*. Thus,* (T, Γ, T_0) *has weakest liberal preconditions iff it has weakest scalar preconditions.*

Proof. Suppose first that $\mathrm{wlp}(f, \beta) = \alpha = [\alpha_1, l, \alpha_n]$ exists in (T, Γ, T_0). Then, by the untupling rule, $\alpha_i f_i \beta \in \mathbf{PC}(T, \Gamma, T_0)$, for each i in $[n]$.

Now fix $i \in [n]$, and suppose that

$$\alpha_i' f_i \beta \ \in \ \mathbf{PC}(T, \Gamma, T_0),$$

for some α_i' in Γ. Then by the tupling rule and the fact that α is maximum,

$$[\alpha_1, l, \alpha_i', l, \alpha_n] \quad \leq \quad \alpha$$

in T, so that $\alpha_i' \leq \alpha_i$. Thus α_i is $\mathrm{wlp}(f_i, \beta)$ in (T, Γ, T_0).

Now suppose that $\alpha_i = \mathrm{wlp}(f_i, \beta)$ exists in (T, Γ, T_0), for each $i \in [n]$. Then

$$[\alpha_1, l, \alpha_n] f \beta \ \in \ \mathbf{PC}(T, \Gamma, T_0),$$

by the tupling rule. As above, it follows that $[\alpha_1, l, \alpha_n]$ is maximum with this property, so that $\mathrm{wlp}(f, \beta)$ exists.

Now consider the guard on the right of a partial correctness assertion.

Proposition 14.9.13 *Let $f : n \to p$ and let $\beta = [\beta_1, l, \beta_p]$ be a p-guard. For each i in $[p]$, let*

$$\widehat{\beta}_i \quad := \quad [\text{TRUE}, l, \beta_i, l, \text{TRUE}]$$

with β_i in the i-th position and TRUE *everywhere else. Then* $\mathrm{wlp}(f, \beta)$ *exists in (T, Γ, T_0) if* $\mathrm{wlp}(f, \widehat{\beta}_i)$ *exists for each i in $[p]$, and in this case*

$$\mathrm{wlp}(f, \beta) \quad = \quad \mathrm{wlp}(f, \widehat{\beta}_1) \wedge l \wedge \mathrm{wlp}(f, \widehat{\beta}_p).$$

Proof. We can assume $p \geq 1$. Note that β is the meet of all $\widehat{\beta}_i$, $i = 1, l, p$. Assume that $\alpha_i = \mathrm{wlp}(f, \widehat{\beta}_i)$ exists, for each i in $[p]$. Then by the conjunction rule,

$$\{\alpha\} \ f \ \{\beta\} \in \mathbf{PC}(T, \Gamma, T_0),$$

where α is the meet of α_i, i in $[p]$. From the monotone rule, it follows that if

$$\{\alpha'\} \, f \, \{\beta\} \in \mathbf{PC}(T, \Gamma, T_0),$$

then

$$\alpha' f \widehat{\beta}_i \in \mathbf{PC}(T, \Gamma, T_0),$$

for each $i \in [p]$, so that $\alpha' \leq \alpha_i$. Hence, $\alpha' \leq \alpha$, showing that $\alpha = \mathrm{wlp}(f, \beta)$.

We state some easily proved facts about weakest liberal preconditions.

Proposition 14.9.14 *Suppose that $f : n \to p$ in (T, Γ, T_0). Then:*

$$\begin{array}{rcl}
\mathrm{wlp}(f,\, \mathrm{wlp}(g, \beta)) & \leq & \mathrm{wlp}(f \cdot g,\, \beta) \\
\mathrm{wlp}(\langle f_1, l, f_n \rangle,\, \beta) & = & [\mathrm{wlp}(f_1, \beta), l, \mathrm{wlp}(f_n, \beta)] \\
\mathrm{wlp}(i_p, \beta) & = & \beta_i \textit{ if } \beta = [\beta_1, l, \beta_p] \\
\mathrm{wlp}(f,\, \beta_1 \wedge \beta_2) & = & \mathrm{wlp}(f, \beta_1) \wedge \mathrm{wlp}(f, \beta_2) \\
\mathrm{wlp}(f,\, \beta_1 \vee \beta_2) & \geq & \mathrm{wlp}(f, \beta_1) \vee \mathrm{wlp}(f, \beta_2) \\
\mathrm{wlp}(f, \textsc{true}_p) & = & \textsc{true}_n \\
\beta \leq \beta' & \to & \mathrm{wlp}(f, \beta) \leq \mathrm{wlp}(f, \beta').
\end{array}$$

When (T, Γ, T_0) has weakest liberal preconditions, the interpolation property has an equivalent form.

Proposition 14.9.15 *Suppose that (T, Γ, T_0) has weakest liberal preconditions. The following conditions are equivalent.*

1. *The triple (T, Γ, T_0) has the interpolation property.*
2. *For all T_0-morphisms $f : n \to p$, $g : p \to q$ and all β in Γ_q,*

$$\mathrm{wlp}(f,\, \mathrm{wlp}(g, \beta)) \;\; = \;\; \mathrm{wlp}(f \cdot g,\, \beta).$$

Proof. Suppose first that (T, Γ, T_0) has the interpolation property. Recalling Proposition 14.9.14, we need to show only that

$$\mathrm{wlp}(f,\, \mathrm{wlp}(g, \beta)) \;\; \geq \;\; \mathrm{wlp}(f \cdot g,\, \beta).$$

Let $\alpha = \mathrm{wlp}(f \cdot g,\, \beta)$, which exists since (T, Γ, T_0) has weakest liberal preconditions. Since (T, Γ, T_0) has the interpolation property, there is some γ with

$$\alpha f \gamma \text{ and } \gamma g \beta \;\; \in \;\; \mathbf{PC}(T, \Gamma, T_0).$$

Hence, $\gamma \leq \mathrm{wlp}(g, \beta)$. By the monotone rule,

$$\alpha f \mathrm{wlp}(g, \beta) \;\; \in \;\; \mathbf{PC}(T, \Gamma, T_0).$$

Thus, by maximality of $\mathrm{wlp}(f,\, \mathrm{wlp}(g, \beta))$,

$$\alpha \;\; \leq \;\; \mathrm{wlp}(f,\, \mathrm{wlp}(g, \beta)),$$

as claimed.

Now suppose that

$$\mathrm{wlp}(f,\, \mathrm{wlp}(g, \beta)) \quad = \quad \mathrm{wlp}(f \cdot g,\, \beta),$$

and that

$$\alpha f \cdot g \beta \quad \in \quad \mathbf{PC}(T, \Gamma, T_0).$$

Let $\alpha' = \mathrm{wlp}(f \cdot g,\, \beta)$, and let $\gamma = \mathrm{wlp}(g, \beta)$. Thus $\alpha \leq \alpha'$, and

$$\alpha' f \gamma \;\in\; \mathbf{PC}(T, \Gamma, T_0).$$

Hence,

$$\alpha f \gamma \text{ and } \gamma g \beta \quad \in \quad \mathbf{PC}(T, \Gamma, T_0),$$

so that (T, Γ, T_0) has the interpolation property. The proof is complete.

There is a nice relationship between the fixed point $f^\dagger$ of the iteration equation for f and another fixed point.

Proposition 14.9.16 *Suppose that (T, Γ, T_0) has weakest liberal preconditions and the interpolation property. Then for each T_0-morphism $f : n \to n + p$ and p-guard β, the n-guard $\mathrm{wlp}(f^\dagger, \beta)$ is the greatest fixed point of the map*

$$\begin{array}{rcl} \Gamma_n & \to & \Gamma_n \\ \xi & \mapsto & \mathrm{wlp}(f,\, [\xi, \beta]). \end{array}$$

Proof. Let $\gamma = \mathrm{wlp}(f^\dagger, \beta)$. Then

$$\begin{array}{rcl} \gamma & = & \mathrm{wlp}(f \cdot \langle f^\dagger, \mathbf{1}_p \rangle,\, \beta) \\ & = & \mathrm{wlp}(f,\, \mathrm{wlp}(\langle f^\dagger, \mathbf{1}_p \rangle,\, \beta)), \end{array}$$

since (T, Γ, T_0) has the interpolation property,

$$\begin{array}{rl} = & \mathrm{wlp}(f,\, [\mathrm{wlp}(f^\dagger, \beta), \mathrm{wlp}(\mathbf{1}_p, \beta)]) \\ = & \mathrm{wlp}(f,\, [\gamma, \beta]), \end{array}$$

by the rules for base morphisms and tupling. Hence γ is a fixed point of the map. If

$$\alpha \quad \leq \quad \mathrm{wlp}(f, [\alpha, \beta]),$$

then

$$\alpha f[\alpha, \beta] \ \in \ \mathbf{PC}(T, \Gamma, T_0),$$

so that by the iteration rule,

$$\alpha f^{\dagger} \beta \ \in \ \mathbf{PC}(T, \Gamma, T_0).$$

Hence, by the maximality of $\mathrm{wlp}(f^{\dagger}, \beta)$,

$$\alpha \quad \leq \quad \mathrm{wlp}(f^{\dagger}, \beta),$$

completing the proof.

The following corollary is one of the main results of this chapter.

Corollary 14.9.17 *Suppose* (T, Γ, T_0) *has (scalar) weakest liberal preconditions and the (scalar) interpolation property. Then its Floyd-Hoare logic is complete.*

Proof. It follows from Proposition 14.9.16 that (T, Γ, T_0) has invariant guards. Indeed, if $\alpha f^{\dagger} \beta$ holds in (T, Γ, T_0) , then for $\gamma = \mathrm{wlp}(f^{\dagger}, \beta)$, both $\alpha \leq \gamma$ and $\gamma f[\gamma, \beta]$ hold. Now, Corollary 14.9.17 follows from the Completeness Theorem.

Exercise 14.9.18 Show that (T, Γ, T_0) has weakest liberal preconditions iff for each T_0-morphism $f : n \to p$ and each β in Γ_p there is some α in Γ_n which is maximal, not necessarily maximum, among the n-guards α' for which

$$\{\alpha'\} \, f \, \{\beta\}.$$

A similar statement holds for strongest liberal postconditions. The proof uses the disjunction and conjunction rules.

14.9.5 The Cook Completeness Theorem

As a corollary to our completeness theorem we can derive the well-known completeness theorem of Cook [Coo78]. Consider again the standard example in Section 14.3. Let $\mathbf{B}$ be a first order structure and let $(T = B^\omega, \Gamma, T_0)$ be the corresponding triple. The triple (T, Γ, T_0) is *expressive* if for each $f : 1 \to p$ in T_0 and each p-guard β, the set

$$\{x \in B^\omega : f \text{ diverges on } x \text{ or } \beta(f(x)) \text{ holds } \}$$

is first order definable. It follows that for each $f : n \to p$ and each p-guard β and each $i \in [n]$, the set

$$\{x \in B^\omega : f \text{ diverges on } (x, i) \text{ or } \beta(f(x, i)) \text{ holds } \}$$

is also first order definable. The Cook completeness theorem says, in effect, that if (T, Γ, T_0) is expressive, its Floyd-Hoare logic is complete. Supposing that (T, Γ, T_0) is expressive, it is obvious that (T, Γ, T_0) has weakest liberal preconditions and the interpolation property. Hence $\mathbf{FH}(T, \varphi, \Sigma_\Pi) = \mathbf{Val}(T, \varphi, \Sigma_\Pi)$, for any appropriate φ, and for the standard homomorphism in particular, by Theorem 14.9.5 above.

Thus, we have provided a completely algebraic proof of the Cook completeness theorem, at least for flowchart programs. Indeed, we have shown that the the standard completeness theorem is derivable from only the equational properties of partial functions. In fact, we have used only Conway theory identities.

Note also that the conditions in Corollary 14.9.17 depend only on the first order properties of the structure $\mathbf{B}$.

14.9.6 The Unwinding Property

We will give yet another more or less well-known condition which, together with the interpolation property, will guarantee that the Floyd-Hoare logic for (T, Γ, T_0) is complete. Recall the definition of the powers of a morphism $f : n \to n + p$ from Definition 5.5.2.4.

Definition 14.9.19 *The triple* (T, Γ, T_0) *has the* **unwinding property** *if, for each partial correctness assertion* $\alpha f^\dagger \beta$ *in* $\mathbf{PC}(T, \Gamma, T_0)$, *with* $f : n \to n + p$, *there is some integer* $k \geq 0$ *with*

$$\alpha f^k [\text{FALSE}_n, \beta] \quad \in \quad \mathbf{PC}(T, \Gamma, T_0).$$

Proposition 14.9.20 *If* (T, Γ, T_0) *has the unwinding and interpolation properties, then* (T, Γ, T_0) *has invariant guards. Thus*

$$\mathbf{FH}(T, \varphi, \Sigma_\Pi) \quad = \quad \mathbf{Val}(T, \varphi, \Sigma_\Pi),$$

for any appropriate homomorphism $\varphi : \Sigma_\Pi \mathbf{TM} \to T$.

Proof. Suppose that

$$\alpha f^\dagger \beta,$$

where $f : n \to n + p$ is a $\Sigma\,\mathbf{TM}$. By the unwinding property, for some k,

$$\alpha f^k[\text{FALSE}_n, \beta].$$

Hence, by the interpolation property, there are n-guard expressions $\gamma_1, l, \gamma_{k-1}$ and p-guard expressions $\delta_1, l, \delta_{k-1}$ such that

$$\begin{array}{rcl} & \alpha f[\gamma_1, \delta_1] & \\ \gamma_i f[\gamma_{i+1}, \delta_{i+1}] \quad (i < k-1) & & \\ & \delta_i & \leq \quad \beta \quad (i < k) \\ & \gamma_{k-1} f[\text{FALSE}_n, \beta]. & \end{array}$$

We let $\gamma := \alpha \vee \gamma_1 \vee l \vee \gamma_{k-1}$. It follows that

$$\{\gamma\}\ f\{[\gamma, \beta]\},$$

so that (T, Γ, T_0) has invariant guards.

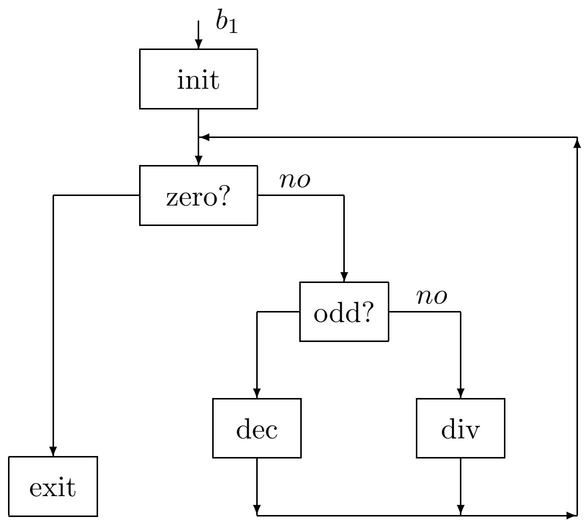
14.10 Examples

This section contains three examples: an example of a correctness proof, an example of a structure which has the interpolation property but does not have invariant guards, and an example of a class of structures which have weakest liberal preconditions but which are not expressive.

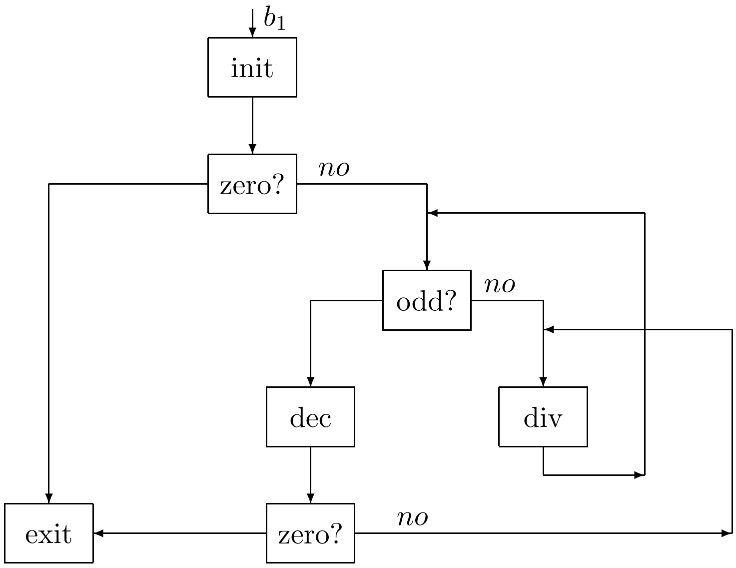
14.10.1 An Example of a Correctness Proof

The flowchart algorithm R below was considered by John Reynolds, in [Rey78]. Reynolds showed why this algorithm is more efficient than the usual while-program to compute x^n, for nonnegative integers x and n. We will show in great detail how our methods enable one to prove the correctness of this algorithm.

The "usual while-program" to compute $y = x^n$ has the following flowchart. The meaning of the expressions in the boxes will be explained below. Nodes labeled b_1, b_2, etc. are "begins".



The flowchart algorithm R is the following:



The programs and predicates used here involve only 5 components of $A = \mathbf{N}^\omega$, where $\mathbf{N}$ is the set of nonnegative integers. Thus we write an element a of A as:

$$a = (x, n, y, z, k).$$

The values of the unnamed components are irrelevant. Here, the functions div, dec and $init : A \to A$ are defined as follows:

$$\begin{array}{rcl} div(x,n,y,z,k) & := & (x,n,y,z^2,k/2) \\ dec(x,n,y,z,k) & := & (x,n,y*z,z,k-1) \\ init(x,n,y,z,k) & := & (x,n,1,x,n). \end{array}$$

The guards $odd?$ and $zero? : A \to A \times [2]$ are defined by:

$$odd?(a) \quad := \quad \begin{cases} (a,1) & \text{if } k \text{ is odd;} \\ (a,2) & \text{otherwise.} \end{cases}$$

$$zero?(a) \quad := \quad \begin{cases} (a,1) & \text{if } k = 0; \\ (a,2) & \text{otherwise.} \end{cases}$$

We consider the triple (T, Γ, T_0), where T is the theory A, where Γ is the collection of first order definable predicates in the structure consisting of $\mathbf{N}$ equipped with at least the operations used to define the functions div, dec, etc., and where T_0 is the flowchart computable partial functions in T.

Now we show how to write R using the iteration theory operations. First, let $G : 2 \to 3$ be the following flowchart program.

```
[begin 1:]  If odd?(a) = (a,1) goto 3 else
           goto begin 2;
[begin 2:]  a := div(a); goto exit 1;
[3:]  a := dec(a); goto 4;
[4:]  if zero(a) = (a,1) goto exit 3 else
           goto exit 2;
[exit 1:]
[exit 2:]
[exit 3:]
```

Then $G^\dagger : 2 \to 1$ is the flowchart program

```
[begin 1:]  If odd?(a) = (a,1) goto 3 else
           goto begin 2;
[begin 2:]  a := div(a); goto begin 1;
[3:]  a := dec(a); goto 4;
[4:]  if zero?(a) = (a,1) goto exit 1 else
           goto begin 2;
[exit 1:]
```

The base morphism $1_2 : 1 \to 2$ can be identified with the following program:

```
[begin 1:]  = exit 1:
[exit 2:]
```

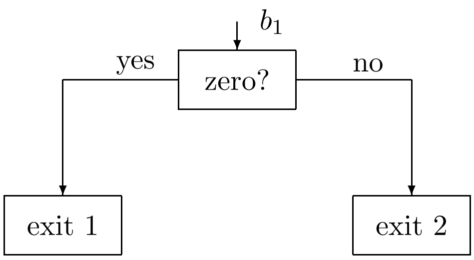
The composite $H := 1_2 \cdot G^\dagger : 1 \to 1$ is the flowchart program:

```
[begin 1:]  If odd?(a) = (a,1) then goto 3 else
            goto 2;
[2:]  a := div(a); goto begin 1;
[3:]  a := dec(a); goto 4;
[4:]  if zero?(a) = (a,1) then goto exit 1 else
            goto 2;
[exit 1:]
```

Let $K : 1 \to 2$ be the program:

```
[begin 1:]  if zero?(a) = (a,1) then goto exit 1
            else goto exit 2;
[exit 1:]
[exit 2:]
```

Note that K is the flowchart algorithm



Lastly, let $F : 1 \to 1$ be the program:

```
[begin 1:]  a := init(a); goto exit 1;
[exit 1:]
```

The algorithm R in question is

$$R \;\; := \;\; F \cdot K \cdot \langle \mathbf{1}_1, H \rangle,$$

which can be written as follows:

```
[begin 1:]  y := 1, z := x, k := n; goto 2;
[2:]  If k is zero then goto exit 1 else goto 3;
[3:]  if k is odd then goto 4 else goto 5;
[4:]  y := y*z, k := k-1; goto 6;
[5:]  z := z*z, k := k/2; goto 3;
[6:]  if k is zero then goto exit 1 else goto 5;
[exit 1:]
```

We would like to prove the partial correctness assertion

$$\alpha F \cdot K \cdot \langle \mathbf{1}_1, H \rangle \beta, \tag{14.39}$$

where $\alpha, \beta : A \to A \times [2]$ are the following guards:

$$\alpha(x, n, y, z, k) \;\; \Leftrightarrow \;\; \text{TRUE}$$

$$\beta(x, n, y, z, k) \;\Leftrightarrow\; (y * z^k \;=\; x^n \wedge k \;=\; 0).$$

We are abusing the notation. We should have written

$$\beta(x, n, y, z, k) \;=\; \begin{cases} ((x, n, y, z, k),\ 1) & \text{if } \ y * z^k = x^n \ \wedge \ k = 0; \\ ((x, n, y, z, k),\ 2) & \text{otherwise.} \end{cases}$$

We can in fact prove easily that

$$\alpha F inv, \tag{14.40}$$

where *inv* is the guard defined by:

$$inv(x, n, y, z, k) \;\Leftrightarrow\; (y * z^k \;=\; x^n).$$

In order to apply the composition rule, we now want to prove

$$invK \cdot \langle \mathbf{1}_1, H \rangle \beta.$$

Note that β is the statement $inv \wedge (k = 0)$. We prove:

$$invK[inv \wedge k = 0,\ inv \wedge k > 0]. \tag{14.41}$$

Trivially,

$$inv \wedge k = 0\mathbf{1}_1\beta. \tag{14.42}$$

Hence, in order to apply the tupling rule, we must show

$$inv \wedge k > 0H\beta, \tag{14.43}$$

i.e.

$$inv \wedge k > 01_2 \cdot G^\dagger \beta.$$

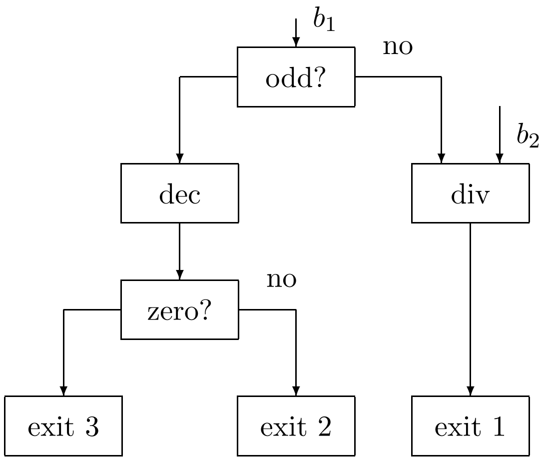
We show

$$\begin{array}{l} \{[inv \wedge k > 0,\ inv \wedge k > 0 \wedge k \text{ even}]\} \\ \quad G \quad \{[inv \wedge k > 0,\ inv \wedge k > 0 \wedge k \text{ even},\ inv \wedge k = 0]\}, \end{array}$$

which means that by the iteration rule,

$$[inv \wedge k > 0,\ inv \wedge k > 0 \wedge k \text{ even}]G^\dagger inv \wedge k = 0.$$

Note that $G : 2 \to 3$ is the flowchart algorithm:



Clearly, by the rule for the distinguished morphism $1_2 : 1 \to 2$,

$$inv \wedge k > 0 1_2 [inv \wedge k > 0,\ inv \wedge k > 0 \wedge k \text{ even }].$$

Hence, by the rule for composition,

$$inv \wedge k > 0 H inv \wedge k = 0.$$

Now, by the tupling rule, (14.42) and (14.43),

$$[inv \wedge k = 0,\ inv \wedge k > 0] \langle \mathbf{1}_1, H \rangle \beta. \tag{14.44}$$

From (14.41) and (14.44) and the composition rule we derive

$$inv K \cdot \langle \mathbf{1}_1, H \rangle \beta. \tag{14.45}$$

Applying the composition rule to (14.45) and (14.40) we obtain (14.39). The correctness of the algorithm is proved.

14.10.2 The Interpolation Property Does Not Imply the Invariant Guard Property

We show that the converse of Proposition 14.9.4 is false. Let B be the collection of finite and cofinite subsets of $\mathbf{N}$. Let $T = \mathbf{N}$ be the iteration theory of all partial functions

$$\mathbf{N} \times [n] \to \mathbf{N} \times [p].$$

Let Γ consist of all finite or cofinite guards, i.e.

$$\alpha \in \Gamma \quad \Leftrightarrow \quad \{x : x\alpha = (x, 1)\} \in B.$$

Let $f : \mathbf{N} \to \mathbf{N} \times [p]$ be any partial function. For each $i \in [p]$ define the set $R_i(f)$ as follows.

$$y \in R_i(f) \quad \Leftrightarrow \quad \exists x(xf = (y, i)).$$

The subtheory T_0 of T is defined by the following property: $f : \mathbf{N} \to \mathbf{N} \times [p]$ belongs to T_0 iff the set $R_i(f)$ belongs to B, for each $i \in [p]$.

Note that f is in T_0 iff for each cofinite set X,

$$\{y : (\exists x \in X)(xf = (y, i))\} \ \in \ B.$$

A vector morphism $\langle f_1, l, f_n\rangle : n \to p$ is in T_0 iff $f_i \in T_0$, for each $i \in [n]$.

Claim. If $f : 1 \to p$, and $g = \langle g_1, l, g_p\rangle : p \to q$ in T_0 then the composite $f \cdot g$ also is in T_0 .
Proof. For i in $[q]$,

$$R_i(f \cdot g) \quad = \quad \{y : \exists x(xfg = (y, i))\}.$$

Then,

$$R_i(f \cdot g) \quad = \quad Z_1 \cup Z_2 \cup l \cup Z_p,$$

where, for each j in $[p]$,

$$Z_j \quad := \quad \{y : \exists x, z(xf = (z, j) \text{ and } zg_j = (y, i))\}.$$

But it is clear that each set Z_i is finite or cofinite. It follows that T_0 is closed under composition.

Since the distinguished morphisms $i_n : 1 \to n$ are clearly in T_0, T_0 is a subtheory of T.

Claim. T_0 is closed under scalar iteration.
Proof. For ease of notation, suppose that $f : 1 \to 2$ in T_0, i.e. $f : \mathbf{N} \to \mathbf{N} \times [2]$. Then

$$xf^\dagger \ = \ y$$

iff there is a finite sequence $x = x_0, l, x_n$, $n \geq 0$, with $x_i f = (x_{i+1}, 1)$ for $i < n$, and $x_n f = (y, 2)$. But note that

$$\{y : \exists x(xf^\dagger = y)\} \quad = \quad \{y : \exists x(xf = (y, 2)\}.$$

Indeed, if $xf^\dagger = y$, for some x, there is an x' with $x'f = (y, 2)$, so that the left-hand set is a subset of the right. But the converse inclusion is immediate. Thus, since the right-hand set is in B if $f \in T_0$, $f^\dagger$ is in T_0.

Hence, by the pairing identity, T_0 is closed under vector iteration and is thus a subiteration theory of T.

We show that the triple (T, Γ, T_0) has the interpolation property. Indeed, suppose that

$$\alpha f \cdot g\beta,$$

and $f, g : 1 \to 1$ are in T_0. Let X and Y be the sets defines as follows:

$$\begin{aligned} X \quad &:= \quad \{x : x\alpha = (x, 1)\} \;=\; \{x : x\alpha \text{ is true }\} \\ Y \quad &:= \quad \{y : \exists x \in X(xf \;=\; (y, 1))\}. \end{aligned}$$

Finally, let γ be the guard which is true exactly on the set Y. Then γ is in Γ, since Y is finite or cofinite. Also

$$\alpha f \gamma \quad \text{and} \quad \gamma g \beta,$$

as is easily seen. (A similar argument works when $f : 1 \to p$ and $g : p \to q$.)

Note also that γ is the strongest liberal post condition of α and f.

Now we show T_0 does not have invariant guards. Define $f : 1 \to 2$ by:

$$xf \quad := \quad \begin{cases} (x+3, 1) & \text{if } x \equiv 0 \bmod 3 \\ (xg, 1) & \text{if } x \equiv 1 \bmod 3 \\ (xh, 2) & \text{if } x \equiv 2 \bmod 3. \end{cases}$$

Here, the functions g and h are defined by:

$$xg \quad := \quad \begin{cases} n & \text{if } x = 3n+1 \\ 0 & \text{otherwise.} \end{cases}$$

$$xh \quad := \quad \begin{cases} n & \text{if } x = 3n+2 \\ 0 & \text{otherwise.} \end{cases}$$

Since $R_1(f) = R_2(f) = \mathbf{N}$, it follows that f is in T_0. Now let $\delta : 1 \to 2$ be the guard which is true exactly on the number 3. Since $f^\dagger$ is not defined at 3,

$$\delta f^\dagger \text{FALSE}.$$

If (T, Γ, T_0) has invariant guards, there is some γ in Γ with $\delta \leq \gamma$ such that

$$\gamma f[\gamma, \text{FALSE}]. \tag{14.46}$$

Now we show that there is no nonzero guard $\gamma \in \Gamma$ such that equation (14.46) holds. Indeed, if (14.46) holds, let

$$X \quad := \quad \{x : x\gamma = (x, 1)\}.$$

Then if $x \in X$, $xf = (x', 1)$ for some $x' \in X$, or else $x'\text{FALSE}$ would have the value $(x', 1)$. Thus, X contains no number which is congruent to 2 mod 3. But X cannot be finite and nonempty nor can X be cofinite. Indeed, suppose X is finite. Then X contains only numbers which are congruent to 1 mod 3. Indeed, we have just seen that X contains no number congruent to 2 mod 3. If $3k \in X$, then $3k + 3$ is in X, by the definition of f, and thus X is infinite. Let $a = 3n + 1$ be the smallest number in X. Then $af = (n, 1)$. But $n < a$ and $n \in X$, a contradiction.

Now suppose that X is cofinite. Then X contains some number of the form $x = 3n + 2$, which is impossible. Thus γ is not in Γ. This shows that (T, Γ, T_0) does not have invariant guards, as claimed.

Remark 14.10.1 This triple (T, Γ, T_0) has strongest liberal post conditions. Indeed, for $f : n \to p$ in T_0 and α an n-guard in Γ_n, the strongest liberal post condition of (α, f) exists, since the following function β is in Γ_p :

$$\beta(y, i) \quad := \quad \begin{cases} (y, i) & \text{if } \exists\, (x, j) \in \mathbf{N} \times [n] \\ & \qquad [\alpha(x, j) = (x, j) \text{ and } f(x, j) = (y, i)] \\ (y, p + i) & \text{otherwise.} \end{cases}$$

Note however, (T, Γ, T_0) cannot have all weakest liberal preconditions, or else it would have the invariant guard property.

14.10.3 A Non-Expressive Structure with Weakest Liberal Preconditions

We give an example of a first order structure **A** with weakest liberal preconditions but, for some f and β, the set

$$\{x \in A : f \text{ diverges on } x \text{ or } \beta(f(x))\}$$

is not first order definable.

Indeed, suppose that **A** and **B** are similar first order structures. Let

$$T_A \;=\; (A^\omega, \Gamma_A, T_0)$$

and

$$T_B \;=\; (B^\omega, \Gamma_B, T_1)$$

be the corresponding triples. Recall the standard homomorphisms

$$\begin{aligned} \varphi_A : \Sigma_\Pi \mathbf{TM} \;&\to\; A^\omega \\ \varphi_B : \Sigma_\Pi \mathbf{TM} \;&\to\; B^\omega \end{aligned}$$

from Section 14.6. It was shown in [BT82] in effect that if **A** and **B** are elementarily equivalent, then

$$\mathbf{Val}(A^\omega, \varphi_A, \Sigma_\Pi) \;=\; \mathbf{Val}(B^\omega, \varphi_B, \Sigma_\Pi). \tag{14.47}$$

Now suppose that **A** and **B** are elementarily equivalent and that T_A has weakest liberal preconditions. We show that T_B does also, by proving that the weakest liberal precondition of $(f\varphi_B, \beta\varphi_B)$ exists in T_B, for any Σ-iteration term $f : 1 \to p$ and any p-guard expression β. Let $\alpha\varphi_A = \mathrm{wlp}(f\varphi_A, \beta\varphi_A)$ in T_A. Then

$$\{\alpha\}\ f\ \{\beta\} \;\in\; \mathbf{Val}(A^\omega, \varphi_A, \Sigma_\Pi);$$

and

$$\alpha' f \beta \;\in\; \mathbf{Val}(A^\omega, \varphi_A, \Sigma_\Pi) \quad\to\quad \alpha'\varphi_A \;\leq\; \alpha\varphi_A.$$

Thus, by equation (14.47),

$$\alpha f \beta \;\in\; \mathbf{Val}(B^\omega, \varphi_B, \Sigma_\Pi).$$

Now, suppose that $\alpha' f \beta \in \mathbf{Val}(B^\omega, \varphi_B, \Sigma_\Pi)$. Since $\alpha'\varphi_A \leq \alpha\varphi_A$, the universal closure γ of $(\alpha' \to \alpha)$ is true in **A**. Thus this same sentence is true in **B**, showing that

$$\alpha'\varphi_B \;\leq\; \alpha\varphi_B.$$

Thus T_B has weakest liberal preconditions also.

Lastly, if $\mathbf{A}$ is the standard model of Peano arithmetic, it is known that $\mathbf{A}$ is expressive and hence has weakest liberal preconditions. So, by the above, any nonstandard model of Peano arithmetic has weakest liberal preconditions. But, as pointed out in [BT82], no nonstandard model is expressive. Indeed, if f is a program which halts exactly on the standard integers, e.g.,

$$\textit{while } x \neq 0 \textit{ do } x := x - 1$$

and if $\beta := \textsc{false}$, then the set

$$\{x \in A : f \text{ diverges on } x \text{ or } \beta(f(x))\}$$

is precisely the nonstandard integers, which is not first order definable.

14.11 Notes

This chapter is based on [BÉ91b]. Theorem 14.1.9 appears in [BÉM90]. Part of this theorem was derived using different hypotheses by Urbanik, [Urb65]. The exercises in Section 14.1 treat material in [BÉ91a]. In Chapter 4.4 of the book [MMT87], the exercises 12 and 13 treat idempotent and diagonal binary functions on lattices. The functions if_b of Definition 14.1.8 are in fact homomorphisms $B^2 \to B$, i.e. they are decomposition operations in the sense of [MMT87], page 302. See also exercise 10 in Chapter 5.6 of that book. Our definition 14.2.1 of a partial correctness assertion as an equation was inspired by a section in the book [MA86], where it was shown how to formulate partial correctness by an equation in a partially additive category. The example in Section 14.10.3 is taken from [BT82]. Cook's completeness theorem can be found in [Coo78]. There are several nice surveys of the history of partial correctness logic, for example [Jon92, Apt81, Apt84].

Index

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