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Random Walks in the Quarter Plane

Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics

Second Edition

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Preface

In recent decades, two-dimensional random walks in domains with non-smooth boundaries became increasingly popular and are of interest to several groups in the mathematical community. In fact, these *Markovian* objects are encountered in pure probabilistic problems, as well as in applications involving queueing theory and, more recently, enumerative combinatorics. This monograph is the continuation of the first edition (1999), aiming to promote original mathematical methods to determine the invariant measure of such processes. Moreover, these methods can also be employed to characterize the transient behavior. *Complex function theory, boundary value problems, Riemann surfaces, functional equations, and Galois theory* are the main mathematical ingredients necessary for our purpose.

Backbone of the Book

This second edition contains 11 chapters and is divided into two parts.

Part I (Chaps. 1–8) presents the theoretical fundamentals of the methods. It essentially corresponds to the content of the first edition of 1999, except for Chap. 7 and Sects. 4.1–4.3, which are new. Chapter 8 briefly makes the link with several related problems.
Part II (Chaps. 9–11) deals with specific case studies borrowed from queueing theory and enumerative combinatorics.

Acknowledgements

- As for the first edition (1999), the authors are indebted to Martine Verneuille, who kindly managed to transform into readable LaTeX, a great deal of the manuscript, which was sometimes—and this is understatement—not easy to decipher, and to Angeal Thomin at Springer-Verlag for her kind assistance.

- This second enlarged edition owes much to the *supporters* of the analytic approach. In this respect, a special tribute goes to Kilian Raschel, for his enthusiasm and his technical skill. In particular, Chaps. 7 and 11 are borrowed from papers written in tight and friendly collaboration with the first author.

We express our gratitude to Irina Kurkova, who, from the very beginning, studied the techniques proposed in this book. Her careful reading helped to correct some errors and led to improvements of the first edition.

We also pay tribute to the memory of Philippe Flajolet, who was deeply interested in the methods involving BVPs, which he tirelessly promoted with his usual sharpness, skill, and humor.

The language and style of several chapters benefited from a sharp and scrupulous proofreading by Richard James, of Inria, and an anonymous copy editor, to whom we express our sincere gratitude.

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Contents

Introduction and History

Historical Comments

Two-dimensional random walks in domains with non-smooth boundaries are of interest to several groups in the mathematical community. In fact, these processes are encountered in pure probabilistic problems, as well as in applications involving queueing theory or enumerative combinatorics. This monograph aims to promote original mathematical methods to determine the invariant measure of such processes. Moreover, as will become apparent later, these methods can also be employed to characterize the transient behavior. To begin, it is worth placing our work in its historical context.

This book has three sources.

1. Boundary value problems (BVP) for functions of one complex variable;
2. Singular integral equations, Wiener–Hopf equations, and Toeplitz operators;
3. Random walks on a half line and related queueing problems.

The first two topics were for a long time at the center of interest of many well-known mathematicians: Riemann, Sokhotski, Hilbert, Plemelj, Carleman, Wiener, and Hopf. This *one-dimensional theory* took its final form in the studies carried out by Krein, Muskhelishvili, Gakhov, Gokhberg, etc.

The third point, and the related probabilistic problems, has been thoroughly investigated by Spitzer, Feller, Baxter, Borovkov, Cohen, etc.

To outline how the thinking evolved, let us recall that many simple problems pertaining to the $M/M/k$ queue amount, very roughly, to solving equations of the form

$$Q(z)\pi(z) = U(z), \quad \forall |z| \leq 1,$$

where U and π are unknown, holomorphic in the unit disc $\mathcal{D}$, continuous on the unit circle; U is a polynomial of degree m, and π is the generating function of a discrete positive random variable, say the number of units in the system. Then, provided the model is meaningful, the known function Q has in general m zeros in $\mathcal{D}$ and this is sufficient to determine U and π.

A basic problem in the $GI/GI/1$ queue was to find the distribution of the stationary waiting time W of an arriving customer, which satisfies the stochastic equation

$$W \stackrel{\mathcal{L}\mathrm{aw}}{=} [W+\sigma]^+, \tag{1}$$

where σ is a known random variable, independent of W and taking real values. Using Laplace transforms, Eq. (1) yields

$$\omega^-(s) = \gamma(s)\omega^+(s), \tag{2}$$

where $\gamma(s) = 1 - E[e^{-s\sigma}]$ and the unknown functions ω^+, ω^- are holomorphic in the right [resp. left] half plane. The function γ is given, but a priori defined only on the imaginary axis. It turns out that (2) is exactly equivalent to the famous *Wiener–Hopf factorization* problem.

After this, there were two main directions of generalization for multidimensional situations.

- The first one is a multidimensional analogue of the Wiener–Hopf factorization, when the problem or equation depends on extra parameters. The theory is then quite similar to the one-dimensional case and the simplest example concerns Wiener–Hopf equations in a half space. Much more involved is the index theory of differential and pseudo-differential operators on manifolds with smooth boundaries. However, when one wants to get an index theory for Wiener–Hopf equations in an orthant of dimension k or another domain with piecewise smooth boundaries, it becomes immediately clear that one needs more knowledge for similar equations in dimension $k-1$. For example, there is an index theory for equations in a quarter plane ($k=2$), thanks to B. Simonenko, because we have a complete control for $k=1$.
- Then, completely new approaches to these problems were discovered by the authors of this book, the goal going far beyond the mere obtention of an index theory for the quarter plane. The main results can be summarized as follows.

  1. Use of generating functions, or Laplace transforms, is quite standard and the resulting functional equations give rise to boundary value problems involving two complex variables. When the jumps of the random walk are bounded by 1 in the interior of the quarter plane, the basic functional equation is written as

$$Q(x,y)\pi(x,y) + q(x,y)\pi(x) + \widetilde{q}(x,y)\widetilde{\pi}(y) + q_0(x,y)\pi_{00} = 0. \tag{3}$$

  2. The first step, which is quite similar to a Wiener–Hopf factorization, consists in considering the above equation on the algebraic curve $Q = 0$ (which is

elliptic in the generic situation), so that we are then left with an equation for two unknown functions of one variable on this curve.

3. Next, a crucial idea is to use Galois automorphisms on this algebraic curve in order to solve Eq. (3). It is clear that we need more information, which is obtained by using the fact that the unknown functions π and $\widetilde{\pi}$ depend solely on x and y respectively, i.e., they are invariant with respect to the corresponding Galois automorphisms of the algebraic curve. It is then possible to prove that π and $\widetilde{\pi}$ can be *lifted* as meromorphic functions onto the *universal covering* of some Riemann surface **S**. Here, **S** corresponds to the algebraic curve $Q = 0$ and the universal covering is the complex plane $\mathbb{C}$.
4. Lifted onto the universal covering π (and also $\widetilde{\pi}$) satisfies a system of non-local equations having the simple form

$$\begin{cases} \pi(t+\omega_1) = \pi(t), & \forall t \in \mathbb{C}, \\ \pi(t+\omega_3) = a(t)\pi(t) + b(t), & \forall t \in \mathbb{C}, \end{cases}$$

where ω_1 [resp. ω_3] are complex [resp. real] constants. The solution can be presented in terms of infinite series equivalent to *Abelian integrals*. The backward transformation (projection) from the universal covering onto the initial coordinates can be given in terms of uniformization functions, which, in the circumstances, are elliptic functions.
5. Another direct approach to solving the fundamental equation consists in working solely in the complex plane. After making the analytic continuation, one shows that the determination of π reduces to a BVP, belonging to the Riemann–Hilbert–Carleman class, the basic form of which can be formulated as follows.

Let $\mathcal{L}^+$ denote the interior of the domain bounded by a simple smooth closed contour $\mathcal{L}$.

Find a function Φ^+ holomorphic in $\mathcal{L}^+$, the limiting values of which are continuous on the contour and satisfy the relation

$$\Phi^+(\alpha(t)) = G(t)\Phi^+(t) + g(t), \quad t \in \mathcal{L},$$

where

- $g, G \in \mathbb{H}_\mu(\mathcal{L})$ (*Hölder condition* with parameter μ on $\mathcal{L}$);
- α, referred to as a *shift* in the sequel, is a function establishing a one-to-one mapping of the contour $\mathcal{L}$ onto itself, such that the direction of traversing $\mathcal{L}$ is changed and

$$\alpha'(t) = \frac{d\alpha(t)}{dt} \in \mathbb{H}_\mu(\mathcal{L}), \quad \alpha'(t) \neq 0, \forall t \in \mathcal{L}.$$

In addition, the function α is most frequently subject to the so-called *Carleman condition*

$$\alpha(\alpha(t)) = t, \quad \forall t \in \mathcal{L}, \quad \text{where typically} \quad \alpha(t) = \bar{t}.$$

6. Analytic continuation gives a clear understanding of possible singularities and thus allows the asymptotics of the solution to be obtained.

All these techniques work quite similarly for Toeplitz operators or for random walk problems. Here, our presentation is given for random walks only for the sake of concreteness. More exactly, we consider the problem of calculating stationary probabilities for ergodic random walks in a quarter plane. But other problems for such random walks can be treated as well: transient behavior, first hitting time problem [50], and calculating the Martin boundary.

The approach relating to points 1, 2, 3, and 4 was mainly settled in the period 1968–1972 (see e.g., [66–69, 76]).

The method in point 5 was proposed in the fundamental study [32], carried out in 1977–1979, which was widely referred to and followed up in many other papers (see [7–9, 14, 16, 17, 21, 25, 29, 30, 31, 33, 35, 42, 79, 82, 83]) and in the book by Cohen and Boxma [24].

Since the first edition was published, numerous studies involving random walks in the quarter plane have been carried out, which were based on the reduction to BVPs. They concerned various topics: queueing systems, analytic combinatorics, diffusion, Brownian motion, asymptotics, etc. It would hardly be possible to cite all of them, nor even the most interesting ones! We deliberately chose to concentrate on ideas and methods, rather than covering or describing a multitude of studies, many of which are only of technical interest.

Contents of the Book

This second edition contains 11 chapters and is organized into two parts.

- **Part I** (Chaps. 1–8) presents the theoretical fundamentals of the methods.
- **Part II** (Chaps. 9–11) borrows specific case studies from queueing theory and enumerative combinatorics.

In Chap. 1, it is explained how the functional equations appear and why they bring complete knowledge about the initial problem.

Chapter 2 presents the foundations of the analytic approach. Section 2.1 contains some material necessary for the rest of the book. In Sect. 2.2, the first step is presented, namely the restriction of the equation to the algebraic curve. This curve is studied in Sects. 2.3 and 2.5, together with the initial domains of analyticity of the unknown functions. Simple basic properties of our Galois automorphisms are given in Sect. 2.4, where the notion of the *group* of the random walk is introduced.

Chapter 3 is exclusively devoted to the analytic continuation of the unknown functions.

When the group of the random walk is finite, a very beautiful algebraic theory exists for solving the fundamental equations (both the homogeneous and non-homogeneous ones). This is the subject of Chap. 4. We provide concrete criteria, in the form of necessary and sufficient conditions, for the solutions to be rational or algebraic. In fact, the solution is obtained in the general case. The analysis contains original results obtained by the authors, many of which were published in the first edition. It is also interesting to note that finiteness of the group holds for some simple queueing systems, which are discussed here as well. A first example of an algebraic solution in a queueing context was found in [43]. Other examples borrowed from combinatorics are also presented in Chap. 11.

The case of an arbitrary group (i.e., not necessarily finite) is made in Chap. 5, by reduction to a Riemann–Hilbert BVP in the complex plane. Necessary and sufficient conditions for the process to be ergodic are obtained in a purely analytical way, by calculating the index of the BVP (which, roughly speaking, gives the number of independent admissible solutions). The unknown functions are given by integral forms, which can be explicitly computed via the Weierstrass $\wp$ function.

Chapter 6 concerns degenerate but in practice important cases (e.g., priority queues, joining the shorter of two queues), when the genus of the algebraic curve is zero [44]. In this case, even simpler solutions exist.

Chapter 7 proposes an explicit criterion for the finiteness of the group in the genus 0 case. The approach consists in viewing the genus 0 as a continuous limit of genus 1 and in substance follows the lines of the article [38].

The purpose of this book was clearly not to cover all questions pertaining to random walks in the quadrant. Some of them, which we believe to be of particular importance, are briefly discussed in Chap. 8. They concern in particular asymptotics of the probability distributions, the Martin boundary, space inhomogeneity, transient behavior, and jumps of arbitrary size.

Chapter 9 shows the first example of a queueing system analyzed by reduction to a BVP in the complex plane.

Chapter 10 deals with the famous so-called *Joining the Shorter Queue* system, where maximal homogeneity conditions clearly do not hold. In the non-symmetrical case, an integral equation is obtained. The results of this chapter are original and provide the first analytic solution to the problem.

In Chap. 11, the generating functions of interest are those obtained when counting random walks with small steps. This is the field of *enumerative combinatorics*. Results so far obtained concern the classification of the functions and their asymptotics.

Part I
The General Theory

Chapter 1
Probabilistic Background

1.1 Markov Chains

For this brief section, we only provide the minimum amount of information necessary; for further details, readers who are not probabilists may refer to any standard textbook on countable Markov chains (MCs) in discrete time.

A discrete time homogeneous Markov chain with a denumerable state space $\mathcal{A}$ is defined by the stochastic matrix

$$\mathbf{P} = \left\| p_{\alpha\beta} \right\|, \quad \alpha, \beta \in \mathcal{A},$$

such that

$$p_{\alpha\beta} \geq 0, \quad \sum_{\beta} p_{\alpha\beta} = 1, \quad \forall \alpha \in \mathcal{A}.$$

The matrix elements of $\mathbf{P}^n$ will be denoted by $p_{\alpha\beta}^{(n)}$.

Definition 1.1.1 An MC is called irreducible if, for any ordered pair α, β, there exits an m, depending on (α, β), such that

$$p_{\alpha\beta}^{(m)} \neq 0.$$

In addition, an irreducible MC is called *aperiodic* if, for some $\alpha, \beta \in \mathcal{A}$, the set $\{n : p_{\alpha\beta}^{(n)} \neq 0\}$ has a greatest common divisor equal to 1. It follows that the same property is true for all ordered pairs α, β.

Definition 1.1.2 An irreducible aperiodic MC is called *ergodic* if, and only if, the equation

$$\pi \mathbf{P} = \pi, \tag{1.1.1}$$

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where π is the row vector $\pi = (\pi_\alpha, \alpha \in \mathcal{A})$, has a unique ℓ_1-solution up to a multiplicative factor, which can be chosen so that

$$\sum_\alpha \pi_\alpha = 1, \quad \text{together with } \pi_\alpha > 0.$$

The π_α's are called *stationary probabilities* (see [15]). The random variable representing the position of the chain at time n will be written X_n and X will denote the random variable with distribution π.

1.2 Random Walks in a Quarter Plane

The class of MCs we shall mainly consider in this book are called *maximally space homogeneous random walks*. They are characterized by the following three properties.

P1 The state space is $\mathcal{A} = \mathbb{Z}_+^2 = \{(i, j) : i, j \geq 0 \text{ are integers}\}$.

P2 *(Maximal space homogeneity)* $\mathbb{Z}_+^2$ is supposed to be represented as the union of a finite number of non-intersecting classes

$$\mathbb{Z}_+^2 = \bigcup_r S_r \,. \tag{1.2.1}$$

Moreover, for each r and for all $\alpha \in S_r$ such that

$$p_{\alpha,\alpha+(i,j)} \neq 0, \quad \alpha + (i, j) \in \mathbb{Z}_+^2 \,,$$

$p_{\alpha,\alpha+(i,j)}$ does not depend on α, and can therefore be denoted by ${}_r p_{ij}$. Throughout most of the book, the classes S_r will have the following structure:

$$\mathbb{Z}_+^2 = S \cup S' \cup S'' \cup \{(0, 0)\} \tag{1.2.2}$$

where

$$\begin{cases} S &= \{(i, j) : i, j > 0\}, \\ S' &= \{(i, 0) : i > 0\}, \\ S'' &= \{(0, j) : j > 0\}. \end{cases}$$

The *internal* parts S' and S'' are called respectively the x-axis and y-axis. In this case, the probabilities ${}_r p_{i,j}$ will be simply written p_{ij}, p'_{ij}, p''_{ij}, p^0_{ij}, according to their respective regions S, S', S'', and $\{(0, 0)\}$. It is worth noting, nevertheless, that in Sect. 1.3 a more general partition of the state space is considered.

The last property deals with the boundedness of the jumps, which will be assumed unless otherwise stated (see e.g., Chaps. 5 and 6).

P3 *(Boundedness of the jumps)* For any $\alpha \in S_r$,

$$p_{\alpha\beta} = 0, \text{ unless } -d_r^- \le (\beta - \alpha)_i \le d_r^+,$$

for some constants $0 \le d_r^\pm < \infty$, where $(\beta - \alpha)_i$ is the i-th coordinate of the vector $\beta - \alpha$, $i = 1, 2$. In addition, the next important assumption will hold throughout this book

$$d_r^\pm = 1, \qquad \text{for the class } S_r = S.$$

The ergodicity conditions for the random walk $\mathcal{L}$ can be given in terms of the mean jump vectors

$$\begin{cases} \vec{\mathbf{M}} &= (M_x, M_y) = \left(\sum i p_{ij}, \sum j p_{ij}\right) \\ \vec{\mathbf{M}}' &= (M'_x, M'_y) = \left(\sum i p'_{ij}, \sum j p'_{ij}\right) \\ \vec{\mathbf{M}}'' &= (M''_x, M''_y) = \left(\sum i p''_{ij}, \sum j p''_j\right). \end{cases} \tag{1.2.3}$$

We shall consider only *irreducible aperiodic random walks.*

Theorem 1.2.1 *When* $\vec{\mathbf{M}} \neq 0$*, the random walk is ergodic if, and only if, one of the following three conditions holds:*

1. $\begin{cases} M_x < 0, \ M_y < 0, \\ M_x M'_y - M_y M'_x < 0, \\ M_y M''_x - M_x M''_y < 0; \end{cases}$
2. $M_x < 0, \quad M_y \ge 0, \quad M_y M''_x - M_x M''_y < 0;$
3. $M_x \ge 0, \quad M_y < 0, \quad M_x M'_y - M_y M'_x < 0.$

■

A probabilistic proof of this theorem exists in [36]. A new and purely analytic proof is presented in Chap. 5 and the analysis of the case $\vec{\mathbf{M}} = 0$ is carried out in detail in Chap. 6.

As stated in the general introduction, this monograph intends to provide a methodology of an essentially analytic nature for constructing and effectively computing the invariant measures associated with the random walks introduced in the present section. In fact, it is worth emphasizing that all these methods can also be employed (up to some additional technicalities) to analyze the transient behavior of the random walk, and to solve explicitly *Kolmogorov's classical equations*, which describe the time-evolution of the semigroup associated to a Markov process, see, e.g., [15].

1.3 Functional Equations for the Invariant Measure

We derive here the fundamental functional equations to be used throughout the book. It seems useful to present them in a more general situation, which means that for now we do not assume any *boundedness of the jumps*. To that end, consider the MC

$$X_n = \left(X_n^1, \ldots, X_n^k\right), \quad n \geq 0, \quad X_n \in \mathbb{Z}_+^k,$$

with state space

$$\mathbb{Z}_+^k = \{z = (z_1, \ldots, z_k) : z_i \geq 0, \quad i = 1, \ldots, k\}$$

which is partitioned into a finite number of classes

$$\mathbb{Z}_+^k = \bigcup S_r,$$

so that the following assumption holds: *two states belong to the same class S_r if, and only if, the probability distributions P_r of the jumps from these states are the same*. The corresponding probability densities are the ${}_r p_{ij}$ introduced previously in Sect. 1.2.

Let us define the vector of complex variables

$$u = (u_1, \ldots, u_k), \quad u_i \in \mathbb{C}, \quad |u_i| = 1, \quad i = 1, \ldots, k,$$

and the jump generating functions

$$P_r(u) = E[u^{(X_{n+1}-X_n})/X_n = z], \quad z \in S_r, \tag{1.3.1}$$

with the standard notation

$$u^z = \prod_{i=1}^{k} u_i^{z_i}.$$

Since by our assumptions $P_r(u)$ does not depend on z, Kolmogorov's equations take the form

$$E[u^{X_{n+1}}] = E\left[u^{X_n} u^{X_{n+1}-X_n}\right] = \sum_r E\left[u^{X_n} \mathbb{1}_{\{X_n \in S_r\}}\right] P_r(u). \tag{1.3.2}$$

To account for the stationary case, we introduce the generating functions

$$\pi_r(u) = E\left[u^X \mathbb{1}_{\{X \in S_r\}}\right] = \sum_{z \in S_r} \pi_z u^z, \tag{1.3.3}$$

where π_z denotes the stationary probability of being in state z. Taking the limit $n \to \infty$ in (1.3.2) and using (1.3.3), we get the basic equation

$$\sum_r [1 - P_r(u)]\,\pi_r(u) = 0. \tag{1.3.4}$$

Note that when the jumps are bounded from below, (1.3.4) is defined for all $u = (u_1, \ldots, u_k)\,, u_i \in \mathbb{C}\,,\ |u_i| \le 1,\ i = 1, \ldots, k$. Since we shall mainly consider the case $k = 2$, it will be convenient to rewrite (1.3.4) in a more explicit way, by means of the notation below, which will be ubiquitous throughout the book.

$$\begin{cases} \pi(x, y) &= \displaystyle\sum_{i,j=1}^{\infty} \pi_{ij} x^{i-1} y^{j-1}, \\ \pi(x) &= \displaystyle\sum_{i\ge 1} \pi_{i0} x^{i-1}, \\ \widetilde{\pi}(y) &= \displaystyle\sum_{j\ge 1} \pi_{0j} y^{j-1}, \\ Q(x, y) &= xy\Big(\displaystyle\sum_{i,j} p_{ij} x^i y^j - 1\Big), \\ q(x, y) &= x\Big(\displaystyle\sum_{i\ge -1}\sum_{j\ge 0} p'_{ij} x^i y^j - 1\Big) \\ \widetilde{q}(x, y) &= y\Big(\displaystyle\sum_{i\ge 0}\sum_{j\ge -1} p''_{ij} x^i y^j - 1\Big), \\ q_0(x, y) &= \Big(\displaystyle\sum_{i,j} p^0_{ij} x^i y^j - 1\Big), \end{cases} \tag{1.3.5}$$

where we have set $p^0_{ij} \stackrel{\text{def}}{=} p_{(0,0),(i,j)}$.

Now Eq. (1.3.4) takes the fundamental form

$$\boxed{-Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \widetilde{q}(x, y)\widetilde{\pi}(y) + \pi_{00} q_0(x, y)}\,. \tag{1.3.6}$$

When property **P3** holds, it is immediate to check that the functions $Q, q, \widetilde{q}$ and q_0 introduced in (1.3.5) are polynomials in x, y. In addition, $\pi(x, y), \pi(x), \pi(y)$ have to be holomorphic in the region $|x|, |y| < 1$. Thus, the analysis of the invariant measure of the random walk amounts to solving the functional Eq. (1.3.6), in agreement with the next theorem.

Theorem 1.3.1 *For the irreducible aperiodic random walk to be ergodic, it is necessary and sufficient that there exist $\pi(x, y), \pi(x), \pi(y)$ holomorphic in $|x|, |y| < 1$, and a constant π_{00}, satisfying the fundamental Eq. (1.3.6) together with the ℓ_1-condition*

$$\sum_{i,j=0}^{\infty} \left|\pi_{ij}\right| < \infty. \tag{1.3.7}$$

In this case these functions are unique. ■

Theorem 1.3.1 proceeds directly from the material given in Definition 1.1.2, asserting existence and uniqueness of a finite invariant measure for irreducible ergodic Markov chains. We shall look for solutions of (1.3.6) from the following point of view:

Find functions $\pi(x, y)$, $\pi(x)$, $\widetilde{\pi}(y)$, *satisfying* (1.3.6), *holomorphic in* $\mathcal{D} \times \mathcal{D}$ *and continuous in* $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, *where*

$$\mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \overline{\mathcal{D}} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leq 1\}.$$

The main idea consists in working on the variety $Q(x, y) = 0$, $(x, y) \in \overline{\mathcal{D}} \times \overline{\mathcal{D}}$ and the content of this book shows, by means of various approaches, that this is sufficient to obtain all the aforementioned functions.

Remark 1.3.2 *A priori*, finding a solution $\pi(x, y)$, holomorphic in $\mathcal{D} \times \mathcal{D}$ and continuous in $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, does not imply the ℓ_1-condition (1.3.7), as it emerges from the theory of functions of several complex variables (see for instance [10]). Furthermore, supposing that the system is not ergodic, we will see that a solution of (1.3.6) exists, holomorphic in $\mathcal{D}_a \times \mathcal{D}_a$, with $a < 1$, where $\mathcal{D}_a$ is the disc $\mathcal{D}_a \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < a\}$.

Chapter 2
Foundations of the Analytic Approach

This chapter presents the essential stepping stones of the analytic approach. They require in particular some basic notions concerning Riemann surfaces and algebraic curves.

2.1 Fundamental Notions and Definitions

We gather here some of the basic notions which appear elsewhere in the book and which are strictly necessary for our purpose, namely the resolution of the Eq. (1.3.6). The logical structure of this section is self-contained, but, for a deeper and more detailed presentation, we refer the reader to the classical monographs [46, 94]. Although the first five chapters of the book demand much less generality, we think that a more abstract understanding is nevertheless useful.

A separable topological space is called a *two-dimensional manifold* $\mathbb{M}$ if each point belongs to a neighborhood which is homeomorphic to an open disc in the complex plane $\mathbb{C}$. A pair (U, φ) formed by some neighborhood $U \subset \mathbb{M}$ and its associated homeomorphism φ is called a *chart*. The mapping $\varphi : U \to \mathbb{C}$ defines a system of local coordinates in U. A collection of charts $\{(U_i, \varphi_i),\ i \in I\}$, where, for some index set I, $\{U_i,\ i \in I\}$ is an open covering of $\mathbb{M}$, is called an *atlas* $\mathcal{A}$.

A connected two-dimensional manifold $\mathbb{M}$ is an (abstract) *Riemann surface* S if there exists an atlas $\mathcal{A}_S$ with the following property:

> *For any pair (U, φ), (V, ψ) of charts in $\mathcal{A}_S$ such that $U \cap V \neq \emptyset$, the mapping $\varphi \circ \psi^{-1}$ is holomorphic in $\psi(U \cap V) \subset \mathbb{C}$.*

The classical notion of holomorphic functions can be generalized to the case of Riemann surfaces. Let S be a Riemann surface, $\mathcal{A}_S$ its atlas, and $Y \subset S$ an open connected set of S. A function $f : Y \to \mathbb{C}$ is said to be *holomorphic in* Y if, for any chart (U, φ) in $\mathcal{A}_S$, the mapping $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic in the normal

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sense in the open set $\varphi(U) \subset \mathbb{C}$. The set of all functions holomorphic in Y builds an algebra over $\mathbb{C}$.

Let now S and T be two Riemann surfaces. The mapping $f : S \to T$ is said to be holomorphic if, for any pair of charts (U, φ), (V, ψ) belonging to $\mathcal{A}_S$ and $\mathcal{A}_T$ respectively, with $f(U) \subset V$, the mapping $\psi \circ f \circ \varphi^{-1}$ is holomorphic in $\varphi(U) \subset \mathbb{C}$. The uniqueness theorem remains valid: if f_1 and f_2 are two holomorphic functions from S to T which are equal on an infinite compact set of S, then they must be equal everywhere.

In a similar way, the definition for a function to be meromorphic can be extended as follows. Let $Y \subset S$ be an open set of the Riemann surface S. A function $f : Y \to \mathbb{C}$ is called *meromorphic* in Y if f is holomorphic in $Y - Y_0$, where $Y_0 \subset Y$ is a set of isolated points such that

$$\lim_{x \in Y,\, x \to p} |f(x)| = \infty, \quad \forall p \in Y_0 .$$

The points of Y_0 are called poles of f. Note that one can show that locally f is the quotient of two holomorphic functions. The set of meromorphic functions in Y form a field. It is classical to remark that, after *compactification*, the complex plane $\mathbb{C}$ is isomorphic to the Riemann sphere $\mathbb{P}^1$, the point at infinity becomes an ordinary point and the meromorphic functions on Y can be considered as holomorphic mappings from Y to $\mathbb{P}^1$.

2.1.1 Covering Manifolds

Let X and Y represent two Riemann surfaces and $h : Y \to X$ a holomorphic mapping of Y *onto* X (i.e. to any point of X, there corresponds at least one point of Y). This assumption is useful only when Y is not compact, since it can be shown that, whenever Y is compact, any holomorphic mapping h is necessarily *onto* and that X is compact [46]. The pair (Y, h) is called a *cover* and Y is a *covering manifold* of X. When h is not constant, $h^{-1}(x)$ is a discrete subset of Y, for any $x \in X$. A point $y \in h^{-1}(x)$ is said to *lie over* x (or to *cover* x) and x is the *projection* of y on X.

Definition 2.1.1 If, for any neighborhood V of y, the restricted mapping $h_{|V}$ is not one-to-one, y is called a *branch point*. More precisely, $y \in V$ is a *branch point of order* $n - 1$ if there exist two charts $(U, \varphi) \in \mathcal{A}_X$, $(V, \psi) \in \mathcal{A}_Y$, $y \in V$, with $h(V) \subset U$, and a finite number n, such that

$$(\varphi \circ h)(v) \equiv \gamma_n \circ \psi(v), \quad \forall v \in V,$$

where the function $\gamma_n : \mathbb{C} \to \mathbb{C}$ satisfies $\gamma_n(z) = z^n$. This means that each point $x \in h(V) \backslash h(y)$ is covered by exactly n points of Y contained in $V \backslash y$, the deleted neighborhood of y. Often n will be referred to as the *multiplicity*. If $n = 1$, i.e. h is a one-to-one mapping $V \to U$, for some V, then y is called a *regular* point.

Since h is holomorphic, it can be shown that there exists an integer s such that every point $x \in X$ is covered by exactly s points of Y, provided that a branch point of order $k - 1$ is counted k times. Then, the covering manifold Y will be called an *s-sheeted covering* of X. It is important to note that s can be infinite, and we shall encounter this case. Among all covering maps without branch points, there exists exactly one, up to an isomorphism, which is simply connected and usually referred to as *the universal covering*.

Definition 2.1.2 (*Proper mapping*) Let (Y, h) be a cover of X. The mapping h is called *proper* if the preimage of every compact set is compact. In particular, if Y is compact then h is always proper.

Suppose that X and Y are Riemann surfaces and $h : Y \to X$ is a proper non-constant holomorphic mapping. Let B be the set of branch-points of h and $A \stackrel{\text{def}}{=} h(B)$. Let also $Y' \stackrel{\text{def}}{=} Y \backslash h^{-1}(A)$ and $X' \stackrel{\text{def}}{=} X \backslash A$. Then the restriction $h_{|Y'} : Y' \to X'$ is a proper unbranched covering, which has a well defined finite number of sheets n. The following more general property is valid.

Proposition 2.1.3 *Let X and Y be Riemann surfaces and $h : Y \to X$ be a proper non-constant holomorphic mapping. Then there exists a natural number n such that h takes every value $c \in X$, counting multiplicities, n times.*

Definition 2.1.4 (*Lifting*) Let us introduce the following objects.

- X, Y, two Riemann surfaces, and Z a topological space;
- $h : Y \to X$, a continuous mapping (see [46]);
- $f : Z \to X$, a continuous function.

A *lifting* of f with respect to h is a continuous mapping $g : Z \to Y$ such that $f = h \circ g$.

Proposition 2.1.5 *Let (Y, h) be a cover of X. Let L be an arc of X having its origin at x (i.e. there exists a continuous mapping $f : [0, 1] \to X$, where $L = f[0, 1]$ and $f(0) = x$). Then, $\forall y \in h^{-1}(x)$, $\exists$ an arc in Y which has its origin at y and lies* over *L.*

Proposition 2.1.6 (Uniqueness of lifting)

(i) Let (Y, h) be a cover without branch-point. Let g_1 and g_2 be two liftings with respect to h. Then either $g_1(z) \neq g_2(z)$, $\forall z \in Z$, or $g_1 \equiv g_2$. In addition, if f is holomorphic, then any lifting g is also holomorphic.

(ii) In general, when there are branch points, the preceding property holds provided that, with the notation introduced just after Definition 2.1.2, X, Y and Z are replaced respectively by X', Y' and $Z' = f^{-1}(X')$.

Remark 2.1.7 This proposition will be used either to lift arcs, in which case $Z = [0, 1]$, or also to compare different mappings, taking then $Z \equiv Y$.

The study of analytic functions in the complex plane frequently entails the introduction of *Riemann surfaces* on which the analytic functions are single-valued. In most of the cases—and particularly in this book—these surfaces are composed of several sheets lying over $\mathbb{C}$ and they are compatible with the definition of abstract Riemann surfaces given above. In fact, they can be described by the *gluing of germs*. The *germ* of a function f at a point $x \in \mathbb{C}$ is the set of all discs $\mathcal{O}_x$ with center x such that there exists a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - x)^n, \quad \forall z \in \mathcal{O}_x. \tag{2.1.1}$$

In general, for given x and $\mathcal{O}_x$, there can exist k_x different series (2.1.1), each of them representing, for instance, the branches of an algebraic function (see definition thereafter). All the pairs $(\mathcal{O}_x, k_x)$, numbered in an arbitrary way, can be pasted together to build an *analytic configuration* which is the Riemann surface of the analytic function f.

The problem of uniqueness of the above analytic continuation process is solved by the following fundamental *monodromy theorem*.

Theorem 2.1.8 *Let X be a Riemann surface. Let u_0 and u_1 two homotopic curves on X, each joining the points A and B, which implies the existence of u_t, $t \in [0, 1]$, a set of curves representing a continuous deformation of u_0 into u_1. Let f be a function holomorphic in a neighborhood of A and admitting an analytic continuation over u_t, $\forall t \in [0, 1]$. Then the two analytic continuations, over u_0 and u_1 respectively, lead to the same holomorphic function in a neighborhood of B. In particular, if X is simply connected, then all closed curves on X are homotopic to one point, and, if f can be continued along any curve starting from A, then there exists one and only one function holomorphic on X and coinciding with f in a neighborhood of A.* ∎

2.1.2 Algebraic Functions

Let X be a Riemann surface and $\mathcal{M}(X)$ be the field of meromorphic functions on X, and let

$$P(T) = \sum_{i=0}^{n} f_i T^i, \quad f_i \in \mathcal{M}(X),$$

be an irreducible polynomial of degree n in one indeterminate T, with coefficients in $\mathcal{M}(X)$. When X is the complex plane $\mathbb{C}$, the equation

$$P(Y) = \sum_{i=0}^{n} f_i Y^i = 0, \quad f_i \in \mathcal{M}(X), \tag{2.1.2}$$

implicitly defines a *multi-valued* function $Y(x)$, $x \in \mathbb{C}$, called an *algebraic function*. Equation (2.1.2) has in the neighborhood of every x, except for a finite number of them called *critical points*, exactly n distinct finite roots $Y_1(x), Y_2(x), \ldots, Y_n(x)$, which are the *branches* of the algebraic function $Y(x)$. In the domain obtained by removing from the complex plane the critical points, each branch of the function $Y(x)$ is *regular*. There are only two sorts of critical points: *poles* and multiple points called *branch points*. The point at infinity in $\mathbb{C}$ does not play a special role and can be both a regular or a branch point, as can be seen by a change of variables.

At any regular point x_0, there exist exactly n different convergent power series (or germs)

$$Y_{x_0,k}(x) = \sum_{l=0}^{\infty} c_l\,(x_0, k)\,(x - x_0)^l\,, \quad k = 1, 2, \ldots n,$$

satisfying (2.1.2). Moreover, in the neighborhood of a branch point, the algebraic function is made of a *cyclical* system of branches, each of them having a power series expansion with fractional exponents. It is possible to construct a Riemann surface, referred to as the *Riemann surface of the algebraic function* $Y(x)$, by putting together the above germs. One of the main goals pursued by Riemann, when he introduced these abstract surfaces, was to render algebraic functions single-valued. This possibility is contained in the following theorem.

Theorem 2.1.9 *There exist a holomorphic n-sheeted proper cover (Y, π) of X and a function $F \in \mathcal{M}(Y)$ such that*

$$\sum_{i=0}^{n} f_i(\pi(y))(F(y))^i = 0, \quad \forall y \in Y. \tag{2.1.3}$$

The triple (Y, π, F) is unique up to isomorphism and is sometimes called (see e.g., [46]) the algebraic function defined by the polynomial P. ■

Up to some notational abuse, Y is simply called the Riemann surface of P, and $Y(x)$ is the corresponding algebraic function. Frequently, X will be the Riemann sphere. In this situation Y is compact, since X is compact. It can be shown that every compact abstract Riemann surface can be realized as the Riemann surface of some algebraic function.

The above theorem is in fact intimately connected to Galois theory, the principles of which will be outlined now.

2.1.3 Elements of Galois Theory

Let K and L denote two arbitrary fields such that $K \subset L$. We say that L is a *finite extension* of K if L is a vector space over K of finite dimension n, which is called the *degree of* L *over* K and is denoted by $[L : K]$.

Let $K[T]$ be the polynomial ring in one indeterminate T over the field K. The extension L of K is said to be *normal* or a *Galois extension* of K if any polynomial $P \in K[T]$, irreducible and having a root in L, has all its roots in L. The necessary and sufficient condition for L to be normal is that there exists an irreducible $P \in K[T]$ such that the roots of P form a basis of L, as a vector space on K.

The *Galois group* $G(L/K)$ of the extension L of K is the group of automorphisms of L leaving all elements of K invariant. The order of this group is denoted by $[L : K]$ and any of its elements realizes a permutation of the roots of any polynomial $P \in K[T]$, provided that P is irreducible and has all its roots in L. Any extension of degree 2 is obviously normal.

Let (Y, π) be a cover of X. An automorphism σ of Y is called a *covering automorphism* if $\pi = \pi \circ \sigma$, which means that $\forall y \in Y$, the points y and $\sigma(y)$ have the same projection on X. The set of all covering automorphisms form a multiplicative group denoted by $\mathrm{Aut}(Y/X)$, which is a subgroup of the group of automorphisms of Y.

Definition 2.1.10 Let (Y, π) be a cover of X without branch point. One says that π is *normal* or *Galois* if, for any $y_0, y_1 \in Y$ such that $\pi(y_0) = \pi(y_1)$, there exists a $\sigma \in \mathrm{Aut}(Y/X)$ such that $\sigma(y_0) = y_1$. From Proposition 2.1.6, it follows that σ is unique, setting $Z = Y$, $f = \pi$ and $\sigma = g$.

Proposition 2.1.11 *Let (Y, π) be a universal cover of X (i.e. simply connected and without branch-point). Then π is* Galois *and Aut (Y/X) is isomorphic to the* fundamental group *of X (i.e. the set of homotopy classes of closed curves going through some arbitrary but fixed point).* (See e.g., [46].) ■

Proposition 2.1.12 *Let (Y, π) a cover of X, the mapping π being proper. Then every covering transformation $\sigma' \in \mathrm{Aut}(Y'/X')$ can be extended to a covering transformation $\sigma \in \mathrm{Aut}(Y/X)$ and π will be called* Galois *if $(Y', \pi_{|Y'})$ is* Galois*, in the sense of Definition 2.1.10, X' and Y' being as in Definition 2.1.2.* ■

Finally, the following important general theorem holds.

Theorem 2.1.13 *Let (Y, π, F) be the algebraic function corresponding to the equation $P(T) = 0$, where P is an irreducible polynomial of degree n, having all its coefficients in $\mathcal{M}(X)$.*

Define the mapping $\pi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ by the relation

$$\left[\pi^*(f)\right](y) = f(\pi(y)), \quad \forall y \in Y \text{ and } f \in \mathcal{M}(X).$$

(In other words, the range of π^* is a subfield of $\mathcal{M}(Y)$ which in this way can be identified with $\mathcal{M}(X)$.) Then the field $\mathcal{M}(Y)$ is an extension of degree n of $\mathcal{M}(X)$ and is isomorphic to the quotient field $\mathcal{M}(X)[T]/P(T)$. Moreover, the groups $\mathrm{Aut}(Y/X)$ *and $G(\mathcal{M}(Y)/\mathcal{M}(X))$ are isomorphic via the isomorphism ϕ defined by*

$$\phi : \sigma \to \hat{\sigma}, \quad with\ \hat{\sigma} \circ f = f \circ \sigma^{-1}, \quad \forall f \in \mathcal{M}(Y).$$

The covering π is Galois *if, and only if, $\mathcal{M}(Y)$ is a* Galois *extension of $\mathcal{M}(X)$.* ■

2.1.4 Universal Covering and Uniformization

Every Riemann surface of a polynomial with coefficients in $\mathcal{M}(\mathbb{P}^1)$ is topologically isomorphic to a sphere with g *handles* attached to it. The number g is called the *genus* of the Riemann surface.

Let S be a compact Riemann surface with its associated polynomial Q and (Ω, λ) its universal covering. λ is unique up to an automorphism of Ω. It is known [53] that only three possible situations can take place.

(i) $\Omega = \mathbb{P}^1$, the Riemann sphere, then $g = 0$.
(ii) $\Omega = \mathbb{C}$, the (finite) complex plane, then $g = 1$.
(iii) $\Omega = \mathcal{D}$, the open unit disc, then $g > 1$.

The next result can be found, for example, in [94].

Proposition 2.1.14 *There exist two functions $f, g \in \mathcal{M}(S)$ such that*

(i) $Q(f, g) = 0$;
(ii) any function belonging to $\mathcal{M}(S)$ is a rational function of f and g. ■

We shall also use the following, see e.g., [46].

Proposition 2.1.15 *Every meromorphic function on $\mathbb{P}^1$ is rational.* ■

Remembering that $\lambda : \Omega \to S$, it follows that the functions $\widetilde{f}$ and $\widetilde{g}$, respectively given by

$$\widetilde{f} = f \circ \lambda, \quad \widetilde{g} = g \circ \lambda,$$

with f, g given in Proposition 2.1.14, are meromorphic on Ω and invariant with respect to the group of covering automorphisms. It is customary to state the following.

Definition 2.1.16 The pair of functions $\widetilde{f}, \widetilde{g}$ gives a *uniformization* of $\mathcal{M}(S)$, or, equivalently, *uniformizes* any $h \in \mathcal{M}(S)$.

2.1.5 Abelian Differentials and Divisors

Here we quickly present two essential notions which will be used in Chap. 4. In general, any holomorphic or meromorphic differential on a Riemann surface S will be called an *abelian differential*.

1. A holomorphic differential on S is called *abelian differential of the first kind*.
2. A meromorphic differential all of whose singularities are poles of order ≥ 2 is called *an abelian differential of the second kind*.
3. The abelian differentials of the *third kind* are taken to be all abelian differentials on S.

Theorem 2.1.17 ([94]) *The vector space of holomorphic differentials on a compact Riemann surface has dimension equal to its genus g.* ■

In particular, when the genus is 1, Abelian differentials of the first kind are unique up to a multiplicative constant. See also Remark 3.3.5 in Sect. 3.3.

An important question is to be able to specify the locations and the orders of poles and zeros of meromorphic functions defined on an arbitrary compact Riemann surface S. To this end, one designates several points $P_1, P_2, \ldots, P_k$ and the associate orders (integers) $m_1, m_2, \ldots, m_k$ of these points. Then the following symbol

$$P_1^{m_1} P_2^{m_2} \ldots P_k^{m_k}$$

will be used to denote the points $P_1, P_2, \ldots, P_k$, with the corresponding associate integers, and will be called a *divisor*. When P_i is a zero (or pole) with multiplicity $m_i, i = 1, \ldots, k$, of some meromorphic function, the corresponding divisor is referred to as a *principal divisor*. For a divisor to be principal, a necessary condition is $\sum m_i = 0$ (see [94]). In Chap. 4, we shall also need Abel's theorem stated hereafter for the sake of completeness.

Theorem 2.1.18 ([94]) *A necessary and sufficient condition for a divisor $\mathfrak{a}$ to be a principal divisor is that there exists a singular 1-chain γ, with boundary $\partial\gamma$, such that*

$$\partial\gamma = \mathfrak{a}$$

and

$$\int_\gamma d\omega = 0,$$

for each differential of the first kind on S. ■

2.2 Restricting the Equation to an Algebraic Curve

In the preceding section, we have introduced a function $\pi(x, y)$ of two complex variables and two functions of one complex variable $\pi(x)$ and $\widetilde{\pi}(x)$ which we would like to find. Here we present the method to get rid of $\pi(x, y)$ allowing us to write a functional equation involving only these two functions of one complex variable. This method has four facets, which can be easily converted into each other.

2.2.1 First Insight (Algebraic Functions)

Consider the multi-valued algebraic function $Y(x)$ satisfying the polynomial equation

$$Q(x, Y(x)) = 0, \quad \forall x \in \mathbb{C}, \tag{2.2.1}$$

where Q is given in (1.3.5). A first naïve approach consists just in instantiating $y = Y(x)$ into the fundamental equation (1.3.6). This substitution is a priori valid only for the pairs $(x, Y_i(x))$ such that

$$|x| \leq 1, \quad |Y_i(x)| \leq 1,$$

where $Y_i(x)$, $i = 0, 1$, are the two branches of $Y(x)$. For these values, we get from (1.3.6)

$$0 = q(x, Y_i(x))\pi(x) + \widetilde{q}(x, Y_i(x))\widetilde{\pi}(Y_i(x)) + \pi_{00}q_0(x, Y_i(x)). \tag{2.2.2}$$

This approach will be used in a self-contained way, allowing us to obtain and solve an integral equation of Fredholm type for $\pi(x)$, by using convenient conformal transformations associated with elliptic functions.

2.2.2 Second Insight (Algebraic Curve)

Let $\Gamma_a \stackrel{\text{def}}{=} \{x \in \mathbb{C} : |x| = a\}$ with, for the sake of shortness,

$$\Gamma \stackrel{\text{def}}{=} \Gamma_1, \quad \text{the unit circle.}$$

Let $\mathcal{B}$ be the algebraic curve in $\mathbb{C}^2$ defined by the fundamental equation

$$Q(x, y) = 0. \tag{2.2.3}$$

As observed at the end of Chap. 1, if there exist functions $\pi(x, y), \pi(x), \widetilde{\pi}(y)$, analytic in $\mathcal{D}^2 \equiv \mathcal{D} \times \mathcal{D}$, continuous in $\overline{\mathcal{D}}^2$ and satisfying the fundamental equation, then we have necessarily, for all $(x, y) \in \mathcal{B} \cap \mathcal{D}^2$,

$$q(x, y)\pi(x) + \widetilde{q}(x, y)\widetilde{\pi}(y) + q_0(x, y)\pi_{00} = 0. \tag{2.2.4}$$

But solutions of (2.2.4) a priori would not imply the continuity of $\pi(x, y)$ in $\overline{\mathcal{D}}^2$. That this continuity indeed holds follows from the next two lemmas.

Lemma 2.2.1 *Assume the conditions of Theorem 1.2.1 hold and the polynomial $Q(x, y)$ is irreducible. Then $\exists \epsilon > 0$ such that the functions π and $\widetilde{\pi}$ can be analytically continued up to the circle $\Gamma_{1+\varepsilon}$ in their respective complex planes. Moreover, in $\mathcal{D}^2_{1+\varepsilon}$ they satisfy Eq. (2.2.4) in $\mathcal{B} \cap \mathcal{D}^2_{1+\varepsilon}$.* ∎

This lemma will be proved later in Sect. 2.5. Let us note that the second assertion follows directly from the first one by the principle of analytic continuation.

It is important to mention right away the case of zero drift, i.e. $\mathbf{M} = 0$, considered later on in Sect. 6.5. Then the algebraic curve has genus zero, but the conditions of Lemma 2.2.1 *do not hold*, since we will show that it is not possible to continue π nor $\widetilde{\pi}$ up to some $\Gamma_{1+\varepsilon}$, $\forall\epsilon$.

Lemma 2.2.2 *Under the conditions of Lemma 2.2.1 the function*

$$\pi(x, y) = -\frac{q(x, y)\pi(x) + \widetilde{q}(x, y)\widetilde{\pi}(y) + q_0(x, y)\pi_{00}}{Q(x, y)} \tag{2.2.5}$$

is analytic in $\mathcal{D}^2_{1+\varepsilon}$. ■

Proof This is just a direct corollary of the famous Weierstrass "*Nullstellensatz*", which claims that, if p is an irreducible polynomial, h is holomorphic in an open domain $V \subset \mathbb{C}^m$ and $h = 0$ on $\{p = 0\} \cap V$, then $\dfrac{h}{p}$ is holomorphic in V. ■

Remark 2.2.3 The reader has already guessed that this projection onto the algebraic curve can be carried out in more general situations and in particular for cases in dimension greater than two.

Remark 2.2.4 The analyticity of the functions $\pi(x, y)$, $\pi(x)$, $\widetilde{\pi}(x)$ in $\mathcal{D}^2_{1+\varepsilon}$ in the ergodic cases of Theorem 1.2.1 can also be derived from a Lyapunov function method, by purely probabilistic arguments (see [36]). It can be shown, in particular, that there exist constants $C > 0$, $0 < \alpha < 1$, which can depend on the parameters p_{ij}, p'_{ij}, p''_{ij}, such that

$$\pi_{ij} < C\alpha^{i+j}.$$

2.2.3 Third Insight (Factorization)

First, we show that, using a one-dimensional factorization of $Q(x, y)$ with respect to either of the two variables x, y, we indeed obtain the projection onto the algebraic curve. For this purpose, we have to introduce the two branches $Y_0(x)$ and $Y_1(x)$ of the algebraic function $Y(x)$ on the unit circle Γ. Exact definitions will be given in the next section.

Choosing $x \neq 1$, $|x| = 1$, we can write

$$Q(x, y) = a(x)Q^+(x, y)Q^-(x, y),$$

where

$$a(x) = p_{11}x^2 + p_{01}x + p_{-1,1}, \quad Q^+(x, y) = y(y - Y_1(x)), \quad Q^-(x, y) = 1 - \frac{Y_0(x)}{y}.$$

This is exactly a Wiener–Hopf factorization of Q considered as a function of y, after setting $y = \exp it$ for all real t. Then the fundamental equation (1.3.6) can be rewritten as

$$a(x)Q^{+}(x,y)\pi(x,y) = \frac{q(x,y)\pi(x) + \widetilde{q}(x,y)\widetilde{\pi}(y) + \pi_{00}q_0(x,y)}{Q^{-}(x,y)}. \tag{2.2.6}$$

Let P_z^+ and P_z^- denote the projection operators in the commutative Banach algebra $\mathcal{B}$ of functions on the unit circle, that is

$$\mathcal{B} = \left\{ \sum_{k=-\infty}^{\infty} a_k z^k : \sum_{k=-\infty}^{\infty} |a_k| < \infty \right\},$$

$$P_z^+\left(\sum_{-\infty}^{\infty} a_k z^k\right) = \sum_{k=0}^{\infty} a_k z^k, \quad P_z^-\left(\sum_{-\infty}^{\infty} a_k z^k\right) = \sum_{k=-\infty}^{-1} a_k z^k, \quad P_z^+ + P_z^- = I_d,$$

where I_d denotes the identity operator. We shall use the formula

$$P_z^-\left(\frac{\omega(z)}{1-\frac{a}{z}}\right) = \omega(a)\frac{a}{z-a}, \quad \forall \omega \in P_z^+(\mathcal{B}), \quad |a| < 1. \tag{2.2.7}$$

Applying now P_y^- (taken as an operator acting on functions of y) to both sides of (2.2.6), using (2.2.7) with $a = Y_0(x)$ and simplifying by $\dfrac{Y_0(x)}{y - Y_0(x)}$, we get exactly Eq. (2.2.2) for the branch $Y_0(x)$.

The point $x = 1$ always requires a more careful analysis, which can be carried out by exploring the non stationary Kolmogorov equations and using generating functions in space-time variables.

2.2.4 Fourth Insight (Riemann Surfaces)

Assuming that the polynomial Q is irreducible, we know from Sect. 2.1.2 that the equations

$$Q(x, Y) = 0 \quad \text{and} \quad Q(X, y) = 0$$

define two Riemann surfaces (with the two corresponding algebraic functions $Y(x)$ and $X(y)$). In fact (see e.g., [94]), these two surfaces have the same genus, and hence are conformally equivalent. Thus we will consider a *single* Riemann surface, denoted by $\mathbf{S}$, which is the Riemann surface for both $Y(x)$ and $X(y)$, but has two different coverings

$$h_x : \mathbf{S} \to \mathbb{C}_x, \quad h_y : \mathbf{S} \to \mathbb{C}_y,$$

$$\begin{tikzcd} & \mathbf{S} \arrow[dl, "x(s)"'] \arrow[dr, "y(s)"] & \\ \mathbb{C}_x \arrow[rr, shift right, "Y(x)"'] & & \mathbb{C}_y \arrow[ll, shift right, "X(y)"'] \end{tikzcd}$$

Fig. 2.1 Lifting the complex planes onto **S**

where $\mathbb{C}_z$ denotes the complex plane with respect to the z-coordinate.

Any function f on a domain $V \subset \mathbb{C}_x$ can be *lifted* onto $h_x^{-1}(V) \subset \mathbf{S}$, thus yielding a new function $\hat{f} \stackrel{\text{def}}{=} f \circ h_x$. Correspondingly, one has $\hat{g} \stackrel{\text{def}}{=} g \circ h_y$. Hence, taking for f and g the identity functions, we are entitled to put

$$x(s) \stackrel{\text{def}}{=} h_x(s), \quad y(s) \stackrel{\text{def}}{=} h_y(s), \quad s \in \mathbf{S}. \tag{2.2.8}$$

The general flowchart is given in Fig. 2.1.

On **S**, we have $Q(x(s), y(s)) = 0$ and it will be convenient to write

$$\begin{aligned} \hat{\pi}(s) &\stackrel{\text{def}}{=} \pi(x(s)), \quad \hat{\tilde{\pi}}(s) \stackrel{\text{def}}{=} \tilde{\pi}(y(s)), \\ \hat{q}(s) &\stackrel{\text{def}}{=} q(x(s), y(s)), \quad etc. \end{aligned}$$

Then, for all $s \in h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D})$, we have the equation

$$\hat{\pi}(s)\hat{q}(s) + \hat{\tilde{\pi}}(s)\hat{\tilde{q}}(s) + \hat{q}_0(s) = 0, \tag{2.2.9}$$

in which $\hat{q}, \hat{\tilde{q}}, \hat{q}_0$ are meromorphic functions on **S**, as resulting from the composition of meromorphic functions. We will reformulate Lemma 2.2.1 on **S** in Sect. 2.5. In fact, there are two constraints of primary importance: the solution $\hat{\pi}$ [resp. $\hat{\tilde{\pi}}$] of (2.2.9) has to be a function of the sole quantity $x(s)$ [resp. $y(s)$]. As shown in the next section, these properties of $\hat{\pi}$ and $\hat{\tilde{\pi}}$ will lead to the central idea, based on the use of *Galois* automorphisms of **S**.

2.3 The Algebraic Curve $Q(x, y) = 0$

The present study relies essentially upon the properties of the algebraic functions $Y(x)$ and $X(y)$, defined by the curve in question. All necessary results will be put together and most of the features of these functions will be formulated in terms of the probabilities p_{ij} of the jumps in the interior of the quarter plane.

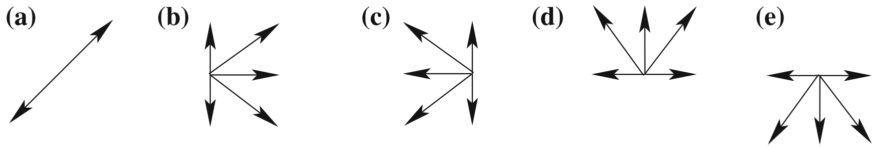


Fig. 2.2 Singular random walks

Definition 2.3.1 The random walk is called *singular* if the associated polynomial Q is either reducible or of degree 1 in at least one of the variables.

Lemma 2.3.2 *The random walk is singular if, and only if, one of the following conditions holds:*

- **A** *There exists* $(i, j) \in \mathbb{Z}^2$, $|i| \le 1$, $|j| \le 1$, *such that only* p_{ij} *and* $p_{-i,-j}$ *are different from 0 (see Fig. 2.2a and the three cases obtained by rotation);*
- **B** *There exists* i, $|i| = 1$, *such that for any* j, $|j| \le 1$, $p_{ij} = 0$ *(Fig. 2.2b, c);*
- **C** *There exists* j, $|j| = 1$, *such that for any* i, $|i| \le 1$, $p_{ij} = 0$ *(Fig. 2.2d, e).*

■

Proof Let us define the triple $a(x), b(x), c(x)$, [resp. $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$] by

$$Q(x, y) \equiv xy\left(\sum p_{ij}x^i y^j - 1\right) = a(x)y^2+b(x)y+c(x) = \widetilde{a}(y)x^2+\widetilde{b}(y)x+\widetilde{c}(y). \tag{2.3.1}$$

We assume $p_{00} < 1$, so that $b(x) \neq 0$, $\widetilde{b}(y) \neq 0$. Then we have the following chain of equivalences:

$$\text{Q is of degree 1 with respect to } y \Leftrightarrow a(x) = 0 \Leftrightarrow p_{i1} = 0, \quad \forall i.$$

This is exactly the cases **A** and **C** of the lemma, with $i = 1$ and $j = 1$ respectively. Analogously the two remaining cases in **B** and **C**, which correspond to $c(x) \equiv 0$ or $\widetilde{c}(y) \equiv 0$, yield the reducibility of Q.

Suppose now that Q is of degree 2 both in x and in y. We shall consider two possibilities.

(a) $Q = f_1 f_2$, *where* f_1 *and* f_2 *are of degree 1 in each variable*. Then we can write, for $i = 1, 2$,

$$\begin{cases} f_i(x, y) = (A_i x + B_i)y - (C_i x + D_i), \\ (A_i, B_i) \neq (0, 0), \quad i = 1, 2, \\ (A_i, C_i) \neq (0, 0). \end{cases}$$

Ex hypothesis, $a(x) \not\equiv 0$. Since also $c(x) \not\equiv 0$ (as mentioned above), we have

$$\prod_{i=1}^{2}(A_i x + B_i) = a(x) \quad \text{and} \quad \prod_{i=1}^{2}(C_i x + D_i) = c(x),$$

which implies that the pair of vectors (A_1, B_1) and (A_2, B_2) [resp. (C_1, D_1), (C_2, D_2)] are simultaneously positive or negative. Thus, ad libitum, we can choose (A_1, B_1) and (A_2, B_2) to be positive, i.e. $A_i, B_i \geq 0, i = 1, 2$.

The relationship

$$(A_1x+B_1)(C_2x+D_2)+(A_2x+B_2)(C_1x+D_1) = (1-p_{00})x-p_{-1,0}-p_{10}x^2 \quad (2.3.2)$$

shows that the vectors (C_1, D_1) and (C_2, D_2) are both positive. This implies that $p_{-1,0} = p_{10} = 0$ and also that both polynomials $(A_1x + B_1)(C_2x + D_2)$ and $(A_2x + B_2)(C_1x + D_1)$ are linear in x. So we obtain

$$A_1C_2 = B_1D_2 = A_2C_1 = B_2D_1 = 0,$$

which yields either

$$B_1 = B_2 = C_1 = C_2 = 0$$

or

$$A_1 = A_2 = D_1 = D_2 = 0.$$

Consequently, there exists a polynomial p of one variable such that $Q(x, y)$ is equal either to $p(xy)$ or to $x^2 p\left(\frac{y}{x}\right)$. Since $Q(1, 1) = 0$, necessarily $p(1) = 0$. Here also, we have to consider two sub-cases:

* $Q(x, y) = p(xy)$, yielding $Q(x, y) = (xy - 1)(p_{11}xy - p_{-1,1})$;
* $Q(x, y) = x^2 p\left(\frac{y}{x}\right)$, yielding $Q(x, y) = (y - x)(p_{-1,1}y - p_{1,-1}x)$.

(b) $Q = f_1 f_2$, where f_1 is of degree 0 with respect to x. Let us prove in fact f_1 is of degree 1 in y. Assume for now that it has degree 2 in y. Then $Q(x, y) = f_1(y) f_2(x)$ and both f_1 and f_2 have degree 2, so that there exist α, β, γ, with

$$a(x) = \alpha f_2(x), \quad b(x) = \beta f_2(x), \quad c(x) = \gamma f_2(x).$$

Hence $\alpha > 0, \gamma > 0$ and f_2 has positive coefficients. But $b(x)$ has one negative coefficient, so that $\beta < 0$, which yields $p_{-10} = p_{10} = 0$ and $f_2(x) = \lambda x$, $\lambda > 0$, contrary to the hypothesis.

Hence f_1 has degree 1 in y and we can write $f_1(y) = y + \beta,\ \beta \neq 0$, so that

$$Q(x, y) = (y + \beta)\left(a(x)y + \frac{c(x)}{\beta}\right),$$

which yields

$$b(x) = \beta a(x) + \frac{c(x)}{\beta}.$$

As $b(x)$ has one strictly negative coefficient, one must have $\beta < 0$ together with $p_{10} = p_{-1,0} = 0$. It also follows that

$$p_{11} = p_{1,-1} = p_{-1,1} = p_{-1,-1} = 0.$$

Similarly, the situation obtained by exchanging the roles of x and y would give

$$Q(x, y) = (x + \widetilde{\beta}) \left(\widetilde{a}(y)x + \frac{\widetilde{c}(y)}{\widetilde{\beta}} \right).$$

The proof of Lemma 2.3.2 is complete. ∎

2.3.1 Branches of the Algebraic Functions on the Unit Circle

Let $\mathcal{P}$ be the simplex

$$\mathcal{P} = \left\{ (p_{-1,-1}, \ldots, p_{11}) : \sum_{i,j} p_{ij} = 1, \;\; p_{ij} \geq 0 \right\}.$$

The points of this simplex are the parameters of a random walk inside the quarter plane. In the simplex $\mathcal{P}$, we shall often consider the set $\mathcal{A}$ of points for which $M_y = 0$. This set subdivides $\mathcal{P}$ into two convex sets $\mathcal{A}_+$ and $\mathcal{A}_-$, for which $M_y > 0$ and $M_y < 0$ respectively.

Definition 2.3.3 A random walk is said to be *simple* if only $p_{10}, p_{01}, p_{-1,0}, p_{0,-1}$ are not equal to 0.

Lemma 2.3.4 *Let the random walk be non-singular and $M_y \neq 0$. Then the algebraic function $Y(x)$ has on Γ two branches, denoted by $Y_0(x)$ and $Y_1(x)$, such that $Y_0(1) < Y_1(1)$. There are two possible cases.*

(i) If $M_y > 0$, then $|Y_0(x)| < 1$ and $|Y_1(x)| \geq 1, \forall |x| = 1$, and the equality takes place only for $x = 1$, where $Y_1(1) = 1$. Moreover, $Y_0(x)$ (resp. $Y_1(x)$), $x \in \Gamma$, describes a real analytic curve, situated inside (resp. outside) the unit circle Γ, for $x \neq 1$.

(ii) If $M_y < 0$, then $|Y_0(x)| \leq 1$, for $|x| = 1$ and equality holds only for $x = 1$, where $Y_0(1) = 1$. Moreover, $|Y_1(x)| > 1$, $\forall x$.

When $M_x \neq 0$, similar properties hold for the algebraic function $X(y)$, the two branches of which are denoted respectively by $X_0(y)$ and $X_1(y)$. ∎

Proof First, we shall state some simple assertions:

(1) Let $P(x, y) = \sum p_{ij} x^i y^j$. Then the equality

$$P(x, y) = 1, \quad |x| = |y| = 1,$$

cannot hold for non-singular random walks, except for $x = y = 1$. This follows from elementary considerations on the sum of complex numbers.

(2) The equation $P(1, y) = 1$ has two roots, 1 and $\dfrac{c(1)}{a(1)}$ (see 2.3.1), and the two following situations can arise:

   (a) if $M_y < 0$, then $Y_0(1) = 1,\ Y_1(1) = \dfrac{c(1)}{a(1)}$;

   (b) if $M_y > 0$, then $0 < Y_0(1) = \dfrac{c(1)}{a(1)} < 1,\ Y_1(1) = 1$.

Take $|x| = 1$ and assume $a(x) \neq 0$. Then the roots in y of $a(x)y^2 + b(x)y + c(x)$ are continuous with respect to $a(x)$, $b(x)$ and $c(x)$. Consequently, the image $Y(\Gamma)$ consists of no more than two connected components. Thus, by *(a)*, this image can intersect Γ only at the point $y = 1$. The case corresponding to the points $\{|x| = 1, a(x) = 0\}$ is obtained by continuity.

Assume, for example, $M_y < 0$. (The argument hereafter would be the same for $M_y > 0$, just replacing $\mathcal{D}$ by $\mathbb{C}\backslash\overline{\mathcal{D}}$.) It follows now from assertion (2) that, if $Y(\Gamma)$ is connected, then $Y(\Gamma) \cap \mathcal{D} = \emptyset$. Hence proving $Y(\Gamma) \cap \mathcal{D} \neq \emptyset$ will show at once that $Y(\Gamma)$ has exactly two connected components: one of which, denoted by $Y_0(\Gamma)$, belongs to $\overline{\mathcal{D}}$ and the other one, $Y_1(\Gamma)$, belongs to $\mathbb{C}\backslash\overline{\mathcal{D}}$. For $x \in \Gamma$, $Y_0(x)$ and $Y_1(x)$ are simple closed analytic curves, and this amounts to saying that $Y(x)$ has no branch point on Γ. This is really the case, because at a branch point $Y(x)$ would take only one value, which is impossible due to *(a)* and *(b)*.

It remains to prove the existence of points x, $|x| = 1$, and y, $|y| < 1$, such that $P(x, y) = 1$. It suffices to check the particular case $x = -1$. To this end, we shall use a powerful continuity argument.

Any point $\rho \in \mathcal{A}_-$ can be connected, along a continuous path $\ell \subset \mathcal{A}_-$, with some point $\rho_0 \in \mathcal{A}_-$ corresponding to a *simple* random walk (see definition above). Moreover, ℓ can be chosen to ensure the continuity of the two roots of $P(-1, y) = 1$ with respect to the parameters of $\mathcal{A}_-$, provided that one does not cross the hyperplane

$$a(-1) = p_{11} - p_{01} + p_{-1,1} = 0.$$

Next one checks easily that, for any *simple* random walk,

$$-1 < Y_0(-1) < 1,$$

whence it follows that $Y_0(-1) \subset \mathcal{D}$ for *any* random walk and the proof of the lemma is complete. ∎

2.3.2 Branch Points

Our concern now is to locate the branch points of the algebraic function, which, in particular, define the genus of the Riemann surface. Starting from *simple* random walks, we will deduce general properties by using continuity arguments with respect to the parameters (i.e. the points of $\mathcal{P}$), as in Lemma 2.3.4.

Lemma 2.3.5 *Consider a simple random walk satisfying $M_y \neq 0$. Then $Y(x)$ has two branch points x_1, x_2 inside $\mathcal{D}$ and two branch points x_3, x_4 outside $\mathcal{D}$. All these four branch points are positive and equal to (with an obvious notation)*

$$x_{1,2} = \frac{1 \pm 2\sqrt{p_{01}p_{0,-1}} - \sqrt{1 \pm 4\sqrt{p_{01}p_{0,-1}} + 4p_{01}p_{0,-1} - 4p_{10}p_{-1,0}}}{2p_{10}},$$
$$x_{3,4} = \frac{1 \pm 2\sqrt{p_{01}p_{0,-1}} + \sqrt{1 \pm 4\sqrt{p_{01}p_{0,-1}} + 4p_{01}p_{0,-1} - 4p_{10}p_{-1,0}}}{2p_{10}}.$$

When $M_y = 0$, $M_x \neq 0$, there are still four positive branch points, which satisfy

$$0 < x_1 < x_2 = 1 < x_3 < x_4, \; if \;\; M_x < 0,$$
$$0 < x_1 < x_2 < 1 = x_3 < x_4, \; if \;\; M_x > 0.$$

■

Proof Factorizing the discriminant $\mathcal{D}_x$ of the equation

$$p_{01}y^2 + y\left(xp_{10} + \frac{p_{-1,0}}{x} - 1\right) + p_{0,-1} = 0,$$

we get the two following relationships which define the branch points, after having set $g(x) \stackrel{def}{\equiv} p_{10}x + p_{-1,0}\dfrac{1}{x}$, namely

$$g(x) = 1 + 2\sqrt{p_{01}p_{0,-1}}\,, \tag{2.3.3}$$

$$g(x) = 1 - 2\sqrt{p_{01}p_{0,-1}}\,. \tag{2.3.4}$$

Since $g(1) = p_{10} + p_{-1,0} < 1 - 2\sqrt{p_{01}p_{0,-1}}$ and $g''(x) > 0$, the Eq. (2.3.3) has two positive zeros, one is bigger and the other is less than one. The same is true for Eq. (2.3.4). Other cases can be treated quite similarly. ■

The next two preliminary lemmas will be used in the proof of the subsequent Lemmas 2.3.8, 2.3.9 and 2.3.10.

Lemma 2.3.6 *The point $x = -1$ is a branch point of $Y(x)$ if $M_y = 0$ and either of the two conditions is satisfied*

$$\begin{cases} p_{10} = p_{01} = p_{-1,0} = p_{0,-1} = 0, \\ p_{11} = p_{1,-1} = p_{-1,1} = p_{-1,-1} = p_{10} = p_{-10} = 0. \end{cases}$$

■

Proof

$$\begin{aligned} D(-1) &= (1 + p_{10} - p_{00} + p_{-1,0})^2 - 4(p_{11} - p_{01} + p_{-1,1})(p_{1,-1} - p_{0,-1} + p_{-1,-1}) \\ &\geq (a(1) + c(1))^2 - 4a(1)c(1) = (a(1) - c(1))^2 = M_y^2. \end{aligned}$$

Thus $D(-1) = 0$ yields first $D(1) = 0$, i.e. the point $x = 1$ is a branch point of $Y(x)$. Moreover, the equalities $D(-1) = D(1) = 0$ imply that

(i) $b(-1) = -b(1) = a(1) + c(1)$,
and, either

(ii) $\begin{cases} a(-1) = a(1), \\ c(-1) = c(1), \end{cases}$

or

(iii) $\begin{cases} a(-1) = -a(1), \\ c(-1) = -c(1). \end{cases}$

It is readily seen that (i) + (ii) [resp. (i) + (iii)] is tantamount to the first [resp. second] condition stated in the lemma. The proof of Lemma 2.3.6 is complete. ■

Lemma 2.3.7 *The hypersurface $p_{10}^2 - 4p_{11}p_{1,-1} = 0$ subdivides $\mathcal{A}_-$ into two pathwise connected domains $\mathcal{A}_{-+}$ and $\mathcal{A}_{--}$, where $p_{10}^2 > 4p_{11}p_{1,-1}$ or $p_{10}^2 < 4p_{11}p_{1,-1}$ correspondingly. An analogous property is true for the hypersurface $p_{01}^2 - 4p_{11}p_{-1,1} = 0$ and the corresponding domains $\mathcal{A}_+$, $\mathcal{A}_{++}$, and $\mathcal{A}_{+-}$.* ■

Proof The proof is easy and left to the reader. ■

Now we shall formulate all information about the general case in the next three lemmas.

Lemma 2.3.8 *For all non-singular r.w. such that $M_y \neq 0$, $Y(x)$ has two branch points x_1 and x_2 (resp. x_3 and x_4) inside (resp. outside) the unit circle. All these branch points lie on the real line.*

∗ *For the pair (x_3, x_4), the following classification holds:*

1. *If $p_{10} > 2\sqrt{p_{11}p_{1,-1}}$, then x_3 and x_4 are positive;*
2. *If $p_{10} = 2\sqrt{p_{11}p_{1,-1}}$, then one point is infinite and the other is positive;*
3. *If $p_{10} < 2\sqrt{p_{11}p_{1,-1}}$, then one point is positive and the other is negative.*

∗ *Similarly, for the pair* (x_1, x_2)*,*

4. *if* $p_{-1,0} > 2\sqrt{p_{-1,1}p_{-1,-1}}$*, then* x_1 *and* x_2 *are positive;*
5. *if* $p_{-1,0} = 2\sqrt{p_{-1,1}p_{-1,-1}}$*, then one point is 0 and the second is positive;*
6. *if* $p_{-1,0} < 2\sqrt{p_{-1,1}p_{-1,-1}}$*, then one point is positive and the other is negative.*

This lemma is also true for $X(y)$*, up to a proper symmetric change of the parameters.* ■

Proof Again let us connect an arbitrary point $\rho \in \mathcal{A}_{-+}$, by a continuous path $\ell \subset \mathcal{A}_{-+}$, to some point $\rho_0 \in \mathcal{A}_{-+}$ corresponding to a non-degenerate *simple* random walk.

Due to Lemma 2.3.5, all zeros of the discriminant at the point ρ_0 are real and mutually different. Along ℓ they will be real and different if a *liaison* does not occur (note that the zeros of the discriminant form complex conjugate pairs). But this discriminant is equal to

$$D(x) = [b(x) - 2\sqrt{a(x)c(x)}][b(x) + 2\sqrt{a(x)c(x)}].$$

At ρ_0, the points x_3 and x_4 are respectively zeros of the two equations

$$b(x) - 2\sqrt{a(x)c(x)} = 0 \text{ or } b(x) + 2\sqrt{a(x)c(x)} = 0.$$

Along ℓ, these zeros remain zeros of these different equations and cannot become equal since otherwise

$$b(x) - 2\sqrt{a(x)c(x)} = b(x) + 2\sqrt{a(x)c(x)} = 0,$$

which is possible only for $x = 0$ or $x = \infty$, but these cases have been considered above.

Let us now consider $\mathcal{A}_{--}$ and a point $\rho_0 \in \mathcal{P}$, corresponding to a r.w. for which $p_{ij} \neq 0$, if, and only if, $|ij| = 1$. The zeros of $D(x)$ in this case satisfy

$$x^2 = \frac{b \pm \sqrt{b^2 - 64p_{11}p_{1,-1}p_{-1,1}p_{-1,-1}}}{8p_{11}p_{1,-1}},$$

where

$$b = 1 - 4p_{11}p_{-1,-1} - 4p_{-1,1}p_{1,-1}.$$

One can prove (see Lemma 2.3.2) that, if either $p_{11} \neq p_{-1,-1}$ or $p_{1,-1} \neq p_{-1,1}$, then

$$b^2 - 64p_{11}p_{-1,1}p_{1,-1}p_{-1,-1} > 0.$$

Clearly, $b \pm \sqrt{b^2 - 64p_{11}p_{1,-1}p_{-1,1}p_{-1,-1}} > 0$. Thus, two zeros of $D(x)$ are positive and two are negative. The fact that two of them lie inside and two other ones lie outside the unit circle follows from the fact (proven above) that, in an arbitrary neighborhood, there are points corresponding to non-degenerate r.w. for which this is true. The rest of the proof is similar, with the simplification that x_3 and x_4 cannot become equal, since one of them is >1 and the other <-1. The sets $\mathcal{A}_{+-}$ and $\mathcal{A}_{++}$ could be handled in the same way. The proof of Lemma 2.3.8 is complete. ■

Lemma 2.3.9 *For all non singular r.w. such that $M_y = 0$, one of the branch points of $Y(x)$ is equal to 1. In addition,*

* *if $M_x < 0$, then two other branch points have a modulus bigger than 1 and the remaining one has a modulus less than 1;*
* *if $M_x > 0$, then two branch points are less than 1 and the modulus of the remaining one is bigger than 1.*

Furthermore, the positivity conditions are the same as in Lemma 2.3.8. ■

Proof Consider the convex set $\mathcal{A}_0$ of the points ρ corresponding to a non-degenerate random walk with $M_y = 0$. The hypersurface defined by $M_x = 0$ divides this set into two convex sets for which, respectively, $M_x > 0$ and $M_x < 0$. The point $x = 1$ is always a branch point of $Y(x)$ and for a *simple* random walk the assertion in question holds, by Lemma 2.3.5. To conclude the proof, we proceed exactly as in Lemma 2.3.8. ■

Lemma 2.3.10 *For all non-singular random walks,* **S** *has genus 0 if, and only if, one of the following relations holds:*

$$M_x = M_y = 0, \tag{2.3.5}$$

$$p_{10} = p_{11} = p_{01} = 0, \tag{2.3.6}$$

$$p_{10} = p_{1,-1} = p_{0,-1} = 0, \tag{2.3.7}$$

$$p_{-1,0} = p_{-1,-1} = p_{0,-1} = 0, \tag{2.3.8}$$

$$p_{01} = p_{-1,0} = p_{-1,1} = 0. \tag{2.3.9}$$

The singular random walks (c) and (e) on Fig. 2.2 also correspond to the genus 0 case and (2.3.5) implies

$$x_2 = x_3 = 1 \quad \text{and} \quad y_2 = y_3 = 1.$$

■

Proof **S** has genus 0 if, and only if, the discriminant

$$D(x) = b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0$$

of the equation $Q(x, y) = a(x)y^2 + b(x)y + c(x) = 0$ has a multiple zero, possibly infinite, where we have put

$$\begin{cases} d_0 &= p_{-1,0}^2 - 4p_{-1,1}p_{-1,-1}\,, \\ d_1 &= 2p_{-1,0}(p_{00}-1) - 4(p_{-1,1}p_{0,-1} + p_{01}p_{-1,-1})\,, \\ d_2 &= (p_{00}-1)^2 + 2p_{10}p_{-1,0} - 4[p_{11}p_{-1,-1} + p_{1,-1}p_{-1,1} + p_{01}p_{0,-1}]\,, \\ d_3 &= 2p_{10}(p_{00}-1) - 4(p_{11}p_{0,-1} + p_{01}p_{1,-1})\,, \\ d_4 &= p_{10}^2 - 4p_{11}p_{1,-1}\,. \end{cases}$$

The point of the proof is that multiple roots can occur only at $x = 0$, $x = 1$ or $x = \infty$. To see this, we use the continuity of the roots with respect to the points of $\mathcal{P}$ and the results of Lemmas 2.3.8 and 2.3.9. Indeed, as $\mathcal{A}$ is convex, any arbitrary point ρ can be connected by a direct line ℓ with a point $\rho_0 \in \mathcal{A}$, which corresponds to a non-degenerate *simple* random walk. The zeros of the discriminant continuously depend on the point of ℓ and cannot intersect Γ: this yields the first assertion of the lemma, when $M_y < 0$. The case $M_y > 0$ can be treated in a similar way.

∗ There is a double root at $x = 1$ if, and only if, $M_x = M_y = 0$. To see this, write

$$\begin{cases} D(1) &= (a(1) - c(1))^2 = M_y^2 = 0, \\ D'(1) &= -4a(1)[b'(1) + a'(1) + c'(1)] = -4a(1)M_x = 0. \end{cases}$$

∗ There is a multiple zero at ∞ if, and only if, $d_3 = d_4 = 0$. It is easy to see that the system of equations $b_3 = b_4 = 0$, with respect to the p_{ij}'s, has admissible solutions if, and only if, (2.3.5) or (2.3.6) or the singular case of Fig. 2.2c hold.
∗ Similarly $D(x)$ has a multiple root at 0 if, and only if, (2.3.7) or (2.3.8) or the singular case of Fig. 2.2e hold.

The proof of Lemma 2.3.10 is complete. ■

2.4 Galois Automorphisms and the Group of the Random Walk

Choose an arbitrary non singular-random walk. This means among other things that $Q(x, y)$, considered as a polynomial in y over the field $\mathbb{C}(x)$ of rational functions of x, is irreducible over this field.

Consider also the vector space over $\mathbb{C}(x)$, generated by the constant 1 and one zero $y(x)$ of Q. This vector space is a field, which is the extension of order 2 of $\mathbb{C}(x)$ and will be denoted in the sequel by $\mathbb{C}(x)[y(x)]$. Each element of $\mathbb{C}(x)[y(x)]$ can be written in a unique way as $u(x)+v(x)y(x)$, where u and v are elements of $\mathbb{C}(x)$. Then it is quite natural to identify $\mathbb{C}(x)[y(x)]$ with the quotient field $\mathbb{C}(x)[T]/Q(x, T)$. Similarly, exchanging x and y, one can define $\mathbb{C}(y)[x(y)]$ and $\mathbb{C}(y)[T]/Q(T, y)$.

Let $\mathbb{C}(x, y)$ be the field of rational functions in (x, y) over $\mathbb{C}$. Since Q is assumed to be irreducible in the general case, the quotient ring of $\mathbb{C}(x, y)$ is in fact a field, which will be denoted by $\mathbb{C}_Q(x, y)$.

Proposition 2.4.1 *The fields* $\mathbb{C}(x)[T]/Q(x,T)$ *and* $\mathbb{C}(y)[T]/Q(T,y)$ *are isomorphic to* $\mathbb{C}_Q(x,y)$.

Proof Noting that $\forall p \in \mathbb{C}_Q(x,y)$, $\exists$ a unique pair $u(x), v(x)$ of elements of $\mathbb{C}(x)$ such that

$$p = u(x) + v(x)y \mod Q.$$

Now the stated isomorphism is simply given by the mapping

$$\{u(x) + v(x)T\} \to \{u(x) + v(x)y\},$$

where the brackets stand for the adequate equivalence classes. Exchanging the roles of x and y, there exists an isomorphism

$$\{u(y) + v(y)T\} \to \{u(y) + v(y)x\},$$

where $u(y)$ and $v(y)$ are elements of $\mathbb{C}(y)$. ■

It is worth summarizing some of the above results by means of the following chain of statements, where the symbol $\cong$ means "isomorphic to":

$$\mathbb{C}_Q(x,y) \cong \mathbb{C}(x)[T]/Q(x,T) \cong \mathbb{C}(y)[T]/Q(T,y) \cong \mathbb{C}(x)[y(x)] \cong \mathbb{C}(y)[x(y)].$$

The Galois group of $\mathbb{C}(x)[y(x)]$ (resp. $\mathbb{C}(y)[x(y)]$) is cyclic of order 2 and its generic element will be denoted by ξ (resp. η), so that

$$\xi(u(x)) = u(x), \qquad \forall u \in \mathbb{C}(x), \tag{2.4.1}$$

$$\xi(y(x)) = \frac{c(x)}{y(x)a(x)} = -\frac{b(x)}{a(x)} - y(x), \tag{2.4.1$'$}$$

$$\eta(w(y)) = w(y), \qquad \forall w \in \mathbb{C}(y), \tag{2.4.2}$$

$$\eta(x(y)) = \frac{\widetilde{c}(y)}{x(y)\widetilde{a}(y)} = -\frac{\widetilde{b}(y)}{\widetilde{a}(y)} - x(y). \tag{2.4.2$'$}$$

It is important to remember that ξ [resp. η] *permutes the two roots* in y [resp. x] of $Q(x,y) = 0$. In fact the above proposition allows us to define two automorphisms $\widetilde{\xi}$ and $\widetilde{\eta}$ of $\mathbb{C}_Q(x,y)$ such that

$$\begin{cases} \widetilde{\xi}(f(x,y)) &= f(x,\widetilde{\xi}(y)) \mod Q, \quad \forall f \in \mathbb{C}(x,y), \\ \widetilde{\eta}(g(x,y)) &= g(\widetilde{\eta}(x),y) \mod Q, \quad \forall g \in \mathbb{C}(x,y). \end{cases} \tag{2.4.3}$$

where

$$\widetilde{\xi}(y) = \frac{c(x)}{ya(x)}, \qquad \widetilde{\eta}(x) = \frac{\widetilde{c}(y)}{x\widetilde{a}(y)}.$$

2.4.1 Construction of the Automorphisms $\hat{\xi}$ and $\hat{\eta}$ on $\mathbf{S}$

We use Theorem 2.1.13, taking $X = \mathbb{P}_x$, $Y = \mathbf{S}$, $n = 2$. Then, by Definition 2.1.10, and Propositions 2.1.12, 2.1.15, $\mathcal{M}(\mathbb{P}_x) = \mathbb{C}(x)$ and $\mathcal{M}(\mathbf{S})$ is an extension of degree 2 of $\mathbb{C}(x)$, so that it is *Galois*.

Moreover, Theorem 2.1.13 and Proposition 2.4.1 imply that the three objects

$$\mathcal{M}(\mathbf{S}),\ \mathbb{C}_Q(x, y),\ \mathbb{C}(x)[y(x)]$$

are isomorphic, and can consequently be identified. According to Sect. 2.2.4 and Eq. (2.2.8), $\mathbb{C}(x)$ can be embedded into $\mathcal{M}(\mathbf{S})$ by the correspondence

$$\hat{f} : \mathbf{S} \to \mathcal{M}(\mathbf{S}), \quad \text{with } \hat{f}(s) \stackrel{\text{def}}{=} f(x(s)), \quad \forall f \in \mathbb{C}(x).$$

Thus ξ, which is a generator of $G\big(\mathbb{C}(x)[y(x)]/\mathbb{C}(x)\big)$, can be viewed as acting on $\mathcal{M}(\mathbf{S})$, and the automorphism $\hat{\xi}$ $\in$Aut$(\mathbf{S}/\mathbb{P}_x)$ will be defined by the relation

$$f \circ \hat{\xi} = \xi^{-1} \circ f, \quad \forall f \in \mathcal{M}(\mathbf{S}).$$

Similarly,

$$f \circ \hat{\eta} = \eta^{-1} \circ f, \quad \forall f \in \mathcal{M}(\mathbf{S}).$$

Letting I_d be the identity operator, the following set of relations hold:

$$h_x \circ \hat{\xi} = h_x \qquad \hat{\xi}^2 = I_d\ , \tag{2.4.4}$$
$$h_y \circ \hat{\eta} = h_y \qquad \hat{\eta}^2 = I_d\ . \tag{2.4.5}$$

The points s and $\hat{\xi}(s)$ [resp. $\hat{\eta}(s)$] have the same projections onto $\mathbb{P}_x$ [resp. $\mathbb{P}_y$]. In particular, the branch points of h_x [resp. h_y] are the fixed points of $\hat{\xi}$ [resp. $\hat{\eta}$], which read $h^{-1}(x_i)$ [resp. $h^{-1}(y_i)$], $i = 1, \ldots, 4$.

Definition 2.4.2 The *group* of the random walk is the group $\mathcal{H}$ of automorphisms of $\mathbb{C}_Q(x, y)$ generated by $\widetilde{\xi}$ and $\widetilde{\eta}$ (given in Eq. (2.4.3), and it depends only on the transition probabilities p_{ij}, $|i|, |j| \le 1$. This group is isomorphic to a subgroup of automorphisms of $\mathbf{S}$ generated by $\hat{\xi}$ and $\hat{\eta}$.

Nota bene: Whenever no ambiguity arises, the following conventions will be adopted throughout the rest of this monograph.

- **The composition of automorphisms will be written as an ordinary product.**
- **As far as global properties are concerned, we shall identify the automorphisms** ξ, η **with their respective counterparts** $\widetilde{\xi}, \widetilde{\eta}, \hat{\xi}, \hat{\eta}$.

The next lemma gives the structure of all groups with generators ξ and η, such that $\xi^2 = \eta^2 = I_d$.

Lemma 2.4.3 *Define*

$$\delta = \eta\xi. \tag{2.4.6}$$

Then $\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^n, n \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. ■

Proof We want to show that $x\delta^n x^{-1} \in \mathcal{H}_0$, for any $x \in \mathcal{H}$ and any integer n. But, since $\eta = \eta^{-1}$ and $\xi = \xi^{-1}$, this is a direct consequence of the following general result [62]. *Let G be a group and H a subgroup of G of index 2; then H is normal in G*. Recall that the *index* is the number of distinct *left cosets*, i.e. subsets of G of the form xH, for all $x \in G$ (here x takes only the values ξ or η). ■

2.5 Reduction of the Main Equation to the Riemann Torus

Here the Riemann surface is supposed to have genus 1 (which excludes non-singular random walks) and we return to Eqs. (2.2.8) and (2.2.9), first studying the geometric properties of the domains $h_x^{-1}(\mathcal{D})$, $h_y^{-1}(\mathcal{D})$ and of their boundaries $h_x^{-1}(\Gamma)$, $h_y^{-1}(\Gamma)$. For the sake of conciseness we shall only consider here the cases

1. $M_y < 0, \quad M_x < 0,$
2. $M_y < 0, \quad M_x > 0,$
3. $M_y < 0, \quad M_x = 0.$

The remaining patterns of interest are obtained by exchanging x and y in the above inequalities. It is also worth noting that the case $M_x > 0$, $M_y > 0$, although yielding a non-ergodic situation, can topologically be handled by means of analogous arguments.

Since **S** has genus 1, $h_x^{-1}(\mathcal{D})$ is connected. To see this, it suffices to check that $h_x^{-1}(\mathcal{D})$ is *arcwise* connected. We have seen in Sect. 2.3.2 that there are two branch points x_1 and x_2 inside $\mathcal{D}$. Choose two arbitrary points u, v in $h_x^{-1}(\mathcal{D})$ on **S** and draw two arcs $[h_x(u), x_1]$ and $[x_1, h_x(v)]$ in $\mathcal{D}$. The arcwise connectivity now follows from Proposition 2.1.5 and the fact that x_1 corresponds to the unique point s_1 on **S**.

Of course, analogous properties hold for $h_y^{-1}(\mathcal{D})$ and $h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D}) \neq \emptyset$, since, by Lemma 2.3.4, $Y(x)$ and $X(y)$ can simultaneously take on values inside $\mathcal{D}$. We will denote by Γ_0 and Γ_1, respectively, the connected components of the boundary $h_x^{-1}(\Gamma)$ of $h_x^{-1}(\mathcal{D})$, such that

$$\begin{cases} \Gamma_0 \subset h_x^{-1}(\Gamma) \cap \{s : |y(s)| \le 1\} = \{|x(s)| = 1\} \cap \{|y(s)| \le 1\}, \\ \Gamma_1 \subset h_x^{-1}(\Gamma) \cap \{s : |y(s)| \ge 1\} = \{|x(s)| = 1\} \cap \{|y(s)| \ge 1\}. \end{cases}$$

Since by hypothesis $M_y \neq 0$, the results of Sect. 2.3 assert that Γ_0 and Γ_1 are closed analytic curves without self-intersections. The curves $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are defined in an analogous way. For example, $\widetilde{\Gamma}_0 \subset h_y^{-1}(\Gamma)$ and, on it, $|y(s)| = 1$, $|x(s)| \le 1$.

Let $M_x \neq 0$. The fact that Γ_0 and Γ_1 are homotopically equivalent follows from the construction of the Riemann surface **S**, after taking into account that, inside and outside the unit circle Γ in the complex plane, there are exactly two branch points. This last property shows, in addition, that the *homology class* of Γ_0 and Γ_1 is one of the normal homology bases on the torus (see [94]), the same being true for $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$.

Assume now that, for example, Γ_0 and $\widetilde{\Gamma}_0$ belong to different homology classes, so that they would intersect. But, for $M_y < 0$, it has been shown in Lemma 2.3.4 that $\Gamma_0 \cap \widetilde{\Gamma}_0 = \emptyset$. [Clearly, if we had supposed $M_y > 0$, by the same argument, we could have written $\Gamma_1 \cap \widetilde{\Gamma}_1$]. Hence, by transitivity, Γ_0, Γ_1, $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$ are defined by the same homology class.

The case $M_x = 0$ can be obtained by continuity, as mentioned in the next lemma, where the situation is more completely summarized.

Lemma 2.5.1 *Define s_0 such that $x(s_0) = y(s_0) = 1$. The three following possibilities exist.*
Case 1 (Fig. 2.3)

$$M_y < 0, \quad M_x < 0.$$

Then $\Gamma_1 \cap \widetilde{\Gamma}_1 = \emptyset$, and $\Gamma_0 \cap \widetilde{\Gamma}_0 = h_x^{-1}(\Gamma) \cap h_y^{-1}(\Gamma)$ consists of the single point s_0.
Case 2 (Fig. 2.4)

$$M_y < 0, \quad M_x > 0.$$

Then $\Gamma_0 \cap \widetilde{\Gamma}_1 = \{s_0\}$.
Case 3 (Fig. 2.5)

$$M_y < 0, \quad M_x = 0.$$

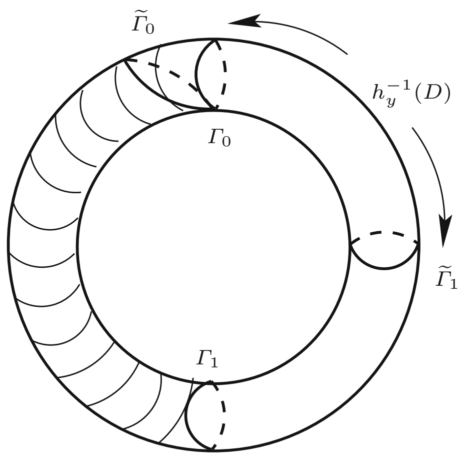


Fig. 2.3 $M_y < 0, \quad M_x < 0$

Fig. 2.4 $M_y < 0, \quad M_x > 0$

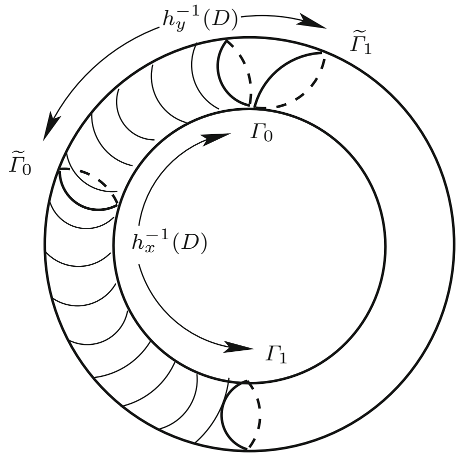
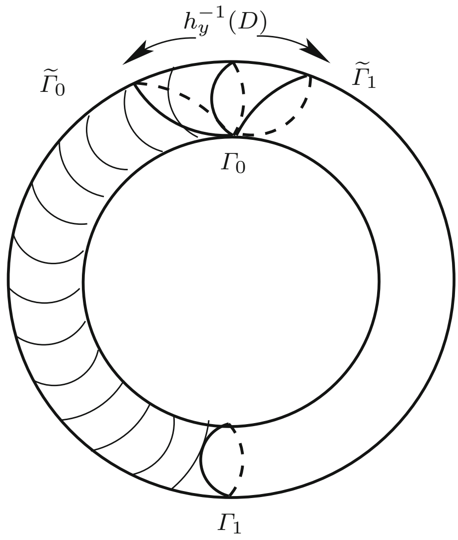


Fig. 2.5 $M_y < 0, \quad M_x = 0$



Then Γ_0 and Γ_1 do not intersect and one can define $\widetilde{\Gamma}_0 = h_x^{-1}(\overline{\mathcal{D}}) \cap h_y^{-1}(\Gamma)$ *and* $\widetilde{\Gamma}_1 = h_y^{-1}(\Gamma) \cap (\mathbf{S}\backslash h_x^{-1}(\mathcal{D}))$*, so that any pair of curves belonging to the triple* $(\Gamma_0, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1)$ *intersect at the point s_0 only.*

The proof in Case 3 is just the same, by letting in Case 1 $\varepsilon \to 0$ in $h_y^{-1}(\Gamma_\varepsilon)$, where $\Gamma_\varepsilon = \{x : |x| = 1 + \varepsilon\}$. ■

As already pointed out above, the case $M_x \geq 0$, $M_y \geq 0$, $M_x + M_y > 0$, is not considered, since then the random walk is transient (see [36]), but the construction is clearly of the same nature. It is a worthwhile exercise (left to the reader) to draw the pictures corresponding to the random walks **B** and **C**, introduced in Lemma 2.3.2.

Thus the functions $\hat{\pi}$ and $\hat{\hat{\pi}}$ are analytic in $h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D})$. They admit a meromorphic continuation to the domains $h_y^{-1}(\mathcal{D})$ and $h_x^{-1}(\mathcal{D})$ respectively, provided that in these domains they are defined by

$$\hat{\pi} = -\frac{\hat{\hat{q}}\,\hat{\hat{\pi}} + \hat{q}_0\pi_{00}}{\hat{q}}, \quad \hat{\hat{\pi}} = -\frac{\hat{q}\,\hat{\pi} + \hat{q}_0\pi_{00}}{\hat{\hat{q}}}.$$

In $h_y^{-1}(\mathcal{D})$ [resp. $h_x^{-1}(\mathcal{D})$], the only possible poles of $\hat{\pi}$ [resp. $\hat{\hat{\pi}}$] are the zeros of $\hat{q}$ [resp. $\hat{\hat{q}}$], since $\hat{q}$ [resp. $\hat{\hat{q}}$] is a meromorphic function on $\mathbf{S}$.

In other words, $\hat{\pi}$ and $\hat{\hat{\pi}}$ are meromorphic in $h_x^{-1}(\mathcal{D}) \cup h_y^{-1}(\mathcal{D})$ and in this region Eq. (2.2.9) still holds. One can now reformulate Lemma 2.2.1, for non-degenerate random walks, as a problem set on the Riemann surface $\mathbf{S}$.

Problem Find two meromorphic functions $\hat{\pi}$ and $\hat{\hat{\pi}}$ in $h_x^{-1}(\mathcal{D})$ and $h_y^{-1}(\mathcal{D})$ respectively, such that the following relations hold:

$$\begin{aligned} \hat{\pi}(s) &= \hat{\pi}(\hat{\xi}(s)), \\ \hat{\hat{\pi}}(s) &= \hat{\hat{\pi}}(\hat{\eta}(s)), \end{aligned}$$

where $(s, \hat{\xi}(s))$ and $(s, \hat{\eta}(s))$ belong to $h_x^{-1}(\mathcal{D}) \cup h_y^{-1}(\mathcal{D})$. Now it is easy to prove Lemma 2.2.1. Assume for instance that Case 1 prevails, i.e. $M_x < 0$, $M_y < 0$. Looking at Fig. 2.3, one can see that $\hat{\pi}$ is also defined in a neighborhood of Γ_0 on S. Now, one *lower* back $\hat{\pi}$ onto $\mathbb{C}$. From the very definition of all unknown functions and the probabilistic context, π must have no pole on Γ and, consequently, in some neighborhood of Γ. But π is a function of the single variable x. Since the function $h_x(s)$ has no branch point and is locally invertible in a neighborhood of Γ_0, it also follows that $\pi(x),\ x \in \mathcal{D}_{1+\varepsilon}$, has no branch point and is holomorphic in this region.

Cases 2 and 3 of Lemma 2.5.1 would give rise to slightly more complicated arguments and they will be considered later, in a more general context in Chaps. 3 and 5.

Remark 2.5.2 The case of genus 0, corresponding to the Riemann sphere, can be treated analogously, as was partially done in [75]. Indeed, it will be completely worked out in the complex plane in Chap. 6.

Chapter 3
Analytic Continuation of the Unknown Functions in the Genus 1 Case

In Sect. 2.5, we have shown, using the Riemann surface $\mathbf{S}$, that the functions π and $\widetilde{\pi}$ could be analytically continued to $\mathcal{D}_{1+\varepsilon}$. In this chapter, we shall propose other methods of analytic continuation, which in a sense are more effective, since they allow in fact a continuation to the whole complex plane. We consider only parameter values ensuring the Riemann surface $\mathbf{S}$ to be of genus 1, according to the study made in Sect. 2.3.

3.1 Lifting the Fundamental Equation onto the Universal Covering

We will use some properties, which are classical only for readers acquainted with the theory of Riemann surfaces, and can be found either in Forster [46] or in Hurwitz and Courant [53]. Since $\mathbf{S}$ is of genus 1, its covering manifold is the complex plane $\mathbb{C}$ and its universal covering has the form $(\mathbb{C}, \lambda)$, where, by Proposition 2.1.11, λ is *Galois* and $\mathrm{Aut}(\mathbb{C}/\mathbf{S})$ is isomorphic to the fundamental group of $\mathbf{S}$, which we denote by $\mathbf{T}$. In addition, it is known that

$$\mathbf{T} = \mathbb{Z}\,\omega_1 \oplus \mathbb{Z}\,\omega_2\,,$$

where ω_1 and ω_2 are two complex numbers, linearly independent on $\mathcal{R}$, acting as generators of $\mathbf{T}$. Thus, any $t \in \mathbf{T}$ can be written in the form

$$t = m_1\omega_1 + m_2\omega_2, \quad m_1, m_2 \in \mathbb{Z}\,,$$

and, $\forall \tau \in \mathrm{Aut}(\mathbb{C}/\mathbf{S})$, there exists a unique $t \in \mathbf{T}$, with

$$\tau(\omega) = \omega + t, \quad \omega \in \mathbb{C}\,.$$

G. Fayolle et al., *Random Walks in the Quarter Plane*, Probability Theory and Stochastic Modelling 40, DOI 10.1007/978-3-319-50930-3_3

In the next section, we will calculate these periods explicitly (they are, in general, defined up to some unimodular transformation, see e.g., [53]).

Along lines quite similar to those of Sect. 2.2.4, any meromorphic function f defined on $\mathbf{S}$ can be lifted onto the universal covering $\mathbb{C}$ by the mapping

$$f^* = f \circ \lambda. \tag{3.1.1}$$

Clearly, any such f^* is meromorphic and satisfies the functional equation

$$f^*(\omega) = f^*(\omega + m_1\omega_1 + m_2\omega_2), \quad \forall m_1, m_2 \in \mathbb{Z}. \tag{3.1.2}$$

Let us recall that a meromorphic function on $\mathbb{C}$ satisfying (3.1.2) is called an *elliptic function* with periods ω_1, ω_2. Thus the field $\mathbb{C}(\mathbf{S})$ of meromorphic functions on $\mathbf{S}$ is isomorphic, by (3.1.1), to the field of elliptic functions with the corresponding periods.

In addition, the period parallelogram

$$\Pi \stackrel{\text{def}}{=} \{\alpha_1\omega_1 + \alpha_2\omega_2 : 0 \le \alpha_i < 1\},$$

provided that its opposite sides are identified, is isomorphic to $\mathbf{S}$. Then $\lambda[0, \omega_1]$ and $\lambda[0, \omega_2]$ constitute a homology basis on the torus.

Let us fix the notation so that $\lambda\{[0, \omega_1]\}$ is homologous to Γ_0 and, therefore, also to the curves $\Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ and consider the domain

$$\Delta \stackrel{\text{def}}{=} h_x^{-1}(\mathcal{D}) \cup h_y^{-1}(\mathcal{D}) \subset \mathbf{S}.$$

Its inverse image $\lambda^{-1}(\Delta)$ belongs to $\mathbb{C}$ and consists of a denumerable number of curvilinear strips, which differ from each other by a translation of vector ω_2. Any such strip is simply connected and invariant by translation of amplitude ω_1. Indeed, Δ is generated by a set of closed curves, say $\mathcal{L}(t)$, $0 \le t \le 1$, homotopic to Γ_0, as was shown in Sect. 2.5 (see e.g., Fig. 2.3). Each curve $\mathcal{L}(t)$ has in Π a preimage homologous to $[0, \omega_1[$.

Denoting by Δ^* the connected component of $\lambda^{-1}(\Delta)$ having a non empty intersection with Π, one can use the mapping λ to lift π and $\widetilde{\pi}$ (which are meromorphic in Δ) onto Δ^*, just setting

$$\pi^* = \hat{\pi} \circ \lambda, \quad \widetilde{\pi}^* = \hat{\widetilde{\pi}} \circ \lambda.$$

In addition, as quoted in Sect. 2.2.4, the coefficients $\hat{q}, \hat{\widetilde{q}}, \hat{q}_0$ are meromorphic functions on $\mathbf{S}$, which in turn can be lifted onto the universal covering and satisfy Eq. (3.1.2).

Notation This is an appropriate place to make a notational convention: each function, whatever its domain of definition may be, will be represented by the same symbol with the adequate arguments, e.g., $\pi(s)$, $\pi(\omega)$, $\widetilde{q}(s)$, $\widetilde{q}(\omega)$, $q(x, y)$, etc.

Hence $\pi(\omega)$ and $\widetilde{\pi}(\omega)$ are well defined and meromorphic in Δ^*, where, from their single valuedness along Γ_0 and $\widetilde{\Gamma}_0$ respectively, they satisfy the equations

$$\pi(\omega + \omega_1) = \pi(\omega), \quad \widetilde{\pi}(\omega) = \widetilde{\pi}(\omega + \omega_1), \tag{3.1.3}$$

$$q(\omega)\pi(\omega) + \widetilde{q}(\omega)\widetilde{\pi}(\omega) + q_0(\omega)\pi_{00} = 0. \tag{3.1.4}$$

Note that it suffices to analyze equation (3.1.3) in the domain $\Delta^* \cap \Pi$.

3.1.1 Lifting of the Branch Points

Starting from the branch points x_i, y_i, previously introduced in Sect. 2.3.2, let $s_i, \widetilde{s}_i \in \mathbf{S}$ be such that

$$h_x(s_i) = x_i, \quad h_y(\widetilde{s}_i) = y_i, \quad \forall i = 1, \ldots, 4.$$

Clearly, s_i and $\widetilde{s}_i$ are uniquely defined by these relations and consequently there exist unique points

$$a_i \stackrel{\text{def}}{=} \lambda^{-1}(s_i) \cap \Pi, \quad b_i \stackrel{\text{def}}{=} \lambda^{-1}(\widetilde{s}_i) \cap \Pi, \quad \forall i = 1, \ldots, 4.$$

Moreover,

- for $i = 1, 2$, the points a_i and b_i belong to $\Delta^* \cap \Pi$, since $s_i, \widetilde{s}_i \in \Delta$;
- for $i = 3, 4$, the points a_i and b_i belong to $\Pi \setminus \Delta^*$.

3.1.2 Lifting of the Automorphisms on the Universal Covering

The goal of this section is to compute explicitly the automorphisms on the universal covering.

Let $\hat{\alpha}$ be an arbitrary automorphism of $\mathbf{S}$. The mapping

$$f \stackrel{\text{def}}{=} \hat{\alpha}^{-1} \circ \lambda : \mathbb{C} \to \mathbf{S}$$

is holomorphic, so that, choosing in Definition 2.1.4

$$X = \mathbf{S}, \quad Y = Z = \mathbb{C}, \quad h = \lambda,$$

we get a lifting of f with respect to λ, which will be denoted by α^* and satisfies the relation $f = \lambda \circ \alpha^*$. Since $(\mathbb{C}, \lambda)$ is a cover of $\mathbf{S}$ without branch point, it follows from Proposition 2.1.6 that α^* is also holomorphic. Taking now

$$\hat{\alpha} \equiv \hat{\xi},$$

where $\hat{\xi}$ has been constructed in Sect. 2.4.1, we get the corresponding automorphism ξ^* on $\mathbb{C}$, which satisfies the relation

$$\lambda \circ \xi^* = \hat{\xi}^{-1} \circ \lambda. \tag{3.1.5}$$

Since

$$\lambda(a_i) = s_i, \quad \hat{\xi}(s_i) = s_i, \quad \forall i = 1, \ldots, 4,$$

it follows that

$$\lambda(a_i) = \lambda(\xi^*(a_i)).$$

Hence, there exist integers $k_{1,i}, k_{2,i} \in \mathbb{Z}$ with

$$\xi^*(a_i) = a_i + k_{1,i}\omega_1 + k_{2,i}\omega_2, \quad \forall i = 1, \ldots, 4.$$

Still using Proposition 2.1.6, we know that ξ^* is unique, provided that one of its values is fixed. It is always possible to use a translation to make a_1 a fixed point of ξ^*, so that

$$\xi^*(a_1) = a_1.$$

Equation (3.1.5) yields immediately

$$\hat{\xi}^{-2} \circ \lambda = \lambda \circ \xi^{*2},$$

whence, since the range of $\xi^{*2} - I_d$ belongs to $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\hat{\xi}^2 = I_d$,

$$\xi^{*2} - I_d = K,$$

where K is a constant. Upon applying this last relation to the fixed point a_i, we indeed get $K = 0$. Thus we have just proved that ξ^* is an automorphism satisfying

$$\xi^{*2} = I_d.$$

Obviously any automorphism ζ^* of the complex plane has the form

$$\zeta^*(\omega) = \alpha\omega + \beta, \quad \forall \omega \in \mathbb{C}.$$

Hence, discarding the trivial automorphism I_d, one easily obtains

$$\xi^*(\omega) = -\omega + 2a_1. \tag{3.1.6}$$

So one can write, $\forall i = 1, \dots, 4$,

$$\xi^*(a_i) = -a_i + 2a_1 = a_i + k_{1,i}\omega_1 + k_{2,i}\omega_2,$$

and, using (3.1.6),

$$a_1 - a_i = \frac{k_{1,i}\omega_1 + k_{2,i}\omega_2}{2},$$

where $k_{1,i}, k_{2,i} \in \{-1, 0, 1\}$, since all a_i's are located in Π. Remembering (see beginning of Sect. 3.1) that $\lambda([0, \omega_1])$ was chosen to be homologous to Γ_0 on $\mathbf{S}$, one concludes that the open curve $\lambda([a_1, a_2])$ is homologous to *one half* of Γ_0.

Thus $|a_1 - a_2| = \dfrac{\omega_1}{2}$ and we will take

$$a_2 - a_1 = \frac{\omega_1}{2}. \tag{3.1.7}$$

Mutatis mutandis, one can prove along the same lines that it is possible to lift η, to obtain an automorphism η^* of $\mathbb{C}$, such that

$$\eta^*(\omega) = -\omega + 2b_1, \quad b_2 - b_1 = \frac{\omega_1}{2}. \tag{3.1.8}$$

Setting

$$\omega_3 = 2(b_1 - a_1), \tag{3.1.9}$$

we can write, in accordance with Eq. (2.4.6),

$$\delta^* = (\eta\xi)^* = \eta^*\xi^*,$$

so that

$$\delta^*(\omega) = \omega + \omega_3. \tag{3.1.10}$$

3.2 Analytic Continuation

Theorem 3.2.1 *π and $\widetilde{\pi}$ can be continued as meromorphic functions to the whole universal covering, where they satisfy Eqs. (3.1.3) and (3.1.4), together with the relations*

$$\pi(\omega) = \pi(\xi^*(\omega)), \quad \widetilde{\pi}(\omega) = \widetilde{\pi}(\eta^*(\omega)), \quad \forall \omega \in \mathbb{C}. \tag{3.2.1}$$

■

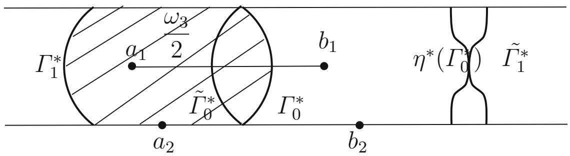


Fig. 3.1 The case $M_x < 0,\ M_y < 0$

Proof The region $\varDelta^*$ is simply connected. Moreover, $\varDelta^* \cap \varPi$ is bounded by the curves $\lambda^{-1}(\varGamma_1) \cap \varPi$ and $\lambda^{-1}(\widetilde{\varGamma}_1) \cap \varPi$. From the analysis made in Sects. 2.4 and 2.5, the following *automorphy* relationships on **S** hold:

$$\hat{\xi}(\varGamma_1) = \varGamma_0, \quad \hat{\delta}(\varGamma_1) = \hat{\eta}(\varGamma_0) \subset \overline{\varDelta}, \quad \hat{\xi}(\widetilde{\varGamma}_0) \subset \overline{\varDelta}, \tag{3.2.2}$$

using in particular the fact that η preserves x and $\varGamma_0 \subset \{s, |x(s)| \leq 1\}$.

In Fig. 3.1, the symbol " $*$ " has been used for all curves, to emphasize that we are on the universal covering, and δ^* is a translation *to the right* of vector $\omega_3 = 2(a_1 - b_1)$ in the system of coordinates (ω_2, ω_1), according to (3.1.10). The detailed computation of $\omega_1, \omega_2, \omega_3$ is carried out in Sect. 3.3. Lifting now the relations in (3.2.2) onto the universal covering, one can write for instance

$$\xi^*(\lambda^{-1}(\varGamma_1) \cap \varPi) = \lambda^{-1}(\varGamma_0) \cap \varPi, \quad \eta^*(\lambda^{-1}(\widetilde{\varGamma}_1) \cap \varPi) = \lambda^{-1}(\widetilde{\varGamma}_0) \cap \varPi,$$

whence

$$\delta^*(\lambda^{-1}(\varGamma_1) \cap \varPi) \subset \overline{\varDelta^*} \cap \varPi, \quad \delta^{*-1}(\lambda^{-1}(\widetilde{\varGamma}_1) \cap \varPi) \subset \overline{\varDelta^*} \cap \varPi. \tag{3.2.3}$$

Thus the domain

$$\varDelta^* \bigcup \delta^*(\varDelta^*)$$

is in fact simply connected and it follows by induction that

$$\bigcup_{n=-\infty}^{\infty} \delta^{*n}(\varDelta^*) \tag{3.2.4}$$

covers the whole complex plane $\mathbb{C}$. Moreover, since on $\mathbb{C}_x$ π is a function of x, $\forall |x| \leq 1$, we have

$$\begin{cases} \pi(\omega) = \pi(\xi^*(\omega)), & \forall (\omega, \xi^*(\omega)) \in \varDelta^*, \\ \widetilde{\pi}(\omega) = \widetilde{\pi}(\eta^*(\omega)), & \forall (\omega, \eta^*(\omega)) \in \varDelta^*, \end{cases}$$

and this equality, by the principle of analytic continuation, indeed holds for all $\omega \in \mathbb{C}$, so that (3.2.1) is proved.

Using now the fundamental Eq. (3.1.4) together with (3.2.1), we obtain after some easy manipulations the following system:

$$\begin{cases} q(\xi^*(\omega))\pi(\omega) + \widetilde{q}(\xi^*(\omega))\widetilde{\pi}(\xi^*(\omega)) + q_0(\xi^*(\omega))\pi_{00} = 0, & \forall(\omega, \xi(\omega)) \in \Delta^*, \\ q(\xi^*(\omega))\pi(\omega) + \widetilde{q}(\xi^*(\omega))\widetilde{\pi}(\delta^*(\omega)) + q_0(\xi^*(\omega))\pi_{00} = 0, & \forall(\omega, \delta^*(\omega)) \in \Delta^*. \end{cases}$$

Eliminating $\pi(w)$ in the above system, we obtain

$$\begin{aligned} \widetilde{\pi}(\delta^*(\omega)) = \widetilde{\pi}(\omega + \omega_3) = & \frac{q(\xi^*(\omega))\widetilde{q}(\omega)}{\widetilde{q}(\xi^*(\omega))q(\omega)}\widetilde{\pi}(\omega) \\ & + \frac{q(\xi^*(\omega))}{\widetilde{q}(\xi^*(\omega))}\left[-\frac{q_0(\xi^*(\omega))}{q(\xi(\omega))} + \frac{q_0(\omega)}{q(\omega)}\right]\pi_{00}. \end{aligned} \tag{3.2.5}$$

Equation (3.2.5) allows us to continue $\widetilde{\pi}(\omega)$, first to the domain $\delta^*(\Delta^*)$, and then, by induction, to the region introduced in (3.2.4). The proof of Theorem 3.2.1 is complete. ■

The next step consists in making the analytic continuation of $\pi(x)$ and $\widetilde{\pi}(y)$ in their respective complex planes $\mathbb{C}_x$ and C_y. To this end, we first go through an intermediate step, namely the analytic continuation of π and $\widetilde{\pi}$ on the Riemann surface **S**. This is the subject of the next theorem.

Theorem 3.2.2 *The functions π and $\widetilde{\pi}$ can be continued to the whole Riemann surface* **S***.* ■

Proof Indeed, since the mapping $\lambda : \mathbb{C} \to \mathbf{S}$ is holomorphic, π and $\widetilde{\pi}$ (which by Theorem 3.2.1 are known to be meromorphic functions on the universal covering $\mathbb{C}$) can be projected onto **S** also as meromorphic, but possibly multi-valued functions (since the torus **S** is not simply connected), according to the usual procedure, as follows.

Any path ℓ on **S**, having its origin at $s \in \Delta$, can be lifted onto a path ℓ^*, which is unique provided that the origin s^* of ℓ^* is fixed in $\lambda^{-1}(s)$, since λ has no branch point: it suffices to impose $s^* \in \lambda^{-1}(s) \cap \Pi$. This choice of s and s^* yields two functions on **S**, which coincide on Δ with π and $\widetilde{\pi}$, respectively, and are defined by

$$\pi \circ \lambda^{-1}(t), \quad \widetilde{\pi} \circ \lambda^{-1}(t), \quad \forall t \in \ell. \tag{3.2.6}$$

From the monodromy theorem,the continuation along any curve homotopic to Γ_0 leads to the same branch. In order to obtain the other branches of the two above-mentioned functions, it suffices to make the analytic continuation along some curve non-homotopic to Γ_0. The proof of the theorem is complete. ■

The pragmatic problem mentioned above can be rephrased as follows: *starting with $\pi(x)$ defined for $|x| \le 1$, we ask whether this function can be continued outside the unit disc.* The solution consists in projecting onto $\mathbb{C}_x$ and $\mathbb{C}_y$ respectively the functions defined on **S** by (3.2.6).

Let us draw a cut in $\mathbb{C}_x$, along the real axis between the points x_3 and x_4. More exactly, this cut is

$$\begin{cases} [x_3, x_4]\,, & \text{if} \quad x_3 < x_4 \le \infty, \\ [x_3, \infty] \cup [-\infty, x_4]\,, & \text{if} \quad x_4 < -1, \end{cases}$$

but in all cases we will write $[x_3, x_4]$.

Theorem 3.2.3 *Under the conditions of Theorem 1.2.1,* $\pi(x)$ *is a meromorphic function in the complex plane* $\mathbb{C}_x$ *cut along* $[x_3, x_4]$. *A similar statement holds for* $\widetilde{\pi}(y)$, *with the corresponding cut* $[y_3, y_4]$ *in* $\mathbb{C}_y$. ■

Proof One can proceed exactly along the lines which were used above to project π from $\mathbb{C}$ onto $\mathbf{S}$. Choose a path ℓ in $\mathbb{C}_x - \mathcal{D}$, not going through x_i, $i = 3, 4$. Since above any $u \in \ell$ there are exactly two points on $\mathbf{S}$, $h_x^{-1}(\ell)$ consists of *two* curves and $h_x(s)$ is *locally* one-to-one $\forall s \neq s_i$, by proposition 2.1.6.
Denoting one of these curves arbitrarily by $\mathcal{L}$, we have $h_x^{-1}(\ell) = \mathcal{L} \cup \hat{\xi}(\mathcal{L})$. On the other hand, from Sect. 2.5 and the construction presented above, it becomes clear that

$$\pi(s) = \pi(\hat{\xi}(s)), \quad \forall s \in \mathbf{S}.$$

Thus, taking $\mathcal{L}$ or $\hat{\xi}(\mathcal{L})$ does not matter and this shows that the analytic continuation of π along any arbitrary path ℓ in $\mathbb{C}_x - \mathcal{D}$ is possible. Here again, the continued function is *a priori* multi-valued. To render it single-valued, it suffices to cut $\mathbb{C}_x$ along $[x_3, x_4]$. Indeed, any simple closed curve ℓ in $\mathbb{C}_x \backslash \{[x_3, x_4] \cup \overline{\mathcal{D}}\}$ is lifted onto $\mathbf{S}$ into a curve $\mathcal{L}$, homotopic either to a single point or to Γ_0 (depending on whether or not $\mathcal{L}$ contains $[x_3, x_4]$ in its interior), but never to a latitude circle on the torus, which would correspond to the segment $[0, \omega_2]$ on the universal covering. Besides, we know that π is single-valued on such $\mathcal{L}$, since we remain on the same branch. The proof of Theorem 3.2.3 is complete. ■

Corollary 3.2.4 *Under the conditions of Theorem 1.2.1,* $\pi(x)$ *and* $\widetilde{\pi}(y)$ *are holomorphic in the neighborhood of the unit circle, in their respective complex planes.*

The proof is obvious since $[x_3, x_4] \subset \mathbb{C}_x - \overline{\mathcal{D}}$. ■

Corollary 3.2.5 *The following conditions are equivalent:*

1. π *is not rational;*
2. $\widetilde{\pi}$ *is not rational;*
3. π *(resp.* $\widetilde{\pi}$*) is not meromorphic in* $\mathbb{C}_x$ *(resp.* $\mathbb{C}_y$*);*
4. x_3 *(resp.* y_3*) is a non-polar singularity of* π *(resp.* $\widetilde{\pi}$).

Proof $4 \to 3 \to 1$ is clear. To show $1 \to 2$, assume π is rational. Then, from the basic functional Eq. (2.2.4), we have

$$\widetilde{\pi}(y) = E(y) + X_0(y)F(y),$$

where E and F are rational functions of y. But $\widetilde{\pi}$ has to be holomorphic in the unit disc, so that, necessarily, $F(y) \equiv 0$, which says that $\widetilde{\pi}$ is then also rational.
For the implication $1 \to 3 \to 4$, two cases have to be considered:

- $x_4 \neq \infty$. One sees easily, using analytic properties of the branches given in Sects. 2.3 and 5.3, that $x = \infty$ is either a regular point or a pole of π. Thus it remains to prove that, whenever x_3 is not a branch point of π, this property also holds for x_4. But there exists no meromorphic function on $\mathbb{C}_x \setminus \{x_4\}$ having x_4 as a branch point.
- $x_4 = \infty$. Then, referring again to Sect. 5.3, we have $y_4 \neq \infty$, provided the random walk is non-singular, and we come to the preceding argument applied to the function $\widetilde{\pi}$. ■

3.3 More About Uniformization

Our purpose here is to get explicit representations for λ, ω_1 and ω_2, which will be used, in particular, at the end of Chap. 5.

First we will recall some classical facts about the Weierstrass elliptic function $\wp(\omega; \omega_1, \omega_2)$, admitting the periods $m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$, and usually denoted by $\wp(\omega)$. All this material can be found, for instance in [53] or [6]. $\wp(\omega)$ is defined by the series

$$\wp(\omega) = \frac{1}{\omega^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(\omega - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right],$$

which is uniformly and absolutely convergent for all $\omega \neq m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$. At the points $m\omega_1 + n\omega_2$, $\wp(\omega)$ has poles of second order. The definition of $\wp(\omega; \omega_1, \omega_2)$ depends on the pair (ω_1, ω_2), up to a unimodular transform of the type

$$\omega_1' = \alpha\omega_1 + \beta\omega_2, \quad \omega_2' = \gamma\omega_1 + \delta\omega_2,$$

where $\alpha, \beta, \gamma, \delta$ are integers satisfying

$$\alpha\delta - \beta\gamma = \pm 1.$$

It is well known that $\wp(\omega)$ satisfies the differential equation

$$\wp'^2(\omega) = 4\wp^3(\omega) - g_2\wp(\omega) - g_3, \tag{3.3.1}$$

where the so-called *invariants* g_2 and g_3 are given by

$$\begin{cases} g_2 = 60 \displaystyle\sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4}, \\ g_3 = 140 \displaystyle\sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6}. \end{cases} \tag{3.3.2}$$

Moreover, any elliptic function with periods ω_1, ω_2 is a rational function of $\wp$ and of its derivative $\wp'$ and any *even* elliptic function is a rational function of the *sole* $\wp(u)$.

In Sect. 2.3, we have introduced the polynomial

$$Q(x, y) = a(x)y^2 + b(x)y + c(x).$$

Setting temporarily, for the sake of shortness,

$$\begin{cases} D(x) = b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ z = 2a(x)y + b(x), \end{cases}$$

with

$$\begin{cases} d_4 = p_{10}^2 - 4p_{11}p_{1,-1}, \\ d_3 = -[2p_{10}(1 - p_{00}) + 4p_{11}p_{0-1} + p_{01}p_{1-1}] < 0, \end{cases}$$

we know from Chap. 2 that $D(x)$ has four real zeros x_i, $i = 1, \dots, 4$. In addition x_2 and x_3 are always positive and we have

$$\begin{cases} -1 \le x_1 < 1 \le x_2 < x_3 < x_4, & \text{if} \quad d_4 > 0, \\ x_4 < -1 \le x_1 < 1 \le x_2 < x_3, & \text{if} \quad d_4 < 0, \\ x_4 = \infty, & \text{if} \quad d_4 = 0. \end{cases}$$

Lemma 3.3.1 *When* $\mathbf{S}$ *is of genus 1, the algebraic curve* $Q(x, y) = 0$ *admits a uniformization given in terms of the Weierstrass* $\wp$*-function and its derivative* $\wp'$ *by the following formulas:*

(i) If $d_4 \neq 0$*, then* $D'(x_4) > 0$ *and*

$$\begin{cases} x(\omega) = x_4 + \dfrac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}, \\ z(\omega) = \dfrac{D'(x_4)\wp'(\omega)}{2\left(\wp(\omega) - \frac{1}{6}D''(x_4)\right)^2}; \end{cases} \tag{3.3.3}$$

(ii) If $d_4 = 0$*, then* $x_4 = \infty$ *and*

$$\begin{cases} x(\omega) = \dfrac{\wp(\omega) - \dfrac{d_2}{3}}{d_3}, \\ z(\omega) = -\dfrac{\wp'(\omega)}{2d_3}. \end{cases} \tag{3.3.4}$$

In both cases, $z(\omega)$ and $\wp'(\omega)$ have the sign.

In the above formulas any x_i might have been chosen, but x_4 is in some sense the best candidate, since it depends on d_4 in a very direct way. ■

Proof The algebraic curve $Q(x, y) = 0$ can be rewritten in the slightly simpler form

$$z^2 = D(x).$$

A classical way of uniformizing such curves, when D has degree 4 in x, consists of a reduction to the *Weierstrass domain*, in which case D would be of degree 3. To achieve this, it suffices to send one of the zeros of D, e.g., x_4, to infinity, using a fractional linear transformation.

Consider next the following Taylor expansion

$$D(x) = (x - x_4)D'(x_4) + \frac{(x - x_4)^2}{2!}D''(x_4) + \frac{(x - x_4)^3}{3!}D^{(3)}(x_4) + \frac{(x - x_4)^4}{4!}D^{(4)}(x_4),$$

where $D^{(j)}(\cdot)$ denotes the derivative of order $j \geq 3$, and let

$$u = \frac{D'(x_4)}{x - x_4}, \quad v = \frac{2zD'(x_4)}{(x - x_4)^2}.$$

We have therefore

$$v^2 = 4u^3 + 2D''(x_4)u^2 + \frac{2u}{3}D^{(3)}(x_4)D'(x_4) + \frac{D^{(4)}(x_4)[D'(x_4)]^2}{6}.$$

Setting now

$$t = u + \frac{1}{6}D''(x_4),$$

yields the well-known Weierstrass canonical form

$$v^2 = 4t^3 - g_2t - g_3. \tag{3.3.5}$$

The case $d_4 = 0$ (and then $d_3 \neq 0$, since the degree of $D(x)$ must be ≥ 3 to have a curve of genus 1) is easier to handle, since it is then possible to eliminate the coefficient of degree 2 in D, by the linear change of variables

$$x = \frac{t - \dfrac{d_2}{3}}{d_3}, \quad z = -\frac{v}{2d_3},$$

which yields at once (3.3.5). Now the Weierstrass domain formed by the pairs (t, v) satisfying (3.3.5) is uniformized by setting directly, see [53],

$$t = \wp(\omega), \quad v = \wp'(\omega).$$

Hence, with these values for t, v, u, the asserted formulas (3.3.3) and (3.3.4) hold and the lemma is proved. ■

It is well known that the periods ω_1, ω_2 of $\wp(\omega)$, which appear in the uniformization of (3.3.5), are uniquely defined in terms of g_2, g_3 by the formulas (3.3.3), since the ratio ω_2/ω_1 is assumed to be non-real. Nonetheless these inversion formulas are not easy to manipulate numerically. Indeed they involve elliptic modular functions (see e.g., [6]). Thus, it seems useful to give a direct explicit form for the periods ω_1, ω_2 in terms of elliptic integrals.

Lemma 3.3.2 *One can choose ω_1, ω_2 so that ω_1 is purely imaginary and ω_2 is real. Such a choice is unique, up to signs. Hence we shall take $\omega_2 > 0$, $\Im(\omega_1) > 0$. Moreover, the ω_j's are given by the following integrals, taken on intervals of the real axis:*

$$\omega_1 = 2i \int_{x_1}^{x_2} \frac{dx}{\sqrt{|D(x)|}}, \quad \omega_2 = 2 \int_{x_2}^{x_3} \frac{dx}{\sqrt{D(x)}}. \tag{3.3.6}$$

■

Proof Using the notation of Lemma 2.3.10, it will be convenient to assume $d_4 = p_{10}^2 - 4p_{11}p_{1,-1} \geq 0$. Then, by Lemma 2.3.8, $\mathcal{D}(x) \leq 0$, for $x_1 \leq x \leq x_2$, and $\mathcal{D}(x) \geq 0$, for $x_2 \leq x \leq x_3$. [Taking $d_4 < 0$ would simply change the sign of $D(x)$ on the respective intervals.]

Let us remark first that if $d_4 \neq 0$, then $D'(x_4) > 0$. Consequently, using the definition of $x(\omega)$ given in Lemma 3.3.1, we show that, if $d_4 \neq 0$, then $\wp$ is a decreasing function of x, by (3.3.3), separately on the intervals $]-\infty, x_4[$ and $]x_4, +\infty[$. If $d_4 = 0$, then $\wp$ is a decreasing function of x on the whole real line $\mathbb{R}$, since $d_3 < 0$. The zeros of $g(\cdot)$, where $g(x) = 4x^3 - g_2 x - g_3$, are usually denoted by e_1, e_2, e_3 (see [6]). They are real and satisfy the following relationships:

$$e_1 + e_2 + e_3 = 0, \quad e_1 > 0, \quad e_3 < 0, \quad e_1 > e_2 > e_3.$$

Let (ω_1, ω_2) be a pair of primitive periods. It is known (see [53]) that

$$\wp\left(\frac{\omega_2}{2}\right) = e_1, \quad \wp\left(\frac{\omega_1}{2}\right) = e_3, \quad \wp\left(\frac{\omega_1 + \omega_2}{2}\right) = e_2.$$

Consider the mapping $h = z \to \dfrac{z}{2}$ in the complex plane. This homothety transforms the period parallelogram into another parallelogram, denoted by H. A property of the Weierstrass function $\wp(\omega)$ says that, when we describe H in the positive direction, starting from the point $\omega = 0$, $\wp(\omega)$ decreases from $+\infty$ to $-\infty$. This yields immediately, from (3.3.3) and (3.3.4),

$$x\left(\frac{\omega_2}{2}\right) = x_1, \quad x\left(\frac{\omega_1+\omega_2}{2}\right) = x_2, \quad x\left(\frac{\omega_1}{2}\right) = x_3, \quad x(0) = x_4. \tag{3.3.7}$$

Now define the elliptic integral

$$I(w) \stackrel{\text{def}}{=} \int_{-\infty}^{w} \frac{dt}{\sqrt{g(t)}}.$$

It is in fact more convenient to consider $I(w)$ on the Riemann surface $\mathbf{S}$, where the integrand becomes a meromorphic function. But then $I(w)$ is a multi-valued function. Fix a value of the integrand at some point a and consider two paths ℓ_1 and ℓ_2 on $\mathbf{S}$, such that the homologous paths L_1 and L_2, inside the period parallelogram in $\mathbb{C}$, join the points a to w. Then there exists integers m_1, m_2 such that

$$\int_{L_1} \frac{dt}{\sqrt{g(t)}} = \int_{L_2} \frac{dt}{\sqrt{g(t)}} + m_1\omega_1 + m_2\omega_2.$$

Taking $a = \infty$, $w = \wp(\omega; \omega_1, \omega_2)$, we get

$$\omega = \int_{\wp(\omega;\omega_1,\omega_2)} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \tag{3.3.8}$$

where ω, for any value of $\wp$, is defined modulo (ω_1, ω_2), noting that $\wp(\omega)$ is single-valued for all ω. It is worth noting that in (3.3.8) the radical is supposed to have a positive value for $\omega \in]0, \frac{\omega_2}{2}[$ and, in fact, the integral should be considered on the torus or, equivalently, in the period parallelogram. Thus at the point $\omega = \frac{\omega_2}{2}$, one passes from one sheet to the other, which means that, for $\omega \in]\frac{\omega_2}{2}, \omega_2[$, the radical is necessarily a negative real quantity.

Let us make now in (3.3.8) the change of variables

$$x = x_4 + \frac{D'(x_4)}{t - \frac{1}{6}D''(x_4)}, \quad \text{if } d_4 \neq 0,$$

$$x = \frac{t - \dfrac{d_2}{3}}{d_3}, \quad \text{if } d_4 = 0.$$

Then

$$\omega = \int_{x_4}^{x(\omega)} \frac{dx}{\sqrt{D(x)}}, \quad \bmod (\omega_1, \omega_2),$$

so that a particular pair of periods $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ is given by

$$\widetilde{\omega}_1 = 2\int_{x_4}^{x_3} \frac{dx}{\sqrt{D(x)}},$$
$$\widetilde{\omega}_2 = 2\int_{x_4}^{x_1} \frac{dx}{\sqrt{D(x)}}.$$

In fact, $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ depend on the choice of the path of integration. In particular, it is always possible to take the path $\overset{\frown}{s_2 s_4}$ instead of $\overset{\frown}{s_1 s_4}$ on $\mathbf{S}$. This means that one can always take

$$\omega_2 = 2\int_{x_2}^{x_3} \frac{dx}{\sqrt{D(x)}} > 0,$$

where the integration is taken over the real interval $[x_2, x_3]$. In addition, the following inequalities hold:

$$2\int_{x_3}^{x_4} \frac{dx}{\sqrt{D(x)}} = \int_{L_2} \frac{dx}{\sqrt{D(x)}} = \int_{L_1} \frac{dx}{\sqrt{D(x)}} = 2\int_{x_2}^{x_1} \frac{dx}{\sqrt{D(x)}},$$

up to the sign, which can be arbitrarily chosen. Since $D(x) \leq 0$, for $x \in [x_1, x_2]$, we get

$$\omega_1 = 2i\int_{x_1}^{x_2} \frac{dx}{\sqrt{-D(x)}}$$

as a possible second primitive period, which is purely imaginary. Since the pair (ω_1, ω_2) is given up to a unimodular substitution, it appears that the above choice of (ω_1, ω_2) is unique, as asserted. The proof of Lemma 3.3.2 is complete. ■

Lemma 3.3.3 *We have*

$$\omega_3 = 2\int_{X(y_1)}^{x_1} \frac{dx}{\sqrt{D(x)}}, \tag{3.3.9}$$

with

$$0 < \omega_3 < \omega_2.$$

■

Proof We shall use the formulae (3.3.7), together with

$$x(a_1) = x_1, \quad a_1 = \frac{\omega_2}{2}, \quad x(b_1) = X(y_1),$$

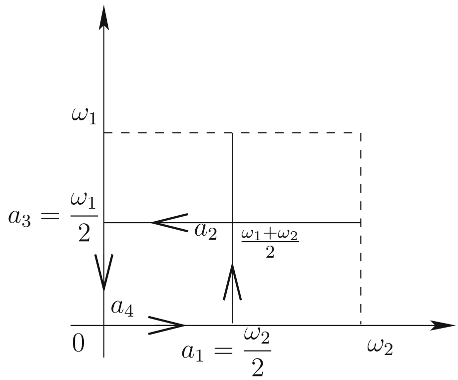


Fig. 3.2 Variation of the Weierstrass $\wp$-function

where the three equations above simply depict the correspondence between the branch points x_i, y_i in the complex plane and their images a_i, b_i, $i = 1, 2$ on the universal covering.

The following properties of the $\wp$-function will be needed (see Fig. 3.2).

(i) $\wp$ is real on the sole intervals $[0, \omega_2[$, $[0, \omega_1[$, $\left[\frac{\omega_1}{2}, \frac{\omega_1}{2} + \omega_2\right[$, $\left[\frac{\omega_2}{2}, \frac{\omega_2}{2} + \omega_1\right[$;
(ii) $\wp$ decreases on $\left[0, \frac{\omega_2}{2}\right]$ from $+\infty$ to e_1 and increases on $\left[\frac{\omega_2}{2}, \omega_2\right[$ from e_1 to $+\infty$. More generally, (see [6]), $\wp$ is real and monotone decreasing along the circuit drawn in Fig. 3.2.

Thus, for $\omega \in \left[0, \frac{\omega_2}{2}\right]$, the function $x(\omega)$ (which is a fractional linear transformation of $\wp(\omega)$) is increasing and takes its values in the following intervals:

$$\begin{cases}]x_4, +\infty[\text{ and }] - \infty, x_1[, \text{ if } d_4 > 0; \\ [x_4, x_1], \text{ if } d_4 < 0; \\ ] - \infty, x_1[, \text{ if } d_4 = 0. \end{cases}$$

We will see later on in Chap. 5 that $X(y_1) < x_1$, which implies first that b_1 is real and also that $b_1 \in]0, \omega_2[$, by the earlier mentioned properties of the function $\wp(\omega)$. Hence only two cases can take place:

$$\begin{cases} b_1 = c, \quad \text{where} \quad c \in \left[0, \frac{\omega_2}{2}\right], \quad x(c) = X(y_1), \quad \text{or} \\ b_1 = \omega_2 - c. \end{cases}$$

To make the right choice, one must know the sign of $\wp'(b_1)$, since $\wp$ decreases on $\left[0, \frac{\omega_2}{2}\right]$ and increases on $\left[\frac{\omega_2}{2}, \omega_2\right]$. By Lemma 3.3.1, $\wp'(b_1)$ and $z(b_1)$ have the same sign. Furthermore, from the very definition of z, we have

$$z(b_1) = 2a(X(y_1))y_1 + b(X(y_1)).$$

Referring again to Sect. 2.3 and to Chap. 5, the branch $Y_0(x)$ of the algebraic function $y(x)$ satisfies $y_1 = Y_0(X(y_1))$. Thus

$$\begin{aligned} z(b_1) &= 2a(X(y_1))Y_0(X(y_1)) + b(X(y_1)) \\ &= a(X(y_1))[Y_0(X(y_1)) - Y_1(X(y_1))] = \pm\sqrt{D(x(y_1))}, \end{aligned}$$

where we have used $Y_0(x) + Y_1(x) = \dfrac{-b(x)}{a(x)}$.

Taking now $x_1 - \varepsilon < x < x_1$, for some $\epsilon > 0$, we will show in Chap. 5 that $Y_0(x) > Y_1(x)$. In addition, $a(x_1) > 0$. This last inequality can be proved by noting that, as $D(x) < 0$ on $]x_1, x_2[$, $a(x)$ has no zero on $[x_1, x_2]$. But $x_2 > 0$ implies $a(x_2) > 0$, so that $a(x_1) > 0$. Finally, the sign of the quantity

$$[Y_0(x) - Y_1(x)]a(x) = \pm\sqrt{D(x)}$$

does not change on the intervals

$$\begin{cases}]x_4, x_1[, \ \text{if } x_4 < 0, \\ ]x_4, +\infty[\cup[-\infty, x_1[, \ \text{if } x_4 > 0. \end{cases}$$

This shows that

$$[Y_0(X(y_1)) - Y_1(X(y_1))]a(X(y_1)) > 0,$$

which yields in turn

$$b_1 = \omega_2 - c.$$

Then

$$\omega_3 = 2(b_1 - a_1) = \omega_2 - 2c = 2\int_{x_4}^{x_1} \frac{dx}{\sqrt{D(x)}} + 2\int_{X(y_1)}^{x_4} \frac{dx}{\sqrt{D(x)}}.$$

Lemma 3.3.3 is proved. ∎

Remark 3.3.4 The calculation of the periods can also be achieved by using Eq. (3.3.2) and the modular functions [53].

Remark 3.3.5 All the derivations concerning the periods and the quantity ω_3 are connected with the fact that there exists a unique Abelian differential of the first kind, up to a multiplicative constant,

$$\frac{dx}{2a(x)y + b(x)}.$$

This differential becomes $d\omega$ when lifted onto the universal covering and $\dfrac{dx}{\sqrt{D(x)}}$ when projected onto the complex plane $\mathbb{C}_x$.

Chapter 4
The Case of a Finite Group

In Sect. 2.4, the group $\mathcal{H}$ of the random walk was shown to be of even order $2n$, $n = 2, \ldots, \infty$. Throughout this chapter, the algebraic curve

$$\mathscr{K} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\},$$

where $Q(\cdot, \cdot)$ has been defined in Eq. (2.3.1), is of genus 1, the number n is assumed to be finite and the functions $q, \widetilde{q}, q_0$ are polynomials. In this case, we will be able to characterize completely the solutions of the basic functional equation, and also to give necessary and sufficient conditions for these solutions to be rational or algebraic. First of all, we propose a concrete criterion ensuring the finiteness of the group. Quite pleasantly, it turns out that this criterion is always tantamount to the cancellation of a single constant, which can be expressed as the *determinant of a matrix of order 3 or 4, and depends in a polynomial way on the coefficients of the walk.* (see [34]).

4.1 Conditions for $\mathcal{H}$ to be Finite

Finding an exact explicit form for n to be finite is a deep question, which has many connections with some classical problems in algebraic geometry. As an immediate consequence of Lemma 2.4.3 and Sect. 3.1.2, the group $\mathcal{H}$ is finite of order $2n$ if, and only if,

$$\delta^n = I \tag{4.1.1}$$

where $\delta = \eta\xi$ is defined by (2.4.6), or equivalently

$$n\omega_3 = 0 \mod (\omega_1, \omega_2),$$

that is, since ω_3 is real,

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$$n\omega_3 = 0 \mod (\omega_2), \tag{4.1.2}$$

where n stands for the minimal positive integer with this property. But the necessary and sufficient condition given by (4.1.2), albeit very simple and theoretically nice, is indeed not easy to check by calculus.

On the *universal covering* $\mathbb{C}$ (the finite complex plane), the automorphisms introduced in Sect. 3.1.2 satisfy the relations

$$\xi^*(\omega) = -\omega + \omega_2, \qquad \eta^*(\omega) = -\omega + \omega_2 + \omega_3, \qquad \delta^*(\omega) = \eta^*\xi^*(\omega) = \omega + \omega_3. \tag{4.1.3}$$

Also, for any $f(x, y) \in C_Q(x, y)$,

$$\delta(f(x, y)) = f(\delta(x), \delta(y)) = f(x(\delta^*(\omega)), y(\delta^*(\omega))), \ \omega \in \mathbb{C}.$$

In particular, using the fact that $x(\omega)$ is an even function of period ω_2,

$$\begin{cases} \eta\xi(x) = x(\eta^*\xi^*(\omega)) = x(\omega + \omega_3), \\ \eta(x) = x(\eta^*(\omega)) = x(-\omega + \omega_2 + \omega_3) = x(\omega - \omega_3). \end{cases} \tag{4.1.4}$$

The main goal of the present section is precisely to transform condition (4.1.2) into closed-form expressions, algebraically tractable. As we shall see, there are some structural differences due to the parity of the number n, remembering that the order of the group is $2n$. Therefore, serving a kind of introductory purpose, we will consider first the groups of order 4, 6, 8.

Recalling that $\mathcal{H}$ is generated by the elements ξ and η, we can define the homomorphism

$$h(R(x, y)) \stackrel{\text{def}}{=} R(h(x), h(y)), \quad \forall h \in \mathcal{H}, \ \forall R \in \mathbb{C}_Q(x, y).$$

Clearly, two elements h_1, h_2 of $\mathcal{H}$ are identical if, and only if,

$$h_1(x) = h_2(x), \quad h_1(y) = h_2(y).$$

In addition, for any $R \in \mathbb{C}_Q(x, y)$, the following important equivalences hold:

$$\begin{cases} \xi(R) = R \iff R \in \mathbb{C}(x), \\ \eta(R) = R \iff R \in \mathbb{C}(y), \end{cases} \tag{4.1.5}$$

so that $\mathbb{C}(x)$ (resp. $\mathbb{C}(y)$) is the set of elements of $\mathbb{C}_Q(x, y)$ invariant with respect to ξ (resp. η). Indeed, we know from Proposition 2.4.1 that R is of the form

$$R(x, y) = A(x) + B(x)y \mod Q(x, y),$$

where $A(x)$ and $B(x)$ are elements of $\mathbb{C}(x)$. But, since $\xi(R) = R$ and $\xi(y) \neq y$, we have necessarily $B(x) \equiv 0$ and the proof is complete. ■

Definition 4.1.1 Introduce the matrix

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{pmatrix}, \tag{4.1.6}$$

and let C_1, C_2, C_3 (resp. D_1, D_2, D_3) denote the column vectors of $\mathbb{P}$ (resp. of $\mathbb{P}^T$, the transpose matrix of $\mathbb{P}$). ■

The following simple property will be very useful.

Proposition 4.1.2 *Assume there exists a positive integer s such that*

$$\delta^s(x) = x. \tag{4.1.7}$$

Then $\delta^s = I$ and the group is of order $2s$, where s stands for the smallest integer with property (4.1.7).

Proof Each of the three following permutations

$$x \Longleftrightarrow y, \quad \delta \Longleftrightarrow \delta^{-1}, \quad \mathbb{P} \Longleftrightarrow \mathbb{P}^T,$$

implies the two other ones. Hence, the quantity $\rho(x, y, k) \stackrel{\text{def}}{=} \delta^k(x).\delta^{-k}(y)$, for any integer $k \geq 1$, remains invariant by permuting $\mathbb{P}$ with $\mathbb{P}^T$.

Assume first $s = 2m$. Then (4.1.7) becomes $\delta^m(x) = \delta^{-m}(x)$, and

$$\rho(x, y, m) = \delta^{-m}(x).\delta^{-m}(y) = \delta^m(x).\delta^m(y),$$

where the second equality is obtained by replacing $\mathbb{P}$ by $\mathbb{P}^T$. Then, comparing with the definition of $\rho(x, y, m)$, we get $\delta^m(y) = \delta^{-m}(y)$, which yields in turn $\delta^s(y) = y$, whence $\delta^s = I$.

The argument works in exactly the same way if s is odd, say $s = 2m + 1$. Indeed, in this case we have

$$\rho(x, y, m) = \delta^{-(m+1)}(x).\delta^{-m}(y) = \delta^m(x).\delta^{m+1}(y),$$

(by exchanging again $\mathbb{P}$ with $\mathbb{P}^T$), which implies $\delta^{m+1}(y) = \delta^{-m}(y)$, that is $\delta^s(y) = y$, concluding the proof of the proposition. ■

Corollary 4.1.3

1. *If there exists an s such that $\delta^s(x) = r(x)$, where $r(x)$ represents a rational fraction of x, then $\delta^{2s}(x) = x$ and the group is of order $4s$.*

2. *If there exists an s such that $\delta^s(x) = t(y)$, where $t(y)$ represents a rational fraction of y, then $\delta^{2s-1}(x) = x$ and the group is of order $4s - 2$.*

In both cases, s stands for the smallest integer with the corresponding property.

Proof Note first the identities $\xi\delta^s\xi = \delta^{-s}$ and $\eta\delta^s\xi = \delta^{-s+1}$.
So, we have the following chain of equalities.

$$\delta^s(x) = r(x) \Longrightarrow \xi\delta^s\xi(x) = \delta^s(x) \Longleftrightarrow \delta^{-s}(x) = \delta^s(x) \Longleftrightarrow \delta^{2s}(x) = x.$$

Similarly

$$\delta^s(x) = t(y) \Longrightarrow \eta\delta^s\xi(x) = \delta^s(x) \Longleftrightarrow \delta^{-s+1}(x) = \delta^s(x) \Longleftrightarrow \delta^{2s-1}(x) = x.$$

In both cases, the conclusion follows from Proposition 4.1.2, in which s is replaced respectively by $2s$ and $2s - 1$. ■

Lemma 4.1.4 *On the algebraic curve $\{Q(x, y) = 0\}$, the following general relations hold:*

$$\begin{cases} \eta(x) = \dfrac{xv(y) - u(y)}{xw(y) - v(y)}, \\[2ex] \xi(y) = \dfrac{y\widetilde{v}(x) - \widetilde{u}(x)}{y\widetilde{w}(x) - \widetilde{v}(x)}, \end{cases} \tag{4.1.8}$$

where u, v, w, h (resp. $\widetilde{v}, \widetilde{v}, \widetilde{w}, \widetilde{h}$) are polynomials of degree ≤ 2.
In particular, there exist affine solutions

$$(u(y), v(y), w(y))^T = A\,y + B, \ (\widetilde{u}(x), \widetilde{v}(x), \widetilde{w}(x))^T = E\,x + F, \tag{4.1.9}$$

with column vectors

$$A = (u_0, v_0, w_0)^T, \ \ B = (u_1, v_1, w_1)^T, \ \ E = (\widetilde{u}_0, \widetilde{v}_0, \widetilde{w}_0)^T, \ \ F = (\widetilde{u}_1, \widetilde{v}_1, \widetilde{w}_1)^T,$$

given by

$$\begin{cases} A = (\alpha\, C_2 + \beta\, C_1) \times \ C_3, \\ B = \ C_1 \times (\alpha\, C_3 + \beta\, C_2), \\ E = (\widetilde{\alpha}\, D_2 + \widetilde{\beta}\, D_1) \times \ D_3, \\ F = \ D_1 \times (\widetilde{\alpha}\, D_3 + \widetilde{\beta}\, D_2), \end{cases} \tag{4.1.10}$$

where $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$ are arbitrary complex constants, and the operator "$\times$" stands for the cross vector product. In addition, when $\mathbb{P}$ is of rank 3, none of the vectors A, B, E, F vanish. Choosing in (4.1.10) $\alpha = \widetilde{\alpha} = 0, \beta = \widetilde{\beta} = 1$, gives

$$\begin{cases} u(y) = y\Delta_{13} - \Delta_{12}, & \widetilde{u}(x) = x\Delta_{31} - \Delta_{21}, \\ v(y) = y\Delta_{23} - \Delta_{22}, & \widetilde{v}(x) = x\Delta_{32} - \Delta_{22}, \\ w(y) = y\Delta_{33} - \Delta_{32}, & \widetilde{w}(x) = x\Delta_{33} - \Delta_{23}, \end{cases} \tag{4.1.11}$$

where Δ_{ij} denotes the cofactor of the $(i, j)^{th}$ entry of the matrix $\mathbb{P}$ given in (4.1.6).

Proof We proceed by construction, assuming $\eta(x)$ is given by the following expression

$$\eta(x) = \frac{xv(y) - u(y)}{xw(y) - h(y)}, \tag{4.1.12}$$

where u, v, w, h are affine functions of y. Then, by using the basic relation

$$\eta(x) = \frac{\widetilde{c}(y)}{x\widetilde{a}(y)}$$

in (4.1.12), we obtain

$$\widetilde{a}(y)v(y)x^2 - [\widetilde{a}(y)u(y) + \widetilde{c}(y)w(y)]x + \widetilde{c}(y)h(y) = 0 \mod Q(x, y),$$

which must be proportional to (2.3.1) written in the form

$$\widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y) = 0.$$

This implies the two identities, $\forall y \in \mathbb{C}$,

$$\begin{cases} \widetilde{a}(y)u(y) + \widetilde{b}(y)v(y) + \widetilde{c}(y)w(y) = 0, \\ h(y) = v(y). \end{cases} \tag{4.1.13}$$

Hereafter, we assume the matrix $\mathbb{P}$ is of rank 3. Indeed, it will be shown in the next section that *this is always the case, except when the group is of order* 4.

So, letting $\mathcal{V}$ denote the vector space of polynomials of degree ≤ 2, the polynomials $\widetilde{a}, \widetilde{b}, \widetilde{c}$ form a base of $\mathcal{V}$, and we can write

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathcal{M} \begin{pmatrix} \widetilde{a} \\ \widetilde{b} \\ \widetilde{c} \end{pmatrix},$$

where $\mathcal{M}$ is an unspecified constant matrix. Then the first equation of (4.1.13) says that one must look for elements $(u, v, w) \in \mathcal{V}^3$ such that the two vectors (u, v, w) and $(\widetilde{a}, \widetilde{b}, \widetilde{c})$ are *orthogonal*. The bilinear mapping $(u, v, w).(\widetilde{a}, \widetilde{b}, \widetilde{c})^T$ gives rise to an associated quadratic form (in the variables $\widetilde{a}, \widetilde{b}, \widetilde{c}$)

$$B(\widetilde{a}, \widetilde{b}, \widetilde{c}) = (\widetilde{a}, \widetilde{b}, \widetilde{c})\, \mathcal{M}^*(\widetilde{a}, \widetilde{b}, \widetilde{c})^T,$$

where $\mathcal{M}^*$ is a symmetric matrix. Consequently, since $\widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y)$ build a base of $\mathcal{V}^3$, $B(\cdot)$ will be identically zero, $\forall y \in \mathbb{C}$, if, and only if, $\mathcal{M}^* = 0$, which implies that $\mathcal{M}$ is a skew-symmetric matrix

$$\mathcal{M} = \begin{pmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{pmatrix}.$$

Next, we can characterize the three basic families $u(y)$, $v(y)$, $w(y)$, which among other things generate (4.1.9) by linear combination.

1. u, v, w of degree 1. It suffices to choose

$$\varepsilon_3 = p_{-1,1}, \ \varepsilon_2 = p_{0,1}, \ \varepsilon_1 = p_{1,1}.$$

2. u, v, w without constant terms, so that u/y, v/y, w/y are admissible and still of degree 1. It suffices to choose

$$\varepsilon_3 = p_{-1,-1}, \ \varepsilon_2 = p_{0,-1}, \ \varepsilon_1 = p_{1,-1}.$$

3. u, v, w of degree 2. It suffices to choose

$$\varepsilon_3 = p_{-1,0}, \ \varepsilon_2 = p_{0,0} - 1, \ \varepsilon_1 = p_{1,0}.$$

The proof of the lemma is complete. ■

Lemma 4.1.5 *Let γ be an endomorphism defined on the algebraic surface $\mathscr{K}$, which is assumed to be invariant on the field $\mathbb{C}(x)$ of rational functions of x, and such that*

$$\gamma(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)}, \tag{4.1.14}$$

where e, f, g, h are polynomials of degree 1 in x. (Note that this is always possible, as shown in Lemma 4.1.4.) Then, for γ to be an involution, the condition $f(x) + h(x) \equiv 0$ is necessary and sufficient.

Proof Applying γ to both terms of (4.1.14) yields the equality

$$\gamma^2(y) = \frac{y[f^2(x) - e(x)g(x)] - e(x)[f(x) + h(x)]}{yg(x)[f(x) + h(x)] - e(x)g(x) + h^2(x)}.$$

If $f(x) + h(x) = 0$, then we have immediately $\gamma^2(y) = y$, and consequently $\gamma^2 = I$, showing at once that $f(x) + h(x) = 0$ is a sufficient condition for γ to be an involution.

On the other hand, we have

$$g(x)(f(x) + h(x))y^2 + (h^2(x) - f^2(x))y + e(x)(f(x) + h(x)) = 0.$$

Comparing the last equation with (2.3.1), we obtain (omitting the variable x)

$$\frac{g(f+h)}{a} = \frac{h^2 - f^2}{b} = \frac{e(f+h)}{c} \mod Q(x, y). \tag{4.1.15}$$

Assume for now that $f + h \not\equiv 0$. Then, by (4.1.15),

$$\begin{cases} bg = a(h - f), \\ cg = ae. \end{cases} \tag{4.1.16}$$

The second degree polynomials $a(x)$ and $b(x)$ are relatively prime. Indeed, the roots of $a(x)$ are either both negative, or complex conjugate. But $b(x)$ does not admit roots with a negative real part, since $p_{00} - 1 < 0$. So, $a(x)$ should divide $g(x)$, which is impossible.

So, in Eq. (4.1.14), we must have $f(x) + h(x) = 0$, $\forall x \in \mathbb{C}$, except when a, b, c are all of degree 1, which corresponds to the *singular random walk* $p_{11} = p_{10} = p_{1,-1} = 0$ given by Lemma 2.3.2. But we note in this latter case that $Q(x, y)$ is of degree 1 in x, and the genus of the algebraic curve is zero. The lemma is proved. ∎

Remark 4.1.6 The result of Lemma 4.1.5 does not hold if polynomials e, f, h are not of degree 1. For instance, one can check directly from (4.1.16) that, if g or e are taken to be of degree 2, then any involution γ necessarily has the form

$$\gamma(y) = \frac{(h - b)y - c}{ay + h}.$$

4.1.1 Criterion for Groups of Order 4

The following proposition gives the simplest form of the announced criterion.

Proposition 4.1.7 *The group $\mathcal{H}$ is of order 4 if, and only if,*

$$\begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0, \tag{4.1.17}$$

and this is the only case where the matrix $\mathbb{P}$ has rank 2.

Proof The equality $\delta^2 = I$ can be rewritten as

$$\xi\eta = \eta\xi,$$

which by Proposition 4.1.2 is, for instance, equivalent to

$$\xi\eta(x) = \eta(x),$$

where we have used $\xi(x) = x$. So, $\eta(x)$ is left invariant by ξ, which implies

$$\eta(x) \in \mathbb{C}(x).$$

Finally, η is an involution ($\eta^2 = I$) and a conformal automorphism on $\mathbb{C}(x)$. Consequently, η is indeed a fractional linear transform of the type

$$\eta(x) = \frac{rx+s}{tx-r},$$

where all coefficients belong to $\mathbb{C}$. The following chain of equivalences hold.

$$\begin{aligned}
\eta(x) = \frac{rx+s}{tx-r} &\Leftrightarrow tx\eta(x) = r(x+\eta(x)) + s \\
&\Leftrightarrow 1,\ x+\eta(x),\ x\eta(x) \ \text{ are linearly dependent on } \mathbb{C} \\
&\Leftrightarrow 1,\ -\frac{\widetilde{b}(y)}{\widetilde{a}(y)},\ \frac{\widetilde{c}(y)}{\widetilde{a}(y)} \ \text{ are linearly dependent on } \mathbb{C} \\
&\Leftrightarrow \widetilde{a}(y), \widetilde{b}(y), \widetilde{c}(y) \text{ are also linearly dependent on } C,
\end{aligned}$$

where Eq. (2.3.1) has been used in the form

$$Q(x, y) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y).$$

It is worth remarking that, starting from $\xi(y)$, the same argument would involve the transpose matrix $\mathbb{P}^T$, thus leading (as expected!) to the same criterion (4.1.17). The proof of the lemma is complete. ∎

4.1.2 *Criterion for Groups of Order* 6

Proposition 4.1.8 *$\mathcal{H}$ is of order* 6 *if, and only if,*

$$\begin{vmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{vmatrix} = 0, \tag{4.1.18}$$

where the Δ_{ij}'s have be given in Lemma 4.1.4.

Proof In this case $(\xi\eta)^3 = I$, which is equivalent to

$$\eta\xi\eta = \xi\eta\xi. \tag{4.1.19}$$

Applying (4.1.19) for instance to x, we get

$$\xi\eta(x) = \eta\xi\eta(x),$$

which shows that $\xi\eta(x)$ is invariant with respect to η and consequently is a rational function of y, remembering we are dealing with the field of rational functions. Similarly, $\eta\xi(y)$, invariant with respect to ξ, is a rational function of x. Hence (4.1.19) is plainly equivalent to

$$\begin{cases} \xi\eta(x) = P(y), \\ \eta\xi(y) = R(x), \end{cases}$$

where P and R are rational. Then

$$y = R(\xi\eta(x)) = R \circ P(y),$$

or, equivalently,

$$R \circ P = I, \tag{4.1.20}$$

so that P and R are fractional linear transforms. Thus (4.1.20) yields the relation

$$\xi(y) = \frac{p\eta(x) + q}{r\eta(x) + s}, \tag{4.1.21}$$

which imposes a linear dependence on $\mathbb{C}$ between the four elements 1, $\xi(y)$, $\eta(x)$, $\xi(y)\eta(x)$, with four unknown constants (in fact three by homogeneity). Our goal is to avoid the pitfall of entering tedious (and harmful!) computations.

Starting from Eq. (4.1.8), we choose $\eta(x)$ by means of (4.1.11), that is

$$\eta(x) = \frac{y(x\Delta_{23} - \Delta_{13}) - x\Delta_{22} + \Delta_{12}}{y(x\Delta_{33} - \Delta_{23}) - x\Delta_{32} + \Delta_{22}}. \tag{4.1.22}$$

Instantiating now (4.1.22) in (4.1.21), we obtain

$$\xi(y) = \frac{y[p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23})] + p(\Delta_{12} - x\Delta_{22}) + q(\Delta_{22} - x\Delta_{32})}{y[r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23})] + r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32})}. \tag{4.1.23}$$

Then, rewriting (4.1.23) in the form proposed in (4.1.14), namely

$$\xi(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)},$$

where

$$\begin{cases} e(x) = p(x\Delta_{22} - \Delta_{12}) + q(x\Delta_{32} - \Delta_{22}), \\ f(x) = p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23}), \\ g(x) = r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23}), \\ h(x) = r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32}), \end{cases} \tag{4.1.24}$$

we are in a position to compare system (4.1.24) with the solution presented in Eq. (4.1.10) of Lemma 4.1.4.

Indeed, letting $\varphi(x) = (e(x), f(x), g(x))^T \stackrel{\text{def}}{=} \varphi_0 + x\,\varphi_1$, we get from (4.1.10) the vector equation

$$\varphi_0 + x\,\varphi_1 = E + x\,F,$$

which, after renaming the constants $\widetilde{\alpha}$ and $\widetilde{\beta}$ as c and $-d$ respectively, yields in turn six linear equations, namely

$$\begin{cases} c\Delta_{11} + d\Delta_{21} + p\Delta_{12} + q\Delta_{22} = 0, \\ c\Delta_{12} + d\Delta_{22} + p\Delta_{13} + q\Delta_{23} = 0, \\ c\Delta_{13} + d\Delta_{23} + r\Delta_{13} + s\Delta_{23} = 0, \\ c\Delta_{21} + d\Delta_{31} + p\Delta_{22} + q\Delta_{32} = 0, \\ c\Delta_{22} + d\Delta_{32} + p\Delta_{23} + q\Delta_{33} = 0, \\ c\Delta_{23} + d\Delta_{33} + r\Delta_{23} + s\Delta_{33} = 0. \end{cases} \tag{4.1.25}$$

In addition, we have to take into account the constraint $f(x) + h(x) = 0$, which is tantamount to

$$\begin{cases} p\Delta_{23} + q\Delta_{33} = r\Delta_{22} + s\Delta_{32}, \\ p\Delta_{13} + q\Delta_{23} = r\Delta_{12} + s\Delta_{22}. \end{cases} \tag{4.1.26}$$

So, the final step is to analyze the feasibility of the global linear system formed by the intersection of (4.1.25) and (4.1.26). Altogether, we are left with eight equations with respect to 6 unknown variables c, d, p, q, r, s.

Equations of system (4.1.25), referred to as $1, 2 \ldots 6$, can be split into two sets:

(a) The set (1, 2, 4, 5), forming an homogeneous linear system of four equations with four unknowns

$$\begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{pmatrix} \begin{pmatrix} c \\ d \\ p \\ q \end{pmatrix} = 0. \tag{4.1.27}$$

(b) The set (3, 6), which can easily be rewritten as

$$\begin{cases} (c + r)\Delta_{13} + (d + s)\Delta_{23} = 0, \\ (c + r)\Delta_{23} + (d + s)\Delta_{33} = 0. \end{cases} \tag{4.1.28}$$

Clearly, system (4.1.27) has a non-trivial solution, if and only if condition (4.1.18) holds, which hence is necessary for the group to be of order 6. To prove its sufficiency, we also have to consider systems (4.1.28) and (4.1.26).

The determinant of system (4.1.28) is equal to $\Delta_{13}\Delta_{33} - \Delta_{23}^2$. But the matrix $\mathbb{P}$, introduced in (4.1.6), has all its entries (i, j) positive, except $(2, 2) = p_{00} - 1 < 0$, so that

$$\Delta_{13}\Delta_{33} - \Delta_{23}^2 \le 0.$$

Moreover, the equality $\Delta_{13}\Delta_{33} - \Delta_{23}^2 = 0$ holds only for special values of the jump probabilities p_{ij}, corresponding to simple singular random walks, which we do not consider here. Consequently (4.1.28) has only the trivial solutions, so that

$$c + r = d + s = 0. \tag{4.1.29}$$

Now, by (4.1.29), one replaces r and s respectively by $-c$ and $-d$ in (4.1.26), and we get two equations coinciding in fact with the equations (2, 5) of system (4.1.25). It is worth remarking that we found the two equations of (4.1.26) are implicitly satisfied, but it was useful to check this fact, just for the sake of completeness ! The proof of the lemma is complete. ■

Now, we shall attack the general situation by splitting into the two possible cases $n = 2m$ and $n = 2m + 1$.

4.1.3 Criterion for Groups of Order 4m

In the rest of this section, we refer without further comment to the notation and formulae of Lemma 3.3.1 and Sect. 4.3.

Proposition 4.1.9 *The group $\mathcal{H}$ is of order $4m$ if and only if the Weierstrass function $\wp$ with periods (ω_1, ω_2) satisfies the equation*

$$\wp(m\omega_3) = \wp(\omega_2/2). \tag{4.1.30}$$

Proof Recalling that $\delta = \eta\xi$, with $\xi(x) = x,\ \eta(y) = y,\ \xi^2 = \eta^2 = I$, we have here $\delta^{2m} = I$, that is

$$(\xi\eta)^m = (\eta\xi)^m. \tag{4.1.31}$$

By applying Eq. (4.1.31) at x (or even at an arbitrary element of $\mathbb{C}(x)$), and replacing $\xi(x)$ by x in the right-hand side, we obtain

$$\xi\delta^m(x) = \delta^m(x),$$

which shows that the involution $\delta^m(x)$ is invariant with respect to ξ, and hence is an element of $\mathbb{C}(x)$. This can be summarized as

$$\delta^m(x) = F(x) = \frac{xf - e}{xg - f}, \tag{4.1.32}$$

where $F(x)$ is a simple *fractional linear transform*, with constants e, f, g, to be determined.

From Proposition 4.1.2 and Corollary 4.1.3, it is worth recalling that (4.1.32) contains exactly the same information as condition (4.1.31).

Thus, Eq. (4.1.32) implies the existence of a linear dependence between the functions

$$x.\delta^m(x),\ x+\delta^m(x),\ \mathbf{1}, \tag{4.1.33}$$

where $\mathbf{1}$ denotes an arbitrary constant function and the symbol "." denotes the ordinary scalar product. Then, after a lifting onto the universal covering and a translation of $-m\omega_3/2$, condition (4.1.33) gives rise to the following lemma.

Lemma 4.1.10 *For the group to be of order* $4m$*, a necessary and sufficient condition is that the three functions*

$$x(\omega - m\omega_3/2).x(\omega + m\omega_3/2),\ x(\omega - m\omega_3/2) + x(\omega + m\omega_3/2),\ \mathbf{1}, \tag{4.1.34}$$

be linearly dependent, $\forall \omega \in \mathbb{C}$. ■

Proof In agreement with (4.3.7), let

$$\begin{aligned} &S = x(\omega + m\omega_3/2) + x(\omega - m\omega_3/2), \quad P = x(\omega + m\omega_3/2).x(\omega - m\omega_3/2),\\ &X = \wp(\omega), \quad Y = \wp(m\omega_3/2). \end{aligned}$$

By (3.3.3), $x(\omega)$ is homographic in X, and hence Lemma 4.1.10 amounts to saying that S, P, $\mathbf{1}$, considered as functions of X, are linearly dependent.

By system (4.3.9) together with definition (4.3.10), one verifies immediately that the existence of constants e, f, g satisfying

$$eS + fP + g = 0, \quad \forall X \in \mathbb{C}, \tag{4.1.35}$$

is synonymous with the linear dependence

$$uA_1 + vB_1 + w(X-Y)^2 = 0, \quad \forall X \in \mathbb{C}, \tag{4.1.36}$$

where the constants u, v, w satisfy the linear system

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2p & p^2 & 1 \\ q-2rp & p(q-rp) & -r \\ 4r(pr-q) & 2(q-rp)^2 & 2r^2 \end{pmatrix} \begin{pmatrix} e \\ f \\ g \end{pmatrix}. \tag{4.1.37}$$

But the determinant of system (4.1.37) is exactly equal to $-2q^3$ (independent of p and r, as expected) and never vanishes (see (4.3.8)). Consequently, Eqs. (4.1.35) and (4.1.36) are in fact equivalent.

Ultimately, we have to extract the vector coefficients of A_1, B_1, D, which from Lemma 4.3.1 are *polynomials of second degree* in X, so that condition (4.1.34) amounts to

$$\Delta(Y) \stackrel{\text{def}}{=} \begin{vmatrix} 4Y & 4Y^2 - g_2 & -(g_2 Y + 2g_3) \\ 2Y^2 & g_2 Y + 2g_3 & 2g_3 Y + g_2^2/8 \\ 1 & -2Y & Y^2 \end{vmatrix} = 0. \tag{4.1.38}$$

The determinant (4.1.38) is equal to

$$8\,Y^6 - 10\,g_2 Y^4 - 40\,g_3 Y^3 - 5/2\,g_2^2\,Y^2 - 2\,g_2\,g_3 Y - 4g_3^2 + 1/8\,g_2^3,$$

but it has the pleasant property of being the product of three explicit second degree polynomials, namely

$$\Delta(Y) = 8[Y^2 - 2e_1 Y - (e_1^2 + e_2 e_3)][Y^2 - 2e_2 Y - (e_2^2 + e_3 e_1)][Y^2 - 2e_3 Y - (e_3^2 + e_1 e_2)], \tag{4.1.39}$$

where e_1, e_2, e_3 are defined in (4.3.2) and (4.3.3). Let P_1, P_2, P_3 denote the three polynomials appearing in (4.1.39), respectively from left to right, with their corresponding reduced discriminants

$$\delta_1 = (e_1 - e_2)(e_1 - e_3) > 0,$$
$$\delta_2 = (e_2 - e_3)(e_2 - e_1) < 0,$$
$$\delta_3 = (e_3 - e_1)(e_3 - e_2) > 0.$$

On the real interval $[0, \omega_2]$, $\wp(\omega)$ is positive and reaches its minimum at $\omega = \omega_2/2$, with $\wp(\omega_2/2) = e_1$. So, we can immediately eliminate P_2, which has two complex roots. As for P_3, which has two real roots, one of them (at least) being negative, we have

$$P_3(e_1) = e_1^2 - 2e_1 e_3 - e_3^2 - e_1 e_2 = (e_1 - e_2)(e_1 - e_3) > 0.$$

But e_1 is larger than the two roots of P_3, which therefore never cancel $\Delta(Y)$. Thus we are left with the real roots in Y of $P_1(Y)$, of which the sole one bigger than e_1 is admissible, namely

$$e_1 + \sqrt{(e_1 - e_2)(e_1 - e_3)} \equiv \wp\left(\frac{\omega_2}{4}\right).$$

Finally, we have shown that (4.1.38) holds if and only if

$$\wp(m\omega_3/2) = \wp(\omega_2/4).$$

To conclude the proof of Proposition 4.1.9, it suffices to remark that, as we are working inside the fundamental parallelogram (ω_1, ω_2), the relations $\frac{m\omega_3}{2} = \pm\frac{\omega_2}{4}$ and $m\omega_3 = \frac{\omega_2}{2}$ are equivalent mod (ω_2). ■

The calculation of $\wp(m\omega_3/2)$ could be carried out from (4.3.4), via the recursive relationship

$$\wp((l+1)\omega_3/2) + \wp((l-1)\omega_3/2) = \frac{(\wp(l\omega_3/2) + \wp(\omega_3/2))(4\wp(l\omega_3/2)\wp(\omega_3/2) - g_2) - 2g_3}{2(\wp(l\omega_3/2) - \wp(\omega_3/2))^2},$$

since the value of $\wp(\omega_3/2)$ is directly obtained from (3.3.3). Yet, one has to admit that the partial fraction giving $\wp(m\omega_3/2)$ in terms of $\wp(\omega_3/2)$ is hardly directly exploitable in Proposition 4.1.9.

Therefore, in the next sections, we shall pursue with the analysis of the crucial condition (4.1.33), having in mind the obtention of concrete formulas by means of *determinants* expressed in terms of the entries of the matrix $\mathbb{P}$ defined in (4.1.6).

4.1.3.1 The Case $m = 2k$

When $m = 2k$, applying the operator δ^{-k} in (4.1.33) is equivalent to saying, after a direct manipulation, that

$$\delta^k(x).\xi\delta^k(x),\ \delta^k(x) + \xi\delta^k(x),\ \mathbf{1}, \tag{4.1.40}$$

are linearly dependent. But we know that $\delta^k(x).\xi\delta^k(x)$ and $\delta^k(x) + \xi\delta^k(x)$ are elements of $\mathbb{C}(x)$, and by (4.3.8), (4.3.4) and (4.3.5), they are in fact ratios of polynomials of degree 2 in x with the same denominator.

In addition, letting $\zeta_j(x) \stackrel{\text{def}}{=} \delta^j(x) + \xi\delta^j(x)$, the following recursive scheme holds.

$$\begin{cases} \zeta_0(x) &= 2x,\ \zeta_1(x) = \delta^{-1}(x) + \delta(x), \\ \zeta_j(x) &= \zeta_{j-1}(\zeta_1(x)) - \zeta_{j-2}(x),\ \forall j \geq 2. \end{cases} \tag{4.1.41}$$

4.1.3.2 The Case $m = 2k - 1$

Upon applying here the operator δ^{-k+1} in (4.1.33) and using the identity $\delta^{-k+1}(x) = \eta\delta^k(x)$, we obtain that

$$\delta^k(x).\eta\delta^k(x),\ \delta^k(x) + \eta\delta^k(x),\ \mathbf{1} \tag{4.1.42}$$

are linearly dependent. One checks immediately that $\delta^k(x).\eta\delta^k(x)$ and $\delta^k(x) + \eta\delta^k(x)$ are invariant with respect to η, and thus are elements of $\mathbb{C}(y)$. Then, exchanging the role of the variables x and y, a uniformization similar to (3.3.3) would show that $\delta^k(x).\eta\delta^k(x)$ and $\delta^k(x) + \eta\delta^k(x)$ are ratios of polynomials of degree 2 in y with the same denominator.

4.1.3.3 Explicit Criterion for Groups of Order 8

This corresponds to $m = 2$ in the preceding section. In this case explicit conditions can be carried out, both manually and also with any good computer algebra system. Hereafter, we only list the main results.

In agreement with Sect. 4.1.3.1, and using again $\xi^2 = I$, the functions

$$\delta^{-1}(x).\delta(x),\ \ \delta^{-1}(x) + \delta(x),\ \ \mathbf{1},$$

are in $\mathbb{C}(x)$ and are sought to be linearly dependent on $\mathbb{C}$. Setting

$$\delta^{-1}(x) + \delta(x) = \frac{P_1(x)}{Q_1(x)}, \quad \delta^{-1}(x).\delta(x) = \frac{R_1(x)}{Q_1(x)},$$

the polynomials P_1, Q_1, R_1 satisfy the relations

$$\begin{cases} P_1 = -2\, S_2\, T_2 + S_3\, T_1 + S_1\, T_3, \\ Q_1 = S_2^2 - S_3\, S_1, \\ R_1 = T_2^2 - T_3\, T_1, \end{cases} \tag{4.1.43}$$

with

$$\begin{aligned}
S_1(x) &= p_{10}\, c(x) - p_{1,-1}\, b(x) = \ \ \Delta_{31}\, x - \Delta_{21}, \\
T_1(x) &= \tfrac{p_{-1,-1}\, b(x) - p_{-1,0}\, c(x)}{x} = -\Delta_{21}\, x + \Delta_{11}, \\
S_2(x) &= p_{11}\, c(x) - p_{1,-1}\, a(x) = -\Delta_{32}\, x + \Delta_{22}, \\
T_2(x) &= \tfrac{p_{11}\, a(x) - p_{-1,1}\, c(x)}{x} = \ \ \Delta_{22}\, x - \Delta_{12}, \\
S_3(x) &= p_{11}\, b(x) - p_{10}\, a(x) = \ \ \Delta_{33}\, x - \Delta_{23}, \\
T_3(x) &= \tfrac{p_{-1,0}\, a(x) - p_{-1,1}\, b(x)}{x} = -\Delta_{23}\, x + \Delta_{13},
\end{aligned}$$

the Δ_{ij}'s being the cofactors of the matrix $\mathbb{P}$ introduced in Lemma 4.1.4. The derivation of (4.1.43) has been obtained by direct algebra. Indeed, the quantities $\delta^{-1}(x) + \delta(x)$ and $\delta^{-1}(x).\delta(x)$ are a priori ratios of polynomials of *4th degree*, which miraculously simplify to ratios of second degree polynomials…

So, the condition ensuring the linear dependence between P_1, Q_1, R_1 given by (4.1.43) leads to the following

Proposition 4.1.11 *The group $\mathcal{H}$ is of order* 8 *if, and only if, the third order determinant*

$$\begin{vmatrix} 2\,\Delta_{22}\Delta_{32} & 2\,(\Delta_{22}^2-\Delta_{12}\Delta_{31}+\Delta_{21}\Delta_{23}) & 2\,\Delta_{12}\Delta_{22} \\ -(\Delta_{21}\Delta_{33}+\Delta_{31}\Delta_{23}) & +\Delta_{11}\Delta_{33}+\Delta_{31}\Delta_{13} & -(\Delta_{11}\Delta_{23}+\Delta_{21}\Delta_{13}) \\ \Delta_{32}^2-\Delta_{31}\Delta_{33} & -2\,\Delta_{32}\Delta_{22}+\Delta_{31}\Delta_{23}+\Delta_{21}\Delta_{33} & \Delta_{22}^2-\Delta_{21}\Delta_{23} \\ \Delta_{22}^2-\Delta_{21}\Delta_{23} & -2\,\Delta_{22}\Delta_{12}+\Delta_{11}\Delta_{23}+\Delta_{13}\Delta_{21} & \Delta_{12}^2-\Delta_{11}\Delta_{13} \end{vmatrix} \tag{4.1.44}$$

is equal to zero. (Due to the size of the printing output, each element of the first line of the matrix in (4.1.44) *has been split vertically as the sum of two terms.)*

■

Remark 4.1.12 It is interesting to note that the polynomials S_i, T_i, $i = 1, 2, 3$, coincide with those appearing in (4.1.11).

Remark 4.1.13 When $m = 2$, i.e. a group of order 8, the line of argument developed in Sect. 4.1.2 could be applied, but at the expense of intricate computations. Indeed, by Lemma 4.1.4,

$$\eta(x) = \frac{xv(y)-u(y)}{xw(y)-v(y)}, \qquad \xi\eta(x) = \frac{xv(\xi(y))-u(\xi(y))}{xw(\xi(y))-v(\xi(y))},$$

whence

$$\xi\eta(x)+\eta(x) = \frac{(xv(\xi(y))-u(\xi(y)))(xw(y)-v(y))+(xv(y)-u(y))(xw(\xi(y))-v(\xi(y)))}{(xw(\xi(y))-v(\xi(y)))(xw(y)-v(y))}, \tag{4.1.45}$$

$$\xi\eta(x).\eta(x) = \frac{(xv(\xi(y))-u(\xi(y)))(xv(y)-u(y)}{(xw(\xi(y))-v(\xi(y)))(xw(y)-v(y))}. \tag{4.1.46}$$

Setting for now in (4.1.45) and (4.1.46)

$$\xi\eta(x).\eta(x) = \frac{K_1}{L}, \quad \xi\eta(x)+\eta(x) = \frac{K_2}{L},$$

we must test a possible linear dependence between the functions

$$K_1,\ K_2,\ L,$$

which are all elements of $\mathbb{C}(x)$. Sketching the global calculus, we have for instance

$$\begin{aligned} L &= (xw(y)-v(y))(xw(\xi(y))-v(\xi(y))) \\ &= w(y)w(\xi(y))x^2 - x[v(y)w(\xi(y))+w(y)v(\xi(y))]+v(y)v(\xi(y)). \end{aligned}$$

A typical term appearing in L is $w(y)w(\xi(y))$, which from Lemma 4.1.4 equals

$$
\begin{aligned}
w(y)w(\xi(y)) &= w_0^2 + w_0 w_1 (y + \xi(y)) + w_1^2 y \xi(y) \\
&= w_0^2 - w_0 w_1 \frac{b(x)}{a(x)} + w_1^2 \frac{c(x)}{a(x)}.
\end{aligned}
$$

Analogously, the other parts of the puzzle give

$$
\begin{aligned}
a(x)L &= a(x)(xw_0 - v_0)^2 - b(x)(xw_0 - v_0)(xw_1 - v_1) + c(x)(xw_1 - v_1)^2, \\
a(x)K_1 &= a(x)(xv_0 - u_0)^2 - b(x)(xv_0 - u_0)(xv_1 - u_1) + c(x)(xv_1 - u_1)^2, \\
a(x)K_2 &= a(x)h_a(x) + b(x)h_b(x) + c(x)h_c(x),
\end{aligned}
$$

where

$$
\begin{aligned}
h_a(x) &= 2[v_0 w_0 x^2 - v_0^2 x + v_0 u_0], \\
h_b(x) &= -(v_1 w_0 - v_0 w_1)x^2 + (2v_0 v_1 + u_0 w_1 + u_1 w_0)x - (u_0 v_1 + u_1 v_0), \\
h_c(x) &= 2[v_1 w_1 x^2 - (v_1^2 + u_1 w_1)x + v_1 u_1].
\end{aligned}
$$

At the very end, this machinery would require us to deal with rational fractions of *4th degree*, implying tedious computations, which are partly avoided by means of the procedure proposed above.

■

4.1.4 Criterion for Groups of Order $4m - 2$

Condition (4.1.1) reads

$$\delta^{2m-1} = I, \tag{4.1.47}$$

which by Proposition 4.1.2 and Corollary 4.1.3 is equivalent to

$$\eta(\delta^m(x)) = \delta^m(x),$$

or, by simple algebra using $\xi(x) = x$, to

$$\delta^m(x) = G(y) \in \mathbb{C}(y). \tag{4.1.48}$$

Similarly, upon applying (4.1.47) to y, we get

$$\delta^{-m}(y) = \delta^{m+1}(y) = \xi(\delta^{-m}(y),$$

so that

$$\delta^{-m}(y) = F(x) \in \mathbb{C}(x).$$

Applying now δ^{-m} to both members of (4.1.48) yields

$$x = \delta^{-m}(G(y)) = G(\delta^{-m}(y)) = G \circ F(x),$$

which shows that $G \circ F = I$, and hence G and F are simple fractional linear transforms.

Setting for instance

$$G(y) = -\frac{py + q}{ry + s},$$

where p, q, r, s are arbitrary complex constants, the problem is to achieve the linear relation

$$r\, y\delta^m(x) + s\, \delta^m(x) + py + q = 0 \quad \mod Q(x, y), \tag{4.1.49}$$

which is necessary and sufficient for the group to be of order $4m - 2$.

4.2 Further General Results

We state below three theorems allowing us to conclude that, for the group to be finite, there is a *unique condition* tantamount to the cancellation of a *determinant*, the elements of which are intricate functions of the coefficients of the transition matrix $\mathbb{P}$, but nonetheless recursively computable.

- The determinant is of order 3, for groups of order $4m$, $m \geq 1$.
- The determinant is of order 4, for groups of order $4m - 2$, $m \geq 1$.

4.2.1 A Theorem About δ^s

The following important fact holds.

Theorem 4.2.1 *For any integer $s \geq 1$, we have*

$$\delta^s(x) = \frac{y\, U_s(x) + V_s(x)}{W_s(x)} \quad \mod Q(x, y), \tag{4.2.1}$$

where U_s, V_s, W_s are second degree polynomials.

Proof For $s = 1$, a direct (however slightly tedious) computation with the (x, y) variables can be worked out. But, already for $s = 2$, one seems to reach the limits of human computational abilities, which are definitely exceeded for $s \geq 3$! A formal verification through the *Maple 18* Computer Algebra System has been carried out for $s = 2, 3$, but seems hardly exploitable as soon as $s > 3$. Hereafter, we propose a simple proof by using again the uniformization (3.3.3).

Let

$$\delta^s(x) - \xi\delta^s(x) \stackrel{\text{def}}{=} 2H,\ \ X = \wp(\omega),\ \ Y = \wp(s\omega_3).$$

Then, with the notation of Sect. 4.3,

$$\delta^s(x) = \frac{S(\omega, s\omega_3)}{2} + H,$$

where $S(\omega, s\omega_3)$ is given by (4.3.9).

Now Eqs. (4.3.4), (4.3.5) and (4.3.8), together with the addition formula for the $\wp$ function, yield

$$\begin{aligned}
H &= \frac{q[\wp(s\omega_3 - \omega) - \wp(s\omega_3 + \omega)]}{2[\wp(s\omega_3 + \omega) - r][\wp(s\omega_3 - \omega) - r]}\\
&= \frac{q\,\wp'(\omega)\wp'(s\omega_3)}{2(X - Y)^2[B(s\omega_3, \omega) - rA(s\omega_3, \omega) + r^2]} = \frac{q\,\wp'(\omega)\wp'(s\omega_3)}{D(X, Y)},
\end{aligned}$$

where $D(X, Y)$, introduced in (4.3.10), is a polynomial of second degree in X and Y. On the other hand, by (3.3.3) and (4.3.8), we have

$$\wp'(\omega) = \frac{2q\,[2a(x)y + b(x)]}{(x - p)^2},$$

and hence

$$H = \frac{2q^2\wp'(s\omega_3)[2a(x)y + b(x)]}{(x - p)^2 D(X, Y)}.$$

Since x is a homographic function of X, it follows that the denominator of H is a second degree polynomial in the variable $(x - p)$, where

$$\begin{aligned}
U_s(x) &= 4q^2\wp'(s\omega_3)a(x),\\
V_s(x) &= 2q^2\wp'(s\omega_3)b(x) + (x - p)^2\left[2pB_1(X, Y) + (q - 2pr)A_1(X, Y) + 4r(pr - q)(X - Y)^2\right],\\
W_s(x) &= (x - p)^2 D(X, Y).
\end{aligned}$$

Moreover, the coefficients involve the Weierstrass $\wp$ function at points of the type $k\omega_3$, k integer, which can be computed via standard recursive schemes. The proof of (4.2.1) is complete. ■

4.2.2 *Form of the General Criterion*

The main results are formulated in the next two theorems.

Theorem 4.2.2 *The finiteness of the group is always equivalent to the cancellation of a* single constant*, which depends on the entries of* $\mathbb{P}$ *in a polynomial way. In* *other words, the group is finite if and only if the non-negative* (p_{ij})*'s belong to the intersection of some algebraic hypersurface with the hyperplane* $\sum p_{ij} = 1$.

Proof By construction, we can a priori write

$$\delta^s(x) = M_s(x)y + N_s(x) \quad \text{mod } Q(x, y), \tag{4.2.2}$$

where M_s and N_s are rational fractions whose numerators and denominators are polynomials of unknown degrees, but with coefficients given in terms of polynomials of the entries of $\mathbb{P}$. Moreover, the decomposition (4.2.2) is unique, so that, comparing with (4.2.1), we have

$$\begin{cases} M_s(x) = \dfrac{4q^2\wp'(s\omega_3)a(x)}{W_s(x)}, \\ N_s(x) = \dfrac{V_s(x)}{W_s(x)}, \end{cases}$$

where V_s, W_s are the second degree polynomials given by Theorem 4.2.1. Moreover, by homogeneity, we can always rewrite

$$M_s(x) = \frac{A_s a(x)}{F_s(x)},$$

where A_s, K_s are real constants with

$$A_s = K_s[4q^2\wp'(s\omega_3)], \quad F_s(x) = K_s W_s(x).$$

In particular, by Corollary 4.1.3, the group is of order $4s$ if and only if $M_s \equiv 0$, or equivalently $A_s = 0$, where now A_s *depends only on the entries of* $\mathbb{P}$ in a complicated polynomial form.

When the group is of order $4s - 2$, we can exchange the role of x and y in the uniformization (3.3.3), by uniformizing $y(\omega)$. Then, *mutatis mutandis*, this yields

$$\delta^s(x) = \frac{\widetilde{A}_s\widetilde{a}(y)x + \widetilde{V}_s(y)}{\widetilde{F}_s(y)} \quad \text{mod } Q(x, y),$$

where $\widetilde{F}_s$, $\widetilde{V}_s$ are second degree polynomials. Referring again to Corollary 4.1.3, we conclude that the group is of order $4s - 2$ if and only if $\widetilde{A}_s = 0$, with $\widetilde{A}_s$ depending on the coefficients of $\mathbb{P}$ in a polynomial way. The proof of the theorem is complete. ■

Theorem 4.2.3 *For the group* $\mathcal{H}$ *to be finite, the necessary and sufficient condition is* $\det(\Omega) = 0$*, where* Ω *is a matrix of order* 3 *(resp.* 4*) when the group is of order* $4m$ *(resp. 4m+2),* $m \geq 1$.

Proof We only present a sketch of the main line of argument.

- When the group $\mathcal{H}$ is of order $4m$, Theorem 4.2.1 and conditions (4.1.40) or (4.1.42) show that we are indeed left with a system of 3 homogeneous linear equations with 3 unknowns.
- The situation for $\mathcal{H}$ to be of order $4m + 2$ is slightly more complicated. Indeed, according to Theorem 4.2.1, in order to satisfy condition (4.1.49), we end up with 6 linear homogeneous equations and only 4 unknowns. By Theorem 4.2.2, the condition for the group to be finite is always equivalent to the cancellation of the constant A_s. This shows that among these 6 relations exactly 4 of them are independent, and so the condition reduces to the cancellation of a determinant of order 4, saying that a linear system of 4 equations with 4 unknowns must have a non trivial solution.
- In both cases, the algebraic hypersurfaces defined by

$$A_s = 0 \text{ and } \det(\Omega) = 0$$

are in fact identical.

The proof of the theorem is complete. ■

4.2.2.1 Miscellaneous Questions

We list below some technical questions, which very likely could be solved with some effort..!

- The degree of the hypersurface appearing in Theorem 4.2.2 can be determined by using the arguments appearing in the proof, together with the recursive relationships involving the Weierstrass functions.
- By continuity with respect to the parameters, we think that it is possible to find random walks having a group of any finite prescribed order. See in this respect the genus 0 case considered in Chap. 7.

4.3 On Some Symmetric Quantities of the $\wp$ Function

From the form of $x(\omega)$ in (3.3.3) and the expressions (4.1.3) of the automorphisms ξ^*, η^*, δ^*, it appears we need to calculate the quantities

$$A(u,v) \stackrel{\text{def}}{=} \wp(u+v) + \wp(u-v), \quad B(u,v) \stackrel{\text{def}}{=} \wp(u+v)\wp(u-v)$$

in terms of rational functions of $\wp(u)$ and $\wp(v)$.

Letting $\wp'(u)$ denote the derivative of $\wp(u)$ with respect to u, it is well known (see e.g., [57]) that $\wp$ is even, $\wp'(z)$ is odd, and they satisfy the algebraic differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{4.3.1}$$

where the quantities g_2, g_3 are the invariants defined in (3.3.2). It is known that Eq. (4.3.1) also admits the classical form

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \tag{4.3.2}$$

where

$$e_1 + e_2 + e_3 = 0,\ g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1),\ g_3 = 4e_1 e_2 e_3,$$

$$e_1 = \wp\left(\frac{\omega_2}{2}\right), \quad e_2 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1}{2}\right).$$

In the present situation, all the e_i's are real and satisfy

$$e_1 > 0, \quad e_3 < 0, \quad e_1 > e_2 > e_3. \tag{4.3.3}$$

We shall also need the *addition theorem* (see [57]), which reads, for all $u, v, u \pm v \neq 0$,

$$\wp(u+v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2 - \wp(u) - \wp(v).$$

It is important to note that $A(u, v)$ and $B(u, v)$ are symmetric and even w.r.t. (u, v).

Let us set for now $X \stackrel{\text{def}}{=} \wp(u)$, $Y \stackrel{\text{def}}{=} \wp(v)$.

Lemma 4.3.1

$$A(u, v) = \frac{(X+Y)(4XY - g_2) - 2g_3}{2(X-Y)^2}, \tag{4.3.4}$$

$$B(u, v) = \frac{(XY)^2 + \frac{g_2}{2}XY + g_3(X+Y) + \frac{g_2^2}{16}}{(X-Y)^2}. \tag{4.3.5}$$

Proof The parity properties allow us to write

$$A(u, v) = \frac{\wp'^2(u) + \wp'^2(v) - 4(\wp^2(u) - \wp^2(v))(\wp(u) - \wp(v))}{2(\wp(u) - \wp(v))^2},$$

which by (4.3.1) yields directly (4.3.4). As for $B(u, v)$, we have

$$B(u, v) = \frac{[(X+Y)(4XY - g_2) - 2g_3]^2 - 4\,(4X^3 - g_2X - g_3)(4Y^3 - g_2Y - g_3)}{16\,(X-Y)^4}. \tag{4.3.6}$$

Then it is quite reasonable to guess the last expression reduces in fact to a rational fraction of second degree. This is clear in our particular context, where we shall take, for example, $u = \omega$ and $v = m\omega_3/2$, but it turns out to be true for arbitrary u, v. Indeed, the numerator of (4.3.6) can be factorized by $(X - Y)^2$, which leads to

(4.3.5) (this last property has been enlighted with the help of the *Maple 18* Computer Algebra System). The proof of the lemma is complete.

∎

Now, by Lemma 4.3.1 and Eq. (3.3.3), we are in a position to compute

$$S(u, v) \stackrel{\text{def}}{=} x(u+v) + x(u-v), \quad P(u, v) \stackrel{\text{def}}{=} x(u+v)x(u-v). \tag{4.3.7}$$

For the sake of brevity, it will be convenient to write

$$x(\omega) = p + \frac{q}{\wp(\omega) - r}, \tag{4.3.8}$$

where p, q, r are known constants appearing in Eq. (3.3.3). Then the following functional algebraic relations, giving $S(u, v)$ and $P(u, v)$ in terms of $A(u, v)$ and $B(u, v)$, are straightforward.

$$\begin{cases} S = \dfrac{2pB + (q - 2pr)A + 2r(pr - q)}{B - rA + r^2}, \\ P = \dfrac{p^2 B + p(q - pr)A + (pr - q)^2}{B - rA + r^2}. \end{cases} \tag{4.3.9}$$

Ad libitum, we shall specify the variables only whenever needed: for instance, instead of S, we shall write $S(X, Y)$, or $S(\omega, s\omega_3)$, etc. So, by using (4.3.4), (4.3.5) and (4.3.9), we obtain the final expressions of S and P, which, as expected, are rational functions of *second degree* with respect to X and Y separately, and thus also with respect to $x(u)$ and $x(v)$.

Moreover, setting

$$A_1 = 2(X - Y)^2 A, \quad B_1 = 2(X - Y)^2 B, \quad D = 2(X - Y)^2 (B - rA + r^2), \tag{4.3.10}$$

one sees immediately that S and P can be expressed as ratios of polynomials of second degree in x through the homography (4.3.8).

4.4 Examples

4.4.1 $\mathcal{H}$ of Order 4

1. The product of two independent random walks inside the quarter plane, so that

$$\sum_{i,j} p_{ij} x^i y^j = p(x)\widetilde{p}(y).$$

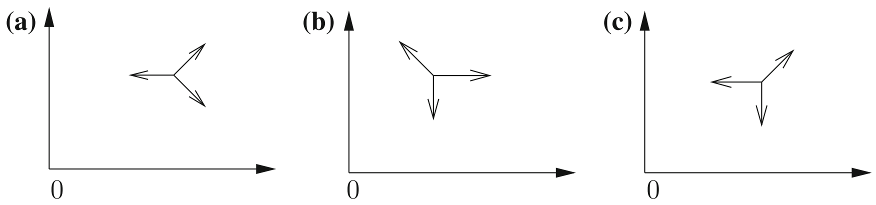


Fig. 4.1 Three examples of groups of order 6

2. The *simple* random walk where $\sum p_{ij} x^i y^j = p(x) + \widetilde{p}(y)$. Thus, in the interior of the quarter plane, $p_{ij} \neq 0$ if, and only if, i or j is zero.
3. Case (a) in Fig. 4.1, which can be viewed as a simple queueing network with parallel arrivals and internal transfers.

4.4.2 $\mathcal{H}$ of Order 6

Among these are the cases studied in [7, 32, 43], which are represented in Fig. 4.1b, c respectively, where, as before, only jumps inside the quarter plane have been drawn.

4.5 Various Comments

As quoted at the beginning of this chapter, the general question of finiteness of the group indeed has intimate links with problems encountered in geometry during the last century. Among them one can cite

1. The problem of abelian integrals;
2. Poncelet's problem, pointed out in private discussions with the authors by L. Flatto.

It is convenient to recall hereafter some fundamental notions and theorems pertaining to compact Riemann surfaces. This material can be found e.g., in [94].

Let us take two arbitrary points u_1, u_2 on the universal covering such that

$$u_2 - u_1 = \omega_3,$$

and let

$$P_1 = \lambda(u_1), \quad P_2 = \lambda(u_2),$$

their corresponding images on $\mathbf{S}$. Then, from Theorem 2.1.17, one can choose on $\mathbf{S}$ a unique (up to a multiplicative constant) abelian differential of the first kind $d\omega$, satisfying

$$\int_{P_1}^{P_2} d\omega = \omega_3. \tag{4.5.1}$$

It follows from (4.1.2), (4.5.1) and Abel's Theorem 2.1.18, that the divisor $\dfrac{P_1^n}{P_2^n}$ is *principal*. In other words, there exists a meromorphic function φ on $\mathbf{S}$, with a unique zero (resp. pole) of n-th order at P_1 (resp. P_2). Clearly, the quantity

$$d(\log \varphi) = \frac{d\varphi}{\varphi}$$

is an abelian differential of the third kind, having P_1 and P_2 as poles of first order.

Lemma 4.5.1 *Condition (4.1.2) holds if, and only if, there exists an abelian integral of the third kind, having logarithmic singularities at the points P_1, P_2 and represented as the logarithm of an algebraic function φ, which belongs to $\mathbb{C}\,(\mathbf{S})$.* ∎

Proof The necessary condition has been proved above. Let us now assume this integral exists and denote it by $\omega_{P_1 P_2}$. Setting

$$\varphi = \exp \omega_{t_1 t_2},$$

and using again Abel's theorem, we come to (4.1.2). Moreover, the corresponding divisor is equal to $\dfrac{P_1^n}{P_2^n}$ and this n is the one we need. In other words, Lemma 4.5.1 reduces the calculation of $\mathcal{H}$ to the following question: when is it possible to integrate a given abelian integral of the third kind in terms of logarithms?

Abel and Chebyshev have studied the latter problem. Its solution for integrals of the form

$$\int (z + C)\frac{dz}{\sqrt{\mathcal{D}(z)}}\,,$$

where $\mathcal{D}(z)$ is the 4^{th} order polynomial with real coefficients, was obtained by E.I. Zolotarev in 1874 [96]. Our situation reduces to integrals of exactly this kind. We shall see in Sect. 4.6 that the group is finite if, and only if, the set $\mathbb{C}\,(x) \cap \mathbb{C}\,(y)$ is non-trivial.

4.6 Rational Solutions

Let the group $\mathcal{H}$ be finite of order $2n$. We shall obtain, by strictly algebraic manipulations, rational solutions of the fundamental equation. These solutions have in general no probabilistic meaning, but they allow us to reduce the problem to finding the solution of a homogeneous equation, i.e. when $q_0(x, y) = 0$.

To find rational solutions, one can simply consider the main equation

$$-Q(x, y)\pi(x, y) = \pi(x)q(x, y) + \widetilde{\pi}(y)\widetilde{q}(x, y) + \pi_{00}q_0(x, y),$$

in $\mathbb{C}_Q(x, y)$. In other words, all calculations will be made *modulo* Q.

Notation We will write $f_\alpha \stackrel{\text{def}}{=} \alpha(f)$, for all automorphisms α and all functions f belonging to $\mathbb{C}_Q(x, y)$.

Then we have

$$q\pi + \widetilde{q}\widetilde{\pi} + q_0\pi_{00} = 0, \tag{4.6.1}$$

$$\pi = \pi_\xi, \tag{4.6.2}$$

$$\widetilde{\pi} = \widetilde{\pi}_\eta. \tag{4.6.3}$$

Applying η to (4.6.1), we get

$$\frac{q_\eta}{\widetilde{q}_\eta}\pi_\eta + \widetilde{\pi} + \frac{(q_0)_\eta}{\widetilde{q}_\eta}\pi_{00} = 0. \tag{4.6.4}$$

Eliminating now $\widetilde{\pi}$ from (4.6.1) and (4.6.4) yields

$$\frac{q_\eta}{\widetilde{q}_\eta}\pi_\eta - \frac{q}{\widetilde{q}}\pi + \left(\frac{(q_0)_\eta}{\widetilde{q}_\eta} - \frac{q_0}{\widetilde{q}}\right)\pi_{00} = 0,$$

or, since $\pi_\eta = \pi_{\eta\xi} = \pi_\delta$,

$$\pi_\delta - f\pi = \psi, \tag{4.6.5}$$

where

$$\begin{cases} \varphi = \dfrac{q}{\widetilde{q}}, \quad f = \dfrac{\varphi}{\varphi_\eta}, \quad \widetilde{f} = \dfrac{\varphi_\xi}{\varphi}, \\ r = \dfrac{\pi_{00}q_0}{\widetilde{q}}, \quad \widetilde{r} = \dfrac{\pi_{00}q_0}{q}, \\ \psi = \dfrac{r - r_\eta}{\varphi_\eta}, \quad \widetilde{\psi} = \dfrac{\widetilde{r} - \widetilde{r}_\xi}{\varphi_\xi}, \\ \widetilde{\delta} \stackrel{\text{def}}{=} \delta^{-1} = \xi\eta. \end{cases} \tag{4.6.6}$$

Now, upon applying δ repeatedly in (4.6.5), π satisfies the following system:

$$\begin{cases} \pi_\delta - f\pi & = \psi, \\ \pi_{\delta^2} - f_\delta\pi_\delta & = \psi_\delta, \\ \cdots & \\ \pi_{\delta^n} - f_{\delta^{n-1}}\pi_{\delta^{n-1}} & = \psi_{\delta^{n-1}}. \end{cases} \tag{4.6.7}$$

$\mathcal{H}$ being of order $2n$, we have $\delta^n = I_d$ and the system (4.6.7) is closed. Indeed, referring to Sect. 2.4, since ex hypothesis π is sought in $\mathbb{C}_Q(x, y)$, we have

$$\pi_{\delta^n} = \pi.$$

Of course analogous relationships could be written for $\widetilde{\pi}$, using the automorphism $\widetilde{\delta}$ and the functions $\widetilde{f}, \widetilde{\psi}$. There are two deeply different situations, which will be analyzed in the next subsections. For this purpose, it is necessary to introduce some definitions, taken from algebra, with the notation employed in Lang [62].

Let us denote by $\mathbb{C}_\delta(x, y)$ the subfield formed by the elements of $\mathbb{C}_Q(x, y)$ which are invariant with respect to δ. Recalling that n is the order of the cyclic group $\mathcal{H}_0 = \{\delta^k,\ k \geq 0\}$ introduced in Lemma 2.4.3, one can define, for any $f \in \mathbb{C}_Q(x, y)$, respectively a *trace* and a *norm* as follows:

$$\begin{cases} Tr(f) \stackrel{\text{def}}{=} Tr_{\mathbb{C}_\delta}^{\mathbb{C}_Q}(f) \stackrel{\text{def}}{=} \sum_{\alpha\in\mathcal{H}_0} f_\alpha = f + f_\delta + \cdots + f_{\delta^{n-1}}, \\ N(f) \stackrel{\text{def}}{=} N_{\mathbb{C}_\delta}^{\mathbb{C}_Q}(f) \stackrel{\text{def}}{=} \prod_{\alpha\in\mathcal{H}_0} f_\alpha = f f_\delta \cdots f_{\delta^{n-1}}. \end{cases}$$

It is worth noting that $Tr(f)$ and $N(f)$ are elements of $\mathbb{C}_\delta(x, y)$, invariant with respect to δ.

4.6.1 The Case $N(f) \neq 1$

Under the assumption

$$N(f) \neq 1, \tag{4.6.8}$$

the linear system (4.6.7) yields directly a rational solution ρ, with

$$\rho = \frac{\displaystyle\sum_{i=0}^{n-1} \psi_{\delta^i} \prod_{k=i+1}^{n-1} f_{\delta^k}}{1 - N(f)}, \tag{4.6.9}$$

with the usual convention that the empty product is equal to 1.

Similarly, under the condition

$$N(\widetilde{f}) \neq 1, \tag{4.6.10}$$

where $N(\widetilde{f})$is computed with the automorphism $\widetilde{\delta}$, we obtain

$$\widetilde{\rho} = \frac{\displaystyle\sum_{i=0}^{n-1} \widetilde{\psi}_{\delta^i} \prod_{k=i+1}^{n-1} \widetilde{f}_{\delta^k}}{1 - N(\widetilde{f})}. \tag{4.6.11}$$

As the reader might have guessed, conditions (4.6.8) and (4.6.10) are in fact equivalent. To prove this property, it suffices to show that

$$N(f)N(\widetilde{f}) = 1. \tag{4.6.12}$$

Since, by (4.6.6),

$$N(f) = \frac{N(\varphi)}{N(\varphi_\eta)} \quad \text{and} \quad N(\widetilde{f}) = \frac{N(\varphi_\xi)}{N(\varphi)},$$

one must simply check $N(\varphi_\xi) = N(\varphi_\eta)$. But

$$N(\varphi_\xi) = [N(\varphi)]_\xi = [N(\varphi)]_\eta = N(\varphi_\eta),$$

where the intermediate equality is easy to derive. Hence (4.6.12) is proved.

Theorem 4.6.1 *Let the order of $\mathcal{H}$ be $2n$ and assume condition (4.6.8) or (4.6.10) holds. Then the system formed by (4.6.1), (4.6.2) and (4.6.3) admits a unique rational solution, ρ, $\widetilde{\rho}$, given by (4.6.9) and (4.6.11), respectively.* ■

Proof We have just proved uniqueness. To establish existence, one has only to show the equality $\rho = \rho_\xi$. Using the relations

$$ff_\eta = 1, \quad \frac{\varphi}{\varphi_\eta} = f, \quad \xi\delta^i = \widetilde{\delta}^{i+1}\eta, \quad \widetilde{\delta}^i = \delta^{n-i}, \quad 0 \le i \le n,$$

we get

$$\begin{cases} \psi_{\xi\delta^i} = \psi_{\widetilde{\delta}^{i+1}\eta} = -\dfrac{\psi_{\widetilde{\delta}^{i+1}}}{f_{\widetilde{\delta}^{i+1}}}, \\ f_{\xi\delta^i} = (f_{\widetilde{\delta}^{i+1}})^{-1} = (f_{\delta^{n-i-1}})^{-1}, \end{cases}$$

whence

$$\rho_\xi = \frac{\left(\displaystyle\sum_{i=0}^{n-1} \psi_{\delta^i} \prod_{k=i+1}^{n-1} f_{\delta^k}\right)_\xi}{(1 - N(f))_\xi} = \frac{-\displaystyle\sum_{i=0}^{n-1} \frac{\psi_{\widetilde{\delta}^{i+1}}}{f_{\widetilde{\delta}^{i+1}}} \prod_{k=i+1}^{n-1} \frac{1}{f_{\widetilde{\delta}^{k+1}}}}{1 - \displaystyle\prod_{i=0}^{n-1} \frac{1}{f_{\delta^i}}} \tag{4.6.13}$$

$$= \frac{\displaystyle\sum_{j=0}^{n-1} \psi_{\delta^j} \left(\prod_{u=0}^{j} \frac{1}{f_{\delta^u}}\right) N(f)}{1 - N(f)} = \rho. \tag{4.6.14}$$

The proof of Theorem 4.6.1 is complete. ■

Remark 4.6.2 We have obtained the unique rational solution, which is non-zero if, and only if, $q_0 \neq 0$. Moreover, whenever this solution has poles inside the unit circle, one can infer that the generating functions of the stationary probabilities cannot be rational.

4.6.2 The Case $N(f) = 1$

Before proceeding to the detailed analysis, we quote in the forthcoming remark three properties, useful in the applications to avoid intricate computations, and which can be derived along the lines which produced (4.6.12).

Remark 4.6.3 The following equivalences hold:

$$\begin{cases} N(f) = 1 \Longleftrightarrow \\ N(\varphi) \in \mathbb{C}(x) \cap \mathbb{C}(y) \Longleftrightarrow \\ N(\varphi) = [N(\varphi)]_\xi = [N(\varphi)]_\eta. \end{cases}$$

In Sect. 4.6.1, it was shown that, whenever condition (4.6.8) holds,

$$\pi = w + \rho, \tag{4.6.15}$$

where $\rho \in \mathbb{C}(x)$ is given by Theorem 4.6.1 and w satisfies the homogeneous equation

$$w_\delta - f w = 0. \tag{4.6.16}$$

Now, whenever

$$N(f) = 1,$$

it will be proved that

$$\pi = cu,$$

where $c \in \mathbb{C}(x)$ is rational and u satisfies the equation

$$u_\delta - u = c_\delta \psi.$$

En passant we shall also find all possible rational solutions.

Notation Let F be an arbitrary field, and h an automorphism of F. Then F_h will denote the subfield of elements of F which are invariant with respect to h.

Lemma 4.6.4 *Let F_h, F be two fields such that*

(i) F is a finite Galois extension of F_h;
(ii) The Galois group $G(F/F_h)$, (i.e. the set of automorphisms of F leaving F_h invariant – see Sect. 2.1.3) is cyclic and generated by h.

Then, for any $\varphi, \psi \in F$ *such that*

$$\begin{cases} N_{F_h}^F(\varphi) \overset{def}{=} \prod_{i=0}^{n-1} \varphi_{h^i} = 1, \\ Tr_{F_h}^F(\psi) \overset{def}{=} \sum_{i=0}^{n-1} \psi_{h^i} = 0, \end{cases}$$

there exist $a, b \in F$ *satisfying respectively*

$$\begin{cases} \varphi = \dfrac{a}{a_h}, \\ \psi = b - b_h. \end{cases}$$

■

Proof The result follows directly from the multiplicative and additive forms of Hilbert's Theorem 90, which we quote now for the sake of completeness (the proofs can be found, for example, in [62]). ■

Theorem 4.6.5 (Hilbert's Theorem 90) *Let* $K \subset L$ *be two arbitrary fields such that the Galois group* $G(L/K)$ *is cyclic of degree* n. *Let* σ *be a generator of* G. *Let* $\beta \in L$.

(Multiplicative form) *The norm* $N_K^L(\beta)$ *is equal to* 1 *if, and only if, there exists an element* $a \neq 0$ *in* L *such that* $\beta = a/a_\sigma$.
(Additive form) *The trace* $Tr_K^L(\beta)$ *is equal to* 0 *if, and only if, there exists an element* b *in* L *such that* $\beta = b - b_\sigma$. ■

Here we shall choose $L = F$ and $K = F_h$. The proof of the multiplicative form of Hilbert's theorem ensures, in a constructive manner, the existence of $\theta \in F$ such that

$$a = \theta + \varphi\theta_h + \varphi\varphi_h\theta_{h^2} + \cdots + \varphi\varphi_h \cdots \varphi_{h^{n-2}}\theta_{h^{n-1}} \neq 0.$$

Such an a is admissible.

Similarly, for any $\gamma \in F$ with $Tr_{F_h}^F(\gamma) \neq 0$, an admissible b is given by

$$b = \frac{1}{Tr_{F_h}^F(\gamma)}[\psi\gamma + (\psi + \psi_h)\gamma_h \cdots + (\psi + \cdots \psi_{h^{n-2}})\gamma_{h^{n-2}}].$$

Note that γ above exists since, $\forall\gamma \in F_h - \{0\}$,

$$Tr_{F_h}^F(\gamma) = n\gamma \neq 0.$$

This concludes the proof of Lemma 4.6.4. ■

Lemma 4.6.6 (Multiplicative decomposition) *Assume now α and β are two automorphisms of F, satisfying $\alpha^2 = \beta^2 = I_d$. Let $h = \alpha\beta$ and $f \in F$ such that*

$$\begin{cases} N_{F_h}^F(f) = 1, \\ N_{F_\alpha}^F(f) = f f_\alpha = 1. \end{cases}$$

Then there exists a $c \in F_\beta$ satisfying

$$f = \frac{c}{c_h}.$$

■

Proof From Lemma 4.6.4, we know that there exists an $a \in F$ with

$$f = \frac{a}{a_h}.$$

Hence

$$f_\alpha = \frac{a_\alpha}{a_\beta} = \frac{1}{f} = \frac{a_h}{a},$$

which yields in turn

$$\frac{a_\alpha}{a_h} = \frac{a_\beta}{a}. \tag{4.6.17}$$

Since

$$\frac{a_\alpha}{a_h} = \left(\frac{a}{a_\beta}\right)_\alpha,$$

Equation (4.6.17) can be rewritten as

$$\left(\frac{a}{a_\beta}\right)_\beta = \left(\frac{a}{a_\beta}\right)_\alpha \quad \text{or} \quad \left(\frac{a}{a_\beta}\right)_h = \frac{a}{a_\beta}.$$

Introduce the sub-field

$$F_{\beta,\alpha} = F_\beta \cap F_\alpha.$$

We shall prove now that F_h is an algebraic extension of degree 2 of $F_{\beta,\alpha}$, generated by β (resp. α) with a group of automorphisms $(1, \beta)$ (resp. $(1, \alpha)$).

- First, if $z \in F_\beta \cap F_\alpha$, then $z = z_\beta = z_\alpha$, so that $z = z_h$, i.e. $F_h \supset F_\beta \cap F_\alpha$.
- Let now $z \in F_h$, i.e. $z = z_h$ or, equivalently, $z_\alpha = z_\beta$. Then $z + z_\beta \in F_\beta$ and $z + z_\beta = z + z_\alpha \in F_\alpha$, so that $z + z_\beta \in F_\beta \cap F_\alpha$.
- Similarly, $z z_\beta \in F_\beta \cap F_\alpha$.

We have thus shown that z is a zero of an equation of order 2, with coefficients in $F_{\beta,\alpha}$. Moreover, fixing $x \in F_h$, $x \notin F_{\beta,\alpha}$, we obtain, for any $z \in F_h$,

$$z = u + vx,$$

where

$$u = \frac{xz_\alpha - x_\alpha z}{x - x_\alpha} \quad \text{and} \quad v = \frac{z - z_\alpha}{x - x_\alpha}.$$

It is easy to check that $u, v \in F_{\beta,\alpha}$. Consequently, F_h is an extension of order 2 of $F_{\beta,\alpha}$.

Since $N_{F_{\beta,\alpha}}^{F_h}\left(\frac{a_\beta}{a}\right) = 1$, we can apply Lemma 4.6.4 to the function $\frac{a_\beta}{a}$ with $h = \beta$. Hence, there exists $b \in F_h$ such that

$$\frac{b}{b_\beta} = \frac{a_\beta}{a},$$

which implies $ab \in F_\beta$. Putting $c = ab$, we get

$$\frac{c}{c_h} = \frac{ab}{a_h b_h} = \frac{a}{a_h} = f.$$

The proof of Lemma 4.6.6 is complete. ∎

Remark 4.6.7 It is also worth noting that, by construction, the element c found in Lemma 4.6.6 has the form

$$c = \sum_{i=0}^{n-1} \theta_{h^i} \prod_{j=0}^{i-1} f_{h^j},$$

with

$$\theta = \theta_\beta.$$

Simpler constructions exist for $n = 2, 3$.

- When $n = 2$ and $f \neq -1$, we put

$$c = 1 + f, \quad \text{since} \quad \frac{1+f}{1+f_h} = f.$$

- For $n = 2$ and $f = -1$, we can take

$$c = x - x_h.$$

- For $n = 3$, we can choose c equal to one of the following elements

$$\begin{cases} 1 + f + f_h, \\ x + x_h f + x_{h^2} f_h, \\ \dfrac{1}{x} + \dfrac{f}{x_h} + \dfrac{1}{x_{h^2}} f_h. \end{cases}$$

Lemma 4.6.8 (Additive decomposition) *Let α, β, h be as in Lemma 4.6.6. Setting $\varepsilon = \pm 1$, let $u \in F$ such that*

$$\begin{cases} Tr^{F}_{F_h}(u) = 0, \\ u + \varepsilon u_\alpha = 0. \end{cases}$$

Then there exists a $\gamma \in F$ satisfying $u = \gamma - \gamma_h$, with $\gamma = \varepsilon \gamma_\beta$. ■

Proof It is sufficient to find a solution V of the equation

$$V - V_h = u,$$

which corresponds to an additive problem in Lemma 4.6.4, and to put

$$\gamma = \frac{1}{2}(V + \varepsilon V_\beta).$$

Then clearly $\gamma = \varepsilon \gamma_\beta$, and

$$\gamma - \gamma_h = \frac{1}{2}\left(V - V_h + \varepsilon(V_\beta - V_\alpha)\right) = \frac{1}{2}(V - V_h + V - V_h) = u,$$

since the condition $u + \varepsilon u_\alpha = 0$ implies, instantiating $u = V - V_h$, that

$$V - V_h + \varepsilon(V_\alpha - V_\beta) = 0.$$

The case $\varepsilon = -1$ will be used only at the end of Sect. 4.7. ■

In the applications of Lemmas 4.6.6 and 4.6.8, we will take

$$\alpha = \eta, \quad \beta = \xi, \quad h = \delta, \quad F = \mathbb{C}_Q(x, y),$$

whence $F_\xi = \mathbb{C}(x)$, $F_\eta = \mathbb{C}(y)$. Now we are in a position to formulate the final result of this section.

Theorem 4.6.9 *Let $N(f) = 1$. Then Eq. (4.6.5) has a rational solution if, and only if,*

$$\sum_{k=0}^{n-1} \psi_{\delta^k} \prod_{i=k+1}^{n-1} f_{\delta^i} = 0. \tag{4.6.18}$$

Moreover, under this condition, the solution is unique up to rational solutions of the system

$$u_\xi = u_\eta = u. \tag{4.6.19}$$

■

Proof Coming back to the basic equation (4.6.5), we get from Lemma 4.6.6

$$(c\pi)_\delta - c\pi = c_\delta \psi, \tag{4.6.20}$$

where $c \in \mathbb{C}(x)$ and $c_\delta \psi \in \mathbb{C}_Q(x, y)$. Setting $w = c\pi$, (4.6.5) finally reduces to

$$w_\delta - w = c_\delta \psi. \tag{4.6.21}$$

From the additive form of Hilbert's theorem, Eq. (4.6.21) has a solution in $\mathbb{C}_Q$ if, and only if, $Tr(c_\delta \psi) = 0$. With $c = f c_\delta$, we have

$$\begin{aligned} Tr(c_\delta \psi) &= \sum_{k=0}^{n-1} (c_\delta \psi)_{\delta^k} = \sum_{k=0}^{n-1} \psi_{\delta^k} c_{\delta^{k+1}} \\ &= c \sum_{k=0}^{n-1} \left(\frac{\psi_{\delta^k}}{\prod_{i=0}^{k} f_{\delta^i}} \right), \end{aligned}$$

so that, using $N(f) = 1$, (4.6.18) is plainly equivalent to $Tr(c_\delta \psi) = 0$, which will be in force in the rest of the proof. Then Lemma 4.6.8 with $\epsilon = 1$ ensures the existence of $\gamma \in \mathbb{C}(x)$, where

$$c_\delta \psi = \gamma - \gamma_\delta,$$

provided that $c_\delta \psi$ satisfies the second condition of Lemma 4.6.8, which is tantamount to checking

$$c_\delta \psi + (c_\delta \psi)_\eta = 0.$$

But, from the definitions given in (4.6.6), we obtain precisely

$$c_\delta \psi + (c_\delta \psi)_\eta = c_\delta \left(\frac{r - r_\eta}{\varphi_\eta} + \frac{\varphi}{\varphi_\eta} \frac{r_\eta - r}{\varphi} \right) \equiv 0.$$

Now Eq. (4.6.21) becomes

$$w_\delta - w = \gamma - \gamma_\delta,$$

or

$$(w + \gamma)_\delta = w + \gamma.$$

Hence

$$w = -\gamma + u,$$

where u satisfies $u_\delta = u$. It follows that the rational solutions of (4.6.20) have the form

$$t = \frac{-\gamma + u}{c},$$

where u is itself a rational solution of the system of Eq. (4.6.19). The proof of Theorem 4.6.9 is complete. ■

One of the main results of Sect. 4.6 is that we could reduce the solution of the non-homogeneous equation to that of a homogeneous one. This is summarized in the next theorem.

Theorem 4.6.10 *All rational solutions of the fundamental Eqs.* (4.6.1)–(4.6.3) *can be classified as follows.*

(i) If $N(f) \neq 1$, then there exists a unique rational solution ρ, which is given by (4.6.9).
(ii) If $N(f) = 1$, there are two cases:
– if condition (4.6.18) is not satisfied, then Eq. (4.6.5) has no rational solution;
– if condition (4.6.18) holds, then any rational solution of (4.6.5) reads as

$$\pi = t_0 + \frac{u}{c},$$

where c is rational, given by Lemma 4.6.6 and Remark 4.6.3, t_0 is the particular rational solution of (4.6.5) given by

$$t_0 = \frac{1}{2n} \sum_{i=1}^{n-1} (i-n) \left[\prod_{j=i}^{n-1} f_{\delta^j} \psi_{\delta^{i-1}} + \left(\prod_{j=i}^{n-1} f_{\delta^j} \psi_{\delta^{i-1}} \right)_\xi \right], \tag{4.6.22}$$

and u is an arbitrary rational function of the composed function

$$\Delta^{(n)} \circ \Delta,$$

where Δ is the fractional linear transform defined by (3.3.3), (3.3.4), and

$$\Delta^{(n)}(x) \stackrel{def}{=} \frac{\prod_{j=0}^{j=n-1} \left(x - \wp\big(e + \frac{j\omega_2}{n}; \omega_1, \omega_2\big) \right)}{\prod_{j=1}^{j=n-1} \left(x - \wp\big(\frac{j\omega_2}{n}; \omega_1, \omega_2\big) \right)}, \tag{4.6.23}$$

denoting by

$$e = \frac{\alpha\omega_1}{2} + \frac{\beta\omega_2}{2n}$$

a zero of $\wp(\omega; \omega_1, \frac{\omega_2}{n})$, with $\alpha, \beta \in]0, 1]$ and either α or β is equal to 1. ■

Proof Assuming $N(f) = 1$ and condition 4.6.18, it is possible by Lemma 4.6.8 to look for t_0 of the form

$$t_0 = -\frac{V + V_\xi}{2c},$$

where, instantiating $\varepsilon = 1$ in Lemma 4.6.8, one chooses

$$\begin{aligned} V &= \frac{1}{n}[(n-1)c_\delta \psi + (n-2)c_{\delta^2}\psi_\delta + \cdots + c_{\delta^{n-1}}\psi_{\delta^{n-2}}] \\ &= \frac{1}{n}\sum_{i=1}^{n-1}(n-i)c_{\delta^i}\psi_{\delta^{i-1}}. \end{aligned}$$

Iterating the relation $c = f c_\delta$, the above choice of V yields easily (4.6.22).

As for u, it follows by condition (4.6.19), since $u \in \mathbb{C}_Q$, that, on the universal covering $\mathbb{C}_\omega$, $u(\omega)$ is an elliptic function with periods ω_1, ω_2, satisfying

$$\begin{cases} u = u_\xi \Rightarrow u(\omega) = u(-\omega) \\ u = u_\delta \Rightarrow u(\omega) = u(\omega + \omega_3). \end{cases}$$

From Chap. 3 and Sect. 4.1, we know that $\omega_3 = \dfrac{k}{n}\omega_2$, where k and n are relatively prime numbers. There exists an integer k' such that

$$kk' = 1 \mod n\,,$$

and hence u is elliptic with periods ω_1 and $\dfrac{\omega_2}{n}$. Since u is even, we know (see e.g., [6]) that

$$u(\omega) = \Delta_0 \circ \wp\left(\omega; \omega_1, \frac{\omega_2}{n}\right),$$

where Δ_0 is an unspecified rational function. Also, using some basic properties of the Weierstrass $\wp$-function (see [6]), we can write

$$\wp\left(\omega; \omega_1, \frac{\omega_2}{n}\right) = \Delta^{(n)} \circ \wp(\omega; \omega_1, \omega_2),$$

where $\Delta^{(n)}$ is given by (4.6.23). Since from the uniformization (3.3.4) $\wp$ is a fractional linear transform of x, the proof of the theorem is complete. ■

Corollary 4.6.11 *The group $\mathcal{H}$ is of finite order n if, and only if,*

$$\mathbb{C}(x) \cap \mathbb{C}(y) \neq \mathbb{C}\,.$$

■

Proof Indeed if n is finite, the set $\mathbb{C}(x) \cap \mathbb{C}(y) \neq \mathbb{C}$ coincides with the set of rational solutions of (4.6.19). On the other hand, if $n = \infty$, then $\omega_1, \omega_2, \omega_3$ are linearly independent on $\mathbb{Z}$ and (4.6.19) admits only constant solutions. ■

4.7 Algebraic Solutions

The general and standard notion of an algebraic function was recalled in Sect. 2.1.2. On the universal covering $\mathbb{C}_\omega$, let us consider the equation

$$t_\delta - ft = \psi, \tag{4.7.1}$$

where t belongs to the field of meromorphic functions with period ω_1, and is subject to the constraint [for technical reasons which will appear later]

$$t = \varepsilon t_\xi, \quad \varepsilon = \pm 1. \tag{4.7.2}$$

Since by (4.6.6) $ff_\eta = 1$, and by (4.7.2) $t_\delta = \varepsilon t_\eta$, a necessary condition for (4.7.1) to be solvable is

$$\psi + \varepsilon \psi_\eta f = 0. \tag{4.7.3}$$

In this section we will assume (4.7.3). The reason for condition (4.7.2) with $\varepsilon = -1$ will appear at the end of subsection 4.7.2. As before, one must separate the cases

$$N(f) = 1 \quad \text{and} \quad N(f) \neq 1.$$

4.7.1 The Case $N(f) = 1$

Recall that by Lemma 4.6.6 there exists a $c \in \mathbb{C}(x)$ such that $f = \dfrac{c}{c_\delta}$.

Theorem 4.7.1 *Equations (4.7.1) and (4.7.2) have a non-zero algebraic solution if, and only if,*

$$\sum_{k=0}^{n-1} \psi_{\delta^k} \prod_{i=k+1}^{n-1} f_{\delta^i} = 0,$$

which is nothing else but condition (4.6.18). Indeed, when (4.6.18) holds, all solutions are algebraic and have the form

$$t = t_1 + \frac{w}{c},$$

where $t_1 \in \mathbb{C}_Q$ is given by

$$t_1 = \frac{1}{2n}\sum_{i=1}^{n-1}(i-n)\left[\prod_{j=i}^{n-1} f_{\delta^j}\psi_{\delta^{i-1}} + \varepsilon\left(\prod_{j=i}^{n-1} f_{\delta^j}\psi_{\delta^{i-1}}\right)_\xi\right], \tag{4.7.4}$$

and w is an algebraic solution of the system

$$w_\delta = \varepsilon w_\xi = w.$$

∎

Proof Setting $w = ct$ and substituting $c = f c_\delta$, Eqs. (4.7.1) and (4.7.2) rewrite as

$$w_\delta - w = c_\delta \psi.$$

Define

$$\begin{cases} \psi_2 &= \dfrac{1}{n} Tr(c_\delta\psi), \\ \psi_1 &= c_\delta\psi - \dfrac{1}{n} Tr(c_\delta\psi). \end{cases}$$

Then all solutions of (4.7.1) and (4.7.2) have the form

$$w = w_1 + w_2 + U, \tag{4.7.5}$$

where

- The function w_i, $i = 1, 2$, is a particular solution of the system

$$\begin{cases} w_\delta - w = \psi_i, \quad i = 1, 2, \\ w_\xi = \varepsilon w. \end{cases}$$

  In addition, from Lemma 4.6.8 and Lemma 4.7.2 to be proved below, w_1 exists in $\mathbb{C}_Q$ since

$$Tr(\psi_1) = 0, \quad \psi_1 + \varepsilon(\psi_1)_\eta = 0.$$

  The computation of w_2 will be carried out after the proof of Lemma 4.7.5.
- The function U satisfies the homogeneous equation

$$U_\delta = \varepsilon U_\xi = U.$$

  In fact, U is algebraic and completely characterized by means of Lemma 4.7.3.

Lemma 4.7.2 *We have*

$$\psi_i + \varepsilon(\psi_i)_\eta = 0, \quad i = 1, 2.$$

∎

Proof There exists a g such that

$$c_\delta \psi = g_\eta - \varepsilon g.$$

Indeed, using $c = f c_\delta = c_\xi$, we get by (4.7.3) $\varepsilon(c_\delta \psi)_\eta + c_\delta \psi = 0$, and one can simply choose

$$g = \frac{c_\delta \psi - \varepsilon(c_\delta \psi)_\eta}{4} = \frac{-\varepsilon c_\delta \psi}{2}.$$

Then

$$\psi_2 = \frac{1}{n} Tr(g_\eta) - \varepsilon \frac{1}{n} Tr(g).$$

Since, from an earlier derivation, $Tr(g_\eta) = (Tr(g))_\eta$, one can write

$$\psi_2 = \frac{1}{n}[Tr(g)_\eta - \varepsilon Tr(g)]$$

and

$$\psi_2 + \varepsilon(\psi_2)_\eta = 0.$$

The proof of the lemma is complete, just remembering that, by definition,

$$\psi_1 = c_\delta \psi - \psi_2.$$

■

Lemma 4.7.3 *On the universal covering $\mathbb{C}_\omega$, let us consider the field of meromorphic functions with period ω_1. Then, in this field, any solution of the equation*

$$w = w_\delta \tag{4.7.6}$$

is an algebraic function of x. ■

Proof On $\mathbb{C}_\omega$, (4.7.6) becomes

$$w(\omega) = w(\omega + \omega_3).$$

Thus w is a function with periods ω_1, ω_3, so that

$$w(\omega) = \Delta_0\big(\wp(\omega; \omega_1, \omega_3)\big) + \wp'(\omega; \omega_1, \omega_3)\Delta_1\big(\wp(\omega; \omega_1, \omega_3)\big),$$

where Δ_0, Δ_1 are rational functions. Since $n\omega_3 = k\omega_2$, using the functions $\Delta^{(n)}$ introduced in the proof of Theorem 4.6.10, we have

$$\begin{cases} \wp(\omega;\omega_1,\omega_3) = \Delta^{(n)} \circ \wp(\omega;\omega_1,k\omega_2), \\ \wp(\omega;\omega_1,\omega_2) = \Delta^{(k)} \circ \wp(\omega;\omega_1,k\omega_2). \end{cases}$$

Since $\wp'$ is given by relation (3.3.1) (with the convenient arguments), and x is a homographic function of $\wp$, Lemma 4.7.3 is proved. ■

Corollary 4.7.4 *There is an effective construction of the algebraic solutions of (4.6.19)* ■

Proof Let

$$\begin{cases} u^*(\omega) = u\left(\omega + \dfrac{\omega_2}{2}\right), \\ x^*(\omega) = x\left(\omega + \dfrac{\omega_2}{2}\right). \end{cases}$$

Then, by (4.6.19), u^* is even, elliptic with periods ω_1, ω_3 and there exists a rational Δ_0 such that

$$u^*(\omega) = \Delta_0(\wp(\omega;\omega_1,\omega_3)).$$

Consequently

$$u(\omega) = \Delta_0 \circ \wp\left(\omega - \frac{\omega_2}{2};\omega_1,\omega_3\right).$$

With $s \stackrel{\text{def}}{=} \wp\left(\omega - \dfrac{\omega_2}{2};\omega_1,k\omega_2\right)$, and using the functions $\Delta^{(k)}$, $\Delta^{(n)}$ introduced in Lemma 4.7.3, we have

$$\begin{cases} u(\omega) = \Delta_0 \circ \Delta^{(n)}(s), \\ x^*(\omega) = \Delta_2 \circ \Delta^{(k)}(s), \end{cases}$$

where Δ_2 is the fractional linear transform given by (3.3.3). Then (see e.g. [6]) there exists a fractional linear transform Δ_3 such that

$$\wp(\omega;\omega_1,\omega_2) = \Delta_3 \circ \wp\left(\omega - \frac{\omega_2}{2};\omega_1,\omega_2\right).$$

Finally

$$\begin{cases} u(\omega) = \Delta_0 \circ \Delta^{(n)}(s), \\ x(\omega) = \Delta_2 \circ \Delta_3 \circ \Delta^{(k)}(s). \end{cases}$$

It is interesting to remark that, when $k = 1$, any solution u of (4.6.18) is in fact a rational function of x, since $\Delta^{(1)}$ is the identity. ■

Continuing with the proof of Theorem 4.7.1, note that, whenever $\psi_2 = 0$ (which corresponds to the condition of the theorem), it suffices to take $w_2 \equiv 0$.
It remains to prove that, for $\psi_2 \neq 0$, (4.7.1) and (4.7.2) have no algebraic solution. To this end, let us first find the function w_2 in considering the equations

$$w_\delta - w = \psi_2, \quad w_\xi = \varepsilon w,$$

on the universal covering $\mathbb{C}_\omega$. We shall show that

$$w_2(\omega) \stackrel{\text{def}}{=} \psi_2(\omega)\widetilde{\Phi}(\omega)$$

is non-algebraic, where

$$\Phi(\omega) \stackrel{\text{def}}{=} \frac{\omega_1}{2i\pi}\zeta(\omega;\omega_1,\omega_3) - \frac{\omega}{i\pi}\zeta\left(\frac{\omega_1}{2};\omega_1,\omega_3\right) \stackrel{\text{def}}{=} \widetilde{\Phi}\left(\omega + \frac{\omega_2}{2}\right), \tag{4.7.7}$$

and $\zeta(\omega)$ denotes the classical Weierstrass ζ-function (see [53]). An intermediate lemma is needed.

Lemma 4.7.5 *The function $\Phi(w)$ given by (4.7.7) is meromorphic in $\mathbb{C}_\omega$, odd, periodic with period ω_1, and satisfies*

$$\Phi(\omega + \omega_3) = \Phi(\omega) + 1.$$

■

Proof Φ is meromorphic and odd since ζ has both these properties. To prove that Φ has period ω_1 and satisfies the equation of the lemma, it suffices to use (see e.g., [6])

$$\zeta(\omega + \omega_i) = \zeta(\omega) + 2\zeta\left(\frac{\omega_i}{2}\right), \quad i \in \{1, 3\},$$

together with Legendre's identity

$$\omega_1\zeta\left(\frac{\omega_3}{2}\right) - \omega_3\zeta\left(\frac{w_1}{2}\right) = \pi i.$$

Lemma 4.7.5 is proved. ■

From the latter two lemmas, it follows that the product $\psi_2\widetilde{\Phi}$ is meromorphic, periodic with period ω_1, and

$$\begin{cases} (w_2)_\delta &= \psi_2(\omega)\widetilde{\Phi}(\omega + \omega_3) = w_2(\omega) + \psi_2(\omega), \\ (w_2)_\xi &= (\psi_2(\omega))_\xi \Phi\left(\dfrac{\omega_2}{2} - \omega\right) = -(\psi_2(\omega))_\xi \widetilde{\Phi}(\omega) = \varepsilon w_2(\omega), \end{cases}$$

where we have used $(\psi_2)_\xi = (\psi_2)_\eta$ and Lemma 4.7.2. Finally, w_2 is a solution of the equation

$$w_\delta - w = \psi_2,$$

subject to the constraint $w_2 = \varepsilon(w_2)_\xi$. On the other hand, in (4.7.5), w_1 and U are algebraic, so that w is algebraic if, and only if, w_2 is algebraic. Consequently, as ψ_2

is rational, it remains to show that $\Phi(\omega - \omega_2/2)$ is not algebraic in $x(\omega)$. To that end, we exploit the fact that Φ has a linear growth in the ω_2-direction. Recalling that $n\omega_3 = k\omega_2$, define, for all $m \in \mathbb{Z}$, the quantities

$$\alpha_m = mk\omega_2 = mn\omega_3.$$

Then, by Lemma 4.7.5, we get, since $x(\omega)$ is elliptic with periods ω_1, ω_2,

$$\begin{cases} x(\omega) = x(\omega + \alpha_m), \\ \Phi(\omega + \alpha_m) = \Phi(\omega) + mn, \quad \forall m \in Z_+. \end{cases}$$

Thus, for a fixed x, Φ takes an infinite number of values, and therefore cannot be algebraic in x. The proof of Theorem 4.7.1 is complete. ■

4.7.2 The Case $N(f) \neq 1$

Theorem 4.7.6 *Equations (4.6.2) and (4.6.5) have an algebraic non-rational solution if, and only if,*

$$N(f) = -1.$$

The solution has the form

$$\pi = \rho + \frac{u}{c},$$

where ρ is given by (4.6.9), u is an algebraic solution of Eq. (4.7.6), and c is algebraic given by

$$c = \sum_{i=0}^{n-1} \left(\theta_{\delta^i} - \theta_{\delta^{i+n}}\right) \prod_{j=0}^{i-1} f_{\delta^j}, \tag{4.7.8}$$

where $\theta \in F_\xi$ is chosen to ensure $c \neq 0$ (in particular, $\theta \notin \mathbb{C}_Q(x, y)$), F being an algebraic extension of $\mathbb{C}_Q(x, y)$ to be precisely stated in the course of the proof. ■

Proof Since $N(f) \neq 1$, then (4.6.2) and (4.6.5) have a rational solution. Hence, we can consider only the homogeneous equation of the type (4.6.16). Let us suppose that the system

$$\begin{cases} t_\delta &= ft, \\ t_\xi &= t, \end{cases} \tag{4.7.9}$$

where f satisfies

$$ff_\eta = 1,$$

has an algebraic solution, i.e. $P(t,x) \equiv 0$ for some polynomial P. The purpose is to find, on the universal covering $\mathbb{C}_\omega$, meromorphic solutions of (4.7.9), with period ω_1.

Notation For any function $g : z \mapsto g(z)$, one will write $g'_z \stackrel{\text{def}}{=} \dfrac{dg}{dz}$ for the derivative of g with respect to z, or simply g' if there is no ambiguity.

By differentiating $P(t,x) = 0$ with respect to x, we see that t'_x is a rational function of t and x, and hence the ratio $\dfrac{t'_x}{t}$ is algebraic in x. Moreover, since $x(\omega)$ is an elliptic function of ω, x'_ω is also elliptic with the same periods, and is therefore algebraic in x. Using then the elementary equality

$$\frac{t'_\omega}{t} = \frac{x'_\omega t'_x}{t},$$

we deduce also that $\dfrac{t'_\omega}{t}$ is algebraic in x. Introducing the following logarithmic derivatives

$$u = \frac{t'_\omega}{t} \quad \text{and} \quad v = \frac{f'_\omega}{f},$$

we get from (4.7.9) that u is an algebraic solution of

$$\begin{cases} u_\delta - u = v, \\ u_\xi + u = 0, \\ v_\eta - v = 0. \end{cases} \tag{4.7.10}$$

Putting in (4.7.1) and (4.7.2), $f = 1$, $\psi = v$, $\varepsilon = -1$, one can apply Theorem 4.7.1, just replacing in its statement f by 1 and ψ by v. This yields

$$Tr(v) = 0.$$

But

$$Tr(v) = Tr\left(\frac{f'}{f}\right) = \frac{N(f)'}{N(f)},$$

since

$$\delta^k(\omega) = \omega + k\omega_3,$$

which in turn implies

$$\frac{d\delta^k(\omega)}{d\omega} = 1.$$

Then $Tr(v) = 0$ implies $N(f) = K$, where K is some constant and we have the following chain of equalities

$$ff_\eta = 1 \Rightarrow N(f)N(f_\eta) = 1 \Rightarrow N(f)(N(f))_\eta = 1 \Rightarrow K^2 = 1 \Rightarrow K = -1,$$

since $K \neq 1$ and this proves the *if* assertion of the theorem.

Now, let us consider the homogeneous equation, with $N(f) = -1$. We intend to prove that this equation has an algebraic solution. For this, let us note that

$$\prod_{i=0}^{2n-1} f_{\delta^i} = \left(\prod_{i=0}^{n-1} f_{\delta^i}\right)^2 = (N(f))^2 = 1,$$

as $f \in \mathbb{C}_Q(x, y)$ and $\delta^n = I_d$.

We would like to get a kind of Hilbert factorization of f. To this end, consider the field F of elliptic functions with periods ω_1 and $2n\omega_3$. Since $n\omega_3 = k\omega_2$, $\mathbb{C}_Q(x, y)$ can be considered as a subfield of this field. For any $u \in F$, δ is then defined by

$$u_\delta(\omega) = u(\omega + \omega_3).$$

The cyclic group generated by δ on F is finite of order $2n$. Clearly, $f \in F$ and moreover

$$N^F_{F_\delta}(f) = 1,$$

where F_δ is the subfield of elliptic functions with periods ω_1 and ω_3. By Lemma 4.6.6, since

$$N^F_{F_\eta}(f) = f f_\eta = 1,$$

there exists a $c \in F_\xi$, where F_ξ is the subfield of elements of F invariant with respect to ξ, such that

$$f = \frac{c}{c_\delta}.$$

Then system (4.7.9) becomes

$$\begin{cases} (tc)_\delta &= tc, \\ (tc)_\xi &= tc, \end{cases}$$

so that tc is elliptic with periods ω_1, ω_3, and, applying Lemma 4.7.3 with $\varepsilon = 1$, algebraic with respect to x.

The last point is to show that c is also algebraic. But this follows directly from the two following properties:

– c is an elliptic function with periods ω_1 and $2n\omega_3 = 2k\omega_2$ which can be obtained as in Remark 4.6.7 and has the form (4.7.8).
– x is an elliptic function with primitive periods ω_1 and ω_2.

The proof of Theorem 4.7.6 is complete. ■

4.8 Final Form of the General Solution

In Sect. 4.7, we proved *en passant* the following theorem which gives the general solution when $N(f) = 1$.

Theorem 4.8.1 *If $N(f) = 1$, then the general solution of the fundamental equations (4.6.2) and (4.6.5) has the form*

$$\pi = w_1 + w_2 + \frac{w}{c}, \tag{4.8.1}$$

where

– *the function c is defined by Lemma 4.6.6;*
– *the function w_1 is given by formula (4.6.22) in which ψ is replaced by*

$$\psi - \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{\psi_{\delta^k}}{\prod_{i=1}^{k} f_{\delta^i}} \right);$$

– *the function w_2 is given by*

$$w_2(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{\psi_{\delta^k}}{\prod_{i=1}^{k} f_{\delta^i}} \right)(\omega)\, \widetilde{\Phi}(\omega); \tag{4.8.2}$$

– *the function w is an algebraic function satisfying $w = w_\xi = w_\delta$.* ■

Now we shall deal with the case $N(f) \neq 1$, proceeding along similar lines. First let us find the solutions of the homogeneous equation

$$t_\delta = f t, \quad \text{with } \; t(\omega) = t(-\omega + \omega_2), \quad \forall \omega \in \mathbb{C}_\omega. \tag{4.8.3}$$

For convenience, we shall put

$$\frac{t'_\omega}{t} = u. \tag{4.8.4}$$

Then u satisfies (take the logarithm in (4.8.4) and differentiate)

$$u_\delta - u = \frac{f'_\omega}{f} \equiv \chi, \quad u_\xi = -u, \tag{4.8.5}$$

with

$$\chi = \chi_\eta. \tag{4.8.6}$$

Remembering that

$$\eta(\omega) = \omega_2 + \omega_3 - \omega,$$

Equation (4.8.6) is a consequence of the following the chain of equivalences (see Lemma 4.6.8 with $\varepsilon = -1$):

$$ff_\eta = 1 \Leftrightarrow f(\omega)f(\omega_2 + \omega_3 - \omega) = 1 \Leftrightarrow \chi(\omega) = \chi(\omega_2 + \omega_3 - \omega) \Leftrightarrow \chi - \chi_\eta = 0.$$

We put now, as in Sect. 4.6,

$$u = \frac{1}{n} Tr(\chi)\widetilde{\Phi} + a \tag{4.8.7}$$

where $\widetilde{\Phi}$ is given in (4.7.7) and a is such that

$$\begin{cases} a_\delta - a = \chi - \dfrac{1}{n} Tr(\chi) \overset{\text{def}}{=} \widetilde{\chi}, \\ a_\xi = -a. \end{cases} \tag{4.8.8}$$

The function a exists, just putting into Lemma 4.6.8, with $\varepsilon = -1$,

$$u = \widetilde{\chi} \quad \text{and} \quad a = -\gamma.$$

Then, as before, we obtain

$$u_\xi = -u. \tag{4.8.9}$$

The solution t of Eq. (4.8.4) is meromorphic if, and only if, all residues of the meromorphic function u are integers. This can be checked by choosing

$$a = \frac{1}{n} \sum_{k=0}^{n-1} k \widetilde{\chi}_{\delta^k}.$$

On the other hand,

$$\widetilde{\chi} = \frac{f'}{f} - \frac{1}{n} Tr\left(\frac{f'}{f}\right),$$

so that a can be rewritten as

$$a = \frac{1}{n} \sum_{k=0}^{n-1} \left(k - \frac{n-1}{2}\right) \left(\frac{f'}{f}\right)_{\delta^k}.$$

The general solution of (4.8.5) is of the form

$$u = \frac{1}{n} Tr(\chi)\widetilde{\Phi} + a - R, \tag{4.8.10}$$

where R is a solution of the homogeneous system

$$\begin{cases} R_\delta = R \iff R(\omega + \omega_3) = R(\omega), \\ R_\xi = -R \iff R(-\omega + \omega_2) = -R(\omega). \end{cases}$$

We will choose R so that the residues become integer-valued.

Lemma 4.8.2 *There exists an elliptic function R, with periods ω_1, ω_3, such that*

$$R(-\omega + \omega_2) = -R(\omega),$$

and all residues of u in (4.8.10) are integer-valued. ■

Proof Let us consider the fundamental rectangle Π with vertices $b, b+\omega_1, b+\omega_3, b+\omega_1+\omega_3$, where b will be fixed later on. We shall proceed in steps.

- First, we will show that the difference of residues at the points z and $z+\omega_3$, for any z, is an integer. Let A be a small circle around z. Then

$$\begin{aligned}\int_{\delta A} u d\omega - \int_A u &= \int_A \frac{1}{n}(Tr(\chi))(\widetilde{\Phi}+1)d\omega - \int_A \frac{1}{n}(Tr(\chi))\widetilde{\Phi}d\omega + \int_A (a_\delta - a)d\omega \\ &= \int_A \frac{1}{n}(Tr(\chi))d\omega + \int_A \widetilde{\chi}d\omega = \int_A \chi d\omega = \int_A \frac{f'}{f}d\omega = 2iK\pi,\end{aligned}$$

  where $|K|$ is the multiplicity of the zero or of the pole.
- Secondly, one must prove the sum of residues of (4.8.7) inside Π is an integer. Take now
$$b = -\frac{\omega_3}{2} - \frac{\omega_2}{2}.$$

  Then, setting $L = \left[\frac{\omega_3}{2} - \frac{\omega_1}{2}, \frac{\omega_3}{2} + \frac{\omega_1}{2}\right]$, we have (since the sum of the integrals along imaginary directions cancel)

$$\int_\Pi u d\omega = \int_L u d\omega - \int_{\delta^{-1}L} d\omega = \int_{\delta^{-1}L} \chi d\omega = 2\pi i\ [\arg(f)]_{\delta^{-1}L} = 2\pi i\ [\arg(f_{\delta^{-1}})]_L,$$

  where $[\arg(z)]_L$ denotes the variation of the argument of z in traversing the contour L in the positive direction. The image of L (which is invariant w.r.t. η) under the projection $x(\lambda(\omega))$ onto $\mathbb{C}_x$ is a closed curve, coinciding with the cut $[y_1, y_2]$. Thus the right-hand side above is an integer.
- Thirdly, it is necessary to show that R exists with $R(-\omega+\omega_2) = -R(\omega)$. But this is clear from (4.8.9) and the proof of Lemma 4.8.2 is complete. ■

In fact, we want to give a somehow more explicit construction of R, noting that u in (4.8.7) is periodic with periods $\omega_1, k\omega_2$, provided that $\omega_3 = \frac{k}{n}\omega_2$. To this end, let us introduce two families a_i, b_i, $1 \le i \le r$, which are respectively the poles and the zeros of f in the rectangle $\widetilde{\Pi} =]0, \omega_1[\times]0, \omega_2[$, and consider the points

$$a_i + \ell\omega_2 \mod \omega_3, \quad \text{and} \quad b_i + \ell\omega_2 \mod \omega_3, \quad \forall 1 \le i \le r,\ 1 \le \ell \le k-1.$$

We choose these points to be poles and zeros of f respectively, and having in addition their real component on the segment $\left]0, \frac{\omega_3}{2}\right[$. Then the residues of R at an arbitrary point α are given by the formula

$$\mathrm{Res}_\alpha R = K_\alpha + \frac{\kappa}{n}\left(\widetilde{\Phi}(\alpha) - \frac{n-1}{2}\right),$$

where

$$\kappa = \begin{cases} -1 & \text{if } \alpha \text{ is a pole of } f, \\ 1 & \text{if } \alpha \text{ is a zero of } f, \end{cases}$$

and

$$\begin{cases} \mathrm{Res}_0 \widetilde{\Phi} = \dfrac{\omega_1}{2i\pi}, \\ \mathrm{Res}_0 R = \dfrac{\mathrm{Res}_0 \widetilde{\Phi}}{n} \left[\dfrac{(Nf)'}{Nf}\right]_{|\omega=0}, \\ \widetilde{R}(\omega) \stackrel{\text{def}}{=} R\left(\omega - \dfrac{\omega_2}{2}\right). \end{cases}$$

Above, α takes one of the values

$$a_i + \ell\omega_2,\ b_i + \ell\omega_2,\ i = 1, \dots, r\ ;\ \ell = 0, \dots, k-1,$$

and the integers K_α are chosen so that

$$\begin{cases} 2(\sum K_{ij} + \sum \widetilde{K}_{ij}) & = & [\arg(f_{\delta^{-1}})], \\ \sum K_{ij} + \sum \widetilde{K}_{ij} & = & -[\arg(\varphi_\xi)], \end{cases}$$

where

$$\begin{cases} K_{ij} = K_\alpha, & \text{for} \quad \alpha = a_i + j\omega_2, \\ \widetilde{K}_{ij} = K_\alpha, & \text{for} \quad \alpha = b_i + j\omega_2, \end{cases}$$

and the variation of the argument is taken along the cut $]y_1, y_2[$ in the direction corresponding to the direction on L. Note that one could get rid of the K_α's, i.e. one could put $K_\alpha \equiv 0$ by introducing at the very beginning, in place of f and φ, the following functions:

$$\hat{\varphi}(x) = \varphi(x)x^{-\arg[\varphi_\xi]}, \quad \hat{f} = \frac{\hat{\varphi}}{\hat{\varphi}_\eta}.$$

Then it is convenient to write $\widetilde{R}$ in the integral form

$$\widetilde{R}(\omega) = \frac{1}{2\pi i}\left[\int_{\Pi} \frac{\wp'(t)\hat{u}(t)dt}{\wp(t)-\wp(\omega)} + \omega_1\zeta(\omega)\right]$$

where $\wp$, ζ are the well-known Weierstrass functions with periods ω_1, ω_3. This yields the more explicit formula

$$\begin{aligned}\widetilde{R}(\omega) = &\sum \frac{\wp'(\omega)}{\wp(\omega)-\wp(b_i+j\omega_2)}\left(K_{ij} + \frac{1}{n}(\widetilde{\Phi}(b_i+j\omega_2) - \frac{n-1}{2})\right)\\ &-\sum \frac{\wp'(\omega)}{\wp(\omega)-\wp(a_i+j\omega_2)}\left(\widetilde{K}_{ij} + \frac{1}{n}(\widetilde{\Phi}(a_i+j\omega_2) - \frac{n-1}{2})\right)\\ &-\frac{1}{n}\left[\frac{(Nf)'}{Nf}\right]_{|\omega=0}\frac{\omega_1}{2\pi i}\frac{\wp'(\omega)}{\wp(\omega)}.\end{aligned}$$

Hence we come to the following result, similar to the one previously obtained in Theorem 4.8.1.

Theorem 4.8.3 *If $N(f) \neq 1$ then the general solution of the fundamental equations (4.7.1), (4.7.2) has the form*

$$\pi = \rho + \theta \exp\left(\int \left(\frac{1}{n}Tr(\chi)\widetilde{\Phi} - \widetilde{R}\right) d\omega\right),$$

where $\rho \in \mathbb{C}(x)$ was found in Sect. 4.6, χ, $\widetilde{\Phi}$, $\widetilde{R}$ are defined above and θ is an algebraic function satisfying

$$\theta = \theta_\xi = \theta_\eta.$$

■

4.9 The Problem of the Poles and Examples

In Sects. 4.6–4.8, we have completely described the structure of the solutions of the fundamental equation in the case of a finite group. However the problem of specifying a unique probabilistic solution was left open. In other words, we must choose solutions π, $\widetilde{\pi}$ without poles in the unit circle.

Consider first the case $N(f) = 1$ (see Theorem 4.8.1), and assume for now the functions c, w_1, w_2 have already been computed. Then we must choose U, so that both functions

$$\pi = w_1 + w_2 + \frac{U}{c}, \quad \widetilde{\pi} = -\frac{q\pi + q_0\pi_{00}}{\widetilde{q}},$$

have no poles in the unit circle. We know that this is possible if, and only if, the system is ergodic (see Chap. 1). In our opinion, the problem of explicitly finding the poles of U is of a computational nature. It is not excluded that there might be deeper insights into the problem of poles, as can be guessed from the examples presented below.

Similarly, when $N(f) \neq 1$, one must find the function θ introduced in Theorem 4.8.3.

4.9.1 Rational Solutions

Here we will analyze miscellaneous random walks which have been encountered in several studies related to computer models.

4.9.1.1 Reversible Random Walks

Take an ergodic random walk satisfying the following reversibility conditions [58]

$$\pi_\alpha p_{\alpha\beta} = \pi_\beta p_{\beta\alpha}, \ \forall \alpha, \beta \in \mathbb{Z}_+^2. \tag{4.9.1}$$

It is easy to prove, using (4.9.1), that the generating functions are rational. For instance, if $p'_{-1,0} \neq 0$ then

$$\pi_{i0} = C \left(\frac{p'_{1,0}}{p'_{-1,0}} \right)^i .$$

4.9.1.2 Simple Examples of Nonreversible Random Walks

Suppose that q and $\widetilde{q}$ can be expressed as

$$\begin{cases} q(x,y) = xa(x,y)a_1(x)a_2(y) \mod Q(x,y), \\ \widetilde{q}(x,y) = ya(x,y)\widetilde{a}_1(x)\widetilde{a}_2(y) \mod Q(x,y), \end{cases} \tag{4.9.2}$$

for some rational functions $a, a_i, \widetilde{a}_i$. Then, putting

$$\pi_1(x) = \frac{x\pi(x)a_1(x)}{\widetilde{a}_1(x)}, \quad \widetilde{\pi}_1(y) = \frac{y\widetilde{\pi}(y)a_2(y)}{\widetilde{a}_2(y)},$$

the fundamental equation takes the form

$$\pi_1(x) + \widetilde{\pi}_1(y) + P(x,y) = 0, \tag{4.9.3}$$

where

$$P(x,y) = \pi_{00} q_0(x,y) \frac{\widetilde{a}_1(x) a_2(y)}{a(x,y)}.$$

Eliminating $\widetilde{\pi}_1$ in (4.9.3), we obtain

$$(\pi_1)_\delta - \pi_1 = P - P_\eta.$$

Let us suppose hereafter the group is of order 4. Since $f = 1 = N(f)$, the latter equation admits one rational solution if, and only if,

$$0 = Tr(P - P_\eta) = P - P_\eta + P_\delta - P_\xi,$$

i.e. $P + P_\delta \in \mathbb{C}(x) \cap \mathbb{C}(y)$. When the latter condition holds, it also occurs frequently that P has the additive representation

$$P(x,y) = s(x) + \widetilde{s}(y),$$

in which case (4.9.3) yields

$$\begin{cases} \pi_1(x) = -s(x) + u(x), \\ \widetilde{\pi}_1(y) = -\widetilde{s}(y) - \widetilde{u}(y), \end{cases}$$

where u satisfies (4.6.19), i.e. $u \in \mathbb{C}(x) \cap \mathbb{C}(y)$, so that $u(x) = \widetilde{u}(y) \mod Q(x,y)$. Then

$$\begin{cases} \pi(x) = \big[-s(x) + u(x)\big] \dfrac{\widetilde{a}_1(x)}{x a_1(x)}, \\ \widetilde{\pi}(y) = -\big[\widetilde{s}(y) + \widetilde{u}(y)\big] \dfrac{\widetilde{a}_2(x)}{y a_2(y)}. \end{cases}$$

When they exist, the functions $u, \widetilde{u}$ are unique and, in this case, π and $\widetilde{\pi}$ must have no pole inside the unit disc.

It is not difficult to construct random walks satisfying (4.9.2), with jumps from the boundaries not exceeding 2, and which are not reversible but have a rational $\pi(x)$. For instance, consider the random walk shown in Fig. 4.2, where

$$\begin{cases} Q(x,y) &= p_{-1,1} y^2 + (p_{10} x^2 - x + p_{-1,0}) y + p_{-1,-1}, \\ q(x,y) &= x q_0(x,y) = x(y^2 p_{02}^0 + y p_{01}^0 - 1), \\ \widetilde{q}(x,y) &= y(x-1). \end{cases}$$

Then we have

$$\frac{x\pi(x)}{x-1} + \frac{y\widetilde{\pi}(y)}{r(y)} + \frac{\pi_{00}}{x-1} = 0,$$

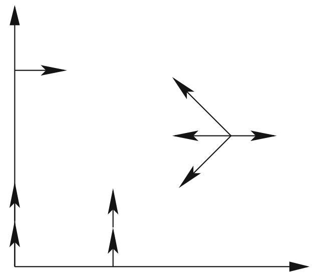


Fig. 4.2 A non-reversible random walk with rational solutions

so that, setting

$$\pi_1(x) = \frac{x\pi(x) + \pi_{00}}{x-1}, \quad \pi_2(y) = \frac{y\widetilde{\pi}(y)}{r(y)},$$

we get

$$\pi_1(x) + \pi_2(y) = 0. \tag{4.9.4}$$

In (4.9.4), we clearly have $N(f) = 1$ and the trace is identically 0. Since one can check that the group is of order 4, i.e. $\omega_3 = \dfrac{\omega_2}{2}$, the remark made at the end of the proof of Corollary 4.7.4 implies directly that

$$\frac{x\pi(x) + \pi_{00}}{x-1} \in \mathbb{C}\,(x) \cap \mathbb{C}\,(y).$$

In addition, the expression of Q shows that $\mathbb{C}\,(x) \cap \mathbb{C}\,(y)$ is generated by the element $yp_{-1,1} + \dfrac{p_{-1,-1}}{y}$, or equivalently by $x(1 - p_{10}x)$. Hence

$$\frac{x\pi(x) + \pi_{00}}{x-1} = \frac{y\widetilde{\pi}(y)}{(1-y)(yp_{02}^0 + 1)} = R\big(x(1 - p_{10}x)\big),$$

where R denotes a rational function, which will be exactly determined. To this end, we must refer to some properties concerning the branch $Y_0(x)$ and the analytic continuation of the functions $\pi, \widetilde{\pi}$ (see Sects. 2.3.1, 5.3 and also [32]). It emerges in particular that the curve $\mathcal{L}$ and $\mathcal{M}$ (introduced in Sect. 5.3) are respectively a circle and a straight line, and also that

$$\frac{x\pi(x) + \pi_{00}}{x-1}$$

cannot have more than one pole, which shows that R is in fact a fractional linear transform of the form

$$R(t) = \frac{at + b}{t + p_{10} - 1}.$$

Straightforward calculations yield

$$\begin{cases} \pi(x) = \dfrac{\pi_{00} p_{10}}{1 - p_{10} - p_{10} x}, \\ \widetilde{\pi}(y) = \dfrac{\pi_{00}(1 - p_{10})(y p_{02}^{0} + 1)}{p_{-1,-1} - p_{-1,1} y}. \end{cases}$$

In fact (4.9.2) gives the simplest example allowing us reduce the problem to the case $f \equiv 1$. To find less evident cases (for groups of order 4), when

$$N(f) = f f_\delta = 1,$$

one must check that

$$A = A_\xi, \quad \text{with} \quad A = \frac{q q_\delta}{\widetilde{q}\, \widetilde{q}_\delta}. \tag{4.9.5}$$

But A can be rewritten as

$$A = \frac{a(x)y + b(x)}{c(x)y + d(x)} \mod Q(x, y),$$

for some $a, b, c, d \in \mathbb{C}$, and we get the following necessary and sufficient condition for (4.9.5) to hold:

$$a(x)d(x) - b(x)c(x) \equiv 0.$$

We can see immediately that this condition yields a set of polynomial equations in the parameters. A particular case arises when $q q_\delta = (q q_\delta)_\xi$, which means that

$$a(x) \equiv 0.$$

The number of corresponding equations is then exactly 3. Setting $\alpha = x\eta(x)$, we have

$$\begin{cases} p'_{11} p'_{-1,0} = p'_{10} p'_{-1,1}, \\ -p'_{11} + \dfrac{p'_{01} p'_{-1,0}}{\alpha} = p'_{10} p'_{01} - \dfrac{p'_{-1,1}}{\alpha}, \\ p'_{01} p'_{10} \alpha - p'_{-1,1} = p'_{-1,0} p'_{01} - p'_{11} \alpha. \end{cases} \tag{4.9.6}$$

Similar equations can be obtained when

$$\widetilde{q}\, \widetilde{q}_\delta = (\widetilde{q}\, \widetilde{q}_\delta)_\xi.$$

We see that polynomial equations even more complicated than (4.9.6) appear and so this is not the right way of tackling the problem.

4.9.1.3 One Parameter Families

A more interesting example for a Markov chain in continuous-time was found in [32]. Let us consider a *simple* random walk such that

$$\begin{cases} \dfrac{Q(x,y)}{xy} = p_{10}(1-x) + p_{01}(1-y) + p_{-1,0}\left(1-\dfrac{1}{x}\right) + p_{0,-1}\left(1-\dfrac{1}{y}\right), \\ \dfrac{q(x,y)}{xy} = t\left(\dfrac{1}{x}-1\right) + p_{0,-1}\left(1-\dfrac{1}{y}\right), \\ \dfrac{\widetilde{q}(x,y)}{xy} = t\left(1-\dfrac{1}{y}\right) + p_{-1,0}\left(1-\dfrac{1}{x}\right). \end{cases}$$

Here the group is of order 4, so that $\delta = \eta\xi = \xi\eta$. Moreover, setting

$$\alpha = \frac{p_{-1,0}}{p_{10}}, \quad \beta = \frac{p_{0,-1}}{p_{01}},$$

we have

$$\alpha = x\eta(x), \quad \beta = y\xi(y).$$

We note that the equations

$$\begin{cases} q = 0 \Longleftrightarrow t = \dfrac{p_{0,-1}x(1-y)}{y(1-x)}, \\ \widetilde{q}_\delta = 0 \Longleftrightarrow t = \dfrac{p_{-1,0}\beta(x-\alpha)}{\alpha(\beta-y)}, \end{cases}$$

provide a parametrization of the curve $Q(x,y)=0$ by means of the parameter t. Indeed, the following equivalence holds

$$\frac{p_{0,-1}x(1-y)}{y(1-x)} = \frac{p_{-1,0}\beta(x-\alpha)}{\alpha(\beta-y)} \Longleftrightarrow Q(x,y)=0. \tag{4.9.7}$$

Then, with f defined in (4.6.6) by

$$f = \frac{q\widetilde{q}_\eta}{\widetilde{q}q_\eta},$$

the next step will be to show that $N(f) = ff_\delta = 1$.

Using (4.9.7), we have

$$\frac{q_\delta}{\widetilde{q}_\delta} = \frac{t\left(\dfrac{x}{\alpha}-1\right) + p_{0,-1}\left(1-\dfrac{y}{\beta}\right)}{t\left(1-\dfrac{y}{\beta}\right) + p_{-1,0}\left(1-\dfrac{x}{\alpha}\right)} = \frac{p_{0,-1}\left[t\left(\dfrac{1}{y}-1\right) + p_{-1,0}\left(\dfrac{1}{x}-1\right)\right]}{p_{-1,0}\left[t\left(\dfrac{1}{x}-1\right) + p_{0,-1}\left(1-\dfrac{1}{y}\right)\right]},$$

that is

$$\frac{q q_\delta}{\widetilde{q}\widetilde{q}_\delta} = -\frac{p_{0,-1}}{p_{-1,0}}. \tag{4.9.8}$$

Upon applying η in (4.9.8), we get

$$\frac{q_\xi q_\eta}{\widetilde{q}_\xi \widetilde{q}_\eta} = -\frac{p_{0,-1}}{p_{-1,0}},$$

so that

$$f f_\delta = -\frac{p_{0,-1}}{p_{-1,0}} \frac{\widetilde{q}_\xi \widetilde{q}_\eta}{q_\xi q_\eta} = 1,$$

which was to be proved. Thus there exists a $c \in \mathbb{C}(x)$ such that $c = f c_\delta$.

On the other hand, if condition (4.6.18) holds, which reads as

$$\psi f_\delta + \psi_\delta = 0,$$

then the generating functions π and $\widetilde{\pi}$ are *rational* and given by

$$\pi(x) = \frac{\pi(0)}{1 - \rho_1 x}, \quad \widetilde{\pi}(y) = \frac{\widetilde{\pi}(0)}{1 - \rho_2 y}, \tag{4.9.9}$$

where ρ_1 and ρ_2 are the positive roots of second degree equations in the parameter space. Chapter 9 presents an explicit model where (4.9.9) holds.

4.9.1.4 Two Typical Situations

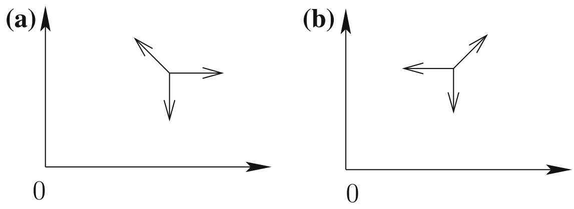
Here we want to show that the above examples are typical in the following sense: a very large number of reasonably simple many-parameter families of rational probabilistic solutions exist, in the case $N(f) = 1$, but there is only a finite number of one-parameter families for which $N(f) \neq 1$. First of all, we shall give immediate necessary conditions for rationality, when π and $\widetilde{\pi}$ have no poles in the unit disc.

Definition 4.9.1 Let a set X and a group G of its one-to-one mappings $g : X \to X$ be given. For an arbitrary but fixed $x \in X$, the set of all $g(x)$, $g \in G$, is called the *orbit* of x under G.

Lemma 4.9.2 *Consider a simple random walk and let* $\mathcal{H} \stackrel{def}{=} \{1, \xi, \eta, \delta\}$. *Then, for* $\pi, \widetilde{\pi}$ *to be rational, it is necessary that there exists a point* $s_0 \in \mathbf{S}$ *such that* $x(s)$ *and* $y(s)$ *are real at all points* $s \in \mathcal{H}s_0 = \{hs_0 : h \in \mathcal{H}\}$, *the orbit of* s_0, *and that at least one of the following conditions is satisfied:*

1. $q(h_1 s_0) = q(h_2 s_0)$, *for some* $h_1 \neq h_2$, $h_1, h_2 \in \mathcal{H}$;
2. $\widetilde{q}(h_1 s_0) = \widetilde{q}(h_2 s_0)$, *for some* $h_1 \neq h_2$, $h_1, h_2 \in \mathcal{H}$;
3. $q(h_1 s_0) = \widetilde{q}(h_2 s_0)$, *for some* $h_1, h_2 \in \mathcal{H}$;

Fig. 4.3 Group of order 6



4. *one of the functions q or $\widetilde{q}$ vanishes at a point $s \in \mathcal{H}s_0$, which is a fixed point of ξ or η.*

■

Before proving this lemma, we shall give hereafter some properties of the orbits and also prove the two auxiliary Lemmas 4.9.3 and 4.9.4. Two cases will be of special interest:

$$\begin{cases} X = \mathbf{S}, \quad G = \mathcal{H}, \quad \text{and} \\ X = \Omega, \quad G = \widetilde{\mathcal{H}}, \quad \text{the lifting of } \mathcal{H} \text{ onto } \Omega. \end{cases}$$

In the second case, it will be convenient to deal with $\Omega/_{\{n\omega_1\}}$, since all functions have the period ω_1.

We are now in a position to quote three useful properties **R1, R2, R3**.

R1 The group $\mathcal{H}$ is finite if, and only if, all the orbits $\mathcal{H}s$, $s \in \mathbf{S}$, are finite.

R2 Setting $I \stackrel{\text{def}}{=} h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D})$, there are situations where

$$\overline{I} \cap \xi\overline{I} = \overline{I} \cap \eta\overline{I} = \emptyset, \tag{4.9.10}$$

e.g., for the groups of order 6 shown in Figs. 4.1 and 4.3, when $M_x < 0$, $M_y < 0$. In fact, if e.g., $\overline{I} \cap \eta\overline{I} \neq \emptyset$, then Γ_0 and $\eta(\Gamma_0)$ have a non-empty intersection. But on Γ_0, $|x| = |y| = 1$ and we get a contradiction, since, for instance in Fig. 4.3b

$$|\eta x| = \left|\frac{p_{-1,0}}{p_{11}xy}\right| = \frac{p_{-1,0}}{p_{11}} > 1.$$

Suppose that (4.9.10) holds. Then, for any $\omega \in \Delta_0$, $\eta\xi\eta(\omega) \notin \Delta_0$,

$$\xi\eta\xi(\omega) \notin \Delta_0.$$

R3 Any orbit $\widetilde{\mathcal{H}}\omega$ in Ω can be mapped in a one-to-one manner onto the set $\mathbb{Z}$. More precisely we can order $\widetilde{\mathcal{H}}\omega$ in the following way:

$$\cdots < \xi\eta(\omega) < \eta(\omega) < \omega < \xi(\omega) < \eta\xi(\omega) < \xi\eta\xi(\omega) < \cdots,$$

where the element ω has number 0 and, in case of equalities, we consider e.g., a and $\xi(a)$ to be formally different. Thus we can call *an interval of the orbit* any inverse of some interval $[m, n] \in \mathbb{Z}$, and the notion of *interval* does not depend on the choice of the point ω in the orbit.

Lemma 4.9.3 *For any ω, the intersection $\{\widetilde{\mathcal{H}}\omega\} \cap \overline{\Delta}_0$ is an interval, both ends of which belong to $\overline{\Delta}_0 \backslash \overline{I}$. All other points of this interval (if there are some) belong to $\overline{I}$.* ■

Proof Let $\omega \in \overline{I}$. Then $\xi(\omega) \in \overline{\mathcal{D}}_1 \subset \overline{\Delta}_0$. If now $\xi(\omega) \in \overline{\mathcal{D}}_1 \backslash \overline{I}$, then $\eta\xi(\omega), \xi\eta\xi(\omega), \ldots$ do not belong to $\overline{\Delta}_0$, as $\omega, \eta(\omega) \in \overline{\Delta}_0$ if, and only if, $\omega, \eta(\omega) \in \{\omega : |y(\omega)| \leq 1\} \cap \overline{\Delta}_0$. If $\xi(\omega) \in \overline{I}$, then similar arguments hold for $\omega_1 = \xi(\omega)$, $\eta(\omega_1)$, etc. ■

From Lemma 4.9.3, one can derive similar properties for the orbit

$$\{\widetilde{\mathcal{H}}_0\omega\} = \{\omega + n\omega_3\}.$$

In Sect. 4.10, it will be shown, for the group of order 6 corresponding to Fig. 4.3b, that

$$\omega_3 = \frac{2}{3}\omega_2.$$

4.9.1.5 Ergodicity Conditions

We can explain in an elementary way the analytical nature of the ergodicity conditions. The functions $q(x, y)$ and $\widetilde{q}(x, y)$ are both assumed to have a zero at the point $s \in \mathbf{S}$, $x(s) = y(s) = 1$.

Lemma 4.9.4 *The function $q(x, y)$ [resp. $\widetilde{q}(x, y)$] has at the point $s \in \mathbf{S}$, $x(s) = y(s) = 1$, a zero of at least second order if, and only if,*

$$M_x M'_y = M_y M'_x \tag{4.9.11}$$

$$[resp.\ M_y M''_x = M_x M''_y] \tag{4.9.12}$$

■

Proof Let $M_y \neq 0$. Then $x(s) - 1$ has a zero of first order at the point $(1, 1) \in \mathbf{S}$, so that q has, at the same point, a zero of order not less than two if, and only if,

$$\frac{dp'(x, y)}{dx}_{|x=y=1} = 0. \tag{4.9.13}$$

But, for $x = y = 1$,

$$\frac{dy}{dx} = -\frac{p_x(x, y)}{p_y(x, y)} = -\frac{M_x}{M_y}$$

and
$$\frac{dp(x,y)}{dx} = \frac{dp'(x,y)}{dy}\frac{dy}{dx} + \frac{dp'(x,y)}{dx} = -M'_y\frac{M_x}{M_y} + M'_x,$$

which shows that (4.9.13) is equivalent to (4.9.11), remembering that the functions $p(x, y)$, $p'(x, y)$, $p''(x, y)$ have been defined in Chap. 1.

In the case $M_y = 0$, $M_x \neq 0$, one should also take into account the quantity $\dfrac{dx}{dy}$.

It will be shown now that, under conditions (4.9.11) or (4.9.12), the system can never be ergodic. In fact, the left-hand side of the fundamental equation

$$q\pi + \widetilde{q}\widetilde{\pi} = -q_0\pi_{00}$$

does have a zero of at least order two at the point (1,1). But it is always possible to choose q_0 (which has no effect on the ergodicity) with a zero of first order at (1,1). The proof of Lemma 4.9.4 is complete. ∎

The non-ergodicity in the other cases can be proved along the same lines. In fact, there is one *travelling* zero of q, which coincides with (1,1) when (4.9.11) holds (a similar travelling zero exists for $\widetilde{q}$, up to a change of parameters) and influences the ergodicity conditions. But it is necessary to have information about other zeros to make sure that π and $\widetilde{\pi}$ have no poles in the unit circle.

4.9.1.6 Proof of Lemma 4.9.2

The power series representing $\pi(x)$ has positive coefficients. Thus if $\pi(x)$ is rational, then it has a positive pole at some point $x_0 > 1$, by the Hadamard–Pringsheim theorem (see e.g., [95]).

Taking a point $s_0 \in \mathbf{S}$ such that $x(s_0) = x_0$, we shall first show that the orbit $\mathcal{H}s_0$ is real, i.e. all values x and y on this orbit are real. For this it is sufficient to prove that when $y(s_0)$ and $y(\xi(s_0))$ are complex conjugate (not real) this leads to a contradiction. Since any orbit contains not more than 4 points and intersects with Δ, then either

$$\text{(a)}\quad |y(s_0)| \leq 1 \qquad\qquad \text{or} \quad \text{(b)}\quad |x(\eta(s_0))| \leq 1. \tag{4.9.14}$$

As for (a), one must have $q(s_0) = 0$, since otherwise $\widetilde{\pi}(s_0) = \infty$, which is impossible. But q is of first degree in y and cannot be zero for non-real y.

In case (b), again $q(s_0) \neq 0$. Hence $\widetilde{\pi}$ has a pole at s_0, so that

$$\widetilde{q}(\eta\xi(x_0)) = \widetilde{q}(\eta(s_0)) = 0.$$

But $\widetilde{q}$ is equal mod Q to a polynomial of first degree in y. Hence the case (b) is again impossible since $x(\eta(s_0)) > 0$.

Continuing with the puzzle, assume now first that $|y(\eta(s_0)) < 1$, for some $\eta \in \mathcal{H}$. We have $y(\eta(s_0)) = y(s_0)$ and $y(\eta\xi(s_0)) = y(\xi(s_0))$. Hence, either $|y(s_0)| < 1$ or

$|y(\xi(s_0))| < 1$. But since π (and also $\widetilde{\pi}$) is rational, any of these points is suitable for our purpose. Take for instance $|y(s_0)| < 1$. But then $q(s_0) = 0$, because otherwise $\widetilde{\pi}$ would have a pole. Hence we are left with two possible situations:

$$\text{(i)} \quad |x(\eta(s_0))| < 1 \qquad\qquad \text{or} \quad \text{(ii)} \quad |x(\eta(s_0))| > 1. \tag{4.9.15}$$

If (i) holds together with $y(\xi(s_0)) \neq 0$, then $\widetilde{\pi}$ has a pole at $s = \xi(s_0)$ and consequently $\widetilde{q}(\eta\xi(s_0)) = 0$, since otherwise π would have a pole at $\xi\eta(s_0)$, which is impossible due to the inequality $|x(\xi\eta(s_0))| < 1$.

In the case (ii), it can be shown along the same lines that, if $q(\xi(s_0)) \neq 0$ and $\widetilde{q}(\eta\xi(s_0)) \neq 0$, then π must have a pole at the point $\xi\eta(s_0)$. Then $q(\eta(s_0)) = 0$, since otherwise $\widetilde{\pi}(\eta(s_0)) = \infty$.

Let $|y(s)| > 1$, for all points s of the orbit. Then $|x(\eta(s_0))| < 1$, as any orbit intersects with Δ. If both $q(s_0) \neq 0$ and $q(\xi(s_0)) \neq 0$, then $\widetilde{\pi}$ has poles at s_0 and $\xi(s_0)$. Thus $\widetilde{q}(\eta(s_0)) = \widetilde{q}(\eta\xi(s_0)) = 0$, still by the same arguments. If only one of the quantities $q(s_0)$ or $q(\xi(s_0))$ vanishes, then $\widetilde{\pi}$ has a pole at the corresponding point and then $\widetilde{q}(hs) = 0$.

Assume now that in $\mathcal{H}s_0$ there exists a fixed point of ξ or η. It is then easy to show that $\mathcal{H}s_0$ consists of exactly two points, one being a fixed point of ξ, the other one a fixed point of η. If, for instance, $s_0 = \xi(s_0)$, then either $q(s_0) = 0$ or $\widetilde{q}(s_0) = 0$. Otherwise, either $\pi(x)$ has a pole at the second point of the orbit, or $\widetilde{\pi}(s_0) = \infty$. But this is impossible since $(\mathcal{H}s_0) \cap \Delta \neq \emptyset$.

Take now the orbit of the point $y(s) = x(s) = 1$ and let

$$\pi\left(\frac{p_{-1,0}}{p_{10}}\right) = \widetilde{\pi}\left(\frac{p_{0,-1}}{p_{01}}\right) = \infty,$$

so that

$$q\left(\frac{p_{-1,0}}{p_{10}}, 1\right) = \widetilde{q}\left(1, \frac{p_{0,-1}}{p_{01}}\right) = 0.$$

But, if $\pi\left(\dfrac{p_{-1,0}}{p_{10}}\right)$ is finite, then π must have a pole at the automorphic point, and similarly for $\widetilde{\pi}$. Therefore the above arguments could be used and the proof of Lemma 4.9.2 is complete. ∎

Corollary 4.9.5 *For given p_{ij}'s, the set of parameters $\{p'_{ij}, p''_{ij}\}$ which produce rational π and $\widetilde{\pi}$ belong to a hypersurface in the parameter space. Incidentally, it is worth noting that this result can be proved in several different ways.*

From Lemma 4.9.2, one sees that two situations prevail:

(i) $q \equiv q_h$ or $\widetilde{q} \equiv \widetilde{q}_h$ or $q \equiv \widetilde{q}_h$, for some $h \in \mathcal{H}$.
This case includes, in particular, reversibility and also (4.9.2). Here many-parameter families exist which produce rational solutions.

(ii) There is a one-parameter algebraic family $q(x, y; c)$, $\widetilde{q}(x, y; c)$ [with a clear notational device], such that e.g., q_h and $\widetilde{q}_h$ have a common zero $(x(c), y(c))$, for any value of the parameter c. Then after eliminating c from the two equations $q_h = \widetilde{q}_h = 0$, we get an algebraic curve, which either coincides with Q or is a degenerate one-point set. The dependency with respect to another parameter (if any) can only be trivial. As remarked earlier, more sophisticated methods might well exist for this problem, which seems to be of a computational nature.

4.10 An Example of an Algebraic Solution by Flatto and Hahn

This is the first example of an algebraic non-rational solution which was encountered and explicitly solved. Moreover it corresponds to a very natural Markovian queueing model, with two servers supplied with parallel arrivals. The analysis of this model gives rise to a random walk in $\mathbb{Z}_+^2$, shown in Fig. 4.4 and having the following parameters:

$$\begin{cases} Q(x,y) &= \dfrac{\left[x^2y^2 - (1+\alpha+\beta)xy + \beta x + \alpha y\right]}{1+\alpha+\beta}, \\ q &= \dfrac{yx^2 - (\alpha+1)x + \alpha}{1+\alpha}, \\ \widetilde{q} &= \dfrac{xy^2 - (\beta+1)y + \beta}{1+\beta}, \\ q_0 &= xy - 1. \end{cases}$$

Setting

$$\pi_1(x) = \frac{\beta x[x\pi(x) + \pi_{00}]}{1-x}, \qquad \widetilde{\pi}_1(y) = \frac{\alpha y[y\widetilde{\pi}(y) + \pi_{00}]}{1-y},$$

the fundamental equation takes the form

$$\pi_1(x) + \widetilde{\pi}_1(y) = 0, \quad \text{on } Q(x,y) = 0. \tag{4.10.1}$$

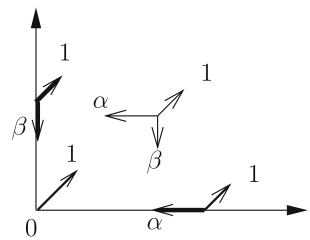


Fig. 4.4 Two queues with parallel arrivals

From Sect. 4.7, it appears that the solution of (4.10.1) (when it exists) is necessarily algebraic, since one can easily check that all conditions of Theorem 4.7.1 are satisfied. To find this solution, we shall proceed to construct the algebraic extension $\mathbb{F}_2$ of $\mathbb{C}_Q(x, y)$, which is defined by the equation

$$v^2 = x_3 - x, \tag{4.10.2}$$

where x_3 is the branch point of $y(x)$ outside the unit circle (here $x_4 = \infty$). The group $\mathcal{H}$ is of order 6 and, in addition, π_1 and $\widetilde{\pi}_1$ belong to

$$\mathbb{C}_{\delta,\xi} = \mathbb{C}_{\delta,\eta} = \mathbb{C}_\delta \cap \mathbb{C}_\xi \subset \mathbb{C}_\delta \subset \mathbb{F}_2.$$

Lemma 4.10.1 *For the random walk shown in Fig. 4.4, we have*

$$\omega_3 = \frac{2\omega_2}{3}. \tag{4.10.3}$$

■

Proof Here

$$\begin{cases} 0 < x_1 < x_2 < 1 < x_3, \quad x_4 = \infty, \\ \alpha \le x_3 \le \beta \le y_3, \quad y_j = \dfrac{\beta x_j}{\alpha}, \quad 1 \le j \le 3. \end{cases}$$

We note immediately that $\omega_3 \in \left\{\dfrac{\omega_2}{3}, \dfrac{2\omega_2}{3}\right\}$. According to the uniformization (3.3.4), we have

$$x(0) = \infty, \quad y(0) = 0, \quad \eta(\omega) = \omega_2 + \omega_3 - \omega, \ \forall \omega \in \mathbb{C}_\omega.$$

Writing for any automorphism $h : \big(x(\omega), y(\omega)\big) \to \big(x(h(\omega)), y(h(\omega))\big)$, we obtain

$$(\eta(\infty), 0) = (0, 0),$$

and, still by (3.3.3),

$$z(\eta(0)) = z(\omega_2 + \omega_3) = \alpha > 0,$$

which implies $\wp'(\omega_2 + \omega_3) > 0$. Using standard properties of the Weierstrass function $\wp$, it follows that $\omega_2 \le 2\omega_3$ and (4.10.3) holds. The proof of the lemma is complete. ■

Lemma 4.10.2

$$\pi \in \mathbb{F}_2. \tag{4.10.4}$$

■

Proof On $\mathbb{C}_\omega$, π_1 is elliptic, with periods ω_1, ω_3, and satisfies

$$(\pi_1)_\xi = \pi_1, \quad \text{or} \quad \pi_1(\omega_2 - \omega) = \pi_1(\omega).$$

Moreover, by (4.10.3), π_1 also admits the period $2\omega_2$. Consider now the function v, defined by (4.10.2), which is algebraic of x, elliptic with periods $\omega_1, 2\omega_2$, and enjoys the following properties:

– two simple poles at $\omega = 0,\ \omega = \omega_2$, with respective residues 1 and -1;
– two simple zeros at $\omega = \dfrac{\omega_1}{2},\ \omega = \dfrac{\omega_1}{2} + \omega_2$, since x_3 is a branch-point.

The residues at 0 and ω_2 having opposite signs, v does not have the period ω_2.

Upon setting for now $\pi^*(\omega) \stackrel{\text{def}}{=} \pi_1(\omega + \omega_2/2)$, one sees immediately that π^* is an even function, elliptic with periods $\omega_1, 2\omega_2$. Consequently, $\pi^*(\omega)$ is a rational function of $\wp(\omega; \omega_1, 2\omega_2)$.

Exactly the same is true for the function $v^*(\omega) \stackrel{\text{def}}{=} v(\omega + \omega_2/2)$. In addition, v^* is of order 2, since it has two poles and two zeros in the parallelogram $[0, 2\omega_2[\times[0, \omega_1[$. Thus $v^*(\omega)$ is fractional linear transform of $\wp(\omega; \omega_1, 2\omega_2)$, and hence π_1 (and consequently π) is also a rational function of v. The proof of the lemma is complete. ■

In fact we have proved that π belongs to the extension of $\mathbb{C}(x)$, of which v is a generator. This extension is nothing else but $\mathbb{C}(v)$.

The final step is to find the rational function mentioned in the proof of Lemma 4.10.2. At that moment, we could proceed as in [43]. Instead, we will use the general machinery developed in this chapter.

First some properties of the branches shown in Sect. 5.3 ensure that

$$\pi_1(x) \sim C/\sqrt{x_3 - x}, \quad \text{for } x \to \infty.$$

Using now (4.10.2) and Corollary 4.7.4 with $k = 2, n = 3$, we deduce easily that Δ_0 (appearing in this corollary) is in fact a fractional linear transform. The function Δ_0 is precisely computed from its values at three specific points, since

$$\pi_1(0) = \pi_1(\infty) = 0, \quad \pi_1(1) = \infty.$$

This yields

$$x\pi(x) + \pi_{00} = \frac{\beta - 1}{\beta} \frac{g(x)}{g(1)},$$

with

$$g(x) = \frac{\sqrt{x_3 - x} + \sqrt{x_3 - 1}}{\left(\sqrt{x_3 - x} + \sqrt{x_3 - \alpha/\beta}\right)\left(\sqrt{x_3 - x} - \sqrt{x_3 - \alpha}\right)}.$$

Similarly

$$y\widetilde{\pi}(y) + \pi_{00} = \frac{\alpha - 1}{\alpha} \frac{\widetilde{g}(y)}{\widetilde{g}(1)} ,$$

with

$$\widetilde{g}(y) = \frac{\sqrt{y_3 - y} + \sqrt{y_3 - 1}}{\left(\sqrt{y_3 - y} + \sqrt{y_3 - \beta/\alpha}\right)\left(\sqrt{y_3 - y} + \sqrt{y_3 - \beta}\right)} .$$

For further computational details, the reader can consult [42, 43].

4.11 Two Queues in Tandem

This random walk is shown in Fig. 4.3a. In the simplest case, when the parameters are continuous on the boundaries, the stationary distribution is given by the famous *product form of Jackson network*, analogous to (4.9.9), which involves very simple rational generating functions. We want to emphasize that rational solutions can exist even when the group for these networks is infinite, although we could not exactly immerse this phenomenon in the theory proposed in this book.

The time-dependent behavior of this system, analyzed by Blanc [8], is much less elementary. It reduces, up to an additional Laplace transform, to the analysis of our fundamental functional equation. The group $\mathcal{H}$ is still of order 6, with $N(f) = 1$, but the corresponding trace is not zero: this shows immediately that the solution cannot be algebraic.

Chapter 5
Solution in the Case of an Arbitrary Group

In Chap. 4, the analysis was based on specific derivations (a closure property in some sense) rendered possible by the finiteness of the order of the group. Hereafter, we shall obtain the complete solution when the order of the group of the random walk is arbitrary, i.e. possibly infinite. The main idea consists in the reduction to a factorization problem on a curve in the complex plane. Generally one first comes up first with integral equations and, in a second step, with explicit integral forms by means of Weierstrass functions.

In Theorem 3.2.2, it has been proved that π [resp. $\widetilde{\pi}$] can be continued as a meromorphic function to the whole complex plane, cut along $[x_3x_4]$ (resp. $[y_3y_4]$). This result was presented as a consequence of analytic continuations on the Riemann surface $\mathbf{S}$ and on the universal covering, by projection and lifting operations. The derivation of Theorem 3.2.2 heavily relied on specific properties of the algebraic curve $Q(x, y) = 0$, given in Lemma 2.3.4.

In this chapter, closed form solutions for the functions π and $\widetilde{\pi}$ will be produced, by reduction to a *boundary value problem* (BVP) of Riemann–Hilbert type in the complex plane, following the approach originally proposed in [32].

5.1 Informal Reduction to a Riemann–Hilbert–Carleman BVP

We shall in fact solve the problem under conditions slightly more general than the ones which produced Eq. (1.3.6). Referring to Eq. (1.3.4) for the stationary probabilities of a piecewise homogeneous random walk, and using the notation of Sect. 1.3, we suppose that the two-dimensional lattice is partitioned into $M + L + 2$ classes S_r such that $\bigcup_{r=1}^{M+L+2} S_r = \mathbb{Z}_+^2$ and

G. Fayolle et al., *Random Walks in the Quarter Plane*, Probability Theory and Stochastic Modelling 40, DOI 10.1007/978-3-319-50930-3_5

$$S_r = \begin{cases} S'_i &= \{(i,0)\}, \quad i = 1, \dots, L-1 \\ S' &= \{(i,0),\ i \geq L\}, \\ S''_j &= \{(0,j)\}, \quad j = 1, \dots, M-1 \\ S'' &= \{(0,j),\ j \geq M\}, \\ S &= \{(i,j)\ ;\ i \geq 1,\ j \geq 1\}, \\ S_0 &= \{(0,0)\}. \end{cases}$$

Incidentally, the generating functions P_r of the jumps in the region r will be, for convenience, denoted by $P_{k\ell}$, for $(k,\ell) \in S_r$, and they depend only on r. Then Eq. (1.3.4) becomes

$$\boxed{-Q(x,y)\pi(x,y) = q(x,y)\pi(x) + \widetilde{q}(x,y)\widetilde{\pi}(y) + \pi_0(x,y)}, \tag{5.1.1}$$

where

$$\begin{cases} \pi(x,y) = \displaystyle\sum_{i,j\geq 1} \pi_{ij} x^{i-1} y^{j-1}, \\ \quad \pi(x) = \displaystyle\sum_{i\geq L} \pi_{i0} x^{i-L}, \quad \widetilde{\pi}(y) = \displaystyle\sum_{j\geq M} \pi_{0j} y^{j-M}, \\ q(x,y) = x^L \left(\displaystyle\sum_{i\geq -L, j\geq 0} p'_{ij} x^i y^j - 1 \right) \equiv x^L (P_{L0}(x,y) - 1), \\ \widetilde{q}(x,y) = y^M \left(\displaystyle\sum_{i\geq 0, j\geq -M} p''_{ij} x^i y^j - 1 \right) \equiv y^M (P_{0M}(x,y) - 1), \\ \pi_0(x,y) = \displaystyle\sum_{i=1}^{L-1} \pi_{i0} x^i (P_{i0}(x,y) - 1) + \sum_{j=1}^{M-1} \pi_{0j} y^j (P_{0j}(x,y) - 1) \\ \qquad\qquad + \pi_{00}(P_{00}(x,y) - 1), \end{cases} \tag{5.1.2}$$

$Q(x,y)$ being given in (1.3.5).

The reader will have noticed that the above partitioning implies that no assumption is made about the boundedness of the upward jumps on the axes, nor at $(0,0)$. In addition, the downward jumps on the x [resp. y] axis are bounded by L [resp. M], where L and M are arbitrary finite integers.

According to Sects. 2.2 and 2.3, the fundamental equation (5.1.1) can be restricted to the algebraic curve $\mathcal{A}$ defined by $Q(x,y) = 0$, $(x,y) \in \mathbb{C}^2$. With the notations of Chap. 2, this yields

$$q(X_0(y),y)\pi(X_0(y)) + \widetilde{q}(X_0(y),y)\widetilde{\pi}(y) + \pi_0(X_0(y),y) = 0, \ \ y \in \mathbb{C}. \tag{5.1.3}$$

We make the additional assumption that the given generating functions P_{i0} and P_{0j} in (5.1.2) have *suitable* analytic continuations, in a sense which will be rendered more precise in Sect. 5.4.

Letting now y tend successively to the upper and lower edge of the slit $[y_1y_2]$, using the fact that $\widetilde{\pi}$ is holomorphic in $\mathcal{D}$ and in particular on $[y_1y_2]$, we can eliminate $\widetilde{\pi}$ in (5.1.3) to get

$$\pi(X_0(y))f(X_0(y),y) - \pi(X_1(y))f(X_1(y),y) = h(y), \text{ for } y \in [y_1y_2], \tag{5.1.4}$$

or, anticipating slightly the results of Sect. 5.3 concerning the functions $X_0(y)$ and $Y_0(x)$,

$$\boxed{\pi(t)A(t) - \pi(\alpha(t))A(\alpha(t)) = g(t), \quad t \in \mathcal{M},} \tag{5.1.5}$$

where

$$\begin{cases} f(x,y) = \dfrac{q(x,y)}{\widetilde{q}(x,y)}, \quad A(x) = f(x, Y_0(x)), \\ h(y) \quad = \dfrac{\pi_0(X_1(y),y)}{\widetilde{q}(X_1(y),y)} - \dfrac{\pi_0(X_0(y),y)}{\widetilde{q}(X_0(y),y)}, \quad g(x) = h(Y_0(x)), \\ \alpha(x) \quad = \overline{x}, \end{cases} \tag{5.1.6}$$

and $\mathcal{M}$ is a simple closed contour $X_0[y_1y_2]$. It turns out that the determination of π, meromorphic in the interior of the domain bounded by $\mathcal{M}$, is possible and is equivalent to solving a BVP of Riemann–Hilbert–Carleman type, on the contour $\mathcal{M}$ in the complex plane. Obviously, as aforementioned, we have assumed that the given functions P_r have meromorphic continuations in the domain bounded by $\mathcal{M}$.

The next section is devoted to a general survey of the basic theoretical results for BVPs in $\mathbb{C}$.

5.2 Introduction to BVPs in the Complex Plane

5.2.1 A Bit of History

In these last fifty years, a huge literature, mainly originating from the still so-called Soviet school, has been devoted to the extensive study of BVPs which arise in the theory of elasticity, hydromechanics and other fields of mathematical physics.

In fact, these problems appeared *in embryo* at the end of the last century. At that time important results were obtained in two directions:

- First, the interpretation of the real and imaginary parts of Cauchy-type integrals as the potential of simple and double layers, respectively (see Harnack, e.g., [80]).
- Secondly, the characterization of the limiting values of Cauchy-type integrals on the contour of integration (Sokhotski and Plemelj, e.g., [80, 87]), involving the so-called *principal* value of singular integrals.

Nevertheless, B. Riemann was probably the first in his dissertation to have mentioned the following general problem:

Find a function which is holomorphic in some domain $\mathcal{D}$ and continuous on the boundary of $\mathcal{D}$, for a given relation between the limiting values of its real and imaginary parts.

This problem was studied by Hilbert in 1905, who gave a partial answer, by reduction to integral equations. Then H. Poincaré in his *Leçons de Mécanique Céleste* [88] came up with a boundary value problem for harmonic functions, which led him to study singular integral equations in more detail.

In this overall presentation, we essentially follow Muskhelishvili [80], Gakhov [47] and Litvintchuk [63].

Let $\mathcal{L}$ be a simple smooth line or curve in the complex plane, i.e. an arc (open or closed) with a continuously varying tangent (see [80]). A function f will be said to satisfy the *Hölder condition* on the curve $\mathcal{L}$ if, for any two points t_1, t_2 of $\mathcal{L}$,

$$|f(t_2) - f(t_1)| \leq A|t_2 - t_1|^{\mu}, \tag{5.2.1}$$

where A and μ are positive constants. Clearly for $\mu > 1$ the above condition (5.2.1) implies $f =$ constant. Hence we only consider the interesting situation $0 < \mu \leq 1$.

When (5.2.1) holds, we shall write, according to a well-established tradition, $\varphi \in \mathbb{H}_{\mu}(\mathcal{L})$, for all φ bounded on $\mathcal{L}$. It is also important to note that $\mathbb{H}_{\mu}(\mathcal{L})$ can be endowed with the norm

$$\|\varphi\| = \sup_{t \in \mathcal{L}} |\varphi(t)| + \sup_{s,t \in \mathcal{L}} \frac{|\varphi(s) - \varphi(t)|}{|s - t|^{\mu}}.$$

5.2.2 *The Sokhotski–Plemelj Formulae*

These formulae were apparently firstly discovered by Sokhotski in 1873 (although not completely rigorously proved, as claimed in Gakhov [47]), who investigated the limiting behavior of Cauchy-type integrals on the contour $\mathcal{L}$. Nevertheless, Muskhelishvili refers to them as *Plemelj formulae*, from the work of Plemelj [87] in 1908, since the arguments were sufficiently rigorous.

Let $\varphi \in \mathbb{H}_{\mu}(\mathcal{L})$. We assume that a positive direction has been chosen on $\mathcal{L}$ (in the case of a closed contour, positive means counter-clockwise). The contour $\mathcal{L}$ is not necessarily bounded. Then the following classical theorem holds (see for example [47, 80]):

Theorem 5.2.1 *The function*

$$\Phi(z) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\varphi(t)dt}{t - z}, \; z \notin \mathcal{L}, \tag{5.2.2}$$

is continuous on $\mathcal{L}$ from the left and from the right, with the exception of the ends. Moreover, the corresponding limiting values, denoted respectively by Φ^+ and Φ^-, are in the class $\mathbb{H}_\mu(\mathcal{L})$, and they satisfy the so-called Sokhotski–Plemelj formulae, for $t \in \mathcal{L}$,

$$\begin{cases} \Phi^+(t) = \dfrac{1}{2}\varphi(t) + \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \dfrac{\varphi(s)ds}{s-t}, \\ \Phi^-(t) = -\dfrac{1}{2}\varphi(t) + \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \dfrac{\varphi(s)ds}{s-t}, \end{cases} \tag{5.2.3}$$

where the integrals are understood in the sense of Cauchy-principal value. Subtracting and adding formulae (5.2.3), one obtains two other equivalent and fundamental formulae, for $t \in \mathcal{L}$, namely

$$\begin{cases} \Phi^+(t) - \Phi^-(t) = \varphi(t), & (5.2.4) \\ \Phi^+(t) + \Phi^-(t) = \dfrac{1}{i\pi}\displaystyle\int_L \dfrac{\varphi(s)ds}{s-t}. & (5.2.5) \end{cases}$$

In fact, (5.2.4) can be shown to be valid for $\varphi(t)$ simply continuous on $\mathcal{L}$, provided some restrictions are put on the way of approaching $\mathcal{L}$ (which should not be, roughly said, too tangential*); on the other hand, cusp points and corner points, when they exist, compel to modify (5.2.3) (see e.g., [80]).* ■

5.2.3 The Riemann Boundary Value Problem for a Closed Contour

Let $\mathcal{L}$ now be a closed contour without self-intersection. It divides the complex plane into two parts: the interior domain within $\mathcal{L}$, denoted by $\mathcal{L}^+$, and the exterior domain (the complement of $\mathcal{L}^+$) denoted by $\mathcal{L}^-$ (when traversing $\mathcal{L}$ in the positive direction, $\mathcal{L}^+$ remains on the left).

We shall say that a function Φ is *sectionally holomorphic* if Φ is holomorphic in every finite region of the plane, except on $\mathcal{L}$ where it has left and right limits Φ^+ and Φ^-. Moreover, this function Φ will be said to have a *finite degree at infinity* if the only singularity at infinity is a pole of finite order. It will be convenient to call Φ sectionally holomorphic whenever $\Phi(\infty) = \text{Constant}$.

The *Riemann* BVP can be stated as follows: *Find a sectionally holomorphic function Φ of finite degree at infinity, under the boundary condition on $\mathcal{L}$*

$$\boxed{\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathcal{L},} \tag{5.2.6}$$

where $G \in \mathbb{C}(\mathcal{L})$, the space of continuous functions on $\mathcal{L}$, and $g \in L_p(\mathcal{L})$. A priori g and G are only defined on $\mathcal{L}$.

For our purpose, we shall only consider the problem for G, $g \in \mathbb{H}_\mu(\mathcal{L})$ and also assume that G does not vanish on $\mathcal{L}$.

Notation We introduce the following important quantity

$$\chi = \mathcal{I}nd[G]_L \stackrel{\text{def}}{=} \frac{1}{2\pi}[\arg G]_{\mathcal{L}} = \frac{1}{2i\pi}[\log G]_{\mathcal{L}}, \tag{5.2.7}$$

which will be called the *index* and represents the variation of the argument of $G(t)$, as t moves along the contour $\mathcal{L}$ in the positive direction.

Without loss of generality, we shall suppose that the origin of the coordinate system lies in the domain $\mathcal{L}^+$. Then the function

$$\log[t^{-\chi}G(t)], \quad t \in \mathcal{L},$$

has zero index, is single-valued and satisfies a Hölder condition.

Let

$$\begin{cases} \Gamma(z) \ = \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \frac{\log(t^{-\chi}G(t))dt}{t-z}, \quad z \notin \mathcal{L}, \\[2ex] X^+(z) = e^{\Gamma(z)}, \quad z \in \mathcal{L}^+, \\ X^-(z) = z^{-\chi}e^{\Gamma(z)}, \quad z \in \mathcal{L}^-. \end{cases} \tag{5.2.8}$$

The functions X^+ and X^- are usually referred to as the *canonical functions* of the BVP. Thus $G(t)$ admits the following factorization by (5.2.4)

$$X^+(t) = G(t)X^-(t), \quad t \in \mathcal{L}, \tag{5.2.9}$$

which gives, *en passant*, a solution of the *homogeneous* BVP (i.e. when $g(t) \equiv 0$) having order $-\chi$ at infinity (i.e. behaving as $z^{-\chi}$). Putting (5.2.9) into (5.2.6), we get

$$\frac{\Phi^+(t)}{X^+(t)} = \frac{\Phi^-(t)}{X^-(t)} + \frac{g(t)}{X^+(t)}, \quad t \in \mathcal{L}. \tag{5.2.10}$$

Since $\dfrac{g}{X^+} \in \mathbb{H}_\mu(\mathcal{L})$, see [80], we can write, using the Sokhotski–Plemelj formula (5.2.4),

$$\psi^+(t) - \psi^-(t) = \frac{g(t)}{X^+(t)}, \quad t \in \mathcal{L},$$

where

$$\psi(z) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{g(s)ds}{X^+(s)(s-z)}, \quad z \notin \mathcal{L}. \tag{5.2.11}$$

Thus (5.2.10) can be rewritten as

$$\frac{\Phi^+(t)}{X^+(t)} - \psi^+(t) = \frac{\Phi^-(t)}{X^-(t)} - \psi^-(t), \quad t \in \mathcal{L}.$$

Two cases must now be distinguished:
(a) $\chi \geq 0$. From the principle of analytic continuation and Liouville's theorem, we obtain

$$\Phi(z) = X(z)\psi(z) + X(z)P_\chi(z), \quad \forall z \notin \mathcal{L}, \tag{5.2.12}$$

where $X(z)$ and $\psi(z)$ are given by (5.2.8) and (5.2.11), and P_χ is an arbitrary polynomial of degree χ. Thus one sees that, in the case of a closed contour $\mathcal{L}$, the solutions belonging to the class of functions bounded at infinity depend on $\chi + 1$ arbitrary complex constants. The second term on the right of (5.2.12) represents the general solution of the homogeneous Riemann BVP.
(b) $\chi < 0$. In this case, the ratio $\dfrac{\Phi^-}{X^-}$ vanishes at infinity, so that $P_\chi \equiv 0$, from Liouville's theorem and

$$\Phi(z) = X(z)\psi(z). \tag{5.2.12bis}$$

But in view of (5.2.8), Φ^- has a pole of order $-\chi - 1$ at infinity. In order for Φ to be holomorphic at infinity (in particular bounded), it is necessary and sufficient that ψ has a zero of order not smaller than $-\chi - 1$.

By expanding the Cauchy-type integral representing ψ (see (5.2.11)) in power series of z around the point at infinity, we get the following conditions of solubility

$$\int_{\mathcal{L}} \frac{g(t)t^{k-1}}{X^+(t)} dt = 0, \quad k = 1, 2, \ldots, -\chi - 1. \tag{5.2.13}$$

One can state the following

Theorem 5.2.2 *The number of linearly independent (non-trivial) solutions of the homogeneous (i.e. $g = 0$) Riemann BVP (5.2.6) is given by*

$$\ell = \begin{cases} \max(0, \chi + 1), & \chi \neq -1, \\ 1, & \chi = -1, \end{cases} \tag{5.2.14}$$

and the number of conditions of solubility is

$$p = \max(0, -\chi - 1). \tag{5.2.15}$$

■

- For $\chi = -1$, the non-homogeneous problem is always soluble and has a unique solution.

- For $\chi < -1$, the non-homogeneous problem has in general no solution. It has exactly one if, and only if, the free term g satisfies the $p = -\chi - 1$ conditions given in (5.2.13).

Let us emphasize that the Riemann BVP (5.2.6) for a closed contour requires us in fact to find two functions Φ^+ and Φ^- which are continuous up to the boundary $\mathcal{L}$. Furthermore, it can be shown easily that, if we were to admit solution functions taking infinite values of integrable order on the contour, the class of solutions would in fact not be extended (see [47], p. 41). This assertion is no longer true for open contours, as will be shown in the next section, since the behavior of a Cauchy-type integral around the ends of an arc is in general more complicated (see e.g., [47, 80]).

5.2.4 The Riemann BVP for an Open Contour

Here we will quote the material directly used to solve the main functional equation (5.1.1).

Let $\mathcal{L}$ denote a smooth non-self-intersecting arc with ends a and b, which can be assumed finite. The positive direction is chosen from a to b. From now on, c will stand for either a or b. The assumptions on G and g and the statement of the BVP are as in Sect. 5.2.3. Since the complex plane cut along $\mathcal{L}$ (the arc $[ab]$) is a simply connected domain, it is important to note that here the BVP is tantamount to finding only one function Φ, holomorphic in the cut plane and having $\mathcal{L}$ as a line of discontinuity, when in the case of a closed contour we had to determine two different holomorphic functions Φ^+ and Φ^-.

The function Φ will be sought in the class of functions holomorphic at infinity, continuous on $\mathcal{L}$ from the left and from the right, with the possible exceptions of the end c, where locally the estimate

$$|\Phi^{\pm}(t)| < \frac{A}{|t-c|^{\alpha}}, \quad 0 \le \alpha < 1, \tag{5.2.16}$$

is to hold. Then the Cauchy-type integral

$$\Omega(z) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\varphi(t)dt}{t-z}, \quad z \in \mathbb{C},$$

with $\varphi \in \mathbb{H}_{\mu}(\mathcal{L})$, has limiting values when z approaches $\mathcal{L}$ from the left or from the right, given from the formulae (5.2.4) and (5.2.5), for all points not coinciding with an end c at which $\varphi(c) \neq 0$.

It can be shown, see e.g., [80], that Ω admits the following representation, in the vicinities of the ends

$$\Omega(z) = \frac{\varepsilon_c \varphi(c)}{2i\pi} \log(z-c) + \Omega_c(z), \tag{5.2.17}$$

where

$$\varepsilon_c = \begin{cases} -1, & \text{if } c \equiv a, \\ +1, & \text{if } c \equiv b, \end{cases}$$

and $\Omega_c(z)$ is a function holomorphic in a neighborhood of c and continuous at c.

Returning to the Riemann BVP, let

$$\Gamma(z) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\log(G(t))dt}{t-z}, \tag{5.2.18}$$

where $\log G(t)$ stands for any value of this multi-valued function, which varies continuously over $\mathcal{L}$. Let

$$\begin{cases} G(a) & = \rho_a e^{i\theta_a}, \\ \Delta & = [\arg G]_{\mathcal{L}}, \\ \log G(t) & = \log|G(t)| + i \arg G(t), \end{cases}$$

so that

$$\arg G(a) = \theta_a \quad \text{and} \quad \arg G(b) = \theta_a + \Delta. \tag{5.2.19}$$

According to (5.2.17), we have at each end c, where $c = a$ or b,

$$\Gamma(z) = \varepsilon_c \frac{\log G(c)}{2i\pi} \log(z-c) + \Gamma_c(z),$$

where $\log G(c)$ is given by (5.2.19). Hence

$$e^{\Gamma(z)} = (z-c)^{\lambda_c} e^{\Omega_c(z)}, \tag{5.2.20}$$

with

$$\lambda_a = \frac{-\theta_a}{2\pi} + \frac{i \log \rho_a}{2\pi}, \tag{5.2.21}$$

$$\lambda_b = \frac{\theta_a + \Delta}{2\pi} - \frac{i \log \rho_b}{2\pi}. \tag{5.2.22}$$

Coming back to the homogeneous BVP (5.2.6) and taking into account condition (5.2.16), one can state the following result:

$$\begin{cases} -2\pi < \theta_a \leq 0 & \Rightarrow e^{\Gamma(z)} \text{ is bounded at } a; \\ \quad 0 < \theta_a < 2\pi & \Rightarrow e^{\Gamma(z)} \text{ is unbounded at } a. \end{cases}$$

Assume we are looking for a solution bounded at a. This will be the situation encountered in this chapter and Chap. 6, in particular when $G(a)$ is a real positive quantity, in which case we shall speak of *automatic boundedness at a*. Let

$$\chi \stackrel{\text{def}}{=} \left\lfloor \frac{\theta_a + \Delta}{2\pi} \right\rfloor. \tag{5.2.23}$$

Then the function

$$X(z) = (z - b)^{-\chi} e^{\Gamma(z)} \tag{5.2.24}$$

is bounded at b. The *index* of the homogeneous BVP (5.2.6) is by definition the quantity χ given in (5.2.23).

If solutions with integrable singularities around the point b were to be admitted, then the index would be $\chi + 1$, with the corresponding solution of the homogeneous BVP

$$X(z) = (z - b)^{-\chi - 1} e^{\Gamma(z)}, \tag{5.2.25}$$

where $\Gamma(z)$ is given by (5.2.18).

Proceeding as in the case of a closed contour, the following result can be established, based on the fact that X, given in (5.2.22), represents a particular solution of the homogeneous BVP, which has order $-\chi$ at infinity, so that in all cases $\lim\limits_{z\to\infty} z^{\chi} X(z) = 1$. The following general result holds.

Theorem 5.2.3 *The general solution of the Riemann BVP (5.2.6), set on a smooth open arc without self-intersection, (in the class of functions bounded at infinity and having possibly singularities of integrable order at the ends), is given by formulae (5.2.12) or (5.2.12bis) and Theorem 5.2.2. Here X is taken from (5.2.24) or (5.2.25) and the index χ is defined in (5.2.23).*

Assuming X given by (5.2.24), then (5.2.12) or (5.2.12bis) hold, where ψ has the integral form (5.2.11) in which

$$X^+(t) = (t - b)^{-\chi} e^{\Gamma^+(t)}, \tag{5.2.26}$$

with, from the Sokhotski–Plemelj formulae,

$$\Gamma^+(t) = \frac{1}{2} \log G(t) + \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\log(G(s))ds}{s - t}, \quad t \in \mathcal{L}. \tag{5.2.27}$$

■

5.2.5 The Riemann–Carleman Problem with a Shift

We shall need the solution of a generalized BVP, sometimes referred to as the Carleman problem, see [47, 63], hereafter presented under the name *Riemann–Carleman problem*.

Let $\mathcal{L}$ be a simple smooth closed contour. The Riemann–Carleman BVP has the following formulation:

Find a function Φ^+ holomorphic in $\mathcal{L}^+$, the limiting values of which on the contour are continuous and satisfy the relation

$$\boxed{\Phi^+(\alpha(t)) = G(t)\Phi^+(t) + g(t), \quad t \in \mathcal{L},} \tag{5.2.28}$$

where

(i) $g, G \in \mathbb{H}_\mu(\mathcal{L}), G(t) \neq 0, \quad \forall t \in \mathcal{L}$;
(ii) $\alpha(t)$, referred to as the *shift* in the sequel, is a function establishing a one-to-one mapping of the contour $\mathcal{L}$ onto itself, such that the direction of traversing $\mathcal{L}$ is changed and

$$\alpha'(t) = \frac{d\alpha(t)}{dt} \in \mathbb{H}_\mu(\mathcal{L}), \quad \alpha'(t) \neq 0, \quad \forall t \in \mathcal{L}.$$

Most of the time, we shall encounter the so-called *Carleman's condition*

$$\alpha(\alpha(t)) = t, \quad \forall t \in \mathcal{L}. \tag{5.2.29}$$

Whenever needed, this condition will be explicitly mentioned. It is worth remarking now that our situation corresponds to $\alpha(t) = \bar{t}, \forall t \in \mathcal{L}$ (see Eq. (5.1.3)).

The BVP (5.2.28) amounts in fact to (5.2.6) and thus allows us to get an integral representation for the solution functions. But the complexity is now increased, because the density of the Cauchy-type integral solution satisfies a Fredholm integral equation of second kind. We follow in this section the method given in Litvinchuk [63]. The main ideas are presented in the next three theorems.

First, one can easily show that, when (5.2.29) holds, a necessary condition of solubility of (5.2.28) is constituted by one of the two following conditions:

1.
$$\partial(t) = \frac{g(\alpha(t)) + g(t)G(\alpha(t))}{1 - G(t)G(\alpha(t))}, \quad G(t)G(\alpha(t)) \neq 1, \quad \forall t \in \mathcal{L},$$

is the boundary value of a function holomorphic in $\mathcal{L}^+$, in which case one has simply

$$\Phi^+(z) = \frac{1}{2i\pi}\int \frac{\partial(t)dt}{t - z}, \quad z \in \mathcal{L}^+.$$

2.
$$G(t)G(\alpha(t)) = 1 \quad \text{and} \quad g(t) + g(\alpha(t))G(t) = 0. \tag{5.2.30}$$

The latter condition (5.2.30) corresponds in fact to a much richer situation, which we shall encounter. Hence, (5.2.30) will be assumed to hold throughout the sequel of this BVP overview.

Lemma 5.2.4 *The homogeneous Riemann–Carleman BVP*

$$\Phi_1^+(\alpha(t)) - \Phi_2^+(t) = 0, \quad t \in \mathcal{L}, \tag{5.2.31}$$

has no solution, but arbitrary constants. ■

Proof From the Sokhotski–Plemelj formulae (5.2.3), the conditions for the analyticity of Φ^+ take the form

$$\frac{1}{2}\Phi_1^+(t) - \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\Phi_1^+(s)ds}{t-s} = 0, \quad t \in \mathcal{L}. \tag{5.2.32}$$

Substituting t by $\alpha(t)$ and s by $\alpha(s)$ in (5.2.32), we have

$$\frac{1}{2}\Phi_1^+(\alpha(t)) + \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\Phi_1^+(\alpha(s))\alpha'(s)ds}{\alpha(s)-\alpha(t)} = 0,$$

or, using (5.2.31),

$$\frac{1}{2}\Phi_2^+(t) + \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\Phi_2^+(s)\alpha'(s)ds}{\alpha(s)-\alpha(t)} = 0. \tag{5.2.33}$$

Replacing Φ_1 by Φ_2 in (5.2.31) and (5.2.32), then summing with (5.2.33), we get precisely

$$\Phi_2^+(t) + \frac{1}{2i\pi}\int \left[\frac{\alpha'(s)}{\alpha(s)-\alpha(t)} - \frac{1}{s-t}\right]\Phi_2^+(s)ds = 0, \quad t \in \mathcal{L}. \tag{5.2.34}$$

As can be easily checked, the kernel of (5.2.34) has for $t = s$ a singularity of order less than one, which is therefore integrable, so that the integral operator is compact. It follows that (5.2.34) is a Fredholm integral equation of the second kind which thus admits only a finite number of linearly independent solutions. Assume that (Φ_1, Φ_2) is a pair of non trivial solutions of (5.2.31). Then (Φ_1^k, Φ_2^k) also form a solution for any arbitrary positive integer k. But then Φ_2^k also satisfies the integral equation (5.2.34), which, since k is arbitrary, would admit an infinite number of independent solutions, contradicting the Fredholm alternative. Thus (5.2.31) admits only the solutions $\Phi_1 = \Phi =$ Constant. Lemma 5.2.4 is proved. ■

Let us introduce the operator $\mathcal{B}$,

$$(\mathcal{B}\varphi)(t) \equiv \varphi(t) + \frac{1}{2i\pi}\int_{\mathcal{L}} \left[\frac{1}{s-t} - \frac{\alpha'(s)}{\alpha(s)-\alpha(t)}\right]\varphi(s)ds. \tag{5.2.35}$$

Then the following lemma holds.

Lemma 5.2.5 *The Fredholm integral equation*

$$(\mathcal{B}\varphi)(t) = 0 \tag{5.2.36}$$

admits only the trivial solution $\varphi \equiv 0$. ■

Proof Let $\varphi(t)$ be a solution of (5.2.36) and set $\beta(t) = \alpha^{-1}(t)$, the inverse mapping. We follow [63]. Consider the two functions

$$\begin{cases} \Phi_1^+(z) & = \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \dfrac{\varphi(\beta(s))ds}{s-z}, \quad z \in \mathcal{L}^+, \\ \Phi_2^+(z) & = \dfrac{-1}{2i\pi}\displaystyle\int_{\mathcal{L}} \dfrac{\varphi(s)ds}{s-z}, \quad z \in \mathcal{L}^+. \end{cases}$$

It follows from (5.2.4) and (5.2.5) that

$$(\mathcal{B}\varphi)(t) = \Phi_1^+(\alpha(t)) - \Phi_2^+(t) = 0, \quad t \in \mathcal{L},$$

whence, using Lemma 5.2.4,

$$\Phi_1^+ = \Phi_2^+ = C,$$

where C is an arbitrary constant. Thus

$$\frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\varphi(\beta(s)) - C}{s-z} ds = \frac{-1}{2i\pi}\int_{\mathcal{L}} \frac{\varphi(s) + C}{s-z} ds = 0, \quad z \in \mathcal{L}^+,$$

so that, by Cauchy's theorem,

$$\varphi(\beta(t)) - C = \psi_1^-(t), \quad t \in \mathcal{L},$$
$$\varphi(t) + C = \psi_2^-(t), \quad t \in \mathcal{L},$$

where $\psi_i^-(t)$ is holomorphic in $\mathcal{L}^-$, with the condition $\psi_i^-(\infty) = 0$, $i = 1, 2$.

Moreover, we have

$$W_1^-(\alpha(t)) = W_2^-(t), \quad t \in \mathcal{L},$$

with

$$\begin{cases} W_1^- = \psi_1^- + C, \\ W_2^- = \psi_2^- - C. \end{cases}$$

Clearly it can be shown, as in Lemma 5.2.4, that the *exterior* problem (5.2.31) (in $\mathcal{L}^-$) has only trivial solutions

$$W_1^-(z) = W_2^-(z) = C_1.$$

Hence, using the conditions $\psi_1(\infty) = \psi_2(\infty) = 0$, we obtain $C = C_1 = -C$.
Consequently, $C_1 = C = 0 = \psi_i(z) = \varphi(t)$. Lemma 5.2.5 is proved. ■

Sometimes it happens that, for special relationships between the parameter values of the random walk (see for example [32, 33]), one has to solve problem (5.2.28) for $G(t) \equiv 1$, which belongs then to the so-called *Dirichlet–Carleman* class described thereafter.

Theorem 5.2.6 *The Carleman–Dirichlet problem*

$$\Phi^+(\alpha(t)) - \Phi^+(t) = g(t), \quad t \in \mathcal{L}, \tag{5.2.37}$$

where $g \in \mathbb{H}_\mu(\mathcal{L})$ satisfies the relation

$$g(t) + g(\alpha(t)) = 0,$$

has a unique solution given up to an arbitrary additive constant by

$$\Phi^+(z) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\varphi(\alpha(s))ds}{s-z} + C, \tag{5.2.38}$$

where C is an arbitrary constant and $\varphi(t)$ is the unique solution of the integral equation

$$(\mathcal{B}\varphi)(t) = g(t),$$

$\mathcal{B}$ being the operator defined in (5.2.35).

Proof This is a direct consequence of Lemmas 5.2.4 and 5.2.5 and of the Fredholm alternative. ■

We return now to the general problem (5.2.28), which corresponds to the main equation (5.1.3). There are several ways to produce a solution having an integral representation. The most direct one (which we will choose) is proposed in [63], and relies on the following theorem of intrinsic interest on *conformal gluing*.

Theorem 5.2.7 *Let $\alpha(t)$ be a Carleman automorphism of the curve $\mathcal{L}$. Then there exists a function w, holomorphic in $\mathcal{L}^+$, except at one point $z = z_0 \in \mathcal{L}^+$, where w has a simple pole, such that*

$$w(\alpha(t)) - w(t) = 0, \quad t \in \mathcal{L}. \tag{5.2.39}$$

Moreover, w establishes a conformal mapping of the domain $\mathcal{L}^+$ onto the domain Δ, which consists of the plane cut along an open smooth arc $\mathcal{U}$.

Proof The proof of this theorem can be found in [63]. ■

From Theorem 5.2.7, one can write

$$w(z) = \frac{1}{z - z_0} + \psi^+(z),$$

where $\psi^+(z)$ is holomorphic in $\mathcal{L}^+$ and satisfies the condition

$$\psi^+(\alpha(t)) - \psi^+(t) = \frac{1}{t - z_0} - \frac{1}{\alpha(t) - z_0} \stackrel{def}{\equiv} g(t), \quad t \in \mathcal{L}.$$

Here (5.2.30) is fulfilled, so that ψ^+ is given by the formula (5.2.38).

It can be shown (see [63]) that α has two fixed points. In the situation we shall encounter, they are simply the intersection points of $\mathcal{L}$ with the real axis (by symmetry arguments) denoted by A and B as in Fig. 5.1.

We denote by $\mathcal{L}_u$ (resp. L_d) the upper (resp. lower) part of $\mathcal{L}$ described in the positive direction, i.e. the one which, on traversing the boundary, leaves the domain $\mathcal{L}^+$ on the left. The respective ends of $\mathcal{U}$ are denoted by a and b.

From Theorem 5.2.7, it follows that the function w, see (5.2.39), has an inverse denoted by z, satisfying

$$\begin{cases} z^+(w) = \alpha(t) \\ z^-(w) = t \end{cases}, \quad \text{for } \omega \in \mathcal{U} \text{ and } t \in \mathcal{L}_d.$$

On $\mathcal{L}_u$, one must exchange z^+ and z^- and $\mathcal{L} \underset{z(w)}{\overset{w(z)}{\rightleftarrows}} \mathcal{U}$.

Now the boundary condition (5.2.28) can be rewritten as

$$\Phi^+[z^+(w)] = G[z^-(w)]\Phi^+[z^-(w)] + g(z^-(w)], \quad w \in \mathcal{U}. \tag{5.2.40}$$

Introduce the composed function $\theta = \Phi^+ \circ z$,

$$\theta(w) \stackrel{\text{def}}{=} \Phi^+[z(w)], \quad w \in \mathbb{C},$$

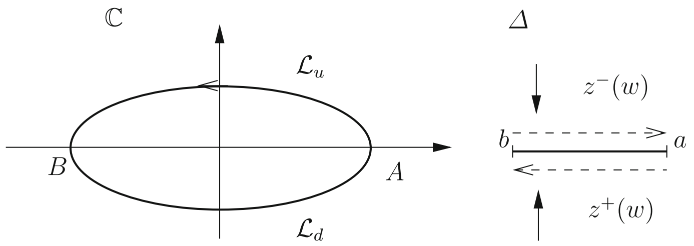


Fig. 5.1 General symmetric form of the curve $\mathcal{L}$

the limiting values of which satisfy, for w approaching $\mathcal{U}$,

$$\begin{cases} \theta^+(w) = \Phi^+[z^+(w)], \\ \theta^-(w) = \Phi^+[z^-(w)]. \end{cases}$$

Then the BVP (5.2.28) on the closed contour $\mathcal{L}$ has been reduced to a Riemann BVP on the arc $\mathcal{U}$ (analyzed in Sect. 5.2.4), with the boundary condition

$$\theta^+(w) = G[z^-(w)]\theta^-(w) + g[z^-(w)], \quad w \in \mathcal{U}. \tag{5.2.41}$$

Keeping in mind that we are interested by solutions of (5.2.40) which are *bounded at both ends*, we shall compute the index of the problem solely in this case.

When w describes $\mathcal{U}$ from a to b in the positive direction, as shown in Fig. 5.1, the point $z^-(w)$ describes the arc $\mathcal{L}_d$ in the negative direction. Thus

$$[\arg G \circ z^-]_{\mathcal{U}} = -[\arg G]_{\mathcal{L}_d}.$$

On the other hand,

$$0 = [\arg G]_{\mathcal{L}_d} + [\arg G \circ \alpha]_{\mathcal{L}_d} = [\arg G]_{\mathcal{L}_d} - [\arg G]_{\mathcal{L}_u},$$

so that

$$[\arg G]_{\mathcal{L}_d} = [\arg G]_{\mathcal{L}_u}.$$

Setting, as in (5.2.7),

$$\chi = \frac{1}{2\pi}[\arg G]_{\mathcal{L}},$$

we have

$$\begin{aligned} [\arg G]_{\mathcal{L}} &= [\arg G]_{\mathcal{L}_d} + [\arg G]_{\mathcal{L}_u} \\ &= 2[\arg G]_{\mathcal{L}_d}, \end{aligned}$$

whence

$$\frac{1}{2\pi}[\arg G \circ z^-]_{\mathcal{U}} = \frac{-\chi}{2}. \tag{5.2.42}$$

Note that χ is even and, in the situation considered above,

$$G(t_A) = G(t_B) = 1,$$

where t_A and t_B are the affixes of A and B, respectively. Moreover,

$$[\arg G]_{\mathcal{L}_d} = \arg G(t_B) - \arg G(t_A) = 2k\pi.$$

On the other hand, the index $\widetilde{\chi}$ of the problem (5.2.41) has been given in (5.2.23)

$$\widetilde{\chi} = \frac{\theta_a + \Delta}{2\pi} = \frac{\Delta}{2\pi} \quad (\text{since } \theta_a = 0),$$

with

$$\Delta = [\arg G(z^-(w))]_{\mathcal{U}} = -\chi\pi.$$

Thus

$$\widetilde{\chi} = \frac{-\chi}{2} \tag{5.2.43}$$

and we can state the following final result, which is a direct consequence of the equivalence of the BVPs (5.2.28) and (5.2.41).

Theorem 5.2.8 *(i) If $\widetilde{\chi} \geq 0$, then the BVP (5.2.28) has $\ell = 1 + \widetilde{\chi}$ linearly independent solutions.*

(ii) If $\widetilde{\chi} < 0$, then the homogeneous problem (5.2.28) has no solution and the non-homogeneous problem is soluble if, and only if, $p = -\widetilde{\chi} - 1$ conditions of the form (5.2.13) are fulfilled. In this case, using (5.2.12bis), the solution of the BVP (5.2.28), which has been reduced to the BVP (5.2.41), is given by

$$\begin{aligned} \Phi^+(z) \equiv \theta(w(z)) &= \frac{X(w(z))}{2i\pi} \int_{[ab]} \frac{g(z^-(s))ds}{X^+(s)(s - w(z))} \\ &= \frac{-X(w(z))}{2i\pi} \int_{\mathcal{L}_d} \frac{w'(t)g(t)dt}{X^+(w(t))(w(t) - w(z))}, \ \forall z \in \mathcal{L}^+. \end{aligned} \tag{5.2.44}$$

■

5.3 Further Properties of the Branches Defined by $Q(x, y) = 0$

With the notation of Sect. 2.3 of Chap. 2, let us recall that

$$Q(x, y) = xy[P(x, y) - 1] \equiv a(x)y^2 + b(x)y + c(x) \equiv \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y).$$

Here we shall analyze more completely the branches $Y_0(x)$ and $Y_1(x)$ of the algebraic function $Y(x)$.

It will be assumed, unless otherwise mentioned, that $Q(x, y)$ is irreducible of degree 2 with respect to x and y, that the Riemann surface has genus 1 and also that there are no branch points on the unit circle Γ. The case of genus 0 will be the subject of Chap. 6.

Lemma 5.3.1 *The equation $Q(x, y) = 0$ has two roots $Y_0(x)$ and $Y_1(x)$ such that, for $|x| = 1$,*

$$\begin{cases} |Y_0(x)| \le 1, \\ |Y_1(x)| \ge 1, \end{cases} \tag{5.3.1}$$

with strict inequalities, except for $x = 1$, where we have

$$\begin{cases} Y_0(1) = \min\left(1, \dfrac{\sum_i p_{i,-1}}{\sum_i p_{i,1}}\right), \\ Y_1(1) = \max\left(1, \dfrac{\sum_i p_{i,-1}}{\sum_i p_{i,1}}\right). \end{cases} \tag{5.3.2}$$

■

Proof Let $|x| = 1, x \neq 1$. Then $\Re[-1 + P(x, y)] < 0$, $\forall y$ such that $|y| = 1$, where $\Re[z]$ denotes the real part of the complex number z, whence

$$[\arg(-1 + P(x, .))]_\Gamma = 0, \tag{5.3.3}$$

where Γ is the unit circle in the complex plane $\mathbb{C}_y$. From the main assumptions made at the beginning of this section, $\sum_i p_{i1} > 0$. Since $P(x, y)$ has a simple pole at $y = 0$, the argument principle in (5.3.3) shows that the equation $Q(x, y) = 0$, $|x| = 1, x \neq 1$ has exactly one root $y = Y_0(x)$ inside the unit disc $\mathcal{D}$. For $x = 1$, one distorts Γ by making a small indentation to the left [resp. right] of the point $y = 1$ inside [resp. outside] Γ, when $M_y > 0$ [resp. $M_y < 0$], so that (5.3.3) still holds on this new contour, say $\widetilde{\Gamma}$. This proves (5.3.2) and Lemma 5.3.1. ■

As an immediate consequence, we have the following:

Corollary 5.3.2 *(a) If $\sum_i p_{i1} = 0$, then $Q(x, y)$ is of first degree in y and the root $Y_0(x)$ satisfies (5.3.1) with $Y_0(1) = 1$.*
(b) If $\sum_i p_{i,-1} = 0$, then $c(x) = 0$ and

$$Q(x, y) = a(x)y^2 + b(x)y = y[a(x) + b(x)].$$

The algebraic curve would then be decomposed, with one trivial root $Y_0(x) = 0$, $\forall x$. The other root Y_1 satisfies (5.3.1) and $Y_1(1) = 1$.

In the next lemmas, specific properties of the functions $Y_0(x)$ and $Y_1(x)$ will be given. In particular, we will study the transforms $Y_i[x_1x_2]$ and $Y_i[x_3x_4]$, $i = 1, 2$, of the real slits $[x_1x_2]$ and $[x_3x_4]$, which will be shown to be two simple non-intersecting closed curves in the complex plane $\mathbb{C}_y$.

Notation

1. From now on, $\mathbb{C}_x$ [resp. $\mathbb{C}_y$], cut along $[x_1x_2] \cup [x_3x_4]$ [resp. $[y_1y_2] \cup [y_3y_4]$] will be denoted by $\widetilde{\mathbb{C}}_x$ [resp. $\widetilde{\mathbb{C}}_y$]. Also, quite naturally, we shall write $Y(x)$, whenever $x = x_i$, $i = 1, \ldots, 4$.
2. Occasionally, the following convention will be ad libitum employed: $\underleftarrow{\overrightarrow{x_1x_2}}$ will stand for the *contour* $[x_1x_2]$, traversed from x_1 to x_2 along the upper edge of the slit $[x_1x_2]$ and then back to x_1, along the lower edge of the slit. Similarly, $\underrightarrow{\overleftarrow{x_1x_2}}$ is defined by exchanging "upper" and "lower".
3. $$\begin{cases} \mathcal{L} &= Y_0[\underleftarrow{\overrightarrow{x_1x_2}}] = \overline{Y}_1[\underrightarrow{\overleftarrow{x_1x_2}}], \\ \mathcal{L}_{ext} &= Y_0[\underleftarrow{\overrightarrow{x_3x_4}}] = \overline{Y}_1[\underrightarrow{\overleftarrow{x_3x_4}}], \\ \mathcal{M} &= X_0[\underleftarrow{\overrightarrow{y_1y_2}}] = \overline{X}_1[\underrightarrow{\overleftarrow{y_1y_2}}], \\ \mathcal{M}_{ext} &= X_0[\underleftarrow{\overrightarrow{y_3y_4}}] = \overline{X}_1[\underrightarrow{\overleftarrow{y_3y_4}}]. \end{cases}$$
4. For any arbitrary simple closed curve $\mathcal{U}$, $G_{\mathcal{U}}$ [resp. $G^c_{\mathcal{U}}$] will denote the interior [resp. exterior] domain bounded by $\mathcal{U}$, i.e. the domain remaining on the left-hand side when $\mathcal{U}$ is traversed in the positive (counter clockwise) direction. This definition remains valid for the case when $\mathcal{U}$ is unbounded but closed at infinity.

Theorem 5.3.3 *The following topological and algebraic properties hold.*

(i) The curves $\mathcal{L}$ and $\mathcal{L}_{ext}$ (resp. $\mathcal{M}$ and $\mathcal{M}_{ext}$) are simple, closed and symmetrical about the real axis in the $\mathbb{C}_y$ [resp. $\mathbb{C}_x$] plane. They do not intersect if the group of the random walk is not of order 4. When this group is of order 4, $\mathcal{L}$ and $\mathcal{L}_{ext}$ [resp. $\mathcal{M}$ and $\mathcal{M}_{ext}$] coincide and form a circle possibly degenerating into a straight line. In the general case they build the two components (possibly identical, in which case the circle must be counted twice) of a quartic curve. Moreover, setting

$$\Delta = \begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{0,0} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix},$$

we have the following topological invariants:
– If $\Delta > 0$, then

$$[y_1y_2] \subset G_{\mathcal{L}} \subset G_{\mathcal{L}_{ext}} \quad and \quad [y_3y_4] \subset G^c_{\mathcal{L}_{ext}};$$

– If $\Delta < 0$, then

$$[y_1y_2] \subset G_{\mathcal{L}_{ext}} \subset G_{\mathcal{L}} \quad and \quad [y_3y_4] \subset G^c_{\mathcal{L}}.$$

– If $\Delta = 0$, then we already know from Chap. 4 that $G_{\mathcal{L}}$ is a circular domain, and in fact

$$[y_1 y_2] \subset G_{\mathcal{L}} \equiv G_{\mathcal{L}_{ext}} \quad \textit{and} \quad [y_3 y_4] \subset G^c_{\mathcal{L}}.$$

Entirely similar results hold for $\mathcal{M}$, $\mathcal{M}_{ext}$, $[x_1 x_2]$ *and* $[x_3 x_4]$.

(ii) *The functions* Y_i [resp. X_i], $i = 0, 1$, *are meromorphic in the cut plane* $\mathbb{C}_x$ [resp. $\mathbb{C}_y$]. *In addition,*

- Y_0 [resp. X_0] *has two zeros and no poles.*
- Y_1 [resp. X_1] *has two poles and no zeros.*
- $|Y_0(x)| \le |Y_1(x)|$ [resp. $|X_0(y)| \le |X_1(y)|$], *in the whole cut complex plane. Equality holds only on the cuts.*

(iii) *The function* Y_0 *can become infinite at a point* x *if, and only if,*

$$p_{11} = p_{10} = 0, \quad \textit{and then} \quad x = x_4 = \infty,$$

or

$$p_{-11} = p_{-10} = 0, \quad \textit{and then} \quad x = x_1 = 0.$$

In addition, most of the properties cited in (ii) and (iii) hold in the genus zero case. ■

Proof of (i). We have seen in Chap. 2 that $Y(x)$ has four real branch points which satisfy either of the following inequalities

$$-1 < x_1 < x_2 < 1 < x_3 < x_4, \tag{5.3.4}$$

$$x_4 < -1 < x_1 < x_2 < 1 < x_3. \tag{5.3.5}$$

It is important to recall that x_2 and x_3 are always positive. When (5.3.4) holds, the $\mathbb{C}_x$ plane is cut along $[x_1 x_2]$ and $[x_3 x_4]$. On the other hand, in the instance (5.3.5), the cut $[x_3 x_4]$ includes the real point at infinity.

As soon as we have proved that $\mathcal{L} \cup \mathcal{L}_{ext}$ is a quartic, the fact that $\mathcal{L}$ and $\mathcal{L}_{ext}$ are simple curves will become immediate. That they do not intersect in the $\mathbb{C}_y$ plane is less obvious and follows from general properties concerning the existence of double points of quartic curves (see e.g., Hartshorne [52] or Kendig [59]). For the sake of completeness we shall give now a direct analytical demonstration of part (i) for the curves $\mathcal{L}$ and $\mathcal{L}_{ext}$.

Let $y = u + iv$ be a point in the complex plane $\mathbb{C}_y$. For $x \in [x_1 x_2] \cup [x_3 x_4]$, $Y_0(x)$ and $Y_1(x)$ are complex conjugate. Thus

$$Y_0(x) Y_1(x) = \frac{c(x)}{a(x)} = u^2 + v^2,$$

$$Y_0(x) + Y_1(x) = \frac{-b(x)}{a(x)} = 2u.$$

Hence, we get

$$\begin{cases} (u^2 + v^2) a(x) - c(x) = 0, \\ 2u a(x) + b(x) = 0. \end{cases} \tag{5.3.6}$$

The two polynomial equations of second degree in system (5.3.6) must have a common root. Recall briefly that two equations in t, say

$$\begin{cases} \alpha_1 t^2 + \beta_1 t + \gamma_1 = 0, \\ \alpha_2 t^2 + \beta_2 t + \gamma_2 = 0, \end{cases}$$

have a common root if, and only if, their coefficients satisfy the relation

$$(\alpha_1\gamma_2 - \alpha_2\gamma_1)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)(\gamma_1\beta_2 - \gamma_2\beta_1) = 0.$$

We apply this relation to (5.3.6), instantiating

$$\begin{cases} \alpha_1 = (u^2 + v^2)p_{11} - p_{1,-1}, \\ \beta_1 = (u^2 + v^2)p_{01} - p_{0,-1}, \\ \gamma_1 = (u^2 + v^2)p_{-1,1} - p_{-1,-1}, \end{cases} \qquad \begin{cases} \alpha_2 = 2up_{11} + p_{1,0}, \\ \beta_2 = 2up_{01} + (p_{00} - 1), \\ \gamma_2 = 2up_{-1,1} + p_{-1,0}. \end{cases}$$

Then the equation of the quartic curve in the $\mathbb{C}_y$ plane, with coordinates (u, v), reads

$$R^2 + ST = 0, \tag{5.3.7}$$

where

$$\begin{aligned} R &= (u^2 + v^2)(p_{11}p_{-1,0} - p_{-1,1}p_{10}) \\ &\quad + 2u(p_{11}p_{-1,-1} - p_{1,-1}p_{-1,1}) + p_{10}p_{-1,-1} - p_{-1,0}p_{1,-1}, \\ S &= (u^2 + v^2)(p_{11}(p_{00} - 1) - p_{01}p_{10}) \\ &\quad + 2u(p_{11}p_{0,-1} - p_{1,-1}p_{01}) + p_{10}p_{0,-1} + (1 - p_{00})p_{1-1}, \\ T &= (u^2 + v^2)(p_{-1,1}(p_{00} - 1) - p_{01}p_{-1,0}) \\ &\quad + 2u(p_{-1,1}p_{0,-1} - p_{-1,-1}p_{01}) + p_{-1,0}p_{0,-1} + (1 - p_{00})p_{-1,-1}. \end{aligned}$$

As mentioned earlier it is possible, from (5.3.7), to write conditions for this quartic to have no double points. Nevertheless, we shall present thereafter a mild force argument proving this assertion directly.

The curves $\mathcal{L}$ and $\mathcal{L}_{ext}$ are symmetrical about the real axis. Assume for now that they have an intersection or a common vertical tangent at a point of the real axis. Then system (5.3.6) holds for two distinct values of x, say x' and x'', but for one and the same pair $(u, |v|)$. Furthermore, one can always find a triple of complex numbers $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that the polynomial

$$F(x) = \alpha a(x) + \beta b(x) + \gamma c(x)$$

has zeros at x' and at an arbitrary fixed point $x_0 \neq x', x''$. But then $F(x)$ also has a zero at x'', since $\dfrac{b(x')}{a(x')} = \dfrac{b(x'')}{a(x'')}$ and $\dfrac{c(x')}{a(x')} = \dfrac{c(x'')}{a(x'')}$, using (5.3.6).

But then, the equation $F(x) = 0$ would have three distinct roots, which is impossible, since $F(x)$ is a polynomial of second degree in x, unless $F(x) \equiv 0$, which means that $a(x)$, $b(x)$ and $c(x)$ are linearly dependent. Consequently the group of the random walk is of order 4, as shown in Lemma 4.8.2 of Chap. 4 and, in this case, the curves $\mathcal{L}$ and $\mathcal{L}_{ext}$ coincide to form a circle (the quartic is then a circle counted twice).

To prove the last point of *(i)* and *(ii)* in Theorem 5.3.3, it is necessary to know the sign of the quantities $Y(x_i)$, $i = 1, \ldots, 4$. In $\widetilde{\mathbb{C}}_x$, $Y_i(x)$, $i = 0, 1$, are meromorphic. Since, by (5.3.4) and (5.3.5), there are no branch points on the real interval $]x_2, x_3[$ and $Y_0(1) < Y_1(1)$, we have

$$Y_0(x) < Y_1(x), \quad \forall x \in]x_2, x_3[\ , \tag{5.3.8}$$

remembering that Y_0 and Y_1 are real on $]x_2, x_3[$. Moreover, Y_0 and Y_1 have no zeros and no poles on $]x_2, x_3[$, since otherwise either $a(x)$ or $c(x)$ would vanish for positive values of x, which is impossible. This argument shows that Y_i, $i = 0, 1$, does not vanish on $R^+ - \{0 \cup \infty\}$. Thus the quantities $Y_i(x_2)$ and $Y_i(x_3)$ have the same sign, which is that of $Y_i(1)$, i.e. positive, and they are finite.

• *Sign of* $Y(x_1)$ First, we observe that $b(x) = p_{10}x^2 + (p_{00} - 1)x + p_{-10}$ has two real positive roots (one being rejected at infinity for $p_{10} = 0$). This follows from

$$b(0) \geq 0, \quad b(1) < 0, \quad b(+\infty) = +\infty.$$

Moreover, since $b(x) = 0 \Rightarrow b^2(x) - 4a(x)c(x) < 0$, one root belongs to the interval $]x_1 x_2[$ and the second one is on $]x_3 x_4[$, noting that $x_4 = \infty$ if $p_{10} = 0$. Thus $Y(x_1) = \dfrac{-b(x_1)}{2a(x_1)} \leq 0$. The borderline case $Y(x_1) = 0$ is obtained when x_1 is at one and the same time a root of $b(x)$ and of $c(x)$. This implies $x_1 = 0$ and $p_{-1,0} = p_{-1,-1} = 0$.

Another special situation arises when $a(x_1) = b(x_1) = 0$, yielding necessarily $x_1 = 0$ and also $p_{-11} = p_{-10} = 0$. Then $Y(x_1) = \infty$ and we have

$$\Re(Y(0)) = \frac{1 - p_{00}}{2p_{01}} > 0,$$

together with

$$\Im(Y(x)) \sim \frac{\pm 1}{\sqrt{x}} \sqrt{\frac{p_{-1,-1}}{P_{01}}},$$

when $x \to 0_+$, and the curve $\mathcal{L}$ is unbounded.

• *Sign of* $Y(x_4)$ The arguments are quite similar. Now we have

$$Y(x_4) = \frac{-b(x_4)}{2a(x_4)}.$$

– If $1 < x_4 < \infty$, then we get $Y(x_4) < 0$, since $b(x_4) > 0$ and $a(x_4) > 0$, using the fact that the possible real zeros of b are necessarily located on the cuts.

– If $-\infty < x_4 < -1$, then the point at infinity on the real line belongs to the cut $[x_3x_4]$ (using the convention made at the beginning of this section), so that

$$Y_0(\infty) = \overline{Y_1(\infty)}.$$

Thus, provided that $p_{10} + p_{11} \neq 0$,

$$\Re(Y(\infty)) = \frac{-b(\infty)}{2a(\infty)} \leq 0,$$

which yields again

$$\Re(Y(x_4)) = Y(x_4) < 0.$$

The limiting case $x_4 = \infty$ implies necessarily $y_4 < \infty$, together with

$$p_{10} = p_{11} = 0 \quad \text{or} \quad p_{10} = p_{1,-1} = 0,$$

which we analyze separately.

- $p_{10} = p_{11} = 0$. Then $x_4 = \infty$ is a common root of $a(x) = b(x) = 0$ and $Y(x_4) = \infty$, which says that the curve $\mathcal{L}_{ext}$ is unbounded. More precisely

$$\begin{cases} \Re(Y(x_4)) = \dfrac{1 - p_{00}}{2p_{01}} & \text{giving a vertical asymptote of } \mathcal{L}_{ext}, \\[2ex] \Im(Y(x)) \sim \pm\sqrt{\dfrac{xp_{1,-1}}{p_{01}}}, & \text{as } x \to x_4 = \infty. \end{cases}$$

- $p_{10} = p_{1,-1} = 0$. Then $x_4 = \infty$ is a common root of $b(x) = c(x) = 0$ and in this case $Y(x_4) = 0$. The behavior of $Y(x)$ can be obtained exactly as above, and details will be left aside.

It might be useful to recall that we omit all *singular* random walks introduced in Sect. 2.3, which occur when e.g.,

$$p_{11} = p_{10} = p_{1,-1} = 0\,.$$

Note also the amusing case

$$p_{11} = p_{10} = p_{-10} = p_{-11}\,,$$

yielding a group of order 4, for which $\mathcal{L} \equiv \mathcal{L}_{ext}$ is simply the vertical line of abscissa $\dfrac{1 - p_{00}}{2p_{01}}$.

Now is the right moment to deal with statement *(iii)* of the theorem, since it overlaps what has been done just above.
Proof of (iii). Assume there exists an x with $|Y_0(x)| = \infty$. Then, anticipating slightly the results of *(ii)*, it follows that $|Y_1(x)| = \infty$. Since

$$Y_0(x) + Y_1(x) = \frac{-b(x)}{a(x)} \quad \text{and} \quad Y_0(x)Y_1(x) = \frac{c(x)}{a(x)},$$

we must have $a(x) = b(x) = 0$, so that either $x = x_1$ or $x = x_4$, because the roots of a (resp. b) cannot be positive (resp. negative).

- $a(x_1) = b(x_1) = 0$. Then $p_{-11} = p_{-10} = 0$ and the curve $\mathcal{L}$ is unbounded.
- $a(x_4) = b(x_4) = 0$. Then $p_{11} = p_{10} = 0$. This case was encountered above, implying $x_4 = \infty$ and the curve $\mathcal{L}_{ext}$ is unbounded.

This result will be used in Chap. 6 for the genus 0 case.

We continue with the last part of assertion *(i)* in the theorem, by using arguments similar to those of Sect. 2.3, namely the continuity with respect to the parameters p_{ij}. First let us denote by $\mathcal{A}$ the set of points in the simplex $\mathcal{P}$ for which $M_x = M_y = 0$. This set $\mathcal{A}$ subdivides $\wp$ into four convex sets $\mathcal{A}_{ij}$, for $i, j = 0, 1$. Now consider the hypersurface $\mathcal{H}$

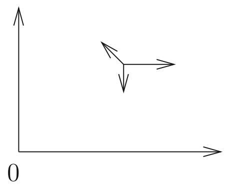
$$\Delta = \begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0.$$

$\mathcal{H}$ divides each $\mathcal{A}_{ij}$ into two pathwise connected subsets $\mathcal{A}_{ij}^{+}$ and $\mathcal{A}_{ij}^{-}$, for which $\Delta > 0$ and $\Delta < 0$, respectively.

It now suffices to check three special cases for which, respectively, $\Delta = 0$, $\Delta > 0$ and $\Delta < 0$:

(a) The usually called *simple* random walk. Here we know that $\Delta = 0$. Moreover, $\mathcal{L} \equiv \mathcal{L}_{ext}$ is the circle centered at the origin, with radius $\sqrt{\frac{p_{0,-1}}{p_{0,1}}}$. It is not difficult to check that the slit $[y_1 y_2]$ (resp. $[y_3 y_4]$) always lies inside (resp. outside) this circle, whenever M_x and M_y do not simultaneously vanish.

(b)



This random walk can depict two $M/M/1$ queues in tandem and its corresponding group is of order 6. Then

$$\Delta = \begin{vmatrix} 0 & p_{10} & 0 \\ 0 & -1 & p_{0-1} \\ p_{-11} & 0 & 0 \end{vmatrix} = p_{-11} p_{10} p_{0-1} > 0.$$

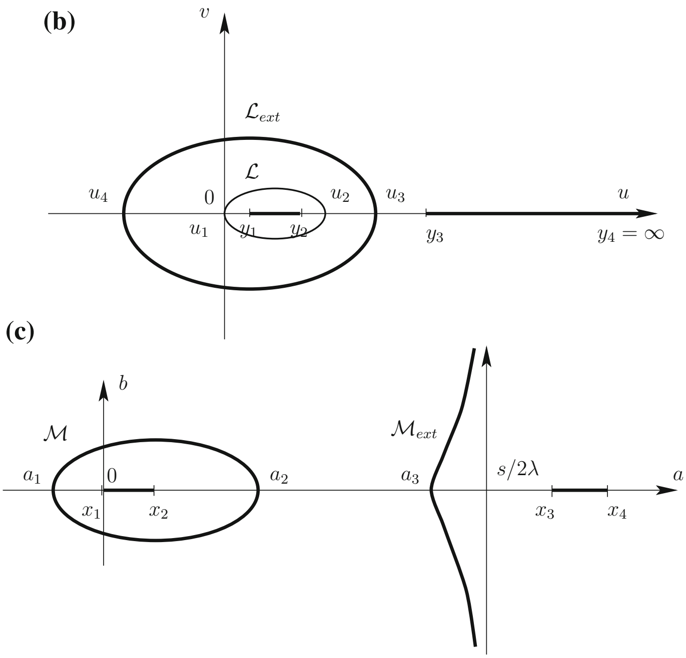
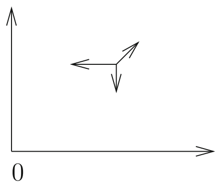


Fig. 5.2 Mappings of the cuts in cases (**b**), (**c**)

Assuming M_x or M_y are different from zero (i.e. genus 1), the curves $\mathcal{L}$, $\mathcal{L}_{ext}$ are shown in Fig. 5.2 $G_{\mathcal{L}} \subset G_{\mathcal{L}_{ext}}$. Here $[y_1 y_2] \subset G_{\mathcal{L}}$ and $[y_3, \infty] \subset G^c_{\mathcal{L}_{ext}}$. Analogous results hold for $\mathcal{M}$ and $\mathcal{M}_{ext}$ in the complex plane $\mathbb{C}_x$.

(c)



Here $\Delta = \begin{vmatrix} p_{11} & 0 & 0 \\ 0 & -1 & p_{0,-1} \\ 0 & p_{-10} & 0 \end{vmatrix} = -p_{11}p_{0,-1}p_{-10} < 0$.

This random walk models a two-server queueing system with parallel arrivals. It was already presented in Sect. 4.10 and also has a group of order 6. Here we have $p_{10} = p_{1-1} = 0$, so that, from the preceding analysis, $x_4 = \infty$ and $Y(x_4) = 0$, assuming again $M_x \neq 0$ or $M_y \neq 0$. As shown in Fig. 5.2, $G_{\mathcal{L}_{ext}} \subset G_{\mathcal{L}}$, $[y_1 y_2] \subset G_{\mathcal{L}_{ext}}$ and $[y_3 y_4] \subset G^c_{\mathcal{L}}$.

Returning to the general random walk, we note that any point $\rho \in \mathcal{A}^+_{ij}$ can be connected along a continuous path $\ell \subset \mathcal{A}^+_{ij}$ to some point $\rho_0 \in \mathcal{A}^+_{ij}$ corresponding to the particular random walk (b). But the respective positions of the curves and the slits can be derived from the argument principle applied to the function

$$\widetilde{D}(y) = \widetilde{b}^2(y) - 4\widetilde{a}(y)\widetilde{c}(y)$$

along the curve $\mathcal{L}_{ext}$. Since $[\arg \widetilde{\mathcal{D}}(y)]_{\mathcal{L}_{ext}}$ is a continuous function of the p_{ij}'s, taking discrete values in the region $\mathcal{A}_{ij}^{+}$, we conclude that, whenever $\Delta > 0$ (in the genus 1 case), $[y_1 y_2] \subset G_{\mathcal{L}} \subset G_{\mathcal{L}_{ext}}$ and $[y_3 y_4] \subset G^c_{\mathcal{L}_{ext}}$.

The case $\Delta < 0$, using the regions $\mathcal{A}_{ij}^{-}$ and the random walk (c) can be treated in a similar way. This concludes part *(i)* of Theorem 5.3.3.

Remark 5.3.4 As shown in Sect. 2.5, the curves $\mathcal{L}$, $\mathcal{L}_{ext}$, $\overleftrightarrow{y_1 y_2}$ and $\overleftrightarrow{y_3 y_4}$ belong to the same homotopy class on the Riemann surface $\mathbf{S}$. This indicates a priori that the curves $\mathcal{L}$ or $\mathcal{L}_{ext}$ contain in their interior at most one slit.

Proof of (ii) We have already shown in the proof of *(i)* that $Y(x_2)$ is positive. Owing to (5.3.8), it follows that $Y_0[\overleftrightarrow{x_1 x_2}]$ is the curve $\mathcal{L}$ traversed in the positive direction. Since on $[x_1 x_2]$, $Y_1(x)$ is the conjugate of $Y_0(x)$, $Y_1[\overleftrightarrow{x_1 x_2}]$ is the curve $\mathcal{L}$ traversed in the negative (clockwise) direction. Similar conclusions hold for the curve $\mathcal{L}_{ext}$. It follows that

$$\begin{cases} \mathcal{I}nd[Y_0(x)]_{\{\overleftrightarrow{x_1 x_2} \bigcup \overleftrightarrow{x_3 x_4}\}} = 2, \\ \mathcal{I}nd[Y_1(x)]_{\{\overleftrightarrow{x_1 x_2} \bigcup \overleftrightarrow{x_3 x_4}\}} = -2. \end{cases} \tag{5.3.9}$$

The functions $Y_i(x)$, $i = 1, 2$, are meromorphic in the cut plane and their poles and zeros are the zeros of $a(x)$ and $c(x)$. Thus it follows from (5.3.9) that $Y_0(x)$ has two zeros and no pole and, conversely, that $Y_1(x)$ has two poles and no zero. On the other hand, from the proof of part *(i)*, Lemma 5.3.1 and the fact that

$$\left|\frac{Y_0(x)}{Y_1(x)}\right| \le 1, \quad x \in \Gamma, \text{ used in particular at } x = 1 \text{ and } x = -1,$$

we have

$$\lim_{x\to\infty} \left|\frac{Y_0(x)}{Y_1(x)}\right| \le 1, \quad \text{with equality only if} \quad x_4 = \infty.$$

Applying now the maximum modulus principle to the function $\frac{Y_0(x)}{Y_1(x)}$, which is holomorphic in the complex plane cut along $[x_1 x_2] \cup [x_3 x_4]$, we immediately get the last assertion of *(ii)*. The proof of Theorem 5.3.3 is complete. ■

Corollary 5.3.5 *1.* $G_{\mathcal{M}} - [x_1 x_2] \underset{X_0(y)}{\overset{Y_0(x)}{\rightleftarrows}} G_{\mathcal{L}} - [y_1 y_2]$ *and the mappings are conformal.*

2. *The values of* Y_0 *belong to* $G_{\mathcal{L}} \bigcup G_{\mathcal{L}_{ext}}$.
3. *The values of* Y_1 *belong to* $G^c_{\mathcal{L}} \bigcup G^c_{\mathcal{L}_{ext}}$.

4. *The singular curves of the composed functions* $X_i \circ Y_j$ *are listed hereafter:*

$$
\begin{aligned}
X_0 \circ Y_0 &: \mathcal{M} \bigcup [x_3 x_4],\\
X_1 \circ Y_0 &: \mathcal{M} \bigcup [x_1 x_2],\\
X_0 \circ Y_1 &: \mathcal{M}_{ext} \bigcup [x_3 x_4],\\
X_1 \circ Y_1 &: \mathcal{M}_{ext} \bigcup [x_1, x_2].
\end{aligned}
$$

Moreover, when $G_{\mathcal{M}} \subset G_{\mathcal{M}_{ext}}$, *the following automorphy relationships hold.*

$$
X_0 \circ Y_0(t) = \begin{cases} t, & \text{if } t \in G_{\mathcal{M}},\\ \neq t, & \text{if } t \in G^c_{\mathcal{M}}. \text{ Then } X_0 \circ Y_0(G^c_{\mathcal{M}}) = G_{\mathcal{M}}. \end{cases}
$$

$$
X_1 \circ Y_0(t) = \begin{cases} t, & \text{if } t \in G^c_{\mathcal{M}},\\ \neq t, & \text{if } t \in G_{\mathcal{M}}. \text{ Then } X_1 \circ Y_0(G_{\mathcal{M}}) = G^c_{\mathcal{M}}. \end{cases}
$$

$$
X_0 \circ Y_1(t) = \begin{cases} t, & \text{if } t \in G_{\mathcal{M}_{ext}},\\ \neq t, & \text{if } t \in G^c_{\mathcal{M}_{ext}}. \text{ Then } X_0 \circ Y_1(G^c_{\mathcal{M}_{ext}}) = G_{\mathcal{M}_{ext}}. \end{cases}
$$

$$
X_1 \circ Y_1(t) = \begin{cases} t, & \text{if } t \in G^c_{\mathcal{M}_{ext}},\\ \neq t & \text{if } t \in G_{\mathcal{M}_{ext}}. \text{ Then } X_1 \circ Y_1(G_{\mathcal{M}_{ext}}) = G^c_{\mathcal{M}_{ext}}. \end{cases}
$$

Proof Assertion 1 follows directly from Theorem 5.3.3 and so do 2 and 3, by application of the maximum modulus principle to the functions $Y_0(x)$ and $\dfrac{1}{Y_1(x)}$ respectively. The last assertion can be checked up to some tedious verifications which will be omitted. The proof of the corollary is complete. ■

Clearly, Corollary 5.3.5 enables us to analytically continue the functions π and $\widetilde{\pi}$. This yields another derivation of Theorem 3.2.2, starting from Eq. (5.1.2).

5.4 Index and Solution of the BVP (5.1.5)

As it emerges from Sect. 5.2, a fundamental quantity in the solution of BVPs is the *index*. Problem (5.1.5) has been obtained from (5.1.3) and we have to combine the two basic constraints imposed on π and $\widetilde{\pi}$, i.e. they must be holomorphic inside their respective unit disc $\mathcal{D}$ and continuous on the boundary Γ (the unit circle). This is to say that in (5.1.5) π might be meromorphic in $G_{\mathcal{M}} \backslash \overline{\mathcal{D}}$ (see Sect. 5.3 for the notation G_U). This leads to the notion of *reduced index*, which is the index of (5.1.5) subject to the constraints on π and $\widetilde{\pi}$ simultaneously. We shall proceed in three steps.

Notation All parameters have to be taken from (5.1.2). For any continuous non-vanishing function $f(t)$ given on a contour $\mathcal{C}$, let $N_Z[f, G]$ denote the number of zeros of $f(t)$ in the open domain G having $\mathcal{C}$ as its boundary.

Theorem 5.4.1 *Let us introduce the following two quantities:*

$$\delta \stackrel{def}{=} \begin{cases} 0, \text{ if } Y_0(1) < 1 \text{ or } \left\{ Y_0(1) = 1 \text{ and } \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} > 0 \right\}, \\ 1, \text{ if } Y_0(1) = 1 \text{ and } \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} < 0. \end{cases} \tag{5.4.1}$$

$$\widetilde{\delta} \stackrel{def}{=} \begin{cases} 0, \text{ if } X_0(1) < 1 \text{ or } \left\{ X_0(1) = 1 \text{ and } \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} > 0 \right\}, \\ 1, \text{ if } X_0(1) = 1 \text{ and } \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} < 0. \end{cases} \tag{5.4.2}$$

Then the functional equation (5.1.4), with boundary condition (5.1.5), reduces to the BVP (5.4.19), the index of which, with the notation (5.2.7)*, is given by*

$$\widetilde{\chi} = -\mathcal{I}nd[K(t)]_{\mathcal{M}} = -L - M + \delta + \widetilde{\delta} - \mathbb{1}_{\{X_0(1)=1, Y_0(1)=1\}}. \tag{5.4.3}$$

Moreover, (5.1.1) admits a probabilistic solution if, and only if,

$$\delta + \widetilde{\delta} = \mathbb{1}_{\{X_0(1)=1, Y_0(1)=1\}} + 1, \tag{5.4.4}$$

which is equivalent to the ergodicity conditions listed in Theorem 1.2.1. ■

Lemma 5.4.2

$$\begin{aligned} \widetilde{\chi}_1 &\stackrel{def}{=} -\mathcal{I}nd[A(t)]_{\mathcal{M}} = -\mathcal{I}nd[q(x, Y_0(x))]_{\mathcal{M}} + \mathcal{I}nd[\widetilde{q}(X_0(y), y)]_{\underset{\longleftarrow}{\overrightarrow{[y_1 y_2]}}} \\ &= \delta + \widetilde{\delta} - L - M - N_Z[q(x, Y_0(x)), G_{\mathcal{M}}] + N_Z[q(x, Y_0(x)), \mathcal{D}] \\ &\quad + N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}]. \end{aligned} \tag{5.4.5}$$

■

Proof As announced at the beginning of Sect. 5.3, the given functions $q(x, y)$ and $\widetilde{q}(x, y)$ are assumed to have suitable analytic continuations with respect to x [resp. y] in the domains $G_{\mathcal{M}}$ [resp. $G_{\mathcal{L}}$]. Then the first equality in (5.4.5) is a direct consequence of the definition of $A(t)$ and of the first *automorphy* property given in Corollary 5.3.5. To establish (5.4.5) we shall compute separately the two terms appearing in the right-hand side, mainly using the argument principle. We have

$$\mathcal{I}nd[\widetilde{q}(X_0(y), y)]_{\underset{\longleftarrow}{\overrightarrow{[y_1 y_2]}}} = -M + N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}] + \widetilde{\delta}, \tag{5.4.6}$$

where $\widetilde{\delta}$ has been defined in (5.4.2). Equation (5.4.6) follows from the argument principle and from the properties of $X_0(y)$.

- When $\widetilde{\delta} = 0$, the same argument shows that, for $\varepsilon > 0$ sufficiently small,

$$N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}] = N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}_{1-\varepsilon}] = M,$$

because on $\Gamma_{1-\varepsilon}$, which denotes here the unit circle with a small indentation to the left of the point $y = 1$, we have

$$\Re\left[\frac{\widetilde{q}(X_0(y), y)}{y^M}\right] < 0, \text{ so that } \mathcal{I}nd\left[\frac{\widetilde{q}(X_0(y), y)}{y^M}\right]_{\Gamma_{1-\varepsilon}} = 0.$$

- When $\widetilde{\delta} = 1$, the same argument shows that

$$N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}] = N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}_{1+\varepsilon}] - 1 = M - 1,$$

using now the circle $\Gamma_{1+\varepsilon}$, having an indentation to the right of the point $y = 1$.

Thus (5.4.6) is proved.

Continuing with the proof of (5.4.5), we have

$$\mathcal{I}nd[q(x, Y_0(x))]_{\mathcal{M}} + \mathcal{I}nd[q(x, Y_0(x))]_{\underset{\longleftarrow}{\overrightarrow{[x_1 x_2]}}} = N_Z[q(x, Y_0(x)), G_{\mathcal{M}}]. \tag{5.4.7}$$

Up to an exchange of the variables in (5.4.6), we can also write

$$\mathcal{I}nd[q(x, Y_0(x))]_{\underset{\longleftarrow}{\overrightarrow{[x_1 x_2]}}} = -L + N_Z[q(x, Y_0(x)), \mathcal{D}] + \delta, \tag{5.4.8}$$

where δ has been defined in (5.4.1). Putting together (5.4.6)–(5.4.8) we obtain (5.4.5) and the proof of Lemma 5.4.2 is complete. ■

The function π is sought to be holomorphic in $\mathcal{D}$ and continuous in $\overline{\mathcal{D}}$, but might have poles in $G_{\mathcal{M}} \cap (\overline{\mathcal{D}})^c$. From Corollary 5.3.5, we know that

$$|Y_0(x)| \leq 1, \text{ for } x \in G_{\mathcal{M}} \cap (\overline{\mathcal{D}})^c,$$

this being a consequence of the maximum modulus principle. Hence, rewriting (5.1.3) as

$$q(x, Y_0(x))\pi(x) + \widetilde{q}(x, Y_0(x))\widetilde{\pi}(Y_0(x)) + \pi_0(x, Y_0(x)) = 0,$$

one sees that the possible poles of $\pi(x)$ in $G_{\mathcal{M}} \cap (\overline{\mathcal{D}})^c$ are necessarily zeros of $q(x, Y_0(x))$ in this region, as $\widetilde{\pi}(y)$ is holomorphic in $\mathcal{D}$ and continuous in $\overline{\mathcal{D}}$. Setting

$$\begin{cases} S(x) = \displaystyle\prod_{r_j \in G_{\mathcal{M}} \cap (\overline{\mathcal{D}})^c} (x - r_j), \\ \pi(x) = \dfrac{\Phi(x)}{S(x)}, \quad G(x) = \dfrac{A(x)}{S(x)}, \end{cases} \tag{5.4.9}$$

where the r_j's stand for all the zeros of $q(x, Y_0(x))$ in $G_{\mathcal{M}} \cap (\overline{\mathcal{D}})^c$, the BVP (5.1.5) takes the form

$$\Phi(t)G(t) - \Phi(\alpha(t))G(\alpha(t)) = g(t), \quad t \in \mathcal{M} \tag{5.4.10}$$

where now $\Phi(x)$ is required to be holomorphic in $G_{\mathcal{M}}$. Since

$$\mathcal{I}nd[G(x)]_{\mathcal{M}} = \mathcal{I}nd[A(x)]_{\mathcal{M}} - N_Z[q(x, Y_0(x)), G_{\mathcal{M}} \cap (\overline{\mathcal{D}})^c],$$

the index of (5.4.10) reads, according to (5.2.43) and (5.4.5),

$$\begin{aligned}\widetilde{\chi}_2 \stackrel{\text{def}}{=} -\mathcal{I}nd[G(x)]_{\mathcal{M}} &= +\delta + \widetilde{\delta} - L - M - \mathbb{1}_{\{1 \in G_{\mathcal{M}}, Y_0(1)=1\}} \\ &+ N_Z[q(x, Y_0(x)), \mathcal{D} \cap (G^c_{\mathcal{M}})] + N_Z(\widetilde{q}(X_0(y), y), \mathcal{D}].\end{aligned} \tag{5.4.11}$$

To proceed further with the *reduction* of the index, we must introduce two additional pieces of information:

(i) First, whenever

$$\widetilde{q}(X_0(y), y) = 0, \quad y \in \overline{\mathcal{D}},$$

the product $\widetilde{q}(X_0(y), y)\widetilde{\pi}(y)$ appearing in (5.1.3) vanishes. Now let us introduce

$$\psi_1(x) = \sum_k a_k \frac{\pi(x) - \pi(X_0(u_k))}{x - X_0(u_k)}, \tag{5.4.12}$$

where the a_k's are coefficients to be specified later, and the summation is taken over the k's such that

$$\widetilde{q}(X_0(u_k), u_k) = 0, \quad (X_0(u_k), u_k) \in \mathcal{D} \times \mathcal{D},$$

with the correct multiplicities. Clearly, $\psi_1(x)$ is holomorphic in the neighborhood of $x = X_0(u_k)$. Moreover in (5.4.12) we have from (5.1.3),

$$\pi(X_0(u_k)) = \frac{-\pi_0(X_0(u_k), u_k)}{q(X_0(u_k), u_k)}, \tag{5.4.13}$$

which gives the $\pi(X_0(u_k))$ in terms of the $\pi_0(X_0(u_k), u_k)$, i.e. linearly with respect to the $L + M - 1$ constants defining $\pi_0(x, y)$ [see Eq. (5.1.2)].

(ii) Secondly, to take into account the zeros of $q(x, Y_0(x))$ in the region $\mathcal{D} \cap G^c_{\mathcal{M}}$, one can start from (5.1.4), which, by analytic continuation, is indeed valid for all y in the complex plane. In particular, it is possible to replace in (5.1.4) y by $Y_0(x)$, so that (5.1.4) takes the form

$$\pi(X_0 \circ Y_0(x))A(X_0 \circ Y_0(x)) - \pi(x)A(x) = g(x), \quad \text{for } x \in \overline{\mathcal{D}} \cap G^c_{\mathcal{M}}. \tag{5.4.14}$$

To obtain (5.4.14) we have replaced in (5.1.4) y by $Y_0(x)$ and have taken into account the automorphy relation of Corollary 5.3.5 $X_1 \circ Y_0(x) = x$, for $x \in G^c_{\mathcal{M}}$. Thus, since by (5.1.6)

$$A(x) = \frac{q(x, Y_0(x))}{\widetilde{q}(x, Y_0(x))},$$

we get from (5.4.14) the additional relationships

$$\pi(X_0 \circ Y_0(v_k)) = \frac{g(v_k)}{A(X_0 \circ Y_0(v_k))}, \tag{5.4.15}$$

for any $v_k \in \mathcal{D} \cap G^c_{\mathcal{M}}$, such that $q(v_k, Y_0(v_k)) = 0$. Then, proceeding as in the derivation of (5.4.12), we set

$$\begin{aligned}\psi(x) &= \psi_1(x) + \sum_{\ell} b_l \frac{\pi(x) - \pi(X_0 \circ Y_0(v_\ell))}{x - X_0 \circ Y_0(v_\ell)} \\ &= \sum_k a_k \frac{\pi(x) - \pi(X_0(u_k))}{x - X_0(u_k)} + \sum_l b_l \frac{\pi(x) - \pi(X_0 \circ Y_0(v_\ell))}{x - X_0 \circ Y_0(v_\ell)},\end{aligned} \tag{5.4.16}$$

where the coefficients a_k and b_l have to be properly chosen.

Now inverting (5.4.16) yields

$$\pi(x) = \psi(x)\frac{R(x)}{P(x)} + T(x), \tag{5.4.17}$$

where, using (5.4.13) and (5.4.15),

$$\begin{cases} R(x) = \prod_{k,\ell}(x - X_0(u_k))(x - X_0 \circ Y_0(v_\ell)), \\ T(x) = \dfrac{R(x)}{P(x)}\left[\sum_{\ell} \dfrac{b_l g(v_\ell)}{(x - X_0 \circ Y_0(v_\ell))A(X_0 \circ Y_0(v_\ell))} - \sum_k \dfrac{a_k \pi_0(X_0(u_k), u_k)}{(x - X_0(u_k))q(X_0(u_k), u_k)}\right], \end{cases} \tag{5.4.18}$$

and $P(x)$ is a polynomial depending linearly on the coefficients a_k and b_l, which are chosen to be real to ensure that $P(x)$ *does not vanish in the domain* $G_{\mathcal{M}} \bigcup \mathcal{D}$. This can clearly be achieved in many ways.

Now joining together (5.4.9), (5.4.10), (5.4.17) and (5.4.18), we obtain the final *reduced* BVP

$$\boxed{\rho(t)K(t) - \rho(\alpha(t))K(\alpha(t)) = k(t), \quad t \in \mathcal{M},} \tag{5.4.19}$$

with

$$\begin{cases} \rho(t) = S(t)\psi(t), \quad K(t) = \dfrac{G(t)R(t)}{P(t)}, \\ k(t) = g(t) + T(\alpha(t))A(\alpha(t)) - T(t)A(t), \\ \alpha(t) = \bar{t}. \end{cases} \tag{5.4.20}$$

In (5.4.19), ρ is sought to be holomorphic in $G_{\mathcal{M}} \bigcup \mathcal{D}$, and all the assumptions concerning π and $\widetilde{\pi}$ have been used. This explains the expression *final reduced index* employed above.

The existence and uniqueness of a solution of (5.4.19) are strictly equivalent to the ergodicity of the random walk defined in Chap. 1, the invariant measure of which satisfies (5.1.1). According to (5.2.43), the index of (5.4.19) is given by

$$\begin{aligned} \mathcal{I}nd[K(t)]_{\mathcal{M}} &= \mathcal{I}nd[G(t)]_{\mathcal{M}} + N_Z[R(x), G_{\mathcal{M}}] \\ &= \mathcal{I}nd[G(t)]_{\mathcal{M}} + N_Z[q(x, Y_0(x)), \mathcal{D} \cap G^c_{\mathcal{M}}] + N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}], \end{aligned}$$

which, using (5.4.11), is exactly the value announced in (5.4.3). The first part of Theorem 5.4.1 is proved.

To derive (5.4.4), note first that, since $L, M \geq 1$, (5.4.3) ensures $\widetilde{\chi} \leq -1$. Thus, by Theorem 5.2.8, a solution of (5.4.19) exists and is unique if, and only if,

$$L + M - \delta - \widetilde{\delta} + \mathbb{1}_{\{X_0(1)=1, Y_0(1)=1\}} - 1$$

conditions of the form (5.2.13) are satisfied. These conditions are linear and involve only the $L+M-1$ unknowns appearing in the definition of $\pi_0(x, y)$. Observing that the existence (or non-existence) of an invariant measure for a Markov chain is not subject to the modification of a *finite* number of parameters (e.g., transition probabilities) of the chain, one can always assume that the conditions are independent. Also taking into account the normalizing condition, (which says that the invariant distribution is *proper*), we will get a non-homogeneous linear system of $L + M - 1$ equations with $L + M - 1$ unknowns, if, and only if, Eq. (5.4.4) holds. Moreover, Lemma 2.2.1 applies. The proof of Theorem 5.4.1 is complete. ■

Theorem 5.4.3 *Under the condition (5.4.4), the function π is given by*

$$\pi(x) = \frac{R(x)H(x)}{2i\pi P(x)S(x)} \int_{\mathcal{M}_d} \frac{k(t)w'(t)dt}{H^+(t)K(\bar{t})(w(t) - w(x))} + T(x), \ \forall x \in G_{\mathcal{M}}, \tag{5.4.21}$$

where

(i) $\mathcal{M}_d$ denotes the portion of the curve $\mathcal{M}$ located in the lower half-plane $\Im z \leq 0$;
(ii) k and K have been introduced in (5.4.20);
(iii) w is solution of the BVP (5.2.39) on the curve $\mathcal{M}$ (see Theorem 5.2.7);

(iv)

$$
\begin{aligned}
H(t) &= (w(t) - X_0(y_2))^{-\widetilde{\chi}} e^{\Gamma(t)}, \ t \in G_{\mathcal{M}}, \\
\Gamma(t) &= \frac{1}{2i\pi} \int_{\mathcal{M}_d} \log \frac{K(\overline{s})}{K(s)} \frac{w'(s)ds}{w(s) - w(t)}, \ t \in G_{\mathcal{M}}, \\
H^+(t) &= (w(t) - X_0(y_2))^{-\widetilde{\chi}} e^{\Gamma^+(t)}, \ t \in \mathcal{M}_d, \\
\Gamma^+(t) &= \frac{1}{2} \log \frac{K(\bar{t})}{K(t)} + \frac{1}{2i\pi} \int_{\mathcal{M}_d} \log \frac{K(\overline{s})}{K(s)} \frac{w'(s)ds}{w(s) - w(t)}, \ t \in \mathcal{M}_d \ .
\end{aligned}
$$

Proof This is just a direct consequence of (5.2.44). ■

5.5 Complements

5.5.1 Analytic Continuation

In fact the functional equation (5.1.5) can in turn be used to analytically continue π, in a finite number of steps, to the whole complex plane cut along $[x_3 x_4]$. Without going into detail, simply observe that the process consists in making use of the automorphy relationships established in Corollary 5.3.5. Consequently, it is also possible to set a BVP on the curve $\mathcal{M}_{ext}$, provided that the generating functions of the jumps on the axes have suitable continuations. An example of this possibility is given in [33].

5.5.2 Computation of w

There are three main possible ways to obtain the function w, which realizes the conformal mapping of $G_{\mathcal{M}}$ onto the complex plane cut along an arc.

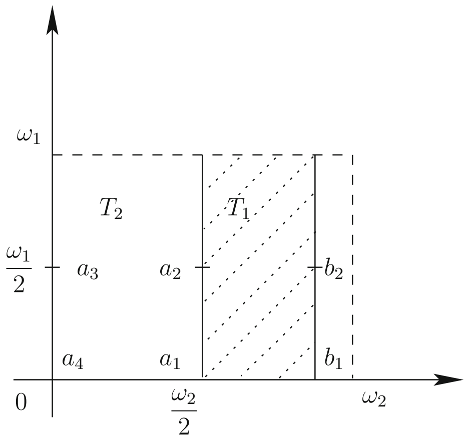
5.5.2.1 An Explicit Form via the Weierstrass $\wp$-Function

Any solution pair (x, y) of the algebraic equation $Q(x, y) = 0$ corresponds to exactly one point s on the Riemann surface **S**. This mapping will be denoted by $(x(s), y(s))$. Indeed, we refer to the uniformization of the algebraic curve, proposed in Chap. 3. It turns out that the surface **S** can be viewed as a semi-open rectangle $[0, \omega_2[\times[0, \omega_1[$, representing the algebraic function $Y(x)$ and shown in Fig. 5.3.

The affixes of $a_i, b_i, i = 1, \dots, 4$ have been determined in Chap. 3: they correspond to (3.1.7)–(3.1.9), and we know that

$$
\omega_3 = 2(b_1 - a_1).
$$

Fig. 5.3 The algebraic surface **S** in the $\mathbb{C}_\omega$-plane



The sheet $\mathbf{T_1}$ in Fig. 5.3 is that of the branch $Y_0(x)$, since it contains the point b_1, which on $\mathbf{S}$ corresponds to $(X(y_1), Y_0(X(y_1)))$, since

$$Y_0(X(y_i)) = y_i, \quad i = 1, 2.$$

We are looking for the function w (appearing in the solution of the BVP presented in the preceding sections), which is defined in the complex plane $\mathbb{C}_x$, holomorphic inside $G_{\mathcal{M}}$, except at one point where it has a simple pole, and subject to the *gluing* condition

$$w(x) = w(\overline{x}), \quad \forall x \in \mathcal{M}. \tag{5.5.1}$$

For $x \equiv x(\omega)$, the homographic function of $\wp$ in (3.3.3), we shall write $w \circ x \stackrel{\text{def}}{=} \widetilde{w}$. The problem for $\widetilde{w}$ can be stated as follows. *Find a function* $\widetilde{w}$ *holomorphic in the shaded area represented in Fig.* 5.3, *in the interior of* $\mathbf{T_1}$, *having one pole (which can be chosen arbitrarily, but not on the segment* $[b_1, b_1 + \omega_1[$), *and satisfying the next two conditions:*

(i) For $\omega \in [b_1, b_1 + \omega_1[$, $\widetilde{w}$ *glues* the two edges of the cut $[y_1, y_2]$, which reads

$$\widetilde{w}\left(b_1 + \frac{\omega_1}{2} + u\right) = \widetilde{w}\left(b_1 + \frac{\omega_1}{2} - u\right), \quad u \in \left]-\frac{\omega_1}{2}, \frac{\omega_1}{2}\right[\tag{5.5.2}$$

and is nothing else but (5.5.1);

(ii) $\widetilde{w}$ also glues the edges of the cut $[x_1, x_2]$, which is located inside $G_{\mathcal{M}}$ in $\mathbb{C}_x$. Thus

$$w^+(x) = w^-(x), \quad \forall x \in [x_1, x_2],$$

or, equivalently,

$$\widetilde{w}(a_2+u)=\widetilde{w}(a_2-u), \quad u\in \left]-\frac{\omega_1}{2}, \frac{\omega_1}{2}\right[. \tag{5.5.3}$$

Let $\psi(\omega) \stackrel{\text{def}}{=} \widetilde{w}(a_2+\omega)$. Since $b_1+\dfrac{\omega_1}{2}=a_2+\dfrac{\omega_3}{2}$, the two conditions (5.5.2) and (5.5.3) can be rewritten as

$$\psi(\omega)=\psi(-\omega), \tag{5.5.4}$$

$$\psi\left(\omega+\frac{\omega_3}{2}\right)=\psi\left(-\omega+\frac{\omega_3}{2}\right), \tag{5.5.5}$$

where ψ has to be holomorphic in $\left]\dfrac{-\omega_1}{2}, \dfrac{\omega_1}{2}\right[\times\left[0, \dfrac{\omega_3}{2}\right[$. Equation (5.5.4) permits us to continue ψ to the rectangle $\left]\dfrac{-\omega_1}{2}, \dfrac{\omega_1}{2}\right[\times\left[\dfrac{-\omega_3}{2}, 0\right]$, as a meromorphic function, so that ψ is even in $\left]\dfrac{-\omega_1}{2}, \dfrac{\omega_1}{2}\right[\times\left[\dfrac{-\omega_3}{2}, \dfrac{\omega_3}{2}\right[$, where it has two simple poles symmetric with respect to the origin. Hence, using (5.5.5), we can write

$$\psi(u)=\psi(u+\omega_1)=\psi(u+\omega_3), \quad \forall u\in \left]\frac{-\omega_1}{2}, \frac{\omega_1}{2}\right[\times\left[\frac{-\omega_3}{2}, \frac{\omega_3}{2}\right[. \tag{5.5.6}$$

Now the relations given in (5.5.6) allow us to continue ψ to the whole complex plane $\mathbb{C}$, as a doubly-periodic meromorphic function $\psi(u)$, with periods ω_1 and ω_3. Moreover, in each rectangle, the two poles can be taken to coincide at the center $u=0$ of such a rectangle. Finally, we have proved that an admissible choice for ψ is simply

$$\psi(u)=\wp(u;\omega_1,\omega_3).$$

5.5.2.2 A Differential Equation

The results obtained in the preceding subsection yield directly

$$\begin{aligned}\frac{d\widetilde{w}}{d\omega}&=\wp'\left(\omega-\frac{\omega_1+\omega_2}{2};\omega_1,\omega_3\right)\\ &=\sqrt{(\widetilde{w}(\omega)-e_1)(\widetilde{w}(\omega)-e_2)(\widetilde{w}(\omega)-e_3)},\end{aligned}$$

where

$$e_1=\wp\left(\frac{\omega_1}{2};\omega_1,\omega_3\right), \quad e_2=\wp\left(\frac{\omega_1+\omega_3}{2};\omega_1,\omega_3\right), \quad e_3=\wp\left(\frac{\omega_1+\omega_2+\omega_3}{2};\omega_1,\omega_3\right).$$

Also,

$$\frac{dw}{dx} = \frac{C}{\sqrt{D(x)}}\,,$$

where $D(x)$ is the discriminant in the uniformization formulas (see Chap. 3), so that, choosing $C = 1$, w satisfies the differential equation

$$\frac{dw}{dx} = \frac{\sqrt{(w-e_1)(w-e_2)(w-e_3)}}{\sqrt{D(x)}}\,.$$

5.5.2.3 An Integral Equation

According to Theorem 5.2.7, we know that w is given by a Cauchy type integral (5.2.38), the density of which satisfies the quasi-Fredholm integral equation (5.2.35). The derivative $\alpha'(t)$, which appears in the kernel of (5.2.35), can be expressed explicitly, noting that

$$\alpha(t) = X_1 \circ Y_0(t), \quad t \in \mathcal{M}.$$

Skipping over the details, we obtain

$$\alpha'(t) \equiv \frac{d\alpha(t)}{dt} = e^{i[\pi - \arg(b^2(t) - 4a(t)c(t))]}, \quad t \in \mathcal{M}\,,$$

where the functions a, b and c are the second degree polynomials of Sect. 5.3.

Chapter 6
The Genus 0 Case

This chapter is devoted to the case when the algebraic curve defined by the equation $Q(x, y) = 0$ has genus 0. The corresponding classification was made in Chap. 2 and exactly five situations have been found, described by the relations (2.3.5) to (2.3.8). In fact, since (2.3.6) and (2.3.8) are equivalent up to a permutation of the variables x and y, we are left with four significantly different cases, which will be treated separately.

In some sense, genus 0 can be viewed, in the complex plane, as a degenerate limiting case of genus 1. Thus many of the results established in Sect. 5.3 about the branches of the algebraic functions X and Y still hold, in particular most of the properties stated in *(i)*, *(ii)* and *(iii)* of Theorem 5.3.3. It is worth recalling (see Remark 2.5.2) that the analytic continuation process could also be carried out on the Riemann sphere, as was done in Chap. 2 on the torus in the genus 1 case. Nevertheless, as a matter of continuity with respect to Chap. 5, we choose to make the complete analysis in the complex plane, since it presents some new interesting features pertaining in particular to automorphic functions [45].

6.1 Properties of the Branches

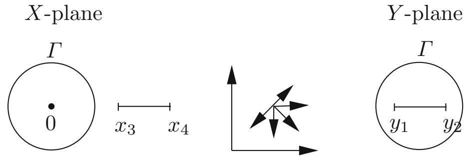
The main properties are summarized in the next theorem.

Theorem 6.1.1

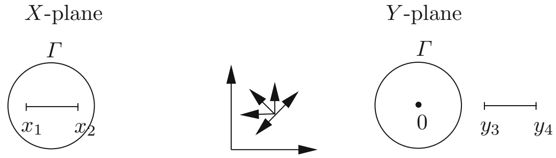
(i) The algebraic curve defined by $Q(x, y) = 0$ *has genus 0 in the following cases.*

G. Fayolle et al., *Random Walks in the Quarter Plane*, Probability Theory
and Stochastic Modelling 40, DOI 10.1007/978-3-319-50930-3_6

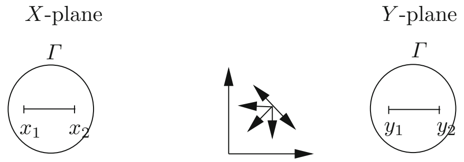
1. $\begin{cases} x_1 = x_2 = 0 \\ y_3 = y_4 = \infty. \end{cases}$



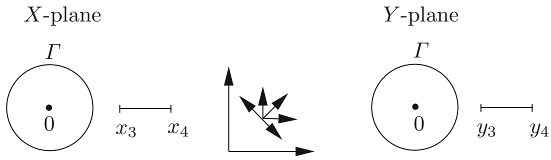
2. $\begin{cases} y_1 = y_2 = 0 \\ x_3 = x_4 = \infty. \end{cases}$



3. $\begin{cases} x_3 = x_4 = \infty \\ y_3 = y_4 = \infty. \end{cases}$



4. $\begin{cases} x_1 = x_2 = 0 \\ y_1 = y_2 = 0. \end{cases}$



5. $\begin{cases} x_2 = x_3 = 1 \\ y_2 = y_3 = 1. \end{cases}$

In addition x_3 and y_3 are always positive, but x_4 and y_4 need not be positive. If for instance $x_4 < 0$, then the plane is cut along $[-\infty, x_4] \cup [x_3, +\infty]$.

(ii) *The cases listed in (i) above correspond respectively to the relations (2.3.9), (2.3.7), (2.3.6), (2.3.8) and (2.3.5) and the following more precise properties hold.*

- *The branches X_i and Y_i, $i = 0, 1$, are meromorphic in their respective complex planes cut along a single slit as shown in part (i) of the theorem.*
- $|X_0(y)| \le |X_1(y)|$ *[resp.* $|Y_0(x)| \le |Y_1(x)|$*] in the whole complex plane properly cut. Equalities can hold only at* $0, \infty$ *or on the cuts.*
- *In case 1, Y_0, Y_1 [resp. X_0, X_1] have a common pole [resp. zero] at $x = 0$ [resp. $y = \infty$] and X_0 is bounded.*

- *In case 2, Y_0, Y_1 [resp. X_0, X_1] have a common zero [resp. pole] at $x = \infty$ [resp. $y = 0$] and Y_0 is bounded.*
- *In case 3, Y_0, Y_1 [resp. X_0, X_1] have a common pole at $x = \infty$ [resp. $y = \infty$].*
- *In case 4, Y_0, Y_1 [resp. X_0, X_1] have a common zero at $x = 0$ [resp. $y = 0$]. The functions X_0 and Y_0 are bounded, but X_1 and Y_1 have two poles.*
- *In case 5, Y_0 [resp. X_0] is holomorphic in the plane cut along $[x_1x_4]$ (resp. $[y_1y_4]$). The branches Y_1 and X_1 have two poles.* ■

Proof All statements of part *(i)* follow directly from Sect. 2.3. The sharper results of *(ii)* can be derived exactly (even in a simpler way) along the lines of Lemma 5.3.1 and Theorem 5.4.1, using the separation of the branches on the unit circle, together with the argument principle and the maximum modulus theorem. Details are omitted. ■

Now we shall analyze in detail the various possibilities listed above. To avoid uninteresting technicalities, the functions π_0, q and $\widetilde{q}$ in (5.1.2) will be assumed to be polynomials with respect to the two variables (x, y). This amounts to saying that the jumps of the random walk on the axes are bounded. On the other hand, the ergodicity conditions are again given by condition (5.4.4), by direct continuity with respect to the parameters $\{p_{i,j}\}$, except perhaps for case 5.

6.2 Case 1: $p_{01} = p_{-1,0} = p_{-1,1} = 0$

Theorem 6.2.1 *Under the above conditions, the functions π and $\widetilde{\pi}$ exist if, and only if, (5.4.4) holds. In this case the following situation holds:*

- *$\widetilde{\pi}$ is a rational function and its poles in the complex plane are the zeros of $\widetilde{q}(X_0(y), y)$ located in G_Γ^c (i.e. outside the unit disc).*
- *π has the form*

$$\pi(x) = U(x)Y_0(x) + V(x),$$

where U and V are rational functions. ■

Proof As in Chap. 5, it is possible to state a BVP of the form (5.1.5), for the function π, on a curve which was denoted $\mathcal{M}$ in Sect. 5.3. The computation of the index of this BVP can be derived exactly as in Theorem 5.4.1 and the ergodicity conditions are still given by (5.4.4). When they hold, it is however not necessary to solve this BVP, as there is a much more direct way to obtain the solution. In fact, writing

$$\widetilde{\pi}(y) = \frac{-q(X_0(y), y)\pi(X_0(y)) - \pi_0(X_0(y), y)}{\widetilde{q}(X_0(y), y)}, \ \forall y \in G_\Gamma^c,$$

we immediately get the analytic continuation of $\widetilde{\pi}$ to the region G_Γ^c, since from the maximum modulus principle and by Theorem 6.1.1, we have

$$|X_0(y)| \leq 1, \ \forall y \in G^c_\Gamma.$$

Moreover, $\forall y \in G^c_\Gamma$ the functions $q(X_0(y), y)$, $\widetilde{q}(X_0(y), y)$ and $\pi_0(X(y), y)$ are analytic and of finite degree at infinity, so that the only singularities of $\widetilde{\pi}$ in the whole complex plane are the potential zeros of $\widetilde{q}(X_0(y), y)$ in G^c_Γ and $\widetilde{\pi}$ is rational.

When $y \to \infty$, $\widetilde{\pi}(y) = \mathcal{O}(y^t)$ where $t = \max(r, s)$, with

$$\frac{q(X_0(y), y)}{\widetilde{q}(X_0(y), y)} = \mathcal{O}(y^r), \qquad \frac{\pi_0(X_0(y), y)}{\widetilde{q}(X_0(y), y)} = \mathcal{O}(y^s).$$

The proof of the theorem is complete. ■

6.3 Case 3: $p_{11} = p_{10} = p_{01} = 0$

Theorem 6.3.1 *Under the above conditions, the functions π and $\widetilde{\pi}$ exist if, and only if, (5.4.4) holds, in which case they can be continued as meromorphic functions, to the whole complex plane. In the domain $G_{\mathcal{M}}$, π has the integral representation given by (5.4.21), where $\mathcal{M}$ is an ellipse, which in the $\mathbb{C}_x$-plane is given by the equation*

$$\frac{(u-u_0)^2}{a^2} + \frac{v^2}{b^2} = 1, \quad with \quad a^2 - b^2 = c^2, \ x = u + iv.$$

The gluing function w, which appears in (5.4.21) and is a solution of (5.2.39), can be chosen as

$$w(z) = \frac{1}{2}\left(h(z) + \frac{1}{h(z)}\right), \quad \forall z \in G_{\mathcal{M}}, \tag{6.3.1}$$

where

$$h(z) = \sqrt{k(\rho)}\,\mathrm{sn}\left(\frac{2K}{\pi}\arcsin\frac{z-u_0}{c}; \rho\right), \ \text{with} \quad \rho = \left(\frac{a-b}{a+b}\right)^2,$$

represents the conformal mapping of the interior of the above ellipse onto the unit disc, $k(\cdot)$ and K being respectively the modulus and the real quarter period of the Jacobi elliptic function sn $(z; k)$. *Similar conclusions hold for $\widetilde{\pi}$.*

Proof To show the first and third claims of the theorem, it suffices to note that the global analysis of Chap. 5 applies *verbatim*, the verification that $\mathcal{L}$ and $\mathcal{M}$ are ellipses being elementary, and the explicit form of $h(z)$ is given in [84, p. 296]. From this integral formula, it is indeed possible to get the analytic continuation of π to the whole complex plane. Nonetheless, it is instrumental to prove the second claim of the theorem directly from the functional Eqs. (5.1.3) or (5.1.5), using some properties of the branches Y_0 and Y_1 given in the next lemma, which is the analogue of Corollary 5.3.5.

Lemma 6.3.2

$$
\begin{aligned}
&|X_1(t)| \geq \frac{|\widetilde{b}(t)|}{2} \quad \text{and} \quad |Y_1(t)| \geq \frac{|b(t)|}{2}, \ \forall t\,.\\
&X_0 \circ Y_1(t) = t \quad \text{and} \quad |X_1 \circ Y_1(t)| > |t|, \ \forall t\,.\\
&X_0 \circ Y_0(t) = \begin{cases} t, & \text{if } t \in G_{\mathcal{M}},\\ \neq t, & \text{if } t \in G^c_{\mathcal{M}} \end{cases} \text{ and } X_0 \circ Y_0(G^c_{\mathcal{M}}) = G_{\mathcal{M}}\,.\\
&X_1 \circ Y_0(t) = \begin{cases} t, & \text{if } t \in G^c_{\mathcal{M}},\\ \neq t, & \text{if } t \in G_{\mathcal{M}} \end{cases} \text{ and } X_1 \circ Y_0(G_{\mathcal{M}}) = G^c_{\mathcal{M}}\,.\\
&|X_1 \circ Y_1(t)| > |t|, \quad \forall t.
\end{aligned}
$$

Proof All statements of the lemma can be proved by standard applications of the maximum modulus principle. The details are omitted. ■

Lemma 6.3.3 *Let Δ_n be the sequence of ring-shaped domains constructed recursively as follows:*

$$\Delta_{n+1} = X_1 \circ Y_1(\Delta_n), \forall n \geq 0\,, \tag{6.3.2}$$

where Δ_0 is the doubly connected domain, with boundary the slit $[x_1x_2]$ and the closed curve $X_1 \circ Y_1([x_1x_2])$. Thus

$$\Delta_0 = G_{X_1 \circ Y_1([x_1x_2])} - [x_1x_2].$$

Let $\mathcal{D}_n \stackrel{def}{=} \bigcup_{k \leq n} \Delta_n$. Then

(i) $\mathcal{D}_{n+1} = \mathcal{D}_n \oplus \Delta_{n+1}$, *where $\oplus$ denotes the direct sum of sets;*
(ii) $\lim_{n\to\infty} \mathcal{D}_n = \mathbb{C} - [x_1x_2]$.

Proof By induction. Let $[G]$ denote the boundary of an arbitrary domain G. Assume (i) holds up to some $n > 0$, but not for $n+1$. Observe first that the *internal* boundary of Δ_{n+1} coincides with the *external* boundary of Δ_n. From the principle of correspondence of the boundaries for conformal mappings, there exist three points z_{n-1}, z_n, t_n such that

$$
\begin{aligned}
z_n &= X_1 \circ Y_1(t_n)\,, \quad z_n\,, t_n \in [\mathcal{D}_n],\\
z_n &= X_1 \circ Y_1(z_{n-1}), \quad z_{n-1}\,, \in [\mathcal{D}_{n-1}].
\end{aligned}
$$

But in this case, it follows from Lemma 6.3.2 that necessarily $t_n = z_{n-1}$, which contradicts $\mathcal{D}_{n-1} \subset \mathcal{D}_n$. So, to prove (i), it suffices to check the initial step, which is indeed straightforward. The point (ii) is also immediate by the first two properties of the branches listed in Lemma 6.3.2. ■

Rewriting for the sake of completeness the basic twin equations

$$q(X_0(y),y)\pi(X_0(y)) + \widetilde{q}(X_0(y),y)\widetilde{\pi}(y) + \pi_0(X_0(y),y) = 0, \ \forall y \in \mathbb{C}, \quad (6.3.3)$$
$$q(x,Y_0(x))\pi(x) + \widetilde{q}(x,Y_0(x))\widetilde{\pi}(Y_0(x)) + \pi_0(x,Y_0(x)) = 0, \quad \forall x \in \mathbb{C}, \quad (6.3.4)$$

we can proceed by induction, in a flip-flop way, as follows.

Assumption: π is meromorphic in $\mathcal{D}_n$.
Conclusion:

1. $\widetilde{\pi}$ is meromorphic in $Y_1(\mathcal{D}_n)$;
2. π has a meromorphic continuation to $\mathcal{D}_{n+1}$.

Assertion 1 of the **Conclusion** is obtained from Eq. (6.3.3), since the functions Y_0 and Y_1 are analytic outside the unit disc, except at $y = \infty$ where they have a single pole; assertion 2 follows by exploiting (6.3.4) and Lemma 6.3.3.

The recursive computation of the poles rely on the forthcoming lemma, which is also another way of getting the meromorphic continuation of the various functions.

Lemma 6.3.4 *Equation (5.1.5) or its reduced form (5.4.19) can be expressed as*

$$\pi(u)f(u,Y_1(u)) - \pi(X_1 \circ Y_1(u))f(X_1 \circ Y_1(u),Y_1(u)) = g(u), \quad u \in [x_1\, x_2], \quad (6.3.5)$$

which gives the meromorphic continuation of π *to the whole complex plane.*

Proof From the analysis carried out in the preceding chapter, Eq. (5.1.4)

$$\pi(X_0(y))f(X_0(y),y) - \pi(X_1(y))f(X_1(y),y) = h(y), \ \forall y \in [y_1 y_2],$$

can be continued to $\mathbb{C}_y$. Writing it in particular for $y \in \mathcal{L}$, it is permitted, by using Lemma 6.3.2, to make the change of variables $y = Y_1(u)$, which implies exactly (6.3.5). ■

Let

$$n_0 = \min_{n\geq 0}\{q(v,Y_0(v))\,\widetilde{q}(v,Y_1(v)) \neq 0, \ \forall v \in \mathcal{D}_n^c\},$$

and $\alpha_0^k,\ k = 1,\ldots,m$, be the poles of π in $\mathcal{D}_{n_0}$. Note that n_0 is finite, since we have assumed that q and $\widetilde{q}$ are polynomials. Then the possible poles of π can be recursively computed from the sequences $\{\alpha_n^k,\ 1 \leq k \leq m,\ n \geq 1\}$, where

$$\alpha_{n+1}^k = X_1 \circ Y_1(\alpha_n^k).$$

Similar results hold for $\widetilde{\pi}$, up to an exchange of parameters.
The proof of Theorem 6.3.1 is complete. ■

6.4 Case 4: $p_{-1,0} = p_{0,-1} = p_{-1,-1} = 0$

Due to the position of the slit $[x_3, x_4]$, the problem here is of a slightly different nature and is connected with the automorphic functions (see [45]), as enlighted in the next theorem.

Theorem 6.4.1 *The functional equation*

$$\pi\big(X_0 \circ Y_0(t)\big) f\big(X_0 \circ Y_0(t), Y_0(t)\big) - \pi(t) f\big(t, Y_0(t)\big) = g(t) \tag{6.4.1}$$

is valid for all $t \in \mathbb{C}$ and provides the analytic continuation of π as a meromorphic function (the number of poles being finite) to the whole complex plane cut along $[x_3\, x_4]$. ■

Proof By an informal topological argument, one can say that the cut $[y_1 y_2]$ has in some sense *shrunk* into a single point, the origin, where the algebraic function $Y(x)$ has a double zero. Hence, Eq. (5.1.4)

$$\pi(X_0(y)) f(X_0(y), y) - \pi(X_1(y)) f(X_1(y), y) = h(y),$$

obtained by elimination of $\widetilde{\pi}$, holds in a neighborhood of the origin in the Y-plane. The claim of the theorem follows from the properties of the branches X_i, Y_i, listed in Theorem 6.1.1. ■

At once, it is important to note that (5.1.4), quoted above, cannot be continued to $[y_3, y_4]$, so that we do not have a BVP on a single curve, but on two curves. In the rest of this section, several methods are proposed to solve (6.4.1).

6.4.1 Integral Equation

We write π in the form

$$\pi(z) = \frac{1}{2\pi} \int_{[x_3 x_4]} \frac{\omega(t)dt}{t - z} + R(z), \quad \forall z \in \mathbb{C} - [x_3 x_4], \tag{6.4.2}$$

where R is a rational function having the same poles and residues as π. Letting now z go to the slit $[x_3 x_4]$ and using formula (5.2.4) in (6.4.1), we get

$$\omega(t) - \frac{1}{\pi} \int_{[x_3 x_4]} \frac{[u K_1(t) + K_2(t)]\omega(u)du}{|u - X_0 \circ Y_0(t)|^2} = K_3(t), \tag{6.4.3}$$

where $K_i, i = 1, 2, 3,$ are real functions, given in terms of f and g, their explicit expression being omitted. Conspicuously, the operator (6.4.3) is strongly contractant and can thus lead to an efficient numerical evaluation, all the more because the unknown density $\omega(t)$ is real.

6.4.2 Series Representation

It follows from the maximum modulus principle that the function $X_0 \circ Y_0$ satisfies

$$\begin{cases} |X_0 \circ Y_0(t)| \le |t|, & \forall t \in \mathbb{C}, \\ |X_0 \circ Y_0(t)| < |t|, & \forall t \in \mathcal{D}, \end{cases}$$

and admits 0 and 1 as fixed points. Thus, denoting by $(X_0 \circ Y_0)^{(n)}$ the n-th iterate of $X_0 \circ Y_0$, we have

$$\lim_{n\to\infty} (X_0 \circ Y_0)^{(n)}(t) = 0, \quad \forall t \in \mathcal{D}.$$

Consequently, by (6.4.1), $\pi(z)$ can be expressed as a series, for $z \in \mathcal{D}$, the terms of which are finite products of analytic functions.

6.4.3 Uniformization

The algebraic curve $Q(x, y) = 0$ is of genus 0 and, consequently, admits a rational uniformization, which will be expressed as

$$\begin{cases} x(s) &= \dfrac{x_3 + x_4}{2} + \dfrac{x_4 - x_3}{4}\left(s + \dfrac{1}{s}\right), \\ y(s) &= \dfrac{y_3 + y_4}{2} + \dfrac{y_4 - y_3}{4}\left(\gamma(s) + \dfrac{1}{\gamma(s)}\right), \end{cases} \tag{6.4.4}$$

where the automorphisms ξ, η, δ, introduced in Chap. 3, take the form

$$\begin{cases} \xi(s) = \dfrac{1}{s}, \\ \eta(s) = \dfrac{as + b}{cs - a}, \\ \delta(s) = \dfrac{bs + a}{-as + c}, \end{cases}$$

and $\gamma(\cdot)$ is a fractional linear transform satisfying

$$\gamma(\eta(s)) = \frac{1}{\gamma(s)}.$$

Clearly, the function $X_0 \circ Y_0$ appearing in Eq. (6.4.1) is, up to the above uniformization, strongly related to the automorphism $\delta = \eta\xi$. More information is given in the next lemma.

Lemma 6.4.2 *Let u_1, u_2 be the two fixed points of $\delta(s)$ in the s-plane. By Lemma 2.3.8 and according to [45], we have the following classification.*

(1) *If $p_{10}^2 - 4p_{11}p_{1,-1} \neq 0$, then*

$$\frac{\delta(s) - u_1}{\delta(s) - u_2} = K \frac{s - u_1}{s - u_2},$$

where u_1, u_2 are the two roots of the equation

$$(x_4 - x_3)u^2 + 2(x_4 + x_3)u + x_4 - x_3 = 0,$$

with $u_1 u_2 = 1$, and the multiplier *K is given by*

$$K = \frac{u_2\delta(0) - 1}{u_1\delta(0) - 1} = \frac{\eta(0) - u_2}{\eta((0) - u_1}.$$

(1.a) *If $p_{10}^2 - 4p_{11}p_{1,-1} > 0$, then x_4 is positive, u_1, u_2 are real and*

$$u_2 < -1 < u_1 < 0.$$

Moreover, K is real, $|K| \neq 1$, so that $\delta(s)$ is of hyperbolic *type.*

(1.b) *If $p_{10}^2 - 4p_{11}p_{1,-1} < 0$, then x_4 is negative, u_1, u_2 are complex conjugate and located on the unit circle. In addition, K is complex, $|K| \neq 1$, so that $\delta(s)$ is of* loxodromic *type.*

(2) *If $p_{10}^2 - 4p_{11}p_{1,-1} = 0$, then $x_4 = \infty$. The two fixed points coincide, with $u_1 = u_2 = -1$, and the uniformization of $x(s)$ in (6.4.4) becomes*

$$x(s) = x_3 - s^2.$$

The transformation $\delta(s)$ is of parabolic *type, and*

$$\frac{1}{\delta(s) + 1} = \frac{1}{s + 1} + C.$$

■

Proof The classification in the lemma relies on classical properties of the group generated by fractional linear transforms. The only technical problem is to check from the uniformization (6.4.4) that $\eta(0)$ is real in case *(1.a)*, and complex in case *(1.b)*. The details are omitted.

■

The above uniformization gives a representation of π as a series of meromorphic functions, the form of which depends on the type of δ, as defined in Lemma 6.4.2.

6.4.4 Setting a Boundary Value Problem (BVP)

Via (5.4.6), the slit $[x_3 x_4]$ in the X-plane is mapped onto the unit circle Γ in the z-plane. Similarly, the curve $X_0 \circ Y_0\big(\overrightarrow{\underleftarrow{[x_3, x_4]}}\big)$ corresponds to two circles in the z-plane, one of them, say $\mathcal{C}$, is located inside Γ. It is now possible to pose a problem of Carleman (see Chap. 5) for π in the annulus between $\mathcal{C}$ and Γ. Moreover, explicit expressions exist for the conformal mapping of this annulus onto simpler regions, allowing to transform the Carleman BVP into a Riemann BVP, and to obtain an integral form solution for π. We will not go more deeply into the analysis, which does not contain theoretical difficulty.

6.5 Case 5: $M_x = M_y = 0$

This is the limiting case where $x_2 = x_3 = 1$ and $y_2 = y_3 = 1$.

Lemma 6.5.1 *The branches X_0 and X_1 of the algebraic function X are, respectively, holomorphic and meromorphic in the complex plane $\mathbb{C}_y$ cut along $[y_1 y_4]$. The image of the cut by $X(y)$ consists of two simple closed curves $\mathcal{M}_1$ and $\mathcal{M}_2$, with*

$$\mathcal{M}_1 = X[\overrightarrow{\underleftarrow{y_1, 1}}], \qquad \mathcal{M}_2 = X[\overrightarrow{\underleftarrow{1, y_4}}],$$

which are symmetrical with respect to the horizontal axis, and smooth *(i.e. the direction of the tangent varies continuously), except at the point $x = 1$, which is a corner point if, and only if, the correlation coefficient r of the random walk in the interior of the quarter plane is not zero. The following situation holds:*

- *If $r \neq 0$, then*

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \{1\}, \quad \mathcal{M}_i \cap \Gamma = \{1\}, \quad i = 1, 2, \tag{6.5.1}$$

$$\begin{cases} \mathcal{M}_1 & \subset \overline{\mathcal{D}}, \\ \mathcal{M}_2 & \subset \mathbb{C} - \mathcal{D}, \end{cases} \quad \text{if } r < 0, \quad \text{and} \quad \begin{cases} \mathcal{M}_1 & \subset \mathbb{C} - \mathcal{D}, \\ \mathcal{M}_2 & \subset \overline{\mathcal{D}}, \end{cases} \quad \text{if } r > 0. \tag{6.5.2}$$

In addition,

$$|X_0(y)| < 1 \quad \text{and} \quad |X_1(y)| > 1, \quad \forall y \in \Gamma - \{1\}.$$

- *If $r = 0$, then*

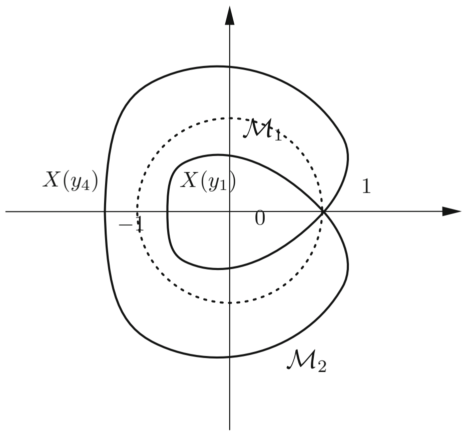


Fig. 6.1 The contour $\mathcal{M}_1 \cup \mathcal{M}_2$, for $r < 0$

$$\mathcal{M}_1 \subset \overline{\mathcal{D}} \subset \mathcal{M}_2 \subset \mathbb{C} - \mathcal{D}.$$

Moreover, the curves $\mathcal{M}_1, \mathcal{M}_2$ are tangent at $x = 1$. They can be identical if, and only if, the group is of order 4 *[see condition (4.1.17)], in which case they coincide with the unit circle Γ. This is in particular always the case for the simple random walk.*

In Fig. 6.1 the dotted curve is the unit circle, and we have drawn the contour $\mathcal{M}_1 \cup \mathcal{M}_2$, which has a self-intersection and is the image of the cut $[\overrightarrow{\overleftarrow{y_1 y_4}}]$ by the mapping $y \to X(y)$, remembering that $X_0[\overrightarrow{\overleftarrow{y_1 y_4}}] = \overline{X}_1[\overrightarrow{\overleftarrow{y_1 y_4}}]$.

Similar properties hold for the algebraic function $Y(x)$, with the corresponding curves $\mathcal{L}_1$ and $\mathcal{L}_2$. ■

Proof To show (6.5.1), observe that if there exists an $x \in \mathcal{M}_1 \cap \Gamma, x \neq 1$, then there exists a $\theta, \theta \in]0, 2\pi[$, such that the pairs $(e^{\varepsilon i\theta}, Y_0(e^{\varepsilon i\theta}))$ and $(e^{\varepsilon i\theta}, Y_1(e^{\varepsilon i\theta}))$, $\varepsilon = \pm 1$, are solutions of the fundamental equation $Q(x, y) = 0$. In addition, $Y_0(e^{i\theta}) \in [y_1, 1]$ and $Y_1(e^{i\theta}) > 1$, since $|Y_0(x)| < 1, \forall x \in \Gamma - \{1\}$.

Setting now $Y_j(e^{i\theta}) = z_j$, $j = 1, 2$, we obtain a system of equations, with respect to the variable x,

$$\begin{cases} a(x)z_1^2 + b(x)z_1 + c(x) &= 0, \\ a(x)z_2^2 + b(x)z_2 + c(x) &= 0. \end{cases}$$

But this system admits only the real solution $x = e^{i\theta}$, because all its coefficients are real. Thus $\theta = 0$ and (6.5.1) is proved.

As for (6.5.2), it suffices to analyze the ratio $\dfrac{\widetilde{c}(y)}{\widetilde{a}(y)}$, $y \in]1 - \varepsilon, 1[$, since on the cut $[y_1, 1]$, X_0 and X_1 are complex conjugate functions. We have

$$1 - \frac{\widetilde{c}(y)}{\widetilde{a}(y)} = \frac{(\tilde{a}'(1) - \tilde{c}'(1))(y-1) + \mathcal{O}(y-1)^2}{\widetilde{a}(y)}.$$

Now, using $M_x = 0$ and letting R denote the covariance of the random walk in the interior of the quarter plane, one easily checks the following equivalence

$$R \stackrel{\text{def}}{=} \tilde{a}'(1) - \tilde{c}'(1) = p_{11} + p_{-1,-1} - p_{-11} - p_{1,-1} < 0 \Longleftrightarrow r < 0,$$

which corresponds exactly to (6.5.2).

The claim of the lemma that $\mathcal{M}_1$ has a corner point at $x = 1$ can be shown by using the first two derivatives with respect to y at $y = 1$ of the equation

$$\widetilde{a}(y)X^2(y) + \widetilde{b}(y)X(y) + \widetilde{c}(y) = 0.$$

After an easy manipulation, it appears that the derivative

$$t \stackrel{\text{def}}{=} \frac{dX_0(y)}{dy}_{|y=1}$$

of the branch X_0, in the cut plane $\mathbb{C} - [y_1, y_2]$, is one of the roots of the second degree equation

$$\widetilde{a}(1)t^2 + Rt + a(1) = 0. \tag{6.5.3}$$

Since $M_x = M_y = 0$, we also have

$$b(1) = -2a(1), \quad \widetilde{b}(1) = -2\widetilde{a}(1).$$

Hence, the discriminant Δ of Eq. (6.5.3) is given by

$$\Delta = R^2 - 4a(1)\widetilde{a}(1) = R^2 - b(1)\widetilde{b}(1).$$

But $|R| < \min(|b(1)|, |\widetilde{b}(1)|)$, so that Δ is negative and the roots of (6.5.3) are finite and complex conjugate. Thus $\mathcal{M}_1$ has a corner point if, and only if, $r \neq 0$.

In a similar way, the derivative $s \stackrel{\text{def}}{=} \frac{dY_0(x)}{dx}_{|x=1}$ satisfies the equation

$$a(1)s^2 + Rs + \widetilde{a}(1) = 0.$$

The assertions of the lemma concerning the case $r = 0$ and the group of order 4 can easily be shown by continuity arguments with respect to the parameters, as in Chap. 5. The proof of Lemma 6.5.1 is complete. ■

Theorem 6.5.2

(i) If $r \leq 0$, then π [resp. $\widetilde{\pi}$] is a holomorphic solution of the Riemann–Carleman problem

$$\pi(t)A(t) - \pi(\alpha(t))A(\alpha(t)) = g(t), \quad t \in \mathcal{M}_1, \tag{6.5.4}$$

where the functions A and g are defined in (5.1.6).

(ii) If $r > 0$, then π [resp. $\widetilde{\pi}$] can be continued as a meromorphic function in the region $G_{\mathcal{M}_1}$ [resp. $G_{\mathcal{L}_1}$]. The poles of π in $G_{\mathcal{M}_1} - \mathcal{D}$ [resp. $\widetilde{\pi}$ in $G_{\mathcal{L}_1} - \mathcal{D}$] are the eventual zeros of q [resp. $\widetilde{q}$].

(iii) The function π and $\widetilde{\pi}$ cannot be analytically continued to a region encompassing the point 1, and hence the conditions of Lemma 2.2.1 are not satisfied. ■

Proof One proceeds exactly as in Chap. 5, by eliminating $\widetilde{\pi}$ which must be continuous on the two edges of the slit $[y_1, 1]$. This yields (6.5.4), exactly as was obtained (5.1.5), and the point (iii) is clear. The technical novelty is that the BVP (6.5.4) belongs to the class of generalized problems (see [47, 80]), since the contour $\mathcal{M}_1$ has a corner point. We will return to this fact later, when computing the index. ■

From now on, we will assume $r < 0$. The reader will convince himself that the forthcoming proofs can easily be transposed to the case $r > 0$, up to some technicalities arising from possible poles in the domain $G_{\mathcal{M}}$, which then contains the unit circle as shown in (6.5.2). Moreover, probabilistic arguments show that the random walk is never ergodic for $r > 0$. In fact, the mean first entrance time of the random walk into the axes, when starting from some arbitrary point with strictly positive coordinates, is infinite for any $r > 0$ (see [36]).

As in Theorem 5.4.1 for the case of genus 1, we will derive the conditions for the functions π and $\widetilde{\pi}$ to be analytic in the open disc $\mathcal{D}$ and continuous on its boundary Γ. The argument will mimic those of Sect. 5.4, and all the *drifts* appearing in the computations have been defined in (1.2.3).

Lemma 6.5.3 *Let*

$$\begin{cases} \lambda_x \stackrel{def}{=} \sum_{ij} i^2 p_{ij} = 2\widetilde{a}(1), \quad \lambda_y \stackrel{def}{=} \sum_{ij} j^2 p_{ij} = 2a(1), \\ \sigma \stackrel{def}{=} \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} = M'_x - \dfrac{RM'_y}{\lambda_y} - i\dfrac{M'_y\sqrt{|\Delta|}}{\lambda_y} \stackrel{def}{=} \rho e^{i\theta}, \\ \widetilde{\sigma} \stackrel{def}{=} \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} = M''_y - \dfrac{RM''_x}{\lambda_x} - i\dfrac{M''_x\sqrt{|\Delta|}}{\lambda_x} \stackrel{def}{=} \widetilde{\rho} e^{i\widetilde{\theta}}. \end{cases} \tag{6.5.5}$$

The BVP corresponding to (6.5.4) has a reduced *form corresponding to (5.4.19), the index of which is given by*

$$\chi = -L - M + \left\lfloor \frac{\theta + \widetilde{\theta} + \arccos(-r)}{\pi} - 1 \right\rfloor. \tag{6.5.6}$$

Moreover, π and $\widetilde{\pi}$ correspond to probabilistic distributions if, and only if,

$$\left\lfloor \frac{\theta + \widetilde{\theta} + \arccos(-r)}{\pi} - 1 \right\rfloor = 1. \tag{6.5.7}$$

■

Proof We follow the method and the notation proposed in Lemma 5.4.2. The main discrepancy with respect to Chap. 5 comes from the discontinuity of the derivative of $X_0(y)$ [resp. $Y_0(x)$] at $y = 1$ [resp. $x = 1$], which renders slightly sharper the calculation of the *reduced index*. Following (5.4.5), the basic step is to estimate the quantity

$$\chi_1 \stackrel{\text{def}}{=} -\mathcal{I}nd[A(t)]_{\mathcal{M}_1} = \zeta + \widetilde{\zeta},$$

with

$$\zeta \stackrel{\text{def}}{=} -\mathcal{I}nd[q(x, Y_0(x))]_{\mathcal{M}_1}, \qquad \widetilde{\zeta} \stackrel{\text{def}}{=} \mathcal{I}nd[\widetilde{q}(X_0(y), y)]_{\underset{\longleftarrow}{\overrightarrow{[y_1,1]}}}.$$

Letting now

$$\begin{cases} \delta = \mathcal{I}nd[q(x, Y_0(x))]_{\Gamma}\,, \\ \widetilde{\delta} = \mathcal{I}nd[\widetilde{q}(X_0(y), y)]_{\Gamma}\,, \end{cases} \tag{6.5.8}$$

we can write

$$\begin{cases} \zeta = N_Z[q(x, Y_0(x)), \mathcal{D}] - N_Z[q(x, Y_0(x)), G_{\mathcal{M}_1}] - L - \delta + 1 - \dfrac{\beta}{\pi}, \\ \widetilde{\zeta} = N_Z[\widetilde{q}(X_0(y), y), \mathcal{D}] - M - \widetilde{\delta} + \dfrac{1}{2}, \end{cases} \tag{6.5.9}$$

where $0 \le \beta \le \pi$ denotes the angle between the two tangents of the curves $\mathcal{M}_1$ and Γ at the point 1. The derivation of (6.5.9) deserves some explanation. We consider the domain between the curve $\mathcal{M}_1$ and Γ. Then the classical *the argument principle* for the function q on the boundary of this domain does not apply without precaution, since q is discontinuous at the corner point 1. Indeed, the variation has to be taken as a Stieltjes integral in principal value, and this requires the generalized form of the Sokhotski-Plemelj formulae (see Sect. 5.2.2 and e.g., [80]):

$$\begin{cases} \Phi^+(t) &= \left(1 - \dfrac{\alpha}{2\pi}\right)\varphi(t) + \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \frac{\varphi(s)ds}{s-t}, \\ \Phi^-(t) &= -\dfrac{\alpha}{2\pi}\varphi(t) + \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \frac{\varphi(s)ds}{s-t}, \end{cases}$$

where $0 \le \alpha \le 2\pi$ is the angle between the two half-tangents at the corner point. We apply these remarks to calculate δ and $\widetilde{\delta}$ defined by (6.5.8).

Choosing the argument of an arbitrary complex number to take its values on the interval $[0, 2\pi[$, we get

$$\begin{cases} \delta = 1 - \dfrac{\theta}{\pi}, \\ \widetilde{\delta} = 1 - \dfrac{\widetilde{\theta}}{\pi}, \end{cases}$$

whence

$$\delta + \widetilde{\delta} = 2 - \frac{\theta + \widetilde{\theta}}{\pi}.$$

For the last point of the lemma concerning probabilistic solutions, we use the result derived in Sect. 5.4 (which still holds, provided the index is properly defined since $\mathcal{M}_1$ is not a smooth Lyapounov contour, see e.g., [47]). Hence, for π and $\widetilde{\pi}$ to be proper probability distributions, it is necessary and sufficient to have

$$\chi = -L - M + 1,$$

which reduces easily to (6.5.7). The proof of the lemma is complete. ∎

Remark 6.5.4 The *ergodicity conditions* for the two-dimensional random walk, obtained in a purely probabilistic way in [36]), read as follows:

$$M'_x < 0, \quad M''_y < 0, \quad \text{and} \quad \lambda_x \frac{M'_y}{M'_x} + \lambda_y \frac{M''_x}{M''_y} > 2R.$$

The reader can check that they do not coincide with (6.5.7) and bring to light a spectacular phenomenon. We have seen in Theorem 6.5.2 the impossibility of using Lemma 2.2.1. Indeed, $\pi(x, y)$ is analytic in the open domain $\mathcal{D} \times \mathcal{D}$, but the continuity on the boundary $\Gamma \times \Gamma$ need not hold, unless additional conditions on the parameters are in force. This also seems to have been observed by probabilistic arguments in [3].

To solve the BVP defined on $\mathcal{M}_1$ in Theorem 6.5.2, in addition to the index analyzed above, we need a convenient gluing function, which will be derived in Sect. 7.2.2 of the forthcoming Chap. 7. Hence, the function $\pi(\cdot)$ can be obtained by the integral formula (5.4.21), using (6.5.6) for the index, and (7.2.8).

6.5.1 Playing with the Uniformization

Define the curve

$$\widetilde{\mathcal{M}} \stackrel{\text{def}}{=} \mathcal{M}_1 \cup [\overrightarrow{\underleftarrow{x_1, 1}}],$$

where $\mathcal{M}_1$ was drawn in Fig. 6.1.

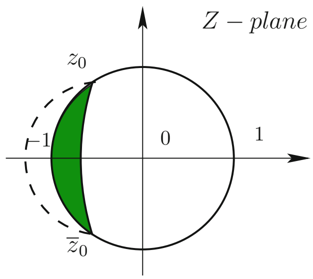


Fig. 6.2 The conformal transform of $\widetilde{\mathcal{M}}$

Lemma 6.5.5 *The simply-connected domain $G_{\widetilde{\mathcal{M}}}$ is conformally mapped by one branch of the function $t \to x^{-1}(t)$, defined in (6.5.11)), onto the lens-shaped domain $\mathcal{H}$ bounded by two circular arcs in the complex z-plane, as shown in Fig. 6.2. In addition, $\mathcal{H}$ is conformally mapped onto the upper half-plane by the function*

$$w(z) = \left(\frac{z - z_0}{z_0 z - 1}\right)^{\frac{\pi}{\psi}}, \tag{6.5.10}$$

where ψ is the angle between the two arcs. ■

Proof The algebraic curve corresponding to $Q(x, y) = 0$ admits a rational uniformization of the following general form, $\forall z \in \mathbb{C}$,

$$\begin{cases} x(z) &= \dfrac{x_4 + x_1}{2} + \dfrac{x_4 - x_1}{4}\left(z + \dfrac{1}{z}\right), \\ y(z) &= \dfrac{y_4 + y_1}{2} + \dfrac{y_4 - y_1}{4}\left(\varepsilon(z) + \dfrac{1}{\varepsilon(z)}\right), \end{cases} \tag{6.5.11}$$

where $\varepsilon(\cdot)$ is a fractional linear transform. Indeed the function in (6.5.11) can be derived by standard arguments (see e.g. [93]). A similar form was encountered in Sect. 6.4, and the reader is referred to Sect. 7.2.2 for full details on the coefficients of the linear transform $\varepsilon(z)$.

By using the properties of the branches $X_0(y)$ and $Y_0(x)$ in conjunction with the uniformization (6.5.11), on checks that the curve $\widetilde{\mathcal{M}}$ corresponds in the z-plane to two circular arcs drawn in Fig. 6.2: one of them is on the unit circle Γ and goes through the point $x = -1$, the second one (dashed line) lying outside. They cross Γ at the conjugate points $z_0, \overline{z}_0$, where

$$x(z_0) = x(\overline{z}_0) = 1.$$

The proof of the lemma is complete. ■

Lastly, as an aside, we note that it would also be quite possible to find an explicit form for the conformal mapping of the finite domain bounded by $\mathcal{M}_1$ onto the interior of the unit disc.

Chapter 7
Criterion for the Finiteness of the Group in the Genus 0 Case

The goal of this chapter is to derive a concrete necessary and sufficient condition for the finiteness of $\mathcal{H}$ (the group of the random walk introduced in Definition 2.4.2) in the genus 0 case. Here the Riemann surface $\mathbf{S}$ is a sphere and its universal covering Ω is the Riemann sphere $\mathbb{P}^1$. Although this part might have been inserted as an additional section in Chap. 6, the approach by continuity from the genus 1 case that we propose plainly justifies the need for an independent chapter. The essence of the results can be found in [38].

In Lemma 2.3.10, it was shown that genus 0 occurs in exactly five situations, which in fact can be classified into *two main categories*, significantly distinct from a probabilistic point of view:

1. Either $\overrightarrow{\mathbf{M}} = 0$;
2. Or one the four relations (2.3.6), (2.3.7), (2.3.8), (2.3.9) holds.

We recall that the so-called *singular random walks* (see Definition 2.3.1) are not considered in this book, their analysis being almost straightforward.

7.1 The Main Theorem

When looking at the results of this chapter, the reader will realize that the crucial part of the analysis is devoted to walks satisfying $\overrightarrow{\mathbf{M}} = 0$ (zero drift), which is really the most difficult (and maybe interesting!) situation.

According to Sect. 6.5, r will denote the *correlation coefficient*, and we set

$$\theta = \arccos(-r). \tag{7.1.1}$$

In particular, θ is the angle of the tangent line with the horizontal axis at the corner point of the curve drawn in Fig. 6.1.

G. Fayolle et al., *Random Walks in the Quarter Plane*, Probability Theory and Stochastic Modelling 40, DOI 10.1007/978-3-319-50930-3_7

Theorem 7.1.1

(a) *When* $\overrightarrow{\mathbf{M}} = 0$*, the group* $\mathcal{H}$ *is finite if and only if* θ/π *is rational, in which case its order is equal to*

$$2 \inf\{\ell \in \mathbb{Z}_+^* : \ell\theta/\pi \in \mathbb{Z}\}. \tag{7.1.2}$$

(b) *When* $\overrightarrow{\mathbf{M}} \neq 0$*, the order of* $\mathcal{H}$ *is* always infinite *in the four possible cases (2.3.6), (2.3.7), (2.3.8), (2.3.9).* ■

In Sect. 7.2.3, we shall propose another theoretical form of the angle (7.1.1), merely involving characteristic values related to the uniformization of curves of genus 0.

7.2 Proof of Part (a) of Theorem 7.1.1

As announced in the preamble, starting from the genus 1 case, we shall resort to a continuity argument. First, even though this is a bit redundant with Theorem 5.2.7, it will be comfortable to recall in a self-contained way the notion of a *conformal gluing function*.

Definition 7.2.1 Let $\mathcal{G} \subset \mathbb{C} \cup \{\infty\}$ be an open and simply connected set, symmetrical with respect to the real axis, and different from $\emptyset$, $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$. A function w is said to be a *conformal gluing function* (CoGF) for the set $\mathcal{G}$ if

- w is meromorphic in $\mathcal{G}$;
- w establishes a conformal mapping of $\mathcal{G}$ onto the complex plane cut along a segment;
- For all t in the boundary of $\mathcal{G}$, $w(t) = w(\bar{t})$.

For instance, $w(t) = t + 1/t$ is a CoGF for the unit disc, and any non-degenerate linear transformation of $w(t)$, namely

$$\frac{ew(t) + f}{gw(t) + h}, \qquad (e, g) \neq (0, 0), \qquad eh - fg \neq 0,$$

is also a CoGF for the unit disc. Incidentally, this implies one can choose arbitrarily the pole of w within the unit disc—in particular, taking $e = 0$, $f = 1$, $g = 1$ and $h = -2$, we get the CoGF $t/(t-1)^2$ with a pole at 1. Conversely, it can be shown that two CoGFs for the same domain are fractional linear transformations of each other (see for instance [63]).

7.2.1 *Limit Conformal Gluing When Passing from Genus 1 to Genus 0*

The approach consists in saying that genus 0 can be obtained as a *topological deformation* of genus 1, when parameters are modified in such a way that $\overrightarrow{\mathbf{M}} \to 0$.

According to the uniformization formulae (3.3.3) or (3.3.4), let

$$f(t) = \begin{cases} \dfrac{D''(x_4)}{6} + \dfrac{D'(x_4)}{t - x_4} & \text{if } x_4 \neq \infty, \\ \dfrac{1}{6}(D''(0) + D'''(0)t) & \text{if } x_4 = \infty. \end{cases}$$

To avoid technicalities of minor importance, we shall assume in the sequel $C \neq 0$, where

$$C \overset{\text{def}}{=} p_{1,0}^2 - 4p_{1,1}p_{1,-1}, \tag{7.2.1}$$

so that, by Lemma 2.3.8, $x_4 \neq \infty$.

It will be convenient to denote by $\wp_{1,\ell}$, for $\ell \in \{2, 3\}$, the Weierstrass elliptic function with periods ω_1, ω_ℓ and series expansion

$$\wp_{1,\ell}(\omega) = \frac{1}{\omega^2} + \sum_{(p_1,p_\ell)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{(\omega - p_1\omega_1 - p_\ell\omega_\ell)^2} - \frac{1}{(p_1\omega_1 + p_\ell\omega_\ell)^2}\right]. \tag{7.2.2}$$

Then, in the genus 1 case, it was shown in Sect. 5.5.2 that a CoGF for the domain $\mathcal{D}_1$, bounded by the curve $\mathcal{M}_1$ of Fig. 6.1, is obtained via the function

$$w(t) = \wp_{1,3}(\wp_{1,2}^{-1}(f(t)) - [\omega_1 + \omega_2]/2).$$

Since any fractional linear transformation of a CoGF is again a CoGF, the well-known addition theorem for $\wp_{1,3}$ involving translation by the half-period $\omega_1/2$ entails that

$$\wp_{1,3}(\wp_{1,2}^{-1}(f(t)) - \omega_2/2) \tag{7.2.3}$$

is also a CoGF for $\mathcal{D}_1$. Now letting $\overrightarrow{\mathbf{M}} \to 0$, so that $x_2, x_3 \to 1$, and by using a continuity argument with respect to the parameters $\{p_{i,j}\}_{-1\le i,j\le 1}$, a direct calculation in the integral formulae (3.3.6) and (3.3.9) shows that $\omega_1 \to i\infty$ and the real periods $\omega_2,\ \omega_3$ converge to certain non-degenerate quantities. More precisely,

$$\begin{cases} \omega_1 \to i\infty, \\ \omega_2 \to \alpha_2 = \dfrac{\pi}{[C(x_4 - 1)(1 - x_1)]^{1/2}}, \\ \omega_3 \to \alpha_3 = \displaystyle\int_{X_0(y_1)}^{x_1} \frac{\mathrm{d}x}{(1 - x)[C(x - x_1)(x - x_4)]^{1/2}}. \end{cases} \tag{7.2.4}$$

Lemma 7.2.2 *Letting* $\overrightarrow{\mathbf{M}} \to 0$, *we have*

$$\frac{\theta}{\pi} = \lim_{\overrightarrow{\mathbf{M}} \to 0} \frac{\omega_2}{\omega_3} = \frac{\alpha_2}{\alpha_3}.$$

Proof At the corner point of the curve $\mathcal{M}_1$, we shall calculate the angle of the tangent line with the horizontal axis in two different ways.

• First, when $\overrightarrow{\mathbf{M}} \to 0$, the derivative $X_0'(1)$ is a root of the second degree polynomial Eq. (6.5.3)

$$\widetilde{a}(1) {X_0'}^2(1) + R X_0'(1) + a(1) = 0,$$

where $R = \sum_{i,j} ijp_{ij}$ is the covariance of the walk in $\mathbb{Z}_+^{*2}$. Then, since

$$2a(1) = \sum_{-1 \le i,j \le 1} j^2 p_{i,j}, \quad 2\widetilde{a}(1) = \sum_{-1 \le i,j \le 1} i^2 p_{i,j},$$

one can check directly the equality

$$\arg(X_0'(1)) = \pm\theta. \tag{7.2.5}$$

• The second way of computing θ relies on the construction of a convenient CoGF, according to the definition recalled in Sect. 7.2.1. Here we shall choose w with $w(1) = \infty$. As we shall see, this implies that, in the neighborhood of $t = 1$, $w(t)$ has the form

$$w(t) = [\alpha + o(1)]/[1 - t]^{\chi}, \tag{7.2.6}$$

for some constants $\alpha \neq 0$ and $\chi > 0$. In addition, the exponent χ must satisfy the relation

$$\chi = \pi/\theta. \tag{7.2.7}$$

Indeed, Eq. (7.2.5) together with the horizontal symmetry of the CoGF yield $\exp(i\theta\chi) = \exp(-i\theta\chi)$, whence $\theta\chi/\pi$ is a positive integer. On the other hand, if $\theta\chi/\pi \geq 2$, then w would not be one-to-one. Identity (7.2.7) follows.

So, we are left with the proof of expansion (7.2.6). To obtain a CoGF in the zero drift case, we could use Lemma 6.5.5, but it will be more convenient here to take the CoGF obtained from the last paragraph, and then to let the drift go to zero.

When $\overrightarrow{\mathbf{M}} \to 0$, by using (7.2.2) and (7.2.4), as well as the well-known identity

$$\sum_{p \in \mathbb{Z}} \frac{1}{(\Omega + p)^2} = \frac{\pi^2}{\sin^2(\pi\Omega)},$$

we obtain, for $\ell \in \{2, 3\}$, uniformly in ω,

$$\wp_{1,\ell}(\omega) \to \left(\frac{\pi}{\alpha_\ell}\right)^2 \left[\frac{1}{\sin^2(\pi\omega/\alpha_\ell)} - \frac{1}{3}\right].$$

In particular, setting

$$u(t) = \sin^2\left(\frac{\alpha_2}{\alpha_3}\left[\arcsin\left\{\left[\frac{1}{3} + f(t)\left(\frac{\alpha_2}{\pi}\right)^2\right]^{-1/2}\right\} - \frac{\pi}{2}\right]\right),$$

and taking the limit in (7.2.3), (7.2.4), it follows that an admissible CoGF for $\mathcal{M}_1$ is given by

$$\left(\frac{\pi}{\alpha_3}\right)^2 \left[\frac{1}{u(t)} - \frac{1}{3}\right]. \tag{7.2.8}$$

As a linear transformation of a CoGF, $u(t)$ itself is a CoGF for $\mathcal{M}_1$. We shall now show the existence of $\alpha \neq 0$ such that, in the neighborhood of $t = 1$,

$$u(t) = [\alpha + o(1)]/[1 - t]^{\alpha_2/\alpha_3}. \tag{7.2.9}$$

By a direct calculation, it can be seen that for $t \in [x_1, 1]$,

$$\frac{1}{3} + f(t)\left(\frac{\alpha_2}{\pi}\right)^2 \tag{7.2.10}$$

belongs to the segment $[0, 1]$, equals 1 at x_1 and 0 at 1. In other words, in order to understand the behavior of $u(t)$ near $t = 1$, it is necessary to analyze the asymptotics, as $T \to \infty$, of the function

$$\sin^2\left(\frac{\alpha_2}{\alpha_3}\left[\arcsin\{T\} - \frac{\pi}{2}\right]\right).$$

For $T \geq 1$, we have

$$\arcsin\{T\} = \int_0^1 \frac{\mathrm{d}u}{(1-u^2)^{1/2}} \pm i \int_1^T \frac{\mathrm{d}u}{(u^2-1)^{1/2}} = \frac{\pi}{2} \pm i \ln\big[T + (T^2-1)^{1/2}\big].$$

Hence, with $\sin(ix) = i\sinh(x)$, we can write

$$\begin{aligned}\sin^2\left(\frac{\alpha_2}{\alpha_3}\left[\arcsin\{T\} - \frac{\pi}{2}\right]\right) &= -\sinh^2\left(\frac{\alpha_2}{\alpha_3}\ln\big[T + (T^2-1)^{1/2}\big]\right)\\ &= -\frac{1}{4}\Big(\big[T + (T^2-1)^{1/2}\big]^{2\alpha_2/\alpha_3}\\ &\qquad + \big[T - (T^2-1)^{1/2}\big]^{2\alpha_2/\alpha_3} - 2\Big).\end{aligned}$$

When $T \to \infty$, $T + (T^2 - 1)^{1/2} = 2T + O(1/T)$ and $T - (T^2 - 1)^{1/2} = O(1/T)$, so that

$$u(t) = -\frac{1}{4}\left(\left[2\left[\frac{1}{3} + f(t)\left(\frac{\alpha_2}{\pi}\right)^2\right]^{-1/2}\right]^{2\alpha_2/\alpha_3} + O(1)\right).$$

As remarked earlier, the function defined in (7.2.10) has a zero at $t = 1$, which is simple. Then Eq. (7.2.9) follows immediately and, as conformal mappings conserve angles, the proof of the lemma is complete. ■

7.2.2 *Limit of the Uniformization When Passing from Genus 1 to Genus 0*

The last step consists in connecting the angle θ to the group $\mathcal{H}$. In fact, we shall deal with the ratio α_3/α_2 and prove that, when $\overrightarrow{\mathbf{M}} = 0$, $\mathcal{H}$ can be interpreted as the group of transformations of $\mathbb{C}/(\alpha_2\mathbb{Z})$

$$\langle \omega \mapsto -\omega + \alpha_2, \omega \mapsto -\omega + \alpha_2 + \alpha_3 \rangle,$$

which is of order $2 \inf\{\ell \in \mathbb{Z}_+^* : \ell\alpha_3/\alpha_2 \in \mathbb{Z}\}$.

Letting $\overrightarrow{\mathbf{M}} \to 0$ by continuity with respect to the parameters $\{p_{i,j}\}_{-1\leq i,j\leq 1}$, the limit of the uniformization (3.3.3) becomes, after some algebra,

$$\begin{cases} x(\omega) = 1 + \dfrac{(x_4 - 1)(1 - x_1)}{1 - x_1 - (x_4 - x_1)\sin^2(\pi\omega/\alpha_2)}, \\ z(\omega) = \dfrac{\pi}{\alpha_2} \dfrac{(x_4 - x_1)(x_4 - 1)(x_1 - 1)\sin(2\pi\omega/\alpha_2)}{[1 - x_1 - (x_4 - x_1)\sin^2(\pi\omega/\alpha_2)]^2}, \end{cases} \tag{7.2.11}$$

where we have used the expression of α_2 given by Eq. (7.2.4). Setting $u = \exp(2i\pi\omega/\alpha_2)$, the obvious identity

$$\sin^2(\pi\omega/\alpha_2) = -(u + 1/u - 2)/4$$

allows us to rewrite in (7.2.11) $x(\omega) = \widetilde{x}(u)$, where

$$\widetilde{x}(u) = \frac{(u - z_1)(u - 1/z_1)}{(u - z_0)(u - 1/z_0)}, \tag{7.2.12}$$

and z_0, z_1 are complex numbers given by

$$\begin{cases} z_0 = \dfrac{2-(x_1+x_4) \pm 2[(1-x_1)(1-x_4)]^{1/2}}{x_4-x_1}, \\ z_1 = \dfrac{x_1+x_4-2x_1x_4 \pm 2[x_1x_4(1-x_1)(1-x_4)]^{1/2}}{x_4-x_1}. \end{cases}$$

Clearly, $\widetilde{y}(u)$ will also be a rational function of the same form as $\widetilde{x}(u)$, but its explicit computation hinges on some general properties of Riemann surfaces. The functions (7.2.12) and $\widetilde{y}(u)$ are *automorphic* and provide a rational uniformization of the algebraic curve $\mathscr{K}$, which is here of genus 0. Moreover, this uniformization is unique *up to a fractional linear transformation* (see, e.g., [94]).

Before deriving the formula for $\widetilde{y}(u)$, let us quote right away a first limit automorphism, namely $\omega \mapsto -\omega + \alpha_2$, since

$$x(\omega) = x(-\omega + \alpha_2).$$

Equivalently, we have

$$\widetilde{x}(u) = \widetilde{x}(1/u),$$

which corresponds precisely to the automorphism of $\mathbb{C}$

$$\xi(u) = \frac{1}{u}. \tag{7.2.13}$$

Similarly, exchanging the roles of x and y, the same continuity argument when $\overrightarrow{\mathbf{M}} \to 0$ in the general uniformization (3.3.3) or (3.3.4) yields

$$y(\omega) = y(-\omega + \alpha_2 + \alpha_3),$$

whence $\omega \mapsto -\omega + \alpha_2 + \alpha_3$ is the second automorphism, noting that $\alpha_3 \neq \alpha_2$.

Indeed, $z(\omega) = -z(-\omega)$, so that $y(\omega) \neq y(-\omega)$. Moreover, $x(\omega) \neq x(-\omega+\alpha_3)$, since from Lemma 3.3.3 $\alpha_3 \leq \alpha_2$, the strict inequality being obtained by comparing the two formulae in (7.2.4) (in this respect see also equation (7.2.20) in the next section).

To derive $\widetilde{y}(u)$ quickly (and nicely!), we make two observations.

- Exchanging the roles of x and y leads to another rational uniformization with some parameter v, such that
$$\widehat{y}(v) = \frac{(v-z_3)(v-1/z_3)}{(v-z_2)(v-1/z_2)}, \tag{7.2.14}$$
where z_2, z_3 are obtained from z_0, z_1, just replacing x_1, x_4 by y_1, y_4, respectively.
- Then, from the above remark, we necessarily have
$$\widetilde{y}(u) = \widehat{y}(\sigma(u)),$$

where
$$\sigma(u) = \frac{eu+f}{gu+h}, \qquad (e,g) \neq (0,0), \qquad eh - fg \neq 0$$

is a linear transformation which will be completely determined.

Since $\widetilde{x}(0) = \widetilde{x}(\infty) = 1$, see (7.2.12), we must have $\widetilde{y}(0) = \widetilde{y}(\infty) = 1$ (recalling that, when $\overrightarrow{\mathbf{M}} = 0$, $Y_0(1) = Y_1(1) = 1$).

Analogously, (7.2.14) yields $\widehat{y}(0) = \widehat{y}(\infty) = 1$. These simple equalities entail at once $\sigma(u) = \rho u$, where ρ is a complex number to be calculated later. Then, by (7.2.14), we obtain

$$\widetilde{y}(u) = \frac{(\rho u - z_3)(\rho u - 1/z_3)}{(\rho u - z_2)(\rho u - 1/z_2)}, \tag{7.2.15}$$

and the second limit automorphism η becomes on $\mathbb{C}$

$$\eta(u) = \frac{1}{\rho^2 u}. \tag{7.2.16}$$

Hence, combining (7.2.13) and (7.2.16), the generator $\delta = \eta\xi$ takes on $\mathbb{C}$ the simple form
$$\delta(u) = (\eta\xi)(u) = \frac{u}{\rho^2}.$$

Therefore, in the genus 0 case, the group $\mathcal{H}$ has the order (possibly infinite)

$$2\inf\{\ell \in \mathbb{Z}_+^* : \rho^{2\ell} = 1\}. \tag{7.2.17}$$

At that moment, we make a brief detour to connect (7.2.12) with the uniformization given in (6.5.11), namely

$$\check{y}(t) = \frac{y_1 + y_4}{2} + \frac{y_4 - y_1}{4}\left(t + \frac{1}{t}\right).$$

Setting
$$T(v) = \frac{vz_2 - 1}{v - z_2},$$

one easily checks the identity $\widehat{y}(v) = \check{y}(T(v))$, which by (7.2.15) yields in particular

$$\widehat{y}(1) = \check{y}(-1) = y_1 = \widetilde{y}(1/\rho),$$

whence, using the properties of the algebraic curve, $\widetilde{x}(1/\rho) = X(y_1)$. Finally, by (7.2.12), we obtain that ρ is a root of the second degree equation

$$\rho + \frac{1}{\rho} = 2\frac{x_1 + x_4 - 2x_1x_4 + (x_1 + x_4 - 2)X(y_1)}{(x_4 - x_1)(1 - X(y_1))}. \tag{7.2.18}$$

The roots of (7.2.18) are complex conjugate and of modulus one. Choosing then the root satisfying $\arg(\rho) \in [0, \pi]$ and letting

$$\Lambda = \frac{x_1 + x_4 - 2x_1x_4 + (x_1 + x_4 - 2)X(y_1)}{2[(X(y_1) - x_1)(X(y_1) - x_4)(1 - x_1)(x_4 - 1)]^{1/2}}, \tag{7.2.19}$$

direct algebra yields

$$\arg(\rho) = \frac{\pi}{2} - \arctan(\Lambda).$$

To conclude the proof of Part (i) of Theorem 7.1.1, it suffices to apply Eq. (7.2.20), derived in Proposition 7.2.3 below, which makes the link between $\arctan(\Lambda)$ and the ratio α_3/α_2. All conclusions so far obtained remain true and are even easier to prove when $C = 0$ (see (7.2.1) and the assumption made at the beginning of Sect. 7.2.1). ■

7.2.3 Second Form of the Criterion in the Zero Drift Case

By calculating the limit ratio α_3/α_2 in a different manner, we shall derive another expression of the angle θ defined in Eq. (7.1.1). The following proposition holds.

Proposition 7.2.3 *With Λ defined in (7.2.19) and its equivalent $\widetilde{\Lambda}$ obtained from Λ by exchanging x and y, we have*

$$\theta = \frac{\pi}{2} - \arctan(\Lambda),$$

so that $\widetilde{\Lambda} = \Lambda$ and the order of the group $\mathcal{H}$ equals

$$2\inf\{\ell \in \mathbb{Z}_+^* : \ell[1/2 - \arctan(\Lambda)/\pi] \in \mathbb{Z}\}.$$

Proof A straightforward calculation carried out along the same lines as for the derivation of formulae (7.2.4) yields, for any $t < x_1$,

$$\int_t^{x_1} \frac{\mathrm{d}x}{(1-x)[C(x-x_1)(x-x_4)]^{1/2}} =$$
$$\frac{1}{[C(x_4-1)(1-x_1)]^{1/2}}\left[\frac{\pi}{2} - \arctan\left(\frac{x_1 + x_4 - 2x_1x_4 + (x_1+x_4-2)t}{2[(t-x_1)(t-x_4)(x_4-1)(1-x_1)]^{1/2}}\right)\right].$$

Instantiating now $t = X(y_1) < x_1$ in the last formula, we get exactly

$$\frac{\alpha_3}{\alpha_2} = \frac{1}{2} - \frac{\arctan(\Lambda)}{\pi}. \tag{7.2.20}$$

■

7.2.4 Another Proof of the Criterion

For the sake of completeness, we present another proof of Theorem 7.1.1, which in comparison with that of the preceding section is simpler and more elementary, but, on the other hand, the arguments given in the original proof are more powerful. The notions of limit conformal gluing (Sect. 7.2.1) and limit uniformization (Sect. 7.2.2), when passing from genus 1 to genus 0, could also be very promising in problems such as finding the first singularities of the generating functions.

As before, we compute in two different ways the angle $\arg(X_0'(1))$ formed by the curve $\mathcal{M}_1$ on Fig. 6.1 at the double point 1.

• The first way has already been given in Sect. 7.2.1, where we have shown $\arg(X_0'(1)) = \pm\theta$, with θ defined in (7.1.1). Note that in the genus 0 case the derivative of the function $X_0(.)$ does not exist at 1, and $X_0'(1)$ is understood as

$$\lim_{y\to 1} \frac{X_0(y) - 1}{y - 1},$$

as $y \to 1$ along the real axis.

• As for the second way, the uniformization of $\mathscr{K}$ is needed. In Sect. 7.2.2 we proved that $\mathscr{K} = \{(\widetilde{x}(u), \widetilde{y}(u)) : u \in \mathbb{C}\}$, where $\widetilde{x}(u)$ and $\widetilde{y}(u)$ are given by (7.2.12) and (7.2.15). Moreover, it follows from Eq. (7.2.16) that the group $\mathcal{H}$ can be interpreted by means of the parameter ρ as

$$\langle u \mapsto 1/u, u \mapsto 1/(\rho^2 u)\rangle, \quad u \in \mathbb{C}. \tag{7.2.21}$$

By (7.2.12), letting $y \to 1$ along the real axis is tantamount to $u/\rho \to 0$, with $u \in \mathbb{R}$ in the uniformization space. Still using (7.2.12), we obtain for real values of u

$$x(u/\rho) = 1 + [(z_1 + 1/z_1) - (z_0 + 1/z_0)]u/\rho + O(u^2).$$

So, we must have $\arg([(z_1 + 1/z_1) - (z_0 + 1/z_0)]/\rho) = \pm\theta$. On the other hand, one can easily check from Sect. 7.2.2 that $[(z_1 + 1/z_1) - (z_0 + 1/z_0)] \in \mathbb{R}$, whence $\arg(1/\rho) = \pm\theta$. It suffices now to combine (7.2.17) and (7.2.21) to complete the proof. ■

7.3 Proof of Part (b) of Theorem 7.1.1

We shall prove that, in all situations (2.3.6)–(2.3.9), the group $\mathcal{H}$ is infinite. First, let us remark that it suffices to show the result for only one of these relationships, since two groups corresponding to jump probability sets obtained from one another by one of the eight symmetries of the square are clearly isomorphic, see Sect. 2.4. Thus we choose to consider the relation (2.3.7). Then the line of argument of Sect. 7.2.1 still holds and genus 0 can again be obtained by continuity from genus 1. Indeed, letting the parameters $\{p_{i,j}\}_{-1\le i,j\le 1}$ be distorted so as to yield (2.3.7), we have by Theorem 6.1.1 that in the limit $x_3 = x_4 = \infty$, and from equations (3.3.6) and (3.3.9)

$$\begin{cases} \omega_1 \to i\alpha_1, \ \text{with } \alpha_1 \in (0,\infty), \\ \omega_2 \to \infty, \\ \omega_3 \to \alpha_3 \in (0,\infty). \end{cases}$$

In particular, for the same reasons as in Sect. 7.2.2, the limit group can be interpreted as the group of transformations

$$\langle \omega \mapsto -\omega, \omega \mapsto -\omega + \alpha_3 \rangle$$

on $\mathbb{C}/(\alpha_1\mathbb{Z})$. This group is obviously infinite, and so is $\mathcal{H}$.

The proof of Theorem 7.1.1 is complete. ■

7.3.1 *Miscellaneous Remarks*

For the sake of completeness, we quote hereafter some facts related to existing works.

- Theorem 7.1.1 is quite simple to check, and therefore provides a really effective criterion.
- A direct calculation gives $\Delta = -a(1)\widetilde{a}(1)\sum_{-1\le i,j\le 1} ijp_{i,j}$, where Δ is the determinant defined in Proposition 4.1.7. Hence, for the group of order 4, our criterion clearly coincides with Eq. (4.1.17).
- From Theorem 7.1.1, it becomes clear that the famous *Gessel's walk* (i.e. with jump probabilities satisfying $p_{1,0} = p_{1,1} = p_{-1,0} = p_{-1,-1} = 1/4$) has a group of order 8. More generally, Theorem 7.1.1 leads to another proof of the (non-)finiteness of the group for all combinatorial models with an underlying genus 0, such as that described in [11, 12].
- As another straightforward consequence of Theorem 7.1.1, one can check the random walk considered in [89], with

$$p_{1,0} = p_{-1,0} = 1/2 - p_{-1,1} = 1/2 - p_{1,-1} = \sin^2(\pi/n)/2,$$

has a group of order $2n$, for all $n \geq 3$.

- The angle θ defined in (7.1.1) gives the angle of the cone in which, after a suitable linear transformation, the random walk with transitions $\{p_{i,j}\}_{-1\leq i,j\leq 1}$ has a covariance which is equal to some multiple of the identity.

Chapter 8
Miscellanea

In the first 7 chapters, we fully described only some of the topics related to the subject. There are, however, many directly related questions which, although not dealt with in the present book, deserve mention in this chapter so as to provide a panoramic overview.

8.1 On Explicit Solutions

Besides considering BVP in the complex plane, there are various alternatives. The first one is to consider a BVP on a Riemann surface, the second one to work on the universal covering, as shown in Sect. 5.5. The direct passage to the universal covering itself, by uniformization, involves Weierstrass functions. In [66], the functions π and $\widetilde{\pi}$ are represented in terms of convergent series (as for meromorphic functions) and convergent products (as for entire functions). In all these approaches, one must localize the zeros and the poles of q and $\widetilde{q}$ in some regions. While this can be easily done for some parameter values, the technical details in the general case are tedious.

The approach via BVP introduced in [32] leads to closed integral forms, involving the Weierstrass $\wp$-function, since the underlying Riemann surface has genus 0 or 1. In particular, these integrals make it possible to find the singularities of the functions $\pi(x)$, $\widetilde{\pi}(y)$, which are the key to getting the asymptotics.

It turns out that the solution method presented in this book also applies to non-stationary problems, where Laplace transforms replace generating functions (see Sect. 8.4 for related examples).

In other problems, especially in many models encountered in queueing theory, jumps inside the quarter plane are not bounded, and it is no longer possible to speak of algebraic curves. Nonetheless, one still can combine a BVP approach in the complex plane with conformal mapping techniques (or Fredholm equations). In this respect, the reader can refer to [5, 24].

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8.2 Asymptotics

Under this title, several important areas are concerned.

8.2.1 Large Deviations

Even if asymptotic problems have not been specifically covered in this book, they have many applications and are mostly of interest in higher dimensions. We recall here the results of [74] concerning the asymptotic behavior of the stationary probabilities. We shall consider the *simple random walk* defined in Chap. 2, i.e. inside the quarter plane all the transition probabilities are zero, but $p_{01}, p_{10}, p_{-1,0}, p_{0,-1}$. In addition the following assumptions are made on the drift vectors $\overrightarrow{\mathbf{M}}, \overrightarrow{\mathbf{M}}', \overrightarrow{\mathbf{M}}''$ defined in (1.2.3):

$$\overrightarrow{\mathbf{M}} < \mathbf{0}, \quad M'_y \neq 0, \quad M''_x \neq 0. \tag{8.2.1}$$

The choice of the simple random walk is not crucial for the applicability of analytical methods, but it simplifies the computations considerably. Also, the case where only one component of $\overrightarrow{\mathbf{M}}$ in (8.2.1) is negative can be analyzed via similar methods.

Some new facts about the Riemann surface will be needed. Let $x(s)$, $y(s)$ be meromorphic functions on $\mathbf{S}$ defining the coverings of the x-plane and y-plane respectively.

1. We know that the algebraic function $Y(x)$ [resp. $X(y)$] has exactly four branch points x_i [resp. y_i], where

$$\begin{cases} 0 < x_1 < x_2 < 1 < x_3 < x_4, \\ 0 < y_1 < y_2 < 1 < y_3 < y_4. \end{cases}$$

2. Let $\mathbf{S}_r = \{s : x(s) \in \mathbb{R}\ , y(s) \in \mathbb{R}\}$ be the set of all real points of $\mathbf{S}$. Then $\mathbf{S}_r$ consists of two disjoint analytic closed curves, homologous to one of the elements of the normal homology basis on $\mathbf{S}$, more exactly to the one different from $x^{-1}(\{x : |x| = 1\})$. These curves will be denoted by F_0, F_1, with F_0 satisfying

$$x_2 \le x(s) \le x_3 \quad \text{and} \quad y_2 \le y(s) \le y_3, \ \forall s \in F_0.$$

   F_0 contains an ordered set of eight characteristic points $s_0, \dots, s_7$, which are defined as follows:

$$\begin{cases} s_0 = (1,1),\ s_1 = \left(\sqrt{\dfrac{p_{0,-1}}{p_{01}}}, y_2\right),\ s_2 = \left(\dfrac{p_{0,-1}}{p_{01}}, 1\right), \\ s_3 = \left(x_3, \sqrt{\dfrac{p_{-1,0}}{p_{10}}}\right),\ s_4 = \left(\dfrac{p_{0,-1}}{p_{01}}, \dfrac{p_{-1,0}}{p_{10}}\right),\ s_5 = \left(\sqrt{\dfrac{p_{0,-1}}{p_{01}}}, y_3\right), \\ s_6 = \left(1, \dfrac{p_{-1,0}}{p_{10}}\right),\ s_7 = \left(x_2, \sqrt{\dfrac{p_{-1,0}}{p_{10}}}\right). \end{cases}$$

3. The function $\phi_\gamma(s) = |xy^\gamma|$, $0 \le \gamma \le 1$, has in the set $\{x \neq 0, y \neq 0\}$ four non-degenerate critical points $s_i(\gamma)$, $i = 1, \ldots, 4$, which are defined by the two equations
$$Q(x,y) = 0, \quad y\frac{\partial}{\partial y}Q(x,y) = \gamma x \frac{\partial}{\partial x}Q(x,y).$$

Each $s_i(\gamma)$ is uniquely defined and depends continuously on γ, noting that $x(s_i(0)) = x_i$.

Moreover, $s_2(\gamma), s_3(\gamma) \in F_0$ and $x_i(\gamma) = x(s_i(\gamma))$, $y_i(\gamma) = y(s_i(\gamma))$ are real. For $\gamma = 1$, one can set $s_1(1) = 0, s_4(1) = \infty$, and hence, for the critical points $s_2(1), s_3(1)$, the above assertions hold. We also have

$$\begin{cases} 1 < x_3(1) < x_3(\gamma) < x_3(0) = x_3, \\ y_3(0) = \sqrt{\dfrac{p_{0,-1}}{p_{01}}} < y_3(\gamma) < y_3(1). \end{cases}$$

It appears that the asymptotics of the stationary probabilities is determined either by the critical point $s_3(\gamma)$ or by the zeros of q_ξ or $\widetilde{q}_\eta$. Note as a reminder that ξ and η are the Galois automorphisms on **S** constructed in Sect. 2.4, and such that

$$\xi(x,y) = \left(x, \frac{p_{0,-1}}{p_{01}y}\right), \quad \eta(x,y) = \left(\frac{p_{-1,0}}{p_{10}x}, y\right).$$

Now in the parameter space $\mathcal{P} \times \{\gamma : 0 < \gamma \le 1\}$, with $\mathcal{P}$ given in Sect. 2.3.1, we introduce the subsets

$$\begin{cases} \mathcal{P}_{--} = \left\{(p,\gamma) : q\left(x_3(\gamma), \dfrac{p_{0,-1}}{p_{01}y_3(\gamma)}\right) \le 0,\ \widetilde{q}\left(\dfrac{p_{-1,0}}{p_{10}x_3(\gamma)}, y_3(\gamma)\right) \le 0\right\}, \\ \mathcal{P}_{+-} = \left\{(p,\gamma) : q\left(x_3(\gamma), \dfrac{p_{0,-1}}{p_{01}y_3(\gamma)}\right) > 0,\ \widetilde{q}\left(\dfrac{p_{-1,0}}{p_{10}x_3(\gamma)}, y_3(\gamma)\right) \le 0\right\}, \end{cases}$$

and $\mathcal{P}_{-+}, \mathcal{P}_{++}$ accordingly. The main result is given by the next theorem.

Theorem 8.1 *Let $m, n \to \infty$ so that $\dfrac{n}{m} \to \gamma$, $0 < \gamma \le 1$. Then we have*

$$\pi(m,n) \sim \begin{cases} \dfrac{C_1}{\sqrt{m}} x_3^{-m}(\gamma) y_3^{-n}(\gamma) & \textit{in } \mathcal{P}_{--}\,, \\ C_2 x_0^{-m}(\gamma) y_0^{-n}(\gamma) & \textit{in } \mathcal{P}_{-+}\,, \\ C_3 x_5^{-m}(\gamma) y_5^{-n}(\gamma) & \textit{in } \mathcal{P}_{+-}\,, \\ C_4 x_0^{-m}(\gamma) y_0^{-n}(\gamma) + C_5 x_5^{-m}(\gamma) y_5^{-n}(\gamma) & \textit{in } \mathcal{P}_{++}, \end{cases} \tag{8.2.2}$$

where the C_i's are constants (depending on γ) and

$$\begin{cases} 1 < x_0(\gamma) < x_3(\gamma), \quad 1 < y_0(\gamma) < \dfrac{p_{0,-1}}{p_{01} y_3(\gamma)}, \\ 1 < x_5(\gamma) < \dfrac{p_{-1,0}}{p_{10} x_3(\gamma)}, \quad 1 < y_5(\gamma) < y_3(\gamma), \end{cases}$$

are solutions of the respective systems

$$\begin{cases} Q(x,y) = 0, \quad q(x, \xi y) = 0, \quad \text{for } x_0(\gamma), y_0(\gamma); \\ Q(x,y) = 0, \quad \widetilde{q}(\eta x, y) = 0, \quad \text{for } x_5(\gamma), y_5(\gamma). \end{cases}$$

For $\gamma = 0$, the asymptotic behavior could be obtained (although this was not done) by means of the method proposed in [86]. ■

We will only give a brief sketch of the methods used to prove the main result (8.2.2). The stationary probabilities can be represented by two-dimensional Cauchy integrals, which, using Leray residues (see e.g., [2]), can be reduced to one-dimensional integrals over some cycle on the Riemann surface. The asymptotics will thus be defined either by the steepest descent point (*saddle-point*) or by a pole, that could be encountered while moving the integration contour on the surface. Both the saddle point and the possible pole belong to the real part of the algebraic curve.

8.2.2 Martin Boundary

Roughly, let us simply say that the so-called *Martin boundary* (MB) describes the way in which a process escapes to infinity (see e.g., [91]). For non-smooth regions, probabilistic methods rarely yield a complete characterization, while complex analysis appears promising. To calculate the MB, one needs the asymptotics of Green functions, and thus of the MB kernel. This requires a little more effort than large deviation asymptotics. Transient random walks in $\mathbb{Z}_+^2$, with the same kind of jumps as in the present book, were considered in [60]. Starting from convenient functional equations, it was possible by analytic continuation arguments to obtain the first singularity and the saddle-point (which form the main contribution to the MB), but still not a complete explicit expression.

8.2.3 *Random Walks Absorbed on the Axes*

These processes appear in many domains (queueing theory, statistical physics, combinatorics). Accordingly they have received a lot of attention over the last 40 years, mainly via probabilistic tools, pure analytic treatments being more seldom. Such an example can be found in [61], for spatially homogeneous random walks in $\mathbb{Z}_+^2$, absorbed when reaching the axes. Here, $\overrightarrow{\mathbf{M}} \neq \mathbf{0}$ and jumps are of size 1. Absorption probabilities of generating functions are obtained, together with their explicit asymptotics along the axes. Asymptotics of the Green functions are also computed along all possible infinite paths, in particular along those approaching the axes tangentially. The key point of the method consists in solving a bi-variate functional equation of the form (1.3.6) satisfied by the Green function $G(x, y)$.

In order to identify the Martin boundary for partially homogeneous random walks on a half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, a natural idea to characterize the so-called *Martin compactification* by using large deviation methods was first proposed by Ignatiouk–Robert in [55]. This approach is applied in [56] to killed random walks in the quadrant $\mathbb{Z}_+^2$, the jumps being unbounded with an exponential decay. In this case, the full Martin compactification is shown to be homeomorphic to the closure of the set $\{w = z/(1 + |z|) : z \in \mathbb{Z}_+^2\}$ in $\mathbb{R}_+^2$. The proofs are primarily of a probabilistic nature, and do not use functional equations.

8.3 Generalized Problems and Analytic Continuation

There are essentially two main possible extensions. First, for finite jumps of arbitrary size. Second, when the *maximal space homogeneity* condition introduced in Chap. 1 does not hold. The reader will observe that these two classes of problems are mathematically not disjoint.

8.3.1 *Arbitrary Finite Jumps*

We have already seen how analytic continuation is crucial in most of the problems, including asymptotics. Undoubtedly the first step toward a generalization, in the case of jumps bounded in modulus by a finite number n, is the analytic continuation process. Here there are $2n$ unknown functions, $\pi_i(s), \widetilde{\pi}_i(s)$, which must be analytic in the connected domain $\mathcal{E} \subset \mathbf{S}$,

$$\mathcal{E} = \{|x(s)| < 1,\ |y(s)| < 1\}.$$

Then a functional equation can be obtained, on a Riemann surface $\mathbf{S}$ of arbitrary genus, which has the form

$$\sum_{i=1}^{n}\left(q_i(s)\pi_i(x(s)) + \widetilde{q}_i(s)\widetilde{\pi}_i(y(s))\right) + q_0(s) = 0 \tag{8.3.1}$$

where $q_i(s), \widetilde{q}_i(s)$ are meromorphic on $\mathbf{S}$. The next results were proved in [75].

1 $\pi_i(s), \widetilde{\pi}_i(s)$ can be continued *infinitely* outside $\mathcal{E}$ and can have only algebraic branch points in the complex plane. The resulting covering surface is isomorphic to the disc but it is not the universal covering of $\mathbf{S}$. The reasons become clear when we consider the sequence of fields

$$F_0 \subset F_1 \subset F_2 \subset \dots, \tag{8.3.2}$$

where

- F_0 is the field of meromorphic functions on $\mathbf{S}$;
- F_{2i} is the minimal Galois extension of $C(y)$, containing $F_{2i-1}, i \geq 1$;
- F_{2i+1} is the minimal Galois extension of $C(x)$ containing $F_{2i}, i \geq 0$.

In the generic situation $\pi_i(s), \widetilde{\pi}_i(s)$ can only be considered as meromorphic functions on the limit of $\mathbf{S}_i$-Riemann surfaces of F_i, or, more exactly, as sections of the inductive limit $\lim_{i\to\infty}$ of the bundles of meromorphic functions on $\mathbf{S}_i$:

- There is a necessary and sufficient condition for the finiteness of the sequence F_i, showing that stabilization rarely occurs. When stabilization does take place, the functions are meromorphic on S_k for some finite k.
- If the problem can be solved by Wiener–Hopf factorization techniques, then the chain stops at F_2.

In the preliminary study [40], finding and classifying branch-points and their associated cuts appear to be two crucial issues. Indeed, the genus of the surface is larger than 1, and one has to deal with hyperelliptic curves. The ultimate goal is to set a generalized BVP on a single curve for a vector of analytic functions.

8.3.2 Space Inhomogeneity

The class of generalized problems contains the famous queueing model referred to as *Joining the Shorter Queue*, which was studied in the non-symmetrical case in [28, 54]. Although it will be the subject of Chap. 10, we will outline its main features here. There is no space homogeneity: the quarter plane is separated into two homogeneous regions, and one must write a functional equation for each region. This gives rise to the following system:

$$\begin{cases} Q_1(x, y)\pi_1(x, y) = q_{11}(x, y)\pi_1(x) + q_{12}(x, y)\pi_2(x) + \widetilde{q}_1(x, y)\widetilde{\pi}_1(y), \\ Q_2(x, y)\pi_2(x, y) = q_{21}(x, y)\pi_1(x) + q_{22}(x, y)\pi_2(x) + \widetilde{q}_2(x, y)\widetilde{\pi}_2(y). \end{cases}$$

There are four unknown functions of one variable. The algebraic curves corresponding to Q_1 and Q_2 are of genus zero. It is possible to write, in some region, a *non-Nœtherian* BVP (i.e. its index is *not finite*). Upon combining Q_1 and Q_2, let us simply say that we are in the situation of non-Galois extension. All functions can be analytically continued as meromorphic functions to the whole complex plane. We refer the reader to Chap. 10 for more details.

8.3.3 Dimension ≥ 3

It is not necessary to insist on the usefulness of getting results for random walks in $\mathbb{Z}_+^N$, $N \geq 3$. For a first step in this direction, see [85]. The general problem of unique continuation for solutions of functional equations on spaces with group actions was considered in [77]. Except for very specia1 cases, a global solution seems out of reach, even computationally. But, this is not too surprising, since, for instance, the ergodicity conditions for random walks in $\mathbb{Z}_+^N$ require finding invariant measures of walks in dimensions $\mathbb{Z}_+^N$!

8.4 Transient Behavior and Laplace Transforms

So far, we have concentrated on the steady state distribution. However, many questions related to performance evaluation require some information about time dependent evolution.

8.4.1 Sojourn Time in a Jackson Network with Overtaking

Consider for instance the distribution of the joint number of customers (M_t, N_t) at time t in coupled queues, under exponential assumptions for the laws of interarrival and service times. Writing Kolmogorov's forward equations, and then taking the Laplace transform

$$F(x, y, s) = \int_0^\infty e^{-st} E[x^{M_t} y^{N_t}] dt, \quad \Re(s) \geq 0,$$

it is possible to get a BVP for $F(x, 0, s)$ or $F(0, y, s)$, observing that s merely plays the role of a complex parameter with a positive real part. Time asymptotics can also be obtained by Tauber's theorems.

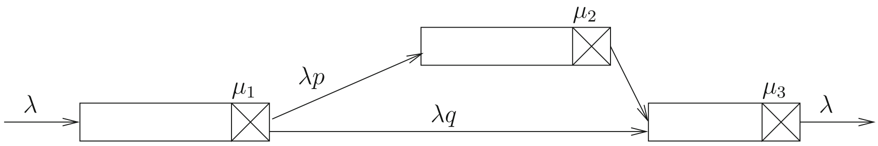

Fig. 8.1 Network with overtaking

A problem analyzed in [33] deals with the sojourn time of a customer in the open 3-node queueing network (of Jackson's type) shown in Fig. 8.1. An inherent *overtaking phenomenon* makes things slightly more complicated. Let us just say that cutting the Gordian Knot amounts to finding the function $G(x, y, z, s)$, which is the Laplace transform of the conditional waiting time distribution of a tagged customer at a departure instant of the first queue. The following non-homogeneous functional equation was obtained and solved.

$$\begin{aligned}K(x, y, z, s)G(x, y, z, s) = {} & \left(\mu_1 - \frac{\lambda}{x}\right) G(0, y, z, s) \\ & + \left(\mu_3 - q\mu_1 \frac{x}{z} - \mu_2 \frac{y}{z}\right) G(x, y, 0, s) \\ & + \frac{\mu_2\mu_3}{(1-x)[s + \mu_3(1-z)]},\end{aligned}$$

where p, q are routing probabilities with $p + q = 1$, λ is the external arrival rate, μ_i is the service rate at queue i, and

$$K(x, y, z, s) = s + \lambda\left(1 - \frac{1}{x}\right) + \mu_1\left(1 - px - q\frac{x}{z}\right) + \mu_2\left(1 - \frac{y}{z}\right) + \mu_3(1 - z).$$

Indeed, considering y and s as parameters, one sees at once that the last equation takes the form

$$K(x, y, z, s)\widetilde{G}(x, z) = A(x)\widetilde{G}(0, z) + B(x, y, z, s)\widetilde{G}(x, 0) + C(x, y, z, s),$$

where K, A, B, C are known functions.

Then, setting $\rho_i = \lambda/\mu_i$, it follows (see [33]) that the stationary sojourn time of an arbitrary customer has a Laplace transform given by

$$(1-\rho_1)(1-\rho_2)(1-\rho_3)\frac{\mu_1}{\mu_1 + s} G\left[\frac{\mu_1}{\mu_1 + s}, \rho_2, \rho_3, s\right].$$

8.4.2 *Diffusion Process*

It is known that a two-dimensional Brownian motion in $\mathbb{R}_+^2$ can be obtained, after a proper scaling, as the limit of a discrete random walk in $\mathbb{Z}_+^2$. In [4], the authors consider a diffusion process in $\mathbb{R}_+^2$, which is supposed to mimic a model of a two-queue Jackson network. The determination of its invariant measure is tantamount to resolving a BVP in the right half-plane for an elliptic operator on a hyperbola (!). As the Riemann surface has genus 0, reasonably simple explicit solutions are obtained for the bivariate Laplace-transform.

8.5 Outside Probability

Operator Factorization and C^*-Algebras. There has been a great deal of activity concerning the *index problem* for pseudo-differential operators in domains with a smooth boundary, and one of the central results is the so-called *Atiyah–Singer index theorem*. In the present book, we have dealt with the simplest case of non-smooth boundaries (one corner), and the derivation of the ergodicity conditions via the index, presented in Sects. 5.4 and 6.5, are new.

In larger dimensions, the index problem, especially for Toeplitz operators in $\mathbb{Z}_+^N$, Wiener–Hopf equations in $\mathbb{R}_+^N$ and BVP for N complex variables, is still largely open. The reason resides in the inductiveness property: dimension N demands much finer properties for the related problems in dimension $N-1$ [78]. However, a formal inductive solution for general N can be given in terms of C^*-algebras and by factorization of functions on the circle, taking their values in the set of compact operators in Hilbert spaces (see [76]).

Physics. Some relationships with the so-called *integrable models* in statistical physics are described in [78], but they definitely require other strong mathematical tools. The methods of the book can be used to tackle a new class of *integrable systems*, such as those appearing in *quantum physics*.

Consider for instance the one-dimensional *Quantum Three-Body* problem. Three particles $a(s), b(s), c(s)$ walk independently on the one-dimensional lattice, except when two or three particles are at the same point P, in which case the jumps may be different. Then the pair of distances

$$(X(s), Y(s)) \stackrel{\text{def}}{=} (b(s) - a(s), c(s) - b(s))$$

form a random walk on the two-dimensional lattice, which is homogeneous everywhere, but on the axes $X(s) = 0$ and $Y(s) = 0$.

Part II
Applications to Queueing Systems and Analytic Combinatorics

Chapter 9
A Two-Coupled Processor Model

This chapter has something of a historical flavor, as it revisits the outlines of the model analyzed in [32], where the method of reduction to a Riemann–Hilbert BVP in the complex plane was originally proposed. It is a concrete application of the methods developed in Part I of this book.

9.1 Model and Equations

Let us consider two parallel M/M/1 queues, with infinite capacities, under the following assumptions.

- Arrivals form two independent Poisson processes with parameters λ_1, λ_2.
- Service times are distributed exponentially with instantaneous service rates S_1 and S_2 depending on the state of the system as follows.

  1. If both queues are busy, then $S_1 = \mu_1$ and $S_2 = \mu_2$.
  2. If queue 2 is empty, then $S_1 = \mu_1^*$.
  3. If queue 1 is empty, then $S_2 = \mu_2^*$.

- The service discipline is FIFO (first-in-first-out) in each queue.

One can directly see that the evolution of the system can be described by the two-dimensional *continuous time Markov process* (M_t, N_t), which stands for the joint number of customers in the queues.

Let $p_t(m, n)$ the probability $\mathbb{P}(M_t = m, N_t = n)$ that, at time t, one finds m jobs in queue 1 and n customers in queue 2. The stationary probabilities

$$p(m, n) \stackrel{\text{def}}{=} \lim_{t\to\infty} p_t(m, n)$$

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satisfy the classical Kolmogorov forward equations, which after setting

$$F(x, y) = \sum_{m\geq 0, n\geq 0} p(m, n)x^m y^n,$$

lead to the basic functional equation (leaving the details to the reader)

$$T(x, y)F(x, y) = a(x, y)F(0, y) + b(x, y)F(x, 0) + c(x, y)F(0, 0), \quad (9.1.1)$$

where

$$\begin{cases} T(x, y) = \lambda_1(1-x) + \lambda_2(1-y) + \mu_1\left(1-\dfrac{1}{x}\right) + \mu_2\left(1-\dfrac{1}{y}\right), \\ a(x, y) = \mu_1\left(1-\dfrac{1}{x}\right) + q\left(1-\dfrac{1}{y}\right), \\ b(x, y) = \mu_2\left(1-\dfrac{1}{y}\right) + p\left(1-\dfrac{1}{x}\right), \\ c(x, y) = p\left(\dfrac{1}{x}-1\right) + q\left(\dfrac{1}{y}-1\right), \\ p = \mu_1 - \mu_1^*, \\ q = \mu_2 - \mu_2^*. \end{cases}$$

Note that

$$\frac{xyT(x, y)}{\lambda_1 + \lambda_2 + \mu_1 + \mu_2}$$

is a probabilistic kernel, analogous to $Q(x, y)$ in equation (1.3.5). Clearly, equation (9.1.1) is of the type (1.3.6), and hence can be solved by the methods described in Chaps. 4 and 5, noting however that $F(x, y)$ takes into account the *full probability measure*, i.e. including the axes, in contrast to $Q(x, y)$ which stands for the measure strictly inside the quarter plane. For instance, the normalizing condition is merely $F(1, 1) = 1$.

From Sect. 2.3.2, the following properties hold.

(a) The algebraic curve defined by $\{T(x, y) = 0\}$ is of genus 1. Moreover, according to definition 2.3.3, the underlying random walk is *simple* and its group is of order 4 by (4.1.17).
(b) The algebraic function $Y(x)$ has four positive real branch points

$$0 < x_1 < x_2 < 1 < x_3 < x_4 < \infty,$$

where

$$x_1 = \frac{\lambda_1 + \mu_1 + (\sqrt{\lambda_2} + \sqrt{\mu_2})^2 - \sqrt{(\lambda_1 + \mu_1 + (\sqrt{\lambda_2} + \sqrt{\mu_2})^2)^2 - 4\lambda_1\mu_1}}{2\lambda_1},$$
$$x_2 = \frac{\lambda_1 + \mu_1 + (\sqrt{\lambda_2} - \sqrt{\mu_2})^2 - \sqrt{(\lambda_1 + \mu_1 + (\sqrt{\lambda_2} - \sqrt{\mu_2})^2)^2 - 4\lambda_1\mu_1}}{2\lambda_1},$$
$$x_3 = \frac{\mu_1}{\lambda_1 x_2}, \quad x_4 = \frac{\mu_1}{\lambda_1 x_1}.$$

(c) $\forall x \in \mathbb{C}_x$, the two roots $Y_0(x)$ and $Y_1(x)$ of $T(x, y) = 0$ satisfy the relation

$$Y_0(x).Y_1(x) = \frac{\mu_2}{\lambda_2}. \tag{9.1.2}$$

9.2 Reduction to a Boundary Value Problem on a Circle

When $x \in [\overrightarrow{\underleftarrow{x_1 x_2}}]$ (see notation in Sect. 5.3), the relation (9.1.2) shows immediately that $Y_0(x)$ and $Y_1(x)$ are complex conjugate and traverse the circle of radius $\sqrt{\frac{\mu_2}{\lambda_2}}$ in opposite directions. It is worth pointing out that, from a computational aspect, the situation is very favorable since there is no need to construct a gluing function.

So, we are looking for a probabilistic solution of the system

$$\begin{cases} T(x, y) = 0, \; |x| \le 1, \; |y| \le 1, \\ a(x, y)F(0, y) + b(x, y)F(x, 0) + c(x, y)F(0, 0) = 0. \end{cases} \tag{9.2.1}$$

From Chap. 5, we know that all functions can be meromorphically continued until the cut $[x_3, x_4]$ and that Theorems 5.4.1 and 5.4.3 apply. However, the form of the coefficients a, b, c in (9.1.1) allows us to distinguish two cases, which differ from both the modeling and technical points of view.

(i) $pq - \mu_1\mu_2 = 0$, which is tantamount to

$$\frac{\mu_1}{\mu_1^*} + \frac{\mu_2}{\mu_2^*} = 1,$$

or, for $0 \le \xi \le 1$,

$$\begin{cases} \mu_1 = \xi\mu_1^*, \\ \mu_2 = (1 - \xi)\mu_2^*. \end{cases} \tag{9.2.2}$$

In a queueing modeling context, this corresponds to the so-called head-of-line processor sharing service discipline (see [32]). Then the second equation of system (9.2.1) becomes

$$\xi F(0,y) - (1-\xi)F(x,0) = \frac{[\mu_1^*(1-\xi)\left(\frac{1}{x}-1\right) - \mu_2^*\xi\left(\frac{1}{y}-1\right)}{\mu_1^*\left(1-\frac{1}{x}\right) - \mu_2^*\left(1-\frac{1}{y}\right)} F(0,0), \tag{9.2.3}$$

where we shall always assume $\mu_1^* \mu_2^* \neq 0$.

(ii) $pq - \mu_1\mu_2 \neq 0$. Setting

$$\begin{cases} F(x,0) = \dfrac{q\mu_1^*}{\mu_1\mu_2 - pq} F(0,0) + G(x), \\ F(0,y) = \dfrac{p\mu_2^*}{\mu_1\mu_2 - pq} F(0,0) + H(y), \end{cases}$$

the second equation in (9.2.1) becomes homogeneous, exactly

$$a(x,y)H(y) + b(x,y)G(x) = 0 \quad \mod T(x,y). \tag{9.2.4}$$

9.3 The Case $pq = \mu_1\mu_2$: Solutions with Elliptic Integrals

When (9.2.2) holds, $F(0,y)$ can be directly obtained by applying Cauchy's formula in the disc $\mathcal{D}\big(\sqrt{\frac{\mu_2}{\lambda_2}}\big)$ bounded by the circle $C\big(\sqrt{\frac{\mu_2}{\lambda_2}}\big)$. Indeed, denoting by $U(x,y)F(0,0)$ the function on the right-hand side of (9.2.3), we eliminate $F(x,0)$ by letting $x \in [x_1, x_2]$ to get the following Dirichlet boundary condition

$$\xi[F(0,y) - F(0,\overline{y}] = [U(X_0(y),y) - U(X_0(y),\overline{y}]F(0,0), \ \forall y \in C\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right). \tag{9.3.1}$$

However the denominator in (9.2.3) might have a pole, and remember the constraint of finding functions analytic in the unit disc $\mathcal{D}$, which is the essence of Theorem 5.4.1 with $L = M = 1$. Setting $\rho_i^* = \frac{\lambda_i}{\mu_i^*}$, $i = 1, 2$, let us simply mention that this ergodicity condition is written

$$1 - \rho_1^* - \rho_2^* > 0,$$

in which case the denominator of $U(X_0(y), y)$ does not vanish in $\mathcal{D}\big(\sqrt{\frac{\mu_2}{\lambda_2}}\big)$. By (9.1.2), we have in (9.3.1) $y\overline{y} = \sqrt{\frac{\mu_2}{\lambda_2}}$, and Cauchy's formula yields, after some algebraic manipulations,

$$F\left(0, \sqrt{\frac{\mu_2}{\lambda_2}}z\right) = \frac{1}{\pi}\int_0^\pi \frac{z\sin\theta\, v(\theta)d\theta}{z^2 - 2z\cos\theta + 1} + F(0,0), \quad |z| < 1,$$

where

$$
\begin{aligned}
v(\theta) &= \frac{-\lambda_2 \sin\theta K(\theta)}{\xi[\rho_1^*(\mu_2^* - \mu_1^*)K^2(\theta) + (\mu_1^* - \mu_2^* + \lambda_1 + \lambda_2)K(\theta) - \mu_1^*]},\\
K(\theta) &= \frac{\lambda_1 + \mu_1 + \beta - \sqrt{[(\sqrt{(\lambda_2} + \sqrt{\mu_2})^2 + \beta][(\sqrt{\lambda_2} - \sqrt{\mu_2})^2 + \beta]}}{2\lambda_1},\\
\beta &= \lambda_2 + \mu_2 - 2\sqrt{\lambda_2\mu_2}\cos\theta.
\end{aligned}
$$

In [32], the functions $F(0, y)$ and $F(x, 0)$ have been completely expressed in terms of *elliptic integrals of the third kind.*

9.4 The Case $pq \neq \mu_1\mu_2$

Here the function $H(y)$ in (9.2.4) is the solution of a Riemann–Hilbert–Carleman BVP (5.2.28) set on the circle $C\left(\sqrt{\frac{\mu_2}{\lambda_2}}\right)$, with a reduced form given by Eqs. (5.4.19) and (5.4.20). Ergodicity conditions can been obtained either probabilistically (Theorem 1.2.1) or analytically (Theorem 5.4.1). We state them without further comment.

Theorem 9.4.1 *The Coupled-Queue System is ergodic if, and only if, the following conditions hold.*

1. $\begin{cases} \lambda_1 < \mu_1, \quad \lambda_2 < \mu_2\,, \\ \mu_1^*(\mu_2 - \lambda_2) > \mu_2\lambda_1 - \mu_1\lambda_2, \\ \mu_2^*(\mu_1 - \lambda_1) > \mu_1\lambda_2 - \mu_2\lambda_1. \end{cases}$
2. $\lambda_1 < \mu_1, \quad \lambda_2 \geq \mu_2, \quad \mu_2^*(\mu_1 - \lambda_1) > \mu_1\lambda_2 - \mu_2\lambda_1.$
3. $\lambda_1 \geq \mu_1, \quad \lambda_2 < \mu_2, \quad \mu_1^*(\mu_2 - \lambda_2) > \mu_2\lambda_1 - \mu_1\lambda_2.$

■

Although starting from a slightly different formulation of the BVP, explicit computations were carried out in [32], which the interested reader may refer to. We do not include here the related integral-form expressions, which are plainly consistent with formula (5.4.21).

9.4.1 $p + q = 0$: Rational Solutions

A very peculiar situation occurs when $p + q = 0$, in which case $pq \neq \mu_1\mu_2$ (complex parameters are not allowed !). From the results of Chap. 4, it is normal to see whether rational solutions may exist, and to do so, it is sufficient to check condition (4.6.18). One sees directly that our model belongs to the so-called *one-parameter family* analyzed in Sect. 4.9.1.3, where the parameter t must be replaced by q, and in this case (4.6.18) holds and the solution takes the simple form (4.9.9). Hence, we necessarily have

$$F(x, y) = \frac{F(0, 0)}{(1 - \alpha x)(1 - \beta y)},$$

where

$$F(0, 0) = (1 - \alpha)(1 - \beta).$$

To find the unknowns α and β, we instantiate $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$ in (9.1.1). This yields the system

$$\begin{cases} \mu_1 - \lambda_1 = \mu_1(1 - \alpha) + p(1 - \beta) - p(1 - \alpha)(1 - \beta), \\ \mu_2 - \lambda_2 = \mu_2(1 - \beta) + q(1 - \alpha) - q(1 - \alpha)(1 - \beta), \end{cases}$$

which after some easy algebra reduces to

$$\begin{cases} \alpha(\mu_1^* + p\beta) = \lambda_1, \\ \beta(\mu_2^* + q\alpha) = \lambda_2. \end{cases} \tag{9.4.1}$$

Taking into account the condition $p + q = 0$ in (9.4.1), we get the conservation law

$$\alpha\mu_1^* + \beta\mu_2^* = \lambda_1 + \lambda_2,$$

where α and β satisfy respectively the following second degree equations

$$\begin{cases} p\mu_1^*\alpha^2 - [p(\lambda_1 + \lambda_2) + \mu_1^*\mu_2^*]\alpha + \lambda_1\mu_2^* = 0, \\ q\mu_2^*\beta^2 - [q(\lambda_1 + \lambda_2) + \mu_1^*\mu_2^*]\beta + \lambda_2\mu_1^* = 0. \end{cases}$$

We leave to the reader the exercise of checking that, under the ergodicity conditions given in Theorem 9.4.1, the system (9.4.1) admits exactly one solution (α, β) such that

$$0 < \alpha < 1, \quad 0 < \beta < 1.$$

Chapter 10
Joining the Shorter of Two Queues: Reduction to a Generalized BVP

This chapter aims to analyze a typical random walk not satisfying the maximal space homogeneity condition **P1**, as stated in Sect. 1.2. The example chosen below is a long-standing problem borrowed from queueing theory. It highlights the huge additional complexity which arises when space-homogeneity is only partial, even for 2-dimensional systems. Indeed, the question often becomes tantamount to solving a *system of functional equations* of the form (1.3.6).

10.1 Model and Historical Remarks

Two queues with exponentially distributed service times of rate α, β, respectively are placed in parallel. The external arrival process is Poisson with parameter λ. The incoming customer always joins the shorter line, or, if the lines are equal, he joins queue 1 or queue 2 with respective probabilities π_1 and $|pi_2$. The basic problem is to analyze the steady state distribution of the joint number of customers in the system.

The symmetrical case, i.e. $\alpha = \beta$, $\pi_1 = \pi_2 = 1/2$, was originally solved in [44], where the unknown functions were proved to be meromorphic in the whole complex plane. Indeed, we shall find an explicit solution by reduction to a BVP on a curve of genus zero.

For unequal service rates (i.e. $\alpha \neq \beta$), the original analysis was made in the Ph.D. theses [28, 54], but never published elsewhere. Two theoretical methods were proposed therein: first by reduction to a boundary value problem (BVP) of non-standard type; secondly, by showing that all unknown functions are meromorphic, with their poles being recursively computed. Inspired by these ideas, J.W Cohen 20 years later gave in [23] an algorithmic procedure to compute these poles and the corresponding residues. Quite surprisingly, he did not refer explicitly to the ideas in

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the above studies, but preferred to mention the article [1], which in turn cites [28, 54] and proposes an approximate iterative algorithm, via the so-called *compensation method*. We propose an analytic solution, the complexity of which is equivalent to a Fredholm integral equation.

10.2 Equations

Here again, the evolution of the system can be depicted by the two-dimensional continuous time Markov process (M_t, N_t), representing the joint number of customers in the queues, but, as we shall see, the story is slightly more intricate, due to the non-spatial homogeneity of the underlying random walk.

Indeed, letting $p_t(m,n)$ denote the probability $\mathbb{P}(M_t = m, N_t = n)$ that at time t there are m jobs in queue 1 and n customers in queue 2, Kolmogorov's equations for the stationary probabilities

$$p(m,n) \stackrel{def}{=} \lim_{t\to\infty} p_t(m,n)$$

have to be written separately in the two distinct regions

$$\mathcal{R}_1 \stackrel{def}{=} \{(m,n), m \le n\} \quad \text{and} \quad \mathcal{R}_2 \stackrel{def}{=} \{(m,n), n \le m\}.$$

In region $\mathcal{R}_1$ we have, $\forall i, j \ge 0$,

$$\begin{aligned}\left[\lambda + \alpha \mathbb{1}_{\{i\ge 1\}} + \beta \mathbb{1}_{\{i+j\ge 1\}}\right] p(i, i+j) &= \alpha p(i+1, j+1) + \beta p(i, i+j+1) \\ &+ \lambda \mathbb{1}_{\{i\ge 1\}} p(i-1, i+j) + \lambda(\mathbb{1}_{\{j=0\}} + \pi_2 \mathbb{1}_{\{j=1\}}) p(i, i+j-1),\end{aligned} \tag{10.2.1}$$

where $\mathbb{1}_{\{\}}$ denotes the classical indicator function. Similarly, in $\mathcal{R}_2$ we have

$$\begin{aligned}\left[\lambda + \alpha \mathbb{1}_{\{i+j\ge 1\}} + \beta \mathbb{1}_{\{j\ge 1\}}\right] p(i+j, j) &= \alpha p(i+1, j+1) + \beta p(i+j+1, j) \\ + \lambda \mathbb{1}_{\{j\ge 1\}} p(i+j, j-1) &+ \lambda(\mathbb{1}_{\{i=0\}} + \pi_1 \mathbb{1}_{\{i=1\}}) p(i+j-1, j).\end{aligned} \tag{10.2.2}$$

Remark 10.2.1 Before exploiting systems (10.2.1) and (10.2.2) by using generating functions, it is worth stressing from the outset that any attempt to solve a functional equation involving several functions of two variables leads to a dead end (to the best of our knowledge of the present state of the art!). The only promising theoretical way consists in constructing a *system of functional equations of the form* (1.3.6).

Define

$$
\begin{cases}
F_1(x,y) = \sum_{i,j\geq 0} p(i,i+j)x^i y^j, \quad P_1(x) = \sum_{i,\geq 0} p(i,i+1)x^i,\\
F_2(x,y) = \sum_{i,j\geq 0} p(i+j,i)x^i y^j, \quad P_2(x) = \sum_{i,\geq 0} p(i+1,i)x^i,\\
Q(x) = F_1(x,0) = F_2(x,0) = \sum_{i,\geq 0} p(i,i)x^i,\\
A_1(x) = (\alpha+\lambda x)P_2(x), \quad A_2(x) = (\beta+\lambda x)P_1(x),\\
G_i(y) = F_i(0,y), \ i=1,2,\\
T_1(x,y) = \lambda\left(1-\frac{x}{y}\right) + \alpha\left(1-\frac{y}{x}\right) + \beta\left(1-\frac{1}{y}\right), \quad R_1(x,y) = xy\,T_1(x,y),\\
T_2(x,y) = \lambda\left(1-\frac{x}{y}\right) + \beta\left(1-\frac{y}{x}\right) + \alpha\left(1-\frac{1}{y}\right), \quad R_2(x,y) = xy\,T_2(x,y),\\
s = \lambda+\alpha+\beta.
\end{cases}
\tag{10.2.3}
$$

Then, after multiplying Kolmogorov's equations by $x^i y^j$, some direct algebra yields the following system.

$$
\begin{cases}
T_1(x,y)F_1(x,y) &= \alpha\left(1-\frac{y}{x}\right)G_1(y) + \left(\frac{\lambda\pi_2 y^2 - \lambda x - \beta}{y}\right)Q(x) + A_1(x),\\
T_2(x,y)F_2(x,y) &= \beta\left(1-\frac{y}{x}\right)G_2(y) + \left(\frac{\lambda\pi_1 y^2 - \lambda x - \alpha}{y}\right)Q(x) + A_2(x),\\
sQ(x) &= A_1(x) + A_2(x).
\end{cases}
\tag{10.2.4}
$$

10.2.1 Reduction of the Number of Unknown Functions

The ergodicity of the process is equivalent to the existence of $F_1(x,y)$ and $F_2(x,y)$ holomorphic in $\mathcal{D}\times\mathcal{D}$ and continuous in $\overline{\mathcal{D}}\times\overline{\mathcal{D}}$. At first sight, system (10.2.4) includes four unknown functions of one variable. In fact, this number immediately boils down to two, by using the zeros of the *kernels* $R_j(x,y)$, $j=1,2$, in $\overline{\mathcal{D}}\times\overline{\mathcal{D}}$. So, we are left with *two unknown functions of one complex variable*, say for instance $G_i(\cdot), i=1,2$, or $A_i(.), i=1,2$. In order to get additional information, we shall combine several BVPs derived from system (10.2.4). As we shall see, determining one function is sufficient to find all the others.

10.3 Meromorphic Continuation to the Complex Plane

It is easy to check that the algebraic curves defined by $\{R_j(x, y) = 0\}$, $j = 1, 2$, correspond to random walks of genus 0, case 3 of Theorem 6.1.1, which was studied in Sect. 6.3.

Notation and Assumption *For convenience and to distinguish between the two kernels, we shall add, either in a superscript or subscript position ad libitum, the pair* $\alpha\beta$ (resp. $\beta\alpha$) *to any quantity related to the kernel* $R_1(x, y)$ (resp. $R_2(x, y)$). *For instance, the branches* $Y_0^{\alpha\beta}$, $X_1^{\alpha\beta}$, *etc. Also, if a property holds both for* $\alpha\beta$ and $\beta\alpha$, *the pair is omitted.* ***From now on, we assume*** $\beta > \alpha$, *the case* $\beta = \alpha$ *being considered in a separate section at the end of this chapter.*

The functions $Y_i^{\alpha\beta}(x)$, $i = 0, 1$, have exactly two branch points, which are always located inside $\mathcal{D}$,

$$x = 0, \quad \text{and} \quad x^*_{\alpha\beta} = \frac{4\alpha\beta}{s^2 - 4\alpha\lambda} < 1. \tag{10.3.1}$$

With the notation of Sect. 5.3 for the contour corresponding to a slit, $\Psi^{\alpha\beta}$ will denote the ellipse (see Sect. 6.3) obtained by the mapping

$$\overrightarrow{\overleftarrow{[0, x^*_{\alpha\beta}]}} \xrightarrow{Y^{\alpha\beta}} \Psi^{\alpha\beta}.$$

In the $\mathbb{C}_y$-plane, setting $y = u + iv$, the equation of $\Psi^{\alpha\beta}$ is

$$\left(u - \frac{\beta s}{s^2 - 4\alpha\lambda}\right)^2 + \frac{s^2 v^2}{s^2 - 4\alpha\lambda} = \frac{\beta^2 s^2}{(s^2 - 4\alpha\lambda)^2}.$$

Similarly for $X_i^{\alpha\beta}(x)$, $i = 0, 1$, with the branch-points

$$y_1^{\alpha\beta} = \frac{\beta}{s + 2\sqrt{\alpha\lambda}} < y_2^{\alpha\beta} = \frac{\beta}{s - 2\sqrt{\alpha\lambda}}, \quad 0 < y_1^{\alpha\beta} < y_2^{\alpha\beta} < 1,$$

and $\Phi^{\alpha\beta}$ will denote the ellipse obtained by the mapping

$$\overrightarrow{\overleftarrow{[y_1^{\alpha\beta}, y_2^{\alpha\beta}]}} \xrightarrow{X^{\alpha\beta}} \Phi^{\alpha\beta}.$$

In the $\mathbb{C}_x$-plane, setting $x = u + iv$, the equation of $\Phi^{\alpha\beta}$ is

$$\left(u - \frac{2\alpha\beta}{s^2 - 4\alpha\lambda}\right)^2 + \frac{s^2 v^2}{s^2 - 4\alpha\lambda} = \frac{\alpha\beta^2 s^2}{\lambda(s^2 - 4\alpha\lambda)^2}.$$

Exchanging the parameters α and β, the respective branch points of $Y_i^{\beta\alpha}(x)$ are

$$x = 0, \quad \text{and} \quad x^*_{\beta\alpha} = \frac{4\alpha\beta}{s^2 - 4\beta\lambda} < 1, \tag{10.3.2}$$

and those of $X_i^{\beta\alpha}(x)$,

$$y_1^{\beta\alpha} = \frac{\alpha}{s + 2\sqrt{\beta\lambda}} < y_2^{\beta\alpha} = \frac{\alpha}{s - 2\sqrt{\beta\lambda}}, \quad 0 < y_1^{\beta\alpha} < y_2^{\beta\alpha} < 1.$$

We shall denote by $\Phi_{int}^{\alpha\beta}$ and $\Phi_{ext}^{\alpha\beta}$, respectively, the interior and exterior domains bounded by the ellipse $\Phi^{\alpha\beta}$.

The assumption $\beta > \alpha$ entails $x^*_{\alpha\beta} < x^*_{\beta\alpha}$. But, with regard to the respective position of $\Phi^{\alpha\beta}$ and $\Phi^{\beta\alpha}$, all situations are possible. Indeed, letting $\overline{OA}_{\alpha\beta}$ and $\overline{OA}_{\beta\alpha}$ be the affixes of the respective rightmost real points of $\Phi^{\alpha\beta}$ and $\Phi^{\beta\alpha}$, so that

$$\overline{OA}_{\alpha\beta} = \frac{sy_2^{\alpha\beta} - \beta}{2\lambda}, \quad \overline{OA}_{\beta\alpha} = \frac{sy_2^{\beta\alpha} - \alpha}{2\lambda}, \tag{10.3.3}$$

then an immediate calculation shows that

$$\overline{OA}_{\alpha\beta} < \overline{OA}_{\beta\alpha},$$

if and only if

$$s > 2\sqrt{\lambda}(\sqrt{\alpha} + \sqrt{\beta}). \tag{10.3.4}$$

But, when $\alpha < \beta$, the following inequality always holds.

$$x^*_{\beta\alpha} < \overline{OA}_{\alpha\beta}, \quad \text{that is} \quad [0, x^*_{\beta\alpha}] \subset \Phi_{int}^{\alpha\beta}. \tag{10.3.5}$$

Indeed, this amounts to studying the sign of the quantity

$$\sqrt{\alpha}s^2 - 4\sqrt{\alpha\lambda}s - 4(\beta - 2\alpha),$$

which is performed by computing the roots in z of the quadratic equation (details are left to the reader).

$$\sqrt{\alpha}z^2 - 4\sqrt{\alpha\lambda}z - 4(\beta - 2\alpha) = 0.$$

In the present context, Theorem 6.1.1 and Lemma 6.3.1 are relevant, yielding in particular the following simple relations [written only for the kernel $R_1(x, y)$], which will be useful for the analytic continuation process of the unknown functions.

$$\begin{cases} Y_0^{\alpha\beta}(1) = \min\left(1, \dfrac{\lambda+\beta}{\alpha}\right), \quad Y_1^{\alpha\beta}(1) = \max\left(1, \dfrac{\lambda+\beta}{\alpha}\right), \\ X_0^{\alpha\beta}(1) = \min\left(1, \dfrac{\alpha}{\lambda}\right), \quad X_1^{\alpha\beta}(1) = \max\left(1, \dfrac{\alpha}{\lambda}\right), \\ |Y_1^{\alpha\beta}(x)| \geq \dfrac{s|x|}{2\alpha}, \quad \forall x \in \mathbb{C}_x, \\ |X_1^{\alpha\beta}(y)| \geq \dfrac{|sy-\beta|}{2\lambda}, \quad \forall y \in \mathbb{C}_y. \end{cases} \tag{10.3.6}$$

We are now in a position to prove the next important result.

Theorem 10.3.1 *The functions $Q(\cdot), G_i(\cdot), A_i(\cdot), i = 1, 2,$ can be continued as meromorphic functions to the whole complex plane.*

Proof We shall present the main lines of the algorithm, the core of which rely on two technical lemmas.

Beforehand, from the preceding chapters and Sect. 6.3, we know there exist two non-empty regions $\mathcal{U}_x^{\alpha\beta}, \mathcal{U}_y^{\alpha\beta}$ such that the solutions of the system

$$R_1(x, y) = 0, \quad |x|, |y| \leq 1, \tag{10.3.7}$$

are necessarily of the form $\big(x, Y_0^{\alpha\beta}(x)\big)$ or $\big(X_0^{\alpha\beta}(y), y\big)$, for $x \in \mathcal{U}_x^{\alpha\beta}$, $y \in \mathcal{U}_y^{\alpha\beta}$. Similar properties hold for the system

$$R_2(x, y) = 0, \quad |x|, |y| \leq 1, \tag{10.3.8}$$

just replacing (α, β) by (β, α).

Lemma 10.3.2 *Let $\mathcal{D}_n$ be the domain recursively defined by*

$$\begin{cases} \mathcal{D}_0 &= \mathcal{D}, \\ \mathcal{D}_{n+1} &= \inf\big\{(X_1 \circ Y_1)^{\alpha\beta}(\mathcal{D}_n), (X_1 \circ Y_1)^{\beta\alpha}(\mathcal{D}_n)\big\}. \end{cases}$$

Then $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ and $\lim_{n\to\infty} \mathcal{D}_n = \mathcal{P}$, where $\mathcal{P}$ denotes the complex plane.

Proof The first part of the lemma is proved by induction. Indeed, the property is true for $n = 0$, from the monotonicity properties of the function $|Y_2|$, and the fact that the curve $Y_2(\Gamma)$ is outside the unit circle Γ.

Suppose $\mathcal{D}_{n-1} \subset \mathcal{D}_n$. Then

$$\mathcal{D}_{n+1} \supset \inf\big\{(X_1 \circ Y_1)^{\alpha\beta}(\mathcal{D}_{n-1}), (X_1 \circ Y_1)^{\beta\alpha}(\mathcal{D}_{n-1})\big\} = \mathcal{D}_n.$$

For the last point, we use the equality

$$(X_1 \circ Y_1)^{\alpha\beta}(t) = \frac{\alpha(Y_1^{\alpha\beta}(t))^2}{\lambda t},$$

which yields, since $|Y_1^{\alpha\beta}(t)| > \dfrac{s|t|}{2\alpha}$, $|(X_1 \circ Y_1)^{\alpha\beta}(t)| \geq \dfrac{s^2}{4\alpha\lambda}|t|$. Hence $\mathcal{D}_n$ contains the disc $m^n|\mathcal{D}|$, where $m = \dfrac{s^2}{4\lambda \max(\alpha, \beta)} > 1$, and

$$\lim_{n\to\infty} m^n \mathcal{D} = \mathcal{P}.$$

The lemma is proved. ■

When the system (10.3.7) is satisfied, we have

$$\alpha\left(1 - \frac{y}{x}\right) G_1(y) + \left(\frac{\lambda\pi_2 y^2 - \lambda x - \beta}{y}\right) Q(x) + A_1(x) = 0. \tag{10.3.9}$$

Analogously, when (10.3.8) holds,

$$\beta\left(1 - \frac{y}{x}\right) G_2(y) + \left(\frac{\lambda\pi_1 y^2 - \lambda x - \alpha}{y}\right) Q(x) + A_2(x) = 0. \tag{10.3.10}$$

Upon combining (10.3.9) and (10.3.10), by a simple calculation we obtain

$$Q(x) = \frac{1}{\Delta(x)}\left[\alpha\left(1 - \frac{Y_0^{\alpha\beta}(x)}{x}\right) G_1(Y_0^{\alpha\beta}(x)) + \beta\left(1 - \frac{Y_0^{\beta\alpha}(x)}{x}\right) G_2(Y_0^{\beta\alpha}(x))\right], \tag{10.3.11}$$

where

$$\Delta(x) = s - \left(\frac{\alpha}{x} + \lambda\pi_2\right) Y_0^{\alpha\beta}(x) - \left(\frac{\beta}{x} + \lambda\pi_1\right) Y_0^{\beta\alpha}(x). \tag{10.3.12}$$

Lemma 10.3.3

(i) $G_1(y)$ is analytic in the annular region $\mathcal{V}^{\alpha\beta}$ located between the unit circle Γ and $Y_1^{\alpha\beta}(\Gamma)$.

(ii) In the unit disc $\mathcal{D}$, we have the following equations.

$$\begin{cases} W_{\alpha\beta}(x)Q(x) &= \alpha\left(1 - \dfrac{Y_1^{\alpha\beta}(x)}{x}\right) G_1(Y_1^{\alpha\beta}(x)) - \alpha\left(1 - \dfrac{Y_0^{\alpha\beta}(x)}{x}\right) G_1(Y_0^{\alpha\beta}(x)), \\ W_{\alpha\beta}(x)A_1(x) &= B(x, Y_0^{\alpha\beta}(x), Y_1^{\alpha\beta}(x)) G_1(Y_1^{\alpha\beta}(x)) \\ & \quad -B(x, Y_1^{\alpha\beta}(x), Y_0^{\alpha\beta}(x)) G_1(Y_0^{\alpha\beta}(x)), \end{cases} \tag{10.3.13}$$

where

$$\begin{cases} W_{\alpha\beta}(x) &= \left(Y_0^{\alpha\beta}(x) - Y_1^{\alpha\beta}(x)\right)\left(\lambda\pi_2 + \dfrac{\alpha}{x}\right), \\ B(x, u, v) &= \alpha\left(1 - \dfrac{v}{x}\right)\left(\dfrac{\lambda x + \beta}{u} - \lambda\pi_2 u\right). \end{cases}$$

The same kind of lemma holds for the triple (Q, A_2, G_2), after a convenient permutation of the adequate parameters.

Proof First, recall that $|X_0^{\alpha\beta}(\Gamma)| \leq 1$ and $X_0^{\alpha\beta} \circ Y_1^{\alpha\beta}(\Gamma) = \Gamma$. Moreover, $Q(X_0^{\alpha\beta}(y))$, $A_1(X_0^{\alpha\beta}(y))$ are analytic for all $y \in \mathcal{V}^{\alpha\beta}$. Hence, by using (10.3.9), we continue analytically $G_1(y)$ to the domain $Y_1^{\alpha\beta}(C)$, noting that the quantity $\alpha\left(1 - \dfrac{y}{X_0^{\alpha\beta}(y)}\right)$ has no zero different from 1.

Then, writing in (10.3.9) that $A_1(x)$ is continuous when traversing the cut $[0, x^*_{\alpha\beta}]$, we obtain the first equation of (10.3.13), a priori only for $x \in [0, x^*_{\alpha\beta}]$. But, from point (i) of the lemma, $G_1(Y_1^{\alpha\beta}(x))$ is also defined in a neighborhood of $[0, x^*_{\alpha\beta}]$. So, by the uniqueness of the analytic continuation, (10.3.13) is valid for $x \in \mathcal{D}$. The lemma is proved. ■

All the pieces of the puzzle are now available to complete the proof of Theorem 10.3.1 by induction, as follows.

1. **Assumption**: $Q(x)$, $A_1(x)$, $A_2(x)$ are meromorphic in $\mathcal{D}_n$, $n \geq 0$.
2. **Conclusion**:

- $G_1(y)$ [resp. $G_2(y)$] can be meromorphically continued to the region $Y_1^{\alpha\beta}(\mathcal{D}_n)$ [resp. $Y_1^{\beta\alpha}(\mathcal{D}_n)$].
- $Q(x)$, $A_1(x)$, $A_2(x)$ can be meromorphically continued to $\mathcal{D}_{n+1} \supset \mathcal{D}_n$.

The first point is immediate by (10.3.9), (10.3.10) and Lemma 10.3.3. As for the second assertion, it suffices to exploit equation (10.3.11) together with the second equation of system (10.3.13), noting that

$$|Y_0(x)| < \sup_{z \in \mathcal{D}_n} |Y_1(z)|, \quad \forall x \in X_1 \circ Y_1(\mathcal{D}_n).$$

■

10.4 A Fredholm Integral Equation for $Q(x)$ When $\alpha \neq \beta$

This section presents entirely new material, which in our opinion provides an effective solution to the problem, including the numerical aspect. The approach is to combine two BVPs, respectively satisfied by the pairs $Q(\cdot)$, $A_1(\cdot)$ and $Q(\cdot)$, $A_2(\cdot)$. Then, taking into account the last conservation equation of system (10.2.4), one is ultimately left with a classical Fredholm integral equation on a real interval.

Multiplying system (10.3.9) by x (for technical reasons which become clear later on), one can eliminate $G_1(y)$ by letting y traverse the cut $[y_1^{\alpha\beta}, y_2^{\alpha\beta}]$, where $X_0^{\alpha\beta}(y) = \overline{X_1}^{\alpha\beta}(y)$. This leads to the BVP

$$
\begin{aligned}
&U_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))Q(x) + V_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))B_1(x) = \\
&U_{\alpha\beta}(\bar{x}, Y_0^{\alpha\beta}(x))Q(\bar{x}) + V_{\alpha\beta}(\bar{x}, Y_0^{\alpha\beta}(x))B_1(\bar{x}), \quad x \in \Phi^{\alpha\beta},
\end{aligned}
\tag{10.4.1}
$$

where

$$
U_{\alpha\beta}(x, y) = \frac{(\lambda\pi_2 x + \alpha)y - sx}{\alpha(x-y)}, \quad V_{\alpha\beta}(x, y) = \frac{(x-1)^{\varepsilon}}{\alpha(x-y)}, \quad B_1(x) = \frac{xA_1(x)}{(x-1)^{\varepsilon}},
$$

and

$$
\varepsilon = \begin{cases} 1, & \text{if } x = 1 \in \Phi_{int}^{\alpha\beta}. \\ 0, & \text{if } x = 1 \in \Phi_{ext}^{\alpha\beta}. \end{cases}
$$

Let $S_{\alpha\beta}(x)$, a function analytic in $\Phi_{int}^{\alpha\beta}$, be the solution of the following auxiliary Riemann–Carleman BVP (see Sect. 5.2.5).

$$
\frac{S_{\alpha\beta}(\bar{x})}{S_{\alpha\beta}(x)} = \frac{V_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))}{V_{\alpha\beta}(\bar{x}, Y_0^{\alpha\beta}(\bar{x}))}, \quad x \in \Phi^{\alpha\beta}. \tag{10.4.2}
$$

Then (10.4.1) becomes

$$
\frac{B_1(x)}{S_{\alpha\beta}(x)} - \frac{B_1(\bar{x})}{S_{\alpha\beta}(\bar{x})} = \frac{U_{\alpha\beta}(\bar{x}, Y_0^{\alpha\beta}(x))Q(\bar{x})}{V_{\alpha\beta}(\bar{x}, Y_0^{\alpha\beta}(x))S_{\alpha\beta}(\bar{x})} - \frac{U_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))Q(x)}{V_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))S_{\alpha\beta}(x)}, \quad x \in \Phi^{\alpha\beta}. \tag{10.4.3}
$$

Before going further, let us fix some technical points right away.

1. By (10.3.11) and the first statement of Lemma 10.3.3, the poles of $Q(x)$ in $\Phi_{int}^{\alpha\beta}$ and $\Phi_{int}^{\beta\alpha}$ correspond to zeros of $\Delta(x)$. Indeed, Eqs. (10.3.11) and (10.3.12) show that $Q(x)$ is analytic in $\mathcal{D}$ and continuous on Γ if and only if $\Delta(x)$ has no zero in the closed unit disc $\overline{\mathcal{D}}$, except at $x = 1$. By (10.3.6), there are two contingencies.

   (a) $\lambda < \beta - \alpha$, then $\Delta(1) \neq 0$.
   (b) $\lambda > \beta - \alpha$, then $\Delta(1) = 0$. Then the derivative

   $$
   \Delta'(1) = \frac{\lambda(\alpha + \beta - \lambda)\left[\lambda + \dfrac{\alpha\beta + \pi_2\alpha^2 + \pi_1\beta^2}{\alpha + \beta}\right]}{\lambda^2 - (\beta - \alpha)^2}
   $$

   is positive if and only if $\lambda < \alpha + \beta$, which appears to be a *necessary condition* for the system to be ergodic (see Theorem 10.5.1 in Sect. 10.5). Indeed, letting $x \to 1$ in Eq. (10.3.11),

   $$
   Q(1) = \frac{\alpha[1 - Y_0'^{\alpha\beta}(1)]G_1(1) + \beta[1 - Y_0'^{\beta\alpha}(1)]G_2(1)}{\Delta'(1)},
   $$

which is positive if and only if (10.5.3) holds, as can easily be verified. Now, distorting Γ by making a small indentation to the right around $x = 1$, we get a new contour $\widetilde{\Gamma}$ on which $\Re[\Delta(x)] > 0$, so that $\arg[\Delta(x)]_{\widetilde{\Gamma}} = 0$. Then, by invoking the argument principle, the function $\Delta(x)$, meromorphic in $\overline{\mathcal{D}}_{ext}$ with a simple pole at infinity, has exactly one zero (hence real) in $\overline{\mathcal{D}}_{ext}$.

2. From the equation of the ellipse $\Phi^{\alpha\beta}$ given in Sect. 10.3, it can be immediately shown that the point $x = 1$ belongs to $\Phi^{\alpha\beta}_{int}$ if and only if

$$\alpha\beta > \sqrt{\alpha\lambda}(s - 2\sqrt{\alpha\lambda}),$$

which reduces to

$$(\sqrt{\alpha} - \sqrt{\lambda})(\beta + \lambda - \sqrt{\alpha\lambda}) > 0. \tag{10.4.4}$$

Hence, when $\beta > \alpha$, inequality (10.4.4) holds if and only if

$$\lambda < \alpha. \tag{10.4.5}$$

Exchanging the roles of α and β, condition (10.4.4) becomes

$$(\sqrt{\beta} - \sqrt{\lambda})(\alpha + \lambda - \sqrt{\beta\lambda}) > 0,$$

which is not satisfied if $\beta < \lambda$. So $\beta > \lambda$ is only *necessary* for the point $x = 1$ to be in $\Phi^{\beta\alpha}_{int}$. Indeed, the point $x = 1$ belongs to $\Phi^{\beta\alpha}_{int}$ if and only if

$$\lambda < \beta < \frac{(\alpha + \lambda)^2}{\lambda}. \tag{10.4.6}$$

3. By formula (5.2.43) and Theorem 5.2.8, the number of solutions of (10.4.2) depends on the index $\widetilde{\chi}$, where

$$\widetilde{\chi} = \frac{-\arg[V_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))]_{\Phi^{\alpha\beta}}}{2\pi}.$$

Here, $V_{\alpha\beta}(x, Y_0^{\alpha\beta}(x))$ is analytic outside $\mathcal{D}$ and, if $Y_0^{\alpha\beta}(1) = 1$,

$$Y_0^{\alpha\beta}(1)' = \frac{\lambda - \alpha}{\lambda - \alpha + \beta}.$$

Whatever the position of $x = 1$ with respect to $\Phi^{\alpha\beta}$, one can check, by using the argument principle around Γ, that

$$\widetilde{\chi} = 0.$$

Hence, the factorization problem (10.4.2) has a unique solution $S_{\alpha\beta}(x)$ analytic in $\Phi_{int}^{\alpha\beta}$, up to a multiplicative constant. Moreover, $S_{\alpha\beta}(x)$ has no zero in $\Phi_{int}^{\alpha\beta}$, as shown by the form of the general solution of the BVP (5.2.28) with index 0.

Lemma 10.4.1 *Suppose $x = 1 \in \Phi_{int}^{\alpha\beta} \cap \Phi_{int}^{\beta\alpha}$, namely inequalities (10.4.5) and (10.4.6) are simultaneously satisfied. Then we have the following properties.*

1. *If inequality (10.3.4) holds, then $Q(x)$ has no pole in $\Phi_{int}^{\alpha\beta}$.*
2. *Inequality (10.3.4) does not hold, then $Q(x)$ has no pole in $\Phi_{int}^{\beta\alpha}$.*
3. *The eventual poles of $Q(x)$, $A_1(x)$, $A_2(x)$ in $\Phi_{int}^{\alpha\beta} \cup \Phi_{int}^{\beta\alpha}$ coincide. Moreover, there is at most one such pole, which is of course real.*

Proof We have seen that the zero of $\Delta(x)$ in $\overline{\mathcal{D}}_{ext}$ is real. In the genus 0 case, the branches (mostly analyzed in Chap. 6) have the following monotonicity properties on the real axis, as the reader can easily check.

(i) On $[-\infty, 0]$, $Y_0^{\alpha\beta}(x)$ increases from $-\infty$ to $y_1^{\alpha\beta} = Y_0^{\alpha\beta}\left(\frac{sy_1^{\alpha\beta}-\beta}{2\lambda}\right)$, and decreases from $y_1^{\alpha\beta}$ to 0.

(ii) On $[x_{\alpha\beta}^*, \infty]$, $Y_0^{\alpha\beta}(x)$ decreases from $\dfrac{sx_{\alpha\beta}^*}{2\lambda}$ to $y_2^{\alpha\beta} = Y_0^{\alpha\beta}(\overline{OA_{\alpha\beta}})$, and increases from $y_2^{\alpha\beta}$ to ∞.

Assertion 1 follows immediately, since in this case $\Re[\Delta(x)] > 0$ on the interval $[x_{\alpha\beta}^*, \overline{OA}_{\alpha\beta}]$, and assertion 2 is obtained by symmetry on the parameters.

As for assertion 3, it suffices to observe, for instance, that the image of the domain $\overline{\mathcal{D}}_{ext} \cap \Phi_{int}^{\alpha\beta}$ by the mapping $Y_0^{\alpha\beta}(x)$ belongs to $\mathcal{D}$. Hence, to ensure the analyticity of $G_1(y)$ in $\mathcal{D}$, one can see by (10.3.9) that the poles of $Q(x)$ and $A_1(x)$ must coincide. The uniqueness is clear since $\Delta(x)$ has at most one zero in $\mathcal{D}_{ext}$. ■

Now let $\theta_{\alpha\beta}$ be a function carrying out the conformal mapping of $\Phi_{int}^{\alpha\beta}$ onto the unit disc (see Theorem 6.3.1). Then $\theta_{\alpha\beta}$ is holomorphic in $\Phi_{int}^{\alpha\beta}$, its inverse function $\theta_{\alpha\beta}^{-1}$ is holomorphic in $\mathcal{D}$, and we have

$$v = \theta(u), \quad |\theta_{\alpha\beta}(u)| = 1, \quad \forall u \in \Phi^{\alpha\beta}.$$

Moreover, by symmetry, one can choose $\theta_{\alpha\beta}(\bar{u}) = \overline{\theta_{\alpha\beta}(u)}$, $\forall u \in \Phi^{\alpha\beta}$, so that

$$\theta_{\alpha\beta}(\bar{u}) = \frac{1}{\theta_{\alpha\beta}(u)}, \quad \forall u \in \Phi^{\alpha\beta}.$$

For $v \in \Gamma$ (the unit circle), $\bar{v} = 1/v$ and $\bar{u} = \theta_{\alpha\beta}^{-1}(1/v)$. Then, since $\theta_{\alpha\beta}^{-1}(1/v)$ is analytic for $|v| > 1$, we can use Cauchy's residue theorem in the domain $\mathcal{D}_{ext}$ in the BVP (10.4.3), the integration variable being v.

By Lemma 10.4.1, we know that $Q(x)$ and $A_1(x)$ have at most one pole (which is then real) at $x = x_0$. In this respect, recalling that $B_1(x)$ was introduced in (10.4.1), let

$$b_1 = \lim_{x \to x_0} (x - x_0) B_1(x), \quad q = \lim_{x \to x_0} (x - x_0) Q(x), \tag{10.4.7}$$

a residue being in general taken to be 0 if the corresponding pole does not exist. We shall first establish two intermediate equations, readily obtained by Cauchy's theorem. Set $w = \theta(x)$, and define

$$I(x) = \frac{1}{2i\pi} \int_\Gamma \frac{B_1(\theta_{\alpha\beta}^{-1}(v))dv}{S_{\alpha\beta}(\theta_{\alpha\beta}^{-1}(v))(v - w)}, \quad J(x) = \frac{1}{2i\pi} \int_\Gamma \frac{B_1(\theta_{\alpha\beta}^{-1}(1/v))dv}{S_{\alpha\beta}(\theta_{\alpha\beta}^{-1}(1/v))(v - w)},$$

noting that $I(x)$ and $J(x)$ are analytic for $x \in \Phi_{int}^{\alpha\beta}$. Then

$$I(x) = \frac{B_1(x)}{S_{\alpha\beta}(x)} + \frac{\varepsilon\theta'_{\alpha\beta}(1)A_1(1)}{S_{\alpha\beta}(1)[\theta_{\alpha\beta}(1) - \theta_{\alpha\beta}(x)]} + \frac{b_1\theta'_{\alpha\beta}(x_0)}{S_{\alpha\beta}(x_0)[\theta_{\alpha\beta}(x_0) - \theta_{\alpha\beta}(x)]},$$
$$J(x) = \frac{\varepsilon\theta'_{\alpha\beta}(1)A_1(1)}{S_{\alpha\beta}(1)\theta_{\alpha\beta}(1)[1 - \theta_{\alpha\beta}(1)\theta_{\alpha\beta}(x)]} + \frac{b_1\theta'_{\alpha\beta}(x_0)}{S_{\alpha\beta}(x_0)\theta_{\alpha\beta}(x_0)[1 - \theta_{\alpha\beta}(x_0)\theta_{\alpha\beta}(x)]},$$

so that (10.4.3) leads to the integral equation

$$\begin{aligned} I(x) - J(x) = {} & \frac{1}{2i\pi} \int_{\Phi^{\alpha\beta}} \left[\frac{U_{\alpha\beta}(\bar{u}, Y_0^{\alpha\beta}(u))Q(\bar{u})}{V_{\alpha\beta}(\bar{u}, Y_0^{\alpha\beta}(u))S_{\alpha\beta}(\bar{u})} \right] \frac{d\theta_{\alpha\beta}(u)}{\theta_{\alpha\beta}(u) - \theta_{\alpha\beta}(x)} \\ & - \frac{1}{2i\pi} \int_{\Phi^{\alpha\beta}} \left[\frac{U_{\alpha\beta}(u, Y_0^{\alpha\beta}(u))Q(u)}{V_{\alpha\beta}(u, Y_0^{\alpha\beta}(u))S_{\alpha\beta}(u)} \right] \frac{d\theta_{\alpha\beta}(u)}{\theta_{\alpha\beta}(u) - \theta_{\alpha\beta}(x)}, \quad \forall x \in \Phi_{int}^{\alpha\beta}. \end{aligned} \tag{10.4.8}$$

As for the first integral, it is convenient to change the variable $\bar{u} \to u$. Since $\theta_{\alpha\beta}(\bar{u}) = 1/\theta_{\alpha\beta}(u)$, $u \in \Phi^{\alpha\beta}$,

$$d\theta_{\alpha\beta}(\bar{u}) = -\frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}^2(u)},$$

and (10.4.8) can be rewritten, for $x \in \Phi_{int}^{\alpha\beta}$, as

$$\begin{aligned} & I(x) - J(x) = \\ & \frac{1}{2i\pi} \int_{\Phi^{\alpha\beta}} \frac{U_{\alpha\beta}(u, Y_0^{\alpha\beta}(u))Q(u)}{V_{\alpha\beta}(u, Y_0^{\alpha\beta}(u))S_{\alpha\beta}(u)} \left[\frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u)[1 - \theta_{\alpha\beta}(u)\theta_{\alpha\beta}(x)]} - \frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u) - \theta_{\alpha\beta}(x)} \right]. \end{aligned} \tag{10.4.9}$$

Let for the moment $C(x)$ denote the integral in (10.4.9). Clearly, the integrand in $C(x)$ is meromorphic in the region $\Phi_{int}^{\alpha\beta} - [0, x_{\alpha\beta}^*]$, where it has simple poles at $u = x$ and possibly at $u = \{x_0, 1\}$. Then Cauchy's theorem allows us to write

$$C(x) = \frac{1}{2\pi} \int_0^{x^*_{\alpha\beta}} Q(u) H_{\alpha\beta}(u) \left[\frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u)[1-\theta_{\alpha\beta}(u)\theta_{\alpha\beta}(x)]} - \frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u)-\theta_{\alpha\beta}(x)} \right]$$
$$+ R_{\alpha\beta}(x), \quad \forall x \in \Phi^{\alpha\beta}_{int} - [0, x^*_{\alpha\beta}],$$

where

$$H_{\alpha\beta}(u) = \frac{(\lambda\pi_2 u + \alpha)\big(Y_1^{\alpha\beta}(u) - Y_0^{\alpha\beta}(u)\big)}{i(u-1)^{\varepsilon} S_{\alpha\beta}(u)}$$
$$= \frac{(\lambda\pi_2 u + \alpha)\sqrt{(s^2 - 4\alpha\lambda)u(x^*_{\alpha\beta} - u)}}{\alpha(u-1)^{\varepsilon} S_{\alpha\beta}(u)},$$

and $R_{\alpha\beta}(x)$ is the sum of the residues at $u = \{1, x_0, x\}$. By direct algebra,

$$R_{\alpha\beta}(x) = \frac{\varepsilon(\lambda\pi_1 + \beta)Q(1)}{S_{\alpha\beta}(1)} \left[\frac{\theta'_{\alpha\beta}(1)}{\theta_{\alpha\beta}(1) - \theta_{\alpha\beta}(x)} - \frac{\theta'_{\alpha\beta}(1)}{\theta(1)[1-\theta_{\alpha\beta}(1)\theta_{\alpha\beta}(x)]} \right]$$
$$+ \frac{qU(x_0, Y_0^{\alpha\beta}(x_0))\theta'_{\alpha\beta}(x_0)}{S_{\alpha\beta}(x_0)\theta_{\alpha\beta}(x_0)[1-\theta_{\alpha\beta}(x_0)\theta_{\alpha\beta}(x)]} - \frac{qU(x_0, Y_0^{\alpha\beta}(x_0))\theta'_{\alpha\beta}(x_0)}{S_{\alpha\beta}(x_0)[\theta_{\alpha\beta}(x_0) - \theta_{\alpha\beta}(x)]}$$
$$- \frac{U(x, Y_0^{\alpha\beta}(x))Q(x)}{S_{\alpha\beta}(x)V(x)}.$$

On the other hand, system (10.3.9) yields

$$A_1(1) = (\lambda\pi_1 + \beta)Q(1),$$

together with a similar relation (details left to the reader) between the residues b_1 and q_1 introduced in (10.4.7). Then, it is easy to check that the first four terms in the expression of $R_{\alpha\beta}$ cancel with the corresponding residues appearing in the expression of $I(x) - J(x)$, so that Eq. (10.4.9) simplifies to

$$\frac{xA_1(x)}{(x-1)^{\varepsilon} S_{\alpha\beta}(x)} = \frac{1}{2\pi} \int_0^{x^*_{\alpha\beta}} Q(u) H_{\alpha\beta}(u) \left[\frac{\theta_{\alpha\beta}(x)\theta'_{\alpha\beta}(u)du}{1-\theta_{\alpha\beta}(u)\theta_{\alpha\beta}(x)} - \frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u)-\theta_{\alpha\beta}(x)} \right]$$
$$- \frac{[(\lambda\pi_2 x + \alpha)Y_0^{\alpha\beta}(x) - sx]Q(x)}{(x-1)^{\varepsilon} S_{\alpha\beta}(x)} + K_{\alpha\beta}, \quad \forall x \in \Phi^{\alpha\beta}_{int} - [0, x^*_{\alpha\beta}], \tag{10.4.10}$$

where $K_{\alpha\beta}$ is a constant.

Similarly, exchanging α and β, we have *mutatis mutandis* the following integral equation giving $A_2(\cdot)$ in terms of $Q(\cdot)$.

$$\frac{xA_2(x)}{(x-1)^\delta S_{\beta\alpha}(x)} = \frac{1}{2\pi}\int_0^{x^*_{\beta\alpha}} Q(u)H_{\beta\alpha}(u)\left[\frac{\theta_{\beta\alpha}(x)\theta'_{\beta\alpha}(u)du}{1-\theta_{\beta\alpha}(u)\theta_{\beta\alpha}(x)} - \frac{\theta'_{\beta\alpha}(u)du}{\theta_{\beta\alpha}(u)-\theta_{\beta\alpha}(x)}\right]$$
$$- \frac{[(\lambda\pi_1 x+\beta)Y_0^{\beta\alpha}(x)-sx]Q(x)}{(x-1)^\delta S_{\beta\alpha}(x)} + K_{\beta\alpha}, \quad \forall x \in \Phi^{\beta\alpha}_{int} - [0, x^*_{\beta\alpha}], \tag{10.4.11}$$

where $K_{\beta\alpha}$ is a constant,

$$H_{\beta\alpha}(u) = \frac{(\lambda\pi_1 u+\beta)\sqrt{(s^2-4\beta\lambda)u(x^*_{\beta\alpha}-u)}}{\beta(u-1)^\delta S_{\beta\alpha}(u)}$$

and the twin of (10.4.2) reads

$$\frac{S_{\beta\alpha}(\bar{x})}{S_{\beta\alpha}(x)} = \frac{V_{\beta\alpha}(x, Y_0^{\beta\alpha}(x))}{V_{\beta\alpha}(\bar{x}, Y_0^{\beta\alpha}(\bar{x}))},\ x \in \Phi^{\beta\alpha}, \quad V_{\alpha\beta}(x,y) = \frac{(x-1)^\delta}{\beta(x-y)}, \tag{10.4.12}$$

with

$$\delta = \begin{cases} 1, & \text{if } x = 1 \in \Phi^{\beta\alpha}_{int}. \\ 0, & \text{if } x = 1 \in \Phi^{\beta\alpha}_{ext}. \end{cases}$$

To prove the claim appearing in the title of this section, three more steps are needed.
1. Extend the integral equation for $A_1(x)$ to the segment $[0, x^*_{\beta\alpha}] \supset [0, x^*_{\alpha\beta}]$. Indeed, this is immediate as $Y_0^{\alpha\beta}(u)$ is continuous on $[x^*_{\alpha\beta}, x^*_{\beta\alpha}]$, and (10.4.11) can be rewritten as

$$\frac{xA_1(x)}{(x-1)^\varepsilon S_{\alpha\beta}(x)} = \frac{1}{2\pi}\int_0^{x^*_{\beta\alpha}} Q(u)\widetilde{H_{\alpha\beta}}(u)\left[\frac{\theta_{\alpha\beta}(x)\theta'_{\alpha\beta}(u)du}{1-\theta_{\alpha\beta}(u)\theta_{\alpha\beta}(x)} - \frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u)-\theta_{\alpha\beta}(x)}\right]$$
$$- \frac{[(\lambda\pi_2 x+\alpha)Y_0^{\alpha\beta}(x)-sx]Q(x)}{(x-1)^\varepsilon S_{\alpha\beta}(x)} + K_{\alpha\beta}, \quad \forall x \in \Phi^{\alpha\beta}_{int} - [0, x^*_{\alpha\beta}], \tag{10.4.13}$$

where $\widetilde{H_{\alpha\beta}}(u) = H_{\alpha\beta}(u)\mathbb{1}_{\{(0\le u\le x^*_{\alpha\beta}\}}$ and $K_{\alpha\beta}$ is a constant.
2. Next, let x go to the cut $[0, x^*_{\beta\alpha}]$. For this purpose, we need the Sokhotski–Plemelj formulae (5.2.3), which are permitted since the integrands in (10.4.11) and (10.4.13) satisfy a Hölder condition. Hence, taking into account the relations

$$Y_0^{\alpha\beta} + Y_0^{\alpha\beta} = \frac{sx}{\alpha}, \quad Y_0^{\beta\alpha} + Y_0^{\beta\alpha} = \frac{sx}{\beta},$$

we obtain

$$
\begin{aligned}
\frac{xA_1(x)}{(x-1)^{\varepsilon}S_{\alpha\beta}(x)} = \frac{1}{2\pi}\int_0^{x^*_{\beta\alpha}} Q(u)\widetilde{H_{\alpha\beta}}(u)\left[\frac{\theta_{\alpha\beta}(x)\theta'_{\alpha\beta}(u)du}{1-\theta_{\alpha\beta}(u)\theta_{\alpha\beta}(x)} - \frac{\theta'_{\alpha\beta}(u)du}{\theta_{\alpha\beta}(u)-\theta_{\alpha\beta}(x)}\right] \\
- \mathbb{1}_{\{(x^*_{\alpha\beta}\le x\le x^*_{\beta\alpha}\}}\frac{[(\lambda\pi_2 x+\alpha)Y_0^{\alpha\beta}(x)-sx]Q(x)}{(x-1)^{\varepsilon}S_{\alpha\beta}(x)} \\
+ \mathbb{1}_{\{(0\le x\le x^*_{\alpha\beta}\}}\frac{sx(\alpha-\lambda\pi_2 x)Q(x)}{2\alpha(x-1)^{\varepsilon}S_{\alpha\beta}(x)} + K_{\alpha\beta}, \quad \forall x\in[0,x^*_{\beta\alpha}],
\end{aligned}
\tag{10.4.14}
$$

together with

$$
\begin{aligned}
\frac{xA_2(x)}{(x-1)^{\delta}S_{\beta\alpha}(x)} = \frac{1}{2\pi}\int_0^{x^*_{\beta\alpha}} Q(u)H_{\beta\alpha}(u)\left[\frac{\theta_{\beta\alpha}(x)\theta'_{\beta\alpha}(u)du}{1-\theta_{\beta\alpha}(u)\theta_{\beta\alpha}(x)} - \frac{\theta'_{\beta\alpha}(u)du}{\theta_{\beta\alpha}(u)-\theta_{\beta\alpha}(x)}\right] \\
+ \frac{sx(\beta-\lambda\pi_1 x)Q(x)}{2\beta(x-1)^{\delta}S_{\beta\alpha}(x)} + K_{\beta\alpha}, \quad \forall x\in[0,x^*_{\beta\alpha}].
\end{aligned}
\tag{10.4.15}
$$

The constants $K_{\alpha\beta}$, $K_{\beta\alpha}$ are computed by instantiating $x = 0$ in (10.4.14) and (10.4.15) respectively.

3. Lastly, the conservation law $sQ(x) = A_1(x) + A_2(x)$ given in (10.2.4) yields the announced integral equation satisfied by $Q(x)$.

For the sake of brevity, set

$$
\begin{aligned}
\Theta_{\alpha\beta}(x,u) &= \widetilde{H_{\alpha\beta}}(u)\theta'_{\alpha\beta}(u)\left[\frac{\theta_{\alpha\beta}(x)}{1-\theta_{\alpha\beta}(u)\theta_{\alpha\beta}(x)} - \frac{1}{\theta_{\alpha\beta}(u)-\theta_{\alpha\beta}(x)}\right], \\
\Theta_{\beta\alpha}(x,u) &= H_{\beta\alpha}(u)\theta'_{\beta\alpha}(u)\left[\frac{\theta_{\beta\alpha}(x)}{1-\theta_{\beta\alpha}(u)\theta_{\beta\alpha}(x)} - \frac{1}{\theta_{\beta\alpha}(u)-\theta_{\beta\alpha}(x)}\right].
\end{aligned}
$$

Then, just adding (10.4.14) and (10.4.15),

$$
\boxed{Z(x)Q(x) = \int_0^{x^*_{\beta\alpha}} Q(u)S(x,u)du + W(x), \quad \forall x\in[0,x^*_{\beta\alpha}]},
$$

with

$$
\begin{cases}
Z(x) &= \lambda s x^2\left(\dfrac{\pi_1}{2\alpha}+\dfrac{\pi_2}{2\beta}\right) + \mathbb{1}_{\{(x^*_{\alpha\beta}\le x\le x^*_{\beta\alpha}\}}(\lambda\pi_2 x+\alpha)\left[\dfrac{sx}{2\alpha} - Y_0^{\alpha\beta}(x)\right], \\
S(x,u) &= (x-1)^{\varepsilon}S_{\alpha\beta}(x)\Theta_{\alpha\beta}(x,u) + (x-1)^{\delta}S_{\beta\alpha}(x)\Theta_{\beta\alpha}(x,u), \\
W(x) &= K_{\alpha\beta}(x-1)^{\varepsilon}S_{\alpha\beta}(x) + K_{\beta\alpha}(x-1)^{\delta}S_{\beta\alpha}(x).
\end{cases}
$$

This integral equation contains both a regular part and a singular part, the latter coming from the kernels $[\theta_{\alpha\beta}(u)-\theta_{\alpha\beta}(x)]^{-1}$ and $[\theta_{\beta\alpha}(u)-\theta_{\beta\alpha}(x)]^{-1}$. We shall not discuss it in full detail here.

10.4.1 Complete Solution of System (10.2.4)

All the unknown functions appearing in system (10.2.4) can be determined once $Q(x)$ is known. Let us consider, for instance, the system (10.3.9), and let x go to the cut $[0, x^*_{\alpha\beta}]$. Since $A_1(x)$ is analytic in the unit disc, we end up with the following non-homogeneous BVP for $G_1(y)$ set on the ellipse $\Psi_{\alpha\beta}$.

$$[\alpha X_0^{\alpha\beta}(y) - y]G_1(y) - [\alpha X_0^{\alpha\beta}(y) - \bar{y}]G_1(\bar{y}) = (\bar{y} - y)[\lambda\pi_2 X_0^{\alpha\beta}(y) + \alpha]Q(X_0^{\alpha\beta}(y), \quad \forall y \in \Psi_{\alpha\beta}.$$

The function $G_2(y)$ is obtained in a similar way, and lastly $A_1(x)$, $A_2(x)$, are in turn derived from (10.3.9) and (10.3.10).

10.4.2 Explicit Integral Forms for Equal Service Rates ($\alpha = \beta$)

In this case, the problem can be simplified in a breathtaking way! The two kernels $T_i(x, y), i = 1, 2$, are equal and we use the same objects as before, without the indices $\alpha\beta$ or $\beta\alpha$. Letting

$$\begin{cases} F(x, y) = F_1(x, y) + F_2(x, y), \\ G(x, y) = G_1(x, y) + G_2(x, y), \\ \Delta(x, y) = s - \left(\dfrac{2\alpha}{x} + \lambda\right) y, \end{cases}$$

we get from (10.2.4) the reduced functional equation

$$F(x, y)T(x, y) = \alpha\left(1 - \frac{y}{x}\right) G(y) - \Delta(x, y)Q(x). \tag{10.4.16}$$

From the methods presented in Chaps. 5 and 6 of this book, the resolution of (10.4.16) is straightforward. We only present the formal result without further comment.

Proposition 10.4.2 *When $\alpha = \beta$, the system is ergodic if and only if $\lambda < 2\alpha$, and in this case*

$$Q(x) = C\, e^{\Gamma(x)}, \tag{10.4.17}$$

where C is a positive constant and

$$\begin{cases} \Gamma(x) &= \dfrac{1}{2\pi}\displaystyle\int_{\Phi} \dfrac{\log w(t)\theta'(t)dt}{\theta(t)-\theta(x)}, \quad x \in \Phi_{int}, \\ w(t) &= \dfrac{i\overline{\xi(t)}}{i\xi(t)}, \quad \xi(t) = \dfrac{\Delta(t)}{1-\frac{Y_0(t)}{t}}, \quad \Delta(t) = \Delta(t, Y_0(t)). \end{cases}$$

Remark 10.4.1 • By Theorem 10.3.1, the integral representing Q(x) can be analytically continued to the whole complex plane as a meromorphic function. The ergodicity condition $\lambda < 2\alpha$ can be directly obtained from Theorem 5.4.1, which also applies in the genus 0 case.

• The first pole of $Q(x)$ belongs to Φ_{ext}. In fact, one can check that, when $\alpha > \lambda$, $\Delta(x, Y_0(x))$ does not vanish in Φ_{int}.

10.5 A Functional Equation for $G_1(y)$

To enhance the richness of the model, we propose an alternative leading to a functional equation for $G_1(y)$. This can be carried out in two steps.

1. Replacing in (10.3.10) $Q(X_0^{\beta\alpha}(y))$ and

$$A_2(X_0^{\beta\alpha}(y)) = sQ(X_0^{\beta\alpha}(y)) - A_1(X_0^{\beta\alpha}(y))$$

   by their expressions in terms of $G_1(y)$, we get an equation giving $G_2(y)$ explicitly in terms of $G_1(y)$.
2. Going to the cut $[y_1^{\beta\alpha}, y_2^{\beta\alpha}] \subset \mathcal{D}$, we eliminate $G_2(y)$, which is sought to be analytic in $\mathcal{D}$.

Then straightforward algebra leads to

$$\begin{aligned} &L_1(z)G_1(Y_1^{\alpha\beta}(z)) - L_1(\bar{z})G_1(Y_1^{\alpha\beta}(\bar{z})) = \\ &K_1(z)G_1(Y_0^{\alpha\beta}(z)) - K_1(\bar{z})G_1(Y_0^{\beta\alpha}(\bar{z})), \ z \in \Phi^{\beta\alpha}, \end{aligned} \tag{10.5.1}$$

where

$$\begin{cases} K_1(z) &= \dfrac{\left(1-\dfrac{Y_0^{\alpha\beta}(z)}{z}\right)\Delta_1(z)}{\left(1-\dfrac{Y_0^{\beta\alpha}(z)}{z}\right)W^{\alpha\beta}(z)}, \quad L_1(z) = \dfrac{\left(1-\dfrac{Y_1^{\alpha\beta}(z)}{z}\right)\Delta(z)}{\left(1-\dfrac{Y_0^{\beta\alpha}(z)}{z}\right)W^{\alpha\beta}(z)}, \\ \Delta_1(z) &= s - \left(\dfrac{\alpha}{z}+\lambda\pi_2\right)Y_1^{\alpha\beta}(z) - \left(\dfrac{\beta}{z}+\lambda\pi_1\right)Y_0^{\beta\alpha}(z). \end{cases}$$

Arguing as before and setting

$$z = X_0^{\beta\alpha}(y), \ \text{so that} \ \bar{z} = X_1^{\beta\alpha}(y), \forall y \in [y_1^{\beta\alpha}, y_2^{\beta\alpha}],$$

Equation (10.5.1) can be rewritten, for $y \in \mathbb{C}_y$, as

$$\begin{aligned} L_1(X_0^{\beta\alpha}(y))G_1(Y_1^{\alpha\beta} \circ X_0^{\beta\alpha}(y)) - L_1(X_1^{\beta\alpha}(y))G_1(Y_1^{\alpha\beta} \circ X_1^{\beta\alpha}(y)) = \\ K_1(X_0^{\beta\alpha}(y))G_1(Y_0^{\alpha\beta} \circ X_0^{\beta\alpha}(y)) - K_1(X_1^{\beta\alpha}(y))G_1(Y_0^{\alpha\beta} \circ X_1^{\beta\alpha}(y)). \end{aligned} \tag{10.5.2}$$

10.5.1 A Non-standard BVP for $G_1(y)$

The following main result holds.

Theorem 10.5.1 *Equation (10.5.1) is equivalent to a generalized BVP of the Riemann type on the closed contour $\mathcal{L} = Y_1^{\alpha\beta}(\Phi^{\beta\alpha})$, which has a unique solution if and only if*

$$\Delta'(1) > 0, \quad \text{orequivalently } \lambda < \alpha + \beta. \tag{10.5.3}$$

Moreover, under the above ergodicity condition, system (10.2.4) has an analytic solution in the unit disc $\mathcal{D}$. ∎

The proof given in [28, 54] is very technical and will be omitted. We just give a brief outline of the main steps.

- To get a BVP on $\mathcal{L}$, it suffices to put $t = Y_1^{\alpha\beta}(z)$ in (10.5.1), in which case, by the properties of the branches established in Lemma 6.3.1, t describes the curve $\mathcal{L}$.
- Under the assumption $\beta \geq \alpha$, the contour $\mathcal{L}$ is closed, since the finite domain bounded by the ellipse $\Phi^{\beta\alpha}$ contains the cut $[0, x^*_{\beta\alpha}]$.
- The key point is the reduction to a BVP of the form

$$U^+(t) - U^-(t) = H(t)\overline{U^+(t)} + C, \quad t \in w(\Psi^{\alpha\beta}), \tag{10.5.4}$$

  where $H(t)$ is known, and $w(z)$ is a conformal gluing of the domain inside the ellipse $\Psi^{\alpha\beta}$ onto the complex plane cut along an open smooth arc (see Theorem 5.2.7). Then the condition for the existence of a function $U(t)$, analytic for $t \notin w(\Psi^{\alpha\beta})$ and bounded at infinity, is precisely (10.5.3). Indeed, applying some general results given in [63], we show that (10.5.4) defines a Noetherian operator with index 0, that is a Fredholm operator.

10.5.2 Poles and Residues of $G_1(y)$: Miscellaneous Issues

From (10.5.2), explicit recursive computations of the poles and residues of $G_1(\cdot)$ can be achieved, but they are tedious and intricate, see [23]. Hereafter, we address some open problems (see [28, 54]).

- It is necessary to know the automorphisms of the algebraic curve whose four branches are the functions $Y_i^{\alpha\beta} \circ X_j^{\beta\alpha}(y)$, $i, j = 0, 1$. That curve is in fact of genus 0, as can be seen by successively uniformizing $R_1(t, z)$ and $R_2(t, y)$, which yields

$$y = \frac{A(u)}{D(u)}, \quad z = \frac{B(u)}{D(u)},$$

where $A(u)$, $B(u)$, $D(u)$ are polynomials in u of respective degrees 4, 4, 3.
Let $E(y, z) = 0$ be the equation obtained by eliminating t in the system

$$R_1(t, z) = R_2(t, y) = 0.$$

- By Galois theory, the splitting field $\mathcal{F}$ of the equation $R_1(t, z) = 0$ is of degree 2. The splitting field $\mathcal{G}$ of $E(y, z) = 0$ is of degree 8 (here y represents any of the four roots), so that its group of automorphisms is also of order 8. It follows that the extension $\mathcal{G}/\mathcal{F}$ is of degree 4. The problem is to find the rational automorphisms of the curve $E(y, z) = 0$ which *change* $\mathcal{F}$, i.e. permutate the roots t_1, t_2 of $R_1(t, z) = 0$. Alas, the automorphisms of the curve do not coincide with those of Galois for 4-th degree equations.
It can be shown that the question is first to simultaneously render the radicals rational

$$\sqrt{(v - v_1)(v - v_2)} \quad \text{and} \quad \sqrt{(1 - vv_1)(1 - vv_2)},$$

which requires elliptic functions. Then

$$\begin{cases} z = F[sn(x)], \\ y = G[sn(x), cn(x)], \end{cases}$$

where $sn(\cdot)$, $cn(\cdot)$ are the classical Jacobi functions, and F, G stand for simple linear fractions. Thus, for a value of z, we obtain four values of x deduced from each other by means of fractional linear transforms. But, *and therein lies the rub*, in order to solve (10.5.2), we also need to find the rational automorphisms (as functions of x), which for a value of y would also produce four values of z. For this purpose, uniformization by means of elliptic functions is no longer sufficient. What about hyperelliptic functions? This remains an open question. Moreover, it is not even clear that there exists a *finite extension* over the field of rational functions, which would be *normal* both with respect to y and z.

Chapter 11
Counting Lattice Walks in the Quarter Plane

Enumeration of planar lattice walks has become a classical topic in combinatorics. For a given set $\mathcal{S}$ of allowed jumps (or steps), it is a matter of counting the number of paths starting from some point and ending at some arbitrary point in a given time, and possibly restricted to some regions of the plane.

Then three important questions naturally arise.

Q1: How many such paths exist?

Q2: What is the nature of the associated *counting generating function* (CGF) of the numbers of walks? Is it *holonomic*,[1] and, in that case, *algebraic* or even *rational*?

Q3: What is the asymptotic behavior, as their length goes to infinity, of the number of walks ending at some given point or domain (for instance one axis)?

If the paths are not restricted to a region, or if they are constrained to remain in a half-plane, the CGFs have an explicit form and can only be rational or algebraic (see [13]). The situation happens to be much richer if the walks are confined to the quarter plane $\mathbb{Z}_+^2$.

Without intending to present an exhaustive account of the numerous results on these subjects, which would lead us beyond the primary scope of this book, we shall give a flavor of how the results of Chaps. 4 and 5 can be directly used to answer questions Q1 and Q2 in the case of a finite group (see [37, 48]).

[1] A function of several complex variables is said to be holonomic if the vector space over the field of rational functions spanned by the set of all derivatives is finite dimensional. In the case of one variable, this is tantamount to saying that the function satisfies a linear differential equation where the coefficients are rational functions (see e.g., [41]).

G. Fayolle et al., *Random Walks in the Quarter Plane*, Probability Theory and Stochastic Modelling 40, DOI 10.1007/978-3-319-50930-3_11

11.1 A Functional Equation for a Tri-Variate CGF

If no restriction is made on the paths, it is well-known that the CGFs are rational and easy to obtain explicitly. As another example, if the walks are assumed to remain in a half-plane, then the CGFs can also be computed and turn out to be algebraic, see e.g., [13].

Next it is quite natural to consider walks evolving in a domain formed by the intersection of two half-planes, for instance the positive quarter plane $\mathbb{Z}_+^2$. It will be convenient to denote by $\mathcal{S}$ the set of admissible *small steps*, included in the set of the eight nearest neighbors, so that

$$\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}.$$

In this case, the problems are more intricate and multifarious results have appeared in various contexts: indeed, some walks admit an algebraic generating function, see e.g., [43], mentioned in Chap. 4, and [48] for the walk with step set $\mathcal{S} = \{(-1, 0), (1, 1), (0, -1)\}$ and starting from $(0, 0)$, whereas others admit a CGF which is not even holonomic, see e.g., [13] for the walk with $\mathcal{S} = \{(-1, 2), (2, -1)\}$ and starting from the point $(1, 1)$.

In this framework, a systematic study was initiated in [12] concerning the nature of the walks confined to $\mathbb{Z}_+^2$, starting from the origin and having small steps of size 1, which means that $\mathcal{S}$ is included in the set $\{(i, j) : |i|, |j| \leq 1\} \setminus \{(0, 0)\}$. Examples of such walks are shown in Figs. 11.1 and 11.2.

A priori, there are 2^8 such models. In fact, after eliminating trivial cases and models equivalent to walks confined to a half-plane, and noting also that some models are obtained from others by symmetry, it is shown in [12] that one is left with 79 inherently different problems to analyze.

A common starting point to deal with these 79 walks relies on the following analytic approach. Let $f(i, j, k)$ denote the number of paths in $\mathbb{Z}_+^2$ starting from $(0, 0)$ and ending at (i, j) at time k (or after k steps). Then the corresponding CGF

$$F(x, y, z) = \sum_{i,j,k \geq 0} f(i, j, k) x^i y^j z^k \tag{11.1.1}$$

satisfies the functional equation (see [12] for the details)

$$K(x, y, z)F(x, y, z) = c(x)F(x, 0, z) + \widetilde{c}(y)F(0, y, z) + c_0(x, y, z), \tag{11.1.2}$$

where

$$\begin{cases} K(x, y; z) = xy\Big[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\Big], \\ c(x) = \sum_{(i,-1)\in\mathcal{S}} x^{i+1}, \\ \widetilde{c}(y) = \sum_{(-1,j)\in\mathcal{S}} y^{j+1}, \\ c_0(x, y, z) = -\gamma F(0, 0, z) - xy/z, \end{cases}$$

with $\gamma = 1$ if $(-1, -1) \in \mathcal{S}$ [i.e. south-west jumps are allowed], $\gamma = 0$ otherwise.

It is worth noting that for $z = 1/|\mathcal{S}|$ (11.1.2) plainly belongs to the generic class of functional equations arising in a probabilistic context, as in most of this book.

For general values of z, the analysis of (11.1.2) for the 79 above-mentioned walks has been carried out in [90], where the integrand of the integral representations is studied in detail via a complete characterization of ad hoc conformal gluing functions.

In [12], the authors consider the group $W \stackrel{def}{=} \langle\alpha, \beta\rangle$ generated by the two birational transformations leaving invariant the generating function $\sum_{(i,j)\in\mathcal{S}} x^i y^j$,

$$\alpha(x, y) = \left(x, \frac{1}{y}\frac{\sum_{(i,-1)\in\mathcal{S}} x^i}{\sum_{(i,+1)\in\mathcal{S}} x^i}\right), \qquad \beta(x, y) = \left(\frac{1}{x}\frac{\sum_{(-1,j)\in\mathcal{S}} y^j}{\sum_{(+1,j)\in\mathcal{S}} y^j}, y\right).$$

Clearly $\alpha^2 = \beta^2 = \mathrm{Id}$, and W is a dihedral group of even order larger than 4.

The difference between the groups W and $\mathcal{H}$ defined in Chap. 4 is not only of a formal character. In fact, W is defined on all of $\mathbb{C}^2$, whereas $\mathcal{H}$ acts only on an algebraic curve of the type (see (11.1.2))

$$\Big\{(x, y) \in \mathbb{C}^2 : R(x, y, z) = xy\Big[\sum_{(i,j)\in\mathcal{S}} r_{i,j} x^i y^j - 1/z\Big] = 0\Big\}. \tag{11.1.3}$$

The notions of order and finiteness in general do not coincide for these respective groups. Indeed, a simple adaptation of the criterion 4.1.17 shows that, for $\mathcal{H}$ to be of order 4, it is necessary and sufficient to have, with the notation in (11.1.3),

$$\begin{vmatrix} r_{1,1} & r_{1,0} & r_{1,-1} \\ r_{0,1} & r_{0,0} - 1/z & r_{0,-1} \\ r_{-1,1} & r_{-1,0} & r_{-1,-1} \end{vmatrix} = 0,$$

and this condition depends on z, while W is independent of z. On the other hand, if W is finite so is $\mathcal{H}$, and conversely if $\mathcal{H}$ is infinite so is W. More exactly,

$$\mathit{Order}(\mathcal{H}) \le \mathit{Order}(W). \tag{11.1.4}$$

11.2 Group Classification of the 79 Main Random Walks

For each of the 79 cases mentioned above, the order of W is calculated in [12]: 56 walks admit an infinite group, while the groups of the 23 remaining ones are finite. It is also proved that among these 23 walks, W has order 4 for 16 walks (the ones with a step set having a vertical symmetry), W has order 6 for 5 walks (the two on the left in Fig. 11.1 and the three on the left in Fig. 11.2), and W has order 8 for the 2 walks on the right in Figs. 11.1 and 11.2. Moreover, for these 23 walks, the answers to both main questions (explicit expression *and* nature of the CGF (11.1.1) are known. In particular, the following results hold.

Theorem 11.2.1 (see [12]) *For the 16 walks with a group of order 4 and for the 3 walks in Fig. 11.1, the formal trivariate series (11.1.1) is holonomic non-algebraic. For the 3 walks on the left in Fig. 11.2, the trivariate series (11.1.1) is algebraic.*

Theorem 11.2.2 (see [11]) *For the so-called* Gessel's walk *on the right in Fig. 11.2, the formal trivariate series (11.1.1) is algebraic.*

Proving Theorem 11.2.1 requires skillful algebraic manipulations together with the calculation of adequate *orbit* and *half-orbit* sums.

As for Theorem 11.2.2, it has been mainly obtained by the powerful computer algebra system *Magma*, which allows dense calculations to be carried out.

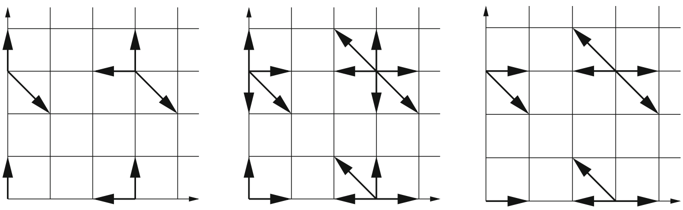

Fig. 11.1 *On the left*, 2 walks with a group of order 6. *On the right*, 1 walk with a group of order 8

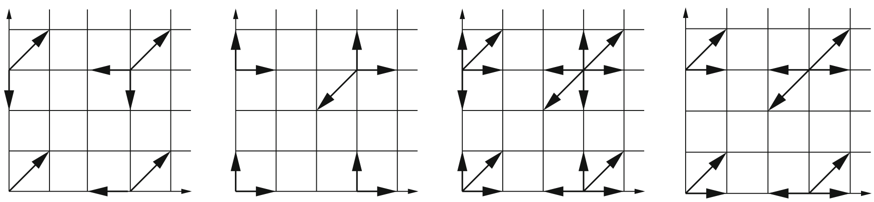

Fig. 11.2 *On the left*, 3 walks with a group of order 6. *On the right*, 1 walk with a group of order 8

11.3 Holonomy and Algebraicity of the Generating Functions

This section proposes another proof of Theorems 11.2.1 and 11.2.2 concerning the bivariate generating function $(x, y) \mapsto F(x, y, z)$ for the 23 walks associated with a finite group. It is a straightforward application of the theory developed in Chap. 4 of this book, which allows us to answer question Q2 for the bivariate generating functions in (11.1.2) in a rather direct way.

For the sake of completeness and readability, we recall hereafter some definitions from Sect. 4.6, when the group $\mathcal{H}$ is of order $2n$.

The *norm* $N(h)$, for any $h \in \mathbb{C}_Q(x, y)$, will be written as

$$N(h) \stackrel{def}{=} \prod_{i=0}^{n-1} h_{\delta^i}. \tag{11.3.1}$$

From Theorem 4.8.1, we know that when $N(f) = 1$ the general solution of the fundamental system (1.3.6) [keeping the same notation up to the replacement of the function c of the Theorem by r] has the form

$$\pi = w_1 + w_2 + \frac{w}{r},$$

where

- the functions w_1 and r are rational;
- the function w_2 is given by

$$w_2 = \frac{\widetilde{\Phi}}{n} \sum_{k=0}^{n-1} \left(\frac{\psi_{\delta^k}}{\prod_{i=1}^{k} f_{\delta^i}} \right); \tag{11.3.2}$$

- the function w is algebraic and satisfies the automorphy conditions

$$w = w_\xi = w_\delta.$$

Corollary 11.3.1 *Assume the group is finite of order* $2n$ *and* $N(f) = 1$. *Then the solution* $\pi(x)$ *of (1.3.6) is holonomic. Moreover, it is algebraic if and only if, on* $\{Q(x, y) = 0\}$,

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^{k} f_{\delta^i}} = 0. \tag{11.3.3}$$

In fact, the proof is an immediate consequence of the next lemma.

Lemma 11.3.2 *In the uniformization given by (3.3.3) or (3.3.4), let us write*

$$\wp(\omega) \stackrel{def}{=} g(x(\omega)),$$

where g is a fractional linear transform. Then the function $w_2(\wp^{-1}(g(x)))$ defined by (11.3.2) is holonomic in x, where $\wp^{-1}$ denotes the elliptic integral inverse function of $\wp$.

Proof Since w_2 is the product of $\widetilde{\Phi}$ by an algebraic function, it suffices to prove that $\widetilde{\Phi}(\wp^{-1}(g))$ is holonomic. It is known that the class of holonomic functions is closed under indefinite integration (see e.g., [41]), thus it is enough to prove that $[\widetilde{\Phi}(\wp^{-1}(g))]'$ is holonomic. In fact, we are going to show that $[\widetilde{\Phi}(\wp^{-1}(g))]'$ is algebraic – we recall that any algebraic function is holonomic. Using, on the one hand, the fact that the derivative of $-\zeta_{1,3}$ is $\wp_{1,3}$, the Weierstrass elliptic function with periods ω_1, ω_3, and on the other hand that

$$[\wp^{-1}]'(u) = 1/[4u^3 - g_2 u - g_3]^{1/2},$$

g_2 and g_3 being the invariants of $\wp$ –for these two properties see e.g., [53]–, we get

$$-2i\pi[\widetilde{\Phi}(\wp^{-1}(g))]' = \frac{g'}{[4g^3 - g_2 g - g_3]^{1/2}}\big[\omega_1\wp_{1,3}(\wp^{-1}(g) - \omega_2/2) + 2\zeta_{1,3}(\omega_1/2)\big].$$

As the function g is rational, $g'/[4g^3 - g_2 g - g_3]^{1/2}$ is algebraic and in order to conclude the argument it is enough to prove that $\wp_{1,3}(\wp^{-1}(g) - \omega_2/2)$ is algebraic. Since $n\omega_3 = k\omega_2$, it was shown in Lemma 4.7.3 that both $\wp$ and $\wp_{1,3}$ are rational functions of the Weierstrass elliptic function with periods $\omega_1, k\omega_2$. Then it is immediately apparent that $\wp_{1,3}$ is also an algebraic function of $\wp$. Moreover, by the well-known *addition theorem* for Weierstrass elliptic functions (see e.g., [53]), $\wp$ is a rational function of $\wp(\cdot + \omega_2/2)$, so that $\wp_{1,3}$ is an algebraic function of $\wp(\cdot + \omega_2/2)$. In particular, $\wp_{1,3}(\wp^{-1}(g) - \omega_2/2)$ is algebraic and Lemma 11.3.2 is proved. ∎

Of course, similar results can be written for $\widetilde{\pi}$. In particular, it is easy to check that if $N(f) = 1$ then also $\widetilde{N}(f) = 1$, where $\widetilde{N}(f) = 1$ is defined in terms of $\widetilde{\delta} \stackrel{def}{=} \xi\eta$. Finally, we are in a position to state the following general result.

Theorem 11.3.3 *Assume the group is finite of order $2n$ and $N(f) = 1$. Then the bivariate series $\pi(x, y)$, the solution of (1.3.6), is holonomic. Furthermore $\pi(x, y)$ is algebraic if and only if (11.3.3) and $(\widetilde{11.3.3})$ hold on $Q(x, y) = 0$, $(\widetilde{11.3.3})$ denoting the condition for $\widetilde{\pi}(y)$ to be algebraic, obtained by symmetry from (11.3.3).*

11.4 The Nature of the Counting Generating Functions

We return now to Eq. (11.1.2) and make use of the machinery of Sect. 11.3 with

$$f = \frac{c\widetilde{c}_\eta}{\widetilde{c}c_\eta}, \quad \psi = \frac{c_0\widetilde{c}_\eta}{c_\eta\widetilde{c}} - \frac{(c_0)_\eta}{c_\eta}. \tag{11.4.1}$$

Beforehand, it is important to note that the variable z plays here the role of a *parameter*. Moreover, the CGF (11.1.1) is well defined and analytic in a region containing the domain

$$\{(x, y, z) \in \mathbb{C}^3 : |x| \leq 1, |y| \leq 1, |z| < 1/|\mathcal{S}|\}.$$

In the previous chapters, we aimed at finding the *stationary measure* of the random walk, in which case $z = 1$. But it is not difficult to check that, *mutatis mutandis*, most of the topological results remain valid at least for $\Re(z) \geq 0$ [see for instance [33] or [90] for time-dependent problems].

Lemma 11.4.1 *For any finite n and f given by (11.4.1), we have $N(f) = 1$.*

Proof Since c [resp. $\widetilde{c}$] is a function solely of x [resp. y], we have $c = c_\xi$ and $\widetilde{c} = \widetilde{c}_\eta$. Hence

$$f = \frac{c}{c_\eta} = \frac{c_\delta}{c_{\delta\eta}} = \frac{c_\delta}{c},$$

which immediately yields $N(f) = 1$ from the definition (11.3.1). ∎

In order to obtain Theorems 11.2.1 and 11.2.2 for the bivariate function $(x, y) \mapsto F(x, y, z)$, it is thus enough, by Theorem 11.3.3 and Lemma 11.4.1, to prove the following proposition.

Proposition 11.4.2 *For the 4 walks in Fig. 11.2, (11.3.3) and $(\widetilde{11.3.3})$ hold in $\mathbb{C}^2$. As for the 19 other walks (3 in Fig. 11.1 and the 16 walks with a vertically symmetric step set $\mathcal{S}$), (11.3.3) nor $(\widetilde{11.3.3})$ hold on $\left\{(x, y) \in \mathbb{C}^2 : K(x, y) = xy\left[\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right] = 0\right\}$, for any z.*

Proof The proof proceeds in three stages.

(i) *The walks in figure* 11.2. Let us check (11.3.3) for the 4 walks in Fig. 11.2. We begin with the popular Gessel's walk, i.e. the rightmost one in Fig. 11.2. Here $c = 1$ in (11.1.2), so that by (11.4.1) $f = 1$ and $\psi = [(xy)_\eta - xy]/z$. Moreover, the order of the group W was shown in [12] to be equal to 8. According to (11.1.4), we know that $Order(\mathcal{H}) \leq 8$. On the other hand, by criteria (4.1.17) and (4.1.18), simple algebra can show that the order of $\mathcal{H}$ cannot be equal to 4 nor to 6, which entails $Order(\mathcal{H}) = 8$, hence $n = 4$. Then

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \sum_{k=0}^{3} \psi_{\delta^k} = \frac{1}{z}\sum_{k=0}^{3}\left[(xy)_{\eta\delta^k} - (xy)_{\delta^k}\right] = -\frac{1}{z}\sum_{\rho\in\mathcal{H}} \operatorname{sign}(\rho)(xy)_\rho. \tag{11.4.2}$$

The last quantity in (11.4.2) can be understood as $(-1/z)$ times the orbit sum of xy through the group $\mathcal{H}$, and it was remarked in [12] that this orbit sum equals zero for Gessel's walk, which is equivalent to (11.3.3) in $\mathbb{C}^2$.

- For the second walk in Fig. 11.2, $Order(W) = Order(\mathcal{H}) = 6$ $(n = 3)$ and still with $c = 1$, so that the above argument applies.
- For the walk on the left in Fig. 11.2, we have

$$\xi = (x, 1/(xy)), \quad \eta = (1/(xy), y),$$

hence $\xi\eta = (1/(xy), x)$. Moreover $c = x$, thus by using (11.4.1) we get $f = x^2 y$ and $\psi = [y - (xy)^2]/z$. Next one easily computes $f_\delta = 1/(xy^2)$, $f_{\delta^2} = y/x$, $\psi_\delta = [x - 1/y^2]/z$ and $\psi_{\delta^2} = [1/(xy) - 1/x^2]/z$. Finally an immediate calculation shows that (11.3.3) holds in $\mathbb{C}^2$.
- As for the last walk in Fig. 11.2, we could check $n = 3$ and again (11.3.3) holds in $\mathbb{C}^2$.

Concerning equation $(\widetilde{11.3.3})$, only Gessel's walk needs some special care, since the other three give rise to symmetric conditions. In fact, from a direct calculation along the same lines as above, it is not difficult to see that $(\widetilde{11.3.3})$ holds in $\mathbb{C}^2$.

(ii) *The 16 walks with a vertical symmetry.*
We will show the equality

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = -\frac{x^2}{zc}(x - x_\eta)(y - y_\xi), \tag{11.4.3}$$

which clearly implies that (11.3.3) will not be satisfied on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$, for any z. For these 16 walks we have $Order(W) = 4 = Order(\mathcal{H})$, that is $n = 2$. Moreover, the vertical symmetry of the jump set $\mathcal{S}$ has two consequences: first, necessarily $\eta = (1/x, y)$; second, the coefficient c in Eq. (11.1.2) must be equal to x, $1 + x^2$ or $1 + x + x^2$. In addition, for any of these three possible values of c, we have $f = c/c_\eta = x^2$. Hence finally

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \psi + \frac{\psi_\delta}{f_\delta} = \frac{(xy)_\eta - xy}{zc_\eta} + \frac{(xy)_\xi - (xy)_\delta}{zc_\xi f_\delta} = \frac{y(x_\eta - x)}{zc_\eta} + \frac{y_\xi(x - x_\eta)}{zc(x^2)_\delta}.$$

Factorizing by x^2/c and using again $c/c_\eta = x^2$, we obtain (11.4.3).

(iii) *The three walks in Fig.* 11.1. We are going to show that (11.3.3) does not hold on $K(x, y) = 0$, for any z. By similar calculations as in (11.4.3), we obtain that for the two first walks in Fig. 11.1 (which are such that $n = 3$),

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \frac{(x - y^2)(1 - xy)(y - x^2)}{zy^3 t},$$

with $t = y$ for the walk at the left and $t = x + y$ for the walk in the middle, so that, clearly, (11.3.3) does not hold on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$, for any z.

Finally, for the walk at the right in Fig. 11.1, we have $n = 4$, and we get

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^{k} f_{\delta^i}} = \frac{(y-1)(x^2-1)(x^2-y)(x^2-y^2)}{xy^4 z},$$

which is clearly not identically zero on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$, for any z. ■

11.5 Some Exact Asymptotics

The end of this chapter is devoted to question Q3 raised in the introduction. As a rule, asymptotics of the coefficients appearing in power series expansions can, most of the time, be derived by analyzing the behavior of the underlying analytic functions in the neighborhood of their first singularities (see e.g., [41]). In compliance with this principle, we will work on the integral-form solution of (11.1.2), as obtained by the methods proposed in Chap. 5.

The kernel $K(x, y, z)$ in (11.1.2) can be rewritten as

$$\widetilde{a}(y)x^2 + [\widetilde{b}(y) - y/z]x + \widetilde{c}(y) = a(x)y^2 + [b(x) - x/z]y + c(x),$$

where

$$\begin{aligned} a(x) &= \textstyle\sum_{(i,1)\in\mathcal{S}} x^{i+1}, \quad b(x) = \sum_{(i,0)\in\mathcal{S}} x^{i+1}, \quad c(x) = \sum_{(i,-1)\in\mathcal{S}} x^{i+1}, \\ \widetilde{a}(y) &= \textstyle\sum_{(1,j)\in\mathcal{S}} y^{j+1}, \quad \widetilde{b}(y) = \sum_{(0,j)\in\mathcal{S}} y^{j+1}, \quad \widetilde{c}(y) = \sum_{(-1,j)\in\mathcal{S}} y^{j+1}. \end{aligned} \tag{11.5.1}$$

Let us also introduce the two discriminants of the kernel, in the respective $\mathbb{C}_x$ and $\mathbb{C}_y$ complex planes,

$$\widetilde{d}(y, z) = [\widetilde{b}(y) - y/z]^2 - 4\widetilde{a}(y)\widetilde{c}(y), \qquad d(x, z) = [b(x) - x/z]^2 - 4a(x)c(x). \tag{11.5.2}$$

From now on, we shall take z to be a *real* variable. This is in no way a restriction, as it will emerge from analytic continuation arguments.

The following inequalities are direct consequences of Sect. 2.3.2 (see also the details in [90]). For $z \in (0, 1/|\mathcal{S}|)$, the polynomial d (resp. $\widetilde{d}$) has four roots (one at most being possibly infinite) satisfying in the x-plane (resp. y-plane)

$$\begin{cases} |x_1(z)| < x_2(z) < 1 < x_3(z) < |x_4(z)| \leq \infty, \\ |y_1(z)| < y_2(z) < 1 < y_3(z) < |y_4(z)| \leq \infty. \end{cases} \tag{11.5.3}$$

Then consider the algebraic functions $X(y, z)$ and $Y(x, z)$ defined by

$$K(X(y, z), y, z) = K(x, Y(x, z), z) = 0.$$

With the notations (11.5.1) and (11.5.2), we have

$$X(y, z) = \frac{-\widetilde{b}(y) + \frac{y}{z} \pm \sqrt{\widetilde{d}(y, z)}}{2\widetilde{a}(y)}, \qquad Y(x, z) = \frac{-b(x) + \frac{x}{z} \pm \sqrt{d(x, z)}}{2a(x)}. \tag{11.5.4}$$

From now on, we shall denote by X_0, X_1 (resp. Y_0, Y_1) the two branches of these algebraic functions defined in the $\mathbb{C}_y$ (resp. $\mathbb{C}_x$) plane. We have proved in Chap. 5 that they can be separated, to ensure $|X_0| \le |X_1|$ in $\mathbb{C}_y$ (resp. $|Y_0| \le |Y_1|$ in $\mathbb{C}_x$), bearing in mind that here the variable z merely plays the role of a parameter.

11.5.1 The Simple Random Walk

To illustrate the method announced in the introduction, in this section we consider the *simple walk* (i.e. $(i, j) \in \mathcal{S}$ if and only if $ij = 0$). Then the following four propositions hold.

Proposition 11.5.1 *For the simple walk,*

$$F(0, 0, z) = \frac{1}{\pi} \int_{-1}^{1} \frac{1 - 2uz - \sqrt{(1 - 2uz)^2 - 4z^2}}{z^2} \sqrt{1 - u^2}\, \mathrm{d}u. \tag{11.5.5}$$

Note that $F(0, 0, z)$ counts the number of excursions starting from (0, 0) and returning to (0, 0).

Proposition 11.5.2 *For the simple walk, as $n \to \infty$,*

$$f(0, 0, 2n) \sim \frac{4}{\pi} \frac{16^n}{n^3}.$$

Proposition 11.5.3 *For the simple walk,*

$$F(1, 0, z) = \frac{1}{2\pi} \int_{-1}^{1} \frac{1 - 2uz - \sqrt{(1 - 2uz)^2 - 4z^2}}{z^2} \sqrt{\frac{1 + u}{1 - u}}\, \mathrm{d}u.$$

The generating function $F(1, 0, z)$ counts the number of walks starting from (0, 0) and ending at the horizontal axis. In addition, by an evident symmetry, $F(0, 1, z) = F(1, 0, z)$.

Proposition 11.5.4 *For the simple walk, as $n \to \infty$,*

$$\sum_{i \geq 0} f(i, 0, n) \sim \frac{8}{\pi} \frac{4^n}{n^2}.$$

Having the expressions of $F(1, 0, z)$ and $F(0, 1, z)$, the series $F(1, 1, z)$ is directly obtained from (11.1.2). Its coefficients represent the total number of walks of a given length, and their asymptotics is the subject of the next result.

Proposition 11.5.5 *For the simple walk, as $n \to \infty$,*

$$\sum_{i, j \geq 0} f(i, j, n) \sim \frac{4}{\pi} \frac{4^n}{n}.$$

Proof of Proposition 11.5.1 The key point is to reduce the computation of $F(x, 0, z)$ to a BVP set on the unit circle $\Gamma = \{t \in \mathbb{C} : |t| = 1\}$. Letting $\mathcal{G}$ be a simply connected domain bounded by a smooth curve $\mathcal{L}$, the BVP we shall encounter below can be stated as follows.

Find a function $G(\cdot)$ of a single complex variable, holomorphic in $\mathcal{G}$ and satisfying a boundary condition of the form

$$G(t) = G(\bar{t}) + g(t), \quad \forall t \in \mathcal{L},$$

where $g(t)$ is a known function satisfying a Hölder condition on $\mathcal{L}$.

According to the method described in Chap. 5, leading essentially to an equation of the form (5.1.5), for any $t \in \Gamma$ we have

$$c(t)F(t, 0, z) - c(\bar{t})F(\bar{t}, 0, z) = \frac{tY_0(t, z) - \bar{t}Y_0(\bar{t}, z)}{z}, \tag{11.5.6}$$

where $\bar{t}$ stands for the complex conjugate of t.

For the simple random walk, a pleasant fact (from a computational point of view) is that, on the unit circle $\bar{t} = 1/t$. Hence, after multiplying both sides of (11.5.6) by $1/(t - x)$ with $|x| < 1$, integrating over Γ and applying Cauchy's formula, we get the following integral form (noting that here $c(x) = x$)

$$c(x)F(x, 0, z) = \frac{1}{2i\pi z} \int_{\Gamma} \frac{tY_0(t, z) - \bar{t}Y_0(\bar{t}, z)}{t - x} \, dt, \quad \forall |x| < 1. \tag{11.5.7}$$

For $t \in \Gamma$, we can proceed as in Sect. 5.3 (see also [90]) to show that for $z \in [0, 1/4]$, $Y_0(t, z)$ is real and belongs to the segment $[y_1(z), y_2(z)]$, the extremities of which are the two branch points located inside the unit circle in the $\mathbb{C}_y$-plane. Then it is not difficult to check that the integral on the right-hand side of (11.5.7) does indeed vanish at $x = 0$. Hence we may write

$$F(0,0,z) = \lim_{x\to 0} \frac{1}{2i\pi xz} \int_{\Gamma} \frac{tY_0(t,z) - \bar{t}Y_0(\bar{t},z)}{t-x}\,\mathrm{d}t = \frac{1}{2i\pi z} \int_{\Gamma} \frac{tY_0(t,z) - \bar{t}Y_0(\bar{t},z)}{t^2}\,\mathrm{d}t, \tag{11.5.8}$$

where the last equality follows from l'Hospital's rule.

To exploit formula (11.5.8), it is convenient to make the straightforward change of variable $t = e^{i\theta}$, which yields

$$F(0,0,z) = \frac{i}{2\pi z} \int_0^{2\pi} \sin\theta\, e^{-i\theta} Y_0(e^{i\theta},z)\,\mathrm{d}\theta = \frac{1}{2\pi z} \int_0^{2\pi} \sin^2\theta\, Y_0(e^{i\theta},z)\,\mathrm{d}\theta, \tag{11.5.9}$$

where we have used the fact that $Y_0(e^{i\theta},z)$ is real and $Y_0(e^{i\theta},z) = Y_0(e^{-i\theta},z)$. Furthermore, we have by (11.5.4)

$$Y_0(e^{i\theta},z) = \frac{1 - 2z\cos\theta - \sqrt{(1-2z\cos\theta)^2 - 4z^2}}{2z}. \tag{11.5.10}$$

Then, instantiating (11.5.10) in (11.5.9), we get after some light algebra

$$F(0,0,z) = \frac{1}{2z^2} - \frac{1}{\pi z^2} \int_0^{\pi} \sin^2\theta \sqrt{(1-2z\cos\theta)^2 - 4z^2}\,\mathrm{d}\theta. \tag{11.5.11}$$

Putting now $u = \cos\theta$ in the integrand, Eq. (11.5.11) becomes exactly (11.5.5). ∎

Proof of Proposition 11.5.2 It emerges easily from Proposition 11.5.1 that the function $F(0,0,z)$ is holomorphic in $\mathbb{C} \setminus ((-\infty,-1/4] \cup [1/4,\infty))$. The asymptotics of its coefficients can thus be derived from the behavior of the function in the neighborhood of $\pm 1/4$. On the other hand $F(0,0,z)$ is even, as can be seen in (11.5.5), or directly since $f(0,0,2n+1) = 0$, for any $n \geq 0$. Consequently, it suffices to focus solely on the point $1/4$.

First, we rewrite $\sqrt{(1-u^2)[(1-2uz)^2 - 4z^2]}$ as the product

$$\sqrt{(1-u)(1-2(u+1)z)}\,\sqrt{(1+u)(1-2(u-1)z)},$$

where the second radical admits the expansion $\sum_{i,j\geq 0} \mu_{i,j}(u-1)^i(z-1/4)^j$. Therefore, for z in a neighborhood of $1/4$, we have

$$F(0,0,z) = -\frac{1}{\pi z^2} \sum_{i,j\geq 0} \mu_{i,j}(z-1/4)^j \int_{-1}^{1} (u-1)^i \sqrt{(1-u)(1-2(u+1)z)}\,\mathrm{d}u. \tag{11.5.12}$$

We shall show below that, for any $i \geq 0$, there exist two functions, say f_i and g_i, which are analytic at $z = 1/4$ and such that, for z in a neighborhood of $1/4$,

$$\int_{-1}^{1} (u-1)^i \sqrt{(1-u)(1-2(u+1)z)}\,\mathrm{d}u = f_i(z)(z-1/4)^{i+2}\ln(1-4z) + g_i(z). \tag{11.5.13}$$

Now, by inserting identity (11.5.13) into (11.5.12), we obtain the existence of a function, say g, analytic at $z = 1/4$ and such that

$$F(0, 0, z) = -\frac{16}{\pi}\mu_{0,0}(z-1/4)^2 \ln(1-4z)[f_0(1/4)+O(z-1/4)]+g(z). \quad (11.5.14)$$

Furthermore, it is easy to prove $f_0(1/4) = 4\sqrt{2}$ and $\mu_{0,0} = \sqrt{2}$. With these values, a classical singularity analysis (see, e.g., [41, 95]) shows that, as $n \to \infty$, the nth coefficient of the function on the right-hand side of (11.5.14) is equivalent to $(16/\pi)4^n/n^3$. It is then immediate to infer that

$$f(0, 0, 2n) \sim \frac{32}{\pi}\frac{4^{2n}}{(2n)^3},$$

noting in the latter quantity the factor 32 (and not 16 !), due to the parity of $F(0, 0, z)$ mentioned earlier. ■

Proof of Equation (11.5.13) Let $\Lambda_i(z)$ denote the integral on the left-hand side of (11.5.13). The change of variable $u = 1/(4z) - [(1 - 4z)/(4z)]v$ gives

$$\Lambda_i(z) = \left(\frac{1-4z}{4z}\right)^i \sqrt{2z}\int_1^{1+8z/(1-4z)} (1 - v)^i \sqrt{v^2 - 1}\,\mathrm{d}v.$$

Then, by letting $v = \cosh t$, we conclude that

$$\Lambda_i(z) = \left(\frac{1-4z}{4z}\right)^i \sqrt{2z}\int_0^{\cosh^{-1}(1+8z/(1-4z))} (1 - \cosh t)^i(-1 + \cosh^2 t)\,\mathrm{d}t. \quad (11.5.15)$$

Let us recall that, by a classical linearization argument, there exist real coefficients $\alpha_0, \dots, \alpha_{i+2}$ such that

$$(1 - \cosh t)^i(-1 + \cosh^2 t) = \sum_{k=0}^{i+2} \alpha_k \cosh(kt). \quad (11.5.16)$$

So, by means of (11.5.16), it is easy to integrate $\Lambda_i(z)$. Then a delinearization argument shows the existence of two functions, g_i and h_i, which are analytic at $z = 1/4$ and satisfy

$$\Lambda_i(z) = g_i(z) + \left(\frac{1-4z}{4z}\right)^i \sqrt{2z}\, h_i(z)\cosh^{-1}(1 + 8z/(1 - 4z)).$$

Finally, remembering that, for all $u \geq 1$, $\cosh^{-1} u = \log(u+\sqrt{u^2 - 1})$, Eq. (11.5.13) follows, since $\cosh^{-1}(1 + 8z/(1 - 4z)) + \ln(1/4 - z)$ is analytic at $z = 1/4$. ■

Proof of Proposition 11.5.3 The argument mimics the one used in Proposition 11.5.2. Instantiating $x = 1$ in Eq. (11.5.7) and using (11.5.10) directly yields

$$F(1,0,z) = \frac{1}{2i\pi z}\int_{\Gamma}\frac{tY_0(t,z)-\bar{t}Y_0(\bar{t},z)}{t-1}\,\mathrm{d}t = \frac{1}{2\pi z}\int_0^{2\pi}(e^{i\theta}+1)Y_0(e^{i\theta},z)\,\mathrm{d}\theta$$
$$= \frac{1}{2\pi z^2}\int_0^{\pi}(1+\cos\theta)(1-2z\cos\theta-\sqrt{(1-2z\cos\theta)^2-4z^2})\,\mathrm{d}\theta,$$

and the proof is complete. ■

Proof of Proposition 11.5.4 This is quite similar to the proof of Proposition 11.5.2, by starting from the integral formulation of $F(1,0,z)$ written in Proposition 11.5.3, and making an expansion of the integrand in the neighborhood of $u = 1$ and $z = 1/4$. Therefore we do not include it here. ■

Proof of Proposition 11.5.5 This is an immediate consequence of Proposition 11.5.4, by using the equality

$$(4-1/z)F(1,1,z) = 2F(1,0,z) - 1/z,$$

which follows from (11.1.2). ■

Remark 11.5.6 In fact $f(0,0,2n)$ is the product of the two Catalan numbers C_n and C_{n+1} and similar expressions exist for the coefficients of $F(1,0,z)$ and $F(1,1,z)$. They can be obtained either by shuffling one-dimensional random walks, or by the reflection principle (see for instance [51]).

11.6 A General Approach for Walks with Small Steps and Eight Possible Neighbors

The following six subsections describe the workflow in some detail.

11.6.1 Outline of the Arguments

This part aims to extend Sect. 11.5.1 to all 79 models. For this, we show in Sect. 11.6.2 that $F(x,0,z)$ and $F(0,y,z)$ satisfy BVPs, which are solved in Sect. 11.6.3. Next, in Sect. 11.6.4 we compute $F(0,0,z)$, $F(1,0,z)$, $F(0,1,z)$ and $F(1,1,z)$. In Sect. 11.6.5, we see that the analysis of Sect. 11.5.1 applies verbatim to the 19 walks that have a group of order 4. In the last Sect. 11.6.6, we analyze the singularities of the generating functions, essentially explaining where they come from. The computation of the exact behavior of the generating functions in the neighborhood of the (dominant) singularities will be undertaken in future work.

11.6.2 Reduction to a BVP

The statements of this section follow again immediately from Chap. 5 and [90]. In the general framework (i.e., for all 79 models), Eq. (11.5.6) holds on the curve, drawn in the complex plane $\mathbb{C}_x$,

$$\mathcal{M}_z = X_0(\overrightarrow{\underleftarrow{[y_1(z), y_2(z)]}}, z) = \overline{X}_1(\overleftarrow{\underrightarrow{[y_1(z), y_2(z)]}}, z)$$

and depicted in the next lemma, assuming the *genus* of the Riemann surface corresponding to the manifold $\{(x, y :\in \mathbb{C}^2 : K(x, y, z) = 0\}$ is equal to 1. Remembering that here z is taken to be real, we let

$$\overrightarrow{\underleftarrow{[y_1(z), y_2(z)]}}$$

stand for a contour, which is the slit $[y_1(z), y_2(z)]$ traversed from $y_1(z)$ to $y_2(z)$ along the upper edge, and then back to $y_1(z)$ along the lower edge.

Lemma 11.6.1 *The curve $\mathcal{M}_z$ is one of the two components of a plane quartic curve with the following properties.*

- *It is symmetrical with respect to the real axis.*
- *It is connected and closed in $\mathbb{C} \cup \{\infty\}$.*
- *It splits the plane into two connected domains, and we shall denote by $\mathscr{G}(\mathcal{M}_z)$ the domain containing the point $x_1(z)$. In addition,*

$$\mathscr{G}(\mathcal{M}_z) \subset \mathbb{C} \setminus [x_3(z), x_4(z)].$$

In the same way, we could define the curve $\mathcal{L}_z$ and the domain $\mathscr{G}(\mathcal{L}_z)$ in the complex plane $\mathbb{C}_y$.

However, in the case of a group $\mathcal{H}$ of order 4, the two components introduced above coincide to form a circle *counted twice*, as in Sect. 11.5.1. Note that, in the *genus* 0 case covered in Chap. 6, the above curve has only one component, which for instance can be an ellipse.

Starting from the formal boundary condition (11.5.6) on the curve $\mathcal{M}_z$, the function $F(x, 0, z)$ can be analytically continued with respect to x from the unit disc to the domain $\mathscr{G}(\mathcal{M}_z)$. All these properties lead to the formulation of the following basic BVP.

Find a function $x \mapsto F(x, 0, z)$ analytic in the domain $\mathscr{G}(\mathcal{M}_z)$, and satisfying condition (11.5.6) on the boundary $\mathcal{M}_z$.

11.6.3 Solution of the BVP

The BVP stated in the previous section can be solved by means of a convenient CGF [see Definition 7.2.1] denoted by $w(t,z)$, $t \in \mathscr{G}(\mathcal{M}_z)$. The curve $\mathcal{M}_z$ splits the plane into two connected components (see Lemma 11.6.1). However, we saw in Chap. 5 that the BPV on $\mathcal{M}_z$ can be reduced to a BVP set on a segment, which is often computationally more convenient.

For instance, the mapping $t \mapsto t + 1/t$ is a CGF for the unit disc centered at 0. It is worth noting that the existence of a CGF (however without any explicit expression) for a generic domain $\mathscr{C}$ is ensured by general results on conformal gluing [63, Chap. 2]. In the following, we shall assume that the unique pole of w is located at $t = 0$. Then the main result of this section is the following.

Proposition 11.6.2 *For $x \in \mathscr{G}(\mathcal{M}_z)$,*

$$c(x)F(x,0,z) - c(0)F(0,0,z) = \frac{1}{2\pi i z}\int_{\mathcal{M}_z} tY_0(t,z)\frac{w'(t,z)}{w(t,z)-w(x,z)}\,\mathrm{d}t, \tag{11.6.1}$$

where $w(x,z)$ is the gluing function for the domain $\mathscr{G}(\mathcal{M}_z)$ in the $\mathbb{C}_x$-plane.

Of course, a similar expression could be written for $F(0,y,z)$.

11.6.4 Computation of $F(0,0,z)$, $F(1,0,z)$, $F(0,1,z)$, $F(1,1,z)$

The expression we shall obtain for $F(0,0,z)$ depends on $c(x)$ or symmetrically on $\widetilde{c}(y)$, see Eq. (11.5.1), in the following respect.

(a) Suppose first $c(0) = 0$ (this equality holds for the simple walk). Then, as in Sect. 11.5.1, we can write

$$F(0,0,z) = \lim_{x\to 0}\frac{1}{2\pi i z c(x)}\int_{\mathcal{M}_z} tY_0(t,z)\frac{w'(t,z)}{w(t,z)-w(x,z)}\,\mathrm{d}t. \tag{11.6.2}$$

(b) If $c(0) \neq 0$ and $c(x)$ is not constant, then it has one or two roots located on the unit circle Γ. More precisely, by (11.5.1), $c(x)$ takes any of the forms

$$x+1, \quad x^2+1, \quad x^2+x+1,$$

with the respective roots -1, $\{i, \bar{i}\}$, $\{j, \bar{j}\}$. Let $\widehat{x}$ denote one of these roots. Then, provided that $\widehat{x} \in \mathscr{G}(\mathcal{M}_z)$, we can write

$$F(0,0,z) = -\frac{1}{2\pi i z}\int_{\mathcal{M}_z} tY_0(t,z)\frac{w'(t,z)}{w(t,z)-w(\widehat{x},z)}\,\mathrm{d}t. \tag{11.6.3}$$

(c) If $c(0) \neq 0$ and $c(x)$ is constant, then we can evaluate the functional equation (11.1.2) at any point (x, y) such that $|x| \leq 1$, $|y| \leq 1$ and $K(x, y, z) = 0$, to obtain an expression for $F(0, 0, z)$.

The computation of $F(1, 0, z)$ and $F(0, 1, z)$ depends on the position of the point 1 with respect to $\mathscr{G}(\mathcal{M}_z)$ and $\mathscr{G}(\mathcal{L}_z)$. Indeed, if 1 belongs to these domains, then $F(1, 0, z)$ and $F(0, 1, z)$ are simply obtained by evaluating the integral formulations (11.6.1) at $x = 1$ and $y = 1$.

We now assume that for a given z, the point 1 does not belong to $\mathscr{G}(\mathcal{M}_z)$. Then, by evaluating (11.1.2) at $(x, Y_0(x, z))$ and $(X_0(Y_0(x, z), z), Y_0(x, z))$, and taking the difference of the two resulting relations, we obtain

$$\begin{aligned} c(x)F(x,0,z) &= c(X_0(Y_0(x,z),z))F(X_0(Y_0(x,z),z),0,z) \\ &\quad + \frac{Y_0(x,z)}{z}[x - X_0(Y_0(x,z),z)]. \end{aligned} \tag{11.6.4}$$

The key point is that, for $x \in \mathscr{G}(\mathcal{M}_z)$, the range of $\mathbb{C}$ through the composite function $X_0(Y_0(x, z), z)$ is $\mathscr{G}(\mathcal{M}_z)$ itself. This fact was proved in Corollary 5.3.5 for $z = 1/|\mathcal{S}|$, but the line of argument easily extends to other values of z. In particular, to compute the right-hand side of (11.6.4), we can use the expression (11.6.1) valid for any x, in particular for $x = 1$.

As for $F(1, 1, z)$, we simply use Eq. (11.1.2), so that

$$(|\mathcal{S}| - 1/z)F(1,1,z) = c(1)F(1,0,z) + \widetilde{c}(1)F(0,1,z) - \delta_{-1,-1}F(0,0,z) - 1/z. \tag{11.6.5}$$

11.6.5 Groups of Order 4

For the 19 walks with a group W and H of order 4 (see the classification given by Theorem 11.2.1) there are two possible ways to compute the exact expression and the asymptotic of the coefficients of $F(0, 0, z)$, $F(1, 0, z)$, $F(0, 1, z)$ and $F(1, 1, z)$.

Firstly, we can use Sect. 11.6.3, as for any of the 79 models. Secondly, the reasoning presented in Sect. 11.5.1 extends directly to the 19 walks that have a group of order 4. Indeed, since in this case the boundary condition (11.5.6) is set on a circle, we can replace $\overline{t}$ by a simple linear fractional transform of t—for example, for the unit circle centered at 0, $\overline{t} = 1/t$. In addition, we know that having a BVP set on a circle is equivalent to the group being of order 4.

As for the example of the simple walk, one can check that these two approaches coincide: take $t + 1/t$ for the CGF $w(t, z)$ in (11.6.1), then make a partial fraction expansion of $w'(t, z)/[w(t, z) - w(x, z)]$, and this will eventually lead to (11.5.7).

11.6.6 On the Singularities of the Generating Functions

As mentioned at the beginning of Sect. 11.6, we shall not provide a complete analysis of the singularities, since this would require a great amount of difficult and lengthy computations. Let us at once remark that only real singularities of $F(0, 0, z)$, $F(1, 0, z)$, $F(0, 1, z)$ and $F(1, 1, z)$ with respect to z play a role. From the expressions obtained in Sect. 11.6.4, we shall explain the main origin of all possible singularities, starting first with $F(0, 0, z)$.

Proposition 11.6.3 *The smallest positive singularity of* $F(0, 0, z)$ *is*

$$z_g = \inf\{z > 0 : y_2(z) = y_3(z)\}. \tag{11.6.6}$$

Remark 11.6.4 We choose to denote the singularity above by z_g, as one alternative definition could be the following: the smallest positive value of z for which the *genus* of the algebraic curve $\{(x, y) \in \mathbb{C}^2 : K(x, y, z) = 0\}$ *switches from* 1 *to* 0. Proposition 11.6.5 also gives other characterizations of z_g.

Outline of the Proof of Proposition 11.6.3 (see [39]) To find the singularities of $F(0, 0, z)$, we start from the expressions obtained in Sect. 11.6.4, especially (11.6.2) and (11.6.3).

- Consider first the case where the group is of order 4. Then the quartic curve $\mathcal{M}_z$ is a circle for any z (see Sect. 11.6.2), and $w(t, z)$ is a rational function of valuation 1 (see Sect. 11.6.3). In fact, the singularities come from $Y_0(t, z)$, since the branch points $x_\ell(z)$ appear in the expression of this function, see (11.5.2) and (11.5.4). Hence the first singularity of $F(0, 0, z)$ is exactly the smallest singularity of the $x_\ell(z)$, and that corresponds to Eq. (11.6.6) by Proposition 11.6.5 below.
- Consider now all the remaining cases (i.e., an infinite group or a finite group of order strictly larger than 4). Then the singularity z_g appears not only for the same reasons as above, but also on account of the CGF $w(t, z)$. Indeed, for $z \in (0, z_g)$, the curve $\mathcal{M}_z$ is smooth, while for $z = z_g$ one can show as in section that $\mathcal{M}_z$ has a non-smooth double point at $X(y_2(z), z)$. Accordingly, $w(t, z)$ has a singularity at z_g, and the behavior of $w(t, z)$ in the neighborhood of z_g is strongly related to the angle between the two tangents at the double point of the quartic curve, as encountered in Sect. 6.5. ■

In Proposition 11.6.3 and in Remark 11.6.4, we gave two different characterizations of z_g. We present below five others.

Proposition 11.6.5 *The value of the first singularity* z_g*, introduced in* (11.6.6)*, can also be characterized by the following five equivalent statements.*

1. $z_g = \inf\{z > 0 : x_2(z) = x_3(z)\} = \inf\{z > 0 : y_2(z) = y_3(z)\}$.
2. z_g *is the smallest positive singularity of the branch points* $x_\ell(z)$.
3. z_g *is the smallest positive singularity of the branch points* $y_\ell(z)$.

4. *z_g is the smallest positive double root of the discriminant $d(x, z)$ considered as a polynomial in x.*
5. *z_g is related to the minimizer of the Laplace transform of the jumps $(i, j) \in \mathcal{S}$ as follows. Define (u, v) as the unique solution in $(0, \infty)^2$ of*

$$\sum_{(i,j)\in\mathcal{S}} i u^i v^j = 0, \qquad \sum_{(i,j)\in\mathcal{S}} j u^i v^j = 0. \tag{11.6.7}$$

Then

$$z_g = \frac{1}{\sum\limits_{(i,j)\in\mathcal{S}} u^i v^j}. \tag{11.6.8}$$

Before proving Proposition 11.6.5, it is worth noting that z_g coincides with the smallest positive singularity of $F(0, 0, z)$. Indeed, this could be checked from the analysis in [26, Sect. 1.5], which matches (11.6.7) and (11.6.8).

In addition, a direct consequence of any of the five points of Proposition 11.6.5 is that z_g is algebraic. In fact, point *4* shows that this degree of algebraicity is at most 7, but the following corollary of Proposition 11.6.5 *(5)* implies that it can be strictly smaller than 7.

Corollary 11.6.6 *We have $z_g = 1/|\mathcal{S}|$ if and only if $\sum_{(i,j)\in\mathcal{S}} i = \sum_{(i,j)\in\mathcal{S}} j = 0$, which happens for 14 models, according to the classification proposed in [12]. Otherwise, $z_g > 1/|\mathcal{S}|$.*

Proof of Proposition 11.6.5 Only point *5* has to be shown, since the others are easy by-products of the definition (11.6.6) of z_g. For this purpose, we use [see Eq. (11.6.6) and assertion *1* in Proposition 11.6.5] the equalities

$$z_g = \inf\{z > 0 : y_2(z) = y_3(z)\} = \inf\{z > 0 : x_2(z) = x_3(z)\}.$$

Then we have (see e.g., Chap. 6 or [61, Sect. 5])

$$X(y_2(z_g), z_g) = x_2(z_g), \qquad Y(x_2(z_g), z_g) = y_2(z_g).$$

Hence, the pair $(u, v) = (x_2(z_g), y_2(z_g))$ satisfies the system

$$K(u, v, z_g) = \frac{\partial}{\partial x} K(u, v, z_g) = \frac{\partial}{\partial y} K(u, v, z_g) = 0,$$

which in turn yields (11.6.7), while (11.6.8) is a consequence of $K(u, v, z_g) = 0$. To see why (11.6.7) has a unique solution in $(0, \infty)^2$, we refer again to [61, Sect. 5].

Since claims 5 and 1 are equivalent to the fact that both discriminants d and $\widetilde{d}$ have a double root, the proof of the proposition is complete. ∎

The main idea for a complete proof of Proposition 11.6.3 would be to proceed along the same lines as in Chap. 7: extract some fine properties of the CGF by continuity with respect to z, when the Riemann surface passes from genus 1 to genus 0. Then one could obtain the precise behavior of $F(0, 0, z)$ near z_g.

As for the singularities of $F(1, 0, z)$ and $F(0, 1, z)$, a key point is to locate the point 1 with respect to the domains $\mathscr{G}(\mathcal{M}_z)$ and $\mathscr{G}(\mathcal{L}_z)$. Indeed, the expressions of $F(1, 0, z)$ and $F(0, 1, z)$ written in Sect. 11.6.4 depend on this location.

If 1 belongs to the sets above for any $z \in (0, z_g)$, then $F(1, 0, z)$ and $F(0, 1, z)$ are simply obtained by evaluating the integrals in Proposition 11.6.2 at $x = 1$ and $y = 1$. Then the first singularity of these functions is again z_g.

Assume now that for some $z \in (0, z_g)$, 1 does not belong to $\mathscr{G}(\mathcal{M}_z)$. Then we use (11.6.4) to define $F(1, 0, z)$. Then the singularities of $F(1, 0, z)$ have to be sought among z_g and the singularities of $Y_0(1, z)$ and $X_0(Y_0(1, z), z)$. But $X_0(Y_0(1, z), z)$ is either equal to 1, or to $\widetilde{c}(Y_0(1, z))/\widetilde{a}(Y_0(1, z))$ by Corollary 5.3.5, and accordingly is either regular or has the same singularities as $Y_0(1, z)$.

On the other hand, Eqs. (11.5.1), (11.5.2) and (11.5.4) imply that the singularities of $Y_0(1, z)$ necessarily satisfy $d(1, z) = 0$. As a consequence, the smallest positive singularity of $Y_0(1, z)$ is given by

$$z_Y = \frac{1}{b(1) + 2\sqrt{a(1)c(1)}}.$$

Lemma 11.6.7 $z_Y \in \left[1/|\mathcal{S}|, z_g\right]$.

Proof The inequality $z_Y \geq 1/|\mathcal{S}|$ is a consequence of the relations

$$2\sqrt{a(1)c(1)} \leq a(1) + c(1), \quad a(1) + b(1) + c(1) = |\mathcal{S}|.$$

Also, z_Y satisfies $d(1, z_Y) = 0$. In other words, z_Y is the smallest positive value of z such that 1 is a root of $d(x, z)$. But we have $x_2(0) = 0$, $x_3(0) = \infty$, and $x_2(z_g) = x_3(z_g)$, see Proposition 11.6.5. Hence, by a continuity argument, one of the points $x_2(z)$ or $x_3(z)$ must become 1 before reaching the other branch point.

Similarly, we conclude that $F(0, 1, z)$ has a singularity at z_g, and possibly at

$$z_X = \frac{1}{\widetilde{b}(1) + 2\sqrt{\widetilde{a}(1)\widetilde{c}(1)}}.$$

Concerning the algebraicity of z_Y and z_X, we merely note that these numbers are either rational or algebraic of degree two, if and only if $a(1)c(1)$ and $\widetilde{a}(1)\widetilde{c}(1)$ are not square numbers.

As for $F(1, 1, z)$, once we know the singularities of $F(0, 0, z)$, $F(1, 0, z)$ and $F(0, 1, z)$, it is straightforward to compute those of $F(1, 1, z)$ by means of Eq. (11.6.5).

Remark 11.6.8 (*Conclusion*) A consequence of the various results of Sect. 11.6 is that the smallest positive singularities of $F(0, 0, z)$, $F(1, 0, z)$, $F(0, 1, z)$ and

$F(1, 1, z)$ are algebraic, and sometimes even rational, for all models of walks. Furthermore, we have the following classification of the singularities. For each of the 74 non-singular models, we need the $\overrightarrow{\mathbf{M}}$ and the covariance

$$C = \sum_{(i,j)\in\mathcal{S}} ij - M_x M_y.$$

The table below gives the smallest positive singularities of all generating functions, in terms of the sign of the coordinates of the drift vector and of the covariance. The notation *"FS"* stands for *"First positive singularity"*.

	FS of $F(1,0,z)$:			FS of $F(0,1,z)$:			FS of $F(1,1,z)$:			
Drift:	z_g	z_Y	$1/\lvert\mathcal{S}\rvert$	z_g	z_X	$1/\lvert\mathcal{S}\rvert$	z_g	z_X	z_Y	$1/\lvert\mathcal{S}\rvert$
$(+,+)$		$\times$			$\times$					$\times$
$(+,0)$		$\times$		$C \geq 0$	$C \leq 0$					$\times$
$(0,+)$	$C \geq 0$	$C \leq 0$			$\times$					$\times$
$(0,0)$	$\times$			$\times$			$\times$			
$(+,-)$		$\times$		$\times$					$\times$	
$(-,+)$	$\times$				$\times$			$\times$		
$(0,-)$	$C \leq 0$	$C \geq 0$		$\times$			$C \leq 0$	$C \geq 0$		
$(-,0)$	$\times$			$C \leq 0$	$C \geq 0$		$C \leq 0$	$C \geq 0$		
$(-,-)$	$\times$			$\times$			$\times$			

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