The World of Generating Functions and Umbral Calculus

D. Loeb
LaBRI (URA CNRS 1304)
Université de Bordeaux I
33405 Talence, France
loeb@geocub.greco-prog.fr

November 16, 1995

Dedicated to Gian-Carlo Rota for his inspiration and his friendship.

1. Incidence Algebras and Generating Functions

1.1. Summary of [DRS 1972]

The idea of a generating function is a powerful tool by which one may calculating the many sequences (even tables) of numbers which arise in combinatorics. [Sloane 1973] For example, let $F(x) = \sum_{n=0}^{\infty} f_n x^n$ where f_n denotes the nth Fibonacci number. $F(x)$ is called the (ordinary) generating function of f_n. Using this generating function, the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and initial conditions $f_0 = f_1 = 1$ can be expressed $(1 - x - x^2)F(x) = 1$ or more simply

$$(1).\ F(x) = -1/(x^2 - x - 1).$$

We then may expand $F(x)$ by partial fractions

$$F(x) = \frac{1}{\sqrt{5}(1 - \gamma_+ x)} - \frac{1}{\sqrt{5}(1 - \gamma_- x)}$$

where $\gamma_\pm = (1 \pm \sqrt{5})/2$ is the golden ratio. We thus derive Binet's formula

$$f_n = (\gamma_+^n - \gamma_-^n)/\sqrt{5}.$$

Other sequences a_n are better expressed by their exponential generating functions $A(x) = \sum_{n=0}^{\infty} a_n x^n/n!$. Here, the sequence is divided by factorials before being used as coefficients of x^n. For example, [Rota 1964] the Bell numbers B_n denote the number of partitions of a set of n elements. We then have the beautiful identity $B(x) = \sum_{n=0}^{\infty} B_n x^n/n! = \exp(\exp(x) - 1)$ from which many properties of the Bell numbers may be derived.

Doubilet, Rota, and Stanley [1972] explain the mathematical underpinnings of such identities. We see for example why sequences such as Fibonacci's are easily expressed in terms of ordinary generating functions, and why the Bell numbers require exponential generating functions. In fact, other classes of generating functions exists, and are occasionally needed for combinatorial calculations. For example, Eulerian generating functions $\sum_{n=0}^{\infty} s_n x^n/[n]!$ where $[n]! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ can be used to study the number of subspaces of a finite vector space.

Thus, Doubilet, Rota, and Stanley have accomplished a *tour de force* — presenting a unified treatment of all known types of generating functions, and justifying our expectations as to what new types are yet to be found. [Stanley 1986] "The explanatory paradigm based on incidence algebras is this: connected with each special algebraic operator on a 'variety' of generating functions is a family of partially ordered sets.... The fundamental operation of *convolution* in the incidence algebra reflect the algebraic operator in questin on generating functions. In this way, the particular algebraic operation acquires a combinatorial interpretation." [HS 1989]

The incidence algebra is a generalization of the algebra of upper triangular matrices. A matrix $M \in K^{n\times n}$ is upper triangular if for all $i,j \in \{1,2,\ldots,n\}$, $M_{ij} = 0$ unless $i \le j$. A "matrix" $M \in K^{P\times P}$ can be defined with rows and columns indexed by elements of some poset P. The condition analogous to upper triangularity is that for all $a,b \in P$, $M_{ab} = 0$ unless $a \le b$. The incidence algebra $\mathbf{I}(P)$ is the set of "matrices" which obey this condition. The closure of $\mathbf{I}(P)$ with respect to multiplication follows immediately from transitivity.

Ordinary upper triangular matrices correspond to the special case where P is an n-element chain. (The other extreme, n-element antichains, corresponds to diagonal matrices.) The incidence algebra encodes all the essential information about the poset; that is, two incidence algebras are isomorphic if and only if their underlying posets are isomorphic. [Stanley 1970] However, not all algebras can be interpreted as incidence algebras. Feinberg [1977] gives a characterization which allows one to determine which algebras are incidence algebras and which coalgebras are incidence coalgebras (cf: Theorem 3.1 for a characterization of the ideals of an incidence algebra).

Using the analogy of upper triangular matrices again, consider the algebra $\mathcal{A}$ of upper triangular infinite Toeplitz matrices M (ie: with constant diagonal $M_{i,i+k} = M_{j,j+k}$). Such matrices are described by their value on each diagonal $d_k = M_{i,i+k}$. Given two such matrices $M, M' \in \mathcal{A}$ with diagonals d_k and d'_k respectively; their product MM' has constant diagonals as well. They are given by the formula $d_0 d'_k + d_1 d'_{k-1} + \cdots + d_k d'_0$ which is exactly the relation between the coefficients of the product of two formal power series. That is to say, $\mathcal{A}$ is isomorphic to the algebra of formal power series.

Another well-known proof of the closed formula for the Fibonacci numbers comes from the matrix identity:

$$\begin{bmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{bmatrix} = M^n \text{ where } M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Diagonalizing M (its characteristic polynomial is $x^2 - x - 1$ from eq. (1) then leads to the closed formula for f_n.

In fact, these two proofs are seen to be one and the same in the light of the isomorphism between formal power series and upper triangular matrices with constant diagonals.

The authors generalize this notion by the way of the reduced incidence algebra. We require $M_{ij} = M_{kl}$ whenever $[i, j]$ and $[k, l]$ are "equivalent" intervals in the poset in question. For example, to rediscover exponential generating functions we take $[A, B] \sim [C, D]$ in the Boolean algebra whenever $B \setminus A = D \setminus C$.

For technical reasons, large reduced incidence algebras must be introduced when the intervals to be considered is no longer form a set but rather a proper class. However, in the majority of cases this use of category theory can be avoided either by considering a single poset which is the disjoint union of the posets under consideration, or by taking the limit of corresponding results calculated in ordinary reduced incidence algebras.

1.2. Coalgebras

Rota and Joni wrote in [1971] about the many applications of coalgebra theory to combinatorics. The combinatorial interpretation of multiplication is "putting things together." Joyal's theory of species in [1981] is essentially a formalization of this important concept. On the other hand, the comultiplication concerns itself with "taking things apart". Amazingly, both combinatorial coalgebra theory and species theory find their roots in [DRS 1972].

Given a poset, the comultiplication $\Delta[u, v]$ is the formal sum over all ways to bisect the interval $[u, v]$,

$$\Delta X^{[u,v]} = \sum_{u \le w \le v} X^{[u,w]} \otimes X^{[w,v]}.$$

Along with the counitary map

$$\epsilon X^{[u,v]} = \begin{cases} 1 & \text{if } u = v, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

this defines the incidence coalgebra on P. The reduced incidence coalgebra can be similarly defined. [Schmitt 1987]

On one hand, these coalgebras are merely the dual of their algebraic equivalents, yet they generate bialgebras, and under certain mild hypotheses even Hopf algebras. These new structures are very rich, and give a new deep interpretation of the theory of Incidence Algebras as well as a combinatorial interpretation of Lagrange inversion. [HS 1989] These results are significant extensions of [DRS 1972] since "computing the antipode is equivalent to computing the convolution inverse of all of the functions of an incidence algebra." [RS 1990]

1.3. The Partition Lattice

The category $\mathbf{\Pi}$ of lattices $\Pi(S)$ of partitions of a finite set S [DRS 1972, p. 270, 285] give rise to the large reduced incidence Hopf algebra $\mathcal{F}$ called the Faà di Bruno Hopf Algebra treated in [Schmitt 1987]. The induced incidence algebra $\mathbf{I}(\mathbf{\Pi})$ is anti-isomorphic to the algebra of all exponential formal power series under composition (See Theorem 5.1).

Why do we have an unaesthetic anti-isomorphism here instead an isomorphism? This is the first case of a morphism of an incidence algebra into a non-Abelian group. Only here for the first time, does an anti-isomorphism mean anything different from an isomorphism. Has the product rule been defined backwards?

I contend that the incidence algebra has been defined quite naturally. It is rather the definition of the partition lattice which is upside-down. Given two partitions π and ϕ we say $\pi \leq \phi$ if π is *finer* than ϕ [DRS 1972, p. 270] On one hand a finer partition has smaller blocks, but on the other hand it has more blocks. More importantly, the equivalence relation for π is contained in the equivalence relation for ϕ if and only if π is coarser than ϕ. If we set $\phi \leq \pi$ when π is *coarses* than ϕ, then we have an isomorphism, for $\mathbf{I}(\mathbf{\Pi}^*)$ is isomorphic to the algebra of exponential formal power series under composition.

1.4. Automated calculation

Some recent work has been done on automated calculations involving generating functions.

For example, the `Gfun` package (short for **G**enerating **Fun**ction) by F. Bergeron, S. Plouffe, B. Salvy, and P. Zimmermann can guess ordinary and exponential generating functions from the first few terms of its generating function.

"Usually, in the physics literature, the distinction between establishing a formula from computer experiments and giving a mathematical proof is not clearly states.... When the number of known elements of the sequence is much bigger than the number of elements needed to guess the exact formula, the probability that the formula is wrong is infinitesimal, and thus the formula can be considered as an experimentally true statement. But mathematically, it is just a conjecture waiting for a proof, and maybe more: a crystal-clear [bijective] understanding." [Viennot 1992, p. 412]

Indeed, bijective combinatorics was introduced in part by Rota [MR 1970, GR 1970]; equations (and inequalities) are proven via the construction of bijections (and injections) between classes of objects having the given number of elements. Algebraic manipulations are replaced with constructions of correspondences. The identities become the reflection of the combinatorial properties of the finite objects involved. The identities are thus seen on a deeper level, allowing us to understand their *raison d'être*. Before creating the correspondence, the first step consists of interpreting the two sides of the identity combinatorially, i.e. finding the "right" combinatorial objects to count.

On the other hand, given a generating function one would like a formula for its coefficients. In theory, Taylor's formula suffices to compute any particular coefficient, but not to compute the general formula.

Actually, a general formula might not be available. However, Flajolet and Odlyzko [1990] have developed techniques (inspired from Hardy and Littlewood [1914]) by which to compute asymptotic series for the coefficients of a series by studying its singularities. For example, the generating function of 2-regular graphs $f(z) = e^{-z/2-z^2/4}/\sqrt{1-z}$ has the expansion $f(z) = e^{-3/4}(1/\sqrt{1-z} + \sqrt{1-z} + \cdots)$ which leads to a matching expansion for number of 2-regular graphs $f_n \sim e^{-3/4}n!(1/\sqrt{\pi n} + c/\sqrt{n^3} + \cdots)$. Indeed, "Generating functions are a bridge between discrete mathematics on the one hand, and continuous analysis ... on the other. The full beauty of the subject emerges only from tuning in on both channels." [Wilf 1990]

Given a grammatical expression of the defining relations of some combinatorial object, the program $\lambda\Upsilon^{\Omega}$ [FSZ 1991] can compute its generating function, and asymptotically analyze the coefficients.

1.5. Open Questions

Sergey Fomin [preprint] has generalized the Robinson-Schensted correspondence of Young tableaux by replacing the problem of enumeration of pairs of paths in the Young's lattice with the more general problem of enumeration of pairs of paths in "(**q**,**r**)-dual" posets P. The number of paths between two points is given in the incidence algebra by $1/(1-\eta_P)$ where η_P represents the covering relation in P. However, the pairs of paths being considered must go in different directions. That is, Fomin's theory if rewritten in the language of incidence algebras would involve a product $1/(1-\eta_P)(1-\eta_{P^*})$ of elements of $\mathbf{I}(P)$ and $\mathbf{I}(P^*)$. What can said about such combined incidence algebras? Does Fomin's theory generalize or simplify in such a context?

Certain generating functions actually have physical interpretations. For example, in the theory of "heaps," the generating function $\rho(t)$ for "pyramids" of segments gives (up to sign) the density of a gas as a function of its activity $-t$.

2. Umbral calculus and binomial enumeration

2.1. Summary of the *Theory of Binomial Enumeration* [MR 1970] and the *Number of Partitions of a Set* [Rota 64].

The theory of binomial enumeration is variously called the calculus of finite differences or the umbral calculus. The term "umbra" means "shadow," since this theory studies the analogies between various sequences of polynomials p_n and the powers sequence x^n. The subscript n in p_n was thought of as the shadow of the superscript n in x^n, and many parallels were discovered between such sequences.

Take the example of the lower factorial polynomials $(x)_n = x(x-1)\cdots(x-n+1)$. Just as x^n counts the number of functions from an n-element set to an x-element set, $(x)_n$ counts the number of injections. Just as the derivative maps x^n to nx^{n-1}, the forward difference operator maps $(x)_n$ to $n(x)_{n-1}$. Just as also polynomials can be expressed in terms of x^n via Taylor's theorem

$$f(x+a) = \sum_{n=0}^{\infty} p_n(a)\frac{D^n f(x)}{n!} = \sum_{n=0}^{\infty} \frac{a^n D^n f(x)}{n!},$$

Newton's theorem allows similar expressions for $(x)_n$

$$f(x+a) = \sum_{n=0}^{\infty} (a)_n \frac{\Delta^n f(x)}{n!}$$

where $\Delta f(x) = f(x+1) - f(x)$. Just as $(x+y)^n$ is expanded using the binomial theorem

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k},$$

$(x+y)_n$ expands by Vandermonde's identity

$$(x+a)_n = \sum_{k=0}^{\infty} \binom{n}{k} (a)_k (x)_{n-k}.$$

And so on. [MR 1970, LR 1991]

This theory is quite classical with its roots in the works of Barrow and Newton—expressed in the belief the some polynomial sequences such as $(x)_n$ really were just like the powers of x. Nevertheless, many doubts arose as to the correctness of such informal reasoning, despite various attempts to set it on an axiomatic base. [Bell 1927, 1940]

The contribution of Rota's school was to first set umbral calculus on a firm logical foundation rooted in coalgebra theory, and category theory (although these connections are often implicit). [MR 1970, and Rota 1964] (See [BB 1956] for a generating function approach.)

That being done, sequences of polynomials of binomial type

$$(2).\ p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$$

could be for once studied systematically thanks to Rota's "operator methods" rather than as a collection of isolated yet philosophically similar results. The sister sequence of divided powers $q_n(x) = p_n(x)/n!$ then obeys identity $q_n(x+y) = \sum_{k=0}^{n} q_k(x) q_{n-k}(y)$.

Umbral substitution [MR 1970, Section 7] is the formalization of the poorly understood operation of raising and lowering indices $x_N \equiv x^N$. [Bell 1940, Axiom 1.5] Indeed, eq. (2) would be written by umbralists as [Bell 1940, Axiom 1.124]

$$\mathbf{p}_n(x+y) = [\mathbf{p}(x) + \mathbf{p}(y)]^n.$$

Indeed the content of [Rota 1964] is essentially that the Stirling numbers provide the "connection constants" necessary to pass from the lower factorial polynomials to the powers of x.

The current direction of research is now to broaden the scope of the theory while retaining its far-reaching power.

2.2. Species

One such generalization is Joyal's theory of species [1981]. We can see the seeds of this theory even in [MR 1970]: See for example "We can associate ... functorially" (p. 173) and the applications in [ibid, section 10]. In the language of species, a rooted tree is a point times a collection of rooted trees; a permutation is a collection of cycles; and endofunctions (functional digraphs) are the composition of permutations with rooted trees. The results in section 10 all follow from these species theoretic considerations. (See related article by Joyal in this volume.)

This approach towards Species was echoed in the theory of prefabs of [BG 1971] and the theory of partitional compositions of [FS 1970, §3.1–3]. A prefab $(S, \circ, f)$ is a set of structures S together with a multivalued binary function $\circ$ which indicates the ways to combine structures, and a real valued function f which measures the underlying symmetries. Though Bender and Goldman do not express their theory in category theoretic terms $S, \circ, f$ must obey certain axioms, and the choice of f reflects the choice between ordinary generating functions, exponential generating functions, etc. or equivalently the choice between linear species, ordinary species, etc.

In [Loeb 1990], the umbral calculus is generalized to symmetric functions. When counting enriched functions (functions, injections, reluctant functions, dispositions, etc.) from N to X, we can assign a weight to each function according to its fiber structure. $w(f) = \prod_{i \in N} f(i) = \prod_{x \in X} x^{|f^{-1}(x)|}$. The total number of such functions is a symmetric function $p_n(X)$ of degree n where $n = |N|$. The elementary and complete symmetric functions are (up to a multiple of $n!$) good examples of such sequences. They obey their own sort of binomial theorem

$$p_n(X \cup Y) = \sum_{k=0}^{n} \binom{n}{k} p_k(X) p_{n-k}(Y).$$

The generating function of $p_n(X)$ is directly related to that of its underlying species. By specializing all x variables to 1, we return to the study to polynomials.

Nevertheless, such a sequence $p_n(X)$ is not a basis for the vector space of symmetric functions. Furthermore, the $p_n(X)$ may not even be algebraically independent. Thus, one would hope for an umbral calculus of *full* sequences of symmetric functions $p_\lambda(X)$ indexed by an integer partition λ. Such a generalization is in progress involving a generalization of species provisionally called genera. Two unresolved (and probably intimately related) questions remain.

First, what is the exact relation between genera and the more elegant theory of colored species exposed by Méndez, and Nava?

Secondly, an important property of any extension of the umbral calculus is that it have its own generalization of Lagrange's inversion formula (as follows from the closed forms for basic polynomials [MR 1970, Theorem 4]). Composition of symmetric functions corresponds to plethysm. Can the plethystic inverse of a genera or its associated symmetric function be calculated as is the case with colored species?

2.3. Coalgebras

A comultiplication over the space of polynomials $K[x]$ is a map from $K[x]$ to $K[x] \otimes K[x]$ or equivalently $K[x, y]$ obeying the conditions of coassociativity and counicity. The translation operator $\Delta f(x) = f(x + y)$ verifies these conditions, since it fixes all constants and $f(x + (y + z)) = f((x + y) + z)$. In fact, along with the usual multiplication of polynomials and the antipode $f(x) \mapsto f(-x)$, the space of polynomials can be seen to be a graded Hopf algebra. Umbral calculus is thus essentially the study of this Hopf algebra.

Coalgebra maps Q are linear functions which commute with comultiplication (and preserve degree). Define $p_n(x) = Qx^n$.

$$p_n(x + y) = Q(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} (Qx^k)(Qy^{n-k}) = \binom{n}{k} p_k(x) p_{n-k}(y).$$

Thus, coalgebra maps are precisely the umbral substitution maps [MR 1970, Section 7] which one sequence of polynomials of binomial type onto another. Similarly, the entire classical theory of umbral calculus can be translated into coalgebraic notation.

However, the translation operator is not the only graded coalgebra map available. Other choices Δ may be made, each leading to its own Δ-Umbral calculus. Instead of studying shift-invariant operators, one studies Δ-invariant operators. A polynomial sequences can thus be said to be "Sheffer" with respect to a certain comultiplication Δ, and pair of Δ-invariant operators S, T. One can show [DL 1993, Levitan 1964, Markowsky 1978, and Viskov 1975] that every polynomial sequence p_n (where $deg(p_n(x)) = n$) can be classified as "Sheffer" via a unique choice of Δ, S, T. One particularly interesting choice of such a generalized shift operator is the q-shift [Andrews 1971, II 1981, et al] and their sequences of "Eulerian" type. More work is probably needed to find a "complete" exposition of the q-umbral calculus and its connections with q-theory.

Seemingly this is the ultimate generalization of the umbral calculus since it includes all polynomial sequences. However, [HR 1992] removes the condition that the coalgebra be graded by considering the divided difference operator

$$\Delta p(x) = \frac{p(x) - p(y)}{x - y}$$

as their generalized shift operator. This "Newtonian" umbral calculus is particularly rich in interesting examples. The expansion formulas here generalize L'Hopital's rule as well as Newton and Lagrange's interpolation formulas.

2.4. Operator Expansions

Rota's operator approach to the calculus of finite difference can be thought of as a systematic study of shift-invariant operators on the algebra of polynomials. The expansion theorem [MR 1970, Theorem 2] states that all shift-invariant operators can be written as formal power series in the derivative D. If $\theta : \mathbf{C}[x] \to \mathbf{C}[x]$ is a shift-invariant operator, then

$$\theta = \sum_{k=0}^{\infty} a_n D^n / n!$$

where $a_n = [\theta x^n]_{x=0}$.

However, a generalization of this by Kurbanov and Maksimov [1986] to arbitrary linear operators has received surprisingly little attention. Any linear operator $\theta : \mathbf{C}[x] \to \mathbf{C}[x]$ can be expanded as a formal power series in X and D where X is the operator of multiplication by x. More generally, let B be any linear operator which reduces the degree of nonzero polynomials by one. (By convention, $\deg(0) = -1$.) Thus, B might be not only the derivative or any delta operator, but also the q-derivative, the divided difference operator, etc. Then θ can be expanded in terms of x and B.

$$\theta = \sum_{k=0}^{\infty} f_n(X) B^n.$$

The coefficients $f_n(X)$ are calculated as follows. Let $p_n(x)$ be a generalized Sheffer sequence for the operator B; i.e.: $Bp_n(x) = np_{n-1}(x)$. For example, if $B = D$, then $p_n(x)$ might equal x^n or any Appell sequence of polynomials. Let $\phi(x, \lambda) = \sum_{k=0}^{\infty} p_n(x)\lambda^n$ be the ordinary generating function for $p_n(x)$. Note that if B is a shift-invariant operator, $B = f(D)$, then $\phi(x, \lambda)$ is easily calculated $p(x, \lambda) = \exp(xf^{(-1)}(\lambda))$ [MR 1970, p. 189] In particular, if $B = D$, then we can take $\phi(x, \lambda) = \exp(x\lambda)$. The generating function for f_n is then given by

$$\sum_{n=0}^{\infty} f_n(x)\lambda^n = \frac{\theta\phi(x, \lambda)}{\phi(x, \lambda)}.$$

In fact, any sequence of polynomials $p_n(x)$ (with $\deg(p_n) = n$) is generalized "Sheffer" with respect to some operator B in the weak sense above. (See [DL 1993, Levitan 1964, Markowsky 1978, and Viskov 1975]. Compare with [BB 1957, §10–11] for an analytic approach.)

Many classical operator identities are simply trivial consequences of [KM 1986]. For example, the integration operator $Intp(x) = \int_0^x p(t)dt$ can be expanded in terms of X and D as follows

$$Int = \sum_{n=0}^{\infty} (-1)^n X^{n+1} D^n/(n+1)!$$

giving an elementary proof of Bourbaki's method of asymptotic integration. [Bourbaki 1949] Further investigation is warranted. In particular, possible applications of the expansion formulas to numerical analysis need to be investigated, and he integral formulas need to be clarified.

2.5. Extended umbral calculi

Part of the original umbral calculus dealt with special functions in general whereas the formulation in [MR 1970] restricted itself to polynomial sequences.

Many researchers have endeavoured to increase the scope of the umbral calculus by allowing sequences of more and more complicated objects.

Roman [1982, 1983, and 1984] developed a version to the umbral calculus for inverse formal power series of negative degree. Most theorems of umbral calculus have their analog in this context. In particular, any shift-invariant operator of degree 1 (delta operator) has a special sequence associated with it—satisfying a type of binomial theorem. Nevertheless, despite its philosophical connections, this theory remained completely distinct from Rota's theory treating polynomials.

Later, in [LR 1989], a theory was discovered which generalized simultaneously Roman and Rota's umbral calculi by embedding them in a logarithmic algebra containing both positive and negative powers of x, and logarithms. A subsequent generalization [Loeb 1991] extends this algebra to a field which includes not only x and $\log(x)$ but also the iterated logarithms, all of whom may be raised to any real power. Sequence of polynomials $p_n(x)$ are then replaced with sequence of asymptotic series p_a^α where the degree a is a real and the level α is a sequence of reals. Rota's theory is the restriction to level $\alpha = (0)$, and degree $a \in \mathbf{N}$. Roman's theory is the restriction to level $\alpha = (1)$ and degree $a \in \mathbf{Z}^-$. Thus, the difficulty in uniting Roman and Rota's theories was essentially that they lay on different levels of some larger yet unknown algebra.

Di Bucchianico [1991] retained the formalism of eq. (2) while allowing the sequence $p_n(x)$ to be composed of arbitrary analytic functions. The resulting sequences can be expressed up to multiplication by e^{cx} for some $c \in \mathbf{R}$ as a polynomial sequence $q_n(x)$ obeying eq. (2). Note however that $q_n(x)$ is not necessarily of degree n.

Another generalization give umbral calculi in several variables (even infinitely many). [BBN 1983, Chen Preprint] At the same time, Barnabei, Brini, and Nicoletti, allow a wider choice of coefficients for the polynomials. The base field is usually taken to be the complex $\mathbf{C}$ or a least some field of characteristic zero.

However, [BBN 1983] allows "polynomials" (called *factorial functions*) taking *values* in any commutative integral domain **A** with unity. For example, for $\mathbf{A} = \mathbf{Z}$, the polynomials $\binom{x}{n}$ form a basis for the Z-module of polynomials $p : \mathbf{Z} \to \mathbf{Z}$. Here, an umbral calculus of polynomial sequences of "integral type" can be devised with $\binom{x}{n}$ playing the role normally played by x^n.

Still further generalization of Rota's umbral calculus abound. would be fruitful to study the relations between them, the generalized translation operators mentioned above, and the so-called Cauchy problems. However, Further generalizations surely remain to be found, since for the moment a great part of the theory of orthogonal polynomials is outside of the scope of any of the current generalizations of the umbral calculus.

3. Bibliography

3.1. Main references.

[DRS 1972] P. Doubilet, G.-C. Rota and R. P. Stanley, *On the Foundations of Combinatorial Theory. VI. The idea of Generating Function* in "Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability," (University of California, Berkeley, 1970/1971), Volume 2, Probability Theory, University of California Press, California (1972) 267–318. (MR53#7796)

[MR 1970] R. Mullin and G.-C. Rota, *On the Foundations of Combinatorial Theory. III. Theory of Binomial Enumeration* in "Graph Theory and its Application," editor: B. Harris, Academic Press, New York and London (1970) 176–213 with loose errata. (MR43#65)

[Rota 1964] G.-C. Rota, *The Number of Partitions of a Set*, American Mathematical Monthly, May 1964, 498–504. (MR28#5009)

3.2. References concerning [DRS 1972].

[BBR 1980, 1981] M. Barnabei, A. Brini, and G.-C. Rota, Sistemi di coefficienti sezionali, Rend. Circ. Mat. Palermo (2) in Italian. (MR84b:05013a,b). Part 1: **29** (1980) 457–484. Part 2: **30** (1981) 161–198.

[Feinberg 1977] R. B. Feinberg, *Characterization of Incidence Algebras*, Discrete Mathematics **17** (1977) 47–70. (MR57#2943)

[FO 1990] P. Flajolet and A. Odlyzko, *Singularity Analysis of Generating Functions,* SIAM Journal of Discrete Mathematics **3** (1990) 216–240. (MR90m:05012)

[FSZ 1991] P. Flajolet, B. Salvy, P. Zimmermann, *Automatic analysis of algorithms,* Journal of Theoretical Computer Science **79** (1991) 37–109. (MR92k:68049)

[Fomin preprint] S. Fomin, *Duality of Graded Graphs*, to appear.

[GR 1970] J. Goldman and G.-C. Rota, *On the Foundations of Combinatorial Theory. IV. Finite Vector Spaces and Eulerian Generating Functions*, Studies in Appl. Math. **49** (1970) 239–258, (MR42#93) Also in "Combinatorial

Theory and Its Applications," (Proc. Colloquiua Mathematica Societatis, Balatonfured 1969), eds. P. Erdøos, A. Renyi, and V. T. Sòs, North-Holland Pub. Co., Amsterdam (1970) 477–507. (MR45-6632)

[GJ 1983] I. P. Goulden and D. M. Jackson, "Combinatorial Enumeration," Wiley-Interscience Series in Discrete Mathematics, John Wiley and Sons (1983). (MR84m:05002)

[HS 1989] M. Haiman and W. Schmitt, *Incidence algebra antipodes and Lagrange inversion in one and several variables*, J. Comb. Theory (A) **50** (1989) 172–185.

[HL 1914] G. H. Hardy and J. E. Littlewood, *Tauberian Theorems concerning power series and Dirichlet's series whose coefficients are positive,* Proc. London Math. Soc. **13** (1914) 174–191.

[JR 1979] S. A. Joni and G.-C. Rota, *Coalgebras and bialgebras in combinatorics,* Studies in Applied Mathematics, **61** (1979) 93–139. (MR81c:05002) See however M. Sweedler, "Hopf Algebras," (1969) (MR40#5705) for the underlying coalgebraic theory.

[Joyal 1981] A. Joyal, *Une Théorie Combinatoire des Séries Formelles*, Advances in Mathematics, **42** (1981), 1–82. (MR84d:05025)

[Plouffe 1992] S. Plouffe, "Approximations de séries génératrices et quelques conjectures," Thesis, Université du Québec à Montréal (1992).

[RS 1990] N. Ray and W. Schmitt, *Coclosure operators and chromatic polynomials*, Proc. Natl. Acad. Sci. USA **87** (1990) 4685–4687.

[Schmitt 1987] W. R. Schmitt, *Antipodes and Incidence Coalgebras*, Journal of Combinatorial Theory (A), **46** (1987) 264–290. (MR88m:05006)

[Sloane 1973] N. J. A. Sloane, "A Handbook of Integer Sequences," Academic Press (1973). The second edition will appear under the title "The New Book of Integer Sequences."

[Stanley 1970] R. P. Stanley, *Structure of Incidence Algebras and Their Automorphism Groups,* Bull. Amer. Math. Soc. **76** (1970) 1236–1239. (MR41#8319)

[Stanley 1978] R. P. Stanley, *Generating functions* in Studies in Combinatorics **17**, editor: G.-C. Rota, Math. Assoc. of America, Washington, DC (1978) 100–141. (MR87j:05003)

[Stanley 1986] R. P. Stanley, "Enumerative Combinatorics," Volume 1, especially Section 3.15 ff., Wadsworth and Brooks/Cole Mathematical Series, Monterey (1986), and Russian translation, Mir, Moscow (1990). (MR87j:05003)

[Viennot 1992] X. G. Viennot, *A Survey of Polyomino Enumeration* in "Actes du 4e colloque Séries formelles et combinatoires algébrique," 15–19 juin 1992, Montréal, Publications du Laboratoire de Combinatoire et d'Informatique Mathématique (1992).

[Wilf 1990] H. Wilf, "Generatingfunctionology," Academic Press, 1990. Especially section 3.13. (MR91g:05008) An excellant introduction to combinatorics written for aspiring "generatingfunctionologists."

3.3. References concerning [MR 1970, Rota 1964].

Note: A. Di Bucchianico maintains an exhaustive list of (over 400) publications concerning the umbral calculus. A partial list can be found in [Di Bucchianico 1992, p. 5–20].

[Andrews 1971] G. E. Andrews, *On the foundations of combinatorial theory V: Eulerian differential operators*, Studies in Applied Mathematics **50** (1971) 345–375. (MR46#8845)

[BBN 1983] M. Barnabei, A. Brini, and G. Nicoletti, *A general umbral calculus in infinitely many variables*, Advances in Mathematics **50** (1983) 49–93. (MR85g:05025) See its bibliography for further references.

[Bell 1927] E. T. Bell, "Algebraic Arithmetic," Amer. Math. Soc. Publications, Vol. 7, (1927) 146–159.

[Bell 1940] E. T. Bell, *Postulational Bases for the Umbral Calculus,* Amer. J. Math. **62** (1940) 717–724. (MR2,99b)

[BG 1971] E. A. Bender, and J. R. Goldman, *Enumerative Uses of Generating Functions*, Indiana Univ. Math. J. **20** (1971) 753–765. (MR42#5814)

[BB 1956] R. P. Boas, Jr. and R. C. Buck, *Polynomials defined by generating relations*, Amer. Math. Monthly **63** (1956) 626–632. (MR18,300)

[BB 1957] R. P. Boas, Jr. and R. C. Buck, "Polynomials Expansions of Analytic Functions," Ergebnisse der Mathematik und Ihrer Grenzgebiete, Neue Folge, Band 19, Springer-Verlag, Berlin-Gøottingen-Heidelberg (1957).

[Bourbaki 1949] N. Borbaki, "Elément de Mathématique I: Les Structures fondamentales de l'analyse. Livre IV: Fonctions d'une Variable Réele (théorie élémentaire). Fasc IX. Chaps 1–3. Dérivées, Primitives et Intégrales, Fonctions Elémentaires." Actualities Sci. Indust. No. 1074, Hermann, Paris, 1st edition (1949), 2nd edition (1958). (MR11,86h, MR13,631a, and MR27#1340)

[Chen preprint] W. Chen, *Compositional Calculus.*

[Di Bucchianico 1987] A. Di Bucchianico, *On Rota's Theory of Polynomials of Binomial Type*, Technical Report TW-820, Rijksuniversiteit Groningen, Holland.

[Di Bucchianico 1991] A. Di Bucchianico, *Banach Algebras, Logarithms, and Polynomials of Convolution Type,* J. Math. Anal. Appl.**156** (1991) 253–273. (MR92d:46123)

[FS 1970] D. Foata and M.-P. Shŭtzenberger, "Théorie Géométrique des Polynômes Eulériens," Lecture Notes in Mathematics **138**, Spring-Verlag, Berlin-Heidelberg-N.Y. 1970. (MR42#7523)

[Garsia 1973] A. M. Garsia, *An exposeé of the Mullin-Rota theory of polynomials of binomial type*, Linear and Multilinear algebra **1** (1973) 47–65 (MR47#9321a).

[HR 1992] P. A. Hirschhorn and L. A. Raphael, *Coalgebraic Foundations of the Method of Divided Difference*, Advances in Mathematics **91** (1992) 75–135. (MR92m:05012)

[II 1981] E. C. Ihrig and M. E. H. Ismail, *A q-umbral calculus*, J. Math. Anal. Appl. **84** (1981) 178–207. (MR82b:13010)

[KM 1986] S. G. Kurbanov and V. M. Maksimov, *Mutual Expansions of Differential Operators and Divided Difference Operators*, Dokl. Akab. Nauk. Uz. S. S. R. **4** (1986) 8–9. (MR87k:05021) Translation by Di Bucchianico to appear.

[Levitan 1964] B. M. Levitan, *Generalized Translation Operators and Some of Their Applications*, Israel Program Sci. Translations, Daniel Davey, Jerusalem (1964). (MR33#2344)

[Loeb 1990] D. E. Loeb, *Sequences of Symmetric Functions of Binomial Type*, Studies in Applied Mathematics **83** (1990) 1–30. (MR92e:05012)

[Loeb 1991] D. E. Loeb, *The Iterated Logarithmic Algebra*. Part 1: Advances in Mathematics **86** (1991) 155–234. (MR92g:05022) Part 2 (Sheffer Sequences): Journal of Mathematical Analysis **156** (1991) 172–183. (MR92d:05013)

[LR 1989] D. E. Loeb and G.-C. Rota, *Formal power series of logarithmic type*, Advances in Mathematics **75** (1989) 1–118. (MR90f:05014)

[LR 1991] D. E. Loeb and G.-C. Rota, *Recent advances in the calculus of finite differences*, Lecture Notes in Pure and Applied Mathematics **132**, Geometry and Complex Variables, Editor: S. Coen, Marcell Dekker (1991) 239–276.

[Markowsky 1978] G. Markowsky, *Differential operators and the theory of binomial enumeration*, J. Math. Anal. Appl. **63** (1978) 145–155. (MR58#21666)

[Roman 1982,1983] S. M. Roman, *The theory of umbral calculus*, J. Math. Anal. Appl. Part 1: **87** (1982) 58–115 (MR84c:05008a). Part 2: **89** (1982) 290–314 (MR84c:05008b). Part 3: **95** (1983), 528–563 (MR85e:05020).

[Roman 1984] S. M. Roman, "The umbral calculus," Academic Press (1984). (MR87c:05015 = Zbl. 536.33001)

[Viskov 1975] O. V. Viskov, *Operator characterization of generalized Appell polynomials*, Dokl. Akad. Nauk SSR **225** (1975), 749-752 English translation in Soviet Math. Dokl. **16** (1975), 1521-1524. (MR52#14416)