ANDRZEJ MOSTOWSKI AND FOUNDATIONAL STUDIES

This page intentionally left blank

Andrzej Mostowski and Foundational Studies

Edited by

A. Ehrenfeucht

Department of Computer Science, University of Colorado, CO, USA

V.W. Marek

Department of Computer Science, University of Kentucky, KY, USA

and

M. Srebrny

Institute of Computer Science, Polish Academy of Sciences, Poland

IOS Press

Amsterdam • Berlin • Oxford • Tokyo • Washington, DC

ISBN 978-1-58603-782-6
Library of Congress Control Number: 2007939570

Publisher
IOS Press
Nieuwe Hemweg 6B
1013 BG Amsterdam
Netherlands
fax: +31 20 687 0019
e-mail: order@iospress.nl

Distributor in the UK and Ireland
Gazelle Books Services Ltd.
White Cross Mills
Hightown
Lancaster LA1 4XS
United Kingdom
fax: +44 1524 63232
e-mail: sales@gazellebooks.co.uk

Distributor in the USA and Canada
IOS Press, Inc.
4502 Rachael Manor Drive
Fairfax, VA 22032
USA
fax: +1 703 323 3668
e-mail: iosbooks@iospress.com

PRINTED IN THE NETHERLANDS

Preface

Andrzej Mostowski was one of the world's leading European scientists, preserving and expanding a national school of research: in his case the Polish School of Logic. This effort took place within the adverse political and social circumstances of post-World War II Warsaw. Having taught at the underground Warsaw University during the War, Mostowski was active in rebuilding the teaching and research infrastructure in Poland, based on the tradition of logic and mathematics research that had already led to the country's outstanding reputation before the War. The country's world-wide contribution to research and teaching in logic, mathematics and computer science after the War bear witness to the impact that this Logic School, Professor Mostowski, his collaborators and students have had. Apart from being a key person in teaching and research management, Mostowski was a profound researcher with contributions in the main areas of mathematical logic and the foundations of mathematics.

This volume is a collection of fifteen research and expository articles, a complete bibliography of Andrzej Mostowski's writings, three biographical and historical articles, and eleven short personal reminiscences, all aimed at illuminating Andrzej Mostowski's ideas and personality. Our main motivation for soliciting and editing these articles was that the name "Mostowski" arouses a great deal of favorable interest in very many people from various countries and in many different research communities.

A starting point for anyone interested in the work of Mostowski is his famous series of lectures "Thirty years of foundational studies", published in 1965, considered by many as an ultimate presentation of the then-current state of mathematical logic, and more generally, the foundations of mathematics. Michael Dunn (logician and computer scientist, Indiana University, Bloomington) told us that he came across "Thirty years ..." as a student in Pittsburgh and found the work so interesting and beautiful that he carried it everywhere for the next two years, trying to convince everyone around him to read it.

It is not clear whether Mostowski would be happy with the title of the current collection. Modest as he was, he could very well argue that the scope of the foundations of mathematics is too large these days to be encompassed by a single volume and a single researcher. The explosion of foundational research resulting from challenges and progress in computer science and artificial intelligence makes it almost impossible to present even their main ideas in a single book. Their impact is even felt further afield, in economics, social science, engineering, cognitive sciences and philosophy.

The contributions included in this collection present a variety of approaches to different areas of logic and the foundations of mathematics. They are written on different levels, by various researchers presenting a variety of views. We believe that these articles can continue to stimulate research in the areas pioneered and mastered by Mostowski.

Reflecting Mostowski's diverse interests, the research contributions in this volume relate to many distinct areas.

Balcar and Jech study a class of Boolean algebras and its relationship with techniques used in set theory, in particular forcing.

Dickmann and Petrovich exhibit an important interaction between three-valued logic and the algebraic theory of diagonal quadratic forms over a kind of rings, called semi-real.

Relating to the original studies of Mostowski in the late 40's and 50's of the 20th century, Friedman discusses the current progress in Gödel phenomena, mainly incompleteness. Friedman lays out a program of significant progress that needs to be made in our comprehension of the scope of incompleteness. The paper challenges the foundational community with specific questions that need to be studied to achieve a deeper understanding of incompleteness.

Grzegorczyk and Zdanowski develop basic decidability results without resorting to Gödel's encoding of formulas as integers. Instead, they work with the elementary theory of strings, introduced by Tarski under the name of "concatenation" in his celebrated paper on the concept of truth. They prove that the theory of strings is essentially undecidable.

Guzicki and Krynicki provide a new result in the area of generalized quantifiers. They define and pursue a quantifier that does not meet the very general definition of a quantifier proposed by Mostowski in 1957.

Keisler extends certain results of Mostowski on the complexity class of the concept of $\lim_{z\to\infty} F(z) = \infty$, showing that in many structures the limit cannot be defined with fewer than three quantifier blocks, but in some more powerful structures the limit property for arbitrary functions can be defined in both two-quantifier forms.

Knight presents the current state of research in the areas of the arithmetic (Kleene–Mostowski) and hyperarithmetic (Davis–Mostowski) hierarchies. In recent years, research in this area has merged with effective aspects of model theory, resulting in a variety of deep results on the complexity of algebraic structures and constructions. The current research in this area, sometimes called "recursive mathematics" is presented.

Kotlarski deals with the interplay of syntactic and semantic arguments in Peano Arithmetic and its fragments. His paper outlines significant progress in the study of formalized arithmetic since Mostowski's death. The results of these investigations have revised our understanding of the nature of independent formulas in Peano Arithmetic, exhibiting sentences that shattered the conviction (current in some mathematical circles) that "ordinary combinatorial sentences" cannot be independent from Peano Arithmetic. A Ramsey-like sentence of this sort was discovered by Paris and Harrington. Numerous other examples were found by Friedman and others.

Makowsky deals both with the reminiscences of Mostowski in the social context of doing mathematics, and with specific mathematical developments grounded in Mostowski's work. Those developments, particularly in model theory and in abstract model theory, where Makowsky contributed significantly, have found application, outside pure mathematics, in theoretical computer science.

Murawski and Woleński investigate the philosophy of mathematics espoused in Mostowski's papers. Mostowski's work was firmly grounded in the philosophical tradition of his teachers, including Tarski and Kotarbiński. The authors show how Mostowski related to the foundational problems in many of his contributions, especially in his work and presentations of Gödel's incompleteness and Cohen's independence results.

Mycielski investigates the issue of mathematical truth, its formal and informal aspects, and its relationship with mathematical practice. The paper discusses Tarski's theory of truth and the question of the presence of the hierarchy of metatheories in everyday mathematics, metatheories determined by some bounds on the allowed lengths of proofs.

L'Innocente and Macintyre present a very nice application of a model-theoretic result of Mostowski on direct products of models to certain fundamental problems of a special Lie algebra and the Diophantine geometry of curves.

The two expository papers by Scott and by Jankowski and Skowron are devoted to the algebraic approach to formal logic. Scott focuses on the quantifiers interpreted as the infs and sups, respectively, in the algebras interpreting various logics. Jankowski and Skowron sketch the past and present perspective of the role of algebraic logic and Pawlak's rough sets in artificial intelligence.

Wells discusses the issue of pseudorecursive varieties, that is, varieties V of algebras such that for each natural number n, the set of equations with at most n variables true in all algebras of V is recursive, but such that the set of equations true in all algebras of V is not recursive. The paper outlines progress in the area as well as a number of important problems.

Some of the contributors pointed out to us that their topics and results are derived from Alfred Tarski's inspiration. We quickly agreed to underline the more general perspective of the Tarski/Mostowski Berkeley/Warsaw school. The reader will find references to Tarski's work, ideas and inspirations in several contributions in this volume. The work and vision of Alfred Tarski, albeit from afar after the WWII, significantly influenced both Mostowski's research and that of his collaborators. It is natural to treat Mostowski as "the Prince of the Tarskian kingdom", as B.F. Wells, the last of Tarski's students, does in his contribution. (In spite of many efforts we have not been able to trace any letter of the postal correspondence between Tarski and Mostowski. We do not believe that the two great mathematicians just disposed of their correspondence. We leave it as an intriguing question to the historians of science.)

The memory section of the volume consists of eleven very nice reminiscence articles by famous logicians (mathematicians, philosophers and computer scientists) Addison, Blikle, Dickmann, Feferman, Hajék, Kowalski, Wells, Vopěnka and others, shedding some interesting light on their own research interests and achievements.

As we contemplate the memory of a giant of foundational research, the main question that presents itself is this:

How did it happen that one man, located in a rather small European country, a country whose rulers deliberately isolated it from the mainstream of world science, a country ravaged by a savage war, was able to create a center for research that extended its reach worldwide?

The answer to this question lies in Mostowski's personality, in his ability to find his way in the complex circumstances of local and world politics on the one hand, and on the other hand in his ability to use the opportunities created by the same circumstances to attract large number of collaborators, individuals who found it appealing to work in a large research group united in its goal of broadening the horizons of human knowledge in logic and the foundations of mathematics.

The historic circumstances demanded such a man. The steady progress in the first half of the 20th century created an opportunity – major problems that were unresolved related to the very nature of computation and reasoning (completeness and incompleteness phenomena, and the study of computable functions), to the fundamental structures underlying mathematics (sets, algorithms and epistemological principles of their existence and manipulation), and to anticipation of the digital revolution that was about to happen.

These opportunities were open to many. It was Mostowski who, consciously or not, transformed this potential into a research program for a large group of people who attached themselves locally, and who came as pilgrims from abroad, to take part in this unique experience. *It is through Mostowski's deep research results and those individuals, their students and their scientific descendants that the ideas of Mostowski live today*.

The editors would like to thank all the contributors. We are sorry we could not invite more experts due to the space limitations. We will gladly welcome any form of continuation of this effort reflecting Andrzej Mostowski's impact. One such example, entirely independent of the present volume, was a conference *50 Years of Generalized Quantifiers – in Honour of Professor Andrzej Mostowski*, Warsaw, June 25 – July 1, 2007 (M. Krynicki, M. Mostowski, J. Väänänen and K. Zdanowski, organizers).

This volume was conceived at a conference in Warsaw and Ruciane in September 2005. That meeting commemorated the contributions of Andrzej Mostowski (on the 30th anniversary of his passing), Helena Rasiowa and Cecylia Rauszer. Several contributions collected in this volume, including D. Scott's and V. Marek's, were presented during the meeting. The editors express their thanks to the organizers of that meeting, including R. Wojcicki, A. Salwicki and G. Mirkowska-Salwicka.

The editors of this volume were Mostowski's students and collaborators during the years (1954–1965 AE), (1962–1975 VWM) and (1969–1975 MS).

We thank many individuals who helped us in the process of editing this volume, including the referees who provided many essential comments and improved presentations of the contributions.

We extend our thanks to E. Fredriksson (like ourselves, a scientific descendant of Mostowski) and his IOS Press of Amsterdam. His long-lasting encouragement, support and efforts made this project possible and enabled this volume to come to fruition. It is not an accident that IOS publishes many titles that relate to the foundational research – see the contribution of E. Fredriksson for a description of his ties to Mostowski.

Our very special thanks go to the IOS Press staff and M. Fröhlich, responsible for final typesetting and the production of the volume.

September, 2007 Andrzej Ehrenfeucht, Victor Marek, Marian Srebrny

Contents

This page intentionally left blank

Section 1

History

This page intentionally left blank

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

On the Life and Work of Andrzej Mostowski (1913–1975)[1]

Stanisław KRAJEWSKI [a] and Marian SREBRNY [b]
[a] *Institute of Philosophy, Warsaw University, Warsaw, Poland*
[b] *Institute of Computer Science, Polish Academy of Sciences, Warsaw, Poland*

Andrzej Stanisław Mostowski was born on November 1, 1913, in Lvov. His father, Stanisław Mostowski, was a medical doctor who worked as an assistant at the Physical Chemistry Department of the University of Lvov; he was conscripted in 1914 as a military doctor and soon after died of typhoid fever. The family had to be provided for by the mother, Zofia née Kramsztyk (1881–1963), who worked for many years in a bank.

Andrzej (called *Staszek* by his mother and close friends, after his father) had one sister Krystyna who after the WWII settled abroad, first in France, and then in Montreal.

In the summer of 1914 Mostowski's mother went with her children to Zakopane to spend holidays there; in the face of the outbreak of war and the death of the father they remained there until about 1920, when they moved to Warsaw. From this time on, Mostowski's whole life was connected with this city. Here, in the years 1923–1931, he attended the well-known Stefan Batory high school. In the higher grades of this school he showed himself to be an outstanding student with particular interests in the sciences. Apparently, his exceptional skills were demonstrated when, after going through an acute strep throat disease when he was 16, he had to start preparing himself to school work. Unfortunately, the illness also resulted in some heart problems, on the basis of which he was later relieved from the duty of military service. The heart problems receded with time, but he was never conscripted anyway. After the war, when he had already become a professor, he was reassigned to the reserves as a so-called "officer with no rank".

In 1931 Mostowski began to study mathematics at the Mathematics and Natural Sciences Department of the University of Warsaw. He soon became interested in the founda-

[1] This paper is based on our original article in Polish published in "Wiadomości matematyczne", Annales Societatis Mathematicae Polonae XXII.1 (1979), pp. 53–64, updated slightly where necessary. It is meant to present mainly the events of Andrzej Mostowski's life. We provide information on his work but do not discuss the content and impact of his scientific achievements.

The information included in this article has been taken from the following sources. Firstly, from the existing publications on the life and scientific output of Andrzej Mostowski. They are listed at the end of this text. The article by S. Hartman, which presents the profile of Mostowski, is of particular interest. We have also used the outline of the biography of A. Mostowski prepared by A. Śródka for the Archive of the Polish Academy of Sciences. The second source of data which we have used is the personal files and other documents of Andrzej Mostowski that have been found in the files of the University of Warsaw and the Institute of Mathematics of the Polish Academy of Sciences. We have also used – the oral information provided by various people and our own memories. Finally, we are grateful to Victor Marek and George Wilmers for many useful comments that have improved our presentation.

Figure 1. Caricature of Andrzej Mostowski by L. Jeśmanowicz, 1946.

tions of mathematics, set theory and logic, which were taught at a very high level at that time there. Among the University lecturers were many eminent scholars who contributed significantly to the history of those fields: Kazimierz Kuratowski, Stanisław Leśniewski, Adolf Lindenbaum, Jan Łukasiewicz, Wacław Sierpiński, Alfred Tarski. It was soon to emerge that Andrzej Mostowski himself was also going to have a deep influence on the development of those disciplines. In the meantime, as a student he intended to obtain an extensive education and already demonstrated clearly that he had wider horizons than the existing research of the contemporary Polish school of mathematics. Among other things he studied the theory of relativity, reading *Raum, Zeit, Materie* by Hermann Weyl and he passed examinations in an extended course on analytical functions (he later used to recall Stanisław Saks with great affection). The theory of analytic functions appealed to him particularly as he lectured on this subject in the last years of his life and insistently urged his assistants to give classes related to this topic.

While studying at the university, Mostowski became a member of the scientific section of the student Mathematical Society whose membership also included Zygmunt Charzyński, Stanisław Hartman, Julian Perkal, Jerzy Słupecki and other people who were longing "to escape from the political tumult and tension".[2] In 1936 Mostowski graduated from the University and obtained the degree of Master of Philosophy in the field of Mathematics.

From among the above mentioned celebrities of the mathematical circles in Warsaw it was Tarski and Lindenbaum who had the biggest influence on the scope of Mostowski's

[2]S. Hartman, 1976, p. 68.

interests and his scientific development. Mostowski used to call Tarski his "master". Lindenbaum once suggested to him during a discussion that he should try to formulate precisely the method of independence proofs[3] which had been sketched by Abraham Fraenkel. This research, conducted initially together with Lindenbaum (which resulted in a joint publication), was the starting point for Mostowski to work out the so-called *Fraenkel–Mostowski permutation models* method. It constituted the content of his doctoral dissertation entitled *On the Independence of the Definition of Finiteness in the System of Logic* written in 1938 and defended in February 1939. The method of permutation models became well-known owing to the publication [3] on the independence of the axiom of choice from the principle of linear ordering; i.e., the statement that every set can be linearly ordered.

Earlier, before the defense of his doctoral dissertation, Mostowski continued his studies abroad as auditor (unenrolled student) taking advantage of the financial support of his uncle, who was an industrialist in Łódź. Mostowski studied in Vienna in 1936/37 and in Zurich in 1937/38. He intended to focus on the applied mathematics because he expected that he would have to earn his living by doing something more practical than working on the foundations of mathematics. At that time nobody knew that the "scientific and technological revolution" was about to come and there was a shortage of research vacancies. However, the classes on applied mathematics turned out to be "horribly boring"[4] mainly owing to the paucity of the mathematical techniques which were being used. Consequently, Mostowski abandoned this field – as it later turned out – forever. What did however particularly arouse his interest were the classes of George Pólya and the seminar of Paul Bernays. Additionally, Mostowski was lucky to have the opportunity of attending to the lectures of some of the greatest masters: Kurt Gödel's (in Vienna) on the consistency of the axiom of choice, Herman Weyl's on symmetry and Wolfgang Pauli's on physics (both of the latter in Zürich). He always spoke of Gödel with utmost respect as of a genius. He once said that Gödel knew the solutions of the most difficult problems – as if he had a direct phone line to God. In reference to Weyl, Mostowski's attitude towards him can be best exemplified by the fact that for years Weyl's portrait hung on the wall of Mostowski's office, on the ninth floor (room 908) of the Palace of Culture and Science in Warsaw, as the sole decoration.

During his stay abroad Mostowski did a lot of intensive research: he studied the axiom of choice and the definitions of the notion of finiteness, he learned recursive function theory by studying the works of Kleene. After returning to Warsaw, in February 1939 he defended his doctoral dissertation. His doctoral thesis was supervised by K. Kuratowski but Mostowski used to stress that it was Tarski who was his research patron, although Tarski was not a professor and therefore could not officially supervise Mostowski's Ph.D. dissertation. (It needs to be mentioned that Tarski never held a professorial position in Poland, for a variety of reasons, including probably antisemitism.)

In January 1939 Mostowski was employed in the scientific section of the State Institute of Meteorology directed by the physicist Jan Blaton. With the onset of the German occupation of Poland, Mostowski lost this job. Nevertheless he continued his research activities throughout the whole period of occupation. Initially he earned his living by giving private lessons and from September 1940 to August 1944 he worked in a small bituminous tile paper factory "Wuko" (M. Bogdani, E. Kossowski and Co.) as a book-

[3]Cf. Crossley, 1975, p. 44.
[4]Crossley, 1975, p. 45.

keeping clerk. At the same time he took part in underground university teaching. In the academic year 1942/43 he lectured on analytic geometry for first year students of mathematics at the underground University of Warsaw (UW) and in 1943/44 he gave a course of lectures in algebra for second year students on Galois theory. Those clandestine study groups, in which he taught, were started on the initiative of Maria Matuszewska – who later became Mostowski's wife – and Klemens Szaniawski, who was later professor of logic at the University of Warsaw. Among Mostowski's students, whose number was never bigger than ten in one group, were also the following future professors: Helena Rasiowa (logician at Warsaw University), Krzysztof Tatarkiewicz (mathematician at Warsaw University, son of a well-known philosopher Władysław Tatarkiewicz) and Jerzy Kroh (chemist at the Polytechnic of Łódz). The teaching took place in different private flats, each of which in practice belonged to one the students, for example in the apartment of Władysław Tatarkiewicz. Professor Krzysztof Szaniawski has told us that apart from Mostowski the group of other lecturers for the first year students also included: Wacław Sierpiński (set theory), Karol Borsuk (analysis) and Zygmunt Waraszkiewicz, and for the second year: Jan Łukasiewicz (logic), Bolesław Sobociński (he filled in for Łukasiewicz) and Stanisław Mazurkiewicz (analytic functions). It is difficult to know whether Mostowski also taught some other clandestine study groups; obviously, there were many of them and also other mathematicians, for example Kazimierz Kuratowski and Witold Pogorzelski, were involved in such underground activity.

At that time Mostowski was academically well on the way to qualifying as a *docent* (roughly, associate professor). We cite below W. Sierpiński's opinion of March 25, 1945, justifying a motion to nominate Mostowski as a senior assistant in Warsaw University's Department of Mathematics: "In July 1944 the process required for Dr Andrzej Mostowski to receive his Habilitation degree in mathematics was well advanced. (...) In order to accomplish it he was only required to give the lecture on his thesis and orally defend it. (...) The completion of these formalities was interrupted by the outbreak of the Warsaw Uprising. (...) In my opinion, which is undoubtedly shared by the rest of my Warsaw colleagues, Dr Andrzej Mostowski should be considered to have obtained the Habilitation degree in mathematics in spite of the fact that the final stages have not formally taken place because of war-time events."

After the Warsaw Uprising of 1944 all of Warsaw's inhabitants were expelled from the city. On several occasions Mostowski succeeded in avoiding being sent to Germany as a slave laborer, and he spent this period hiding "in different dens and hovels to evade the street round-ups", according to the description of Stanisław Hartman,[5] then Mostowski's cousin and a companion of his wartime vicissitudes.

Also during that period Mostowski married Maria Irena Matuszewska. The wedding took place on September 26, 1944 in Tarczyn, some 30 km south of Warsaw. In October 1944 they both found employment on an agricultural research estate of the Main School of Farming in Skierniewice where until January 1945 Mostowski worked again as a junior book-keeping clerk.

Mostowski had left behind at his home in Warsaw his notebook in which he had made notes on all of his mathematical discoveries since 1942. According to his reminiscences[6] it was a nice and very thick notebook and when, during the Uprising, the Germans ordered him to leave his home at Słoneczna street in which he lived with his

[5] S. Hartman, 1976, p. 68.
[6] Cf. Crossley, 1975, p. 32.

mother, he had had to choose whether to take the notebook or a loaf of bread. He chose the bread. The notebook was burned in a fire, which undoubtedly was a loss from the point of view of science, because later Mostowski never managed to find time to recreate its entire content. The research on the decidability of the theory of well-orderings, included in the notebook, was published with no proof in 1949 in a joint paper with Tarski – its proof appeared only in 1978 as a paper written together with a third co-author. Moreover, the notebook also contained, among other things, the results of the research on the analogy between the projective hierarchy and the arithmetically definable sets as well as the proofs of some of the consequences of the axiom of constructibility in descriptive set theory.

From January 1945 for several months Mostowski had no job and lived on the money "from odd private lessons and from the sale of a ring and a watch, as well as from a single subsistence allowance from the University (in the amount of 500 złotys)".[7] In May 1945 he found employment as a senior lecturer at the section of mathematics of the Electrical Department of the Polytechnic of Silesia which had a temporary location in Cracow. Soon thereafter he formally obtained a habilitation degree in mathematical logic at the Department of Mathematics of the Jagiellonian University by successfully defending his thesis [4], which is of fundamental significance for the study of the mutual connections of various forms of the axiom of choice for families of finite sets.

The story of Mostowski's subsequent life after the war is rather simple. In December 1945 Mostowski joined Warsaw University and continued there until the end of his days – for almost thirty years. Initially he was a so-called deputy professor in the section of philosophy of mathematics, while in 1947 he was appointed an associate professor of the philosophy of mathematics in the Mathematics and Natural Sciences Department of Warsaw University and then in 1951 was appointed a full (ordinary) professor. During the academic year of 1950/51 he was the dean of the Mathematics and Natural Sciences Department and from 1966 until his death he acted as the Deputy Director of the Institute of Mathematics of Warsaw University. He also was the first director of the Ph.D. programme in the Department of Mathematics and Mechanics of Warsaw University, a programme which was established in 1970. Additionally, between January and September 1946 he used to commute to Łódź, where he was a deputy professor at the university, in order to lecture there. When the National Institute of Mathematics (which later became the Institute of Mathematics of the Polish Academy of Sciences) was established, i.e. from July 1, 1949, he headed the Section of the foundations of mathematics, and from October 1956 to January 1964 he was also the Institute's deputy director. In 1970, when the government no longer allowed more than one position to be held by a scientist, his association with the Mathematical Institute of the Academy of Science ended. Also, he served on many committees of diverse kinds. Inter alia, he was a member of the Commission for the Support of the Scientific and Artistic Creativity attached to the Presidium of the Government Cabinet (in the years 1949–1952), the team of mathematical experts in the Principal Advisory Council for the Minister of Education (1963–1966), the Principal Disciplinary Commission at the Ministry of Education (1965–1968), and the Mathematical Sciences Committee from the time of its establishment in 1960. Moreover, he was the secretary of the Polish Mathematical Society (1946–1948) and its vice-chairman and the chairman of its Warsaw Section (1952–1955).

[7] According to the biography from 1950 (from the Warsaw University files).

It has to be emphasized here that Professor Mostowski clearly did not like to hold official posts and did it against his natural preferences. Since the end of the war he had tried to devote all of his time to science and his family life. He was actively involved in the actions of many different scientific organizations, and he was a member of only one non-scientific organization, namely the Polish Teachers' Union (since 1946). On the other hand, his scientific activity encompassed an exceptional variety of topics: he did research and helped others to do so by inspiring them and suggesting ideas, he taught many subjects and wrote student textbooks, he organized scientific life in Poland and international co-operation in the field, he was the editor of mathematical periodicals and he took part in creating new university syllabuses.

The situation of mathematical logic in Warsaw after the war was very difficult. Not much was left of what had formerly been one of the foremost logic centers in the world. Mostowski was virtually left alone, except for Sierpiński and Kuratowski whose activity in the field of foundations of mathematics, however, was in practice limited to set theory. Mostowski started essentially alone, without the benefit of the advice an academic scholar may normally expect to get from his elders, having essentially no formal teaching or managerial experience. Nonetheless he felt he had a duty to rebuild a center in the field of mathematical logic and that he was partly responsible for renewing Polish mathematical life in general. What he had in mind did not solely refer to the foundations of mathematics. For example, even though his primary interests were not in algebra, but in foundations of mathematics, for many years, basically because of the shortage of qualified teachers in algebra, he lectured on this discipline. In 1950s, together with M. Stark, he wrote the textbooks: *Higher Algebra* (three parts), *Elements of Higher Algebra* (translated into English in 1963) and *Linear Algebra*, which were well-known and widely appreciated in Poland. For sixteen years, from 1953 to 1969, Mostowski held the post of the chair of the algebra section and only after that he became the director of the newly created Section of the Foundations of Mathematics at the Institute of Mathematics of Warsaw University. Several important algebraists started their studies in this field by attending his seminars. Mostowski often gave lecture courses on Galois theory. One of the consequences of his longer visit to Berkeley was to stimulate research in differential algebra in Poland. He himself, however, never dealt with algebra in a creative way.

He used his talent original mathematical discovery mainly in research in what is broadly understood as mathematical logic and the foundations of mathematics, i.e. set theory, recursion theory (based on the concept of a computable function), model theory, logical calculi and proof theory. Already in the first years after the war, his successive papers significantly broadened the scope of research in those subjects. Apart from the above mentioned habilitation thesis, among his most important publications of that period are: [5] – introducing the so-called Kleene–Mostowski hierarchy, [6] – introducing an algebraic method of demonstrating non-deducibility in the intuitionistic logic and [8] – results related to Gödel's theorem, a topic which always fascinated him. Mostowski expressed his deep understanding of this theorem in the book [13] and the paper [8] which was a far-reaching generalization of the theorem. The fundamental issues of undecidability of various versions of formalized arithmetic were analyzed by Mostowski in several papers and studied in the collection of articles [15] by Tarski, Mostowski and Raphael M. Robinson.

Throughout the period after the war Mostowski gave courses and seminars on the foundations of mathematics. In 1948 he published the textbook on mathematical logic [7] and four years later a monograph on set theory [12], which was written together with K. Kuratowski. The logic textbook served many generations of students, but nevertheless the author never agreed for it to be translated into other languages. He claimed that the book did not reflect the state of research at the moment of its creation, and that in this sense it was obsolete.

However, the above mentioned monograph on set theory was translated into English in 1967 (a new, revised edition was published in 1976) and all the other books and major papers (except the algebra textbooks) were published in the so-called congress languages and after the war primarily in English. Among them there are also other systematic lectures on set theory – the subject that Mostowski particularly liked. Additionally, the method of forcing, introduced in 1963 by Paul J. Cohen, which was a major step forward in the independence proofs, was among other issues the topic of the series of Mostowski's lectures published in print at the end of the sixties: [27,31], and also of the later publication [33], as well as of the monograph [32].

Mostowski continued to study set theory throughout his life. Apart from the already mentioned publications let us add the paper [10], important for the research on Gödel–Bernays set theory, or the paper [28] on constructible sets. The first of those papers was probably an early sign of Mostowski's particular interest in the axiomatic theory of classes and its stronger version – the Kelley–Morse theory – which he popularized and pursued mainly in the seventies. Special attention should also be called to the series of papers in which, starting in the late fifties, he introduced fragments of the set theories related to weaker systems of the theory of types based on natural numbers (cf. [24]). Mostowski studied those higher order arithmetics and their various classes of models in the sixties and seventies. Before that he initiated research into another kind of strengthening of first order arithmetic, described in the fundamental paper [22] written together with A. Grzegorczyk and Cz. Ryll-Nardzewski.

In the fifties the following important papers contributed to model theory: [11] – starting investigations of the theory of products of models, [20] (together with A. Ehrenfeucht) – initiating the research on models with the so-called indiscernible elements and [21] – introducing the generalized quantifiers, which was a precursor to the development of the theory of abstract logics that became very fashionable fifteen years later. Furthermore, in the sixties Mostowski conducted, and often also initiated, investigations in model theory of various non-classical logics, for example in [25] on axiomatizability in weak second order logic or in [30] on interpolation in various logics.

The above list of topics gives an idea of the influence that Mostowski had on the development of the foundations of mathematics. His deep knowledge of all the issues in mathematical logic which were current in the mid-sixties emanates from the survey papers and, above all, from the excellent lectures [27] which Jaakko Hintikka described as "about as good a survey as anyone can hope to find".[8] It has to be mentioned, however, that with the immense increase in the quantity of publications in the Foundations of Mathematics, with the corresponding growth of solved and open problems, and with the variety of new proof techniques developed both in the Foundations of Mathematics and Computer Science in recent decades, presentation of the panorama of Foundations is a task which has today become beyond the capabilities of a single person. It is unlikely that anyone, including Mostowski, would be able to present picture of the entire Foundations today.

[8] In the introduction to *The Philosophy of Mathematics*, Oxford University Press, 1969, which he edited.

Mostowski's bibliography lists 119 publications. Teaching courses, systematic lectures and monographs comprise 13 of them, the papers containing original mathematical results – about 80. Some of the entries are repeated owing to translations and multiple editions.

Mostowski quickly gained international recognition. It is enough to mention that as early as in 1948/49 he was invited as a "temporary member" to the prestigious Institute for Advanced Studies in Princeton. There, he again had the opportunity of meeting Gödel. Later, he used to repeatedly travel abroad to take part in different symposiums and lectures in many countries, and was always welcomed anywhere he went. In 1958/59 he was a "visiting professor" at the University of California at Berkeley, where the world famous center for the foundations of mathematics had been created by Tarski after World War II. In 1969/70 Mostowski was a member of All Souls College in Oxford.

Mostowski enjoyed immense intellectual and moral prestige in the mathematical community. Thus it is no surprise that he was a member of the Council and the Executive Committee of the European section of the Association for Symbolic Logic and, additionally, in 1964–1968 the Deputy Chair of the Section of Logic, Methodology and Philosophy of Science in the International Union of History and Philosophy of Science (IUHPS), and from 1972 – the chairman of this section. In 1974 Mostowski became a member of the International Committee responsible for granting the Fields medals (that year the Committee awarded medals to Enrico Bombieri and David Mumford). At the end of the sixties, a discussion on the future of the Mathematical Reviews and Zentralblatt für Mathematik periodicals was conducted in the Notices of American Mathematical Society. One of the arguments used in favor of the continuation of publishing the scientific papers' reviews was the possibility of losing such "masterly reviews" as, for example, Mostowski's review of the results of P.J. Cohen.

On Mostowski's 60th birthday, Fundamenta Mathematicae published a special volume commemorating this occasion. Leading researchers of Foundations contributed to the volume; among them were Keith J. Devlin, Petr Hajek, Edgar G.K. Lopez-Escobar, Yiannis N. Moschovakis, Ken McAloon, Kenneth Jon Barwise, Mihaly Makkai, Wojciech Guzicki, Solomon Feferman, Dirk van Dalen, Wiktor Marek, Henk Barendregt, C.E. Michael Yates, Leo Bukovsky, Sergei R. Kogalovskii, Robert L. Vaught, Hao Wang, Anne S. Troelstra, Roland Fraisse, Ulrich Felgner, Wolfram Schwabhauser, Lesław W. Szczerba, Fred Galvin, Jean Larson, Leon Henkin.

Mostowski died prematurely at 62. After his death the following publications and events were dedicated to him to honour his memory: a volume of Proceedings of the international conference in Bierutowice in 1975, a special session at a congress of the Section of Logic, Methodology and Philosophy of Science of IUHPS, a conference in Bierutowice in September 1976, a symposium at the annual Logic Colloquium conference in Oxford in 1976 and another such conference in Wrocław in 1977.

Mostowski eagerly traveled to different mathematical centers[9] but at the same time many people visited him in Warsaw in order to attend his seminars, listen to his com-

[9]The more important short scientific foreign visits of Andrzej Mostowski were the following: Berkeley (3 weeks, 1963), Helsinki (3 weeks, 1964), London (10 days, 1965), Jerusalem (4 weeks, 1966), Montreal (5 weeks, 1966), Los Angeles (4 weeks, 1967), Brussels (7 days, 1968), Varenna (17 days, 1968), Helsinki (2 weeks, 1969), Oberwolfach (7 days, 1969), Paris (2 weeks, 1972), Genova (3 weeks, 1972), Waterloo (2 weeks, 1973), Melbourne (3 weeks, 1973), Kiel (3 days, 1974), Berkeley (2 months, 1975).

ments, discuss with him the research issues or just learn from him. A longer visit was thus, inter alia, made by John W. Addison, Miroslav Benda, Maurice Boffa, Max Dickmann, Einar Fredriksson, Donato Giorgetta, Peter G. Hinman, Finn V. Jensen, Robert Kowalski, Edgar G.K. Lopez-Escobar, Moshe Machover, Janos Makowsky, Kan C. Ng, Karel Prikry, Diane Resek, Ladislav Rieger, Hidemitsu Sayeki, Yoshindo Suzuki, Benjamin F. Wells, George Wilmers. It also has to be emphasized here that the mathematicians from the now world famous Prague school of Petr Vopěnka and Petr Hájek owe Mostowski a much appreciated help in introducing them to major world centers of logic.

The exceptional friendliness and personal charm of Mostowski caused our country to stand exceptionally high in the favour of foreigners. Many of them continued to maintain lively scientific relations with Polish mathematicians. Some of them even learned to speak Polish. This fact was occasionally the source of amusing situations. Once a conversation between a number of people took place after Mostowski's lecture in Montreal. Mostowski fluently switched from language to language answering different questions. At one point Sayeki, a Japanese, asked him about something in Polish and Mostowski answered the question straight away. The astonished listeners congratulated him on such proficient command of Japanese.

Mostowski had obviously the biggest impact on the shape of mathematical logic and the foundations of mathematics in Poland. At the time of his death, all the people who were active in this discipline in Warsaw and most such persons in other cities were directly or indirectly his students. Amongst Polish mathematicians, the following scholars co-authored research papers with Mostowski: Andrzej Ehrenfeucht, Andrzej Grzegorczyk, Kazimierz Kuratowski, Adolf Lindenbaum, Jerzy Łoś, Wiktor Marek, Helena Rasiowa, Czesław Ryll-Nardzewski, Alfred Tarski. Also Stanisław Jaśkowski, Stanisław Mazur and Roman Sikorski, who collaborated in the writing of the expository article [18], should be mentioned here.

Mostowski devoted significant effort to furthering scientific publishing. He was the editor of the mathematical, astronomical and physical series of the Bulletin of the Polish Academy of Sciences (since 1956), a member of the editorial committees of Fundamenta Mathematicae, Dissertationes Mathematicae, Studia Logica and Journal of Symbolic Logic, one of the editors (since 1966) of the well-known series of the Studies in Logic and the Foundations of Mathematics published by the North Holland Publishing Company (moreover, since its establishment in 1951, he remained in contact with the publisher, M. Frank, and significantly contributed to the increase of importance and reach of this series), and lastly, he also was a co-creator and co-editor of the Annals of Mathematical Logic, whose objective was to publish long articles that were normally distributed on a small scale in a mimeographed form and therefore were available to a limited extent to the people from outside the main centers.

Mostowski's wife, Maria Mostowska, was for many years, until her retirement in 1982 the head librarian at the Mathematical Institute of the Polish Academy of Sciences. All those who visited the library during her tenure recall her efforts to provide the best support for the creative work of the mathematicians. The passion to mathematics passed from Mostowski to his sons: the older one, Tadeusz (b. 1947), is a mathematician at Warsaw University; and the younger one, Jan (b. 1949) – a physicist at the Institute of Physics of the Polish Academy of Sciences. His daughter Maria (b. 1956), called Isia, graduated in medicine and currently is active in business in Warsaw.

One of the life-long passions of Mostowski was astronomy, and in particular stargazing. As a student he became a member of the Society of Friends of Astronomy. He visited some of the best-known observatories and in 1958 brought home from the United States a telescope. From that time on, together with his sons, he carried out systematic observations of the stars. He would often climb the roof of his neighbours house for better visibility.

While Mostowski was well known for his modesty, his scientific and organizational achievements resulted in many honors and prizes. In 1956 he was elected a Corresponding Member of the Polish Academy of Sciences and in 1963 a full member of the Academy (for the period of 1960–1963 he served as a member of the Scientific Secretariat of the III Division of the Academy of Sciences). Mostowski twice received Polish State Prizes for Science: the lower (second class) prize in 1952, and in 1966 the First Class State Prize. The prize was awarded for his contributions to the Foundations of Mathematics. The Polish non-governmental organization in the United States, the Jurzykowski Foundation, awarded its prize to Mostowski in 1972. In 1973 Mostowski was elected to the Finnish Academy of Sciences. Since Mostowski did not keep track of distinctions we could not find other awards which were, certainly, bestowed upon him. Those who knew him were well aware that he did not care.

The relentless activity of Professor Mostowski did not decrease even after his long-lasting illness in 1973 when he was diagnosed with diabetes. He continued to be an example of reliability and the sense of duty. He also suffered from high blood pressure. The refusal to take the drugs that, as he felt, limited his capacity to think and work, was among the reasons for his untimely death.

Mostowski travelled to Berkeley and Stanford for the Summer of 1975. On his way back he was travelling to London, ONT, for the Fifth Congress of Logic, Methodology and Philosophy of Science. On his way there he visited his student Malgorzata Dubiel-Lachlan in Vancouver, BC. On August 20, 1975 he gave his last lecture there, and half an hour after its completion he suffered a stroke. He died two days later, without regaining consciousness. Accommodating his wish, his ashes were dispersed over the ocean.

A list of early publications on Andrzej Mostowski

[1970] A. Grzegorczyk, *Andrzej Mostowski*, Nauka Polska 4 (1971), pp. 80–84.

[1975] J.N. Crossley (ed.), Algebra and Logic, Lecture Notes in Mathematics 450, Springer, 1975, a record of a discussion with the participation of Andrzej Mostowski.

[1975] K. Kuratowski, *An extensive obituary of A. Mostowski*, Życie Warszawy, 29.08.1975.

[1975] *An extensive obituary of A. Mostowski,* London's The Times, 23.09.1975.

[1976] S. Hartman, *Andrzej Mostowski*, Matematyka 2 (1976), pp. 67–70.

[1976] W. Marek, M. Srebrny, A. Zarach (eds.), *Set theory and hierarchy theory. A memorial tribute to Andrzej Mostowski,* Lecture Notes in Mathematics 537, Springer, 1976, Curriculum vitae of A. Mostowski.

[1976] L. Pacholski, *O działalności naukowej Andrzeja Mostowskiego*, Matematyka 2 (1976), pp. 67–70.

[1976] *An extensive obituary of A. Mostowski*, Review of the Polish Academy of Sciences 4 (1976), pp. 147–150.

[1976] H. Rasiowa, *Andrzej Mostowski* (1913–1975), Nauka Polska 9–10 (1976), pp. 233–237.

[1976] H. Rasiowa, *Andrzej Mostowski*, Studia Logica 36 (1977), pp. 1–3.

[1977] H. Rasiowa, *A tribute to Andrzej Mostowski*, in: Logic Colloquium '76, eds. R.O. Gandy and J.M.E. Hyland, North-Holland Publ. Co., 1977, pp. 139–144.

[1978] *A. Mostowski, Foundational Studies, Selected Works*, volume I, Polish Scientific Publishers and North-Holland Publ. Co., 1978 (a short biography, full bibliography of publications and reports on the results of Andrzej Mostowski in various areas of the foundations of mathematics).

[1979] S. Krajewski, M. Srebrny, *The life and activities of Andrzej Mostowski* (in Polish), Wiadomości Matematyczne, Annales Societatis Mathematicae Polonae XXII.1 (1979), pp. 53–64.

References to publications by Andrzej Mostowski

[1] *O niezależności definicji skończoności w systemie logiki*, Addendum to Roczniki Polskiego Towarzystwa Matematycznego, Vol. XI (1938), pp.1–54.

[2] (with A. Lindenbaum) *Über die Unäbhangigkeit des Auswahlaxioms und einiger seiner Folgerungen*, Sprawozdania Towarzystwa Naukowego Warszawskiego XXXI (1939), pp. 27–32.

[3] *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, Fundamenta Mathematicae XXXII (1939), pp. 201–252.

[4] *Axiom of choice for finite sets*, Fundamenta Mathematicae XXXIII (1945), pp. 137–168.

[5] *On definable sets of positive integers*, Fundamenta Mathematicae XXXIV (1947), pp. 81–112.

[6] *Proofs of non-deducibility in intuitionistic functional calculus*, The Journal of Symbolic Logic 13 (1948), pp. 193–203.

[7] *Logika Matematyczna*, Monografie Matematyczne, Vol. XVIII (1948), Warszawa-Wrocław, viii+338 pp.

[8] *An undecidable arithmetical statement*, Fundamenta Mathematicae XXXVI (1949), pp. 143–164.

[9] (with A. Tarski) *Arithmetical classes and types of well-ordered systems*, Bulletin of the American Mathematical Society 55 (1949), p. 65 (1192).

[10] *Some impredicative definitions in the axiomatic set theory*, Fundamenta Mathematicae XXXVII (1951), pp. 111–124. Correction: ibidem XXXVIII (1952), p. 238.

[11] *On direct product of theories*, The Journal of Symbolic Logic 17 (1952), pp. 1–31.

[12] (with K. Kuratowski) *Teoria Mnogości*, ix+311 pp., Monografie Matematyczne XXVII, Warszawa-Wrocław 1952.

[13] *Sentences undecidable in formalized arithmetic*, Studies in Logic and the Foundations of Mathematics, Amsterdam (1952), vii+117 pp.

[14] (with M. Stark) *Algebra Wyższa*, Part I, Biblioteka Matematyczna I 3, Warszawa 1953, vi+308 pp.

[15] (with A.Tarski and R.M. Robinson) *Undecidable theories*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1953, ix+98 pp.

[16] (with M. Stark) *Algebra Wyższa*, Part II, Biblioteka Matematyczna III, Warszawa 1954, vi+173 pp.

[17] (with M. Stark) *Algebra Wyższa*, Part III, Biblioteka Matematyczna IV, Warszawa 1954, vi+262 pp.

[18] (in collaboration with A. Grzegorczyk et al.) *The present state of investigations of the foundations of mathematics*, Rozprawy Matematyczne (Dissertationes Mathematicae) IX (1955), pp. 1–48 (German translation: Die Hauptreferate des 8. Polnischen Mathematikerkongreses, Berlin 1954, pp. 11–44.

[19] (with M. Stark) *Elementy Algebry Wyższej*, Biblioteka Matematyczna XVI, first edition, Warszawa 1958 (and further editions), 367 pp.

[20] (with A. Ehrenfeucht) *Models of axiomatic theories admitting automorphisms*, Fundamenta Mathematicae XLIII (1956), pp. 50–68.

[21] *On a generalization of quantifiers*, Fundamenta Mathematicae XLIV (1957), pp. 12–36.

[22] (with A. Grzegorczyk and Cz. Ryll-Nardzewski) *The classical and ω-complete arithmetic*, The Journal of Symbolic Logic 23 (1958), pp. 188–206.

[23] (with M. Stark) *Algebra Liniowa*. Biblioteka Matematyczna XIX, first edition, Warszawa 1958 (and further editions).

[24] *Formal system and analysis based on an infinitary rule of proof*, Proceedings of Warsaw Symposium on Infinitistic Methods, Polish Scientific Publishers, Warsaw, and Pergamon Press, New York-Oxford-London-Paris 1961, pp. 141–166.

[25] *Concerning the problem of axiomatizability of the field of real numbers in the weak second order logic*, Essays on the Foundations of Mathematics, Jerusalem 1961, Magnes Press, pp. 269–286.

[26] (with M. Stark) *Introduction to Higher Algebra*. Oxford-Warszawa 1963, 474 pp.

[27] *Thirty years of foundational studies*, Acta Philosophica Fennica 17 (1965), pp. 1–180.

[28] *Models of Zelmero-Fraenkel set-theory satisfying the axiom of constructibility*, Acta Philosophica Fennica 18 (1965), pp. 135–144.

[29] (with K. Kuratowski) *Set theory*, (translated from the Polish second edition) Studies in Logic and the Foundations of Mathematics, Amsterdam-Warszawa 1967, 417 pp.

[30] *Craig Interpolation Theorem in some extended systems of logic*, in: Logic, Methodology and Philosophy of Sciences III, Studies in Logic and the Foundations of Mathematics, Amsterdam 1968, pp. 87–103.

[31] *Models of Set Theory*, C.I.M.E. (Italy), 1968 (Lectures delivered in Varenna, September 1968), pp. 67–179.

[32] *Constructible sets with applications*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1969, ix+269 pp.

[33] *An exposition of forcing*, in: Algebra and Logic, edited by J. Crossley, Lecture Notes in Mathematics 450 (1975), Springer-Verlag, pp. 220–282.

[34] (with K. Kuratowski) *Set theory*, second revised edition, Studies in Logic and the Foundations of Mathematics, Vol. 86, Amsterdam-Warszawa, 1976, xiv+514 pp.

[35] (with J. Donner and A. Tarski) *Decidability of the theory of well-orderings,* Logic Colloquium '77, North-Holland Publishing Co, 1978.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Mathematical Logic in Warsaw in the 60's and 70's, or the Interaction of Logic and Life

Victor W. MAREK
Department of Computer Science, University of Kentucky, Lexington, KY 40506, USA
marek@cs.uky.edu

To the memory of Andrzej Mostowski,
Helena Rasiowa, and Cecylia Rauszer

The purpose of these notes is to describe the atmosphere of Mathematical Logic research in Warsaw in the years 1963–1975. Particular attention will be paid to the research of the group of logicians headed by Professor Andrzej Mostowski. It is, as always with personal memories, a biased report. It is not based on archive research (although both Warsaw University and the Mathematical Institute of the Academy of Sciences must have extensive archival materials devoted to the period). The beginning and the end of the reported time period coincide with my joining of logic research group (still as a student), and the death of Professor Mostowski in August 1975.

First, I need to put myself, the rapporteur, in the picture. During the years 1960–1964 I studied mathematics at Warsaw University. I attended Professor Mostowski's course in Linear Algebra (he was the head algebraist on the Faculty at that time) during 1962 Winter semester. It was a very good course full of beautiful mathematics and with an exquisite presentation. But that contact with Professor Mostowski was not how I got into logic research. My attraction to mathematical logic and foundations of mathematics came through a different route. It so happened that I met Andrzej Ehrenfeucht (now at Boulder, Colorado) socially, and soon I was attending a seminar codirected by him and Professor Zdzisław Pawlak at the Mathematical Institute of the Academy of Sciences. The topic of the seminar at the time was "Automated Theorem Proving". We were reading the seminal papers of Hao Wang, Davis and Putnam, Robinson, and others devoted to the use of computers in logic. It was all very exciting. Zdzisław was an electrical engineer (in fact computer engineer). Andrzej was Zdzisław's principal mathematical advisor (in fact, after Andrzej left Poland I fulfilled that task for a number of years). It was not clear what was the science we were doing. It was not mathematics as we knew it, it was something else. Computer Science was not yet a well-defined discipline.

The social program after the seminars involved drinking coffee (Ehrenfeucht could afford and was drinking other beverages too). The social interaction allowed for mathematics. So one day, as we were sitting in a cafe on Constitution Square – a couple of years ago it still served same coffee and cakes, but today it serves Thai food – Andrzej said: "You need to write an M.Sc. thesis, right? What about trying this problem, and by the way, here is the line of attack." The problem was (as I later learned) a minor Erdös problem in combinatorial set theory. I was just taking the first half of Professor Mostowski's two year course on set theory and its metamathematics, so I knew what it was all about. In a matter of hours the matter was finished. As was common with Ehrenfeucht, the proof worked beautifully, I had an impression I was just connecting the dots.

But the solution of the problem of M.Sc. thesis opened another problem. Namely, what was I supposed to do with the rest of my life. It was quite clear that it was to be research in logic. But where and with whom?

At the time, Andrzej had two young collaborators. A year older than me was Grzegorz Rozenberg (now, and for many years a distinguished computer scientist at the Leiden University, Holland) and myself. Grzegorz became Andrzej's assistant at the Mathematical Institute of the Academy. But I think that the combinatorial set theory problem was no accident, for as the result I got the second best (I thought at the time) offer: a position of an assistant to Professor Andrzej Mostowski at the Department of Algebra (yes!) of the Faculty of Mathematics and Physics.

The administrative structure of the University was different at the time; the Mathematical Institute of the University was not yet in existence. The probational period was, roughly, one year. In addition to teaching I took courses in metamathematics of set theory (with Mostowski) and functional analysis (with Professor Stanisław Mazur). Then there was an obligatory visit with one of senior researchers (she happened to be Professor Mazur's wife) and this dialogue: "When are you going to start seriously studying functional analysis?" "I suspect never, Ma'am Professor", I was firmly in orbit around Professor Mostowski.

There were at least four groups of mathematical logicians in Warsaw at the time; three were associated with the University, one with the Academy of Sciences. At the University, the Department of Algebra had a strong contingent of logicians. Professor Mostowski was the head; he was usually teaching Linear Algebra. Then, there was the Department of Logic, headed by Professor Rasiowa. This was a place where an algebraic approach to logic was followed. Professor Rasiowa also taught algebra often. I recall sitting beside Cecylia (Ina) Rauszer at Rasiowa's freshman algebra course. Logic was not taught at all at the undergraduate level. This in Warsaw! The third group was the foundations of geometry group, headed by Professor Szmielew.

Not surprisingly, the prevailing interests of all three groups at the University were directly grounded in the work of Alfred Tarski (since WW II at the University of California, Berkeley), who pioneered the techniques used by those groups and (as I learned soon) was an ultimate arbiter of Warsaw logic (by mail, of course). Surprisingly, Tarski was referred to as "Fredzio" (I did not invent this term), on the account of his nickname before WW II. At times he was also referred to as "Master Alfred". For those who want to learn more, a biography of Tarski by Anita and Solomon Feferman presents a sanitized version of Tarski's life.

As usual, there was a dynamic relationship between the three groups at the University, a natural competition for resources, students and special topics lectures. The whole picture was dominated by Professor Mostowski. He was Professor Rasiowa's Ph.D. advisor (but not Professor Szmielew's), and for a variety of reasons was the best known of the three.

The group at the Mathematical Institute, headed formally by Professor Mostowski, with Professors Ehrenfeucht and Pawlak, and with some involvement of Professor Jerzy Łoś (who was migrating into the area of mathematical economics) was more concerned at the time with the issues of computation. True, access to computers was restricted, and what was available was primitive, but they had a vision. This was a vision of mathematical logic, and more generally of foundations of mathematics applied to computation. This vision was later inherited by Professor Rasiowa. Anyway, I joined Professor Mostowski's group, and Ina Rauszer joined Professor Rasiowa's.

There was a shortage of teaching assistants at the University, and my class finished a year early. We took all sorts of additional classes, and I recall passing an examination in a special topics course on algebraic logic with Professor Rasiowa. The examination was at Professor Rasiowa's apartment on Wiejska street. It was known that knowledge of the proof of the Rasiowa–Sikorski lemma was necessary to pass the examination. But not of any proof (there are several), but of the original proof. The Rasiowa–Sikorski lemma asserts the existence of certain filters in Boolean Algebras. These filters are closed under a denumerable family of joins. This is a beautiful example of interaction between different areas of mathematics. On the one hand it is the Baire Category Theorem in disguise; on the other it is an abstract version of Henkin's proof of the completeness of predicate calculus. A real mathematical gem. Fortunately I knew the lemma, the proof[1] and a couple of other things. Tea and cakes were served by the domestic servant *after* I successfully proved the completeness of the first order logic.

It looked like algebraic methods were a good topic to follow, but my heart was in another branch of foundations at the time. Set theory was all the rage. In 1963 Paul J. Cohen of Stanford University obtained a result that electrified the logic community. He proved that Cantor's Continuum Hypothesis, a problem open for approximately 70 years, was actually independent of Zermelo–Fränkel set theory. The story of the Continuum Hypothesis was full of great names (Cantor, Hilbert, von Neumann, Sierpiński, Kuratowski, Gödel), and it was a major problem of the foundations of mathematics. It really made a difference to basic mathematical specialties such as real analysis. After all, it is a fundamental question: How many real numbers are there? Formally, the Continuum Hypothesis says that there are precisely $\aleph_1$ real numbers. Here $\aleph_1$ is the size of the set of all denumerable von Neumann ordinals. This is a beautiful statement. Yes, there are nondenumerably many reals (we knew that since Cantor). But the continuum (reals) is really not that huge. The Continuum Hypothesis has all sort of interpretations. In a naive, kind of *Occam's razor* view, the *highest authorities* created nondenumerably many reals, but as few as they could. Before WW II, Lindenbaum and Tarski studied cardinal arithmetic in detail, establishing many equivalents of the Continuum Hypothesis (and its generalizations). It was a real Warsaw topic.

Here is how I learned about the work of Cohen. Andrzej Ehrenfeucht did not open his mail. I think this phenomenon persists till today. Now, there was an ancient book case in the Room 1 of the Mathematical Institute of the Academy. Andrzej was throwing into this book case the content of his mailbox. Even though I worked at the University, I got a privilege of opening that mail, specifically, envelopes with scientific papers. Two

[1] The alternative proof, due to Feferman and Tarski was not acceptable.

envelopes – I recall – contained reprints of Cohen's papers on Continuum Hypothesis, and a preprint on independence of the axiom of choice.

In a fundamental break with the mathematical practice, Kurt Gödel proved in 1936 that the Continuum Hypothesis is consistent. That is, adding it to the commonly accepted axioms of set theory would not lead to inconsistency (assuming set theory is consistent). The point is that he did not prove the Continuum Hypothesis itself; what he proved was that using the Continuum Hypothesis will not lead to inconsistency if the remaining axioms of set theory do not entail inconsistency. Likewise, Gödel proved that the axiom of choice (another "suspicious" but commonly used property of sets) was consistent. The reason why this approach was so unorthodox was that by showing the Continuum Hypothesis consistent Gödel implicitly suggested that maybe the Continuum Hypothesis is not provable. Now, the same Gödel demonstrated in 1931 that there are true arithmetic statements φ such that both φ and $\neg\varphi$ are not provable in (say) set theory. This statement is an arithmetized version of "Liar Paradox", and so mathematicians often claim that *it is philosophy, not mathematics*. Even stronger, Gödel announced (the proof was finally presented by Bernays and Hilbert a couple of years later) that every axiomatizable theory containing arithmetic is incomplete and that the arithmetized version of Liar Paradox (in the form "I am unprovable") is undecidable for such theories. But Continuum Hypothesis is a concrete statement, of obvious mathematical meaning. We can explain it easily to every mathematician. To claim that such statement may be undecidable was really revolutionally at the time.

Even today, most mathematicians (and it is, sort of, based on their experience) believe that the incompleteness phenomenon does not touch their practice. All these undecidable statements (they think) are really kind of strange, formalizing philosophical paradoxes appearing on the fringes of mathematics. The prevailing point of view supports more similar fallacies, for instance a common belief that simply formulated statements must definitely be solvable in an elementary manner. Of course we now know (for instance due to the results of Martin Davis, Yuri Matiyasevich and Julia Robinson, and also of Harvey Friedman) that this is patently false, yet the idea lives on.

In 1936, after Mostowski completed his studies of Mathematics in Warsaw University (and before he started to work at the Meteorological Institute in Warsaw), he received a gift from an affluent uncle – a fully paid one-year trip to Vienna and Zürich. Mostowski visited right at the time when Gödel presented his ideas. The result was that Mostowski was very familiar with Gödel's work and later contributed to several of the topics Gödel pioneered. In 1940, after several trips back and forth from Austria (which in the meanwhile was annexed by Germany) Gödel moved to the Institute of Advanced Studies in Princeton, NJ. Mostowski visited him there in 1949. Gödel's work introduced a wealth of ideas that influence set theory research to this day.

Professor Mostowski kept a button in his desk. This button, he told me, was found on the porch of Gödel's house in Princeton, NJ. Few years ago I happened to visit Princeton, and went to see that house (it was not even clear if it was the same house, but my research established that Gödel definitely lived at that address before his death). I was wondering if I could maybe find another button. There was none. I asked the owners to be let in. The spirit of Gödel was absent; I felt no vibes. Later on that evening my wife and I decided to celebrate the event and went to buy a bottle of wine. The person selling me wine was the same man who let me in to the former Gödel home.

In Vienna, with encouragement of Gödel, Mostowski prepared his Ph.D. dissertation on permutation models of set theory. The paper with these results was published in German in Fundamenta Mathematicae, right before the Germans invaded Poland in 1939 and closed all secondary and higher education schools (this, of course included forbidding mathematical research) as unnecessary for subhumans. Limited vocational education was allowed.

Coming back to the Continuum Hypothesis. A similar kind of breakthrough happened with Cohen's proof. A new technique called *forcing* was introduced. As common in such situations, forcing at the beginning appeared mysterious. The technique worked, we just did not know why. Many groups of logicians rushed into the area (and many people migrated into foundations in the process). Numerous proposals for explanation and an improved presentation were made. Most exotic tools (for instance, sheafs and other tools from algebraic geometry) were proposed first. Poland was not absent from the quest for explanation – there was a proposal by Professor Ryll-Nardzewski (he presented it in Warsaw), and then Professor Mostowski wrote a book with his presentation of Cohen's work. But then the most unexpected thing happened: Professors Dana Scott and Robert Solovay were able to cast Cohen's argument in terms of algebraic semantics for logic; the semantics proposed by Rasiowa and Sikorski! This presentation (which has since become standard) finally fully explained what was going on in Cohen's proof. Dr. Einar Fredriksson (one of many visitors to Mostowski's group) presented the Scott–Solovay paper at the Warsaw seminar. Professor Rasiowa, for whatever reasons, was absent at those talks.

Today (with enough of preparation, but not that much) we can explain Cohen's argument to well-qualified sophomores. This is a common fate of great discoveries; when the dust settles, a new perspective is gained, and many people would say: "Really, I could do this." In hindsight, the Baire Category theorem and similar arguments on closure under denumerable unions were in Cohen's and other forcing arguments from the start. But Scott and Solovay, either by an act of intuition or by incredible insight, got it just right! The gist of Cohen's proof, besides of the correct construction of the algebraic model, is based on a combinatorial lemma, actually due to Monika Karłowicz and Ryszard Engelking (it so happened that the combinatorial lemma mentioned above is a special case). So many connections to Warsaw!

The discoveries of Cohen and then other major discoveries in foundations (the issue of measurable cardinals in set theory, the work of Scott, and then Solovay, Gaifman, Silver and many others on indiscernibles in constructible universe), dominated the Warsaw logic scene until 1970. Every couple of weeks we were getting new fascinating papers. We felt that the world was spinning around us and major discoveries were being made.

But the reality of Poland of late 60s was more complex. The local communist dictator, Władysław Gomułka, a fellow whose education was limited to 6 years of basic education (in the Austro–Hungarian empire) plus one-year terrorists' school in Moscow in the 30's, was facing increased opposition in the country. The opposition came both from nationalistic and democratic circles. The economy was also going very badly. A scapegoat was needed. In 1967, with the six-day war between Israel and its Arab neighbors, Gomułka found scapegoats. Those were (who else!) Zionists. The hot summer of 1967 was when the hunting of Zionists started.

During the spring of 1967, at the invitation of Professor Rabin, Professor Mostowski spend a couple of months in Jerusalem. As we will see, this had a small effect on my life. Professor Mostowski was working on a variety of topics. One was a statement called the

Fraïsse Conjecture. There are many such conjectures, Professor Roland Fraïsse of Marseilles (who found many fundamental results and techniques of Model Theory) routinely generated conjectures. This one was on the second-order theories of denumerable ordinals. Mostowski has shown that this conjecture is unprovable in set theory. Five years later I proved that it is consistent with set theory (actually a consequence of Gödel's axiom of constructibility), too.

The fall of 1967 was hot, but this time it was politically hot. There were all sorts of petitions to sign about students who were considered by the authorities to be politically unreliable and hence suspended from the university. The witch hunt started. The tone of newspapers moved from Orwell's newspeak to Streicher's "Der Stürmer". But before it all started, Professor Mostowski decided to take his two assistants (Dr. Janusz Onyszkiewicz, today a member of the European Parliament) and myself for trips. Dr. Onyszkiewicz went with Mostowski to Montreal, I went to Amsterdam for the Congress of Logic, Methodology and Philosophy of Science. Here is how it was decided. As a result of his stay in Jerusalem, Mostowski had a small aluminum Israeli coin. Onyszkiewicz was to guess. Heads was Montreal; tails was Amsterdam.

The Congress was held in Grand Hotel Krasnapolsky. All I could afford was the YMCA. The "Y" was located in the "Red-light district", full of houses of disrepute. But the linen at the "Y" were clean, and, more importantly, a filling breakfast was included in the price of the hostel.

As I went to see Professor Mostowski, he fished out from his pocket 400 Dutch guilders, an equivalent of $100.00 (a lot of money at the time) and told me: "You need to eat too. And these trousers are horrible, buy some. You have to be presentable. And buy a small present for your wife – you travel, she suffers". So, to save on food, Professor Grzegorczyk (who also attended the congress) and I ate Indonesian fast food and enjoyed watching the great logicians who attended the meeting. Among those was Professor Alfred Tarski, and this turned out to be the goal of my "presentability".

One day I spotted Mostowski and Tarski in animated discussion. I was sure that they were discussing the current great progress in logic, but when I approached it turned out that instead they were hotly discussing politics. I thought that such great people with all their tremendous insights were wasting their time. I got introduced to Tarski. Of course, with my provincial upbringing I was not that eloquent. But it was good enough, and right there, as an outcome to that conversation, it was decided that in the fall I would start in the Ph.D. program at Berkeley.

In Amsterdam, Mostowski talked about the Craig Lemma in some extensions of first-order logic. Mostowski gave a sufficient condition for a logic *not* to have the interpolation property. As was common in Warsaw, the argument had a strong real-function component. Today interpolation property is important in many computer science applications of logic. Thus, from this perspective, it is important to have abstract conditions that imply its presence or its absence.

It looked like 1967 had Berkeley in store for me, but fate and politics intervened. There is no control sample, so we will never know if it was good or bad for me. In the fall my application for leave had been turned down. Instead I received a one-year paid leave so I could work at the Mathematical Institute of the Academy. I believe that Professor Mostowski wanted to save me from the political troubles that were coming. And it worked. The winter was getting even hotter than the summer. I recall Professor Mostowski telling me that I needed to finish my dissertation promptly. Here is what he told me, in his typical fashion: "I gave to our colleague B. (an algebraist, later emigrated to Sweden) a fountain pen and told him to write a dissertation. Do you need a fountain pen?" I took the cue and left for a couple of weeks to a mountain hostel and came back with a dissertation. In the morning I worked with tree-loggers; writing was in the afternoons. After I came back it was just typing and correcting. With almost no teaching duties I certainly could focus on the dissertation.

The year 1968 brought student revolts all over the place: in Berkeley, New York (Columbia), Paris, Amsterdam, and West Berlin. We had student revolts in Eastern Europe, too. Good things were happening in Prague. In Warsaw we had all sort of events. But there were ominous signs. Here was one: University or no university, I taught a special topic course on set theory (planned well in advance, and the plan was the most important thing at the time). I was lecturing in the evenings at the Palace of Culture (where the Mathematics faculty was located). As we were leaving class (maybe 6 of us) early one evening in March, walking the square before the Palace, I noticed three police thugs following us. It was not me, of course, but some of my students that had been followed for some reason.

On March 8, the Warsaw University students held a political meeting. Police, disguised as workers, came and beat students up. This event, called commonly "March" in Polish, had various consequences. The third grade of mathematics was disbanded, and some of our students, including Marian Srebrny, later my Ph.D. student (Professor Srebrny is with the Computer Science Institute of the Academy), were drafted (membership in the armed forces was considered a severe punishment). The colleagues with Jewish names were let go. For a moment the names were a criterion. On Sunday March 10, I went to see Professor Mostowski in his home. He was scared, and very worried. He offered to help me find a position if I had to leave Poland. By pure coincidence two student leaders unexpectedly came while I was present at Mostowski's apartment. He talked to them in my presence and was pessimistic about their goals.

I thought about this event later, many times, and very hard. Professor Mostowski was a true moral authority for us, his students. But then I figured it out, based on many stories about the Nazi occupation he told us, all about humiliation and about the underground university (He taught risking his life. The punishment, at the most lenient, was concentration camp). But one story stayed with me especially. It was the story about choice: In August 1944, during the Warsaw uprising against the Nazis, the Germans forced civilians out of Warsaw (and then destroyed the city completely). When they came to Mostowski's apartment, all he could take with him was one kilogram of goods (certainly they had *rules*). The choice was this: the black notebook with all the results he proved during five years of German occupation, or a loaf of bread. Mostowski chose the bread. The notebook burned with the apartment. The punch line of this story was that in the transit camp at Pruszków, Professor Mostowski met Professor Władysław Tatarkiewicz, the philosopher. Tatarkiewicz was also carrying just one item: a manuscript of the book "On happiness". I heard this story several times, usually in the context of results proven by other people, but which were first in the "black notebook". Then I realized: if the choice was: bread for his family or the notebook; family came before mathematics. And there were not many things that could compete.

It was so typical of Professor Mostowski: it was mathematics and family that made sense, but not much more. I recall that once, after my colleagues and I left a cigarette in the logic assistants' room after a long evening of card playing, resulting in the visit by the fire brigade (and an official investigation), Professor Mostowski called the entire department personnel to his office. He had us standing, and, pacing in front of us, told us the following: "What an utmost stupidity! Playing cards instead of proving theorems! This is horrible. And this alcohol (some of us had this problem). This is totally unacceptable! And this spelunking, Don't you know how many mathematicians perished in the mountains? (Here he looked at Janusz Onyszkiewicz, a well-known climber and speleologist). And smoking, pipe included (here he looked at me)! And even worse, some of you have no family. And these boats, even worse, ice boating! (here he looked at Paweł Zbierski). There will be severe consequences." And indeed there were. One of our colleagues, Y, was let go. The fellow was the most conventional of all of us; he did not smoke, he had family, we never saw him playing cards. After some time I discreetly asked Professor Mostowski why he let Y go. "He did not prove theorems" was the answer.

Metamathematics of set theory was, doubtless, Professor Mostowski's greatest love. He returned to it throughout his career. Both his dissertations (Ph.D. and Habilitationsschrift) were devoted to it; he studied the issues related to consistency and independence (in effect the same thing) of various set-theoretic sentences throughout his life. It was no wonder that Cohen's proof was such a jolt for us. This was truly new mathematics, and we, his collaborators, took to it like fish to water. We wanted to study it in greater detail than the official seminar allowed us to do. Onyszkiewicz, Jaegermann, Wiśniewski and I decided to have an unofficial, side seminar devoted to forcing alone. Much to our surprise Mostowski came to the organizational meeting and requested that we assign him a paper to present. The implied statement was that he gladly joined our initiative on an equal basis and did not want to "steal it"!

We carried on. Professor Rasiowa, a person with infinite patience and a lot of common sense, rescued our colleagues who were thrown out of the University. She was the Dean of Mathematics and Physics (later of Mathematics and Mechanics). Both she and Mostowski believed that only by acting behind the scenes could they save mathematics from politics. Much later, during one of her visits to Lexington, Professor Rasiowa told us (there was a group of us, Warsaw mathematicians in Lexington) the story of her studies of mathematics during the German occupation and her hair-raising description of being buried under the ruins of a church bombed by Germans. She was pulled from the rubble and survived. My generation did not have this experience, but I felt I understood something about her and her generation that evening.

During the summer, Professor Mostowski would spend a month or so in Zawoja, a resort in the mountains. This could be July or August. He was sending us postcards with instructions, or with some new mathematics. One of those years, in July, a state prize was awarded. He got a prize for his contributions to logic. A fellow from the official government newspaper came to the department in need of information about Mostowski's work. I had to provide an explanation. "Could you provide something that a worker could understand?" the newsman said. "Maybe the correct analogy is a kind of occupational safety. You see," I said, "mathematicians are always worried about contradiction. A kind of occupational safety problem. This is what logic is concerned with." And of course, the next day (fortunately deeply within the newspaper, but still in quite large letters) there was a piece entitled "Occupational safety in Mathematics". The gist of the story was that not only workers were concerned with occupational safety, gloves, eye-goggles etc. – mathematicians were concerned too. The most concerned was (who else) Professor Mostowski, and hence the state prize. I was not quoted as a source, but, of course

Mostowski knew. This really scared me. I thought: he can forgive me all sorts of behavior, long hair, pink shirts, etc. – but this? Fortunately, Professor Mostowski had a sense of humor about it.

The excitement with forcing abated; we were moving on. Slowly, but unmistakably, Mostowski's seminars were moving in a different direction; that of formalized second-order theories.

Let me first discuss second-order arithmetic. Second-order arithmetic is, in fact, a theory of natural numbers and their subsets. It can be formalized in first-order logic. When we do so, it is sometimes called analysis (Hilbert used this term). There are fundamental reasons why it is worthwhile to study second-order arithmetic. First, via definable mappings with reals, it gives us deep insights into the properties of real numbers. But there are deeper and even more profound reasons. When we limit second-order arithmetic to some parts (too technical to be discussed here), it turns out that we capture some of the most fundamental theorems of mathematics (for instance the König theorem on finitely splitting trees or the Baire Category Theorem). Amazing results of Harvey Friedman, Jeffrey Remmel, Stephen Simpson and their students have shown that, in fact, the number of these important fragments appears to be limited. This is the area of Reverse Mathematics. Anyway, studies of models of second-order arithmetic were, and still are, important.

When we limit ourselves to models of that theory that have "true" natural numbers, we get ω-models. With further restrictions, namely that the notion of well-ordering is preserved (a concept introduced by Mostowski) we get β-models. Mostowski's group studied these models intensely for a number of years. It is also quite easy to see that second-order arithmetic is closely related to set theory without the power-set axiom. We were exploiting these connections, especially connections to so-called hereditarily-denumerable sets. Deep connections with Gödel constructible sets were found. For a number of years, we studied a strange phenomenon occurring in constructible sets called "gaps". The topic was pioneered by great American analytic philosophers George Boolos and Hilary Putnam. Marian Srebrny wrote a dissertation on this topic.

The seminar grew, some people were leaving (Jaegermann, Wiśniewski, A.W. Mostowski), but many new people were coming (in no special order: Paweł Zbierski, Zygmunt Ratajczyk, Krzysztof Apt, Zofia Smólska, Wojciech Guzicki, Małgorzata Dubiel, Marian Srebrny, Adam Krawczyk, Roman Kossak, Zygmunt Vetulani, Roman Murawski, Andrzej Pelc, Stanisław Krajewski, Michał Krynicki, Konrad Bieliński, Henryk Kotlarski, Andrzej Zarach and others). A lot of good work was done. A constant stream of shorter- or longer-term visitors (Einar Fredriksson, Robert Kowalski, Judith Ng, B.F. (Pete) Wells, Max Dickmann, Johann (Janos) Makowsky, Donato Giorgetta, C.C. Chang, Hendrik (Henk) Barendregt, Dirk van Dalen, Petr Hajek, Petr Vopenka, Yoshindo Suzuki, George Wilmers, Peter Hinman, Moshe Machover, Kenneth McAloon and many, many others) came to talk to or spend some time with Mostowski. If we could not travel because receiving a coveted passport was a big deal, many people visited and brought the news with them. We were so used to the constant stream of visitors that we could not imagine that it could be different. Of course it could have been, and whenever I think about it I am forever grateful to all those who came to visit and brought with them knowledge and excitement. I can only hope we were able to repay in kind.

The seminar rules were quite interesting. Regardless who was the speaker (this included Professor Mostowski) the principal goal of the audience was to prove that: (a) the speaker did not know what s/he was speaking about, (b) even if s/he knew, the results were trivial, and (c) even if the results were not trivial, there were much better proofs. The net effect of this was that these meetings, usually held on Wednesdays, 5 p.m., in the Tarski tradition, were the most fascinating experience I (and I believe most of my colleagues, especially if they were not on the receiving end) had. I still keep a couple of notebooks (both mine and few of Mostowski's – Mrs. Mostowska gave me those after Professor's death). It reminds me of the happiest moments of my life. The visitors usually sat beside Mostowski. In their honor the language was often English, at least with the English written on the blackboard. If not, Mostowski himself translated the controversy, which was the norm rather than exception. Of course this mode of seminar was not accidental. I realized that later, when right before his death I helped Mostowski with the seminar. The idea, and it was even spelled out quite openly, was that when people participate, the learning process is much stronger. I recall that once, with Mostowski presenting (thus on the receiving end), I was especially nagging about part of the argument. Mostowski grew furious. "You need to read Spinoza's 'Ethics'," he finally erupted. But by the unwritten rules, this did not affect our relationship. Sometimes the rules were suspended. This happened if the Chair of Logic in Military-Political Academy (yes, there was such academy, and such chair), general Adam Uźiębło visited. He would come in general's uniform (Chairs in military academies were generals by default) and with two bodyguards. Uźiębło would sit in the first row, with Mostowski. They would stand at the back of the room. We were strangely quiet on such occasions.

Especially if Professors Bogdan (Kali) Węglorz or Jan Mycielski from Wrocław (now in Boulder) visited, and if the seminar was held in the Mathematical Institute, we would strengthen our spirits after the seminar in a restaurant called "The Rafter," where the only food served was beef tripe soup, hot pepper powder, and several kinds of beer. Surprisingly, the place still serves the tripe soup, but the menu is much larger. Needless to say, Professor Mostowski did not attend nor approve.

Ehrenfeucht and Onyszkiewicz played a game. The bet was that a specific word would have to be used in the presentation. Offensive words were not the subject. When the word was "glass" or even "shoe", it was not that hard. But once Ehrenfeucht[2] was supposed to use the word "crocodile". The seminar was 90 minutes, with 15 minutes break in the middle. Sometime around the 80th minute, those of us who knew the bet got really excited. But then, suddenly, Ehrenfeucht wrote one of those amazing facts he was so good at, on the blackboard. "How do you know that?" Mostowski asked, 'Oh, J. Crocodile, Pacific Journal of Mathematics, I forgot the exact title" was the answer.

So, in 1968 I got my Ph.D. and now worked as a beginner assistant professor. In 1969, Professor Mostowski went to Oxford, to All Souls College, for a semester. I managed the ship, seminar and all. There was a generational gap in Mostowski's group. Andrzej Ehrenfeucht, a man with supernatural powers when it came to logic, had left a couple of years earlier for Los Angeles and then Boulder, CO. His supernatural power was that he essentially was able to see the contents of the envelopes with papers sent to him. He also somehow guessed the proofs. These were proofs in the same papers I mentioned above. I do not know if he still has those powers. In preparation for this paper, I asked Andrzej how he did this. "Very simple" he answered, "I would know what they were doing, maybe from Mostowski who got those papers earlier, and once I knew the results, I could provide the proofs." This included (I believe) Morley's work on Łoś conjecture.

[2]But other people claim it was Ryll.

Professor Grzegorczyk was working at the Mathematical Institute of the Academy. Professor Rasiowa, with her growing interest in the applications of logic in computer science, was not going in the same direction as Mostowski. Janusz Onyszkiewicz, a couple of years older than me, was already in politics. For whatever reason, I had to step in. But before this happened, during his stay in Oxford, Professor Mostowski got in touch with Professor Dirk van Dalen of Utrecht in Holland. I went there as a postdoc in the fall 1970. The appointment brought me together again with Grzegorz Rozenberg; we were sharing the office. The logic group in Utrecht consisted of van Dalen, Barendregt (a star even though still a Ph.D. student, and already writing his λ-calculus book) and a couple of younger students. We were reading Azriel Levy's book on the hierarchy of formulas of set theory. I was also looking for a topic for Habilitationsschrift.

But things in Poland got complicated again. Gomułka's regime was in its death throes, and one morning the landlady from whom I was renting a room showed me the local newspaper with a photo of looters in Gdańsk. A worker's revolt against the rise in prices of basic food was in progress. The blood of Gdańsk workers eventually brought a coup d'etat. Gomułka was deposed, and a former miner named Gierek (at least he was fluent in French . . .) became the new dictator. But there was a sense of optimism in the air. At least the propaganda smelling of Nazi Germany disappeared overnight, we hoped for good.

I finished my Habilitationsschrift in Utrecht (and defended it two years later in Warsaw). The topic was related to Mostowski's ideas, this time not on second-order arithmetic, but on the Kelley–Morse theory of classes. It is a close relative of second-order arithmetic, except that here we talk about sets and classes. A class is a collection of sets, but unlike sets does not have to be an element of other objects. When a full comprehension schema for classes is adopted, we get the Kelley–Morse theory of classes. It was not obvious that such a theory is consistent with the axiom of constructibility, and even what the forms of constructibility for Kelley Morse should be (there are two). Without going into much detail, I succeeded in proving the consistency of Kelley–Morse theory with the axiom of constructibility. In fact the method used in this argument, now commonly called bi-simulation (and pioneered by our group in the context of second-order arithmetic), is often used in computer science. The analogies with second-order arithmetic (as predicted by Mostowski) were very deep.

There was a marked difference in the letters of Professor Mostowski before and after I sent him the first draft of the dissertation (all the lemmas, no proofs). Those latter letters were full of relief. It was clear that at least one problem was solved. Later I learned that right after my letter Mostowski found a completely different argument (based on techniques of "ramified analysis" pioneered by Hilary Putnam and Robin Gandy) for the same results. Professor Solovay told me later that he (and also Leslie Tharp) got this result too.

I was leaving for Utrecht without a clear idea what I was going to do, and it was a happy coincidence that I found a problem that was worth doing. Professor Mostowski wrote me many letters (the originals are in the Polish Academy of Sciences Archives) when he or I traveled or from vacations (Professor Mostowski liked to hike, not the Tatra mountains like Tarski, but the lower Beskidy mountains). In the summer of 1971, one of his postcards introduced me to the Logic Semester at the newly opened Banach Center. Today, special semesters, special years etc. are common. It was not so in early 70's. There were summer schools and special sessions, but not long periods of sustained effort, at least not in Eastern Europe. The leaders of the Mathematical Institute of the Academy opened the Banach Center, and the first semester was devoted to logic (an earlier, informal semester was Professor Łoś' semester on mathematical economics).

Meanwhile very good things were happening in the neighboring department, too. Professor Rasiowa developed her group in several directions. The principal techniques were algebraic, developed together with Sikorski and presented in their monograph "Mathematics of Metamathematics", in several, constantly improving editions. Investigations of many-valued logics, and especially algebraic methods, were carried by Rasiowa and her many collaborators, including Ina Rauszer. But one topic of the late 60's was special: algorithmic logic, a multi-modal logic explaining various phenomena of programming languages. This work was renamed later to dynamic logic (with the obligatory renaming of main contributors, too), but the fundamental ideas and, more importantly, proofs came from Warsaw. Algorithmic logic was proposed by Professors Andrzej Salwicki, Grażyna Mirkowska-Salwicka and Antoni Kreczmar and their collaborators. In fact, it was a herald of things to come.

Professors Rasiowa and Sikorski studied algebraic semantics of various logics, not only classical logics. One instance was classical modal logic with one or two modalities, called $S4$. This and many other modal logics (later on they mystically reappeared in the context of formalized common-sense reasonings) were introduced by analytical philosophers of the first quarter of the 20th century. Topological/algebraic methods pioneered by Tarski and then Rasiowa and Sikorski applied here as well. But the new application, given by Salwicki and collaborators, was to think about computers in similar terms. Come to think about it, it is very natural. Let us think about a computer as a device which has *states*. When we execute instructions, state changes. We can associate with instructions modal operators and investigate what modal formulas hold after we execute an instruction. In the age where computers facilitate "fly-by-wire", control atomic power stations and lung ventilators, we have a vested interest to see what will happen as a result of instructions. Many proposals have been made for the logic of programs, and the work of Robert Floyd, Anthony (Tony) Hoare, Edsger Dijkstra and others resulted in techniques for proving properties of hardware and software. Of course there is no magic bullet. After all the halting problem is undecidable, so we can be only so sure. But we have no choice, we have to eliminate faults and bugs as much as possible, and Warsaw logicians were in the forefront of these studies.

The Logic Semester was a huge success. We had many visitors, both short term and long term, with Professors Alistair Lachlan, Kenneth Bowen and Adrian Mathias for long visits, and many other important visitors, including Professors Gerald Sacks, Melvin Fitting, Petr Hajek and many others. I recall walking with Adrian Mathias on the main throughway of Warsaw in search of cherry color shoe shine (Mathias could not live without such shoe shine). There was none in entire Warsaw. Mathias in his full Cambridge uniform, cape and all, was an instant sensation.

The day-to-day management of the semester was in Professor Grzegorczyk's hands. Phenomenal breathtaking presentations. It looked like we made it. But nothing is free. For a variety of reasons, as before, politics intervened. There must have been a confluence of several problems (certainly the presence of a large body of visitors from Western Europe, Eastern Europe, and US must have been a factor), and we suddenly found ourselves in the cross hairs of the police. The gossip had it that one of visiting members of Soviet Academy of Sciences personally wrote a denunciation and that someone from the leadership of the Mathematical Institute of the Academy saw it. There were small signs, for instance we were not be allowed to hold our seminars at Banach Center (the custodian told us that she was forbidden to let us in). Clearly something was going on, in a totalitarian system one should be careful when there are such signs.

In January 1973 I again, and this time against the express will of Professor Mostowski, left the University for a three-year stint at the Institute. (Great opportunity, just do research, salary deposited at your bank account. I actually opened such an account for the first time in my life.) Mostowski believed that the University offered a better protection against the troubles of the day.

On the surface, nothing has changed. We had a seminar as usual (Wednesday, 5pm, like Tarski in his days). But then one day, as I was talking with Professor Mostowski on some issues related to Kelley Morse theory of classes (in his office in room 908 on the 9th floor of the Palace of Culture) we had a visit of a colleague, Professor X. A person with the same first and last name has been recently revealed as an unofficial but paid collaborator of the secret police. It transpired that the faculty had been informed about some kind of political conspiracy. Allegedly some of our younger colleagues (this included Konrad Bieliński, later hero of "underground Solidarity", Staszek Krajewski and Henryk Kotlarski) were meeting with some others and agitating against bringing the students' union into the fold of communist youth. The alleged conspiracy consisted of the assistants and senior personnel of the department of foundations. It was formed of Stalinists (I am not kidding!), Ukrainian nationalists, Zionists and all sort of deviationists. The head of this alleged conspiracy was Professor Mostowski. This was dubbed by us "conspiracy of logicians", and it would have been more than funny, but it had practical and immediate consequences. Besides of unpleasant consequences for myself (you can guess who I was in the roster of culprits, and who were the others) there was one consequence which may or may not have been triggered by the event. In a couple of months we were visiting Professor Mostowski in a sanitarium where he was fighting heart ailments and hypertension. I guess the shock of the not so subtle interrogation made Professor Mostowski ill. It took two more years. In May of 1975, as we were sitting at the Academy of Sciences cafeteria on the 24th floor of the Palace of Culture, Professor Mostowski told me quietly: "I stopped taking those pills". I saw the ramifications immediately, for it was quite clear that it put him in danger of a stroke. 'Why, Professor, sir?" I asked. "I cannot think when I take them", he said. We were just writing our first paper together. Not that it was the first paper on which we collaborated; just that previously Mostowski refused to put his name on the papers to which he contributed. I was very proud, Professor Mostowski had few coauthors (but Professors Łoś, Rasiowa, Ryll-Nardzewski, Grzegorczyk, and of course Tarski made the list).

Professor Mostowski spent the summer of 1975 in Berkeley and then visiting Małgorzata Dubiel-Lachlan and Alistair Lachlan in Vancouver. It was there, en route to the Congress in London, Ont., that he had a stroke and died. He was about to step down as the president of the Union of Logic, Methodology and Philosophy of Sciences. It was again hot in August 1975 in Warsaw. No air-conditioning. I was sitting with Paweł Zbierski in my apartment talking mathematics when the phone-call from Wojtek Guzicki came. He just heard from Małgorzata Dubiel about Mostowski's death. Our world ended. There is no grave where we could come and pay homage. His ashes were spread in the wind.

In the middle ages, philosophers and theologians studied a branch of science (since then forgotten) called teratology. It is a science of monsters, of a kind you see in your dreams if you eat too much pizza in the evening. It is not that surprising that the last paper of Mostowski, a witness to horrible excesses of 20th century, was called "Contribution to teratology". Actually, the paper, his first in a Russian journal, was about truly strange models of second-order arithmetic. We now know that the "standard part" of a nonstandard denumerable model of second-order arithmetic is characterized as a Scott set (this is a technical term, and the result is due to Friedman). There are Scott sets that are not models of second-order arithmetic. So the restriction to natural numbers does not necessarily generate a model of second-order arithmetic. But Mostowski explicitly constructed a nonstandard model that was elementarily equivalent to the full model of second-order arithmetic, and yet its standard part was not a model. That did not follow from Friedman's work. A real monster of a model!

His interest in monsters was not accidental. Actually, Mostowski told me a number of times about the German artist, Sascha Schneider (a modernist and a professor of Beaux Arts Academy in Münich in the late 19th century and the beginning of the 20th century). One of Schneider's drawings was memorable, Mostowski said: A horrible creature (a bird?) is in the process of preparing for biting of a male figure (Prometheus?) liver. I went to the library of the Warsaw Academy of Arts to investigate and see it for myself. There was a portfolio of Schneider's drawings. The picture in question, "Das Gefühl der Abhängigkeit" (meaning: "The feeling of dependence") was missing.[3]

Next year, in Oxford, I was attending Logic Colloquium. Many friends and students of Mostowski were present, among them Professor Rasiowa. The morning of my talk, Zbyszek Raś (today at Charlotte, NC) came to my room and told me: "Mother told me to lend you a tie". As usual, she was right – I had none.

Meanwhile, the grand vision of logic in Warsaw was spreading in a variety of directions. Professor Rasiowa realized that the foundational problems had shifted; while the foundations of mathematics were important, even more important were the foundations of computer science. All that mathematical logic did for mathematics in a hundred of years had to be done for computer science in a much faster pace. A similar realization pulled me in another area of application of logic – the theory of databases. Professor Pawlak was pushing me in that direction, and even before Professor Mostowski's death I was spending a significant portion of my effort on the issues related to databases and finite combinatorics. Several of my colleagues were studying foundations of arithmetic, with an eye on computer science applications. This trend was even more visible in Professor Rasiowa's group, with particular attention paid to the issues of various logical systems for approximation (Professors Lech Polkowski and Andrzej Skowron). But mathematics works in mysterious ways. After I emigrated to United States and moved to computer science, it turned out that my interests became very close to those of Professor Rasiowa's. She visited us in Lexington and we even wrote a couple of papers together. Even stranger was the evolution of Ina Rauszer. Both she and I got interested in the formal systems for distributed databases. She visited us for several semesters in Lexington.

There is no doubt in my mind that this account of mathematical life of logicians is biased. But this is how I saw it – very exciting, full of unexpected events and the turning of the wheel of fortune. A splendid time, a marvelous time, and a sad time too.

But then, there is a question I often ask myself and my colleagues. What would be happening to Mostowski if he had lived longer? Would he have persisted in the direction

---

[3]Thanks for Professor Raphael Finkel of my Department for pointing me to the depository where this drawing can be seen: http://www.avenarius.sk/pics/schneider/abhaengig.jpg.

he was pursuing or would he, like Professor Rasiowa, have turned in the direction of the foundations of computer science? Around that time, after the results of Cook and the explosion of computational complexity theory, after the results of Scott on denotational semantics and on λ-calculus, many of us turned sharply to the foundations of computer science. There were and still are fundamental applications of logic in all sorts of areas: databases, programming languages, complexity theory, electronic design automation, program correctness and other areas. As computers become truly ubiquitous with processors in our cars, refrigerators, atomic power stations, TV sets and soon within ourselves, the role of logic grows. Theorems are still proved, but quite often those are motivated not by pure and inapplicable mathematics, but by need to understand the conceptual power of the notion of computation. If one accepts the idea that computer science is nothing but applied logic, then it is very natural to turn to computer science for inspiration. The roll-call of the "big names" of logic throughout the last 50 years is full of people who turned to computer science for at least inspiration, if not for applications of their work. Rasiowa and Rauszer did exactly that. Whether Mostowski would do the same we will never know.

In 1964, Professor Salwicki wrote the first SAT solver in Poland. Some time later Professors Grzegorczyk and Mostowski (and myself) went for a demonstration. I believe the solver accepted formulas up to 10 variables and provided a counterexample if there was one. Professor Mostowski (I believe, Professor Salwicki says it was Grzegorczyk) wrote a formula consisting of equivalences only. It is an old fact, known to Leśniewski, that there is a simple algorithm for testing such formulas for validity. But the solver could not handle this because such formulas grow quickly when transformed into conjunctive normal form. Older logicians were happy: The machine did not promise to make them jobless. But today equivalence reasoning, and similar artificial intelligence techniques has been incorporated in the solvers. Today the solvers are handling huge problems, testing very large circuits for safety and correctness. A true application of logic, and a great triumph. What would the Masters say today?

Acknowledgements

Several individuals read this paper at various stages of its writing. Those include: Andrzej Ehrenfeucht, Rafi Finkel, Grażyna Mirkowska-Salwicka, Tadeusz Mostowski, Andrzej Salwicki, Marian Srebrny, Mirek Truszczyński, my wife Elizabeth and daughter Natalia. They pointed to many errors and suggested corrections. Whatever factual errors remain – those are all mine.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Mathematical Logic in Warsaw: 1918–1939[1]

Jan WOLEŃSKI
Jagiellonian University, Institute of Philosophy, Kraków

1. Prehistory (Before 1915)

Normal academic life began in Warsaw in the 19th century with the establishment of the Warsaw Society of Friends of Science (1800) and the University of Warsaw (1816). Both institutions were closed after the November Uprising (1830–1831). The University was renewed as the Main School in 1862 and closed once again in 1869; after that Warsaw had the Imperial University (Russian) until 1914. The Warsaw Scientific Society was established in 1907 as a continuation of the Warsaw Society of Friends of Science. Still another institution should be mentioned in this context, namely the Flying University (1885), an illegal college, meeting in private houses; it was transformed into the Society of Scientific Courses (1905), which carried out its activities officially. At the University of Warsaw in 1816–1831, logic was taught (as a part of philosophy) by Ignacy Adam Zaballewicz (1784–1831), a philosopher very strongly influenced by Kant. Henryk Struve (1840–1912) was perhaps the most remarkable person in logic in Warsaw in the 19th century. He was a professor of the Main School and the Imperial University. He taught logic systematically and wrote a valuable history of logic published in 1911. Classes in logic were conducted by Adam Mahrburg (1855–1913) at the Flying University; Tadeusz Kotarbiński (1886–1981) was one of his students. The Warsaw Scientific Society had the Division of Anthropology, Social Sciences, History and History of Philosophy. Some of its meetings included talks in logic, which were published in its Proceedings (for example, the papers by Władysław Biegański (1857–1917); more on him below).

Philosophical life in Warsaw was not restricted to academic institutions. Warsaw was a stronghold of positivism in Poland (so-called Warsaw positivism). In general, positivism created a favourable climate for the philosophy of science, which was closely related to logic at that time, at least in philosophy. Mahrburg, already mentioned, was one of the central figures of Warsaw positivism, and the Kozłowskis (Władysław, 1832–1899, Władysław Mieczysław, 1858–1932) were other persons interested in logic. The Warsaw positivists paid a great attention to publishing scientific books, translations, as

[1] This paper is related to [22,23,25,26]; see also [2,8,9] as other sources about the development of logic and mathematics in Poland, particularly in Warsaw. Since my task consists in a general overview, I do not include here detailed bibliographical data about the works of particular authors, except explicit quotations. The dates of birth and death are given only for Polish mathematicians, logicians and philosophers. If other editions than originals or translations are mentioned in the bibliography at the end of the paper, page-references are to them.

well as books by Polish authors. In fact, Warsaw became perhaps the most important publishing centre in science in Poland, at that time divided into three parts, occupied by Russia, Germany (Prussia until 1871) and Austro-Hungary (Austria until 1867). Restricting ourselves to the period after 1860, the following books in logic are to be mentioned: translations of *Logic* by Alexander Baine (1878) and *System of Logic* by John Stuart Mill, and textbooks of logic by Struve (1868–1870, 1907) and Biegański (1903, 1907); the importance of teaching logic was commonly recognized in Warsaw and stressed by the fact that *The Guide for Autodidacts*, published (1st ed., Warsaw 1902) a special chapter devoted to logic (written by Mahrburg). Biegański, sometimes called the Polish Claude Bernard, was a doctor of medicine by profession (he worked for a hospital in Częstochowa, a provincial city) but a philosopher (one of the most distinguished philosophers of medicine of his time) and logician by passion. Except textbooks on logic, he published an extensive treatise about the theory of logic (1911), in which he reported on some works in the algebra of logic (in fact, it was one of the earliest information about this stage of the development of mathematical logic in Polish). Last but not least, the rise of *Przegląd Filozoficzny* (Philosophical Review) has to be mentioned. This journal was established by Władysław Weryho (1868–1916) in Warsaw in 1898 and very soon became the most important Polish philosophical journal. *Przegląd* published logical papers from its first issue and many later leading Polish logicians made their debut as writers in it; these include Kazimierz Ajdukiewicz (1890–1963), Tadeusz Czeżowski (1889–1981), Kotarbiński, Stanisław Leśniewski (1886–1939), Jan Łukasiewicz (1878–1956) and Zygmunt Zawirski (1882–1948).

The preceding remarks are mostly connected to logic as a part of philosophy. In mathematical logic (or rather logical and foundational works inspired by mathematics). Samuel Dickstein (1851–1939) was the Polish pioneer of these interests. Although he did not have an official academic position until 1915, he was very active in scientific enterprises. He published a very interesting monograph about the concepts and methods of mathematics (in 1891), in which he competently informed about the foundational works of Bolzano, Cantor, Dedekind, Frege, Grassmann, Hankel, Helmholtz, Kronecker, Peano, Riemann, and Weierstrass. Dickstein also established a new journal, *Wiadomości Matematyczno-Fizyczne* (Mathematical-Physical News) as well as a series of books, "Prace Matematyczno-Fizyczne" (Mathematical-Physical Works). This series as well as other publications included translations of works by Riemann (1877), Klein (1899), Helmholtz (1901), Poincaré (two in 1908, 1911), Dedekind (1914), Italian geometers (1914, 1915) and Whitehead (about 1915). Although it is difficult to evaluate the real influence of these publications, they certainly contributed to increasing of interest in the abstract approach to mathematics in Poland. Zygmunt Janiszewski (1888–1920) delivered lectures in classes organized by The Society of Scientific Courses. He also organized private scientific meetings in his apartment, among other, devoted to logic, in which Kotarbiński and Leśniewski participated (both studied in Lvov, but lived in Warsaw). Janiszewski wrote two papers on logic and the philosophy of mathematics for the 2nd edition of the *Guide for Autodidacts* (see [5,6]). There is a great difference between the treatment of logic in both editions of this work. Mahrburg's paper is entirely traditional and connected logic with epistemology, whereas Janiszewski presented logic as closely related to mathematics. He saw logic as somehow independent of its applications [5, p. 454]:

> We note that logistics has no practical profits as its aim, at least directly. Logistic symbolism and the analysis of concepts by their reduction to their primitive elements are not introduced in order to think, argue and write in this way. Similarly, physicists do not create the theory of sounds in order to help musicians in composing or to write notes as mathematical equations (however, this does not exclude an indirect use of acoustic theories in the development of music). Logistics can contribute to the development of other sciences by discovering new forms of reasoning and by training thinking, but this is only a possibility and would be its by-product. The direct aim of logistics in application to other sciences can only consist in explaining their logical structure.

This is an interesting passage, because it shows that Janiszewski, who studied in France, did not accept skepticism about logic, very characteristic of French mathematicians, and explicitly rejected Poincaré's objections against logic as pointless. As far as the matter concerns the philosophy of mathematics, Janiszewski, limited its scope to (a) the nature of objects and theorems of mathematics (are they *a priori* or not?), and (b) the problem of the existence of mathematical objects and the correctness of some modes of reasoning in mathematics. There is an essential difference between (a) and (b) [6, p. 470]:

> The problems considered in previous sections (that is, concerning (a) – J. W.] are, so to speak, outside of the scope of a mathematician's activity. Independently of any view about these questions or its lack, this fact has no influence, at least no direct one, on work inside mathematics and does not prevent communication between mathematicians. Disregarding what mathematicians think about the essence of natural numbers or mathematical induction, they will use them in the same way. On the other hand, there are controversial problems which have a direct influence on mathematical activity. They concern the validity of some mathematical arguments and the objective side of some mathematical concepts.

Janiszewski mentions the character of mathematical definitions (predicative or not) and the admissibility of the axiom of choice as examples of group (b). He reports controversies in the philosophy of mathematics rather than proposing solutions. As we will see these views became canonical in the Warsaw school of logic and the foundations of mathematics.

2. Warsaw 1915–1939: People, Institutions, International Relations

In order to relate to important historical events, I inserted the dates 1918 and 1939 into the title of this section. However, the golden period of logic in Poland, mostly related to Warsaw, started in 1915. Russian troops left Warsaw very soon after the beginning of World War I and the city was under German occupation until 1918. The German authorities, in order to gain the sympathies of Poles, agreed to reopen the University of Warsaw in 1915. Since Warsaw had no real academic community of its own, most professors were brought from other universities, mostly from Lvov. Łukasiewicz and Kotarbiński obtained positions in philosophy, one of the professorships in mathematics was given to Dickstein (he was a local person). Łukasiewicz began lectures in logic. Kazimierz Kuratowski (1896–1980), one of the most distinguished Polish mathematicians, a student at this time, wrote an interesting report about Łukasiewicz's classes [8, pp. 23–24]:

> Jan Łukasiewicz was another professor who greatly influenced the interests of young mathematicians. Besides lectures on logic, Professor Łukasiewicz conducted more specialized lectures which shed new light on the methodology of deductive sciences and the foundations of

> mathematical logic. Although Łukasiewicz was not a mathematician, he had an exceptionally good sense of mathematics and therefore his lectures found a particularly strong response among mathematicians. [. . .]. I remember a lecture of his on the methodology of deductive sciences in which he analysed, among other things, the principles which any system of axioms should satisfy (such as consistency and independence of axioms). The independence of axioms in particular was not always observed by writers and even in those days was not always exactly formulated. Łukasiewicz submitted to detailed analysis Stanisław Zaremba's *Theoretical Analysis* [1915] which was well-known at the time, questioning a very complicated principle formulated in that work, which was supposed to replace the rule of the independence of axioms. The criticism was crushing. Nevertheless, it brought about a polemical debate in which a number of mathematicians and logicians took part in the pages of the *Philosophical Review* (1915–1918). I mention this because a by-product of Łukasiewicz's idea in our country was the exact formulations of such notions as those of quantity, the ordered set, and the ordered pair (the definition of the ordered pair which I proposed during the discussion was to find a place in world literature on the subject). This illustrates the influence brought by Jan Łukasiewicz, philosopher and logician, on the development of mathematical concepts."

The debate to which Kuratowski alluded led to results much more important than the above-mentioned definition of the ordered pair. The discussion originated in an extensive paper of Łukasiewicz (see [11]) in which he substantially analysed Zaremba's views, in particular his definition of magnitude. Zaremba replied, but his arguments were criticized by Kuratowski, Czeżowski and Leon Chwistek (1884–1944; he was active in Cracow, later in Lvov). Zaremba became personally offended by this polemic, which brought about a very deep and hot conflict between Polish mathematicians working in Warsaw and Lvov, and those grouped around Zaremba (1863–1942) in Cracow (see also below).

This conflict was very closely related to the Janiszewski program for the development of mathematics in Poland. In 1916 the Committee of the Mianowski Fund, a charitable foundation supporting Polish science, invited scholars from various fields to formulate remarks and more extensive projects concerning the most effective activities aiming at improving the organization of research. The organizers collected 44 papers, published in the first volume of *Nauka Polska* (Polish Science), a newly established journal devoted to various aspects of scientific research in Poland. Mathematics was represented by the voices of Zaremba and Janiszewski. Although the former paper is almost forgotten, Janiszewski's contribution (see [7]) gained more fame than any other from the rest of the submitted comments. The Janiszewski program is commonly regarded as the decisive factor of the rise and the subsequent development of modern mathematics in Poland, particularly in the years 1918–1939. The main idea of the program consisted in promoting various activities for achieving an autonomous position by Polish mathematics. Let me quote the very end of [7, p. 18]:

> If we do not like to always "lag behind", we must apply radical means and go to the fundamentals of what is wrong. We must create a [mathematical] "workshop" at home! However, we may achieve this by concentrating the majority of our mathematicians in working in one selected branch of mathematics. In fact, this takes place automatically nowadays, but we have to help this process. Doubtless, establishing in Poland a special journal devoted to the only selected branch of mathematics will attract many to research in this field. [. . .]. Yet there is also another advantage of such a journal in building the mentioned "workshop" in ourselves: we would become a technical center for publications in the related field. Others would send manuscripts of new works and have relations with us. [. . .]. If we want to capture the proper position in the world of science, let us come with our own initiative.

Literally understood, Janiszewski's words are quite cryptic or mysterious, because he did not point out which field should be chosen as the only branch of mathematics on which Poles could concentrate in the future, although he clearly alluded to something that "takes place automatically". However, his project became understood univocally: Polish mathematicians should concentrate on set theory and topology as well as their applications to other branches of mathematics.

It is possible that Janiszewski's formulations were so mysterious because he wanted to avoid a direct clash with Zaremba, a great enemy of new tendencies in the foundations of mathematics. It is quite understandable that open conflicts about how science or its particular branches should be done were felt to be inappropriate in the discussion which was considered to be of the utmost importance in Poland; in fact, the construction of normal academic life in a country which had recovered independence became one of the principal national enterprises. Although the mentioned personal matters certainly seriously deepened Zaremba's negative attitude against logic and logicians, still another important aspect of this controversy should be mentioned. Zaremba was strongly influenced by the French style of doing mathematics, according to which logic was quite marginal in mathematics and regarded as a servant of mathematics. Although Zaremba's writings contained a lot of logical and foundational topics, he considered them as introductory for mathematics. As I already remarked, Janiszewski, who had also studied in France, represented an opposite view. The division of the Polish mathematical community into two hostile camps was also perceived and commented abroad. Nicolai Luzin, a great Russian mathematician reported on the situation in Poland in a letter to Armand Denjoy (September 30, 1926). This letter notes a conflict, scientific and personal, between the classical school (Zaremba and his circle) and more contemporary tendencies (mathematicians in Warsaw and Lvov). Luzin even spoke about the war of ideas concerning mathematics in Poland in which the trends proposed by Janiszewski in his program decisively won. In Warsaw it was documented by appointing Janiszewski, Stefan Mazurkiewicz (1888–1945) and Wacław Sierpiński (1882–1969) as professors of mathematics just after 1918.

Since, due to the execution of the Janiszewski program, mathematical logic and the foundations belonged to the hearts of mathematics, the University of Warsaw wanted to appoint a professor responsible for these fields. Łukasiewicz was a natural candidate for this job, but he became a member of the national government. Sierpiński recommended Leśniewski, who became Professor of the Philosophy of Mathematics in 1919. Łukasiewicz returned to the university in 1920 and was appointed Professor of Philosophy at the Faculty of Mathematics and Natural Sciences. In fact, his position was in mathematical logic. It is a remarkable fact that two philosophers occupied Chairs in the community of mathematicians. The journal which was proposed by Janiszewski in his programmatic paper and which materialized as the *Fundamenta Mathematica*, was devoted to the areas of mathematical studies selected by his program as leading. Since it was published in foreign languages from the start, Janiszewski's expectation and hope as far as the matter concerned its importance were fully realized, because the *Fundamenta* became very soon the principal international journal of set-theoretical mathematics. Now, according to the Janiszewski program, logic and the foundations belonged to the very heart of mathematics. This was documented by the fact that the Editorial Board of the journal consisted of two mathematicians (Wacław Sierpiński, Stefan Mazurkiewicz; Janiszewski died in 1920, before the first volume appeared) and two logicians (Stanisław Leśniewski, Jan Łukasiewicz). At the beginning, the editors wanted to publish two separate volumes, one devoted to logic and the foundations, and another to set theory and its applications, but this idea was finally abandoned. To be faithful to history, we should note that the harmony between mathematicians and logicians came to an end in 1929, when Sierpiński made very critical and sarcastic comments about Leśniewski's paper; it is probable that this criticism was provoked by Leśniewski's earlier aggressive remarks about set theory. In any case, Leśniewski resigned from his membership in the Editorial Board; Łukasiewicz felt himself obliged to follow this move. However, this incident did not decrease the continuing importance of mathematical logic and the foundations of mathematics in Warsaw.

Łukasiewicz was certainly the *spiritus movens* of the logical community in Warsaw. He was a great teacher, continuing the tradition of his teacher Kazimierz Twardowski (1866–1938) in Lvov, as well as a very efficient organizer of various scientific enterprises. He and Leśniewski, supported by professors of mathematics, programmatically sympathetic to logic, were able to attract many young mathematicians and philosophers to logical research. Results came very soon. Alfred Tarski (1901–1983), later recognized as one of the greatest logicians of all times, decided to specialize in logic (he graduated and obtained his PhD under Leśniewski) and became the third pillar of the Warsaw School of Logic (WSL for brevity). In the 1920s and early 1930s these Big Three were joined by (I list them in alphabetical order) Stanisław Jaśkowski (1906–1965), Adolf Lindenbaum (1904–1941?; the question marks in the dates indicate that exact data are uncertain or even unknown), Andrzej Mostowski (1913–1975), Moses Presburger (1904–1943(?)), Jerzy Słupecki (1904–1987), Bolesław Sobociński (1906–1980) and Mordechaj Wajsberg (1902–1943?). All of them except for Sobociński were mathematicians by training. Lindenbaum came to the WSL after his PhD in topology, Mostowski and Presburger studied with Tarski (the latter never finished his doctor thesis), Łukasiewicz was the supervisor of the rest. Henryk Hiż (1917–2006) and Czesław Lejewski (1913–2001) became the last students of the WSL; the latter graduated in 1939, while the former began his studies, but both began their active scientific careers after 1945. The WSL as a working group had eleven members at its peak, that is, around 1937. Is this large or small? Of course, everything depends on the point of reference. Evaluating from the contemporary point of view, about a dozen people working together in logic is perhaps not so many. However, if one looks at this group from a broader international perspective, one should remember that no other place in the world in which logic was actively done had even one third of this amount. Thus, at the time Warsaw was the place most populated by professional logicians in the world.

The Warsaw logical community was much larger than the WSL *sensu stricto*. Some mathematicians, already mentioned above, like Kuratowski, should be included. Ajdukiewicz lectured in Warsaw in 1926–1928 (he temporarily replaced Łukasiewicz, who withdrew from the university). Some people were connected with the WSL rather loosely, like Zygmunt Kobrzyński (?–?), others, like Jerzy Billig (?–?), Jerachmiel Brykman (?–?) and Zygmunt Kruszewski (?–?) are mentioned by various sources as fairly active and even as having some interesting results, but not very much is known about them. Kazimierz Pasenkiewicz (1897–1995), later professor of logic at the Jagiellonian University in Cracow, did not work in logic before 1939. Logic was fairly successfully popularized by Kotarbiński among students of philosophy. Although they were mostly interested in the philosophy of science, some of them, in particular Jan Drewnowski (1896–1978), Janina Hosiasson-Lindenbaum (1899–1942), Edward Poznański (1901–1976; he was of the pioneers of logic in the Hebrew University in Jerusalem) and Dina Sztejnbarg (later

Janina Kotarbińska, wife of Tadeusz Kotarbiński, 1902–1997), were always very close to mathematical logic and its problems as well as applied logical tools in the philosophy of science. Father Jan Salamucha (1903–1944) came to logic from the Faculty of Theology. Also Father Józef M. Bocheński (1902–1995), who, although permanently living in Italy or Switzerland, was in close and frequent contact with the WSL. Both Bocheński and Salamucha, together with Drewnowski and Sobociński, formed a group, called the Cracow Circle, which intended to modernize scholastic philosophy by means of mathematical logic. In general, the Warsaw logical circle counted dozens of people. This is documented by reports about the number of participants at logical talks.

The teaching of logic was very intensive in Warsaw, especially for mathematicians. Everybody had to pass an elementary course in formal logic, either separate one or being a part of the introduction to philosophy. Usually, the students had a choice between classes conducted by Łukasiewicz and Kotarbiński. The separate course in logic had a small section devoted to the philosophy of science in order to offer something interesting to non-mathematicians. Students who wanted to specialize in mathematical logic could attend the more advanced classes (lectures or seminars) of Łukasiewicz, Leśniewski (he usually did not teach elementary courses) and later of Tarski. Typically, logical education in Warsaw lasted three full academic years. For example, the list of classes (at the Faculty of Mathematics and Natural Sciences) in the 1933/1934 academic year lists the following lectures, exercises, proseminars (elementary seminars) and advanced seminars related to mathematical logic and the foundations of mathematics:

– elements of mathematical logic (Łukasiewicz);
– an outline of the methodology of the deductive sciences (Łukasiewicz);
– an outline of the methodology of the natural sciences (Łukasiewicz);
– selected problems of mathematical logic (Łukasiewicz);
– the history of ancient logic (Łukasiewicz);
– exercises in logic and methodology for mathematicians (Łukasiewicz, Tarski);
– exercises in logic for students of the natural sciences (Łukasiewicz);
– proseminar in logic (Łukasiewicz, Tarski);
– seminar in logic (Łukasiewicz);
– foundations of protothetics (Leśniewski);
– arithmetic of real numbers (Leśniewski);
– theoretical arithmetic (Tarski);
– methodology of the deductive sciences (Tarski);
– a discussion class about the methodology of the deductive sciences (Tarski).

It is interesting that the teaching of Łukasiewicz as well as his classes conducted together with Tarski were classified as philosophical (this was probably caused by the fact that Łukasiewicz was formally professor of philosophy), the rest was listed under mathematics. All of these classes covered 34 hours per week. Moreover, Sierpiński lectured about set theory and Kotarbiński about logic at the Faculty of Humanities.

At the beginning, there was a problem with textbooks. Because the matter concerned not only logic but practically all fields, a special series of textbooks and lecture notes was established. A Polish translation of Louis Couturat's *L'algebre de la logique* (originally published in 1905) appeared in 1918 (Łukasiewicz initiated that) as the first textbook for students of mathematics in the University of Warsaw; this fact additionally shows that logic was really considered as very important in that mathematical circle.

Bronisław Knaster (1893–1980), later a well-known mathematician, who translated this book, wrote in the preface that this work does not satisfy the recent standards of logical precision. Although he did not refer to the new patterns of exactness, he probably had in his mind the development of logic after Frege and the *Principia Mathematica* of Whitehead and Russell (1st edition, 1910–1913). It is almost sure that this evaluation was inspired by Łukasiewicz himself, who in his paper against Zaremba (see above) referred to Frege as a paragon of precision. Kotarbiński's lectures notes were published twice (1924, 1925), and those by Ajdukiewicz in 1928. In 1929, two very important books appeared, Łukasiewicz [12] and Kotarbiński's textbook of philosophy. The former, published as lecture notes, contained the material lectured on by Łukasiewicz and gives a very good picture of the logical elementary curriculum for mathematicians, but the latter was primarily addressed to philosophers. Finally, Tarski [20] should be mentioned. This small book was published in the "Small Mathematical Library" series. On the back cover one can read:

> Small Mathematical Library provides mathematical readings that exceed the school program. It gives materials for school mathematical circles, and helps teachers of mathematics to supplement their lessons in secondary schools and make them more interesting.

Thus, this book, which became a standard university textbook in many countries, started as a text for students at the pre-university level.

It is interesting to refer to the booklet *Mathematical-Physical Study. Information Book for Newcomers*, published by the Students' Mathematical-Physical Circle of the University of Warsaw in 1926 and intended to provide general and bibliographical information for new students. This guide informs at the beginning that set theory, topology, functions of real variables and mathematical logic are considered as privileged fields at the University of Warsaw. In the part devoted to the foundations of mathematics its author (Lindenbaum) lists various papers and books, and recommends the *Principia Mathematica* as a principal and epoch-making work in mathematical logic, although he immediately adds that it is accessible only for advanced students. Remembering Russell's well-know remark that only six persons (including three Poles) read all of the *Principia*, the recommendation of this massive work for students, even advanced, seems as shocking. In fact, Leśniewski wanted to translate this work into Polish. He contacted Russell in this respect at the beginning of the 1920s, but the latter replied that the project should be postponed for the planned 2nd edition (it appeared in 1925). Leśniewski had a special seminar on the *Principia* and since he became disappointed by its imprecision at some points, this caused him to abandon the idea of translation. Other foreign books were also recommended; in particular, Tarski [20], lists Russell's *Introduction to Mathematical Philosophy* (1918, and its German translation) and Carnap's *Abriss der Logistik* (1929).

Although the main scientific activities of the WSL concentrated around the seminars and Chairs of Leśniewski and Łukasiewicz, other institutions also participated in logical enterprise in the Polish capital. The Warsaw Philosophical Society had a special logic section. It had regular meetings with talks and discussions devoted to logical problems. In the Warsaw Scientific Society, more precisely in its Third Faculty (Mathematics, Physics, Chemistry) several important papers in logic were introduced before their publication in the proceedings of this organization (see below). April 22, 1936 was the day of the opening meeting of the Polish Logical Society, initiated by Łukasiewicz. Its first board included Łukasiewicz (President), Tarski (Deputy of the President), Lindenbaum (Secretary), Sobociński (Treasurer) and Mostowski (a member). The tasks of this society consisted in supporting logical research and popularizing it by organizing talks, publication activities, opening libraries, recognizing results, creating scholarships, securing adequately remunerated jobs for logicians, collecting money for financial support for members of the Society and taking care of international co-operation. Warsaw logicians published their papers in various journals, namely *Przegląd Filozoficzny*, *Fundamenta Mathematicae*, *Comptes rendus de la Société des Sciences et des Lettres de Varsovie*, *Classe III*, *Wiadomości Matematyczne* and *Studia Philosophica*. The Warsaw Scientific Society edited a special series of works in mathematics and physics in which Tarski's famous monograph on truth was published. Although Warsaw logicians had no problems with finding places to publish their work, Łukasiewicz wanted to establish a professional logical journal in order to create suitable conditions for publications in logic. According to him, it should publish papers in Polish, as well as their translations in foreign languages. *Studia Logica*, a projected series of small monographs, became the first attempt, but it ended with one issue that appeared in 1934 only in English (Jaśkowski's study on natural deduction). Finally, Łukasiewicz succeeded in his efforts and two volumes of *Collectanea Logica* were prepared by 1939; unfortunately, all copies were destroyed in September 1939 and only preprints has been preserved. Since the 1930s Warsaw logicians began to publish their papers in various international journals, particularly in *Monatshefte für Mathematik und Physik* (Tarski, Wajsberg), *Mathematische Annalen* (Wajsberg), *Erkenntnis* (Łukasiewicz, Tarski) as well as in proceedings of congresses and conferences in which they participated.

The WSL as a social group was considerably complex from a sociological point of view. Some of its members, for example, Lindenbaum or Jaśkowski, were quite rich, others, like Mostowski, Słupecki or Sobociński came from the middle class, but others, like Tarski, were rather poor; some, like Lindenbaum, were close to communism, others, like Łukasiewicz or Leśniewski (after 1918) shared quite conservative or rightist views. Łukasiewicz was twice elected as the Rector Magnificus of the University of Warsaw, but Tarski, besides his (non-permanent) university appointment, taught in a pedagogical college and a secondary school, Wajsberg in a Jewish high school, while Pressburger worked as a modest clerk. Lindenbaum, Presburger, Tarski and Wajsberg were Jews, but Leśniewski and Sobociński (later also Łukasiewicz) were strongly antisemitic. This last point created various tensions, also concerning academic positions. In general, Jews, even if assimilated, had difficulties in obtaining appointment as professors in Poland; anti-Jewish tendencies became quite strong in Poland in the 1930s. Since Tarski's case is known and still discussed, let me make some comments on this matter. He consciously assimilated, changed his name (from Tajtelbaum; he used Tajtelbaum-Tarski for a while) and married a woman of Polish origin. Leśniewski, his doctoral promoter, advised him to assimilate to have better chances for an academic promotion. However, Tarski never got a professorship in Poland. He applied in Lvov, but the University chose Chwistek; Tarski's attempt to get the professorship in Poznan was also not successful. Certainly, his Jewish origin did not help him, particularly in the second case; according to information from Hiż, the University of Poznan closed the position in logic in order to prevent Tarski's appointment. As far as the matter concerns the University of Warsaw it was difficult to expect it would agree to establish the third professorship in logic, having already two positions in this field. There is an interesting letter of Leśniewski to Twardowski (September 8, 1935; quoted after Feferman, Feferman 2004 [2, p. 101]; I skip the emphasis added by the authors):

1) I am inclined to think that it could be immensely useful to create a chair in our university for Tarski, whose specialty differs significantly from those of Łukasiewicz and myself, since it would enable Tarski to conduct his scientific and pedagogical operations in a significantly broader and more independent field that, at any rate, subordinate position of an adjunct professor permit.

2) When I suggested this past year creating a chair of [. . .] metamathematics I was acting in favor of the scientific interests of the University of Warsaw, and not in favor of the scientific needs, though very much justified, of Tarski.

3) In connection of a series of facts in recent years which I could tell you sometime if you are interested, I feel a sincere antipathy toward Tarski [Leśniewski probably alludes here to his not specifically stated accusations concerning the unauthorized employment of his ideas by Tarski and other Jews; these complaints were indirect and never confirmed – J. W.] and though I intend, for the reasons of which I spoken above, to do everything in my power so that he can get a chair in Warsaw; however, I admit [because I can be responsible only for my deeds, but not my feelings – this fragment is omitted by the Fefermans – J. W.] that I would be extraordinarily pleased if some day I were read in the newspapers that he was being offered a full professorship, for example in Jerusalem, from where he could send us offprints of his valuable work for our profit.

This passage well documents the complexity of the situation. Leśniewski, on the one hand, was full of respect for Tarski as a logician, but, on the other hand, he did not hide his anti-Semitic feelings. Leśniewski's alleged accusations of plagiarism or the like that document still one possible cause of tensions, namely sensitivity for priority in achieved results, were very strong among the most prominent members of the WSL; in the late 1920s Leśniewski accused Ajdukiewicz of plagiarism concerning the theory of syntactic categories, and Łukasiewicz had objections (this happened after 1945) that Tarski did not sufficiently credited Łukasiewicz's priority in discovering many-valued logic.

The WSL, in spite of all differences in the social status and position of its members and the tensions connected with Jewish problem in Poland, was very strongly unified by the common scientific enterprise which was uncompromisingly undertaken as the ultimate vocation. This certainly tempered conflicts (as is clear from the quoted passage from Leśniewski's letter to Twardowski) and made collective work possible. Tarski once said (I know this from a personal communication by Hiż) that religion divides people, logic brings them together. If we replace 'religion' by 'ideology', we obtain a good perspective as to why social, national and political differences did not prevent cooperation inside the WSL. This collective style of work is documented by the following two quotations ([12, p. 9], [16]):

I owe most, however, to the scientific atmosphere which had developed in Warsaw University in the field of mathematical logic. In discussions with my colleagues, especially Professor S. Leśniewski and Dr. A. Tarski, and often in discussions with their and my own students, I have made clear to myself many concepts, I have assimilated many ways of formulating ideas and I have learned about many new results, about which I am today not in position to say to whom the credit of authorship goes.

In the course of the years 1920–1930 investigations were carried out in Warsaw belonging to the part of metamathematics – or better metalogic – which has as its field the study of the simplest deductive discipline, namely the sentential calculus. These investigations were initiated by Łukasiewicz, the first results originated both with him and with Tarski. At the seminar for mathematical logic which was conducted by Łukasiewicz in the University of Warsaw from 1926, most of the results stated below of Lindenbaum, Sobociński and Wajsberg

> were found and discussed. The systematization of all the results and the clarification of the concepts concerned was the work of Tarski.

This style of work resulted in several joint papers (also with Polish mathematicians) written by Tarski with Banach (1892–1945; Stefan Banach belonged to the Lvov mathematical circle, but he represented the same general attitude toward mathematics as his Warsaw colleagues), Tarski with Lindenbaum, Tarski with Kuratowski, Tarski with Sierpiński, Tarski with Mostowski and Lindenbaum with Mostowski. Several important problems were set forth at the above mentioned seminars, for example, Jaśkowski's system of natural deduction or Słupecki's work on functionally complete systems of many-valued logic were answers to problems stated by Łukasiewicz to his students.

The first appearance of Warsaw logicians on the international scene took place at the mathematical congress in Bologna in which Łukasiewicz and Tarski participated. In the late 1920s and early 1930s Warsaw was visited by Karl Menger, Rudolf Carnap, Ernst Zermelo, Heinrich Scholz and Willard van Orman Quine. Menger was the first who came to Poland. He was very impressed by his visit [17, p. 143]:

> In the autumn of 1929, the mathematicians of the University of Warsaw invited me to deliver a lecture. [. . .]. As I observed during this and subsequent visits, Warsaw between the two worlds wars had a marvelous scientific atmosphere. The interest of the mathematicians in their own as well as their colleagues' and students' work was on an intensity that I have rarely observed in other mathematical centers. I discovered the same spirit in the Warsaw School of Logic. But up to that time the Polish logicians had been somewhat isolated.

Menger's visit contributed essentially to ending the mentioned isolation. He invited Tarski to visit Vienna. Tarski came to Vienna in 1930 and delivered two lectures. He also met Gödel and Carnap, inviting the latter to Warsaw. This visit initiated permanent relations between the WSL and logical empiricism; Tarski came to Vienna several times, where he always had very fruitful discussions about logic and philosophy, among others with Karl Popper in 1935. Several Warsaw logicians (Łukasiewicz, Tarski and Lindenbaum) participated in congresses of scientific philosophy organized by the Vienna Circle in Prague (1934) and Paris (1935); this second event also became famous because of Tarski's two lectures on semantics and its applications to the concept of logical consequence, were acclaimed as the highlights of this meeting. Łukasiewicz and Tarski were appointed as members of the advisory committee of the famous *Encyclopedia of the Unified Science*, a special collection intended to popularize logical empiricism as scientific philosophy. Particularly close relations linked the WSL and the group in Münster, where Scholz established the first German department of mathematical logic. Leśniewski and Łukasiewicz revisited Münster; Scholz received honorary doctorate in Warsaw, and Łukasiewicz in Münster. When Quine asked Carnap in the early 1930s where he could learn logic, the answer was very simple: go to Warsaw. The WSL also helped foreign logicians to publish their works. Two important studies of Jacques Herbrand appeared in publications of the Warsaw Scientific Society, probably because they could not find a place in France.

3. The Scientific Ideology of WSL

Łukasiewicz and Leśniewski came to mathematical logic from philosophy. Both considered this passage as a kind of conversion and salvation ([a] Łukasiewicz 1936, pp. 227–228, [b] [10, pp. 197–198]):

> [a] My critical appraisal of philosophy as it has existed so far is the reaction of a man who, having studied philosophy and read various philosophical books to the full, finally came into contact with the scientific method not only in theory, but also in the direct practice of his own creative work. This is the reaction of a man who experienced that specific joy which is a result of a correct solution of a uniquely formulated scientific problem, a solution which at any moment can be checked by a strictly defined method and about which one simply knows that it must be that and no other and that it will remain in science once and for all as a permanent result of methodical research. This is, it seems to me, a normal reaction of every scientist to philosophical speculation. Only a mathematician or a physicist who is not versed in philosophy and comes into casual contact with it usually lacks the courage to express aloud his opinion of philosophy. But he who has been a philosopher and has become a logician and has come to know the most precise methods of reasoning which we have at our disposal today, has no such scruples.
>
> [b] Living intellectually beyond the sphere of the valuable achievements of the exponents of 'Mathematical Logic', and yielding to many destructive habits resulting from the one-sided, 'philosophical' grammatical culture, I struggled [. . .] with a number of problems which were beyond my powers at that time, discovering already-discovered Americas on the way. I have mentioned those works [that is, published in the years 1911–1915 – J. W.] desiring to point out that I regret that they have appeared in print, and I formally 'repudiate' them herewith, though I have already done this within the university faculty, affirming the bankruptcy of the 'philosophical' grammatical – work of the initial period of my work.

Although Łukasiewicz and Leśniewski became professors in the mathematical environment, they did not do "hard" mathematics. They did mathematical logic, which was at that time relatively simple as compared with other areas of mathematics, like, for example, analysis, algebra or geometry. This was accepted by Warsaw mathematicians, like Janiszewski, Mazurkiewicz or Sierpiński.

However, neither Łukasiewicz nor Leśniewski forgot their philosophical inheritance and interests. The same also concerns Tarski to some extent. He once remarked [21, p. 20]:

> Almost all researchers, who pursue the philosophy of exact sciences in Poland, are indirectly or directly the disciples of Twardowski, although his own work could hardly be counted within this domain.

In this situation the relations between philosophy and mathematics were a delicate and fairly important question. Łukasiewicz saw the problem in the following way ([12, p. 425]; observe almost the same formulation as in the last quotation from Tarski):

> By a happy coincidence mathematicians as well as philosophers cooperated in strengthening Polish mathematical logic. This circumstance augurs well as far as the matter concerns the future development of this science in Poland, because mathematicians will not allow that logic would change into philosophical speculation, but philosophers will defend this science against servile applications of mathematical methods in it and limiting it to the role of an auxiliary field. In fact, logic in Poland, particularly in Warsaw, is considered as an autonomous science having its own tasks and goals. The deductive systems belonging to logic are, according to our view, more important and more fundamental than various deductive systems included into mathematics. We understand the peculiarity of logical problems and do not consider them only from the point of view of their usefulness for mathematicians. [. . .] our mathematicians as well as philosophers, who became occupied with logic as the first began to work in mathematical logic, brought with them a framed feeling of scientific preciseness. Almost all philosophers, doing mathematical logic in Poland are students of prof. Twardowski and belong to the so-

> called “Lvov philosophical school”, in which they learned how to think clearly, scrupulously and methodically. Due to this fact, Polish mathematical logic achieved a much higher degree of exactness than mathematical logic abroad.

Both of Łukasiewicz’s views, namely that mathematical logic is autonomous with respect to mathematics and philosophy, and that it provides the most perfect pattern of the scientific method, certainly did not pass the test of time, because it is now considered as a part of mathematics and philosophy and the method generated by logic is not always evaluated as perfect, but, in the concrete historical situation in Poland in the 1920s and 1930s, it served well the development of logical research in Poland.

Although Łukasiewicz and Leśniewski abandoned philosophy for mathematical logic, they had different attitudes concerning how logical investigations should be realized. This was reported as follows [19, pp. 42–43]:

> There is an interesting contrast [. . .] between the two great figures of the Warsaw School of Logic, Łukasiewicz and Leśniewski. The latter was also a philosopher by training; he too moved away from philosophy and avoided even philosophical ‘asides’ in his published work. But, unlike Łukasiewicz, he held that one could find a ‘true’ system in logic and in mathematics. His systematization of the foundations of mathematics was not meant to be merely postulational; he wished to give, in deductive form, the most general laws according to which reality is built. For this reason, he had little use for any mathematical or logical theory which even though consistent, he did not consider to be in accord with the fundamental structural view of reality. [. . .]. Thus, in a sense, though he never mentions philosophy, Leśniewski may be regarded as a philosopher of logic [. . .]. Łukasiewicz had no such preoccupation. He did not try to construct a definite system of the foundations of the deductive sciences. His aims were, on the other hand, to provide exact and elegant structures for many domains of our thinking where such had either been wanting or insufficient; and on the other hand, to restore the vital historical dimension to logic.

Łukasiewicz’s view became prevalent in the WSL, perhaps for its very close similarity to the common attitude of Warsaw mathematicians toward philosophical problems in mathematics. In general, they insisted that mathematical research should not be limited by any *a priori* adopted view in the foundations of mathematics. Thus, Warsaw logicians did not accepted any of the foundational Big Three: logicism, intuitionism or formalism. On the contrary, mathematicians should use all of the methods of ordinary mathematics (see more about this in Murawski and Woleński, in this volume). Although this practical view can be questioned from a theoretical point of view, it resulted in a fairly workable general background for concrete logical investigations. This does not mean that philosophy did not enter into logical work in the WSL. Omitting the exceptional case of Leśniewski and his logical systems, let me recall that such important investigations as those concerning many-valued logic or the semantic concept of truth were directly inspired by philosophical insights. Once again, it proved that a reasonable philosophical pluralism is a very good environment for a success in science.

4. An (Incomplete) Survey of Results

A. Propositional calculus (PC)

- bracket-free notation (Łukasiewicz);
- various axiomatic formulations of PC (Łukasiewicz, Wajsberg, Sobociński);
- partial systems of PC (implicative, equivalential) (Łukasiewicz, Leśniewski, Tarski, Wajsberg);

- PC with quantifiers (Łukasiewicz);
- natural deduction system for PC (Jaśkowski);
- metalogic of PC: conceptual apparatus, matrix semantics, completeness, consistency, Post-completeness, independence of axioms (Łukasiewicz, Tarski, Lindenbaum, Wajsberg);

B. Predicate logic (QL)

- QL for finite domains (Wajsberg);
- natural deduction system for QL (Jaśkowski);

C. Non-classical and modal logics

- many-valued logic and its metalogical properties (Łukasiewicz, Tarski, Wajsberg, Słupecki, Lindenbaum);
- intutionistic logic (Tarski, Jaśkowski, Wajsberg);
- modal logic based on many-valued PC (Łukasiewicz, Tarski);
- modal logic based on classical PC (Wajsberg);

D. General metamathematics and model theory

- conceptual framework (Tarski);
- general theory of definability (Tarski, Lindenbaum);
- theory of consequence operation and the calculus of systems (Tarski);
- model theory, in particular the semantic definition of truth, the undefinability of truth in rich formal theories, the semantic method of proving incompleteness (Tarski);
- the method of the elimination of quantifiers (Tarski, Presburger);
- special results: the deduction theorem (Tarski), Lindenbaum's maximalization theorem, the Lindenbaum–Tarski algebra, degrees of completeness (Tarski, Mostowski);

E. Specific metamathematics and the foundations of concrete mathematical theories

- the completeness of the arithmetic of natural numbers with addition (Presburger);
- the completeness and decidability of the elementary algebra of real numbers and geometry, the axiomatization of elementary geometry (Tarski);
- Boolean algebra (Tarski);
- the foundations of set theory and topology, in particular: the axiom of choice, inaccessible cardinals, the definition of finiteness, definable sets of real numbers (Tarski, Lindenbaum, also Sierpiński, Kuratowski, Banach);

F. Leśniewski's systems

- protothetic (Leśniewski, Tarski, Sobociński, Wajsberg);
- ontology (Leśniewski, Sobociński);
- mereology (Leśniewski, Tarski);
- the theory of syntactic categories (Leśniewski).

G. History of logic (Łukasiewicz)

5. Concluding Remarks

Let us suppose that someone were to try to make a prediction of the development of mathematical logic at the beginning of the 20th century. We can expect that he or she would point to Germany, England or even France or USA as world logical centers, but certainly not Poland, and not only because there was no such country at the time. Thirty years later one could read [18, p. 73] that

> Poland has lately become the main country and Warsaw the main bastion of research in symbolic logic by virtue of the work of Jan Łukasiewicz.

Another evaluation is this [3, p. 200]:

> There is probably no country which has contributed, relative to the size of its population, so much to mathematical logic and set theory as Poland.

A fairly strange remark about the relation between the size of the population and the intensity of logical investigations performed by its members is perhaps a sign of surprise. However, there is not very much reason for being surprised. What was done by the WSL resulted from a well-conceived and well organized scientific enterprise. Of course, some additional causes occurred, for example, the enthusiasm of the Polish people after recovering independence and the appearance of extremely talented persons, like Tarski or Lindenbaum. However, a bit of luck is a necessary condition of success in any serious activity.

Looking at the survey in Section 4 we immediately realize that it mentions many results and ideas which very strongly influenced the subsequent development of logic. This particularly concerns many-valued logic, general metamathematics, model theory and the history of logic. Also Leśniewski's systems, though nonstandard, are permanently investigated in all parts of the world. Let me mention one evaluation [4, p. 135]:

> But before you dismiss him [Tarski – J. W.] as a mere theorem prover, you should ask yourself what your grandsons and granddaughters are likely to study when they settle down to their 'Logic for computing' class at 9.30 after school assembly. Will it be syllogisms? Just possibly it could be the difference between saturated objects and unsaturated concepts, though I doubt it. I put my money on Tarski's definition of truth for formalized languages. It has already reached the university textbooks of logic programming, and another ten years should see it safely into the sixth forms. This is a measure of how far Tarski has influenced the whole framework of logic.

Warsaw logicians very considerably influenced the development of logic and philosophy on an international scale, not only in Poland, where ideas of Ajdukiewicz and Kotarbinski (radical conventionalism, reism) used very extensively logical devices developed in Warsaw. In particular, Tarski's semantic definition of truth re-oriented logical empiricism and Popper, and Leśniewski's ideas attract philosophers working in formal ontology. Łukasiewicz's analysis of logical determinism is commonly considered as a masterpiece of logical analysis.

World War II brought terrible and tragic consequences for the WSL. Its Jewish members perished in ghettos, camps or other places. Łukasiewicz, Tarski, Sobociński and Lejewski left Poland and never returned; Leśniewski died in May 1939, Salamucha was killed in 1944. The war stopped normal logical research. As I already mentioned, two first volumes of *Collectanea Logica* were destroyed in September 1939. All manuscripts left by Leśniewski were burned, including his book on logical antinomies. In his recollections Mostowski (see [1, p. 33]) says that he had the following choice before leaving Warsaw after the Warsaw uprising in 1944: to take bread or his notes concerning the arithmetical hierarchy and constructive sets – he chose to take bread. Despite all of the difficulties, Poles succeeded in organizing a powerful system of clandestine education comprising secondary schools as well as universities. Thus, this time was not entirely lost, also in the case of logic. It was taught at both levels, also by the members of the WSL and its associates (Łukasiewicz – until his emigration in 1944, Kotarbiński, Sobociński, Słupecki, Mostowski, Salamucha and Hiż). Although scientific investigations were difficult, they were continued, for example Łukasiewicz on Aristotelian logic, Mostowski on set theory or Słupecki about many-valued logic. Some later well-known logicians studied or graduated during the war at the underground university, for example, Andrzej Grzegorczyk (1922 –), Jan Kalicki (1922–1955) and Helena Rasiowa (1917–1994). Thus, the Polish logical tradition was not interrupted by the dark times in 1939–1945 and became renewed after the war, mainly by Andrzej Mostowski.

References

[1] J. Crossley, Reminiscences of Logicians, in: *Algebra and Logic*, J. Crossley, ed., Springer Verlag, 1975, pp. 3–57.

[2] A.B. Feferman and S. Feferman, *Alfred Tarski. Life and Logic*, Cambridge University Press, Cambridge, 2004.

[3] A. Fraenkel and Y. Bar-Hillel, *The Foundations of Set Theory*, North-Holland Publishing Company, Amsterdam, 1958.

[4] W. Hodges, Truth in a Structure, *The Proceedings of the Aristotelian Society. New Series* **86** (1985/1986), 135–151.

[5] Z. Janiszewski, Logistyka (Logistics), in: *Poradnik Dla Samouków* (A Guide for Autodidacts), A. Heflich i S. Michalski, eds, Warszawa, 1915, pp. 449–461.

[6] Z. Janiszewski, Zagadnienia filozoficzne matematyki (Philosophical Problems of Mathematics), in: *Poradnik Dla Samouków* (A Guide for Autodidacts), A. Heflich i S. Michalski, eds, Warszawa, 1915, pp. 462–489.

[7] Z. Janiszewski, O potrzebach matematyki w Polsce (On Needs of Mathematics in Poland), *Nauka Polska, jej potrzeby, organizacja i rozwój* **I** (1918), 11–18.

[8] K. Kuratowski, *A Half Century of Polish Mathematics*, Państwowe Wydawnictwo Naukowe, Warszawa, 1980.

[9] M.G. Kuzawa, *Modern Mathematics. The Genesis of a School in Poland*, College and University Press, New Haven, 1968.

[10] S. Leśniewski, O podstawach matematyki, *Przegląd Filozoficzny* **30** (1927) 163–204; Eng. tr. by D.I. Barnett, On the Foundations of Mathematics, in S. Leśniewski, *Collected Works*, Kluwer Academic Publishers, Dordrecht, 1992, pp. 174–226.

[11] J. Łukasiewicz, O pojęciu wielkości (On the Concept of Magnitude), *Przegląd Filozoficzny* **19** (1916), 1–70; partial Eng. tr. by O. Wojtasiewicz in [15, pp. 65–83].

[12] J. Łukasiewicz, *Elementy Logiki Matematycznej*, Nakładem Komisji Wydawniczej Koła Matematyczno–Fizycznego Słuchaczów Uniwersytetu Warszawskiego, Warszawa, 1929; Eng. tr. by O. Wojtasiewicz, *Elements of Mathematical Logic*, Pergamon Press, Oxford, 1963.

[13] J. Łukasiewicz, O znaczeniu i potrzebach logiki matematycznej (On the Importance and Needs of Mathematical Logic), *Nauka Polska* **10** (1929), 604–620; repr. in J. Łukasiewicz, *Logika i metafizyka* (Logic and Metaphysics), Wydział Filozofii i Socjologii Uniwersytetu Warszawskiego, Warszawa, 1998, pp. 424–436.

[14] J. Łukasiewicz, Logistyka a filozofia (Logistics and philosophy), *Przegląd Filozoficzny* **39** (1933), 115–131; English tr. by O. Wojtasiewicz in [16], pp. 218–235.

[15] J. Łukasiewicz, *Selected Works*, North-Holland Publishing Company, Amsterdam, 1970.

[16] J. Łukasiewicz and A. Tarski, Untersuchungen über Aussagenkalkül, *Comptes rendus de la Société des Sciences et des Lettres de Varsovie, Classe III* **23** (1930), 30–50; Eng. tr. by J.H. Woodger in A. Tarski, *Logic, Semantics, Metamathematics*. *Papers from 1923 to 1938*, Clarendon Press, Oxford, 1956, pp. 38–59.

[17] K. Menger, *Reminiscences of the Vienna Circle and the Mathematical Colloquium*, Kluwer Academic Publishers, Dordrecht, 1994.

[18] H. Scholz, *Abriss der Geschichte der Logik*, Junker & Dünnhaupt, Berlin, 1931; Eng. tr. by K.L. Leidecker, *A Concise History of Logic*, Philosophical Library, New York, 1961.

[19] B. Sobociński, In Memoriam Jan Łukasiewicz (1878–1956), *Philosophical Studies* (Maynoorh, Ireland) **6** (1956), 3–49.

[20] A. Tarski, *O Logice Matematycznej i Metodzie Dedukcyjnej* (On Mathematical Logic and the Deductive Method), Książnica–Atlas, Lwów–Warszawa, 1936; Eng. ed., *Introduction to Logic and to the Methodology of Deductive Sciences*, Oxford University Press, Oxford, 1941.

[21] A. Tarski, Drei Briefe an Otto Neurath, *Grazer Philosophische Studien* **43** (1992), 1–32 (Eng. tr. by J. Tarski).

[22] J. Woleński, *Logic and Philosophy in the Lvov-Warsaw School*, Kluwer Academic Publishers, Dordrecht, 1989.

[23] J. Woleński, Mathematical Logic in Poland 1900–1939: People, Circles, Institutions, Ideas, *Modern Logic* **5** (1995), 363–405; repr. in J. Woleński, *Essays in the History of Logic and Logical Philosophy*, Jagiellonian University Press, Kraków, 1999, pp. 59–84.

[24] J. Woleński, Lvov, in: *In Itinere. European Cities and the Birth of Modern Scientific Philosophy*, R. Poli, Rodopi, ed., Amsterdam, 1997, pp. 161–176.

[25] J. Woleński, The Achievements of Polish School of Logic, in: *The Cambridge History of Philosophy 1870–1945*, Th. Baldwin, ed., Cambridge University Press, Cambridge, 2003, pp. 401–416.

[26] J. Woleński, Polish Logic, *Logic Journal of the IGPL* (2004), 399–428.

Section 2

Mathematical Contributions

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

The Three-Valued Logic of Quadratic Form Theory over Real Rings

Max DICKMANN [a] and Alejandro PETROVICH [b]

[a] *Equipe de Logique Mathématique (UMR 7056), CNRS – Université Paris VII, and Projet Topologie et Géométrie Algébriques, Institut de Mathématiques de Jussieu (UMR 7586), 2, Place Jussieu – Case 7012, 75251 Paris Cedex 05, France*

[b] *Departamento de Matemáticas, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, 1428 Buenos Aires, Argentina*

To Prof. Andrzej Mostowski, *in memoriam*.

Abstract. This paper is a survey of the interaction between the three-valued propositional calculus of Łukasiewicz-Post and the axiomatic versions of quadratic form theory over (semi-real) rings known as abstract real spectra and real semigroups.

Keywords. Abstract real spectrum, real semigroup, Post algebra, special group, abstract order space.

Introduction

"A l'heure actuelle il n'est pas possible d'emettre une opinion définitive si les logiques multivalentes ne resteront qu'une construction théorique philosophique ou si elles trouveront des applications en dehors de la logique la plus abstraite."

A. Mostowski, 1957 [1]

Though Mostowski's cautious appraisal of many-valued logics has not been decisively refuted in the intervening fifty years, in this paper we present some ideas and results exhibiting an interaction between *three-valued propositional logic* and an area of current mathematical investigation. Namely, the theories of *abstract real spectra*—an axiomatic version of the real spectra of rings—and of *real semigroups*—an algebraic theory encompassing the reduced theory of (diagonal) quadratic forms over semi-real rings. [Note. Throughout this paper "ring" means commutative ring with unit.]

We have discovered these connections during research work carried out since 1999–2000. Parts of this work have already appeared in print, [2,3]. The present paper—which contains the motivations, explains the material involved, and gives precise statements of results, *but omits proofs*—is a summary of Chapter 2 of a forthcoming monograph, [4], which also deals with other aspects of the above mentioned theories.

Our interest in this area of research leading, in particular, to the results reported below, stems from:

[1] [1], p. 5.

(1) The recognition of the inherently three-valued nature of the real spectra of rings.

(2) The development by M. Marshall in [5] of a theory of *abstract real spectra* (abbreviated ARS), a notion first considered by Bröcker in [6]. In his monograph Marshall shows the intimate relationship of ARSs with quadratic forms, both in the abstract and over rings, a relationship that he systematically used to develop the theory. Under the name of "spaces of signs", the book [7] employs ARSs as a tool in real geometry (cf. especially Chapter IV).

(3) Work by Miraglia and the first author, [8], introducing and systematically developing the theory of *special groups* (SG), a first-order axiomatic theory of quadratic forms comprising, amongst its models:

   (i) Quadratic form theory over fields of characteristic $\neq 2$ and, more generally,
   (ii) Quadratic form theory over an extensive class of rings with many units ([9], Thm. 3.16, p. 17).
   (iii) Boolean algebras.

The paragraphs that follow review the main features of this background material for the benefit of a non-specialized mathematical readership, and may help to place in context the results presented in later sections.

1. The three-valued character of the real spectrum of a ring. Real spectra of rings were discovered by Coste and Roy [10] in the late 1970s following a reconstruction for real closed fields of the abstract machinery (ringed spaces, schemes, toposes and the like) earlier developed by Grothendieck in order to make possible the study of algebraic varieties over algebraically closed fields. It turned out that real spectra have remarkable geometric properties—far better than those of their predecessors, the Zariski spectra of rings—, a fact that led to the realization, *a posteriori*, that much of the Grothendieck-style abstract machinery was not needed for the development of real algebraic geometry. Appropriate references for detailed presentations of the real spectrum of a ring, the study of its properties and its applications in real algebraic geometry are [10–14].

Briefly stated, the real spectrum of a (commutative, unitary) ring A, $\mathrm{Spec}_{\mathrm{R}}(A)$, is defined as follows:

Objects: All pairs $(P, <_P)$ where P is a prime ideal of A and $<_P$ is a total order of the quotient ring A/P ("total order" is meant here in the ring-theoretic sense, i.e., a total order compatible with the algebraic operations). The pairs $(P, <_P)$ can be subsumed into single subsets of A with suitable properties, called *prime cones*, facilitating the algebraic manipulations; cf. [12], Def. 6.1, p. 106. Note that $\mathrm{Spec}_{\mathrm{R}}(A)$ may be empty. In fact, $\mathrm{Spec}_{\mathrm{R}}(A) \neq \emptyset$ if and only if -1 is not a sum of squares in A; rings with this property are called *semi-real*; cf. Section 5 below.

Topology: The so-called Harrison topology, where a basis of opens consists of the sets of the form

$$H(a_1, \ldots, a_n) = \{(P, <_P) \mid a_i/P >_P 0 \quad \text{for } i = 1, \ldots, n\},$$

for all finite sequences $a_1, \ldots, a_n \in A$, $n \geqslant 1$. If non-empty, $\mathrm{Spec}_{\mathrm{R}}(A)$ is a spectral space, as it should (see any of the references [10–14]).

This definition can obviously be restated as follows:

Objects: $\mathrm{Spec}_{\mathrm{R}}(A)$ consists of all functions $f : A \longrightarrow \mathbf{3} = \{0, 1, -1\}$ such that:

(i) $f^{-1}[0]$ is a prime ideal of A;
(ii) $\{a/f^{-1}[0] \mid f(a) = 1\} \cup \{0\}$ is the positive cone of a (ring) total order of the domain $A/f^{-1}[0]$.

Topology: $H(a_1, \ldots, a_n) = \{f \in \mathrm{Spec}_{\mathrm{R}}(A) \mid f(a_i) = 1 \quad \text{for } i = 1, \ldots, n\}$.

With this presentation the three-valued character of $\mathrm{Spec}_{\mathrm{R}}(A)$ becomes manifest.

One more point ought to be underlined in this connection. Dually, the elements of the ring A can be conceived of as functions from $\mathrm{Spec}_{\mathrm{R}}(A)$ into **3**. For each $a \in A$ we define a map $\overline{a} : \mathrm{Spec}_{\mathrm{R}}(A) \longrightarrow \mathbf{3}$ as follows: for $\alpha = (P, <_P) \in \mathrm{Spec}_{\mathrm{R}}(A)$,

$$\overline{a}(\alpha) = \begin{cases} 1 & \text{if } a/P >_P 0 \\ 0 & \text{if } a \in P \\ -1 & \text{if } a/P <_P 0. \end{cases} \tag{1}$$

To be honest, the correspondence $a \longmapsto \overline{a}$ is not injective (for example, the elements a ($\neq 0$) and $b = a \cdot \sum_{i=1}^{n} x_i^2$, where $\sum_{i=1}^{n} x_i^2 \neq 1$ and at least one of the x_i is not in P, are different but have the same "bars"). Further, the set $G_A = \{\overline{a} \mid a \in A\}$ does not have the structure of a ring: although the product in A induces a well-defined operation in G_A, the addition of A does not (think of $A = \mathbb{Q}$). However, a "shadow" of the addition of A will still survive in G_A, making it a *real semigroup* in the sense of Section 2 below.

2. Abstract real spectra. This is a topological axiomatic theory modelled on the real spectra of rings; it is expounded in Chapters 6–8 of [5]. It has a predecessor, modelled on the spaces of orders of fields, named *abstract spaces of orderings* (AOS), that Marshall developed in the late 1970's; cf. Chapters 1–4 of [5], where references to his original papers can be found. Note that the real spectrum of a field is just its space of orders (which is not empty only if the field is *orderable*, also called *formally real*). In both cases there is a close connection between real spectra and quadratic form theory, where the latter is to be suitably interpreted in the abstract context. Although the axioms for ARSs look quite weak, the resulting theory is surprisingly rich.

Marshall's abstract framework is as follows (cf. [5], Ch. 6): an ARS is a pair (X, G), where X is a set and G a unitary subsemigroup (submonoid) of $\{1, 0, -1\}^X$ under pointwise product, containing the constant functions with values 1, 0, -1, and separating points in X (for all $x, y \in X,\ x \neq y$, there is $g \in G$ such that $g(x) \neq g(y)$). Two ternary relations on G, called *representation* and *transversal representation* (by forms of dimension 2) are defined for all $a, b, c \in G$ by:

$$\text{[R]} \quad c \in D_X(a, b) \quad \Leftrightarrow \quad \forall x \in X[c(x) = 0 \vee a(x)c(x) = 1 \vee b(x)c(x) = 1],$$

$$\text{[TR]} \quad c \in D^t_X(a, b) \quad \Leftrightarrow \quad \forall x \in X[(c(x) = 0 \wedge a(x) = -b(x)) \vee a(x)c(x) = 1 \vee b(x)c(x) = 1].$$

Two additional axioms are imposed, giving respectively:

— An algebraic characterization of the sets $\{a \in G \mid a(x) \in \{0, 1\}\},\ x \in X$, in terms of D, product and the constants of G (Axiom AX2, [5], p. 99).

— Associativity of the relation D^t: for $a_1, a_2, a_3, b, c \in G$,

$$b \in D^t(a_1, c) \wedge c \in D^t(a_2, a_3) \quad \Rightarrow \quad \exists d\big(d \in D^t(a_1, a_2) \wedge b \in D^t(d, a_3)\big).$$

(Axiom AX3, [5], p. 100.)

Both representation relations are extended by induction to forms of arbitrary dimension—i.e., n-tuples from G, for any integer $n \geqslant 1$; cf. 2.4(a) below.

How do the abstract notions of representation introduced above relate to the algebraic operations in the case of rings?. Here, $X = \mathrm{Spec}_{\mathrm{R}}(A)$ and $G = G_A$. More generally, we can fix a (proper) preordering T of A (see [5], p. 84) and consider $X = \mathrm{Spec}_{\mathrm{R}}(A, T) = \{\alpha \in \mathrm{Spec}_{\mathrm{R}}(A) \mid \alpha \supseteq T\}$ $(= X_T)$ and $G = G_{A,T} = \{\overline{a} \mid a \in A\}$ where, with a slight abuse of notation, for $a \in A$, $\overline{a}$ denotes the restriction to $\mathrm{Spec}_{\mathrm{R}}(A, T)$ of the map defined in (1) above. In particular, for $T = \sum A^2$ we get back $\mathrm{Spec}_{\mathrm{R}}(A)$ and G_A. Then, representation and transversal representation (by forms of arbitrary dimension) take the following form:

$$\begin{aligned}
\overline{b} \in D_{X_T}(\overline{a_1}, \ldots, \overline{a_n}) \quad &\Leftrightarrow \quad t_0 b = \sum_{i=1}^{n} t_i a_i \text{ for some } t_0, \ldots, t_n \in T \\
&\qquad \text{such that } \overline{t_0} \cdot \overline{b} = \overline{b}. \\
\overline{b} \in D^t_{X_T}(\overline{a_1}, \ldots, \overline{a_n}) \quad &\Leftrightarrow \quad \text{There exist } b', a'_1, \ldots, a'_n \text{ such that } \overline{b} = \overline{b'}, \\
&\qquad \overline{a_i} = \overline{a'_i}\ (i = 1, \ldots, n), \text{ and } b' = \sum_{i=1}^{n} a'_i.
\end{aligned}$$

(Cf. [5], Prop. 5.5.1(5), p. 95, and p. 96.) The representation relations (on forms of arbitrary dimension) are quantifier-free definable in terms of each other:

$$\begin{aligned}
a \in D^t(b_1, \ldots, b_n) \quad &\Leftrightarrow \quad a \in D(b_1, \ldots, b_n) \text{ and} \\
&\qquad -b_i \in D(b_1, \ldots, b_{i-1}, -a, b_{i+1}, \ldots b_n) \\
&\qquad \text{for all } i = 1, \ldots, n. \\
a \in D(b_1, \ldots, b_n) \quad &\Leftrightarrow \quad a \in D^t(a^2 b_1, \ldots, a^2 b_n).
\end{aligned}$$

The first of these equivalences implies

$$\begin{aligned}
a \in D^t(b_1, \ldots, b_n) \quad &\Rightarrow \quad -b_i \in D^t(b_1, \ldots, b_{i-1}, -a, b_{i+1}, \ldots b_n) \\
&\qquad (i = 1, \ldots, n),
\end{aligned}$$

a crucial property precisely known as *transversality* in quadratic form theory.

If $A = k$ is a formally real field, the semigroup G_k becomes a group, namely $G_k = k^\times/_{\sum k^{\times 2}}$, the quotient of the multiplicative group $k^\times$ by the subgroup of (non-zero) sums of squares, and both representation relations boil down to

$$\overline{a} \in D_k(\overline{b}, \overline{c}) \quad \Leftrightarrow \quad \exists s, t \in \Sigma k^2 \quad (a = bs + ct), \tag{2}$$

for $a, b, c \in k^\times$, where $\overline{x} = x/_{\Sigma k^{\times 2}}$.

A reformulation of the setting just described—and, in fact, of the notion of a real semigroup in Section 2 below—in the language of (commutative, unitary) "multirings", i.e., rings with a multivalued addition, has recently appeared in [15].

3. Special groups. This theory was initially formulated in 1991 and developed at length in the monograph [8]; see also [16] and [17].

One point of departure was to give a direct axiomatic treatment of quadratic form theory in the structures (G_k, D_k) of the preceding example (called $G_{red}(k)$ in the notation used in [8]), and even more general structures, without taking a detour through the space of orders of the underlying field k (in many important examples this space is empty!). For example, if k is *any* field of characteristic $\neq 2$, one may consider the group $G(k) = k^{\times}/_{k^{\times 2}}$ with a representation relation defined as in (2) above, requiring only that s, t be squares of k, rather than sums of squares.

Another source of motivation was to construct an algebraic, first-order theory dual to Marshall's abstract order spaces, modelled on Stone's duality between Boolean algebras and Stone spaces. The theory of *special groups* gives an answer to both these prayers. It works as follows.

The language is that of groups (of exponent 2), augmented with an extra constant -1 and a binary relation which, following traditional use in quadratic form theory, we write as $a \in D(1, b)$. In [8] a 4-ary relation $\langle\, a, b \,\rangle \equiv \langle\, c, d \,\rangle$, axiomatizing the notion of *isometry* of binary forms, is used instead of binary representation; these formulations are equivalent, but the latter makes the axioms more transparent (the axioms for special groups in terms of binary representation are given in [18], Prop. 1.2, p. 30). Special groups are defined by a seven simple axioms in this language, to which an extra axiom can be added to get the *reduced special groups* (RSG); see [8], Def. 1.2, p. 2. The duality between RSGs and AOSs was proved in [17] and appears in [8], Thm. 3.19, pp. 57–58.

Another interesting feature of this setting is that Boolean algebras (BA) become reduced special groups [2] under the interpretation:

$\cdot\ = \triangle$ (= symmetric difference, i.e., addition in the Boolean ring B);
$1 = \bot$ (= the first element of B, usually denoted 0);
$-1 = \top$ (= the last element of B, usually denoted 1);
$a \in D(1, b) \quad \Leftrightarrow \quad a \leqslant b$ ($\leqslant$ is the order of B).

Binary isometry is then given by:

$$\langle\, a, b \,\rangle \equiv \langle\, c, d \,\rangle \quad \Leftrightarrow \quad a \triangle b = c \triangle d \quad \text{and} \quad a \wedge b = c \wedge d.$$

The AOS dual to the RSG associated in this way to a BA is just its Stone space. Thus the Duality Theorem 3.19 of [8] is literally a generalization of Stone's duality theorem. Incidentally, note that the RSG associated to any Boolean algebra is isomorphic to the RSG $G_{red}(k)$ of some formally real field k (in fact, many of them); this is a well-known result of Craven, see [19], Thm. 6.9, pp. 97–98.

Further, every RSG, G, is canonically and functorially embedded in a certain BA, its *Boolean hull* B_G, namely the BA of clopens of the AOS X_G dual to G. For details and related results, see [8], Ch. 4, especially pp. 60–66 and Thm. 4.17. Using the richer structure of BAs it is possible to read off information concerning the behaviour of quadratic

[2] In the dual language of spaces of orders this relationship was known to quadratic form theorists, but was not systematically explored.

forms over a RSG, G, from its Boolean hull B_G; cf. [8], Ch. 7. For deeper connections between the Boolean hull construction and Milnor's mod 2 K-theory of fields, see [20] and [21].

4. Summary of contents. The methodology leading to the results presented below is parallel to that outlined in paragraphs 2 and 3 above, bringing into the picture the three-valued nature of the ARSs, as sketched in paragraph 1. The roles played by abstract order spaces, reduced special groups and Boolean algebras are taken up in the three-valued context by abstract real spectra, real semigroups and Post algebras, respectively.

In Section 1 we briefly review the algebraic counterpart of the system of three-valued propositional calculus that applies here, namely

- The so-called Łukasiewicz-Post algebras of order 3, that we presently call *Post algebras*.

The next section gives, for the reader's benefit, a short summary of some notions and results from [3] necessary to understand the material that follows. We emphasize two basic points:

- The notion of a *ternary semigroup*, a class of enriched semigroups underlying the whole theory.
- The axioms for *real semigroups* and some of their properties. These are ternary semigroups with extra—axiomatically given—relations for binary representation and transversal representation, which carry the quadratic form theory involved.

Our program properly begins with

- Section 3. **The real semigroup structure of Post algebras**, where we discuss the way to define our representation relations in terms of the Post algebra operations, and derive some consequences. This is the analog, in the present context, of the definition of a RSG structure in Boolean algebras, indicated above.

In analogy to the Boolean hull of a RSG, in Section 4 we construct

- The Post hull of a real semigroup, showing that its functorial properties are similar to those of the Boolean hull of a RSG. By means of this construction we are able to get information about the behaviour in real semigroups of a particularly important class of quadratic forms—the so-called *Pfister forms*—out of the richer algebraic structure of their Post hulls.

Finally, in Section 5,

- We characterize those rings whose associated real semigroup is a Post algebra, and give some concrete examples. We also mention an analog of Craven's theorem, showing that every Post algebra is "realized" by some ring, i.e., its associated RS is isomorphic to that of some ring.

1. Łukasiewicz-Post Algebras of Order 3

The structures of the title are the algebraic counterparts of the n-valued propositional calculus developed independently by Łukasiewicz and Post in the early 1920s. They were introduced by Moisil and Rosenbloom in the early 40s. The monograph [22] contains an exhaustive study of these algebraic structures; Chapters X and XI of [23] give an account of the basic results. Here we shall only consider the case $n = 3$. We begin by summarizing some basic notions and results about Łukasiewicz and Post algebras of order 3.

Definition 1.1. A *three-valued Łukasiewicz algebra* is a structure $(L, \wedge, \vee, \neg, \nabla, \top)$ fulfilling the following requirements:

[L1] $(L, \wedge, \vee, \top)$ is a distributive lattice with last element $\top$.

[L2] The unary operation $\neg$ (*negation*) verifies the De Morgan laws:

(i) $\neg\neg x = x$.
(ii) $\neg(x \wedge y) = \neg x \vee \neg y$.

[L3] The unary operation ∇, called the *possibility operator*, verifies:

(i) $\neg x \vee \nabla x = \top$.
(ii) $x \wedge \neg x = \neg x \wedge \nabla x$.
(iii) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

A *Post algebra* of order 3 is a three-valued Łukasiewicz algebra with a *center*, that is, a distinguished element $\mathbf{c}$ verifying $\neg\mathbf{c} = \mathbf{c}$.

Henceforth we omit the words “three-valued” and “order 3”.

Remarks 1.2. (a) The De Morgan laws [L2] imply the dual law $\neg(x \vee y) = \neg x \wedge \neg y$. The first element of a Łukasiewicz algebra is $\bot = \neg\top$.

(b) In addition, every Łukasiewicz algebra satisfies the so-called *Kleene inequality*:

$$x \wedge \neg x \leqslant y \vee \neg y. \tag{3}$$

This inequality implies that the center in a Post algebra is necessarily unique, and $x \wedge \neg x \leqslant \mathbf{c} \leqslant y \vee \neg y$.

A number of other, conceptually important operations are definable in terms of the operator ∇. We mention, for instance:

(i) The *necessity operator* Δ defined by $\Delta x = \neg\nabla\neg x$.
(ii) The *Łukasiewicz implication*, a binary operation defined by

$$x \to y = ((\nabla\neg x) \vee y) \wedge ((\nabla y) \vee \neg x).$$

(iii) The *arithmetical* (or MV-algebra) operations

$$x \oplus y = \neg x \to y, \qquad x \odot y = \neg(\neg x \oplus \neg y).$$

To be sure, the presentation of Łukasiewicz algebras given above is one of several equivalent alternatives. For example, implication or truncated sum ($\oplus$) are frequently used as primitive notions, instead of ∇.

Examples 1.3. (a) Every Boolean algebra B with its usual negation becomes a Łukasiewicz algebra upon defining $\nabla x = x$. However, Boolean algebras are never Post algebras. Indeed, the existence of a center $\mathbf{c}$ such that $\neg\mathbf{c} = \mathbf{c}$, together with the laws $\mathbf{c} \vee \neg\mathbf{c} = \top$ and $\mathbf{c} \wedge \neg\mathbf{c} = \bot$, leads to a collapse: $\mathbf{c} = \bot = \top$.

(b) The simplest Post algebra is the three-element chain $\mathbf{3} = \{\bot, \mathbf{c}, \top\}$ with $\bot < \mathbf{c} < \top$, $\neg\bot = \top$, $\neg\top = \bot$, $\neg\mathbf{c} = \mathbf{c}$ and the operator ∇ defined by $\nabla\bot = \bot$ and $\nabla\mathbf{c} = \nabla\top = \top$.

(c) Further examples of Post algebras—in fact, all possible examples—are given by the Representation Theorem 1.6 below.

Note. It is customary to denote the elements $\bot, \mathbf{c}, \top$ of a Post algebra by the symbols $0, \frac{1}{2}, 1$, respectively. We have changed this usual notation in order to prevent confusion with our notation for the distinguished elements of ternary semigroups and real semigroups, which in due course will also enter into the picture.

Proposition 1.4. *Let L be a three-valued Łukasiewicz algebra. Then the modal operators ∇ and Δ satisfy the following conditions for all $x, y \in L$:*

(a) $\Delta x \leqslant x \leqslant \nabla x$.
(b) $\Delta\bot = \Delta\,\mathbf{c} = \bot$, $\Delta\top = \top$, *and* $\nabla\bot = \bot$, $\nabla\mathbf{c} = \nabla\top = \top$.
(c) Δ *and* ∇ *are lattice homomorphisms, i.e.,* $\Delta(x \vee y) = \Delta x \vee \Delta y$ *and* $\Delta(x \wedge y) = \Delta x \wedge \Delta y$; *similar identities hold for* ∇.
(d) $\Delta^2 x = \Delta x$ *and* $\nabla^2 x = \nabla x$.
(e) $\nabla\Delta x = \Delta x$ *and* $\Delta\nabla x = \nabla x$.
(f) $\nabla x \wedge \neg\nabla x = \bot$ *and* $\Delta x \vee \neg\Delta x = \bot$.
(g) $\nabla x = x$ *if and only if* $x \vee \neg x = \top$.
(h) *If* $\Delta x = \Delta y$ *and* $\nabla x = \nabla y$, *then* $x = y$.
(i) *If L has a center,* $\mathbf{c}$, *then* $x = (\mathbf{c} \wedge \nabla x) \vee \Delta x = (\mathbf{c} \vee \Delta x) \wedge \nabla x$.

Elements of a Łukasiewicz algebra verifying $x \wedge \neg x = \bot$ (equivalently, $x \vee \neg x = \top$) are called *complemented* or *Boolean*. Under the induced operations the set of Boolean elements is a Boolean algebra, denoted by $B(L)$. Note that $\nabla x, \Delta x \in B(L)$ for all $x \in L$ (1.4(f)).

Remark 1.5. (Post-algebra characters.) A *PA-character* is a Post algebra homomorphism with values in $\mathbf{3}$; it is therefore required to preserve all the primitives (constants and operations) of the language described above.

The Kleene inequality (3) implies that the prime filters (in the usual lattice sense) of PAs come in pairs when ordered under inclusion: every minimal prime filter P of L is strictly contained in one and only one (maximal) prime filter, $g(P) = \{a \in L \mid \neg a \notin P\}$.

If $h : L \longrightarrow \mathbf{3}$ is a character, then $h^{-1}(\{\top\})$ is a minimal prime filter, and $h^{-1}(\{\top, \mathbf{c}\}) = g(h^{-1}(\{\top\}))$ is the maximal prime filter containing it. All PA-characters are obtained this way: if P is a minimal prime filter of L, the map $h_P : L \longrightarrow \mathbf{3}$

$$h_P(x) = \begin{cases} \top & \text{if } x \in P \\ \mathbf{c} & \text{if } x \in g(P)\backslash P \\ \bot & \text{if } x \notin g(P) \end{cases}$$

is a PA-character. The correspondence $P \longmapsto h_P$ is a bijection.

The set of characters of a Post algebra L will be denoted by X_L. The topology induced on X_L by $\mathbf{3}^L$ (product of the discrete topology in $\mathbf{3}$) makes it a Boolean space.

A central result in the theory of Post algebras is an analog of Stone's representation theorem for Boolean algebras; this result is repeatedly (though implicitly) used in the sequel.

Theorem 1.6 (Representation Theorem for Post algebras; [23], Thm. X.4.5).

(i) *Let X be a Boolean space. Then, the set $\mathcal{C}(X) = \mathcal{C}(X, \mathbf{3})$ of continuous functions of X into $\mathbf{3}$ under pointwise operations is a Post algebra.*

(ii) *If L is a Post algebra, the evaluation map $ev : L \longrightarrow \mathcal{C}(X_L)$ defined by $ev(a)(h) = h(a)$ for $a \in L,\ h \in X_L$, is a Post-algebra isomorphism. Hence, every Post algebra is isomorphic to one of the form $\mathcal{C}(X)$, where X is a Boolean space.*

It follows from this theorem that every finite Post algebra is of the form 3^n for some positive natural number n. Further, if L is a Post algebra and a, b are elements of L such that $a \nleq b$, then there exists a character $h \in X_L$ such that $h(a) \nleq h(b)$, i.e., $h(a) > h(b)$. This remark is used in many proofs below.

2. Real Semigroups

In [3] we introduced a class of structures called *real semigroups*. These algebraic structures, given by a simple set of first-order axioms in a mathematically natural language, model a fragment of the reduced theory of diagonal quadratic forms common to a fairly large class of rings, the semi-real rings. Real semigroups also turn out to be a natural generalization of the *reduced special groups* introduced in [8]. Theorem 4.1 of [3], pp. 115ff, establishes a functorial duality between the category of real semigroups and the category of abstract real spectra.

We first define the class of (enriched) semigroups that underlie the whole setting.

Definition 2.1 ([3], Def. 1.1, p. 100). A *ternary semigroup* is a structure $(S, \cdot, 1, 0, -1)$ with individual constants $1, 0, -1$ and a binary operation $\cdot$ verifying the following conditions:

[T1] $(S, \cdot, 1)$ is a commutative semigroup with unit.
[T2] $x^3 = x$ for all $x \in S$.
[T3] $-1 \neq 1$ and $(-1)(-1) = 1$.
[T4] $x \cdot 0 = 0$ for all $x \in S$.
[T5] For all $x \in S$, $x = -1 \cdot x$ implies $x = 0$.

Additional information on ternary semigroups can be found in Section 1 of [3] (pp. 100–105), and in Ch. 1, Section 1 of [4].

To define the notion of a real semigroup we enrich the language of ternary semigroups with a ternary relation D. Following traditional notation, we write $a \in D(b, c)$ instead of $D(a, b, c)$, and set:

$$a \in D^t(b, c) \quad \text{iff} \quad a \in D(b, c) \wedge -b \in D(-a, c) \wedge -c \in D(b, -a). \tag{4}$$

Definition 2.2. A *real semigroup* (abbreviated RS) is a ternary semigroup $(G, \cdot, 1, 0, -1)$ together with a ternary relation D on G satisfying the following axioms:

[RS0] $c \in D(a, b)$ implies $c \in D(b, a)$.
[RS1] $a \in D(a, b)$.
[RS2] $a \in D(b, c)$ implies $ad \in D(bd, cd)$.
[RS3] (Strong associativity). If $a \in D^t(b, c)$ and $c \in D^t(d, e)$, then there exists $x \in D^t(b, d)$ such that $a \in D^t(x, e)$.
[RS4] $e \in D(c^2a, c^2b)$ implies $e \in D(a, b)$.
[RS5] If $ad = bd,\ ae = be$ and $c \in D(d, e)$, then $ac = bc$.
[RS6] $c \in D(a, b)$ implies $c \in D^t(c^2a, c^2b)$.
[RS7] (Reduction) $D^t(a, -b) \cap D^t(b, -a) \neq \emptyset$ implies $a = b$.
[RS8] $a \in D(b, c)$ implies $a^2 \in D(b^2, c^2)$.

The relations D and D^t are respectively called *representation* and *transversal representation*.

Examples 2.3. (1) The motivating and, obviously, most important example of real semigroup is the semigroup G_A associated to a ring A, defined at the end of paragraph 1 of the Introduction, and its relative $G_{A,T}$, the RS associated to a preorder T of a ring A, defined in paragraph 2 therein.

(2) Adding an element 0 to the underlying group (with $-1 \neq 1$) of a reduced special group, G, and extending the group operation by $x \cdot 0 = x$ for all $x \in G \cup \{0\}$, gives raise to a ternary semigroup G^*. Extending in turn the representation relation of G by

$$D_{G^*}(a, b) = \begin{cases} \{a, b\} & \text{if } a = 0 \text{ or } b = 0 \\ D_G(a, b) \cup \{0\} & \text{if } a, b \in G, \end{cases} \tag{5}$$

gives a representation relation verifying the axioms for RSs, as shown by straightforward checking. Since in an RSG we have $a \in D(b, c) \Rightarrow -b \in D(-a, c)$ ([8], pp. 2, 3), it follows from (4) above that the relations D and D^t coincide on binary forms with entries in G.

(3) The ternary semigroup $\mathbf{3} = \{1, 0, -1\}$ has a unique structure of real semigroup, with representation given by:

$$D_{\mathbf{3}}(0,0) = \{0\}; \qquad D_{\mathbf{3}}(0,1) = D_{\mathbf{3}}(1,0) = D_{\mathbf{3}}(1,1) = \{0,1\};$$
$$D_{\mathbf{3}}(0,-1) = D_{\mathbf{3}}(-1,0) = D_{\mathbf{3}}(-1,-1) = \{0,-1\};$$
$$D_{\mathbf{3}}(1,-1) = D_{\mathbf{3}}(-1,1) = \mathbf{3};$$

and transversal representation given by:

$$D^t_{\mathbf{3}}(0,0) = \{0\}; \qquad D^t_{\mathbf{3}}(0,1) = D^t_{\mathbf{3}}(1,0) = D^t_{\mathbf{3}}(1,1) = \{1\};$$
$$D^t_{\mathbf{3}}(0,-1) = D^t_{\mathbf{3}}(-1,0) = D^t_{\mathbf{3}}(-1,-1) = \{-1\};$$
$$D^t_{\mathbf{3}}(1,-1) = D^t_{\mathbf{3}}(-1,1) = \mathbf{3}.$$

(4) In the next section we will show that Post algebras are naturally endowed with a structure of real semigroup.

Definition and Notation 2.4. We shall use the basic concepts and notation from quadratic form theory as introduced in [5], Section 6.2, pp. 105ff; they apply verbatim to our context. We explicitly use the following:

(a) Representation by forms of dimension $n \geqslant 3$ is inductively defined by:

$$D\big(\langle a_1, \ldots, a_n\rangle\big) = \bigcup \big\{D(a_1, b) \mid b \in D\big(\langle a_2, \ldots, a_n\rangle\big)\big\},$$

and similarly for transversal representation (for $n = 1$, $D(\langle a\rangle) = \{b^2 a \mid b \in G\}$, $D^t(\langle a\rangle) = \{a\}$).

(b) A *Pfister form* (*of degree n*) is a form of the shape $\bigotimes_{i=1}^{n} \langle\, 1, a_i \,\rangle$, abbreviated $\langle\langle\, a_1, \ldots, a_n \,\rangle\rangle$. As well-known, Pfister forms play a central role in quadratic form theory over fields and over RSGs; cf. [24], Ch. X; [25]; [5]; or [8]; they do too in the present abstract setting.

(c) A suitable version of Witt-equivalence: if $\phi = \langle\, a_1, \ldots, a_n \,\rangle$, $\psi = \langle\, b_1, \ldots, b_m \,\rangle$ are forms over a RS, G (possibly of different dimensions n, m), we set:

$$\phi \sim_G \psi \quad \Leftrightarrow \quad \text{For all } h \in X_G, \sum_{i=1}^{n} h(a_i) = \sum_{j=1}^{m} h(b_j),$$

(sum in $\mathbb{Z}$).

Remark 2.5. In a reduced special group the binary relation $a \in D(1, b)$ is a partial order <u>not</u> compatible with the group operation (only a very weak form of compatibility holds, cf. [18], Prop. 1.2, p. 30). In the case of RSs, none of the relations $a \in D(1, b)$ or $a \in D^t(1, b)$ defines a partial order. However, RSs do carry a natural partial order defined by

$$a \leqslant b \quad \Leftrightarrow \quad a \in D(1, b) \quad \text{and} \quad -b \in D(1, -a);$$

we call it the *representation partial order*. This is the order induced on a RS by the natural order of its Post hull (cf. Proposition 3.5(i)); for more details see [2], Section 2.4, pp. 13–15, or [4], Ch. I, Section 5.

3. The Real Semigroup Structure of a Post Algebra

We shall now describe how every Post algebra can be endowed with a ternary relation which makes it into a real semigroup. Before doing this, we define its ternary semigroup structure.

As in the case of Boolean algebras, **symmetric difference**, $\triangle$, is defined in a Post algebra as follows:

$$x \triangle y = (x \wedge \neg y) \vee (y \wedge \neg x).$$

It is easy to check that $\triangle$ has the following properties:

(i) $\triangle$ is commutative; (ii) $x \triangle \perp = x$; (iii) $x \triangle \top = \neg x$.

With **c** denoting the center, see 1.1, Kleene's inequality also yields:

(iv) $x \bigtriangleup \mathbf{c} = \mathbf{c}$; (v) $x = \neg x$ implies $x = \mathbf{c}$; (vi) $\bigtriangleup$ is associative; and (vii) $x \bigtriangleup x \bigtriangleup x = x$.

These observations amount to:

Proposition 3.1. *Let P be a Post algebra. Then $(P, \bigtriangleup, \bot, \mathbf{c}, \top)$ is a ternary semigroup where $\bot$ is the unit 1, $\mathbf{c}$ is the absorbent element 0, and $\top$ is the distinguished element -1 (cf. Definition 2.1).*

Remarks. (a) In view of this Proposition, for notational consistency we will identify the chain $\{\bot, \mathbf{c}, \top\}$ with the chain $1 < 0 < -1$ (in that order). The distinguished elements in general Post algebras continue to be denoted $\bot, \mathbf{c}, \top$.

(b) Note that in a Post algebra an element x is invertible for symmetric difference if and only if it is Boolean: $x \bigtriangleup x = \bot$ if and only if $x \wedge \neg x = \bot$.

Definition 3.2. For a Post algebra P, we define a ternary relation as follows: for $x, y, z \in P$,

$$x \in D_P(y, z) \quad \Leftrightarrow \quad y \wedge z \wedge \mathbf{c} \leqslant x \leqslant y \vee z \vee \mathbf{c}.$$

Remark. In the particular case that $x \in B(P)$, the formula above reduces to:

$$x \in D_P(y, z) \quad \Leftrightarrow \quad y \wedge z \leqslant x \leqslant y \vee z.$$

Thus, restricting the representation relation D_P to the Boolean algebra $B(P)$ we retrieve the formula given in [8], Cor. 7.13, p. 149, for binary representation in BAs.

The next Proposition is crucial for the proof of many subsequent results and in particular for the proof of Theorem 3.4 below.

Proposition 3.3. *Let P be a Post algebra, and let $x, y, z \in P$. Then,*

(i) *$x \leqslant y$ if and only if for every PA-character h, $h(x) \leqslant h(y)$ (in $\mathbf{3}$).*
(ii) *$x \in D_P(y, z)$ if and only if for every PA-character h, either $h(x) = 0$, or $h(x) = h(y)$, or $h(x) = h(z)$.*
(iii) *$x \in D^t_P(y, z)$ if and only if for every PA-character h, either $h(x) = 0$ and $h(y) = \neg h(z)$, or $h(x) = h(y)$, or $h(x) = h(z)$.*

Remark. By Proposition 3.3 one may use the familiar method of "truth-table checking"—in this case on three-valued tables—to prove the validity in Post algebras of quantifier-free formulas of the language of RSs (representations, transversal representations, etc.) by checking their validity in $\mathbf{3}$. This is done in the proof of several results below, e.g., Theorem 3.4(i) and Proposition 3.6.

Theorem 3.4. *Let P be a Post algebra. Then,*

(i) *Transversal representation takes the following form: for $x, y, z \in P$,*

$$x \in D^t_P(y, z) \quad \Leftrightarrow \quad (y \wedge \nabla z) \vee (z \wedge \nabla y) \leqslant x \leqslant (y \vee \Delta z) \wedge (z \vee \Delta y).$$

(ii) *The structure* $(P, \triangle, \perp, \mathbf{c}, \top, D_P)$ *is a real semigroup.*
(iii) *Its associated abstract real spectrum is the Post-algebra character space* X_P *(cf. 1.5).*

The next result exhibits some elementary relations between Post-algebra operations and the representation relation defined in 3.2.

Proposition 3.5. *Let* P *be a Post algebra and let* $x, y, z \in P$. *Then:*

(i) $x \leqslant y$ *if and only if* $x \in D_P(\perp, y)$ *and* $\neg y \in D_P(\perp, \neg x)$.
(ii) $x \wedge y \in D_P(x, y)$ *and* $x \vee y \in D_P(x, y)$.
(iii) $x \in D_P(\nabla x, \nabla x)$ *and* $x \in D_P(\Delta x, \Delta x)$.

In the category of Post algebras homomorphisms coincide with RS-morphisms:

Proposition 3.6. *Let* P_1, P_2 *be Post algebras and let* $f : P_1 \longrightarrow P_2$ *be a map. The following are equivalent:*

(i) f *is a morphism of real semigroups.*
(ii) f *is a Post-algebra homomorphism.*

The proof of (i) $\Rightarrow$ (ii) is delicate.

It is natural to ask whether there is a first-order axiomatization of Post algebras within the class of real semigroups and, if this is the case, to get first-order definitions of the Post algebra operations in the language $L = \{\cdot, 1, 0, -1, D\}$ for RSs. Both questions are answered in the affirmative by:

Proposition 3.7. *Let* G *be a RS and let* $G^\times$ *denote the set of invertible elements of* G. *Then,* G *is* (*the RS of*) *a Post algebra if and only if* G *verifies:*

(1) *For all* $a \in G$ *there is* $x \in G^\times$ (*necessarily unique*) *such that* $a \in D_G(x, x)$ *and* $D_G(1, -a) \cap G^\times = D_G(1, -x) \cap G^\times$.

Write ∇a *for such* x *and set* $\Delta a = -\nabla(-a)$.

(2) *For all* $a, b \in G$ *there exists* $y \in G$ (*necessarily unique*) *such that:*

$$\nabla y \in D_G(\nabla a, \nabla b), \qquad \Delta y \in D_G(\Delta a, \Delta b),$$

$$D^t_G(1, \nabla y) = D^t_G(1, \nabla a) \cap D^t_G(1, \nabla b), \quad \textit{and}$$

$$D^t_G(1, \Delta y) = D^t_G(1, \Delta a) \cap D^t_G(1, \Delta b).$$

If these conditions hold, then $y = a \wedge b$.

The proof of this result is rather involved. A different characterization is given in Theorem 5.1(4) below; however, the latter does not give explicit definitions of the modal and the lattice operators.

4. The Post Hull of a Real Semigroup

The aim of this section is to show how every RS, G, can be functorially embedded into (the real semigroup associated to) a certain, canonically determined, Post algebra P_G.

This is the analog for RSs of the main result (Thm. 4.17) in Ch. 4 of [8]. The pattern of the construction is similar to that of the Boolean hull of a RSG, with some additional hurdles.

The **Post hull** P_G of a real semigroup G is the RS of the Post algebra $\mathcal{C}(X_G)$ of continuous functions of the character space X_G into $\mathbf{3}$, given by Theorem 1.6; here, the set X_G of RS-characters *is endowed with the topology inherited from* $\mathbf{3}^G$ (called the **patch** or **constructible** topology), having as a *sub*-basis the (clopen) sets $[a = \delta] := \{g \in X_G \mid g(a) = \delta\}$, for arbitrary $a \in G$ and $\delta \in \mathbf{3} = \{1, 0, -1\}$. A useful observation, required in the proofs of several results below, is that the sets

$$[b = 0] \cap \bigcap_{i=1}^{n} [a_i = 1], \tag{6}$$

with $a_1, \ldots, a_n, b \in G$, are a basis of clopens for this topology; cf. [5], p. 111, Note 1. The canonical embedding $\varepsilon_G : G \longrightarrow P_G$ is the evaluation map: for $a \in G$ and $h \in X_G$,

$$\varepsilon_G(a)(h) = h(a).$$

The following Proposition gives the basic properties of the Post hull.

Proposition 4.1. *If G is a RS, then $\varepsilon_G : G \longrightarrow P_G$ has the following properties:*

(i) *ε_G is well-defined, i.e., $\varepsilon_G(a) \in P_G$ for all $a \in G$.*
(ii) *$\varepsilon_G(1)$, $\varepsilon_G(0)$ and $\varepsilon_G(-1)$ are the constant maps with values $\bot$, $\mathbf{c}$ and $\top$, respectively.*
(iii) *ε_G is a RS-morphism verifying in addition: for $a, b, d \in G$,*

$$a \in D_G(b, d) \textit{ if and only if } \varepsilon_G(a) \in D_{P_G}(\varepsilon_G(b), \varepsilon_G(d)).$$

In particular, ε_G is injective.
(iv) *P_G is generated by* $\mathrm{Im}(\varepsilon_G)$ *as a Post algebra.*

Let $\mathbf{P}$ denote the category of Post algebras and Post-algebra homomorphisms. The correspondence $G \longmapsto P_G$ can be extended to morphisms by composition: given a RS-homomorphism $f : G \longrightarrow H$, the map $f^* : X_H \longrightarrow X_G$ defined by $f^*(\sigma) = \sigma \circ f$ for $\sigma \in X_H$, is continuous in the constructible topology. Next we define $P(f) : P_G \longrightarrow P_H$ by $P(f)(\gamma) = \gamma \circ f^*$ for $\gamma \in P_G$. This correspondence is functorial and has the very important property given in item (ii) of the following:

Theorem 4.2. *Let G, H be real semigroups. Then*

(i) *The correspondence $G \longmapsto P_G$ and $f \longmapsto P(f)$ is a covariant functor from the category* $\mathbf{RS}$ *of real semigroups (with RS-morphisms) to the category* $\mathbf{P}$.
(ii) *If $f : G \longrightarrow H$ is a RS-morphism, then $\varepsilon_H \circ f = P(f) \circ \varepsilon_G$, i.e., the diagram (D) below left is commutative:*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\varepsilon_G"'] \arrow[dr, phantom, "(D)"] & H \arrow[d, "\varepsilon_H"] \\ P_G \arrow[r, "P(f)"'] & P_H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\varepsilon_G"] \arrow[d, "f"'] & P_G \arrow[dl, "\varepsilon_L^{-1} \circ P(f)"] \\ L & \end{tikzcd}$$

Moreover, $P(f)$ is the unique Post-algebra homomorphism which makes the square above commutative.

(iii) *The pair (P_G, ε_G) is a hull for G in the category $\mathbf{P}$; that is, given a Post algebra L, any RS-morphism $f : G \longrightarrow L$ factors through ε_G, i.e. the triangle above right is commutative.*

A conceptually important, and rather powerful by-product of the Post-hull construction is Theorem 4.4, an analog for RSs of Theorem 5.2 of [8] (pp. 75 ff.).

Definition 4.3. A RS-homomorphism $f : G \longrightarrow H$ between RS's, G, H, is called a **complete embedding** if for every pair of forms ϕ, ψ, over G (possibly of different dimensions), and with Witt-equivalence $\sim$ defined as in 2.4(c),

$$\phi \sim_G \psi \quad \Leftrightarrow \quad f * \phi \sim_H f * \psi.$$

Remarks. (a) Complete embeddings preserve and reflect binary representation: for $a, b, c \in G$,

$$a \in D_G(b, c) \quad \text{if and only if} \quad f(a) \in D_H(f(b), f(c)).$$

A similar result holds for transversal representation.

(b) Complete embeddings are monomorphisms for the representation partial order; cf. 2.5. In particular, they are injective.

Theorem 4.4. *Let $f : G \longrightarrow H$ be a RS-morphism. Let $f^* : X_H \longrightarrow X_G$ be the map of ARSs dual to f, and let $P(f) : P_G \longrightarrow P_H$ denote the Post-algebra homomorphism associated to f. The following are equivalent:*

(1) *f^* is surjective.*
(2) $\mathrm{Im}(f^*)$ *is dense in X_G.*
(3) *$P(f)$ is injective.*
(4) *$P(f)$ is a Post-algebra isomorphism from P_G onto the Post subalgebra of P_H generated by* $\mathrm{Im}(\varepsilon_H \circ f)$.
(5) *For every Pfister form φ over G and every $a \in G$,*

$$f(a) \in D_H(f * \varphi) \quad \Rightarrow \quad a \in D_G(\varphi).$$

(6) *f is a complete embedding.*

The most delicate points in the proof of this theorem are the implications (5) $\Rightarrow$ (2) and (6) $\Rightarrow$ (2).

Combining Theorem 4.4 and Proposition 4.1, in the case $H = P_G$, $f = \varepsilon_G$, we get:

Corollary 4.5. *Let G be a RS, and let $\varepsilon_G : G \longrightarrow P_G$ be its Post-hull embedding. Then:*

(1) *The dual map $\varepsilon_G^* : X_{P_G} \longrightarrow X_G$ is a homeomorphism.*
(2) *Every RS-character $h \in X_G$ extends uniquely to a Post algebra character $\hat{h} : P_G \longrightarrow \mathbf{3}$, i.e., $h = \hat{h} \circ \varepsilon_G$.*
(3) *ε_G is a complete embedding. It preserves and reflects representation by arbitrary binary forms and by (multiples of) Pfister forms: if φ is such a form with entries in G, and $a \in G$,*

$$a \in D_G(\varphi) \quad \Leftrightarrow \quad \varepsilon_G(a) \in D_{P_G}(\varepsilon_G * \varphi).$$

A similar equivalence holds for transversal representation.

Note. In [5], Section 8.8, pp. 176 ff, Marshall constructs, in the dual category of ARSs, a real closure, $\tilde{G}$, for any real semigroup, G; his construction is functorial. The dual ARSs of both these RSs are identical: $X_{\tilde{G}} = X_G$; it is then clear that $\tilde{G}$ and G have the same Post hull. Post algebras are real closed RSs (cf. [5], Prop. 8.8.3, pp. 177–178).

Remark 4.6. Beyond their RS structure—revealed by the results of Section 3—Post algebras possess a richer algebraic structure. This additional structure yields finer information concerning the behaviour of quadratic forms. For example, one can characterize representation and transversal representation by arbitrary quadratic forms over a Post algebra purely in terms of the lattice operations, the modal operators ∇ and Δ, and its center ([4], Ch. II, Thm. 5.1), yielding additional information valid in this case. By considering the Post hull $\varepsilon_G : G \longrightarrow P_G$ of a RS, G, one may ask whether it is possible to get information concerning the behaviour of quadratic forms in G from the additional data available in its Post hull P_G. This depends on the validity of the implication

$$\varepsilon_G(x) \in D_{P_G}(\varepsilon_G * \phi) \quad \Rightarrow \quad x \in D_G(\phi),$$

which, in general, does not hold for arbitrary forms ϕ. However, Corollary 4.5(3) shows that this implication holds (at least) for multiples of Pfister forms and for arbitrary binary forms.

Using this descent technique from P_G to G one gets, for instance, the following analogs of results well-known in the theory of quadratic forms over fields and over RSGs:

Corollary 4.7. *Let G be a RS, $b, a_1, \ldots, a_n \in G$, and let $\varphi = \langle\langle a_1, \ldots, a_n \rangle\rangle$ be a Pfister form. Then,*

(1) *$D_G(\varphi)$ and $D_G^t(\varphi)$ are subsemigroups, downwards closed under the representation partial order of G. Further,*
(2) *$D_G(\varphi)$ is saturated, i.e.,*

$$a \in D_G(b, c) \quad \text{and} \quad b, c \in D_G(\varphi) \quad \Rightarrow \quad a \in D_G(\varphi),$$

(but $D_G^t(\varphi)$ not necessarily so).
(3) *If, in addition, G is a Post algebra, then $D_G(\varphi)$ and $D_G^t(\varphi)$ are lattice ideals; $D_G^t(\varphi)$ is also closed under ∇.*

(4) $0 \in D^t_G(\varphi) \quad \Rightarrow \quad D^t_G(\varphi) = G$.
(5) $b \in D_G(\varphi) \quad \Rightarrow \quad D_G(b\,\varphi) = D_G(\varphi) \cap \{x \in G \mid x = b^2 x\}$.
(6) $b \in D^t_G(\varphi) \quad \Rightarrow \quad D^t_G(b\,\varphi) = b \cdot D^t_G(\varphi) = D^t_G(\varphi) \cap \{x \in G \mid x = b^2 x\}$.
(7) $b \in D^t_G(\varphi)$ *and* $x \in D^t_G(b\,\varphi) \quad \Rightarrow \quad x \in D^t_G(\varphi)$.

5. Rings and Post Algebras

Throughout this section A stands for a *semi-real* ring, i.e., a ring such that $-1 \notin \sum A^2$. This requirement is known to be equivalent—among other useful conditions—to the existence of a real prime ideal in A, and hence to $\mathrm{Spec}_{\mathrm{R}}(A) \neq \emptyset$; cf. [26]. Though we deal only with RSs of the form $G_A = G_{A,\,\Sigma A^2}$, all our relevant results carry over to the more general case of RSs of the form $G_{A,T}$, T a preorder of A; their generalization is left to the interested reader.

Our first result gives a characterization of those ARSs whose associated real semigroup is a Post algebra.

Theorem 5.1. *Let (X, G) be an ARS. The following are equivalent:*

(1) *G is a Post algebra.*
(2) *If $h_1, h_2, h_3 \in X$ and $h_1 h_2 h_3 \in X$, then $h_1 = h_2 = h_3$.*
(3) *If Y_1, Y_2 are (non-empty) disjoint closed subsets of X, there are $a, b \in G$ such that $a \restriction Y_1 = b \restriction Y_1 = 1$, $a \restriction Y_2 = -1$, and $b \restriction Y_2 = 0$.*
(4) i) *For all $x \in G$ there exists $y \in D^t_G(1, -x^2)$ such that $xy = 0$.*
    ii) *For each $a \in G$ there are elements $x \in D^t_G(a^2, -a)$ and $y \in D^t_G(a^2, a)$ such that $xy = 0$.*

Theorem 5.1 yields a pure ring- (and order-) theoretic characterization of those semi-real rings whose associated RS is a Post algebra:

Proposition 5.2. *Let A be a semi-real ring. The following are equivalent:*

(1) *G_A is a Post algebra.*
(2) a) *Any two real prime ideals of A are incomparable under inclusion.*
    b) *For every real prime ideal P of A, the fraction field K_P of A/P has a unique order.*

Proposition 5.2 gives raise to some natural examples of rings whose associated RSs are Post algebras.

Example 5.3. Let $A = \prod_{i=1}^n K_i$ be a finite product of fields (i.e., A is a reduced semi-local ring). Assume that at least one of the fields K_i is formally real and that, whenever K_i is formally real, it has a unique order. Then, G_A is a Post algebra, which clearly is finite.

Generalizing this example, we have:

Example 5.4. Let A be a von Neumann-regular ring [3]. If A has at least one real prime and $A/P = K_P$ is uniquely ordered for every real prime P, then G_A is a Post algebra.

[3](Commutative) von Neumann-regular rings are those in which every principal ideal is generated by an idempotent. For more information, see [27].

Coment. One may think that Post algebras behave, as real semigroups (and rings), in basically the same way as Boolean algebras do in the context of reduced special groups (and fields). Proposition 5.2 and the ensuing examples show that this is not quite the case. In fact, in [8], p. 59 and pp. 78 ff, it is shown that the (formally real) fields K whose associated RSG, $G_{red}(K)$, is a Boolean algebra are exactly the so-called SAP fields; these may have many orders. As Proposition 5.2 shows, the situation is more restrictive in the context of RSs and rings. This behaviour can be traced to the fact that adding a zero to a Boolean algebra (viewed as a RSG, cf. 2.3(2)) does not produce a Post algebra (except for the 2-element algebra). Indeed, a Boolean algebra, B, is just the set of invertible elements of the Post hull of the real semigroup $B^* = B \cup \{0\}$ obtained by adding 0 to B.

Finally, we state a "realizability" result for Post algebras analogous to, but easier than, Craven's realizability theorem for Boolean algebras mentioned in paragraph 3 of the Introduction.

Proposition 5.5. *Let P be a Post algebra and let $X_P = X$ be its character space (cf. 1.5). Then P is RS-isomorphic to real semigroup of the ring $\mathcal{C}(X, \mathbb{Z})$ of integer-valued continuous functions on X (discrete topology in $\mathbb{Z}$; pointwise operations). Further, the same result holds replacing $\mathbb{Z}$ by any uniquely ordered field endowed with the discrete topology.*

Remark that the ring $\mathcal{C}(X, F)$, X a Boolean space and F a field with the discrete topology, is von Neumann-regular, while $\mathcal{C}(X, \mathbb{Z})$ is not.

References

[1] A. Mostowski, L'æuvre scientifique de Jan Łukasiewicz, *Fundamenta Math.* **44** (1957), 1–11.

[2] M. Dickmann and A. Petrovich, Abstract Real Spectra, Real Semigroups and Three-valued Logic, in: *Séminaire de Structures Algébriques Ordonnées, 2001–2002*, F. Delon, M. Dickmann and D. Gondard, eds, Paris 7–CNRS Logique, vol. 73, February 2002, 33 pp. (Extended abstract).

[3] M. Dickmann and A. Petrovich, Real Semigroups and Abstract Real Spectra. I, *Contemporary Math.*, vol. 344, Amer. Math. Soc., 2004, pp. 99–119.

[4] M. Dickmann and A. Petrovich, *Real Semigroups and Abstract Real Spectra*, in preparation, approx. 140 pp. (tentative title).

[5] M. Marshall, *Spaces of Orderings and Abstract Real Spectra*, *Lecture Notes Math.*, vol. 1636, Springer-Verlag, Berlin, 1996.

[6] L. Bröcker, Abstract real spectra, in: *Séminaire de Structures Algébriques Ordonnées, 1992–1993*, F. Delon, M. Dickmann, D. Gondard, eds, Paris 7–CNRS Logique, Prépublications, vol. 49, Paris, 1994, 4 pp.

[7] C. Andradas, L. Bröcker and J. Ruiz, *Constructible Sets in Real Geometry*, *A Series of Modern Surveys in Mathematics*, vol. 33, Springer-Verlag, Berlin, Heidelberg, New York, 1996.

[8] M. Dickmann and F. Miraglia, Special Groups. Boolean-Theoretic Methods in the Theory of Quadratic Forms, *Memoirs Amer. Math. Soc.* **689** (2000).

[9] M. Dickmann and F. Miraglia, Rings with Many Units and Special Groups, in: *Séminaire de Structures Algébriques Ordonnées, 2003–2004*, F. Delon, M. Dickmann, D. Gondard, eds., Paris 7–CNRS Logique, Prépublications, vol. 77, Paris, 2005, 25 pp.

[10] M. Coste and M.-F. Roy, La topologie du spectre réel, in: *Ordered Fields and Real Algebraic Geometry*, D. Dubois, T. Recio, eds, *Contemporary Math.*, vol. 8, Amer. Math. Soc., 1982, pp. 27–59.

[11] J. Bochnak, M. Coste and M.-F. Roy, Géométrie algébrique réelle, *Ergeb. Math.* **12**, Springer-Verlag, Berlin, Heidelberg, New York (1987). [English translation: Real Algebraic Geometry, *Ergeb. Math.* **36**, Springer-Verlag (1998).]

[12] M. Dickmann, Applications of model theory to real algebraic geometry; a survey, *Lecture Notes Math.*, vol. 1130, Springer-Verlag, 1985, pp. 76–150.

[13] E. Becker, On the real spectrum of a ring and its applications to semi-algebraic geometry, *Bulletin Amer. Math. Soc.* **15** (1986), 19–60.

[14] M. Knebusch, An invitation to real spectra, in: *Quadratic and Hermitean forms, Conf. Hamilton/Ont. 1983*, CMS Conf. Proc., vol. 4, Amer. Math. Soc., 1984, pp. 51–105.

[15] M. Marshall, Real reduced multirings and multifields, *J. Pure Appl. Algebra* **205** (2006), 452–468.

[16] M. Dickmann, Anneaux de Witt abstraits et groupes spéciaux, in: *Séminaire de Structures Algébriques Ordonnées, 1991–1992*, F. Delon, M. Dickmann, D. Gondard, eds, Paris 7–CNRS Logique, Prépublications, vol. 42, Paris, 1993.

[17] A. Lira, Les groupes spéciaux, Ph.D. Thesis, Univ. of Paris VII, 1995.

[18] M. Dickmann, M. Marshall and F. Miraglia, Lattice-ordered reduced special groups, *Annals Pure Appl. Logic* **32** (2005), 27–49.

[19] A. Prestel, *Lectures on Formally Real Fields*, Monografias de Matemáticas, vol. 22, IMPA, Rio de Janeiro, 1975. [Reprinted in *Lecture Notes Math.*, vol. 1093, Springer, 1984].

[20] M. Dickmann and F. Miraglia, On quadratic forms whose total signature is zero mod 2^n, *Inventiones Math.* **133** (1998), 243–278.

[21] M. Dickmann and F. Miraglia, Lam's Conjecture, *Algebra Colloquium* **10** (2003), 149–176.

[22] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Łukasiewicz-Moisil Algebras*, *Annals of Discrete Mathematics*, vol. 49, North Holland, Amsterdam, 1991.

[23] R. Balbes and Ph. Dwinger, *Distributive Lattices*, Univ. of Missouri Press, 1974.

[24] T.Y. Lam, *Introduction to Quadratic Forms over Fields*, *Graduate Series in Mathematics*, vol. 67, Amer. Math. Soc., 2004.

[25] W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, Berlin, 1985.

[26] T.Y. Lam, An Introduction to Real Algebra, *Rocky Mountain J. Math.* **14** (1984), 767–814.

[27] R.S. Pierce, Modules over Commutative Regular Rings, *Memoirs Amer. Math. Soc.* **70** (1967).

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Remarks on Gödel Phenomena and the Field of Reals *,†

Harvey M. FRIEDMAN
Department of Mathematics, Ohio State University
friedman@math.ohio-state.edu

Professor Andrzej Mostowski devoted a number of books and papers [1–4] to the incompleteness phenomena. He presented streamlined versions of Gödel Incompleteness Theorem and contributed to the popularization of Gödel's results both among the mathematicians and philosophers. It can be safely said that on par with Löb, Rosser, and Tarski he introduced the necessary rigor and formalism to the incompleteness area, starting the mathematical investigations that continue even today.

While mathematicians, philosophers and more generally scientists push the limits of unknowable finding relationships of Gödel incompleteness with physics, computer science and other areas of hard science we will focus on the relationship of incompleteness to the hard core of mathematics; the theory of reals and other number systems. I wish to make some remarks on the Gödel phenomena generally, and on the Gödel phenomena within the field of real numbers.

A lot of the well known impact of the Gödel phenomena is in the form of painful messages telling us that certain major mathematical programs cannot be completed as intended. This aspect of Gödel – the delivery of bad news – is not welcomed, and defensive measures are now in place:

1. In Decision Procedures. "We only really wanted a decision procedure in less generality, closer to what we have worked with successfully so far. Can you do this for various restricted decision procedures?"
2. In Decision Procedures. "We only really wanted a decision procedure in less generality, closer to what we have worked with successfully so far. Here are restricted decision procedures covering a significant portion of what we are interested in."
3. In Incompleteness. "This problem you have shown is independent is too set theoretic, and pathology is the cause of the independence. When you remove the pathology by imposing regularity conditions, it is no longer independent."
4. In Incompleteness. "The problem you have shown is independent has no pathology, but was not previously worked on by mathematicians. Can you do this for something we are working on?"

---

* This research was partially supported by NSF Grant DMS 0245349.

† This text is a modified and extended version of a paper that was prepared at the request of the organizers of the Gödel Centenary, in case Professor Georg Kreisel was unable to deliver his talk. Professor Kreisel gave his talk as scheduled, and this talk was not delivered. Excerpts were presented at our regularly scheduled talk later in the meeting.

Of these, number 3 is most difficult to answer, and in fact is the one where I have real sympathy. So I will focus on 1, 2, 4.

I regard these objections as totally natural and expected.

When the Wright Brothers first got a plane off the ground for long enough to qualify as "flight", obvious natural and expected reactions are:

> Can it be sustained to really go somewhere? If it can go somewhere, can it go there in a reasonable amount of time? If it can go there in a reasonable amount of time, can it go there safely? If it can go there safely, can it go there economically?

The answer to these and many other crucial questions, is YES. In fact, a bigger, more resounding YES then could have ever been imagined at that time.

But to establish yes answers, there had to be massively greater amounts of effort by massively more people, involving massive amounts new science and engineering, than were involved in the original breakthrough.

And so it is with much of Gödel. To reap anything like the full consequences of his great insights, it is going to take far greater efforts over many years than we have seen. Consider Diophantine equations. A decision procedure for Diophantine equations over Z or Q has been one of the holy grails of mathematics. We know that this is impossible for Z and suspect it is impossible for Q.

Already this bad news represents a rather substantial body of work by many people over many years, far more than what it took for Gödel to show this for some class of almost Diophantine equations over Z.

The number of variables needed presently for this is 9. For 9, the degrees needed are also gigantic.

An absolutely fantastic improvement would be, say, that the Diophantine equations over Z with 5 variables of degree at most 10 is recursively unsolvable.

Want to get very dramatic? Cubics in three integer variables, quartics in two integer variables, or cubics in two rational variables. (A decision procedure for the existence of solutions to cubics in two integer variables has been given.)

Such things, assuming they are true, will take massively more effort than has been devoted thus far.

Specifically, nobody thinks that the present undecidability proof for Diophantine equations over Z is even remotely the "right" proof. Yet there has not been a serious change in this proof since 1970, when it appeared.

Yet, of the so called "leading logicians" in the world, how many have made a sustained effort to find a better proof? How many of the leading recursion theorists, set theorists, proof theorists, foundationalists, and, yes, model theorists? Almost none.

We now turn to the incompleteness phenomena. The fact that the plane flies at all comes from the original Gödel first incompleteness theorem. That you can fly somewhere comes from the Gödel second incompleteness theorem and Gödel/Cohen work. Upon reflection after many years, we now realize that we want very considerable flexibility in where we can fly.

In fact, there will be a virtually unending set of stronger and stronger requirements as to where we want to go with incompleteness.

I have merely scratched the surface of non set theoretic destinations for incompleteness, for 40 years. Almost alone – I started in the late 60's: in 1977 I was not alone (Paris/Harrington [5] for PA).

The amount of effort devoted to unusual destinations for the incompleteness phenomena is trivial. Well, I might be exhausted from working on this, but what does that amount to compared to, say, the airline industry after the Wright Brothers? Zero.

Most of my efforts have been towards finding that single mathematically dramatic Π^0_1 sentence whose proof requires far more than ZFC. Recently, I have shifted to searching for mathematically dramatic finite sets of Π^0_1 sentences all of which can be settled only by going well beyond the usual axioms of ZFC.

In the detailed work, perfection remains elusive. So far, the Π^0_1 (and other very concrete) statements going beyond ZFC still have a bit of undesirable detail. There is continually less and less undesirable detail. The sets of Π^0_1 sentences clearly have substantially less undesirable detail. I strongly believe in this:

Every interesting substantial mathematical theorem can be recast as one among a natural finite set of statements, all of which can be decided using well studied extensions of ZFC, but not all of which can be decided within ZFC itself.

Recasting of mathematical theorems as elements of natural finite sets of statements represents an inevitable general expansion of mathematical activity. This applies to any standard mathematical context. This program has been carried out, to some very limited extent, by my Boolean Relations Theory – details will be presented in my forthcoming book.

Now concerning the issue of: who cares if it is independent if it wasn't worked on before you showed it independent?

In my own feeble efforts on Gödel phenomena, sometimes it was worked on before. Witness Borel determinacy (Martin), Borel selection (Debs/Saint Raymond), Kruskal's tree theorem (J.B. Kruskal), and the graph minor theorem (Robertson/Seymour).

Mathematics as a professional activity with serious numbers of actors, is quite new. Let's say 100 years old – although that is a stretch.

Assuming the human race thrives, what is this compared to, say, 1000 more years? Probably a bunch of minor trivialities in comparison.

Now 1000 years is absolutely nothing. A more reasonable number is one million years. And what does our present mathematics look like compared to that in one million years time?

There is not even the slightest expectation that what we call mathematics now would be even remotely indicative of what we call mathematics in 1M years time. The same can be said for our present understanding of the Gödel phenomena.

Of course, one million years time is also absolutely nothing. This Sun has several billion good years left. Mathematics in one billion years time?? I'm speechless.

I now come to the field of real numbers. The well known decision procedure of Tarski is often quoted as a deeply appreciated safe refuge from the Gödel phenomena.

However, a very interesting and modern close look reveals the Gödel phenomena in force.

It is known that the theory of the reals is nondeterministic exponential time hard, and exponential space easy. I have not heard that the gap has been eliminated.

This lower bound is proved in a Gödelian way, drenched with Turing machines and interpretations and arithmetizations.

Furthermore, there is another aspect that is also very Gödelian: lengths of proofs. I think that the least length or size of the proof/refutation of any sentence in the field of reals has a double exponential upper and lower bound.

We can go further. Given a sentence in the field of reals, what can we say about the least length/size of a proof/refutation in ZFC? This has an exponential lower bound.

In fact, the situation even supports the kind of Gödelian project underway for the usual systems of foundations of mathematics.

For instance, we can ask for a short sentence in the field of reals all of whose proofs in ZFC with abbreviation power are ridiculously long. We can even more ambitiously ask that the sentence be of clear mathematical interest.

The computational complexity of the field of reals is just barely high enough to support such results.

However, the "tameness" of the field of reals is most commonly applied not to sentences in primitive notation, but rather to sentences with mathematically convenient abbreviations.

For example, we can add quantifiers over all polynomials in n variables of degree $\leqslant d$, for every fixed n, d. Or we can add quantifiers over all semialgebraic functions in n dimensions, made up of $\leqslant r$ algebraic components of degree $\leqslant d$, with n, r, d fixed.

Presumably the resulting complexity will be far higher, involving a substantial increase in the height of the exponential stack.

References

[1] A. Mostowski, An undecidable arithmetical statement, *Fundamenta Mathematicae* **36** (1949), 143–164.

[2] A. Mostowski, On undecidable sentences in formalized mathematical systems, in: *American Mathematical Society Translations*, series 2, vol. 29, pp. 1–50, (English translation of Mostowski's paper of 1946).

[3] A. Mostowski, Sentences Undecidable in Formalized Arithmetic. An exposition of the theory of Kurt Gödel, in: *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1952.

[4] A. Mostowski, Undecidability and essential undecidability in arithmetic (with R.M. Robinson and A. Tarski), part II of *Undecidable Theories*, pages 39–74, by A. Tarski in collaboration with A. Mostowski and R.M. Robinson. *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1953.

[5] J. Paris and L. Harrington, A mathematical incompleteness in Peano arithmetic, in: *Handbook of Mathematical Logic*, J. Barwise, ed., North-Holland Publishing Co., Amsterdam, 1977, pp. 1133–1142.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Undecidability and Concatenation

Andrzej GRZEGORCZYK and Konrad ZDANOWSKI

Abstract. We consider the problem stated by Andrzej Grzegorczyk in "Undecidability without arithmetization" (Studia Logica 79 (2005)) whether certain weak theory of concatenation is essentially undecidable. We give a positive answer for this problem.

1. Introduction and Motivations

The present paper is devoted to proving the essential undecidability of the theory TC of concatenation of words. The paper may be treated as a continuation of the paper [2] written by the first author. We adopt some of the style, notation, abbreviations and results of [2].

We consider the theory of concatenation as presented in [2]. This is a weak theory of words over two letter alphabet $\Sigma = \{a, b\}$. The axioms are the following:

$\mathrm{TC1}$ $x\frown(y\frown z) = (x\frown y)\frown z$,
$\mathrm{TC2}$ $x\frown y = z\frown w \Rightarrow$
$\big((x = z \wedge y = w) \vee \exists u\big((x\frown u = z \wedge y = u\frown w) \vee (x = z\frown u \wedge u\frown y = w)\big)\big)$,
$\mathrm{TC3}$ $\neg(\alpha = x\frown y)$,
$\mathrm{TC4}$ $\neg(\beta = x\frown y)$,
$\mathrm{TC5}$ $\neg(\alpha = \beta)$.

Here α and β denote one letter words a and b respectively. Concatenation may be understood as a function defined on arbitrary texts and such that: *if x and y are some texts then $x\frown y$ is a text composed of the texts x and y in such a manner that the text y follows immediately the text x.*

$\mathrm{TC1}$ and $\mathrm{TC2}$ are due to A. Tarski. The other axioms say only that one letter words a and b are indivisible and different. Let us note that we adopt the convention from [2] and we do not have in our universe the empty word. This is inessential convention since it is easily seen that both theories of concatenation of words: with and without the empty word are interpretable one into the other one.[1]

---

[1]Indeed, if we have the empty word in the universe then the formula $\exists y(x\frown y \neq y \vee y\frown x \neq y)$ defines the submodel with the universe consisting of the nonempty words. And moreover, each model of TC can be obtained as such restriction.

It is a bit harder to define a model with the empty word in a model of TC. If $M = (U, \frown, a, b)$ is a model of TC then we can define the set of pairs $\{(a, a)\} \cup \{b\} \times U$ and treat the element (a, a) as the unique empty word and the elements (b, a), (b, b) as two one letter words. The concatenation operation can be defined from the concatenation in M. Again, each model of the theory or words with the empty word can be obtained in this way.

We call the axiom TC2 as the editor axiom. The intuition for that name is the following. Let us assume that two editors, A and B, divide the same book into two volumes: A into x and y and B into z and w. Then, one of the two things happens. Either they divided the book into the same volumes or there is a part of the book u such that their partitions differ exactly with the respect to the placement of u into the end of the first volume or into the beginning of the second one.

The above theory TC is proved undecidable in [2]. Here we prove that it is essentially undecidable.

Now, let us say a bit more about motivations. In [2] there are mentioned some arguments why the theory TC seems to be interesting. One can add an argument more. The theory TC may be understood as the theory of concatenation of texts which may be also infinite ordered strings having any arbitrary power of symbols and which are ordered in substrings of any arbitrary order and any power. Hence concatenation may be conceived as an operation on order-types, which have an additional peculiarity. The conception of order-types has been defined by George Cantor. Cantor defined also the operation which he called addition of order-types. The Cantor's addition satisfies the axioms TC1 and TC2. Cantor's order type is a class of abstraction obtained by means of arbitrary similarity (which is an isomorphic mapping of order). A text may be conceived as a type of an order R, but we also suppose that the considered similarity identifies the elements which have the peculiarities of a given set P.

Thus a formal definition of text as a class of abstraction may be the following: Let U be an universe and let P be a set (of letters). We consider pairs $\langle R, F\rangle$ such that R is an order of U and F is a mapping from U into P. Then, we can set $\mathrm{Texts}(\langle R, F\rangle)$ as the following class of abstraction:

$$\mathrm{Texts}\big(\langle R, F\rangle\big) = \big\{\langle R', F'\rangle : \exists s\colon U \longrightarrow U\big[\, s \text{ is 1–1 and “onto” and}$$
$$\forall x \in U\, \forall y \in U\, \big(xRy \equiv s(x)R's(y)\big) \wedge \forall x \in U\big(F(x) = F'(s(x))\big)\big]\big\}.$$

A simple example is when the set P contains only two elements and a mapping F is such that

$$\forall x \in U(F(x) = a \vee F(x) = b),$$

where a and b are different.

One can also imagine strings as geometrically continuous entities which cannot be understood as composed of separate atoms (like points). Thinking in TC we are not obliged to think about something similar to real finite texts of letters. These facts open also some new perspectives larger than the arithmetic of natural numbers.

2. Basic Notions

We extend our language with new definable relations. So, we write $x \subseteq y$ as a shorthand for

$$x = y \vee \exists u_1 \exists u_2 (u_1 \frown x = y \vee u_1 \frown x \frown u_2 = y \vee x \frown u_2 = y).$$

Of course, the intuitive meaning of $x \subseteq y$ is that the word x is a subword of y. We write $x \nsubseteq y$ for the negation of $x \subseteq y$.

For a given alphabet Γ, we write Γ^+ to denote the set of all finite nonempty words over Γ. With some abuse of notation we write also $u \subseteq w$, for $u, w \in \Gamma^+$, to state that u is a subword of w. It should be always clear from the context whether we mean a formula of our formal theory or the relation between words. These two things should not be mixed together. The formula, like $x \subseteq y$, is just a sequence of symbols in our language which can be interpreted in many various ways, the relation $\subseteq$, which is a subset of $\Gamma^+ \times \Gamma^+$, is a set theoretical object which refers to a property of words (from the standard model of TC). Therefore, the reader is asked for a constant attention to differentiate the situation when we talk about words $u, w \in \Gamma^+$ such that $u \subseteq w$ from that when we use $x \subseteq y$ as a formula of TC.

We use also $x \subseteq_e y$ as a shorthand for

$$x = y \vee \exists u\,(x \frown u = y).$$

The intuitive meaning of the above formula is that x is a word which is an initial subword of y. However, in contrast to $x \subseteq y$, we treat $x \subseteq_e y$ only as a definitional extension, not as a new atomic formula.

We will often skip the concatenation symbol between two words just as it is in case of the multiplication symbol in arithmetic. So xy should be read as $x \frown y$. We do this especially often when we concatenate one letter words. Thus, e.g. $\alpha\beta\alpha$ should be read as a term $\alpha \frown \beta \frown \alpha$ and the intuitive meaning of this term is the word aba. Let us note that we did not write parenthesis in $\alpha \frown \beta \frown \alpha$. Thus, this term can be read either as $(\alpha \frown \beta) \frown \alpha$ or $\alpha \frown (\beta \frown \alpha)$. However, these two terms are provably equal under $\mathrm{TC1}$. We will skip parenthesis in terms whenever possible, that is everywhere.

In our proof we use a theory of concatenation over 3 letters alphabet $\Sigma' = \{a, b, c\}$. So, the language of our theory has one binary function symbol $\frown$ for concatenation and constants α, β and γ for denoting one letter words a, b and c, respectively. Accordingly, we need to extend the set of axioms. We change $\mathrm{TC5}$ into

$$\mathrm{TC5}' \ \neg(\alpha = \beta \vee \alpha = \gamma \vee \beta = \gamma)$$

and we add

$$\mathrm{TC6} \ \neg(\gamma = x \frown y).$$

We call this theory TC'. This small extension simplifies our reasoning and does not affect the main result since this theory is easily interpretable in the theory TC of concatenation over two letters alphabet. Indeed, working with the two letter alphabet one can consider only words from the set $\{aba, abba, abbba\}^+$. Then, the formula $\varphi_U(x)$ of the form

$$\begin{aligned}&\beta\alpha\beta \not\subseteq x \wedge \alpha\alpha\alpha \not\subseteq x \wedge \beta\beta\beta\beta \not\subseteq x \wedge \\ &\quad \exists u \subseteq x(\alpha\beta \frown u = x) \wedge \exists u \subseteq x(u \frown \beta\alpha = x)\end{aligned}$$

defines within Σ^+ the universe of the words from $\{aba, abba, abbba\}^+$. Moreover, it is straightforward to show that we can prove in TC all axioms of TC' if we define α, β and γ as terms $\alpha\beta\alpha$, $\alpha\beta\beta\alpha$ and $\alpha\beta\beta\beta\alpha$ and restrict quantification to the set defined by $\varphi_U(x)$.

We say that a formula φ is Δ_0 if all quantifiers in φ are of the form $Qx \subseteq y$, where $Q \in \{\exists, \forall\}$. This means that all quantifiers are relativized to subwords of other words.

This relativization is a special case of relativizations of quantifications to a set definable by a formula $\varphi(x)$. In such situations we use notation

$$\big(\forall x : \varphi(x)\big)\psi(x) \text{ which is } \forall x\big(\varphi(x) \Rightarrow \psi(x)\big)$$

and

$$\big(\exists x : \varphi(x)\big)\,\psi(x) \text{ which is } \exists x\big(\varphi(x) \wedge \psi(x)\big).$$

Let us define Δ_0 formulae which will be used most often as formulae to which we relativize quantifiers:

1. $\alpha\beta(x) = (\gamma \not\subseteq x)$ which says that x is a word which does not contain c;
2. $\alpha(x) = (\beta \not\subseteq x \wedge \gamma \not\subseteq x)$ which says that x is a word which does not contain b and c;
3. $\beta(x) = (\alpha \not\subseteq x \wedge \gamma \not\subseteq x)$ which says that x is a word which does not contain a and c.

All words in Σ'^{+} have their names in TC'. We take a convention from arithmetic and for $u \in \Sigma'^{+}$ by $\underline{u}$ we denote the term whose denotation is u. E.g. $\underline{abca} = \alpha\beta\gamma\alpha$.

We use also an ω-type ordering on the set of words from $\{a, b\}^{+}$.

Definition 1. For $u, w \in \{a, b\}^{+}$, $u < w$ if either the length of u is less than the length of w or u and w have equal length and u is smaller than w in the usual lexicographic ordering. More formally, for $u, w \in \{a, b\}^{+}$, $u = u_n \ldots u_0$ and $w = w_k \ldots w_0$, where $u_i, w_j \in \{a, b\}$, for $i \leqslant n$, $j \leqslant k$, it holds that $u < w$ if

$$n < k \vee \big[n = k \wedge \exists i \leqslant n(\forall j \leqslant n(i < j \Rightarrow u_j = w_j) \wedge u_i = a \wedge w_i = b)\big].$$

An initial segment of this ordering is the following: a, b, aa, ab, ba, bb, aaa, aab, aba. We fix this enumeration as $\{u_i\}_{i\in\omega}$, so e.g. $u_0 = a$, $u_1 = b$, $u_2 = aa$. We will show later that this ordering is definable by a formula $\varphi_{<}(x, y)$ of TC' and that TC' proves useful properties of $\varphi_{<}(x, y)$. This formula uses all three symbols: α, β and γ. We could write this definition already in TC without the use of the third letter (see e.g. [4] for such a definition). However, this would make unnecessary complications.

3. The Essential Undecidability of TC

Our line of the proof of the essential undecidability of TC is classical. Firstly, we take TC', an extension of TC. By the remark made in Section 2 it is enough to show the essential undecidability of TC'. Then, we show the Σ_1-completeness of TC'. Namely, we show for any Σ_1-formula φ which is true in the standard model $\{a, b, c\}^{+}$ that φ is also provable in TC'. Next, we provide a Σ_1 formula $\varphi_{<}(x, y)$ which defines the ω-type ordering $<$ on the words from $\{a, b\}^{+}$ in the standard model $\{a, b, c\}^{+}$. Then, we show that for any word $u \in \{a, b\}^{+}$,

$$\mathrm{TC}' \vdash \big(\forall x : \alpha\beta(x)\big)\Big(\varphi_{<}(x, \underline{u}) \iff \bigvee_{w<u} x = \underline{w}\Big).$$

Moreover, there is a formula, which we call $\mathrm{HasWit}(x)$, such that for all $u \in \{a, b\}^{+}$,

$$\mathrm{TC}' \vdash \mathrm{HasWit}(\underline{u})$$

and

$$\mathrm{TC}' \vdash \forall x\big(\mathrm{HasWit}(x) \Rightarrow \big(\varphi_<(x,\underline{u}) \vee x = \underline{u} \vee \varphi_<(\underline{u},x)\big)\big).$$

Having the above facts it is easy to show, using some coding of computations as finite words, that TC' is essentially undecidable. We show this final argument at the end of the paper.

3.1. Σ_1*-Completeness of* TC'

Each element $u \in \{a,b,c\}^+$ has its name $\underline{u}$ in TC'. Thus, to show that TC' proves all Σ_1-sentences true in $\{a,b\}^+$ it is enough to show this only for Δ_0-sentences. We do this in a series of lemmas. This result itself was proven in [2]. So we do not include the full proof here but we only write the basic steps of the proof. We decided to do this because our present formalization differs from that of [2].

Lemma 2. *For each* $u, w \in \{a,b,c\}^+$,

$$\mathrm{TC}' \vdash \neg(\underline{u} = \underline{w}) \textit{ if and only if } u \neq w.$$

Proof. The proof is by induction on (the length of) u. For the base case one should use $\mathrm{TC5}'$. For the induction step one should also use associativity and the editor axiom. We skip the proof here. □

Now, we show that TC' can handle $x \subseteq y$ relation.

Lemma 3. *For all* $u \in \{a,b,c\}^+$,

$$\mathrm{TC}' \vdash \forall x\Big(x \subseteq \underline{u} \iff \bigvee_{w \subseteq u} x = \underline{w}\Big).$$

Proof. Let us observe that the implication from the right to the left is obvious. Thus, we prove only the converse.

The proof is by induction on the word u. For $u = a$ the thesis follows from the fact that $\forall x \neg(\alpha = x \frown y)$. Similar arguments are used for $u = b$ or $u = c$.

Now, let us assume that for some $u \in \{a,b,c\}^+$

$$\mathrm{TC}' \vdash \forall x\Big(x \subseteq \underline{u} \iff \bigvee_{w \subseteq u} x = \underline{w}\Big).$$

and consider $u \frown a$. Now, we work in TC'. Let us assume that $x \subseteq \underline{u} \frown \alpha$. If $x = \underline{u} \frown \alpha$ then there is nothing to prove. So, let us consider three remaining cases:

1. $\exists z\, x \frown z = \underline{u} \frown \alpha$,
2. $\exists z\, z \frown x = \underline{u} \frown \alpha$,
3. $\exists z \exists w\, z \frown x \frown w = \underline{u} \frown \alpha$.

Let us consider the first case. Then, by the editor axiom, either $x = \underline{u}$ and $z = \alpha$, and we are done, or there exists w such that

$$x \frown w = \underline{u} \text{ and } z = w \frown \alpha$$

or

$$x = \underline{u} \frown w \text{ and } w \frown z = \alpha.$$

The latter case is impossible by TC3. The only remaining case is when $x \frown w = \underline{u}$ and $z = w \frown \alpha$. In this case $x \subseteq \underline{u}$ and we may use our inductive assumption.

In the second case, $z \frown x = \underline{u} \frown \alpha$, if $x = \alpha$ then we are done. If $x \neq \alpha$, then there is w such that

$$z \frown w = \underline{u} \text{ and } x = w \frown \alpha.$$

Because $z \frown w = \underline{u}$, we have that $w \subseteq \underline{u}$. Thus, by the inductive assumption,

$$\bigvee_{r \subseteq u} w = \underline{r}.$$

It follows that

$$\bigvee_{r \subseteq u} x = \underline{r} \frown \alpha.$$

In the third case either $w = \alpha$ or there is r such that

$$z \frown x \frown r = \underline{u} \wedge w = r \frown \alpha.$$

In both cases $x \subseteq \underline{u}$ and we can use our inductive assumption. □

As a corollary we obtain the following.

Corollary 4. *For each* $u, w \in \{a, b, c\}^+$*,*

$$\mathrm{TC}' \vdash (\underline{u} \subseteq \underline{w}) \textit{ if and only if } u \subseteq w$$

and

$$\mathrm{TC}' \vdash \neg(\underline{u} \subseteq \underline{w}) \textit{ if and only if } u \not\subseteq w$$

Proof. Let $u, w \in \{a, b, c\}^+$. By Lemma 3 we can reduce in TC' all formulae $\underline{u} \subseteq \underline{w}$ to equations between words. Then, by Lemma 2, we can prove or disprove all equations between words in $\{a, b, c\}^+$. □

Now, we can easily prove the theorem on Σ_1-completeness (originally proven in [2]).

Theorem 5 ([2]). TC' *proves all true* (*in* $\{a, b, c\}^+$) Σ_1*-sentences.*

Proof. It is enough to show the theorem only for Δ_0-sentences. This follows from the fact that each element of $\{a, b, c\}^+$ has its name in TC'.

The proof is by induction on the complexity of a formula $\varphi(\underline{r_1}, \ldots, \underline{r_k})$, where φ is Δ_0 and $r_1, \ldots, r_k \in \{a, b, c\}^+$. For basic cases, $\underline{u} = \underline{w}$, $\neg(\underline{u} = \underline{w})$, $\underline{u} \subseteq \underline{w}$ and $\neg(\underline{u} \subseteq \underline{w})$ one should use Lemma 2 and Corollary 4.

We consider the hardest case when φ is of the form $\exists x \subseteq \underline{u}\, \psi(x, \underline{r_1}, \ldots, \underline{r_k})$ for $u, r_1, \ldots, r_k \in \{a, b, c\}^+$. For this case we can write φ in the following equivalent form using Lemma 3:

$$\bigvee_{w \subseteq u} \psi(\underline{w}, \underline{r_1}, \ldots, \underline{r_k}).$$

Now, it is enough to observe that, for each $w \subseteq u$, our inductive assumption holds for $\psi(\underline{w}, \underline{u_1}, \ldots, \underline{u_k})$. $\square$

3.2. The Construction of the ω-Type Ordering

Now, we present the main new ingredient of this work. We show how to define the ω-type ordering on $\{a, b\}^+$ in such a way that its properties will be provable in TC'. The inspiration for the formula $\varphi_<(x, y)$ which defines the ordering $<$ is from the article by Quine, [4]. However, the constructions of [4] are of semantical character and work in the standard model. We put the main stress on the provability of some properties of $\varphi_<$ in a weak theory TC'.

Definition 6. Let $\{u_i\}_{i \in \omega}$ be the enumeration of words from $\{a, b\}^+$ in the ordering $<$. The witness for a word $u = u_i$ is the word of the form

$$c \frown u_0 \frown c \frown u_1 \frown c \frown u_2 \frown \ldots \frown c \frown u_{i-1} \frown c \frown u_i \frown c.$$

We denote the witness for u_i by w_i.

We also want to provide a recursive definition of the witness w_i for a word u_i. The witness for $u_0 = a$ is $w_0 = cac$ and the witness for u_{i+1} is a word $w_i \frown u_{i+1} \frown c$, where w_i is the witness for u_i.

Now, we define the formula $\mathrm{Next}(z, x, y, w)$ which states that x and y are successive parts of w such that $x \in \{a\}^+$, $y \in \{b\}^+$, $cxc \subseteq w$ and $cyc \subseteq w$ and such that there is no word in w of the form csc, for $s \in \{a\}^+ \cup \{b\}^+$, between x and y. Moreover, to fix a position of x in w we use an additional word z such that $z \frown x \subseteq_e w$. Thus, x is a subword of w which starts just after z.

Definition 7. Let the formula $\mathrm{Next}(z, x, y, w)$ be the following

$$\begin{aligned}
&\alpha(x) \wedge \beta(y) \wedge zx \subseteq_e w \wedge \big(z = \gamma \vee \exists t \subseteq z(t\gamma = z)\big) \wedge zx\gamma \subseteq_e w \wedge \\
&\quad \big\{zx\gamma y\gamma \subseteq_e w \vee \exists t \subseteq w\big[zx\gamma t\gamma y\gamma \subseteq_e w \wedge \\
&\quad \big(\forall s : \alpha\beta(s)\big)\big(\gamma s\gamma \subseteq \gamma t\gamma \Rightarrow \big(\neg\alpha(s) \wedge \neg\beta(s)\big)\big)\big]\big\}.
\end{aligned}$$

Now, we define a formula $\mathrm{Wit}(x, y)$ which defines a relation "y is the witness for x".

Definition 8. The formula $\mathrm{Wit}(x, y)$ is the following formula

$$(x = \alpha \wedge y = \gamma\alpha\gamma) \vee (x = \beta \wedge y = \gamma\alpha\gamma\beta\gamma) \vee$$
$$\{\alpha\beta(x) \wedge x \neq \alpha \wedge x \neq \beta \wedge \gamma\alpha\gamma\beta\gamma \subseteq_e y \wedge \exists z \subseteq y(z\gamma x\gamma = y \wedge \neg(x \subseteq z)) \wedge$$
$$\forall z \subseteq y(\forall z_1{:}\alpha\beta(z_1))(\forall z_2{:}\alpha\beta(z_2))(z\gamma z_1\gamma z_2\gamma \subseteq_e y \Rightarrow \mathrm{Succ}(z\gamma, z_1, z_2, y)) \wedge$$
$$\forall z \subseteq y(\forall z_1{:}\alpha\beta(z_1))((z\gamma z_1\gamma \subseteq_e y \wedge z_1 \neq x) \Rightarrow$$
$$(\exists z_2{:}\alpha\beta(z_2))(z\gamma z_1\gamma z_2\gamma \subseteq_e y \wedge \mathrm{Succ}(z\gamma, z_1, z_2, y)))\},$$

where $\mathrm{Succ}(z, z_1, z_2, y)$ states (with the help of y) that z_2 is a successor of z_1 in the ordering $<$. It assumes that $zz_1\gamma z_2 \subseteq_e y$. It has the form

$$(z_1 = \alpha \wedge z_2 = \beta) \vee$$
$$\exists u \subseteq z_1(u\alpha = z_1 \wedge u\beta = z_2) \vee$$
$$(\beta(z_1) \wedge \exists s \subseteq_e z\, \exists t \subseteq z(\mathrm{Next}(s, t, z_1, y) \wedge z_2 = \alpha t)) \vee$$
$$\exists u \subseteq z_1(\beta(u) \wedge \alpha u = z_1 \wedge$$
$$\exists s \subseteq_e z\, \exists t \subseteq w(\alpha(t) \wedge \mathrm{Next}(s, t, u, w) \wedge z_2 = \beta t))) \vee$$
$$\exists u \subseteq z_1(\beta(u) \wedge \exists r \subseteq z_1(r\alpha u = z_1 \wedge$$
$$\exists s \subseteq_e z\, \exists t \subseteq z(\alpha(t) \wedge \mathrm{Next}(s, t, u, w) \wedge z_2 = r\beta t))).$$

Before we comment on formulae Wit and Succ we present a simple lemma which will be useful in our analysis.

Lemma 9. *Let* $u, w \in \{a, b, c\}^+$ *such that* $(\{a, b, c\}^+, \frown, a, b, c) \models \mathrm{Wit}[u, w]$. *Then, for all* $s \in \{a, b, c\}^+$ *and for all* $i, j > 0$ *such that* $(\{a, b, c\}^+, \frown, a, b, c) \models \mathrm{Next}[s, a^i, b^j, w]$ *it holds that* $i = j$.

Proof. Let us assume that $(\{a, b, c\}^+, \frown, a, b, c) \models \mathrm{Wit}[u, w]$. We prove by induction on the length of s that for all $i, j > 0$, if $(\{a, b, c\}^+, \frown, a, b, c) \models \mathrm{Next}[s, a^i, b^j, w]$ then $i = j$.

If the length of s is 1 then $s = c$ and, by the first line of Wit, $i = j = 1$.

Now, let $s \subseteq_e w$ be such that $(\{a, b, c\}^+, \frown, a, b, c) \models \mathrm{Next}[s, a^i, b^j, w]$ and let us assume that the lemma holds for all $s' \subseteq_e w$ such that the length of s' is less than s.

There are words $x_1, \ldots, x_n \in \{a, b\}^+ - (\{a\}^+ \cup \{b\}^+)$ such that

$$sa^i cx_1c \ldots cx_ncb^jc \subseteq_e w.$$

Let $x_0 = a^i$ and $x_{n+1} = b^j$. By the third line of the formula Wit, for each $0 \leqslant k \leqslant n$,

$$\big(\{a, b, c\}^+, \frown, a, b, c\big) \models \mathrm{Succ}[sx_0c \ldots cx_{k-1}c, x_k, x_{k+1}, w]. \tag{1}$$

Then, by our inductive assumption, for each $s' \subseteq_e s$ such that $s' \neq s$, if

$$\big(\{a, b, c\}^+, \frown, a, b, c\big) \models \mathrm{Next}[s', a^r, b^s, w]$$

then $r = s$. It follows that the only case in which the lengths of x_k and x_{k+1} in (1) may be different is described by the third line of the formula Succ. But this line never holds because for each $0 \leqslant k \leqslant n$, $x_k \notin \{b\}^+$. Thus, the lengths of $x_0, x_1, \ldots, x_{n+1}$ are equal and, in consequence, $i = j$. $\square$

Now, let us comment a bit on $\mathrm{Succ}(z, z_1, z_2, y)$. In the first line we just consider the case which is a kind of the starting point. In the second line of $\mathrm{Succ}(z, z_1, z_2, y)$ we consider the simple situation when z_1 is a word which ends with the letter a. Then, z_2 has to be the same word but with the last a changed to b. Then, we consider the situation when z_1 consists only of letters b and we find, using the formula $\mathrm{Next}(s, t, z_1, y)$, the word t which has the same length as z_1 and consists only of a's (here we need Lemma 9). In this situation z_2 is just t extended by one letter a. In the fourth and the fifth line we consider the situation when z_1 is of the form ab^n that is when it is the letter a followed by some number of b's. Then, z_2 has the form ba^n. The last lines are just a generalization of this case to the word z_1 of the form rab^n, where $r \in \{a, b\}^+$. Let us remark that we have to consider these cases separately because we have no empty word in our universe.

The formula $\mathrm{Wit}(x, y)$ is Δ_0 thus TC' proves all true instances of $\mathrm{Wit}(\underline{u}, \underline{w})$, for $u, w \in \{a, b\}^+$. However, we need the following stronger statement.

Lemma 10. *For all* $u \in \{a, b\}^+$,

$$\mathrm{TC}' \vdash \exists^{=1} w \mathrm{Wit}(\underline{u}, w).$$

Proof. Let $u = u_i$ in $<$-enumeration and let w_i be a witness for u. Then, since $\mathrm{Wit}(\underline{u_i}, \underline{w_i})$ is a true Δ_0-formula we have

$$\mathrm{TC}' \vdash \mathrm{Wit}(\underline{u_i}, \underline{w_i}).$$

Now, let t be a new constant. Then, by induction on $k < i$, we can prove the following statement

$$\mathrm{TC} + \mathrm{Wit}(\underline{u_i}, t) \vdash \exists r(\underline{w_k} \frown r = t).$$

Then, by the last line of the formula $\mathrm{Wit}(x, y)$ we have

$$\underline{w_i} = t \vee \exists r(\underline{w_i} \frown r = t).$$

However the second disjunct, $\exists r(\underline{w_i} \frown r = t)$, is impossible by the following argument. In the second line of the formula $\mathrm{Wit}(x, y)$ we state that y ends with $\gamma x \gamma$ and that there is no x before this occurrence. Thus, if $\mathrm{Wit}(\underline{u_i}, t)$ then

$$\exists z\big(z\gamma\underline{u_i}\gamma = t \wedge \neg(\underline{u_i} \subseteq z)\big).$$

But, we know that a witness for u_i is an initial segment of t and that this witness ends with a word $\gamma\underline{u_i}\gamma$. So, the only possibility is that $t = \underline{w_i}$. □

Definition 11. We write a formula $\varphi_<(x, y)$ which defines the $<$ ordering:

$$\exists z \exists w\big(\mathrm{Wit}(x, z) \wedge \mathrm{Wit}(y, w) \wedge \exists r(z \frown r = w)\big).$$

We show that each finite fragment of the ordering $<$ which is below some $u \in \{a, b\}^+$ is properly described in TC' by the formula $\varphi_<(x, y)$.

Lemma 12. *For all* $u \in \{a, b\}^+$,

$$\mathrm{TC}' \vdash \big(\forall x : \alpha\beta(x)\big)\Big(\varphi_<(x, \underline{u}) \iff \bigvee_{r<u} x = \underline{r}\Big).$$

Proof. The direction from the right to the left is obvious. For the other direction let us assume that $\varphi_<(x, \underline{u})$. But then there is a witness z for x and there is r such that $z \frown r = \underline{w}$, where w is the unique witness for u. Thus, x has to be a subword of w, since z ends with $\gamma x \gamma$. But subwords of $\underline{w}$ from $\{a, b\}^+$ which occur in $\underline{w}$ before $\gamma \underline{u} \gamma$ are words less than $\underline{u}$. Thus, $\bigvee_{r<u} x = \underline{r}$. □

The important fact in the essential undecidability proof of a given arithmetical theory T is often the following:

$$\text{for each } n \in \omega, T \vdash \forall x(x < \underline{n} \vee x = n \vee \underline{n} < x),$$

where $\underline{n}$ is the term for the number n. Unfortunately, we cannot prove that for any $u \in \{a, b\}^+$, $\mathrm{TC}' \vdash \forall x(\varphi_<(x, \underline{u}) \vee x = \underline{u} \vee \varphi_<(\underline{u}, x))$. Indeed, in some nonstandard models M for TC' there are elements d for which there is no witness e such that $M \models \mathrm{Wit}(d, e)$. Such models can be constructed from the models for the arithmetic $I\Delta_0 + \neg \exp$.[2] Since this line of proof cannot be carried out we take the second option and we define within our universe the set of words for which there are witnesses.

Definition 13. Let $\mathrm{HasWit}(x)$ be the formula $\exists y \, \mathrm{Wit}(x, y)$.

By Lemma 10, for each $u \in \{a, b\}^+$, $\mathrm{TC}' \vdash \mathrm{HasWit}(\underline{u})$. Moreover, we can prove that words x which satisfy $\mathrm{HasWit}(x)$ have good properties in our ordering.

Theorem 14. *For each $u \in \{a, b\}^+$,*

$$\mathrm{TC}' \vdash \forall x\big(\mathrm{HasWit}(x) \Rightarrow \big(\varphi_<(x, \underline{u}) \vee x = \underline{u} \vee \varphi_<(\underline{u}, x)\big)\big).$$

Proof. Let $u \in \{a, b\}^+$ be the i-th word in our ω-type enumeration, that is $u = u_i$. Then, for $k \leqslant i$, the word $w_k \in \{a, b, c\}^+$ is, by Lemma 10, the unique word such that

$$\mathrm{TC}' \vdash \mathrm{Wit}(\underline{u_k}, \underline{w_k}).$$

Now, let us assume that $\mathrm{HasWit}(x)$ and that w is such that $\mathrm{Wit}(x, w)$. Now, to prove the theorem we need to show in TC' that

$$\varphi_<(x, \underline{u}) \vee x = \underline{u} \vee \varphi_<(\underline{u}, x).$$

So, we assume that $\neg x = \underline{u}$ and $\neg \varphi_<(x, \underline{u})$. By Lemma 12 this is equivalent to

$$\bigwedge_{k \leqslant i} \neg x = \underline{u_k}.$$

To finish the proof we need to show that $\varphi_<(\underline{u}, x)$ what is equivalent to

$$\exists z(\underline{w_i} \frown z = w).$$

In order to do this, we prove by induction on k, that for all $k \leqslant i$,

$$\exists z(\underline{w_k} \frown z = w). \tag{2}$$

[2] Indeed, we can treat the elements of a given model $M \models I\Delta_0$ as words over some fixed k-ary alphabet, e.g. 0 could be the empty word, 1 the word a, 2 the word b, 3 is the word aa, etc. Then, the concatenation operation is definable by a Δ_0 formula and has the rate of growth of multiplication (see e.g. [3]). However, it could be proven, in $I\Delta_0$, that if a number w interprets a witness for a number u, then it has to be exponentially bigger than u. Thus, if our model does not satisfy the totality of exp then not all its elements will have witnesses.

Of course we have no induction axioms among axioms of TC'. However, the induction we need involves only finitely many steps (to be exact i steps). Thus, we can carry out this reasoning within TC'.

Firstly, we rule out some easy cases. If $u = \underline{u_0} = \alpha$ and $x \neq \underline{u}$ then (2) follows from the first two lines of the formula Wit. Indeed, in this case either $w = \gamma\alpha\gamma\beta\gamma$ or $\gamma\alpha\gamma\beta\gamma \subseteq_e w$. In both cases $\exists z(\gamma\alpha\gamma\frown z = w)$. Thus, we assume that $x \neq \alpha$.

For $k = 0$ and $k = 1$ equation (2) follows just from the definition of the formula $\mathrm{Wit}(x, y)$ (the first two lines of this formula) and from the fact that $x \neq \alpha$ and $x \neq \beta$.

Now, let us assume that for some $k < i$,

$$\exists z(\underline{w_k}\frown z = w).$$

Then, by the last line of a formula $\mathrm{Wit}(x, y)$, for $z_1 = \underline{u_k}$ there is z_2 such that $\mathrm{Succ}(\underline{u_k}, z_2, w)$ and $\underline{w_k}\frown z_2\frown\gamma \subseteq w$. But, z_2 is determined uniquely by the initial segment of w which is just w_k. So, $z_2 = \underline{u_{k+1}}$. Moreover, it is not the case that $\underline{w_k}\frown z_2\frown\gamma = w$ since otherwise $x = \underline{u_{k+1}}$, the last word from $\{a, b\}^+$ in w. Thus,

$$\exists z(\underline{w_k}\frown\underline{u_{k+1}}\frown\gamma\frown z = w).$$

But $\underline{w_k}\frown\underline{u_{k+1}}\frown\gamma = \underline{w_{k+1}}$ and we obtain

$$\exists z(\underline{w_{k+1}}\frown z = w)$$

what is just (2) for $k + 1$. □

3.3. The Main Result

In this subsection we prove our main theorem. Our proof is in the spirit of many proofs of the essential undecidability of some theory T. We assume that the reader has some knowledge of basic concepts from the recursion theory as presented in [5] or [1].

Before we state the main theorem we discuss how one can code computations in the standard model for finite words.

Lemma 15. *There is a method of coding Turing machines, their inputs and computations as finite words over an alphabet $\{a, b\}$ such that there are Δ_0 formulae $\mathrm{Accept}(x, y, z)$ and $\mathrm{Reject}(x, y, z)$ with the following property: for all words $M, w, C \in \{a, b, c\}^+$,*

$$\big(\{a, b\}^+, \frown, a, b\big) \models \mathrm{Accept}(\underline{M}, \underline{w}, \underline{C}) \iff$$

C is an accepting computation of a machine M on the input w

and

$$\big(\{a, b\}^+, \frown, a, b\big) \models \mathrm{Reject}(\underline{M}, \underline{w}, \underline{C}) \iff$$

C is a rejecting computation of M on w.

Proof. We give only a sketch of a proof. The reader interested in an example of such a coding may consult e.g. the appendix of [6]. Moreover, let us observe that in the formulation of the lemma we referred to *words* coding computations or Turing machines as to computations and Turing machines. This identification is assumed during the rest of this section. For simplicity, we consider only deterministic Turing machines using only two letter alphabet $\{a, b\}$.

We present the coding for words with more letters than $\{a, b\}$. This is not essential since we can interpret in TC the words over arbitrary finite alphabet and $(\{a, b\}^+, \frown, a, b)$ is, of course, a model of TC.

We can code a Turing machine by a concatenation of words of the form $[[q^n x]DCq^i]$, where [,] and q are letters from the alphabet, $x \in \{a, b\}$, $D \in \{L, R\}$ and $C \in \{a, b\}$. Then, the word $[[q^n x]DCq^i]$ indicates that being in the state n and reading the symbol x the machine writes C on the tape, moves its head left (L) or right (R) and enters the state i. It is straightforward to observe that we can write a Δ_0 formula which checks that a given word M is a concatenation of words as above and that no subword of the form $[q^n x]$ repeats twice (the machine is deterministic).

Next, we can code one state of the computation of a machine M as $\#u_1 q^i u_2\#$, where $\#$ is a letter from the alphabet and $u_1, u_2 \in \{a, b\}^+$ describe the content of the working tape and the string q^i indicates the position of the head and the state of the machine. Again, it is easy to write a Δ_0 formula which defines a set of states. Then, we can code a computation C of M as the word

$$MC_1 \dots C_m,$$

where M is a code of the machine and C_i is the i-th configuration during the computation C. Let us make a convention that the initial state is always 1, the accepting state is 2 and the rejecting state is 3 and that before M stops it writes a's everywhere on the tape and goes to the beginning of the tape. Thus, the initial configuration on the input w is just a word qw, and the accepting and rejecting configurations are qqa^m and $qqqa^k$, for some $m, k \in \omega$, respectively. These configurations are easily characterized by Δ_0 formulae.

Now, the only difficulty is to write a Δ_0 formula which states that each configuration C_{r+1} is the next configuration after C_r, for $r < m$. Fortunately, M makes only local changes on the tape. Thus the formula which expresses this should simply find a word of the form $[[q^n x]DCq^i]$ in M such that the word $q^n x$ appears in C_r. Then, it should check that C_{r+1} can be constructed from C_r by applying changes according to $[[q^n x]DCq^i]$. The disjunction over all subwords of M of the form $[[q^n x]DCq^i]$ expresses that C_{r+1} is the next configuration after C_r. This disjunction is easily expressible by an existential quantifiers restricted to subwords of M. $\square$

Let us stress here that in the last lemma we require only that Accept and Reject are Δ_0 formulae which define the suitable concepts in the standard model for finite words. Nevertheless, by Theorem 5, we can state the following: for all words $M, w, C \in \{a, b\}^+$,

$$\mathrm{TC}' \vdash \mathrm{Accept}(\underline{M}, \underline{w}, \underline{C}) \iff$$
$$C \text{ is an accepting computation of a machine } M \text{ on the input } w$$

and

$$\mathrm{TC}' \vdash \mathrm{Reject}(\underline{M}, \underline{w}, \underline{C}) \iff C \text{ is a rejecting computation of } M \text{ on } w.$$

This fact will be used in the proof of Theorem 16.

Let us remark here that although TC' is over the three letters alphabet $\{a, b, c\}$ it proves exactly the same Δ_0 formulae with parameters from $\{a, b\}^+$ as TC. This is so because bounded quantification restricted to words from $\{a, b\}^+$ does not take us from $\{a, b\}^+$. So, in the above equations we may freely use TC or TC$'$.

Now, we formulate our main theorem.

Theorem 16. TC *is essentially undecidable.*

Proof. We present the proof for the theory TC' which is interpretable in TC.

To show the essential undecidability of TC' we follow the usual reasoning. We take M to be a Turing machine such that the sets

$$A = \{u \in \{a, b\}^+ : M \text{ accepts } u\}$$

and

$$B = \{u \in \{a, b\}^+ : M \text{ rejects } u\}$$

are recursively inseparable. We define two formulae, $\gamma_A(x) :=$

$$\exists y\big(\mathrm{HasWit}(y) \wedge \mathrm{Accept}(\underline{M}, x, y) \wedge \forall z\big(\varphi_<(z, y) \Rightarrow \neg\mathrm{Reject}(\underline{M}, x, z)\big)\big)$$

and $\gamma_B(x) :=$

$$\exists y\big(\mathrm{HasWit}(y) \wedge \mathrm{Reject}(\underline{M}, x, y) \wedge \forall z\big(\varphi_<(z, y) \Rightarrow \neg\mathrm{Accept}(\underline{M}, x, z)\big)\big),$$

where Accept and Reject are formulae from Lemma 15. Then, for each $u \in \{a, b\}^+$,

$$\text{if } u \in A \text{ then } \mathrm{TC}' \vdash \gamma_A(\underline{u}) \tag{3}$$

and

$$\text{if } u \in B \text{ then } \mathrm{TC}' \vdash \gamma_B(\underline{u}). \tag{4}$$

We show (3). If $u \in A$ then there is a computation of M which accepts u. Let $C \in \{a, b\}^+$ be the word coding this computation. Then,

$$\mathrm{TC}' \vdash \mathrm{HasWit}(\underline{C}) \wedge \mathrm{Accept}(\underline{M}, \underline{u}, \underline{C}).$$

Moreover, we have that

$$\mathrm{TC}' \vdash \forall x\Big(\varphi_<(x, \underline{C}) \iff \bigvee_{w<C} x = \underline{w}\Big).$$

But for all such $w < C$, w is not a rejecting computation of M on the input u. Thus,

$$\mathrm{TC}' \vdash \forall z\big(\varphi_<(z, \underline{C}) \Rightarrow \neg\mathrm{Reject}(\underline{M}, \underline{u}, z)\big).$$

It follows that $\mathrm{TC}' \vdash \gamma_A(\underline{u})$.

In the same way one can prove (4).

Now, we know that γ_A and γ_B represents sets A and B in TC'. However, essential features of these formulae are the following equations

$$\text{if } u \in A, \text{ then } \mathrm{TC}' \vdash \neg\gamma_B(\underline{u}) \tag{5}$$

and

$$\text{if } u \in B, \text{ then } \mathrm{TC}' \vdash \neg\gamma_A(\underline{u}). \tag{6}$$

To show (5) let us assume that $u \in A$ and let C be an accepting computation of M on the input u. We want to show that

$$\mathrm{TC}' \vdash \forall y\big(\big(\mathrm{HasWit}(y) \wedge \mathrm{Reject}(\underline{M}, \underline{u}, y)\big) \Rightarrow \exists z\big(\varphi_<(z, y) \wedge \mathrm{Accept}(\underline{M}, \underline{u}, z)\big)\big).$$

Now, we work in TC'. Let y be such that $\mathrm{HasWit}(y)$ and $\mathrm{Reject}(\underline{M}, \underline{u}, y)$. It suffices to show that

$$\varphi_<(\underline{C}, y) \wedge \mathrm{Accept}(\underline{M}, \underline{u}, \underline{C}).$$

By the definition of C we have $\mathrm{Accept}(\underline{M}, \underline{u}, \underline{C})$. So, it is enough to show that $\varphi_<(\underline{C}, y)$. Since C is a standard word we have, by Theorem 14,

$$\varphi_<(\underline{C}, y) \vee \underline{C} = y \vee \varphi_<(y, \underline{C}).$$

Then, by Lemma 12,

$$\mathrm{TC}' \vdash \forall x\Big(\varphi_<(x, \underline{C}) \iff \bigvee_{r<C} x = \underline{r}\Big).$$

None of the words $r \leqslant C$ is a rejecting computation of M on the input u. Moreover, because Reject is a Δ_0 formula, this fact is detected by TC': for all $r \leqslant C$,

$$\mathrm{TC}' \vdash \neg\mathrm{Reject}(\underline{M}, \underline{u}, \underline{r}).$$

Since we have $\mathrm{Reject}(\underline{M}, \underline{u}, y)$, it follows that y cannot be a word which is less than or equal to C. So, we conclude that $\varphi_<(\underline{C}, y)$. The proof of (6) can be carried out in the very same manner.

Now, we show that there is no consistent T which extends TC' and has a decidable set of consequences. Let us assume, for the sake of contradiction, that we have such a T. Then, let us consider the set

$$S = \big\{u \in \{a, b\}^+ : T \vdash \gamma_A(\underline{u})\big\}.$$

Since T is decidable, S is decidable, too. By (3), $A \subseteq S$ and, by (6) and the fact that T is consistent, $B \cap S = \emptyset$. Thus, we have separated A and B by a recursive set what is impossible by the definition of A and B. We conclude that there is no decidable and consistent extension of TC'. □

4. Final Remarks

We have proven that TC is essentially undecidable. Let us observe that if we drop one of the axioms from TC2 to TC5 then we obtain a theory which has a decidable extension. Indeed, if we drop TC5 then we can interpret all axioms in the model for arithmetic of addition without zero $(\omega - \{0\}, +, 1, 1)$. By Presburger theorem this model has a decidable theory. Similarly, if we drop TC4, then this theory is satisfied in the model $(\omega - \{0\}, +, 1, 2)$. Finally, if we drop the editor axiom then such a theory is satisfied in a finite model $(\{a, b, c\}, f, a, b)$, where f is a constant binary function which maps everything to c. We conjecture that also TC without the first axiom has a decidable extension, so TC is an example of a minimal essentially undecidable theory.

We do not know whether our style of defining the ω-type ordering on words poses good properties provably in TC'. Indeed, we showed rather some basic facts which are needed in the proof of Theorem 16. In particular, we would like to ask whether there is a formula $\psi_{\leqslant}(x,y)$ such that $\psi_{\leqslant}(x,y)$ defines ω-type ordering on $\{a,b\}^+$ in the standard model for concatenation and such that TC' would prove that the set of words for which $\psi_{\leqslant}$ defines the linear ordering is closed on concatenation. Namely, let $\gamma(x)$ be the following formula

$$\forall z_1\forall z_2\forall z_3\Big\{\bigwedge_{i\leqslant 3}\psi_{\leqslant}(z_i,x)\Rightarrow\big[\big(\psi_{\leqslant}(z_1,z_2)\wedge\psi_{\leqslant}(z_2,z_3)\big)\\ \Rightarrow\psi_{\leqslant}(z_1,z_3)\big)\wedge\big(\psi_{\leqslant}(z_1,z_2)\vee\psi_{\leqslant}(z_2,z_1)\big)\big]\Big\}.$$

Thus, $\gamma(x)$ states that $\psi_{\leqslant}(y,z)$ defines a linear ordering below x. We want to ask whether there exists such a formula $\psi_{\leqslant}(y,z)$ for which the set defined by $\gamma(x)$ constructed for this $\psi_{\leqslant}$ is closed on concatenation, provably in TC':

$$\mathrm{TC}'\vdash\forall x\forall y\big(\big(\gamma(x)\wedge\gamma(y)\big)\Rightarrow\gamma(x\frown y)\big).$$

We conjecture that the answer is positive.

As for the final remark, in [2] the conjecture was stated that TC is essentially weaker than the Robinson arithmetic Q. Let us formulate this question in a bit more precise form: whether Q is interpretable in TC. Now, it seems to us that the answer is also positive.

Appendix. An Outline of the Possibility of Proving the Essential Undecidability of TC in Another Way

In our collaboration on the essential undecidability of TC we both have agreed to use the ω-type ordering $<$ of words, but we differ in the manner of applying it. Konrad Zdanowski wanted to apply the concept of the Turing machine, Andrzej Grzegorczyk would like to continue his analysis of discernible relations started in [2] and tried to use the property of representability of GD relations. Konrad Zdanowski has accomplished his task earlier and his solution is exhibited in Section 3. Now we draw up the other way of proving essential undecidability. We assume that the reader has in hand a copy of [2]. Much of notational conventions in this appendix is from this article.

Formally we can proceed in TC using Definition 8 of the present paper, or (perhaps easier) by writing a little different definition based on an intuition of comparison of two words u and w in the manner step by step: 'one symbol after another symbol'. Namely we assume that every word begins by its first symbol and if there are two words u and w:

$u=u_1u_2\dots u_k$ and $w=w_1w_2\dots w_k\dots w_n$

then we consider the sequence of pairs of the initial subwords:

$\langle u_1,w_1\rangle,\langle u_1u_2,w_1w_2\rangle,\dots$ and so on.

Then u is shorter or equal w (in the length) when there is such a sequence of pairs, in which for any subword of u there is a corresponding subword in w. This sequence of pairs should be defined as a text (word) by means of an inductive condition. To describe this procedure and the whole sequence we need to use a code, especially if we do not want to add a new constant γ. If the two words u, w have the same length then they have a common initial part and we put that $u < w$ when on the first place where they differ, there is the sign α in u and the sign β in w.

In this way of proving of our Essential Undecidability Theorem we also make use of several properties of the relations $<$ and $\leqslant$, which may be proved in TC. These properties bind the relations $<$ and $\leqslant$ with their defining formulae: $\varphi_<(u, w)$ and $\varphi_\leqslant(u, w)$. The properties which are necessary are mentioned below as lemmas LA–LE:

LA For any $m \in \{a, b\}^+, TC \vdash \forall u(\varphi_<(u, \underline{m}) \iff \bigvee_{r<m} u = \underline{r})$.
LB For any $n, m \in \{a, b\}^+$, $n < m \equiv \mathrm{TC} \vdash \varphi_<(\underline{n}, \underline{m})$.
LC $\mathrm{TC} \vdash \forall y \forall u(\varphi_<(u, y) \Rightarrow \varphi_\leqslant(u, u))$
LD For any $m \in \{a, b\}^+$, $\mathrm{TC} \vdash \varphi_\leqslant(\underline{m}, \underline{m})$.
LE For any $m \in \{a, b\}^+$, $\mathrm{TC} \vdash \forall u(\varphi_\leqslant(u, u) \Rightarrow (\varphi_\leqslant(u, \underline{m}) \vee \varphi_\leqslant(\underline{m}, u)))$.

Let us recall from [2] that ED is the smallest class of relations (of arbitrary arity) which contains: the binary relation of identity, the ternary relation of concatenation and which is closed under the logical constructions:

1. of the classical propositional calculus,
2. of the operation of quantification relativized to the subtexts of a given text.

Let us note that it follows from the definition that each relation in ED is definable in the standard model for concatenation by a Δ_0 formula.

The class GD satisfies the same conditions and is closed also under the operation of dual quantification. This means that:

If $S, T \in$ GD and the relation R satisfies two equivalences:

$R(x, \ldots) \equiv \exists y S(y, x, \ldots)$ and $R(x, \ldots) \equiv \forall y T(y, x, \ldots)$

then also $R \in$ GD. The class GD (of General Discernible relations) in the domain of texts corresponds to the class of General Recursive relations of integers.

We need the following lemma.

Lemma 17 (The Normal Form Lemma). GD *is the class of all relations which may be defined from the relations of the class* ED *by at most one application of the operation of dual quantification. The application of dual quantification may be the last step in defining a given* GD *relation.*

Proof. We shall show that the operation of dual quantification may be always the last step of the defining-process of a GD relation. (In this proof all definitions and symbols are taken from [2].) Suppose that $S, T \in$ GD and are defined by dual quantification:

$$S(x, \ldots) \equiv \forall y A(y, x, \ldots) \text{ and } S(x, \ldots) \equiv \exists y B(y, x, \ldots), \tag{7}$$

$$T(x, \ldots) \equiv \forall y C(y, x, \ldots) \text{ and } T(x, \ldots) \equiv \exists y D(y, x, \ldots). \tag{8}$$

where $A, B, C, D \in$ ED.

A new GD relation R (according to definition 6 (of [2])) may be defined by S and T by using propositional connectives or quantifications limited or dual. Hence we shall consider the following 4 cases.

1. R is defined by negation:

$$R(x,\ldots) \equiv \neg S(x,\ldots).$$

Then by (7) the relation R may be presented in the dual form (according to the de Morgan rules) as follows:

$$R(x,\ldots) \equiv \forall y B(y,x,\ldots) \text{ and } R(x,\ldots) \equiv \exists y A(y,x,\ldots).$$

If $A, B \in$ ED then according to definition 5 also $A, B \in$ ED.

2. R is defined by means of conjunction:

$$R(x,\ldots) \equiv \big((S(x,..) \wedge T(x,\ldots)\big) \tag{9}$$

Then by (7) and (8) R may be presented in the dual form as follows:

$$R(x,\ldots) \equiv \forall y\big(A(y,x,\ldots) \wedge C(y,x,\ldots)\big) \tag{10}$$

and

$$R(x,\ldots) \equiv \exists w\big(\exists y\big(y \subseteq w \wedge B(y,x,\ldots)\big) \wedge \exists u\big(u \subseteq w \wedge D(u,x,\ldots)\big)\big). \tag{11}$$

The formula (10) follows from (9), (7) and (8) by logic of quantifiers.

The formula (11) follows from (9), (7) and (8) because in the theory of concatenation one can easily prove the following:

$$\forall y \forall u \exists w (y \subseteq w \wedge u \subseteq w),$$

namely $y \frown u$ is such an element w. If $A, B, C, D \in$ ED, then according to definition 5 we have that:

$$\big\{\langle y,x,\ldots\rangle : A(y,x,..) \wedge C(y,x,..)\big\} \in ED$$

and also:

$$\big\{\langle w,x,\ldots\rangle : \exists y\big(y \subseteq w \wedge B(y,x,\wedge)\big) \wedge \exists u\big(u \subseteq w \wedge D(u,x,..)\big)\big)\big\} \in \text{ED}.$$

3. R is defined by means of limited quantification:

$$R(x,u,\ldots) \equiv \forall z\big(z \subseteq u \Rightarrow S(z,x,..)\big). \tag{12}$$

According to (7) the above definition (12) of R implies that:

$$R(x,u,\ldots) \equiv \forall z\big(z \subseteq u \Rightarrow \forall y A(y,z,x,\ldots)\big) \tag{13}$$

and

$$R(x,u,..) \equiv \forall z\big(z \subseteq u \Rightarrow \exists y B(y,z,x,\ldots)\big). \tag{14}$$

From (13) we easily get the following:

$$R(x,u,\ldots) \equiv \forall w\big[\forall z\big(z \subseteq w \Rightarrow \forall y\big(y \subseteq w \Rightarrow \big(z \subseteq u \Rightarrow A(y,z,x,\ldots)\big)\big)\big)\big] \tag{15}$$

Namely as w we take the element $z \frown y$. The formula (15) shows that R is defined by application of the general quantifier to an ED relation.

To show that R may be presented also as an application of the existential quantifier to an ED relation we shall to use in the metatheory a stronger non-elementary construction.

First let notice that for every $x, \ldots$ and every $z \subseteq u$ there may be infinitely many y such that $B(y, z, x, \ldots)$. We should choose one of them. But we may use the fact that the set of all finite texts is well ordered by the ordering $<$. Hence there is the first such $y^{(z,x,\ldots)}$, that $B(y^{(z,x,\ldots)}, z, x, \ldots)$. For the fixed $u, x, \ldots$ the set of $z \subseteq u$ is finite, thus also the set of tuples $\langle z, x, \ldots \rangle$ for $z \subseteq u$ is finite and the set of $\{y^{(z,x,\ldots)} : z \subseteq u\}$ is also finite. Hence there exist also a text y_0, which contains all texts of the set $\{y^{(z,x,\ldots)} : z \subseteq u\}$, this means such that for any $z \subseteq u$ it is true that $y^{(z,x,\ldots)} \subseteq y_0$. Thus we can assert that:

$$R(x, u, \ldots) \equiv \exists y_0 \forall z \big(z \subseteq u \Rightarrow \exists y \subseteq y_0\, B(y, z, x, \ldots)\big). \tag{16}$$

The formula (16) is dual to (15). The formulae (15) and (16) prove that R is defined using ED relations by one operation of dual quantification.

4. R is defined by dual quantification.

Now we suppose that:

$$R(\ldots) \equiv \forall x S(x, \ldots), \tag{17}$$

$$R(\ldots) \equiv \exists x T(x, \ldots), \tag{18}$$

where S and T satisfy the formulae (7) and (8). Then we shall prove that two applications of dual quantification (one (7), (8) and the second (17), (18)) can be condense to one. Indeed from (17) and (7) we get that:

$$R(\ldots) \equiv \forall x \forall y A(y, x, \ldots) \equiv \forall z \big(\forall x \subseteq z \forall y \subseteq z A(y, x, \ldots)\big) \tag{19}$$

and from (18) and (8) we infer that:

$$R(\ldots) \equiv \exists x \exists y D(y, x, \ldots) \equiv \exists z \big(\exists x \subseteq z\, \exists y \subseteq z\, D(y, x, \ldots)\big). \tag{20}$$

The proofs of (19) and (20) are elementary.

This accomplishes the proof of the lemma. □

Now we can prove the Representability Theorem:[3]

Theorem 18 (Representability Theorem). *If $R \in$ GD, TC $\subseteq T$ and T is consistent, then R is represented in T.*

Proof. Let $R \in$ GD. According to the Normal Form Lemma there are two relations $A, B \in$ ED such that for any word x, $R(x) \equiv \forall y A(x, y)$ and $R(x) \equiv \exists y B(x, y)$. Hence, we can also say that:

for each $R \in$ GD there are two relations $A, B \in$ ED such that for any word x,

$$R(x) \equiv \exists y A(x, y) \text{ and } \neg R(x) \equiv \exists y B(x, y). \tag{21}$$

According to theorem 12 of [2] there are two formulae Φ and Ψ of TC such that for all x, y:

[3]We recall that a relation R is said to be represented in the theory T by a formula φ if and only if for all words $x_1, \ldots, x_n$,

$$R(x_1, \ldots, x_n) \equiv T \vdash \varphi(\underline{x_1}, \ldots, \underline{x_n}).$$

$$A(x,y) \equiv T \vdash \Phi(\underline{x},\underline{y}) \text{ and } \neg A(x,y) \equiv T \vdash \neg\Phi(\underline{x},\underline{y}), \tag{22}$$

$$B(x,y) \equiv T \vdash \Psi(\underline{x},\underline{y}) \text{ and } \neg B(x,y) \equiv T \vdash \neg\Psi(\underline{x},\underline{y}).^4 \tag{23}$$

Starting from the rule: $R \vee \neg R$ we get, from: (21)–(23), that:

$$\forall x \exists y \big(A(x,y) \vee B(x,y)\big).$$

Hence there is also the smallest element $m(x)$ (with respect to the relation $<$ which is of the type ω). Thus, let us define

$$m(x) = \min\big\{y : A(x,y) \vee B(x,y)\big\}, \tag{24}$$

and thus $A(x, m(x)) \vee B(x, m(x))$ and

$$\forall x \forall y \big(y < m(x) \Rightarrow \neg\big(A(x,y) \vee B(x,y)\big)\big), \tag{25}$$

$$\exists y\, A(x,y) \equiv A\big(x, m(x)\big) \text{ and } \exists y\, B(x,y) \equiv B\big(x, m(x)\big). \tag{26}$$

According to definition 7 of [2] and adjusting to the style assumed in the present paper we shall prove that: there is a formula TC such that: for every $x \in \{a,b\}^+$,

$$R(x) \equiv T \vdash \Xi(\underline{x}).$$

For the relation R the corresponding formula may be the following:

$$\Xi(x) = \exists y \big(\Phi(x,y) \wedge \forall u \big(\varphi'_<(u,y) \Rightarrow \neg\big(\Phi(x,u) \vee \Psi(x,u)\big)\big)\big),$$

where the formula $\varphi'_<(u,y)$ is the following:

$$\big(\varphi_\leqslant(u,y) \wedge \neg(u=y) \wedge \varphi_\leqslant(y,y)\big) \vee \big(\varphi_\leqslant(u,u) \wedge \neg\varphi_\leqslant(y,y)\big). \tag{$\varphi'_<$}$$

Indeed, for $x \in \{a,b\}^+$ suppose that $R(x)$. Hence from (21), (26) and (22) we get that:

$$T \vdash \Phi(\underline{x}, \underline{m(x)}) \tag{27}$$

From ($\varphi'_<$), lemmas LA, LB, LD and (25), (22), (23) we get that:

$$T \vdash \forall u \big(\varphi'_<(u, \underline{m(x)}) \Rightarrow \neg\big(\Phi(\underline{x},u) \vee \Psi(\underline{x},u)\big)\big) \tag{28}$$

The premises (27) and (28) give the implication: $R(x)$ implies $T \vdash \Xi(\underline{x})$.

Now, to prove the converse implication suppose that: $T \vdash \Xi(\underline{x})$. If (on the ground of T) we suppose that:

$$\Phi(\underline{x}, y), \tag{29}$$

$$\forall u \big(\varphi'_<(u,y) \Rightarrow \neg\big(\Phi(\underline{x},u) \vee \Psi(\underline{x},u)\big)\big). \tag{30}$$

Then on the ground of T we may compare the element y with the element which has the name: $\underline{m(x)}$. From (22), (23), (24) we get that:

$$T \vdash \big(\Phi\big(\underline{x}, \underline{m(x)}\big) \vee \Psi\big(\underline{x}, \underline{m(x)}\big)\big) \tag{31}$$

---

[4] Instead of using theorem 12 of [2] one may note that since $A \in$ ED then there is a Δ_0 formula which defines A in the standard model for finite words. If we choose Φ to be that formula, we will obtain (22) by the fact that the negation of Φ is also a Δ_0 formula which defines the complement of A and by Theorem 5 which gives that all true Δ_0 sentences are decided in TC.

In the same way one can show (23).

Hence from (30) and (31) on the ground of T we get that:

$$\neg\varphi'_{<}\big(\underline{m(x)}, y)\big) \tag{32}$$

On the other hand $m(x) \in \{a,b\}^{+}$. Then by LD, (32) and ($\varphi'_{<}$) we get that:

$$\varphi_{\leqslant}(\underline{m(x)}, y) \Rightarrow \underline{m(x)} = y, \tag{33}$$

$$\varphi_{\leqslant}(y, y). \tag{34}$$

By (33), (34) and LE we have that:

$$\varphi_{\leqslant}(y, \underline{m(x)}) \vee \underline{m(x)} = y. \tag{35}$$

But according to LA the first possibility of (35) is contradictory to (25), (22), and (29). Hence it remains that

$$\underline{m(x)} = y. \tag{36}$$

The conclusion (36) which is obtained in the consistent theory T together with the premise (29) give that $T \vdash \Phi(\underline{x}, \overline{m(x)})$. Hence by (26), (22), (21) we get that $R(x)$. This accomplishes the proof of representability. □

Having the Representability Theorem we can repeat Gödel's diagonal procedure exactly in the same way as in the proof of theorem 17 from [2]. Of course now we take T not as true in the standard model but as an arbitrary consistent extension of TC.

Acknowledgments

We would like to thank to the two referees for their insightful reading of the paper and comments which significantly improved the presentation.

References

[1] N. Cutland, *Computability: An Introduction to Recursive Function Theory*, Cambridge University Press, 1980.

[2] A. Grzegorczyk, Undecidability without arithmetization, *Studia Logica*, **79** (2005), 163–230.

[3] P. Hájek and P. Pudlák, *Metamathematics of First-Order Arithmetic*, Springer Verlag, 1993.

[4] W. Quine, Concatenation as a basis for arithmetic, *Journal of Symbolic Logic*, **11** (1946), 105–114.

[5] H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill Education, 1967.

[6] K. Zdanowski, *Arithmetics in Finite But Potentially Infinite Worlds*, PhD thesis, Warsaw University, 2005. Available at: http://www.impan.gov.pl/~kz/files/KZ_PhD.pdf.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Witness Quantifiers and 0-1 Laws

Wojciech GUZICKI [a] and Michał KRYNICKI [b]
[a] *Institute of Mathematics, University of Warsaw, Warsaw, Poland*
guzicki@mimuw.edu.pl
[b] *Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University, Warsaw, Poland*
m.krynicki@uksw.edu.pl

Abstract. We consider the problem of 0-1 law for logics with additional quantifiers which do not meet the Mostowski definition of generalized quantifier.

Keywords. generalized quantifiers, 0-1 law, weak quantifiers, relational quantifiers, witness quantifiers

In 1957 in a fundamental paper [23] Andrzej Mostowski formulated a general definition of a quantifier. According to that definition a *quantifier* or *generalized quantifier* is a functor Q which associates with every structure $\mathcal{A}$ a family $\mathrm{Q}(\mathcal{A})$ of subsets of the universe of $\mathcal{A}$. Moreover, the functor Q commutes with every isomorphism between structures. Particularly, the family $\mathrm{Q}(\mathcal{A})$ is closed under permutations of the elements of the universe of the structure $\mathcal{A}$. Such a functor Q determines a meaning of the quantifier symbol Q of a language. A part of a definition of the satisfaction relation concerning the quantifier symbol Q is the following:

$$\mathcal{A} \models Qx\varphi[\mathbf{a}] \quad \text{if and only if} \quad \{a \in |\mathcal{A}| : \mathcal{A} \models \varphi[\mathbf{a}(x/a)]\} \in \mathrm{Q}(\mathcal{A}).$$

In the sense of this definition the classical existential quantifier is the functor $\mathrm{Q}_\exists(\mathcal{A}) = \{X \subseteq |\mathcal{A}| : X \neq \emptyset\}$. The quantifier *there exists infinitely many* is the functor $\mathrm{Q}_{\aleph_0}(\mathcal{A}) = \{X \subseteq |\mathcal{A}| : X \text{ is infinite}\}$. In a similar way we can define such quantifiers as for example *there exists uncountable many*, *there exists exactly countable many* etc. It should be observed that the above Mostowski definition of quantifier can be used to define only quantifiers which bind exactly one variable in one formula. So, for example the so called Härtig quantifier *the same number of elements is … as …* as well as Rescher quantifier *more elements is … than …* cannot be defined by this way. Motivated by these examples Lindström in [22] extended the definition of generalized quantifier so that it allowed quantifiers to bind a finite number of variables in one or more formulas. We omit this definition here because in our paper we will consider quantifiers binding one variable only.

During by last two decades of twentieth century logicians were specially interested in logics of finite structures. In this context the concept of generalized quantifier is very useful. For example in a natural way it can be used to formalize expression like *for at least half*, *for even number* etc. There are also other areas of research in Computer Sciences where the concept of a generalized quantifier plays important role. For review of recent work in this subject, see [28].

In [8] and independently in [10] it was proved that elementary logic satisfies the so called 0-1 law. This means that for each sentence without functions and constant symbols either this sentence is asymptotically true in all finite structures or the negation of this sentence is asymptotically true in all finite structures. In the eighties several papers studied this law in extensions of elementary logic. For example in [3] and [27] it is shown that the 0-1 law holds for fixpoint logic. In [13] it is proved that 0-1 law holds for iterative logic which extend fixpoint logic. Finally this result was extended in [15], where it is proved that 0-1 law holds for the infinitary logic $L^{\omega}_{\infty,\omega}$. There are also several papers (e.g. [25,14]) where the 0-1 law for some fragments of second order logic is considered. The 0-1 law was also discussed in the context of logics with additional quantifiers. For example in [12] the logic with the quantifier *there is at least a fraction r of the elements of the domain satisfying $\varphi(x)$* is studied. Knyazev shows there that some fragments of the logic with these quantifiers satisfies 0-1 law. In [9] the logic with Härtig and Rescher quantifier are considered. Authors show there that only some small fragments of the logic with Härtig quantifier have 0-1 law. Finally, the problem of 0-1 law for logic with generalized quantifiers is studied in [6]. There quantifiers connected to some graph's properties are considered.

There are still several open question concerning the subject of 0-1 law and this subject remains in the field of interest of several logicians (see e.g. [2]).

The main aim of this paper is to consider the problem of 0-1 laws in logics with additional quantifiers which do not meet the Lindström definition of generalized quantifier. According to our knowledge, up to now no one considered this problem, though there are many examples of quantifiers of this type (e.g. topological quantifiers, probability quantifiers, quantifiers of measure etc. cf. [1] or [20]). Logic with such quantifiers usually have the same properties as elementary logic, as for example completeness, compactness etc. It is natural to ask if 0-1 laws holds for these logics. The other aim of this paper is to provide an easy introduction to this subject.

In this paper we give some observations concerning weak quantifiers, relational quantifiers and, so called, witness quantifier introduced here. Structures for logic with such quantifiers usually have form $(\mathcal{A}, F)$, where F is a family of sets. In consequence we have to define precisely what we mean by 0-1 law in such a case. We will do this separately for each kind of quantifiers. We show that 0-1 law holds in many cases of witness quantifiers but usually does not hold for weak and relational quantifiers. Some questions are still left open.

All logics we will consider have the same syntax. The language of these logics is the usual first order language with one additional symbol for quantifier binding one variable in one formula. We denote such a language by $L(Q)$.

Considering finite structures we assume that the universa of structures are initial segments of the natural numbers i.e. if $card(\mathcal{A}) = n$ then $|\mathcal{A}| = \{1, 2, \ldots, n\}$. For a set X the symbol $|X|$ denotes the number of elements of X.

1. Logics with Weak Quantifiers

In [23] Andrzej Mostowski observed that the meaning of a quantifier is determined by a family of subsets of the universe. For example the quantifier *there exists infinitely many* corresponds to the family of infinite sets. Hence structures of the logic with additional quantifier have form $(\mathcal{A}, q)$, where q is a family of subsets of the universe $\mathcal{A}$. The definition of the satisfaction relation is as follows:

$$(\mathcal{A}, q) \models Qx\varphi[\mathbf{a}] \quad \text{if and only if} \quad \{a \in |\mathcal{A}|;\ (\mathcal{A}, q) \models \varphi[\mathbf{a}(x/a)]\} \in q$$

In his definition Andrzej Mostowski in [23] required that the family q is closed under permutations of the universe of the structure $\mathcal{A}$. However now, there are known several natural examples of quantifiers which do not satisfy this requirement (see e.g. [20]). So, it is interesting to consider cases when the meaning of the quantifier symbol in a given structure is determined by any family of subsets of the universe of this structure or family satisfying some special, other than given in [23], conditions.

In [11] Keisler uses structures of the form $(\mathcal{A}, q)$, where q is a family of subsets of the universe of $\mathcal{A}$, as a tool in the proof of the completeness theorem for the logic with the quantifier *there exists uncountably many*. Keisler calls such structures *weak models*. He considers logic having as a language the language with an additional quantifier binding one variable in one formula and weak models as possible structures. Keisler calls this logic the logic with a *weak quantifier*. Similarly, if we assume that q is a filter or ultrafilter then we obtain the logic with *filter* or *ultrafilter quantifier*, respectively. In [11] it is observed that logics with quantifiers mentioned above are axiomatizable and have compactness and Skolem–Löwenheim properties. It means that they have similar properties as the elementary logic. It is interesting to ask if this similarity covers also the 0-1 laws.

Let $\mathbf{M}_w$ ($\mathbf{M}_f$, $\mathbf{M}_{uf}$) denote a class of all weak (filter, ultrafilter – respectively) models. If φ is a sentence of $L(Q)$ we define:

$$\mu_n(\varphi) = \frac{|\{(\mathcal{A}, q) \in \mathbf{M} : (\mathcal{A}, q) \models \varphi \quad \text{and} \quad card(\mathcal{A}) = n\}|}{|\{(\mathcal{A}, q) \in \mathbf{M} : card(\mathcal{A}) = n\}|},$$

where $\mathbf{M}$ denotes $\mathbf{M}_w$, $\mathbf{M}_f$ or $\mathbf{M}_{uf}$, depending on considered logic.

Finally, $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$.

We say that the logic with weak (filter, ultrafilter) quantifier *has limit property* if for every sentence $\varphi \in L(Q)$ without functions and constant symbols the limit $\mu(\varphi)$ exists. Similarly, we say that such a logic *has 0-1 law* (or *satisfies 0-1 law*) if for arbitrary sentence $\varphi \in L(Q)$ without functions and constant symbols either $\mu(\varphi) = 0$ or $\mu(\varphi) = 1$.

1.1. Weak Quantifier

It is easy to observe that for arbitrary elementary formula (i.e. without additional quantifier Q) φ we have $\mu_n(Qx\varphi) = \frac{1}{2}$. In particular $\mu(Qx(x = c)) = \frac{1}{2}$, where c is a constant, and $\mu(Qx(V_\alpha(x))) = \frac{1}{2}$, where $\alpha \in \{0,1\}^n$, and V_α is a formula $U_0^{\alpha(0)}(x) \wedge \ldots \wedge U_{n-1}^{\alpha(n-1)}(x)$, where $U_i^0(x)$ denotes $U_i(x)$ and $U_i^1(x)$ denotes $\neg U_i(x)$ and U_i is a unary predicate. Moreover, the same holds for other formulas as for example $QxQy(x = y)$, $QxQy(\neg x = y)$, or $QxQyQz(x = y \wedge x = z)$. On the other hand if φ is the sentence $QxU(x) \wedge QxV(x)$ then we have:

$$\mu_n(QxU(x) \wedge QxV(x)) = \frac{(2^{2n} - 2^n) \cdot 2^{2^n - 2} + 2^n \cdot 2^{2^n - 1}}{2^{2n} \cdot 2^{2^n}} = \frac{1}{4} + \frac{1}{2^{n+2}}.$$

Hence $\mu(QxU(x) \wedge QxV(x)) = \frac{1}{4}$.

Generalizing our observations given in the last example we can prove the following

Proposition 1.1. *Logic with weak quantifier does not have the 0-1 law. Moreover, for arbitrary natural numbers m, k such that $m \leqslant 2^k$ there is a sentence φ of unary signature of the logic with weak quantifier such that $\mu(\varphi) = \frac{m}{2^k}$.*

Proof. Let m and k be fixed and take l such that $k \leqslant 2^l$. Let $\alpha_1, \ldots, \alpha_k \in 2^l$ be pairwise different and $\varepsilon_1, \ldots, \varepsilon_m \in 2^k$ be also pairwise different. For $i = 1, \ldots, m$ by Φ_i we denote the formula $\bigwedge_{j=1}^{k} \varphi^j_{\varepsilon_i(j)}$, where φ^j_0 denotes the formula $QxV_{\alpha_j}(x)$, φ^j_1 denotes its negation and $QxV_\alpha(x)$ is as in the beginning of this subsection. Now, in an easy way we may calculate that $\mu(\Phi_i) = \frac{1}{2^k}$. Thus $\mu(\bigvee_{i=1}^{m} \Phi_i) = \Sigma_{i=1}^{m} \mu(\Phi_i) = \frac{m}{2^k}$. □

In effect the set $l_\mu^{\mathbf{M}_w} = \{\mu(\varphi);\ \varphi \in L(Q)\}$ is dense in the interval [0,1]. It seems to be interesting to find a more precise description of $l_\mu^{\mathbf{M}_w}$.

1.2. Ultrafilter Quantifier

At first we note that in finite set each ultrafilter is generated by one-element set. This means that in a structure with n elements there are exactly n different ultrafilters. Finally, if $\mathcal{A}$ is a finite structure and q is an ultrafilter on the universe of $\mathcal{A}$ then the sentence $Qx\varphi(x)$ is true in $(\mathcal{A}, q)$ if and only if the element generating the ultrafilter q satisfies $\varphi(x)$ in $(\mathcal{A}, q)$. Using this observation we can calculate in an easy way $\mu(\varphi)$ for several sentences φ. For example, we can see that $\mu_n(Qx(x = c)) = \frac{1}{n}$ and hence $\mu(Qx(x = c)) = 0$. Similarly, $\mu_n(QxQyS(x, y)) = \frac{1}{n^2}$ and hence $\mu(QxQyS(x, y)) = 0$. For the same reason $\mu(QxQy(x = y)) = 1$. On the other hand it is easy to see that for arbitrary n, $\mu_n(QxU(x)) = \frac{n \cdot 2^{n-1}}{n \cdot 2^n} = \frac{1}{2}$. This allows us to prove the following

Proposition 1.2. *Logic with ultrafilter quantifier does not have the 0-1 law. For arbitrary natural numbers m, k such that $m \leqslant 2^k$ there is a sentence φ of unary signature of the logic with ultrafilter quantifier such that $\mu(\varphi) = \frac{m}{2^k}$.*

Proof. Let for $i = 0, 1, \ldots, k-1$, let φ^0_i denote the formula $QxU_i(x)$ and φ^1_i denote the formula $\neg QxU_i(x)$, where U_i are unary predicates. For $\alpha \in 2^k$ we denote by Φ_α the sentence $\varphi_0^{\alpha(0)} \wedge \ldots \wedge \varphi_{k-1}^{\alpha(k-1)}$. So, we have $\mu(\Phi_\alpha) = \mu(\varphi_0^{\alpha(0)} \wedge \ldots \wedge \varphi_{k-1}^{\alpha(k-1)}) = \Pi_{i<k} \mu(\varphi_i^{\alpha(i)}) = \frac{1}{2^k}$. Thus if $\alpha_1, \ldots, \alpha_m \in 2^k$ are pairwise different functions, then $\mu(\Phi_{\alpha_1} \vee \ldots \vee \Phi_{\alpha_m}) = \Sigma_{i=1}^{m} \mu(\Phi_{\alpha_i}) = \frac{m}{2^k}$. □

Let us denote $l_\mu^{\mathbf{M}_{uf}} = \{\mu(\varphi);\ \varphi \in L(Q)\}$.

Proposition 1.3. *Logic with ultrafilter quantifier has the limit property. Moreover, $l_\mu^{\mathbf{M}_{uf}} = \{\frac{m}{2^k};\ m, k \in \mathbb{N}, m \leqslant 2^k\}$.*

Proof. In a finite set every ultrafilter is generated by a one-element set. So, the logic with ultrafilter quantifier is interpreted in elementary logic in such a way that a formula of the form $Qx\varphi(x)$ is translated to the formula $\varphi(c)$,where c is a new constant. Moreover, there is a one-one correspondence between structures for the logic with ultrafilter quantifier and structures for elementary logic of the signature extended by a one constant. Thus we have:

$$l_\mu^{\mathbf{M}_{uf}} = \{\mu(\varphi);\ \varphi \in FO_{\sigma\cup\{c\}} \text{ for relational } \sigma\}.$$

By the results of [10] for every sentence $\varphi \in FO_{\sigma\cup\{c\}}$, $\mu(\varphi) = \frac{l}{2^s}$ for some natural numbers l and s such that $l \leqslant 2^s$. Now, the application of the Proposition 1.2 finishes the proof. $\square$

1.3. Filter Quantifier

Note that each filter on finite set is a principal filter. This means that if F is a filter on a finite set A then there exists a subset $B \subseteq A$ such that $F = \{X \subseteq A;\ B \subseteq X\}$. So, we can interpret a logic with filter quantifier in elementary logic in such a way that a formula of the form $Qx\varphi(x)$ is translated to the formula $\forall x(U(x) \longrightarrow \varphi(x))$, where U is a new unary predicate. The important thing is that there is a one–one correspondence between structures for the logic with filter quantifier and structures for elementary logic of the signature extended by a one unary predicate (i.e. a structure $(\mathcal{A}, q)$ corresponds to the structure $(\mathcal{A}, U)$, where U generate the filter q). This allows to reduce a calculation of $\mu_n(\varphi)$ for $\varphi \in L(Q)$ to a calculation of $\mu_n(\overline{\varphi})$ for some elementary sentence $\overline{\varphi}$ of the signature extended by one unary predicate. Thus we have the following observed in [29]:

Proposition 1.4. *Logic with filter quantifier has 0-1 law.*

2. Logics with Relational Quantifiers

Let $\mathbf{K}$ be a class of binary relations. The logic with relational quantifier determined by the class $\mathbf{K}$ is defined as follows. Structures for this logic have the form $(R; \mathcal{A})$, where $\mathcal{A}$ is a usual structure and R is a binary relation on the universe of $\mathcal{A}$ from the class $\mathbf{K}$. Such structure is denoted by $\mathcal{A}^R$ and the class of all such structures is denoted by $St(\mathbf{K})$. The language of this logic is the language with additional quantifier – $L(Q)$. The semantics of the additional quantifier is defined as follows:

$$\mathcal{A}^R \models Qx\varphi[\mathbf{a}] \quad \text{if and only if} \quad \exists a \in |\mathcal{A}|\ \forall b\ (R(a,b) \Rightarrow \mathcal{A}^R \models \varphi[\mathbf{a}(x/b)]).$$

A quantifier defined in such way is called *relational quantifier determined by a class* $\mathbf{K}$. The relational quantifier determined by the class of all equivalence relations is called *rough quantifier* and was introduced in [26]. Axiomatization of the logic with rough quantifier is presented in [21]. Logics with relational quantifiers determined by different kind of classes of linear or partial ordering are studied in [16] and [17]. Logic with relational quantifier determined by the class of all binary relations is studied in [19]. Logics with relational quantifiers determined by classes of relations are studied from a general point of view in [18].

Logic with relational quantifier is interpretable in the elementary logic. To prove this interpretability we define a translation F from the set of formulas of $L(Q)$ into the set of first order formulas of the language of the same signature extended by one symbol for binary relation P. The definition goes by induction. For atomic formula φ put $\mathsf{F}(\varphi) = \varphi$. The function F commutes with connectives and classical quantifiers. The inductive clause for additional quantifier is as follows: $\mathsf{F}(Qx\varphi) =$ "$\exists y \forall x(P(y,x) \Rightarrow \mathsf{F}(\varphi))$", where y is not free in φ.

It is easy to verify that for arbitrary structure $\mathcal{A}^R$, a valuation $\mathbf{a}$ in $\mathcal{A}^R$ and a formula φ, we have

$$\mathcal{A}^R \models \varphi[\mathbf{a}] \quad \text{if and only if} \quad (\mathcal{A}, R) \models \mathsf{F}(\varphi)[\mathbf{a}].$$

Using the above interpretability it can be observed that under some general assumption on the class of binary relation $\mathbf{K}$ the logic with relational quantifier determined by $\mathbf{K}$ is axiomatizable and have compactness and Skolem–Löwenheim properties.

If φ is a sentence of $L(Q)$ we define:

$$\mu_n(\varphi) = \frac{|\{\mathcal{A}^R \in St(\mathbf{K}) : \mathcal{A}^R \models \varphi \quad \text{and} \quad card(\mathcal{A}) = n\}|}{|\{\mathcal{A}^R \in St(\mathbf{K}) : card(\mathcal{A}) = n\}|}.$$

Finally, $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$.

We say that the logic with relational quantifier determined by a class $\mathbf{K}$ *has 0-1 law* (or *satisfies 0-1 law*) if for arbitrary sentence $\varphi \in L(Q)$ without functions and constant symbols either $\mu(\varphi) = 0$ or $\mu(\varphi) = 1$. In a similar way we define a *limit property* for logics with relational quantifiers.

Using the interpretability of the logic with relational quantifier in elementary logic we can easily obtain the following

Proposition 2.1. *Logic with relational quantifier determined by the class of all binary relations has 0-1 law.*

There is a natural connection between logic with relational quantifiers and logics with several kinds of weak quantifiers. Really, let $\mathcal{A}^R$ be a structure with relational quantifier. We define:

$$\mathsf{q}_R = \{X \subseteq |\mathcal{A}|;\ \exists a \in |\mathcal{A}| \forall b\ ((a,b) \in R \Rightarrow b \in X)\}.$$

An easy proof by induction on a construction of a formula φ gives us the following

Proposition 2.2. *For each structure $\mathcal{A}^R$, formula φ and valuation $\overline{a}$ in $\mathcal{A}$ we have*

$$\mathcal{A}^R \models \varphi[\overline{a}] \quad \textit{if and only if} \quad (\mathcal{A}, \mathsf{q}_R) \models \varphi[\overline{a}].$$

The above observation can be applied to the problem of the 0-1 law in the case of the logic with relational quantifier determined by the class of all linear ordering. Namely, we can easily observe that if a relation R in a structure $\mathcal{A}^R$ is a linear ordering then q_R is an ultrafilter. Moreover if $\mathcal{A}$ has n elements then there is exactly $(n-1)!$ different orderings R determining the same ultrafilter q_R. This allows us to conclude that for arbitrary formula φ the number $\mu_n(\varphi)$ for the logic with relational quantifier determined by linear orderings is the same as the number $\mu_n(\varphi)$ for the logic with weak ultrafilter quantifier. Thus we have

Proposition 2.3. *Logic with relational quantifier determined by the class of all linear orderings does not have 0-1 law. However it has limit property. Moreover, the set of all numbers $\mu(\varphi)$ for this logic is the set of all numbers of the form $\frac{m}{2^k}$, where m and k are natural numbers such that $m \leqslant 2^k$.*

The problem of 0-1 law for logics with relational quantifiers is connected with the same problem for a class of structures (for exact definition see [5,7] or [24]). In [4] (see also [5]) Compton proved that the class of all partial orderings has 0-1 law. This implies that in the case of the logic with relational quantifier determined by the class of all partial orderings for every formula φ of the empty signature either $\mu(\varphi) = 0$ or $\mu(\varphi) = 1$. But it does not solve the general problem of 0-1 law for that logic. So, we have an interesting open problem: does the logic with relational quantifier determined by the class of all partial orderings has 0-1 law?

The same open question we can put in the case of the logic with relational quantifier determined by the class of all equivalence relations. Here calculation of the fraction $\mu_n(\varphi)$, even for simply formulas φ is connected with calculation or at least estimation of Bell numbers.

3. Logics with Witness Quantifiers

A typical quantifier phrase has form "there exist many elements such that …". Of course the word "many" could be interpreted in different ways. For example, in the case of the existential quantifier it means – at least one. In the case of the relational quantifier determined by a class of equivalence relations it means – all elements from one equivalence class. In a similar way we define the so called *witness quantifier*.

3.1. Witness Quantifier

In the case of the witness quantifier the word "many" means: at least one element from a special set. The elements of this set are called witnesses. We assume that in structures with n elements the set of witnesses has at least $f(n)$ elements, where f is a nondecreasing sequence of naturals numbers such that for each n, $f(n) \leqslant n$, and moreover, for some k, $f(k) > 0$. So, if such function is bounded then it is constant from some points. We will call such a function *almost constant*. In the opposite case the function f increases in infinite many places and in consequence $\lim_{n\to\infty} f(n) = \infty$.

Possible structures of the logic with witness quantifier have form $(U; \mathcal{A})$, where $\mathcal{A}$ is a usual structure of a given signature and U is a subset of a universe of $\mathcal{A}$. Moreover, we assume that if the structure $\mathcal{A}$ is finite and its universe has n elements then the set U has at least $f(n)$ elements and if the structure $\mathcal{A}$ is infinite then for every n the set U has at least $f(n)$ elements. We will denote such structure by $\mathcal{A}^U$.

The satisfaction relation is defined as follows:

$$\mathcal{A}^U \models Qx\varphi[\overline{a}] \quad \text{if and only if} \quad \text{for some } a \in U \ \mathcal{A}^U \models \varphi[\overline{a}(x/a)]$$

Thus we can consider a structure $\mathcal{A}^U$ as a weak structure (in the sense of [11]) $(\mathcal{A}, q_U)$ where $q_U = \{X \subset |\mathcal{A}|;\ X \cap U \neq \emptyset\}$.

Such defined logic we will call a *logic with witness quantifier determined by* f and denote by $L(Q_w^f)$. Observe that if f is identity function then the meaning of the witness quantifier is the same as the meaning of the existential quantifier. So in that case $L(Q_w^f)$ is equivalent to elementary logic. However in general we have the following

Proposition 3.1. *$L(Q_w^f)$ is interpretable in elementary logic.*

Proof. In order to show this interpretability we define a translation F from the set of formulas of $L(Q)$ to the set of elementary formulas of the language enriched by one unary predicate V. This definition goes by induction. For atomic φ, $\mathsf{F}(\varphi) = \varphi$. F commutes with connectives and existential and universal quantifiers. Finally, $\mathsf{F}(Qx\varphi(x)) = \exists x(V(x) \wedge \mathsf{F}(\varphi(x)))$. □

By induction with respect of a construction of a formula φ, we can easily prove that for arbitrary structure $\mathcal{A}^U$, a valuation $\overline{a}$ in $\mathcal{A}^U$ and a formula φ, we have

$$\mathcal{A}^U \models \varphi[\overline{a}] \quad \text{if and only if} \quad (\mathcal{A}, U) \models \mathsf{F}(\varphi)[\overline{a}].$$

From the last lemma follows easily the following

Corollary 3.2. *$L(Q_w^f)$ is axiomatizable and has compactness as well as downward and upward Skolem–Löwenheim property.*

3.2. An Axiomatization

Let us consider the following schemas of formulas:

A0. Each substitution of the elementary tautology.
A1. $Qx\varphi(x) \Rightarrow Qy\varphi(y)$, if y is not free in $\varphi(x)$ and x does not occur as a free variable in $\varphi(x)$ in the scope of the quantifier binding y.
A2. $\forall x(\varphi \Rightarrow \psi) \Rightarrow (Qx\varphi \Rightarrow Qx\psi)$.
A3. $Qx\varphi(x) \Rightarrow \exists x(\varphi(x) \wedge Qy(x = y))$.
A4. $\exists^{\geqslant n} x(x = x) \Rightarrow \exists x_1 \dots x_{f(n)}(\bigwedge_{i=1}^{f(n)} Qy(y = x_i) \wedge \bigwedge_{i \neq j} x_i \neq x_j)$.

We take sentences **A0.–A4.** as axioms of a proposed logical system. As a rules of inferences we take modus ponens and generalization. The notion of the proof and consistency of a set of sentences is as usual. Now, we can prove a completeness theorem for logic $L(Q_w^f)$.

Theorem 3.3. *Let T be a set of sentences of $L(Q)$. Then T is consistent in $L(Q_w^f)$ if and only if T has a model.*

Proof. We can easily verify that each sentence provable from T is true in each model for T. Thus if T has a model then T is consistent. Now let us assume that T is consistent. We can assume that T is complete and has Henkin property. First we construct in a standard way a model $\mathcal{A}$ for elementary part of T. Then we define $U = \{[\tau]_\sim;\ T \vdash Qx(x = \tau)\}$. Thus we have:

$$\mathcal{A}^U \models Qx(x = y)[[\tau]_\sim] \quad \text{if and only if} \quad T \vdash Qx(x = \tau).$$

By induction on the construction of a formula φ we show the following equivalence:

$$\mathcal{A}^U \models \varphi[[\tau_1]_\sim, \dots, [\tau_n]_\sim] \quad \text{if and only if} \quad T \vdash \varphi(\tau_1, \dots, \tau_n).$$

It suffices to consider the case of the additional quantifier. If $\mathcal{A}^U \models Qx\varphi$ then for some term τ we have $\mathcal{A}^U \models \varphi[[\tau]_\sim]$ and $\mathcal{A}^U \models Qx(x = y)[[\tau]_\sim]$. By induction hypothesis $T \vdash \varphi(\tau)$ and $T \vdash Qx(x = \tau)$. By **A2.** we get $T \vdash Qx\varphi(x)$. Now, assume $T \vdash Qx\varphi(x)$. By **A3.** we obtain $T \vdash \exists x(\varphi(x) \wedge Qy(x = y))$. Because T has Henkin property then there exists a term τ such that $T \vdash \varphi(\tau)$ and $T \vdash Qy(\tau = y)$. By induction hypothesis $\mathcal{A}^U \models \varphi[[\tau]_\sim]$ and $[\tau]_\sim \in U$. Thus $\mathcal{A}^U \models Qx\varphi(x)$.

This finishes the proof that every sentence from T is true in $\mathcal{A}^U$. By **A4.** this structure is the appropriate structure for the logic $L(Q_w^f)$. □

3.3. 0-1 Law and Witness Quantifiers

If φ is a sentence of $L(Q)$ we define:

$$\mu_n(\varphi) = \frac{|\{\mathcal{A}^U : \mathcal{A}^U \models \varphi \quad \text{and} \quad card(\mathcal{A}) = n\}|}{|\{\mathcal{A}^U : card(\mathcal{A}) = n\}|}.$$

Finally, $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$.

As in the preceding cases we say that the logic $L(Q_w^f)$ *has 0-1 law* (or *satisfies 0-1 law*) if for arbitrary sentence $\varphi \in L(Q)$ without functions and constant symbols either $\mu(\varphi) = 0$ or $\mu(\varphi) = 1$. If for every such sentence φ, the limit $\mu(\varphi)$ exists then we say that logic $L(Q_w^f)$ *has limit property*.

Example 3.1. Let us consider the sentence $QxV(x)$, where V is an unary predicate. We have:

$$\mu_n(QxV(x)) = \frac{\sum\limits_{k\geqslant f(n)} (2^n - 2^{n-k}) \cdot \binom{n}{k}}{2^n \cdot \sum\limits_{k\geqslant f(n)} \binom{n}{k}} = 1 - \frac{\sum\limits_{k\geqslant f(n)} \frac{1}{2^k}\binom{n}{k}}{\sum\limits_{k\geqslant f(n)} \binom{n}{k}} \geqslant 1 - \frac{1}{2^{f(n)}}.$$

Thus if $\lim_{n\to\infty} f(n) = +\infty$ then $\mu(QxV(x)) = 1$. However, some easy estimation allows us to prove that the same holds for almost constant function f. So, $\mu(QxV(x)) = 1$ for arbitrary f.

Example 3.2. Let us consider the sentence $Qx(x = a)$, where a is a constant. Then we have

$$\mu_n(Qx(x = a)) = \frac{\sum\limits_{k\geqslant f(n)} \binom{n}{k} \cdot k}{n \cdot \sum\limits_{k\geqslant f(n)} \binom{n}{k}} = \frac{\sum\limits_{k\geqslant f(n)} \binom{n-1}{k-1}}{\sum\limits_{k\geqslant f(n)} \binom{n}{k}}$$

$$= \frac{2^{n-1} - \sum\limits_{0<k<f(n)} \binom{n-1}{k-1}}{2^n - \sum\limits_{k<f(n)} \binom{n}{k}}.$$

So, for example, in the case of the function $f(n) = [\frac{n+1}{2}]$ as well as in the case of almost constant function f we have $\mu(Qx(x = a)) = \frac{1}{2}$.

Theorem 3.4. *If a function f is such that there exists N and $\varepsilon > 0$ such that for every $n > N$ the following inequality holds*

$$\sum_{k \geqslant f(n)} \binom{n-1}{k-1} > \varepsilon \sum_{k \geqslant f(n)} \binom{n}{k}$$

then the logic $L(Q_w^f)$ has 0-1 law.

Proof. We should show that if φ is a sentence of $L(Q)$ with no function or constants symbols then either $\mu(\varphi) = 0$ or $\mu(\varphi) = 1$. We proceed as in the proof of the 0-1 law for elementary logic (see [8]). Let S be the signature of the sentence φ. Let $X = \{x_1, \ldots, x_m\}$. By a *complete open description* we mean a conjunction $\bigwedge\{\psi;\ \psi \in A\}$, where for each predicate P in S, let say k-ary, and $(z_1, \ldots, z_k) \in X^k$, the set A contains exactly one of $P(z_1, \ldots, z_k)$ or $\neg P(z_1, \ldots, z_k)$, and for every $z \in X$ the set A contains exactly one of $Qx(x = z)$ or $\neg Qx(x = z)$. We say that a complete open description $N(x_1, \ldots, x_m, y)$ *extends* a complete open description $M(x_1, \ldots, x_m)$ if every conjunct occurring in M also occurs in N. Similarly as in [8], we consider the set of all sentences of the form:

$$\begin{aligned}&\forall \overline{x}(((\bigwedge_{i<j} x_i \neq x_j) \wedge M(x_1, \ldots, x_m)) \\ &\quad \Longrightarrow \exists y((\bigwedge_i y \neq x_i) \wedge N(x_1, \ldots, x_m, y))),\end{aligned} \tag{$*$}$$

where $N(x_1, \ldots, x_m, y)$ is a complete open description extending $M(x_1, \ldots, x_m)$. Let us denote this set of sentences by T. In order to show that the set T is a consistent and complete set of sentences we use the interpretability of the logic with witness quantifier in elementary logic (see Proposition 3.1). We translate the set of sentences T to an elementary theory T^* and show in a standard way that T^* is consistent and complete. This implies consistency and completeness of T. Now, we will prove the other important property of T. □

Lemma 3.5. *For each $\psi \in T$, $\mu(\psi) = 1$*

Proof of the Lemma. Again we proceed as in [8]. For convenience we assume that the signature of ψ contains only one binary predicate P. Let ψ be a sentence $(*)$ and let $\varphi(x_1, \ldots, x_m)$ be

$$M(x_1, \ldots, x_m) \wedge \forall y((\bigwedge_i y \neq x_i) \Longrightarrow \neg N(x_1, \ldots, x_m, y)).$$

Let $\varphi(y)$ be $\neg(A_1 \wedge \ldots \wedge A_{3m+2})$, where A_j for $j = 1, \ldots, 2m+1$ are defined as in [8] and for each $i = 1, \ldots, m$

$$A_{2m+i+1} = \begin{cases} Qx(x = i) & \text{if } Qx(x = x_i) \text{ is a conjunct of } N(x_1, \ldots, x_m, y) \\ \neg Qx(x = i) & \text{if } \neg Qx(x = x_i) \text{ is a conjunct of } N(x_1, \ldots, x_m, y) \end{cases}$$

and finally

$$A_{3m+2} = \begin{cases} Qx(x = y) & \text{if } Qx(x = y) \text{ is a conjunct of } N(x_1, \ldots, x_m, y) \\ \neg Qx(x = y) & \text{if } \neg Qx(x = y) \text{ is a conjunct of } N(x_1, \ldots, x_m, y) \end{cases}$$

Assume now that $n > m$. As in [8] we obtain

$$\mu_n(\neg\psi) \leqslant n^m \prod_{j=m+1}^{n} \mu_n(\varphi(j)) = n^m\left(1 - \left(\frac{1}{2^{2m+1}} \cdot a_n^{m+1}\right)\right)^{n-m},$$

where $a_n = \frac{\sum_{k \geqslant f(n)} \binom{n-1}{k-1}}{\sum_{k \geqslant f(n)} \binom{n}{k}}$. Now, using our assumption about the function f we can deduce that for some $\alpha < 1$ and sufficiently big n, $\mu_n(\neg\psi) \leqslant n^m \alpha^{n-m}$. Thus by l'Hospital's rule $\mu_n(\neg\psi) \to 0$ when $n \to \infty$. Because $\mu_n(\psi) + \mu_n(\neg\psi) = 1$ then $\mu_n(\psi) \to 1$ when $n \to \infty$. This finishes the proof of the Lemma 3.5.

The rest of the proof of the theorem goes in a standard way (see [8]). Finally using compactness argument and completeness of the theory T we deduce that for arbitrary sentence φ, $T \vdash \varphi$ if and only if $\mu(\varphi) = 1$. □

Let us observe that the condition occurring in the hypothesis of the Theorem 3.4 is satisfied by rather wide class of functions. For example this condition is satisfied by every almost constant function. Moreover, we have also the following

Corollary 3.6. *If there exists $\varepsilon > 0$ and k such that for every $n \geqslant k$ we have $\frac{f(n)}{n} > \varepsilon$ then $L(Q_w^f)$ has 0-1 law.*

Proof. Let $\frac{f(n)}{n} > \varepsilon$ for sufficiently big n. For such ε we have

$$\varepsilon \sum_{k \geqslant f(n)} \binom{n}{k} < \frac{f(n)}{n} \sum_{k \geqslant f(n)} \binom{n}{k} = \sum_{k \geqslant f(n)} \frac{f(n)}{k} \binom{n-1}{k-1} \leqslant \sum_{k \geqslant f(n)} \binom{n-1}{k-1}.$$

So, the hypothesis of the Theorem 3.4 is satisfied. □

4. Logics with Narrow Witness Quantifier

In the case of witness quantifier determined by a function f we assumed that the set of witnesses in structure having n element has at least $f(n)$ elements. Now, we modify this definition assuming that in structures with n elements this set has exactly $f(n)$ elements. A quantifier defined in such a way we call *narrow witness quantifier*. Let us remind that we assume that the function f is a nondecreasing sequence of naturals numbers such that for each n, $f(n) \leqslant n$, and moreover, for some k, $f(k) > 0$.

The logic with narrow witness quantifier determined by a function f will be denoted by $L(Q_{nw}^f)$. A possible structures of this logic have form $(U; \mathcal{A})$, where $\mathcal{A}$ is a usual structures of a given signature and U is a subset of the universe of $\mathcal{A}$. Moreover, we assume that if the structure $\mathcal{A}$ is finite and its universe has n elements then the set U has exactly $f(n)$ elements and if the structure $\mathcal{A}$ is infinite then for every n the set U has at least $f(n)$ elements. We will denote such structure by $\mathcal{A}^U$. The definition of satisfaction relation is the same as in the case of the logic with witness quantifier.

It easy to see that we can repeat our considerations from the last section to obtain some basic results about the logic $L(Q_{nw}^f)$. We need the following schemas of formulas.

A5. $\exists^{\leqslant n} x(x = x) \Rightarrow \forall x_0 \ldots x_{f(n)} (\bigwedge_{i=0}^{f(n)} Qy(y = x_i) \to \bigvee_{i \neq j} x_i = x_j)$.

Thus we can observe the following

Proposition 4.1. *$L(Q^f_{nw})$ is axiomatizable and has compactness as well as downward and upward Skolem–Löwenheim properties. Moreover, the set of axioms* **A0.–A5.** *forms a complete axiomatization of $L(Q^f_{nw})$.*

Now, we will consider the problem of 0-1 law for the logic $L(Q^f_{nw})$.

Example 4.1. Let us consider the sentence $QxV(x)$. We have:

$$\mu_n(QxV(x)) = \frac{(2^n - 2^{n-f(n)}) \cdot \binom{n}{f(n)}}{2^n \cdot \binom{n}{f(n)}} = 1 - \frac{1}{2^{f(n)}}.$$

Thus if $\lim_{n\to\infty} f(n) = +\infty$ then $\mu(QxV(x)) = 1$ and for almost constant function f, $\mu(QxV(x)) = 1 - \frac{1}{2^p}$ for some positive natural p. Hence we have

Proposition 4.2. *If f is an almost constant function then the logic $L(Q^f_{nw})$ does not have 0-1 law.*

Example 4.2. Let us consider the sentence $Qx(x = a)$, where a is a constant. Then we have

$$\mu_n(Qx(x = a)) = \frac{\binom{n}{f(n)} \cdot f(n)}{n \cdot \binom{n}{f(n)}} = \frac{f(n)}{n}.$$

So, in the case of an almost constant function f we have $\mu(Qx(x = a)) = 0$. For other functions the limit $\mu(Qx(x = a))$ can be any real number from the interval $[0, 1]$.

Repeating the proof of the Theorem 3.4 we can prove the following.

Theorem 4.3. *If for every natural number m, $\lim_{n\to\infty}(1 - \frac{f(n)}{n})^n n^m = 0$ then the logic $L(Q^f_{nw})$ has 0-1 law.*

So, for many cases of functions f the logic $L(Q^f_{nw})$ has 0-1 law. For example, from the last theorem follows that if $f(n) = \Omega(n^\alpha)$, where $0 < \alpha \leqslant 1$, then $L(Q^f_{nw})$ has 0-1 law. An open question concerns cases when the function f is unbounded but is growing very slowly, as for example in the case of the function $f(n) = [\log n]$.

5. Logics with Bounded Witness Quantifiers

Now, we modify the definition of the witness quantifier in such a way that we assume that in structures with n elements the set of witnesses has at most $f(n)$ elements. A quantifier defined in such a way we call *bounded witness quantifier*.

The logic with bounded witness quantifier determined by a function f will be denoted by $L(Q^f_{bw})$. Again a possible structures of this logic have form $\mathcal{A}^U$, and if the structure $\mathcal{A}$ is finite and its universe has n elements then the set U has at most $f(n)$ elements. In the case when $\mathcal{A}$ is infinite we assume nothing about the set U. The definition of satisfaction relation is the same as in the case of the logic with witness quantifier.

It is easy to see that we can repeat our considerations from the last section to obtain some basic results about the logic $L(Q^f_{bw})$. So, we have the following

Proposition 5.1. *$L(Q^f_{bw})$ is axiomatizable and has compactness as well as downward and upward Skolem–Löwenheim properties. Moreover, the set of axioms* **A0.–A3.**, **A5.** *forms a complete axiomatization of $L(Q^f_{bw})$.*

To say something about the problem of 0-1 law for the logic $L(Q^f_{bw})$ we consider some examples.

Example 5.1. Let us consider the sentence $QxV(x)$. We have:

$$\mu_n(QxV(x)) = \frac{\sum\limits_{k\leqslant f(n)} (2^n - 2^{n-k}) \cdot \binom{n}{k}}{2^n \cdot \sum\limits_{k\leqslant f(n)} \binom{n}{k}} = 1 - \frac{\sum\limits_{k\leqslant f(n)} \frac{1}{2^k}\binom{n}{k}}{\sum\limits_{k\leqslant f(n)} \binom{n}{k}}.$$

From the above equality it can be proved that if $\lim_{n\to\infty} f(n) = \infty$ then $\mu(QxV(x)) = 1$. But in the case of an almost constant function an easy calculation shows that $\mu(QxV(x)) = 1 - \frac{1}{2^p}$. Hence we have

Proposition 5.2. *If f is an almost constant function then the logic $L(Q^f_{bw})$ does not have 0-1 law.*

Example 5.2. Let us consider the sentence $Qx(x = a)$, where a is a constant. Then we have

$$\mu_n(Qx(x = a)) = \frac{\sum\limits_{k\leqslant f(n)} \binom{n}{k} \cdot k}{n \cdot \sum\limits_{k\leqslant f(n)} \binom{n}{k}} \leqslant \frac{f(n)}{n}.$$

So, if $\lim_{n\to\infty} \frac{f(n)}{n} = 0$ then $\mu(Qx(x = a)) = 0$.

Finally, repeating the proof of the Theorem 3.4 we obtain

Theorem 5.3. *If a function f is such that there exist N and $\varepsilon > 0$ such that for every $n > N$ the following inequality holds*

$$\sum_{k\leqslant f(n)} \binom{n-1}{k-1} > \varepsilon \sum_{k\leqslant f(n)} \binom{n}{k},$$

then the logic $L(Q^f_{bw})$ has 0-1 law.

It can be observed that the condition occurring in the assumption of the last theorem is satisfied by function $f(n) = [\frac{n}{2}]$ as well as by every function of the form $f_k(n) = \max\{n - k, 0\}$.

References

[1] J. Barwise and S. Feferman, *Model–Theoretic Logics*, Springer Verlag, 1985.

[2] A. Blass and Y. Gurevich, New Zero-One Law and Strong Extension Axioms, *Bulletin of the European Association for Theoretical Computer Science* **72** (2000), 103–122.

[3] A. Blass, Y. Gurevich and D. Kozen, A zero-one law for logic with a fix-point operator, *Information and Control* **67** (1985), 70–90.

[4] K.J. Compton, The computational complexity of asymptotic problems I: partial order, *Information and Computation* **78** (1988), 108–123.

[5] K.J. Compton, 0-1 laws in logic and combinatorics, *Algorithms and Order*, J. Rival, ed., 1989, pp. 353–383.

[6] A. Dawar and E. Grädel, Generalized quantifiers and 0-1 laws, in: *Proc. 10th IEEE Symp. on Logic in Computer Science*, 1995, pp. 54–64.

[7] H.-D. Ebbinghaus and J. Flum, *Finite Model Theory*, Springer-Verlag, 1999 (second edition).

[8] R. Fagin, Probabilities on finite models, *The Journal of Symbolic Logic* **41** (1976), 50–58.

[9] G. Fayolle, S. Grumbach and Ch. Tollu, *Asymptotic probabilities of languages with generalized quantifiers*, preprint, 1993.

[10] Y. Glebskii, D. Kogan, M. Liogonkii and V. Talanov, Range and degree of realizability of formulas in the restricted predicate calculus, *Cybernetics* **5** (1969), 142–154.

[11] H.J. Keisler, Logic with the quantifier "there exist uncountable many", *Annals of Mathematical Logic* **1** (1970), 1–93.

[12] V. Knyazev, Zero-one law for an extension of first order predicate logic, *Cybernetics* **26** (1990), 292–296.

[13] P. Kolaitis and M.Y. Vardi, The decision problem for the probabilities of higher-order properties, in: *Proc. 19th ACM Symp. on Theory of Computing*, 1987, pp. 425–435.

[14] P. Kolaitis and M.Y. Vardi, $0/1$ laws and decision problems for fragments of second order logic, *Information and Computation* **87** (1990), 302–338.

[15] P. Kolaitis and M.Y. Vardi, $0/1$ laws for infinitary logic, in: *Proc. 5th IEEE Symp. on Logic in Computer Science*, 1990, pp. 156–167.

[16] M. Krynicki, Linearly ordered quantifiers, *Bulletin of the Polish Academy of Sciences. Mathematics* **37** (1989), 295–303.

[17] M. Krynicki, Quantifiers determined by partial orderings, *Zeitschrift fur Math. Logik und Grundlagenforschung* **36** (1990), 79–86.

[18] M. Krynicki, Quantifiers determined by classes of binary relations, *Quantifiers: Logics, Models and Computation*, Volume Two: Contributions, M. Krynicki, M. Mostowski and L.W. Szczerba, eds, Kluwer Academic Publishers, 1995, pp. 125–138.

[19] M. Krynicki, Relational quantifier, *Dissertationes Mathematicae* **347** (1995).

[20] M. Krynicki and M. Mostowski, Ambiguous quantifiers, *Logic at Work*, E. Orłowska, ed., Springer-Verlag, 1999, pp. 548–565.

[21] M. Krynicki and H.-P. Tuschik, An axiomatization of the logic with the rough quantifier, *The Journal of Symbolic Logic* **56** (1991), 608–617.

[22] P. Lindström, First order predicate logic with generalized quantifiers, *Theoria* **32** (1966), 12–36.

[23] A. Mostowski, On a generalization of quantifiers, *Fundamenta Mathematicae* **44** (1957), 12–36.

[24] W. Oberschelp, Asymptotic 0-1 laws in combinatorics, *Combinatorial theory*, D. Jungnickel, ed., Lectures Notes in Mathematics, vol. 969, 1982, pp. 276–292.

[25] L. Pacholski and W. Szwast, The 0-1 law fails for the class of existential second-order Gödel sentences with equality, in: *Proc. 30th IEEE Symp. on Foundations of Computer Science*, 1989, pp. 160–163.

[26] L.W. Szczerba, Rough Quantifiers, *Bulletin of the Polish Academy of Sciences. Mathematics* **35** (1987), 251–254.

[27] V.A. Talanov and V.V. Knyazev, The asymptotic truth value of infinite formulas, in: *Proc. All-Union seminar on discrete mathematics and its applications*, (1984), 56–61.

[28] J. Väänänen, ed., *Generalized Quantifiers and Computation*, Lecture Notes in Computer Science, vol. 1754, 1999.

[29] A.S. Wołowski, *Prawdopodobieństwa asymptotyczne w* $L(Q)$, MSc Thesis, Warsaw University, 1997.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Logic for Artificial Intelligence: A Rasiowa–Pawlak School Perspective

Andrzej JANKOWSKI [a,b] and Andrzej SKOWRON [c]
[a] *Institute of Decision Processes Support*
[b] *AdgaM Solutions Sp. z o.o., Wąwozowa 9 lok. 64, 02-796 Warsaw, Poland*
andrzejj@adgam.com.pl
[c] *Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland*
skowron@mimuw.edu.pl

"The problem of understanding of intelligence is said to be the greatest problem in science today and "the" problem for this century – as deciphering the genetic code was for the second half of the latest one.

Arguably, the problem of learning represents a gateway to understanding intelligence in brains and machines, to discovering how the human brain works and to making intelligent machines that learn from experience and improve their competence…"

THE MATHEMATICS OF LEARNING:
DEALING WITH DATA
T. Poggio, S. Smale, Notices AMS, Vol. 50, May 2003 [185]

Abstract. The Rasiowa–Pawlak school was established during the second half of the twentieth century. The school concentrates on studies in logics, foundations of computer science and artificial intelligence (AI). Its formation has been greatly influenced by the logician Andrzej Mostowski, a professor at Warsaw University [110,111], who, in particular, directed the doctoral dissertation of Helena Rasiowa. Nowadays, the disciples of the Rasiowa–Pawlak school are active in many research-development centres worldwide. The school founded its own journal, Fundamenta Informaticae. In this paper, we present selected trends in the studies of the school concerning applications of logic in AI. At the beginning, we briefly describe the genesis of the Rasiowa–Pawlak school. We then present the understanding, currently dominating within the school, on such basic concepts as AI and logic. Since the beginning of the 1950's, the focus of the research by Helena Rasiowa and her associates has been the application of algebraic and topological methods to the investigation of crucial problems of logic from an AI perspective. Amongst them are the completeness theorem, construction of deduction systems, construction of models, especially models for constructive mathematics [136,241,67] and related logics such as intuitionistic, intermediate, modal, and approximation logics. In the paper, we discuss the fundamental, in our opinion, ideas underlying these roots of the Rasiowa–Pawlak school. A great importance in the studies of the school is assigned to the search for optimal tools for reasoning about complex vague concepts, construction of knowledge representation systems, reasoning about knowledge as well as for the application of logics in learning, communication, perception, planning, action, cooperation, and competition.

It should be noted that as is the case with many other research centers, the Rasiowa– school studies pertaining to the application of logics in AI have also undergone an evolution which we present in this paper. We include extensive references to the literature on the approach presented in this paper.

Keywords. Logic, AI, algebraic logic, abstract logic, approximation, wisdom technology, adaptive rough-granular computing, rough sets, machine learning.

1. The Genesis of the Rasiowa–Pawlak School

Since Poland regained its independence during the XX century after about 150 years of annexation by Russia, Germany and Austria, the Polish intelligentsia (both political as well as academic) has placed great importance on the design and the deployment of many action plans aiming at establishing a firm position for Poland internationally. These plans were, to a great extent, results of the considerations and actions of intellectuals affiliated with positivism in the nineteenth century on Polish territories under foreign rules. One of many such action plans focused on designing a research stimulation program in mathematics, logic and philosophy in the free Poland. The most important components of this program were published in a work by Zygmunt Janiszewski [83]. The consistent realization of Janiszewski's program led to the birth of one of the most powerful research centres in mathematics and mathematical logic during that time. Leading figures of the pre-WWII Polish mathematical school were[1] Stefan Banach, Samuel Eilenberg, Kazimierz Kuratowski, Jan Łukasiewicz, Stanisław Mazur, Stanisław Saks, Juliusz Paweł Schauder, Wacław Sierpiński, Hugo Steinhaus, Alfred Tarski, Antoni Szczepan Zygmund, and many others. On the other hand, leaders of the pre-WWII logical school were Kazimierz Ajdukiewicz, Leon Chwistek, Stanisław Jaśkowski, Tadeusz Kotarbiński, Stanisław Leśniewski, Adolf Lindenbaum, Bolesław Sobociński, Alfred Tarski, Kazimierz Twardowski to name a few.[2]

The refulgent expansion of Polish pre-WWII mathematical and logical schools was tragically interrupted with the outbreak of the World II. Many Poles were striving to continue the education and research work within undercover structures organized by the Polish underground state during the war. In such a way, Helena Rasiowa studied logic under the supervision of Jan Łukasiewicz, Bolesław Sobociński, Andrzej Mostowski, Karol Borsuk. Her first Master's thesis supervised by Jan Łukasiewicz and Bolesław Sobociński, burnt up during the Warsaw Uprising. After the war, for a short time, she had worked as a teacher of mathematics and then, following the advice of Andrzej Mostowski, she returned to Warsaw University. In 1950, she defended under the supervision of Andrzej Mostowski her PhD thesis on algebraic methods in logics. During his lectures at Warsaw University, Andrzej Mostowski frequently recalled the previously known vision of building a *thinking machine*, i.e., a device capable of not only calculating arithmetic expressions but also of thought process computation. While at it, he used to say the following after Gottfried Wilhelm Leibniz:

[1] In alphabetical order.
[2] In alphabetical order.

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, and say to each other: 'Let us calculate' [115].

The idea to replace the intuitive process of reasoning with a process of formal evaluations of algebraic expressions was considered by Andrzej Mostowski as crucial and in this context he used to recall another quote from Leibniz:

No one else, I believe, has noticed this, because if they had ... they would have dropped everything in order to deal with it; because there is nothing greater that man could do ([146, p. 57]).

While investigating the problem of undecidability of intuitionistic predicate calculus, Andrzej Mostowski proposed a novel approach to semantics by means of algebraic models with logical values in a pseudo-Boolean algebra [147]. This approach was further studied and extended to investigation of the properties of logics by Helena Rasiowa, Roman Sikorski [196] and many of their disciples (see, e.g., [193,194,19,20,37,60,4,142, 39,162,188]).

After the Second World War, Kazimierz Kuratowski also played a key role in the reconstruction of the Polish mathematical and logical school. He founded the State Institute of Mathematics (PIM)[3] and was its director from 1948 to 1968. At the very beginning of the PIM existence, Kazimierz Kuratowski came up with an initiative to build the first computer in Poland. To do that, he organized the 23rd December 1948 historical meeting attended by Andrzej Mostowski, Krystyn Bochenek, Henryk Greniewski, Leon Łukaszewicz and Romuald Marczyński and created within PIM the Group of Mathematical Apparatuses (GAM). During the meeting, Kazimierz Kuratowski announced that Polish mathematics should occupy itself with computing machines and that the meeting goal was to discuss the possibilities and plans for the construction of the first computer in Poland. As a result of the decisions made afterwards, a preliminary program started in 1952 aimed at building the first Polish computer within GAM [131]. In this project supervised by Romuald Marczyński, Zdzisław Pawlak participated. The first step in developing Polish electronic computers was the construction of mercury-based ultrasonographic memory. The choice of this kind of memory was influenced by the intention to build a computer with sufficiently high speed. Mercury memory influenced the construction of a sequence of Polish computers (EMAL, XYZ, EMAL-2, and BINEG) that continued until 1959. As a result of these experiments, works on first Polish computers were also started up at the Warsaw University of Technology, where a project and a prototype of a first generation vacuum-tube computer was developed in 1960. The prototype was later improved and initiated the UMC-1, the first serial production of computers in Poland. These computers were designed on the basis of an original arithmetic with base '-2' proposed by Zdzisław Pawlak, who engaged in research on models of computer architecture and the summary of his research results was published in 1963 in his habilitation thesis entitled *"Organization of Address-Less Machines"*. In this thesis, Zdzisław Pawlak proposed the investigation of parenthesis-free languages, a generalization of Polish notation introduced by Jan Łukasiewicz in 1924 (see, e.g., [23]). In further stages of his research on computational models, Zdzisław Pawlak paid more and more attention to logical as-

[3]Later incorporated into the system of Polish Academy of Sciences and currently called Mathematical Institute of Polish Academy of Sciences.

pects of computational models. His interests, at that time, were greatly influenced by a disciple of Andrzej Mostowski, namely, Andrzej Ehrenfeucht, who had particular interests in applications of games to problems of definability and decidability (especially in the theory of ordinals) between the 1950s and the 1960s. The results obtained by Andrzej Ehrenfeucht have many applications in, *e.g.*, studies in modern complexity theory (Ehrenfeucht–Fraïssé games).

Since the 1970s, Zdzisław Pawlak's interests were particularly focused on information retrieval, knowledge representation systems and, next, on the logical foundations of design and construction of algorithms devised to represent and process complex and vague concepts using computers operating on data in two-valued logics.

In the early 1970s Zdzisław Pawlak, in cooperation with Victor Marek and Witold Lipski, started investigations on mathematical foundations of information retrieval [166]. Intensive investigations led to deep results in the area (see., e.g., [132,121–123,81]). The close cooperation of Zdzisław Pawlak with Victor Marek continued for more than a decade. In particular, in the early 1980s, Victor Marek was a member of a research group at the Institute of Computer Science of the Polish Academy of Sciences, where Zdzisław Pawlak discovered rough sets and the idea of classifying objects by means of their attributes [167–169]. Zdzisław Pawlak also closely cooperated with many other researchers. We would like to mention here his close cooperation with Ewa Orłowska, Erhard Konrad, and Cecylia Rauszer on knowledge representation systems and rough sets (see, e.g., [157,158,102,103,160,159,171,134,87,217,201]). During the succeeding years, Zdzisław Pawlak refined and amplified the foundations of rough sets and their applications,[4] and nurtured worldwide research in rough sets that has led to over 4000 publications.[5] As a result of this research, rough set theory and its diverse applications, especially in representing and handling complex, vague concepts and perceptions, has emerged and flourished during the recent years (see, e.g., [170,172–174]).

The Rasiowa–Pawlak school has been created on the basis of nearly forty years of seminars along with lectures at Warsaw University given by Helena Rasiowa from the 1950s to the 1990s. The principal threads of the school consisted of the so-called Tuesday seminars on logic and Thursday seminars on application of logic to foundations of computer science. Many people participated actively in these seminars. The majority of the participants are currently scattered across the globe. The authors would like to emphasize the special and unique atmosphere of beneficent cooperation amongst participants in these events as well as the ample and fruitful discussions on the research problems current during that time. The school has managed its own international journal entitled Fundamenta Informaticae, initiated principally by Helena Rasiowa and Zdzisław Pawlak. Since 1977, Fundamenta Informaticae has been one of the main research presentation platforms of the Rasiowa–Pawlak school. Its topical scope is quite broad and includes the majority of research trends in artificial intelligence, logic, mathematics, and theoretical computer science. It is, hence, difficult to describe, even briefly, all these trends. We therefore do not pretend to provide all streams of the Rasiowa–Pawlak school research in this paper. Instead, we aim at surveying of certain aspects of this research related to the application of logic in AI, so characteristic to the Rasiowa–Pawlak school. As a result, many important trends of the school such as algorithmic logics, natural de-

[4]For example, Zdzisław Pawlak received the Best Paper Award for the paper on rough sets at the ACM 23rd Annual Conference on Computer Science in Nashville, TN, USA in 1995.

[5]See http://rsds.univ.rzeszow.pl/.

duction and non-classical-logic-resolution-based reasoning systems, logical aspects of concurrent processes, application of universal algebra to computational models, algebraic aspects of non-Fregean logics will be omitted. While selecting materials for this paper, we would like, first of all, to underscore the research directions on logics in the Rasiowa–Pawlak school, which we deem particularly important for the further advancement of AI. The interested readers are encouraged to consult many other detailed works, especially on the algebraic approach (see [193,194,196,19,20,4,162,37,198–200] and http://rsds.univ.rzeszow.pl/).

In the context of the studies on application of logic to computer science conducted in the 60s and 70s, in one of these trends, a special stress was placed on understanding algorithms and computer programs in purely logical terms. For this task, a special logic called algorithmic logic [4,142] has been developed within the school, and the programming language LOGLAN based on this logic was invented [105].

It is difficult to present a complete list of researchers whose investigations were substantially influenced in different periods of their scientific activity by Helena Rasiowa or Zdzisław Pawlak. Among them are Lech Banachowski, Mohua Banerjee, Wiktor Bartol, Jan Bazan, Malcolm Beynon, Leonard Bolc, Gianpiero Cattaneo, Mihir Kumar Chakraborty, Newton da Costa, Andrzej Czyżewski, Wiktor Dańko, Piotr Dembiński, Patrick Doherty, Jan Doroszewski, Albert Dragalin, Didier Dubois, Ivo Düntsch, George Epstein, Anna Gomolińska, Jerzy Grzymała-Busse, Petr Hajek, Tsutomu Hosoi, Masahiro Inuiguchi, Andrzej Jankowski, Jouni Järvinen, Jan Komorowski, Beata Konikowska, Bożena Kostek, Antoni Kreczmar, Churn J. Liau, Tsau Young Lin, Witold Lipski, Wing Liu, Witold Łukaszewicz, Larisa Maksimowa, Victor Marek, Antoni Mazurkiewicz, Ernestina Menasalvas, Grażyna Mirkowska-Salwicka, Mikhail Moshkov, Adam Mrózek, Maciej Mączyński, Daniele Mundici, Hung Son Nguyen, Cat Ho Nguyen, Sinh Hoa Nguyen, Tuan Trung Nguyen, Damian Niwiński, Hiroakira Ono, Ewa Orłowska, Sankar K. Pal, Eleonora Perkowska, James F. Peters, Jan Plaza, Lech Polkowski, Henri Prade, Andrzej Proskurowski, Halina Przymusińska, Slavian Radev, Sheela Ramanna, Zbigniew Raś, Cecylia Rauszer, Grzegorz Rozenberg, Leszek Rudak, Andrzej Salwicki, Giovanni Sambin, Dana Scott, Maria Semeniuk-Polkowska, Roman Sikorski, Dimiter Skordev, Andrzej Skowron, Roman Słowiński, Jerzy Stefanowski, Jarosław Stepaniuk, Zbigniew Suraj, Roman Suszko, Piotr Synak, Roman Swiniarski, Andrzej Szałas, Marcin Szczuka, Dominik Ślęzak, Helmut Thiele, Jerzy Tiuryn, Tadeusz Traczyk, Boris Trakhtenbrot, Shusaku Tsumoto, Paweł Urzyczyn, Dimiter Vakarelov, Alicja Wakulicz-Deja, Stanisław Waligórski, Quoyin Wang, Anita Wasilewska, Jakub Wróblewski, Wei-Zhi Wu, Urszula Wybraniec-Skardowska, JingTao Yao, YiYu Yao, Marek Zawadowski, Ning Zhong, Wojciech Ziarko, Tomasz Zieliński.

2. The Concept of Artificial Intelligence

Ever since its inception, the notion of *artificial intelligence* (AI) has been understood in a variety of ways. Along with advances in knowledge and studies in the world, the viewpoint on its understanding within the Rasiowa–Pawlak school has been changing as well. Currently the generally accepted understanding is in accordance with the classic introductory textbook to this field [207], viz.,

Premises and observations				Decisions and further actions		
Temperature	Visibility	Fuel	Clear road	Acceleration	Deceleration	Turn
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...

Figure 1. Decision table.

[we] define AI as the study of agents that receive percepts from the environment and perform actions.... The AI enterprise is based around the idea of intelligent agents – systems that can decide what to do and then do it.

There are many extensions and modifications of this classical definition of AI (see, e.g., [247]). The Rasiowa–Pawlak school concentrates especially on learning and improving of approximations of vague complex concepts (used on relevant levels of real-life problem solving) in dynamically changing environments (in which we have cooperating, communicating, and competing agents) by using uncertain and insufficient knowledge or resources.

The multitude in the number of different definitions of AI is a consequence of the divergence in the understanding of the concept of *intelligence* itself, both in regard to humans as well as machines. In this paper, *intelligence* is understood in accordance with the definition put forward by *Mainstream Science on Intelligence* and signed by 52 intelligence researchers in 1994 (*Wall Street Journal*):

[A] very general mental capability that, among other things, involves the ability to reason, plan, solve problems, think abstractly, comprehend complex ideas, learn quickly and learn from experience. It is not merely book learning, a narrow academic skill, or test-taking smarts. Rather, it reflects a broader and deeper capability for comprehending our surroundings – "catching on", "making sense" of things, or "figuring out" what to do (reprinted in [64, p. 13]).

It should be noted that there is a formal view of intelligence that has its origins in natural language and philosophy.[6] It is worthwhile mentioning that, as a natural consequence of the understanding of an agent's intelligence by Stuart Russell and Peter Norvig [207], the essence of this intelligence could be described by a decision function, represented in the form of a very large table with first columns describing attributes related to an agent's observations (*percepts*) and assumptions, and then last columns related to the proposed agent actions. In other words, the table could be as in Figure 1.

This kind of table may represent a driver's behavior on a road, or a physician's actions while treating a patient. In this case, the decision of *intelligent agents – systems that can decide what to do and then do it* is represented by rough concepts which can be employed to implement algorithms solving specific problems by means of the advanced rough set techniques proposed by Zdzisław Pawlak [167,170,172–174]. For instance, in [39], the application of such techniques in a control algorithm for unmanned helicopters, e.g., monitoring road traffic, is well illustrated. From another perspective that

[6]Intelligence. 1. The faculty of understanding. 2. Understanding as a quality admitting of degree. 3. The action or fact of mentally apprehending something [161].

keys on a principal notion associated with intelligence (i.e., understanding), there is *an individual's perception or judgment of a situation* [161] to consider. The original work by Zdzisław Pawlak on classification of objects and perception that has inspired research concerning what might best be described as perceptual intelligence, adaptive learning and what is known as rough ethology (see, e.g., [175,176]).

The difficulty with approximation of vague concepts or perceptions lies in, among other things, the fact that we do not have precise mathematical definitions for those concepts, but only partial information based on limited knowledge of the features of objects and the concepts. Moreover, this information is usually imprecise, noisy, biased, insufficient, and is a subject of dynamic changes.

3. The Concept of Logic

The concept of *logic* has been intensively studied since ancient times. The dominating understanding of this concept within the Rasiowa–Pawlak school is due to Alfred Tarski [233], who in the first half of the 20th century initiated studies on abstract understanding of the consequence operation and of the satisfiability relation. The classic paper [234] contained the following statement:

> *The term "semantics" denotes certain relations between a language's expressions and objects.*

In [235] as well as in many other papers, Tarski wrote:

> *Semantics is a discipline which, speaking loosely, deals with certain relations between expressions of a language and the objects (or "states of affairs") "referred to" by those expressions.*

Elsewhere in the paper, he wrote:

> *... the words "designates", "satisfies", and "defines" express relations (between certain expressions and the objects "referred to" by these expressions)...*

Since the early works [233], Alfred Tarski had paid much attention to deduction theory in his research. He even invented theories in which primary notions were propositions and a consequence operation satisfying the so-called Tarski axioms for consequence.

Taking into consideration the traditions of the Polish school of logic already mentioned, by the term *logic* we intuitively understand a structure of the form

$$< M, L, \models, D >, \tag{1}$$

where $\models$ is any relation between the class M of models and the class L of language expressions, and D is a deduction system. Łukasiewicz's disciples stress here that the relation $\models$ is not necessarily a (partial) function from the product $M \times L$ into the two-element Boolean algebra. In addition, $\models$ can have values in some multi-valued structure. In this context, a little more general concept is investigated within the Rasiowa–Pawlak school. Namely, by an abstract *multi-valued logic structure* we understand a system consisting of the following components:

1. A class of *admissible worlds* including real or imaginary objects which could be used as models representing our knowledge. We assume that each admissible

world has an assigned set of *logical values*; in classical mathematics only two logical values *true* and *false* are considered.
2. A set of *expressions* used as a language for representation of our thoughts about properties and phenomena in the admissible worlds.
3. A *truth function* being a (partial) function which for every admissible world A assigns another function from the set of all relevant expressions into the set of logical values of A; this function enables us to estimate of the degree of credibility and verifiability of expressions in each admissible world. Put another way, Tarski writes that this means that *the truth or falsehood of any sentence obtained from that function by substituting whole sentences for variables depends exclusively on the truth or falsity of the sentences that have been substituted* [239]. In general, a truth function is a propositional function of truth values [34].
4. A *deductive system* that enables us to draw inferences about an admissible world, based on credibility and verifiability of knowledge about that world; in general, we can assume that it is a closure operation [144].

It is easy to imagine many examples of abstract multi-valued logical structures and related research problems such as completeness, compactness, theorem proving, concept approximation, as well as interpolation.

Example 1. Let us consider as expressions of a logical structure a set of sentences (i.e., formulas without free variables) of a theory T of the classical predicate calculus. The class of admissible worlds could be just the class of all relational structures for the theory T. Logical values are: *true* and *false*, i.e., the elements of two-element Boolean algebra.

Example 2. Let us consider as expressions the set of all formulas with free variables from an infinite set of variables V of a theory T of classical predicate calculus. Then, the admissible worlds can be once again the relational structures. However, the set of logical values and the truth function should be slightly more complex. Namely, for each relational structure $\mathcal{A}$, let S be the set of all valuations of free variables V into the universe of $\mathcal{A}$ (i.e., functions from V into the universe of $\mathcal{A}$). Then, the set of logical values of $\mathcal{A}$ is the set of elements of the Boolean algebra of all subsets of S. Let the truth function for $\mathcal{A}$ assign to each formula p, the set of all valuations v, making p true in the structure $\mathcal{A}$. Thus, for the classical predicate calculus, the class of admissible worlds may contain more than two logical values.

Although Alfred Tarski was probably the first to investigate the general concept of satisfiability as a binary relation, and Jan Łukasiewicz and Emil Post were the first to investigate multi-valued semantical structures, it may be worthwhile mentioning that these concepts are currently widely used in mathematics and AI under various names. Notice that ubiquitous mathematical concepts such as matrix and table may be considered as multi-valued semantical structures where

- **Worlds**: matrix row indices $\Rightarrow$ *Admissible worlds*,
- **Expressions**: matrix column indices $\Rightarrow$ *Set of expressions*,
- **Truth Values**: matrix values $\Rightarrow$ *truth values*.

From the viewpoint of the Rasiowa–Pawlak school, such tables represent the principal concept of rough sets called *Pawlak's information system* [170,172], in which *admis-*

sible worlds are the objects of the system, while *expressions* are functions *representing* attributes. The *truth function* can be used in the context of a truth table to evaluate the truth or falsity of a value associated with the function values associated with an attribute of a particular sample object. Notice that if we treat decision tables (such as presented in Figure 1) as an essence of AI then multi-valued logical structures could be also treated as an essence of AI.

However, putting aside the philosophical context, from a purely formal viewpoint, if a language possesses denotations for logical values and a logical equivalence relation, then abstract multi-valued logical structures may be considered as logical structures with two logical values, in which the fact that a formula has an intermediate value v can be expressed as it is true that this formula is equivalent to v. Obviously, this maneuver has a merely formal character and usage of many logical values directly is more convenient in many situations. For example, Boolean multi-valued models could be very convenient for an interpretation of the Heisenberg uncertainly principle (see [13, pp. 156–157]) and for proving of independence of axioms of set theory (see [13]).

In the case of abstract multi-valued logical structures having only two logical values, we simply refer to them as abstract logical structures [84,85].

The approach to the semantics of classical predicate calculus presented in the second example is a very special case of a more general semantics known as the Rasiowa–Sikorski Boolean models [193,194,196]. The concept of Rasiowa–Sikorski Boolean models for set theory was applied by Dana Scott and Robert Solovay in an elegant method in the proof of independence of the axiom of choice and the continuum hypothesis from the axioms of the ZF set theory [208,209,13]. However, the original proof of the independence of the axiom of choice done by Paul Cohen [35] used Kripke semantics. In 1973, Denis Higgs, and independently Dana Scott and his students [50], generalized the Rasiowa–Sikorski Boolean models to the case of category theory, and especially to topos theory [73,50]. In particular, Denis Higgs defined the notion of Ω-set and established for a complete Boolean algebra B, the equivalence of the topos of B-sets both with the category of sets and maps in the Boolean extension $V(B)$ of the universe of sets and with the category of canonical set-valued sheaves on B. Topos-based semantics is now used as a uniform generalization of the Rasiowa–Sikorski models and Kripke-style semantics [241,130]. For example, in [130, p. 277], one can find the concept of Cohen topos which is a powerful tool for analysis of the independence of axioms of set theory. There are many other applications of Rasiowa–Sikorski Boolean models. From the point of view of reasoning under uncertainty in AI, there is an interesting and yet not very popular application of the Rasiowa–Sikorski Boolean models to interpretation of uncertainty in terms of quantum theory (see [36] and [13, pp. 156–157]). This approach could be a starting point for a better understanding of imprecise and vague complex concepts by means of the Rasiowa–Sikorski Boolean models. For authors of this article, especially interesting is the next step in this direction, namely, exploring possible applications of the Rasiowa–Sikorski Boolean models in the discovery of the ontology of patterns in time series based on combination of wavelets, quantum mechanics, granularity, and fractal geometry [25, p. 25]. Next, these patterns are used for approximation of concepts or percepts that paves the way toward making predictions and economical or financial decisions.

One of the research directions in the Rasiowa–Pawlak school is a the characterization of relationship of classical logic to other logic. For better understanding of the role of classical logic, it is important to identify relationships between logics, in particular, between classical logic and non-classical logics. It is worthy to note that important stimuli to these studies in the Rasiowa–Pawlak school were papers by Jan Łukasiewicz [23,196] (interpretation of classical logic in intuitionistic logic by double negation), Andrzej Mostowski [148] (introduction of the concept of generalized quantifiers), and Rasiowa–Sikorski (relationships of classical predicate calculus with intuitionistic and modal predicate calculus (see, e.g., [196, pp. 408, 485]). From the point of view of applications of logic to AI, especially interesting are intuitionistic logic and its relation to classical logic [241]. For many years, intuitionistic logic has been explored as a framework for computer science foundations. Using societies of intelligent agents for modeling in AI is strongly related to a very interesting principle known as the Brouwer–Heyting–Kolmogorov interpretation (of intuitionistic logic). Under this principle, intuitionistic proofs of implicative formulas are functions and the existence of proofs requires witnesses – agents [241]. Another idea interesting for the AI foundations is Kolmogorov's interpretation of intuitionistic implication as a reduction problem [241, p. 31]. The history of relationships between intuitionistic logic and classical logic is very old. In particular, Glivenko's Theorem [61] (discovered independently by Jan Łukasiewicz [23]) says that: *An arbitrary propositional formula A is classically provable if and only if $\neg\neg A$ is intuitionistically provable*. For other translations of propositional calculus see, e.g., [241] and [152]. Glivenko's Theorem cannot directly be extended to predicate calculus, although there are some forms of this theorem which use special types of modification of the Glivenko negative translation (i.e., Gödel-Gentzen [241], Gödel [241], Kolmogorov [241], Kuroda [241], Kleene [99], Rasiowa–Sikorski [196]). Andrzej Mostowski [148] stimulated a different dimension of characterization of classical logic, viz., a characterization in terms of extension of this logic by some infinite logical connectives. Examples of results in this direction can be found in [120,6,7,16,28].

There is a characterization of classical logic obtained in the Rasiowa–Pawlak school in terms of relationships to other logics. Namely, *any logical structure $\mathcal{L}$ with countable set of formulas is embeddable into a classical logical structure if and only if $\mathcal{L}$ satisfies compactness and completeness theorems* [84,85]. This result, in some sense, could be treated as a reverse theorem to Glivenko's Theorem for propositional calculus and its modifications for the predicate calculus.

4. Roots of the Algebraic Approach to Logic by the Rasiowa–Pawlak School

The core techniques that constitute the roots of the algebraic approach to logic by the Rasiowa–Pawlak school were developed in the 1950s and 1960s, and have been further extended in later years. Since the very beginning, these techniques have been focused on the following topics:

1. Development of algebraic methods used in search of the most relevant semantical structures, i.e., structures that enable to search efficiently for constructive (algorithmic) problem solutions in specific types of domains (e.g., applications of modal logic to knowledge representation in multiagent systems, logical reasoning about rough concepts, logical models for quantum computation), in particular research on semantics and inference rules.

2. Algebraic construction of *canonical world* for a consistent set of expressions and application of this technique to the study of equivalence between deductive inference and inference based on the truth relation (the completeness theorem).
3. Construction and analysis of alternative deductive systems for a given logic, for example based on modification of approaches invented by Stanisław Jaśkowski, Gerhard Gentzen, David Hilbert, Jacques Herbrand, and Helena Rasiowa together with Roman Sikorski [196,128].
4. Analysis of the fundamental model-theoretic properties of the admissible worlds for a given logical structure (e.g., the Skolem–Löwenheim theorem, the compactness theorem, the omitting-types theorem).
5. Research on the *geometric* properties of the *space of models* [54,205,250]. In this framework, by a space of models we understand a space which has models (Q-filters in the Lindenbaum–Tarski algebra) as points and a topology generated by Stone's representation theorem. One can consider also a distance between points measuring similarity of models. This is a metaphor of Stone's representation theorem for Boolean algebras [211]. One of the most exciting intellectual experiences for the authors of this article is a proof of the Rasiowa–Sikorski Lemma using topological properties of the *space of models* (using the Baire property). Beside of this application of topological methods to proving the completeness theorem, there are many other very interesting logical properties of the *space of models* implied by the topological properties of this space. They concern results characterizing open theories and mechanisms for the construction of Herbrand alternatives [196].

Basically, tools employed by the Rasiowa–Pawlak school are, in a natural way, a continuation of the idea of concept calculus proposed by Gottfried Wilhelm Leibniz, and later developed by George Boole and his disciples, in particular by Alfred Tarski and Adolf Lindenbaum. The evolution of tools employed in the algebraic research of different aspects of logic can be summarized by the scheme presented in Figure 2.

To introduce with acuity the key elements characteristic to the algebraic approach to logic, typical for many trends in the Rasiowa–Pawlak school, let us assume that Q denotes a set of abstract logical operators representing some logical connectives. In the case of classical logic, the most important ones are conjunction, disjunction, implication, and negation. As it was noticed by Adolf Lindenbaum and Alfred Tarski, if we glue together all the sentences which represent the same thought in a deductively closed theory in classical logic, we get a Boolean algebra. Intuitively, this is an algebra of thoughts for the theory. Algebraic operators in the algebra of thoughts correspond to logical connectives. In particular, provable implication corresponds to a partial order of thoughts.

Greater value in the order intuitively means *more true*. Disjunction corresponds to supremum in the order generated by the implication, and conjunction corresponds to infimum in the same order. If we treat the existential quantifier of classical logic as infinite disjunction, then its algebraic interpretation is the supremum. Similarly, if we treat the universal quantifier as infinite conjunction, then it corresponds to the infimum.

The set of abstract logical operators represented by propositional connectives can be generalized to a set Q consisting of more logical operators. In the standard way, we define the concept of Q-algebra that includes all operators from the set Q [193,194], and concept of Q-homomorphism between two Q-algebras which preserves all Q-operators. In this paper, we assume that Q is enumerable and each Q-algebra has a special constant

Domain & Operators	*Natural Artthmetic*	*Algebra of subsets*	*Boolean Algebra*	*Logical concepts in Lindenbaum – Tarski algebra*	*Semantical models for constructive mathematics*	*Topoi*	*Wisdom Granular Computing for a given application domain*
X < Y	X is smaller than Y	X is a subset of Y	X is smaller than Y in Boolean algebra	Y could be deduced from X	Logical value of X is smaller than logical value of Y in a Heyting algebra	Morphism from X to Y	Wisdom granule Y is a consequence of wisdom granule X in the domain
0	Zero	Empty set	The smallest element	False	0 in Heyting algebra	Initial element	Smallest wisdom granule in the domain
1	One	Full set	The biggest element	True	1 in Heyting algebra	Terminal element	Biggest wisdom granule in the domain
+	Addition	Join of two sets	Maximum	Disjunction	Maximum	Coproduct	Relative coproduct of two wisdom granule
*	Multiplication	Intersection of two sets	Minimum	Conjunction	Minimum	Product	Relative product of two wisdom granule
X^Y	Exponentia-tion X to power Y	Join of (–Y) and X	Join of (–Y) and X	Implication (Y implies X)	Relative pseudo – complementation Y→X in Heyting algebra	Object corresponding to all morphisms from Y to X	Granule corresponding to all consequences from granule Y to granule X
Mod (X)	Modulo X calculus	Quotient algebra of the filter generated by set X	Quotient Boolean algebra of the filter generated by set X	Lindenbaum – Tarski algebra for a theory generated by a set of axioms X	Models for a theory generated by axioms X	Category of sheaves over X	All consequences from a given granule X
Logical values	True False	True False	True False	Algebra of logical values	Elements of Heyting algebra	Subobject classifier	Identification of subgranules of granules

ANCIENT	**CONTEMPORARY**	**FUTURE**

Figure 2. Evolution of computational models of logical concepts from the Rasiowa–Pawlak school perspective (the last column is hypothetical for further research).

1 for the logical value of truth. An interesting introduction to logics based on Q-algebras with implication (extensions of implicative algebras) can be found in [193,194].

We say that an abstract multi-valued logical structure is a Q-*logical structure*, if

– the *algebra of thoughts* for any theory is a Q-algebra,
– the logical values for each admissible world form a Q-algebra,
– the truth function is defined using a Q-homomorphism from the set of expressions into a Q-algebra of logical values for each admissible world.

The main idea of the algebraic approach to predicate calculus in the research papers of Helena Rasiowa and her students is based on two concepts:

– Q-*representability*,
– Q-*characterizability*.

These concepts have been used in several variants and have been applied as the main tool for solving of several important problems.

In order to introduce both concepts let us assume that we have two classes M and K of Q-algebras. We say that M is Q-*representable* by K, if for any algebra A from the class M there exists a Q-isomorphism from A to a Q-algebra in K. For example, if Q includes standard finite Boolean operators, then Stone's representation theorem for Boolean algebras means that the class of Boolean algebras is Q-representable by the class of all of Q-fields of sets. The Stone representation theorem has been generalized for infinite Boolean operators by Rasiowa and Sikorski. If Q is a set of standard finite Boolean operators and a countable set of operators corresponding to infinite infima and suprema in such Q-Boolean algebras, then the Rasiowa–Sikorski representation theorem [196,197] says that the class of all such Q-Boolean algebras is Q-representable by the class of all Q-fields of sets. By the definition it means that for any countable set Q of infima and suprema in a Boolean algebra, there exists a Q-isomorphism to a Q-field of sets.

We say that M is Q-*characterizable* by K, if for any algebra A from the class M and for any element x of A such that x is different from 1, there exists a Q-homomorphism from A to a Q-algebra from the class K such that the value of x after the transformation by the Q homomorphism is different than 1.

For example, if Q includes standard finite Boolean operators and K has only one two-element Boolean algebra, then each Q-homomorphism from a Boolean algebra A into the class K can be identified with a Q-prime filter of A, and any element a of A can be identified with a set of all Q-prime filters including a. Originally, the Rasiowa–Sikorski Lemma was expressed as follows: If Q is a set of standard finite Boolean operators and a countable set of operators corresponding to infinite infima and suprema in such Q-Boolean algebras, then the class of all such Q-Boolean algebras is Q-characterizable by the class which has only one two-element Boolean algebra.

The logical meaning of the concept of Q-representability of the class of Q-algebras of thoughts by the Q-algebras of logical values for admissible worlds of a logical structure L is that L satisfies the completeness theorem. The concept of Q-representability of the class of Q-algebras of thoughts by the Q-algebras of logical values for admissible worlds of L is used for construction of special admissible worlds called *canonical models*.

We say that a Q-logical structure L is a *Rasiowa–Sikorski logical structure* if the class of Q-algebras of thoughts is Q-characterizable by the Q-algebras of logical values for admissible worlds of L. There are several examples of Rasiowa–Sikorski logical structures based on logics such as: intuitionistic [196], Heyting–Brouwer [198], intermediate [199], modal [196,193,194], Post [193], semi-Post [194] and algorithmic [4,142], autoepistemic [200], knowledge for groups of agents [202,203,53,183].

It is possible to apply several schemes of analysis to the research of Rasiowa–Sikorski logical structures. An excellent compendium of such schemes can be found in [196,193,194]. The schemes can be used for typical model-theoretic applications such as the completeness theorem or construction of the canonical model, and also for not so typical ones like natural deduction systems in the Rasiowa–Sikorski style [196,128], or the proof of decidability of formulas in *prenex form* for constructive intuitionistic theories [196].

Along with the emergence of the concept of topos, generalizing the semantics of Boolean and pseudo-Boolean models, as well as the appearance of other approaches like Kripke semantics or Beth tableaux, interests to these methods have also grown within the Rasiowa–Pawlak school. On one hand, opportunity to generalize techniques characteristic for algebraic and Kripke models to topos has been investigated [60]. On the other hand, internal representations of some logics in others have also been researched [88].

5. Logic for Reasoning and Knowledge

In the early stages, reasoning was understood within the Rasiowa–Pawlak school as natural deduction systems by Stanisław Jaśkowski, Genzen's sequent calculus systems, Herbrand disjunctions, and their modifications [19,20,192,240]. In particular, in the book by Helena Rasiowa and Roman Sikorski [196], an elegant deduction system for predicate calculus, called in literature the Rasiowa–Sikorski deduction system [128], has been presented. Much effort has also been devoted to the deduction systems in non-classical logics. An exemplary survey containing many results in this field can be found in [19,20]. Various aspects of logic programming and deduction in quantum logics have also been studied [138,154].

Together with the advancement in understanding the specifications of complex vague concepts, a gradual shift of the center of studies has taken place, toward approximation logic and approximate reasoning (see, e.g., [194,133,195,200,202,19,20,188,162]).

In practical applications, there has been a still-increasing need for approximate reasoning about concepts and objects on the basis of incomplete, partial, biased and ever changing information about them. The concepts themselves are usually vague. Methods for approximate reasoning about such concepts based on rough sets have been developed. Problems considered in applications can usually be reduced to the construction of objects adhering to vague specifications to a satisfactory degree (e.g., fuzzy sets [252–254], rough sets [167,170,172–174], rough mereology [187,189]). As a result, constructed approximate solutions are significantly easier to obtain than exact solutions, which often prove impossible to attain due to the lack of exact model or the inhibiting costs associated with their computing. This way of thinking seems to be popular in humans while solving problems. The research on searching for approximate solutions adhering to vague specifications to a satisfactory degree is therefore considered one of the principal advanced directions in developing better intelligent systems (see, e.g., [177,178]). Rough set based methods have proven effective in relatively simple applications such as searching for reducts and relevant features for a given phenomenon, as well as in highly complex situations such as image recognition and processing (see, e.g., [21,22,180]), biologically-inspired adaptive learning controllers (see, e.g., [178,180]) or control and cooperation among autonomous robots monitoring and repairing power lines.[7] An interesting application is the design for a control system of an autonomous helicopter, where a joint approach between rough sets and nonmonotonic reasoning has been presented [39].

Reasoning considered as formal operations on features of linguistic features has high complexity and thus identification of formulas (or abstraction classes understood as granules), to which the reasoning can be limited to, is of great concern. However, this kind of applications requires specific knowledge and wisdom representation systems, as well as the resignation from hitherto existing formalizations of language in classical logic. In this context, an interesting trend in the seminars by Rasiowa consisted of attempts referring to pre-WWII ideas by Stefan Banach and his collaborators to use geometrical methods (relying on a switch from language to topological concept space) [196,54,205,250]. Another, also appealing direction, pertaining to Wittgenstein's proposal, was to look at language and satisfiability relation by means of the apparatus of game theory [251,210, 74]. These works laid the ground for a proposal of algebraic models for non-Fregean logics by Roman Suszko [232,18]. Another application of game theory, from a bit different perspective, to investigate features and reasoning in intuitionistic logic can be found in [60,74]. Game theory can also be applied to approximate reasoning about granular computing [91].

[7]For more details on applications the reader is referred to the bibliography in [172–174] and http://rsds.univ.rzeszow.pl/.

By the term *knowledge*, we understand a system consisting of information, its internal relations and inference rules making it possible to reason about phenomena occurring in a domain of the world. Ever since its inception, one of the main problems in AI has been the construction of effective knowledge representation systems. There have been many paradigms of knowledge representation in both structured systems (e.g., frames, rule-based systems, semantic networks, logic programming) and non-structured systems (e.g., neural networks, genetic algorithms, ant systems). With the unprecedented expansion of the Internet, there is a growing demand for effective systems to represent, process, search and provide information in texts, images, audio and video recordings. One of the many effective tools for knowledge representation, in particular for vague concepts description, are rough set based systems and their generalizations. Some of such tools were employed by the authors to build in 2000 an Internet search engine called Excavio capable of communication with users through simple dialogs using phrases. Recently, this kind of technology has became more and more popular in text mining [1,3, 32,48,69,72]. The main idea of these applications relies on using automatic document clustering algorithms in regard to sets of sequences of phrases potentially interesting to searchers. The precursors of an interesting direction of application of rough set methods to Internet search engines were the Japanese [76]. This paper was based on ideas presented in [218]. The algorithms were also developed and published by members of the Rasiowa–Pawlak school [150]. Vastly interesting systems applying paradigms of rough sets and algebraic aspects of logic have also appeared in Japan, China, India, Sweden, and Canada [172–174].

Within the Rasiowa–Pawlak school a system for representing knowledge from many fields and for reasoning about this knowledge primarily by means of rough set paradigms has been designed and implemented. The system can be accessed on the server of Warsaw University and is presented at the Internet address: http://logic.mimuw.edu.pl/~rses/.

Independently from knowledge representation systems using the rough set approach, another important trend in the Rasiowa–Pawlak school is a series of research on autoepistemic logics and application of logic to the negotiation in a society of cooperating or competing agents. These works are in a natural way related to the fundamental intuitions of modal operators in modal logic. Interesting results in this area can be found in [202,195,183].

6. Mathematical Approach to Vagueness

One of the fundamental motivations for Jan Łukasiewicz in inventing multi-valued logics was his belief that not all propositions can be considered as either true or false, due to the impossibility of humans to predict certain events. A standard example frequently used in Łukasiewicz's works is the sentence *Jan will be in Warsaw next year* [23]. Jan Łukasiewicz, referring to the indetermination principle in quantum mechanics, stated that assigning to this sentence either truth or falsehood is abusive. His works initiated, intensely developed until now, a research direction pertaining to *uncertainty* by means of multi-valued logics. Among others there is an interesting attempt to combine approximation logic based on rough set paradigm with Post's multi-valued logics [194].

Mathematics requires that all mathematical concepts (including set) must be exact, otherwise precise reasoning would be impossible. However, philosophers and, recently, computer scientists as well as other researchers have become interested in vague (imprecise) concepts.

We would like to remind that modern understanding of the notion of vague (imprecise) concept has a quite firmly established meaning context including the following issues [98]:

1. The presence of borderline cases.
2. Boundary regions of vague concepts are not crisp.
3. Vague concepts are susceptible to sorites paradoxes.

Moreover, it is usually assumed that the understanding (approximation) of vague concepts (their semantics is determined by the satisfiability relation) depends on the agent's knowledge, which is often changing. Hence the approximation by agent of vague concepts should also be considered changing with time (this is known as the concept drift). In the 20th century, it became obvious that new specialized logical tools need to be developed to investigate and implement practical problems involving vague concepts. One of such trends are rough sets.

Rough set theory, proposed by Zdzisław Pawlak in 1982 [167,170] can be seen as a new mathematical approach to vagueness. The rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, measurements, observations, patterns, knowledge). For example, if objects are patients suffering from a certain disease, symptoms of the disease from clinical observations and diagnoses about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take on them identical values (Leibniz's Law of Indiscernibility: The Identity of Indiscernibles) [117]. However, in the rough set approach indiscernibility is defined relative elementary sets of objects with matching descriptions based on values of functions representing selected attributes thought to be inherent in objects or feature-based measurements based on the appearances of objects [182]. Any set of all indiscernible (similar) objects is called an elementary set, and forms a basic granule (atom) of knowledge about the universe. Any union of some elementary sets that is a subset of a set of interest (e.g., set of objects X that are considered somehow acceptable) is referred to as crisp (precise) set; otherwise, for those elementary sets that have a non-empty intersection with a set X of interest and its complement, the attention turns to the boundary of X. Whenever the boundary is not empty, then the set X is considered rough. The size of the boundary provides a measure of the vagueness of our knowledge about X. Consequently, each rough set has boundary-line cases, i.e., objects which cannot with certainty be classified either as members of the set or of its complement. Obviously crisp sets have no boundary-line elements at all. This means that boundary-line cases cannot be properly classified by employing available knowledge. Thus, the assumption that objects can be *seen* only through the information available about them leads to the view that knowledge has granular structure. Due to the granularity of knowledge, some objects of interest cannot be discerned and appear as the same (or similar). As a consequence, vague concepts, in contrast to precise concepts, cannot be characterized in terms of information about their elements. Therefore, in the proposed approach, we assume that any vague concept is replaced by a lower and the upper approximation of the vague concept. The lower approximation consists of all objects which surely belong to the concept and the upper approximation contains all objects which possibly belong to the concept. The difference between the upper and the lower approximation constitutes the boundary region of the vague concept. Approximations are two basic operations in rough set theory. Hence, rough set theory expresses vagueness not by means of membership, but by employing a boundary region of a set. If the boundary region of a set is empty it means that the set is crisp, otherwise the set is rough (inexact). A non-empty boundary region of a set means that our knowledge about the set is insufficient to define the set precisely. Rough set theory is not an alternative to classical set theory but is embedded in it. Rough set theory can be viewed as a specific implementation of Frege's idea of vagueness, i.e., imprecision in this approach is expressed by a boundary region of a set. The boundary region is defined relative to a given set of attributes, i.e., it could change if the set of features changes. Moreover, the boundary region is changing according to data access. Hence, one can see that boundary region cannot be defined by only one crisp concept. In [204,179], sorites paradoxes in the framework of rough sets are discussed.

Rough set theory has attracted attention of many researchers and practitioners all over the world, who have contributed essentially to its development and applications. Rough set theory overlaps with many other theories. Still, rough set theory may be considered as an independent discipline in its own right. The rough set approach seems to be of fundamental importance in artificial intelligence and cognitive sciences, especially in research areas such as machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, knowledge discovery, decision analysis, and expert systems. The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data like probability distributions in statistics, basic probability assignments in Dempster-Shafer theory, a grade of membership or the value of possibility in fuzzy set theory. It should also be observed that functions representing either attributes or features of objects do reflect *a priori* reflection about the appropriate choice of functions that represent selected features of observed objects such as widgets and organisms such as beetles or ants or attributes of conceptual objects such as population demographics or numbers in a statistical study. In effect, this signals a strong tie between fuzzy set and rough set theory, since each feature (*e.g.*, *high, medium, low* relative to observed or anticipated distributions of values in an experiment) is represented by a membership function chosen beforehand in fuzzy set theory. One can observe the following aspects of the rough set approach:

- partition of sets of sample objects using some form of the indiscernibility relation,
- identifying neighborhoods of sample objects relative to elementary sets to facilitate perception and object recognition,
- approximation of each set of objects of interest with another set,
- introduction of efficient algorithms for finding hidden patterns in data,
- determination of optimal sets of data (data reduction),
- evaluation of the significance of data,
- generation of sets of decision rules from data,
- easy-to-understand formulation,
- straightforward interpretation of obtained results,
- suitability of many of its algorithms for parallel processing.

When recalling the sources motivating the creation of multi-valued logic, which inspired Jan Łukasiewicz during his analysis of uncertain concepts by means of intermediate logical values, we would like to stress that many works on rough sets were dedicated to the relations between rough sets and multi-valued logics by Jan Łukasiewicz and others (see, e.g., [37,135,162]).

Despite of the efforts to build computers operating on multi-valued logics (electronic as well as optical), currently the model based on two-valued Boolean logic is dominating. It is, hence, not the best architecture to implement concepts described in multi-valued non-classical logics. Although there are announcements on research on building quantum computers based on quantum logic which is a non-classical logic, they have not yet been realized for real-life applications [33,75,228]. Given this context, especially significant are numerous works attempting to find handy and effective computing models for the treatment of complex vague concepts. Solving this problem is the essence of the idea by Zdzisław Pawlak who, we remind, was the architect of one of the first of the world computers, based on a system different from the typical binary system, to enter serial production. Zdzisław Pawlak postulated that we should accept the fact that our conceptual apparatus is limited and we are able to describe the reality only by means of expressions comprehensible to us, whose truth value (as understood by two-valued logic) can be easily verified. Apart from such expressions, however, there exist expressions concerning complex vague concepts that cannot be easily verified (e.g., *a financial operation bears an unacceptable risk*, *Kowalski contracted disease X*, *the situation on a road is dangerous*). In such cases, we generally can express, in a language of easily verifiable concepts, which objects with certainty fall into the scope of the described concept (lower approximation) as well as which objects may fall into this scope with some certainty (upper approximation). This simple and obvious idea has become foundation for constructing highly effective tools supporting the representation and processing of vague concepts in practically all fields of applications [167,170,172–176].

Vague complex concepts are very often related to one another through a hierarchy induced by abstraction levels of these concepts. Such hierarchies occur, for instance, when some concepts are components of other concepts. In such a context, there is a great interest in investigating the relation *being a part-of*. It is a different approach from the ontology of modern mathematics (also known as *Cantor ontology*), which is based on the relation *being an element-of*. Such alternative ontology for mathematics was proposed by Stanisław Leśniewski [118,119] in 1929 and became the inspiration for an important research within the Rasiowa–Pawlak school called *rough mereology* [187,162]. Within this approach, the notion *rough inclusion relation* plays a central role. It describes to what degree some concepts are parts of other concepts. A rough mereological approach is based on the relation to be a part to a degree. It is interesting to note here that Jan Łukasiewicz in 1913 investigated the inclusion to a degree of concepts in his discussion on relationships between probability and logical calculi [23].

Certainly, the mereological approach is not the only one attempt for establishing links between vague concepts and rough sets. Among others are links based on treating vague concepts by means of logical values. In this case, one can build an algebra of such vague concepts as an algebra of logical values. Usually, this kind of algebra is a pseudo-Boolean algebra and the relationships between vague concepts can be expressed as relationships of logical values of an intermediate logic. Notice, that if we would like to prove relationships between some concepts in intuitionistic logic, then we should be able to use constructive evidence. For example, we cannot affirm that any two concepts x and y are either equal or unequal. In intuitionistic logic we have two kinds of *inequality*:

- *classical inequality* which means that x is equal to y leads to contradiction,
- *intuitionistic inequality* which is stronger and means that we can provide a constructive evidence which proves that x and y are not equal.

In particular, if x and y are real numbers, then in order to show that x and y are intuitionistically unequal it is necessary to provide a constructive evidence for that. In this case, we can construct an example of a rational number which separates x and y. In other words, our knowledge about relationships between concepts in intuitionistic logic requires constructive evidences. For example, if we show that the statement *John does not have pneumonia* leads to contradiction, then in intuitionistic logic it does not mean that *John has pneumonia* holds.

Having in mind the above remarks it is interesting to represent our knowledge in the framework of intuitionistic logic. In general, it is not easy because the free Heyting algebra is infinite, even in the case of one generator only. However, if we assume that:

- each concept can be represented by a finite set of objects and
- the universe of objects which can be used for representation of concepts is finite,

then some intermediate logics may be used instead of intuitionistic logic [87].

Naturally, the remarks above do not cover all aspects of techniques employed by the Rasiowa–Pawlak school in treating vague concepts. At the current stage, particularly intensive are studies on rough granular computing as well as in the direction of working out techniques for learning of complex vague concepts. We discuss these issues more broadly in the following section.

7. Logic for Learning, Communication, Perception, Planning, Action, Cooperation, and Competition

From the viewpoint of understanding of the definition of AI along the lines of Russell and Norvig [207], the essence of artificial intelligence is to equip an agent with a decision table using one complex concept that allows to implement the idea:

> *… intelligent agents – systems that can decide what to do and then do it.*

In Section 6, we outlined the idea of construction and description of vague concepts using rough sets. In other words, a solution to learning of complex vague concepts is also a solution to the fundamental goals in AI. In this context, developing techniques applicable to the AI field called machine learning is of primary concern. Unfortunately, we are still very far from fully satisfactory solutions, and the currently developed approaches to learning are suitable to deal with a *predictable* environment, where one can foresee which of the existing machine learning paradigms may perform the best. In practice, so far, it has not been possible to develop satisfactory methods for autonomous systems [124], neither to work out universal techniques that allow machine learning to be applied in environments with high unpredictability (we mean an unpredictability degree comparable to that of an ecosystem in which living organisms have, and successfully could, learn various concepts in order to adapt and survive) [57]. From the perspective of further advances in AI, it would be handy to have tools allowing to design, construct, and development of machines capable of:

1. Initialization of input knowledge provided in a simplified natural language based on an expert's domain knowledge.
2. Automatic planning of experiments and learning of new, complex vague concepts that allow better actions within the domain of operation of the machine.
3. The skilful and effective use of the possessed wisdom in order to perform best possible actions.

Basically speaking, these ambitious goals have come around in AI many times since its inception. They can be observed in various attempts from the discussions of Alan Turing's test [242] through programs that generate winning strategies in games [46,79, 223], discussion of key problems in knowledge discovery systems [114], to many other modern research trends in this direction [100]. Within the Rasiowa–Pawlak school there have also been many approaches to solving this crucial problem. The authors of this paper are supervising applied projects in medicine and economy, where they attempt to use a new, possibly universal approach to a better understanding of the path leading to solving the three problems mentioned above. The research concerns mainly construction of a hierarchical architecture for networks of multi-valued logical structures (see, e.g., [9–11,90,91,149,151,150,12,224]). These works intensively employ rough set paradigms, though it is worth mentioning that they bear meta-AI characteristics (analogously to meta-mathematical research within the Rasiowa–Pawlak school). We call this technology wisdom technology, in short. In a simplification, these techniques refer to one of the directions in KDD dubbed hierarchical multi-search discovery systems (see also, e.g., [92]). These issues are related to the fundamental problems of pattern recognition, machine learning and KDD, involving methods for the extraction of new features [100, 182,181].

In any case, the research trends mentioned previously were not equipped with two essentially new elements derived from wisdom technology, concerning simulation of judgment processes [68,96,184,211] and of processes that construct, adaptively refine and handle complex vague concepts by way of interaction with the environment [127]. These processes are present at each level of the hierarchical structure mentioned above. Such a hierarchical structure constitutes a metaphoric view of a well-known psychological concept – *Maslow Hierarchy of Human Needs* (see Figure 3).

The wisdom technology can be explained by the so-called the *wisdom equation* [90] which can be expressed metaphorically as follows:

$$wisdom = KSN + AJ + IP,$$

where KSN, AJ, IP denote knowledge sources network, adaptive judgment, and interactive processes, respectively. For the context of the wisdom equation see Figure 4.

Combination of the technologies represented in the wisdom equation offers an intuitive starting point for a variety of approaches to design and implementation of computational models for wisdom technology. In this sense, wisdom is a concept of a higher level than knowledge, information and data in the context of DIKW hierarchy suggested in a poem by T.S. Eliot (see Figure 5).

The basic concepts of wisdom technology outlined above do not constitute a complete solution of the three fundamental problems of AI as posed at the beginning of the

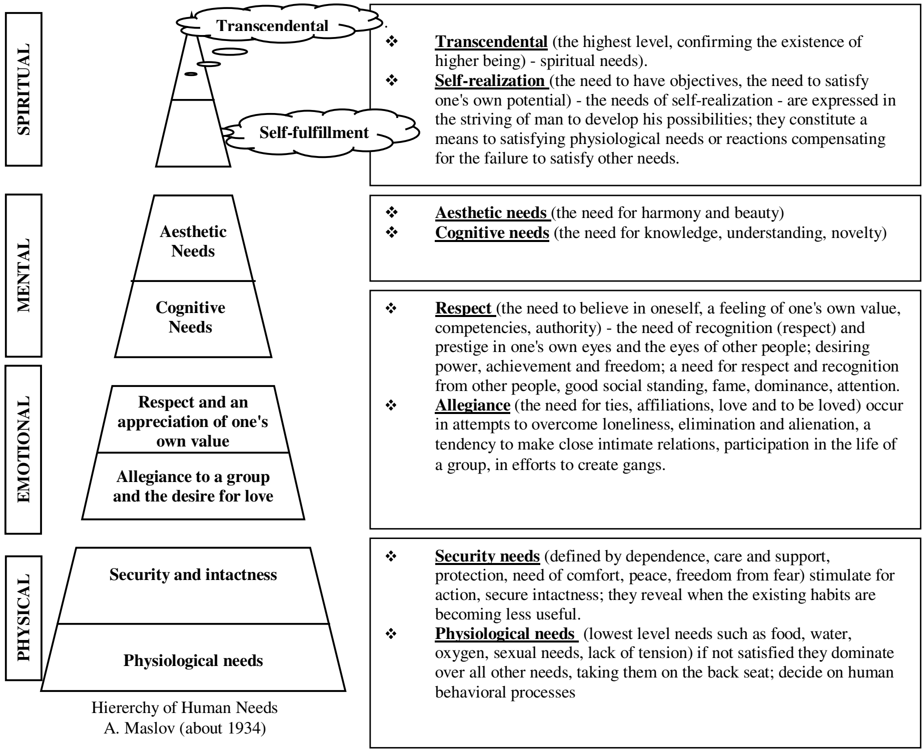
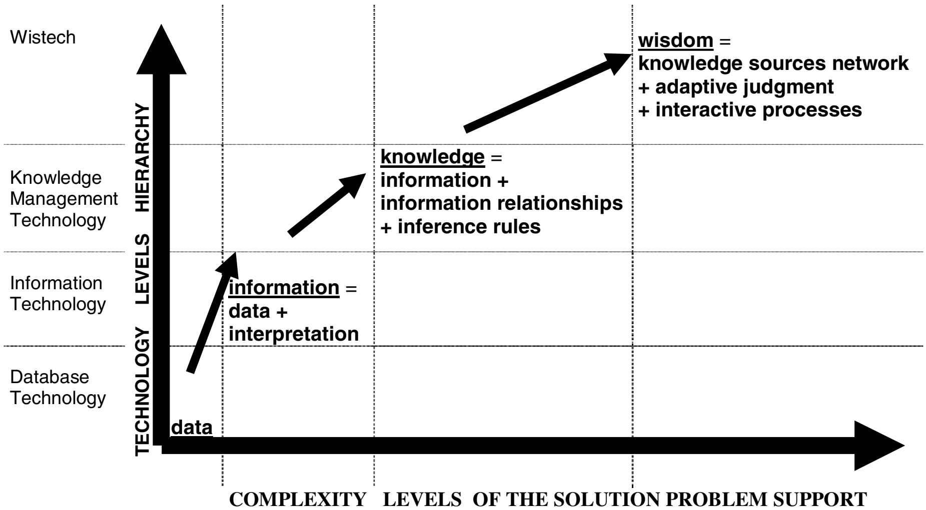


Figure 3. The Maslow Hierarchy of human needs (about 1934) as an example of judge hierarchy of habit controls.

section. Wisdom technology should rather be considered as a proposal indicating a potential research direction. Without doubt, in these studies, apart from mechanisms for the construction of metaphorically understood Maslow hierarchy for intelligent agents (in order to implement processes of judgment and of steering interactions with the environment), it will be critical to work out effective mechanisms for learning of new complex and vague concepts in the prospective domain of operation of an intelligent agent. Two directions may be distinguished in the learning mechanisms just mentioned:

1. Learning of concept hierarchies (including the metaphorically understood Maslow hierarchy for an intelligent agent), i.e., strategies for discovery of levels of the hierarchy, including relevant languages for each level and methods for selection of relevant concepts.
2. Learning concepts at particular nodes of the hierarchy.

No doubt, both kinds of processes are strongly interlaced with each other. However, a better understanding of the mechanism interlacing both groups would be more easily attained if a better understanding of specific features of each separated group was possible. From the viewpoint of studies conducted within the Rasiowa–Pawlak school, we consider the approach to understanding both process groups through a combination of rough set techniques and evolutionary programming [77–79,113] to the construction of abstract hierarchical multi-valued logical structures to be particularly essential. The simplest examples of such combination of rough sets and evolutionary programming can be



	Understanding	**Perception**	**Prediction**
Questions:	**Questions:** about data values and *additionally questions about data context, like:* ***Who? What? When? Where? How Much?***	**Questions:** information type questions and *additionally questions about explanation and prediction, like:* ***How? Why? What if?***	**Questions:** knowledge type *questions and additionally questions about correct judgments and decisions, action/ interaction planning / executing and justification, like:* ***What to do? Why to do it? When to do it? How to do it?***
Objects:	**Objects**: *Data and data explanation by a description, picture or other presentations.*	**Objects:** *Information and rules for information transformation (reasoning, ...), constrains, relationships between concepts, ideas and thought patterns.*	**Objects:** *Knowledge and correct judgments, decisions based on a hierarchy of being values or beliefs, action plans, incorporation of vision, design, plans and implementation standards based on being preferences.*
Time context:	**Time context:** *Usually information is a posteriori, known after the fact.*	**Time context:** *Usually knowledge is a priori, known before the fact and provides its meaning.*	**Time context:** *Usually deals with management interactions with environment to achieve future objectives.*
Measures:	**Measures:** *Logical values, uncertainty, completeness, amount.*	**Measures:** *Efficiency of problem solutions by applying theory to information, quality of problem solutions.*	**Measures:** *Priorities, culture values, profits, quality of action results and plan implementation.*

Figure 4. Wisdom equation context.

put straightforward. Namely, let us assume that our goal is to construct a very complex and vague concept for a decision table (as understood by rough sets described in the beginning of the paper) satisfying the idea:

> … *intelligent agents – systems that can decide what to do and then do it.*

At first one can assume that the table consisted of rules is provided by experts and that each rule is assigned a weight determined by means of rough set techniques (or of other paradigms). Next, the rules over a population of chromosomes constitute an input to the genetic algorithms under consideration. In the sequel, the approximation of our complex vague concept is refined using the implemented genetic algorithm and interactions with the environment and external experts. Although this demarche is described for a single

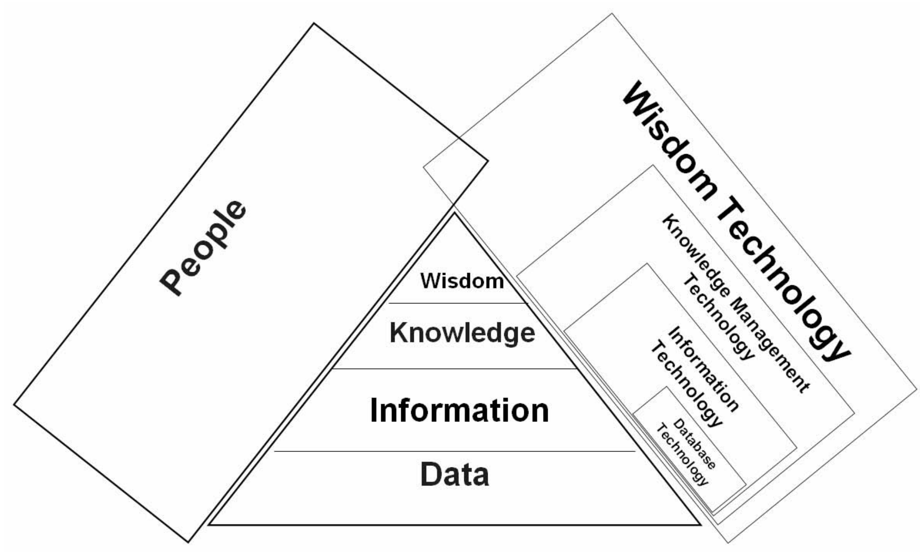
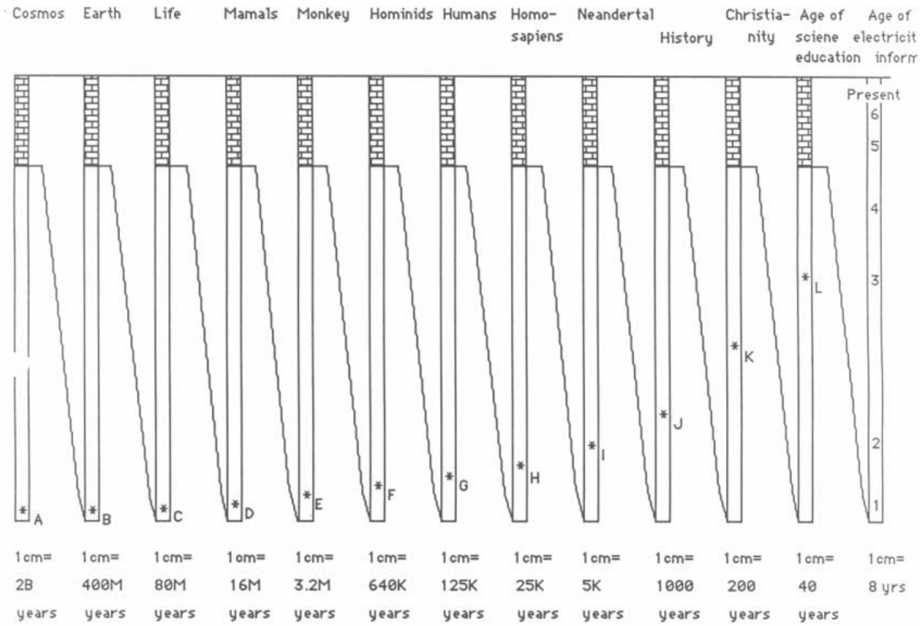


Figure 5. DIKW hierarchy.

node in the hierarchy, it is not difficult to imagine a similar construction (combining Pawlak decision tables with genetic algorithms) aiming at making best possible decisions for the architecture of the developed hierarchy of concepts. The combination of rough set techniques, genetic algorithms and simulations of interactions by agents with one another and with the environment using game theory provides a certain framework to granular computing (see [91]). The essence of these algorithms is searching for the descriptions of concept granules *optimal* for an agent, which can be later effectively employed by the agent to solve specific problems posed to it by the environment or by other agents (rules of the game). This search process is known to be highly complex and one should not expect that it will yield very precise granules. The goal here is merely to search for granular approximations sufficiently useful to the agent to construct solutions suitable to a satisfactory degree. The process leading to better and better approximation of granules can be based on genetic algorithms or on many other evolutionary models. However, due to high complexity of the concepts being approximated (e.g., because of the huge search space for relevant features [26,245]), this process is usually highly time-consuming in practical applications. Its acceleration is mainly possible by implementation of the broadest possible knowledge provided by domain experts [9–11,90,91,149,151,150,12,224], and in the case of genetic algorithms, by application of various techniques for speeding-up the evolution. This kind of evolution control techniques is sometimes referred to as *evolution of evolution* techniques [89,43]. This concept is a metaphor of observable phenomena for refining and accelerating evolution in the nature (see Figure 6, [86,89]).

To sum up, we would like to mention that the described proposals for learning mechanisms should be closely synchronized with specific agent's tasks in the environment and should effectively support its such fundamental functions as communication, perception, planning, action, cooperation, and competition [2,30,31,38,42,49,57,58,65,80, 94,100,104,124,125,164,207,216,226,227,230,238,240,243].



*A origin of the Universe and first chemical elements
*B origin of rocks on the Earth
*C origin of many-celled animals
*D origin of placetitals
*E origin of the propliopitehecus
*F origin of Australopithecus
*G origin of Homo erectus
*H origin of Homo sapiens
*I origin of Cro Magnan man
*J origin of the first civilizations
*K beginnings of the Renaissance
*L origin of modern science and technology of production

1 – first plane
2 – theory of quantum
3 – first atomic explosion; first computer
4 – first man in cosmos
5 – VLSI chips
6 – semiconductivity in high temperatures

Figure 6. The extra exponential growth in the speed of evolution.

8. Examples of Future Research Directions: From the Tarski Concept of Truth, Galois Connections, and Adjoint Functors to Adaptive Rough-Granular Computing by Agents

Alfred Tarski, in his research on satisfiability relation and concept of truth, investigated features of a class $Mod(A)$ of models satisfying some set A of expressions of the language as well as the set of features $Th(M)$ of a class of models M [235–237,70,238]. Clearly, by the definition, these functions Mod and Th are *adjoint*, i.e., they satisfy the condition:

$$M \subseteq Mod(A) \text{ if and only if } Th(M) \vdash A, \tag{2}$$

where

- M denotes models, worlds, memorized sequences of receptors receiving stimuli from the environment; sometimes subclasses of M are called *scenes*,
- A denotes a set of expressions of the language; the expressions are used to represent or denote concepts,

- $M_1 \subseteq M_2$ asserts inclusion of the model M_1 in M_2; for instance, a *scene reasoning* process may involve, for some reasons, certain models (e.g., when planning a trip by car, we consider all possible access paths at the beginning; then, depending on some other conditions, constraints, and other criteria, we gradually rule out irrelevant models and, as a consequence, we consider only a class of models M_1 with actual prospective access route models containing only the necessary information to make the trip),
- $Mod(A)$ denotes the class of all models satisfying expressions A,
- $Th(M)$ denotes the class of all language expressions that hold true in all models belonging to the class M,
- $A_1 \vdash A_2$ states that the all expressions belonging to A_2 can be derived in accordance with considered deduction rules from the set of formulas A_1.[8]

Condition (2) is usually expressed in the formalism of Galois connections or, more generally, adjoint functors as follows:

$$\frac{M \subseteq Mod(A)}{Th(M) \vdash A}, \tag{3}$$

and we say that Th is the left adjoint to functor Mod, whereas Mod is the right adjoint to functor Th. Using the language of category theory [130,129], we can also say that Th and Mod are adjoint to each other, and simply write

$$Th \dashv Mod. \tag{4}$$

Intuitively, $Th(M)$ can be regarded as a verbal description in the language of features of the models belonging to the class M. On the other hand, $Mod(A)$ can be intuitively considered as a projection of an agent's understanding of the expressions A about the class of models satisfying these expressions. In this way, we obtain the following metaphor of the adjoint functor above mentioned:

Linguistic description in a language of a class of models M	$\dashv$	*Imaginated models of a given set of language expressions A*

Similarly, a metaphoric evaluation of the aforementioned conjugation allows us to obtain, for a class of *perceived worlds* M and *expressions* A describing M, the following metaphors in natural language:

Description of M in a language	$\dashv$	*Imaginated interpretation of* A,
Symbolic reasoning about M	$\dashv$	*Scene reasoning described by expressions from* A,
Deductive reasoning in a language of M	$\dashv$	*Inductive generalization of properties specified by expressions* A,

[8] $\vdash$ (called a turnstile, first used by Gottlob Frege [51]) reads "provable from" or "is derivable from".

Judgments concerning M	$\dashv$	*Action plans derived from expressions A,*
Analysis of features of models in M	$\dashv$	*Synthesis of classes of models satisfying given features from A,*
Judgments concerning M	$\dashv$	*Emotions concerning expressions from A,*
Logical functions of the left hemisphere (e.g., word computing) as perception effects by sensory organs of models in M,	$\dashv$	*Imaginative functions of the right hemisphere (scene calculus) as effects of understanding of propositions from A by the brain.*

It can be seen from the metaphors above that adjoint functors can be regarded in a natural way as a generalization of the concept of semantics understood as a binary relation between model and language. It particularly concerns the Cartesian-closed categories, intensively investigated by Joachim Lambek and others [112]. In these categories, morphisms are considered as deductive reasoning operators. Hence, adjoint categories can be considered as pairs of categories corresponding to language and models. This kind of approach to semantics by means of adjoint functors could be a starting point to research about formula-less and model-less semantics by an analogy to point-less topology where *points* are represented by open sets including points (see, [93]). In other words, this kind of semantics deals with approximations of formulas and approximations of models instead of dealing directly with formulas and models only.

The above metaphors are only to draw attention to the adjoint relations between symbolic reasoning category and imaginative-scene reasoning category. It is worth stressing that it has become common in the cognitive studies and multi-agent interaction [227] to pay more and more attention to the role of the duality, illustrated above, in processes involving cognition and intelligence [226]. The intuitions illustrated here are basic for the implementation of granular computing [5,90,91,219,220,222,225], where an *intelligent* agent is equipped with two hemispheres (*of a brain*) – the left one is used for describing things in a symbolic language and for symbolic reasoning, while the right hemisphere deals with imagining by the agent the acceptable models satisfying certain features and with reasoning on possibilities of traversing from one model to the other (scene reasoning). Such an agent communicates with the world through sensors attached to the both brains hemispheres (see, e.g., [91]). Intuitively, the essence of granular computing is to *construct* the best possible ontological tools that are helpful in discovery of information granules which may help us wisely solve practical problems. In the context of rough sets and approximation spaces, granulation is rooted in the discovery of elementary sets that identify neighborhoods of objects of interest and give substance to our search for instances of classes of objects (i.e., concepts [161]) and which serve to establish a ground for perception [253]. A classical example illustrating these intuitions is the history of solving by our civilization, through nearly 2000 years since the ancient Greeks, problem of geometric constructions. This problem has become relatively easy due to a granulation method initiated by Évariste Galois. Using the Galois theory, certain problems in field theory may be reduced to group theory, which is in some sense simpler and better understood. In order to demonstrate the impossibility of such geometric constructions as squaring the circle, angle trisection or doubling the cube, we can express the problem in an algebraic language (instead of a geometric one) by means of the following observation: A given number can be constructed using a ruler and a compass if and only if its rank over the field of rationals is a natural power of 2. At the same time, using the Galois theory, one can express an algebraic problem by means of group theory, which can be solved relatively more easily [248,56].

In some sense, the Galois idea has been generalized by Garrett Birkhoff. He noticed in 1940 [17] that any binary relation yields two inverse dual isomorphism called *polarities*. He introduced this name because they also generalize the dual isomorphism between *polars* in analytic and projective geometry. This concept has been further generalized by Oystein Ore [153] and Cornelius J. Everett [44] to any partial order, and was called *Galois connections*. Everett showed that all Galois connections can be obtained by contraction from suitable polarities. Next, the concept of Galois connections has been generalized to category theory by Daniel M. Kan [97], who introduced the concept of *adjoint functors*. In 1946, Cornelius J. Everett and Stanisław Ulam[9] [45], using projective geometry, showed how to use this kind of connection to an algebraic interpretation of semantical meaning of quantifiers. In Ulam's opinion [244], it was the first algebraic interpretation of the universal and existential quantifiers semantic. This interpretation was based on an analogy with the projection operators in projective geometry. However, probably the first quantifier manipulation in an algebraic style could be find in early papers related to descriptive set theory (see [106–108]). This kind of algebraic quantifier manipulation is a basis for a certain type of measure of concept complexity, called the Kleene–Mostowski hierarchy (see [99,145]). For example, the well known Tarski–Kuratowski algorithm provides an easy way to get an upper bound on the classifications assigned to a formula and the set it defines. Notice that existentional and general quantifiers could be treated as adjoint functors. Having in mind this, one can generalize the Kleene–Mostowski hierarchy to arbitrary adjoint functors. This new hierarchy creates a measure of the granule complexity. The intuitions of algebraic interpretations of semantical issues of universal and existential quantifiers by an analogy with the projection operators in projective geometry, were further extended by Alfred Tarski and his research group [70]. Within the Rasiowa–Pawlak school an algebraic interpretation of quantifiers semantic is by means of *suprema* (existential quantifier) and *infima* (universal quantifier) (see [196]). William F. Lawvere and his successors unified all these ideas on the ground of category theory (see [130]) by showing how quantification can be constructed in suitable categories by using the idea of adjoint functors (i.e., by a construction which is a generalization of Galois connections).

In computer science, Galois connections are associated with several other names such as classification [8], contexts [55], or the Chu spaces [8,190]. It is interesting that applications of the Chu spaces to parallel and concurrent models were intensively studied [191]. These works are directly related to the idea of the use of algebraic approaches, developed by the Rasiowa–Pawlak school, to modeling of parallel and concurrent processes [63,137].

One of the key research directions within the Rasiowa–Pawlak school is the investigation of algebraic features of various kinds of logics. Especially important is the method

[9] See [206,244] for more details about Stanisław Ulam's life and work.

of proving completeness by means of the so-called *canonical models* (which are built of terms, language's expressions and a filter representing a given theory), described in the following section. The method shows how, having at disposal a description, to imagine models for it (to use the aforementioned metaphor, it is a cycle of back and forth shifts from the left hemisphere to the right one, and reverse). On the other hand, having models that can be transformed by suitable functors into models expressible in another language, one can build an approximation language for new models (a cycle of back and forth shifts from the right hemisphere to the left one, and reverse).

It is worth reminding that in modeling of concept approximations using rough sets, it is frequently necessary to discover relevant semantic granular structures (in right-hemisphere), syntactic granular structures (in left-hemisphere), and plans of interactions with environment and other agents. Next, based on the results of interactions it is necessary to upgrade the world perception, estimate the distance to the planned goals, and reconstruct plans. Intuitively speaking, this approach to adaptive granular computing is very similar to the quality improvement cycle known as PDCA cycle (Plan Do Check Act) or the Deming cycle. This is a fundamental challenge for the methods concerning approximate (inductive) defining of concepts from acquired information (e.g., sensor measurements represented by a given information system). This approach could be also applied to data mining (see, e.g., [100]), text mining (see, e.g., [100,15,47,66,249]), machine learning (see, e.g., [139,140,101,141,143,52,176–178,180]), and pattern recognition (see, e.g., [41,71,181]).

9. Conclusions

In this paper, some research trends within the Rasiowa–Pawlak school concerning the application of logic to AI have been selected and discussed. At the beginning, a concise genesis of the school was presented. In the next part, the understanding of AI and logic currently dominating within the school, along with the characteristic algebraic and topological tools employed by the school were described. We exposed rough set methods introduced by Zdzisław Pawlak and indicated by some trends in current studies concerning learning of complex vague concepts and their treatment. Particular attention was paid to wisdom technology and granular computing. It should be noted that this paper does not claim to be a complete and exhaustive presentation of research methods in application of logic to AI conducted within the Rasiowa–Pawlak school. These methods have significantly broader applications and have undergone a remarkable evolution during the past couple decades. This evolution, however, indicates certain directions for the future studies. For a better understanding of the evolutionary scope of the research directions conducted within the Rasiowa–Pawlak school, readers are encouraged to consult Figure 7.

Acknowledgments

The research of Andrzej Jankowski was supported by Institute of Decision Process Support. The research of Andrzej Skowron was supported by the grant from Ministry of Scientific Research and Information Technology of the Republic of Poland.

Many thanks to Dr Anna Gomolińska, Professor Victor Marek, Professor James Peters and Mr. Tuan Trung Nguyen for their incisive comments and for suggesting many helpful ways to improve this article.

Evolution of AI models of computing in the Rasiowa-Pawlak School

Mathematical concepts and tools

	Many-valued and non-classical logic ⇒	**Approximate reasoning in distributed environments & natural computing: perception-based computing**
Logic	➢ Computability, uncertainty, natural deduction, algebraic semantics and language algebraic properties of different types of logic, especially in: • intuitionistic, • modal, • Post, intermediate, • with strong negation • implicative • algorithmic • program • non-Fregean • with infinite logical operators ➢ Abstract logics, relationship between them and characterization of classical and other logics ➢ Hierarchy of metalogics ➢ Logical aspects of programming paradigms ➢ Interpretation of logical operators in models of computation (generalized quantifiers, model operators, Post operators)	➢ Evolution of concepts: hierarchy of metalogics created by interactions with environment. ➢ Society of agents, represented by a set of modal operators. ➢ Consensus and emotional states as modal operators over non-classical truth values. ➢ Logic for distributed systems ➢ Reasoning under uncertainty in distributed systems. ➢ Vague concept approximation ➢ Boolean approximate reasoning: RSES ➢ Conflicts, negotiations, cooperation ➢ RS, FS, combination with nonmonotonic reasoning ➢ Approximate reasoning about knowledge ➢ Common sense reasoning ➢ Perception logic: evolving system of interacting local logics ➢ Computational models based on perception ➢ Computational models of behavior ➢ Learning and adaptation ➢ Autonomous computing
Algebra	➢ Algebraic models for non-classical and abstract predicate calculus (Q-algebras), generalization of Rasiowa – Sikorski Lemma ➢ Lattice theory, Boolean, Heyting, Brouwer, Post and other algebras ➢ Syntax and semantics as adjoint: concepts (Galois connections) ➢ Topos theory approach ➢ Internal representation of deduction by sheaves over closure spaces	➢ Algebraic structures for reasoning under uncertainty ➢ RS algebras, FS algebras ➢ Relational calculi ➢ Partial algebras ➢ Calculi of approximation spaces ➢ Mereological calculi of information granules
Geometry	➢ Topological properties of spaces of models and concepts ➢ “Distance” between theories which represent knowledge of agents ➢ Geometry of computations ➢ Cantor Space, as a geometric space of models for classical propositioinal calculus ➢ Topological interpretation of modal operators ➢ Closure spaces as generalized geometric spaces ➢ Heuristics based on geometry of computation space	➢ Measures of proximity (similarity): states and set of states of computations and concepts ➢ Similarity of cases and case-based reasoning ➢ Geometry of concepts ➢ Similarity of theories ➢ Granular space, information granulation and granular computing ➢ Discovery of granularity levels from data, e.g., relevant multi-valued logics.
	Philosophy, CS, Biology, Psychology, Sociology, … ⇒	**Approximate reasoning about complex vague concepts and objects in distributed & dynamically changing environments**

Inspirations outside of mathematics

Figure 7. Evolution of logical approaches to AI in the Rasiowa–Pawlak school.

References

[1] G. Antoniou and F. van Harmelen, *A Semantic Web Primer. Cooperative Information Systems*, The MIT Press, Massachusetts, 2004.

[2] R. Axelrod, *The Complexity of Cooperation*, Princeton University Press, Princeton, NJ, 1997.

[3] R. Baeza-Yates and B. Ribeiro-Neto, *Modern Information Retrieval*, ACM Press Series/Addison Wesley, New York, 1999.

[4] L. Banachowski, A. Kreczmar, G. Mirkowska, H. Rasiowa and A. Salwicki, *An introduction to Algorithmic Logic. Mathematical Investigations in the Theory of Programs*, Mathematical Foundations of Computer Science, Banach Center Publications, PWN, Warszawa, 1977, pp. 7–99.

[5] A. Bargiela and W. Pedrycz, *Granular Computing: An Introduction*, Kluwer Academic Publishers, Dordrecht, 2003.

[6] J. Barwise, Absolute logics and $L_{\infty,\omega}$, *Annals of Mathematical Logic* **4** (1972), 309–340.

[7] J. Barwise, Axioms for abstract model theory, *Annals of Mathematical Logic* **7** (1974), 221–265.

[8] J. Barwise and J. Seligman, *Information Flow: The Logic of Distributed Systems*, Cambridge University Press, Cambridge UK, 1997.

[9] J.G. Bazan, J.F. Peters and A. Skowron, *Behavioral pattern identification through rough set modelling*, in: D. Ślęzak et al. [231], pp. 688–697.

[10] J. Bazan and A. Skowron, *On-line elimination of non-relevant parts of complex objects in behavioral pattern identification*, in: S.K. Pal et al. [163], pp. 720–725.

[11] J. Bazan and A. Skowron, *Classifiers based on approximate reasoning schemes*, in: B. Dunin-Kęplicz et al. [41], pp. 191–202.

[12] J. Bazan, A. Skowron and R. Swiniarski, *Rough sets and vague concept approximation: From sample approximation to adaptive learning*, Transactions on Rough Sets V: Journal Subline, LNCS, vol. 3100, Springer, Heidelberg, 2006, pp. 39–63.

[13] J.L. Bell, Set Theory, *Boolean-Valued Models and Independence Proofs*, Clarendon Press, Oxford, 2005.

[14] J. Benthem, G. Heinmann, M. Rebuschi and H. Visser, *The Age of Alternative Logic*, Springer, Berlin, 2006.

[15] M.W. Berry, *Survey of Text Mining: Clustering, Classification, and Retrieval*, Springer, Berlin, 2003.

[16] J.Y. Béziau, *Logica Universalis: Towards a General Theory of Logic*, Birkhäuser Basel, 2005.

[17] G. Birkhoff, *Lattice Theory*, AMS Colloquium Publications 25, (3rd ed.) American Mathematical Society, Providence, 1967.

[18] W.S. Bloom, Investigations into the sentential calculus with identity, *Notre Dame Journal of Formal Logic* **13**(3) (1972), 289–308.

[19] L. Bolc and P. Borowik, *Many-Valued Logics 1, Theoretical Foundations*, Springer-Verlag, Berlin, 1992.

[20] L. Bolc and P. Borowik, *Many-Valued Logics **2**, Automated Reasoning and Practical Applications*, Springer, Berlin, 2003.

[21] M. Borkowski, *Digital Image Processing in Measurement of Ice Thickness on Power Transmission Lines: A Rough Set Approach*, M.Sc. Thesis, Supervisor: J.F. Peters, Department of Electrical and Computer Engineering, University of Manitoba, 2002.

[22] M. Borkowski and J.F. Peters, *Matching 2D image segments with genetic algorithms and approximation spaces*, Transactions on Rough Sets V, LNCS, vol. 4100, 2006, pp. 63–101.

[23] L. Borkowski, ed., *Jan Łukasiewicz – Selected Works*, North Holland, PWN, Amsterdam, Warsaw, 1970.

[24] R. Brachman and H. Levesque, *Knowledge Representation and Reasoning*, Morgan Kaufmann, Stanford, 2004.

[25] O. Bratteli and P.E.T. Jorgensen, *Wavelets Through a Looking Glass: The World of the Spectrum by Applied and Numerical Harmonic Analysis*, Birkhäuser, Boston, 2002, see also http://www.cs.uiowa.edu/~jorgen/excerpt2.pdf.

[26] L. Breiman, Statistical modeling: The two cultures, *Statistical Science* **16**(3) (2001), 199–231.

[27] S. Brin and L. Page, *The Anatomy of a Large-Scale Hypertextual Web Search Engine*, Stanford University, 1998.

[28] D.J. Brown and R. Suszko, Abstract Logics, *Dissertationes Mathematicae* **102** (1973), 9–42.

[29] C. Carpineto and G. Romano, *Concept Data Analysis: Theory and Applications*, John Wiley, New York, 2004.

[30] N.L. Cassimatis, A cognitive substrate for achieving human-level intelligence, *AI Magazine* **27**(2) (2006), 45–56.

[31] N.L. Cassimatis, E.T. Mueller and P.H. Winston, Achieving human-level intelligence through integrated systems and research, *AI Magazine* **27**(2) (2006), 12–14.

[32] S. Chakrabarti, *Mining the Web: Analysis of Hypertext and Semi Structured Data*, The Morgan Kaufmann Series in Data Management Systems, Morgan Kaufmann, San Francisco, 2002.

[33] M.D. Chiara, R. Giuntini and R. Greechie, *Reasoning in Quantum Theory, Sharp and Unsharp Quantum Logics*, Kluwer Academic Publisher, Dordrecht, 2004.

[34] A. Church, *Introduction to Mathematical Logic*, Princeton University Press, NJ, 1956.

[35] P. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, New York, 1966.

[36] M. Davis, A relativity principle in quantum mechanics, *Int. J. Theor. Phys.* **16** (1977), 867–874.

[37] S. Demri and E. Orłowska (Eds.), *Incomplete Information: Structure, Inference, Complexity*, Monographs in Theoretical Computer Science, Springer-Verlag, Heidelberg, 2002.

[38] A. Desai, Adaptive complex enterprises, *Communications ACM* **48**(5) (2005), 32–35.

[39] P. Doherty, W. Łukaszewicz, A. Skowron and A. Szałas, *Knowledge Representation Techniques: A Rough Set Approach*, Studies in Fuzziness and Soft Computing 202, Springer, Heidelberg, 2006.

[40] T. Calishain and R. Dornfest, *Google Hacks*, OReilly Media, Inc., 2004.

[41] R. Duda, P. Hart and R. Stork, *Pattern Classification*, John Wiley, New York, 2002.

[42] B. Dunin-Kęplicz, A. Jankowski, A. Skowron and M. Szczuka, *Monitoring, Security, and Rescue Tasks in Multiagent Systems (MSRAS2004)*, Series in Soft Computing, Springer, Heidelberg, 2005.

[43] A.E. Eiben and J.E. Smith, *Introduction to Evolutionary Computing*, Natural Computing Series, Springer, Berlin, 2003.

[44] C.J. Everett, Closure operations and Galois theory in lattices, *Trans. AMS* **55** (1944), 514–525.

[45] C.J. Everett and S. Ulam, Projective Algebra, *Amer. J. Math.* **68** (1946), 77–88.

[46] E. Feigenbaum and J. Feldman, eds, *Computers and Thought*, McGraw Hill, New York, 1963.

[47] S. Feldman and J. Sanger, *The Text Mining Handbook, Advanced Approaches in Analyzing Unstructured Data*, Cambridge University Press, New York, 2006.

[48] D. Fensel, *Ontologies: A Silver Bullet for Knowledge Management and Electronic Commerce*, Springer, Berlin, 2003.

[49] K.D. Forbus and T.R. Hinrisch, Companion cognitive systems: A step toward human-level AI, *AI Magazine* **27**(2) (2006), 83–95.

[50] M.P. Fourman and D.S. Scott, Sheaves and logic, in: M.P. Fourman, C.J. Mulvey and D.S. Scott, eds, *Applications of Sheaves. Proceedings L.M.S. Durham Symposium 1977*, Springer Lecture Notes in Mathematics, vol. 753, Springer-Verlag, Berlin, 1999, pp. 302–401.

[51] G. Frege, *Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle, 1879.

[52] J.H. Friedman, T. Hastie and R. Tibshirani, *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, Springer, Heidelberg, 2001.

[53] D.M. Gabbay and J. Woods, *Handbook of The History of Logic, Volume 7*, Logic and The Modalities In The Twentieth Century, Elsevier North Holland, 2006.

[54] P. Gärdenfors, *Conceptual Spaces, The Geometry of Thought*, The MIT Press, 2000.

[55] B. Ganter, G. Stumme and R. Wille, eds, *Formal Concept Analysis, Foundations and Applications*, LNCS, vol. 3626, Springer, Heidelberg, 2005.

[56] D.J.H. Garling, *A Course in Galois Theory*, Cambridge University Press, New York, 1987.

[57] M. Gell-Mann, *The Quark and the Jaguar – Adventures in the Simple and the Complex*, Brown and Co., London, 1994.

[58] M. Ghallab, D. Nau and P. Traverso, *Automated Planning: Theory and Practice*, Elsevier, Morgan Kaufmann, San Francisco, 2004.

[59] S. Ghemawat, H. Gobioff, *Shun-Tak Leung*, The Wistech Network File System, Wistech Network, 2005.

[60] S. Ghilardi and M. Zawadowski, *Sheaves, Games, and Model Completions, A Categorical Approach to Nonclassical Propositional Logics*, Trends in Logic 14, Kluwer, Dordrecht, 2002.

[61] V. Glivenko, Sur quelques points de la logique de M. Brouwer, Academie Royale de Belgique, *Bulletins de la classe des sciences* **5**(15) (1929), 183–188.

[62] K. Gödel, Zur intuitionistischen Arithmetik und Zahlentheorie, *Ergebnisse eines mathematischen Kolloquiums* **4** (1933), 34–38.

[63] J. Grabowski. On partial languages, *Fundamenta Informaticae* **4**(2) (1981), 427–498.

[64] L.S. Gottfredson, Editorial: Mainstream science on intelligence. An editorial with 52 signatories, history, and bibliography, *Intelligence* **24**(1) (1997), 13–24.

[65] R. Granger, Engines of the brain: The computational instruction set of human cognition, *AI Magazine* **27**(2) (2006), 15–31.

[66] S. Grimes, *The developing text mining market*, A white paper prepared for Text Mining Summit 2005, Boston, June 7–8, 2005. Alta Plana Corporation, 2005.

[67] A. Grzegorczyk, Some approaches to constructive analysis, in: A. Heyting, ed., *Constructivity in Mathematics, Studies in Logic and the Foundations of Mathematics*, Colloquium at Amsterdam 1957, North-Holland, Amsterdam, 1959, pp. 43–61.

[68] A.O. Hagan, C.E. Buck, A. Daneshkhah, J.R. Eiser, P.H. Garthwaite, D.J. Jenkinson, J.E. Oakley and T. Rakow, *Uncertain Judgements: Eliciting Expert Probabilities*, Wiley, New York, 2006.

[69] J. Heaton, *Programming Spiders, Bots, and Aggregators in Java*, Sybex, 2002.

[70] L. Henkin, J.D. Monk and A. Tarski, *Cylindric Algebras, part I*, North-Holland, Amsterdam, 1971.

[71] C. Henry and J.F. Peters, Image Pattern Recognition Using Approximation Spaces and Near Sets, *Proc. 2007 Joint Rough Set Symposium (JRS07)*, Toronto, Canada, 14–16 May, 2007.

[72] M. Henzinger and S. Lawrence, *Extracting knowledge from the World Wide Web*, Wistech Network, 2004.

[73] D. Higgs, *A category approach to Boolean-valued set theory*, University of Waterloo, Canada, 1973.

[74] J. Hintikka, *The Principles of Mathematics Revisited*, Cambridge University Press, Cambridge, UK, 1996.

[75] M. Hirvensalo, *Quantum Computing*, Springer-Verlag, Heidelberg 2001.

[76] T.B. Ho and N.B. Nguyen, Nonhierarchical document clustering based on a tolerance rough set model and its application to information retrieval, *International Journal of Intelligent Systems* **17** (2002), 199–212.

[77] J.H. Holland, *Adaptation in Natural and Artificial Systems, An Introductory Analysis with Applications to Biology, Control, and Artificial Intelligence*, The University of Michigan Press, 1975.

[78] J.H. Holland, *Hidden Order, How Adaptation Builds Complexity*, Addison Wesley, Reading, MA, 1995.

[79] J.H. Holland, *Emergence from Chaos to Order*, Oxford University Press, 1998.

[80] M.N. Huhns and M.P. Singh, *Readings in Agents*, Morgan Kaufmann, 1998.

[81] T. Imieliński and W. Lipski, Incomplete information in relational databases, *Journal ACM* **31**(4) (1984), 761–791.

[82] P. Jackson and I. Moulinier, *Natural Language Processing for Online Applications: Text Retrieval, Extraction, and Categorization*, Natural Language Processing 5, John Benjamins Publishing Co, 2002.

[83] Z. Janiszewski, *On needs of mathematics in Poland (O potrzebach matematyki w Polsce)* (in Polish), in: Nauka Polska, Jej Potrzeby, Organizacja i Rozwój, Warszawa, 1918; see also reprint in *Wiadomości Matematyczne* **VII** (1963), 3–8.

[84] A. Jankowski, An alternative characterization of elementary logic, *Bull. Acad. Pol. Sci., Ser. Math. Astr. Phys.* **XXX**(1–2) (1982), 9–13.

[85] A. Jankowski, Galois structures, *Studia Logica* **44**(2) (1985), 109–124.

[86] A. Jankowski, *Proposal of Evolution Programming Methodology for Learning in an Uncertain Environment*, Research & Development Technical Report, Department of Computer Science, University of North Carolina, Charlotte, 1986.

[87] A. Jankowski and C. Rauszer, Logical foundation approach to users' domain restriction in data bases, *Theoretical Computer Science* **23** (1983), 11–36.

[88] A. Jankowski and M. Zawadowski, Sheaves over Heyting Lattices, *Studia Logica* **44**(3) (1985), 237–356.

[89] A. Jankowski, Z. Michalewicz, Z.W. Ras and D. Shoff, *Evolution Programming as a Programming Methodology for Learning in an Uncertain Environment*, Department of Computer Science, University of North Carolina, Charlotte, 1988.

[90] A. Jankowski and A. Skowron, *A wistech paradigm for intelligent systems*, Transactions on Rough Sets VI: Journal Subline, LNCS, vol. 4374, Springer, Heidelberg, 2007, pp. 94–132.

[91] A. Jankowski and A. Skowron, *Wisdom granual computing*, In Handbook of Granular Computing, John Wiley & Sons, New York, 2007 (to appear).

[92] A. Jankowski and J. Żytkow, Hierarchical control and heuristics in multisearch systems, *Proceedings of the Fourth International Symposium (ISMIS 1989), Charlotte, NC, USA, 11–14 October 1989*, North Holland, 1989.

[93] P. Johnstone, *Stone Spaces*, Cambridge University Press, Cambridge UK, 1986.

[94] R.M. Jones and R.E. Wray, Comparative analysis of frameworks for knowledge intensive intelligent agents, *AI Magazine* **27**(2) (2006), 57–70.

[95] P.E.T. Jorgensen, *Analysis and Probability: Wavelets, Signals, Fractals*, Graduate Texts in Mathematics 234, Springer, Berlin, 2006.

[96] D. Kahneman, P. Slovic and A. Tversky, eds, *Judgement under Uncertainty: Heuristics and Biases*, Cambridge University Press, New York, 1982.

[97] D.M. Kan, Adjoint functors. *Trans. Am. Math. Soc., Soc.* **87** (1958), 294–329.

[98] R. Keefe, *Theories of Vagueness*, Cambridge Studies in Philosophy, Cambridge, UK, 2000.

[99] S.C. Kleene, *Introduction to Metamathematics*, Van Nostrand, Princeton, 1952.

[100] W. Kloesgen and J. Żytkow, *Handbook of Knowledge Discovery and Data Mining*, Oxford University Press, New York, 2002.

[101] Y. Kodratoff and R.S. Michalski, eds, *Machine Learning: An Artificial Intelligence Approach, Vol. III*, Morgan Kaufmann, San Mateo, CA, 1990.

[102] E. Konrad, E. Orłowska and Z. Pawlak, *Knowledge Representation Systems. Definability of Information*, Research Report PAS 433, Institute of Computer Science, Polish Academy of Sciences, 1981.

[103] E. Konrad, E. Orłowska and Z. Pawlak, *On Approximate Concept Learning*, Report 81-07, Fachbereich Informatik, TU Berlin, Berlin 1981; short version in: Collected Talks, European Conference on Artificial Intelligence 11/5, Orsay/Paris, 1982, pp. 17–19.

[104] S. Kraus, *Strategic Negotiations in Multiagent Environments*, The MIT Press, Massachusetts, 2001.

[105] A. Kreczmar, A. Salwicki and M. Warpechowski, *Loglan'88 – Report on the Programming Language*, LNCS, vol. 414, Springer-Verlag, Heidelberg, 1990.

[106] K. Kuratowski and A. Tarski, Les opérations logiques et les ensembles projectifs, *Fundamenta Mathematicae* **17** (1931), 240–248.

[107] K. Kuratowski, *Topology, Vol. I*, Academic Press, New York–London, 1966.

[108] K. Kuratowski, *Topology, Vol. II*, Academic Press, New York–London, 1968.

[109] K. Kuratowski, *Pół Wieku Matematyki Polskiej 1920–1970*, Wspomnienia i refleksje, Wiedza Powszechna, Warszawa, 1973 (in Polish).

[110] K. Kuratowski, W. Marek, L. Pacholski, H. Rasiowa, C. Ryll-Nardzewski and P. Zbierski, eds, *A. Mostowski, Foundational Studies, Selected Works Vol. I*, North-Holland, Amsterdam–New York; PWN–Polish Scientific Publishers, Warsaw, 1979.

[111] K. Kuratowski, W. Marek, L. Pacholski, H. Rasiowa, C. Ryll-Nardzewski and P. Zbierski, eds, *A. Mostowski, Foundational Studies. Selected Works Vol. II*, North-Holland, Amsterdam–New York, PWN–Polish Scientific Publishers, Warsaw, 1979.

[112] J. Lambek and P.J. Scott, *Introduction to Higher-Order Categorical Logic*, Cambridge Studies in Advanced Mathematics 7, Cambridge University Press, 1986.

[113] W.B. Langdon and R. Poli, *Foundations of Genetic Programming*, Springer, Berlin, 2002.

[114] P. Langley, H.A. Simon, G.L. Bradshaw and J.M. Żytkow, *Scientific Discovery: Computational Explorations of the Creative Processes*, The MIT Press, 1987.

[115] G.W. Leibniz, *Dissertio de Arte Combinatoria*, Leipzig, 1666.

[116] G.W. Leibniz, *New Essays on Human Understanding (1705)*, Translated and edited by Peter Remnant and Jonathan Bennett, Cambridge UK, Cambridge, 1982.

[117] G.W. Leibniz, Discourse on metaphysics, in: R. Ariew, D. Garber, eds, Leibniz, G.W., *Philosophical Essays*, Hackett Publishing Company, Indianapolis, 1989, pp. 35–68.

[118] S. Leśniewski, Grundzüge eines neuen Systems der Grundlagen der Mathematik, *Fundamenta Mathematicae* **14** (1929), 1–81.

[119] S. Leśniewski, On the foundations of mathematics, *Topoi* **2** (1982), 7–52.

[120] P. Lindström, On extensions of elementary logic, *Theoria* **35** (1969), 1–11.

[121] W. Lipski, Information storage and retrieval – Mathematical foundations II (combinatorial problems), *Theoretical Computer Science* **3**(2) (1976), 183–211.

[122] W. Lipski and V.W. Marek, File organization: An application of graph theory, in: J. Loeckx, ed., *Proceedings of the Second Colloquium on Automata, Languages and Programming, University of Saarbrücken, July 29 – August 2, 1974 (ICALP 1974)*, LNCS, vol. 14, Springer, Heidelberg, 1974, pp. 270–279.

[123] W. Lipski, On databases with incomplete information, *Journal ACM* **28**(1) (1981), 41–70.

[124] J. Liu, *Autonomous Agents and Multi-Agent Systems: Explorations in Learning, Self-Organization and Adaptive Computation*, World Scientific Publishing, Singapore, 2001.

[125] J. Liu and L.K. Daneshmend, *Spatial Reasoning and Planning: Geometry, Mechanism, and Motion*, Springer, Berlin, 2003.

[126] J. Liu, X. Jin and K.C. Tsui, *Autonomy Oriented Computing: From Problem Solving to Complex Systems Modeling*, Kluwer/Springer, Heidelberg, 2005.

[127] M. Luck, P. McBurney and C. Preist, *Agent Technology. Enabling Next Generation Computing: A Roadmap for Agent Based Computing*, AgentLink, 2003.

[128] R.C. Lyndon, *Notes on Logic*, D. Van Nostrand Company, Inc., 1967.

[129] S. MacLane, *Categories for the Working Mathematicians. Graduate Texts in Mathematics*, Springer, Berlin, 1997.

[130] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Universitext, Springer, Berlin, 1994.

[131] R.W. Marczyński, The first seven years of Polish digital computers, *Annals of the History of Computing* **2**(1) (1980), 37–48.

[132] V. Marek and Z. Pawlak, Information storage and retrieval systems: Mathematical foundations, *Theoretical Computer Science* **1** (1976), 331–354.

[133] V. Marek and H. Rasiowa, Approximating sets with equivalence relations, *Theoretical Computer Science* **48**(3) (1986), 145–152.

[134] V. Marek and C. Rauszer, Query optimization in the databases distributed by means of product equivalence relations, *Fundamenta Informaticae* **11** (1988), 241–266.

[135] V. Marek and M. Truszczyński, Contributions to the theory of rough sets, *Fundamenta Informaticae* **39**(4) (1999), 389–409.

[136] S. Mazur, *Computable Analysis* (edited by A. Grzegorczyk, H. Rasiowa), Rozprawy Matematyczne **33**, Warsaw, 1963.

[137] A. Mazurkiewicz, *Concurrent program schemes and their interpretations*, Technical Report, DAIMI Report PB-78, Aarhus University, Aarhus, 1977.

[138] M.J. Mączyński, Non-classical logics in quantum mechanics and physics, *Bulletin of the Section of Logic* **25**(3/4), University of Łódź, Department of Logic, 1996, 161–165.

[139] R.S. Michalski, T.J. Carbonell and T.M. Mitchell, eds, *Machine Learning: An Artificial Intelligence Approach*, Palo Alto, TIOGA Publishing Co., Palo Alto, CA, 1983.

[140] R.S. Michalski, T.J. Carbonell and T.M. Mitchell, eds, *Machine Learning: An Artificial Intelligence Approach. Vol. II*, Morgan Kaufmann, Los Altos, CA, 1986.

[141] R.S. Michalski and G. Tecuci, eds, *Machine Learning – A Multistrategy Approach. Vol. IV*, Morgan Kaufmann, San Mateo, CA, 1994.

[142] G. Mirkowska and A. Salwicki, *Algorithmic Logic*, PWN & D. Reidel Publ., Warsaw & Dordrecht, 1987.

[143] T.M. Mitchell, *Machine Learning*, Mc Graw-Hill, Portland, 1997.

[144] E.H. Moore, *Introduction to a form of general analysis*, AMS Colloquium **2**, New Haven, 1910.

[145] A. Mostowski, On definable sets of positive integers, *Fundamenta Mathematicae* **34** (1947), 81–112.

[146] A. Mostowski, *Logika matematyczna, Kurs uniwersytecki*, Monografie matematyczne, 1948 (in Polish).

[147] A. Mostowski, Proof of non-deducibility in intuitionistic functional calculus, *The Journal of Symbolic Logic* **13** (1948), 204–207.

[148] A. Mostowski, On a generalization of quantifiers, *Fundamenta Mathematicae* **44** (1957), 12–36.

[149] H.S. Nguyen, J. Bazan, A. Skowron, and S.H. Nguyen, *Layered learning for concept synthesis*, Transactions on Rough Sets I: Journal Subline, LNCS, vol. 3100, Springer, Heidelberg, 2006, pp. 187–208.

[150] Ch.L. Ngo and H.S. Nguyen, *A method of Web search result clustering based on rough sets*, in: A. Skowron et al. [215], pp. 673–679.

[151] T.T. Nguyen, *Eliciting domain knowledge in handwritten digit recognition*, in: S. Pal et al. [163], pp. 762–767.

[152] M. O'Connor, Embeddings into free Heyting algebras and translations into intuitionistic propositional logic, *Proceedings of Logical Foundations of Computer Science*, 2007 (to appear), see http://arxiv.org/abs/math.LO/0702651.

[153] O. Ore, Galois connexions, *Trans. AMS* **55** (1944), 494–513.

[154] E. Orłowska, Resolution systems and their applications II, *Fundamenta Informaticae* **3** (1980), 333–362.

[155] E. Orłowska, Kripke semantics for knowledge representation logics, *Studia Logica* **49**(2) (1990), 255–272.

[156] E. Orłowska (Ed.), *Incomplete Information: Rough Set Analysis*, Studies in Fuzziness and Soft Computing 13, Springer-Verlag/Physica-Verlag, Heidelberg, 1997.

[157] E. Orłowska and Z. Pawlak, *Expressive Power of Knowledge Representation Systems*, Research Report PAS 432, Institute of Computer Science, Polish Academy of Sciences, April (1981).

[158] E. Orłowska, *Semantics of Vague Concepts. Applications of Rough Sets*, Institute for Computer Science, Polish Academy of Sciences, Report 469, March 1982.

[159] E. Orłowska and Z. Pawlak, Expressive power of knowledge representation systems, *International Journal of Man-Machine Studies* **20**(5) (1984), 485–500.

[160] E. Orłowska and Z. Pawlak, Representation of non–deterministic information, *Theoretical Computer Science* **29** (1984), 27–39.

[161] *The Oxford English Dictionary*, Oxford University Press, London, 1933.

[162] S.K. Pal, L. Polkowski and A. Skowron, eds, *Rough-Neural Computing: Techniques for Computing with Words*, Cognitive Technologies, Springer, Heidelberg, 2004.

[163] S.K. Pal, S. Bandoyopadhay and S. Biswas, eds, *Proceedings of the First International Conference on Pattern Recognition and Machine Intelligence (PReMI'05), December 18–22, 2005, Indian Statistical Institute, Kolkata*, LNCS, vol. 3776, Springer, Heidelberg, 2005.

[164] S. Parsons, P. Gmytrasiewicz and M. Wooldridge, *Game Theory and Decision Theory in Agent-based Systems*, Kluwer, Dordrecht, 2002.

[165] T.B. Passin, Explorers Guide to the Semantic Web Mining Publications, 2004.

[166] Z. Pawlak, Mathematical foundations of information retrieval, *Proceedings of Symposium of Mathematical Foundations of Computer Science, September 3–8, 1973, High Tatras*, pp. 135–136; see also: Mathematical Foundations of Information Retrieval. Computation Center, Polish Academy of Sciences, Research Report CC PAS 101, 1973.

[167] Z. Pawlak, *Classification of Objects by Means of Attributes*, Research Report CC PAS 429, Institute of Computer Science, Polish Academy of Sciences, 1981.

[168] Z. Pawlak, *Rough Sets*, Research Report CC PAS 431, Institute of Computer Science, Polish Academy of Sciences, January 1981.

[169] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences* **11** (1982), 341–356.

[170] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data. System Theory*, Knowledge Engineering and Problem Solving 9, Kluwer, Dordrecht, 1991.

[171] Z. Pawlak and C. Rauszer, Dependency of attributes in information systems, *Bull. Polish Acad. Sci. Math.* **33** (1985), 551–559.

[172] Z. Pawlak and A. Skowron, Rudiments of rough sets. Information Sciences, *An International Journal* **177**(1) (2007), 3–27.

[173] Z. Pawlak and A. Skowron, Rough sets: Some extensions. Information Sciences, *An International Journal* **177**(1) (2007), 28–40.

[174] Z. Pawlak and A. Skowron, Rough sets and Boolean reasoning. Information Sciences, *An International Journal* **177**(1) (2007), 41–73.

[175] J.F. Peters, *Rough ethology: Towards a biologically-inspired study of collective behaviour in intelligent systems with approximation spaces*, Transactions on Rough Sets III: Journal Subline, LNCS, vol. 3400, Springer, Heidelberg, 2005, pp. 153–174.

[176] J.F. Peters and C. Henry, Reinforcement learning with approximation spaces, *Fundamenta Informaticae* **71**(2–3) (2006), 323–349.

[177] J.F. Peters and C. Henry, Approximation spaces in off-policy Monte Carlo learning, *Engineering Applications of Artificial Intelligence* **20**(5) (2007), 667–675.

[178] J.F. Peters, C. Henry and D. Gunderson, Biologically-Inspired Adaptive Learning Control Strategies: A Rough Set Approach. *International Journal of Hybrid Intelligent Systems* (2007), *in press*.

[179] J.F. Peters, Near sets. Special theory about nearness of objects, *Fundamenta Informaticae* **75**(1–4) (2007), 407–433.

[180] J.F. Peters, M. Borkowski, C. Henry and D. Lockery, Monocular vision system that learns with approximation spaces, in: A. Ella, P. Lingras, D. Ślezak, Z. Suraj, eds, *Rough Set Computing: Toward Perception Based Computing*, Idea Group Publishing, Hershey, PA, 2006, pp. 1–22.

[181] J.F. Peters, Near Sets. Toward Approximation Space-Based Object Recognition, *Proc. 2007 Joint Rough Set Symposium (JRS07), Toronto, Canada, 14–16 May*, 2007.

[182] J.F. Peters, Classification of objects by means of features, in: D. Fogel, J. Mendel, X. Yao and T. Omori, eds, *Proc. First IEEE Symposium on Foundations of Computational Intelligence (FOCI'07), 1–5 April*, 2007, pp. 1–8.

[183] J.A. Plaza, Logics of public communications, in: M.L. Emrich, M.S. Pfeifer, M. Hadzikadic and Z.W. Ras, eds, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems held in Charlotte*, NC, North-Holland, New York, 1989, pp. 201–216.

[184] S. Plous, *The Psychology of Judgement and Decision Making*, McGraw-Hill, New York, 1993.

[185] T. Poggio and S. Smale, The mathematics of learning: Dealing with data, *Notices of the AMS* **50**(5) (2003), 537–544.

[186] R. Poli and P. Simons, *Formal Ontology*, Kluwer, Dordrecht, 1996.

[187] L. Polkowski and A. Skowron, Rough mereology: A new paradigm for approximate reasoning, *International Journal of Approximate Reasoning* **15**(4) (1996), 333–365.

[188] L. Polkowski, *Rough Sets: Mathematical Foundations. Advances in Soft Computing*, Physica-Verlag, Heidelberg, 2002.

[189] L. Polkowski, S. Tsumoto and T.Y. Lin, eds, *Rough Set Methods and Applications: New Developments in Knowledge Discovery in Information Systems*, Studies in Fuzziness and Soft Computing 56, Physica-Verlag, Heidelberg, 2000.

[190] V. Pratt, *Chu Spaces*, Notes for the School on Category Theory and Applications. University of Coimbra, July 13–17, 1999.

[191] V.R. Pratt, Transition and cancellation in concurrency and branching time, *Math. Struct. in Comp. Science* **13**(4) (2003), 485–529.

[192] D. Prawitz, *Natural Deduction, A Proof-Theoretical Study*, Mineola, New York, Dover Publications, Inc., 2006.

[193] H. Rasiowa, *An Algebraic Approach to Non-Classical Logics*, PWN Warsaw, 1974.

[194] H. Rasiowa, *Algebraic Models of Logics*, Warsaw University, 2001.

[195] H. Rasiowa and W. Marek, On reaching consensus by groups of intelligent agents, in: Z.W. Ras, ed., *Methodologies for Intelligent Systems*, North-Holland, Amsterdam, 1989, pp. 234–243.

[196] H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, Monografie Matematyczne 41, PWN Warsaw, 1963.

[197] C. Rauszer, Notes on the Rasiowa–Sikorski lemma, *Studia Logica* **34** (1975), 265–268.

[198] C. Rauszer, An algebraic approach to the Heyting-Brouwer predicate calculus, *Fundamenta Mathematicae* **96** (1977), 127–135.

[199] C. Rauszer and H. Ono, On algebraic and Kripke semantics for intermediate logicism, in: *Universal Algebra and Applications*, Series of Banach Center Publications 9, PWN, Warszawa, 1982, pp. 431–438.

[200] C. Rauszer, Algebraic considerations of autoepistemic logic, *Fundamenta Informaticae* **15** (1991), 168–169.

[201] C. Rauszer, Dependencies in relational databases: Algebraic and logical approach, *Fundamenta Informaticae* **19**(3/4) (1993), 235–274.

[202] C. Rauszer, Knowledge representation systems for groups of agents, in: J. Woleński, ed., *Philosophical Logic in Poland*, Kluwer, Dordrecht, 1994, pp. 217–238.

[203] C. Rauszer and H. de Swart, Different approaches to knowledge, common knowledge and Aumann's theorem, in: A. Lanux, H. Wansing, eds, *Knowledge and Belief in Philosophy and Artificial Intelligence*, Akademie Verlag, Berlin, 1995, pp. 87–101.

[204] S. Read, *Thinking About Logic. An Introduction to the Philosophy of Logic*, Oxford University Press, Oxford, New York, 1995.

[205] K. van Rijsbergen, *The Geometry of Information Retrieval*, Cambridge University Press, 2006.

[206] G.-C. Rota, *Indiscrete Thoughts* (edited by Fabrizio Palombi, reviewed by Andrew Leah), Birkhäuser, Boston, 1997.

[207] S. Russel and P. Norvig, *Artificial Intelligence: A Modern Approach*, Prentice Hall, 2002.

[208] D. Scott and R. Solovay, *Boolean-valued models for set theory*, Proceedings of UCLA Set Theory Conference, 1967.

[209] D. Scott, A proof of independence of continuum hypothesis, *Mathematical Systems Theory* **1** (1967), 89–111.

[210] S. Shanker, *Wittgenstein's Remarks on the Foundations of AI*, Routledge, London, 2006.

[211] R. Sikorski, *Boolean Algebras*, Springer-Verlag, Berlin, 1964.

[212] A. Skowron, Perception logic in intelligent systems, in: S. Blair et al., eds, *Proceedings of the 8th Joint Conference on Information Sciences (JCIS 2005), July 21–26, 2005, Salt Lake City, Utah, USA, pp. 1–5*. X-CD Technologies: A Conference & Management Company, ISBN 0-9707890-3-3, Toronto, Ontario, Canada, 2005.

[213] A. Skowron, *Rough sets in perception-based computing (keynote talk)*, in: S.K. Pal et al. [163], pp. 21–29.

[214] A. Skowron, ed., *Logic, Algebra and Computer Science: H. Rasiowa and Cecylia Rauszer in Memoriam*, Warsaw, June 1994. Bulletin of the Section of Logic **25**(3/4), Department of Logic of the University of Łódź, October – December 1996.

[215] A. Skowron, R. Agrawal, M. Luck, T. Yamaguchi, O. Morizet-Mahoudeaux, J. Liu and N. Zhong, eds, *Proceedings of the 2005 IEEE/WIC/ACM International Conference on WEB Intelligence, Compiegne, France, September 19–22, 2005*, IEEE Computer Society Press, Los Alamitos, CA, 2005, pp. 1–819.

[216] A. Skowron, J.-P. Barthes, L. Jain, R. Sun, P. Morizet-Mahoudeaux, J. Liu and N. Zhong, eds, *Proceedings of the 2005 IEEE/WIC/ACM International Conference on Intelligent Agent Technology, Compiegne, France, September 19–22, 2005*, IEEE Computer Society Press, Los Alamitos, CA, 2005, pp. 1–766.

[217] A. Skowron and C. Rauszer, The discernibility matrices and functions in information systems, in: R. Słowiński, ed., *Intelligent Decision Support – Handbook of Applications and Advances of the Rough Sets Theory*, System Theory, Knowledge Engineering and Problem Solving **11**, Kluwer, Dordrecht, 331–362, 1992.

[218] A. Skowron and J. Stepaniuk, Tolerance approximation spaces, *Fundamenta Informaticae* **27** (1996), 245–253.

[219] A. Skowron and J. Stepaniuk, Information granules: Towards foundations of granular computing, *International Journal of Intelligent Systems* **16**(1) (2001), 57–86.

[220] A. Skowron and J. Stepaniuk, *Information granules and rough-neural computing*, in: S.K. Pal et al. [162], pp. 43–84.

[221] A. Skowron and P. Synak, Complex patterns, *Fundamenta Informaticae* **60**(1–4) (2004), 351–366.

[222] A. Skowron, R. Swiniarski and P. Synak, *Approximation spaces and information granulation*, Transactions on Rough Sets III: Journal Subline, LNCS, vol. 3400, Springer, Heidelberg, 2005, pp. 175–189.

[223] J.M. Smith, *Evolution and the Theory of Games*, Cambridge University Press, 2006.

[224] J. Stepaniuk, J. Bazan and A. Skowron, Modelling complex patterns by information systems, *Fundamenta Informaticae* **67**(1–3) (2005), 203–217.

[225] J. Stepaniuk, A. Skowron, J. Peters and R. Swiniarski, Calculi of approximation spaces, *Fundamenta Informaticae* **72**(1–3) (2006), 363–378.

[226] R. Sun, *Duality of the Mind: A Bottom-up Approach Toward Cognition*, Lawrence Erlbaum Associates, 2001.

[227] R. Sun (Ed.), *Cognition and Multi-Agent Interaction. From Cognitive Modeling to Social Simulation*, Cambridge University Press, New York, NY, 2006.

[228] K. Svozil, *Quantum Logic*, Springer, Heidelberg, 1998.

[229] W. Swartout, J. Gratch, R.W. Hill, E. Hovy, S. Marsella, J. Rickel and D. Traum, Towards virtual humans, *AI Magazine* **27**(2) (2006), 96–108.

[230] K. Sycara, Multiagent systems, *AI Magazine* (Summer 1998), 79–92.

[231] D. Ślęzak, J.T. Yao, J.F. Peters, W. Ziarko and X. Hu, eds, *Proceedings of the 10th International Conference on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing (RSFDGrC'2005), Regina, Canada, August 31 – September 3, 2005, Part II*, LNAI 3642, Springer, Heidelberg, 2005.

[232] R. Suszko, Ontology in the Tractatus of L. Wittgenstein, *Notre Dame Journal of Formal Logic* **9** (1968), 7–33.

[233] A. Tarski, *Fundamental concepts of the methodology of deductive science* (1930), in: S.R. Givant, R.N. McKenzie [237].

[234] A. Tarski, *Grundlegung der wissenschaftlichen Semantik. Actes du Congrès International de Philosophie Scientifique*, vol. III, Paris, 1936, pp. 1–8.

[235] A. Tarski, The semantical concept of truth and the foundations of semantics, *Philosophy and Phenomenological Research* **4** (1944), 341–375.

[236] A. Tarski, Contributions to the theory of models, I and II, *Indag. Math.* **16** (1954), 572–588.

[237] A. Tarski, Contributions to the theory of models, III, *Indag. Math.* **17** (1954), 56–64.

[238] A. Tarski, *The Collected Papers of Alfred Tarski*, 4 vols. S.R. Givant and R.N. McKenzie, eds, Birkhäuser, 1986.

[239] A. Tarski, *Introduction to Logic and to the Methodology of Deductive Sciences*, Oxford University Press, UK, 1963.

[240] A.S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, Cambridge University Press, 2000.

[241] A.S. Troelstra and D. Van Dalen, *Constructivism in Mathematics: An Introduction*, Studies in Logic and the Foundations of Mathematics Vol. 1 & 2, Elsevier Science Publishing Company, 1988.

[242] A. Turing, Computing machinery and intelligence, *Mind* **LIX**(236) (October 1950), 433–460.

[243] W. Van Wezel, R. Jorna and A. Meystel, *Planning in Intelligent Systems: Aspects, Motivations, and Methods*, Wiley, Hoboken, New Jersey, 2006.

[244] S.M. Ulam, *Adventures of a Mathematician*, University of California Press (reprint edition), 1991.

[245] V. Vapnik, *Statistical Learning Theory*, Wiley, New York, 1998.

[246] T.L. Vincent and J.S. Brown, *Evolutionary Game Theory, Natural Selection, and Darwinian Dynamics*, Cambridge University Press, 2005.

[247] P. Wang, *What do you mean by "AI"?*, http://nars.wang.googlepages.com/home.

[248] L. Wantzel, Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas, *Journal de Mathématiques Pures et Appliquées* **1**(2) (1837), 366–372.

[249] S. Weiss, N. Indurkhya, T. Zhang and F. Damerau, *Text Mining: Predictive Methods for Analyzing Unstructured Information*, Springer, Heidelberg, 2004.

[250] D. Widdows, *Geometry and Meaning*, CSLI Publications, 2004.

[251] L. Wittgenstein, *Philosophical Investigations*, Cambridge University Press, Cambridge, UK, 2004.

[252] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 333–353.

[253] L.A. Zadeh, From computing with numbers to computing with words – from manipulation of measurements to manipulation of perceptions, *IEEE Transactions on Circuits and Systems – I: Fundamental Theory and Applications* **45**(1) (1999), 105–119.

[254] L.A. Zadeh, Toward a generalized theory of uncertainty (GTU) – An outline, *Information Sciences* **171** (2005), 1–40.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Contributions to the Theory of Weakly Distributive Complete Boolean Algebras

Bohuslav BALCAR and Thomas JECH [1]
Mathematical Institute, AS CR, Žitná 25, CZ - 115 67 Praha 1, Czech Republic
balcar@cts.cuni.cz, jech@math.cas.cz

Abstract. We investigate complete Boolean algebras that carry a continuous submeasure.

Keywords. Boolean algebra, Maharam algebra, measure algebra, forcing, weak distributivity, diagonal properties

1. Introduction

A complete Boolean algebra B is a *measure algebra* if it carries a strictly positive σ-additive measure. B is a *Maharam algebra* if it carries a strictly positive continuous submeasure. Every measure algebra is a Maharam algebra and every Maharam algebra is weakly distributive and satisfies the countable chain condition (ccc).

The 1937 problem of von Neumann from the Scottish book [1] asks if every weakly distributive ccc Boolean algebra is a measure algebra. In [2], Balcar, Jech and Pazák proved that it is consistent that every weakly distributive complete ccc Boolean algebra is a Maharam algebra, and in [3], Talagrand proved that there exists a Maharam algebra that is not a measure algebra.

For more details on the history see [4].

The present paper investigates *diagonal properties* of complete Boolean algebras. These properties are related to the weak distributive law and to the existence of a continuous submeasure. We apply these properties to get a closer look at Maharam algebras, in particular those that are not measure algebras.

2. Diagonal Properties

A *Boolean algebra* is a set B with Boolean operations $a \vee b$ and $-a$ and constants $\mathbf{0}$ and $\mathbf{1}$. If every subset A of B has a least upper bound $\bigvee A$ and the greatest lower bound $\bigwedge A$ then B is *complete* Boolean algebra. An *antichain* in B is a nonempty set $A \subset B$ such that distinct $a, b \in A$ are *disjoint*, i.e. $a \wedge b = \mathbf{0}$. B satisfies the *countable chain condition* (*ccc*) if it has no uncountable antichains.

[1] Supported in part by the GAAV Grant IAA100190509 and by the Research Program CTS MSM 0021620845.

Let B be a complete Boolean algebra. B is *weakly distributive* (more precisely, (ω,ω)-weakly distributive), if whenever $A_0, A_1, \ldots, A_n \ldots$ $(n \in \omega)$ are countable maximal antichains, then there exists a dense set D such that each $d \in D$ meets only finitely many elements of each A_n.

Definition. *Let B be a complete Boolean algebra.*

(1) *B has the* diagonal property *if whenever $A_0, A_1, \ldots, A_n, \ldots$ $(n \in \omega)$ are countable maximal antichains, then each A_n has a finite subset E_n such that*

$$\lim_n \bigvee\{a : a \in E_n\} = \mathbf{1}.$$

(2) *B has the* strategic diagonal property *if Player II has a winning strategy in the infinite game where Player I plays maximal antichains $A_0, A_1, A_2, \ldots$, Player II plays finite sets $E_0 \subset A_0, E_1 \subset A_1, E_2 \subset A_2, \ldots$ and Player II wins if and only if*

$$\lim_n \bigvee\{a : a \in E_n\} = \mathbf{1}.$$

(3) *B has the* uniform diagonal property *if there exist functions $F_0, F_1, \ldots, F_n, \ldots$ $(n \in \omega)$ acting on maximal antichains such that whenever $A_0, A_1, A_2, \ldots$ are maximal antichains and for each n, $E_n = F_n(A_n)$, then*

$$\lim_n \bigvee\{a : a \in E_n\} = \mathbf{1}.$$

Clearly, the uniform diagonal property implies the strategic diagonal property, which in turn implies the diagonal property.

It has been long well known that if B satisfies ccc then the diagonal property is equivalent to weak distributivity.

If B satisfies ccc then either the strategic diagonal property or the uniform diagonal property is equivalent to the existence of a continuous submeasure (see Fremlin [5] or Balcar–Jech [4]).

In [6] (Lemma 3.5) it is shown that the diagonal property implies the $\mathfrak{b}$-chain condition, where $\mathfrak{b}$ is the *bounding number*, the least cardinal $\mathfrak{b}$ of a family $\mathcal{F}$ of functions from ω to ω such that for every $g \in \omega^\omega$ there is some $f \in \mathcal{F}$ such that $g(n) \leqslant f(n)$ for infinitely many n:

Theorem 1 ([6]). *A complete Boolean algebra B satisfies the diagonal property if and only if it is weakly distributive and satisfies the $\mathfrak{b}$-chain condition.*

Proof. First let B be weakly distributive and $\mathfrak{b}$-cc. Let $A_n = \{a_{nk} : k \in \omega\}$, $n \in \omega$, be countable maximal antichains. By weak distributivity,

$$\bigvee_{f \in \omega^\omega} \liminf_n \bigvee\bigl\{a_{nk} : k \leqslant f(n)\bigr\} = \mathbf{1},$$

and by $\mathfrak{b}$-cc there is a family $\mathcal{F}$ of size less then $\mathfrak{b}$ such that

$$\bigvee_{f \in \mathcal{F}} \liminf_n \bigvee\bigl\{a_{nk} : k \leqslant f(n)\bigr\} = \mathbf{1}.$$

Let $g \in \omega^\omega$ be an upper bound of $\mathcal{F}$ under eventual domination. It follows that

$$\liminf_n \bigvee\{a_{nk} : k \leqslant g(n)\} = \mathbf{1}$$

proving that B has the diagonal property.

Conversely, if B does not have the $\mathfrak{b}$-chain condition then B contains $\mathcal{P}(\mathfrak{b})$ as a complete subalgebra, and it suffices to prove that $\mathcal{P}(\mathfrak{b})$ does not have the diagonal property.

Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded family of functions from ω to ω, and let, for each $n, k \in \omega$,

$$a_{n,k} = \{\alpha < \mathfrak{b} : f_\alpha(n) = k\},$$
$$A_n = \{a_{nk} : k \in \omega\}.$$

Each A_n is a maximal antichain in $\mathcal{P}(\mathfrak{b})$, and if $g \in \omega^\omega$, then there exist an $\alpha < \mathfrak{b}$ such that $g(n) < f_\alpha(n)$ for infinitely many n. Hence for infinitely many n,

$$\alpha \notin \bigcup\{a_{nk} : k \leqslant g(n)\},$$

and so $\alpha \notin \liminf_n \bigcup\{a_{nk} : k \leqslant g(n)\}$. So $\mathcal{P}(\mathfrak{b})$ does not have the diagonal property. $\square$

We now show that the strategic diagonal property implies ccc, and so both the strategic diagonal property and the uniform diagonal property are equivalent to B being Maharam algebra.

If B does not have ccc then it contains $\mathcal{P}(\omega_1)$ as a complete subalgebra, so it is enough to show that $\mathcal{P}(\omega_1)$ does not have the strategic diagonal property.

Theorem 2. *The algebra $\mathcal{P}(\omega_1)$ does not have the strategic diagonal property. Consequently, if B is a complete Boolean algebra with the strategic diagonal property then B satisfies the countable chain condition.*

Proof. For each $\beta < \omega_1$, let f_β be a one-to-one function from β into ω. For $\alpha < \omega_1$ and $n \in \omega$, let

$$a_{\alpha,n} = \{\beta < \omega_1 : f_\beta(\alpha) = n\},$$
$$A_\alpha = \{a_{\alpha,n} : n \in \omega\}.$$

Now assume that σ is a strategy for Player II in the game *(2)*. We shall find α_n, $n \in \omega$, such that if Player I plays A_{α_n} and Player II uses σ, then II loses.

Let $\omega_1^{<\omega}$ denote the set of all increasing finite sequences of countable ordinals. For $s \in \omega_1^{<\omega}$, $s = \langle\alpha_0, \dots, \alpha_{n-1}\rangle$, let $F(s) = k$ be such that if Player II applies σ to $\langle A_{\alpha_0}, \dots A_{\alpha_{n-1}}\rangle$ resulting in a finite set $E \subset A_{\alpha_{n-1}}$, then $E \subset \{a_{\alpha_{n-1},0}, \dots, a_{\alpha_{n-1},k}\}$. For each $s \in \omega_1^{<\omega}$ there exists some k such that the set

$$W_s = \{\alpha < \omega_1 : F(s^\frown\alpha) = k\}$$

is uncountable. Let C_s be the set of all $\beta < \omega_1$ such that $W_s \cap \beta$ is unbounded in β. C_s is a closed unbounded set.

Let C be the diagonal intersection of the C_s, i.e.

$$C = \{\beta < \omega_1 : (\forall s \in \beta^{<\omega})\beta \in C_s\}.$$

C is closed unbounded, so let $\beta > 0$ be some $\beta \in C$.

We shall construct $\{\alpha_n\}_n$ such that for every n, if $E_n = \sigma(A_{\alpha_0}, \dots A_{\alpha_n})$ then $\beta \notin \bigcup\{a : a \in E_n\}$. This witnesses that II loses the game and so σ is not a winning strategy.

We construct α_n by induction. For $n \in \omega$, assume that $\alpha_0, \dots, \alpha_{n-1}$ have been found, and let $s = \langle \alpha_0, \dots, \alpha_{n-1} \rangle$. Let k be such that $F(s^\frown\alpha) = k$ for all $\alpha \in W_s$. Since $W_s \cap \beta$ is infinite and f_β is one-to-one, there exists some $\alpha \in W_s$ such that $f_\beta(\alpha) > k$. We let $\alpha_n = \alpha$. Since $\beta \notin a_{\alpha,0} \cup \dots a_{\alpha,k}$ we have $\beta \notin \bigcup\{a : a \in E_n\}$ where $E_n = \sigma(A_{\alpha_0}, \dots A_{\alpha_n})$.

Thus if Player I plays A_{α_n}, $n \in \omega$, Player II loses: we have

$$\beta \notin \bigcup_{n=0}^{\infty} \bigcup\{a : a \in E_n\}$$

proving that it is not the case that $\lim_n \bigcup\{a : a \in E_n\} = \omega_1$. $\square$

3. Forcing Iteration

Let us consider the operation on complete Boolean algebras corresponding to iterated forcing: if B is a complete Boolean algebra and $\dot{C}$ is a complete Boolean algebra in V^B, then $B * \dot{C}$ is a complete Boolean algebra that produces the iterated forcing model $(V^B)^{\dot{C}}$. For the details, see [7].

It is well known that if B is a measure algebra and if $\dot{C}$ is a measure algebra in V^B then $B * \dot{C}$ is a measure algebra. Similarly, if B is a ccc and weakly distributive and $\dot{C}$ is ccc and weakly distributive in V^B, then $B * \dot{C}$ is ccc and weakly distributive. We prove the same for Maharam algebras:

Theorem 3. *If B is a Maharam algebra and if $\dot{C}$ is a Maharam algebra in V^B, then $B * \dot{C}$ is a Maharam algebra.*

According to Fremlin's notes [5], Theorem 3 was also proved by Farah by a different argument.

If f and g are functions from ω to ω, we say that g dominates f, $f <^* g$, if for some $N \in \omega$, $f(n) < g(n)$ for all $n \geqslant N$.

A Boolean-valued name $\dot{a}$ for a natural number corresponds to a (countable indexed) partition of $\mathbf{1}$, namely

$$\big\{\|\dot{a} = k\| : k \in \omega\big\}.$$

A Boolean-valued name $\dot{f}$ for a function from ω to ω corresponds to a matrix of partitions $\{A_n : n \in \omega\}$, where

$$A_n = \big\{\|\dot{f}(n) = k\| : k \in \omega\big\}.$$

If $\dot{f}$ is a name for a function from ω to ω and if $g : \omega \to \omega$, then

$$\|\dot{f} <^* g\| = \mathbf{1} \quad \text{iff} \quad \lim_n \big\|\dot{f}(n) <^* g(n)\big\| = \mathbf{1}.$$

Using these observations, we can reformulate the diagonal properties as follows, obtaining another characterization of ccc weakly distributive and Maharam algebras (see [4, pp. 258, 259 and 261]).

Let B be a complete Boolean algebra. Then

(1) B has the diagonal property if and only if for every name $\dot{f}$ for a function from ω to ω there exists a function $g : \omega \to \omega$ such that $\|\dot{f} <^* g\| = \mathbf{1}$.

(2) B has the strategic diagonal property if Player II has a winning strategy in the game where I plays names $\dot{f}(0), \dot{f}(1), \dot{f}(2), \dots$ for integers and II plays integers $g(0), g(1), g(1), \dots$ and II wins if and only if $\|\dot{f} <^* g\| = \mathbf{1}$.

(3) B has the uniform diagonal property if and only if there exist functions F_n, $n \in \omega$, acting on names for natural numbers such that for every name $\dot{f}$ for a function from ω to ω, we have

$$\|\dot{f} < g\| = \mathbf{1},$$

where $g : \omega \to \omega$ is obtained as follows: for every n, $g(n) = F_n(\dot{f}(n))$.

Theorem 4. *The diagonal properties are preserved under two-step forcing iteration:*

If B has the diagonal property (resp. the uniform diagonal property) and if $\dot{C}$ has, in V^B, the diagonal property (resp. the uniform diagonal property) then $B * \dot{C}$ has the diagonal (resp. uniform diagonal) property.

Theorem (3) now follows.

Proof. First we give a proof for the diagonal property which is somewhat simpler.

Let $D = B * \dot{C}$ and assume that both B and $\dot{C}$ (in V^B) have the diagonal property. Then if $\dot{f}$ is a D-name, then (since $\dot{f}$ corresponds to a B-name for a $\dot{C}$-name) V^B satisfies that there is some $\dot{g}$ such that $\|\dot{f} <^* \dot{g}\|_{\dot{C}} = \mathbf{1}$. Now $\dot{g}$ is a B-name, so there exists some h such that $\|\dot{g} <^* h\|_B = \mathbf{1}$. It follows that $\|\dot{f} <^* \dot{g}\|_D = \mathbf{1}$ and $\|\dot{g} <^* h\|_D = \mathbf{1}$, hence $\|\dot{f} <^* h\|_D = \mathbf{1}$.

Now let $D = B * \dot{C}$ and assume that both B and $\dot{C}$ (in V^B) have the uniform diagonal property.

There exist functions F_n acting on B-names for natural numbers that witness the domination of each B-name $\dot{f}$, and similarly, in V^B there are functions $\dot{G}_n$ acting on $\dot{C}$-names for natural numbers. We obtain functions H_n for the algebra $D = B * \dot{C}$ as follows: If $\dot{a}$ is a D-name for a natural number, apply $\dot{G}_n$ inside V^B to get a B-valued natural number $\dot{b}$, and then let $H_n(\dot{a}) = F_n(\dot{b})$. (In other words, $H_n(\dot{a}) = F_n(\dot{G}_n(\dot{a}))$ where $\dot{a}$ on the right hand side is considered a B-name for a $\dot{C}$-name.) Then we verify that the functions H_n witness the uniform diagonal property of D. ☐

4. Pathological Maharam Algebras

Let us call a Maharam algebra *pathological* if it is not a measure algebra. Talagrand's construction produces an example of pathological Maharam algebra B of size $2^{\aleph_0}$. The following theorem implies that there are arbitrary large pathological Maharam algebras.

Theorem 5. *If B is a measure algebra and $\dot{C}$ is a pathological Maharam algebra in V^B then $B * \dot{C}$ is pathological.*

Moreover, $B * \dot{C}$ has a continuous submeasure that extends the given measure on B.

The algebra $B * \dot{C}$ contains a measure algebra (B) as a complete subalgebra. It is not known whether there exists a Maharam algebra that does not contain a measure algebra as a complete subalgebra.

Proof. Let $D = B * \dot{C}$. The algebra D is a Maharam algebra by Theorem (3). If D were a measure algebra, then $\dot{C} = D : B$ would also be a measure algebra: this is proved e.g. in Kunen's paper [8]. Since Theorem (5) claims a little more, we give a proof that D is pathological by explicitly describing the pathological submeasure.

Let B be a measure algebra. There is a measure space M and σ-additive measure μ on M such that B is isomorphic to the σ-algebra of Borel sets in M mod the ideal of μ-null sets. An argument due to Dana Scott gives a representation of real numbers in V^B by real-valued measurable function on M: a name $\dot{r} \in V^B$ for a real number corresponds to a measurable function $f : M \to \mathbb{R}$ such that for every real q

$$\|\dot{r} \geqslant q\|_B = \{x \in M : f(x) \geqslant q\}.$$

Now let $\dot{C}$ be a B-name for a pathological Maharam algebra, and let $\dot{m}$ be a name for a continuous submeasure on $\dot{C}$ that is not uniformly exhaustive. That is, for every $q \in B^+$ there exist a $p \in B^+$, $p < q$, and an $\varepsilon > 0$ such that for every $k \in \omega$,

$$p \Vdash \exists \text{ disjoint } c_1, \ldots, c_k \text{ with } \dot{m}(c_i) \geqslant \varepsilon,\ i = 1, \ldots, k. \tag{1}$$

Let $D = B * \dot{C}$. We define a function $\nu : D \to [0, 1]$ as follows: An element of D is a B-name $\dot{c}$ for an element of $\dot{C}$. The value $\dot{m}(\dot{c})$ is a B-name for a real number in $[0, 1]$, hence represented by a measurable function $f : M \to [0, 1]$. We let

$$\nu(\dot{c}) = \int_M f \,\mathrm{d}\mu.$$

It is a routine argument to verify that ν is a subadditive function on D, and $\nu(\dot{c}) > 0$ when $\dot{c} \neq \mathbf{0}$. Moreover, if $\|\dot{c} \in B\| = \mathbf{1}$, then let b be the unique $b \in B$ such that $\|\dot{c} = \mathbf{1}\| = b$ and $\|\dot{c} = \mathbf{0}\| = -b$. We have $\nu(\dot{c}) = \int_b \mathrm{d}\mu = \mu(b)$.

We'll show that ν is continuous and is not uniformly exhaustive.

First let $\dot{c}_0 \geqslant \dot{c}_1 \geqslant \cdots \geqslant \dot{c}_n \geqslant \ldots$ be a sequence in D such that $\bigwedge_{n\in\omega} \dot{c}_n = \mathbf{0}$. Let $\varepsilon > 0$. We shall find an $N \in \omega$ such that $\nu(\dot{c}_n) < 2\varepsilon$, for every $n \geqslant N$.

In V^B, the sequence $\{\dot{c}_n\}_{n\in\omega}$ is a decreasing sequence and $\| \bigwedge_{n\in\omega} \dot{c}_n = \mathbf{0}\| = \mathbf{1}$, and since $\dot{m}$ is continuous, we have $\| \lim_n \dot{m}(\dot{c}_n) = \mathbf{0}\| = \mathbf{1}$. Thus there exists a maximal antichain $\{p_k : k \in \omega\}$ and for each k a number n_k such that

$$p_k \Vdash \dot{m}(\dot{c}_n) < \varepsilon, \quad \text{for all } n \geqslant n_k.$$

Now let K be such that

$$\mu\Bigg(\bigvee_{k=K+1}^{\infty} p_k \Bigg) < \varepsilon,$$

and let $N = \max\{n_k : k \leqslant K\}$. Let $a = p_0 \vee \cdots \vee p_k$ and $b = \bigvee_{k=K+1}^{\infty} p_k$. For each n, let f_n be a measurable function representing $\dot{m}(\dot{c}_n)$. If $n \geqslant N$, we have

$$\nu(\dot{c}_n) = \int_M f_n \,\mathrm{d}\mu = \int_a f_n \,\mathrm{d}\mu + \int_b f_n \,\mathrm{d}\mu < \int_a \varepsilon \,\mathrm{d}\mu + \int_b 1 \,\mathrm{d}\mu \leqslant \varepsilon + \varepsilon = 2\varepsilon.$$

Thus ν is continuous.

In order to show that ν is not uniformly exhaustive, we first find some $p \neq \mathbf{0}$ and $\varepsilon > 0$ such that Eq. (1) holds for every $k \in \omega$. Thus for every k there exist $\dot{c}_1, \dots, \dot{c}_k \in D$ such that $p \Vdash \dot{c}_1, \dots, \dot{c}_k$ are pairwise disjoint and $p \Vdash \dot{m}(\dot{c}_i) \geqslant \varepsilon$ for $i = 1, \dots, k$. We may assume that $-p \Vdash \dot{c}_i = \mathbf{0}$ for all i, and so $\dot{c}_1, \dots, \dot{c}_k$ are pairwise disjoint. Moreover, for each $i = 1, \dots, k$,

$$\nu(\dot{c}_i) \geqslant \int_p \varepsilon \, \mathrm{d}\mu = \varepsilon \cdot \mu(p).$$

Thus ν is not uniformly exhaustive. □

References

[1] R.D. Mauldin, ed., *The Scottish Book*, Birkhäuser Boston, Mass., 1981. Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

[2] B. Balcar, T. Jech and T. Pazák, Complete CCC Boolean algebras, the order sequential topology, and a problem of von Neumann, *Bull. London Math. Soc.* **37**(6) (2005), 885–898.

[3] M. Talagrand, Maharam's problem. *Annals of Math.*, to appear.

[4] B. Balcar and T. Jech, Weak distributivity, a problem of von Neumann and the mystery of measurability, *Bull. Symbolic Logic* **12**(2) (2006), 241–266.

[5] D.H. Fremlin, Maharam algebras, *Unpublished notes*, December, 2004.

[6] B. Balcar, W. Główczyński and T. Jech, The sequential topology on complete Boolean algebras, *Fund. Math.* **155**(1) (1998), 59–78.

[7] T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[8] K. Kunen, Some points in βN, *Math. Proc. Cambridge Philos. Soc.* **80**(3) (1976), 385–398.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Quantifiers in Limits

H. Jerome KEISLER
University of Wisconsin, Madison

Abstract. The standard definition of $\lim_{z\to\infty} F(z) = \infty$ is an $\forall\exists\forall$ sentence. Mostowski showed that in the standard model of arithmetic, these quantifiers cannot be eliminated. But Abraham Robinson showed that in the nonstandard setting, this limit property for a standard function F is equivalent to the one quantifier statement that $F(z)$ is infinite for all infinite z. In general, the number of quantifier blocks needed to define the limit depends on the underlying structure $\mathcal{M}$ in which one is working. Given a structure $\mathcal{M}$ with an ordering, we add a new function symbol F to the vocabulary of $\mathcal{M}$ and ask for the minimum number of quantifier blocks needed to define the class of structures $(\mathcal{M}, F)$ in which $\lim_{z\to\infty} F(z) = \infty$ holds.

We show that the limit cannot be defined with fewer than three quantifier blocks when the underlying structure $\mathcal{M}$ is either countable, special, or an o-minimal expansion of the real ordered field. But there are structures $\mathcal{M}$ which are so powerful that the limit property for arbitrary functions can be defined in both two-quantifier forms.

1. Introduction

An important advantage of the nonstandard approach to elementary calculus is that it eliminates two quantifiers in the definition of a limit. For example, the standard definition of

$$\lim_{z\to\infty} F(z) = \infty$$

requires three quantifier blocks,

$$\forall x\, \exists y\, \forall z\big[y \leqslant z \Rightarrow x \leqslant F(z)\big].$$

Mostowski showed that in the standard model of arithmetic, these quantifiers cannot be eliminated. But Abraham Robinson showed that in the nonstandard setting, this limit property is equivalent to the one quantifier statement

$$\forall z\big[z \in I \Rightarrow F(z) \in I\big],$$

where F is a standard function and I is the set of infinite elements. Because of the quantifiers, beginning calculus students cannot follow the standard definition but have no trouble with the nonstandard definition. Since all the basic notions in the calculus depend on limits, students often find the nonstandard approach to the calculus to be easier to understand than the standard approach (see [7,12]).

In general, the number of quantifier blocks needed to define the limit depends on the underlying structure $\mathcal{M}$ in which one is working. Given a structure $\mathcal{M}$ with an ordering, we add a new function symbol F to the vocabulary of $\mathcal{M}$ and ask for the minimum number of quantifier blocks needed to define the class of structures $(\mathcal{M}, F)$ in which $\lim_{z\to\infty} F(z) = \infty$ holds.

We show that in the standard setting the limit cannot be defined with fewer than three quantifier blocks when the underlying structure $\mathcal{M}$ is not too powerful. We obtain this result in the case that $\mathcal{M}$ is countable, and in the case that $\mathcal{M}$ is an o-minimal expansion of the real field $\mathcal{R} = (\mathbb{R}, \leqslant, +, \cdot)$.

As is usual in the literature, we consider the quantifier hierarchy which takes into account both n-quantifier forms. For each n, one can ask whether a property is Π_n, Σ_n, both Π_n and Σ_n (called Δ_n), or Boolean in Π_n. Each of Σ_n and Π_n implies Boolean in Π_n, which in turn implies Δ_{n+1}. The standard definition of limit is Π_3.

In Section 4 we will see that the limit property can never be Boolean in Π_1 sentences. However, if $\mathcal{M} = (\mathbb{R}, \leqslant, \mathbb{N}, g)$ where g maps $\mathbb{R}$ onto $\mathbb{R}^{\mathbb{N}}$, then the limit property is Δ_2. What happens here is that there is a standard definition of limit which uses one less quantifier block than the usual definition, but needs a function which codes sequences of real numbers by real numbers, and is therefore beyond the scope of an elementary calculus course.

In Section 5 we show that when $\mathcal{M}$ is countable, the limit property is not Σ_3. In Section 6 we prove that the limit property is not Σ_3 when $\mathcal{M}$ is the real ordering with an embedded structure with universe $\mathbb{N}$. In Section 7 we prove that the limit property is not Σ_3 when $\mathcal{M}$ is a saturated or special structure, even when one adds a predicate for the set of infinite elements. This shows that Robinson's result for standard functions does not carry over to arbitrary functions.

In Section 8 we consider infinitely long sentences. In an ordered structure $\mathcal{M}$ with universe set $\mathbb{R}$ and at least a constant symbol for each natural number, the limit property can be expressed naturally by a countable conjunction of countable disjunctions of Π_1 sentences. We show that the limit property cannot be expressed by a countable disjunction of countable conjunctions of Σ_1 sentences.

In Section 9 we prove our main result: If $\mathcal{M}$ is an o-minimal expansion of the real ordered field, then the limit property is not Boolean over Π_2. We leave open the question of whether one can improve this result by showing that the limit property is not Σ_3.

We also consider the similar but simpler property of a function being bounded. The standard definition of boundedness is Σ_2, or alternatively, a countable disjunction of universal sentences. We show that when $\mathcal{M}$ is countable, special, or an o-minimal expansion of the real ordered field, the boundedness property is not Π_2. When $\mathcal{M}$ has universe set $\mathbb{R}$, boundedness cannot be expressed by a countable conjunction of existential sentences.

2. Preliminaries

We introduce a general framework for the study of quantifiers required for defining limits.

We assume throughout that $\mathcal{M}$ is an ordered structure with no greatest element. That is, $\mathcal{M}$ is a first order structure for a vocabulary $L(\mathcal{M})$ that contains at least the order relation $\leqslant$, and $\leqslant$ is a linear ordering with no greatest element. We will consider structures $\mathcal{K} = (\mathcal{M}, f)$ where $f : \mathcal{M} \to \mathcal{M}$ is a unary function. The vocabulary of $(\mathcal{M}, f)$ is $L(\mathcal{M}) \cup \{F\}$ where F is an extra unary function symbol. For a formula $\varphi(\vec{x}, \vec{y})$ of $L(\mathcal{M}) \cup \{F\}$, the notation $\varphi(f, \vec{x}, \vec{y})$ means that $(\mathcal{M}, f) \models \varphi(\vec{x}, \vec{y})$.

We use the notation $|X|$ for the cardinality of X, $\equiv$ for elementary equivalence, $\prec$ for elementary substructure, and $\cong$ for isomorphic.

Quantifier-free formulas are called Π_0 formulas and also called Σ_0 formulas. A Π_{n+1} formula is a formula of the form $\forall \vec{x}\, \theta$ where θ is a Σ_n-formula. A Σ_{n+1} formula is a formula of the form $\exists \vec{x}\, \theta$ where θ is a Π_n-formula. The negation of a Π_n formula is equivalent to a Σ_n formula, and vice versa. A formula is said to be **Boolean in** Π_n if it is built from Π_n formulas using $\wedge, \vee, \neg$. Any formula which is Boolean in Π_n is equivalent to both a Π_{n+1} formula and a Σ_{n+1} formula.

Definition 2.1. In the language $L(\mathcal{M}) \cup \{F\}$, BDD is the Σ_2 sentence which says that f is bounded,

$$BDD = \exists x \forall y\, F(y) \leqslant x.$$

LIM is the Π_3 sentence which says that $\lim_{x\to\infty} f(x) = \infty$,

$$LIM = \forall x \exists y \forall z \big[y \leqslant z \Rightarrow x \leqslant F(z) \big].$$

We will say that a sentence θ of $L(\mathcal{M}) \cup \{F\}$ is Π_n **over** $\mathcal{M}$ if there is a Π_n sentence of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to θ in every structure $(\mathcal{M}, f)$. Similarly for Σ_n. We say that θ is Δ_n **over** $\mathcal{M}$ if it is both Π_n and Σ_n over $\mathcal{M}$. We will say that θ is **Boolean in** Π_n **over** $\mathcal{M}$, or B_n **over** $\mathcal{M}$, if there is a sentence of $L(\mathcal{M}) \cup \{F\}$ which is Boolean in Π_n and is equivalent to θ in every structure $(\mathcal{M}, f)$. With this terminology, there are two quantifier hierarchies of sentences over $\mathcal{M}$,

$$\Delta_1 \subset \Pi_1 \subset B_1 \subset \Delta_2 \subset \Pi_2 \subset B_2 \subset \Delta_3 \subset \Pi_3,$$

$$\Delta_1 \subset \Sigma_1 \subset B_1 \subset \Delta_2 \subset \Sigma_2 \subset B_2 \subset \Delta_3 \subset \Sigma_3.$$

By the **quantifier level** of a sentence over $\mathcal{M}$ we mean the lowest class in these hierarchies to which a sentence belongs over $\mathcal{M}$. Note that the level of a sentence over an expansion of $\mathcal{M}$ is at most its level over $\mathcal{M}$.

In this paper we will consider the following problem.

Problem. Find the quantifier level of BDD and LIM over a given structure $\mathcal{M}$.

For any $\mathcal{M}$, BDD is at most Σ_2 and LIM is at most Π_3 over $\mathcal{M}$.

We remark that whenever $\mathcal{M}$ is an expansion of an ordered field, the limit property $\lim_{x\to 0} f(x) = c$ will be at the same quantifier level as LIM. This can be seen by the change of variables $z = 1/x$. Similar remarks can be made for other limit concepts in the calculus.

3. Results of Mostowski and Robinson

In order to compare the results in this paper to earlier results of Mostowski and Robinson, we define the quantifier level of a sentence over a structure $\mathcal{M}$ relative to a set $\mathcal{F} \subseteq \mathcal{M}^{\mathcal{M}}$, where $\mathcal{M}^{\mathcal{M}}$ is the set of all functions from $\mathcal{M}$ into $\mathcal{M}$. We way that a sentence θ of $L(\mathcal{M}) \cup \{F\}$ is Π_n **over** $\mathcal{M}$ **relative to** $\mathcal{F}$ if there is a Π_n sentence of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to θ in every structure $(\mathcal{M}, f)$ with $f \in \mathcal{F}$. Similarly for Σ_n. Thus a sentence is Π_n over $\mathcal{M}$ in our original sense iff it is Π_n over $\mathcal{M}$ relative to $\mathcal{M}^{\mathcal{M}}$.

Note that if a sentence is Π_n over $\mathcal{M}$, then it is Π_n over $\mathcal{M}$ relative to every set $\mathcal{F} \subseteq \mathcal{M}^{\mathcal{M}}$, and similarly for Σ_n. Thus for ever $\mathcal{M}$ and $\mathcal{F}$, BDD is at most Σ_2 over $\mathcal{M}$ relative to $\mathcal{F}$, and LIM is at most Π_3 over $\mathcal{M}$ relative to $\mathcal{F}$.

In the paper [10], Mostowski showed that several limit concepts, including BDD and LIM, have the highest possible quantifier level over the standard model of arithmetic. In fact, over this particular structure, he obtains the stronger result that BDD and LIM have the highest possible quantifier level relative to the set of primitive recursive functions.

Theorem 3.1 (Mostowski [10]). *Let $\mathcal{F}$ be the set of all primitive recursive functions, and let $\mathcal{N}$ be the standard model of arithmetic with a symbol for every function in $\mathcal{F}$.*

(i) *BDD is not Π_2 over $\mathcal{N}$ relative to $\mathcal{F}$.*
(ii) *LIM is not Σ_3 over $\mathcal{N}$ relative to $\mathcal{F}$.*

The proof of Theorem 2 in [10] shows that for every Σ_2 formula $\theta(y)$ in $L(\mathcal{N})$ there is a primitive recursive function $g(x, y)$ such that $\theta(y)$ defines the set of y for which $g(\cdot, y)$ is bounded. By the Arithmetical Hierarchy Theorem of Kleene [8] and Mostowski [9], θ may be taken to be Σ_2 but not Π_2 over $\mathcal{N}$, and (i) follows.

Similarly, the proof of Theorem 3 in [10] shows that for every Π_3 formula $\psi(y)$ in $L(\mathcal{N})$ there is a primitive recursive function $h(x, y)$ such that $\psi(y)$ defines the set of y for which $\lim_{x\to\infty} h(x, y) = \infty$, and (ii) follows by taking ψ to be Π_3 but not Σ_3 over $\mathcal{N}$.

Abraham Robinson's characterization of infinite limits with one universal quantifier uses an elementary extension of $\mathcal{M}$.

Definition 3.2. In an elementary extension ${}^*\mathcal{M}$ of $\mathcal{M}$, an element is **infinite** (over $\mathcal{M}$) if it is greater than every element of $\mathcal{M}$. By a **hyperextension** of $\mathcal{M}$ we mean a structure $({}^*\mathcal{M}, I)$ where ${}^*\mathcal{M}$ is an elementary extension of $\mathcal{M}$ with at least one infinite element, and I is a unary predicate for the set of infinite elements. A function ${}^*f : {}^*\mathcal{M} \to {}^*\mathcal{M}$ is **standard** if there is a function $f : \mathcal{M} \to \mathcal{M}$ such that $({}^*\mathcal{M}, {}^*f)$ is an elementary extension of $(\mathcal{M}, f)$.

Theorem 3.3 (A. Robinson). *Let $({}^*\mathcal{M}, I)$ be a hyperextension of $\mathcal{M}$. Then BDD and LIM are Π_1 over $({}^*\mathcal{M}, I)$ relative to the set standard functions.*

In fact, Robinson shows that for every standard function ${}^*f : {}^*\mathcal{M} \to {}^*\mathcal{M}$, $({}^*\mathcal{M}, {}^*f)$ satisfies

$$BDD \Leftrightarrow \forall x\big[\neg I(F(x))\big],$$

$$LIM \Leftrightarrow \forall x\big[I(x) \Rightarrow I(F(x))\big].$$

In the nonstandard treatment of elementary calculus, one works in a hyperextension of the system $\mathbb{R}$ of real numbers or the system $\mathbb{N}$ of natural numbers. The extra predicate I for infinite elements eliminates one quantifier block in the definition of boundedness and two quantifier blocks in the definition of limit. The question we address here is whether one can also eliminate these quantifiers in the original first order language $L(\mathcal{M}) \cup \{F\}$. That is, are there any "quantifier shortcuts" for the statements BDD or LIM in $\mathcal{M}$? This question is completely standard in nature, but is motivated by Robinson's results in nonstandard analysis.

4. Cases with Low Quantifier Level

In this section we show that when the underlying structure $\mathcal{M}$ is sufficiently powerful, the properties BDD and LIM are equivalent to sentences in both two quantifier forms. All that is needed is a symbol for a particular function which is first order definable in the real field with a predicate for the natural numbers. This gives a warning that some restrictions are needed on $\mathcal{M}$ in order to show that BDD and LIM are higher than Δ_2 over $\mathcal{M}$. We also show that BDD and LIM can never be B_1 over $\mathcal{M}$.

Theorem 4.1. *Let $\mathcal{M} = (\mathbb{R}, \leqslant, \mathbb{N}, g)$ be the ordered set of real numbers with a predicate for the natural numbers and a function $g : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ such that $x \mapsto g(x, \cdot)$ maps $\mathbb{R}$ onto $\mathbb{R}^{\mathbb{N}}$, that is, for each $y \in \mathbb{R}^{\mathbb{N}}$ there exists $x \in \mathbb{R}$ such that $y = g(x, \cdot)$.*

(i) *BDD is Δ_2 over $\mathcal{M}$.*
(ii) *LIM is Δ_2 over $\mathcal{M}$.*

Proof. The language $L(\mathcal{M})$ has the vocabulary $\{\mathbb{N}, \leqslant, G\}$.

(i) BDD is itself a Σ_2 sentence. In every structure $(\mathcal{M}, f)$, the negation of BDD is equivalent to the Σ_2 sentence

$$\exists x (\forall n \in \mathbb{N}) n \leqslant F\big(G(x, n)\big).$$

(ii) In every structure $(\mathcal{M}, f)$, LIM is equivalent to the Σ_2 sentence

$$\exists x (\forall n \in \mathbb{N}) \forall y \big[G(x, n) \leqslant y \Rightarrow n \leqslant F(y)\big],$$

and the negation of LIM is equivalent to the Σ_2 sentence

$$(\exists m \in \mathbb{N}) \exists x (\forall n \in \mathbb{N}) \big[n \leqslant G(x, n) \wedge F\big(G(x, n)\big) \leqslant m\big]. \quad \square$$

It is well known that in the above theorem, the function g can be taken to be definable by a first order formula in the structure $(\mathbb{R}, \leqslant, +, \cdot, \mathbb{N})$. One way to do this is to let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a definable pairing function, and for irrational x, let $h(x, \cdot) \in \mathbb{N}^{\mathbb{N}}$ be the continued fraction expansion of x, and take $g(x, n) = z$ iff $\forall m\, h(z, m) = h(x, \pi(m, n))$.

We now show that for an arbitrary $\mathcal{M}$, LIM and BDD cannot be Boolean in Π_1.

Theorem 4.2. *Let $\mathcal{M}$ be an ordered structure with no greatest element.*

(i) *BDD is not Boolean in Π_1 over $\mathcal{M}$.*
(ii) *LIM is not Boolean in Π_1 over $\mathcal{M}$.*

Proof. Let φ be a sentence of $L(\mathcal{M}) \cup \{F\}$ which is Boolean in Π_1. Since conjunctions and disjunctions of Π_1 sentences are equivalent to Π_1 sentences, and similarly for Σ_1, φ is equivalent to a sentence

$$(\alpha_1 \vee \beta_1) \wedge \cdots \wedge (\alpha_n \vee \beta_n)$$

where each α_i is Σ_1 and each β_i is Π_1. We may assume that α_i and β_i have the form

$$\alpha_i = \exists\vec{x}\,\exists\vec{y}\big[F(\vec{x}) = \vec{y} \wedge \overline{\alpha}_i(\vec{x},\vec{y})\big],$$

$$\beta_i = \forall\vec{x}\,\forall\vec{y}\,\big[F(\vec{x}) = \vec{y} \Rightarrow \overline{\beta}_i(\vec{x},\vec{y})\big]$$

where $\overline{\alpha}_i$ and $\overline{\beta}_i$ are quantifier-free formulas of $L(\mathcal{M})$. (This can be proved by induction on the number of occurrences of F).

Let us say that a pair of tuples $(\vec{a},\vec{b})$ in $\mathcal{M}$ decides φ if for any $f, g : \mathcal{M} \to \mathcal{M}$ such that $f(\vec{a}) = g(\vec{a}) = \vec{b}$, φ holds in either both or neither of the models $(\mathcal{M}, f)$, $(\mathcal{M}, g)$.

Claim. *There is a pair* $(\vec{a},\vec{b})$ *which decides* φ.

Proof of Claim. The proof is by induction on n. The claim is trivial for $n = 0$, where the empty conjunction is taken to be always true. Suppose $n > 0$ and the claim holds for $n-1$. Then there is a pair $(\vec{a},\vec{b})$ which decides

$$(\alpha_1 \vee \beta_1) \wedge \cdots \wedge (\alpha_{n-1} \vee \beta_{n-1}).$$

A pair $(\vec{c},\vec{d})$ is called compatible with $(\vec{a},\vec{b})$ if $b_i = d_j$ whenever $a_i = c_j$, that is, there exists a function f with $f(\vec{a},\vec{c}) = (\vec{b},\vec{d})$.

Case 1. There is a pair $(\vec{c},\vec{d})$ compatible with $(\vec{a},\vec{b})$ such that $\overline{\alpha}_n(\vec{c},\vec{d})$. Then $f(\vec{c}) = \vec{d}$ implies α_n.

Case 2. Case 1 fails but there is a pair $(\vec{c},\vec{d})$ compatible with $(\vec{a},\vec{b})$ such that $\neg\overline{\beta}_n(\vec{c},\vec{d})$. Then $f(\vec{a},\vec{c}) = (\vec{b},\vec{d})$ implies $\neg(\alpha_n \vee \beta_n)$.

Case 3. For every pair $(\vec{c},\vec{d})$ compatible with $(\vec{a},\vec{b})$,

$$\neg\overline{\alpha}_n(\vec{c},\vec{d}) \wedge \overline{\beta}_n(\vec{c},\vec{d}).$$

Then $f(\vec{a}) = \vec{b}$ implies β_n.

In each case, $(\vec{a} \cup \vec{c}, \vec{b} \cup \vec{d})$ decides φ, completing the induction.

Now let $(\vec{a},\vec{b})$ decide φ. There are functions f, g such that $f(\vec{a}) = g(\vec{a}) = \vec{b}$, but BDD holds in $(\mathcal{M}, f)$ and fails in $(\mathcal{M}, g)$. Therefore φ cannot be equivalent to BDD over $\mathcal{M}$. A similar argument holds for LIM. □

5. Countable Structures

In this section we consider the case that the universe of $\mathcal{M}$ is countable. In that case, we apply Mostowski's Theorem 3.1 to show that BDD and LIM have the highest possible quantifier level. We first observe that Mostowski's proof of Theorem 3.1 also gives a relativized form of the result.

Theorem 5.1. *Let α be a finite tuple of finitary functions on $\mathbb{N}$, let $\mathcal{F}(\alpha)$ be the set of all functions which are primitive recursive in α, and let $\mathcal{N}$ be the standard model of arithmetic with an extra symbol for each function in the set $\mathcal{F}(\alpha)$.*

(i) *BDD is not Π_2 over $\mathcal{N}$ relative to $\mathcal{F}(\alpha)$.*
(ii) *LIM is not Σ_3 over $\mathcal{N}$ relative to $\mathcal{F}(\alpha)$.*

Corollary 5.2. *Let $\mathcal{N}$ be an expansion of the standard model of arithmetic with the natural ordering $<$.*

(i) *BDD is not* Π_2 *over* $\mathcal{N}$.
(ii) *LIM is not* Σ_3 *over* $\mathcal{N}$.

Theorem 5.3. *Suppose $\mathcal{M}$ is an ordered structure with no greatest element and the universe of $\mathcal{M}$ is countable.*

(i) *BDD is not* Π_2 *over* $\mathcal{M}$.
(ii) *LIM is not* Σ_3 *over* $\mathcal{M}$.

Proof. We may assume that the vocabulary of $\mathcal{M}$ is finite, since only finitely many symbols occur in a Σ_3 formula. We may also assume that the universe of $\mathcal{M}$ is the set $\mathbb{N}$ of natural numbers. Let $<_{\mathcal{M}}$ be the ordering of $\mathcal{M}$ (which may be different from the natural order $<$ of $\mathbb{N}$). Since $\mathcal{M}$ has no greatest element, there is a function $h : \mathbb{N} \to \mathbb{N}$ such that $h(n) <_{\mathcal{M}} h(n+1)$ for each n, and $\forall x \exists n\, x \leqslant_{\mathcal{M}} h(n)$. For each x let $\lambda(x)$ be the least n such that $x \leqslant_{\mathcal{M}} h(n)$.

We prove (ii). The proof of (i) is similar. We observe that for each function $g : \mathbb{N} \to \mathbb{N}$, $\lim_{x\to\infty} h(g(\lambda(x))) = \infty$ with respect to $<_{\mathcal{M}}$ if and only if $\lim_{x\to\infty} g(x) = \infty$ with respect to $<$. If LIM is not Σ_3 over an expansion of $\mathcal{M}$, then it is not Σ_3 over $\mathcal{M}$. We may therefore assume that $\mathcal{M}$ has symbols for $<, +, \cdot, h$, and λ.

Now suppose that LIM is Σ_3 over $\mathcal{M}$, that is, there is a Σ_3 sentence $\exists\vec{x}\,\forall\vec{y}\,\exists\vec{z}\,\varphi(\vec{x}, \vec{y}, \vec{z})$ of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to LIM in all structures $(\mathcal{M}, f)$. To complete the proof we will get a contradiction.

Let $\psi(\vec{x}, \vec{y}, \vec{z})$ be the formula obtained from $\varphi(\vec{x}, \vec{y}, \vec{z})$ by replacing each term $F(u)$ by $h(F(\lambda(u)))$. By our observation above, for any function $g : \mathbb{N} \to \mathbb{N}$, we have $\lim_{x\to\infty} g(x) = \infty$ with respect to $<$ if and only if $(\mathcal{M}, g)$ satisfies the Σ_3 sentence $\exists\vec{x}\,\forall\vec{y}\,\exists\vec{z}\,\psi(\vec{x}, \vec{y}, \vec{z})$. This contradicts Corollary 5.2 and completes the proof.

We also give a second argument, which will be useful later. Instead of Corollary 5.2, this argument uses the following analogous fact from descriptive set theory (see [5, Exercise 23.2]):

In $\mathbb{N}^{\mathbb{N}}$, the set L of functions g with $\lim_{x\to\infty} g(x) = \infty$ is $\mathbf{\Pi}_3$ but not $\mathbf{\Sigma}_3$ in the Borel hierarchy.

We have

$$L = \bigcup_{\vec{x}\in\mathbb{N}} \bigcap_{\vec{y}\in\mathbb{N}} \bigcup_{\vec{z}\in\mathbb{N}} \big(\{g : (\mathcal{M}, g) \models \psi(\vec{x}, \vec{y}, \vec{z})\}\big).$$

For each $(\vec{x}, \vec{y}, \vec{z})$, the set $\{g : (\mathcal{M}, g) \models \psi(\vec{x}, \vec{y}, \vec{z})\}$ depends only on finitely many values of g and thus is clopen in $\mathbb{N}^{\mathbb{N}}$. It follows that L is $\mathbf{\Sigma}_3$ in $\mathbb{N}^{\mathbb{N}}$, contrary to the preceding paragraph. $\square$

The next corollary follows by the Löwenheim-Skolem theorem.

Corollary 5.4. *Let $\mathcal{M}$ be an ordered structure with no greatest element.*

(i) *There is no* Π_2 *sentence of* $L(\mathcal{M}) \cup \{F\}$ *which is equivalent to BDD in all models of* $Th(\mathcal{M})$.
(ii) *There is no* Σ_3 *sentence of* $L(\mathcal{M}) \cup \{F\}$ *which is equivalent to LIM in all models of* $Th(\mathcal{M})$.

A result in this direction was previously obtained by Kathleen Sullivan in [12]. She showed that there is no Σ_2 sentence of $L(\mathcal{M})\cup\{F\}$, and no Π_2 sentence of $L(\mathcal{M})\cup\{F\}$, which is equivalent to LIM in all models of $Th(\mathcal{M})$.

6. The Real Line

The sentence LIM is a Π_3 sentence in the vocabulary with the single relation symbol $\leqslant$. It is therefore natural to ask whether LIM is Σ_3 over the real ordering $(\mathbb{R},\leqslant)$. In this section we show that the answer is no. Given a structure $\mathcal{N}$ with universe $\mathbb{N}$, we let $(\mathbb{R},\leqslant,\mathcal{N})$ be the structure formed by adding to $(\mathbb{R},\leqslant)$ a symbol for $\mathbb{N}$ and the relations of $\mathcal{N}$.

Theorem 6.1. *Let $\mathcal{N}$ be a structure with universe* $\mathbb{N}$.

(i) *BDD is not Π_2 over* $(\mathbb{R},\leqslant,\mathcal{N})$.
(ii) *LIM is not Σ_3 over* $(\mathbb{R},\leqslant,\mathcal{N})$.

Proof. We prove (ii). The proof of (i) is similar. We may assume that $\mathcal{N}$ has a countable vocabulary. Let $\mathbb{Q}$ be the set of rational numbers, and let $\lambda(x)=\min\{n\in\mathbb{N} : x\leqslant n\}$.

Claim 1. $(\mathbb{Q},\leqslant,\mathcal{N})$ *is an elementary substructure of* $(\mathbb{R},\leqslant,\mathcal{N})$.

Proof of Claim 1. We note that for any two increasing n-tuples $\vec{a},\vec{b}$ in $\mathbb{R}$ such that $\vec{a}\cap\mathbb{N}=\vec{b}\cap\mathbb{N}$ and $\lambda(\vec{a})=\lambda(\vec{b})$, there is an automorphism of $(\mathbb{R},\leqslant,\mathcal{N})$ which sends $\vec{a}$ to $\vec{b}$. Claim 1 now follows from the Tarski-Vaught test ([1, Proposition 3.1.2]).

Define the function $\alpha:\mathbb{N}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{R}}$ by $(\alpha(g))(x)=g(\lambda(x))$, and let $\beta(g)$ be the restriction of $\alpha(g)$ to $\mathbb{Q}$.

Claim 2. *For each function $g\in\mathbb{N}^{\mathbb{N}}$, $(\mathbb{Q},\leqslant,\mathcal{N},\beta(g))$ is an elementary substructure of* $(\mathbb{R},\leqslant,\mathcal{N},\alpha(g))$.

To see this, note that by Claim 1, $(\mathbb{Q},\leqslant,(\mathcal{N},g))$ is an elementary substructure of $(\mathbb{R},\leqslant,(\mathcal{N},g))$. It is easily seen that λ is definable in $(\mathbb{R},\leqslant,\mathbb{N})$, and hence $\alpha(g)$ is definable in $(\mathbb{R},\leqslant,(\mathcal{N},g))$. Claim 2 follows.

Let L be the set of $g\in\mathbb{N}^{\mathbb{N}}$ such that $(\mathcal{M},\alpha(g))\models LIM$. We note that $\lim_{n\to\infty}g(n)=\infty$ in $\mathbb{N}^{\mathbb{N}}$ if and only if $g\in L$, and hence that L is not $\mathbf{\Sigma_3}$ in $\mathbb{N}^{\mathbb{N}}$. Now suppose that LIM is Σ_3 over $(\mathbb{R},\leqslant,\mathcal{N})$, and take a Σ_3 sentence

$$\theta=\exists\vec{x}\,\forall\vec{y}\,\exists\vec{z}\,\varphi(\vec{x},\vec{y},\vec{z})$$

of $L(\mathbb{R},\leqslant,\mathcal{N})\cup\{F\}$ which is equivalent to LIM in all structures $(\mathbb{R},\leqslant,\mathcal{N},f)$. By Claim 2, for each $g\in\mathbb{N}^{\mathbb{N}}$, $(\mathbb{R},\leqslant,\mathcal{N},\alpha(g))$ satisfies θ if and only if $(\mathbb{Q},\leqslant,\mathcal{N},\beta(g))$ satisfies θ. Therefore

$$L=\bigcup_{\vec{x}\in\mathbb{Q}}\bigcap_{\vec{y}\in\mathbb{Q}}\bigcup_{\vec{z}\in\mathbb{Q}}\big(\{g:\varphi(\beta(g),\vec{x},\vec{y},\vec{z})\}\big).$$

For each $(\vec{x},\vec{y},\vec{z})$, the set $(\{g:\varphi(\beta(g),\vec{x},\vec{y},\vec{z})\})$ depends only on finitely many values of g and thus is clopen in $\mathbb{N}^{\mathbb{N}}$. Since $\mathbb{Q}$ is countable, it follows that L is $\mathbf{\Sigma}_3$ in $\mathbb{N}^{\mathbb{N}}$. This contradiction proves (ii). $\square$

We remark that the above theorem also holds, with the same proof, when $\mathbb{R}$ is replaced by any dense linear ordering with a cofinal copy of $\mathbb{N}$.

7. Saturated and Special Structures

In this section we show that BDD and LIM have the highest possible quantifier level if the underlying structure $\mathcal{M}$ is saturated, or more generally, special. This happens even if one adds a symbol for the set I of infinite elements to the vocabulary. Thus Robinson's Theorem 3.3 for standard functions cannot be extended to the set of all functions on a special structure.

We recall some basic facts (See [1, Chapter 5]). By definition, a structure $\mathcal{M}$ is κ-**saturated** if every set of fewer than κ formulas with parameters in $\mathcal{M}$ which is finitely satisfiable in $\mathcal{M}$ is satisfiable in $\mathcal{M}$. $\mathcal{M}$ is **saturated** if it is $|\mathcal{M}|$-saturated. $\mathcal{M}$ is **special** if it is the union of an elementary chain of λ^+-saturated structures where λ ranges over all cardinals less than $|\mathcal{M}|$.

Let us call a cardinal κ **nice** if $\kappa = \sum_{\lambda<\kappa} 2^\lambda$. For example, every inaccessible cardinal is nice, every strong limit cardinal is nice, and every cardinal $\lambda^+ = 2^\lambda$ is nice.

Facts 7.1.

(i) *If $\omega \leqslant |\mathcal{M}| < \kappa$ and κ is nice then $\mathcal{M}$ has a special elementary extension of cardinality κ.*
(ii) *Any two elementarily equivalent special models of the same cardinality are isomorphic.*
(iii) *Reducts of special models are special.*
(iv) *If $\mathcal{M}$ is special, then $(\mathcal{M}, \vec{a})$ is special for each finite tuple $\vec{a}$ in $\mathcal{M}$.*

Theorem 7.2. *Suppose $\mathcal{M}$ is an ordered structure with no greatest element, $\mathcal{M}$ is special, $|\mathcal{M}|$ is nice, and $|L(\mathcal{M})| < |\mathcal{M}|$.*

(i) *BDD is not Π_2 over $\mathcal{M}$.*
(ii) *LIM is not Σ_3 over $\mathcal{M}$.*

Proof. We prove (ii). The proof of (i) is similar. Let T be the set of all Σ_3 consequences of $Th(\mathcal{M}) \cup \{LIM\}$. Consider a finite subset $T_0 \subseteq T$. The conjunction of T_0 is equivalent to a Σ_3 sentence θ, and $Th(\mathcal{M}) \cup \{LIM\} \models \theta$. By Corollary 5.4, we cannot have $Th(\mathcal{M}) \cup \{\theta\} \models LIM$, and therefore we cannot have $Th(\mathcal{M}) \cup T_0 \models LIM$. By the Compactness Theorem, there is a model $(\mathcal{M}_0, g)$ of $Th(\mathcal{M}) \cup T \cup \{\neg LIM\}$. Now let U be the set of all Π_3 sentences which hold in $(\mathcal{M}_0, g)$. By the Compactness Theorem again, there is a model $(\mathcal{M}_1, f)$ of $Th(\mathcal{M}) \cup U \cup \{LIM\}$. Then every Σ_3-sentence holding in $(\mathcal{M}_1, f)$ holds in $(\mathcal{M}_0, g)$. Let $\kappa = |\mathcal{M}|$. Since $|L(\mathcal{M})| < \kappa$ and κ is nice, it follows from Fact (i) that we may take $(\mathcal{M}_0, g)$ and $(\mathcal{M}_1, f)$ to be special models of cardinality κ. By Fact (ii), $\mathcal{M}_0$ and $\mathcal{M}_1$ are isomorphic to $\mathcal{M}$, so we may take $\mathcal{M}_1 = \mathcal{M}_0 = \mathcal{M}$. Then LIM holds in $(\mathcal{M}, f)$ and fails in $(\mathcal{M}, g)$, but every Σ_3 sentence holding in $(\mathcal{M}, f)$ holds in $(\mathcal{M}, g)$. This proves (ii). The proof of (i) is similar. $\square$

Given a hyperextension $({}^*\mathcal{M}, I)$ of $\mathcal{M}$, we will say that a function $g : {}^*\mathcal{M} \to {}^*\mathcal{M}$ is **standard** if $({}^*\mathcal{M}, g)$ is an elementary extension of $(\mathcal{M}, f)$, or in other words, if $({}^*\mathcal{M}, g, I)$ is a hyperextension of $(\mathcal{M}, f)$. Robinson's Theorem 3.3 shows that BDD and LIM are Π_1 for standard functions over any hyperextension $({}^*\mathcal{M}, I)$ of $\mathcal{M}$.

Our next theorem will show that Robinson's result does not carry over to nonstandard functions. In fact, we will show that Theorem 7.2 holds even when a symbol for the set I of infinite elements is added to the vocabulary. Thus for nonstandard functions, one cannot lower the quantifier level of BDD or LIM by adding a symbol for I.

Lemma 7.3. *Let $\mathcal{M}$ be an ordered structure with no greatest element, such that every element of $\mathcal{M}$ is a constant symbol of $L(\mathcal{M})$. Suppose that $({}^*\mathcal{M}, I)$ a hyperextension of $\mathcal{M}$ of cardinality $\kappa = |{}^*\mathcal{M}|$ such that $|L(\mathcal{M})| < \kappa$, κ is nice, $({}^*\mathcal{M}, f, g)$ is special, and every Π_n sentence of $L(\mathcal{M}) \cup \{F\}$ true in $({}^*\mathcal{M}, f)$ is true in $({}^*\mathcal{M}, g)$. Then every Π_n sentence of $L(\mathcal{M}) \cup \{F, I\}$ true in $({}^*\mathcal{M}, f, I)$ is true in $({}^*\mathcal{M}, g, I)$. Similarly for Σ_n.*

Proof. The result for Σ_n follows from the result for Π_n by interchanging f and g. Let $\kappa = |{}^*\mathcal{M}|$. By the Keisler Sandwich Theorem ([1, Proposition 5.2.7 and Exercise 5.2.7]), there is a chain

$$(\mathcal{M}_0, h_0) \subseteq (\mathcal{M}_1, h_1) \subseteq \cdots (\mathcal{M}_n, h_n)$$

such that

$$(\mathcal{M}_0, h_0) \equiv (\mathcal{M}, f), \quad (\mathcal{M}_1, h_1) \equiv (\mathcal{M}, g),$$

and for each $m < n - 1$,

$$(\mathcal{M}_m, h_m) \prec (\mathcal{M}_{m+2}, h_{m+2}).$$

By Fact (i) we may take each $(\mathcal{M}_m, h_m)$ to be a special structure of cardinality κ. For each $m \leqslant n$ let I_m be the set of elements of $\mathcal{M}_m$ greater than each element of $\mathcal{M}$. We then have

$$(\mathcal{M}_0, h_0, I_0) \subseteq (\mathcal{M}_1, h_1, I_1) \subseteq \cdots (\mathcal{M}_n, h_n, I_n).$$

Since each element of $\mathcal{M}$ has a constant symbol in $L(\mathcal{M})$, it follows from Fact (i) that

$$(\mathcal{M}_0, h_0, I_0) \cong ({}^*\mathcal{M}, f, I) \quad \text{and} \quad (\mathcal{M}_1, h_1, I_1) \cong ({}^*\mathcal{M}, g, I).$$

By Fact (iv), $(\mathcal{M}_m, h_m, \vec{a})$ is still special for each m and finite tuple $\vec{a}$ in $\mathcal{M}_m$. Then by Fact (ii), for each $m < n - 1$ and finite tuple $\vec{a}$ in $\mathcal{M}_m$, there is an isomorphism from $(\mathcal{M}_m, h_m)$ onto $(\mathcal{M}_{m+2}, h_{m+2})$ which fixes $\vec{a}$ and each element of $\mathcal{M}$. This isomorphism also sends I_m onto I_{m+2}, so

$$(\mathcal{M}_m, h_m, I_m, \vec{a}) \cong (\mathcal{M}_{m+2}, h_{m+2}, I_{m+2}, \vec{a}).$$

Using the Tarski-Vaught criterion for elementary extensions ([1, Proposition 3.1.2]), it follows that

$$(\mathcal{M}_m, h_m, I_m) \prec (\mathcal{M}_{m+2}, h_{m+2}, I_{m+2}).$$

By the Keisler Sandwich Theorem again, every Π_n-sentence true in $(\mathcal{M}_0, h_0, I_0)$ is true in $(\mathcal{M}_1, h_1, I_1)$, and hence every Π_n-sentence true in $({}^*\mathcal{M}, f, I)$ is true in $({}^*\mathcal{M}, g, I)$. $\square$

Theorem 7.4. *Let $\mathcal{M}$ be an ordered structure with no greatest element, and let $({}^*\mathcal{M}, I)$ be a hyperextension of $\mathcal{M}$ such that ${}^*\mathcal{M}$ is special and $|{}^*\mathcal{M}|$ is uncountable and nice.*

(i) *BDD is not Π_2 over $({}^*\mathcal{M}, I)$.*
(ii) *LIM is not Σ_3 over $({}^*\mathcal{M}, I)$.*

Proof. We prove (ii). The proof of (i) is similar. By the proof of Theorem 7.2, there are functions f, g such that $(^*\mathcal{M}, f)$ and $(^*\mathcal{M}, g)$ are special, LIM holds in $(^*\mathcal{M}, f)$ and fails in $(^*\mathcal{M}, g)$, and every Σ_3 sentence true in $(^*\mathcal{M}, f)$ is true in $(^*\mathcal{M}, g)$. By Facts (i) and (ii), we may take f and g so that $(^*\mathcal{M}, f, g)$ is special. Then by the preceding lemma, every Σ_3 sentence true in $(^*\mathcal{M}, f, I)$ is true in $(^*\mathcal{M}, g, I)$. Therefore LIM is not Σ_3 over $(^*\mathcal{M}, I)$. □

8. Infinitely Long Sentences

In this section we consider the quantifier levels of BDD and LIM in the infinitary logic $L_{\omega_1\omega}$ formed by adding countable conjunctions and disjunctions to first order logic. See [6] for a treatment of the model theory of this logic.

Definition 8.1. Let Q be a set of sentences of $L_{\omega_1\omega}$ in the vocabulary $L(\mathcal{M})$. $\bigwedge Q$ denotes the set of countable conjunctions of sentences in Q. $\bigvee Q$ denotes the set of countable disjunctions of sentences in Q. BQ denotes the set of finite Boolean combinations of sentences in Q.

For example, $\bigvee \bigwedge B \bigwedge \Pi_1$ is the set of sentences of the form $\bigvee_m \bigwedge_n \theta_{mn}$ where each θ_{mn} is a finite Boolean combination of countable conjunctions of universal first order sentences.

We say that $\mathcal{M}$ has **cofinality** ω if $\mathcal{M}$ has a countable increasing sequence $a_0, a_1, \ldots$ which is unbounded, that is, $\forall x \bigvee_n x \leqslant a_n$. Every countable $\mathcal{M}$ and every structure $\mathcal{M}$ with universe $\mathbb{R}$ has cofinality ω. In this section our main focus will be uncountable $\mathcal{M}$ with cofinality ω.

Suppose that $\mathcal{M}$ has universe $\mathbb{R}$ and a constant symbol for each natural number. Then BDD is $\bigvee \Pi_1$ over $\mathcal{M}$, because

$$(\mathcal{M}, f) \models BDD \Leftrightarrow \bigvee_n \forall z\, F(z) \leqslant n.$$

LIM is $\bigwedge \bigvee \Pi_1$ over $\mathcal{M}$, because

$$(\mathcal{M}, f) \models LIM \Leftrightarrow \bigwedge_m \bigvee_n \forall z \big[n \leqslant z \Rightarrow m \leqslant F(z)\big].$$

LIM is also $\bigwedge \Sigma_2$ over $\mathcal{M}$, because

$$(\mathcal{M}, f) \models LIM \Leftrightarrow \bigwedge_m \exists y \forall z \big[y \leqslant z \Rightarrow m \leqslant F(z)\big].$$

The next theorem shows that if $\mathcal{M}$ has cofinality ω, then BDD is not $\bigwedge \Sigma_1$ over $\mathcal{M}$ and LIM is not $\bigvee \bigwedge \Sigma_1$ over $\mathcal{M}$. Thus if the outer quantifiers $\exists$ and $\forall$ are replaced by $\bigvee$ and $\bigwedge$, then Theorem 5.3 holds for uncountable structures $\mathcal{M}$ of cofinality ω. In fact, we get a stronger result with $B \bigwedge \Pi_1$ in place of Σ_1.

Theorem 8.2. *Suppose that $\mathcal{M}$ has cofinality ω.*

(i) *BDD is not $\bigwedge B \bigwedge \Pi_1$ over $\mathcal{M}$.*
(ii) *LIM is not $\bigvee \bigwedge B \bigwedge \Pi_1$ over $\mathcal{M}$.*

Proof. We prove (ii). The proof of (i) is similar.

Let a_n be an unbounded strictly increasing sequence in $\mathcal{M}$. Since each sentence of $L_{\omega_1\omega}$ has countably many symbols, we may assume without loss of generality that the vocabulary $L(\mathcal{M})$ is countable. We may also assume that $L(\mathcal{M})$ has a constant symbol, say $\mathbf{n}$, for each a_n, and a symbol for the function $\lambda(x) = \min\{a_n : x \leqslant a_n\}$. As in the proof of Theorem 5.3, we let $\alpha : \mathbb{N}^{\mathbb{N}} \to \mathcal{M}^{\mathcal{M}}$ be the function defined by $(\alpha(g))(x) = g(\lambda(x))$, and let L be the set of $g \in \mathbb{N}^{\mathbb{N}}$ such that $(\mathcal{M}, \alpha(g)) \models LIM$.

Claim. *For each first order quantifier-free formula $\varphi(F, \vec{x})$ in the vocabulary $L(\mathcal{M}) \cup \{F\}$, the set*

$$\{g \in \mathbb{N}^{\mathbb{N}} : (\mathcal{M}, \alpha(g)) \models \exists \vec{z}\, \varphi(F, \vec{z})\}$$

is $\mathbf{\Sigma}_1$ in the Borel hierarchy.

Proof of Claim. We may assume without loss of generality that $\exists \vec{z}\, \varphi(F, \vec{z})$ has the form

$$\exists \vec{x}\, \exists \vec{y} \big[F(\vec{x}) = \vec{y} \wedge \psi(\vec{x}, \vec{y})\big]$$

where $\psi(\vec{x}, \vec{y})$ is a first order quantifier-free formula of $L(\mathcal{M})$. Since $(\alpha(g))(x) = (\alpha(g))(\lambda(x))$ for all g and x, $(\mathcal{M}, \alpha(g))$ satisfies $\exists \vec{z}\, \varphi(F, \vec{z})$ if and only if it satisfies

$$\bigvee_{\vec{\mathbf{p}}} \bigvee_{\vec{\mathbf{q}}} \big[F(\vec{\mathbf{p}}) = \vec{\mathbf{q}} \wedge \exists \vec{x} \big[\lambda(\vec{x}) = \vec{\mathbf{p}} \wedge \psi(\vec{x}, \vec{\mathbf{q}})\big]\big].$$

Let

$$U = \big\{(\vec{\mathbf{p}}, \vec{\mathbf{q}}) : \mathcal{M} \models \exists \vec{x} \big[\lambda(\vec{x}) = \vec{\mathbf{p}} \wedge \psi(\vec{x}, \vec{\mathbf{q}})\big]\big\}.$$

Then $(\mathcal{M}, \alpha(g))$ satisfies $\exists \vec{z}\, \varphi(F, \vec{z})$ if and only if $g(\vec{\mathbf{p}}) = \vec{\mathbf{q}}$ for some $(\vec{\mathbf{p}}, \vec{\mathbf{q}}) \in U$. For each pair $(\vec{\mathbf{p}}, \vec{\mathbf{q}})$, the set $\{g : g(\vec{\mathbf{p}}) = \vec{\mathbf{q}}\}$ is clopen in $\mathbb{N}^{\mathbb{N}}$, so the union of these sets over U is $\mathbf{\Sigma}_1$ in the Borel hierarchy, as required.

Now suppose to the contrary that there is a $\bigvee \bigwedge B \bigwedge \Pi_1$ sentence

$$\theta = \bigvee_m \bigwedge_n \varphi_{mn}(F)$$

which is equivalent to LIM in all models $(\mathcal{M}, f)$, where $\varphi_{mn}(F)$ is $B \bigwedge \Pi_1$ in the vocabulary $L(\mathcal{M}) \cup \{F\}$.

We observe that any finite conjunction or disjunction of $\bigwedge \Pi_1$ sentences is equivalent to a $\bigwedge \Pi_1$ sentence, and the negation of a $\bigwedge \Pi_1$ sentence is equivalent to a $\bigvee \Sigma_1$ sentence. It follows that θ is equivalent to a sentence

$$\bigvee_m \bigwedge_n \Big[\bigwedge_p \forall \vec{z} \varphi_{mnp}(F, \vec{z}) \vee \bigvee_q \exists \vec{z}\, \psi_{mnq}(F, \vec{z})\Big]$$

where each $\varphi_{mnp}(F, \vec{z})$ and $\psi_{mnq}(F, \vec{z})$ is a first order quantifier-free formula of $L(\mathcal{M}) \cup \{F\}$.

Using the claim, it follows that

$$L = \bigcup_m \bigcap_n \Big(\bigcap_p A_{mnp} \cup \bigcup_q B_{mnq}\Big)$$

where each A_{mnp} and B_{mnq} is clopen in $\mathbb{N}^{\mathbb{N}}$. By renumbering and rearranging, we can get

$$L = \bigcup_m \bigcap_n \bigcup_q (A_{mnq} \cup B_{mnq}).$$

Therefore L is $\mathbf{\Sigma}_3$ in $\mathbb{N}^{\mathbb{N}}$. This is a contradiction, and proves (ii). $\square$

9. O-minimal Structures

In this section we show that BDD has the highest possible quantifier level, and that LIM is not Boolean in Π_2, in the case that $\mathcal{M}$ is an o-minimal expansion of the ordered field of real numbers. We leave open the question whether LIM can be Σ_3. We will use two lemmas from [4] concerning fast and indiscernible functions.

An ordered structure is **o-minimal** if every set definable with parameters is a finite union of intervals and points. There is an extensive literature on the subject (e.g. see [2,3]). An example of an o-minimal structure is the ordered field of reals with analytic functions restricted to the unit cube and with the (unrestricted) exponential function. It is easily seen that if $\mathcal{M}$ is o-minimal, then every ordered reduct of $\mathcal{M}$ is o-minimal, and the expansion of $\mathcal{M}$ formed by adding a symbol for each definable relation is o-minimal.

Throughout this section we let $\mathcal{M}$ be an o-minimal expansion of the real ordered field $\mathcal{R} = (\mathbb{R}, \leqslant, +, \cdot)$. "Definable" will always mean first order definable in $\mathcal{M}$ with parameters from $\mathcal{M}$.

When working with o-minimal structures, one often restricts attention to definable functions. It is easily seen that LIM is Π_2 over an o-minimal structure $\mathcal{M}$ relative to the set of definable functions, since LIM fails for a definable function f if and only if $\exists x \exists y \forall z [x \leqslant z \Rightarrow f(z) \leqslant y]$. However, here we consider arbitrary functions on $\mathcal{M}$.

Definition 9.1. A strictly increasing sequence s of positive integers is **$\mathcal{M}$-fast** if for each definable function $f : \mathbb{R} \to \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $f(s(k)) < s(k+1)$ for all $k > N$. For convenience we also require that $s(0) = 0$.

Lemma 9.2 ([4, 3.3]). *If $L(\mathcal{M})$ is countable then there exists an $\mathcal{M}$-fast sequence.*

Given $\vec{u}, \vec{v} \in \mathbb{N}^n$ and $N \in \mathbb{N}$, we write $\vec{u} =_N \vec{v}$ if $\min(u_i, N) = \min(v_i, N)$ for $i = 1, \ldots, n$, and $\vec{u} \sim \vec{v}$ if $\vec{u}, \vec{v}$ are order isomorphic, that is, $u_i \leqslant u_j$ iff $v_i \leqslant v_j$ for $i, j = 1, \ldots, n$. We write $\vec{u} \sim_N \vec{v}$ if $\vec{u} =_N \vec{v}$ and $\vec{u} \sim \vec{v}$. Note that $\sim_N$ is an equivalence relation on $\mathbb{N}^n$ with finitely many classes.

Given an $\mathcal{M}$-fast sequence s and a tuple $\vec{u} \in \mathbb{N}^n$, we write $s(\vec{u})$ for $(s(u_1), \ldots, s(u_n))$.

Lemma 9.3 ([4, 3.6]). *Suppose s is $\mathcal{M}$-fast, $h : \mathbb{R}^n \to \mathbb{R}$ is definable, and $g : \mathbb{N}^n \to \mathbb{R}$ is the function given by $\vec{u} \mapsto h(s(\vec{u}))$. Then there exists $N \in \mathbb{N}$ such that*

$$g(\vec{u}) \leqslant g(\vec{v}) \Leftrightarrow g(\vec{w}) \leqslant g(\vec{z})$$

whenever $\vec{u} =_N \vec{v}$ and $(\vec{u}, \vec{v}) \sim_N (\vec{w}, \vec{z})$.

In [4], the function g is called **almost indiscernible**. We will call N an **indiscernibility bound** for h.

Corollary 9.4. *Suppose s is $\mathcal{M}$-fast, $A \subseteq \mathbb{R}^n$ is definable, and*

$$B = \{\vec{u} \in \mathbb{N}^n : A(s(\vec{u}))\}.$$

There exists $N \in \mathbb{N}$ such that $B(\vec{u}) \Leftrightarrow B(\vec{v})$ whenever $\vec{u}, \vec{v} \in \mathbb{N}^n$ and $\vec{u} \sim_N \vec{v}$.

Proof. The characteristic function h of A is definable. Let N be an indiscernibility bound for h, and let g be as in Lemma 9.3. Suppose $\vec{u}, \vec{v} \in \mathbb{N}^n$ and $\vec{u} \sim_N \vec{v}$. Let $k \in \mathbb{N}$ be greater than N and each u_i and v_i. By replacing the u_i by $k + u_i$ whenever $N \leqslant u_i$, we obtain a tuple $\vec{w}$ such that $\vec{w} \sim_N \vec{u}$ and $(\vec{u}, \vec{w}) \sim_N (\vec{v}, \vec{w})$. By Lemma 9.3, $g(\vec{u}) \leqslant g(\vec{w})$ iff $g(\vec{v}) \leqslant g(\vec{w})$, and $g(\vec{u}) \geqslant g(\vec{w})$ iff $g(\vec{v}) \geqslant g(\vec{w})$. It follows that $g(\vec{u}) = g(\vec{v})$, so $B(\vec{u}) \Leftrightarrow B(\vec{v})$. ☐

By an indiscernibility bound for a formula ψ of $L(\mathcal{M})$ with parameters from $\mathcal{M}$ we mean an indiscernibility bound for the characteristic function of the set defined by ψ. It is clear that if N is an indiscernibility bound for ψ, then any $M \geqslant N$ is also an indiscernibility bound for ψ.

Theorem 9.5. *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, \leqslant, +, \cdot)$. Then BDD is not Π_2 over $\mathcal{M}$.*

Proof. We may assume that $\mathcal{M}$ has a countable vocabulary. By Lemma 9.2 there is an $\mathcal{M}$-fast sequence s. Suppose to the contrary that there is a Π_2 sentence $\forall\vec{x}\varphi(F, \vec{x})$ of $L(\mathcal{M}) \cup \{F\}$ which is equivalent to *BDD* in all structures $(\mathcal{M}, f)$. We may assume that $\forall\vec{x}\varphi(F, \vec{x})$ has the form

$$\forall\vec{x}\,\exists\vec{y}\,\exists\vec{z}\big[F(\vec{x}, \vec{y}) = \vec{z} \wedge \theta(\vec{x}, \vec{y}, \vec{z})\big]$$

where θ is a quantifier-free formula in which F does not occur, $|\vec{z}| = |\vec{x}| + |\vec{y}| = j + k$, and

$$F(\vec{x}, \vec{y}) = \big(F(x_1), \dots, F(x_j), F(y_1), \dots, F(y_k)\big).$$

(This can be proved by induction on the number of occurrences of F in φ). Let $f : \mathbb{R} \to \mathbb{R}$ be the function such that $f(x) = x$ if $x = s(4i)$ for some $i \in \mathbb{N}$, and $f(x) = 0$ otherwise. For each $m \in \mathbb{N}$ let $f_m : \mathbb{R} \to \mathbb{R}$ be the function such that $f_m(x) = f(x)$ for $x \leqslant s(m)$ and $f_m(x) = 0$ for $x > s(m)$. Then each f_m is bounded but f is unbounded. Therefore $\forall\vec{x}\,\varphi(f_m, \vec{x})$ holds for each $m \in \mathbb{N}$, but $\forall\vec{x}\varphi(f, \vec{x})$ fails. (Recall that $\forall\vec{x}\,\varphi(f, \vec{x})$ means that $(\mathcal{M}, f)$ satisfies $\forall\vec{x}\,\varphi(F, \vec{x})$.) Hence there exists a tuple $\vec{a}$ such that $\neg\varphi(f, \vec{a})$.

The notation $\vec{w}_1 < \vec{w} < \vec{w}_2$ means that each coordinate of $\vec{w}$ is strictly between each coordinate of $\vec{w}_1$ and $\vec{w}_2$, that is, $\vec{w}$ is in the open box with vertices $\vec{w}_1$ and $\vec{w}_2$. We will deal with variables which are outside the range of s by putting them into an open box with vertices in the range of s.

By Corollary 9.4, there is an $N \in \mathbb{N}$ which is an indiscernibility bound for each formula of the form

$$\exists\vec{w}\big[\vec{w}_1 < \vec{w} < \vec{w}_2 \wedge \theta(\vec{a}, \vec{y}, \vec{z})\big]$$

where $\vec{w} \subseteq \vec{y}$. We also take N so that $s(N) > \max(\vec{a})$.

By hypothesis, we have

$$\exists\vec{y}\,\exists\vec{z}\big[f_N(\vec{a}, \vec{y}) = \vec{z} \wedge \theta(\vec{a}, \vec{y}, \vec{z})\big].$$

Choose $\vec{b}$ in $\mathcal{M}$ such that

$$\exists \vec{z}\big[f_N(\vec{a},\vec{b}) = \vec{z} \wedge \theta(\vec{a},\vec{b},\vec{z})\big].$$

Then $\theta(\vec{a},\vec{b},f_N(\vec{a},\vec{b}))$.

We wish to find $\vec{e}$ in $\mathcal{M}$ such that $\theta(\vec{a},\vec{e},f(\vec{a},\vec{e}))$. This will show that $\varphi(f,\vec{a})$, contrary to hypothesis, and complete the proof that BDD is not Π_2 over $\mathcal{M}$. We cannot simply take $\vec{e}=\vec{b}$, because b could have a coordinate b_i such that $f(b_i) \neq f_N(b_i)$. This happens when $b_i = s(4m)$ where $4m > N$, so that $f(b_i) = b_i$ but $f_N(b_i) = 0$.

Claim. *For each $\vec{u} \in \mathbb{N}^n$ there exists $\vec{v} \in \mathbb{N}^n$ such that $\vec{v} \sim_N \vec{u}$, $f(s(\vec{v})) = f_N(s(\vec{v}))$, and if $u_j = u_i + 1$ then $v_j = v_i + 1$.*

Proof of Claim. Take $\vec{v}$ such that $\vec{v} \sim_N \vec{u}$, and v_i is not a multiple of 4 when $v_i > N$, and $v_j = v_i + 1$ when $u_j = u_i + 1$. Then $v_i = u_i$ below N, and $f(s(v_i)) = f(s(u_i)) = f_N(s(v_i)) = f_N(s(u_i)) = 0$ when $u_i > N$. It follows that $f(s(\vec{v})) = f_N(s(\vec{v}))$.

We may rearrange $\vec{b}$ into a k-tuple $(\vec{c},\vec{d})$ such that the terms of $\vec{c}$ belong to the range of s and the terms of $\vec{d}$ do not. Then $f_N(\vec{d}) = \vec{0}$ is a tuple of 0's. Let $(\vec{t},\vec{w})$ be the corresponding rearrangement of $\vec{y}$. Then $|\vec{c}| = |\vec{t}| = n$. Let $\vec{d}_1,\vec{d}_2$ be the vertices of the smallest open box containing $\vec{d}$ with coordinates in the range of s. That is, $\vec{d}_1 < \vec{d} < \vec{d}_2$ and the coordinates of $\vec{d}_1,\vec{d}_2$ are consecutive in the range of s. Then

$$\exists \vec{w}\big[\vec{d}_1 < \vec{w} < \vec{d}_2 \wedge \theta\big(\vec{a},\vec{c},\vec{w},f_N(\vec{a},\vec{c}),\vec{0}\big)\big]$$

holds in $\mathcal{M}$. We have $(\vec{c},\vec{d}_1,\vec{d}_2) = s(\vec{u})$ for some $\vec{u} \in \mathbb{N}^n$. Take $\vec{v}$ as in the claim and let $(\vec{e}_0,\vec{d}_3,\vec{d}_4) = s(\vec{v})$. Then by indiscernibility,

$$\exists \vec{w}\big[\vec{d}_3 < \vec{w} < \vec{d}_4 \wedge \theta\big(\vec{a},\vec{e}_0,\vec{w},f_N(\vec{a},\vec{e}_0),\vec{0}\big)\big].$$

Note that $s(N) > \max(\vec{a})$, so $f(\vec{a}) = f_N(\vec{a})$. Now let $\vec{e} = (\vec{e}_0,\vec{e}_1)$ where $\vec{e}_1$ is a witness for $\vec{w}$ in the above formula. Since the coordinates of $\vec{d}_3$ and $\vec{d}_4$ are consecutive in the range of s, the coordinates of $\vec{e}_1$ must be outside the range of s, where f and f_N are 0. It follows that $f(\vec{a},\vec{e}) = f_N(\vec{a},\vec{e})$. Therefore $\theta(\vec{a},\vec{e},f(\vec{a},\vec{e}))$, and hence BDD is not Π_2 over $\mathcal{M}$. □

Corollary 9.6. *Suppose that $\mathcal{M}$ is an o-minimal expansion of* $(\mathbb{R},\leqslant,+,\cdot)$. *Then*

(i) *BDD is not $\bigwedge\Pi_2$ over $\mathcal{M}$.*
(ii) *LIM is not $\bigwedge\Pi_2$ over $\mathcal{M}$.*

Proof. This follows from the proof of Theorem 9.5. For (i), we suppose that there is a $\bigwedge\Pi_2$ sentence $\bigwedge_i \forall\vec{x}\varphi_i(F,\vec{x})$ which is equivalent to BDD in all structures $(\mathcal{M},f)$, choose $i \in \mathbb{N}$ such that $\forall\vec{x}\,\varphi_i(f,\vec{x})$ fails, and get a contradiction as in the proof of Theorem 9.5.

For part (ii), we first observe that all the functions $f:\mathcal{M}\to\mathcal{M}$ used in the proof of (i) satisfy the Π_1 sentence

$$\forall x\big[F(x) = x \vee F(x) = 0\big].$$

Call this sentence $\beta(F)$. It follows that $\beta(F) \wedge BDD(F)$ is not $\bigwedge\Pi_2$ over $\mathcal{M}$. We also note that

$$\beta(F) \Leftrightarrow \beta\big(x - F(x)\big)$$

and

$$\beta(f) \Rightarrow \big[BDD(F) \Leftrightarrow LIM\big(x - F(x)\big)\big].$$

Now suppose that $LIM(F)$ is $\bigwedge \Pi_2$ over $\mathcal{M}$. Then $\beta(F) \Rightarrow LIM(x - F(x))$ is also $\bigwedge \Pi_2$ over $\mathcal{M}$. Hence $\beta(F) \wedge BDD(F)$ is $\bigwedge \Pi_2$ over $\mathcal{M}$, a contradiction. $\square$

Theorem 9.7. *Suppose that $\mathcal{M}$ is an o-minimal expansion of* $(\mathbb{R}, \leqslant, +, \cdot)$. *Then LIM is not $\bigvee B_2$ over $\mathcal{M}$.*

Proof. We may assume that $\mathcal{M}$ has a countable vocabulary. By Lemma 9.2 there is an $\mathcal{M}$-fast sequence s. Suppose to the contrary that there is a $\bigvee B_2$ sentence ψ which is equivalent to LIM in all structures $(\mathcal{M}, f)$. Then $\neg\psi$ is equivalent to a $\bigwedge B_2$ sentence. Since finite conjunctions and disjunctions of Π_2 sentences are equivalent to Π_2 sentences, and similarly for Σ_2, $\neg\psi$ is equivalent to a sentence

$$\bigwedge_m (\alpha_m \vee \beta_m)$$

where each α_m is Σ_2 and each β_m is Π_2. We may assume that α_m and β_m have the form

$$\alpha_m = \exists\vec{x}\,\forall\vec{y}\,\forall\vec{z}\big[F(\vec{x},\vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{x},\vec{y},\vec{z})\big],$$

$$\beta_m = \forall\vec{x}\,\exists\vec{y}\,\exists\vec{z}\big[F(\vec{x},\vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{x},\vec{y},\vec{z})\big]$$

where $\overline{\alpha}_m$ and $\overline{\beta}_m$ are quantifier-free formulas of $L(\mathcal{M})$.

We now define a family of functions from $\mathbb{N}$ into $\mathbb{N}$ which we will later use to build functions from $\mathbb{R}$ into $\mathbb{R}$. We begin with a sequence $h : \mathbb{N} \to \mathbb{N}$ with the following properties:

(a) For each $m \in \mathbb{N}$, $h(m) \leqslant m$ and $h(m)$ is either 0 or a power of 2.
(b) Each power of 2 occurs infinitely often in the sequence h,
(c) If $h(m) > 0$, them m is halfway between two powers of 2.

Note that $h(i)$ is "usually" 0, and that whenever $i < j$ and $h(i) > 0$, $h(j) > 0$, we have $h(h(i)) = 0$ and $2i \leqslant j$. For each finite or infinite sequence of natural numbers σ, let h_σ be the function obtained from h by putting $h_\sigma(i) = 0$ if $h(i) = 2^n$ and $i > \sigma(n)$, and putting $h_\sigma(i) = h(i)$ otherwise. Thus when n is in the domain of σ, $h_\sigma^{-1}\{2^n\}$ is the finite set $h^{-1}\{2^n\} \cap \{0, \ldots, \sigma(n)\}$. When σ is finite and n is outside its domain, $h_\sigma^{-1}\{2^n\}$ is the infinite set $h^{-1}\{2^n\}$.

For each σ, define the function $f_\sigma : \mathbb{R} \to \mathbb{R}$ by putting $f_\sigma(s(i)) = s(h_\sigma(i))$ if $h_\sigma(i) > 0$, and $f_\sigma(x) = x$ for all other x. Note that for each infinite sequence σ we have $\lim_{x\to\infty} f_\sigma(x) = \infty$, so LIM holds in $(\mathcal{M}, f_\sigma)$. But for each finite sequence σ we have $\liminf_{x\to\infty} f_\sigma(x) < \infty$, so LIM fails in $(\mathcal{M}, f_\sigma)$.

We will now build an infinite sequence $\sigma = (\sigma(0), \sigma(1), \ldots)$ such that $(\mathcal{M}, f_\sigma)$ satisfies $\neg\psi$. This will give us the desired contradiction, since $(\mathcal{M}, f_\sigma)$ satisfies LIM. We will simultaneously build a sequence of tuples $\vec{a}_m, m \in \mathbb{N}$, and a strictly increasing "growth sequence" $g(0) < g(1) < \cdots$. We will form an increasing chain

$$\sigma[0] \subset \sigma[1] \subset \cdots$$

of finite sequences, and take σ to be their union. This chain will have the property that each term of $\sigma[m] \setminus \sigma[m-1]$ will be $\geqslant g(m-1)$. Whenever possible, $\sigma[m]$ will be chosen so that α_m holds in $(\mathcal{M}, f_{\sigma[m]})$, and $\vec{a}_m$ will be a witness for the initial existential quantifiers of α_m.

For convenience we let $\sigma[-1]$ denote the empty sequence and put $g(-1) = 1$. Suppose $m \in \mathbb{N}$ and we already have $\sigma[i]$, $\vec{a}_i$, and $g(i)$ for each $i < m$. We have two cases.

Case 1. There is a finite sequence $\eta \supset \sigma[m-1]$ such that α_m holds in $(\mathcal{M}, f_\eta)$, and each term of $\eta \setminus \sigma[m-1]$ is $\geqslant g(m-1)$. In this case we take $\sigma[m]$ to be such an η, and take $\vec{a}_m$ to be a tuple in $\mathbb{R}$ which witnesses the initial existential quantifiers of α_m in $(\mathcal{M}, f_{\sigma[m]})$, that is,

$$\forall \vec{y} \forall \vec{z} \big[f_{\sigma[m]}(\vec{a}_m, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}, \vec{z}) \big].$$

Case 2. Otherwise. In this case we take $\sigma[m]$ to be an arbitrary finite sequence such that $\sigma[m] \supset \sigma[m-1]$ and each term of $\sigma[m] \setminus \sigma[m-1]$ is $\geqslant g(m-1)$, and let $\vec{a}_m$ be an arbitrary tuple in $\mathbb{R}$.

We now define $g(m)$. By Corollary 9.4, there is an $N \in \mathbb{N}$ which is an indiscernibility bound for each formula of the form

$$\forall \vec{w} \big[\vec{w}_1 < \vec{w} < \vec{w}_2 \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}_1, \vec{w}, \vec{z}_1, \vec{w}) \big]$$

where $\vec{w} \subseteq \vec{y}$, $\vec{y}_1$ is the part of $\vec{y}$ outside $\vec{w}$, and $\vec{z}_1$ is the corresponding part of $\vec{z}$.

Let p be the number of variables in the sentences $\alpha_i, \beta_i, i \leqslant m$. Take $g(m)$ so that:

(d) $g(m) \geqslant N$, $g(m) \geqslant 2p$, $g(m) > g(m-1)$, $g(m) > \max(\sigma[m])$,
(e) $s(g(m)) > \max(\vec{a}_m)$,
(f) The condition stated before Claim 2.

Finally, we define σ to be the union $\sigma = \bigcup_m \sigma[m]$. To complete the proof we prove two claims, Claim 1 concerning α_m and Claim 2 concerning β_m. Condition (f) will not be used in Claim 1, but will be needed later for Claim 2.

Claim 1. *Suppose α_m holds in $(\mathcal{M}, f_{\sigma[m]})$ (Case 1 above). Then α_m holds in $(\mathcal{M}, f_\sigma)$.*

Proof of Claim 1. We show that

$$\forall \vec{y} \forall \vec{z} \big[f_\sigma(\vec{a}_m, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}, \vec{z}) \big].$$

Suppose not. Then there is a tuple $\vec{b}$ in $\mathcal{M}$ such that

$$\neg \forall \vec{z} \big[f_\sigma(\vec{a}_m, \vec{b}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_i, \vec{b}, \vec{z}) \big].$$

As in the proof of Theorem 9.5, we may rearrange $\vec{b}$ into a k-tuple $(\vec{c}, \vec{d})$ such that the terms of $\vec{c}$ belong to the range of s and the terms of $\vec{d}$ do not. Then $f_\sigma(\vec{c})$ is also in the range of s, and $f_\sigma(\vec{d}) = \vec{d}$. Let $(\vec{t}, \vec{w})$ be the corresponding rearrangement of $\vec{y}$. Let $\vec{d}_1, \vec{d}_2$ be the vertices of the smallest open box containing $\vec{d}$ with coordinates in the range of s. Then

$$\neg \forall \vec{w} \big[\vec{d}_1 < \vec{w} < \vec{d}_2 \Rightarrow \overline{\alpha}_m \big(\vec{a}_m, \vec{c}, \vec{w}, f_\sigma(\vec{a}_m, \vec{c}), \vec{w} \big) \big].$$

Let $\vec{u}$ be the sequence in $\mathbb{N}$ such that $(\vec{c}, \vec{d}_1, \vec{d}_2) = s(\vec{u})$. Since the terms of σ beyond $\sigma[m]$ are greater than $g(m)$, we have $h_\sigma(i) = h_{\sigma[m]}(i)$ for all $i \leqslant g(m)$ and $f_\sigma(x) = f_{\sigma[m]}(x)$ for all $x \leqslant s(g(m))$. For $i > g(m)$ we have either $h_\sigma(i) = h_{\sigma[m]}(i)$, or $h_\sigma(i) = 0$ and $h_{\sigma[m]}(i) > 0$. Recall that $g(m) \geqslant 2p \geqslant 2|\vec{u}|$, where p is the number of variables in

$\alpha_m \vee \beta_m$. It follows that the sequence $h_{\sigma[m]}$ has enough zeros to insure that there is a sequence $\vec{v}$ in $\mathbb{N}^p$ such that:

(g) For each i, $(\vec{u}, 2^i) \sim_{g(m)} (\vec{v}, 2^i)$,
(h) $v_i = u_i$ whenever $h_\sigma(i) > 0$,
(i) If $u_i = u_i + 1$ then $v_i = v_i + 1$,
(j) $h_{\sigma[m]}(\vec{v}) = h_\sigma(\vec{u})$.

Therefore $(\vec{v}, h_{\sigma[m]}(\vec{v})) \sim_{g(m)} (\vec{u}, h_\sigma(\vec{u}))$. Let $(\vec{e}_0, \vec{d}_3, \vec{d}_4) = s(\vec{v})$. Then by (d), (e), (g), and indiscernibility, we have

$$\neg\forall\vec{w}\big[\vec{d}_3 < \vec{w} < \vec{d}_4 \Rightarrow \overline{\alpha}_m\big(\vec{a}_m, \vec{e}_0, \vec{w}, f_{\sigma[m]}(\vec{a}_m, \vec{e}_0), \vec{w}\big)\big].$$

We may therefore extend $\vec{e}_0$ to a tuple $\vec{e} = (\vec{e}_0, \vec{e}_1)$ such that

$$\vec{d}_3 < \vec{e}_1 < \vec{d}_4 \wedge \neg\, \overline{\alpha}_m\big(\vec{a}_m, \vec{e}, f_{\sigma[m]}(\vec{a}_m, \vec{e}_0), \vec{e}_1\big).$$

The coordinates of $\vec{d}_1$ and $\vec{d}_2$ are consecutive in the range of s, so by (i) the coordinates of $\vec{d}_1$ and $\vec{d}_2$ are consecutive in the range of s. Therefore the coordinates of $\vec{e}_1$ are outside the range of s, so $f_{\sigma[m]}(\vec{e}_1) = \vec{e}_1$. Then

$$\neg\, \overline{\alpha}_m\big(\vec{a}_m, \vec{e}, f_{\sigma[m]}(\vec{a}_m, \vec{e})\big).$$

This contradicts the fact that

$$\forall\vec{y}\forall\vec{z}\big[f_{\sigma[m]}(\vec{a}_i, \vec{y}) = \vec{z} \Rightarrow \overline{\alpha}_m(\vec{a}_m, \vec{y}, \vec{z})\big].$$

and completes the proof of Claim 1.

We now state the postponed condition (f) for the growth sequence g. Recall that p is the number of variables in the sentences $\alpha_i, \beta_i, i \leqslant m$.

(f) For each $\vec{u} \in \mathbb{N}^p$ there exists $\vec{v} \in \mathbb{N}^p$ with the following properties:
 (f1) $\max(\vec{v}) < g(m)$,
 (f2) If $u_j = u_i + 1$ then $v_j = v_i + 1$,
 (f3) $(\vec{v}, h_{\sigma[m]}(\vec{v})) \sim_m (\vec{u}, h_{\sigma[m]}(\vec{u}))$.

There is a $g(m)$ with these properties because the equivalence relation $\sim_m$ has only finitely many classes.

Claim 2. *The sentence $\neg\psi$ holds in $(\mathcal{M}, f_\sigma)$.*

Proof of Claim 2. We must show that for each $m \in \mathbb{N}$, $\alpha_m \vee \beta_m$ holds in $(\mathcal{M}, f_\sigma)$. Since *LIM* fails in $(\mathcal{M}, f_{\sigma[m]})$, $\alpha_m \vee \beta_m$ holds in $(\mathcal{M}, f_{\sigma[m]})$. In Case 1 above, by definition α_m holds in $(\mathcal{M}, f_{\sigma[m]})$, and by Claim 1, α_m holds in $(\mathcal{M}, f_\sigma)$.

In Case 2, α_m fails in $(\mathcal{M}, f_{\sigma[r]})$ for each $r \geqslant m$, and therefore β_m holds in $(\mathcal{M}, f_{\sigma[r]})$ for each $r \geqslant m$. In this case we prove that β_m holds in $(\mathcal{M}, f_\sigma)$. We fix a tuple $\vec{a}$ in $\mathcal{M}$ and prove

$$\exists\vec{y}\,\exists\vec{z}\big[f_\sigma(\vec{a}, \vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{a}, \vec{y}, \vec{z})\big].$$

By Corollary 9.4 there is an $M \in \mathbb{N}$ which is an indiscernibility bound for each formula of the form

$$\exists\vec{w}\big[\vec{w}_1 < \vec{w} < \vec{w}_2 \wedge \overline{\beta}_m(\vec{a}, \vec{y}_1, \vec{w}, \vec{z}_1, \vec{w})\big]$$

where $\vec{w} \subseteq \vec{y}$, $\vec{y}_1$ is the part of $\vec{y}$ outside $\vec{w}$, and $\vec{z}_1$ is the corresponding part of $\vec{z}$.

Take r large enough so that $r \geqslant m$ and $r \geqslant M$, and $s(r) > \max(\vec{a})$. Since β_m holds in $(\mathcal{M}, f_{\sigma[r]})$, we have

$$\exists \vec{y}\, \exists \vec{z} \big[f_{\sigma[r]}(\vec{a}, \vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{a}, \vec{y}, \vec{z}) \big].$$

We may therefore choose $\vec{b}$ in $\mathcal{M}$ so that

$$\overline{\beta}_m\big(\vec{a}, \vec{b}, f_{\sigma[r]}(\vec{a}, \vec{b})\big).$$

As in the proof of Theorem 9.5, we may rearrange $\vec{b}$ into a tuple $(\vec{c}, \vec{d})$ such that the terms of $\vec{c}$ belong to the range of s and the terms of $\vec{d}$ do not. Then $f_{\sigma[r]}(\vec{d}) = \vec{d}$. Let $(\vec{t}, \vec{w})$ be the corresponding rearrangement of $\vec{y}$, and let $\vec{d}_1, \vec{d}_2$ be the vertices of the smallest open box containing $\vec{d}$ with coordinates in the range of s. Then

$$\exists \vec{w} \big[\vec{d}_1 < \vec{w} < \vec{d}_2 \wedge \overline{\beta}_m\big(\vec{a}, \vec{c}, \vec{w}, f_{\sigma[r]}(\vec{a}, \vec{c}), \vec{w}\big) \big].$$

Let $\vec{u}$ be the tuple in $\mathbb{N}$ such that $s(\vec{u}) = (\vec{c}, \vec{d}_1, \vec{d}_2)$. Then there is a tuple $\vec{v}$ in $\mathbb{N}$ such that conditions (f1)–(f3) hold with r in place of m. Here p is the number of variables in $\alpha_i, \beta_i, i \leqslant r$, so β_m has at most p variables, and $\vec{u}$ can be taken with length p.

Let $s(\vec{v}) = (\vec{e}_0, \vec{d}_3, \vec{d}_4)$. Since $r \geqslant M$, by (f3) and indiscernibility we have

$$\exists \vec{w} \big[\vec{d}_3 < \vec{w} < \vec{d}_4 \wedge \overline{\beta}_m\big(\vec{a}, \vec{e}_0, \vec{w}, f_{\sigma[r]}(\vec{a}, \vec{e}_0), \vec{w}\big) \big].$$

Take $\vec{e}_1$ in $\mathcal{M}$ such that

$$\vec{d}_3 < \vec{e}_1 < \vec{d}_4 \wedge \overline{\beta}_m\big(\vec{a}, \vec{e}_0, \vec{e}_1, f_{\sigma[r]}(\vec{a}, \vec{e}_0), \vec{e}_1\big).$$

By (f2), every coordinate of $\vec{e}_1$ is outside the range of s, so $f_{\sigma[r]}(\vec{e}_1) = \vec{e}_1$. Putting $\vec{e} = (\vec{e}_0, \vec{e}_1)$, we have

$$\overline{\beta}_m\big(\vec{a}, \vec{e}, f_{\sigma[r]}(\vec{a}, \vec{e})\big).$$

It remains to prove that $f_\sigma(\vec{a}, \vec{e}) = f_{\sigma[r]}(\vec{a}, \vec{e})$. Suppose $i \leqslant g(r)$ and $h_{\sigma[r]}(i) = 2^j$. If j is in the domain of $\sigma[r]$, then $h_\sigma(i) = 2^j$ because $\sigma(j) = \sigma[r](j)$. Otherwise $\sigma(j) \geqslant g(r)$, hence $i \leqslant g(r) < \sigma(j)$ and $h(i) = 2^j$, and again $h_\sigma(i) = 2^j$. If $h_{\sigma[r]}(k) = 0$ then $h_\sigma(k) = 0$. Therefore $h_\sigma(k) = h_{\sigma[r]}(k)$ for all $k \leqslant g(r)$.

By (f1), $\max(\vec{v}) < g(r)$. Therefore $h_\sigma(\vec{v}) = h_{\sigma[r]}(\vec{v})$, and hence $f_\sigma(\vec{v}) = f_{\sigma[r]}(\vec{v})$. Since we also have $\max(\vec{a}) < s(r)$, it follows that $f_\sigma(\vec{a}, \vec{e}) = f_{\sigma[r]}(\vec{a}, \vec{e})$. Therefore

$$\overline{\beta}_m\big(\vec{a}, \vec{e}, f_\sigma(\vec{a}, \vec{e})\big),$$

and we have the required formula

$$\exists \vec{y}\, \exists \vec{z} \big[f_\sigma(\vec{a}, \vec{y}) = \vec{z} \wedge \overline{\beta}_m(\vec{a}, \vec{y}, \vec{z}) \big].$$

This establishes Claim 2 and completes the proof. □

We conclude with a problem which we state as a conjecture.

Conjecture 9.8. *Suppose that $\mathcal{M}$ is an o-minimal expansion of* $(\mathbb{R}, \leqslant, +, \cdot)$. *Then LIM is not Σ_3 over $\mathcal{M}$.*

10. Conclusion

Given an ordered structure $\mathcal{M}$, the quantifier level of a sentence θ of $L(\mathcal{M}) \cup \{F\}$ over $\mathcal{M}$ is the lowest class in the hierarchies $\Delta_1 \subset \Pi_1 \subset B_1 \subset \Delta_2 \subset \Pi_2 \ldots$ and $\Delta_1 \subset \Sigma_1 \subset B_1 \subset \Delta_2 \subset \Sigma_2 \ldots$ which contains a sentence equivalent to θ in all structures $(\mathcal{M}, f)$. We investigate the quantifier levels of the Σ_2 sentence BDD, which says that f is bounded, and the Π_3 sentence LIM, which says that $\lim_{z\to\infty} f(z) = \infty$, over a given ordered structure $\mathcal{M}$. This work is motivated by Mostowski's result that BDD is not Π_2 and LIM is not Σ_3 relative to the primitive recursive functions over the standard model of arithmetic, and Abraham Robinson's result which characterizes BDD and LIM for standard functions by Π_1 sentences in a language with an added predicate for the set of infinite elements.

We show that BDD and LIM can never be B_1 over a structure $\mathcal{M}$, but if $\mathcal{M}$ is an expansion of the real ordered field with a symbol for $\mathbb{N}$ and each definable function, then BDD and LIM are at the lowest possible level Δ_2 over $\mathcal{M}$. We show that BDD is at its highest possible level, Σ_2 but not Π_2, and that LIM is at its highest possible level, Π_3 but not Σ_3, in the following cases: $\mathcal{M}$ is countable, $\mathcal{M}$ is the real ordering with an embedded structure on the natural numbers, and $\mathcal{M}$ is special of nice cardinality with an extra predicate for the infinite elements.

When $\mathcal{M}$ has universe $\mathbb{R}$, we obtain analogous results with the outer quantifiers replaced by countable disjunctions and conjunctions. In that case we show that BDD cannot be expressed by a countable conjunction of existential sentences, and LIM cannot be expressed by a countable disjunction of countable conjunctions of existential sentences.

The most interesting case is where $\mathcal{M}$ is an o-minimal expansion of the real ordered field. In that case we show that BDD is at the maximum level, and that LIM is not B_2. Moreover, BDD and LIM are not $\bigwedge \Pi_2$, and LIM is not $\bigvee B_2$. We leave open the question of whether LIM is at its maximum level, not Σ_3, in that case.

References

[1] C.C. Chang and H. Jerome Keisler, *Model Theory*, Third Edition, Elsevier, 1990.

[2] L. van den Dries, o-minimal structures, in: *Logic: From Foundations to Applications*, W. Hodges et al., eds, Oxford, 1996, pp. 137–185.

[3] L. van den Dries, *Tame Topology and o-minimal Structures*, Cambridge University Press, 1998.

[4] H. Friedman and C. Miller, Expansions of o-minimal structures by fast sequences, *Journal of Symbolic Logic* **70** (2005), 410–418.

[5] A. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[6] H. Jerome Keisler, *Model Theory for Infinitary Logic*, North-Holland, 1971.

[7] H. Jerome Keisler, *Elementary Calculus. An Infinitesimal Approach*, Second Edition, Prindle, Weber and Schmidt, 1986. Online Edition, http://www.math.wisc.edu/~keisler/, 2000.

[8] S.C. Kleene, Recursive predicates and quantifiers, *Transactions of the American Mathematical Society* **53** (1943), 41–73.

[9] A. Mostowski, On definable sets of positive integers, *Fundamenta Mathematicae* **34** (1947), 81–112.

[10] A. Mostowski, Examples of sets definable by means of two and three quantifiers, *Fundamenta Mathematicae* **42** (1955), 259–270.

[11] A. Robinson, *Non-standard Analysis*, North-Holland, 1966.

[12] K. Sullivan, *The Teaching of Elementary Calculus: An Approach Using Infinitesimals*, Ph.D. Thesis, University of Wisconsin, Madison, 1974.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

The Kleene–Mostowski Hierarchy and the Davis–Mostowski Hierarchy

Julia F. KNIGHT
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, U.S.A.
julia.f.knight.1@nd.edu

1. Introduction

The *Kleene–Mostowski hierarchy* was defined by Mostowski [21] and Kleene [18] independently. The sets and relations in the hierarchy are the ones obtained from computable relations by finitely many applications of the operations of projection and complement. The hierarchy classifies these sets and relations as Σ^0_n, Π^0_n, and Δ^0_n for various n, based on the number of applications of the two operations. Mostowski and Kleene each showed that the hierarchy is *proper*. For each n, there is a set S that is Σ^0_{n+1} and not Π^0_{n+1}, and since any Σ^0_n or Π^0_n set is Σ^0_{n+1}, S is neither Σ^0_n nor Π^0_n.

The Kleene–Mostowski hierarchy is often called the *arithmetical hierarchy*. This name reflects the fact that the sets and relations in the hierarchy are exactly those definable in the *standard model of arithmetic* $\mathcal{N} = (\omega, +, \cdot, S, 0)$. In fact, for all $n \geqslant 1$, a set or relation is Σ^0_n, or Π^0_n, if and only if it is definable in $\mathcal{N}$ by a finitary Σ_n, or Π_n, formula. The arithmetical hierarchy is useful in measuring the complexity of index sets such as $Tot = \{e : \varphi_e \text{ is total}\}$ and $Cof = \{e : W_e \text{ is co-finite}\}$.

The *Davis–Mostowski hierarchy* was defined by Mostowski [22] and Davis [7,8] independently. This hierarchy, often called the *hyperarithmetical hierarchy*, extends the arithmetical hierarchy through transfinite levels. We have Σ^0_α, Π^0_α, and Δ^0_α sets and relations, for all computable ordinals α. The definition involves Kleene's system of ordinal notation. In this system, each finite ordinal has a unique notation, while each infinite ordinal has infinitely many. Davis had asked whether the class of Σ^0_α sets and relations is independent of the notation for α, and Spector [25] showed that it is.

We consider *computable infinitary* formulas. Roughly speaking, these are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are computably enumerable. To make this precise, we assign indices to the formulas, based on ordinal notation. The formulas are classified as computable Σ_α and computable Π_α, for various computable ordinals α. The analogue of Spector's Theorem holds—what we can express by a computable Σ_α, or Π_α, formula is independent of the notation for α. In a structure $\mathcal{A}$, the relation defined by a computable Σ_α, or computable Π_α, formula is Σ^0_α, or Π^0_α, relative to $\mathcal{A}$.

There is a large body of work in computable structure theory, all based on the Kleene–Mostowski hierarchy and the Davis–Mostowski hierarchy. We give a small sample. The structures that we consider have universe a subset of ω, and the language of each structure is computable. We identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$, where this may be thought of as a subset of ω (via Gödel numbering). Thus, $\mathcal{A}$ is a computable structure if $D(\mathcal{A})$ is a computable set.

There is a collection of results that explain, in terms of definability, bounds on relative complexity which persist under isomorphism. We describe three of these results, one from [20] and the other two from [1]. The first result gives conditions on a set S and a structure $\mathcal{A}$ guaranteeing that for $\mathcal{B} \cong \mathcal{A}$, S is Σ^0_α relative to $\mathcal{B}$. The second result gives conditions on a computable structure $\mathcal{A}$ guaranteeing that for all $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$ such that F is Δ^0_α relative to $\mathcal{B}$. The third result gives conditions on a structure $\mathcal{A}$ and a relation R guaranteeing that for any isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is Σ^0_α relative to $\mathcal{B}$.

There is quite a lot of work on computable structures of familiar kinds. We describe some results on computable Abelian p-groups. N. Khisamiev [16,17], characterized in an algebraic way the reduced Abelian p-groups, of length less than ω^2, which have computable copies. For a structure $\mathcal{A}$, the *index set*, denoted by $I(\mathcal{A})$, is the set of indices for computable copies of $\mathcal{A}$. The complexity of $I(\mathcal{A})$ measures how difficult it is to distinguish computable copies of $\mathcal{A}$ from other computable structures. In [5], there are results calculating the complexity of the index sets for all Abelian p-groups of length $< \omega^2$. For a class K, closed under isomorphism, the *computable isomorphism problem*, denoted by $E(K)$, is the set of pairs (a, b) of indices for isomorphic computable elements of K. The complexity of $E(K)$ provides one measure of how difficult it is to classify the computable members of K, up to isomorphism. Calvert [3], calculated, for each computable limit ordinal α, the complexity of the isomorphism problem for computable reduced Abelian p-groups of length α.

In Section 2, we define the arithmetical hierarchy and prove some basic facts about it. In Section 3, we describe Kleene's system of ordinal notation, and we show that the ordinals with notations are the same as the computable ordinals. In Section 4, we define the hyperarithmetical hierarchy and prove some facts about it, including Spector's Theorem, saying that the complexity class Σ^0_α is independent of the notation for α. In Section 5, we describe computable infinitary formulas. We give the analogue of Spector's Theorem, saying that what is expressible by a computable Σ_α formula is independent of the notation for α. In Section 6, we give the first collection of applications to computable structure theory, explaining bounds on complexity relative to arbitrary (non-computable) copies of a given structure $\mathcal{A}$. In Section 7, we give the second collection of applications, on computable reduced Abelian p-groups.

1.1. Notation and Background

We assume that the reader has basic knowledge of computable functions and computably enumerable (c.e.) sets. Our notation comes from Rogers' classic text [23]. We use φ_e to denote the partial function computed by Turing machine number e in some standard list. We write $f(k) \downarrow$, or $f(k) \uparrow$, to indicate that k is, or is not, in the domain of f. We write $\varphi_{e,s}(k) \downarrow$, or $\varphi_{e,s}(k) \uparrow$, to indicate that the computation of φ_e on input k, halts, or fails to halt, within s steps. We write W_e for the c.e. set that is the domain of φ_e, and we write $W_{e,s}$ for the finite approximation $\{k : \varphi_{e,s}(k) \downarrow\}$.

We shall use the $s - m - n$ Theorem, saying that for any partial computable function $g(\overline{x}, \overline{y})$, there is a total computable function $f(\overline{x})$ such that

$$\varphi_{f(\overline{x})}(\overline{y}) = g(\overline{x}, \overline{y}).$$

We shall also use the Recursion Theorem. This, in its most basic form, says that for any total computable function $f(x)$, there is a number e such that

$$\varphi_e = \varphi_{f(e)}.$$

Sometimes the Recursion Theorem is illustrated by results that look like recreational mathematics. There are serious applications in defining partial computable functions by recursion on ordinals, or their notations.

We recall a familiar definition.

Definition 1. The *halting set* is $K = \{e : \varphi_e(e) \downarrow\}$.

Proposition 1.1. *K is c.e. and not computable.*

Proof. It is easy to see that K is c.e. To see that it is not computable, we recall that a set is computable if it is both c.e. and co-c.e. If K is computable, then there is some e such that $W_e = \overline{K}$ (the complement of K). For this e, we should have $e \in W_e$ iff $e \notin K$, but for any e, we have $e \in W_e$ iff $e \in K$. This is a contradiction. □

The set K lies at the top of the class of c.e. sets, under "m-reducibility".

Definition 2. For $A, B \subseteq \omega$, A is *m-reducible* to B, written $A \leqslant_m B$, if there is a total computable function f such that for all n, $n \in A$ iff $f(n) \in B$.

Proposition 1.2. *For any c.e. set A, $A \leqslant_m K$.*

Proof. Let g be the partial computable function such that

$$g(x, y) = \begin{cases} y & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

By the $s - m - n$ Theorem, there is a total computable function k such that $\varphi_{k(x)}(y) = g(x, y)$. Note that the domain of $\varphi_{k(x)}$ is either ω or $\emptyset$, depending on whether $x \in A$. Then $x \in A$ iff $k(x) \in K$. □

We suppose that the reader has seen oracle machines. We write φ_e^X for the partial function computed by Turing machine e equipped with an oracle for the set X, and we write W_e^X for the domain of this function. For any set X, the *jump* of X is $X' = \{e : \varphi_e^X(e) \downarrow\}$. The proof that the halting set K is c.e. and not computable (Proposition 1.1) also shows the following.

Proposition 1.3. *For any set X, X' is c.e. relative to X and not computable relative to X.*

The proof that for all c.e. sets A, $A \leqslant_m K$ (Proposition 1.2) also shows the following.

Proposition 1.4. *For any sets X and A, if A is c.e. relative to X, then $A \leqslant_m X'$.*

2. The Arithmetical Hierarchy

2.1. Definition

We define the classes of Σ^0_n and Π^0_n relations, proceeding by induction.

Definition 3.

1. A relation is Σ^0_0 and Π^0_0 if it is computable.
2. A relation is Σ^0_{n+1} if it can be expressed in the form $(\exists y)R(\overline{x}, y)$, where $R(\overline{x}, y)$ is Π^0_n.
3. A relation $S(\overline{x})$ is Π^0_{n+1} if the complementary relation, $\neg\, S(\overline{x})$, is Σ^0_{n+1}.
4. For $n \geqslant 1$, a relation is Δ^0_n if it is both Σ^0_n and Π^0_n.
5. A relation is *arithmetical* if it is Δ^0_n, for some n.

From the definition, it is easy to see the following.

Proposition 2.1. *The relation $S(\overline{x})$ is Σ^0_n iff it has a definition of the form $(\exists y_1)(\forall y_2)\cdots R(\overline{x}, \overline{y})$, where $R(\overline{x}, \overline{y})$ is computable, and there are n alternating quantifiers, starting with $(\exists)$. Similarly, the relation $S(\overline{x})$ is Π^0_n iff it has a definition of the form $(\forall y_1)(\exists y_2)\ldots R(\overline{x}, \overline{y})$, where $R(\overline{x}, \overline{y})$ is computable, and there are n alternating quantifiers, starting with $(\forall)$.*

If we have two existential (or universal) quantifiers together, we can replace them by a single one (using the familiar computable pairing function). Thus, S is Σ^0_n, or Π^0_n, if it has a definition as above, but with n alternating *blocks* of quantifiers.

The following result characterizes the Σ^0_1, Π^0_1, and Δ^0_1 relations in more familiar terms.

Proposition 2.2.

1. *A relation is Σ^0_1 iff it is c.e.*
2. *A relation is Π^0_1 iff it is co-c.e. (i.e., the complementary relation is c.e.).*
3. *A relation is Δ^0_1 iff it is computable.*

Proof. For Statement 1, we note that a c.e. relation may be expressed in the form $(\exists s)\varphi_{e,s}(\overline{x}) \downarrow$, where $\varphi_{e,s}(\overline{x}) \downarrow$ is a computable relation on $\overline{x}$ and s. $\square$

2.2. Simple Facts

If is easy to see that if a relation is either Σ^0_n or Π^0_n, then it is both Σ^0_{n+1} and Π^0_{n+1}. We get an expression of the desired form by adding redundant quantifiers. If we apply Boolean operations and quantifiers to arithmetical relations, the resulting relation is always arithmetical. We use familiar tricks, moving the quantifiers to the outside, changing variables as necessary, and combining like quantifiers, to arrive at the following simple facts.

Proposition 2.3.

1. *If R is Σ^0_n or Π^0_n, then it is Δ^0_m for all $m > n$.*
2. *The family of Σ^0_n sets, or relations with a fixed number of places, is closed under finite union and intersection.*

3. *The family of Σ^0_n relations is closed under projection; i.e., if $R(\overline{x}, y)$ is Σ^0_n, then $(\exists y)R(\overline{x})$ is Σ^0_n.*
4. *The family of Σ^0_n relations is closed under bounded quantification; i.e., if $R(\overline{x}, y)$ is Σ^0_n, then $(\forall y < z)R(\overline{x}, z)$ is Σ^0_n.*

Similarly, the family of Π^0_n sets, or relations with a fixed number of places, is closed under finite union and intersection, and if $R(\overline{x}, y)$ is Π^0_n, then $(\forall y)R(\overline{x}, y)$ and $(\exists y < z)R(\overline{x}, y)$ are Π^0_n.

Remark. An arithmetical relation on several variables can be replaced by a set at the same level in the arithmetical hierarchy. Let $u = \langle y_1, \ldots, y_m\rangle$ be the standard code for the finite tuple $(y_1, \ldots, y_m)$, and let $(u)_i$ be the i^{th} component of the tuple coded by u. Suppose

$$S(y_1, \ldots, y_m) \text{ iff } (Q_1 z_1) \cdots (Q_n z_n) R(y_1, \ldots, y_m; z_1, \ldots, z_n),$$

and let

$$A = \big\{\langle y_1, \ldots, y_m\rangle : S(y_1, \ldots, y_m)\big\}.$$

We have $u \in A$ iff $(Q_1 z_1) \cdots (Q_n z_n) R((u)_1, \ldots, (u)_m; z_1, \ldots, z_n)$. Thus, if S is Σ^0_n, or Π^0_n, or Δ^0_n, the same is true of A.

2.3. Familiar Examples

We locate some familiar sets in the arithmetical hierarchy. In each case, we show that the set lies "on top" of the appropriate class under m-reducibility.

Definition 4. Let Γ be a complexity class, such as Σ^0_1 or Π^0_3. We say that S is m-complete Γ if S is in Γ and for all A in Γ, $A \leqslant_m S$.

Example 1. The set K is m-complete Σ^0_1.

Proof. This is immediate from Propositions 1.1 and 1.2. $\square$

Example 2. The set $E = \{e : W_e = \emptyset\}$ is m-complete Π^0_1.

Proof. To show that E is Π^0_1, we note that $e \in E$ iff $(\forall x)(\forall s)[\varphi_{e,s}(x)\ \uparrow]$, where the expression in brackets is computable. To show hardness, suppose S is Π^0_1. Let

$$g(n, x) = \begin{cases} x & \text{if } n \text{ is in the complement of } S \\ \uparrow & \text{otherwise} \end{cases}$$

The $s - m - n$ Theorem yields a total computable function f such that $\varphi_{f(n)}(x) = g(n, x)$, and $n \in S$ iff $f(n) \in E$. $\square$

Example 3. The set $Tot = \{e : \varphi_e \text{ is total}\}$ is m-complete Π^0_2.

Proof. To show that Tot is Π^0_2, we note that $e \in Tot$ iff $(\forall x)(\exists s)[\varphi_{e,s}(x)\ \downarrow]$, and the condition in brackets is computable. To show hardness, suppose S is Π^0_2, say $n \in S$ iff

$(\forall x)(\exists y)R(n,x,y)$, where R is computable. We define a partial computable function

$$g(n,x) = \begin{cases} x & \text{if } (\exists y)R(n,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

By the $s-m-n$-Theorem, there is a total computable function f such that $\varphi_{f(n)}(x) = g(n,x)$. Then $n \in S$ iff $f(n) \in Tot$. □

In the same way, we could show that the set $Inf = \{e : W_e \text{ is infinite}\}$ is m-complete Π^0_2.

Example 4. The set $Cof = \{e : W_e \text{ is co-finite}\}$ is m-complete Σ^0_3.

Proof. To show that Cof is Σ^0_3, we note that

$$e \in Cof \text{ iff } (\exists n)(\forall x > n)(\exists s)\, x \in W_{e,s}.$$

To show hardness, let S be Σ^0_3, say $n \in S$ iff $(\exists x)(\forall y)(\exists z)R(n,x,y,z)$, where R is computable. Without loss, we may suppose that $R(n,x,0,0)$. We define a function k, which enumerates the pairs $\langle x,y\rangle$ such that $(\forall y' \leqslant y)(\exists z)R(n,x,y,z)$. We start by letting $k(0) = \langle 0,0\rangle$. Having defined $k|(s+1) = \{(n,k(n)) : n \leqslant s\}$, we define $k(s+1)$ as follows. If there is a new pair $\langle x,y\rangle$, not in $ran(k|(s+1))$, such that $x,y \leqslant s$ and $(\forall y' \leqslant y)(\exists z \leqslant s)R(n,x,y,z)$, let $k(s+1)$ be the first such pair. If there is no such pair, let $k(s+1) = k(s)$.

Let g be the partial computable function such that $g(n,s) = s$ if $k(s)$ has the form $\langle x,y\rangle$ and there exists $t > s$ such that $k(t) = \langle x',y\rangle$, where $x' \leqslant x$, $y' \geqslant y$, and $\langle x,y\rangle \neq \langle x',y'\rangle$. Let f be a total computable function such that $\varphi_{f(n)}(s) = g(n,s)$. Then $W_{f(n)}$ is the set of stages s such that $k(s)$ has the form $\langle x,y\rangle$, and at some stage $t > s$, we have a new $k(t) = \langle x',y'\rangle$, where $x' \leqslant x$ and $y' \geqslant y$. The complement of $W_{f(n)}$ is the set of stages s after which we never see the current first component x, or any $x' \leqslant x$, with a new second component $y' \geqslant y$. If $n \in S$, then there is some x^* that is the first component in infinitely many pairs. In this case, the complement of $W_{f(n)}$ must be finite, consisting of some of the stages where we see the greatest y for some $x < x^*$. If $n \notin S$, then there are infinitely many stages s at which we see some x with its last y, and for each $x' \leqslant x$, the last y' is no larger than this y. Since these stages are all in the complement of $W_{f(n)}$, then this must be an infinite set; i.e., $f(n) \notin Cof$. □

2.4. Non-Collapse

Our next goal is to show that the arithmetical hierarchy does not collapse. There is more than one way to do this. The proof that we give involves universal *enumeration relations*.

Theorem 2.4. *For each $n \geqslant 1$ and each tuple of variables $\overline{x}$,*

1. *there is a Σ^0_n relation $E_{\Sigma^0_n}(e,\overline{x})$ which, for different values of e, yields all of the m-ary Σ^0_n relations in the variables $\overline{x}$,*
2. *there is a Π^0_n relation $E_{\Pi^0_n}(e,\overline{x})$ which, for different values of e, yields all Π^0_n relations in the variables $\overline{x}$.*

Proof. We give the proof for Statement 1. The Σ^0_n relations, in the variables $\overline{x}$, have the form $(Q_1 y_1)\cdots(Q_n y_n)R(\overline{x};y_1,\ldots,y_n)$, where $R(\overline{x};y_1,\ldots,y_n)$ is computable and $Q_1,\ldots,Q_n$ is an appropriate alternating sequence of quantifiers.

Case 1. Suppose Q_n is $\exists$.

Then $(Q_n y_n)R(\overline{x}; y_1, \ldots, y_n)$ is $(\exists y_n)R(\overline{x}; y_1, \ldots, y_n)$. This is a c.e. relation in $\overline{x}$, $y_1, \ldots, y_{n-1}$, so it is the domain of the partial computable function $\varphi_e(\overline{x}; y_1, \ldots, y_{n-1})$, for some e. The relation $T(e, \overline{x}, y_1, \ldots, y_{n-1}, s)$, which holds iff $\varphi_{e,s}(\overline{x}, y_1, \ldots, y_{n-1}) \downarrow$, is computable. We have $(\exists y_n)R(\overline{x}, y_1, \ldots, y_n)$ iff $(\exists s)T(e, \overline{x}, y_1, \ldots, y_{n-1}, s)$. We let $E_{\Sigma^0_n}(e, \overline{x})$ be

$$(Q_1 y_1) \cdots (Q_{n-1} y_{n-1})(\exists s)T(e, \overline{x}; y_1, \ldots, y_{n-1}, s).$$

Case 2. Suppose Q_n is $\forall$.

In this case, $(Q_n y_n)R(\overline{x}; y_1, \ldots, y_n)$ is $(\forall y_n)R(\overline{x}; y_1, \ldots, y_n)$, which is logically equivalent to $\neg(\exists y_n)\neg R(\overline{x}; y_1, \ldots, y_n)$. This is a co-c.e. relation, so it can be expressed in the form $(\forall s)\neg T(e, \overline{x}, y_1, \ldots, y_{n-1}, s)$, for some e. We let $E_{\Pi^0_n}(e, \overline{x})$ be

$$(Q_1 y_1) \cdots (Q_{n-1} y_{n-1})(\forall s)\neg T(e, \overline{x}; y_1, \ldots, y_{n-1}, s). \qquad \square$$

The enumeration relations from Theorem 2.4 provide examples showing the following.

Corollary 2.5. *For each $n \geqslant 1$, there is a set S which is Σ^0_n and not Π^0_n. (Then $\overline{S}$ is Π^0_n and not Σ^0_n.)*

Proof. Consider the Σ^0_n enumeration relation $E_{\Sigma^0_n}(e; x)$ for Σ^0_n sets. Let $R(x)$ be the relation $E_{\Sigma^0_n}(x, x)$. Clearly, $R(x)$ is Σ^0_n. Suppose it is also Π^0_n. Then $\overline{R}$ is Σ^0_n. Let e be an index. By the definition of R, we have $e \in R$ iff $E_{\Sigma^0_n}(e, e)$. Since e is a Σ^0_n index for $\overline{R}$, we have $e \in \overline{R}$ iff $E_{\Sigma^0_n}(e, e)$. This is a contradiction. $\square$

Corollary 2.5 implies that the arithmetical hierarchy does not collapse.

Corollary 2.6. *For each n, there is a set which is Δ^0_{n+1} and not Δ^0_n.*

Proof. If S is Σ^0_n and not Π^0_n, then it is Δ^0_{n+1} and not Δ^0_n. $\square$

The method that we have used to show non-collapse of the arithmetical hierarchy, which involved universal enumeration relations, can be applied in other settings. In particular, it can be used to show non-collapse of the *analytical hierarchy*, defined by Kleene [19], or the *difference hierarchy*, defined by Ershov [10–12].

2.5. An Alternative Definition

It is possible to give an alternative definition of the arithmetical hierarchy in terms of iterated jumps of $\emptyset$. We consider the sequence of sets $\emptyset^{(n)}$, where $\emptyset^{(0)} = \emptyset$, and $\emptyset^{(n+1)} = (\emptyset^{(n)})'$. We will see that a relation is Σ^0_n (or Π^0_n, or Δ^0_n) just in case it is c.e. (or co-c.e., or computable) relative to $\emptyset^{(n-1)}$. This alternative definition is the one that we use to extend the arithmetical hierarchy to transfinite levels. We need some further facts.

Theorem 2.7. *For a relation S,*

1. *S is Σ^0_1 if and only if it is c.e. relative to a computable set (or finite tuple of computable sets).*
2. *for $n > 1$, the following are equivalent:*

(a) *S is Σ^0_n,*
(b) *S is c.e. relative to a Π^0_{n-1} set (or finite tuple of sets, each of which is Π^0_{n-1} or Σ^0_{n-1}),*
(c) *S is c.e. relative to a Δ^0_n set.*

Proof. Statement 1 clear. We prove Statement 2. To show that (a) implies (b), suppose S is Σ^0_n. Then it is the projection of some Π^0_{n-1} relation R, so it is c.e. relative to R. To show that (b) implies (c), suppose S is c.e. relative to a set or tuple of sets, each of which is Π^0_{n-1} or Σ^0_{n-1}. Each Σ^0_{n-1} or Π^0_{n-1} set is Δ^0_n, and a tuple of Δ^0_n sets can be replaced by a single set which is Δ^0_n. Finally, to show that (c) implies (a), suppose that S is c.e. relative to A, where A is Δ^0_n, say $S = W^A_e$. Let D_k be the k^{th} finite set in a standard list (given k, we can effectively determine the size of D_k, in addition to deciding membership). Let $R(x, s, u, v)$ be the computable relation that holds iff oracle machine e halts on input x, within s steps, using oracle information that is positive for elements of D_u and negative for elements of D_v. Then $x \in S$ iff $(\exists v)(\exists u)(\exists s)[R(\overline{x}; u, v, s)\ \&\ D_u \subseteq A\ \&\ D_v \subseteq \overline{A}]$. Since A is Δ^0_n, the relations $D_u \subseteq A$, $D_v \subseteq \overline{A}$ are both Σ^0_n, since A and $\overline{A}$ are both Σ^0_n. Then the relations $(\forall z \in D_u)\, z \in A$, $(\forall z \in D_v)\, z \in \overline{A}$ are also Σ^0_n. □

Corollary 2.8. *For $n \geqslant 1$, if a set S is computable relative to some Δ^0_n set R, then S is Δ^0_n.*

Proof. Both S and $\overline{S}$ are c.e. relative R, so by Theorem 2.7, both are Σ^0_n. Then by definition, S is Δ^0_n. □

Lemma 2.9.

1. *For $n \geqslant 1$, $\emptyset^{(n)}$ is m-complete Σ^0_n.*
2. *For $n \geqslant 1$, $\emptyset^{(n-1)}$ is Δ^0_n, and any Δ^0_n set S is computable relative to $\emptyset^{(n-1)}$.*

Proof. We use induction. First, let $n = 1$. Since $\emptyset'$ is essentially the same as K, the proof of Propositions 1.1 and 1.2 give Statement 1. Statement 2 is trivial. Supposing the statements for n, we consider $n + 1$. By Proposition 1.3, $\emptyset^{(n+1)}$ is c.e. relative to $\emptyset^{(n)}$, and by the Induction Hypothesis, $\emptyset^{(n)}$ is Σ^0_n. Therefore, we can apply Theorem 2.7 to conclude that $\emptyset^{(n+1)}$ is Σ^0_{n+1}. If S is Σ^0_{n+1}, then by Theorem 2.7, S is c.e. relative to some Π^0_n set B, and it is also c.e. relative to $\overline{B}$, a Σ^0_n set. By the Induction Hypothesis, B and $\overline{B}$ are computable relative to $\emptyset^{(n)}$. Then S is c.e. relative to $\emptyset^{(n)}$, and by Proposition 1.4, $S \leqslant_m \emptyset^{(n+1)}$. For Statement 2, note that $\emptyset^{(n-1)}$ is Σ^0_{n-1}, so it is Δ^0_n. Moreover, if S is Δ^0_n, then S and $\overline{S}$ are each c.e. relative to some Σ^0_{n-1} set. Using the Induction Hypothesis, we may suppose for both that the Σ^0_{n-1} set is $\emptyset^{(n-1)}$. Therefore, $S \leqslant_T \emptyset^{(n-1)}$. □

The result below gives alternative definitions for the classes of Σ^0_n, Π^0_n, and Δ^0_n relations in terms of the sets $\emptyset^{(n)}$.

Theorem 2.10. *For $n \geqslant 1$,*

1. *the Σ^0_n relations are those which are c.e. relative to $\emptyset^{(n-1)}$,*
2. *the Π^0_n relations are those which are co-c.e. relative to $\emptyset^{(n-1)}$,*
3. *the Δ^0_n relations are those which are computable relative to $\emptyset^{(n-1)}$.*

Proof. This is immediate from Theorem 2.7 and Lemma 2.9. □

We have seen that for all $n \geqslant 1$, the universal Σ^0_n enumeration relation $E_{\Sigma^0_n}(e, x)$ is Σ^0_n and not Π^0_n. By Theorem 2.10, $\emptyset^{(n)}$ also has this property.

Corollary 2.11. *For each $n \geqslant 1$, $\emptyset^{(n)}$ is Σ^0_n and not Π^0_n.*

Proof. Since $\emptyset^{(n)}$ is c.e. relative to $\emptyset^{(n-1)}$, it is Σ^0_n. If $\emptyset^{(n)}$ is Π^0_n, then by Theorem 2.10, it is co-c.e. relative to $\emptyset^{(n-1)}$. Therefore, it is computable relative to $\emptyset^{(n-1)}$. By Proposition 1.3, this is a contradiction. □

2.6. Arithmetical Functions

We can locate *functions* as well as relations at various levels in the arithmetical hierarchy. Recall that a partial function is computable if and only if its graph is c.e. We extend this idea, saying, for $n > 0$, that a function f is *partial* Δ^0_n if the graph of f is Σ^0_n. By Theorem 2.10, this holds just in case the graph of f is c.e. relative to $\emptyset^{(n-1)}$, or f is partial computable relative to $\emptyset^{(n-1)}$.

2.7. Relativizing

We may relativize the arithmetical hierarchy to an arbitrary set X and consider sets and relations Σ^0_n, Π^0_n, and Δ^0_n relative to X. The only change in the definition is at the bottom level; a set is Σ^0_0 and Π^0_0 relative to X if it is computable relative to X.

2.8. Sets Definable in $\mathcal{N}$

The *standard model of arithmetic* is $\mathcal{N} = (\omega, +, \cdot, S, 0, <)$. The next result connects the arithmetical hierarchy with definability in $\mathcal{N}$.

Proposition 2.12. *For $n \geqslant 1$, a relation is Σ^0_n, or Π^0_n, iff it is defined in $\mathcal{N}$ by a finitary Σ_n, or Π_n, formula.*

Proof sketch. A simple inductive argument shows that a relation defined in $\mathcal{N}$ by a Σ_n, or Π_n, formula must be Σ^0_n, or Π^0_n. In his negative solution of Hilbert's Tenth Problem, Matijasevic showed that any c.e. set is definable in $\mathcal{N}$ by a Σ_1 formula of a special form—saying that a certain polynomial equation has a solution in the integers. (For a nice account of Matijasevic's Theorem, see [9] or [13].) Since any Σ^0_1 relation is defined by a Σ_1 formula, it follows that any Π^0_1 relation is defined by a Π_1 formula. Now, proceeding by induction, we see that for $n \geqslant 1$, any Σ^0_n, or Π^0_n, relation is defined by a finitary Σ_n, or Π_n, formula. □

3. Computable Ordinals and Kleene's $\mathcal{O}$

To extend the arithmetical hierarchy through the transfinite levels, we need names, or *notations*, for ordinals. In this section, we describe Kleene's system of ordinal notation, and we show that the ordinals with notations are the same as the computable ordinals.

Definition 5. An ordinal α is *computable* if there is a computable well ordering of type α.

Proposition 3.1. *The computable ordinals form a countable initial segment of the ordinals.*

Proof. It is clear that there are only countably many computable ordinals. If $\mathcal{A}$ is a computable ordering, then for any element a, the restriction of the ordering to $pred(a) = \{b : \mathcal{A} \models b < a\}$ is computable. □

3.1. Kleene's $\mathcal{O}$

Kleene assigned *notations* to ordinals in such a way that the notation tells how the ordinal has been built up from below. We define, simultaneously, the set $\mathcal{O}$ of notations, a function $|\ |_{\mathcal{O}}$ taking each $a \in \mathcal{O}$ to the ordinal α for which a is a notation, and a strict partial ordering $<_{\mathcal{O}}$ on $\mathcal{O}$.

We let 1 be the notation for 0. If a is a notation for α, then 2^a is a notation for $\alpha+1$. In the partial ordering, we let $b <_{\mathcal{O}} 2^a$ if either $b <_{\mathcal{O}} a$ or $b = a$. For a limit ordinal α, the notations are the numbers $3 \cdot 5^e$, where φ_e is a total computable function, with values in $\mathcal{O}$, such that

$$\varphi_e(0) <_{\mathcal{O}} \varphi_e(1) <_{\mathcal{O}} \varphi_e(2) <_{\mathcal{O}} \ldots,$$

and α is the least upper bound of the sequence of ordinals $\alpha_n = |\varphi_e(n)|_{\mathcal{O}}$. In the partial ordering, we let $b <_{\mathcal{O}} 3 \cdot 5^e$ if there exists n such that $b <_{\mathcal{O}} \varphi_e(n)$.

We can see that $1,_2, 2^2, 2^{2^2}, \ldots$ are the unique notations for the finite ordinals $0, 1, 2, 3, \ldots$. When we come to ω, there are infinitely many notations. Only countably many ordinals can have notations. These are called *constructive* ordinals. If α is constructive, then so is $\alpha + 1$. Therefore, the first non-constructive ordinal must be a limit ordinal.

3.2. The Constructive Ordinals are the Computable Ones

Kleene and Spector showed that an ordinal is constructive if and only if it is computable.

Lemma 3.2. *Suppose α is a constructive ordinal, say $|a|_{\mathcal{O}} = \alpha$. Then the set $\{b : b <_{\mathcal{O}} a\}$ consists of exactly one notation for each $\beta < \alpha$. Moreover, the restriction of $<_{\mathcal{O}}$ to this set is isomorphic to $pred(\alpha)$, under the isomorphism $|\ |_{\mathcal{O}}$.*

Proof. This is easily shown by induction on α. □

The next result says that every constructive ordinal is computable.

Proposition 3.3. *For all $a \in \mathcal{O}$, if $|a|_{\mathcal{O}} = \alpha$, then there is a computable well ordering of type α.*

Proof. Given $a \in \mathcal{O}$, the set $pred(a) = \{b : b <_{\mathcal{O}} a\}$ is c.e., uniformly in a. The restriction of $<_{\mathcal{O}}$ to $pred(a)$ is c.e., again uniformly in a. We have a c.e. well ordering of type α. The universe of the ordering is c.e. Given a pair of elements, we can effectively determine which is larger, using comparability. Let f be the function taking $b \in pred(a)$ to $\langle b, s\rangle$, where s is the stage at which b first appears in $pred(a)$. Then f is an isomorphism from $pred(a)$ onto a computable ordering of type α. □

We have seen that every constructive ordinal is computable. We must prove the converse.

Theorem 3.4. *Every computable ordinal is constructive.*

Proof. Let α be a computable ordinal. There is a computable well ordering of type α, and there is also a computable well ordering of type $\alpha + 1$, call it $\mathcal{A}$. From $\mathcal{A}$, we pass to a computable well ordering $\mathcal{B}$ of order type $\omega(\alpha + 1)$ (replacing each element of $\mathcal{A}$ by a copy of ω). We let the universe of $\mathcal{B}$ be the set B of numbers of the form $\langle a, n\rangle$, for $a \in \mathcal{A}$ and $n \in \omega$. We let $\langle a, n\rangle <_B \langle a', n'\rangle$ iff either $a <_{\mathcal{A}} a'$ or else $a = a'$ and $n < n'$.

Lemma 3.5. *We have uniform effective procedures that do the following:*

1. *decide, for a given element, whether it is first, a successor, or a limit element,*
2. *for any successor element b, find the immediate predecessor,*
3. *for any limit element b, determine an increasing sequence $\ell(b, i)$ converging to b.*

Proof of Lemma 3.5. It is easy to check that $(B, <_B)$ is a computable ordering satisfying Conditions 1 and 2. For Condition 3, let $b_0, b_1, b_2, \ldots$ be the elements of B, listed according to the usual ordering on ω. For an element b which is a limit element according to the ordering $<_B$, we define $\ell(b, i)$ by induction on i. For each i, we take the first n such that $b_n <_B b$ and $b_n >_B \ell(b, j)$ for all $j < i$, and we let $\ell(b, i)$ be b_n. □

To complete the proof of Theorem 3.4, it is enough to prove the following.

Lemma 3.6. *Let $\mathcal{B}$ be as above. There is a partial computable function g, defined on B, such that $g(b)$ is a notation in $\mathcal{O}$ for the order type of $pred(b)$. Moreover, $b' <_B b$ iff $g(b') <_{\mathcal{O}} g(b)$.*

Proof of Lemma 3.6. The proof uses the Recursion Theorem. We begin by defining a partial computable function f such that if e is a computable index for a function with the behavior we want for g on all $y <_B x$, then $f(e, x)$ has the behavior we want on all $y \leqslant_B x$. We consider three cases.

Case 1. Suppose x is the first element of $\mathcal{B}$. Then we let $f(e, x) = 1$.

Case 2. Suppose x is the successor of y in $\mathcal{B}$. Say $\varphi_e(y) = z$. Then we let $f(e, x) = 2^z$.

Case 3. Suppose x is a limit element of $\mathcal{B}$. Let $\ell(x, i)$ be the computable increasing sequence with limit x, defined just before Lemma 3.6. Let $h(e, x)$ be the total computable function, given by the $s - m - n$ Theorem, such that $\varphi_{h(e,x)}(i) = \varphi_e(\ell(x, i))$. Then we let $f(e, x) = 3 \cdot 5^{h(e,x)}$.

It is clear that for any e, for all $x \in B$, if for all $y <_B x$, $\varphi_e(y)$ has the value we want for $g(y)$, then $f(e, x)$ has the value we want for all $y \leqslant_B x$, Using the $s - m - n$ Theorem again, we get a total computable function k such that

$$f(e, x) = \varphi_{k(e)}(x).$$

By the Recursion Theorem, there exists n such that $\varphi_{k(n)} = \varphi_n$.

Claim 1. *For all $x \in B$, $\varphi_n(x)$ is defined.*

This is an easy induction on the well ordering $\mathcal{B}$. If $\varphi_n(y)\downarrow$ for all $y <_B x$, then $f(n,x)\downarrow$, where $f(n,x) = \varphi_{k(n)}(x) = \varphi_n(x)$.

Claim 2. *If $pred(x)$ has order type β, then $\varphi_n(x)$ is a notation b for β. Moreover, φ_n maps $\{y : y \leqslant_B x\}$ isomorphically onto $\{c : c <_{\mathcal{O}} b\}$.*

This is easy to show, by induction on β. Thus, φ_n is the desired function g, completing the proof of Lemma 3.6. □

This is all we needed to prove Theorem 3.4. □

Remark. With a little added effort, the proof of Theorem 3.4 can be made to give a partial computable function f such that if e is an index for a computable linear ordering $\mathcal{A}$, then $f(e)$ is defined, with value in $\mathcal{O}$ just in case $\mathcal{A}$ is a well ordering (see [2]).

4. The Hyperarithmetical Hierarchy

The first step in defining the hyperarithmetical hierarchy is to extend the sequence of sets $\emptyset, \emptyset', \emptyset'', \ldots$.

4.1. Extending the Sequence $\emptyset^{(n)}$

While each finite ordinal has a unique notation in $\mathcal{O}$, each infinite ordinal has infinitely many notations. We define sets $H(a)$, for all $a \in \mathcal{O}$, such that if $|a|_{\mathcal{O}} = \alpha$, then $H(a)$ has the properties we want for $\emptyset^{(\alpha)}$.

Definition 6 (The special sets $H(a)$).

1. $H(1) = \emptyset$,
2. $H(2^a) = H(a)'$,
3. $H(3 \cdot 5^e) = \{\langle u,v\rangle : u <_{\mathcal{O}} 3 \cdot 5^e \;\&\; v \in H(u)\}$—this is the same as the set $\{\langle u,v\rangle : (\exists n)(u \leqslant_{\mathcal{O}} \varphi_e(n) \;\&\; v \in H(u))\}$.

Remarks. By 3 above, if a is a notation for a limit ordinal, then $H(a)$ gives a "uniform" upper bound for the sets $H(u)$, for $u <_{\mathcal{O}} a$, in the sense that the sets $H(u)$ are uniformly computable in $H(a)$. It is easy to check that if a is the unique notation for a finite ordinal n, then $H(a) = \emptyset^{(n)}$.

We have seen that for finite $n \geqslant 1$, a set is Σ^0_n, or Π^0_n, if it is c.e., or co-c.e. relative to $\emptyset^{(n-1)}$.

Definition 7. Suppose α is an infinite computable ordinal, and let a be a notation for α. Let S be a set or relation.

1. We say that S is Σ^0_α, or Π^0_α, if it is c.e., or co-c.e. relative to $H(a)$.
2. We say that S is Δ^0_α if it is both Σ^0_α and Π^0_α.

4.2. Spector's Theorem

We need to know that the definition we have just given is independent of our choice of the notation for α. The proof of this, due to Spector, requires several lemmas. No step is difficult, but it is worthwhile thinking about how the steps fit together. Spector's Theorem is important. Moreover, when we talk about computable infinitary formulas, we will prove another important fact using essentially the same proof.

Lemma 4.1. *There is a partial computable function that assigns to each pair (a,b) of elements of $\mathcal{O}$ such that $a \leqslant_{\mathcal{O}} b$, an index for $H(a)$ as a set computable in $H(b)$.*

Proof. First, note that if $a, b \in \mathcal{O}$, where $a \leqslant_{\mathcal{O}} b$, then by shortening the stack of 2's in b, we can find $c \leqslant_{\mathcal{O}} b$ and $n < \omega$ such that $|c|_{\mathcal{O}} + n = |b|_{\mathcal{O}}$ and one of the following holds:

Case 1. $c = a$,

Case 2. $|c|_{\mathcal{O}}$ is a limit ordinal and $a <_{\mathcal{O}} c$.

For any $n \in \omega$, we can find an index for X as a set computable in $X^{(n)}$—the index is computed just from n, independent of X. This takes care of Case 1. In Case 2, we have $u \in H(a)$ if and only if $\langle a, u\rangle \in H(c)$. We can find an index for $H(a)$ as a set computable in $H(c)$. Then we obtain an index for $H(a)$ as a set computable in $H(b)$. $\square$

Lemma 4.2 (Spector). *There is a partial computable function f such that for each $a \in \mathcal{O}$, $f(a)$ is an index for $\{b \in \mathcal{O} : |b|_{\mathcal{O}} < |a|_{\mathcal{O}}\}$ as a set computable in $H(a)'$.*

Proof. The function f is defined by computable transfinite recursion on ordinal notations. We must give $f(1)$ and say how to compute $f(2^b)$ from $f(b)$ and how to compute $f(3 \cdot 5^e)$ from the restriction of f to $\{b : b <_{\mathcal{O}} 3 \cdot 5^e\}$. The Recursion Theorem then gives us an index for the desired partial computable function f, defined on all of $\mathcal{O}$.

Case 1. Suppose $a = 1$. We let $f(1)$ be an index for $\emptyset$ relative to $\emptyset'$.

Case 2. Suppose $a = 2^b$. We say how $f(a)$ is defined in terms of $f(b)$. Note that

$$(d \in \mathcal{O} \;\&\; |d|_{\mathcal{O}} < |2^b|_{\mathcal{O}})$$

if and only if one of the following holds:

1. $d = 1$,
2. d has the form 2^c, where $c \in \mathcal{O}$ and $|c|_{\mathcal{O}} < |b|_{\mathcal{O}}$,
3. d has the form $3 \cdot 5^{e'}$, where $d \in \mathcal{O}$ and $|d|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}}$; i.e., for all n, $\varphi_{e'}(n) \in \mathcal{O}$, $\varphi_{e'}(n) <_{\mathcal{O}} \varphi_{e'}(n+1)$, and $|\varphi_{e'}(n)|_{\mathcal{O}} < |b|_{\mathcal{O}}$.

Determining whether d satisfies 1 is trivial. Assuming that $f(b)$ is an index for $\{c \in \mathcal{O} : |c|_{\mathcal{O}} < |b|_{\mathcal{O}}\}$ relative to $H(b)'$, we can determine whether $d = 2^c$ satisfies 2, using $H(2^b)$, or, by Lemma 4.1, using $H(2^b)'$. We can determine whether $d = 3 \cdot 5^{e'}$ satisfies 3, using $H(2^b)'$. For each n, we can determine whether $\varphi_{e'}(n) \in \mathcal{O}$ and $|\varphi_{e'}(n)|_{\mathcal{O}} < |b|_{\mathcal{O}}$, using $H(2^b)$, and we can determine whether $\varphi_{e'}(n) <_{\mathcal{O}} \varphi_{e'}(n+1)$, using $H(2)$ or, by Lemma 4.1, using $H(2^b)$. We can find an index for $\{d \in \mathcal{O} : |d|_{\mathcal{O}} < |2^b|_{\mathcal{O}}\}$ as a set computable in $H(2^b)'$, and this is $f(2^b)$. We have described the procedure for deciding whether d is in the set, so, in principle, we know the index. This is $f(2^b)$.

Case 3. Let $a = 3 \cdot 5^e$. Assuming that we know how to compute $f(b)$, for $b <_{\mathcal{O}} a$, we must say how to compute $f(a)$. We have $(d \in \mathcal{O}\ \&\ |d|_{\mathcal{O}} < |3 \cdot 5^e|_{\mathcal{O}})$ iff $(\exists n)(d \in \mathcal{O}\ \&\ |d|_{\mathcal{O}} < |\varphi_e(n)|_{\mathcal{O}})$. Say $\varphi_e(n) = b_n$. We suppose $f(b_n)$ is an index for $I_n = \{d \in \mathcal{O} : |d|_{\mathcal{O}} < |b_n|_{\mathcal{O}}\}$ as a set computable in $H(b_n)'$. Now, $f(3 \cdot 5^e)$ is supposed to be an index for $\cup_n I_n$ as a set computable in $H(3 \cdot 5^e)'$. For any n, we can determine whether $d \in I_n$ using oracle $H(b_n)' = H(2^{b_n})$, or, by Lemma 4.1, using oracle $H(3 \cdot 5^e)$. The procedure is uniform in n. Then we have a procedure for determining whether $d \in \cup_n I_n$ using $H(3 \cdot 5^e)'$. Since we know the procedure, we can compute an index. This is $f(3 \cdot 5^e)$. □

Lemma 4.3 (Spector). *There are partial computable functions that assign to each $b \in \mathcal{O}$ indices for $\{a \in \mathcal{O} : |a|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}}\}$ and $\{a \in \mathcal{O} : |a|_{\mathcal{O}} = |b|_{\mathcal{O}}\}$ as sets computable in $H(2^b)'$.*

Proof. For $b \in \mathcal{O}$, we have $(a \in \mathcal{O}\ \&\ |a|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}})$ iff $(a \in \mathcal{O}\ \&\ |a|_{\mathcal{O}} < |2^b|_{\mathcal{O}})$, and $(a \in \mathcal{O}\ \&\ |a|_{\mathcal{O}} = |b|_{\mathcal{O}})$ iff $(a \in \mathcal{O}\ \&\ |a|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}}\ \&\ |a|_{\mathcal{O}} \not< |b|_{\mathcal{O}})$. □

Theorem 4.4 (Spector). *There is a partial computable function f such that for $a, b \in \mathcal{O}$ with $|a|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}}$, $f(a, b)$ is an index for $H(a)$ as a set computable in $H(b)$.*

Proof. Formally, we define, by computable transfinite recursion on ordinal notation, a partial computable function that assigns to each $b \in \mathcal{O}$ an index relative to $H(b)$ for a function with values appropriate for $f(a, b)$. Let u_0 be an index for $\emptyset$ as a set computable in all sets X.

Case 1. Suppose $b = 1$. Then we let $f(1, 1) = u_0$.

Case 2. Suppose $b = 2^d$.

(a) Suppose $a = 1$. Then we let $f(1, 2^d) = u_0$.
(b) Suppose $a = 2^c$. Assuming that $f(c, d)$ is an index for $H(c)$ relative to $H(d)$, we determine an index for $H(c)'$ relative to $H(d)'$ (from an index for $H(c)'$ as a set c.e. relative to $H(d)$). We let this be $f(2^c, 2^d)$.
(c) Suppose $a = 3 \cdot 5^e$. Assuming that $f(a, d)$ is an index for $H(a)$ relative to $H(d)$, we determine an index for $H(a)$ relative to $H(d)'$. We let this be $f(3 \cdot 5^e, 2^d)$.

Case 3. Suppose $b = 3 \cdot 5^e$.

(a) Suppose $a = 1$. Then we let $f(1, b) = u_0$.
(b) Suppose $a = 2^c$. Assuming that we know $f(a, d)$ for $d <_{\mathcal{O}} 3 \cdot 5^e$, we must say what is $f(a, b)$. We describe a procedure for determining whether $x \in H(a)$, using $H(3 \cdot 5^e)$, and we let $f(a, b)$ be the index of this procedure. We first enumerate the set $\{d \in \mathcal{O} : d <_{\mathcal{O}} 3 \cdot 5^e\}$, searching for the unique element d such that $|a|_{\mathcal{O}} = |d|_{\mathcal{O}}$. For a given $d <_{\mathcal{O}} 3 \cdot 5^e$, we can check whether $|d|_{\mathcal{O}} = |2^c|_{\mathcal{O}}$ using $H(2^d)'$, by Lemma 4.3, or using $H(3 \cdot 5^e)$, by Lemma 4.1. Having found the desired d, we get an index $f(a, d)$ for $H(a)$ relative to $H(d)$, and we can also find an index i for $H(a)$ relative to $H(3 \cdot 5^e)$, using Lemma 4.1. Finally, we check whether x is in the set with this index i. Note that $f(a, b)$ is not i—we need the oracle for $H(3 \cdot 5^e)$ to compute i, and f is supposed to be computable. We let $f(a, 3 \cdot 5^e)$ be an index for the whole procedure, which first locates d, then determines i, and then does the calculation.

(c) Suppose $a = 3 \cdot 5^{e'}$. We can determine whether $\langle u, v\rangle \in H(3 \cdot 5^{e'})$, using $H(3 \cdot 5^e)$ as follows. First, check that $u <_{\mathcal{O}} 3 \cdot 5^{e'}$. We can do this using $H(2) = \emptyset'$, so we can also use $H(3 \cdot 5^e)$. Next, we find $d <_{\mathcal{O}} 3 \cdot 5^e$ such that $|u|_{\mathcal{O}} = |d|_{\mathcal{O}}$, as we did above (when $a = 2^c$). Assuming that $f(u, d)$ is an index for $H(u)$ relative to $H(d)$, we find an index i for $H(u)$ relative to $H(3 \cdot 5^e)$, as in Lemma 4.1. Finally, we apply procedure i to see if $v \in H(u)$. Again, we let $f(3 \cdot 5^{e'}, 3 \cdot 5^e)$ be an index for the whole procedure. □

We have proved Spector's theorem saying that here is a partial computable function f such that for $a, b \in \mathcal{O}$ with $|a|_{\mathcal{O}} \leqslant |b|_{\mathcal{O}}$, $f(a, b)$ is an index for $H(a)$ as a set computable in $H(b)$. As a corollary, we have the fact that the Turing degree of $H(a)$ depends only on the ordinal $|a|_{\mathcal{O}}$.

Corollary 4.5. *If $a, b \in \mathcal{O}$ and $|a|_{\mathcal{O}} = |b|_{\mathcal{O}}$, then $H(a) \equiv_T H(b)$.*

4.3. More on the Definition

We have defined the hyperarithmetical hierarchy in two cases, first the finite levels, and then the infinite ones.

1. For finite n, we have $\emptyset^{(n-1)} = H(a)$, where a is the unique notation for $n - 1$. A set or relation is Σ^0_n, or Π^0_n, or Δ^0_n, if it is c.e., co-c.e., or computable, relative to the set $\emptyset^{(n-1)}$.
2. For an infinite computable ordinal α, a set or relation is Σ^0_α, Π^0_α, or Δ^0_α if it is, respectively, c.e., co-c.e., or computable relative to $H(a)$ for some, or all, notations a for α. (Spector's Theorem justifies this definition.)

There is a lack of uniformity in the definition. We might think of changing the definition of the arithmetical hierarchy, to say that for finite as well as infinite α, the Σ^0_α, Π^0_α, and Δ^0_α relations are those which are c.e., co-c.e., or computable, relative to $H(a)$, where $|a|_{\mathcal{O}} = \alpha$. However, the reasons for leaving the definition as is are more compelling. The Σ^0_n and Π^0_n relations are expressible with n alternations of quantifiers. The pleasing match of quantifier complexity with level extends naturally through all levels of the hyperarithmetical hierarchy.

It is convenient to have a canonical Δ^0_α set associated with a given notation a for α, whether α is finite or infinite.

Definition 8. For $a \in \mathcal{O}$, we let

$$\Delta^0_a = \begin{cases} H(a) & \text{if } |a|_{\mathcal{O}} \geqslant \omega, \\ H(b), \text{ where } |b|_{\mathcal{O}} + 1 = a, & \text{if } 0 < |a|_{\mathcal{O}} < \omega. \end{cases}$$

The set Δ^0_a is *Turing complete* Δ^0_α; i.e., Δ^0_a is Δ^0_α, and all Δ^0_α sets are computable relative to Δ^0_a. In terms of this set, we have a uniform definition. A set is Σ^0_α or Π^0_α, if it is c.e., or co-c.e., relative to Δ^0_a.

We may define the hyperarithmetical functions and partial functions as follows.

Definition 9.

1. A total function is said to be Δ^0_α if it is computable relative to Δ^0_a, for some, or all notations a for α.
2. A partial function $f(\overline{x})$ is *partial* Δ^0_α if the relation $f(\overline{x}) = y$ is Σ^0_α.

We may relativize the hyperarithmetical hierarchy to any set X. We begin by relativizing the sets $H(a)$, letting $H(1) = X$, and keeping the other clauses as before. Next, we define the special oracles $\Delta^0_a(X)$.

Definition 10. Suppose a is a notation for α.

1. If α is finite, let $\Delta^0_a(X) = H(b)$, where $2^b = a$.
2. If α is infinite, let $\Delta^0_a(X) = H(a)(X)$

We complete the definition of the classes of relatively Σ^0_α, Π^0_α, and Δ^0_α sets and relations as follows.

Definition 11. A set or relation is Σ^0_α, Π^0_α, or Δ^0_α *relative to* X if it is c.e., co-c.e., or computable, relative to $H(a)(X)$.

4.4. Non-Collapse

The fact that the hyperarithmetical hierarchy does not collapse is proved in the same way as the corresponding fact for the arithmetical hierarchy. We begin with the following lemma.

Lemma 4.6. *Let α, β be computable ordinals with $\beta < \alpha$. Then any Σ^0_β or Π^0_β set is Δ^0_α.*

Proof. Fix a path through $\mathcal{O}$, and let a and b be notations for α and β on this path. Then $\Delta^0_{2^b}$ is computable in Δ^0_a. If S is Σ^0_β or Π^0_β, it is computable relative to $\Delta^0_{2^b}$, and also relative to Δ^0_a. Therefore, it is Δ^0_α. □

Proposition 4.7 (Proper Hierarchy Theorem). *For each computable ordinal α, there is a Σ^0_α set that is not Π^0_α, so it is not Σ^0_β or Π^0_β for any $\beta < \alpha$.*

Proof. For $\alpha = n$, we use $\emptyset^{(n-1)}$. For $\alpha \geqslant \omega$, we take $H(a)'$, where $|a| = \alpha$. This is c.e. and not co-c.e. relative to $H(a)$, so it is Σ^0_α and not Π^0_α. By Lemma 4.6, it is not Σ^0_β or Π^0_β for any $\beta < \alpha$. □

5. Computable Infinitary Formulas

Before describing the computable infinitary formulas, we recall just a little about the larger class of $L_{\omega_1\omega}$ formulas. For more about $L_{\omega_1\omega}$, see the excellent book by Keisler [15]. We fix a predicate language L. The $L_{\omega_1\omega}$ formulas are infinitary formulas in which the disjunctions and conjunctions are countable. We consider formulas in *normal form*. These formulas have only finitely many free variables, and the negations are brought inside. We cannot bring all of the quantifiers to the front. We classify the formulas as Σ_α or Π_α, for countable ordinals α.

Definition 12.

1. The Σ_0 and Π_0 formulas are the finitary quantifier-free formulas of L.
2. The Σ_α formulas (for $\alpha > 0$) have the form $\varphi(\overline{x}) = \bigvee\!\!\!\bigvee_i (\exists \overline{u}_i)\psi_i(\overline{x}, \overline{u}_i)$, where ψ_i is Π_β for some $\beta < \alpha$.

3. The Π_α formulas (for $\alpha > 0$) have the form $\varphi(\overline{x}) = \bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\psi_i(\overline{x}, \overline{u}_i)$, where ψ_i is Σ_β for some $\beta < \alpha$.

Example 1. There is a Π_2 sentence of $L_{\omega_1\omega}$ saying of an ordered field that it is Archimedean. The sentence says $(\forall x) \bigvee\!\!\!\bigvee_n x < n$. The class of Archimedean ordered fields cannot be axiomatized using finitary (elementary) first order sentences. It is an easy exercise, using the Compactness Theorem, that if T is the set of elementary first order sentences true in all Archimedean ordered fields, then T has a model with an element greater than all finite n. The reciprocal of this element is infinitesimal.

Example 2. There is a Π_2 sentence of $L_{\omega_1\omega}$ saying of an Abelian group that it is a p-group; i.e., all elements (except 0) have order a power of p. The sentence says $(\forall x) \bigvee\!\!\!\bigvee_n p^n x = 0$. If T is the set of elementary first order sentences true in all Abelian p-groups, then T has a model with an element of infinite order.

For each formula φ in normal form, there is a formula $neg(\varphi)$, also in normal form, which is logically equivalent to the negation of φ. We obtain $neg(\varphi)$ by taking the dual, switching disjunctions and existential quantifiers for conjunctions and universal quantifiers.

5.1. Computable Infinitary Formulas and Their Indices

For computable structure theory, it is useful to consider *computable infinitary formulas*. Roughly speaking, these are infinitary formulas in which the infinite disjunctions and conjunctions are over c.e. sets. We consider only formulas in normal form. To each formula, we associate a finite tuple of variables including all those which occur free. We classify the formulas as computable Σ_α and computable Π_α, for various computable ordinals α. To make precise what is a c.e. set of formulas, we need a system of indices for the formulas. If $a \in \mathcal{O}$, where $|a|_{\mathcal{O}} = \alpha$, we define computable sets S_a^Σ and S_a^Π of indices for computable Σ_α and computable Π_α formulas, and we say which formula corresponds to a given index.

The computable Σ_0 and Π_0 formulas $\varphi(\overline{x})$ are just the finitary quantifier-free formulas. The variables of the formula $\varphi(\overline{x})$ are all those which appear in the formula. We may suppose that $\varphi(\overline{x})$ is in complete disjunctive normal form; i.e., each atomic sub-formula occurs, positively or negatively, in each disjunct. Let $S_1^\Sigma = S_1^\Pi$ be the set of Gödel numbers for these formulas. The formula $\varphi(\overline{x})$ corresponding to index i is the one with Gödel number i.

Suppose $\alpha > 0$. The computable Σ_α formulas $\varphi(\overline{x})$ are the c.e. disjunctions of formulas of the form $(\exists \overline{u})\psi(\overline{u}, \overline{x})$, where $\psi(\overline{u}, \overline{x})$ is computable Π_β for some $\beta < \alpha$. The computable Π_α formulas $\varphi(\overline{x})$ are the c.e. conjunctions of formulas of the form $(\forall \overline{u})\psi(\overline{u}, \overline{x})$, where $\psi(\overline{u}, \overline{x})$ is computable Σ_β for some $\beta < \alpha$. Let a be a notation for α. Then S_a^Σ is the set of tuples of the form $\langle \Sigma, a, \overline{x}, e\rangle$, where $\overline{x}$ is a tuple of variables and $e \in \omega$, and S_a^Π is the set of tuples of the form $\langle \Pi, a, \overline{x}, e\rangle$. Suppose $i = (\Sigma, a, \overline{x}, e)$ is an index in S_a^Σ. The computable Σ_α formula with this index is the disjunction of the formulas $(\exists \overline{u})\psi$, where ψ has an index $j \in W_e \cap (\cup_{b <_{\mathcal{O}} a} S_b^\Pi)$, and $\overline{u}$ is the set of variables of ψ which are not in $\overline{x}$. Now, suppose $i = (\Pi, a, \overline{x}, e)$ is an index in S_a^Π. The computable Π_α formula with this index is the conjunction of the formulas $(\forall \overline{u})\psi$, where ψ has an index $j \in W_e \cap (\cup_{b <_{\mathcal{O}} a} S_b^\Sigma)$, and $\overline{u}$ is the set of variables of ψ which are not in $\overline{x}$.

For each computable infinitary formula φ, there is a naturally associated formula $neg(\varphi)$, defined below, such that $neg(\varphi)$ is logically equivalent to the negation of φ. Moreover, given an index for φ, in S_a^{Σ}, or S_a^{Π}, we can find an index for $neg(\varphi)$, in S_a^{Π}, or S_a^{Σ}.

Definition 13 ($neg(\varphi)$).

1. If φ is computable Σ_0 and Π_0, then $neg(\varphi)$ is the formula in complete disjunctive normal form that is logically equivalent to $\neg\varphi$.
2. If φ is computable Σ_α (for $\alpha > 0$), of the form $\bigvee\!\!\!\bigvee_i (\exists \overline{u}_i)\psi_i$, then $neg(\varphi)$ has the form $\bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i) neg(\psi_i)$.
3. If φ is computable Π_α, of the form $\bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\psi_i$, then $neg(\varphi)$ has the form $\bigvee\!\!\!\bigvee_i (\exists \overline{u}_i) neg(\psi_i)$.

5.2. Examples of Computable Infinitary Formulas

Looking back at our examples of $L_{\omega_1\omega}$ sentences, we see that there is a computable Π_2 sentence saying of an ordered field that it is Archimedean. Similarly, there is a computable Π_2 sentence saying of an Abelian group that it is a p-group.

We include in our languages the logical constants $\top$ (truth) and $\bot$ (falsity). These are useful in results such as the following.

Proposition 5.1. *Let L be a countable relational language. Then there is a computable infinitary sentence φ, in the language $L \cup \omega$, characterizing the computable L-structures with universe ω.*

Proof. First, there is a computable infinitary sentence

$$\psi = (\forall x) \bigvee\!\!\!\bigvee_{b \in B} x = n$$

saying that all elements are named by constants from ω. Next, for any e and any atomic sentence α in the language $L \cup \omega$, let $d(e, \alpha, 1)$ be the disjunction, over $c \in \omega$, of sentences

$$d_c^1 = \begin{cases} \top & \text{if } c \text{ is a computation of } \varphi_e(\alpha) \text{ with value } 1 \\ \bot & \text{otherwise} \end{cases}$$

Let $d(e, \alpha, 0)$ be the disjunction of sentences

$$d_c^0 = \begin{cases} \top & \text{if } c \text{ is a computation of } \varphi_e(\alpha) \text{ with value } 0 \\ \bot & \text{otherwise} \end{cases}$$

Now,

$$\psi \ \& \ \bigvee\!\!\!\bigvee_e \bigwedge\!\!\!\bigwedge_\alpha \big[\big(\alpha \to d(e,\alpha,1)\big) \ \& \ \big(\neg\alpha \to d(e,\alpha,0)\big)\big]$$

has the desired meaning and is logically equivalent to a computable infinitary sentence. $\square$

5.3. Facts About Computable Infinitary Formulas

The class of computable infinitary formulas, has the same expressive power as the least "admissible fragment" of $L_{\omega_1\omega}$. What is useful about the computable infinitary formulas is the classification as computable Σ_α or computable Π_α, for computable ordinals α, which gives the following result.

Proposition 5.2 (Ash). *For a computable Σ_α, or computable Π_α formula $\varphi(\overline{x})$, the relation defined by $\varphi(\overline{x})$ in a computable structure $\mathcal{A}$ is Σ^0_α, or Π^0_α. Moreover, from an index for $\varphi(\overline{x})$ and an index for $\mathcal{A}$, we can effectively determine an index for the relation. For an arbitrary structure (not computable), the relation defined by $\varphi(\overline{x})$ is Σ^0_α, or Π^0_α relative to $\mathcal{A}$.*

Idea of proof. We consider the case where $\varphi(\overline{x})$ is computable Σ_1. From an index for $\varphi(\overline{x})$, we obtain a c.e. index for the set of disjuncts $(\exists u)\psi(\overline{x},\overline{u})$ of $\varphi(\overline{x})$. Given the diagram of $\mathcal{A}$, we enumerate $\overline{a}$, into $\varphi^{\mathcal{A}}$, when we find evidence that $\mathcal{A} \models \psi(\overline{a},\overline{c})$, for some $\overline{c}$, where $(\exists x)\psi(\overline{x},\overline{u})$ is one of the disjuncts of $\varphi(\overline{x})$. What we are doing is obviously uniform, in the structure $\mathcal{A}$, as well as in the index for the formula $\varphi(\overline{x})$. The general proof is a straightforward induction, and we omit it. □

Spector's proof that the classes Σ^0_α and Π^0_α are independent of the notation for α, provides also the proof of the result below, saying that what is expressible by computable Σ_α, or Π_α, formulas is independent of the notation used in the index. Moreover, if a and b are two notations for α, then we can pass effectively from an index involving a to an index involving b such that the formulas represented by the indices are logically equivalent.

Proposition 5.3. *There is a partial computable function that assigns to any pair of notations a, b with $|a|_\mathcal{O} = |b|_\mathcal{O}$, indices for functions $f^\Sigma_{a,b}$ mapping S^Σ_a to S^Σ_b and $f^\Pi_{a,b}$ mapping S^Π_a to S^Π_b such that the corresponding formulas are logically equivalent.*

Proof. We proceed by computable transfinite recursion on b. It is enough to describe $f^\Sigma_{a,b}$. Given an index i for φ in S^Π_a, we can pass effectively to an index for $neg(\varphi)$ in S^Σ_a, and if $f^\Sigma_{a,b}$ gives us the index S^Σ_b for the corresponding formula ψ, then we can pass effectively to an index j in S^Π_b for $neg(\psi)$. We let $f^\Pi_{a,b}(i) = j$.

Case 1. $b = 1$. We let $f^\Sigma_{1,1}$ be the identity on the set of Gödel numbers of finitary quantifier-free formulas.

Case 2. $b = 2^d$. For $a = 2^c$, if $i = \langle \Sigma, a, \overline{x}, e\rangle$, we pass to an index e' for the set of $f_{d,c}$ images of elements of $W_e \cap S^\Pi_c$, and we let $f_{a,b}(i) = \langle \Sigma, b, \overline{x}, e'\rangle$.

Case 3. $b = 3\cdot 5^e$. Suppose $a = 3\cdot 5^{e'}$. For $i \in W_{e'} \cap \cup_{d'<_\mathcal{O} a} S^\Pi_{d'}$, there is some $d' <_\mathcal{O} b$ such that $i \in S^\Pi_{d'}$. There is a unique $d <_\mathcal{O} a$ such that $|d'|_\mathcal{O} = |d|_\mathcal{O}$. Say $f_{d,d'}(i)$ is an index for $\psi_{d,d'}(\overline{u})$, and let $\overline{v}$ be the tuple of variables in $\overline{u}$ apart from those in $\overline{x}$. We let $\nu_{d',d}$ be a sentence with index in $S^\Sigma_{2^{2^{d'}}}$ that is logically equivalent to $\top$ if $|d|_\mathcal{O} = |d'|_\mathcal{O}$ and to $\bot$ otherwise. We can find an index in S^Σ_a for a formula logically equivalent to the disjunction of the formulas $\nu_{d',d}$ & $(\exists \overline{v})\psi_{d,d'}(\overline{u})$, for $d' <_\mathcal{O} b$. This is $f_{a,b}(i)$. □

Proposition 5.4. *Given $b <_{\mathcal{O}} a$ and $i \in S_b^{\Sigma} \cup S_b^{\Pi}$, we can find an index i' in S_a^{Σ} (or in S_a^{Π}) such that the formulas with indices i and i' are logically equivalent.*

Proof. For the first step, given i, with second component b, we enumerate those $d <_{\mathcal{O}} a$, searching for the one such that $|d|_{\mathcal{O}} = |b|_{\mathcal{O}}$. Using $H(2^a)$, when we see $d <_{\mathcal{O}} a$, we know an index for $H(2^{2^b})$ as a set computable in $H(2^a)$, and by Corollary 4.3, we can tell whether $|b|_{\mathcal{O}} = |d|_{\mathcal{O}}$. We can use Proposition 5.3 to find an index i', with second component d, such that the sentences with indices i and i' are logically equivalent. □

5.4. Computable Infinitary Propositional Formulas

It is sometimes useful to have computable infinitary propositional formulas as well as predicate formulas. Fix a computable propositional language P. A computable Σ_0 or Π_0 formula is a finitary formula of the language. For $\alpha > 0$, a computable Σ_α formula is a c.e. disjunction of formulas each of which is computable Π_β for some $\beta < \alpha$. A computable Π_α formula is a c.e. conjunction of formulas each of which is computable Σ_β for some $\beta < \alpha$. The changes in the indices are straightforward.

6. Forcing and Expressible Completeness

There are a number of results giving definability conditions to account for bounds on complexity relative to an arbitrary copy of a given structure. We give three such results. The first, from [20], gives conditions on a structure $\mathcal{A}$ and a set S, guaranteeing that S is Σ_α^0 relative to all copies of $\mathcal{A}$.

6.1. Relatively Intrinsically Σ_α^0 Sets

The result below gives syntactical conditions under which a set S is Σ_α^0 relative to all copies of a given structure.

Theorem 6.1. *Let $\mathcal{A}$ be a structure with universe ω, and let $S \subseteq \omega$. Then the following are equivalent:*

1. *For all $\mathcal{B} \cong \mathcal{A}$, S is Σ_α^0 relative to $\mathcal{B}$.*
2. *There is a tuple $\overline{c}$, and there is a computable sequence of computable Σ_α sentences $(\varphi_n(\overline{c}))_{n\in\omega}$, such that $n \in S$ iff $\mathcal{A} \models \varphi_n(\overline{c})$.*

We could replace Condition 2 above by the statement that the set S is "enumeration reducible" to the computable Σ_α type of the tuple $\overline{c}$, where this means that we have a uniform effective procedure for turning enumerations of the set of computable Σ_α formulas true of $\overline{c}$ into an enumeration of S.

Proof of Theorem 6.1. To see that 2 implies 1, take the sequence $(\varphi_n(\overline{c}))_{n\in\omega}$ of computable Σ_α sentences such that $n \in S$ iff $\mathcal{A} \models \varphi_n(\overline{c})$. Let $(\mathcal{B}, \overline{d}) \cong (\mathcal{A}, \overline{c})$. We have $n \in S$ iff $\mathcal{B} \models \varphi_n(\overline{d})$. Therefore, S is Σ_α^0 relative to $\mathcal{B}$.

To prove that 1 implies 2, we use forcing. We build a generic copy $\mathcal{A}^*$ of $\mathcal{A}$, with universe ω. The forcing conditions are finite partial permutations, which we think of as finite partial isomorphisms from $\mathcal{A}^*$ to $\mathcal{A}$. The forcing language describes the copy $\mathcal{A}^*$.

Let L be the predicate language of $\mathcal{A}$. We describe $\mathcal{A}^*$ using a propositional language PL in which the propositional variables are the atomic sentences of $L \cup \omega$. The propositional variable $\varphi(\bar{b})$ has the intended meaning $\mathcal{A}^* \models \varphi(\bar{b})$. Similarly, if $\gamma(\bar{b})$ is a finite Boolean combination of propositional variables, then $\gamma(\bar{b})$ says that $\mathcal{A}^* \models \gamma(\bar{b})$.

The forcing language includes computable Σ_α formulas $\psi_{e,n}$ with the meaning "n is in the set with Σ^0_α index e relative to $\mathcal{A}^*$". We describe $\psi_{e,n}$ in the special case where $\alpha = 1$. Here $\psi_{e,n}$ is a computable Σ_1 formula with the meaning: $n \in \varphi_e^{\mathcal{A}^*}$.

Let C be the set of halting computations c of φ_e, with input n, using oracle information α_c (positive) and β_c (negative), where α is a finite set of sentences $\pm\varphi$ for $\varphi \in PL$, and β is a finite set consisting of $\pm\varphi$ for $\varphi \in PL$ and other k, not in PL (the fact that such k are not in $D(\mathcal{A}^*)$ is always true). For each $c \in C$, let γ_c be the conjunction of the formulas

$$\begin{array}{rl} \pm\varphi & \text{for } \pm\varphi \in \alpha_c \\ \varphi & \text{for } \neg\varphi \in \beta_c \\ \neg\varphi & \text{for } \varphi \in \beta_c \end{array}$$

Then $\bigvee\!\!\!\bigvee_{c\in C} \gamma_c$ is the desired computable Σ_1 formula $\psi_{e,n}$ saying that $n \in W_e^{\mathcal{A}^*}$.

We define forcing in the usual way for the computable infinitary formulas of PL. We write $p \Vdash \varphi$ to indicate that p forces φ. Note that a computable Σ_0 or Π_0 formula of the forcing language is a finite Boolean combination of atomic sentences in the language L with constants from ω.

Definition 14.

1. For a finitary propositional formula φ, $p \Vdash \varphi$ if the constants in φ are all in $dom(p)$, and by interpreting those constants in $\mathcal{A}$, p makes φ true in $\mathcal{A}$.
2. For $\alpha > 0$, if φ is computable Σ_α, of the form $\bigvee\!\!\!\bigvee_i \psi_i$, where ψ_i is computable Π_β for some $\beta < \alpha$, then $p \Vdash \varphi$ if for some i, $p \Vdash \psi_i$.
3. For $\alpha > 0$, if φ is computable Π_α, of the form $\bigwedge\!\!\!\bigwedge_i \psi_i$, where ψ_i is computable Σ_β for some $\beta < \alpha$, then $p \Vdash \varphi$ if for all $q \supseteq p$, it is not the case that $q \Vdash neg(\varphi)$, where $neg(\varphi) = \bigvee\!\!\!\bigvee_i neg(\psi_i)$.

We have versions of the usual lemmas (as in Cohen [6]). These are all proved by induction on the formulas in the forcing language.

Lemma 6.2 (Extension). *If p forces φ and $q \supseteq p$, then q forces φ.*

Lemma 6.3 (Consistency). *For any p and φ, p does not force both φ and $neg(\varphi)$.*

Lemma 6.4 (Density). *For any p and φ, some $q \supseteq p$ forces either φ or $neg(\varphi)$.*

Definition 15 (R_a and D_e).

1. For each $a \in \omega$, let R_a be the set of forcing conditions q with $a \in ran(q)$.
2. Let D_e be the set of forcing conditions q that satisfy one of the following three conditions:

   (a) for some $n \notin S$, q forces $\psi_{e,n}$,
   (b) for some $n \in S$, q forces $neg(\psi_{e,n})$,
   (c) q has no extension satisfying (a) or (b) above.

Since S may be of arbitrary complexity, there may be no formula of our forcing language saying that S is the Σ^0_α set with index e relative to $\mathcal{A}^*$. Even so, we may think of Condition (c), in the definition of D_e as saying that q forces this. For q satisfying Condition (c), for each $n \in S$, there exists $r \supseteq q$ such that r forces $\psi_{e,n}$, and for each $n \notin S$, there is no such r.

Note. The sets D_e and R_a are *dense*; i.e., for any p, there exists $q \supseteq p$ such that q is in the set.

Definition 16. A *complete forcing sequence* is a sequence $(p_n)_{n\in\omega}$ of forcing conditions such that

1. $p_n \subseteq p_{n+1}$,
2. for each φ in the forcing language, there is some n such that $p_n \Vdash \varphi$ or $p_n \Vdash neg(\varphi)$,
3. for each a, there is some n such that $p_n \in R_a$,
4. for each e, there is some n such that $p_n \in D_e$.

We choose a complete forcing sequence $(p_n)_{n\in\omega}$. Let $F = \cup_n p_n$. It is easy to see that F is a permutation of ω. Let $\mathcal{A}^*$ be the structure induced by F such that $\mathcal{A}^* \cong_F \mathcal{A}$. Now, $\mathcal{A}^*$ is a predicate structure. There is a propositional structure consisting of the atomic sentences in $D(\mathcal{A}^*)$. We call this $\mathcal{A}^*$ as well.

We have two further lemmas (as in Cohen).

Lemma 6.5 (Truth and Forcing). *For each φ in the forcing language, $\mathcal{A}^* \models \varphi$ iff there is some n such that $p_n \Vdash \varphi$.*

Lemma 6.6 (Definability of Forcing). *For any formula φ of the forcing language, any tuple $\overline{b} = (b_1, \ldots, b_n)$ in ω, and corresponding tuple $\overline{x} = (x_1, \ldots, x_n)$ of variables, we can find a predicate formula $Force_{\overline{b},\varphi}(\overline{x})$ such that for any $\overline{a} = (a_1, \ldots, a_n)$, the following are equivalent:*

1. *$\mathcal{A} \models Force_{\overline{b},\varphi}(\overline{x})$,*
2. *the set p of pairs (b_i, a_i) is a forcing condition (i.e., $b_i = b_j$ iff $a_i = a_j$), and $p \Vdash \varphi$.*

We are about to complete the proof of the theorem. Since $\mathcal{A}^* \cong \mathcal{A}$, S is Σ^0_α relative to $\mathcal{A}^*$, say e is an index. Some $p = p_n$ is in D_e. Say p takes $\overline{d}$ to $\overline{c}$. Using Truth and Forcing, we see that Condition (c) (from the definition of D_e)must hold. Then for $n \in \omega$, we have $n \in S$ iff $(\exists q \supseteq p) q \Vdash \psi_{e,n}$. We let $\varphi_n(\overline{c})$ be a computable Σ_α sentence (in the predicate language) saying $(\exists q \supseteq p) q \Vdash \psi_{e,n}$. We take the disjunction over k, and $\overline{b} = (b_1, \ldots, b_k)$ of formulas saying that there exists $\overline{u} = (u_1, \ldots, u_n)$ such that $Force_{\overline{d},\overline{b},\psi_{e,n}}(\overline{c}, \overline{u})$. □

6.2. Further Results Obtained by Forcing

The next two results are from [1] and [4] (see also [2]).

Theorem 6.7. *Let $\mathcal{A}$ be a computable structure, and let R be a relation on $\mathcal{A}$. Then the following are equivalent:*

1. *For all isomorphisms F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is Σ^0_α relative to $\mathcal{B}$.*
2. *R is definable in $\mathcal{A}$ by a computable Σ_α formula $\varphi(\overline{c},\overline{x})$, where $\overline{c}$ is a finite tuple of parameters.*

Theorem 6.8. *Let $\mathcal{A}$ be a computable structure. Then the following are equivalent:*

1. *For all $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$ such that F is Δ^0_α relative to $\mathcal{B}$.*
2. *There is a tuple $\overline{c}$ of elements and there is c.e. set Φ of computable Σ^0_α formulas, with parameters $\overline{c}$, such that*
   (a) *each tuple in $\mathcal{A}$ satisfies some formula in Φ, and*
   (b) *if tuples $\overline{a}$ and $\overline{a}'$ satisfy the same formula from Φ, then there is an automorphism of $(\mathcal{A},\overline{c})$ taking $\overline{a}$ to $\overline{a}'$.*

7. Computable Abelian p-Groups

In this section, we look at three kinds of problems on computable structures.

1. For a mathematically interesting class of structures, which members have computable copies?
2. For a given structure $\mathcal{A}$, how difficult is it to distinguish between the computable copies of $\mathcal{A}$ and other computable structures?
3. For a class K of computable structures, closed under isomorphism, how hard is it to classify the computable elements, up to isomorphism?

There are many results on these problems. We consider the setting of Abelian p-groups. We say very little about the proofs.

7.1. Ulm's Theorem

We begin by recalling some algebra.

Definition 17. Let p be a fixed prime. An *Abelian p-group* is an Abelian group in which each non-zero element has order p^n, for some n.

The structure of countable Abelian p-groups is well-understood (see [14]). Let $\mathcal{G}$ be an Abelian p-group. We define a nested sequence of subgroups $\mathcal{G}_\alpha$, where

1. $\mathcal{G}_0 = \mathcal{G}$,
2. $\mathcal{G}_{\alpha+1} = p\mathcal{G}_\alpha$, and
3. for limit α, $\mathcal{G}_\alpha = \cap_{\beta<\alpha}\mathcal{G}_\beta$.

If $\mathcal{G}$ is a countable Abelian p-group, then there is a countable ordinal α such that $\mathcal{G}_\alpha = \mathcal{G}_{\alpha+1}$. The *length* of $\mathcal{G}$, denoted by $\lambda(\mathcal{G})$, is the first such α. An element a has *height* β if $a \in \mathcal{G}_\beta - \mathcal{G}_{\beta+1}$. The group $\mathcal{G}$ is *reduced* if every non-zero element has a height; i.e., if $\mathcal{G}_{\lambda(\mathcal{G})} = \{0\}$. Let $P = \{a \in \mathcal{G} : pa = 0\}$. Note that P, consisting of 0 and the elements of order p, is a subgroup of $\mathcal{G}$. For each $\beta < \lambda(\mathcal{G})$, $(\mathcal{G}_\beta \cap P)/(\mathcal{G}_{\beta+1} \cap P)$ is essentially a vector space over Z_p. Let $u_\beta(\mathcal{G})$ be the dimension. Then the *Ulm sequence* of $\mathcal{G}$ is the sequence $(u_\beta(\mathcal{G}))_{\beta<\lambda(\mathcal{G})}$.

Theorem 7.1 (Ulm). *If $\mathcal{G}$ and $\mathcal{G}'$ are countable reduced Abelian p-groups with the same Ulm sequence, then $\mathcal{G} \cong \mathcal{G}'$.*

An arbitrary countable Abelian p-group $\mathcal{G}$ can be expressed as a direct sum $\mathcal{G}_{\lambda(\mathcal{G})} \oplus \mathcal{G}_R$, where $\mathcal{G}_{\lambda(\mathcal{G})}$ consists of the elements that are infinitely divisible, and $\mathcal{G}_R$ is reduced. The first direct summand, $\mathcal{G}_{\lambda(G)}$ is unique as a set, and it is characterized up to isomorphism by a "dimension"—the maximal number of algebraically independent elements. The other direct summand, $\mathcal{G}_R$, is unique up to isomorphism, although not as a set. By Ulm's Theorem, the isomorphism type of $\mathcal{G}_R$ is determined by the Ulm sequence.

7.2. Results of Khisamiev

N. Khisamiev [16,17] characterized the reduced Abelian p-groups, of length $< \omega^2$, with computable copies. Suppose $\mathcal{G}$ is a computable reduced Abelian p-group. We can locate $\mathcal{G}_\omega$, $\mathcal{G}_{\omega \cdot n}$ in the arithmetical hierarchy.

Lemma 7.2. *If $\mathcal{G}$ is a computable reduced Abelian p-group, then $\mathcal{G}_\omega$ is Π^0_2, and in general, $\mathcal{G}_{\omega \cdot m}$ is Π^0_{2m}.*

Proof. We have $a \in \mathcal{G}_\omega$ iff for all n, there exists b such that $p^n b = a$. Therefore, $\mathcal{G}_\omega$ is Π^0_2. Similarly, $a \in \mathcal{G}_{\omega \cdot (m+1)}$ iff for all n, there exists b such that $p^n b = a$ and $b \in \mathcal{G}_{\omega \cdot m}$. If $\mathcal{G}_{\omega \cdot m}$ is Π^0_{2m}, then this is Π^0_{2m+2}. □

Lemma 7.3. *If $\mathcal{G}$ is a computable reduced Abelian p-group, then for each m, the relation $\{(n,k) : u_{\omega \cdot m+n}(\mathcal{G}) \geqslant k\}$ is Σ^0_{2m+2}.*

Proof. We have $u_{\omega \cdot m+n}(\mathcal{G}) \geqslant k$ iff there exist $a_1, \ldots, a_k \in \mathcal{G}_{\omega \cdot m+n} \cap P$ such that for all $z_1, \ldots, z_k \in Z_p$, not all 0, $z_1 a_1 \ldots z_k a_k \notin \mathcal{G}_{\omega \cdot m+n+1}$. □

Here is Khisamiev's result for groups of length ω.

Theorem 7.4 (Khisamiev). *Let $\mathcal{G}$ be a reduced Abelian p-group of length ω. Then $\mathcal{G}$ has a computable copy iff the following conditions hold.*

1. *the relation $R(\mathcal{G}) = \{(n,k) : u_n(\mathcal{G}) \geqslant k\}$ is Σ^0_2,*
2. *there is a computable function f such that for each n, $f(n,s)$ is non-decreasing in s, with limit $n^* \geqslant n$, where $u_{n^*}(\mathcal{G}) \neq 0$.*

Idea of proof. To construct a computable copy of $\mathcal{G}$, we guess the pairs in $R(\mathcal{G})$. The function $f(n,s)$ is used in taking care of mistakes. □

Here is Khisamiev's more general result.

Theorem 7.5 (Khisamiev). *Let $\mathcal{G}$ be a reduced Abelian p-group such that $\lambda(\mathcal{G}) < \omega^2$. Then $\mathcal{G}$ has a computable copy if and only if for each i such that $\omega \cdot (i+1) \leqslant \lambda(\mathcal{G})$, the following conditions hold:*

1. *$R_i(\mathcal{G}) = \{(n,k) : u_{\omega \cdot i+n}(G) \geqslant k\}$ is Σ^0_{2i+2},*
2. *there is a Δ^0_{2i+1} function f_i such that for all n, $f_i(n,s)$ is non-decreasing in s, with limit $n^* \geqslant n$, where $u_{\omega \cdot i+n^*}(\mathcal{G}) \neq 0$.*

Outline of proof. The proof is inductive. Groups of length ω are handled in Theorem 7.4. The following lemma (in relativized form) is used for the inductive step.

Lemma 7.6 (Khisamiev). *Suppose $\mathcal{G}$ is a reduced Abelian p-group such that*

1. *$\mathcal{G}_\omega$ has a Δ^0_3 copy,*
2. *the relation $R(\mathcal{G}) = \{(n,k) : u_n(\mathcal{G}) \geqslant k\}$ is Σ^0_2, and there is a computable function f such that for each n, $f(n,s)$ is non-decreasing in s, with limit $n^* \geqslant n$, where $u_{n^*}(\mathcal{G}) \neq 0$.*

Then $\mathcal{G}$ has a computable copy. □

7.3. Index Sets

For a structure $\mathcal{A}$, the *index set* for $\mathcal{A}$, denoted by $I(\mathcal{A})$, is the set of indices for computable copies of $\mathcal{A}$. Let K be a class of structures closed under isomorphism. The *index set* for K, denoted by $I(K)$, is the set of computable indices for members of K. Often we want to measure the difficulty in separating copies of $\mathcal{A}$ from other members of K. For $\mathcal{A} \in K$, we are interested in the complexity of $I(\mathcal{A})$ *within* $I(K)$.

Definition 18. Let Γ be a complexity class (such as Π^0_3). Let $A \subseteq B$.

1. A is Γ within B if there is a set $S \in \Gamma$ such that $A = B \cap S$.
2. A is Γ-hard within B if for any $S \in \Gamma$, there is a computable function $f : \omega \to B$ such that $n \in S$ iff $f(n) \in A$.
3. A is m-complete Γ within B if A is Γ within B and Γ-hard within B.

We state results from [5] on index sets for reduced Abelian p-groups of length $< \omega^2$. We write K_α for the class of groups of length α. We note that $I(K_\omega)$ is Π^0_3.

Proposition 7.7. *Let K_ω be the class of reduced Abelian p-groups of length ω, and let $\mathcal{G}$ be a computable member of K_ω. Then $I(\mathcal{G})$ is m-complete Π^0_3 within $I(K_\omega)$.*

Idea of proof. To show that $I(\mathcal{G})$ is Π^0_3 within $I(K_\omega)$, it is enough to give a computable Π^0_3 sentence σ such that for computable $\mathcal{C} \in K_\omega$, $\mathcal{C} \models \sigma$ iff $\mathcal{C} \cong \mathcal{G}$. To show that $I(\mathcal{G})$ is Π^0_3 hard within $I(K_\omega)$, we show that for any Π^0_3 set S, there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ of elements of K_ω such that $\mathcal{C}_n \cong \mathcal{G}$ iff $n \in S$. □

More generally, we have the following.

Theorem 7.8. *Let $\mathcal{G} \in K_{\omega\cdot M+N}$, where $M, N \in \omega$.*

1. *If $\mathcal{G}_{\omega M}$ is minimal for the given length; i.e., of the form $\mathbb{Z}_{p^N}$, then $I(\mathcal{G})$ is m-complete Π^0_{2M+1} within $I(K_{\omega\cdot M+N})$.*
2. *If $\mathcal{G}_{\omega M}$ is finite but not minimal for the given length, then $I(\mathcal{G})$ is m-complete d-Σ^0_{2M+1} (difference of Σ^0_{2M+1}) within $I(K_{\omega\cdot M+N})$.*
3. *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{G}) = \infty$, and for all $m < k$, $u_{\omega M+m}(\mathcal{G}) = 0$, then $I(\mathcal{G})$ is m-complete Π^0_{2M+2} within $I(K_{\omega\cdot M+N})$.*
4. *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{G}) = \infty$ and for some $m < k$ we have $0 < u_{\omega M+m}(\mathcal{G}) < \infty$, then $I(\mathcal{G})$ is m-complete d-Σ^0_{2M+2} within $I(K_{\omega\cdot M+N})$.*
5. *If there exist $m < k < N$ such that $u_{\omega M+m}(\mathcal{A}) = u_{\omega M+k}(\mathcal{A}) = \infty$, then $I(\mathcal{G})$ is m-complete Π^0_{2M+3} within $I(K_{\omega\cdot M+N})$.*

7.4. Isomorphism Problems

For a class K of structures, closed under isomorphism, the *computable isomorphism problem*, $E(K)$, is the set of pairs (a, b) such that a, b are computable indices for isomorphic members of K. The complexity of $E(K)$ is one possible measure of the difficulty of classifying computable members of K, up to isomorphism. We state results from [3] on reduced Abelian p-groups of various lengths. Again we let K_α be the class of groups of length α.

Proposition 7.9. *The isomorphism problem $E(K_\omega)$ is m-complete Π^0_3 within $I(K_\omega)^2$.*

Idea of proof. To see that $E(K_\omega)$ is Π^0_3 (within $I(K_\omega)^2$), note that if $\mathcal{G}$ and $\mathcal{G}'$ are reduced Abelian p-groups of length ω, then $\mathcal{G} \cong \mathcal{G}'$ iff for all pairs (n, k), we have $(n, k) \in R(\mathcal{G})$ iff $(n, k) \in R(\mathcal{G}')$. To see that $E(K_\omega)$ is Π^0_3-hard within $I(K_\omega)^2$, we note that for a particular computable group $\mathcal{G}$ in K_ω, say the one with $u_n(\mathcal{G}) = \infty$ for all n, $I(\mathcal{G})$ is Π^0_3-hard within $I(K_\omega)^2$. □

Here is Calvert's more general result.

Theorem 7.10 (Calvert). *Let $\hat{\alpha} = \sup_{\gamma<\alpha}(2\gamma + 3)$. Then $E(K_{\omega\cdot\alpha})$ is m-complete $\Pi^0_{\hat{\alpha}}$ within $I(K_{\omega\cdot\alpha})^2$.*

References

[1] C.J. Ash, J.F. Knight, M. Mannasse and T. Slaman, Generic copies of countable structures, *Annals of Pure and Applied Logic* **42** (1989) 195–205.

[2] C.J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, Elsevier, 2000.

[3] W. Calvert, The isomorphism problem for computable Abelian p-groups of bounded length, *J. Symb. Logic* **70** (2005) 331–345.

[4] J. Chisholm, Effective model theory versus recursive model theory, *J. Symb. Logic* **55** (1990) 1168–1191.

[5] W. Calvert, V.S. Harizanov, J.F. Knight and S. Miller, Index sets of computable structures, *Algebra and Logic* **45** (2006) 306–325.

[6] P.J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, 1966.

[7] M. Davis, *On the Theory of Recursive Unsolvability*, Ph.D. Thesis, Princeton University, 1950.

[8] M. Davis, Relative recursive functions and the extended Kleene hierarchy, *Proc. of the International Congress of Mathematicians, 1950, Cambridge, Mass., U.S.A*, vol. 1, 1952, p. 723.

[9] M. Davis, Hilbert's tenth problem is unsolvable, *Amer. Math. Monthly* **80** (1973) 233–269.

[10] Yu.L. Ershov, A hierarchy of sets, I, *Algebra and Logic* **7** (1968) 15–47 (English translation).

[11] Yu.L. Ershov, On a hierarchy of sets, II, *Algebra and Logic* **7** (1968) 47–74 (English translation).

[12] Yu.L. Ershov, On a hierarchy of sets, III, *Algebra and Logic* **9** (1970) 34–51 (English translation).

[13] J.P. Jones and Yu.V. Matijasevic, Proof of recursive unsolvability of Hilbert's tenth problem, *Amer. Math. Monthly* **98** (1991) 689–709.

[14] I. Kaplansky, *Infinite Abelian Groups*, revised ed., University of Michigan Press, 1969.

[15] H.J. Keisler, *Model Theory for Infinitary Logic*, North-Holland, 1971.

[16] N.G. Khisamiev, The arithmetic hierarchy of Abelian groups, *Sibirsk. Mat. Zh.* **29** (1988) 144–159.

[17] N.G. Khisamiev, Constructive Abelian groups, in: *Handbook of Recursive Mathematics*, vol. II, Ershov et al., eds, Elsevier, 1998, pp. 1177–1231.

[18] S.C. Kleene, Recursive predicates and quantifiers, *Trans. of the Amer, Math. Soc.* **53** (1943) 41–73.

[19] S.C. Kleene, Arithmetical predicates and function quantifiers, *Trans. of the Amer. Math. Soc.* **79** (1955) 312–340.

[20] J.F. Knight, Degrees coded in jumps of orderings, *J. Symb. Logic* **51** (1986) 1034–1042.

[21] A. Mostowski, On definable sets of positive integers, *Fund. Math.* **34** (1947) 81–112.

[22] A. Mostowski, A classification of logical systems, *Studia Philosophica* **4** (1951) 237–274.
[23] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967.
[24] R.I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, 1987.
[25] C. Spector, Recursive well-orderings, *J. Symb. Logic* **20** (1955) 151–163.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

A Model Theoretic Approach to Proof Theory of Arithmetic

Henryk KOTLARSKI
Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University, ul. Dewajtis 5, 01–815 Warszawa, Poland
hkl@impan.gov.pl

Abstract. I review Ratajczyk's version of model theoretic approach to proof theory for arithmetic. I develop the idea of the Hardy hierarchy and its application to construction of initial segments satisfying PA. As a sample of applications of these ideas I give a version of the Paris–Harrington theorem and a sketch of independence of Goodstein theorem.

Professor Andrzej Mostowski died in 1975, just before the breakthrough in the area of models of Peano arithmetic, PA. In the time when I knew him as a student he was interested in ω-models and β-models of second order arithmetic, this area has different problems and methods. But he was interested in models of PA, he showed his interest in private talks. It should be also noticed that one of his very influential papers [8] was inspired by a question attributed to G. Hasejaeger, the question whether PA has a model with nontrivial automorphisms.

In this paper I shall review an approach to proof theory for arithmetic due to one of the late Mostowski's students in Warsaw, Zygmunt Ratajczyk [33], and, at about the same time, Richard Sommer [36,37] in Berkeley. Both of them distilled these ideas from earlier papers, most notably [30,32,18] and [31]. I shall work merely below ε_0, despite the fact that the ideas were developed further, see [1,25,26] and [2]. Recently P. D'Aquino and J. Knight used Sommer's version of these ideas to obtain initial segments satisfying some elementarity condition, see [6].

I assume the reader to be familiar with some model theory for arithmetic, Kaye's book [17] is more than enough. Let me also note that Hájek and Pudlák [16] also prove consistency and reflection of PA by induction on ε_0, but their approach is different (they use Herbrand theorem, this approach goes back to [7,35] and [12], see also [13]). A more traditional approach is known in proof theory, from many papers in this direction let me cite only [38]. Let me also remark that the recent book [21] is entirely devoted to model–theoretic problems, and hence all the material on the border of proof theory and model theory is omitted.

I organized the paper as follows. In Section 1 I work out a system of notations for ordinals below ε_0. Of course, this material is well–known, the reader familiar with some proof theory for arithmetic may skip this part without troubles with reading further parts of the paper. In later part of Section 1 I work out a system of fundamental sequences to work with. In Section 2 I introduce the Hardy hierarchy and work out its properties.

In Section 3 I prove the main lemma on approximating functions. In Section 4 I state some partition results for the notion of largeness determined by the Hardy hierarchy. In Section 5 I give the idea (due to J. Paris) of an indicator. In Section 6 I work out a particular indicator given by some purely logical machinery. In Section 7 I work out transfinite induction in PA. In Section 8 I sketch a proof of the recursion theorem, which is needed to perform definitions by transfinite induction in PA. Once again, the material of these two sections is well known in proof theory. In Section 9 I prove totality of functions in the Hardy hierarchy. The main construction appears in Section 10. In this section I also derive some corollaries, in particular a version of the theorem of Paris–Harrington [32]. In the final Section 11 I sketch the idea of Goodstein sequences and sketch the proof of the main result of L. Kirby and J. Paris [20].

1. Ordinals Below ε_0

In this section I give a treatment of ordinals below ε_0. To be more specific, I develop directly a system of notations of ordinals below ε_0, see, e.g., [27] for a set–theoretic treatment. Moreover I work out some machinery, the so–called fundamental sequences, this will be needed in Section 2 to define the Hardy hierarchy of quickly growing functions and prove its main properties.

The idea is as follows. It is well known that the set of all polynomials with non negative integer coefficients is well ordered by the relation of domination between functions: $f \prec g \equiv \exists x\, \forall y > x\; f(y) < g(y)$. This is equivalent to the ordering obtained by comparing at first degrees of polynomials f and g, if they are equal, then one compares the coefficients at $x^{\deg(g)}$, etc. As a matter of fact we must work with a slightly more complicated family of functions. It is customary to denote the variable by ω in these considerations, thus we have in mind polynomials in variable ω.

We define two sequences Pol_n of families of functions and relations $\prec_n$ on Pol_n for natural n. We let Pol_0 be the family of all constant functions, where constants are non negative integers, and we let $\prec_0$ be the ordering of constant functions (the order between values). We let Pol_{n+1} be the set of all functions of the form

$$\omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_{r-1}} \cdot m_r + \beta,$$

where $\alpha_0, \ldots, \alpha_r$ are elements of $\mathrm{Pol}_n \setminus \mathrm{Pol}_{n-1}$, $\beta \in \mathrm{Pol}_n$, $\alpha_0 \succ_n \cdots \succ_n \alpha_r$ and $m_0 \neq 0, \ldots, m_r \neq 0$. We define the relation $\prec_{n+1}$ in the natural manner, that is $\alpha \succ_{n+1} \gamma$ if whenever α is written as above and

$$\gamma = \omega^{\gamma_0} \cdot b_0 + \cdots + \omega^{\gamma_t} \cdot b_t + \delta$$

then either there exists $i \leqslant r$ such that

$$\begin{aligned}&(\forall k < i\; \alpha_k = \gamma_k \;\&\; a_k = b_k) \;\&\; \big[\alpha_i \succ_n \gamma_i \vee (\alpha_i = \gamma_i \;\&\; a_i > b_i)\big]\\ &\quad \text{or} \quad (r = t \;\&\; \forall k \leqslant r\; \alpha_k = \beta_k \;\&\; b_k = a_k) \;\&\; \beta \succ_n \delta.\end{aligned}$$

Elements of Pol_n are called *polynomials of order* n. We shall treat elements of $\cup_{n\in\mathbb{N}}\mathrm{Pol}_n$ as ordinals, or, to be more specific, notations for ordinals. It is immediate to check that the set of ordinals (in the above sense) is well ordered by $\cup_{n\in\mathbb{N}} \prec_n$, its order type is denoted in the literature as ε_0. (The reader knowing some theory of well orderings will easily check that ε_0 is the least ordinal solution to the equation $\alpha = \omega^\alpha$.)

Of course, 0 is an ordinal (as a trivial polynomial). Every $0 < \alpha < \varepsilon_0$ may be written uniquely as

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_s} \cdot a_s \tag{1}$$

for some $\alpha_0 > \alpha_1 > \cdots > \alpha_s$ with $\alpha > \alpha_0$ and $a_0, \ldots, a_s \in \mathbb{N} \setminus \{0\}$. It is customary to refer to a_i's as to *coefficient*, and to α_i's as to *exponents* in the Cantor normal form of α. We shall refer to (1) as to the *Cantor normal form* of an ordinal below ε_0. (In more advanced theories of ordinals one works with ordinals greater than ε_0, then one has a similar Cantor normal form expansion, but one cannot require $\alpha > \alpha_0$.)

Observe that if α is written in its Cantor normal form (1) then all exponents may be written in their Cantor normal forms, etc. This process (when iterated sufficiently many times so that only natural numbers occur in the expansion) yields the *full* Cantor normal form of α.

The relation $\prec$ between ordinals written in the form (1) is given just by comparing exponents and coefficients. From now on we write the usual inequality sign $<$ rather than $\prec$ to denote inequality between ordinals below ε_0.

If $0 < \alpha < \varepsilon_0$ we denote by $\mathrm{LM}(\alpha)$ the *leftmost exponent* in its Cantor normal form, i.e., α_0 in (1). Similarly, by $\mathrm{RM}(\alpha)$ we denote the *rightmost exponent*, that is α_s in (1).

We say that the Cantor normal form (1) is *trivial* if $s = 0$ and $a_0 = 1$ (i.e., α is of the form ω^δ) and *nontrivial* otherwise.

We shall need several notions concerning ordinals below ε_0. The first two ones are those of a limit ordinal and successor. An ordinal $\alpha < \varepsilon_0$ is *successor* iff it is of the form $\beta + 1$ for some β. Observe that α is a successor iff in its normal form expansion (1), $\alpha_s = 0$. An ordinal α is *limit* iff it is not 0 and $\alpha_s > 0$. Observe that an ordinal is a successor (of β) iff $\alpha > \beta$ and there is no ordinal γ with $\beta < \gamma < \alpha$. Also, α is limit iff for every $\beta < \alpha$ there exists γ such that $\beta < \gamma < \alpha$.

We used above the addition $\beta + 1$. This is a general notion of *addition* of ordinals. It is defined by transfinite induction. We let $\alpha + 0 = \alpha$, $\alpha + 1$ is the successor of α (that is, if α is written in its normal form (1)), then the Cantor normal form of $\alpha + 1$ is the expansion of α followed by $1 = \omega^0$ (or if $\alpha_s = 0$ then the Cantor normal form of $\alpha + 1$ is the same as that of α but with $a_s + 1$ in place of a_s). If λ is limit then $\alpha + \lambda$ is the smallest ordinal greater than all $\alpha + \beta$ where $\beta < \lambda$. Observe that addition of ordinals is not commutative (e.g. $1 + \omega = \omega \neq \omega + 1$), but it is associative. Moreover if we are given two ordinals α, β then the Cantor normal form of $\alpha + \beta$ need not be the expression obtained from the expansion of α by writing the expansion of β as following the expansion of α. In order to specify the assumptions under which it is so we introduce the next notion. We write $\alpha \gg \beta$ if $\alpha = 0$ or $\beta = 0$ or all exponents in the Cantor normal form of α are $\geqslant$ all the exponents in the Cantor normal form of β. We remark that $\alpha \gg \beta$ does not imply $\alpha \geqslant \beta$, indeed, $\omega^\alpha \cdot m \gg \omega^\alpha \cdot k$ for all m, k.

We shall also use the relation $\alpha \ggg \beta$, which is defined like $\alpha \gg \beta$, but with strict inequality.

One defines also *multiplication* of ordinals by induction. We let $\alpha \cdot 0 = 0$, $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ and for limit λ, $\alpha \cdot \lambda =$ the smallest ordinal exceeding all $\alpha \cdot \beta$ for $\beta < \lambda$. We point out that addition and multiplication of ordinals are associative but not commutative.

One more remark should be added here. Every $0 < \alpha < \varepsilon_0$ may be written in the form

$$\alpha = \beta + \omega^\rho, \quad \text{where} \quad \beta \gg \omega^\rho. \tag{2}$$

This form is obtained from (1) by putting

$$\beta = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_{s-1}} \cdot a_{s-1} + \omega^{\alpha_s} \cdot (a_s - 1) \quad \text{and} \quad \rho = \alpha_s.$$

We shall refer to (2) as to the *short Cantor normal form* of α. Observe that the Cantor normal form of α is trivial iff $\beta = 0$ in (2), moreover if $\beta \neq 0$ then $\beta < \alpha$ and $\omega^\rho < \alpha$.

It will be convenient to have some more notation concerning ordinals. Let $\omega_0 = \omega$ and $\omega_{n+1} = \omega^{\omega_n}$. We shall also use the following generalization. We let $\omega_m(\alpha)$ the sequence of ordinals defined by induction in the following manner: $\omega_0(\alpha) = \alpha$, $\omega_{m+1}(\alpha) = \omega^{\omega_m(\alpha)}$. Thus, $\omega_0(\omega) = \omega$, more generally $\omega_m(\omega) = \omega_{m+1}$ and $\omega_m(1) = \omega_m$ in this notation. The sequence $\omega_m(\alpha)$ is defined, in particular, for finite α.

The next notion we need is that of a *fundamental sequence*. For every limit $\lambda < \varepsilon_0$ we choose a sequence $\{\lambda\}(n)$ converging to λ from below. We let $\{\omega\}(m) = m$, more generally $\{\omega^{\alpha+1}\}(m) = \omega^\alpha \cdot m$. For α limit we let $\{\omega^\alpha\}(m) = \omega^{\{\alpha\}(m)}$. Finally, if $\alpha = \varrho + \omega^\mu$ with $\varrho \gg \omega^\mu$ then we let $\{\alpha\}(m) = \varrho + \{\omega^\mu\}(m)$. It is easy to see that these conditions determine exactly one sequence $\{\lambda\}(m)$ for each limit $\lambda < \varepsilon_0$, the *fundamental sequence for* λ.

We shall need several other notions. We extend the notion of a fundamental sequence to non limit ordinals by putting $\{0\}(n) = 0$ and $\{\alpha + 1\}(n) = \alpha$.

Observe that $\{\alpha\}(b) < \alpha$ and whenever $\gamma \gg \alpha$ then $\gamma \gg \{\alpha\}(b)$ for all b. We shall use these observations without explicit mention.

For $\beta, \alpha < \varepsilon_0$ we write $\beta \to_n \alpha$ iff there exists a finite sequence $\alpha_0, \ldots, \alpha_k$ of ordinals such that $\alpha_0 = \beta$, $\alpha_k = \alpha$ and for every $m < k$ there exists $j_m \leqslant n$ such that $\alpha_{m+1} = \{\alpha_m\}(j_m)$. We write $\beta \Rightarrow_n \alpha$ if there exists a sequence as above, but with each $j_m = n$. Observe that both relations $\to_n, \Rightarrow_n$ are transitive and imply $\beta \geqslant \alpha$.

In the following lemma we collect the main properties of the relation $\Rightarrow_b$. The lemma is known from Ketonen and Solovay [18], they write that part 6 of it is related to some earlier work of Diana Schmidt.

Lemma 1.1. [18]

1. *For every* α, b $\alpha \Rightarrow_b 0$.
2. *If* $\beta \gg \alpha$ *and* $\alpha \Rightarrow_b \gamma$ *then* $\beta + \alpha \Rightarrow_b \beta + \gamma$.
3. *If* $k < l$ *and* $b \geqslant 0$ *then* $\omega^\alpha \cdot l \Rightarrow_b \omega^\alpha \cdot k$.
4. *If* $\beta \Rightarrow_b \alpha$ *and* $b > 0$ *then* $\omega^\beta \Rightarrow_b \omega^\alpha$.
5. $\alpha \Rightarrow_b \{\alpha\}(j)$ *and* $\{\alpha\}(b) \Rightarrow_b \{\alpha\}(j)$ *for* $j \leqslant b$.
6. $\{\alpha\}(b) \Rightarrow_1 \{\alpha\}(j)$ *for* $0 < j \leqslant b$.
7. *If* $n \leqslant b$ *and* $\alpha \Rightarrow_n \beta$ *then* $\alpha \Rightarrow_b \beta$.
8. $\beta \Rightarrow_n \alpha$ *iff* $\beta \to_n \alpha$.
9. *If* $\alpha < \beta$ *then there exists* b *such that* $\beta \Rightarrow_b \alpha$.

Proof. Part 1 is immediate by induction on α. Indeed, for $\alpha = 0$ there is nothing to prove, if $\alpha = \beta + 1$ and $\beta_0, \ldots, \beta_{r-1}$ is a sequence witnessing the relation $\beta \Rightarrow_b 0$ then the sequence $\alpha, \beta_0, \ldots, \beta_{r-1}$ witnesses $\beta + 1 \Rightarrow_b 0$. If α is limit then $\alpha, \{\alpha\}(b)$, the sequence witnessing $\{\alpha\}(b) \Rightarrow_b 0$ does the job.

For part 2 let $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ be the sequence witnessing $\alpha \Rightarrow_b \gamma$. Then it is easy to see that the sequence $\beta + \alpha_0, \beta + \alpha_1, \ldots, \beta + \alpha_{r-1}$ witnesses $\beta + \alpha \Rightarrow_b \beta + \gamma$.

For part 3 let α be fixed. Then $\{\omega^\alpha \cdot l\}(b) = \omega^\alpha \cdot (l-1) + \{\omega^\alpha\}(b)$. By part 1 there exists a sequence $\gamma_0, \ldots, \gamma_{r-1}$ witnessing the relation $\{\omega^\alpha\}(b) \Rightarrow_b 0$. By part 2, the sequence $\omega^\alpha \cdot (l-1) + \gamma_0, \ldots, \omega^\alpha \cdot (l-1) + \gamma_{r-1}$ witnesses the relation $\omega^\alpha \cdot l \Rightarrow_b \omega^\alpha \cdot (l-1)$. Thus, if $k = l-1$, we are done. Otherwise we continue the same procedure, that is we go down to $\omega^\alpha \cdot (l-2)$ etc.

Let us go to the next part, i.e., to 4. Let $\beta_0, \ldots, \beta_{r-1}$ be the sequence witnessing the relation $\beta \Rightarrow_b \alpha$. Consider the sequence $\omega^{\beta_0}, \ldots, \omega^{\beta_{r-1}}$. We shall put between items of this sequence many more items to obtain the right one. Consider two consecutive elements of this sequence. These elements look like $\omega^\gamma, \omega^{\{\gamma\}(b)}$ for some γ. If γ is limit then we do not put anything between these two items. Otherwise $\gamma = \delta + 1$ for some δ. Then $\{\omega^{\delta+1}\}(b) = \omega^\delta \cdot b$. We put between the two items all the sequence witnessing the relation $\omega^\delta \cdot b \Rightarrow_b \omega^\delta$, its existence follows from parts 1 and 3. Clearly, the sequence obtained in this manner has the required properties.

Let us go to the main part of the lemma, i.e, to 5. Obviously, it suffices to prove the second claim. We proceed by induction on α. If $\alpha = 0$ there is nothing to prove (one element sequence works). Also the nonlimit step is trivial. So assume that α is limit and the claim holds for all $\beta < \alpha$. If α admits a nontrivial Cantor normal form then we may write $\alpha = \delta + \beta$, where $\beta < \alpha$ and $\delta \gg \beta$. By the inductive assumption there exists a sequence $\beta_0, \ldots, \beta_{r-1}$ witnessing the relation $\{\beta\}(b) \Rightarrow_b \{\beta\}(j)$. Then the sequence $\delta + \beta_0, \ldots, \delta + \beta_{r-1}$ witnesses $\{\alpha\}(b) \Rightarrow_b \{\alpha\}(j)$ by part 2. So let us go to the case when the Cantor normal form of α is trivial, $\alpha = \omega^\delta$. If δ is nonlimit, $\delta = \rho + 1$ we must show $\{\omega^{\rho+1}\}(b) = \omega^\rho \cdot b \Rightarrow_b \{\omega^{\rho+1}\}(j) = \omega^\rho \cdot j$. This is a direct consequence of part 3. Finally, let δ be limit. Then by the inductive assumption $\{\delta\}(b) \Rightarrow_b \{\delta\}(j)$ and we may apply part 4.

The proof of part 6 is very similar to that of part 5, so is left to the reader.

Let us go to the next item of the lemma, i.e., to part 7. Let $T(\alpha)$ denote the property[1]

$$\forall \beta, n, b\big[(n \leqslant b \,\&\, \alpha \Rightarrow_n \beta) \Rightarrow (\alpha \Rightarrow_b \beta)\big]$$

and we prove $\forall \alpha < \varepsilon_0\, T(\alpha)$ by induction on α. If $\alpha = 0$ then $\beta = 0$ and the sequence with only one item equal 0 witnesses the relation $\alpha \Rightarrow_b \beta$. Also the implication $T(\alpha) \Rightarrow T(\alpha+1)$ is easy. Indeed, if $\alpha + 1 \Rightarrow_n \beta$ then $\beta \leqslant \alpha + 1$. If $\beta = \alpha + 1$ then one element sequence with the only item $\alpha + 1$ works, otherwise $\beta \leqslant \alpha$ and $\alpha \Rightarrow_n \beta$. We apply the inductive assumption to α. So let us concentrate at the limit step. Thus, let α be limit and assume the conclusion for all ordinals strictly smaller than α. We write both of these ordinals in their Cantor normal forms:

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_s} \cdot a_s$$

and

$$\beta = \omega^{\beta_0} \cdot b_0 + \cdots + \omega^{\beta_t} \cdot b_t.$$

By $\alpha \Rightarrow_n \beta$ we have $\alpha \geqslant \beta$. If they are equal then one element sequence works, so assume that $\beta < \alpha$ and consider two cases.

[1]The $\Rightarrow$ sign without index denotes implication; the relation $\Rightarrow_b$ is as above. I am taking notation from different sources, I hope that this will not lead misunderstanding.

Case 1. $s > t \,\&\, \forall i \leqslant t[a_i = b_i \,\&\, \alpha_i = \beta_i]$, that is $\alpha = \beta + \delta$ for some $\delta \ll \beta$, $\delta \neq 0$. By part 1 $\delta \Rightarrow_b 0$, hence $\alpha = \beta + \delta \Rightarrow_b \beta$ by part 2.

Case 2. $\exists j \leqslant s\{(\forall k < j\, a_k = b_k \,\&\, \alpha_k = \beta_k) \,\&\, [\alpha_j > \beta_j \vee (\alpha_j = \beta_j \,\&\, a_j > b_j)]\}$. Fix such j. Let δ be the common initial part of Cantor normal forms of α and β. Thus we may write

$$\alpha = \delta + \omega^{\alpha_j} \cdot a_j + \cdots + \omega^{\alpha_s} \cdot a_s$$

and

$$\beta = \delta + \omega^{\beta_j} \cdot b_j + \cdots + \omega^{\beta_t} \cdot b_t.$$

We may assume that $\alpha_j > \beta_j$, otherwise we shall move the part $\omega^{\alpha_j} \cdot (a_j - b_j)$ to the left (i.e., we change δ). Consider the sequence $\gamma_0, \ldots, \gamma_{r-1}$ witnessing the relation $\alpha \Rightarrow_n \beta$. It is determined uniquely by conditions $\gamma_0 = \alpha$, $\gamma_{i+1} = \{\gamma_i\}(n)$ for $i < r - 1$. It is strictly decreasing and $\gamma_{r-1} = \beta$. It follows that $\gamma_i = \delta + \rho_i$ for some ρ_i. The sequence $\rho_0, \ldots, \rho_{r-1}$ witnesses the relation

$$\omega^{\alpha_j} \cdot a_j + \cdots + \omega^{\alpha_s} \cdot a_s \Rightarrow_n \omega^{\beta_j} \cdot b_j + \cdots + \omega^{\beta_t} \cdot b_t.$$

It follows that since these ordinals are strictly smaller than α and β respectively (i.e., if $\delta \neq 0$), we may apply the inductive assumption and get a sequence $\xi_0, \ldots, \xi_{m-1}$ witnessing the relation

$$\omega^{\alpha_j} \cdot a_j + \cdots + \omega^{\alpha_s} \cdot a_s \Rightarrow_b \omega^{\beta_j} \cdot b_j + \cdots + \omega^{\beta_t} \cdot b_t.$$

Then the sequence $\delta + \xi_0, \ldots, \delta + \xi_{m-1}$ witnesses the relation $\alpha \Rightarrow_b \beta$. Thus it remains to show what to do if $\delta = 0$, i.e., $\alpha_0 > \beta_0$. We may assume that $a_0 = 1$, for the sequence $\gamma_0, \ldots, \gamma_{r-1}$ witnessing the relation $\alpha \Rightarrow_n \beta$ must go thru ω^{α_0}, hence $\alpha \Rightarrow_n \omega^{\alpha_0} \Rightarrow_n \beta$ and we may apply the inductive assumption in both places to change to $\Rightarrow_b$ (if $b = 0$ then we apply part 3). So assume that $\alpha = \omega^{\alpha_0}$. If $\alpha_0 = \zeta + 1$ then $\alpha \Rightarrow_b \omega^{\zeta} \cdot b$, so by part 3 $\alpha \Rightarrow_b \omega^{\zeta} \cdot n$ and we may apply the inductive assumption $T(\omega^{\zeta} \cdot n)$. So assume that α_0 is limit. Consider the sequence $\gamma_0, \ldots, \gamma_{r-1}$ witnessing the relation $\alpha \Rightarrow_n \beta$. Thus, $\gamma_1 = \omega^{\{\alpha_0\}(n)}$. Let $\delta_0 = \omega^{\alpha_0}$, $\delta_1 = \omega^{\{\alpha_0\}(b)}$. Further part of the required sequence is the whole sequence witnessing the relation $\omega^{\{\alpha_0\}(b)} \Rightarrow_b \omega^{\{\alpha_0\}(n)}$, whose existence is a consequence of parts 5 and 4. Further part of the required sequence δ_i is given by the inductive assumption, applied to obtain the sequence witnessing the relation $\omega^{\{\alpha_0\}(n)} \Rightarrow_b \beta$.

Part 8 is now immediate by part 7. Indeed, one direction is immediate (one may take each $j_m = n$), for the converse we observe that we may increase each j_m if necessary using part 7.

For part 9 we proceed by induction on β. Thus the induction thesis is $T(\beta)$:

$$\forall \alpha < \beta \,\exists b \,\beta \Rightarrow_b \alpha,$$

and we prove $\forall \beta < \varepsilon_0\, T(\beta)$. If $\beta = 0$ then the conclusion holds vacuously (there is no α to consider). Also the nonlimit step is immediate. Let β be limit and assume the conclusion for all ordinals smaller than β. Pick $\alpha < \beta$, then $\alpha < \{\beta\}(n)$ for some n. By $T(\{\beta\}(n))$ there exists d such that $\{\beta\}(n) \Rightarrow_d \alpha$. By part 7 $b = \max(n, d)$ has the required property. □

It turns out that part 9 of Lemma 1.1 may be strengthened considerably, namely the b to be found may be chosen to depend only on α (see Lemma 1.3). But in order to obtain this strengthening we need an auxiliary notion.

Let $\alpha < \varepsilon_0$. We define *pseudonorm* of α as the greatest natural number which occurs in its (full) Cantor normal form. Technically we define the function psn sending ordinals below ε_0 into $\mathbb{N}$ by putting $\mathrm{psn}(n) = n$ for $n < \omega$ and

$$\mathrm{psn}(\alpha) = \max\bigl(\mathrm{psn}(\alpha_0), \ldots, \mathrm{psn}(\alpha_s), a_0, \ldots, a_s\bigr)$$

where α is written in its Cantor normal form (1). (We write "pseudonorm" because Ketonen and Solovay [18] use a slightly another function, the *norm* of α, for similar purpose.)

The following properties of pseudonorm are easy:

1. If $\mu \gg \nu$ then $\mathrm{psn}(\mu + \nu) \geqslant \mathrm{psn}(\mu), \mathrm{psn}(\nu)$,
2. $\mathrm{psn}(\omega^\rho) = \mathrm{psn}(\rho)$

We ask the reader to check these properties. We merely point out that the inequality in point 1 may be strict (and if this happens then $\mathrm{RM}(\mu) = \mathrm{LM}(\nu)$).

Lemma 1.2.

1. *If* $\alpha < \beta < \varepsilon_0$, $m \geqslant 2$, $\mathrm{psn}(\alpha) < m$ *and* β *is limit then* $\alpha \leqslant \{\beta\}(m)$.
2. *For every* α *and every* β, *if* $\mathrm{RM}(\beta) > \alpha$ *and* $a = \mathrm{psn}(\alpha)$ *then* $\{\beta\}(a) \gg \omega^\alpha$ *and* $\{\beta\}(a) + \omega^\alpha < \beta$.

Proof. The first part is proved by induction on β. That is we denote by $T(\beta)$ the assertion

$$\forall \alpha < \beta\, \forall m \geqslant 2\bigl[\mathrm{psn}(\alpha) < m \,\&\, \mathrm{Lim}(\beta) \Rightarrow \alpha \leqslant \{\beta\}(m)\bigr]$$

and prove it by transfinite induction.

The first limit ordinal is ω. But if $\beta = \omega$ and $\mathrm{psn}(\alpha) \leqslant m$ then $\alpha \leqslant m = \{\beta\}(m)$ and we are done.

Let β be limit and assume the conclusion for each ordinal $< \beta$. Let $\alpha < \beta$. We select the (leftmost) common parts of Cantor normal forms of both of these ordinals, i.e., we write $\alpha = \delta + \omega^\gamma + \mu$ and $\beta = \delta + \omega^\xi + \nu$, where $\mu \ll \omega^\gamma \ll \delta$ and $\nu \ll \omega^\xi \ll \delta$. We have $\gamma < \xi$. If $\delta \neq 0$ then we may apply directly the inductive assumption to $\beta' = \omega^\xi + \nu$ (and $\alpha' = \omega^\gamma + \mu$). So assume that $\delta = 0$, that is $\beta = \omega^\xi + \nu$ and $\alpha = \omega^\gamma + \mu$. If $\nu \neq 0$ then $\{\beta\}(m) = \omega^\xi + \{\mu\}(m) > \alpha$ because $\xi > \gamma$. So assume that $\nu = 0$, i.e., $\beta = \omega^\xi$ and $\alpha = \omega^\gamma + \mu$. Then $\xi > \gamma$ because $\beta > \alpha$. If ξ is nonlimit, say $\xi = \zeta + 1$ then $\{\beta\}(m) = \omega^\zeta \cdot m > \omega^\gamma + \mu = \alpha$ (recall that $m \geqslant 2$), so assume that ξ is limit. But then $\gamma < \{\xi\}(m)$ by the inductive assumption $T(\xi)$. It follows that $\{\omega^\xi\}(m) = \omega^{\{\xi\}(m)} > \omega^\gamma + \mu = \alpha$ because $\omega^\gamma \gg \mu$ and we are done.

We prove the second part. We write $\beta = \delta + \omega^{\beta'}$ where $\delta \gg \omega^{\beta'}$. By the assumption, $\beta' > \alpha$. If $\beta' = \beta'' + 1$ then $\{\beta\}(a) = \delta + \omega^{\beta''} \cdot a$ with $\beta'' \geqslant \alpha$, so the first conclusion is immediate. The second one follows from the fact that in the decisive step the exponent β' was changed to the smaller one, i.e. β''. If β' is limit then $\{\delta + \omega^{\beta'}\}(a) = \delta + \omega^{\{\beta'\}(a)}$. By part 1 $\{\beta'\}(a) \geqslant \alpha$, so the first conclusion holds. The second does as well because in the decisive step the exponent β' was lowered to $\{\beta'\}(a)$. $\square$

The following lemma is the promised strengthening of part 9 of Lemma 1.1.

Lemma 1.3. *If* $\alpha < \beta < \varepsilon_0$, $b \geqslant 1$, $\mathrm{psn}(\alpha) \leqslant b$ *then* $\beta \Rightarrow_b \alpha$ *and* $\{\beta\}(b) \Rightarrow_b \alpha$.

Proof. By induction on β. Thus the inductive thesis is $T(\beta)$:

$$\forall\alpha, b\big[\big(b \geqslant 1 \,\&\, \alpha < \beta \,\&\, \mathrm{psn}(\alpha) \leqslant b\big) \Rightarrow (\beta \Rightarrow_b \alpha)\big].$$

For $\beta = 0$ there is nothing to prove. Assume the conclusion for β and let $\alpha < \beta + 1$. Then $\alpha = \beta$ or $\alpha < \beta$. If $\alpha = \beta$ then the one element sequence, whose only item is β, works, and if $\alpha < \beta$ then we take the sequence witnessing the relation $\{\beta\}(b) \Rightarrow_b \alpha$ and add to it β as the first item.

Let β be limit and assume the conclusion for all $\beta' < \beta$. Pick α, b such that $b > 1$, $\alpha < \beta$ and $\mathrm{psn}(\alpha) \leqslant b$. Then $\alpha \leqslant \{\beta\}(b)$ by Lemma 1.2, so we may apply the assumption to $\beta' = \{\beta\}(b)$. We add to the sequence obtained in this manner the item $\{\beta\}(b)$ as the first item. $\square$

2. Hardy Hierarchy

In this section we describe the Hardy hierarchy and the notion of largeness determined by it. The technical version of our considerations were distilled from the material of Ketonen–Solovay [18] by Zygmunt Ratajczyk [33], see also [25] and Ratajczyk's final [34]. But the notion of Hardy hierarchy is much older, see, e.g. [9] or [10] for an exposition and history.

Let h be a finite increasing function (in the usual sense of the word, that is $\forall x, y \in \mathrm{Dom}(h)[x < y \Rightarrow h(x) < h(y)]$). Assume moreover that it increases the argument, i.e. $\forall x\, x < h(x)$. As an example, let $A \subseteq \mathbb{N}$ and let $h = h^A$ be the successor in the sense of A, i.e., the function defined on A if A is infinite and defined on $A \setminus \{\max(A)\}$ if A is finite, which associates with every a in its domain the next element of A.

For every $\alpha < \varepsilon_0$ we define a function h_α, by induction on α. We put $h_0(x) \simeq x$ and $h_{\alpha+1}(x) \simeq h_\alpha(h(x))$. We let $h_\lambda(x) \simeq h_{\{\lambda\}(x)}(x)$ for λ limit. We point out that the domain of h is a subset of $\mathbb{N}$, which may be finite or infinite. The family $h_\alpha : \alpha < \varepsilon_0$ is called *Hardy hierarchy* based on h. We remark that Odifreddi [29], vol. 2 p. 308 calls it *moderate growing hierarchy* (and uses another notation). As usual, $\simeq$ means "either both sides are undefined or both are defined and equal", but we shall use directly the equality sign in this situation.

In order to see the exact mechanism of Hardy hierarchy, the reader is suggested to check the following. Let $h(x) = S(x) = x + 1$ be the usual successor function. Then $S_{\omega\cdot n}(x) = 2^n \cdot x$, and, hence, $S_{\omega^2}(x) = 2^x \cdot x$. Moreover one should write the formula for $S_{\omega^3}(x)$ in order to see that this function grows quicker than $\beth_x(x)$, where the *iterated exponentiation* $\beth_i(x)$ is defined by $\beth_0(x) = x$, $\beth_{i+1}(x) = 2^{\beth_i(x)}$.

The notion of Hardy hierarchy allows us to define a set A of natural numbers to be *α-large*. That is A is α-large iff $(h^A)_\alpha(a)$ is defined, where h^A denotes the successor in the sense of A (i.e., the function with domain $A \setminus \{\max(A)\}$ which associates with every b in its domain the next element of A) and $a = \min(A)$. We shall write just h if the meaning of A is clear from the context.

A set A is *α-small* if it is not α-large. It will be also convenient to say that a set A is *exactly α-large* if it is α-large but $A \setminus \{\max(A)\}$ is α-small. We shall also say that A is *at most α-large* if it is either α-small or exactly α-large.

One can restate the definition of largeness in the following manner. A set A is 0-large iff it is nonempty. A is $\alpha+1$-large iff $A \setminus \{\min(A)\}$ is α-large. A is λ-large, λ limit, iff it is $\{\lambda\}(\min(A))$-large. We remark that Ketonen and Solovay [18] use a slightly different notion of largeness. See also [23] for some similar notions of largeness.

The following proposition shows that this notion of largeness has the expected property.

Proposition 2.1. *For every $\alpha < \varepsilon_0$ and every infinite $A \subseteq \mathbb{N}$ there exists $u \in A$ such that $\{a \in A : a \leqslant u\}$ is α-large.*

Proof. By induction on α. For $\alpha = 0$ $u = a_0 = \min(A)$ satisfies our demand. Assume the lemma for α. Let A be infinite. Then $A \setminus \{\min(A)\}$ is infinite. Let $u \in A \setminus \{\min(A)\}$ be such that $\{a \in A \setminus \{\min(A)\} : a \leqslant u\}$ is α-large. Then u satisfies our demand for the set A and $\alpha + 1$. We leave the limit step to the reader. □

Below we write $f(x)\downarrow$ for "$f(x)$ is defined", i.e., $x \in \mathrm{Dom}(f)$.

Lemma 2.2. *Let h be a function as above. Then for every $\alpha < \varepsilon_0$*

1. *h_α is increasing.*
2. *For every β, b if $\alpha \Rightarrow_b \beta$ then if $h_\alpha(b)$ exists then $h_\beta(b)$ exists and $h_\alpha(b) \geqslant h_\beta(b)$.*

Proof. By simultaneous induction on α. For $\alpha = 0$ both claims are evident. Assume both claims for α, we prove them for $\alpha+1$. Then $h_{\alpha+1} = h_\alpha \circ h$ is increasing as a composition of two increasing functions. Further, if $\alpha + 1 \Rightarrow_b \beta$ and $h_{\alpha+1}(b)\downarrow$, then $\alpha + 1 \Rightarrow_{h(b)} \beta$ by Lemma 1.1 part 7, and $h_\alpha(h(b))\downarrow$. By inductive assumption for α and $h(b)$ we infer that $h_\beta(h(b))\downarrow$, so also $h_\beta(b)\downarrow$.

Assume both claims for all $\alpha' < \alpha$, where α is limit. We check firstly the second claim. Pick β, b such that $\alpha \Rightarrow_b \beta$ and $h_\alpha(b)\downarrow$. Then $h_{\{\alpha\}(b)}(b)\downarrow$ and $h_{\{\alpha\}(b)}(b) \geqslant h_\beta(b)$ by part 2 applied to $\alpha' = \{\alpha\}(b)$. Let us show part 1 for α. So let $x < y$. Then $\{\alpha\}(y) \Rightarrow_y \{\alpha\}(x)$ by Lemma 1.1, part 5, hence

$$h_\alpha(y) = h_{\{\alpha\}(y)}(y) \geqslant h_{\{\alpha\}(x)}(y)$$

by part 1, and this expression is $\geqslant h_{\{\alpha\}(x)}(x) = h_\alpha(x)$ because $h_{\{\alpha\}(x)}$ is increasing by the inductive assumption. □

Below if we write $A = \{a_0, \ldots, a_{\mathrm{Card}(A)-1}\}$ we assume that this enumeration is the natural one, i.e., in increasing order.

Lemma 2.3.

1. *For every α if A, B are finite sets of the same cardinality and such that for every $i < \mathrm{Card}(A)$ $b_i \leqslant a_i$ then for every $i < \mathrm{Card}(A)$ if $(h^A)_\alpha(a_i)$ exists then $(h^B)_\alpha(b_i)$ exists and $(h^A)_\alpha(a_i) \geqslant (h^B)_\alpha(b_i)$.*
2. *If A, B are finite sets, A is α-large, $\mathrm{Card}(A) = \mathrm{Card}(B)$ and for every $i < \mathrm{Card}(A)$ $b_i \leqslant a_i$ then B is α-large.*
3. *If $A \subseteq B$ and A is α-large then B is α-large.*

Proof. Part 1 is immediate by induction on α and is left to the reader. The second part is a direct consequence of the first one. The third part follows from the observation that if $A \subseteq B$ then B has an initial segment of cardinality $\mathrm{Card}(A)$. But obviously, if a set has an α-large initial segment then it is α-large itself, so the second part may be applied. □

The following is a minor variant of Lemma 2.3 in which we speak of sets of different cardinalities.

Lemma 2.4. *For every α and every D, E, if $D \subseteq E$, $x \in D$ and $(h^D)_\alpha(x)\downarrow$ then $(h^E)_\alpha(x)\downarrow$ and $(h^E)_\alpha(x) \leqslant (h^D)_\alpha(x)$.*

Proof. By induction on α. If $\alpha = 0$ the conclusion is obvious. Assume the conclusion for α, we derive it for $\alpha + 1$. So let D, E satisfy the assumption. Let $x \in D$ be such that $(h^D)_{\alpha+1}(x)$ exists. Then $(h^D)_{\alpha+1}(x) = (h^D)_\alpha((h^D)(x))$. Let $y = (h^D)(x)$. We apply the inductive assumption to y. Thus we infer $(h^E)_\alpha(y)\downarrow$ and $(h^E)_\alpha(y) \leqslant (h^D)_\alpha(y)$. But $(h^E)(x) \leqslant (h^D)(x) = y$, hence $(h^E)_{\alpha+1}(x) = (h^E)_\alpha((h^E)(x)) \leqslant (h^E)_\alpha((h^D)(x)) \leqslant (h^D)_\alpha((h^D)(x)) = (h^D)_{\alpha+1}(x)$ because $(h^E)_\alpha$ is increasing by Lemma 2.2. We leave the limit step to the reader. □

The following lemma is, up to our knowledge, due to S. Wainer.

Lemma 2.5 (On composition). *Let h be as above. Then for every α and every $\beta \gg \alpha$ $h_{\beta+\alpha} = h_\beta \circ h_\alpha$.*

Proof. By induction on α. For $\alpha = 0$ there is nothing to prove. Assume the conclusion for α. Let $\beta \gg \alpha + 1$. Then $\beta \gg \alpha$, hence

$$\begin{aligned} h_{\beta+(\alpha+1)} &= h_{(\beta+\alpha)+1} = h_{\beta+\alpha} \circ h = (h_\beta \circ h_\alpha) \circ h = h_\beta \circ (h_\alpha \circ h) \\ &= h_\beta \circ h_{\alpha+1}. \end{aligned}$$

Assume the conclusion for all $\alpha' < \alpha$, where α is limit. Let $\beta \gg \alpha$. Then $\beta \gg \{\alpha\}(b)$ for all b. It follows that

$$h_{\beta+\alpha}(b) = h_{\beta+\{\alpha\}(b)}(b) = h_\beta \circ h_{\{\alpha\}(b)}(b) = h_\beta \circ h_\alpha(b)$$

as desired. □

Let us restate this fact in the following manner.

Lemma 2.6. *Let A be a finite set and let $\beta \gg \alpha$. Then A is $\beta + \alpha$-large iff there exists $u \in A$ such that $\{x \in A : x \leqslant u\}$ is α-large and $\{x \in A : u \leqslant x\}$ is β-large.*

Observe that it may happen that $h_\alpha(b) = h_\beta(b)$ also if $\alpha \neq \beta$, e.g., $h_\omega(b) = h_b(b)$. It will be convenient to be able to choose some ordinal uniquely in such situations. This is done as follows. Let a set A be given. Let $\mu < \varepsilon_0$. We define two sequences μ_j, b_j by the following induction. We let $\mu_0 = \mu$ and $b_0 = a_0 = \min(A)$. Assume that μ_j and b_j are constructed. If $\mu_j = 0$ then the construction terminates. If $\mu_j > 0$ and μ_j is limit we let $\mu_{j+1} = \{\mu_j\}(b_j)$ and $b_{j+1} = b_j$. If μ_j is nonlimit then the construction terminates if $b_j = a_{r-1} = \max(A)$, otherwise we let $\mu_{j+1} = \mu_j - 1$ and $b_{j+1} = h^A(b_j)$, the next element of A. This completes the definition of the sequences μ_j, b_j.

Observe that the sequence μ_j is decreasing. We remark also that the construction terminates in two cases: $\mu_j = 0$ or μ_j is nonlimit and $b_j = \max(A)$. It should be also noticed that in the process of this construction we passed from b_j to the next element of A only if μ_j is nonlimit.

The following proposition is (essentially) the original definition of a μ-large set, cf. Ketonen–Solovay [18].

Proposition 2.7. *Under the notation introduced above, A is μ-large iff there exists j such that $\mu_j = 0$.*

Proof. We claim that the following are equivalent:

1. A is μ-large,
2. for every j the set $\{x \in A : b_j \leqslant x\}$ is μ_j-large,
3. for some j the set$\{x \in A : b_j \leqslant x\}$ is μ_j-large.

Both nonobvious implications are proved by induction on j. They follow immediately from the definitions and are left to the reader.

Granted the claim we see that if A is μ-large and no $\mu_j = 0$ then the construction cannot terminate. Indeed, let μ_t, b_t be its last item. Consider the sequence $\gamma_0 = \mu_t$, $\gamma_{i+1} = \{\gamma_i\}(b_t)$ till we get a nonlimit γ_i. Then the set $\{x \in A : b_i \leqslant x\}$ is γ_i-large, where $\gamma_i > 0$, so it has an element $> b_t$ and we have b_{j+1} and μ_{j+1}. But this is impossible as the sequence μ_j would be then a decreasing sequence of ordinals.

If A is μ-small and $\mu_j = 0$ for some j then the set $\{x \in A : b_j \leqslant x\}$ being nonempty is 0-large. □

This proposition allows one to associate with every $a \in A$ an ordinal. That is, given a fixed μ such that A is μ-small (or at least $A \setminus \{\max(A)\}$ is μ-small) we associate with every $a \in A$ the nonlimit μ_j such that $a = b_j$. But, of course, this assignment of ordinals to elements of A depends on μ. We shall write $\mathrm{KS}(\mu; a)$ (or $\mathrm{KS}^A(\mu; a)$ if necessary) for the last μ_j with $a = b_j$. Observe that for some μ, $\mathrm{KS}(\mu; a)$ is not defined for all elements of A, but they are defined if A is at most μ-large.

We generalize slightly this idea. We let $\mathrm{RKS}^A(\mu; \alpha, a)$ be an abbreviation for $\exists j[\alpha = \mu_j \;\&\; a = b_j]$. Thus, $\mathrm{KS}(\mu; a)$ is the last α for which $\mathrm{RKS}^A(\mu; \alpha, a)$ holds. Once again, we omit the superscript A unless it is really necessary.

As the reader has noticed, every $h_\alpha(a)$ is $h_m(a)$ for some finite m. Let us make this point precise. We define $\mathrm{Nit}(h; \alpha, a)$, the *number of iteration function* by induction on α: $\mathrm{Nit}(h; 0, a) = 0$, $\mathrm{Nit}(h; \alpha+1, a) = \mathrm{Nit}(h; \alpha, h(a)) + 1$, and $\mathrm{Nit}(h; \lambda, a) = \mathrm{Nit}(h; \{\lambda\}(a), a)$ for limit λ. As usual, we shall write Nit^A if necessary.

Lemma 2.8. *Under the above notation we have $h_\alpha(a) = h^{\mathrm{Nit}(h;\alpha,a)}(a)$.*

Proof. By induction on α, left to the reader. □

Let us introduce an additional hierarchy of quickly growing functions. Let G be a function of which we assume, as usual, that it is increasing and increases the argument, that is $\forall x\, x < G(x)$. We let $G_0(x) = G(x)$, $G_{\alpha+1}(x) = (G_\alpha)^x(x)$ and $G_\lambda(x) = G_{\{\lambda\}(x)}(x)$ for limit λ. Here F^m denotes the mth iterate of F, i.e., $F^0(x) = x$, $F^{m+1}(x) = F^m(F(x))$. This hierarchy is called the *Grzegorczyk–Wainer hierarchy* or the *Schwichtenberg–Wainer hierarchy* or the *fast–growing hierarchy* in the literature.

Proposition 2.9. *Let G be as above, let h_α denote the Hardy hierarchy based on G. Then for every $\alpha < \varepsilon_0$ $G_\alpha = h_{\omega^\alpha}$.*

Of course, one can define a set $A \subseteq \mathbb{N}$ to be *α-large in Grzegorczyk–Wainer sense* if $G_\alpha(\min(A))\downarrow$, where G_α denotes the αth function in Grzegorczyk–Wainer hierarchy based on the successor in the sense of A.

Yet another hierarchy of quickly growing functions is defined as follows. Let h satisfy the same assumptions as above, i.e., it is increasing and increases the argument. We define a sequence sl_α of functions exactly like in the definition of Hardy hierarchy, but in nonlimit step we reverse the order of composition, that is $\mathrm{sl}_{\alpha+1} = h \circ \mathrm{sl}_\alpha$. The hierarchy gotten in this manner is called *slow growing hierarchy based on* h. We point out that this small change causes a drastic difference in the behaviour of slow growing hierarchy and that of Hardy hierarchy. Once again, we get the notion of a set A being α*-large in the sense of slow growing hierarchy*.

From now on we write α-large for the notion given by Hardy hierarchy, otherwise we make explicit the hierarchy we are working with.

It will be also convenient to have the following notion. Say that $E \subseteq \mathbb{N}$ is f*-scattered* if for every $e \in E \setminus \{\max(E)\}$, $f(e) \leqslant h^E(e)$.

We shall need some more lemmata on the Hardy hierarchy. The lemma below is due to Teresa Bigorajska.

Lemma 2.10. *Let λ be a limit ordinal smaller than ε_0. Then if $\beta \gg \mathrm{LM}(\lambda)$ then for every $n \in \omega$, $\{\omega^\beta \cdot \lambda\}(n) = \omega^\beta \cdot \{\lambda\}(n)$.*

Proof. Let λ be limit and let ϱ be the smallest exponent in the Cantor normal form expansion of λ. Then $\lambda = \delta + \omega^\varrho$ for some $\delta \gg \omega^\varrho$. Let $\beta \gg \mathrm{LM}(\lambda)$ and $n \in \omega$. We have

$$\{\omega^\beta \cdot \lambda\}(n) = \big\{\omega^\beta(\delta + \omega^\varrho)\big\}(n) = \{\omega^\beta \cdot \delta + \omega^{\beta+\varrho}\}(n) = \omega^\beta \cdot \delta + \{\omega^{\beta+\varrho}\}(n).$$

The last equality holds because $\omega^\beta \cdot \delta \gg \omega^{\beta+\varrho}$. Obviously $\beta \gg \varrho$, hence if $\varrho = \alpha + 1$ for some α then

$$\begin{aligned}\omega^\beta \cdot \delta + \{\omega^{\beta+\varrho}\}(n) &= \omega^\beta \cdot \delta + \{\omega^{\beta+\alpha+1}\}(n) = \omega^\beta \cdot \delta + \omega^{\beta+\alpha} \cdot n \\ &= \omega^\beta \cdot (\delta + \omega^\alpha \cdot n) = \omega^\beta \cdot \big(\delta + \{\omega^{\alpha+1}\}(n)\big) \\ &= \omega^\beta \cdot \{\delta + \omega^\varrho\}(n) = \omega^\beta \cdot \{\lambda\}(n).\end{aligned}$$

Let ϱ be limit. By the assumption $\beta \gg \varrho$ we get

$$\begin{aligned}\omega^\beta \cdot \delta + \{\omega^{\beta+\varrho}\}(n) &= \omega^\beta \cdot \delta + \omega^{\{\beta+\varrho\}(n)} = \omega^\beta \cdot \delta + \omega^{\beta+\{\varrho\}(n)} \\ &= \omega^\beta \cdot \big(\delta + \omega^{\{\varrho\}(n)}\big) = \omega^\beta \cdot \big(\delta + \{\omega^\varrho\}(n)\big) = \omega^\beta \cdot \{\lambda\}(n).\end{aligned}$$

□

Fix ϱ and iterate h_ϱ in the Hardy style. Thus, $(h_\varrho)_0 = \mathrm{id}$, $(h_\varrho)_{\alpha+1} = (h_\varrho)_\alpha \circ h_\varrho$, and for limit λ, $(h_\varrho)_\lambda(b) = (h_\varrho)_{\{\lambda\}(b)}(b)$. For ϱ of the form ω^β we have the following information.

Lemma 2.11. *If $\alpha, \beta < \varepsilon_0$ and $\beta \gg \mathrm{LM}(\alpha)$ then $(h_{\omega^\beta})_\alpha = h_{\omega^\beta \cdot \alpha}$.*

Proof. By induction on α, using Lemma 2.10 in limit steps. □

In particular, $(h_{\omega^k})_{\omega^k} = h_{\omega^{2k}}$, that is, $(h_{\omega_1(k)})_{\omega_1(k)} = h_{\omega_1(2k)}$. For $m > 1$ the result is less exact.

Lemma 2.12. *For $m, b \geqslant 2$ we have $(h_{\omega_m(k)})_{\omega_m(k)}(b) \leqslant h_{\omega_m(k+1)}(b)$.*

Proof. By Lemma 2.11

$$(h_{\omega_m(k)})_{\omega_m(k)}(b) = h_{\omega_m(k)^2}(b) = h_{\omega_m(k)^2}(b) = h_{\omega^{\omega_{m-1}(k)\cdot 2}}(b),$$

hence by Lemma 2.2 it suffices to show that

$$\omega^{\omega_{m-1}(k+1)} \Rightarrow_b \omega^{\omega_{m-1}(k)\cdot 2}. \qquad (*)$$

We proceed by induction on m. Let $m = 2$. Thus we must show that $\omega^{\omega^{k+1}} \Rightarrow_b \omega^{\omega^k \cdot 2}$. Consider the sequence $\omega^{\omega^{k+1}}, \omega^{\omega^k \cdot b}$. If $b = 2$ then this sequence has the desired properties. Otherwise $\omega^k \cdot b \Rightarrow_b \omega^k \cdot 2$ by Lemma 1.1, part 3, hence by part 4 of the same lemma we have $\omega^{\omega^k \cdot b} \Rightarrow_b \omega^{\omega^k \cdot 2}$ and we may take concatenation of these two sequences.

Assume $(*)$ for m. By the inductive assumption and part 4 of Lemma 1.1 we have

$$\omega^{\omega^{\omega_{m-1}(k+1)}} \Rightarrow_b \omega^{\omega^{\omega_{m-1}(k)\cdot 2}},$$

hence it suffices to show that we may go down one step with this 2, i.e., to show that

$$\omega^{\omega^{\omega_{m-1}(k)\cdot 2}} \Rightarrow_b \omega^{\omega^{\omega_{m-1}(k)}\cdot 2}.$$

By Lemma 2.14 part 1 we have $\omega_{m-1}(k) \cdot 2 \Rightarrow_b \omega_{m-1}(k) + 1$, hence $\omega^{\omega_{m-1}(k)\cdot 2} \Rightarrow_b \omega^{\omega_{m-1}(k)+1}$, so $\omega^{\omega^{\omega_{m-1}(k)\cdot 2}} \Rightarrow_b \omega^{\omega^{\omega_{m-1}(k)}\cdot b}$. Once again we go down from b to 2 as above. □

Let us restate these facts in terms of large sets.

Lemma 2.13.

1. *A is ω^{2k}-large iff it has an ω^k-large subset B such that for every $b \in B \setminus \{\max B\}$, the interval $A \cap [b, h^B(b)]$ of A is ω^k-large.*
2. *If $m > 1$ and A is $\omega_m(k+1)$-large then it has an $\omega_m(k)$-large subset B such that for every $b \in B \setminus \{\max B\}$ the interval $A \cap [b, h^B(b)]$ is $\omega_m(k)$-large.*
3. *If $m > 0$ and A is $\omega_m(k+1)$-large and C is the set of all elements of A which have even numbers (in the increasing enumeration of A), then C is $\omega_m(k)$-large.*

Proof. The first two parts are just restatements of previous two lemmas, the third one is immediate by the second part. Indeed, let B be as in the second part. Let $b = \mathrm{Card}(B)$ and let $C' = \{a_{2i} : i < b\}$. Then $a_{2i} \downarrow$ for $i < b$ because $a_{2i} \leqslant b_i$, hence C' is $\omega_m(k)$-large by Lemma 2.3, so C is $\omega_m(k)$-large as well by the same lemma. □

Lemma 2.14.

1. $\forall \gamma > 0\, \forall b > 0\; \gamma \Rightarrow_b 1$.
2. *For all limit α we have $\forall u > b > 1\; [\{\alpha\}(u) \Rightarrow_b \{\alpha\}(b) + 1]$.*
3. $\forall \alpha \gg \omega\, \forall \delta \gg \omega^\alpha\, \forall u > b > 1\; \delta + \omega^{\{\alpha\}(u)} \Rightarrow_b \delta + \omega^{\{\alpha\}(b)} \cdot b$.
4. *If a set D is $\delta + \omega^{\{\alpha\}(u)}$-large, $b = \min D$ satisfies $u > b > 1$ then D is $\delta + \omega^{\{\alpha\}(b)} \cdot b$-large.*

Proof. Part 1 is immediate by induction on γ. Part 2 is proved by induction on α. Cases $\alpha = \omega$ and $\alpha \to \alpha + \omega$ are immediate, so we show only the step $\alpha \gg \omega^2$. Write $\alpha = \delta + \omega^\tau$, where $\delta \gg \omega^\tau$. Thus, $\tau > 1$. If $\tau = \varrho + 1$ then

$$\{\alpha\}(u) = \{\delta + \omega^{\varrho+1}\}(u) = \delta + \omega^\varrho \cdot u = \delta + \omega^\varrho \cdot b + \omega^\varrho \cdot (u - b).$$

We use part 1 to infer $\{\alpha\}(u) \Rightarrow_b \{\alpha\}(b)+1$ as required. So let τ be limit. Then $\{\tau\}(u) \Rightarrow_b \{\tau\}(b)+1$ by the inductive assumption, so by Lemma 1.1, part 4,

$$\{\alpha\}(u) = \{\delta+\omega^\tau\}(u) = \delta+\omega^{\{\tau\}(u)} \Rightarrow_b \delta+\omega^{\{\tau\}(b)+1}.$$

Moreover, $\omega^{\{\tau\}(b)+1} \Rightarrow_b \omega^{\{\tau\}(b)}\cdot b = \omega^{\{\tau\}(b)}+\omega^{\{\tau\}(b)}\cdot(b-1)$ and the same argument as above works.

Part 3 follows from part 2 and part 3 of Lemma 1.1.

In order to prove part 4, let D, u, b satisfy the assumption. That is, we have $h_{\delta+\{\omega^\alpha\}(u)}(\min D)\downarrow$. By part 3 and Lemma 2.2, $h_{\delta+\omega^{\{\alpha(b)\}}\cdot b}(\min D)\downarrow$ as required. $\square$

Of course, all these ideas make sense if $h(x) = S(x) = x+1$, in this case we shall write S_α for h_α.

3. Approximating Functions

In this section we work out a combinatorial notion which we shall use in the proofs of independence results. Let $B \subseteq \mathbb{N}$ and let g be a function (defined on a subset of $\mathbb{N}$, which may be finite or infinite). We say that A *approximates* g if

$$\forall a \in A\, \forall x < a-2\big\{g(x)\downarrow \Rightarrow \big[g(x) < h^A(a) \vee g(x) \geqslant \max(A)\big]\big\}.$$

Of course, if A is infinite then the clause $g(x) \geqslant \max(A)$ should be omitted.

We define a sequence of families of finite sets of natural numbers. We say that $A \subseteq \mathbb{N}$ is *approximatively* 0*-large* if $\mathrm{Card}(A) > \min(A)$. Say that A is *approximatively* $n+1$*-large* if for every finite function g there exists an approximatively n-large $B \subseteq A$ which approximates g.

The goal of this section is the following result.

Theorem 3.1. *For every* $a, n \in \mathbb{N}$ *there exists an approximatively* n*-large set* $A \subseteq \mathbb{N}$ *with* $\min(A) \geqslant a$*.*

Soon after the famous paper by J. Paris and L. Harrington [32] appeared, a variant of the statement of Theorem 3.1 was shown to be unprovable in arithmetic by Pavel Pudlák (unpublished), who refers the idea to H. Friedman (unpublished). Since then, it is called *Pudlák's principle*, see, e.g. [15].

We shall see later that this is meaningful and unprovable in Peano arithmetic (though for each fixed n it is provable). In fact, our exposition of the Paris–Harrington result will be based on some variation of Theorem 3.1, cf. Theorem 10.6. The reformulation of Pudlák's principle in this manner and the proof of Theorem 3.2 and its application to construct initial segments in models of PA are due to Zygmunt Ratajczyk [33,25]. Richard Sommer [36,37] uses a similar tool (see [37], Lemma 5.24).

Theorem 3.2. *If* A *is* ω^α*-large and* $\min(A) \geqslant 2$*, then for every finite function* g *there exists an* α*-large* $B \subseteq A$ *which is an approximation of* g*.*

Of course, Theorem 3.2 implies Theorem 3.1 immediately. Precisely let $\omega_1 = \omega$ and $\omega_{m+1} = \omega^{\omega_m}$.

Corollary 3.3. *Every* ω_{n+1}*-large set is* $n-1$*-approximatively large.*

In order to prove 3.2 we need a minor generalization of a set being an approximation for a function. Let (A, B) be a pair of finite sets, where $\min(A) \geqslant 2$. We say that the pair (A, B) is an *approximation* for a function g if $\max(A) = \min(B)$ and

$$\forall a \in A \setminus \{\max(A)\}\, \forall x < a - 2\{g(x){\downarrow} \Rightarrow [g(x) < h^A(a) \vee g(x) \geqslant \max(B)]\}.$$

For ordinals $\alpha, \beta, \gamma < \varepsilon_0$ we write $\alpha \to (\beta, \gamma)$ for "whenever A is an α-large set and g is a function there exist $B_1, B_2 \subseteq A$ such that

1. $\min(B_1) = \min(A)$,
2. $\max(B_1) = \min(B_2)$,
3. the pair (B_1, B_2) is an approximation for g,
4. B_1 is β-large and B_2 is γ-large".

The idea here is as follows. As a matter of fact we want to work with B_1. The parameter B_2 is just a reserve for further steps of induction needed for the following lemma.

Lemma 3.4. *If $\alpha \gg \beta$ and $\beta > 0$ then $\omega^{\alpha+\beta} \to (\beta, \omega^\alpha)$.*

Observe that Lemma 3.4 implies Theorem 3.2 immediately (just put $\alpha = 0$).

Proof of Lemma 3.4. We denote by $T(\beta)$ the formula

$$\beta > 0 \Rightarrow \forall \alpha \gg \beta [\omega^{\alpha+\beta} \to (\beta, \omega^\alpha)]$$

and prove it by induction on β. In each case the subcase $\alpha = 0$ is the same as the subcase when $\alpha \neq 0$, so we do not separate these subcases.

Case 1 $\beta = 1$. Let A be $\omega^{\alpha+1} = \omega^\alpha \cdot \omega$-large. Let g be a function. Let $a_0 = \min(A)$ and $b_0 = \max(A)$. Hence $h^A_{\omega^{\alpha+1}}(a_0) = h^A_{\omega^\alpha \cdot a_0}(a_0) \leqslant b_0$. Let $a_k = h^A_{\omega^\alpha \cdot k}(a_0)$ for $k = 0, \ldots, a_0$. Of course, $a_0 < \cdots < a_{a_0}$. By the pigeon–hole principle there exists j_0 with $1 \leqslant j_0 \leqslant a_0 - 1$ and $[a_{j_0}, a_{j_0+1}) \cap g{*}[0, a_0 - 2) = \emptyset$ because there are at most $a_0 - 2$ images of $x < a_0 - 2$ and there are $a_0 - 1$ intervals $A \cap [a_1, a_2), \ldots, A \cap [a_{a_0-1}, a_{a_0})$ of the set A. We put $B_1 = \{a_0, a_{j_0}\}$ and $B_2 = A \cap [a_{j_0}, a_{j_0+1}]$. Then B_1 is 1-large as $j_0 \neq 0$, so B_1 has exactly two elements. B_2 is ω^α-large since $h^{B_2}_{\omega^\alpha}(a_{j_0}) = a_{j_0+1}$, so the pair (B_1, B_2) has the desired properties.

Case 2 The successor step $T(\beta) \Rightarrow T(\beta + 1)$. Let $\alpha \gg \beta + 1$, so $\alpha \gg \beta$. Assume $\omega^{\alpha+\beta} \to (\beta, \omega^\alpha)$. Let A be $\omega^{\alpha+\beta+1}$-large and let a function g be given. By the initial step there exists a pair (A_1, A_2) of subsets of A approximating g and such that $\min(A_1) = \max(A_2)$, $\mathrm{Card}(A_1) = 2$ and A_2 is $\omega^{\alpha+\beta}$-large. By the inductive assumption $\omega^{\alpha+\beta} \to (\beta, \omega^\alpha)$ there exists a pair (A_3, A_4) approximating g and such that $\max(A_3) = \min(A_4)$, A_3 is β-large and A_4 is ω^α-large. Consider the pair $(A_1 \cup A_3, A_4)$. Clearly, it is an approximation for g. Thus it remains to show that $A_1 \cup A_3$ is $\beta + 1$-large. Let $a_0 = \min(A) = \min A_1$. Then $A_1 \cup A_3 = \{a_0\} \cup A_3$. Thus $h^{A_1 \cup A_3}(a_0) = \min(A_3)$, so $h^{A_1 \cup A_3}_{\beta+1}(a_0) \simeq h^{A_1 \cup A_3}_\beta(\min(A_3)) \simeq h^{A_3}_\beta(\min(A_3)){\downarrow}$.

Case 3 β *limit*. The inductive assumption is

$$\forall \gamma < \beta\, \forall \alpha \gg \gamma [1 \leqslant \gamma \Rightarrow \omega^{\alpha+\gamma} \to (\gamma, \omega^\alpha)].$$

Let $\alpha \gg \beta$. Then $\alpha \gg \{\beta\}(n)$ and $\{\beta\}(n) < \beta$ for all n. Let A be $\omega^{\alpha+\beta}$-large and let a function g be given. Let $a_0 = \min(A)$. Then A is $\{\omega^{\alpha+\beta}\}(a_0)$-large, i.e., $\omega^{\alpha+\{\beta\}(a_0)}$-large. By the inductive assumption there exists an approximation (A_1, A_2) for g such that $\min(A_1) = a_0$, A_1 is $\{\beta\}(a_0)$-large and A_2 is ω^α-large. Thus, A_1 is β-large, so the same pair (A_1, A_2) has the required properties. $\square$

Let me remark that in [23] we work out several similar notions of largeness and prove the approximation lemma for each of them.

4. Some Combinatorics Involving Large Sets

In this section I cite some partition theorems for large sets. All the ideas go back to J. Ketonen and R. Solovay [18] and were reworked to sets large in the sense of Hardy by T. Bigorajska and the author.

For $n > 1$ let

$$\alpha \to (\beta)^n_c$$

be an abbreviation for: for every α-large set A with $\min(A) > c$ and every partition $P : [A]^n \to (< c)$ there exists a β-large homogeneous set. For $n = 1$ we are speaking directly about partitioning elements. When partitioning elements the lower index in the Ramsey relation $\to$ may be an ordinal. Let A be a finite subset of $\mathbb{N}$. We say that the partition $A = \cup_{0 \leqslant i \leqslant e} B_i$ of A is *α-large* if the set $E = \{\min(B_0), \ldots, \min(B_e)\}$ is α-large. A partition is *α-small* otherwise. Thus, if $n = 1$ we write

$$\alpha \to (\beta)^1_\gamma$$

if for every α-large set A with $\min(A) > 0$ and every partition $A = \cup_{0 \leqslant i \leqslant e} B_i$ of A which is γ-small, there exists $i \leqslant e$ such that B_i is β-large.

Theorem 4.1. *If $\alpha, \beta < \varepsilon_0$, $\alpha \geqslant 1$ and $\beta \gg \mathrm{LM}(\alpha)$ then $\omega^\beta \cdot \alpha \to (\omega^\beta)^1_\alpha$.*

Proof. See [3]. $\square$

For every $\alpha < \varepsilon_0$ and every $c \in \mathbb{N} \setminus \{0\}$ we define $\omega_{(0)}(\alpha, c) = 1$, $\omega_{(1)}(\alpha, c) = \omega^\alpha \cdot c$, $\omega_{(2)}(\alpha, c) = \omega^{\omega_{(1)}(\alpha,c)}$, $\omega_{(n+1)}(\alpha, c) = \omega^{\omega_{(n)}(\alpha,c)\cdot 3}$.

Theorem 4.2. *Let $n \in \mathbb{N} \setminus \{0\}$, let A be a set $\omega_{(n)}(\alpha, c)$-large, where $\alpha < \varepsilon_0$, $c < \min(A)$. If $P : [A]^n \to (< c)$ is a partition of the set $[A]^n$ into at most c parts then there exists an ω^α-large homogeneous set.*

Proof. See [4]. $\square$

Here are some lower bounds.

Theorem 4.3. *Let $k \geqslant 3$. Let $A \subset \mathbb{N}$ be such that for every partition $L : [A]^k \to 3^{k-2}$ there exists a monochromatic $D \subseteq A$ with strictly more than $\min(D) + \frac{(k-2)\cdot(k-1)}{2} + 1$ elements. Then A is ω_{k-1}-large.*

Proof. See [5]. $\square$

The proof of Theorem 4.3 given in [4] is just a rework of the argument given in Section 6.3 in [14].

In the above result the homogeneous set is supposed to be ω-large (and have some minor number of additional elements). Here is a version in which we have some information about larger monochromatic sets. Let a function $F\colon \varepsilon_0 \to \varepsilon_0$ be defined by the following conditions:

1. $F(0) = 0$,
2. $F(\alpha + 1) = F(\alpha) + 1$,
3. $\beta \gg \alpha \Rightarrow F(\beta + \alpha) = F(\beta)(+)F(\alpha)$,
4. $F(\omega^n) = \omega^n + \omega^{n-1} + \cdots + \omega^0$ for $n < \omega$,
5. $F(\omega^\alpha) = \omega^\alpha \cdot 2 + 1$ for $\alpha \geqslant \omega$.

Theorem 4.4. *Let $k \geqslant 3$ and $m > 1$. Let $A \subseteq \mathbb{N}$ be at most ω_{m+k-2}-large with $\min(A) \geqslant k$. Then there exists a partition $L_k : [A]^k \to 3^{k-2}$ such that every $D \subseteq A$ monochromatic for L_k is at most $F(\omega_m) + \frac{(k-1)(k-2)}{2}$-large.*

Proof. See [5]. □

Corollary 4.5. *Let $m, k \in \mathbb{N}$. Let $\alpha = F(\omega_m) + \frac{(k-2)(k-1)}{2} + 1$. Let A be such that $A \to (\alpha)^k_{3^{k-2}}$ and $3 \leqslant k \leqslant \min(A)$. Then A is ω_{m+k-2}-large.*

Proof. See [5]. □

We remark that if we multiply by 3 the number of sets in the partition, then we may require the homogeneous set to be considerably smaller than in Theorem 4.4, see [24]. J. Ketonen and R. Solovay work very hard to obtain partitions into 8 elements. By a result of A. Weiermann, the number of parts may be always reduced to 2, but for the price of extending the dimension by 1, see [24].

5. Weak Indicators

Before going further we work out a variant of the notion of an indicator. The idea is due to H. Friedman [11], it was worked out further by J. Paris, L. Kirby (cf. [19]) and L. Harrington and led the famous theorem on independent combinatorial statement, see [32] and Theorem 10.6. More directly, the indicator described below was used in several papers on nonstandard satisfaction, see e.g. [22].

Before going to the main construction we define the notion of *quantifier rank* of a formula. We let $Q_0 = \Delta_0$ and Q_{n+1} = the closure of $Q_n \cup \exists Q_n$ under conjunction, negation and bounded quantification. We write $\mathrm{qr}(\varphi) \leqslant n$, *the quantifier rank of φ is $\leqslant n$*, if $\varphi \in Q_n$. Obviously, this is a primitive recursive notion, so we may work with it freely in T. It is easy to write the universal formula Tr_{Q_n} for Q_n-formulas, but it should be noticed that Tr_{Q_n} is of class Q_{n+1}.

We begin the basic construction. Let T be a recursive consistent theory in the language of PA. At first we write down a formula $A(n, b, c, w)$ which expresses "b is a complete and immediately consistent set of substitutions of the form $\varphi(S^{u_0}0, \ldots, S^{u_{m-1}}0)$, where $\varphi \in Q_n$ and $\varphi, u_0, \ldots, u_{m-1} \leqslant c$, and each substitution which is in w is in b", provided other conditions are satisfied, i.e., $\varphi \in Q_n$ and $\varphi, u_0, \ldots, u_{m-1} \leqslant c$. In order to construct this formula, let us write that a substitution of the form $\varphi(S^{u_0}0, \ldots, S^{u_{m-1}}0)$ is n, c-*admissible* if $\varphi \in Q_n$ and $\varphi, u_0, \ldots, u_{m-1} \leqslant c$. Then $A(n, b, c, w)$ is an abbreviation for the conjunction of

1. For all $x \in b$, x is a sentence and there exist $\varphi \in Q_n$ and $u_0, \ldots, u_{m-1} \leqslant c$ such that $\varphi \leqslant c$ and $x = \varphi(S^{u_0}0, \ldots, S^{u_{m-1}}0)$.
2. For each $\varphi, u_0, \ldots, u_{m-1}$, if the substitution $\neg\varphi(S^{u_0}0, \ldots, S^{u_{m-1}}0)$ is n, c-admissible then one of $\varphi(S^{u_0}0, \ldots, S^{u_{m-1}}0)$, $\neg\varphi(S^{u_0}0, \ldots, S^{u_{m-1}}0)$ is in b, but not both.
3. For every φ, ψ if the conjunction $\varphi \mathbin{\&} \psi$ is n, c-admissible, then it is in b iff both conjuncts are in b.
4. For every sentence $\varphi(S^u 0) \in b$ and every m, if the sentence $\exists v_m\, \varphi(v_m)$ is n, c-admissible then this statement is in b.
5. For every x, y, z, if the statement $S^x 0 + S^y 0 = S^z 0$ is n, c-admissible then this statement is in b iff $x + y = z$.
6. The same for other atomic formulas.
7. For every sentence $\varphi \in w$, if φ is n, c-admissible then it is in b.
8. For every n, c-admissible axiom φ of T, $\varphi \in b$.

The idea here is the following. If $A(n, b, c, w)$ then b is a candidate for truth up to c and all statements in w are supposed to be true; w is just a starting point. But the definition gives the quantifier step in one direction, the other direction will be treated below.

Let $B(n, b, c, d, e, w)$ be an abbreviation for the conjunction of the following formulas:

$$A(n, b, c, w) \mathbin{\&} A(n, d, e, w) \mathbin{\&} b \subseteq d$$

and

"for all $\varphi, u_0, \ldots, u_{m-1}$ if the statement $\exists v\, \varphi(v, S^{u_0}0, \ldots, S^{u_{m-1}}0)$ is in b then there exists u such that the statement $\varphi(S^u 0, S^{u_0}0, \ldots, S^{u_{m-1}}0)$ is in d".

We let $\mathrm{Ext}(j, n, \bar{b}, \bar{c}, w)$ be an abbreviation of

$$\mathrm{Seq}(\bar{b}) \mathbin{\&} \mathrm{Seq}(\bar{c}) \mathbin{\&} \mathrm{lh}(\bar{b}) = \mathrm{lh}(\bar{c}) = j \mathbin{\&} \bigwedge_{i<j-1} B(n, \bar{b}_i, \bar{c}_i, \bar{b}_{i+1}, \bar{c}_{i+1}, w).$$

Here Ext stands for extension, we shall extend the information in the argument below.

We let $Y(n, x, y, w) = \max\{j : \exists \bar{b}, \bar{c}[\mathrm{Ext}(j, n, \bar{b}, \bar{c}, w) \mathbin{\&} x \leqslant \bar{c}_0 \mathbin{\&} \bar{c}_{j-1} \leqslant y]\}$. We assert that Y is a Σ_1 formula. Indeed, $Y(n, x, y, w) \geqslant j$ iff there exist $\bar{b}, \bar{c}$ with

$$\begin{aligned}&\big[\mathrm{Ext}(j, n, \bar{b}, \bar{c}, w) \mathbin{\&} x \leqslant \bar{c}_0 \mathbin{\&} \bar{c}_{j-1} \leqslant y\big] \mathbin{\&}\\ &\quad \neg\exists \bar{b}', \bar{c}'\big[\mathrm{Ext}(j+1, n, \bar{b}', \bar{c}', w) \mathbin{\&} x \leqslant \bar{c}'_0 \mathbin{\&} \bar{c}'_j \leqslant y\big].\end{aligned}$$

One may bound the quantifier $\neg\exists$ in the above by bounding it to maximum of all sequences of length $\leqslant y$ with all items $\leqslant y$ (this gives a bound for $\bar{c}'$) and binding $\bar{b}'$ by maximum of all sets of sequences (of length $\leqslant y$) of subsets of the interval $(< y)$. This causes no serious difficulty, so we omit the details here.

We shall also write $A^{\mathrm{T}}, B^{\mathrm{T}}, \mathrm{Ext}^{\mathrm{T}}, Y^{\mathrm{T}}$ if necessary. (Observe that condition 8 depends on T.) Moreover we shall omit the variable w in most cases. This variable is just to have the possibility of having a starting point of the construction. From now on if we omit this variable we mean that $w = \emptyset$.

Theorem 5.1 (H. Friedman [11]). *Let $\mathcal{M}$ be a nonstandard model of* T*, where* T *contains $I\Sigma_1$. Then $\mathcal{M}$ has arbitrarily large initial segments satisfying* T*. In fact, if $\mathcal{M} \models \mathrm{PA}$ and $n \in \mathbb{N}$, then $\mathcal{M}$ has arbitrarily large Σ_n-elementary initial segments satisfying* T.

Proof. We begin with the second part. Pick $a \in \mathcal{M}$, we find an initial segment of $\mathcal{M}$ satisfying T and containing a (as element). Fix $n \in \mathbb{N}$. Let, as usual, Tr_{Π_n} denote the universal formula for Π_n formulas. We let $\phi_m(x, y)$ be

$$\begin{aligned} &\forall \varphi, u \leqslant x \big\{ \big[\varphi \in \Pi_m \mathbin{\&} \exists w \, \mathrm{Tr}_{\Pi_m}(\varphi(S^u 0, S^w 0))\big] \\ &\quad \Rightarrow \exists w < y \, \mathrm{Tr}_{\Pi_m}\big(\varphi(S^u 0, S^w 0)\big)\big\}. \end{aligned} \tag{3}$$

Clearly, we have $\mathrm{T} \vdash \forall x \exists y \, \phi_m(x, y)$. It follows that for every $k \in \mathbb{N}$ we have in $\mathcal{M}$: “there exists a sequence $b_0, \ldots, b_k$ such that for each $i < k - 1$ $\phi_{m-1}(b_i, b_{i+1})$”. We let $c_i = \{\varphi(S^u 0) \in Q_m : \varphi, u \leqslant a_i \mathbin{\&} \mathrm{Tr}_{Q_m}(\varphi(S^u 0))\}$. Then we have, in $\mathcal{M}$, $\mathrm{Ext}(k + 1, m, \bar{b}, \bar{c})$. Summing up, we obtained: for every $k, m \in \mathbb{N}$, $\mathcal{M}$ satisfies $\exists \bar{b}, \bar{c} \, \mathrm{Ext}(k, m, \bar{b}, \bar{c})$. By overspill lemma this holds also for all $k, m < m_0$ for some nonstandard $m_0 \in \mathcal{M}$. Pick $\bar{b}, \bar{c} \in \mathcal{M}$ such that $\mathcal{M}$ thinks $\mathrm{Ext}(m_0 - 1, m_0 - 1, \bar{b}, \bar{c})$ and let $I = \{i \in \mathcal{M} : i < c_m$ for some $m \in \mathbb{N}\}$. Then it is easy to check that for $\bar{e} \in I$, $I \models \varphi(\bar{e})$ iff $\varphi(\bar{e}) \in b_m$ for some $m \in \mathbb{N}$, all the definitions above were given just to ensure this. The heart of the matter is the fact that for standard φ, $\varphi \in Q_{m_0 - 1}$.

The first part is proved in the same manner, working with ϕ_0, we leave the details to the reader. □

Let me also remark that H. Friedman proved also that every countable nonstandard $\mathcal{M} \models \mathrm{PA}$ has arbitrarily large initial segments isomorphic with $\mathcal{M}$.

6. Indicators

Let T be a recursive theory in the language of PA containing $I\Sigma_1$. Observe that the construction of Section 5 may be carried out inside any model of T, hence the idea of a weak indicator works for T. We should remark that the proof of Theorem 5.1 requires more than $T \supseteq I\Delta_0 + \mathrm{Exp}$ (we used induction in $\mathcal{M}$ applied to a more complicated formula than merely bounded one), and, indeed, there exist models of $I\Delta_0 + \mathrm{Exp}$ with no arbitrarily large initial segments satisfying T. One such model may be obtained by the following trivial construction. Begin with any nonstandard $\mathcal{K} \models \mathrm{PA}$, pick a nonstandard $a \in \mathcal{K}$ and let $\mathcal{M} = \sup\{\beth_n(a) : n \in \mathbb{N}\}$.

Let T satisfy the above assumption. A recursive function $Y(x, y) = z$ is an *indicator* (for models of T) if for every countable $M \models \mathrm{T}$ and any $a, b \in \mathcal{M}$ we have: there exists an initial segment I of $\mathcal{M}$ such that $a \in I < b$ and $I \models \mathrm{T}$ iff the value $Y(a, b)$, calculated in $\mathcal{M}$, is nonstandard. A healthy way of thinking of this is the following. The value $Y(a, b)$ is a sort of distance from a to b (though not in the sense of metric spaces, it is not symmetric), hence the main property of indicators is: there exists an I with the properties stated in the definition iff the distance from a to b is nonstandard.

Proposition 6.1 (J. Paris). *Let* T *satisfy the above assumptions. Let* $Y(x, y)$ *be an indicator for models of* T. *Then*

1. *If* $\mathbb{N} \models \mathrm{T}$ *then* $\mathbb{N} \models \forall x, z\, \exists y\, Y(x, y) \geqslant z$,
2. *if* T *is consistent then there exists a model of* $\mathrm{T} + \neg\forall x, z\, \exists y\, Y(x, y) \geqslant z$,
3. *for each* $z \in \mathbb{N}$, T *proves that the function* $x \mapsto \min\{y : Y(x, y) \geqslant z\}$ *is total,*
4. *if* F *is a recursive function such that for some formula representing* F, T *proves "*F *is total" then there exists* $z \in \mathbb{N}$ *such that* F *is dominated by the function* $x \mapsto \min\{y : Y(x, y) \geqslant z\}$.

It follows that if $\mathbb{N} \models \mathrm{T}$, then the statement $\forall x, z\, \exists y\, Y(x, y) \geqslant z$ is independent from T.

Proof. For the first part pick $x, z \in \mathbb{N}$. It suffices to show that every countable nonstandard model $\mathcal{M}$ of T models also $\exists y\, Y(x, y) \geqslant z$. So let $\mathcal{M}$ be given and pick a nonstandard $y \in \mathcal{M}$. Then the value of $Y(x, y)$ is nonstandard (because some model of T, namely $\mathbb{N}$, is between x and y), so this value is greater than z.

For the second part we pick any countable nonstandard $\mathcal{M} \models \mathrm{T}$ and any nonstandard $x \in \mathcal{M}$. Assume that $\mathcal{M} \models \forall x, z\, \exists y\, Y(x, y) \geqslant z$, for otherwise $\mathcal{M}$ has the desired property. We let $y = \min\{y : Y(x, y) \geqslant x\}$. Then there exists $I \models \mathrm{T}$ between x, y as x was chosen to be nonstandard. If $I \models \forall x, z\, \exists y\, Y(x, y) \geqslant z$, then we pick $y' \in I$ such that $I \models Y(x, y') \geqslant x$. Then y' satisfies the same formula in $\mathcal{M}$ (because Y is recursive, i.e., Σ_1). This is a contradiction with minimality of y, indeed, $y' < I < y$. The third part is immediate by Theorem 5.1. The fourth part follows from the fact that in the proof of the second part z was arbitrarily small in the nonstandard part of $\mathcal{M}$. □

As pointed out above, one may construct an indicator for models of T using the idea presented in Section 5. We simply write down the formula $\mathrm{Ext}(j, n, \bar{b}, \bar{c})$ as above (but we change the definition of the formula A by requiring each n, c-admissible axiom of T be in b) and let $Y(x, y) = \max\{j : [\mathrm{Ext}(j, j, \bar{b}, \bar{c})\, \&\, x \leqslant \bar{c}_0\, \&\, \bar{c}_{j-1} \leqslant y]\}$, where $\bar{c}_0$ denotes the first item of $\bar{c}$ and $\bar{c}_{j-1}$ denotes its $j-1$-st item. Thus we do the same as in Section 5, but we identify the length j of appropriate sequences with the bound n on admissible formulas. We leave verification that $Y(x, y)$ is, in fact, an indicator for models of T to the reader.

Observe that Proposition 6.1 gives an independent statement (for a given T satisfying the assumptions) in a way drastically different than the proof of Gödel's theorem. Moreover, this statement is Π_2, whilst Gödel's consistency statement is Π_1.

7. Transfinite Induction in PA

Observe that there is no problem with formalizing the definition of an ordinal (see Section 1) in PA. The reason is that as a matter of fact we obtained there a primitive recursive function which associates with every n the (Gödel number of a) Σ_1 formula Pol_n (and the auxiliary definition of $\prec_n$), hence one can use the formula $\exists n\, \mathrm{Tr}_{\Sigma_1}(\mathrm{Pol}_n(\alpha))$

as the definition of an ordinal below ε_0. We shall write that α is an ordinal in the sense of the model under consideration, or that $\alpha < \varepsilon_0$.

Our goal is to prove transfinite induction principle below ε_0 in PA. The exact formulation is in Theorem 7.1 and Corollary 7.2.

For (a notation for) an ordinal ϱ and a natural number m we define $L(\varrho, \Sigma_m)$ to be the scheme of minimum for Σ_m formulas over ϱ, i.e. the scheme

$$\bigl[\exists\alpha < \varrho\, A(\alpha)\bigr] \to \exists\alpha\bigl[A(\alpha)\,\&\,\forall\beta < \alpha\,\neg A(\beta)\bigr],$$

where A is a Σ_m formula, possibly with other free variables.

The following result is a variant of Gentzen's proof of transfinite induction in PA. This variant is due to Mints [28], see also [16], [25] and [36,37].

Theorem 7.1. *For all $l \geqslant 1$ and all k, m*

$$I\Sigma_{m+l} \vdash L(\omega_{m+1}(k), \Sigma_l).$$

Let $\mathrm{TI}(\varrho, \Pi_m)$ denote the scheme of transfinite induction for Π_m formulas over ϱ, that is

$$\bigl[\psi(0)\,\&\,\forall\beta < \varrho\bigl(\forall\gamma < \beta\,\psi(\gamma) \Rightarrow \psi(\beta)\bigr)\bigr] \Rightarrow \forall\beta < \varrho\,\psi(\beta),$$

where $\psi \in \Pi_m$. Observe that the scheme $\mathrm{TI}(\varrho, \Pi_m)$ is equivalent to $L(\varrho, \Sigma_m)$.

Corollary 7.2. *For every $k, m \in \mathbb{N}$ PA proves $\mathrm{TI}(\omega_m, \Sigma_k)$.*

Lemma 7.3. *For each k, l $I\Sigma_l \vdash L(\omega^k, \Sigma_l)$.*

Proof. Let $A(\cdot)$ be a Σ_l formula. Work in $I\Sigma_l$. Assume $\exists\alpha < \omega^k\, A(\alpha)$; we construct the smallest such α. An α given by the assumption must be of the form $\alpha = \omega^{k-1} \cdot m_{k-1} + \cdots + \omega^0 \cdot m_0$. We shall simply choose the required sequence $m_{k-1}, \ldots, m_0$ by applying the scheme of minimum to a sequence $C_{k-1}, \ldots, C_0$ of formulas. We let $C_{k-1}(m_{k-1})$ be

$$\exists\langle m_{k-2}, \ldots, m_0\rangle\, \exists\alpha\bigl[\alpha = \omega^{k-1} \cdot m_{k-1} + \cdots + \omega^0 \cdot m_0\,\&\,A(\alpha)\bigr].$$

By the assumption $\exists m_{k-1}\, C_{k-1}(m_{k-1})$ and by the scheme of minimum for Σ_l formulas (which is equivalent to Σ_l induction) there exists a smallest m_{k-1} such that $C_{k-1}(m_{k-1})$. This m_{k-1} is a parameter in further formulas. We find m_{k-2} in the same manner. That is we write $C_{k-2}(m_{k-2})$:

$$\exists\langle m_{k-3}, \ldots, m_0\rangle\, \exists\alpha\bigl[\alpha = \omega^{k-1} \cdot m_{k-1} + \cdots + \omega^0 \cdot m_0\,\&\,A(\alpha)\bigr]$$

and get the smallest m_{k-2}. Iterate this procedure k times. This gives the sequence $m_{k-1}, \ldots, m_0$, so gives an ordinal $\alpha = \omega^{k-1} \cdot m_{k-1} + \cdots + \omega^0 \cdot m_0$; clearly this is the smallest witness for $A(\cdot)$. $\square$

Observe that the argument gives a proof which heavily depends on k, indeed, the greater k is the longer procedure is applied in the process of choosing the sequence of coefficients $m_{k-1}, \ldots, m_0$, therefore of α.

Lemma 7.4. *Let $\nu \geqslant \omega$ be an ordinal. Then for each l $I\Delta_0 + L(\nu, \Sigma_{l+1}) \vdash L(\omega^\nu, \Sigma_l)$.*

Observe that Theorem 7.1 is an immediate consequence of Lemmas 7.3 and 7.4.

The proof of Lemma 7.4 is similar to that of Lemma 7.3, but slightly more delicate. The reason is that we cannot work so freely; indeed, the α to be found may have an expansion of nonstandard length (we are working in a fixed model for $I\Delta_0+L(\nu,\Sigma_{l+1})$).

Let $A(\cdot)$ be a Σ_l formula; we shall obtain a proof by reducing the problem to Σ_{l+1} formulas and minimum principle over ν.

We define recursively two sequences $B_s(\cdot)$ and $C_s(\cdot)$ of Σ_{l+1} formulas. $B_0(\alpha_0)$ is

$$\begin{aligned}&\big[\exists m_0 \neq 0\, \exists r\, \exists\langle m_1,\dots,m_{r-1}\rangle\, \exists\langle \alpha_1,\dots,\alpha_{r-1}\rangle\, \exists\alpha\\ &\quad \nu > \alpha_0 > \dots > \alpha_{r-1} \,\&\, \alpha = \omega^{\alpha_0}\cdot m_0 + \dots + \omega^{\alpha_{r-1}}\cdot m_{r-1} \,\&\, A(\alpha)\big] \,\&\\ &\quad \big[\forall\gamma < \alpha\, \forall r\, \forall\langle m_0,\dots,m_{r-1}\rangle\, \forall\langle \alpha_1,\dots,\alpha_{r-1}\rangle\, \forall\alpha\\ &\quad (\nu > \gamma > \alpha_1 > \dots > \alpha_{r-1} \,\&\, \alpha = \omega^{\gamma}\cdot m_0 + \omega^{\alpha_1}\cdot m_1 + \dots + \omega^{\alpha_{r-1}}\cdot m_{r-1})\\ &\quad \Rightarrow \neg A(\alpha)\big].\end{aligned}$$

Thus, $B_0(\alpha_0)$ is just a description of the smallest possible exponent in the expansion $\alpha = \omega^{\alpha_0}\cdot m_0+\dots+\omega^{\alpha_{r-1}}\cdot m_{r-1}$, provided $\exists\alpha < \omega^\nu\, A(\alpha)$. Observe that B_0 is Σ_l & Π_l, so is Σ_{l+1}. The formula $C_0(m_0)$ is defined similarly; this will be the description of the smallest possible coefficient m_0. $C_0(m_0)$ is

$$\begin{aligned}&\exists\alpha_0\big\{B_0(\alpha_0) \,\&\, \big[\exists r\, \exists\langle m_1,\dots,m_{r-1}\rangle\, \exists\langle \alpha_1,\dots,\alpha_{r-1}\rangle\, \exists\alpha\\ &\quad \nu > \alpha_0 > \dots > \alpha_{r-1} \,\&\, \alpha = \omega^{\alpha_0}\cdot m_0 + \dots + \omega^{\alpha_{r-1}}\cdot m_{r-1} \,\&\, A(\alpha)\big] \,\&\\ &\quad \big[\forall k < m_0\, \forall r\, \forall\langle m_1,\dots,m_{r-1}\rangle\, \forall\langle \alpha_1,\dots \alpha_{r-1}\rangle\, \forall\alpha\\ &\quad (\nu > \alpha_0 > \dots > \alpha_{r-1} \,\&\, \alpha = \omega^{\alpha_0}\cdot k + \omega^{\alpha_1}\cdot m_1 + \dots \omega^{\alpha_{r-1}}\cdot m_{r-1})\\ &\quad \rightarrow \neg A(\alpha)\big]\big\}.\end{aligned}$$

Once again, $C_0(\cdot)$ is Σ_{l+1}. Observe moreover that in $I\Delta_0 + L(\nu,\Sigma_{l+1})$ we have

$$\big[\exists\alpha < \omega^\nu\, A(\alpha)\big] \rightarrow \exists\alpha_0, m_0\, \big[B_0(\alpha_0) \,\&\, C_0(m_0)\big]. \tag{$*$}$$

Now assume that $B_0,\dots,B_{s-1}$ and $C_0,\dots,C_{s-1}$ are constructed, we write down B_s and C_s. $B_s(\alpha_s)$ is

$$\begin{aligned}&\exists\langle \alpha_0,\dots,\alpha_{s-1}\rangle\, \exists\langle m_0,\dots,m_{s-1}\rangle\big\{\big[\forall i < s\, \mathrm{Tr}_{\Sigma_{l+1}}\big(B_i(\alpha_i) \,\&\, C_i(m_i)\big)\big] \,\&\\ &\quad \big[\exists r\, \exists m_s\, \exists\langle \alpha_{s+1},\dots,\alpha_r\rangle\, \exists\langle m_{s+1},\dots,m_r\rangle\, \exists\alpha\\ &\quad \alpha_s > \alpha_{s+1} > \dots > \alpha_r \,\&\, \alpha = \omega^{\alpha_0}\cdot m_0 + \dots + \omega^{\alpha_r}\cdot m_r \,\&\, A(\alpha)\big] \,\&\\ &\quad \big[\forall\gamma < \alpha_s\, \forall r\, \forall m_s\, \forall\langle \alpha_{s+1},\dots,\alpha_r\rangle\, \forall\langle m_{s+1},\dots,m_r\rangle\, \forall\alpha\\ &\quad (\alpha_{s+1} > \dots > \alpha_r \,\&\, \alpha = \omega^{\alpha_0}\cdot m_0 + \dots + \omega^{\gamma}\cdot m_s + \dots \omega^{\alpha_r}\cdot m_r)\\ &\quad \rightarrow \neg A(\alpha)\big]\big\}.\end{aligned}$$

Observe that this is still a Σ_{l+1} formula, because by the collection principle the quantifier $\forall i < s\, \mathrm{Tr}_{\Sigma_{l+1}}$ does not extend the class of this formula in $I\Sigma_{l+1}$.

$C_s(m_s)$ is constructed in a similar manner, namely it is

$$\exists\langle\alpha_0,\ldots,\alpha_{s-1}\rangle\,\exists\langle m_0,\ldots,m_{s-1}\rangle\big\{\big[\forall i<s\,\mathrm{Tr}_{\Sigma_{l+1}}\big(B_i(\alpha_i)\,\&\,C_i(m_i)\big)\big]\;\&$$
$$\exists\alpha_s B_s(\alpha_s)\;\&\;\big[\exists r\,\exists\langle\alpha_{s+1},\ldots,\alpha_r\rangle\,\exists\langle m_{s+1},\ldots,m_r\rangle\,\exists\alpha$$
$$\alpha_s>\alpha_{s+1}>\cdots>\alpha_r\;\&\;\alpha=\omega^{\alpha_0}\cdot m_0+\cdots+\omega^{\alpha_r}\cdot m_r\;\&\;A(\alpha)\big]\;\&$$
$$\big[\forall k<m_s\,\forall r\,\forall\langle\alpha_{s+1},\ldots,\alpha_r\rangle\,\forall\langle m_{s+1},\ldots,m_r\rangle\,\forall\alpha(\alpha_s>\alpha_{s+1}>\cdots>\alpha_r\;\&$$
$$\alpha=\omega^{\alpha_0}\cdot m_0+\cdots+\omega^{\alpha_s}\cdot k+\cdots+\omega^{\alpha_r}\cdot m_r)\to\neg A(\alpha)\big]\big\}.$$

Once again, this is a Σ_{l+1} formula. Let $D(\eta)$ be the formula $\exists i\,\mathrm{Tr}_{\Sigma_{l+1}}(B_i(\eta))$. It is still Σ_{l+1}.

Granted this we see that there exists η such that $D(\eta)$, hence there exists the smallest η so that $D(\eta)$. For this η there exists i such that $\mathrm{Tr}_{\Sigma_{l+1}}(B_i(\eta))$. Pick sequences $\langle\alpha_0,\ldots,\alpha_i\rangle$ and $\langle m_0,\ldots,m_i\rangle$ as above, that is such that for each $j\leqslant i$ $\mathrm{Tr}_{\Sigma_{l+1}}(B_j(\alpha_j)\,\&\,C_j(m_j))$. Obviously, the ordinal $\alpha=\omega^{\alpha_0}\cdot m_0+\cdots+\omega^{\alpha_i}\cdot m_i$ is the smallest such that $A(\alpha)$ and the proof of Lemma 7.4 and, hence, of Theorem 7.1, is finished.

8. Definitions by Transfinite Induction

Having shown a theorem on *provability* by transfinite induction in PA we give some material about *definability* of relations and functions by transfinite induction.[2] In set theory the appropriate theorems are much more intuitive because one may write the desired definition explicitly as "there exists a sequence with such and such properties". This does not work in the case of PA and related theories, indeed, the desired sequence would have to be an infinite object.

Traditionally, the above difficulty is overcome by use of some sort of diagonalization (the so-called *recursion theorem*, or *fixed-point theorem*, cf. 8.1). We give an exposition of some version of this approach below. But in many cases we shall be able to write the desired definition explicitly.

Theorem 8.1 (Kleene) (Fixed–point theorem). *For every total recursive function f there exists a fixed point of f, i.e. there exists $e\in\mathbb{N}$ such that $I\Delta_0+\mathrm{Exp}$ proves* $\forall a\,[\mathrm{Tr}_{\Sigma_1}(e;a)\equiv\mathrm{Tr}_{\Sigma_1}(f(e);a)]$.

Proof of Theorem 8.1. We begin with the function which sends a pair $\langle\varphi(\cdot,\cdot),y\rangle$ to the substitution $\varphi(S^y0,v)$, i.e. $\gamma(\varphi,y)=\mathrm{Sub}(\varphi(w,v),S^y0)=\varphi(S^y0,v)$. Let a natural number r be a definition of the function $x\mapsto f(\gamma(x,x))$. Thus, working in $I\Delta_0+\mathrm{Exp}$, we have $\mathrm{Tr}_{\Sigma_1}(r;\varphi,u)\equiv u=f(\gamma(\varphi,\varphi))$. Finally, let $e=\gamma(r,r)$. Then

$$\mathrm{Tr}_{\Sigma_1}(e;a)\equiv\mathrm{Tr}_{\Sigma_1}(\gamma(r,r);a)\equiv\mathrm{Tr}_{\Sigma_1}(f(\gamma(r,r));a)\equiv\mathrm{Tr}_{\Sigma_1}(f(e);a),$$

i.e. e has the desired property. □

[2] I would like to thank Konrad Zdanowski for discussions concerning parts of the material of this section.

Suppose we want to define a recursive function $G(\alpha) = a$. For every α, $G(\alpha)$ is supposed to depend on $G{\restriction}(<\alpha)$, say, we would like it to be $F(G{\restriction}(<\alpha))$, where F is recursive. The set $G{\restriction}(<\alpha)$ need not be finite, so we shall work with its definition rather than with the set itself. It follows that F is supposed to be a (recursive) function which associates with an ordinal α (the Gödel number of) a Σ_1 formula. This is the motivation for the statement of the following theorem.

Theorem 8.2 (Theorem on definitions by transfinite induction). *Let F be a recursive function. Then there exists a function G such that for every ordinal α, $G(\alpha) = F(z)$, where $z = \ulcorner \exists\beta[\beta \prec S^{\alpha}0 \mathbin{\&} \mathrm{Tr}_{\Sigma_1}(G(\beta) = S^{b}0)]\urcorner$.*

Here $S^{\alpha}0$ is the numeral corresponding to the (notation for) α, hence z is the Gödel number of a formula.

In the situation described in Theorem 8.2 we have *defined the function G by transfinite induction from F.*

Proof of Theorem 8.2. Given a recursive function h we transform it to

$$A(h)(\alpha) = F(z), \text{ where } z = \ulcorner \exists\beta, b\big[\beta \prec S^{\alpha}0 \mathbin{\&} \mathrm{Tr}_{\Sigma_1}(h(\beta) = S^{b}0)\big]\urcorner$$

and apply Theorem 8.1 to the transformation A. □

This result was formulated for the standard model. We shall use it in the following way. We shall use this to express the appropriate notion and obtain the existence conditions, then we shall prove the uniqueness conditions by transfinite induction. See the next section for a sample.

9. Totality of Functions in Hardy Hierarchy

In this section we show how to work with Hardy hierarchy in $I\Sigma_n$ and related theories.

We sketch below the possibility of formalizing this idea using the fixed–point theorem (i.e., 8.1). But one of our goals is to avoid diagonalization (which was used in the proof of fixed–point theorem), so we give a direct treatment.

The first difficulty is that the definition of fundamental sequences involved transfinite induction: $\{\omega^{\gamma}\}(b) = \omega^{\{\gamma\}(b)}$. This is avoided as follows. Let λ be a limit ordinal. Using its short normal form (2) we write it as $\lambda = \beta_1 + \omega^{\delta_1}$, where $\beta_1 \gg \omega^{\delta_1}$. Now we write δ_1 in its short Cantor normal form: $\delta_1 = \beta_2 + \omega^{\delta_2}$, where $\beta_2 \gg \omega^{\delta_2}$. We continue in the same fashion till we get δ_r of the form $\delta_r = m$, where $0 < m < \omega$. Then we let $\{\delta_{r-1}\}(b) = \omega^{m-1} \cdot b$. Then we come back to the original ordinal λ. This process is easily formalized in $I\Sigma_1$. Granted this, we see that $I\Sigma_{m+1}$ defines correctly fundamental sequences up to each $\omega_{m+1}(k)$. The argument above gives the existence conditions and Theorem 7.1 gives uniqueness of fundamental sequences.

Our next goal is to indicate how to formalize in PA the definition of Hardy hierarchy. Let h be a function. We write the formula describing the process sketched in Lemma 2.7. That is we write $h_{\alpha}(u) = w$ iff there exist two sequences $\alpha_0, \ldots, \alpha_{r-1}$ and $u_0, \ldots, u_{r-1}$ such that $\alpha_0 = \alpha$, $u_0 = u$, for each $j < r-2$ if α_j is limit then $\alpha_{j+1} = \{\alpha_j\}(u_j)$ and $u_{j+1} = u_j$ and if α_j is nonlimit then $\alpha_{j+1} = \alpha_j - 1$ and $u_{j+1} = h(u_j)$, and $\alpha_{r-1} = 0$ and $u_{r-1} = w$. Clearly this can be written in PA.

Before going further we give some more details in order to show that if h is Σ_1 then we obtain also a Π_1 definition. Let $B(\alpha, u, \gamma)$ be the following formula:

$$\begin{aligned}&\exists r, \langle \alpha_0, \dots \alpha_{r-1}\rangle, \langle u_0, \dots, u_{r-1}\rangle \big\{\alpha = \alpha_0 \,\&\, \gamma = \alpha_{r-1} \,\&\, u = u_0 \,\&\\ &\quad \forall j < r-2\big\{\alpha_j \neq 0 \,\&\, \big[\mathrm{Lim}(\alpha_j) \Rightarrow \alpha_{j+1} = \{\alpha_j\}(u_j)\big] \,\&\\ &\quad \big[\mathrm{Succ}(\alpha_j) \Rightarrow \alpha_{j+1} = \alpha_j - 1 \,\&\, u_{j+1} = h(u_j)\big]\big\}\big\}.\end{aligned}$$

We assert that for each $k \in \mathbb{N}$ $I\Sigma_{m+1}$ proves "for all $\alpha < \omega_{m+1}(k)$ and all u there exists the smallest γ such that $B(\alpha, u, \gamma)$". Once again, this follows from Theorem 7.1 because transfinite induction for Π_s formulas is equivalent to the scheme of minimum for Σ_s formulas. Obviously, we must have $\gamma = 0$ (otherwise the conditions determine the next items of the sequences of α's and u's). Then we see that if r is such that $\alpha_{r-1} = \gamma = 0$ then $w = u_{r-1}$ is just $h_\alpha(u)$. In other words, for each $k \in \mathbb{N}$, $I\Sigma_{m+1}$ proves that $h_{\omega_{m+1}(k)}$ is total. It follows that the formula $h_{\omega_{m+1}(k)}(u) = w$ is Π_1 as well, i.e., we have two definitions: one of the class Σ_1 and another one of the class Π_1 (their equivalence is provable in $I\Sigma_{m+1}$).

As usual, we say that h is a Δ_m-function in T if there exists Σ_m formula and a Π_m formula, whose equivalence is provable in T and both define h. Our goal is to show

Theorem 9.1. *Let h be a Δ_1 function. Let $m \in \mathbb{N} \setminus \{0\}$ and let $I\Sigma_m$ prove that h is total. Let $k \in \mathbb{N}$. Then $I\Sigma_m$ proves $\forall \alpha < \omega_m(k)\, \forall a\, h_\alpha(a){\downarrow}$.*

Of course, the most interesting case is when h is the usual successor function, $h(x) = S(x) = x + 1$. Observe that the statements to be proved are Π_2. Moreover one cannot prove them by Σ_1 induction (with parameter a) because of the successor step in the definition of Hardy hierarchy. Therefore we proceed differently.

Proof of Theorem 9.1. At first we prove the following version lemma on compositions (cf. Lemma 2.5):

$$\begin{aligned}&\forall \alpha < \omega_m(k)\, \forall \beta < \omega_m(k)\, \forall b, c, d \big\{\big[\beta \gg \alpha \,\&\, h_\alpha(b) = c \,\&\, h_\beta(c) = d\big]\\ &\quad \Rightarrow h_{\beta+\alpha}(b) = d\big\}.\end{aligned} \tag{4}$$

For every $k \in \mathbb{N}$ this is shown exactly like in the proof of Lemma 2.5, using Π_1 transfinite induction $\mathrm{TI}(\omega_m(k), \Pi_1)$, which is provable by Theorem 7.1 and the remark following it. In order to see that this is, indeed, a Π_1 formula we use the remark before the statement of the theorem. In the predecessor of the implication we use the Σ_1 form of the definition of the Hardy hierarchy and in the successor we use its Π_1 form.

Now let $m = 1$. By induction on k we prove

$$\text{for every } k \in \mathbb{N},\ I\Sigma_1 \vdash \forall j, a\, h_{\omega^k \cdot j}(a){\downarrow}. \tag{5}$$

Fix any model $\mathcal{M}$ of $I\Sigma_1$, we prove that the above statements hold in $\mathcal{M}$. For $k = 0$ we must prove that $\mathcal{M} \models \forall j, a\, h_j(a){\downarrow}$ and this is immediate by induction on j (with parameter a). So assume the conclusion for k. Then we apply induction on j to see that $\mathcal{M}$ satisfies $\forall j, a\, h_{\omega^{k+1} \cdot j}(a){\downarrow}$. For $j = 0$ there is nothing to check. For $j = 1$ this is just the conclusion for k and $j = a$. In order to obtain the conclusion for $j + 1$ we apply case $j = 1$ to obtain $b = h_{\omega^{k+1}}(a)$ and apply the conclusion for j to b. Observe that formal induction on j was applied to the formula $h_{\omega^k \cdot j}(a){\downarrow}$ with parameter a.

Now let $m > 1$. This time we shall show that for every $k \in \mathbb{N}$, $I\Sigma_m$ proves the statement

$$\forall \alpha < \omega_{m-1}(k)\, h_{\omega^\alpha}(a)\downarrow. \tag{6}$$

Observe that by Theorem 7.1, $I\Sigma_m$ proves transfinite induction for Π_2 formulas over $\omega_{m-1}(k)$. (6) is obvious for $\alpha = 0$ and for limit α. Also, the limit step is evident by the same argument as for (5), but with α instead of k. □

We assumed in Theorem 9.1 that $m > 0$. For $m = 0$ the situation is different because of the following result.

Theorem 9.2 (R. Parikh). *Let $H(x, y)$ be a Δ_0 formula such that $I\Delta_0$ proves $\forall x\, \exists y\, H(x, y)$. Then there exists $m \in \mathbb{N}$ such that $I\Delta_0 \vdash \forall x\, \exists y < x^m\, H(x, y)$.*

Proof (J. Paris). Assume the contrary, for every $m \in \mathbb{N}$ the theory $I\Delta_0 + \exists x\, \forall y < x^m\, \neg H(x, y)$ is consistent and let $\mathcal{M}_m$ be its model. We apply the ultraproduct construction. Let $\mathcal{F}$ be any nonprincipal ultrafilter in the Boolean algebra $P(\mathbb{N})$ and let $\mathcal{M}$ be the ultraproduct $\times_{m\in\mathbb{N}}\mathcal{M}_m/\mathcal{F}$. Then by Łoś's theorem $\mathcal{M}$ satisfies $I\Delta_0$. Choose $d \in \mathcal{M}$ which is (the equivalence class of) any sequence $d_m \in \mathcal{M}_m$ such that $\mathcal{M}_m \models \forall y < d_m^m\, \neg H(d_m, y)$. Let I be the initial segment of $\mathcal{M}$ defined as $I = \sup\{d^k : k \in \mathbb{N}\}$. Then $I \models I\Delta_0$, indeed, it is a cut in a model of $I\Delta_0$. In order to derive a contradiction it suffices to show that I cannot satisfy $\exists y\, H(d, y)$. In order to see this we define $e_m = \min\{t : H(d_m, t) \text{ in } \mathcal{M}_m\}$. Let e denote the element of $\mathcal{M}$ determined by the sequence $e_m : m \in \mathbb{N}$. For any fixed $k \in \mathbb{N}$, the set $\{m \in \mathbb{N} : M_m \models \exists y < d_m^k\, H(d_m, y)\}$ is finite, hence its complement is in $\mathcal{F}$, so for every $k \in \mathbb{N}$, $\mathcal{M} \models d^k < e$, so $e \notin I$. But if y were an element of I such that $I \models H(d, y)$, then $y < I < e$. Moreover the same y would satisfy $H(d, y)$ in $\mathcal{M}$ (because H is Δ_0). On the other hand, again, using Łoś's theorem, e is the smallest element of $\mathcal{M}$ satisfying this formula. Contradiction. □

Our following goal is to give some information about iterations of quickly growing Skolem functions F_k. They are defined as follows.

$$F_k(x) = \min\bigl\{y : \phi_k(x, y)\bigr\}, \tag{7}$$

where the formula $\phi_k(x, y)$ is given by (3). We give some information about Hardy iterations of the function F_k, for a fixed $k \in \mathbb{N}$.

It will be convenient to work directly with the relation ϕ_k rather than with the function F_k determined by it (because the use of minimum puts the formula to a higher class of complexity). Say that Z is *ϕ_k-scattered* if for all $a \in Z \setminus \{\max(Z)\}$, $\phi_k(a, h^Z(a))$ holds. Let $D_k(\alpha)$ be an abbreviation for

$$\forall a\, \exists Z \bigl[\min(Z) \geqslant a \;\&\; Z \text{ is } \alpha\text{-large} \;\&\; \phi_k\text{-scattered}\bigr]. \tag{8}$$

Here is a generalization of Theorem 9.1.

Theorem 9.3. *For every $m > 0$ and every $t, r \in \mathbb{N}$, $I\Sigma_{m+t+1}$ proves $\forall \alpha < \omega_m(r)\, D_t(\alpha)$.*

Once again, if one wants to use Theorem 7.1, then one obtains a weaker result (because the formula to be proved by transfinite induction is Π_{t+3}), hence we shall perform some additional work.

Proof of Theorem 9.3. Exactly as in the proof of Theorem 9.1 our starting point is a reformulation of the lemma on compositions, i.e., Lemma 2.5. It is as follows:

$$\begin{aligned}&\text{Let } A, B \text{ be sets, such that } \min(B) = \max(A) \text{ and } A \text{ is } \alpha\text{-large and}\\ &B \text{ is } \beta\text{-large, where } \beta \gg \alpha. \text{ Then } A \cup B \text{ is } \beta + \alpha\text{-large.}\end{aligned} \tag{9}$$

This has nothing to do with being ϕ_t-scattered. The restriction of (9) to $\alpha, \beta < \omega_m(r)$ is provable in $I\Sigma_m$. Of course, this holds if A, B are ϕ_t-scattered.

Observe that the proof of $\forall a\, \exists b\, \phi_t(a, b)$ requires some induction. This fact was proved as follows. Pick a. Enumerate as $\varphi_i(S^{u_i}0, v) : i < s$ all substitutions of the form $\varphi(S^u 0, v)$ with $\varphi, u \leqslant a$ and $\varphi \in \Pi_t$. Then we apply induction on j in

$$\begin{aligned}j \leqslant s \Rightarrow \big\{\exists z\, \forall i \leqslant j\big[\big(\exists w\, \mathrm{Tr}_{\Pi_t}(\varphi_i(S^{u_i}0, S^w 0))\big) \Rightarrow \exists w\\ \leqslant z\, \mathrm{Tr}_{\Pi_t}\big(\varphi_i(S^{u_i}0, S^w 0)\big)\big]\big\}.\end{aligned} \tag{10}$$

This formula is Σ_{t+2}.

Now we assert that

$$\forall r \in \mathbb{N} \quad I\Sigma_{t+2} \vdash \forall \alpha < \omega^r\, D_t(\alpha). \tag{11}$$

This is proved exactly as (5). Thus, we proceed by induction on r. For $r = 1$ we apply (10), in the inductive step we apply, again (10) and the analogue of lemma on compositions in the form stated as (9). We leave the details to the reader.

Of course, (11) gives Theorem 9.3 for $m = 1$. In order to prove it for $m > 1$ we shall prove

$$I\Sigma_{m+t+1} \vdash \forall \alpha < \omega_{m-1}(r)\, D_t(\omega^\alpha).$$

This time we proceed by induction on α. For $\alpha = 0$ there is nothing to prove as above, in the successor case one uses induction on the exponent and the analogue of lemma on compositions in the form stated as (9), the limit step is obvious. The whole induction is legitimate by (a variant of) Theorem 7.1. $\square$

Our following goal is to indicate the possibility of formalizing the proof of the approximation lemma (i.e., Theorem 3.2) in $I\Sigma_m$. In the proof we used the auxiliary notion of a pair of sets approximating given function, see the proof of Lemma 3.4. The proof was by transfinite induction. Observe that the quantifier "there exist two sets B_1, B_2" may be bounded, indeed, the coding of sets by means of binary expansions has the property that $B \subseteq A \Rightarrow B \leqslant A$, hence we may bound this quantifier to A. Thus, the statement of Lemma 3.4 is of the class Π_1.

In the proof of Lemma 3.4 we used the following version of the pigeonhole principle:

$$\begin{aligned}&\big[\forall i < a_0 - 2\, \exists z < a_0 - 1\, \xi(i, z)\big]\\ &\quad \Rightarrow \big[\exists s < a_0 - 1\, \forall i < a_0 - 2\, \exists z < a_0 - 1[\xi(i, z)\ \&\ z \neq s]\big],\end{aligned}$$

where ξ is a Δ_0 formula. This principle is easily proved by Σ_1-induction. Summing up, we obtain:

Lemma 9.4. *Let $m > 0$. Then for every $k \in \mathbb{N}$, $I\Sigma_m$ proves "if A is $\omega_m(k)$-large set, then for every finite function g there exists an $\omega_{m-1}(k)$-large subset B of A which is an approximation for g".*

10. Hardy Largeness and Indicators

In this section we present the main part of the construction which yields initial segments satisfying PA and related theories. As pointed out earlier, this is Ratajczyk's [33] approach to the Pudlák's principle (hence, to the Paris–Harrington's result, cf. Theorem 10.6); Sommer's [36,37] approach to proof–theoretic problems via models is slightly different. We begin with case $m = 1$ for motivational purpose.

Let $\mathcal{M}$ be a nonstandard model of $I\Sigma_1$. By Theorem 9.1, for every standard $k \in \mathcal{M}$,

$$\mathcal{M} \models \forall a\, h_{\omega^k}(a){\downarrow},$$

hence by the argument of overspill lemma the same statement holds for some nonstandard $k \in \mathcal{M}$. Fix such $k \in \mathcal{M}$. Pick also nonstandard $a \in \mathcal{M}$. Then the set $A = [a, h_{\omega^k}(a)]$ is ω^k-large. Thus, there exists an ω^k-large set in $\mathcal{M}$, pick such one and denote it by A. From now on we denote by h^A the successor function in the sense of A. By Lemma 9.4, given a function $g \in \mathcal{M}$, A has a k-large subset B that is an approximation for g. Consider the function

$$g(x) = \begin{cases} \min\{w \text{ such that } \mathrm{Tr}_0\big(\varphi(S^u0, S^w0)\big)\} & \text{if } x \text{ is of the form } \varphi(S^u0, v) \\ & \text{with } \varphi \in \Delta_0 \text{ and} \\ & \exists w\, \mathrm{Tr}_0\big(\varphi(S^u0, S^w0)\big) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

and restrict its domain to $(< \max(A))$ to make it $\mathcal{M}$-finite. Let B be a k-large subset of A approximating g. Let us show the reason of working with sets approximating such a function. We have: for every $b \in B \setminus \{\max(B)\}$ for every substitution $\varphi(S^u0, v) < b-2$ with $\varphi \in \Delta_0$, if $\exists w\, \varphi(S^u0, S^w0)$, then the smallest such w is either strictly below $h^B(b)$ or is above $\max(B)$. Pick any initial segment I of $\mathcal{M}$ closed under successor and smaller than $\mathrm{Card}(B)$ (as k is nonstandard, many such cuts exist, one of them is just $\mathbb{N}$). We let $J = \sup\{b_i : i \in I\}$, where, as usual $B = \{b_0, \dots, b_k\}$ in increasing order. This is a natural candidate for a model for $I\Sigma_1$ (the property of B will ensure this), but there are two minor difficulties here. We must make sure that J is a substructure of $\mathcal{M}$, that is, J is closed under multiplication. Moreover, we must ensure that (at least for standard) φ and $u \in J$, the substitution $\varphi(S^u0, v)$ is in J.

These difficulties are overcome as follows. We change the construction slightly. We let $r = \max\{r : 2r \leqslant k\}$ and $C = \{h^A_{\omega^r \cdot i}(a_0) : i \text{ is such that } h^A_{\omega^r \cdot i}(a_0){\downarrow}\}$. Then the set C is ω^r-large and $h^A_{\omega^r}$-scattered. Well, we did not require A to be of the form $[a, b]$, but clearly, $h(a) \geqslant a + 1$ for all a, and hence, $h^A_{\omega^r}(a) \geqslant S_{\omega^r}(a) \geqslant S_{\omega^2}(a) \geqslant 2^a \cdot a$ for all a, hence C is scattered with respect to exponentiation. It follows that each cut J as defined above (but with C rather than A) is closed under exponentiation, and hence it is closed under addition and multiplication. Thus we apply the approximation lemma to C and obtain an r-large set $B \subseteq C$.

This trick allows us to overcome the second difficulty as well. The reason is Grzegorczyk's theorem (which states that each primitive recursive function is dominated by S_{ω^k} for some k), or, rather a formalized version of it. Indeed, the function $\varphi, u \mapsto \varphi(S^u0, v)$, i.e., the substitution function, being primitive recursive, is dominated by h_{ω^r} (because r

is nonstandard), hence if $\varphi, u \leqslant c$, where $c \in C \setminus \{\max(C)\}$, then $\varphi(S^u 0, v) \leqslant h^C(c)$ and we may apply the approximation lemma.

Let us show how the property of B ensures that each cut J as above satisfies $I\Sigma_1$. For $b \in B \setminus \{\max(B)\}$ we have in $\mathcal{M}$: for $\varphi, u \leqslant b-2$ if $\exists w \operatorname{Tr}_0(\varphi(S^u 0, S^w 0))$ then either $\exists w < h_2^B(b) \operatorname{Tr}_0(\varphi(S^u 0, S^w 0))$ or $\forall w < \max(B) \operatorname{Tr}_0(\neg\varphi(S^u 0, S^w 0))$. It follows that for each cut J as above

$$J \models \exists w\, \varphi(u, w) \qquad \text{iff} \qquad \mathcal{M} \models \exists w < \max(B) \big[\varphi(u, w) \,\&\, \forall z < w\, \neg\varphi(u, z)\big]$$

for all $\varphi \in \Delta_0$ and $u \in J$.

Granted this reduction of Σ_1-truth in J to truth in $\mathcal{M}$ we argue as follows. Assume that

$$J \models \big[\exists x\, \varphi(0, x, u)\big] \,\&\, \forall t \big[\big(\exists x\, \varphi(t, x, u)\big) \Rightarrow \exists x\, \varphi(t+1, x, u)\big].$$

It follows that $\mathcal{M} \models \exists x < \max(B)\, \varphi(0, x, u)$ and for each $t \in J$,

$$\mathcal{M} \models \big(\exists x < \max(C)\, \varphi(t, x, u)\big) \Rightarrow \exists x < \max(B)\, \varphi(t+1, x, u).$$

By overspill lemma, this holds for each $t < t_0$ for some $t_0 \in \mathcal{M} \setminus J$. In particular, $J \models \forall t\, \exists x\, \varphi(t, x, u)$ as required.

Let us sum up. Below we call an indicator for T a function which has the decisive property: $Y(x, y)$ is nonstandard iff there exists an initial segment of the model under consideration satisfying T. We formulate the results for models of $I\Sigma_m$ with an extra subset (named by a new predicate).

Theorem 10.1.

1. *If $\mathcal{M}$ is a nonstandard model of $I\Sigma_1$ then there exist a set $C \in \mathcal{M}$ which is scattered with respect to exponentiation and ω^s-large for some nonstandard $s \in \mathcal{M}$.*
2. *If $C \in \mathcal{M}$ satisfies the properties stated in part 1 then the function $Y(x, y) = \max\{z : h^C_{\omega^z}(x) \leqslant y\}$ is an indicator for initial segments J satisfying $I\Sigma_1$ such that J is closed under the successor in the sense of C. Moreover, if $Y(x, y)$ is nonstandard then the appropriate I may be chosen in such a way that $C \cap I$ is unbounded in I and the structure $(I, C \cap I)$ satisfies Σ_1 induction in its language.*
3. *If F is a recursive function such that (for some formula representing F) $I\Sigma_1$ proves "F is total", then there exists $s \in \mathbb{N}$ such that F is dominated by S_{ω^s}.*

Proof. The first claim and the main part of claim 2 were proved above. For the moreover clause of claim 2 we simply remark that we could apply the same construction but for the enumeration $\varphi(S^u 0, v) : r$ of formulas with parameter A and do the same. For the third part one should only remark that in particular we obtained an ω^k-large set $A \in \mathcal{M}$ being of the form $[a, S_{\omega^k}(a)]$ with nonstandard k and were able to find initial segments J of $\mathcal{M}$ such that J is not closed under S_{ω^r}, provided $r \in \mathcal{M}$ is nonstandard. $\square$

Before going to the case $m > 1$ let us observe that the heart of the matter in the construction presented above was the following: if $\varphi, u \leqslant b$, where $b \in B$, then either all statements of the form $\exists w < z\, \varphi(u, w)$ for $z \in B$, $z \geqslant h_2^B(b)$ are true in $\mathcal{M}$ or all these statements are false. That is we may bound quantifier to any (large enough) element of B without changing truth of the statement obtained in this manner. It is convenient to introduce two sets: E, the set of these sentences as above that are true and A, the set of negations of elements of E. These sets play the role of Σ_1 and Π_1 truth (in initial segments of $\mathcal{M}$ determined as above). It is convenient to introduce one more set $D = \{b_{2i} : i$ is such that b_{2i} exists$\}$, where $B = \{b_i : i\}$ in increasing order. Then D is $k/2$-large (or $(k-1)/2$-large if $\mathrm{Card}(B)$ is odd) and has the decisive property: for all $d \in D \setminus \{\max(D)\}$ for all $\varphi, u \leqslant d$ if $\varphi \in \Delta_0$ then for each $t, s \in D$ strictly greater than d, the equivalence

$$\exists w < t\, \mathrm{Tr}_0\big(\varphi(S^w 0, S^u)\big) \equiv \exists w < s\, \mathrm{Tr}_0\big(\varphi(S^w 0, S^u)\big)$$

holds in $\mathcal{M}$.

We shall use analogues of these sets when working with case $m > 1$ in order to state inductive conditions for the construction. Let us go to the case when $m > 1$. The result will be as follows.

Theorem 10.2. *Let $m > 0$ be fixed.*

1. *For every $k \in \mathbb{N}$, $I\Sigma_m$ proves $\forall a\, \exists b[a, b]$ is $\omega_m(k)$-large.*
2. *Let $\mathcal{M} \models I\Sigma_m$ and let $A \in \mathcal{M}$ be $\omega_m(k)$-large for all standard $k \in \mathcal{M}$. Then there exists an initial segment I of $\mathcal{M}$ such that $A \cap I$ is unbounded in I and the structure $(I, A \cap I)$ satisfies $I\Sigma_m$ in its language.*
3. *The function $Y(x, y) = \max\{k : S_{\omega_m(k)}(x) \leqslant y\}$ is an indicator for models of $I\Sigma_m$.*
4. *If f is a recursive function such that (for some Σ_1 definition of f) $I\Sigma_m$ proves totality of f, then there exists $k \in \mathbb{N}$ such that $f(x)$ is dominated by $S_{\omega_m(k)}$.*

Similar result holds for full PA:

Theorem 10.3.

1. *For every $k \in \mathbb{N}$* PA *proves $\forall a\, \exists b[a, b]$ is ω_k-large.*
2. *Let $\mathcal{M} \models$* PA *and let $A \in \mathcal{M}$ be ω_k-large for all standard $k \in \mathcal{M}$. Then there exists an initial segment I of $\mathcal{M}$ such that $A \cap I$ is unbounded in I and the structure $(I, A \cap I)$ satisfies* PA *in its language.*
3. *The function $Y(x, y) = \max\{k : S_{\omega_k}(x) \leqslant y\}$ is an indicator for models of* PA.
4. *If f is a recursive function such that for some Σ_1 definition of f,* PA *proves that f is total, then f is dominated by S_{ω_k} for some $k \in \mathbb{N}$.*

Proof of Theorem 10.2. Fix $m \in \mathbb{N}$, $m > 0$. The first part of the theorem is just a restatement of Theorem 9.1. Let us go to the main construction. Fix $\mathcal{M} \models I\Sigma_m$. Let c be a nonstandard element of $\mathcal{M}$. Then for every standard k $\mathcal{M}$ satisfies "there exists an $\omega_m(k+1)$-large set U with $c \leqslant \min(U)$", so by overspill lemma the same holds in $\mathcal{M}$ for some nonstandard $k \in \mathcal{M}$. Pick such a nonstandard k and an appropriate set $U \in \mathcal{M}$. By Lemma 2.13 there exist a subset C of U which is $\omega_m(k)$-large and $\omega_m(k)$-scattered. In particular, C is ω^ω-scattered. We construct a sequence $D_1, \dots, D_m$ of sets in the sense of $\mathcal{M}$ such that

1. $C \supseteq D_1 \supseteq \cdots \supseteq D_m$,
2. D_j is $\omega_{m-j}(k-j)$-large for $j < m$ and D_m is $\frac{k-m}{2}$-large if $k - m$ is even and is $\frac{k-m-1}{2}$-large if $k - m$ is odd,

3. for every $j \leqslant m$ and every $d \in D_j$ we have: for every $\varphi, u_j, \dots, u_m \leqslant d$ if $\varphi \in \Delta_0$ and $Q_{j-1}, \dots, Q_0$ is a sequence of quantifiers, then for every $\langle f_{j-1}, \dots, f_0\rangle, \langle g_{j-1}, \dots, g_0\rangle \in [D_j]^j$ with $d < f_{j-1}$ and $d < g_{j-1}$, we have

$$\begin{aligned} &Q_{j-1}w_{j-1} < f_{j-1}\, Q_{j-2}w_{j-2} < f_{j-2} \cdots Q_0 w_0 \\ &\qquad < f_0\, \mathrm{Tr}_0\big(\varphi(S^{w_0}0, \dots, S^{w_{j-1}}0, S^{u_j}0, \dots, S^{u_m}0)\big) \\ &\text{iff} \\ &Q_{j-1}w_{j-1} < g_{j-1}\, Q_{j-2}w_{j-2} < g_{j-2} \cdots Q_0 w_0 \\ &\qquad < g_0\, \mathrm{Tr}_0\big(\varphi(S^{w_0}0, \dots, S^{w_{j-1}}0, S^{u_j}0, \dots, S^{u_m}0)\big) \end{aligned} \tag{13}$$

holds in $\mathcal{M}$.

The construction is as follows. We begin with the function which differs only notationally from (12). That is we let

$$g_1(x) = \begin{cases} \min\big\{w_0 : \mathrm{Tr}_0\big(\varphi(S^{w_0}0, S^{u_1}0, \dots, S^{u_m}0)\big)\big\} \\ \qquad\qquad \text{if } x \text{ is of the form } \varphi(v_0, S^{u_1}0, \dots, S^{u_m}0), \text{with} \\ \qquad\qquad \varphi \in \Delta_0 \text{ and } \exists w_0\, \mathrm{Tr}_0\big(\varphi(S^{w_0}0, S^{u_1}0, \dots, S^{u_m}0)\big) \\ 0 \qquad\qquad \text{in other cases.} \end{cases} \tag{14}$$

We let B_1 be an $\omega_{m-1}(k)$-large subset of C which is an approximation of g_1. Writing $B_1 = \{b_0, \dots, b_{\mathrm{Card}(B)-1}\}$ we take as D_1 the set consisting of these $b \in B_1$ which have even indices. By Lemma 2.13, this set is $\omega_{m-1}(k-1)$-large. Part 3 of (13) is checked exactly as above.

We let

$$\begin{aligned} A_1 = \big\{&\forall v_0\, \varphi(v_0, S^{u_1}0, \dots, S^{u_m}0) : \varphi \in \Delta_0, \varphi, u_1, \dots, u_m \leqslant \max(D_1), \\ &\forall w_0\, \mathrm{Tr}_0(\varphi(S^{w_0}0, S^{u_1}0, \dots, S^{u_m}0))\big\} \end{aligned}$$

and

$$\begin{aligned} E_1 = \big\{&\exists v_0\, \varphi(v_0, S^{u_1}0, \dots, S^{u_m}0) : \varphi \in \Delta_0, \varphi, u_1, \dots, u_m \leqslant \max(D_1), \\ &\exists w_0\, \mathrm{Tr}_0\big(\varphi(S^{w_0}0, S^{u_1}0, \dots, S^{u_m}0)\big)\big\}. \end{aligned}$$

We use these sets, or, more exactly, A_1, to define the function g_2. We let

$$g_2(x) = \begin{cases} \min\{w_1 : \ulcorner\forall v_0\, \varphi(v_0, S^{w_1}0, S^{u_2}0, \cdots, S^{u_m}0)\urcorner \in A_1\} \\ \qquad\qquad \text{if } x \text{ is } \forall v_0\, \varphi(v_0, v_1, S^{u_2}0, \dots, S^{u_m}0), \varphi \in \Delta_0 \\ \qquad\qquad \text{and there exists } w_1 \text{ as above} \\ 0 \qquad\qquad \text{in other cases.} \end{cases} \tag{15}$$

Let B_2 be an $\omega_{m-2}(k-1)$-large subset of D_1 that is an approximation of g_2 and let D_2 be the set of these $b \in B_2$ which have even numbers in the increasing enumeration of B_2. Then D_2 is $\omega_{m-2}(k-2)$-large.

Pick $b \in B_2$ such that $h_2^{B_2}(b)\downarrow$. Then for $\varphi, u_2, \dots, u_m \leqslant b$, the substitution $\forall v_0\, \varphi(v_0, v_1, S^{u_2}0, \dots, S^{u_m}0)$ is smaller than $h^{B_2}(b) - 2$, hence the appropriate w_1 is either below $h_2^{B_2}(b)$ or is greater or equal $\max(B_2)$. Thus, for $d \in D_2$ and $\varphi, u_2, \dots, u_m \leqslant d$ and every $t_1, s_1 \in D_1$, if $d < t_1, s_1$ and $\varphi \in \Delta_0$, then

$$\exists w_1 < t_1 \ulcorner \forall v_0\, \varphi(v_0, S^{w_1}0, S^{u_2}0, \ldots, S^{u_m}0)\urcorner \in A_1$$
$$\text{iff} \tag{16}$$
$$\exists w_1 < s_1 \ulcorner \forall v_0\, \varphi(v_0, S^{w_1}0, S^{u_2}0, \ldots, S^{u_m}0)\urcorner \in A_1.$$

It follows that for each $d \in D_2$ and each $\varphi, u_2, \ldots, u_m \leqslant d$ and each $t_1 < t_0$ and $s_1 < s_0$ in D_2 and strictly greater than d, if $\varphi \in \Delta_0$, then

$$\exists w_1 < t_1\, \forall w_0 < t_0\, \mathrm{Tr}_0\big(\varphi(S^{w_0}0, S^{w_1}0, S^{u_2}0, \ldots, S^{u_m}0)\big)$$
$$\text{iff} \tag{17}$$
$$\exists w_1 < s_1\, \forall w_0 < s_0\, \mathrm{Tr}_0\big(\varphi(S^{w_0}0, S^{w_1}0, S^{u_2}0, \ldots, S^{u_m}0)\big).$$

A similar reasoning with $\neg\varphi$ establishes part 3 of the required properties of D_2, i.e., (13).

We iterate this construction. So let D_j be given and have the properties listed above. We let A_j be the set of all sentences

$$\forall v_{j-1}\, \exists v_{j-2} \cdots Q v_0\, \varphi(v_0, \ldots, v_{j-1}, S^{u_j}0, \ldots, S^{u_m}0)$$

(where Q is the appropriate quantifier) such that $\varphi \in \Delta_0$, and there exists a sequence $d < t_{j-1} < \cdots < t_0$ of elements of D_j such that $\varphi, u_j, \ldots, u_m \leqslant d$ and

$$\forall w_{j-1} < t_{j-1}\, \exists w_{j-2} < t_{j-2} \cdots Q w_0 \\ < t_0\, \mathrm{Tr}_0(\varphi(S^{w_0}0, \ldots, S^{w_{j-1}}0, S^{u_j}0, \ldots, S^{u_m}0)).$$

We let $g_j(x) =$

$$\min\big\{w_j : \forall v_{j-1}\, \exists v_{j-2} \cdots Q v_0\, \varphi(v_0, \ldots, v_{j-1}, S^{w_j}0, S^{u_{j+1}}0, \ldots, S^{u_m}0) \in A_j\big\}$$
$$\text{if } x \text{ is } \forall v_{j-1} \exists v_{j-2} \cdots Q v_0\, \varphi(v_0, \ldots, v_{j-1}, v_j, S^{u_{j+1}}0, \ldots, S^{u_m}0) \text{ with } \varphi \in \Delta_0$$
$$\text{and } w_0 \text{ exists} \tag{18}$$

and $g_j(x) = 0$ otherwise.

Let B_{j+1} be an $\omega_{m-j-1}(k-j)$-large subset of D_j which is an approximation for g_{j+1} and let D_{j+1} be the set of these $b \in B_{j+1}$ which have even numbers in the increasing enumeration of B_{j+1}. Then conditions 1 and 2 are obvious, so let us indicate why 3 holds. Pick $d \in D_{j+1}$ and $\varphi, u_{j+1}, \ldots, u_m \leqslant d$ be given. Then the substitution $\forall v_{j-1} \cdots Q v_0\, \varphi(v_0, \ldots, v_j, S^{u_{j+1}}0, \ldots, S^{u_m}0)$ is below the next element of B_{j+1}, hence for $t_j, s_j \in D_{j+1}$ strictly greater than d we have: $\exists w_j < t_j \ulcorner \forall v_{j-1}\, \exists v_{j-2} < t_j \cdots Q v_0\, \varphi(v_0, \ldots, v_{j-1}, S^{w_j}0, S^{u_{j+1}}0, \ldots, S^{u_m}0)\urcorner \in A_j$ iff $\exists w_j < s_j \ulcorner \forall v_{j-1}\, \exists v_{j-2} s_j \cdots Q v_0\, \varphi(v_0, \ldots, v_{j-1}, S^{w_j}0, S^{u_{j+1}}0, \ldots, S^{u_m}0)\urcorner$ belongs to A_j, hence we may apply the inductive assumption.

The last of these sets, D_m, is of nonstandard finite cardinality. Let I be any initial segment of $\mathcal{M}$, closed under successor and smaller than $\mathrm{Card}(D_m)$. Let $J_I = \{e \in \mathcal{M} : e < d_i \text{ for some } i \in I\}$, where $D_m = \{d_0, \ldots, d_{\mathrm{Card}(D_m)-1}\}$ in increasing order. In order to show that $J_I \models I\Sigma_m$ we show a reduction of Σ_m-truth in J_I to truth in $\mathcal{M}$.

We claim that for each $j \leqslant m$ we have:

$$\text{for each } \Delta_0 \text{formula } \varphi \text{ and each } u_j, \ldots, u_m \in J_I \\ J_I \models \forall w_{j-1}\, \exists w_{j-2} \cdots Q w_0\, \varphi(w_0, \ldots, w_{j-1}, u_j, \ldots, u_m) \tag{19}$$

iff

> there exists $d \in D_j$ greater than all parameters $u_j, \ldots, u_m$ and a sequence $t_{j-1} < \cdots < t_0$ of elements of D_m strictly greater than d such that
> $$\mathcal{M} \models \forall w_{j-1} < t_{j-1}\, \exists w_{j-2} < t_{j-2} \cdots Q w_0 < t_0\, \varphi(w_0, \ldots, w_{j-1}, u_j, \ldots, u_m). \tag{20}$$

This fact is proved by induction on j, case $j = 1$ being immediate. Assume the fact for j, we show it for $j + 1$. Assume

$$J_I \models \forall w_j\, \exists w_{j-1} \cdots Q w_0\, \varphi(w_0, \ldots, w_j, u_{j+1}, \ldots, u_m).$$

Pick $d \in D_j \cap J_I$ greater than all parameters $u_{j+1}, \ldots, u_m$. Let $t_j \in D_j$ be the next element of D_j. Consider the formula obtained from the above by bounding the first quantifier to t_j. This formula is true in J_I, so by the inductive assumption there exists a sequence $t_{j-1}, \ldots, t_0$ as desired. For the converse assume that d and a sequence $t_j, \ldots, t_0$ like on the right hand side exists. Then each sequence $s_j, \ldots, s_0$ of elements of D_{j+1} has the same property. Pick $w_j \in J_I$. Find $d' \in D_{j+1}$ which is greater than d and w_j. Pick a sequence $s_j < s_{j-1} < \cdots < s_0$ of elements of D_{j+1} with $s_j > d'$. Then

$$\mathcal{M} \models \exists w_{j-1} < s_{j-1}\, \cdots\, Q w_0 s_0\, \varphi(w_0, \ldots, w_{j-1}, w_j, u_{j+1}, \ldots, u_m)$$

by (13). By the inductive assumption

$$J_I \models \exists w_{j-1}\, \cdots\, Q w_0\, \varphi(w_0, \ldots, w_{j-1}, w_j, u_{j+1}, \ldots, u_m).$$

Let us show how to use the above reduction to show that $J_I \models I\Sigma_m$. Let $\varrho(u, v)$ be a Π_m formula and assume that $J_I \models \exists v\, \varrho(u, v)$. Write ϱ in the form $\forall w_{m-1}\, \exists w_{m-2} \cdots \varphi(w_0, \ldots, w_{m-1}, u, v)$ with $\varphi \in \Delta_0$. Pick an appropriate $v \in J_I$ and let $u_m = \langle u, v \rangle$. Pick a sequence $t_{m-1} < \cdots < t_0$ of elements of D_m such that $u_m \leqslant d < t_{m-1}$ for some $d \in D_j$. By the above reduction, $\mathcal{M} \models \forall w_{m-1} < t_{m-1}\, \exists w_{m-2} < t_{m-2} \cdots Q w_0\, \varphi(w_0, \ldots, w_{m-1}, u, v)$. But the smallest such v exists in $\mathcal{M}$ (because this is a model for $I\Delta_0$), and this element is the smallest which satisfies ϱ in J_I.

Granted this construction we see that part 1 of Theorem 10.2 is just a reformulation of Theorem 9.1. Part 2 was proved above. Let us merely point out that in order to obtain initial segments satisfying $I\Sigma_m$ in the language with the additional predicate letter for A one applies the same construction, but allows formulas with parameter A in the definition of functions g_j. Part 3 is immediate by parts 1 and 2, and part 4 is a direct consequence of part 2. □

Proof of Theorem 10.3. This is the same argument as that for Theorem 10.2. The only difference is that we begin with a set $A \in \mathcal{M}$ which is ω_m-large for some nonstandard $m \in \mathcal{M}$ and perform the above construction $m - 1$ times. We leave the details to the reader. □

The following result is known in Warsaw as Wainer's theorem.

Theorem 10.4.

1. *For all* $m \in \mathbb{N}$, PA *proves that* S_{ω_m} *is total,*
2. *if* f *is a recursive function such that for some* Σ_1 *definition of* f, PA *proves that* f *is total, then* f *is dominated by* S_{ω_m} *for some* $m \in \mathbb{N}$.

But the conclusion is much stronger. In order to state it we extend the notion of a fundamental sequence to ε_0. We let $\{\varepsilon_0\}(m) = \omega_m$. Of course, this allows us to speak about ε_0-large sets. Thus, A is ε_0-large iff it is ω_{a_0}-large, where $a_0 = \min(A)$. Let $\mathrm{RFN}_{\mathrm{PA}}(\Sigma_t)$ denote the scheme of reflection (with parameters) for Σ_t-formulas. In the statement of the theorem below if $t = 0$, say that Z is ϕ_{-1}-scattered if it is scattered with respect to the usual successor function.

Theorem 10.5. *For every t,* PA *proves that the scheme* $\mathrm{RFN}_{\mathrm{PA}}(\Sigma_t)$ *is equivalent with the statement* $\forall a\, \exists Z[\min(Z) > a$ *& Z is ϕ_{t-1}–scattered and ε_0–large*$]$.

Proof. Left to the reader. □

Here is a version of the main result of J. Paris and L. Harrington [32].

Theorem 10.6. *Let $k \geqslant 2$. Then $I\Sigma_{k-1}$ does not prove the following statement: "for every a there exists b such that for every partition $L : [a,b]^{k+1} \to 3^{2k-3}$ there exists a set $D \subseteq [a,b]$ which is homogeneous for L and satisfies* $\mathrm{Card}(D) > \min(D) + \frac{(k-2)\cdot(k-1)}{2} + 1$*".*

Proof. By formalizing the proof of Theorem 4.3 we see that otherwise $I\Sigma_{k-1}$ would prove the statement "for every a there exists b such that the interval $[a,b]$ is ω_k-large". But this contradicts Theorem 10.2. □

11. Goodstein Sequences

One of the most interesting independent statements was found by L. Kirby and J. Paris [20]. Below I give a sketch of an exposition. Suppose we are given two natural numbers, m, n, where $n > 1$. We write m to the base n:

$$m = n^k \cdot a_k + n^{k-1} \cdot a_{k-1} + \cdots + n \cdot a_1 + a_0. \tag{21}$$

We do the same with all exponents and repeat this until we obtain an expression in which only natural numbers strictly smaller than n occur as exponents and coefficients, and n itself as a basis of exponentiation. The expression obtained in this way is called the *full expansion of n to the base n*. We define the number $G_n(m)$ as follows. If $m = 0$ we let $G_n(m) = 0$. Otherwise we let $G_n(m)$ be the number obtained by substituting $n+1$ for n in the full expansion of m to the base m, subtracting 1 from the number obtained in this way. Now we define the *Goodstein sequence for m starting at n*, $\mathrm{Good}_{n,k}(m)$, by induction on k. We let $\mathrm{Good}_{n,0}(m) = m$, $\mathrm{Good}_{n,k+1}(m) = G_{n+k}(\mathrm{Good}_{n,k}(m))$. Observe that this changing bases of exponentiation seems to force this sequence to be increasing, because this subtracting of 1 seems to be relatively unimportant. But we have:

Theorem 11.1 (L. Kirby, J. Paris [20]).

1. *If $m, n > 1$, then the sequence* $\mathrm{Good}_{n,k}(m)$ *eventually hits* 0*,*
2. *if $n > 1$, then the statement* $\forall m\, \exists k\, \mathrm{Good}_{n,k}(m) = 0$ *is independent from* PA.

Kirby and Paris give the following estimate: if $n = 2$ and $m = 266$, then $\mathrm{Good}_{2,3}(266) \simeq 10^{10\,000}$. This slightly anthropomorphic description of Goodstein sequences suffices to convince the reader of truth of the first part of Theorem 11.1. The reason is as follows. Each $\mathrm{Good}_{n,k}(m)$ is a full expansion of some natural number to the base $n+k$. Replace the base in each item of the Goodstein sequence by ω. Clearly we obtain a sequence of ordinals $\gamma_{n,k}(m)$ (cf. Section 1). Moreover, a moment of contemplation yields: the sequence of ordinals obtained in this way is strictly decreasing, so must be finite because of transfinite induction. But the only reason is that this sequence reached 0 and the first part of Theorem 11.1 follows.

The second part is be proved by showing that the set $\{n, n+1, \ldots, n+r\}$, where r is the first place where our Goodstein sequence hits 0, of bases used in the Goodstein sequence is α-large, where α depends on n and m (then we apply part 4 of Theorem 10.3).

Let the function $\mathrm{tow}_k(n)$ associate $n^{\cdot^{\cdot^{n}}}$ with n (the tower of k n's). Of course, the definition is inductive on k: we let $\mathrm{tow}_0(n) = 1$, and $\mathrm{tow}_{k+1}(n) = n^{\mathrm{tow}_k(n)}$. Observe that

Lemma 11.2. *If $m = \mathrm{tow}_s(n)$, then $\gamma_{n,0}(m) = \omega_s$.*

Now we identify the greatest ordinal $< \alpha$ whose pseudonorm is $\leqslant a$. We define the symbol $\mathrm{GO}(a,\alpha)$ for $a > 0$ and $\alpha > 0$ by induction on α. We let $\mathrm{GO}(a,1) = 0$, $\mathrm{GO}(a,\omega) = a$. Other cases are as follows.

$$\mathrm{GO}(a, \alpha+1) = \begin{cases} \alpha & \text{if } \mathrm{psn}(\alpha) \leqslant a \\ \mathrm{GO}(a,\alpha) & \text{if } \mathrm{psn}(\alpha) > a. \end{cases}$$

Before giving the general limit step we put

$$\mathrm{GO}(a, \omega^{\nu}) = \omega^{\mathrm{GO}(a,\nu)} \cdot a + \mathrm{GO}(a, \omega^{\mathrm{GO}(a,\nu)}).$$

Finally, if $\alpha = \xi + \omega^{\nu}$ in short Cantor normal form and $\xi \neq 0$, then

$$\mathrm{GO}(a, \alpha) = \begin{cases} \xi + \mathrm{GO}(a, \omega^{\nu}) & \text{if } \mathrm{psn}(\xi) \leqslant a \\ \mathrm{GO}(a,\xi) & \text{if } \mathrm{psn}(\xi) > a. \end{cases}$$

It is easy to check (by induction on α) that $\mathrm{GO}(a,\alpha) < \alpha$ and $\mathrm{psn}(\mathrm{GO}(a,\alpha)) \leqslant a$. In fact, $\mathrm{GO}(a,\alpha)$ is the greatest ordinal with these properties. Now an analysis of the construction yields the following fact

Lemma 11.3. *When passing from $\mathrm{Good}_{n,k}(m)$ to $\mathrm{Good}_{n,k+1}(m)$, the ordinals associated with them satisfy $\gamma_{n,k+1}(m) = \mathrm{GO}(n+k+1, \gamma_{n,k}(m))$.*

Actually, both Lemmas 11.2 and 11.3 require some work. The reason is that we described the notion of the Goodstein sequence in a slightly anthropomorphic way. Here is a more precise description of Goodstein sequences. Let n, m be as above and consider the expansion (21) of m to the base n. Put $f^{m,n}(x) = \sum_{i=0}^{k} a_i \cdot x^{f^{i,n}(x)}$. This definition is inductive on m, beginning with $f^{m,n}(x) = 0$. Observe that $f^{m,n}(x) \in \mathbb{N}$ for $x \in \mathbb{N}$ and if we substitute $x = \omega$, then $f^{m,n}(\omega)$ is an ordinal below ε_0. This gives the following description of Goodstein sequences: $\mathrm{Good}_{n,0}(m) = m = f^{m,n}(n)$ and

$$\mathrm{Good}_{n,k+1}(m) = G_{n+k}(\mathrm{Good}_{n,k}(m)) = f^{\mathrm{Good}_{n+k,k}(m)}(n+k+1) - 1.$$

This description is used in the appropriate analysis. Summing up, we have

Lemma 11.4. *If* $m = \mathrm{tow}_s(n)$*, and* r *is the smallest natural number such that* $\mathrm{Good}_{n,r}(m) = 0$*, then the set* $A = \{m, m+1, \ldots, m+r\}$ *is* ω_s*-large.*

Idea of the proof. This comes out from the discussion in Lemma 2.7. In that lemma we used a function which associates with each element of A an ordinal which is smaller than the one associated by the construction used in this section and one has to compare these two associations. □

Of course, Theorem 11.1 follows from Lemma 11.4 because of Theorem 10.3.

References

[1] J. Avigad and R. Sommer, A model theoretic approach to ordinal analysis, *Bulletin of Symbolic Logic* **3**(1) (March 1997).

[2] J. Avigad and R. Sommer, The model–theoretic ordinals analysis of theories of predicative strength, *Jour. Symb. Log.* **64** (1999), 327–349.

[3] T. Bigorajska and H. Kotlarski, A partition theorem for α-large sets, *Fund. Math.* **160** (1999), 27–37.

[4] T. Bigorajska and H. Kotlarski, Some combinatorics involving ξ-large sets, *Fund. Math.* **175** (2002), 119–125.

[5] T. Bigorajska and H. Kotlarski, Partitioning α-large sets: some lower bounds, *Trans. Amer. Math. Soc.* **358**(11) (2006), 4981–5001.

[6] P. D'Aquino and J. Knight, *Strong initial segments of models of* $I\Delta_0$, to appear.

[7] B. Dreben and J. Denton, Herbrand–style consistency proofs, in: *Intuitionism and proof theory*, J. Myhill, A. Kino and R.E. Vesley, eds, North Holland Publ. Comp., 1970, pp. 419–433.

[8] A. Ehrenfeucht and A. Mostowski, Models of axiomatic theories admitting automorphisms, *Fund. Math.* 43 (1956), 50–68. Reprinted in: A. Mostowski, *Foundational studies*, vol. I, North Holland Publ. Comp. and Państwowe Wydawnictwo Naukowe, 1979, pp. 494–512.

[9] M. Fairtlough and S. Wainer, Ordinal complexity of recursive definitions, *Information and computation* **99**(2) (1992), 123–153.

[10] M. Fairtlough and S. Wainer, Hierarchies of provably recursive functions, in: *Handbook of Proof Theory*, S. Buss, ed., North Holland Publ. Comp., 1998, pp. 149–207.

[11] H. Friedman, Countable models of set theories, in: *Cambridge Summer School in Mathematical Logic*, Lecture Notes in Math., vol. 337, Springer Verlag, 1973, pp. 539–573.

[12] W.D. Goldfarb and T.M. Scanlon, The ω-consistency of number theory via Herbrand's theorem, *Jour. Symb. Log.* **39** (1974), 678–692.

[13] W.D. Goldfarb, Ordinal bounds for k-consistency, *Jour. Symb. Log.* **39** (1974), 693–699.

[14] R. Graham, B. Rothschild and J. Spencer, *Ramsey theory*, 2–nd edition, Wiley, 1990.

[15] P. Hájek and J. Paris, Combinatorial principles concerning approximations of functions, *Archive für Mathematische Logik und Grundlagenforschung* **26** (1986/87), 13–28.

[16] P. Hájek and P. Pudlák, *Metamathematics of First–Order Arithmetic*, Perspectives in Mathematical Logic, Springer–Verlag, 1993.

[17] R. Kaye, *Models of Peano arithmetic*, Oxford Logic Guides, vol. 15, Oxford University Press, 1991.

[18] J. Ketonen and R. Solovay, Rapidly growing Ramsey functions, *Annals of Mathematics* **113** (1981), 267–314.

[19] L. Kirby, *Initial segments in models of Peano Arithmetic*, Ph.D. Thesis, University of Manchester, 1977.

[20] L. Kirby and J. Paris, Accessible independence results for Peano arithmetic, *Bull. London Math. Soc.* **14** (1982), 285–293.

[21] R. Kossak and J. Schmerl, *The Structure of Nonstandard Models of Arithmetic*, Oxford Logic Guides, Oxford University Press, 2006.

[22] H. Kotlarski, Bounded induction and satisfaction classes, *Zeitschr. Math. Log.* **32** (1986), 531–544.

[23] H. Kotlarski and B. Piekart, Some variations of the Hardy hierarchy, *Math. Log. Quart.* **51**(4) (2005), 417–434.

[24] H. Kotlarski, B. Piekart and A. Weiermann, *More on lower bounds for partitioning α-large sets*, submitted to Annals of Pure and Applied Logic.

[25] H. Kotlarski and Z. Ratajczyk, Inductive full satisfaction classes, *Annals of Pure and Applied Logic* **47** (1990), 199–223.

[26] H. Kotlarski and Z. Ratajczyk, More on induction in the language with a satisfaction class, *Zeitschrift Math. Log.* **36**(5) (1990), 441–454.

[27] K. Kuratowski and A. Mostowski, *Set theory*, North Holland Publ. Comp. and Państwowe Wydawnictwo Naukowe, 1976.

[28] G.E. Mints, *Exact estimate of the provability of transfinite induction in initial parts of arithmetic*, Zap. Naučn. Sem. Leningr. Otdel. Mat. In-ta V.A. Steklova, vol. 49, 1971, pp. 115–133.

[29] P.G. Odifreddi, *Classical recursion theory*, North Holland Publ. Comp., vol. I, 1989, vol. II, 1999.

[30] J. Paris, Some independence results for Peano arithmetic, *Journal of Symb. Log.* **43** (1978), 752–731.

[31] J. Paris, A hierarchy of cuts in models of arithmetic, in: *Model theory of algebra and arithmetic*, Springer Lecture Notes in Math., vol. 834, Springer–Verlag, 1890, pp. 312–337.

[32] J. Paris and L. Harrington, A mathematical incompleteness in Peano arithmetic, in: *Handbook of mathematical logic* (J. Barwise, editor), North Holland Publ. Comp., 1977, pp. 1133–1142.

[33] Z. Ratajczyk, A combinatorial analysis of functions provably recursive in $I\Sigma_n$, *Fund. Math.* **130** (1988), 191–213.

[34] Z. Ratajczyk, Subsystems of true arithmetic and hierarchies of functions, *Ann. Pure and Applied Logic* **64** (1993), 95–152.

[35] T.M. Scanlon, The consistency of number theory via Herbrand's theorem, *Jour. Symb. Log.* **38** (1973), 29–58.

[36] R. Sommer, *Transfinite induction and hierarchies of functions generated by transfinite recursion within Peano arithmetic*, Ph.D. thesis, University of California, Berkeley, 1990.

[37] R. Sommer, Transfinite induction within Peano Arithmetic, *Ann. Pure and Appl. Log.* **76** (1995), 231–289.

[38] A. Weiermann, A classification of rapidly growing Ramsey functions, *Proc. Amer. Math. Soc.* **132**(2) (2003), 553–561.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Towards Decidability of the Theory of Pseudo-Finite Dimensional Representations of $sl_2(k)$; I

Sonia L'INNOCENTE [a,1] and Angus MACINTYRE [b]

[a] *Department of Mathematics and Computer Science, University of Camerino, Camerino, Italy*
sonia.linnocente@unicam.it

[b] *School of Mathematics, Queen Mary University of London, London, England*
angus@dcs.qmul.ac.uk

1. Introduction

In this paper, we refine the analysis begun in Ivo Herzog's paper [7] on representations of the Lie algebra $sl_2(k)$, where k is an algebraically closed field of characteristic 0. Our principal contribution is to bring out a connection to fundamental problems in the diophantine geometry of curves. We expect to show, in a subsequent paper, that the theory of finite dimensional representations of $sl_2(k)$ is decidable, modulo some widely believed conjectures in diophantine geometry. It should be noted that Prest and Puninski [13] showed that the theory of all $sl_2(k)$-modules is undecidable (this important result seems not to be well-known).

Our model theory and definability are relative to the formalism of left R-modules for a ring R [12]. In particular, we tacitly identify the theory of representations of the Lie algebra $sl_2(k)$ with the theory of modules over U_k, the universal enveloping algebra of $sl_2(k)$. We follow Herzog in calling a U_k-module M finite dimensional if it is finite dimensional over k. (Note that k is fixed throughout the paper. We discuss the effect of varying k in the sequel.)

Then M is pseudo-finite dimensional (henceforward PFD) if it satisfies all sentences of the language of U_k-modules true in all finite dimensional modules. By classical model theory [4], M is PFD if and only if M is elementarily equivalent to an ultraproduct of finite dimensional modules.

The study of finite dimensional U_k-modules M is dominated by the classical result due to Lie and Study (see, for instance, [6], Theorem 8.7) showing that any such M is uniquely a finite direct sum of simple finite dimensional modules and there is exactly one of the latter for each finite dimension. We write V_λ for the unique simple finite

[1]This research was carried out between October and December of 2006, when the first author visited the second in London with financial support from Indam, Istituto di Alta Matematetica "Francesco Severi", Roma, Italy.

dimensional module of dimension $\lambda + 1$, and recall that it has a beautiful presentation as the additive group of homogeneous polynomials $F(X, Y)$ of degree λ in X, Y, with U_k acting as certain differential operators ([6], § 8.1). Part of the motivation for studying PFD-modules is to isolate the model-theoretic uniformities in the $V(\lambda)$, as $\lambda \to \infty$.

The ring U_k is left and right Ore domain (for instance, see [5], § 2.3), and thus belongs to a well-studied class. To understand PFD-modules, Herzog introduced an exotic epimorphic extension U'_k of U_k and showed:

1. U'_k is a von Neumann regular ring;
2. PFD-modules are naturally U'_k-modules;
3. the model theory of U'_k-modules is interpretable in that of U_k-modules;
4. there is an elegant axiomatization of the class of PFD-modules as a subclass of the class of U'_k-modules.

Unfortunately, the very abstract nature of the construction of U'_k leaves some basic questions unanswered. We remedy this by giving a "recursive" construction of U'_k, building it from U_k in stages. This should enable us to prove decidability of the theory of PFD-modules (assuming some plausible conjectures about the decision problem for integer points on curves). We are obliged to describe the structure of the sets

$$\{\lambda : V(\lambda) \models \Phi\},$$

for Φ a sentence of the language of U_k-modules. In this paper we bring out some basic new information about the case when Φ concerns the nontriviality of certain kernels. This is where diophantine geometry is relevant.

For both of us it is an honour to dedicate a paper to the memory of Andrzej Mostowski. The junior author (S. L'I.) did not exist at the time of Mostowski's untimely death, and the model theory of modules was just beginning, but she is well aware of the lasting importance of his ideas (and we use essentially some of his work in this paper).

The senior author (A. M.) would like to make a more personal statement:

I began reading Mostowski's books and papers when I was an isolated teenager in Scotland, and was taken by their range and clarity. Throughout graduate school I continued to learn more of his work, and the paper with Andrzej Ehrenfeucht has remained one of my favorites. I first met Mostowski in 1968 (in Warsaw and in Italy), and was very much encouraged by the interest he showed in my work. We met a few times before 1975, and he became for me one of the most admired figures on logic, both for his work and for the strength and generosity of his personality. His death had many of us fearing for the future of logic in Poland, And yet, after thirty years, Poland is rich in young researchers, and the ideas of Mostowski's generation (and later mine) have evolved very far, exactly the right memorial to an outstanding teacher and researcher.

Both authors are deeply grateful to the anonymous referee for a careful reading of the paper, and constructive criticism.

2. The Basic Structure and Formalism

2.1. The Lie Algebra $sl_2(k)$

Fix, for the rest of the paper, an algebraically closed field k of characteristic 0. $sl_2(k)$ is the Lie algebra (over k) of 2×2 matrices of trace 0. Throughout we consider the basis of $sl_2(k)$ over k given by x, y, h where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy the following relations:

$$[x, y] = h,$$
$$[h, x] = 2x,$$
$$[h, y] = -2y.$$

(and indeed all other Lie algebra relations are generated by these).

The study of left modules over the Lie algebra $sl_2(k)$ is naturally equivalent to the study of left modules over U_k the universal enveloping algebra [3]. U_k is well-understood. Via the natural embedding $sl_2(k) \to U_k$, we construe U_k as freely generated over k by elements x, y, h satisfying:

$$xy - yx = h,$$
$$hx - xh = 2x,$$
$$hy - yh = -2y.$$

U_k is a left and right Ore domain ([5], § 2.3), and so has a field of fractions. We follow Herzog in calling this field K. Its U_k-module structure turns out to be important.

Before looking at U_k-modules, we review the basics about the ring structure of U_k. Here, and in most of the paper, we are presenting, with a different emphasis, ideas from Herzog's paper [7].

Recall that U_k has a $\mathbb{Z}$-grading as k-algebra, induced by defining:

$$gr(h) = gr(\alpha) = 0 \quad \text{for all } \alpha \in k$$
$$gr(x) = 1,$$
$$gr(y) = -1.$$

Let $U_{k,n}$ be the subalgebra of elements of grade n. We have

$$U_k = \bigoplus_{n \in \mathbb{Z}} U_{k,\,n};$$

$$\text{for } n > 0,\ U_{k,\,n} = x^n U_{k,\,0} = U_{k,\,0} x^n;$$
$$\text{for } n < 0,\ U_{k,\,n} = y^n U_{k,\,0} = U_{k,\,0} y^n.$$

These results are proved by simple manipulations of the basic relations connecting x, y and h.

Of fundamental importance is the *Casimir operator*, defined as

$$c = xy + yx + \frac{1}{2}h^2 \quad \in U_{k,\,0}.$$

Lemma 2.1. *c is central in U_k.*

Proof. It is enough to prove that c commutes with x, y and h.

i) $xc = x\big(xy + yx + \frac{1}{2}h^2\big) = xyx + x(yx+h) + \frac{1}{2}(hx-2x)h = 2xyx + \frac{1}{2}hxh,$

$$\begin{aligned} cx &= (xy + yx + \tfrac{1}{2}h^2)x = xyx + (xy-h)x + \tfrac{1}{2}h^2x \\ &= 2xyx - hx + \tfrac{1}{2}h(xh+2x) = 2xyx + \tfrac{1}{2}hxh = xc; \end{aligned}$$

ii) $yc = cy$ is proved similarly;

iii) $ch = \big(xy + yx + \frac{1}{2}h^2\big)h = \big(h + 2yx + \frac{1}{2}h^2\big)h = h^2 + \frac{1}{2}h^3 + 2yxh$

$$\begin{aligned} &= h^2 + \tfrac{1}{2}h^3 + 2y(hx-2x), \\ hc &= h\big(xy + yx + \tfrac{1}{2}h^2\big) = h\big(h + 2yx + \tfrac{1}{2}h^2\big) = h^2 + \tfrac{1}{2}h^3 + 2hyx \\ &= h^2 + \tfrac{1}{2}h^3 + 2(yh-2y)x = ch. \end{aligned}$$ □

Lemma 2.2. *$U_{k,0} = k[h,c]$, the polynomial ring on the two commuting generators h, c.*

Proof. See [7] for a brief sketch, using the Poincaré-Birkhoff-Witt Theorem. □

Similarly, one can prove.

Lemma 2.3. *The center of U_k is $k[c]$.*

2.2. The Simple Finite Dimensional Modules

Let λ be an integer $\geqslant 1$. Let V_λ be the k-vectorspace of homogeneous polynomials of degree λ over k in the two variables X and Y. V_λ has dimension $\lambda + 1$, and a natural basis is given by the monomials

$$\{X^i \cdot Y^{\lambda-i}, \quad 0 \leqslant i \leqslant \lambda\}.$$

V_λ is given a U_k-module structure by having:

$$x \text{ act as } X\frac{\partial}{\partial Y}$$

$$y \text{ act as } Y\frac{\partial}{\partial X},$$

$$h \text{ act as } X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

See [6], § 8.1., for the details.

Lemma 2.4. *V_λ is a simple U_k module. Every simple finite dimensional U_k-module is isomorphic to a unique V_λ. Furthermore, every finite dimensional U_k-module is isomorphic to a direct sum of simple modules, uniquely up to relabelling.*

Proof. See [6], Theorem 8.2 and Theorem 8.5 for the first two statements and Theorem 8.7 for the last statement. □

Of crucial importance for us are the eigenvalues of h, c, x and y on a finite dimensional module M. Here, let us summarize the most basic definitions and facts that we will use later:

i) Any simple finite dimensional U_k-module V_λ decomposes as follows:

$$V_\lambda = \bigoplus_{0 \leqslant i \leqslant \lambda} V_{\lambda,\, i},$$

where each $V_{\lambda,\, i}$ equals the one dimensional h-invariant subspace $\{v \in V_\lambda : hv = (\lambda - 2i)v\}$ of V_λ; more precisely, we have $V_{\lambda,\, i} = Ker(h - (\lambda - 2i) \cdot 1)$.

ii) On V_λ, c acts as scalar multiplication by $\frac{\lambda(\lambda+2)}{2}$.

iii) If we make the convention that $V_{\lambda,\, i} = \{0\}$ if $i \notin [-\lambda, \lambda]$, then:

$$x \text{ maps } V_{\lambda,\, i} \text{ to } V_{\lambda,\, i-1},$$

and

$$y \text{ maps } V_{\lambda,\, i} \text{ to } V_{\lambda,\, i+1}.$$

iv) $Ker(x) = V_{\lambda,0}$, and
$Ker(y) = V_{\lambda,\lambda}$.

v) x and y act nilpotently.

vi) $V_\lambda = Ker(x) \oplus Image(y)$
$V_\lambda = Image(x) \oplus Ker(y)$.
The $V_{\lambda,\, i}$ are called the *weight spaces*, $V_{\lambda,\, 0}$ is called the *highest weight* space and $V_{\lambda,\, \lambda}$ is called the *lowest weight* space. (The terminology will be suitably adjusted once we deal with general U_k-module M.)

vii) Let M be a finite dimensional U_k-module. For $\lambda \in k$, define $Cas(\lambda, M)$ to be $Ker(c - \frac{\lambda(\lambda+2)}{2} \cdot 1)$. Then

$$M = \bigoplus_{\lambda \in k} Cas(\lambda, M),$$

with $Cas(\lambda, M) = \{0\}$ unless $\lambda \in \mathbb{N}$. Indeed, $Cas(\lambda, M)$ is isomorphic to some finite power of V_λ.
Note that for $\lambda \geqslant 0$, $\frac{\lambda(\lambda+2)}{2}$ determines λ.

viii) Define $Cas(M)$ as

$$\big\{\lambda : \; Cas(\lambda, M) \neq \{0\}\big\}.$$

2.3. Basic Model Theory and PFD Modules

See [12] for all the basics on the formalism for model theory of modules. Our language is that of abelian groups, with a unary function symbol for (the endomorphism given by) each element of U_k.

We begin by noting the following:

Lemma 2.5. *If M is a U_k-module, then*

$$M \equiv M \oplus M.$$

Proof. By the Baur-Monk criteria for elementary equivalence [18] (generalizing that of Szmielew [16]), the elementary type of M is determined by the cardinality (modulo ∞) of all $\varphi(M)/\psi(M)$, where φ, ψ are pp-formulas defining subgroups of M. But since the infinite field k is in the center of U_k, these groups are k-subspaces, and so the above indices are always 1 or infinite. Thus M and $M \oplus M$ have the same elementary invariants. $\square$

Corollary 2.6. *The class of finite dimensional U_k-modules is not closed under elementary equivalence.*

Proof. Lemma 2.5 shows that $M \equiv M^{(\omega)}$. $\square$

One has already reached an interesting question.

Question 2.7. *Which "U_k-modules" are elementary equivalent to a finite dimensional module?*

It is worth noting that the theory of finite dimensional U_k-modules (for k a recursive algebraically closed field) is co-r.e. (that is, it has a recursively enumerable complement). The basic structure theory of finite dimensional modules gives a recursive enumeration of them using explicit matrix representations of the actions of h, x and y on the spaces V_λ. But for any fixed M of finite dimension we can test truth in M using this matrix representation and the decidability of the theory of algebraically closed fields. Thus if a sentence is not in the theory, it will be enumerated at some finite stage.

Herzog defines any U_k-module M to be pseudo-finite dimensional (PFD) if it is a model of the elementary theory of all finite dimensional modules. By general model theory, M is PFD if and only if M is elementary equivalent to an ultraproduct of finite dimensional modules (see [4], Exercise 4.1.18).

One should note that various "pseudo-finite" structures have been studied, notably fields and groups in [1] and [17] respectively. The flavour here is different (as described in § 2.2), as it is the dimension which is pseudofinite, and not the cardinality.

Before entering on the more delicate details of the analysis, we discuss some fairly superficial aspects of ultraproducts of finite dimensional modules.

Lemma 2.8. *Let M be a PFD R-module. Then, the following properties hold:*

i) $Cas(\lambda, M) = \{0\}$ *unless* $\lambda \in \mathbb{N}$.
ii) *If M is finite dimensional then $Cas(M)$ is finite.*
iii) *If $Cas(M)$ is finite, then M is elementary equivalent to a finite dimensional U_k-module if and only if*

$$M = \sum_{\lambda \in Cas(M)} Cas(\lambda, M) \quad \left(= \bigoplus_{\lambda \in Cas(M)} Cas(\lambda, M) \right),$$

and this is a first order property of M, for fixed finite $Cas(M)$.

Proof.

i) Obvious, since the property is expressible in the language and is true for finite dimensional M.
ii) This follows from the decomposition into a sum of V_λ.
iii) Suppose M is finite dimensional. So, $Cas(M) = E$ for some finite subset $E \subseteq \mathbb{N}$. Obviously, each $Cas(\lambda, M)$, for $\lambda \in E$, is definable, and so one can express by a first-order sentence that $M = \sum_{\lambda \in E} Cas(\lambda, M)$.

The direct sum representation follows.

Conversely, if M has this form, then

$$M \cong \bigoplus_{\lambda \in E} Cas(\lambda, M) \equiv \bigoplus_{\lambda \in E} V_\lambda,$$

by Mostowski's Theorem [10] and the fact that in all finite dimensional M_0, for all λ and all sentences Φ

$$Cas(\lambda, M_0) \models \Phi \Leftrightarrow V_\lambda \models \Phi. \qquad \square$$

Corollary 2.9. *M is elementary equivalent to a finite dimensional M_0 if and only if $Cas(M)$ is finite and $M = \sum_{\lambda \in Cas(M)} Cas(\lambda, M)$*

Proof. This is immediate by Lemma 2.8 iii). $\square$

Remarks 2.10.

1. For M a nonzero PFD module, $Cas(M)$ may be $\{0\}$. To see this, take M equal to the ultraproduct $\prod_{\lambda \in \mathbb{N}} V_\lambda / D$, where D is nonprincipal.
   We will show in a later paper that there are $2^{\aleph_0}$ complete theories of PFD U_k-modules M with $Cas(M) = \{0\}$.
2. For any $E \subseteq \mathbb{N}$ with $0 \in E$, there is a PFD M with $Cas(M) = E$. This is almost immediate from the compactness theorem, for if $\lambda_1, \ldots, \lambda_n \in E$ (with n a nonzero natural number) and $\mu_1, \ldots, \mu_m \in \mathbb{N} \backslash E$ (with m a nonzero natural number), then

   $$V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_n}$$

   is a finite dimensional module M with $\mu_1, \ldots, \mu_m \notin Cas(M)$ and $\lambda_1, \ldots, \lambda_n \in Cas(M)$.

3. The Appropriate Definable Scalars

Let $FinDim$ be the class of all finite dimensional U_k-modules. We consider the ring U_k' of definable scalars attached to $FinDim$. In concrete terms, one consider pp-formulas $p(u, v)$ in two variables (modulo equivalence in all $M \in FinDim$) such that p defines a (necessarily additive) map $M \to M$ for all $M \in FinDim$. Addition and composition are defined in the obvious way, giving a ring structure on the equivalence classes. U_k' is the resulting ring (Herzog gives in [7] several equivalent definitions).

Remarks 3.1. Some things are immediately clear.

i) There is a natural ring homomorphism $U_k \to U_k'$;
ii) $FinDim$ is naturally a class of U_k'-modules;
iii) Each U_k'-module formula is naturally equivalent, for $M \in FinDim$, to a U_k-formula.

Less obvious is the result of Harish-Chandra.

Lemma 3.2. *$U_k \to U_k'$ is 1-1.*

Proof. See [5]. $\square$

Understanding U'_k is naturally a prerequisite for understanding PFD modules. Herzog revealed some striking facts about U'_k, and in particular that it is von Neumann regular. Probably because of the rather abstract account he gave of U'_k, he did not answer the following questions:

Question 3.3. *What are the elementary types of PFD modules over U_k?*

Question 3.4. *Is the elementary theory of PFD modules decidable, if k is countable and given a natural recursive presentation?*

We hope to answer both questions. We will make full use of Herzog's work, but will reorganize the analysis so that U'_k is constructed in stages, and the diophantine information is used systematically.

3.1. Duality and the Action of the Weyl Group

The Weyl group of $sl_2(k)$ is cyclic of order 2 and its generator induces an involution σ on U_k via:

$$\sigma(x) = -y;$$
$$\sigma(y) = -x;$$
$$\sigma(h) = -h.$$

Herzog shows σ extends to an involution of U'_k. His argument, though expressed abstractly, is really quite concrete. He then considers a related canonical anti-isomorphism $\theta : U_k \to U_k^{opp}$ between U_k and U_k^{opp} defined by:

$$\theta(x) = -x;$$
$$\theta(y) = -y;$$
$$\theta(h) = -h.$$

$\theta\sigma$ induces an antihomomorphism of the lattice of pp subgroups of M, uniformly for all U_k-modules M. (Herzog states this in functorial terms.) The action is written as $\varphi \to \varphi^-$, and is explicitly described in [7]. He goes on to show that it respects equivalence modulo $FinDim$, that is, $\varphi \to \psi$ on $FinDim$ implies $\varphi^- \to \psi^-$ on $FinDim$, and this becomes a key tool in his analysis. We remark that $\varphi \to \varphi^-$ is entirely constructive, uniformly in any k-scalars.

We note the following fact:

Lemma 3.5. *For $\alpha, \beta \in k$, with $\alpha \neq 0$, $\alpha + \beta \cdot x$ is invertible in U'_k.*

Proof. x acts nilpotently on each $M \in FinDim$, so $\alpha + \beta \cdot x$ acts invertibly on M, with inverse $\alpha^{-1} \cdot \sum_{m=0}^{\infty} (-1)^m (\alpha^{-1}\beta x)^m$, which is a finite sum, of length depending M. To see the inverse uniformly as a definable scalar, consider

$$\phi(u, v) : u = (\alpha + \beta \cdot x) \cdot v. \qquad \square$$

The above fact is given only as a simple example, and has no special place in the order of our analysis.

The heart of the matter is the generation of idempotents, and especially those corresponding to annihilators (which we prefer, for reasons connected to the duality of Section 3.1, to call kernels) of elements of $U_{k,\,0}$.

For $p \in U_k$, we consider the following k-subspace:

$$Ker(p)(M) = \{m \in M : p \cdot m = 0\}.$$

We want to have in U'_k an (associated) idempotent e_p corresponding to the projection from M to this subspace, but this has no real meaning in terms of definable scalars unless we have a pp complement for $Ker(p)$. It is not obvious that such exists, and Herzog's proof that it does depends on the anti-isomorphism $\varphi \to \varphi^-$ discussed above.

3.2. The Centralizer of h

There is a direct (and obvious) connection between elements q of U'_k commuting with h, and elements of U'_k preserving weight spaces for M in $FinDim$.

Recall that such M are the direct sum of their pp-definable subspaces $Cas(\lambda, M)$. $Cas(\lambda, M)$ is isomorphic to a sum of copies of V_λ. Just as in V_λ we have the h-invariant weight spaces $V_{\lambda,\,i}$ ($0 \leqslant i \leqslant \lambda$), where $V_{\lambda,\,i} = ker(h - (\lambda - 2i))$ (in the sense of V_λ), we can define $Cas(\lambda, M)_i$ as $Ker(h - (\lambda - 2i))$ (in the sense of $Cas(\lambda, M)$), and we have $Cas(\lambda, M) = \sum_i Cas(\lambda, M)_i$.

Now, if q commutes with h, each $Ker(h - (\lambda - 2i))$ is closed under q and, since the Casimir element c commutes with h, each $Cas(\lambda, M)_i$ is closed under q.

If $M \cong V_\lambda$ and $qh = hq$, q leaves each 1-dimensional weight space $V_{\lambda,\,i}$ invariant.

"Conversely", if q, on every V_λ, leaves the weight spaces $V_{\lambda,\,i}$ invariant then $hq - qh$ is the zero map on V_λ, for each λ. It follows that $hq - qh$ is the zero map on every $M \in FinDim$, so $hq - qh = 0 \in U'_k$.

We take Herzog's arguments, and add some number theory, to get the basic insight needed to answer Questions 3.3 and 3.4.

Note that on V_λ the elements of $U_{k,\,0}$ ($= k[h, c]$) have the common basis of eigenvectors $X^i Y^{\lambda - i}$. In particular for each p in $U_{k,0}$, we have

$$V_\lambda = Ker(p) \oplus Image(p),$$

and then obviously $M = Ker(p) \oplus Image(p)$ for every $M \in FinDim$. Since $Image(p)$ is uniformly pp-definable, we thus see our first idempotent, $e_p \in U'_k$, for $p \in U_{k,\,0}$ defined by

$$e_p(m_1 + m_2) = m_1$$

where $m_1 \in Ker(p)$ and $m_2 \in Image(p)$.

So, e_p is the projection onto $Ker(p)$ relative to the decomposition $M = Ker(p) \oplus Image(p)$.

Remark 3.6. $1 - e_p$ is the corresponding projection onto $Image(p)$.

We note that all the e_p above commute with $U_{k,0}$ (they, too, have on V_λ the $X^iY^{\lambda-i}$ as eigenvectors).

Some are 0, for example e_c (where c is the Casimir element), since $Ker(c) = \{0\}$ on any $M \in FinDim$. Note that $e_h \neq 0$ in U_k', since for even λ, h has a nonzero kernel in V_λ (and for odd λ it has not).

Note that for $p = c - \frac{\lambda(\lambda+2)}{2}$, e_p gives the projection onto $Cas(\lambda, M)$ for $M \in FinDim$.

3.3. *Standard and Nonstandard e_p*

Here we go significantly beyond [7]. Let $p \in U_{k,0}$, so $p = p(c,h)$ where $p(u,v) \in k[u,v]$. As before, for $M \in FinDim$, $Ker(p)$ is a sum of eigenspaces $Cas(\lambda, M)_i$, where

$$Cas(\lambda, M)_i = \left\{ m : c \cdot m = \frac{\lambda(\lambda+2)}{2} \cdot m, h \cdot m = (\lambda - 2i) \cdot m \right\}.$$

Then, as Herzog shows, $Cas(\lambda, M)_i \subseteq Ker(p)$ if and only if $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$.

The subsequent analysis in [7], essentially goes as follows.

Case 1. $p \notin k[u]$.

Then for all but finitely many λ ($\notin \{\lambda_1, \ldots, \lambda_r\}$) $p(\frac{\lambda(\lambda+2)}{2}, v)$ is not the zero polynomial. If $p(\frac{\lambda(\lambda+2)}{2}, v) \neq 0 \in k[v]$, then $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$ has no more than d solutions (λ, i), where d is the v-degree of p.

If however $\lambda \in \{\lambda_1, \ldots, \lambda_r\}$ and $p(\frac{\lambda(\lambda+2)}{2}, v) = 0 \in k[v]$, then $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$ has no more than $|\lambda|$ solutions with $0 \leqslant i \leqslant |\lambda|$, and there are no more than $2d$, such λ, where d, is the u-degree of p.

Case 2. $p \in k[u]$.

Then there are no more than $2d$ λ with $p(\frac{\lambda(\lambda+2)}{2}) = 0$, and for each such λ no more than $|\lambda|$ i with $0 \leqslant i \leqslant |\lambda|$. This proves the following fact.

Lemma 3.7. *If $p \neq 0$, $p \in U_{k,0}$ there is a bound $B(p)$, computable semi-algebraically from p, on the dimension of $V_\lambda \cap Ker(p)$, independently of λ.*

Proof. Done above (modifying slightly that in [7]). □

But much more is true!

Consider the affine plane curve $\mathcal{C}_p$ defined by $p(u,v) = 0$.

Suppose first p is an absolutely irreducible polynomial, and that $\mathcal{C}_p$ has genus $\geqslant 1$. Then by Siegel's Theorem (see [8]) there are only finitely many pairs $(a,b) \in \mathbb{Z}^2$ such that $p(\frac{a}{2}, b) = 0$. Thus there are only finitely many pairs (λ, i) so that $(\lambda, i) \in \mathbb{Z}^2$ and $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$. If follows that uniformly across all $M \in FinDim$, $Ker(p) \cap Cas(\lambda, M) = \{0\}$ except for λ in a finite set *supp*(p) which is independent of M.

Moreover, there is a finite set I so that for $\lambda \in$ *supp*(p), $Cas(\lambda, M) \cap Ker(p) \subseteq \oplus_{i \in I} Cas(\lambda, M)_i$. The existence of such a *supp*(p) and I (which we will obtain below for p much more general than those just considered) will lead us to call p (and the e_p) standard.

We can relax the hypothesis on p significantly. First, factor p into absolutely irreducible factors p_r. The Herzog argument identifies $Ker(p)$ as $\oplus_{\lambda,\, i}\, Cas(\lambda, M)_i$, where the summation is over all (λ, i) in $\mathbb{Z}^2$, with $0 \leqslant i \leqslant \lambda$, and $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$. Thus we are reduced to considering for each r the condition $p_r(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$.

It may happen that p_r is not defined over $\mathbb{Q}$. But then suppose $(\lambda, i) \in \mathbb{Z}^2$ and

$$p_r\left(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i\right) = 0.$$

Then for any automorphism σ of k,

$$p_r^{\sigma}\left(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i\right) = 0,$$

where p_r^{σ} is got from p_r by acting on its coefficients by σ.

As far as $Ker(p_r)$ is concerned, we can assume that one of the coefficients of p_r is 1, and then if p_r is not defined over $\mathbb{Q}$, there exists σ so that $p_r^{\sigma} = 0$ defines a different curve. Then, by Bezout's Theorem, $\mathcal{C}_{p_r}$ and $\mathcal{C}_{p_r^{\sigma}}$ have an absolutely bounded number of points of intersection. So, for p_r not defined over $\mathbb{Q}$ (and even of genus 0) p_r is standard in the sense outlined before.

Thus, we can see that the interesting p are those for which one of the p_r is defined over $\mathbb{Q}$, and has infinitely many (λ, i) in $\mathbb{Z}^2$ with $p_r(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i) = 0$.

In particular, $\mathcal{C}_{p_r}$ must have genus 0.

But this is far from enough to guarantee infinitely many zeros (λ, i) from $\mathbb{Z}^2$ with $0 \leqslant i \leqslant \lambda$. Siegel himself showed (see [14] for a particularly clear treatment) that if p is an absolutely irreducible polynomial over $\mathbb{Q}$, with $p = 0$ defining a genus 0 curve with $\geqslant 3$ points at infinity, then the curve has only finitely many points of the form $(\frac{a}{2}, b)$, $(a, b) \in \mathbb{Z}^2$. A fairly recent subsequent literature has completely clarified which plane curves $\mathcal{C}_p$ have infinitely many points [11,15].

In a later paper we will discuss the fine detail of this, in connection with decidability. For now, an example is provided to show that there are p which are not standard (and we call these *nonstandard*).

Example 3.8. Let $p(u, v) = u - v^2$. Then, $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = \frac{(\lambda+1)^2 - 1}{2} - (\lambda - 2i)^2$. So, $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0 \Leftrightarrow (\lambda + 1)^2 - 2(\lambda - 2i)^2 = 1$.

So, we are asking for infinitely many points $(\lambda + 1, \lambda - 2i)$ with $0 \leqslant i \leqslant \lambda$, on the genus zero curve corresponding to the Pell equation $X^2 - 2Y^2 = 1$.

The only thing to check is every integer solution (X, Y) with $X \geqslant 1$, $Y \leqslant 0$ is of the form $(\lambda + 1, \lambda - 2i)$ with $0 \leqslant i \leqslant \lambda$. This of course requires $X - Y$ to be odd, but this is automatic, since if $X - Y$ were even, that is $X - Y = 2W$ (for some integer W), we would have: $(Y + 2W)^2 - 2Y^2 - 1 = 4YW + 4W^2 - Y^2 - 1 \not\equiv 0 \quad mod\, 4$.

What we need is that for X and Y solutions with $X \geqslant 1$, $Y \leqslant -3$, we have

$$0 \leqslant \frac{X - Y - 1}{2} \leqslant X - 1,$$

i.e $0 \leqslant X - Y - 1 \leqslant 2X - 2$,
i.e $0 \leqslant -Y - 1 \leqslant X - 2$,
i.e $3 + Y \leqslant 0 \leqslant X - Y$,

but this is automatic, proving that every nontrivial integer solution of $X^2 - 2Y^2 = 1$ with $X \geqslant 1$, $Y \leqslant -3$ is of the form $(\lambda + 1, \lambda - 2i)$ with $0 \leqslant i \leqslant \lambda$.

Let us consider p as in the Example 3.8, we can see that $e_p \cdot e_{c-\frac{\lambda(\lambda+2)}{2}} \neq 0$ for infinitely many λ, and the possible λ are determined by integer solutions of the Pell equation $X^2 - 2Y^2 = 1$, with $X \geqslant 1$.

Furthermore, we ask how many i exist for each λ with $0 \leqslant i \leqslant \lambda$ for which the following relations holds:

$$\frac{\lambda(\lambda+2)}{2} = (\lambda - 2i)^2 \quad 0 \leqslant i \leqslant \lambda.$$

In fact, there are two such i, say i_1, i_2, with $i_1 + i_2 = \lambda$, distinct unless $\lambda = \frac{\lambda}{2}$, when $\lambda = 0$ is the only possibility. Thus, when $e_p \cdot e_{c-\frac{\lambda(\lambda+2)}{2}} \neq 0$, its action on V_λ is to project onto a 2-dimensional sum of weight spaces $(V_\lambda)_i \oplus (V_\lambda)_{\lambda-i}$.

Now, we give a formal definition of "standard".

Definition 3.9.

i) $p \in U_{k,0}$ is standard if there is a finite set $supp(p) \subseteq \mathbb{N}$, such that $e_p \cdot e_{c-\frac{\lambda(\lambda+2)}{2}} = 0$ for $\lambda \notin supp(p)$;
ii) The λ such that $e_p \cdot e_{c-\frac{\lambda(\lambda+2)}{2}} \neq 0$ form the support of p;
iii) p is *nonstandard* if the support of p is infinite.

We can note that by Herzog's argument, there is an absolute finite bound on the dimension of $e_p V_\lambda$.

4. Redundancy in the Preceding Construction of Idempotents

Suppose $p(u,v) = \prod p_r(u,v)^{m_r}$ is the decomposition of p $(\in U_{k,0})$ into powers of distinct irreducible factors. Then, by Herzog's basic argument,

$$M \cap Ker(p) = \bigoplus_{\lambda,i} Cas(\lambda, M)_i,$$

where the direct sum is taken over all i such that, for some r, $p_r(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$.

Note that $e_{c-\frac{\lambda(\lambda+2)}{2}} \cdot e_{h-(\lambda-2i)}$ gives the projection onto $Cas(\lambda, M)_i$. Now suppose each p_r is standard.

Then the different idempotents:

$$e_{c-\frac{\lambda(\lambda+2)}{2}} \cdot e_{h-(\lambda-2i)}$$

are pairwise orthogonal, so the finite sum

$$\sum_{\lambda,\, i} e_{c-\frac{\lambda(\lambda+2)}{2}} \cdot e_{h-(\lambda-2i)}$$

gives the projection onto $Ker(p)$.

So, $e_p = \sum_{\lambda,\, i} e_{c - \frac{\lambda(\lambda+2)}{2}} \cdot e_{h-(\lambda-2i)}$. Note that $e_{c - \frac{\lambda(\lambda+2)}{2}} \cdot e_{h-(\lambda-2i)}$ gives on any V_λ the projection onto a subspace of dimension $\leqslant 1$, so these are what Herzog calls "pseudoweights".

Suppose however some p_r are nonstandard, say exactly $p_1, \ldots, p_l$ (with $l \in \mathbb{N}$). Our convention is that the p_r are distinct. There seems now no possibility of defining $e_{p_1}, \ldots, e_{p_l}$ ("nonstandard idempotents") in terms of pseudoweights. We simply show how to define e_p in terms of the $e_{p_1}, \ldots, e_{p_l}$ and various pseudoweights.

Then $e_{p_1}, \ldots, e_{p_l}$ are not quite orthogonal to each other, but nearly so. Consider, for example, e_{p_1} and e_{p_2}, and the corresponding plane curves $\mathcal{C}_{p_1}$ and $\mathcal{C}_{p_2}$. These intersect in finitely many (algebraic) points, so there are only finitely many (λ, i) which are common zeros of $p_1(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i) = 0$ and $p_2(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i) = 0$.

By the same technique as above, one can write the projection onto $Ker(p_1) \cap Ker(p_2)$ as a finite sum of orthogonal pseudoweights. One does the same for all finite intersections of $p_1, \ldots, p_l$ and then by a routine formal argument with commuting idempotents writes the projection onto $Ker(p_1) + \ldots + Ker(p_l)$ as a polynomial in $c_{p_1}, \ldots, c_{p_l}$ and certain pseudoweights as above. Finally, taking account of the finitely (λ, i) so that $p(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i) = 0$ but $p_j(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i) \neq 0$ for $1 \leqslant j \leqslant l$, one easily writes e_p as a polynomial in $e_{p_1}, \ldots, e_{p_l}$ and definite pseudoweights.

We have shown that the ring generated over $U_{k,0}$ is generated by the various $e_{c - \frac{\lambda(\lambda+2)}{2}}, e_{h-(\lambda-2i)}$ and the e_p for p nonstandard. Obviously we will not understand U_k' till we understand the latter.

4.1. The Elements of $U_{k,0}$ Inverted in U_k'

Lemma 4.1. *$p(c,h)$ is invertible in U_k' if and only if $p(\frac{\lambda(\lambda+2)}{2},\ \lambda - 2i) = 0$ has only the solution $(0,0)$ with $0 \leqslant i \leqslant \lambda$.*

Proof. The condition just given is equivalent to $Ker(p) = \{0\}$, and is clearly equivalent to p being invertible on all $M \in FinDim$, hence to p being invertible in U_k'. $\square$

Note that the condition just given is equivalent to $e_p = 0$.

Incorporating the inverses just considered we have a commutative ring generated over $U_{k,0}$ by the idempotents just discussed, together with all $\frac{1}{p}$ where $e_p = 0$.

One can go a bit further. Because of the decomposition $M = Ker(p) \oplus Image(p)$, we have in general "$p$ invertible modulo e_p" corresponding to the existence of a map which is the identity on $Ker(p)$, and the inverse of p on $Image(p)$. It can be written naturally as $e_p + p^{-1}(1 - e_p)$, and it satisfies $p \cdot (e_p + p^{-1}(1 - e_p)) = (1 - e_p)$.

Adding all of these gives as a commutative extension of $U_{k,0}$. Note that $p \cdot (e_p + p^{-1}(1 - e_p)) \cdot p = p$. So, $e_p + p^{-1}(1 - e_p)$ "regularizes" p in the sense of making p satisfy the axiom for von Neumann regularity.

5. More on h-Invariance

To go further, even before bringing x and y into the picture, one needs to consider not only U_k' but also the lattice (evidently modular) of pp-definable subgroups modulo equivalence in all $FinDim$. Herzog presents this in several ways, with emphasis on the lattice of finitely generated subobjects of the localization, corresponding to $FinDim$, of the element H (the forgetful functor) of the free abelian category over U_k [7]. Herzog uses crucially the anti-isomorphism of this lattice, $\varphi \to \varphi^-$, which we mentioned already in Section 3.1. It is very important that $\varphi \to \varphi^-$ is given completely explicitly.

In view of the basic importance of the above lattice in what follows, we should fix a suggestive notation for it. On the other hand, we do not wish to enter into a detailed discussion of Herzog's various equivalent definitions of the lattice. We have no doubt that the most elegant and fundamental approach is via categories of functors and localisation, but, given our emphasis on decidability, our immediate purposes are best met by a "Lindenbaum algebra" formulation as in the first sentence of this section. We will simply take over Herzog's notation $Latt\, H_S$, where S is the Serre subcategory of coherent functors that vanish on all finite-dimensional representations of $sl_2(k)$. We will need also the extended notation $Latt\, \varphi_S$ for the lattice of subobjects of the localisation at this S of the subobject of H given by the formula φ. This of course has a perspicuous meaning in the Lindenbaum algebra formulation, in terms of equivalence modulo $FinDim$ of formulas intersected with φ.

For the above lattice, we have a natural notion of h-invariance, namely that φ is h-invariant if and only if $h\varphi \subseteq \varphi$ in $FinDim$. This is readily seen to be equivalent to φ being, in each simple finite dimensional U_k-module V_λ, a sum of weight spaces.

The fundamental result about h-invariance is the following result.

Lemma 5.1. *If φ is h-invariant, so is φ^- and $M = \varphi(M) \oplus \varphi^-(M)$ for all M in $FinDim$, so φ is complemented in the above lattice.*

Proof. See [7], page 260. □

This gives us more idempotents. For clearly if φ is h-invariant we can define the idempotent e_φ corresponding to the projection, with kernel φ^-, onto the subgroup (defined by) φ. Note however, that this is quite an abstract procedure. It is not clear at this stage how to tell if $e_\varphi = e_\psi$ in U_k'.

By the way, the preceding discussion is enough to show that the centralizer of h in U_k' is a commutative von Neumann regular ring. From r in U_k' and commuting with h, one passes to φ defining the image of r, and then to e_φ, which is h-invariant, and generates the same ideal in U_k' as r does.

The pseudoweights are the h-invariant elements φ of the lattice which in all V_λ they define spaces of dimension $\leqslant 1$. (We have already see many of them.) For such φ it follows that the lattice $Latt\, \varphi_S$ is not merely complemented (as follows from what said above) but actually uniquely complemented and is a Boolean algebra. (One should note that if φ is a pseudoweight and $\psi \subseteq \varphi$, then ψ is pseudoweight too.)

Among the pseudoweights, some special ones arise as follows.

The highest weight space of V_λ is $Ker(x)$, which is of course h-invariant. We have e_x, the corresponding projection onto $Ker(x)$ (which is complemented by $Image(y)$). Herzog gives an important generalization for any idempotent e in the centralizer of h (in U_k'). He gives, explicitly in his Proposition 18, a definition of a pseudoweight e_0 corresponding to the projection of the highest weight subspace of the image of e. e_0 is called the *highest* pseudoweight of e. More precisely, on V_λ, $e_0 V_\lambda = \{0\}$ if $e = \{0\}$, and otherwise is $V_{\lambda,\, j}$ where j is minimal with $0 \leqslant j \leqslant \lambda$ such that $V_{\lambda,\, j} \subseteq eV_\lambda$.

6. Uniformly Bounded φ

A pp-formula φ is said to be uniformly bounded if there is some n so that for all λ the dimension of φ in V_λ is bounded by n. (There is now no condition of h-invariance.)

We have already seen some h-invariant examples, namely φ defining $Ker(p)$, for p in $U_{k,0}$. Now, (all this is in [7]) one generalizes to U_k. The basic result, with a constructive proof, is the following.

Lemma 6.1. *Suppose $q \in U_k$, $q \neq 0$. Then there is p in $U_{k,0}$ and a nonnegative integer such that*

$$Ker(q) \cap Image(y^n) \cap Image(p) \cap Image(x^n) = \{0\},$$

for all $M \in FinDim$,

Proof. See [7], page 265. □

Herzog's discussion brings into view further basic information connected to the standard/nonstandard distinction.

Suppose q, p, n are as in the preceding lemma, and $M \in FinDim$. Then $Image(p)$ is complemented by $Ker(p)$, $Image(x^n)$ is complemented by $Ker(y^n)$, and $Image(y^n)$ by $Ker(x^n)$.

We have already considered e_p, and now we consider e_{x^n}, e_{y^n} corresponding to the projections onto $Ker(x^n)$, $Ker(y^n)$ respectively. Note that $Ker(x^n)$ and $Ker(y^n)$ are h-invariant, so e_{x^n} and e_{y^n} are in the centralizer of h, and e_p, e_{x^n}, e_{y^n} pairwise commute. The idempotent $1-(1-e_p)(1-e_{x^n})(1-e_{y^n})$ gives the projection onto the complement of $Image(p) \cap Image(x^n) \cap Image(y^n)$, and this complement has dimension bounded by the sum of those of $Ker(p)$, $Ker(x^n)$ and $Ker(y^n)$.

In particular, if M is reduced, that is, is a sum of V_λ without repetitions (which is no restriction as far as elementary equivalence is concerned) both $Ker(x^n)$ and $Ker(y^n)$ have dimension $\leqslant n$ in each $Cas(\lambda, M) \neq \{0\}$, and then the above complement has dimension $\leqslant$ dimension of $Ker(p) + 2n$.

By [7] Lemma 21, we know that the projection from $Ker(q)$ to the subspace given by $1-(1-e_p)(1-e_{x^n})(1-e_{y^n})$ is injective.

Now suppose p is standard. Then for all but finitely many λ, $Ker(p) \cap Cas(\lambda, M) = \{0\}$, and in this case, on $Cas(\lambda, M)$, the projection of $Ker(q)$ to $Ker(x^n) + Ker(y^n)$ is injective. For the other λ, one has only that the dimension of $Ker(p)$ in $Cas(\lambda, M)$ is uniformly bounded, giving the same result for $Ker(q)$.

When p is nonstandard, one has only the uniform boundedness result from the lemma above. In the sequel, we will look more closely at the nature of the kernels of general q.

For future reference, we note that Herzog's proof of the lemma above actually gives more useful information than he states.

Lemma 6.2. *Let $q = x^n a_n + x^{n-1} a_{n-1} + \ldots + x a_1 + c_0 + y b_1 + \ldots + y^m b_m$ where the a's, b's and c's are in $U_{k,0}$. Let $w \in V_\lambda$ with $q \cdot w = 0$. For $0 \leqslant i \leqslant \lambda$, let w_i be the projection of w onto the i-th weight space.*

Let i_0 be the least i with $w_i \neq 0$, and let i_1 be the greatest i with $w_i \neq 0$. Then

i) *If $a_n \neq 0$, either $a_n \cdot w_{i_0} = 0$ or $x^n \cdot w_{i_0} = 0$;*
ii) *If $b_m \neq 0$, either $b_m \cdot w_{i_1} = 0$ or $y^m \cdot w_{i_1} = 0$;*

iii) *If all* $b_j = 0$ *and* $c_0 \neq 0$ *and* $a_n \neq 0$ *then* $q \cdot w \neq 0$*;*
iv) *If all* $a_j = 0$ *and* $c_0 \neq 0$ *and* $b_m \neq 0$ *then* $q \cdot w \neq 0$*;*
v) *If all* $a_j = 0$ *and* $c_0 = 0$*, then* $y^{m_1} \cdot w = 0$*, where* m_1 *is minimal such that* $b_{m_1} \neq 0$*;*
vi) *If all* $b_j = 0$ *and* $c_0 = 0$*, then* $x^{n_1} \cdot w = 0$*, where* n_1 *is minimal such that* $a_{n_1} \neq 0$.

Proof. (i) and (ii) are seen by inspection of what Herzog does on page 265. For (iii) and (iv) observe that in these cases q is $c_0 + (q - c_0)$, and $(q - c_0)$ acts nilpotently on the finite dimensional modules, so q is invertible. (v) and (vi) are done similarly, this time expressing q as a power of y (respectively x) times an invertible element. □

7. U_k' is von Neumann Regular: Rearranging the Proof

We review the last stages of the proof quickly, indicating which points are less constructive. Firstly, Lemma 25 in [7] shows that if φ is an h-invariant uniformly bounded pp-formula, then *Latt* φ_S (its corresponding lattice of subobjects modulo equivalence in $FinDim$) is complemented. The proof is by induction on the least n such that the dimension of $V_\lambda \cap \varphi \leqslant n$, and is constructive in this n. By this we mean that once we know n there is an explicit recursive procedure of length n that allows us to write down complements for (formulas defining) subobjects of ϕ. But note that even for φ defining $Ker(p)$ with $p \in U_0$, it is not completely clear how constructive the exact bound n limiting the dimension is, though in that special case a constructive upper bound for n is clear. If in the uniform boundedness of φ, an upper bound for the dimension n is given constructively, the rest of the proof is constructive, as is seen by inspection of what Herzog writes.

To complete the proof one has to drop the uniform boundedness assumption. This is done very beautifully by Herzog, using the duality and the model theory of one special U_k-module, K, the field of fractions of U_k.

K, as a left U_k'-module, is simple as a module over its endomorphism ring ([7], page 251), and thus induces a fundamental partition of the lattice of pp-definable subgroups. By the remark on simplicity, since each φ defines a module over the endomorphism ring, then either φ defines (0) in K or φ defines K in K. Moreover, the φ which define (0) form an ideal $\mathcal{I}$ in the lattice and those defining K form a complementary filter $\mathcal{F}$.

Herzog identifies $\mathcal{I}$ very neatly as given by the φ such that (modulo the theory of U_k-modules) φ is bounded by a (nontrivial) torsion condition $rv = 0$ (with $r \neq 0$). With rather more work he shows ([7], page 253) that $\mathcal{F}$ can be characterized "dually" as the set of φ which contain a nontrivial divisibility condition, that is, contain a nontrivial $Image(r)$, with $r \in U_k' - \{0\}$.

If the field k is given recursively (as it can be if k is countable) one may combine these characterizations of $\mathcal{I}$ and $\mathcal{F}$ to show that $\mathcal{I}$, $\mathcal{F}$ and the theory of K are *recursive*.

Note that the preceding characterizations show that $\mathcal{I}$ consists of the uniformly bounded φ, and $\mathcal{F}$ consists of the φ whose codimension is uniformly bounded in the sense that the dimensions of the V_λ/φ are uniformly bounded.

Let us observe that despite its importance for the theory of PFD modules, K itself is not PFD. Indeed, $Ker(x) = \{0\}$ in K, since x is invertible in the ring K. But in every (nonzero) PFD U_k'-module, $Ker(x) \neq \{0\}$.

The proof ends by showing that each pp-formula φ has a complement in *Latt* H_S. There are two cases (decidable by the above discussion).

Case 1. $\varphi \in \mathcal{F}$. Even constructively, it suffices to find an h-invariant $\psi \in \mathcal{I}$ with $\varphi + \psi = H_S$. (This depends on the earlier proof that *Latt* ψ_S is complemented.) Herzog argues that one can assume that φ is a divisibility condition, and then the proof is routine using his Lemma 21 (our 6.2). In fact, it all works constructively. For φ contains a divisibility condition modulo the recursively enumerable theory of U_k-modules, and for any such divisibility condition we can effectively bound its codimension (and so that of φ) uniformly for all $V(\lambda)$.

Case 2. $\varphi \in \mathcal{I}$. This is done constructively, by duality.

We are still some distance from any "constructive presentation" of U_k'. For example, we have emphasized above the constructive aspects of the proof that *Latt* H_S is complemented. It is not quite immediate to get (von Neumann) regularity of U_k' (see Herzog's discussion on pages 254–256).

If we have constructed any element r in U_k', we need to explain the procedure for finding the element s with $rsr = r$ (and then it is formal to show that sr is idempotent and generates the same left ideal as r). What is needed is the following. Let e_1 correspond to projection onto $Image(r)$, and e_2 correspond to projection onto $Ker(r)$, and note that, unlike what happens when r is in the centralizer of h, in general $Image(r) \cap Ker(r) \neq \{0\}$. Let us define the element s as follows:

$$s\big(e_1(m)\big) = (1 - e_2) \cdot m_0, \quad \text{where } rm_0 = e_1(m),$$
$$s\big((1 - e_1)m\big) = 0.$$

Note that if $rm_0 = rm_1 = e_1(m)$, then $m_0 - m_1 \in Ker(r)$ so $(1-e_2)m_0 = (1-e_2)m_1$. So, s is a section of r. For,

$$\begin{aligned} r\big(s(e_1(m))\big) &= r\big((1 - e_2) \cdot m_0\big) \\ &= rm_0 - r(e_2 m_0) \\ &= rm_0 = e_1(m). \end{aligned}$$

Clearly, $rs((1 - e_1)m) = 0$. So, we have $rsr = r$, and s is obtained constructively from r.

7.1. Building U_k'

It is should be clear from Herzog's analysis that the "fundamental" idempotents are the e_p ($p \in U_{k,0}$), followed first by the more general e_q ($q \in U_k$), and then by the e_{x^n}.

For generating more idempotents the highest weight idempotents e_0 associated to (previously constructed) e are crucial. Finally, the "sections" s (described above) are used systematically.

For a general $\varphi \in$ *Latt* H_S, the corresponding e_φ is got relatively easily from the preceding using the test whether $\varphi \in \mathcal{I}$ or $\varphi \in \mathcal{F}$, and the corresponding dominating $Ker(q)$ or dominated $Image(q)$. So, we are now in a position to generate U_k' constructively, for k countable.

8. Constructive Presentation of U'_k

Henceforward k is countable (although it is not hard to give a sensible meaning to what follows for general k). That U_k is a computable domain is clear, using the defining relations, the grading $\bigoplus_{n\in\mathbb{Z}} U_{k,\,n}$, and the unique representations of $U_{k,\,0}$ in terms of c and h, and of $U_{k,\,n}$ $(n \neq 0)$ in terms of $x^n \cdot U_{k,\,0}, U_{k,\,0} \cdot x^n, y^n \cdot U_{k,\,0}, U_{k,\,0} \cdot y^n$ as before.

Now we add the idempotents e_p, corresponding to projection onto $Ker(p)$, for $p \in U_{k,\,0}$. Such p are written uniquely as $p(c, h)$, where $p(u, v) \in k[u, v]$.

Fix $p(u, v)$ and factor it constructively as a product of a constant and powers of monic irreducible $p_l(u, v)$ over k (for some positive integer l). It is clear ([7]) that $Ker(p)$ is the sum of the $Ker(p_l)$, and that e_p can be written equationally in terms of the (pairwise commuting) e_{p_l}. So, it is enough to add the e_{p_l}.

As in [7], we have to consider solutions in integers (λ, i) with $0 \leqslant i \leqslant \lambda$ of $p_l(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$.

There are only finitely many solutions if either:

i) p_l is not in $\mathbb{Q}[u, v]$ (the argument was given in Section 3.3),

or

ii) p_l defines a curve of genus $\geqslant 1$ over $\mathbb{Q}$.

or

iii) p_l defines a curve of genus 0 over $\mathbb{Q}$ and certain suitable conditions are satisfied (see Section 3.3).

Now a moment's reflection shows that to decide such questions as $e_{p_l} = 0$ in the theory of PFD modules, we need to be able to decide compatibilities between the various e_p and to know, in the cases *i*), *ii*), *iii*) above, what are the finitely many solutions. In Case (i), we can readily decide that p_l is not defined over $\mathbb{Q}$, exhibit constructively a normal number field E over which it is defined, together with a real subfield F so that $E = F(i)$. Moreover, we can obtain effectively an automorphism σ of E so that $p_l \neq {p_l}^{\sigma}$. Then Bezout applied to $C = R(i)$ gives us bounds for the absolute values of the common zeros of p_l and ${p_l}^{\sigma}$, in terms of the absolute values of the coefficients of p_l and ${p_l}^{\sigma}$, and so allows us to bound the common integral zeros, and thus the integral zeros of p_l.

For *ii*), the problem is very profound, and no unconditional algorithm is known (though one is expected). For a thorough discussion, see [8]. It is known that if the Mordell-Weil Theorem can be constructivized then above problem is decidable [14]. Here we shall simply assume that the decision problem for curves of nonzero genus is decidable, in the sense that we can decide if a plane curve of nonzero genus over $\mathbb{Q}$ has an integer point (α, β) with $\alpha, \beta > 0$, and then find the finitely many solutions.

Case *iii*) involves subtleties not fully appreciated in earlier discussions [11,15] of the genus 0 case. It is now known, using Baker's method, how to decide if a genus 0 curve has only finitely many points, and then how to find these points.

Henceforward we assume an algorithm for testing which irreducible monic p in $k[u, v]$ have only finitely many zeros $(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i)$ with $0 \leqslant i \leqslant \lambda$ (and $\lambda, i \in \mathbb{Z}$), and then listing those zeros.

In 3.3 we called a $p(u, v)$, which is monic, irreducible over $\mathbb{Q}$, and has only finitely many solutions $(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i)$ as above, *standard*. We now have some axioms about

e_p, for p standard. An example is: $e_p = 0$, if there are no solutions $(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i)$. More generally, if $(\frac{\lambda_r(\lambda_r+2)}{2}, \lambda_r - 2i_r)$, where $r = 1, \ldots, R$ (for some nonzero positive integer R) are all the solutions, one has an axiom: $e_p = \sum_{r=1}^{R} e_{c-\frac{\lambda_r(\lambda_r+2)}{2}} \cdot e_{h-(\lambda_r - 2i_r)}$.

So, in fact, one can define e_p for the very special idempotents on the right-hand side of the equation. For economy of notation, let us do so (that is, dispense with the general e_p as primitive).

We note that all e_p (p in $U_{k,0}$) pairwise commute. In addition, we have obvious orthogonality axioms. Firstly, $e_{c-\frac{\lambda(\lambda+2)}{2}} \cdot e_{c-\frac{\mu(\mu+2)}{2}} = 0$, for $\lambda \neq \mu$. Secondly, for distinct weight-spaces with $Cas(\lambda)$, we have: $e_{c-\frac{\lambda(\lambda+2)}{2}} \cdot e_{h-(\lambda-2i)} \cdot e_{h-(\lambda-2j)} = 0$ for $i \neq j$.

The other dramatis personae at this stage are the e_p, where p is nonstandard. (We already know that there are interesting such e_p, connected to Pell equations.) Such p are over $\mathbb{Q}$, monic and irreducible. Here is the first axiom: $e_{p_1} \cdot e_{p_2} = 0$ for p_1, p_2 nonstandard if there is no (λ, i) with $p_1(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = p_2(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$. More generally, for $p_1 \neq p_2$, $e_{p_1} \cdot e_{p_2} = \sum_{r=1}^{R} e_{c-\frac{\lambda_r(\lambda_r+2)}{2}} \cdot e_{h-(\lambda_r-2i_r)}$ if $(\frac{\lambda_1(\lambda_1+2)}{2}, \lambda_1 - 2i_1), \ldots, (\frac{\lambda_R(\lambda_R+2)}{2}, \lambda_R - 2i_R)$ are the common zeros.

Finally, (because of the uniform boundedness phenomenon), we have $e_p \cdot e_{c-\frac{\lambda(\lambda+2)}{2}} = e_{c-\frac{\lambda(\lambda+2)}{2}} \cdot \sum_{i=1}^{s} e_{h-(\lambda-2i)}$ where $i_1, \ldots, i_s$ are the solutions of $p(\frac{\lambda(\lambda+2)}{2}, \lambda - 2i) = 0$ (this includes $e_p \cdot e_{c-\frac{\lambda(\lambda+2)}{2}} = 0$ if there are no solutions). So, now we have a fragment of U'_k, generated by $U_{k,0}$ and the idempotents $e_{c-\frac{\lambda(\lambda+2)}{2}}$, the $e_{h-(\lambda-2i)}$, and the e_p for p nonstandard. Any element of this ring can be represented in the form

$$\begin{aligned}
\alpha_0 + \alpha_1 \cdot e_{c-\frac{\lambda_1(\lambda_1+2)}{2}} \quad & + \ldots + \alpha_m \cdot e_{c-\frac{\lambda_m(\lambda_m+2)}{2}} + \\
+ \alpha_{m+1} \cdot e_{h-\mu_{m+1}} \quad & + \ldots + \alpha_{m+n} \cdot e_{h-\mu_{m+n}} + \\
+ \alpha_{m+n+1} \cdot e_{c-\frac{\lambda_{m+n+1}(\lambda_{m+n+1}+2)}{2}} \cdot e_{h-\mu_{m+n+1}} \quad & + \ldots + \\
+ \alpha_{m+n+s} \cdot e_{c-\frac{\lambda_{m+n+s}(\lambda_{m+n+s}+2)}{2}} \cdot e_{h-\mu_{m+n+s}} \quad & + \\
+ \alpha_{m+n+s+1} \cdot e_{p_1} \quad & + \ldots + \alpha_{m+n+s} \cdot e_{p_t}
\end{aligned}$$

where the α are in $U_{k,0}$, the λ and μ are integers, $\lambda \geqslant 0$, and the $p_1, \ldots, p_t$ are nonstandard.

Using the relations we gave above one readily sees closure under multiplication. The crucial issue is uniqueness of the above representation. This has to be an essential part of our decision procedure. Note a slight ambiguity, as the $h - \mu$ are nonstandard according to our definition. So, we should assume that the $p_1, \ldots, p_t$ are not of this form. Note too that we should obviously assume that $\lambda_1, \ldots, \lambda_m$ are distinct and $\mu_{m+1}, \ldots, \mu_{m+n}$ are distinct.

Suppose the sum above represents the zero element in U'_k.

First, suppose $\alpha_0 \neq 0$. Then let d be a bound on the dimension of $V_\lambda \cap Ker(\alpha_0)$. First restrict to (λ, μ) distinct from the (λ_r, μ_r) appearing in the sum. Then restrict further to the (λ, μ) such that $V_{\lambda,\mu}$ is not included in any of $Ker(p_1), \ldots, Ker(p_t)$. This leaves infinitely many λ to choose from. Again using uniform boundedness, one sees that there exists D so that if for some μ the pair (λ, μ) is not yet eliminated then there are $\geqslant \lambda - D$ such μ with (λ, μ) not yet eliminated. For any such (λ, μ) all terms except α_0 from the above sum vanish on $V_{\lambda,\mu}$. But then α_0 does too. But then if $\lambda - D > d$ we have a contradiction to uniform boundedness of α_0. So $\alpha_0 = 0$.

So now we put $\alpha_0 = 0$ in the above. As before restrict to (λ, μ) distinct from the (λ_r, μ_r) in the sum. The effect of this is that for such (λ, μ) the action of the sum on $V_{\lambda,\,\mu}$ is equal to that of $\alpha_{m+n+s+1} \cdot e_{p_1} + \ldots + \alpha_{m+n+s} \cdot e_{p_t}$.

Now, recall that the different $Ker(p_i)$ intersect in finite dimensional subspaces, and each $Ker(p_i)$ meets infinitely many $Cas(\lambda)$ nontrivially. Thus there are infinitely many λ so that $Ker(p_1)$ meets $Cas(\lambda)$ nontrivially, but no other $Ker(p_2)$ meets $Cas(\lambda)$ nontrivially. For each such λ choose a μ so that $p_1(\frac{\lambda(\lambda+2)}{2},\, \lambda - 2\mu) = 0$.

Now, $p_2(\frac{\lambda(\lambda+2)}{2},\, \lambda - 2\mu) \neq 0, \ldots, p_t(\frac{\lambda(\lambda+2)}{2},\, \lambda - 2\mu) \neq 0$, so $e_{p_1}, \ldots, e_{p_t}$ vanish on $V_{\lambda\,\mu}$. So $\alpha_{m+n+s+1} \cdot e_{p_1}$ vanishes on $V_{\lambda,\,\mu}$. Thus, $\alpha_{m+n+s+1}$ and e_{p_1} have infinitely many common integer zeros, and so p_1 divides $\alpha_{m+n+s+1}$. But now we note that $p \cdot e_p = 0$ in general. This should be added to our defining relations, and so if p divides $q \in U_{k,\,0}$, then $q \cdot e_p = 0$. So now we should assume in our sum representation that no p_i divides $\alpha_{m+n+s+i}$. Then we conclude that each $\alpha_{m+n+s+i} = 0$ if the sum is zero.

So, finally we return to that sum under the assumption that $\alpha_0 = \alpha_{m+n+s+1} = \ldots = \alpha_{m+n+s+t} = 0$.

If λ is greater than all the λ_j that occur in the sum, then on V_λ the sum is equal to $\alpha_{m+1} \cdot e_{h-\mu_{m+1}} + \ldots + \alpha_{m+n} \cdot e_{h-\mu_{m+n}}$. Now assume, in addition, that λ is bigger than each of $|\mu_{m+1}|, \ldots, |\mu_{m+n}|$. This leaves, for each j, infinitely many λ such that $\lambda - \mu_{m+j}$ is even. It follows that for each j, $V_{\lambda,\, \frac{\lambda-\mu_{m+j}}{2}} \subseteq Ker\, \alpha_{m+j}$ for infinitely many λ such that $\lambda - \mu_{m+j}$ is even. By the same argument as for the nonstandard p above, $h - \mu_{m+j}$ divides α_{m+j}. So again we have an instance of $p \cdot e_p = 0$. As before we conclude that $\alpha_{m+1} = \ldots = \alpha_{m+j} = 0$ provided no $h - 2\mu_j$ divides α_{m+j}. We can in any case ignore such terms because of our defining relations, and so we come down to the case of a sum

$$\begin{aligned}\alpha_1 \cdot e_{c-\frac{\lambda_1(\lambda_1+2)}{2}} + \ldots + \alpha_m \cdot e_{c-\frac{\lambda_m(\lambda_m+2)}{2}} &\qquad + \\ + \ldots + \alpha_{m+n+1} \cdot e_{c-\frac{\lambda_{m+n+1}(\lambda_{m+n+1}+2)}{2}} \cdot e_{h-\mu_{m+n+1}} &+ \\ + \ldots + \alpha_{m+n+s} \cdot e_{c-\frac{\lambda_{m+n+s}(\lambda_{m+n+s}+2)}{2}} \cdot e_{h-\mu_{m+n+s}} &\end{aligned}$$

which we suppose to be 0 (in U_k'). We can clearly assume $\lambda_1, \ldots, \lambda_m$ distinct, but perhaps not the distinctness of the list $\lambda_{m+n+1}, \ldots, \lambda_{m+n+s}$. There may be some overlap between the two lists. Suppose first λ_1 does not occur in the second list. Then the idempotent $e_{c-\frac{\lambda_1(\lambda_1+2)}{2}}$ is orthogonal to all the other $e_{c-\frac{\lambda(\lambda+2)}{2}}$ occurring, and we conclude that $\alpha_1 \cdot e_{c-\frac{\lambda_1(\lambda_1+2)}{2}} = 0$, but notice that this relation expresses exactly that $\alpha_1(\frac{\lambda_1(\lambda_1+2)}{2},\, \lambda_1 - 2i) = 0$ for all i with $0 \leqslant i \leqslant \lambda_1$. And then the relation is simply a consequence, by our defining relations, of this fact about the number λ_1. Thus we may discard it.

So, provided we discard (as our relations permit) terms $\alpha \cdot e_{c-\frac{\lambda_i(\lambda_i+2)}{2}}$ with $c - \frac{\lambda_1(\lambda_1+2)}{2}$ dividing α_i, we can assume that all of $\lambda_1, \ldots, \lambda_m$ occur in second list too.

Dually, if some λ_{m+n+i} occurs in the second list, but not in first, we get $\alpha_{m+n+i} \cdot e_{c-\frac{\lambda_{m+n+i}(\lambda_{m+n+i}+2)}{2}} \cdot e_{h-\mu_{m+n+i}} = 0$. If μ_{m+n+i} is not of form $\lambda_{m+n+i} - 2\gamma_i$ with γ_i integral and $0 \leqslant \gamma_i \leqslant \lambda_{m+n+i}$, the above equation follows from an obvious relation on the e's, and so can be discarded. Thus we assume $\lambda_{m+n+i} - \mu_{m+n+i}$ even, with γ_i as above, and then the above equation says that $V_{\lambda_{m+n+i}\,\gamma_i} \subseteq Ker(\alpha_{m+n+i})$, and we can deduce that $(\frac{\lambda_{m+n+i}(\lambda_{m+n+i}+2)}{2}, \mu_{m+n+i})$ is a zero of $\alpha_{m+n+i}(u, v)$.

But conversely this forces, by an obvious relation, the equation. Thus we may discard the term $\alpha_{m+n+i} \cdot e_{c-\frac{\lambda_{m+n+i}(\lambda_{m+n+i}+2)}{2}} \cdot e_{h-\mu_{m+n+i}}$. The only remaining (notational) complication is that there may be repetitions in the second list, so that some λ_{m+n+i} occurs with both μ_{m+n+i} and at least one different μ_{m+n+j} attached.

Using orthogonality, this leads to m equations $(\alpha_i - \alpha_{m+n+i} \cdot e_{h-\mu_{m+n+i}}) \cdot e_{c-\frac{\lambda_i(\lambda_i+2)}{2}} = 0$. There are two cases (for each i).

Case 1. $\lambda_i - \mu_{m+n+i}$ even, say $= 2\gamma_i$, with $0 \leqslant \gamma_i \leqslant \lambda_i$, γ_i integral. Thus $\alpha \cdot e_{c-\frac{\lambda_i(\lambda_i+2)}{2}}$ So we deduce that $(\frac{\lambda_i(\lambda_i+2)}{2}, \mu_{\lambda_i - 2\gamma_i})$ is a root of $(\alpha_i - \alpha_{m+n+i})(u, v)$, and for all γ with $0 \leqslant \gamma \leqslant \lambda_i$ and $\gamma \neq \gamma_i$, $(\frac{\lambda_i(\lambda_i+2)}{2}, \mu_{\lambda-2\gamma})$ is a root of α_i.

Conversely, in Case 1, these two conditions imply the equation, using the obvious relations.

Case 2. Not Case 1.

Then the equation becomes $\alpha_i \cdot e_{c-\frac{\lambda_i(\lambda_i+2)}{2}} = 0$, and as usual this follows from formal relations.

Thus, after a long argument we have given a unique normal form for all elements of the ring generated by $U_{k,0}$ and the basic idempotents e_p for $p \in U_{k,0}$.

8.1. The Centralizer of h Again

The ring already described is a subring of the centralizer $Z'(h)$ of h in U_k'. Herzog has a fairly simple argument to show that $Z'(h)$ is a commutative von Neumann regular ring. He uses the notion of h-invariant pp-formula, that is pp-formula φ such that $h\varphi \subseteq \varphi$ for all $M \in FinDim$.

From the standpoint of constructivity there is a problem, for it is certainly not clear at this stage that the set of h-invariant formulas is recursively enumerable (its complement clearly is).

The basic examples of h-invariant formulas are (those defining) $Ker(p)$ and $Image(p)$ for p in $U_{k,0}$. For these we have $M = Ker(p) \oplus Image(p)$ for any $M \in FinDim$. This observation yielded the h-invariant e_p for projection onto $Ker(p)$ and $1 - e_p$ for projection onto $Image(p)$.

Other examples are $Ker(x^n)$, $Ker(y^n)$, $Image(x^n)$, $Image(y^n)$, with related decompositions

$$\begin{aligned} M &= Ker(x^n) \oplus Image(y^n) \\ &= Ker(y^n) \oplus Image(x^n) \end{aligned}$$

giving idempotents (in $Z'(h)$) e_{x^n}, e_{y^n}.

More generally we can use Lemma 6.1 to get information about $ker(q)$ for q in U. Let us use the notation of Lemma 6.1.

Then the point is that $Image(y^n) \cap Image(p) \cap Image(x^n)$ is h-invariant. Let φ be a pp-formula defining it. Since $Ker(p)$ is uniformly bounded, we have a uniform (constructive) bound on the codimension of the set defined by φ. Now by [7], Proposition 13, φ^- is also h-invariant, and $M = \varphi \oplus \varphi^-$.

Thus we have a pp-definable injection from $Ker(q)$ into φ^-, which is (constructively) uniformly bounded.

Now we can use Herzog's beautiful "highest pseudoweight" construction. Choose a bound d for the dimension of $V_\lambda \cap \varphi^-$ and construct $e_0, \ldots, e_{d-1}$ as follows. e is the idempotent for projection onto φ^- (one can write it down explicitly and constructively in terms of e_{x^n}, e_{y^n}, e_p).

By [7], Proposition 18, there is an explicit formula uniformly defining the highest weight space of φ^-, yielding an idempotent e_0 in $Z'(h)$ (in some model M for certain λ $e_0\, Cas(\lambda) = \{0\}$).

Now replace φ^- by $(1-e_0)\varphi^-$ and get e_1 defining the highest weight space for this. Again, $e_1 \in Z'(h)$. Repeat as far as the construction of e_{d-1}, and we have uniformly $\varphi^- = e_0 \cdot \varphi^- \oplus e_1 \cdot \varphi^- \oplus \ldots \oplus e_{d-1} \cdot \varphi^-$ (and the e_i are of course pairwise orthogonal).

Herzog adds a refinement, again entirely constructive. Namely, for any pseudoweight e, he writes down a pp injection from eM to $e_x M$, uniformly for $M \in FinDim$ (p. 269, end of proof of Theorem 30). This, in terms of U'_k, corresponds to having α, β, in $Z'(h)$, with

$$e_x\, \alpha\, e = \alpha e$$

$$\beta\, e = e.$$

This implicitly contains a decomposition of the form $e_x = e_x \cdot (1 - e_\alpha) \oplus e_x \cdot e_\beta$.

Note too that it shows that e is in the ideal generated by e_x in $Z'(h)$, and so in both the left and right ideals generated by e_x in U'_k.

Thus we see that constructively we have for each $q \neq 0$ in U_k a pp injection of $Ker(q)$ into $Ker(x^n)$ (n, q as Lemma 8.2).

One can then generalize this to get, for a pp-formula φ in $\mathcal{I}$, constructively and uniformly a pp injection of φ into $Ker(x^n)$ for some n.

A crucial point now is the constructive content of Herzog's argument, which shows that the lattice of subobjects of an h-invariant $\varphi \in \mathcal{I}$ is complemented. The proof is by induction on a bound for the dimension in the uniform boundedness condition. To do this constructively is nontrivial, as we still lack a proof that h-invariance is a recursively enumerable condition.

What we do is in fact quite reminiscent of the sort of unwindings pioneered in logic by Kreisel [9], although his unwindings are not generally associated to presentations of structures.

8.2. *An Enumeration of U'_k*

We have repeatedly stressed that the theory of PFD is co-re, and that we have not yet improved this. We expect to do so in the sequel, by bringing more number theory to bear. What we have done, in the preceding, is to give a normal form, and in particular a recursive enumeration, for the elements of the ring generated by $U_{k,0}$ and the basic idempotents e_p for $p \in U_{k,0}$. The "enumeration" we now give of the whole of U'_k is much weaker, and we explain how.

The elements of U'_k are associated, not at all uniquely, to pp-formulas $\varphi(u, v)$ which satisfy the not obviously r.e condition of defining maps on each M in PFD. It is perfectly clear that there are recursive operations $+$, $-$ and $\cdot$ on the class of all pp-formulas

$\varphi(u,v)$ which when restricted to those in U'_k give the operations of U'_k. That is, the ring operations on U'_k lift naturally to operations on the set of all $\varphi(u,v)$. What remains to be proved, hopefully in the sequel, is that the equality in U'_k lifts to a recursive operation on the set of all $\varphi(u,v)$. Thus, in this paper, when we talk of presenting U'_k we have in mind a set of pp-formulas, with recursive operations $+$, $-$ and . (and some other recursive operations with algebraic significance), but we make no assumptions about the equality. We do not wish to enter into formalities of recursive model theory here. What we intend should be clear from what follows. Any time we have an enumeration as above, with liftings of the ring operations (and maybe others) recursively enumerable, but the equality not assumed r.e, we say we have a weakly enumerated structure. Note, however, that we do not regard the lifted structures in our case as rings. They simply become so modulo an equivalence relation which can be very complicated. When we want to refer to the liftings of the ring operations we call our structure a pre-ring.

The ring generated by $U_{k,0}$ and the basic idempotents e_p for $p \in U_{k,0}$, for which we have given a genuine recursive presentation, is a subring of the centralizer $Z'(h)$ of h in U'_k. Note that we have a recursive enumeration of certain pp formulas defining these elements (although we certainly do not have all such pp formulas).

Now, following Herzog, we construct a weakly enumerated structure which lifts, as above, a von Neumann regular ring $Z^+(h)$ which is a subring of $Z'(h)$ and contains the ring generated by $U_{k,0}$ and the basic idempotents e_p for $p \in U_{k,0}$. Basically, we have to start with the latter ring (note that for it we have identified the collapsing congruence as recursively enumerable, and so we can without danger conflate the pre-ring and the ring) and close, in the sense of pre-rings under

1. going from r to $Image(r)$ to the idempotent (Herzog, page 261) corresponding to projection onto $Image(r)$;
2. the ring operations.

Note that these operations have clear meaning at the level of pp-formulas.

In this way we see clearly that $Z^+(h)$ is weakly presented. Moreover, it has a "section" operator as defined in Section 7. In this context this means that we have an operation taking an r to an s so that $(r - rsr, 0)$ is in the congruence, and this operation is recursive.

8.3. From $Z^+(h)$ Towards the Lifting of U'_k, via Lattice Considerations

Here we follow Herzog's page 265. As we pointed out already, $\mathcal{I}$ and $\mathcal{F}$ are recursive. Indeed, his argument shows that we can recursively find, for φ in $\mathcal{I}$, a bound n so that $dim_k \varphi(V_\lambda) \leqslant n$ (enumerate the theory of U_k-modules to get φ bounded by a $Ker(q)$, for $q \in U_k$). There is of course a very serious issue of getting optimal n, but this can be bypassed, for now, by using Herzog's fundamental "highest pseudoweight space" operator, which is recursive at the pre-ring level, if suitably interpreted. First note that we can recursively bound the dimension of $Ker(q)$ using the workhorse Lemma 6.1 from the section on uniformly bounded φ.

We are going to make crucial use of the details of Herzog's work on his page 262 on the highest pseudoweight space construction. This takes one constructively from any (definition of an) idempotent e in $Z'(h)$ to a definition, by an h-invariant pp-formula, of the highest weight space of eV_λ, uniformly in λ. Then in turn one gets (a pp definition of) the idempotent e_0 corresponding to projection onto this highest weight space. We will use the notation $hw(e)$ as being more memorable than e_0.

Now we first pass to a bigger pre-ring $Z^{++}(h)$, got from $Z^{+}(h)$ by closing off, in the obvious recursively enumerable way, under hw and all the operations previously used to construct $Z^{+}(h)$. Again we have a pre-ring which is weakly enumerated and, again, we have closure under the section operation, so that we have a lifting of a von Neumann regular subring of U_k'.

Now we prove a constructive analogue of Herzog's Lemma 25. Suppose $\varphi(u)$ corresponds to an idempotent $1-e$ in $Z^{++}(h)$, and $\varphi(u)$ is in $\mathcal{I}$. Find a recursive bound n for $dim_k \varphi(V(\lambda))$. Form successively:

$$f_0 = e - hw(e), f_1 = f_0 - hw(f_0), \ldots$$

proceeding through $n+1$ steps (the series may well stabilize before this, but it will stabilize by $n+1$ steps). Each f_i is in $Z^{++}(h)$. An easy constructive argument (based on Herzog's proposition 16) shows that the lattice of subobjects (relative to $FinDim$) of each $Im(f_i)$ is a Boolean algebra, and then by easy and explicit Boolean algebra we get

Lemma 8.1. *The lattice of subobjects of φ as above is complemented and the corresponding idempotents are in $Z^{++}(h)$.*

One should really emphasize the constructive aspects of this prior to collapsing by the lattice congruence coming from $FinDim$. The lattice operations, and the complementation, are constructive at the level of pp-formulas. The congruence itself is not yet fully analyzed from a constructive viewpoint.

Now we consider a general φ. As in Herzog there are two cases.

Case 1. φ in $\mathcal{F}$. Get constructively $q \in U$ so that φ dominates $Im(q)$ in the theory of U-modules. Then, by Herzog's Lemma 21, get p in U_0, so that for all V in $FinDim$

$$Im(q) + Ker(x^n) + Ker\big(\theta\sigma(\rho)\big) + Ker(y^n) = 0$$

so that in Herzog's notation for the lattice of subobjects of H

$$\varphi + \psi = H,$$

where ψ is the sum of the three kernels in the above equation. But evidently ψ is in $Z^{++}(h)$, whence the lattice of subobjects of ψ is constructively complemented, by the preceding lemma. So, constructively, as in Herzog, we get a complement for φ in H.

Case 2. φ in $\mathcal{I}$. Now work with φ^- in $\mathcal{F}$.

What has been proved constructively? In terms analogous to those used on "liftings" of U_k', we have considered the set of pp-formulas in one free variable, and put on it recursive liftings of the lattice operations, the sum operation, and a relative complement operation, which, modulo the congruence associated to FinDim, become the Herzog operations on subobjects of H.

We stress that we are fully aware of the sketchy nature of our discussion. This is typical of the unwinding of proofs. We do not expect understanding from a reader who is not already familiar with Herzog's precise but non-effective construction. In the sequel we will be more precise, depending on the demands of the situation.

8.4. Reaching U'_k as a Constructive von Neumann Regular Semiring

If one wants to proceed constructively, then the preceding arguments are not quite enough to get a presentation of a lifting of U'_k. On page 266 of Herzog, he can conclude directly that U'_k is von Neumann regular. We have to do more, because we have used, as Herzog does without comment, only binary $\varphi(u, v)$ that define maps from H to H. We should, and can, get round the prima facie nonconstructive nature of this restriction.

Lemma 8.2. *We can attach constructively to every pp-formula φ in two free variables a pp-formula φ^f in the same variables such that*

- *φ^f does define a function on H;*
- *φ^f defines the same function as φ on H if φ defines a function on H.*

Proof. Let $\varphi(u, v)$ be pp-formula. Let $\chi(v)$ be $\varphi(0, v)$. Then $\varphi(u, v)$ fails to be the graph of a(partial) function only if $\chi(V) \neq 0$ for some V in FinDim. If $\varphi(u, v)$ does define a function then the function is total on V if and only if $\theta(V) = V$, where θ is $(\exists w)\varphi(v, w)$. Now we have constructively the (liftings of) definable idempotents e_χ and e_θ corresponding to projection to the respective subspaces $\chi(V)$ and $\theta(V)$ (the projections got from constructive definable complements in the set of pp-formulas in one variable).

Now we define a pp-formula φ^{pf} in two variables by

$$(\exists w)\big(\varphi(u, v) \text{ and } (1 - e_\chi(w)) = v\big).$$

Clearly this is the graph of a partial function, for if $\varphi(u, w_1)$ and $\varphi(u, w_2)$ and $(1 - e_\chi)(w_1) = v_1$ and $(1 - e_\chi)(w_2) = v_2$ in V then $\chi(w_1 - w_2)$ and so $(1 - e_\chi)(w_1) = (1 - e_\chi)(w_2)$ since $w_1 - w_2$ lives on χ.

Equally clearly, if φ is the graph of a partial function on V then e_χ annihilates V, so $1 - e_\chi$ is the identity on V so φ and φ^{pf} define the same partial function. Finally, let $\varphi^f(u, v)$ be $\varphi^{pf}(e_\theta(u, v)$, and we clearly have the required conclusion. □

What we have done above replaces Herzog's terse argument. For now we can take as our domain for the lifting the set of all φ in two variables, replacing φ systematically by φ^f to get a prering structure which is obviously recursive. Moreover, by going from φ^f to its image (construed as in the set of all pp-formulas in one variable), we get, constructively, the section operator on our domain, and thus a proof that when we mod out the congruence coming from $FinDim$, we have a von Neumann regular ring, Herzog's U'_k.

8.5. Concluding Remarks

Our ultimate goal is to exhibit U'_k as a genuine recursive von Neumann regular ring, and thereby to get decidability of the theory of $FinDim$, as well as a clear understanding of the algebra of sets

$$\{\lambda : V_\lambda \models \Phi\},$$

for Φ a sentence of the language of U_k-modules. Till now we have considered in detail only the variation of kernels of p for p in U_0. In the next paper we will go on to consider the finer detail of the structure of the $Ker(q)$ for general q in U_k. Various uses will be made of deeper diophantine geometry. Thus we will give a normal form for the nonstandard p, and analyze, for pairs of nonstandard elements p_1 and p_2, the set of λ for which each has a nonzero kernel in V_λ. A deep theorem relating to this is that of Bilu and Tichy [2]. We expect that this analysis will lead us to a recursive presentation of U'_k.

References

[1] J. Ax, The elementary theory of finite fields, *Annals of Math.* **88** (1968), 239–271.
[2] Y.F. Bilu, R.F. Tichy, The Diophantine equation $f(x) = g(y)$, *Acta Arith.* **95**(3) (2000), 261–288.
[3] N. Bourbaki, *Lie groups and Lie Algebras*, Springer, 2002.
[4] C.C. Chang, H.J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.
[5] J. Dixmier, *Enveloping Algebras*, North Holland, 1977.
[6] K. Erdmann, M. Wildon, *Introduction to Lie algebras*, Springer, SUMS series, 2006.
[7] I. Herzog, The pseudo-finite dimensional representations of $sl(2,k)$, *Selecta Mathematica* **7** (2001), 241–290.
[8] M. Hindry, J. Silverman, *Diophantine geometry. An introduction*, in: Graduate Texts in Mathematics, vol. 201, Springer, 2000.
[9] A. Macintyre, The mathematical significance of proof theory, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **363**(1835) (2005), 2419–2435.
[10] A. Mostowski, On direct products of theories, *J. Symbolic Logic* **17** (1952), 1–31.
[11] D. Poulakis, Affine curves with infinitely many integral points, *Proc. Amer. Math. Soc.* **131**(5) (2003), 1357–1359.
[12] M. Prest, *Model theory and modules*, Cambridge UP, 1988.
[13] M. Prest and G. Puninski, Pure injective envelopes of finite length modules over a Generalised Weyl Algebra, *J. Algebra* **251** (2002), 150–177.
[14] J.P. Serre, *Lie algebras and Lie groups: 1964 Lectures given at Harvard University*, in: LNM, vol. 1500, Second Edition, 1992.
[15] J. Silverman, On the distribution of integer points on curves of genus zero, *Theoretical Computer Science* **235** (2000), 163–170.
[16] W. Szmielew, Elementary properties of abelian groups, *Fund. Math.* **41** (1955), 203–271.
[17] J.S. Wilson, On pseudofinite simple groups, *J. London Math. Soc.* (2) **51** (1995), 471–490.
[18] M. Ziegler, Model theory of modules, *Ann. Pure Appl. Logic* **26** (1984), 149–213.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Encounters with A. Mostowski

J.A. MAKOWSKY
Department of Computer Science, Technion–Israel Institute of Technology, 32000 Haifa, Israel
janos@cs.technion.ac.il

Abstract. We trace our encounters with Prof. A. Mostowski, in person and in his papers. It turns out that he and his papers left a continuous mark on my own work, from its very beginning, and till today.

1. The Zürich Logic Seminar

I do not quite remember when I first heard the name of A. Mostowski, but it was in the early stages of my undergraduate studies at the mathematics and physics department of the ETH (Swiss Federal Institute of Technology) in Zürich, around 1967–1968.

This is how I got into Mathematical Logic at an early stage of my studies. When still at the Gymnasium, I spent much of my time studying philosophy. Plato, Leibniz, Kant, Cassirer, the Neokantians, and the Vienna Circle attracted my special interest. I also attended university seminars on Kant and Chomsky, before my matriculation exam (Matura). Originally I wanted to study mathematics and philosophy, but following good advise, I finally registered in 1967 to study mathematics and physics, leaving philosophy to my free time.

So it was only natural that in my first semester I participated in a seminar of the general studies department dedicated to Wittgenstein's *Tractatus Logico-Philosophicus*, organized by E. Specker[1] and G. Huber[2]. Wittgenstein was just slowly being rediscovered in German Language Academia[3], and the seminar turned out to be a major event. E. Specker was also the lecturer in my first course on algebra, and he impressed me by

[1]E. Specker, born 1920, Professor of Mathematics and Mathematical Logic at ETH, Zürich, 1955–1987. Ernst Specker has made decisive contributions towards shaping directions in topology, algebra, mathematical logic, combinatorics and algorithms over the last 60 years. His collected works are published as [50].

[2]G. Huber, born 1923, was a disciple of H. Barth and K. Jaspers. 1956–1990 he held the chair of philosophy at ETH. His interest were in metaphysics, epistemology, ethics, history of philosophy (especially ancient Greek philosophy), contemporary social environmental problems, and politics of science. His magnum opus "Eidos und Existenz" was published in 1995.

[3]Notably through Ingeborg Bachmann's essay [4] and articles published in the "Kursbuch", a then newly founded highly influential periodical published by H.M. Enzensberger [28]. Both Ingeborg Bachmann and H.M. Enzensberger are outstanding literary figures of German language literature of the post-war period, and both are among its most distinguished poets. Bachmann's poetry is available in English [5]. For more information, cf. http://en.wikipedia.org/wiki/Ingeborg_Bachmann. and http://en.wikipedia.org/wiki/Hans_Magnus_Enzensberger. Enzensberger wrote in 1962 a poem on Gödel's incompleteness theorem, which also serves as the text of H.W. Henze's second violin concerto, [70]. For Wittgenstein's influence on poetry, cf. [89].

his approach to mathematics, by his non-conventional personality, and his civil courage in the politically charged time of the Cold War and the international student revolts. In my second semester I took E. Specker's course in Mathematical Logic, and also started to attend the Logic Seminar. The Logic Seminar had a long tradition, originally started by P. Bernays and F. Gonseth in 1936. After the war E. Specker, J.R. Büchi, C. Böhm and G. Müller were regulars, and W. Craig, R. Sikorski and H. Wang were among its guests. In 1955 E. Specker joined as newly appointed professor, and in 1966 H. Läuchli.

The seminar was always held on Monday between 5pm and 7pm. When I studied between 1967 and 1974 it was known as the Läuchli-Specker Seminar, but Prof. P. Bernays, who was born in 1887, still attended and commented the sessions. The seminar was also an undergraduate seminar, where students could get credit, but most of the participants were regulars, undergraduates, graduates and post-graduates, including other logicians from Zürich (H. Bachmann, E. Engeler, P. Finsler, R. Fittler and B. Scarpellini) and other occasional guests. The seminar continued its activities after Specker's retirement under H. Läuchli. After Läuchli's premature death in 1997, it continued again under E. Specker, who participated uninterruptedly till the official end of the seminar in 2002. From 1973 on, E. Specker and V. Strassen also organized a joint seminar of the University of Zürich and ETH on Logic and Algorithmics[4]. Each semester the seminars were dedicated to a topic. During my studies we had seminars on such diverse topics as many-valued logics, decidable und undecidable theories after [24,103], model theory after [94, Chapter 5], categoricity after [73], the spectrum problem, Hilbert's 10th problem, finite versions of the axiom of choice after [75], non-standard analysis, intuitionism, and complexity theory, cf. [96]. In these seminars I was particularly impressed by the omnipresence of Polish logicians. The names and works of A. Ehrenfeucht, K. Kuratowski, J. Łoś, J. Łukasiewicz, A. Mostowski, H. Rasiowa, C. Ryll-Nardzewski, R. Sikorski, R. Suszko, and A. Tarski became all familiar to me in this period. In E. Specker's work on set theory [95], and later also in H. Läuchli's early papers [57], A. Mostowski's [74,75] plays an important role. E. Specker met A. Mostowski several times at conferences, definitely in Warsaw 1959 and in Jerusalem 1966. They held each other in very high esteem.

So it happened later that I did most of my Ph.D. work in Warsaw while being immatriculated in Zürich, much like A. Mostowski did most of his Ph.D. work in Zürich, while being immatriculated in Warsaw.

2. Personal Encounters

I first met A. Mostowski personally in October 1971, I saw him last in May 1973, and all our personal encounters took place in Warsaw. As described in the previous section I knew of A. Mostowski first from the literature. In 1970 I had already read the famous monograph on set theory by K. Kuratowski and A. Mostowski [56], I was familiar with most of Mostowski's work in model theory, but I had been most impressed by his booklet *Thirty years of Foundational studies*, [81].

In the summer of 1971 I attended the ASL Logic Colloquium in Cambridge and presented my results on the finite axiomatizability of categorical theories which constituted

[4] Among the participants of these seminars in the years 1970–1974 there were quite a few who are or were still active researchers more than twenty years later: W. Baur, C. Christen (1943–1994), W. Deuber (1942–1999), M. Fürer, J. von zur Gathen, J. Heintz, M. Sieveking, and E. Zachos.

my M.Sc. thesis. It was there that I met first S. Shelah, although I had corresponded with him before. I also met W. Marek, P. Zbierski[5] and M. Srebrny. W. Marek suggested I should try to visit Warsaw University and spend some time there. Upon my return I discovered the existence of a Swiss-Polish student exchange program and applied, together with my friend D. Giorgetta, for a grant to spend a year in Warsaw. The Swiss office in charge of the program informed us that only one of us was awarded the grant for mathematics, and it was D. Giorgetta[6]. But D. Giorgetta discovered upon his arrival, that the Polish side had selected both of us for a stay with the Logic Group in Warsaw and immediately wrote me a letter about this discovery. I traveled to Warsaw to see for myself what was going on, and with Prof. Mostowski's and Prof. H. Rasiowa's help the Swiss authorities were persuaded to change their decision. Officially, I was to be a guest to H. Rasiowa's group with A. Wasilewska as my formal host. It was then, in October 1971, that I first met A. Mostowski personally.

I spent the period from April 1972 till July 1972 as a Swiss-Polish exchange student during the preparation of my Ph.D. thesis in Warsaw. I was accompanied by my first wife. Prof. Mostowski took personal care of us. He took us himself to register at the police at the Mostowski Palace. I remember the astonished face of the clerk at the information desk, when A. Mostowski introduced himself. "Are you, is it possible, Panie Professorze, are you a relative of the original owner of this palace?" "No, no, I am not, just a coincidence of names", he replied, and inquired where we had to register. Only recently I learned that this was not quite true. A. Mostowski was a remote relative of Tadeusz Mostowski, after whom the palace is named[7].

At that time I was working in model theory and tried to settle the question asked by M. Morley in [73], whether there are finitely axiomatizable complete theories which were categorical in uncountable powers. I was working hard. I have been working on this before, but now had no new results. I had made some progress on this before, and the results were the content of my M.Sc. thesis [64]. I just had submitted two papers with these results to Polish journals, [65,68]. During my first stay in Warsaw I also studied generalized quantifiers, and finally decided to change topic. I read a lot, discussed a lot of mathematics with A. Mostowski whom I saw weekly, with W. Marek, and attended the logic seminar. The seminar was held in Polish, but as there were also other foreign guests,

[5] 1944–2002, another dear colleague who left us prematurely.

[6] Donato Giorgetta finished his Ph.D. under E. Specker in Mathematical Logic, and became later an actuarian for ZURICH Insurance. As a result of his stay in Warsaw he also became W. Marek's brother-in-law. Two more grants were awarded the same year: For studies in Slavic languages to D. Weiss, now professor at the University of Zürich. Among his many scientific activities he was also leader of a research project "Towards a History of Verbal Propaganda in Soviet Union and Socialist Poland", and for art studies to Roman Signer, by now a leading avant-garde artist. Roman Signer is well known for his explosions, his moving everyday objects and motorized vehicles. He is regarded as a representative of an expanded concept of sculpture and work and has meanwhile become a role model for a younger generation of artists.

[7] The building, at present the seat of Warsaw Metropolitan Police, is dated from 18th century. In 1762 Jan Hilzen, Minsk Governor, purchased the area on which the Palace is situated today and after demolition of a hitherto existing building he constructed a new mansion. In 1795 Tadeusz Mostowski, Hilzen's grandson, inherited the residence. In 1822 he ceded it to the Government of Polish Kingdom. In 1823 the Palace was reconstructed and adapted as a seat of Federal Commission of Internal Affairs and the Police. A well-known architect, Antoni Corazzi, designed the new shape of the Palace. The classical, monumental facade of the Palace is characterized by a beautiful break with bas-reliefs and four-column Corinthian portico. The interior hides a real architectural pearl, so called "White Ballroom", embellished with wonderful architectural profiles and ornamentation.

the speakers often chose to write English on the blackboard. I studied the beginnings of abstract model theory and read everything I could get my hands on about the model theory of extensions of First Order Logic. I had become an expert in the field, but still had no serious new results. A. Mostowski's wife, Maria Mostowska, then the director of the excellent library of the Mathematical Institute of the Polish Academy of Sciences, was always very helpful and knowledgeable when I tried to dig in the hidden treasures of this library.

In this period I also came to the conclusion that my marriage reached the end of its course. When I confessed to Professor Mostowski[8] in June that my work was stalling due to a marriage crisis, he asked politely and with empathy whether it was serious. I tried to explain a bit, but after my first attempts he just remarked that he was married since before the War, and in these days there were much bigger crises and none of them did lead to the dissolution of his marriage. Neither did these crises hinder his mathematical work. Mathematics, it seemed, was his anchor and his rescue island. Mathematics was the only rock solid secure thing on earth. I remember A. Mostowski once coming into the room I shared with W. Marek, J. Onyszkiewicz and P. Zbierski, telling us calmly about an incredible incident which just had occurred. During an advanced exam a student known for his excellence had failed when asked to sketch a proof of the Fundamental Theorem of Algebra. We suggested that the student may have been nervous and scared and as a consequence may have had a blackout. "Scared ?, blackout ?", A. Mostowski asked, "even if I were woken up in the middle of the night from deep sleep, and I were asked at gun point to prove the Fundamental Theorem of Algebra, I would just go ahead and prove it". It seemed that his maxim was "I prove therefore I am".

When my marriage crisis reached its peak in July, I asked Professor Mostowski for permission to interrupt my stay in Warsaw. He was very understanding. He also told me of the planned Logic Year to be held at the newly founded Banach Center, and suggested I should be one of its invited guests and lecture about abstract model theory.

In the following months my reading and my conversations with A. Mostowski and W. Marek bore fruit. I finally managed to prove some theorems in abstract model theory and got encouragement and suggestions from G. Kreisel, whom I had the pleasure to meet first in Zürich in the autumn of 1972. When I returned to Warsaw in February 1973 I had already most of the results which would later constitute my Ph.D. thesis, cf. [66,69].

My stay at the Logic Year at the Banach Center was one of the most formative periods of my scientific life. I lectured on abstract model theory for two months. Many scientific friendships were formed there and last till today. In the days where the Cold War was still the order of the day, the Logic Year provided a rare opportunity where established and young scientists from the Eastern Block and the West could dedicate their time to do mathematics and socialize without the interference of world politics. Both the research conditions and the social life were excellent and left a mark on all the participants. All were extremely happy about this. So it seemed at least. We later learned that A. Mostowski was reprimanded by the political authorities in Poland for having created a too liberal atmosphere. Be that as it may, A. Mostowski and H. Rasiowa were excellent hosts, and in spite of his various duties and obligations, A. Mostowski attended my lectures and continued to show interest for my work and discuss it with me. During that period I also had a very intensive correspondence with G. Kreisel in Stanford, and in

[8] When I used to speak to him alone, we spoke German and I addressed him as "Herr Mostowski", as he insisted that I use this more casual form.

April 1973 Kreisel sent a telegram offering me a visiting assistant professor position in Stanford starting in October 1973. I left Warsaw in June 1973. I never thought that this was the last time I would see A. Mostowski. He had given me a lot as a human being and as scientist. He incorporated the old ideal of a humble but proud person serving science and respecting people as they are. He was always ready to encourage curiosity and true talent. He was suspicious of showmanship and preferred to let mathematics shine in its own light. His understatement concerning his own work reaches a peak in his booklet *Thirty years of Foundational studies*, [81], where according to the index of authors he refers only five times to his own work, among which once for pointing out an error in one of his papers.

In the following sections I will discuss those of A. Mostowski's papers in more detail which had a direct influence on my own work.

3. Categoricity in Power

A first-order theory T is categorical if all of its models are isomorphic. Because of the compactness theorem for First Order Logic this can only be the case if the model unique up to isomorphism is finite. For infinite models, J. Łoś, and independently R. Vaught, [106], introduced the notion of categoricity in power: A first-order theory T is categorical in an infinite cardinal κ if all of its models of cardinality κ are isomorphic. R. Vaught uses the notion to prove decidability of various theories. His method is now known under the heading of Vaught's Test. In [63] J. Łoś gives examples for this newly defined notion and states some unsolved problems concerning it.

The first papers relating to these questions all come from the Polish school. In 1956 A. Ehrenfeucht and A. Mostowski [25] introduced the notion of indiscernibles. These ideas were further developed by A. Ehrenfeucht, [21], where he shows (using the general continuum hypothesis) that a theory categorical in an uncountable successor cardinal cannot define a connected antisymmetric relation of any finite arity. In 1959 C. Ryll-Nardzewski [90], gave a characterization of countable theories categorical in $\aleph_0$. This result was also independently found by E. Engeler [27] and L. Svenonius [97].

In 1969 in the Zürich Logic Seminar we have read the fundamental paper by M. Morley on categoricity in power [73]. This is probably the most influential paper in model theory written so far. It builds on technical work of the Polish schools in Warsaw (Mostowski), Wrocław (Ryll-Nardzewski) and Berkeley (Tarski, Ehrenfeucht, Vaught). In it, M. Morley, proves a truly deep theorem:

Theorem 1 (Morley, 1965). *A countable first-order theory is categorical in one uncountable power iff it is categorical in all uncountable powers.*

In proving this, the beginnings of a rich structure theory for models of first-order theories is laid. M. Morley also formulated extremely fruitful *open problems* emanating from his main results. The continuation of this line of research has become an important chapter in the history of logic[9]. Model theory from 1965 till today has been strongly influenced by Morley's paper, with S. Shelah its most dominant figure. But at the core of Morley's work we have his precursors, A. Ehrenfeucht and A. Mostowski.

[9] We have no space tell this fascinating story. The reader may consult the excellent book by W. Hodges [48].

I also tried my luck with one of these problems, the question of finite axiomatizability of complete κ-categorical theories. At first my modest attempts looked promising, [65,68]. An almost strongly minimal theory is a simple case of a theory categorical in all uncountable powers. I managed to show[10]:

Theorem 2 (Makowsky, 1971). *There are no complete finitely axiomatizable $\aleph_0$-categorical strongly minimal theories.*

A theory categorical in all uncountable powers is superstable, but the converse is not true. Using an idea of H. Läuchli, my supervisor, I managed to show:

Theorem 3 (Makowsky, 1971). *There is a complete finitely axiomatizable superstable theory.*

H. Läuchli also had suggested a sufficient group-theoretic hypothesis which would enable one to construct a complete finitely axiomatizable theory categorical in all uncountable powers. The group-theoretic hypothesis is still an open problem in combinatorial group theory, cf. [68], but it did not attract the attention of the group theorists, in spite of my various efforts to popularize the problem among combinatorial group theorists. After 1972 I was stuck with these results, but so were all the many others who tried. Only after 1980 the questions were completely answered. M. Peretyat'kin [88] showed

Theorem 4 (Peretyat'kin, 1980). *There are complete finitely axiomatizable theories categorical in all uncountable powers.*

B. Zilber [107] and G. Cherlin, L. Harrington and A. Lachlan [13] showed

Theorem 5 (Zilber, 1980). *There are no complete finitely axiomatizable theories categorical in all infinite powers.*

Very recently I wrote a paper with B. Zilber, [87], relating categorical theories to graph polynomials. In it, indiscernibles again play a crucial role. So we are back to A. Ehrenfeucht and A. Mostowski, with [25].

4. Generalized Quantifiers and Infinitary Logics

Inspired by A. Mostowski's lovely booklet [81], I already got interested in generalized quantifiers and extensions of First Order Logic during my undergraduate studies. I also read [47], where L. Henkin introduced partially ordered quantifier prefixes and infinitary formulas, and presented his ideas at the Symposium on Foundations of Mathematics held in Warsaw in the Fall of 1959. It was W. Marek who pointed out to me P. Lindström's work [60,61]. P. Lindström proved in these papers:

Theorem 6 (Lindström, 1966). *First Order Logic is the only model theoretic logic which both has the compactness and the Löwenheim-Skolem-Tarski property.*

[10] R. Vaught told me, when I presented my result in the Berkeley Logic Seminar in 1973, that he also proved it much earlier, but he had not published his proof.

The exact framework for this theorem is a bit complicated. For an elaborate statement of Lindström's Theorems in their proper setting, the reader may consult the introductory chapters of [11] by J. Barwise, H.-D. Ebbinghaus and J. Flum [7,19,34].

The origins of this theorem go back to A. Mostowski. In [79] a generalization of quantifiers is proposed, and a few structural characterizations are given for quantifiers on pure sets, the cardinality quantifiers. In the paper, A. Mostowski also states several open problems, mostly asking whether classical theorems for First Order Logic have their corresponding counterparts for First Order Logic augmented with a generalized quantifier. This paper triggered extensive research into various generalized quantifiers, and in the following years an abundance of generalized quantifiers appeared in the literature, establishing completeness and compactness theorems, or exhibiting counterexamples. A. Mostowski approached the question of Craig's interpolation theorem for such extensions, [82]. However, no general theory evolved till the work of P. Lindström.

I once asked P. Lindström how he had found his celebrated theorem. He answered that he was looking for a non-trivial application of the Ehrenfeucht-Fraïssé games, cf. [22]. What he really proved was

Theorem 7. *Given two countable structures $\mathcal{A}, \mathcal{B}$ over a finite vocabulary which are elementarily equivalent, then there exists a countable structure $\mathcal{C}$ which is an elementary extension of both $\mathcal{A}$ and $\mathcal{B}$.*

In his proof he used the ingenious encoding of the back-and-forth property of Ehrenfeucht-Fraïss'e games within the two-sorted pair $[\mathcal{A}, \mathcal{B}]$ and applied the compactness and the Löwenheim-Skolem-Tarski property. He noted further that if the vocabularies of the two structures are made disjoint, the same argument could be used to show that no generalized quantifier, as defined in Mostowski's paper, could be added to the logic. Further analyzing his argument he realized that an even more general definition of generalized quantifier would allow the same trick. So he worked out his definition of abstract logics.

As already suggested by A. Mostowski in [79,82], it is a natural question to ask whether there were analogous characterizations of logics involving the Craig interpolation or the Beth definability theorem. Besides generalized quantifiers one can also introduce the logic $\mathcal{L}_{\kappa,\lambda}$ with disjunctions and conjunctions of size $< \kappa$ and strings of standard quantifiers of length $< \lambda$. E. Engeler in his thesis was the first to study such infinitary first-order languages, [26]. C. Karp proved a completeness theorem for the language $\mathcal{L}_{\omega_1,\omega}$, [54]. E. Lopez-Escobar, [58] proved the Craig interpolation theorem for this logic. D. Scott, [92], studied the model theory of $\mathcal{L}_{\omega_1,\omega}$ further, showing, among other results, the following theorem:

Theorem 8 (Scott, 1965). *For every countable structure $\mathcal{A}$ with finite vocabulary there is a sentence $\sigma(\mathcal{A})$ in $\mathcal{L}_{\omega_1,\omega}$ which has, up to isomorphisms, only $\mathcal{A}$ as a model.*

The sentence $\sigma(\mathcal{A})$ is today called the *Scott sentence* of $\mathcal{A}$. J. Barwise, in his thesis [6], showed

Theorem 9 (Barwise, 1967). *There are infinitely many sublogics of $\mathcal{L}_{\omega_1,\omega}$ which satisfy the Craig interpolation theorem.*

On the other hand, A. Mostowski, [82], was the first to show how to prove that certain logics do not satisfy the Craig interpolation theorem. In 1970 H. Friedman rediscovered Lindström Theorems and discussed in several widely circulated manuscripts, [37,38], the status of the Craig's interpolation theorem and Beth's definability theorem in various logics, and finally published his [39].

In the light of this situation I proved, [66]:

Theorem 10 (Makowsky, 1973). *$\mathcal{L}_{\omega_1,\omega}$ is the smallest model-theoretic logic which satisfies the Craig interpolation theorem and contains all the Scott sentences of countable structures.*

Actually, the theorem holds for a property strictly weaker than Craig interpolation, the so called Δ-interpolation, first discussed in an abstract framework by S. Feferman, [32,33]. Both these papers appeared in the special issue of Fundamenta Mathematicae dedicated to A. Mostowski's 60th birthday[11]. My own Ph.D. thesis [67,86,69] deals with Δ-interpolation, and I owe a lot to A. Mostowski and S. Feferman, for their interest and encouragement which I received during its preparation.

By 1975 the subject of abstract model theory was well established and attracted many researchers, and I continued working in the field in collaboration with S. Shelah till 1985, [83–85]. The monumental book [11] was written when the field most flourished. It contains essentially all the results obtained in this field till 1984. Then the interest in the subject faded, as it became clear that many theorems depend on set-theoretic assumptions, such as the existence of large cardinals, cf. [71]. However, the concepts developed in abstract model theory found new life in finite model theory, a discipline developed in theoretical computer science, under the name of *Descriptive Complexity*. The books [20,49,59] bear witness to this revival. But this is another story.

5. The Spectrum Problem

Descriptive complexity has its origin in the spectrum problem. A. Mostowski was among the first to study this problem as well.

We denote by $Spec(\phi)$ the set of natural numbers n such that ϕ has a model of size n. A first-order spectrum is a set of natural numbers S such that there is a first-order sentence ϕ_S over some finite vocabulary such that $S = Spec(\phi_S)$. In 1952 H. Scholz, [91], posed the following problem.

Scholz's Question: Characterize the sets of natural numbers which are first-order spectra.

This problem inaugurated a new column of *Problems* to be published in the Journal of Symbolic Logic and edited by L. Henkin. Other questions published in the same issue were authored by G. Kreisel and L. Henkin. They deal with a question about interpretations of non-finitist proofs dealing with recursive ordinals and the no-counter-example interpretation (Kreisel), the provability of formulas asserting the provability or independence of provability assertions (Henkin), and the question whether the ordering principle

[11] My paper [68] also appears in this issue.

is equivalent to the axiom of choice (Henkin)[12]. All in all 9 problems were published, the last in 1956.

The context in which Scholz's question was formulated is given by the various completeness and incompleteness results for First Order Logic which were the main concern of logicians of the period. An easy consequence of Gödel's classical completeness theorem of 1929 states that validity of first-order sentences in all (finite and infinite) structures is recursively enumerable, whereas Church's and Turing's classical theorems state that it is not recursive. In contrast to this, it was shown in 1950 by B. Trakhtenbrot [104], that validity of first-order sentences in all finite structures (f-validity) is not recursively enumerable, and hence satisfiability of first-order sentences in some finite structure (f-satisfiability) is not decidable, although it is recursively enumerable. Thus, what H. Scholz was really asking is whether one could prove anything meaningful about f-satisfiability besides its undecidability.

The first to publish a paper in response to H. Scholz's problem was G. Asser [3]. He did give a rather intricate necessary and sufficient condition for an arithmetical function $f_S(n)$ to be the characteristic function of a spectrum S. The condition shows that such a function is elementary in the sense of Kalmar. In his thesis written under the supervision of A. Mostowski in 1950, A. Grzegorczyk defined a hierarchy of low complexity recursive functions, today known as the Grzegorczyk Hierarchy. The thesis was published as [43]. As Kalmar's elementary functions correspond to the third level $\mathcal{E}^3$ of the Grzegorczyk Hierarchy, we have

Theorem 11 (Asser, 1955). *If f_S is the characteristic function of a first-order spectrum S, then $f_S \in \mathcal{E}^3$.*

Furthermore, G. Asser showed that not every characteristic function $f \in \mathcal{E}^3$ is the characteristic function of a first-order spectrum. Asser also noted that his characterization did not establish whether the complement of a spectrum is a spectrum.

Asser's Question: Is the complement of a first-order spectrum a first-order spectrum?

About the same time, A. Mostowski [78] also considered the problem. He showed

Theorem 12 (Mostowski, 1956). *All sets of natural numbers, whose characteristic functions are in the second level of the Grzegorczyk Hierarchy $\mathcal{E}^2$, are first-order spectra.*

Further important work on the spectrum problem is contained in J. Bennett's thesis, [8], which was never published and contains a wealth of interesting results. Scholz's original question was finally answered in 1972, after twenty years, when N. Jones and A. Selman related first-order spectra to non-deterministic time bounded Turing machines. Their result was first published in a conference version in 1972, [52], and the journal version appeared in 1974, [53]. Let $\mathbf{NE} = \bigcup_{c \geqslant 1} \mathbf{NTIME}(2^{c \cdot n})$ the family of sets of natural numbers recognizable in exponential time by non-deterministic Turing machines.

[12] Incidentally, A. Mostowski had published a paper answering this question in 1939, [74]. The paper appeared in Fundamenta Mathematicae, but was overlooked and is not referenced in the Mathematical Reviews. But it was reviewed in the Journal of Symbolic Logic the same year, JSL 4 (1939) pp. 129–130 by A.A. Bennett. This is not surprising, as the Nazis effectively stopped the functioning of Fundamenta Mathematicae in 1939, and its circulation was made almost impossible till the end of World War II. In the first number of Fundamenta Mathematicae after the War the victims of the War and the Nazi atrocities are listed.

coNE denotes the family of sets of natural numbers S such that $\mathbb{N} - S \in \mathbf{NE}$. N. Jones and A. Selman showed

Theorem 13 (Jones and Selman, 1972). *A set $S \subseteq \mathbb{N}$ is a first-order spectrum if and only if $S \in \mathbf{NE}$.*

Corollary 14. *First order spectra are closed under complementation if and only if $\mathbf{NE} = \mathbf{coNE}$.*

This shows why Asser's question could not be answered easily. The question whether $\mathbf{NE} = \mathbf{coNE}$ is an outstanding open problem of complexity theory, cf. [42,51]. It is related in spirit to the more famous question of whether $\mathbf{NP} = \mathbf{coNP}$, and for both it is widely believed that the answer is negative. However, a positive answer to $\mathbf{NE} = \mathbf{coNE}$ would have less dramatic consequences.

The characterization of spectra using non-deterministic complexity classes was independently found also by C. Christen and R. Fagin. Claude Christen[13]'s thesis, [14] remains unpublished, and only a small part was published in German [15]. Christen discovered all his results independently, and only in the late stage of his work his attention was drawn to Bennett's work [8] and the paper of Jones and Selman [52]. It turned out that most of his independently found results were already in print or published by R. Fagin after completion of C. Christen's thesis.

Ronald Fagin's thesis is a treasure of results dealing also with generalized spectra, which are models of existential second order sentences. For a published account of his work, cf. [29,30].

R. Fagin's main result is:

Theorem 15 (Fagin, 1975). *A class of finite structures K is recognizable by a polynomial time non-deterministic Turing machine if and only if it is definable by an existential sentence in Second Order Logic.*

This theorem is rightly seen as the beginning of a new discipline: Descriptive Complexity Theory, [20,49,59]. A very personal account of its evolution and of Fagin's work in finite model theory is given in [31]. Its main purpose is to characterize low level complexity classes in terms of logical definability. Scholz's question and the early answers of G. Asser's and A. Mostowski's papers pioneered this line of investigations. So did the papers by J.R. Büchi, B. Trakhtenbrot and C.C. Elgot, [12,23,105]. They showed

Theorem 16 (Büchi, Elgot, Trakhtenbrot, 1960). *The regular languages (sets of finite words) are exactly those languages which are definable in Monadic Second Order Logic.*

The corresponding problem for more complex sets of integers, initiated by S. Kleene, [55], was also investigated by A. Mostowski already in a series of papers starting in 1947, [76].

And where is my encounter with A. Mostowski in all this? It is twofold. C. Christen did his work while I was studying in Zürich and we became good friends. Both he and I learned about the spectrum problem in the Zürich Logic Seminar, where also A. Mostowski's paper was discussed. C. Christen finally presented his work in the

[13] Claude Christen, born 1943, joined the faculty of CS at the University of Montreal in 1976 and died there, a full professor, prematurely, on April 10, 1994.

Specker-Strassen seminar, which I also attended. It took another thirty years till I returned to the spectrum problem. In [18] the following was shown:

Theorem 17 (Durand, Fagin and Loescher, 1997). *Let $S \subset \mathbb{N}$. S is ultimately periodic iff there is a first-order sentence ϕ_S with one unary function symbol and a finite number of unary relation symbols such that $S = Spec(\phi_S)$.*

Y. Gurevich and S. Shelah have generalized this to Monadic Second Order Logic, [44]. In [35], E. Fischer and I generalized this further to arbitrary vocabularies, provided the class of models is of some bounded width.

6. The Feferman-Vaught Theorem

One theorem which accompanies my mathematical work throughout, is the so called Feferman-Vaught theorem which tells us how to reduce the first-order properties of a generalized sum or product of structures to the first-order properties of its summands, respectively factors. The theorems actually extends, in the case of generalized sums, to Monadic Second Order Logic. The theorem, and especially its proof, has its origin in A. Mostowski's [77]. This theorem has far reaching applications in establishing the decidability of first-order and monadic second-order theories, in automata theory, and in algorithmics. The latter two applications where thoroughly discussed in [72]. Some passages of this section are taken from [72].

I first explain the theorem, and for this we need some technical notation. For a vocabulary τ, $FOL(\tau)$ denotes the set of τ-formulas in First Order Logic. $SOL(\tau)$ and $MSOL(\tau)$ denote the set of τ-formulas in Second Order and Monadic Second Order Logic, respectively. A *sentence* is a formula without free variables. For a class of τ-structures K, $Th_{FOL}(K)$ is the set of sentences of $FOL(\tau)$ that are true in all $\mathfrak{A} \in K$. We write $Th_{FOL}(\mathfrak{A})$ for $K = \{\mathfrak{A}\}$. Similarly, $Th_{SOL}(K)$ and $Th_{MSOL}(K)$ denote the corresponding sets of sentences for SOL and $MSOL$. For a set of sentences $\Sigma \subseteq SOL(\tau)$ we denote by $Mod(\Sigma)$ the class of τ-structures which are models of Σ.

The Feferman-Vaught Theorem stands at the beginnings of model theory. W. Hodges, in his delightful book, [48], very carefully traces the history of early model theoretic developments. Most of the references in the sequel are taken from it.

A. Tarski published four short abstracts on model theory in 1949 in the Bulletin of the American Mathematical Society [100,101,99,98]. He also had sent his manuscript of *Contribution to the theory of models, I* to E.W. Beth for publication as [102]. Seemingly inspired by these, E.W. Beth published two papers on model theory [9,10]. In [10] he, and independently R. Fraïssé in [36], showed, among other things, that

Theorem 18 (Beth 1954, Fraïssé 1955). *Let $\mathfrak{A}, \mathfrak{B}$ be linear orders, $\mathfrak{C} = \mathfrak{A} \sqcup_{<} \mathfrak{B}$ their ordered disjoint union. Then $Th_{FOL}(\mathfrak{C})$ is uniquely determined by $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$, and can be computed from $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.*

In the early fifties A. Tarski had many young researchers gathered in Berkeley, among them A.Ehrenfeucht, S. Feferman, R. Fraïssé and R. Vaught. One of the many questions studied in this early period of logic, and the one which interests us here is the following:

Question: Let $\mathfrak{A}, \mathfrak{B}$ be τ-structures, $\mathfrak{A} \times \mathfrak{B}$ the cartesian product and $\mathfrak{A} \sqcup \mathfrak{B}$ the disjoint union. Assume we are given $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.

What can we say about $Th_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$?

What happens in the case of infinite sums and products?

This question triggered many landmark papers and also led to the study of ultraproducts.

The Feferman-Vaught Theorem evolved as follows. In [77] Mostowski proves, among other things[14] the analogue of Theorem 18 for products

Theorem 19 (Mostowski 1952). *Let $\mathfrak{A}, \mathfrak{B}$ relational structures or algebras, $\mathfrak{C} = \mathfrak{A} \times \mathfrak{B}$ their cartesian product. Then $Th_{FOL}(\mathfrak{C})$ is uniquely determined by, and can be computed effectively from, $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.*

Finally, S. Feferman and R. Vaught answered the question in the outermost generality, [40].

S. Feferman recalls[15], A. Tarski did not really appreciate the answer given. A special case of this answer reads as follows.

Theorem 20 (Feferman and Vaught, 1959). *Let $\langle \mathfrak{A}_i, i \in I \rangle$ be structures of the same similarity type. Then the theory of the infinite cartesian product $Th_{FOL}(\prod_{i \in I} \mathfrak{A}_i)$ and the theory of the disjoint union $Th_{FOL}(\bigsqcup_{i \in I} \mathfrak{A}_i)$ are uniquely determined by the theories of $\langle Th_{FOL}(\mathfrak{A}_i), i \in I \rangle$.*

A. Mostowski's proof already contains all the ingredients of the proof via reduction sequences, as used later in [40].

Another version also allows for the index structure to vary.

Theorem 21 (Feferman and Vaught, 1959). *Let $\mathfrak{A}$ be structure and I an index set. Then the theory of the infinite cartesian product $Th_{FOL}(\prod_{i \in I} \mathfrak{A})$ and the theory of the disjoint union $Th_{FOL}(\bigsqcup_{i \in I} \mathfrak{A})$ are uniquely determined by $Th_{FOL}(\mathfrak{A})$ and $Th_{MSOL}(I)$.*

By combining Theorems 20 and 21 with transductions and interpretations, similar results can be stated for a wide variety of *generalized products*. In the original paper [40] the transductions or interpretations are hidden in an unfortunate lengthy definition of generalized products.

In a sequence of papers by A. Ehrenfeucht, H. Läuchli, S.Shelah and Y. Gurevich, cf. [22,62,93,45,46] it emerged that Theorems 20 and 21 remain true for $MSOL$ rather than FOL in the case of the sum $Th_{MSOL}(\bigsqcup_{i \in I} \mathfrak{A}_i)$ and the multiple disjoint union $Th_{MSOL}(\bigsqcup_{i \in I} \mathfrak{A})$ but not for products.

All these extensions of A. Mostowski's approach from 1952 had far-reaching applications. The Feferman-Vaught Theorem played an important role in, or can be used to simplify, the proofs of decidability of various theories. This includes the first-order theories of Abelian groups, linear orders, and the monadic second order theories of various classes of orders and trees. There is no space here to discuss all this.

[14] In his own words: "The paper deals with the notion of direct product in the theory of decision problems. (...) [It discusses] a theory of which the primitive notions are representable as powers of certain base-relations and [reduces] all the problems concerning this theory (in particular the decision problem) to problems concerning the theory of the base relations".

[15] Personal communication, December 2000.

More recently, I used, together with B. Courcelle and U. Rotics, the Feferman-Vaught Theorem to simplify and extend results in algorithmic graph theory, knot theory, and the computational complexity of graph polynomials, [16,17,72].

7. Epilogue and Conclusions

I have met A. Mostowski regularly only for a relatively short time in his last years. His last student, with whom I became friends, was K. Apt, who worked on infinitistic rules of proof, cf. [2], a topic initiated by A. Mostowski in [80]. K. Apt later worked in program semantics, where some of these ideas seemed to be on the back of his mind, [1]. At this time I also worked on related problems, [41]. These papers have to do with proving termination of non-deterministic programs under various assumptions of fair execution. Termination of programs is not provable in general in the framework of first order logic, as it depends heavily on the existence of suitable well-orderings. The proof rules used in these papers reflect upon this. Is it a coincidence, that we both got involved with such questions, or is it yet another example of A. Mostowski's influence? Be that, as it may.

My own published work started in the model theory of first order theories, and then moved on to abstract model theory. In both these areas A. Mostowski's papers played an important role. From 1978 on I tried my hand in various questions in the foundations of Computer Science, mostly in Database Theory, Program Semantics and Verification. Here, A. Mostowski's influence is rather indirect. He did not live to see the importance of Logic for Computer Science. It was left to H. Rasiowa and her students to play an important role in these aspects of Logic. However, from 1995 on I returned to Logic, dealing mostly with applications of Logic to Combinatorics and Algorithmics. In these areas I found myself rereading papers by A. Mostowski and E. Specker, and finding there sources of inspiration.

I had several seminal encounters with great teachers. I only list here those related to mathematical logic. E. Specker was the first, in 1967. His lessons on mathematics and life still have an impact on me now. I owe a lot to S. Feferman, G. Kreisel, A. Lachlan, W. Marek, and A. Mostowski who directly influenced my early work. I owe a lot to my co-authors, especially S. Shelah, J. Stavi and B. Courcelle, and to my own graduate students. In this paper I tried to show, what scientific impact A. Mostowski and his papers had on my own work. I think the evidence speaks for itself.

Acknowledgements

I am indebted to W. Marek for his encouragement, insistence, patience and editorial care during the writing of this memorial essay.

References

[1] K.R. Apt and E.-R. Olderog, Proof rules and transformations dealing with fairness, *Sci. Comput. Programming* **3** (1983), 65–100.

[2] K.R. Apt, Infinitistic rules of proof and their semantics, *Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys.* **21** (1973), 879–886.

[3] G. Asser, Das Repräsentationsproblem im Prädikatenkalkül der ersten Stufe mit Identität, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **1** (1955), 252–263.

[4] I. Bachmann, Ludwig Wittgenstein – Zu einem Kapitel der jüngsten Philosophiegeschichte, *Frankfurter Hefte* **8** (1953), 7.

[5] I. Bachmann, *Darkness Spoken. The collected poems. English and German. Translated by P. Filkins*, Zephyr Press, 2006.

[6] K.J. Barwise, *Infinitary logic and admissible sets*, PhD thesis, Department of Philosophy, Stanford University, Stanford, CA, USA, 1967.

[7] K.J. Barwise, Model-theoretic logics: Background and aims, in: *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 1, Springer Verlag, 1985.

[8] James H. Bennett, *On spectra*, PhD thesis, Princeton University, 1962.

[9] E.W. Beth, On Padoa's method in the theory of definitions, *Indagationes Mathematicae* **15** (1953), 330–339.

[10] E.W. Beth, Observations métamathématiques sur les structures simplement ordonneés, in: *Applications scientifiques de la logique mathématique*, volume 5 of *Collection de Logique Mathématique, Serie A*, Paris and Louvain, 1954, pp. 29–35.

[11] J. Barwise and S. Feferman, eds, *Model-Theoretic Logics*, Perspectives in Mathematical Logic. Springer Verlag, 1985.

[12] J.R. Büchi, Weak second–order arithmetic and finite automata, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **6** (1960), 66–92.

[13] G. Cherlin, L. Harrington, and A. Lachlan, $\aleph_0$-categorical $\aleph_0$-stable structures, *Annals of Pure and Applied Logic* **18** (1980), 227–270.

[14] C.-A. Christen, *Spektren und Klassen elementarer Funktionen*, PhD thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1974.

[15] C.A. Christen, Spektralproblem und Komplexitätstherie, in: *Komplexität von Entscheidungsproblemen*, E. Specker and V. Strassen, ed., volume 43 of *Lecture Notes in Computer Science*, Springer, 1976, pp. 102–126.

[16] B. Courcelle, J.A. Makowsky, and U. Rotics, Linear time solvable optimization problems on graph of bounded clique width, extended abstract, in: *Graph Theoretic Concepts in Computer Science, 24th International Workshop, WG'98*, J. Hromkovic and O. Sykora, eds, volume 1517 of *Lecture Notes in Computer Science*, Springer Verlag, 1998, pp. 1–16.

[17] B. Courcelle, J.A. Makowsky, and U. Rotics, On the fixed parameter complexity of graph enumeration problems definable in monadic second order logic, *Discrete Applied Mathematics* **108**(1–2) (2001), 23–52.

[18] A. Durand, R. Fagin, and B. Loescher, Spectra with only unary function symbols, in: *CSL'97*, M. Nielsen and W. Thomas, eds, volume 1414 of *Lecture Notes in Computer Science*, Springer, 1997, pp. 189–202.

[19] H.D. Ebbinghaus, Extended logics: The general framework, in: *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 2, Springer Verlag, 1985.

[20] H. Ebbinghaus and J. Flum, *Finite Model Theory*, Springer Verlag, 1995.

[21] A. Ehrenfeucht, On theories categorical in power, *Fund. Math.* **44** (1957), 241–248.

[22] A. Ehrenfeucht, An application of games to the completeness problem for formalized theories, *Fund. Math.* **49** (1961), 129–141.

[23] C.C. Elgot, Decision problems of finite automata design and related arithmetics, *Trans. Amer. Math. Soc.* **98** (1961), 21–52.

[24] Yu. Ershov, L. Lavrov, I.A. Taĭmanov, and M.A. Taĭtslin, Elementary theories, *Russian Mathematical Surveys* **20**(4) (1965), 35–105.

[25] A. Ehrenfeucht and A. Mostowski, Models of axiomatic theories admitting automorphisms, *Fund. Math.* **43** (1956), 50–68.

[26] Erwin Engeler, *Untersuchungen zur Modelltheorie*, PhD thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1958.

[27] E. Engeler, A characterization of theories with isomorphic denumerable models, *Not. Amer. Math. Soc.* **6** (1959), 161. Abstract.

[28] H.M. Enzensberger, ed., *Kursbuch*, vol. 8, Suhrkamp, 1967.

[29] R. Fagin, Generalized first-order spectra and polynomial time recognizable sets, *American Math. Society Proc.* **7** (1974), 27–41.

[30] R. Fagin, Monadic generalized spectra, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **21** (1975), 89–96.

[31] R. Fagin, Finite model theory – a personal perspective, in: *ICDT'90*, S. Abiteboul and P.C. Kannelakis, eds, volume 470 of *Lecture Notes in Computer Science*, Springer Verlag, 1990, pp. 3–24.

[32] S. Feferman, Two notes on abstract model theory, I: Properties invariant on the range of definable relations between structures, *Fund. Math.* **82** (1974), 153–165.

[33] S. Feferman, Two notes on abstract model theory, II: Languages for which the set of valid sentences is s.i.i.d., *Fund. Math.* **89** (1975), 153–165.

[34] J. Flum, Characterizing logics, in: *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 3, Springer Verlag, 1985.

[35] E. Fischer and J.A. Makowsky, On spectra of sentences of monadic second order logic with counting, *Journal of Symbolic Logic* **69**(3) (2004), 617–640.

[36] R. Fraïssé, Sur quelques classifications des relations, basées sur les isomorphisms restraints, I: Études générale, *Publications Scientifiques de l'Université d'Alger, Serie A* **2** (1955), 15–60.

[37] H. Friedman, Back and forth, $L(q)$, $L_{\infty,\omega}(q)$ and Beth's theorem, Mimeographed notes, Stanford University, 1971.

[38] H. Friedman, The Beth and Craig theorems in infinitary languages, Mimeographed notes, Stanford University, 1971.

[39] H. Friedman, Beth's theorem in cardinality logics, *Israel Journal of Mathematics* **14** (1973), 205–212.

[40] S. Feferman and R. Vaught, The first order properties of algebraic systems, *Fund. Math.* **47** (1959), 57–103.

[41] O. Grumberg, N. Francez, J.A. Makowsky, and W.P. de Roever, A proof rule for fair termination of guarded commands, *Information and Control* **66**(1–2) (1985), 83–102.

[42] M.G. Garey and D.S. Johnson, *Computers and Intractability*, Mathematical Series, W.H. Freeman and Company, 1979.

[43] A. Grzegorczyk, Some classes of recursive functions, *Rozprawy Mat.* **4** (1953), 1–46.

[44] Y. Gurevich and S. Shelah, Spectra of monadic second-order formulas with one unary function, in: *LICS'03*, IEEE, 2003, pp. 291–300.

[45] Y. Gurevich, Modest theory of short chains, I, *Journal of Symbolic Logic* **44** (1979), 481–490.

[46] Y. Gurevich, Monadic second order theories, in: *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 14, Springer Verlag, 1985.

[47] L. Henkin, Some remarks on infinitely long formulas, in: *Infinitistic Methods*, Polish Scientific Publishers, 1961, pp. 87–103.

[48] W. Hodges, *Model Theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 1993.

[49] N. Immerman, *Descriptive Complexity*, Graduate Texts in Computer Science, Springer, 1999.

[50] G. Jäger, H. Läuchli, B. Scarpellini, and V. Strassen, eds, *E. Specker, Selecta*, Birkhäuser, 1990.

[51] D.S. Johnson, A catalog of complexity classes, in: *Handbook of Theoretical Computer Science*, J. van Leeuwen, ed., volume 1, chapter 2, Elsevier Science Publishers, 1990.

[52] N.D. Jones and Alan L. Selman, Turing machines and the spectra of first-order formulas with equality, in: *STOC '72: Proceedings of the fourth annual ACM symposium on Theory of computing*, ACM Press, New York, NY, USA, 1972, pp. 157–167.

[53] N.D. Jones and A.L. Selman, Turing machines and spectra of first order formulas, *Journal of Symbolic Logic* **39** (1974), 139–150.

[54] C. Karp, *Languages with expressions of infinite length*, Studies in Logic, North Holland, 1964.

[55] S.C. Kleene, Recursive predicates and quantifiers, *Trans. Amer. Math. Soc.* **53** (1943), 41–73.

[56] K. Kuratowski and A. Mostowski, *Set Theory*, Studies in Logic, North Holland, 1967.

[57] H. Läuchli, Auswahlaxiom in der algebra, *Comment. Math. Helv.* **37** (1962), 1–18.

[58] E.G.K. Lopez-Escobar, An interpolation theorem for denumerably long formulas, *Fund. Math.* **57** (1965), 253–272.

[59] L. Libkin, *Elements of Finite Model Theory*, Springer, 2004.

[60] P. Lindström, First order predicate logic with generalized quantifiers, *Theoria* **32** (1966), 186–195.

[61] P. Lindström, On extensions of elementary logic, *Theoria* **35** (1969), 1–11.

[62] H. Läuchli and J. Leonhard, On the elementary theory of order, *Fund. Math.* **59** (1966), 109–116.

[63] J. Łoś, On the categoricity in power of elementary deductive systems and some related problems, *Colloq. Math.* **3** (1954), 58–62.

[64] J.A. Makowsky, Kategorizität und endliche axiomatisierbarkeit, Master's thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1971.

[65] J.A. Makowsky, Note on almost strongly minimal theories, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **20** (1972), 529–534.

[66] J.A. Makowsky, Langages engendrés à partir des formules de Scott, *C.R. hebd. Acad. Sc. Paris* **276** (1973), 1585–1587.

[67] J.A. Makowsky, Δ-*logics and generalized quantifiers*, PhD thesis, Department of Mathematics, ETH-Zurich, Switzerland, 1974.

[68] J.A. Makowsky, On some conjectures connected with complete sentences, *Fund. Math.* **81** (1974), 193–202.

[69] J.A. Makowsky, Securable quantifiers, κ-unions and admissible sets, in: *Logic Colloquium,93*, A. Rose et al., eds, volume 80 of *Studies in Logic*, North Holland, 1975, pp. 409–428.

[70] J.A. Makowsky, Münchhausen's theorem und seine bedeutung für die musik und musikwissenschaft, in: *Die Zeichen, Neue Aspekte der musikalischen Aesthetik II*, H.W. Henze, ed., volume 2, S. Fischer, Frankfurt, 1981, pp. 278–307.

[71] J.A. Makowsky, Compactness, embeddings and definability, in: *Model-Theoretic Logics*, Perspectives in Mathematical Logic, chapter 18, Springer Verlag, 1985.

[72] J.A. Makowsky, Algorithmic uses of the Feferman-Vaught theorem, *Annals of Pure and Applied Logic* **126** (2004), 1–3.

[73] M. Morley, Categoricity in power, *Trans. Amer. Math. Soc.* **114** (1965), 514–538.

[74] A. Mostowski, Über die Unabhängigkeit des Wohlordnungssatzes vom ordnungsprinzip, *Fund. Math.* **32** (1939), 201–252.

[75] A. Mostowski, Axiom of choice for finite sets, *Fund. Math.* **33** (1945), 137–168.

[76] A. Mostowski, On definable sets of positive integers, *Fund. Math.* **34** (1947), 81–112.

[77] A. Mostowski, On direct products of theories, *Journal of Symbolic Logic* **17** (1952), 1–31.

[78] A. Mostowski, Concerning a Problem of H.Scholz, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **12** (1956), 210–214.

[79] M. Mostowski, On a generalization of quantifiers, *Fund. Math.* **44** (1957), 12–36.

[80] A. Mostowski, Formal system of analysis based on an infinitistic rule of proof, in: *Infinitistic Methods*, Polish Scientific Publishers, 1961, pp. 141–166.

[81] A. Mostowski, *Thirty Years of Foundational Studies*, Acta Philosophica Fennica, Basil Blackwell, Oxford, 1966.

[82] A. Mostowski, Craig's interpolation theorem in some extended systems of logic, in: *Logic, Methodology and Philosophy of Science, III*, Studies in Logic, North Holland, 1968, pp. 87–103.

[83] J.A. Makowsky and S. Shelah, The theorems of beth and Craig in abstract model theory, I: The abstract setting, *Trans. Amer. Math. Soc.* **256** (1979), 215–239.

[84] J.A. Makowsky and S. Shelah, The theorems of beth and Craig in abstract model theory, II: Compact logics, *Archiv für Mathematische Logik und Grundlagen der Mathematik* **21** (1981), 13–35.

[85] J.A. Makowsky and S. Shelah, Positive results in abstract model theory: A theory of compact logics, *Annals of Pure and Applied Logic* **25**(3) (1983), 263–300.

[86] J.A. Makowsky, S. Shelah, and J. Stavi, Δ–logics and generalized quantifiers, *Annals of Mathematical Logic* **10** (1976), 155–192.

[87] J.A. Makowsky and B. Zilber, Polynomial invariants of graphs and totally categorical theories, *XXX* **xx** (200x), xx–yy. Preliminary version available at www.cs.technion.ac.il/$\sim$janos.

[88] M.G. Peretyat'kin, An example of an ω_1-categorical complete finitely axiomatizable theory, *Algebra and Logic* **19** (1980), 202–229.

[89] M. Perloff, ed., *Wittgenstein's ladder*, University of Chicago Press, 1996.

[90] C. Ryll-Nardzewski, On the categoricity in power $\leqslant \aleph_0$, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **7** (1959), 545–548.

[91] H. Scholz, Problem # 1: Ein ungelöstes Problem in der symbolischen Logik, *Journal of Symbolic Logic* **17** (1952), 160.

[92] D. Scott, Logic with denumerably long formulas and finite strings of quantifiers, in: *Theory of Models*, Studies in Logic, 1965, pp. 329–341.

[93] S. Shelah, The monadic theory of order, *Annals of Mathematics* **102** (1975), 379–419.

[94] J. Shoenfield, *Mathematical Logic*, Addison-Wesley Series in Logic, Addison-Wesley, 1967.

[95] E. Specker, Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom), *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **3** (1957), 173–210.

[96] E. Specker and V. Strassen, eds, *Komplexität von Entscheidungsproblemen*, volume 43 of *Lecture Notes in Computer Science*, Springer Verlag, 1976.

[97] L. Svenonius, $\aleph_0$-categoricity in first order predicate calculus, *Theoria* **25** (1957), 173–178.

[98] A. Tarski, Arithmetical classes and types of algebraically closed and real-closed fields, *Bulletin of the AMS* **55** (1949), 63.

[99] A. Tarski, Arithmetical classes and types of boolean algebras, *Bulletin of the AMS* **55** (1949), 64.

[100] A. Tarski, Arithmetical classes and types of mathematical systems, *Bulletin of the AMS* **55** (1949), 63.

[101] A. Tarski, Metamathematical aspects of arithmetical classes and types, *Bulletin of the AMS* **55** (1949), 63–64.

[102] A. Tarski, Contribution to the theory of models, I, *Indagationes Mathematicae* **16** (1954), 572–581.

[103] A. Tarski, A. Mostowski, and R. M. Robinson, *Undecidable Theories*, North-Holland Publishing Company, 1953.

[104] B. Trakhtenbrot, On the impossibility of an algorithm for the decision problem in finite domains, *Doklady Akademii Nauk SSSR* **70** (1950), 569–572.

[105] B. Trakhtenbrot, Finite automata and the logic of monadic predicates, *Doklady Akademii Nauk SSSR* **140** (1961), 326–329.

[106] R. Vaught, Applications to the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability, *Indagationes Mathematicae* **16** (1957), 467–472.

[107] B. Zilber, *Uncountably Categorical Theories*, volume 117 of *Translations of Mathematical Monographs*, American Mathematical Society, 1993.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Observations on Truth, Consistency and Lengths of Proofs

Jan MYCIELSKI
Department of Mathematics, University of Colorado, Boulder, CO 80302
jan.mycielski@colorado.edu

Introduction

Mathematicians often leave the physical interpretations of their concepts and theorems to natural scientists. In particular those who work in logic and foundations rarely indicate the ontological, epistemological or psychological significance of their work. Of course every application of mathematics and every description of reality must accept a correspondence between some of its concepts (thoughts) and other real things or processes without fully defining this correspondence. Sometimes mathematicians avoid this problem pretending that they are formalists, and that they care only about sufficiently difficult proofs, or that they are Platonists, i.e., that they study an ideal world which is not physical but exists independently of them. However, in the eyes of natural scientists this is only a pose since it is clear that, contrary to formalism, pure mathematics and its logical form stems from experience and from natural intelligence, and that Platonism is a kind of fairy tale.

In particular Tarski's theory of truth (TTT) is interesting mainly because it is a study of certain physical connections between sentences and some other structures. Thus TTT describes three real things: descriptions, their objects, and the relations of meaning and truth between them. Some critics objected that this mathematical model is too abstract or that correspondence too vague to be interesting. We reject this criticism since we think that this model is important for our understanding of natural intelligence and of knowledge, and no better model is known. (However TTT does not explain the efficiency with which organisms learn and use their knowledge.) Other critics of TTT make a distinction between scientific knowledge and philosophy, and they place TTT into the first category denying that it belongs to the second. It appears to me that the belief in such a distinction is unfounded and very destructive for philosophy (and TTT does not corroborate that distinction).

In this paper I will discuss not only TTT but also some concepts and theories less abstract than TTT. Since this paper belongs to applied mathematics, its significance could be easily contested if one ignored the intended meanings of the concepts which I use.

I will argue in Section 1 that TTT elucidates our uses of the concept of truth in common languages. This is not a new idea, but in the papers collected in [2] it appears to be forgotten, and this stimulated me to stress it once again.

I will argue in Section 2 that the concept of truth (assuming the common sense of the word true) cannot apply to pure mathematics (assuming the common sense of the word pure). Let me mention that a number of philosophers implicitly or explicitly claim the opposite (see e.g. [2] and [4]).

I will consider in Section 3 the question why mathematicians believe that set theory (ST) is consistent. The statement "ST is consistent" will express the physical prediction that the process of derivation of theorems of ST from its axioms will never yield a contradiction. My answer is that, those who do not base this belief on authority, derive it from a certain mental experience, which I will try to describe. This is a personal experience of many set theorists (in particular mine), which is independent from the knowledge that mathematicians have not yet discovered any contradiction in logic and set theory. (This is not to deny the importance of the latter fact.)

Remark 1. There exists another attempt to answer this question, the answer of Platonists: ST is consistent because it is true in an ideal reality. However, as mentioned above, this explanation cannot convince more critical people. Indeed, in the minds of instinctive rationalists *Ockham's principle of economy rules over convictions*. Hence they believe that there is no Platonic reality and that those who think that they have an inner sense which makes them see such a reality are mistaken. (But we can explain how their illusion arises. It derives from the personal mental experience mentioned above.)

I will discus in Section 4 the practical significance of the mathematical definitions of lengths of proofs. I will define an extension of first-order logic intended to be such that its proofs will not be longer than the complete informal proofs of mathematical practice. (The proofs in the usual first-order logic are, as a rule, much longer.)

I will discuss in Section 5 a theorem of W. Reinhardt (as much as I know first published in [12]), in the context of the definition of Section 4. The theorem asserts the existence of a theorem α of ST which is short enough to be effectively written down but whose shortest proof in set theory is far too long to be written down. This theorem shows that the general concept of a convincing informal proof is like the concept of truth, if we wish to make sense of it we must rank it. (I think that this resolves the following paradox of R. Penrose [11]: If a human being M is a Turing machine and M can effectively define itself, then M can create sentences of which he is quite sure but he cannot prove; see Scholium at the end of this paper.)

1. TTT, Sense and Nonsense

I wish to stress here the significance of TTT for the theory of meaning of certain sentences of common language. This significance of TTT appears to be forgotten or underappreciated in the current literature. Let us summarize what TTT taught us about meaning. In his theory Tarski proved the surprising result:

The truth of sentences of a language L (talking about a structure A with a universe rich enough to represent L itself) is not expressible in L. In other words the set of Gödel numbers of sentences true in A is not a definable subset of the universe of A.

This implied a corollary (perhaps not stressed by other authors) that in any sentence S of common languages *that make sense* the relation of truth must be ranked. The ranks of all appearances of the word *true* in S (which I will indicate by integer subscripts) must be made clear by the grammatical structure of S or by the context or situation in which S was expressed. Thus we make two observations (in mathematical notation):

(a) *An expression of the form*

$$\mathrm{True}_m(\mathrm{S}) \quad (m = 1, 2, \dots)$$

makes sense iff the expression S *makes sense and all the ranks* n *of the relations* True_n, *which may appear in* S, *satisfy* $n < m$.

(b) *If both sides of the equivalence*

$$\mathrm{True}_m(\mathrm{S}) \leftrightarrow \mathrm{True}_n(\mathrm{S})$$

make sense, then this equivalence is valid.

As Tarski wrote, (a) implies that the self-referential sentence *This sentence is not true* makes no sense. Indeed it cannot be ranked. Thus TTT is an important contribution to a theory of common languages. TTT was published in 1933, and it seems to me that other theories of truth that were published later, contributed neither to our knowledge of common languages nor to our knowledge of the relation of truth.

Remark 2. In his original papers Tarski suggested to rank languages. Thus a language should be discussed only in it metalanguage, the metalanguage only in it metametalanguage, etc. It seems to be simpler to talk about one universal language in which the relation of truth is ranked and to accept the syntactic restriction (a) and the axiom schema (b).

Recall that TTT plays a basic role in a chapter of pure mathematics called model theory. However, this theory studies often the truth of sentences in imaginary structures. Thus I claim that there exists another interpretation of TTT which we can call *internal*. To explain it briefly, notice that the human mind (brain) stores not only sentences but also representations of structures (imagined structures). Thus the relation of truth described in TTT can be interpreted as a relation between sentences and imaginary structures. (Many structures of pure mathematics are imaginary, e.g. a well ordering of the real line.) Hence we can talk about *external* and *internal* interpretations of TTT. These remarks will lead us to the ideas of Section 2.

For the sake of completeness we have to mention a third natural meaning of truth. Namely, the agreement of a model with reality. Indeed, man and animals that do not have languages form mental models of reality. Thus we can speak of the truth or falsehood of *mental models*. Of course this sense of truth is different from the former ones. (A priori it may appear that this sense of truth has only external significance, but in human imagination we can find sometimes more abstract models of more detailed models. Hence these models of models can be also true or false. So there exist also an internal interpretation of truth between models.) A mathematical definition of mental models was outlined in [8].

2. Is There any Truth in Pure Mathematics?

If we stick to the common sense of the words true and pure, the answer to this question is negative. As long as mathematics talks about ideal, i.e., imaginary structures the common (external) concept of truth does not apply to it. Let us say that an object of pure mathematics is *real* if it has the potential for representing a real object or process. Otherwise I call it *imaginary* (see [5] for a mathematical definition of the class of real objects). Recall again that in set theory there are plenty of imaginary ones (e.g. non-measurable sets, sufficiently large infinite cardinals, etc.)

However, mathematicians usually say that their axioms and their theorems are true. Why is it so? The status of pure mathematics in the brain of a mathematician is almost the same as the status of his knowledge about physical reality. Thus he uses the term true in the internal sense explained at the end of the previous section. However, a philosopher or a natural scientist should be more conscious of the difference between the external and internal meanings. My critical attitude toward the ideas of the philosophers talking about truth in mathematics (like in [2] and [4]) derives from the fact that they do not seem to pay attention to this basic distinction.

Thus if we accept the usual, i.e., the external, meaning it makes no sense to talk about the truth of mathematical statements until these statements are understood as talking about physical objects or processes. And yet, the mathematician's focus upon the internal meaning is well-motivated since ST uses very often imaginary objects to describe or to explain the real ones. This is a surprising and important property of mathematics. For example, many functions, that occur in the equations of physical laws, and hence are real, are analyzed and explained by means imaginary sets of ordered pairs. (A systematic application of this idea is due to K. Kuratowski.) This goes so far that in practice we are unable to understand or explain physical reality without using imaginary objects. (For a more elaborate discussion see [5] and [1].)

We think that nature (evolution) endowed us with the rules and axioms of logic and set theory as a framework to analyze and describe reality, to make useful predictions and classifications (and this natural ability entails the use of imaginary objects (see [5] and [6])). This assertion is visible from the fact that mathematicians of all cultures know instinctively how to use correctly the rules and axioms of logic and set theory, *including* those who have not learned these rules and axioms. [While these who learned them feel that they are obvious, with the exception of the tiny minority of intuitionists.] It may be surprising that this natural logical-set-theoretic language and framework of thought contains the axiom of infinity. This is so, since it is natural to think in terms of unending discrete processes and in terms of continua, and to think that space-time is continuous (and hence infinite). (And it happens that these natural logico-set-theoretic laws of thought yield many imaginary objects, that is objects which do not have any direct physical interpretations.)

The many meanings (external and internal) of the word true may be confusing, so let me summarize the above thoughts:

In mathematics we talk about the truth of such and such theories in such and such models (in the sense of Tarski's definition of truth). But the model can be real or imaginary. In the first case this is the usual external sense of truth. In the second case it is the internal sense. In scientific practice there is little or no confusion. As a rule the speaker and the audience know if the model is intended to be real or only imaginary. In pure mathematics the word true is used in the internal sense; true means nothing more than proved in some theory implicitly indicated by the context. This is so common that some mathematicians and philosophers forget about the external sense, which is the only sense of general interest. There exist also other senses of truth, namely the external and internal truth of models.

3. Consistency

Being an opponent of Platonism I have to consider the problem why I (and others) believe in the consistency of the system ZFC of axioms of set theory (and its natural extension with, say, strongly compact cardinals). I will propose now a solution of this problem (for additional remarks see [6]).

First, as explained in the previous sections, only such statements which have a clear intended meaning can be believed or disbelieved. But the consistency of ZFC has an external meaning, namely the physical prediction that the process of deriving theorems from the axioms of set theory will never yield $0 = 1$. Thus, I understand here consistency as a physical law and not an assertion in pure mathematics. Hence it can be true or false in the external sense, believed or disbelieved.

I think that beliefs of sane people are essentially the same as their knowledge, and, like knowledge, they can be more or less firm (knowledge has degrees of certainty). Empirical inductive evidence yields general beliefs, which become stronger if we accumulate a large corroborating experience. (There are cases when a few sufficiently independent examples appears sufficient. It is so for equations or equivalences that are sufficiently simple and have no known counterexamples. We appear to be hard-wired to believe such equations.)

Of course the collective experience of set theorists, and of all mathematicians, corroborates the belief that ZFC is consistent. But I wish to point out another more direct and more personal cause of this belief. This is a series of mental events which arise automatically when we read with understanding the rules of logic and the axioms of set theory. I believe that this is the experience of many people. Let me try to describe it.

First, I divide the axioms of set theory into two groups. (1) *Constructive axioms*: union, powerset, replacement-schema, choice, infinity and the existence of strongly compact cardinals. These are tools for imagining sets. (2) *Descriptive-simplifying axioms*: extensionality and regularity. These are simplifying assumptions removing from set theory objects that are inessential for mathematics (urelements and non-well founded sets).

{In [6] I pointed out that it is reasonable to add $V = \mathrm{OD}$, GCH, and Souslin's Hypothesis to our list of descriptive-simplifying axioms at least until somebody finds some important mathematics using objects eliminated by these axioms. I see no reason to think that this will ever happen, but set theorists seem overly conservative harking back to the times when the consistency of these additions was problematic. ST will denote that extension of ZFC (it is fully defined in [6]).}

The constructive axioms (of group (1)) generate in our imaginations certain finite structures. The elements of these structures are called sets, but in fact they are like boxes intended to contain other boxes, intended to contain other boxes, and so on..., and one box, called the empty set, intended to remain empty. Some of them, called infinite sets, are intended to contain infinitely many boxes, but they are never filled up since we can actually imagine only finitely many things. Thus, in our imaginations, we can observe only potential infinity. The constructive axioms generate in our minds a finite number of constant or variable, i.e., incompletely defined sets, which can be denoted by the terms of an ε-extension of the first-order language of set theory, and instances of membership and equality between these sets, which can be expressed by atomic formulas or their negations. This is a finite part (which is real) of a diagram of an infinite structure (which is imaginary). We call it a *mental model* (see [8]). Mental models change with time as we build in our minds more and more sets and more and more atomic formulas and their negations (we also forget some objects that we have imagined).

This constructive process defined by the group of constructive axioms is so simple that we infer inductively that it can be continued as long as we wish and we will never produce an atomic statement and its negation. This is equivalent to the prediction that set theory is consistent. We observe that if we restrict this process so as to produce only sets=boxes that are necessary or desirable for the constructions of mathematics, our boxes will also satisfy the descriptive-simplifying axioms of ST. Finally logic without quantifiers but with the Hilbert ε-symbols gives us the most natural basis for the process of generating these finite structures of boxes. (*Notice that the notion of impredicativity does not appear in a language without quantifiers*.)

Our brains appear to be made for classifying reality, that is decompose objects into sets, forming certain sets, sets of sets, etc. Thus the former constructions are natural. They are not arbitrary games with symbols. (Thus we reject formalism.)

Let me add some points. In their encounters with their environments children develop in their brains the rules of logic and most of the constructive axioms of ST, and are able to use them without being conscious of the form of these mental tools. Evolution gave them the specific ability to classify and to use logic in order to build useful mental models of reality. It gave them also the potential for learning and developing languages that can communicate, i.e., induce similar structures in the brains of other people. In particular, even such an abstruse mental construction as that of potentially infinite sets, is given to us by nature. (As mentioned above we need it to perceive or imagine unending discrete processes and continua.) Thus when people learn the definitions of the rules of logic and the constructive axioms of set theory, these things resonate in their minds such that they call them obvious. And even mathematicians who have not learned them explicitly know how to apply them in their work. Finally, mathematical experience allows them to accept the descriptive-simplifying axioms (group (2)) since these are useful simplifications.

However, I think that the above claim about the natural character of the constructive axioms of ZFC does not apply to the constructive axiom of existence of strongly compact cardinals (see [6]). And yet, for set theorists, it is easy to imagine arbitrarily large strongly compact cardinals (by analogy with the least infinite cardinal ω). (This axiom is basic for a rich and beautiful theory which was developed within the last two decades by W.H. Woodin and others. When writing [6] I forgot that this axiom is so strong that it implies $\mathrm{AD}^{L(\mathbb{R})}$, a theorem of J. Steel and W.H. Woodin.) That analogy of strongly compact cardinals with ω makes us believe that this axiom cannot cause inconsistency.

However, having declared that the sentences we believe stem from mental models, inductive generalizations and analogy, let me recall that such things can lead us astray. For example, I think that, if somebody unfamiliar with set theory learns the axioms of ZFC and the theorem of Zermelo about the determinacy of finite games with perfect information, this may convince him that ZFC + AD is consistent. Indeed, it takes a fairly long construction to show that this theory is inconsistent. And yet the following speaks in favor of the mental heuristics described above. In the past, whenever inconsistent theories were proposed (there are a few such examples, the best known one is that of Frege and I will discuss it in a moment), it did not take long to discover their inconsistency; as a matter of fact all set theories that were actively studied and appeared consistent for a few years have survived all later attempts to prove their inconsistency.

To be still more convincing about the reality of the mental experience described above, I will add two examples of variants of set theory for which we do not have any such a heuristic. These are Frege's original set theory with the unrestricted set existence schema and Quine's set theory NF. In both theories the class of all sets V is a member of itself. Of course we cannot put a box into itself and we lack any clear finite mental models supporting these theories. I guess that this was Cantor's basic idea which allowed him to dismiss out of hand the criticism of his theory based on Russell's or his own proofs that certain classes are not sets (see [9]). Surely he would have immediately accepted the axiom that no set can belong to itself (I do not know if he actually made such a statement). *As well known, Russell's simple argument showed that Frege's theory was inconsistent, while the problem of consistency of Quine's NF is still unsolved.*

Remark 3. The above considerations suggests that it is unlikely that reverse mathematics can yield additional support to the belief that set theory is consistent. For example, I see no direct support for the consistency of the theory FIN(ZFC) (see [7] and [10]), other than the mental experience supporting the consistency of ZFC itself (which was described above). When I realised this, a paper announced in [7] became unnecessary and it was never written. (An argument in S. Lavine [9] that FIN(ZFC) is more intuitive than ZFC does not convince me.)

4. What Is a Complete Informal Proof?

As mentioned at the beginning of this paper, people working in Mathematical Logic sometimes intend their contributions to be applied, but they do not solve explicitly additional problems which arise in these applications. Such is the case with the problem of defining lengths of complete informal proofs. This length is hopelessly overestimated if it is defined to be the length of the sequence of symbols constituting the corresponding formal proof in, say, the first-order language of set theory. Thus I will propose another definition which appears to be more realistic. This will be applied in Section 5.

Recall that the rules of first-order logic and the axioms of set theory are universal for mathematics (since all mathematical theories are interpretable in set theory). However, first-order logic constitutes a very small fragment of common language. This explains why proofs in the first-order language of set theory are so much longer than informal proofs.

Following a theoretical fashion, the versions of first-order logic most commonly taught and studied introduce only primitive relation and function symbols, while the rules of proof which are used in practice include additional schemata that allow us to form new relation and function symbols given by explicit or inductive definitions, and even function symbols naming Skolem functions. It appears that if we extend first-order logic and the language of set theory L to a language L^* including the symbols generated by these schemata we have a formal language as powerful as common language, i.e., the informal complete proofs translate into formal proofs in L^* that are no longer than the originals. Of course a vast mathematical vocabulary would have to be translated from common language into L^* in order to do this in reality, and there is no urgent reason to work out such a dictionary. But L^* is useful since it appears that we can formalize most informal proofs in L^* without extending their lengths.

L^* is an extension of the first-order language L of ZF with some operators δ, Δ and ε, for creating new symbols, and by rules of proof that explain their roles.

L^* will have the same individual variables $x_1, x_2, \ldots$ as L. The operator δ will formalize the procedure of introducing new relation symbols. If ϕ is a formula of L^* whose free variables are $x_1, \ldots, x_n$, then δ_ϕ is a new (one letter) n-ary relation symbol, for which we add the axiom

$$\delta_\phi(x_1, \ldots, x_n) \leftrightarrow \phi(x_1, \ldots, x_n). \tag{δ}$$

The operator Δ will formalize a procedure for introducing recursively defined functions in set theory: Let $\phi(x, y)$, be a formula of L^* whose free variables are x and y. Then Δ_ϕ is a new function symbol which satisfies the following axiom (called the schema of recursive definitions):

$$\forall x \exists y \phi(x, y) \to \forall x \phi\big(\Delta_\phi | x, \Delta_\phi(x)\big), \tag{Δ}$$

where $\Delta_\phi | x$ denotes the restriction of Δ_ϕ to the domain x.

Remark 4. The above makes sense only if we assume, as we do, that all the new symbols are allowed in the axiom-schema of replacement of set theory (otherwise $\Delta_\phi | x$ could fail to denote a set).

Remark 5. For sake of simplicity I have expressed only the schema of recursive definitions relative to the membership relation. One could do the same for all extensional well-founded relations.

The operator ε (due to Hilbert [3]) will formalize the introduction of Skolem functions. Let X and Y be disjoint strings of distinct individual variables of lengths m and n respectively. Let $\phi(X, Y)$ be a formula of L^* whose free variables are those of X and Y, then the expression $\varepsilon_{Y,\phi}$ denotes a string of n new m-ary function symbols, which satisfies the following axiom:

$$\phi(X, Y) \to \phi\big(X, \varepsilon_{Y,\phi}(X)\big). \tag{ε}$$

Remark 6. With this extension we could omit from the usual rules of proof of first-order logic the rules which pertain to quantifiers. Indeed, we can define quantifiers as abbreviations:

$$\exists Y \phi(X, Y) \leftrightarrow \phi\big(X, \varepsilon_{Y,\phi}(X)\big),$$

and

$$\forall Y \phi(X, Y) \leftrightarrow \phi\big(X, \varepsilon_{Y,\neg\phi}(X)\big),$$

and we can derive the rules about quantifiers from the other rules of logic. However, such an elimination of quantifiers induces a polynomial extension of lengths of sentences and proofs. But we do not wish to elongate proofs, thus we allow quantifiers in L^*.

This concludes the definition of the extension L^* of the language of ZF.

Of course it is understood that the operators δ_ϕ, Δ_ϕ, and $\varepsilon_{Y,\phi}$, yield one-letter symbols, and that

$$\delta_\varphi = \delta_\psi, \quad \Delta_\varphi = \Delta_\psi, \quad \text{and} \quad \varepsilon_{Y,\varphi} = \varepsilon_{Z,\psi},$$

if and only if $\varphi = \psi$ and $(Y, \varphi) = (Z, \psi)$, respectively.

Now I can state the main thesis of this section. (ST is the set theory already discussed in Section 3, and defined in [6].)

Thesis. A complete informal mathematical proof from the axioms of ST can be formalized in L^*, with the rules of first-order logic plus (δ), (Δ) and (ε), without any extension of its length, i.e., of the number of letters of the informal proof.

This thesis implies that ST developed in L^* is a natural formalization of ST developed in any natural language.

Caveat. The Thesis is true if I have not overlooked any natural linguistic deductive device that is important for mathematics. (I do not know if the possible extension of Δ mentioned in Remark 5 is needed.)

Remark 7. The logic L^* is recursively axiomatized and its formal proofs can be checked in polynomial time.

Remark 8. L^* is so saturated that if a first-order theory T is interpretable in ST, them T is a subtheory of ST. In other words, there exists a translation of the one-letter symbols of T into appropriate one-letter symbols of ST (symbols of logic remain the same) such that the theorems of T turn into theorems of ST.

5. A Theorem of W. Reinhardt

By Gödel's incompleteness theorems there exist convincing statements which cannot be proved in ST in L^*. For example the natural arithmetical statement Con(ST) expressing the consistency of ST is convincing but not provable in ST. Thus if the Thesis is to be true the concept of a complete informal proof must be restricted accordingly. Assuming such a restriction (see Scholium at the end of this Section), we will state and prove an interesting theorem about complete informal proofs.

Assume that ST is consistent. Then we can write effectively a theorem α of ST such that we cannot write down any proof of α in ST in L^* since all such proofs are much too long for human beings or computers.

This theorem is due to W. Reinhardt (see [12], where a variant of Reinhardt's proof is given in Section 11). Now I will present his original proof with the minor modification that PA formalized in first-order logic is replaced by ST formalized in L^*.

Let us say that a formula φ *is n-effective* iff there exists a full system of definitions of the symbols occurring in φ such that φ plus this system can be written down as a sequence of less than n symbols. The *length of a proof* is the length of the sequence of symbols constituting that proof.

Theorem. *If ST is consistent then there exists a theorem α of ST such that α is* 300,000-*effective but every proof of α in ST in L^* is of length at least* 10^{100} *(thus it is far too long to be written down).*

Proof. Let $\beta(x)$ be a 100,000-effective formula of ST, which expresses the following:

> x is a Gödel number of a sentence ξ of PA such that there
> exists no proof of ξ in ST in L^* with less than 10^{100} symbols.

We accept without proof that such a β exists. Then, by the Fixed Point Theorem of Logic, there exists a sentence α in the language L*, such that

$$\text{ST proves} \quad \alpha \leftrightarrow \beta(\#\alpha), \tag{\#}$$

where $\#\alpha$ denotes the Gödel number of α.

The usual construction of an α satisfying (#) yields the three properties required in the theorem. Namely:

Property I. α is 300,000-effective. Indeed,

$$\alpha = \beta(f(\#\beta(f(x)))),$$

where f is a function symbol defined in such a way that

$$\text{ST proves } f(\#\phi(x)) = \#\phi(\#\phi(x)),$$

for all unary formulas ϕ. (Notice that (#) follows immediately from this definition of α.) Since β is 100,000-effective it is clear that α is 300,000-effective.

Property II. α is a theorem of ST.

Suppose to the contrary that α is not a theorem of ST. Then, by the definition of $\beta(x)$, $\beta(\#\alpha)$ is a theorem of ST. Hence, by (#), α is also a theorem of ST. This contradiction proves Property II.

Property III. α has no proof in ST of length less than 10^{100}.

Suppose to the contrary that there exists such a proof. Then $\neg\beta(\#\alpha)$ is a theorem of ST. So, by (#), $\neg\alpha$ is also a theorem of ST and, by Property II, ST is inconsistent. This contradicts the assumption of the theorem, whence Property III is proved.

This concludes the proof. □

Remark 9. The above theorem provides an example of a sentence α such that there exists a short proof that α has a proof, but there is no short proof of α itself. Thus, if we added to L* the rule (There exists a proof of σ in ST) $\rightarrow \sigma$, for any sentence σ, we would have a short proof of α. But, of course, we could prove a theorem similar to the former for the theory ST extended with that rule.

Remark 10. Several set-theorists made the observation that the axioms of ZFC + $\mathrm{AD}^{L(\mathbb{R})}$ yield a theory of the structure $\langle L(\mathbb{R}), \in\rangle$, which appears complete except for the independent statements of the kind discovered by Gödel. I do not know how to express this observation in a precise way. But the paradox of R. Penrose [11], which was stated at the end of the Introduction, appears to be resolved by the following:

Scholium. The Penrose paradox is a fact. The concept of a complete informal proof is like the concept of truth; in order to define it precisely we must rank it and M can produce proofs of many ranks, but M cannot define the set of theorems that it can prove by proofs of arbitrary ranks. For example, formal provability in ST in L* constitutes one of these ranks, a larger rank may include the rule expressed in Remark 9, etc.

References

[1] R. Dougherty and J. Mycielski, Canonical universes and intuitions about probabilities, *Dialectica* **60**(4) (2006), 357–368.

[2] W.D. Hart, ed., *The Philosophy of Mathematics*, Oxford Univ. Press, 1996.

[3] D. Hilbert, Die Logischen Grundlagen der Mathematik, *Math. Annalen* **88** (1923), 191–165.

[4] L. Kolakowski, *Metaphysical Horror*, Univ. of Chicago Press, 2001.

[5] J. Mycielski, Pure mathematics and physical reality, *Journal of Mathematical Sciences* 2006 (to appear); with a Russian translation in *Fundamentalnaya i Prikladnaya Matematika* **11**(5) (2005), 151–168.

[6] J. Mycielski, A system of axioms of set theory for the rationalists, *Notices of the Amer. Math, Soc.* **53**(2) (2006), 206–213.

[7] J. Mycielski, Locally finite theories, *Journal for Symbolic Logic* **51** (1986), 59–62.

[8] J. Mycielski, On the tension between Tarski's nominalism and his model theory (definitions for a mathematical model of knowledge), *Annals of Pure and Applied Logic* **126** (2004), 215–224.

[9] S. Lavine, *Understanding the infinite*, Harvard Univ. Press, 1994.

[10] J. Pawlikowski, Remarks on locally finite theories, *Abstracts of Papers Presented to the Amer. Math. Soc.* **10** (1989), 172.

[11] R. Penrose, *The Emperor's New Mind*, Penguin Books, 1989.

[12] S. Świerczkowski, Finite sets and Gödel's incompleteness theorems, *Dissertationes Mathematicae* **422** (2003), Polska Akademia Nauk, Instytut Matematyczny.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

The Algebraic Interpretation of Quantifiers: Intuitionistic and Classical

Dana S. SCOTT
Hillman University Professor of Computer Science,
Philosophy, and Mathematical Logic (Emeritus)
Computer Science Department, Carnegie Mellon University
Pittsburgh, Pennsylvania, USA
dana.scott@cs.cmu.edu

Keywords. Algebraic logic, completeness, Boolean and Heyting algebras

"*Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache und dann ist es alsobald ganz etwas Anderes.*"

J.W. von Goethe

1. Introduction

In their 1963 book, *The Mathematics of Metamathematics* [Rasiowa and Sikorski, 1963], the authors present an approach to completeness theorems of various logics using algebraic methods. Perhaps they put the quotation from Goethe over their preface to suggest that mathematicians can be forgiven if they introduce new interpretations of old ideas.

The idea of an algebra of logic can of course be traced back to Boole, but it was revived and generalized by Stone and Tarski in the 1930s; however, the most direct influence on the work of Rasiowa and Sikorski came from their well-known colleague, Andrzej Mostowski, after WW II. Mostowski's interpretation of quantification [Mostowski, 1948] can as well be given for intuitionistic as classical logic.

The present expository paper will briefly review the history and content of these ideas and then raise the question of why there was at the time no generalization made to higher-order logic and set theory. Entirely new light on this kind of algebraic semantics has more recently been thrown by the development of topos theory in category theory. Some reasons for pursuing this generalization will also be discussed in Section 12.

In order to dispel any confusion that might arise about the title of this paper, the author would like to point out that there are other meanings to "algebraic" as relating to the semantics of Logic. Perhaps the first interpretation of quantifiers in an algebraic style goes back to consideration of early work in Descriptive Set Theory. Quantifier manipulation was explained in this regard in the well known book on Topology by K. Kuratowski (also quoting Tarski). In 1946 C.J. Everett and S. Ulam published their paper on Projective Algebras. A. Tarski, building on his famous paper on Truth in formalized languages and on his studies of Relation Algebras (along with J.C.C. McKinsey, L. Henkin and many collaborators) developed an elaborate equational theory of Cylindric Algebras (duly quoting Everett and Ulam). In a related but different style, P.R. Halmos and collaborators studied Polyadic Algebras. A generalization of Relation Algebras was proposed by P.C. Bernays and that research has been continued over many years by W. Craig. Most recently, F.W. Lawvere unified many of these ideas – along with the lattice-theoretic semantics explained in the present paper – by showing how quantification can be construed in suitable categories by using the idea of adjoint functors. None of these works are cited in our bibliography here, because the literature is too vast to be explained in such a short paper.

2. Preliminaries, Terminology, Notation

The following standard terminology and notation will be used throughout the paper.

A complete lattice (cLa) is a partially ordered set (poset) $\langle A, \leqslant\rangle$ where every subset has a least upper bound (lub) under $\leqslant$. (Dually we can say, every subset has a greatest lower bound (glb).) A broader notion of a lattice requires only every finite subset to have a lub and a glb. A lub is also called a sup; and a glb an inf.

Theorem 2.1. *If a poset has all lubs, it has all glbs; and conversely.*

The proof is well known: the greatest lower bound of a family is the least upper bound of all the lower bounds of the family.

Theorem 2.2. *Using the following notation, lubs and glbs can be uniquely characterized as follows:*

(LUB) $y \leqslant \bigvee X \Longleftrightarrow \forall z[\forall x \in X.\, x \leqslant z \Longrightarrow y \leqslant z]$
(GLB) $\bigwedge X \leqslant y \Longleftrightarrow \forall z[\forall x \in X.\, z \leqslant x \Longrightarrow z \leqslant y]$
(TOP) $1 = \bigvee A = \bigwedge \emptyset$
(BOT) $0 = \bigwedge A = \bigvee \emptyset$

Theorem 2.3. *Lattices (not necessarily complete) are characterized by these axioms:*

- $\leqslant$ *is a partial order*
- $x \leqslant z \,\&\, y \leqslant z \Longleftrightarrow x \vee y \leqslant z$
- $z \leqslant x \,\&\, z \leqslant y \Longleftrightarrow z \leqslant x \wedge y$
- $0 \leqslant x \leqslant 1$

Examples of complete lattices abound in set theory. Here is an example of a direct consequence of Theorem 2.1.

Theorem 2.4. *The powerset $\mathcal{P}(A)$ is always a cLa; hence, if a family $\mathcal{C} \subseteq \mathcal{P}(A)$ is closed under all intersections (unions), then $\langle \mathcal{C}, \subseteq\rangle$ is also a cLa.*

We need to remember that in the cLa $\mathcal{P}(A)$ the intersection of the empty family is taken to be A.

3. Closures on a Power Set $\mathcal{P}(A)$

Closure operations $C : \mathcal{P}(A) \to \mathcal{P}(A)$ give interesting examples of complete lattices. There are several kinds of closures axiomatized as follows.

General:

- $X \subseteq C(X) = C(C(X))$
- $X \subseteq Y \Longrightarrow C(X) \subseteq C(Y)$

Topological:

- $X \subseteq C(X) = C(C(X))$
- $C(\emptyset) = \emptyset$
- $C(X \cup Y) = C(X) \cup C(Y)$

Algebraic:

- $X \subseteq C(X) = C(C(X))$
- $C(\bigcup_{X \in \mathcal{W}} X) = \bigcup_{X \in \mathcal{W}} C(X)$ for directed $\mathcal{W} \subseteq \mathcal{P}(A)$

For any closure operation we say that a set $X \subseteq A$ is *closed* provided $C(X) \subseteq X$.

Theorem 3.1. *The family of closed sets of a general closure operation is closed under arbitrary intersections and so forms under inclusion a complete lattice. Moreover, an isomorph of any complete lattice can be found in this way.*

For the easy proof see [Rasiowa and Sikorski, 1963]. The complete lattices formed from topological or algebraic closure operations have additional properties. We will discuss the topological case further below.

4. Implication and Distribution

Definition 4.1. *A lattice operation $\to$ is called an* implication *if and only if it satisfies this axiom:*

- $x \wedge y \leqslant z \Longleftrightarrow x \leqslant y \to z$

Note. The axiom uniquely determines an implication on a lattice if it exists. There is an equational axiomatization where we can define the partial ordering in several ways.

- $$\begin{aligned} x \leqslant y &\Longleftrightarrow x \vee y = y \\ &\Longleftrightarrow x \wedge y = x \\ &\Longleftrightarrow x \to y = 1 \end{aligned}$$

See [Rasiowa and Sikorski, 1963] for details. A lattice with implication is called a *Heyting algebra*. (Ha is used for short.)

Theorem 4.1. *All Ha's are distributive lattices.*

Proof. From basic lattice properties we have $x \wedge y \leqslant (x \wedge y) \vee (x \wedge z)$. Then $y \leqslant x \to ((x \wedge y) \vee (x \wedge z))$. Similarly we have $x \wedge z \leqslant (x \wedge y) \vee (x \wedge z)$. Thus $z \leqslant x \to ((x \wedge y) \vee (x \wedge z))$. It follows that $y \vee z \leqslant x \to ((x \wedge y) \vee (x \wedge z))$. But then $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$. This is half of the Distributive Law. The other half holds in all lattices. $\square$

Definition 4.2. *A Heyting algebra which has all glbs and lubs (= complete lattice) is called a complete Heyting algebra (or cHa, for short).*

Theorem 4.2. *A complete lattice is a cHa if, and only if, it satisfies:*

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i).$$

The proof of the above equation is a direct generalization of the proof of Theorem 4.1. For the converse we can use the following definition of implication:

$$y \to z = \bigvee \{x \mid x \wedge y \leqslant z\}.$$

The results about distributivity of Heyting algebras were proved independently by Stone [Stone, 1937] and Tarski [Tarski, 1938].

5. Boolean Algebras

The difference between classical and intuitionistic logic lies in the fact that the former satisfies the law of the Excluded Middle. In algebraic terms we have the following.

Definition 5.1. *A Ha (resp. cHa) is a Boolean algebra (or Ba) (resp. complete Boolean algebra or cBa) iff it satisfies* $x \vee (x \to 0) = 1$.

The element $x \to 0$ is called the *negation* of x and is denoted by $\neg x$.

Perhaps we should pause to note this passage from [Rasiowa and Sikorski, 1963] on pp. 8–9:

> The inclusion of two chapters on intuitionism is not an indication of the authors' positive attitude towards intuitionistic ideas. Intuitionism, like other non-classical logics, has no practical application in mathematics. Nevertheless many authors devote their works to intuitionistic logic. On the other hand, the mathematical mechanism of intuitionistic logic is interesting: it is amazing that vaguely defined philosophical ideas concerning the notion of existence in mathematics have lead to the creation of formalized logical systems which, from the mathematical point of view, proved to be equivalent to the theory of lattices of open subsets of topological spaces. Finally, the formalization of intuitionistic logic achieved by Heyting and adopted in this book is not in agreement with the philosophical views of the founder of intuitionism, Brouwer, who opposed formalism in mathematics. Since in treating intuitionistic logic we have limited ourselves to problems which are directly connected with general algebraic, lattice-theoretical and topological methods employed in the book, we have not included the latest results of Beth and Kreisel concerning other notions of satisfiability which we have adopted.

As a matter of fact the topological modeling of intuitionistic formal rules is not really to be regarded as "amazing". For, in view of Theorem 4.2, we conclude at once this theorem and corollary.

Theorem 5.1. *Every sublattice of a cBa (also cHa) closed under* $\wedge$ *and* $\bigvee$ *is a cHa.*

Corollary 5.1. *The lattice of open subsets of a topological space forms a cHa.*

There is also a reverse connection between Heyting algebras and Boolean algebras. An element x of a Heyting algebra is called *stable* (sometimes regular) if it satisfies $x = \neg\neg x$.

Theorem 5.2. *The stable elements of a Ha (cHa) form a Ba (cBa).*

A proof is given in [Rasiowa and Sikorski, 1963], pp. 134–135. Another connection is an embedding theorem.

Theorem 5.3. *Every cHa can be embedded in a cBa so as to preserve* $\wedge$ *and* $\bigvee$.

For a proof see [Johnstone, 1982].

6. Finite Lattices

A finite lattice is complete. Hence, we conclude

Theorem 6.1. *A finite distributive lattice* $\langle A, \leqslant\rangle$ *is a cHa, as is the dual* $\langle A, \geqslant\rangle$.

Perhaps we should note here that 6.1 is not true constructively. (See [Fourman–Scott, 1977] for the explanation.)

The finite Ha's can be analyzed in terms of a special kind of elements. These results are well known. Detailed references can be found in [Rasiowa and Sikorski, 1963].

Definition 6.1. *The set of join irreducible elements of a lattice* $\langle A, \leqslant\rangle$ *is defined as* $\mathit{Irr}(A) = \{x \in A \mid \forall y, z \in A\ [x \leqslant y \vee z \Longrightarrow x \leqslant y \text{ or } x \leqslant z]\}$.

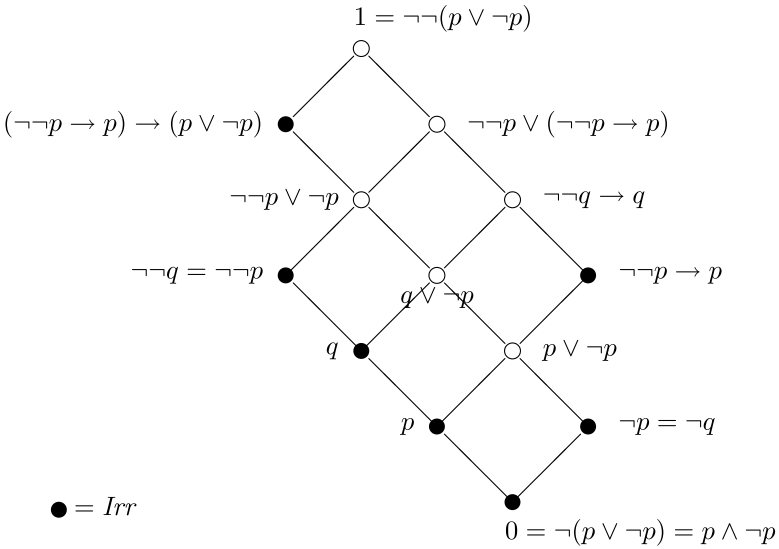
If an element is not irreducible in a distributive lattice, it can be written as the join of two strictly smaller elements. This remark makes it possible to give an inductive proof of the next theorem.

Theorem 6.2. *If* $\langle A, \leqslant\rangle$ *is a finite Ha, then for all elements* $x \in A$,

$$x = \bigvee\{y \in \mathit{Irr}(A) \mid y \leqslant x\}.$$

Here is an example of a finite distributive lattice with the irreducible elements marked.

Example 6.1.



We note that some authors exclude 0 as Irr.

The irreducible elements relate directly to what now are known as *Kripke Models*. The following is a classic result.

Theorem 6.3. *Every finite poset* $\langle I, \leqslant \rangle$ *with 0 is isomorphic to the join irreducibles of a finite Ha.*

Proof. Let $A = \{X \subseteq I |\, 0 \in X \wedge \forall x \in X. \{y \in I |\, y \leqslant x\} \subseteq X\}$. The poset $\langle A, \subseteq \rangle$ is closed under $\cap$ and $\cup$ and hence is a cHa. It is easy to check that $Irr(A) = \{\{y \in I |\, y \leqslant x\} |\, x \in I\}$. $\square$

The connection to Kripke Semantics can be explained as follows. For elements $P, Q \in A$, $x \in I$, we have these formulae:

$$\begin{aligned}
P = I \;&\Longleftrightarrow \forall x \in I.\, x \in P \\
P \subseteq Q \;&\Longleftrightarrow \forall x \in P.\, x \in Q \\
P \cap Q \quad &= \quad \{x \in I \mid x \in P \wedge x \in Q\} \\
P \cup Q \quad &= \quad \{x \in I \mid x \in P \vee x \in Q\} \\
P \to Q \quad &= \quad \{x \in I \mid \forall y \leqslant x [y \in P \Longrightarrow y \in Q]\}
\end{aligned}$$

Working with the finite Ha's – as is well known – leads to a proof of the decidability of intuitionistic propositional logic.

Theorem 6.4. *If a polynomial equation*

$$\sigma(x, y, z, \ldots) = \tau(x, y, z, \ldots)$$

fails in some Ha, then it fails in some finite Ha.

Proof. Let $[\![x]\!]_A$, $[\![y]\!]_A$, $[\![z]\!]_A, \ldots$ be the valuation in a Ha A such that

$$[\![\sigma(x,y,z,\ldots)]\!]_A \neq [\![\tau(x,y,z,\ldots)]\!]_A$$

Let $p_0, p_1, \ldots, p_n$ be all the subpolynomials of σ and of τ. Let B be the finite $\wedge$,$\vee$-sublattice of A generated by the elements $[\![p_i]\!]_A$ for $i \leqslant n$. We need to show

Lemma 6.1. *If $x, y \in B$ and $x \underset{A}{\rightarrow} y \in B$, then $x \underset{B}{\rightarrow} y = x \underset{A}{\rightarrow} y$.*

The easy proof goes back to Definition 4.1. Then, because all implications in σ and τ are in B, we derive

Corollary 6.1. $[\![\sigma]\!]_B = [\![\sigma]\!]_A$ *and* $[\![\tau]\!]_B = [\![\tau]\!]_A$.

Hence, the equation fails in the finite Ha B. This proof was first given in [McKinsey–Tarski, 1948]. □

7. Algebras of Formulae

In this section we recall Mostowski's interpretation of quantifiers as found in algebras of formulae. The *Lindenbaum algebra* of a theory T in classical first-order logic is the algebra formed by equivalence classes of formulae under provable equivalence in the theory. If we denote this provability by $\vdash_T \Phi$, then the equivalence class of Φ is defined as

$$[\![\Psi]\!]_T = \{\Psi \mid \quad \vdash_T \Phi \rightarrow \Psi \quad \text{and} \quad \vdash_T \Psi \rightarrow \Phi\}.$$

These equivalence classes form a partially ordered set by defining

$$[\![\Phi]\!]_T \leqslant [\![\Psi]\!]_T \quad \text{iff} \quad \vdash_T \Phi \rightarrow \Psi.$$

We employ here formulae Φ with possibly *free variables*, but such variable symbols could be taken as additional constants about which the theory T has no special axioms. The theory T might also have other constants and operation symbols. We let $Term$ denote the collection of all individual expressions formed from the variables and constants with the aid of the operation symbols.

In general the Lindenbaum algebra is not a complete Boolean algebra, but it enjoys an amount of completeness shown in crucial result of [Rasiowa–Sikorski, 1950] and of [Henkin, 1949] which we state in 7.2.

Theorem 7.1. *The Lindenbaum algebra of a classical first-order theory is a Ba.*

The proof of this fact in the formulation we have given here is very simple. The axioms for Boolean algebras exactly mimic well known *rules of inference* of logic. Thus, when we define lattice operations in the Lindenbaum algebra, such as the following:

$$[\![\Phi]\!]_T \cap [\![\Psi]\!]_T = [\![\Phi \wedge \Psi]\!]_T,$$

then we have a direct translation between algebraic relationships and deduction rules.

Theorem 7.2. *In the Lindenbaum algebra of a classical first-order theory T (using free variables) we have:*

$$[\![\exists x.\Phi(x)]\!]_T = \bigvee_{\tau \in \mathit{Term}} [\![\Phi(\tau)]\!]_T \quad \textit{and} \quad [\![\forall x.\Phi(x)]\!]_T = \bigwedge_{\tau \in \mathit{Term}} [\![\Phi(\tau)]\!]_T.$$

Proof. The proof of the quantification results is also quite simple. First, it is clear from rules of logic that

$$[\![\Phi(\tau)]\!]_T \leqslant [\![\exists x.\Phi(x)]\!]_T$$

holds for all terms τ. In other words, the equivalence class of the quantified formula is an *upper bound*. Next, suppose that Ψ provides *any other* upper bound, in the sense that

$$[\![\Phi(\tau)]\!]_T \leqslant [\![\Psi]\!]_T$$

holds for all terms τ. Because we have infinitely many free variables, we can pick one, v, say, not free in Ψ or otherwise in Φ. This means that $\vdash_T \Phi(v) \rightarrow \Psi$. But then by logical deduction $\vdash_T \exists x.\Phi(x) \rightarrow \Psi$. In terms of equivalence classes we have $[\![\exists x.\Phi(x)]\!]_T \leqslant [\![\Psi]\!]_T$. The LUB property is thus established. The GLB property is proved by a dual argument (or by using negation). □

8. Axiomatizing Free Logic

Before discussing models of intuitionistic predicate logic using cHa's, we need to broaden the idea of *existence* in axiomatic theories. The need to do this is not urgent in classical theory, for we can always say that under normal conditions the value of a function, $f(x)$, say, is given in a natural way; but then if the conditions on x do not hold, then the value of $f(x)$ can be a conventional value, 0, say. This only makes sense, however, using Excluded Middle. In other words, values of terms τ in the formal language are not always going to be required *to exist*.

The method the author has found to be practical is to build existence conditions into the deduction rules for the quantifiers. This has been called *free logic* ([Lambert, 1969], [Mostowski, 1951], [Quine, 1995], [Scott, 1977], [Lambert, 2003]), meaning that logic is freed of implicit existence assumptions. And in connection with this point of view, it seems most convenient to allow free variables also to be used without existence assumptions. Hence, *substitution* for free variables will be allowed for arbitrary terms.

A second move is to weaken equality statements $x = y$ to equivalence statements $x \equiv y$, meaning roughly that if one or the other of x and y exists, then they are identical. This seems to give the most liberal set of rules. Of course, the axioms for propositional logic remain the same as commonly known, so we do not repeat them here.

Definition 8.1. *The axioms and rules for (intuitionistic) equality and quantifiers are as follows:*

(Sub) $\qquad \dfrac{\Phi(x)}{\Phi(\tau)}$

(Ref) $\qquad x \equiv x$

(Rep) $\quad x \equiv y \wedge \Phi(x) \Longrightarrow \Phi(y)$

$(\forall Ins)$ $\quad (\forall x)\Phi(x) \wedge (\exists x)[x \equiv y] \Longrightarrow \Phi(y)$

$(\forall Gen)$ $$\frac{\Phi \wedge (\exists x)[x \equiv y] \Longrightarrow \Psi(y)}{\Phi \Longrightarrow (\forall y)\Psi(y)}$$

$(\exists Ins)$ $\quad \Phi(y) \wedge (\exists x)[x \equiv y] \Longrightarrow (\exists x)\Phi(x)$

$(\exists Gen)$ $$\frac{\Phi(y) \wedge (\exists x)[x \equiv y] \Longrightarrow \Psi}{(\exists y)\Phi(y) \Longrightarrow \Psi}$$

Existence and strict identity can then be defined.

Definition 8.2.

- $Ex \Longleftrightarrow (\exists y)[x \equiv y]$
- $x = y \Longleftrightarrow Ex \wedge Ey \wedge x \equiv y$

Conversely, existence and weak equality can be defined from strong identity.

Theorem 8.1.

- $x \equiv y \Longleftrightarrow [Ex \vee Ey] \Longrightarrow x = y$
- $Ex \Longleftrightarrow x = x$

Read "$E\tau$" as "the value of τ exists", and "$\tau = \sigma$" as "the values of τ and σ are existing and are identical". It actually is a matter of taste which notions to take as primitive as $\equiv$ and $=$ are interdefinable under our conventions, as we have seen.

9. Heyting-Valued Semantics

We take a definition from [Fourman–Scott, 1977] which we use for model theory over a cHa. The work reported in that paper was developed over several years in Scott's graduate seminars at Oxford. Independently, D. Higgs proposed similar definitions – especially for Boolean-valued models. See [Fourman–Scott, 1977] for references.

Definition 9.1. *Let A be a cHa. An A-set is a set M together with a mapping $e : M \times M \to A$ such that for all $x, y, z \in M$, $e(x, y) = e(y, x)$ and*

$$e(x, y) \wedge e(y, z) \leqslant e(x, z).$$

It is called total iff additionally $e(x, x) = 1$ for all $x \in M$. An n-placed predicate on M is a mapping $p : M^n \to A$ where

$$e(x_1, y_1) \wedge \cdots \wedge e(x_n, y_n) \wedge p(x_1, \ldots, x_n) \leqslant p(y_1, \ldots, y_n)$$

holds for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$.

We wish to show how A-sets together with predicates as a structure

$$\langle M, e, p_1, \dots p_m \rangle$$

give A-*valued models* for intuitionistic first-order free logic.

Structures for logic with operation symbols can be similarly set up, but to interpret operations as functions it is better to consider *complete* A-sets. The details can be found in [Fourman–Scott, 1977].

Definition 9.2. *Given an A-structure*

$$\mathcal{M} = \langle M, e, p_1, \dots p_m \rangle$$

and a given valuation $\mu : Var \to M$ of free variables, the A-valuation of formulae of free logic is defined recursively as follows:

$$\begin{aligned}
[\![P_i(v_1, \dots, v_{n_i}]\!]_{\mathcal{M}}(\mu) &= p_i(\mu(v_1), \dots, \mu(v_{n_i}));\\
[\![v_i = v_j]\!]_{\mathcal{M}}(\mu) &= e(\mu(v_i), \mu(v_j));\\
[\![\Phi \wedge \Psi]\!]_{\mathcal{M}}(\mu) &= [\![\Phi]\!]_{\mathcal{M}}(\mu) \wedge [\![\Psi]\!]_{\mathcal{M}}(\mu);\\
[\![\Phi \vee \Psi]\!]_{\mathcal{M}}(\mu) &= [\![\Phi]\!]_{\mathcal{M}}(\mu) \vee [\![\Psi]\!]_{\mathcal{M}}(\mu);\\
[\![\Phi \Rightarrow \Psi]\!]_{\mathcal{M}}(\mu) &= [\![\Phi]\!]_{\mathcal{M}}(\mu) \to [\![\Psi]\!]_{\mathcal{M}}(\mu);\\
[\![\neg\Phi]\!]_{\mathcal{M}}(\mu) &= \neg[\![\Phi]\!]_{\mathcal{M}}(\mu);\\
[\![\exists v_i \Phi(v_i)]\!]_{\mathcal{M}}(\mu) &= \bigvee_{\nu} (e(\nu(v_i), \nu(v_i)) \wedge [\![\Phi(v_i)]\!]_{\mathcal{M}}(\nu));\\
[\![\forall v_i.\Phi(v_i)]\!]_{\mathcal{M}}(\mu) &= \bigwedge_{\nu} (e(\nu(v_i), \nu(v_i)) \to [\![\Phi(v_i)]\!]_{\mathcal{M}}(\nu)).
\end{aligned}$$

In the above equations, the lattice operations on the right-hand side of the semantic equations are those of the cHa A. And in the last two equations the lubs and glbs are taken over all valuations ν where $\nu(v_j) = \mu(v_j)$ for all variables v_j different from the v_i in the quantifier.

Note that the intention here is that we have a derived semantics for these other two kinds of formulae:

$$\begin{aligned}
[\![Ev_i]\!]_{\mathcal{M}}(\mu) &= [\![v_i = v_i]\!]_{\mathcal{M}}(\mu), \quad \text{and}\\
[\![v_i \equiv v_j]\!]_{\mathcal{M}}(\mu) &= [\![Ev_i \vee Ev_j \to v_i = v_j]\!]_{\mathcal{M}}(\mu).
\end{aligned}$$

And again, because we have defined cHa to exactly mimic the laws and rules of deduction of intuitionistic logic, the proof of the next theorem is obvious.

Theorem 9.1. *The universally valid formulae (with free variables) of an A-structure form a theory in intuitionistic free logic. If A is Boolean, then we have a classical theory.*

10. Completion of Lattices and Completeness of Logic

Lindenbaum algebras give us a certain "formal" interpretation of intuitionistic theories in Ha's. But a Lindenbaum algebra is not usually complete. What is required to obtain structures over cHa's is an application of a method called *MacNeille Completion* [MacNeille, 1937].

Theorem 10.1. *Every Ha can be isomorphically embedded into a cHa preserving all existing glbs and lubs.*

When this theorem is proved we will have at once this conclusion.

Corollary 10.1. *Intuitionistic first-order logic is complete with respect to structures over cHa's; that is, every theory has such a model with the same valid formulae.*

Note that the cHa to be used is not fixed in this statement, as it depends on the choice of theory to make a Lindenbaum algebra.

Proof of Theorem 10.1. Let $\langle A, \leqslant\rangle$ be a Ha. For $X \subseteq A$, define the upper bounds and lower bounds of X by

$$ub(X) = \{y \in A \mid \forall x \in X.\, x \leqslant y\} \quad \text{and} \quad lb(X) = \{x \in A \mid \forall y \in X.\, x \leqslant y\}.$$

Lemma 10.1. *The map $X \mapsto lb(ub(X))$ is a monotone closure on $\mathcal{P}(A)$.*

Proof. Clearly $X \subseteq lb(ub(X))$. Also as ub and lb are antimonotone, so the composition is monotone. We need next to show that

$$ub(X) \subseteq ub(lb(ub(X))).$$

Suppose $y \in ub(X)$ and $x \in lb(ub(X))$. Then, by definition, $x \leqslant y$. Thus $y \in ub(lb(ub(X)))$. Hence, we have a closure operation. □

It is interesting to note in this proof that no special properties of $\leqslant$ were needed. By the general theorems on closures, we can conclude the following.

Lemma 10.2. *$\overline{A} = \{L \subseteq A \mid lb(ub(L)) \subseteq L\}$ is a complete lattice closed under arbitrary intersections.*

Note that $\{\emptyset\}$ is the least element of $\overline{A}$, and A itself $= lb(ub(\{1\}))$ is the largest element. This however does not prove yet that $\overline{A}$ is a cHa. To this end, for $L, M \subseteq A$, define $L \to M = \{a \in A \mid \forall b \in L.\, a \wedge b \in M\}$.

Lemma 10.3. *For K, L, $M \in \overline{A}$ we have $K \cap L \subseteq M$ iff $K \subseteq L \to M$.*

Proof. Assume first that $K \cap L \subseteq M$. Suppose $a \in K$ and $b \in L$. Then $a \wedge b \in K \cap L$. Thus $K \subseteq L \to M$. Conversely, assume $K \subseteq L \to M$. Suppose $a \in K \cap L$. Then $a \in L \to M$. But $a = a \wedge a \in M$. Thus $K \cap L \subseteq M$. □

Lemma 10.4. *If L, $M \in \overline{A}$, then $L \to M \in \overline{A}$.*

Proof. $L \to M = \bigcap_{b \in L} \{a \in A |\ a \wedge b \in M\}$. So, it is sufficient to show for each $b \in L$ that the set

$$K = \{a \in A |\ a \wedge b \in M\} \in \overline{A}.$$

As a shorthand let us write $y \geqslant K$ for $y \in ub(K)$. Suppose $x \in lb(ub(K))$. Then $\forall y \geqslant K.\ x \leqslant y$. Suppose $z \geqslant M$. If $a \in K$, then $a \wedge b \in M$. So $a \wedge b \leqslant z$. Thus $a \leqslant b \to z$. So $b \to z \geqslant K$. Hence, $x \leqslant b \to z$ and then $x \wedge b \leqslant z$. Therefore, $x \wedge b \in lb(ub(M)) \subseteq M$. So $x \in K$. Thus $K \in \overline{A}$. □

Lemma 10.5. *The map $x \mapsto \downarrow x = lb(\{x\})$ is a Ha isomorphism of A into $\overline{A}$ which preserves existing glbs and lubs.*

Proof. The map is one-one and monotone. Also, it is easy to check the following conditions providing a homomorphism.

$$\begin{aligned}
\downarrow (x \wedge y) &= \downarrow x \cap \downarrow y; \\
\downarrow (x \vee y) &= lb(ub(\downarrow x \cup \downarrow y)); \\
\downarrow x \to \downarrow y &= \{a |\ \forall b \leqslant x.\ a \wedge b \leqslant y\} = \{a |\ a \wedge x \leqslant y\} = \downarrow (x \to y); \\
\downarrow \bigwedge_{i \in I} x_i &= \bigcap_{i \in I} \downarrow x_i; \\
\downarrow \bigvee_{i \in I} x_i &= lb(ub(\bigcup_{i \in I} \downarrow x_i)).
\end{aligned}$$

The MacNeille theorem is thus proved. □

The corresponding completeness theorem for classical logic comes down to the next theorem. We use in this proof the well known fact that classical logic results from intuitionistic logic by assuming that every proposition is equivalent to its double negation.

Theorem 10.2. *If A is a Ba, then so is $\overline{A}$.*

Proof. For $L \in \overline{A}$, we calculate

$$\begin{aligned}
\neg\neg L &= \{a |\ \forall c \in L.\ b \wedge c = 0 \Rightarrow a \wedge b = 0\} \\
&= \{a |\ \forall b[\forall c \in L.\ c \leqslant \neg b \Rightarrow a \leqslant \neg b]\} \\
&= \{a |\ \forall b[L \leqslant b \Rightarrow a \leqslant b]\} = lb(ub(L)) = L
\end{aligned}$$ □

Therefore, in view of Theorem 7.2, every theory T in classical first-order logic has a model in cBa with the same set of valid formulae. This, however, is *not* Gödel's Completeness Theorem. For this we need the famous lemma [Rasiowa–Sikorski, 1950]. But note to this point we did not make use of ultrafilters or the Stone Representation Theorem.

Lemma 10.1 (Rasiowa–Sikorski). *Given a countable number of infs and sups in a non-trivial Ba, there is an ultrafilter preserving them.*

An ultrafilter is basically a homomorphism from the Ba to the two-element Ba. Without going through the MacNeille completion, however, we can apply the lemma directly to the Lindenbaum algebra of a classical first-order theory to prove

Corollary 10.2 (Gödel). *Every consistent, countable first-order classical theory has a two-valued model.*

Turning our attention now to intuitionistic logic we need the analogue of Theorem 7.2. The proof is essentially same, except that the rules of quantifiers of Section 8 have to be employed.

Theorem 10.3. *In the Lindenbaum algebra of a theory T in intuitionistic first-order free logic (using free variables) we have:*

$$[\![\exists x.\Phi(x)]\!]_T = \bigvee_{\tau \in Term} [\![E\tau \wedge \Phi(\tau)]\!]_T \quad \textit{and}$$

$$[\![\forall x.\Phi(x)]\!]_T = \bigwedge_{\tau \in Term} [\![E\tau \to \Phi(t)]\!]_T.$$

To avoid any misunderstanding, we should stress here (and in 7.2) that the axioms of the theories T should *not* be given with any free variables. The reason is that the proof of 7.2 and 10.3 requires at one point choosing a variable not free in T (as well as in certain formulae). Note that this step also requires that the formulae be finite, while the stock of free variables is infinite.

Theorem 10.4. *Every theory T in intuitionistic first-order free logic has a model in a cHa with the same set of valid formulae.*

Note that if the theory were inconsistent we would have to use the trivial one-element Ha. We also note that in intuitionistic logic there is no Rasiowa–Sikorski Lemma to give us models in "simpler" cHa's other than the one obtained from completing the Lindenbaum algebra.

There are two recent papers on the MacNeille completion method that have been brought to the attention of the author. They are [Harding–Bezhanishvili, 2004] and [Harding–Bezhanishvili, 2007]. As regards completion of Ha's, the first paper has the interesting result that the only non-trivial varieties of Ha's for which the desired result holds are Ha and Ba. Their paper gives many details and also historical references about the method. It should be noted, however, that they are especially interested in topological representations, and in this paper we have not used any representation theory. The second paper gives many results about Boolean modal operators, but undoubtedly their ideas can be adapted to Ha's in some cases. The next section has one such result found by the author.

11. Modal Logic

Starting perhaps with [McKinsey–Tarski, 1944], the algebraic interpretation of logic has been applied to investigations of modal logic. Here we shall not take the time to set out logical axioms but will be content to state a result about cHa's and modal-like operators. The first point to emphasize is that Heyting algebras give us many possibilities (or distinctions) for operators not available in the Boolean world.

We consider four algebraic axioms systems, ($\Box$), ($\Diamond$), (∇), (Δ), named by the symbols used for the operator.

($\Box$)
- $\Box 1 = 1$
- $\Box(x \wedge y) = \Box x \wedge \Box y$
- $\Box\Box x = \Box x \leqslant x$

($\Diamond$)
- $\Diamond 0 = 0$
- $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$
- $\Diamond\Diamond x = \Diamond x \geqslant x$

(∇)
- $\nabla(x \wedge y) = \nabla x \wedge \nabla y$
- $\nabla\nabla x = \nabla x \geqslant x$

(Δ)
- $\Delta(x \vee y) = \Delta x \vee \Delta y$
- $\Delta\Delta x = \Delta x \leqslant x$

It is clear that system ($\Diamond$) is the dual of system ($\Box$). In Boolean algebra, as is well known, we can pass from one system to the other by defining

$$\Diamond = \neg\Box\neg \quad \text{and} \quad \Box = \neg\Diamond\neg.$$

However, in Heyting algebra the failure of the law of Double Negation blocks this neat conversion (just as we cannot say $\exists = \neg\forall\neg$). As far as the author knows there is no simple correspondence between models of the one system and the other. The same remarks apply to comparing system (∇) to system (Δ).

What we shall prove here is that—in case there is any interest in quantified modal logic—the MacNeille Completion procedure can be applied to Lindenbaum algebras to get algebraic completeness proofs over cHa's for the system ($\Box$). The question for the other systems is left open.

Theorem 11.1. *If A is a Ha and if the operator $\Box : A \to A$ satisfies the ($\Box$) axioms, then so does the MacNeille completion $\overline{A}$ with the operator $\Box : \overline{A} \to \overline{A}$ defined by*

$$\Box L = \bigvee_{x \in L} \downarrow \Box x$$

for all $L \in \overline{A}$. Moreover, the lattice embedding $x \mapsto \downarrow x$ preserves the $\Box$-operator.

Proof. On A the operator $\Box$ is monotone, and on $\overline{A}$ the corresponding operator $\Box$ is as well. Note that we can also write

$$\Box L = lb(ub(\{\Box x | \, x \in L\}))$$

for all $L \in \overline{A}$.

First, $\Box \downarrow 1 = \Box A = A$, as required.

Next, note that since $\Box x \in L$ for all $x \in L$, we conclude $\Box L \subseteq L$. By monotonicity we also have $\Box\Box L \subseteq \Box L$.

To prove $\Box L \subseteq \Box\Box L$, we need to show $\Box x \in \Box\Box L$ for all $x \in L$. By definition we have

$$\Box\Box L = lb(ub(\{\Box y | \, y \in \Box L\})).$$

Suppose $x \in L$. Then if $z \geqslant \{\Box y | \, y \in \Box L\}$, we conclude $z \geqslant \Box\Box x = \Box x$. Therefore $\Box x \in \Box\Box L$ as desired.

Finally, we calculate using the $\wedge$-$\bigvee$-Distributive Law as follows:

$$\begin{aligned}\Box L \cap \Box M &= \bigvee_{\substack{x \in L \\ y \in M}} (\downarrow \Box x \cap \downarrow \Box y) \\ &= \bigvee_{\substack{x \in L \\ y \in M}} \downarrow \Box(x \wedge y) \\ &= \bigvee_{z \in L \cap M} \downarrow \Box z \\ &= \Box(L \cap M)\end{aligned}$$

Thus the defined $\Box : \overline{A} \to \overline{A}$ satisfies the ($\Box$)-equations. $\Box$

We remark in passing that to obtain a $\Box$-operator on any cHa A, all one needs is a subset $U \subseteq A$ such that $1 \in U$ and $x \wedge y \in U$ for all $y \in U$. We than define

$$\Box x = \bigvee \{y \in U | y \leqslant x\}.$$

The verification of the ($\Box$)-axioms is straightforward. This, unsurprisingly, generalizes how the interior operator is defined in any topological space in terms of basic open sets. Of course, any $\Box$-operator can be so defined in terms of the set

$$U = \{\Box x | \, x \in A\}.$$

The system (∇) at first looks like $(\Box)$, but the last inequality is *reversed*. Such operators on cHa's are variously called *j-operators* or *nuclei* (see [Fourman–Scott, 1977] or [Johnstone, 2002–2007], vol. 2, pp. 480ff. The significance of such operators is that they correspond to $\wedge$-$\bigvee$-preserving congruence relations E on a cHa A where we can define ∇_E, given E, by

$$\nabla_E x = \bigvee \{y \in A | \, xEy\}.$$

The other way around, give ∇, we can define E_∇ by

$$xE_\nabla y \quad \text{iff} \quad \nabla x = \nabla y,$$

It follows that $E = E_{\nabla_E}$ and $\nabla = \nabla_{E_\nabla}$ for such ∇ and E.

The ∇-operators on cHa A can also be characterized in terms of fixed-point set

$$U_\nabla = \{x \in A | \, \nabla x = x\} = \{\nabla x | \, x \in A\},$$

Indeed, as for any general closure operation, we can obtain

$$\nabla x = \bigwedge \{y \in U_\nabla | \, x \leqslant y\}.$$

And of course the subset $U_\nabla \subseteq A$ is a subset closed under arbitrary glbs in A, as it is for general closures. However, the first equation in (∇) gives U_∇ an additional property:

$$x \to y \in U_\nabla \quad \text{for all } x \in A \text{ and all } y \in U_\nabla.$$

To prove this, it is sufficient to prove in system (∇) the equation:

$$\nabla(x \to \nabla y) = x \to \nabla y$$

On the one hand we have $x \to \nabla y \leqslant \nabla(x \to \nabla y)$. On the other hand we see:

$$\nabla(x \to \nabla y) \wedge \nabla x = \nabla(x \wedge (x \to \nabla y)) \leqslant \nabla\nabla y = \nabla y.$$

It follows that

$$\nabla(x \to \nabla y) \leqslant \nabla x \to \nabla y$$

Inasmuch as $x \leqslant \nabla x$, we conclude

$$\nabla x \to \nabla y \leqslant x \to \nabla y,$$

which is what we needed to prove.

Suppose in a cHa A we have $U \subseteq A$ with the $\bigwedge$-closure property and the $\to$-property above. Define

$$\nabla_U x = \bigwedge\{y \in U \,|\, x \leqslant y\}.$$

The key step is to prove the first equation of (∇). Because ∇_U is a closure operation, we have at once:

$$\nabla_U(x \wedge y) \leqslant \nabla_U x \wedge \nabla_U y.$$

Also we see:

$$x \wedge y \leqslant \nabla_U(x \wedge y).$$

But then

$$x \leqslant y \to \nabla_U(x \wedge y) \in U,$$

so

$$\nabla_U x \leqslant y \to \nabla_U(x \wedge y).$$

We conclude

$$y \leqslant \nabla_U x \to \nabla_U(x \wedge y) \in U,$$

so

$$\nabla_U y \leqslant \nabla_U x \to \nabla_U(x \wedge y).$$

This establishes

$$\nabla_U x \wedge \nabla_U y \leqslant \nabla_U(x \wedge y)$$

as required.

If A were a cBa, then ∇-operators are not very interesting. In fact, set $0^* = \nabla 0$, then $\nabla x = 0^* \vee x$, as is easily proved.

System (Δ) is introduced as a "dual" of (∇), but the author has no idea whether it is at all interesting to study.

12. Higher-Order Logic

Here is a principle the author regards – with hindsight – as self-evident.

Principle. *If you understand what first-order models with* arbitrary *predicates are, then you can interpret second-order logic with* predicate quantifiers. *And then you can go on to models of* higher-order logic*, even set theory.*

In what is perhaps the first modern text book in mathematical logic [Hilbert–Ackermann, 1928], second-order logic is presented as a natural step by introducing quantifiers on predicates and relations. Even though the Hilbert–Ackermann book did not define a formal semantics (as was to become standard from the later works of Tarski and Carnap) "meaning" and "logical validity" were intuitively understood. Indeed, Hilbert–Ackermann clearly presented the completeness problems Gödel was soon to solve. Of course, at that stage interpretations of formulae were classical and two valued.

Inasmuch as cHa's generalize cBa's, we will first consider in this section higher-order logic in intuitionistic free logic. For a cHa A, we defined in Section 9 general A-structures containing arbitrary A-valued predicates and relations. For simplicity, in order to fix ideas, let us restrict attention to one-placed predicates. These are functions $p : M \to A$ which have to satisfy an *extensionality condition* requiring $e(x, y) \leqslant (p(x) \leftrightarrow p(y))$, for all $x, y \in M$, and a *strictness condition* requiring

$$p(x) \leqslant e(x, x), \quad \text{for all } x \in M.$$

Let $\mathcal{P}_A(M)$ be the collection of all such p, and call this the A*-valued powerset* of M. To make this an A-set, we need to define (Leibniz) equality on $\mathcal{P}_A(M)$. The definition is just a translation into algebraic terms what we know from logic:

$$e(p, q) = \bigwedge_{x \in M} (p(x) \leftrightarrow q(x))$$

for $p, q \in \mathcal{P}_A(M)$. We note that

$$e(p, p) = 1$$

for all $p \in \mathcal{P}_A(M)$, a "totality" condition that need not hold in M.

Perhaps, to avoid confusion, we should notate the equality on M as e_0, and the equality on $\mathcal{P}_A(M)$ as e_1. We can then use subscripts for the equalities in higher powersets.

Another side remark concerns *totality* in A-sets. Experience (and the study of category theory and topos theory) has shown that for models of intuitionistic logic it is essential to take into account *partial elements* $x \in M$ where $e(x, x) \neq 1$. The reason is that these models come up naturally, and a key example is discussed below.

Given an A-set M, the *completion* $\overline{M}$ of M can be taken to be the collection of all "singletons" $s \in \mathcal{P}_A(M)$; that is to say those one-place predicates such that

$$s(x) \wedge s(y) \leqslant e(x, y)$$

for all $x, y \in M$. There is a natural embedding of M into $\overline{M}$ which we will denote by $\varepsilon : M \to \overline{M}$ and define by

$$\varepsilon(x)(y) = \varepsilon_0(x, y),$$

for all $x, y \in M$. It is clear that $\varepsilon(x) \in \overline{M}$ for all $x \in M$. However, we have to be a little more careful to say how $\overline{M}$ is an A-set, as it is not correct to use e_1 as the equality. Here is the appropriate definition:

$$\overline{e}(s,t) = \bigvee_{x \in M} s(x) \ \wedge \bigvee_{x \in M} t(x) \wedge e_1(s,t).$$

The point here is our definition of being a singleton does not guarantee that an $s \in M$ is *non empty*. The condition to be so is $\overline{e}(s,s) = 1$. But for $x \in M$ we will only have

$$\overline{e}(\varepsilon(x), \varepsilon(y)) = e_0(x,y),$$

for all $x, y \in M$. If $x \in M$ is a properly partial element, then $e_0(x,x) \neq 1$. More details about this notion of completeness are to be found in [Fourman–Scott, 1977], Section 4. We say that an A-set M is *complete* if, and only if, for all $s \in M$ there is a unique $x \in M$ such that $s = \varepsilon(x)$; this means

$$s(y) = e_0(x,y),$$

for all $y \in M$. In general $\overline{M}$ proves to be complete and we can restrict attention to complete A-sets.

The next question is in what sense does the multi-sorted structure

$$\langle M, e_0, p_1, \ldots, \mathcal{P}_A(M), e_1, \alpha_1 \rangle$$

form a model for a second-order logic? Here $\langle M, e_0, p_1, \ldots \rangle$ is a model for first-order logic to which we are adding a new sort, namely, the power set. As a connection between M and $\mathcal{P}_A(M)$ is needed, we let $\alpha_1 : M \times \mathcal{P}_A(M) \to A$ stand for "application" or "membership" defined by

$$\alpha_1(x,p) = p(x)$$

for all $x \in M$ and $p \in P_A(M)$. Then the answer to the question is obvious (it is hoped), because in $\mathcal{P}_A(M)$ there is by design a function p to represent the mapping

$$x \mapsto [\![\Phi(v)]\!]_A(\mu^v_x),$$

where Φ is any formula, of a first- or second-order logic, where μ is a given valuation of the variable, and where μ^v_x is the alteration of μ to make $\mu^v_x(v) = x$. The semantics of multi-sorted formulae of course evaluates quantifiers as glbs or lubs over the appropriate ranges of the variables.

We may recall that early formulation of high-order logic used an inference rule of *substitution of formulae* for predicate variables. The paper [Henkin, 1953] was perhaps the first to show that an axiomatization using a *comprehension axiom* suffices for higher-order logic. Especially as it is difficult to formulate the substitution rule correctly, the Henkin-style has been used ever since.

The author is claiming here that the passage from models of first-order logic to models of second-order logic by adding a new type of objects is an easy one: we just make sure that the new type has the maximal number of objects agreeing with the concept of an A-valued predicate. This is the principle articulated at the head of this section.

By the same token then, it should be just as easy to continue to *third-order* logic. (As a technical convenience it is better to keep to complete A-sets, and so $\mathcal{P}_A(M)$ needs to be replaced by $\overline{P_A(M)}$.) And then fourth-order and higher-order models will follow by the same principle.

Some examples will perhaps show how the A-valued semantics works out. As a cHa we can take the lattice Ω of open subsets of a convenient topological space T. We then use as first-order elements continuous, real-valued functions f on open sets. That is $f : Ef \to \mathbb{R}$ continuously, where Ef, the domain of definition of f, is in Ω. Equality on such function is defined by

$$e(f, g) = Int\{t \in Ef \cap Eg |\, f(t) = g(t)\}.$$

Using the presentation of [Fourman–Scott, 1977], Section 8, one shows that this set, which we can call $\mathbb{R}_\Omega$, becomes a complete Ω-set. Moreover the usual arithmetic operations of $+$ and $\cdot$ make sense on $\mathbb{R}_\Omega$. The analogue of *ordering* is defined by

$$< (f, g) = \{t \in Ef \cap Eg |\, f(t) < g(t)\}.$$

As a first-order structure $\mathbb{R}_\Omega$ models (most of) the intuitionistic theory of the reals. When using the general method of passing to a second-order structure, one obtains a model also satisfying a very suitable version of *Dedekind completeness* of the ordering. See [Scott, 1968] for an early version.

Now let us modify the construction by letting B be the cBa of measurable sets modulo set of measure zero of a measure space T. The new "reals", $\mathbb{R}_B$, are equivalence classes of real-valued measurable functions (functions made equivalent if they agree up to a set of measure zero). Equality is defined by

$$e([f], [g]) = \{t |\, f(t) = g(t)\}/Null,$$

where $Null$ is the ideal of sets of measure zero. In the Boolean B-valued logic, $\mathbb{R}_B$ not only becomes a model of the classical first-order theory of the reals (= real closed fields), but in second-order logic it becomes a model for the Dedekind Completeness Axiom. (One exposition can be found in [Scott, 1967a].) In third-order logic (and higher) it becomes possible to give formally a version of the Continuum Hypothesis. As Solovay discovered, we can give a Boolean-valued version of Paul Cohen's independence proof in this way—provided we take T to be a product space (with a product measure) so that there are a very large number of independent "random variables" (= measurable functions). See [Scott, 1967a] or [Bell, 2005] for further details.

Not only can both classical and intuitionistic higher-order logic be modeled in cHa's (or cBa's) but the iteration of he power set can be pushed into the transfinite obtaining models (and independence proofs) for Zermelo–Fraenkel Set Theory. As was said, in hindsight (after Cohen's breakthrough) this all looks easy and natural. The technical facts about measures, topologies, and Boolean algebras were well set out in [Sikorski, 1960] and were known earlier. So the question the author cannot answer is: Why did not Mostowski or Rasiowa or Sikorski or one of their students in Warsaw extend the algebraic interpretations of the quantifiers to higher-order logic?

13. Conclusion

Our discussion in this paper has concentrated on the lattice-theoretic semantical interpretations of classical and intuitionistic logics championed by the Polish schools of Mostowski (Warsaw) and Tarski (Berkeley). A main point of our exposition is that the well-known algebraic characterizations of cHa's and cBa's exactly mimic the rules of deduction in the respective logics; hence the algebraic completeness proofs are not all that surprising. (The step to classical two-valued models, of course, needs a further argument.) Moreover, once the formal laws are put forward in this way, the connection between intuitionistic logic and topology are not surprising either (*pace* Rasiowa and Sikorski). (But the steps to *special* topological interpretations need further investigation and have not been considered here.)

A second main point here concerns higher-order logic. The author contends that once the situation of first-order is understood, the generalization of the algebraic interpretations is clear. As mentioned in the last section, he does not understand why the Polish schools did not see this already in about 1955. Being a member at that time of the Tarski school, the author also does not understand why *he* did not see this, since the generalization is based on very simple general principles. If this had been done at the time, the question of how the Continuum Hypothesis fares in, say, Boolean-valued interpretations would have come up quickly. Moreover the Polish schools had all the necessary technical information about special cBa's at hand to solve the problem of independence. Ah, well, history, unlike water, does not always find the most direct path.

As a curious sidelight, the idea of the algebraic interpretation in cBa's was already suggested by Alonzo Church in 1953 [Church, 1953]. But neither he nor his later students (including the author) were inspired to look into it further. Instead, it took the breakthrough of *forcing* of Paul Cohen (based on quite different intuitions), which was reinterpreted in terms of Boolean-valued logic by Robert Solovay to close the circle.

And it should not be forgotten that a completely different intuition stemming from *algebraic geometry* was being developed by Grothendieck and his followers. One result was the definition of an *elementary topos* by F. William Lawvere and Myles Tierney around 1970 (for history see [Johnstone, 2002–2007]) which gives the most general notion of the *algebra of higher types* in intuitionistic logic. Topos theory includes not only the interpretations in cHa's, but also the interpretations generalizing Kleene's *realizability*. However, there are many other aspects of topos theory coming from abstract algebra and from algebraic geometry and topology that go far beyond what logicians have ever imagined. The results and literature are too vast to survey here, but the multi-tome synthesis of Peter Johnstone will provide an excellent and coherent overview. The challenge to logicians now is to use these mathematical techniques to draw (or discover) new conclusions significant for the foundations of mathematics.

14. A Note of Acknowledgement

A first version of this survey was presented at the conference *75 Years of Predicate Logic*, held at the Humboldt University, Berlin, Germany, September 18–21, 2003, on the occasion of the anniversary of the publication of [Hilbert–Ackermann, 1928]. Unfortunately owing to his retirement and subsequent move from Pittsburgh, PA, to Berkeley, CA, the author was unable to complete a manuscript for the proceedings volume, which was eventually published under the title *First-Order Logic Revisited*, Logos-Verlag, Berlin, 2004. The author would, however, like to thank the organizers of that conference for his being able to attend a very interesting workshop and for very warm hospitality in Berlin.

Subsequently the invitation to Poland for the *Trends in Logic* conference in 2005, which was dedicated to the memories of Polish colleagues in Logic whom he knew so well, gave the author the chance to rework the presentation. An invitation to lecture in the Philosophy Department of Carnegie Mellon University in the Spring of 2006 led to further improvements, particularly those concerning the MacNeille Completion and the application to Modal Logic. Thanks go to Steven Awodey for providing that opportunity – and for many years of collaboration in related areas.

Finally, the publisher and editors of this volume made it possible for Marian Srebrny to travel to Berkeley for a week of intense work putting the author's conference notes into publishable form. Without this help this paper would not exist, and the author is especially grateful for this stimulating and essential collaboration. Thanks go also to Marek Rynćko, Warsaw, and Stefan Sokołowski, Gdańsk, for their expertise in TEX typesetting. Warm thanks, too, go to the organizers of the 2005 conference for the chance to visit Poland again and to meet so many old friends.

Working on this paper and going over the lists of references gave the author a chance to review how much he owes to the Polish Schools of Logic. He wishes to dedicate this survey to the memories of these teachers and colleagues.

A BRIEF, SELECTIVE, HISTORICAL BIBLIOGRAPHY

1928

Hilbert, D. and Ackermann, W., ***Grundzüge der theoretischen Logik***, Springer-Verlag, 1928.

1930

Gödel, K., *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatshefte für Mathematik und Physik, vol. 37(1930), pp. 349–360.

Heyting, A., *Die formalen Regeln der intuitionistischen Logik*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys. mathem. Klasse. 1930, pp. 42–56.

1931–32

Gödel, K., *Zum intuitionistischen Aussagenkalkül*, Ergebnisse eines mathematischen Kolloquiums, vol. 4 (for 1931–32, pub. 1933), p. 40.

1934

Hilbert, D., and Bernays, P., ***Grundlagen der Mathematik***, Vol. 1, Springer-Verlag, 1934.

Stone, M.H., *Boolean algebras and their relation to topology*, Proc. of the National Academy of Sciences, vol. 20(1934), pp. 197–202.

1935

Tarski, A., *Zur Grundlegung der Boole'schen Algebra*, Fundamenta Mathematicae, vol. 24(1935), pp. 177–198.

1936

Stone, M.H., *The theory of representations for Boolean algebras*, Transactions of the American Mathematical Society, vol. 40(1936), pp. 37–111.

1937

MacNeille, H., *Partially ordered sets*, Transactions of the American Mathematical Society, vol. 42(1937), pp. 416–460.

Stone, M.H., *Applications of the theory of Boolean rings to general topology*, Transactions of the American Mathematical Society, vol. 41(1937), pp. 21–364.

Stone, M.H., *Topological representations of distributive lattices and Brouwerian logics*, Casopis pro pestovani matematiky a fysiky, vol. 67(1937–38), pp. 1–25.

1938

Tarski, A., *Der Aussagenkalkül und die Topologie*, Fundamenta Mathematicae, vol. 31(1938), pp. 103–134.

1939

Hilbert, D., and Bernays, P., ***Grundlagen der Mathematik***, Vol. II, Springer-Verlag, 1939.

1944

Everett, C.J., *Closure operators and Galois Theory in Lattices*, Transactions of the American Mathematical Society, vol. 55(1944), pp. 514–525.

McKinsey, J.C.C., and Tarski, A., *The algebra of topology*, Annals of Mathematics, vol. 45(1944), pp. 141–191.

1946

McKinsey, J.C.C., and Tarski, A., *On closed elements in closure algebras*, Annals of Mathematics, vol. 47(1946), pp. 122–162.

1948

Birkoff, G., ***Lattice Theory***, American Mathematical Society, 1948.

McKinsey, J.C.C., and Tarski, A., *Some theorems about the sentential calculi of Lewis and Heyting*, The Journal of Symbolic Logic, vol. 13(1948), pp. 1–15.

Mostowski, A., *Proofs of non-deducibility in intuitionistic functional calculus*, The Journal of Symbolic Logic, vol. 13(1948), pp. 204–207.

1949

Henkin, L., *The completeness of the first order functional calculus*, The Journal of Symbolic Logic, vol. 14(1949), pp. 159–166.

Rieger, L., *A note on topological representations of distributive lattices*, Cas. pro p'est. mat. a fys. 74(1949), pp. 55–61.

Rieger, L., *On the lattice theory of Brouwerian propositional logic*, Acta Facultatis Rerum Naturalium Universitatis Carolinae 189(1949), pp. 3–40.

1950

Henkin, L., *An algebraic characterization of quantifiers*, Fundamenta Mathematicae, vol. 37(1950), pp. 63–74.

Henkin, L., *Completeness in the theory of types*, Journal of Symbolic Logic, vol. 15(1950), pp. 81–91.

Rasiowa, H., and Sikorski, R., *A proof of the completeness theorem of Gödel*, Fundamenta Mathematicae, vol. 37(1950), pp. 193–200.

1951

Beth, E.W., *A topological proof of the theorem of Löwenheim–Skolem–Gödel*, Indagationes Mathematicae, vol. 13(1951), pp. 436–444.

Mostowski, A., *On the rules of proof in the pure functional calculus of the first order*, Journal of Symbolic Logic, vol. 13(1951), pp. 107–111.

Rasiowa, H., *Algebraic treatment of the functional calculi of Heyting and Lewis*, Fundamenta Mathematicae, vol. 38(1951), pp. 99–126.

Rasiowa, H., and Sikorski, R., *A proof of the Skolem–Löwenheim theorem*, Fundamenta Mathematicae, vol. 38(1951), pp. 230–232.

Rieger, L., *On countable generalized σ-algebras, with a new proof of Gödel's completeness theorem*, Czechoslovak Math. Journal, vol. 1 (76)(1951), pp. 29–40.

1953

Church, A., *Non-normal truth-tables for the propositional calculus*, Boletin de la Sociedad Matematica Mexicana, vol. 10(1953), pp. 41–52.

Henkin, L., *Banishing the Rule of Substitution for Functional Variables*, The Journal of Symbolic Logic, vol. 18(1953), pp. 201–208.

Hasenjaeger, G., *Eine Bemerkung zu Henkin's Beweis fur die Vollstandigkeit des Pradikatenkalkuls der Ersten Stufe*, The Journal of Symbolic Logic, vol. 18(1953), pp. 42–48

1955

Łoś, J., *Quelques remarques, theoremes et problemes sur les classes definissables d'algebres*, In: Mathematical Interpretations of Formal systems, North-Holland, 1955, pp. 98–113.

1960

Sikorski, R., ***Boolean Algebras***, Ergebnisse der Mathematik, vol. 25, Springer-Verlag, 1960.

1962

Frayne, T., Morel, A.C., and D.S. Scott, *Reduced direct powers*, Fundamenta Mathematicae, vol. 51(1962), pp. 195–228.

1963

Henkin, L., *A Theory of Propositional Types*, Fundamenta Mathematicae, vol. 52(1963), pp. 323–344.

Rasiowa, H., and Sikorski, R., ***The Mathematics of Metamathematics***, Monografie Matematyczne, vol. 41, Warszawa 1963, 519 pp.

1963–64

Cohen, P.J., *The Independence of the Continuum Hypothesis I and II*, Proceedings of the National Academy of Sciences, vol. 50(1963), pp. 1143–1148, and vol. 51(1964), pp. 104–110.

1966

Solovay, R.M., *New proof of a theorem of Gaifman and Hales*, Bulletin of the American Mathematical Society, vol. 72(1966), pp. 282–284.

1967

Rieger, L., ***Algebraic Methods of Mathematical Logic***, Prague-New York-London, 1967.

Scott, D., *Existence and description in formal logic*, In: Bertrand Russell: Philosopher of the Century, edited by R. Schoenman, Allen and Unwin, 1967, pp. 181–200.

Scott, D., *A proof of the independence of the continuum hypothesis*, Mathematical Systems Theory, vol. 1(1967), pp. 89–111.

1968

Scott, D., *Extending the topological interpretation to intuitionistic analysis*, Compositio Mathematica, vol. 20(1968), pp. 194–210.

1969

Lambert, K., ed. ***The Logical Way of Doing Things***, Yale University Press, New Haven, Conn., 1969, Reprinted Ann Arbor: University Microfilms International, 1980.

1970

Scott, D., *Extending the topological interpretation to intuitionistic analysis, II*, In: Proceedings of Conference on Proof Theory and intuitionism, edited by J. Myhill, North-Holland, 1970, pp. 235–255.

1971

Lawvere, F.W., *Quantifiers and sheaves*, International Congress of Mathematicians, Nice 1970, vol. 1, Gauthier-Villars, 1971, pp. 329–334.

1972

Hasenjäger, G., ***Introduction to the basic concepts and problems of modern logic***, Dordrecht-Holland, D. Reidel Pub. Co., 1972, 180 pp.

1974

Rasiowa, H., ***An algebraic approach to nonclassical logics***, Studies in Logic and the Foundations of Mathematics, vol. 78, North-Holland, Amsterdam, 1974, xv + 403 pp.

1979

Grayson, R.J., *Heyting-valued models for intuitionistic set theory*, In: Applications of Sheaves, Durham Proceedings (1977), edited by M. Fourman et al., LNM 753, 1979, Springer-Verlag, 1979, pp. 402–414.

Fourman, M., and Scott, D., *Sheaves and logic*, In: Applications of Sheaves, Durham Proceedings (1977), edited by M. Fourman, Springer-Verlag, LNM 753, 1979, pp. 302–401.

Scott, D., *Identity and existence in intuitionistic logic*, In: Applications of Sheaves, edited by M. Fourman, Durham Proceedings (1977), Springer-Verlag, LNM 753, 1979, pp. 660–696.

1982

Johnstone, P.T., ***Stone Spaces***, Cambridge University Press, 1982, xxi + 370 pp.

1995

Quine, W.V.O., ***Free logic, description and virtual classes***, In: Selected Logic Papers, Harvard University Press, 1995, xii + 307 pp.

1996

Henkin, L., *The Discovery of My Completeness Proofs*, The Bulletin of Symbolic Logic, Vol. 2(1996), pp. 127–158.

2002–2007

Johnstone, P.T., ***Sketches of an Elephant: A Topos Theory Compendium***, Oxford University Press, vol. 1(2002), 562 pp., vol. 2(2002), 715 pp., vol. 3(2007?)), ???pp.

2003

Lambert, K., ***Free logic: Selected essays***, Cambridge University Press, 204 pp., 2003.

2004

Harding, J., and Bezhanishvili, G., *MacNeille completions of Heyting algebras*, Houston Journal of Mathematics, vol. 30 (2004), pp. 937–952.

2005

Bell, J.L., ***Set Theory: Boolean-Valued Models and Indpendence Proofs***, Third Edition, Clarendon Press, 2005, 216 pp.

2007

Harding, J., and Bezhanishvili, G., *MacNeille completions of modal algebras*, Houston Journal of Mathematics, vol. 33 (2007), pp. 355–384.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Abstracting and Generalizing Pseudorecursiveness

Benjamin WELLS
University of San Francisco

Abstract. Pseudorecursive varieties [16] strongly express a lack of recursive uniformity related to equational logic. This article suggests an abstraction of the notion and its integration with algebraic and computational studies—steps that may support or at least explicate Tarski's claim that the nonrecursive pseudorecursive equational theories are nonetheless decidable.

Keywords. Pseudorecursive varieties, equational logic, decidable theories, computability
MR Classification. Primary 03D35; secondary 03D15, 68Q05, 68Q10, 68Q15

Prelude

After a gap of several years, I visited Alfred Tarski and stated that I would work in any field he nominated in order to complete a dissertation under his direction. He suggested that equational logic was of current interest to him. "So, in the end it's pseudorecursive varieties?" He nodded, pleased.

1. Introduction

Recall the following concepts of equational logic:

Definition 1.1.

(1) A *variety* is a class of algebras of a given similarity type closed under the homomorphic images operator H, the subalgebras operator S, and direct products (of two algebras) operator P.
(2) Let B be a set of formal equations $\sigma \approx \tau$ (or identities, i.e., positive universal Horn sentences) of a given algebraic similarity type. The *equational theory* $\mathrm{Th}(B)$ *generated by the base* B is the smallest set E of equations including $\{x \approx x\} \cup B$ and closed under (i) substitution of a variable (at every instance in an equation) with a given term and (ii) replacement of any subterm α by a term β where $\alpha \approx \beta$ is in E.
(3) The set of all equations true in every member of a set of algebras M of given type is denoted by $\mathrm{Eq}(M)$.

(4) The set of algebras in which every equation in the set B is true is denoted by $\mathrm{Mod}(B)$.

Assuming B is a nonempty set of equations of given type, and M is a nonempty set of algebras of a given type, then the basic properties of these notions are:

(A) $\mathrm{Th}(B) = \mathrm{Th}(\mathrm{Th}(B)) = \mathrm{Eq}(\mathrm{Mod}(B))$;
(B) $\mathrm{Mod}(B) = \mathrm{Mod}(\mathrm{Th}(B))$ is a variety;
(C) $\mathrm{Eq}(M) = \mathrm{Th}(\mathrm{Eq}(M))$;
(D) $\mathrm{HSP}(M) = \mathrm{Mod}(\mathrm{Eq}(M))$ is the variety generated by M;
(E) If V is a variety, then $\mathrm{Mod}(\mathrm{Eq}(V)) = V$, and $\mathrm{Eq}(V) = \mathrm{Th}(\mathrm{Eq}(V))$, which we write as $\mathrm{Th}(V)$.

Pseudorecursive varieties were announced vaguely and anonymously in [6] (p. 395/note 1, and p. 473/note 4) and first constructed in [11]. Such a variety V, or really its equational theory $\mathrm{Th}(V)$, exhibits lack of uniformity in its decision problems. On one hand, the set $\mathrm{Th}_n(V)$ of equations that are true in the variety and have the number of their variables bounded by n is recursive for each bound $n \in \omega$. On the other hand, the entire theory $\mathrm{Th}(V)$ is not recursive. In other words, if Tm is the class of Turing machines, then:

Definition 1.2. An equational theory $\mathrm{Th}(V)$ is *pseudorecursive* iff

$$(\forall n \in \omega)(\exists M \in \mathrm{Tm})(M \text{ decides } \mathrm{Th}_n(V)), \tag{$*$}$$

but

$$\neg(\exists M \in \mathrm{Tm})(M \text{ decides } \cup_{n\in\omega} \mathrm{Th}_n(V)),$$

where the union in the latter is just $\mathrm{Th}(V)$.

Many-one reducibility (or m-reducibility) is a preorder on subsets of ω strictly finer than Turing reducibility. We write $A \leqslant_{\mathrm{m}} B$ to mean that there is a recursive function $f : A \to B$ such that for all $x \in \omega$, $x \in A$ iff $f(x) \in B$. We write $A \equiv_{\mathrm{m}} B$ to mean that both $A \leqslant_{\mathrm{m}} B$ and $B \leqslant_{\mathrm{m}} A$ hold.

In [16] a finite equational base $\Psi 1$ is constructed from a specialized Turing machine that computes a function whose domain is exactly a chosen but arbitrary nonrecursive, recursively enumerable set X of positive integers. $\mathrm{Th}(\Psi 1)$ is an equational theory of semigroups with 0 and finitely many additional unary operations. The main result is quoted here.

Theorem 1.3 (Theorem 10.4 in [16].). $\mathrm{Th}(\Psi 1)$ *is pseudorecursive; indeed,* $\mathrm{Th}(\Psi 1) \equiv_{\mathrm{m}} X$.

As the Prelude suggests, Tarski was attracted to this problem from the time I mentioned it to him in 1970; he remained enthusiastic even when I was not. Indeed, as reported in [11,16,18], he took the result as an argument for the importance of formal language. Yet in 1982 when he asked me my thoughts on why he liked my discovery/construction of pseudorecursive theories, he did not have in mind the indispensability of formal language in mathematics. Instead he asserted,

"Because they are decidable." $(**)$

This astonishing claim is quoted in [11, pp. xii–xiv] and [16, pp. 460–461]; it is discussed further in [18,20].

Some clarification is needed. In the conversation mentioned, Tarski used a word he created to capture the positive part of the characterization of pseudorecursiveness. Two versions have appeared in print, the second with two definitions:

Definition 1.4.

(1) (from [11, p. xii]; see also [16, p. 460] and [18, p. 320, note 17]) An equational theory $\mathrm{Th}(V)$ is *quasidecidable* iff for every natural number n, there is a decision procedure for the set $\mathrm{Th}_n(V)$ of equations in the theory with n variables.

(2A) An equational theory $\mathrm{Th}(V)$ is *quasirecursive* iff for every natural number n, there is a Turing machine T_n that decides the set $\mathrm{Th}_n(V)$ of equations in the theory with n variables. This is equivalent to the more symbolically expressed $(*)$, which is used in the recounting of this conversation in [19, p. 226] (see also [18, p. 305]).

(2B) (from [11, p. xvii], see also [16, p. 465] and [15,18,20]) An equational theory $\mathrm{Th}(V)$ is *quasirecursive* iff for every natural number n, the set $\mathrm{Th}_n(V)$ of equations in the theory with n variables is recursive.

Definition 1.4(1) is an informal definition, because "decision procedure" is considered here to be a commonsense term. Conventionally, it refers to a procedure mediated or implemented by a Turing machine, as in (2A), but this too is an informal definition, because "decides" is not defined here. The reader will easily accept that (2A) is formalized in (2B).[1]

If we take the conventional meaning of a decision procedure in (1),[2] the quasidecidable theories are the same as the quasirecursive ones. Strictly speaking, though, "quasidecidable" is vaguer, broader than "quasirecursive." Whether Tarski used one term or the other in the conversation, two things are observable: (i) [11, p. xii] uses "quasidecidable"; (ii) Tarski read and accepted that account of our conversation when he carefully read the draft of the dissertation [11]. For this reason, "quasidecidable" is the term of choice here.

During our discussion of the claim $(**)$ above, Tarski argued that quasidecidable was tantamount to decidable (see [18] for more about his argument, and mine with him). But once one unlinks decidable from recursive and admits some new decision procedure weaker than Tm computation, then the definition of quasidecidability itself could use it and also be weakened, either once and for all, or possibly in some hierarchy of increasingly broader notions.

For equational theories (of, say, the type of semigroups with additional unary operations and constants), there are therefore several candidates for the Tarski extension of decidability:

(A) the countable collections of finitely based pseudorecursive theories directly constructed in [11,16,17];

[1] The standard view is that (1) is also formalizable as (2B) and $(**)$ is a contradiction. Tarski did not agree. Please see [18] for more discussion.

[2] But that could entail that Tarski was claiming the theories in $(*)$ and $(**)$ to be recursive, which he did not. See [18].

(B) all finitely based pseudorecursive theories;
(C) all recursively based pseudorecursive theories;
(D) all r.e. pseudorecursive theories;[3]
(E) all pseudorecursive theories (that is, the nonrecursive quasirecursive theories);
(F) all nonrecursive quasidecidable theories (using one or more new characterizations of decidability à la Tarski in the definition of quasidecidable).

In addition to these nested collections, there are varieties with properties of interest that appear among (B–E):

(G) all pseudorecursive theories of specialized varieties.

An algebraic example of specialization is being locally finite (where every finitely generated member of the variety is finite). For pseudorecursive locally finite varieties, see [11,16,5], and, most recently, [3]. An algorithmic example of specialization is a property proposed by Tarski in 1981:

$$\textit{All word problems in the variety are recursive, but not uniformly.} \qquad (***)$$

When the variety is also pseudorecursive, it is said to be *strongly pseudorecursive* in [5].[4] Observe that all locally finite varieties also have $(***)$. Thus the first strongly pseudorecursive varieties constructed (or identified) were the varieties of [11, Remark 13.5] (also appearing in [16, Remarks 11.2.3,4]).[5] Next were the author's specific constructions in [7] and [16, Theorems 10.7 and 10.9]. For strongly pseudorecursive varieties and others satisfying $(***)$, also see [12] and [1–4].

It should now be clear that Tarski implicitly held all of (A–G) to be decidable. The last three groups are uncountable (see Remarks 11.2.3,4 in [16, p. 507]). Therefore the collection of newly decidable theories is uncountable. So it is unlikely that the collection of required decision procedures or "mechanical methods," not to say machines, will be countable. This raises even more anguishing questions.

For the present article, only the first four groups (A–D) are of interest.

2. Results on Extending the Notion of Pseudorecursiveness

The following result sharpens Theorem 1.3 and improves an earlier formulation in [19, pp. 232–233] that only mentions finitely based varieties but covers others.[6] Here, P is the class of problems solvable by deterministic Turing machines halting in time bounded by polynomial functions of the length of the input.

[3]This group is listed for completeness and later generalization. In fact, (C) and (D) coincide by Remark 13.6 in [11, p. 201] using the pleonastic variable rewriting technique, as it is called in [16, p. 504].

[4]The problem of existence of strongly pseudorecursive varieties (and especially finitely based ones) was proposed to me by Tarski in 1981, mentioned in a broader context in [11, Remark 12.12.1], incorporated as an example attributed to me in [7], discussed explicitly in [16], and finally settled for the finite base by Dejan Delić in [4].

[5]Credit for the two nonconstructive constructions must go to Juri Kleĭman and Ralph McKenzie. Note that the error in [11] concerning no recursive base for Kleĭman's varieties is corrected in [16] to no finite base (continuum many varieties cannot be co-r.e.)

[6]This was first extended by the editor's correction in [21] that removes "finitely based."

Theorem 2.1 (extending Theorem 3.1 in [19]). *Each pseudorecursive theory* $\mathrm{Th}(\Psi)$ *discussed in* [11, Section 11], [16, Section 10] *and* [17] *has* $\mathrm{Th}_n(\Psi) \in \mathrm{P}$ *for every* $n \in \omega$.

Proof. For [16, Section 10] the proof is given in [19]. For the additional pseudorecursive theories given in [11,17], the result for $\Psi 1$ (see Theorem 1.3) extends to them because all the codings are polynomial bounded. □

The lack of uniformity revealed by pseudorecursive theories is reflected at lower complexity than recursive functions. Two examples are given below. The first concerns a theory that is nondeterministic-polynomial-time complete (NPC), and the second relies on a notion of parallel computation introduced in [19]. We assume P ⊂ NP.

Let W be an NPC subset of ω^+. Applying the constructions that led to the finite pseudorecursive base $\Psi 1$ [16] to W instead of the nonrecursive r.e. set X yields the following result.

Theorem 2.2 (Theorem 4.1 in [19]). *There is a finite base* Ψ_{NPC} *for an equational theory of semigroups with additional unary operations and distinguished elements such that*

$$\mathrm{Th}(\Psi_{\mathrm{NPC}}) \in \mathrm{NPC},$$

and for all $n \in \omega$,

$$\mathrm{Th}_n(\Psi_{\mathrm{NPC}}) \in \mathrm{P}.^7$$

Remark 2.3. In Theorem 2.2, the set of NPC problems can be replaced with other complexities higher than P, such as exponential time, NP space, superexponential time, etc.

The following extension of P/NP is a working definition characterizing sequential vs. parallel computation.

Definition 2.4. By a *K-sequential* computation we shall mean a Turing machine operating in Kalmár elementary time. Ackermann's function $A(m, n, p)$—recursive, but not primitive recursive—may be characterized by:

$$\begin{aligned} A(m,n,0) &= m+n \\ A(m,n+1,p+1) &= A(m,A(m,n,p+1),p) \qquad (m,n,p \geqslant 0), \end{aligned}$$

with two special cases:

$$\begin{aligned} A(m,0,1) &= 0 \\ A(m,0,p) &= 1 \qquad (m \geqslant 0, p \geqslant 2). \end{aligned}$$

Ackermann's diagonal function Acd(n) = $A(n, n, n)$ is obviously recursive but also fails to be primitive recursive. A function (or set) is *K-parallelizable* if it can be computed in Kalmár elementary time t by Turing machines operating simultaneously, sequentially, but not necessarily independently, where the number of machines grows as Acd(t). Acd and A are K-parallelizable and not K-sequential. Let NSP be the class of non-K-sequential K-parallelizable problems, and let Ksq be the class of K-sequential problems.

[7] The expansion here and in Theorem 2.5 only requires finitely many unary operations or distinguished elements. If infinitely many are desired, then the base constructed will be infinite and can be P.

Theorem 2.5 (Theorem 4.4 in [19]). *There is a finite base Ψ_{NSP} for an equational theory of semigroups with additional unary operations and distinguished elements such that*

$$\mathrm{Th}(\Psi_{\mathrm{NSP}}) \in \mathrm{NSP},$$

but for all $n \in \omega$,

$$\mathrm{Th}_n(\Psi_{\mathrm{NSP}}) \in \mathrm{Ksq}\,.$$

3. Abstract Pseudorecursiveness via §-Structures

Briefly outlined in this section is an abstraction of pseudorecursiveness in the form of ramified families of sets called §-structures. The goal is to extract a portion of the structure of the equational theories, the less the better, but enough to assure that some pseudorecursive equational theories are represented and some nonrecursive unions are not. The definition given is provisional and must be refined in future work.

Informally, a §-structured set will be a union $A = \bigcup A_m$ of primitive recursive sets $A_m \subset \omega \times \omega$ with auxiliary sets, relations, functions that reflect corresponding structures and behaviors in equational logic. These associations are indicated by parenthetical comments below. Recall the notions of Te_n, the terms involving at most n of the variables $v_0, v_1, \ldots$, and Eqn_n, the equations $\alpha \approx \beta$ such that $\alpha\beta \in \mathrm{Te}_n$.

Definition 3.1.

(1) The following sets A, the A_m, B, C, and the C_m form the *center*.

- (i) There are countable primitive recursive sets $B \subset C \subset \omega$. ($B$ represents the variables and C, the *compact* terms, whose variable indices form an interval in ω.)
- (ii) If $m \in \omega$, then $C_m \subset C$. (C_m would be the compact terms in Te_m using variables among $v_0, v_1, \ldots, v_{m-1}$.)
- (iii) $A_m \subseteq C_m \times C_m$ and $A_m \subseteq A_{m+1}$ for each m. (A_m is confined to the portion of Eqn_m of equations $\alpha \approx \beta$ with $\alpha, \beta, \alpha\beta$ compact, with variables among $v_0, v_1, \ldots, v_{m-1}$.)
- (iv) There is a unique element $b \in B$ such that $\langle b, b\rangle \in A_1$. (The equation $v_0 \approx v_0$ is represented by $\langle b, b\rangle$.)
- (v) $A = \bigcup A_m$ is an equivalence relation. (An equational theory yields an equivalence on terms.)

(2) The maps below and the set D form the *cable*.

- (vi) $\mathrm{vst} : C \to {}^{\omega}2$ (gives the set of variable indices occurring in a term),
- (vii) $\mathrm{mnv}, \mathrm{mxv} : C \to \omega$ (gives the minimum and maximum index of the variables occurring in a term),
- (viii) In the following, the index m is suppressed on the maps.
  $r_k : C_{m+1} \to C_m$ (replaces all v_m with v_k for $k < m$ and $k+1 \geqslant$ the value of mnv),
  $s_{jk} : C_m \to C_m$ (swaps all v_j and v_k for $j, k < m$ as long as the term remains compact),

$\mathrm{sub}_t : C_m \to C_{m+1}$ for $t \in C_{m+1}$ (substitutes a compact term t with variables among $v_0, v_1, \ldots, v_m$ for v_{m-1} as long as the term remains compact),
$\mathrm{rep}_{c,d} : C_m \to C_{m+k}$ for $k = 0$ or ± 1 (replaces the first occurrence of the subterm c with a term d, where $c \approx d$ or $d \approx c$ belongs to Eqn_{m+1}, as long as the term remains compact),
$\mathrm{id} : C_m \to C_{m+1}$ (inclusion mapping).
These are extended to the A_m and A in the obvious way.

(ix) The finite, P, K-sequential, primitive recursive, recursive or (at worst) recursively enumerable set $D \subset A$ generates A under the maps in (vii). (D represents a finite, P, K-sequential, primitive recursive, recursive, or r.e. axiom base; there is an obvious extension to other specialized axiom bases.)

(3) The cable has the following properties.

(x) mnv $\leqslant$ mxv and mnv = mxv on B
vst, mnv, mxv are used to characterize C_m and A_m (so that the sides of an equation form a compact term),
$\mathrm{mnv}^{-1}\{0,1\} \subseteq \mathrm{Fld}(A)$ (for the constructed pseudorecursive theories, this supports the zero),
$r_k \circ r_k = r_k \circ r_{m-1}; s_{jk} \circ s_{kj} = \mathrm{id}$,
r, s definable from sub, mnv, mxv (according to the logic parallel),
$\mathrm{rep}_{\mathrm{sub}_t[c,d]} = \mathrm{rep}_{c',d'}$ (that is, the result is defined),
$\mathrm{sub}_t \circ \mathrm{rep}_{c,d} = \mathrm{rep}_{\mathrm{sub}_t(c,d)} \circ \mathrm{sub}_t$,
$\mathrm{sub}_{\mathrm{sub}_t(u)} = \mathrm{sub}_t \circ \mathrm{sub}_u$,
$b \notin \mathrm{Fld}(\mathrm{Rng}(\mathrm{sub}))$ (eliminates circularity that could defeat the layering of the Th_m).

(4) The A-classes have the following invariance:

(xi) $A[\mathrm{sub}^i_u(t)] = \mathrm{sub}^i_u[A[t]]$ (sub acting on C_i),
$\mathrm{rep}_{c,A[c]}[A[t]] = A[t]$,

(5) The center and cable together form the §-*structured set* (or §-structure) $\S A$.

It may seem that the versions of substitution and replacement modeled here are not general enough, yet they are sufficient to entail pseudorecursive behavior in some equational partial theories. A better criticism is that an abstraction should be simpler, sparer, cleaner than the concrete—otherwise, the exercise is hard to justify. But this definition is complex and unnatural. To some extent it reveals the complexity inherent in a pseudorecursive theory, like the layering properties exploited in the interpretation theory of [17]. The numerous requirements in the definition have indeed been simplified over its precursors. The following discussion supports trying harder.

Definition 3.2.

(1) A *polynomial* §-structured set is a §-structure with a P-computable center and cable.
(2) A *K-sequential* §-structured set is a §-structure with a K-sequentially computable center and cable.

The next theorem suggests the provisional definition of §-structures has some merit.[8] It will extend work announced in [13–15]. We introduce the specific §-structures derived from pseudorecursive theories. They can be viewed as canonical.

Definition 3.3.

(1) Given an equational theory T, define the collection of sets and maps suggested in Definition 3.1 as the intentions of the center and cable; we call the result a *logical prestructure*. If this also has the computability properties of a §-structure, then we call it a *logical §-structure*, or log§-structure.
(2) Extend (1) to the polynomial case in Definition 3.2.1 to obtain *polynomial log§-structures.*
(3) Extend (1) to the K-sequential case in Definition 3.2.2 to obtain *K-sequential log§-structures.*

Theorem 3.4.

(1) *There are pseudorecursive* (NPC, NSP) *equational theories that yield nonrecursive (*NPC *polynomial,* NSP *K-sequential, respectively) logical §-structures.*
(2) *There are recursively enumerable* (NPC, NSP) *equational theories that do not yield (*NPC *polynomial,* NSP *K-sequential, respectively) log§-structures.*
(3) *Every §-structured set is recursively enumerable; every polynomial §-structure is* NP*; every K-sequential §-structure is K-parallelizable.*

Proof.

(1) A sufficient fragment of the argument from [11,16] can be reflected through properties of the given mappings for the pseudorecursive case. The other two then follow in the manner of Theorems 2.2 and 2.5.
(2) Classical embeddings of Turing machines in equational theories provide trivial counterexamples.
(3) We can rely on the strength of the computability requirements for the cable and center. □

4. Reflection on Tarski's Claim of Decidability

As mentioned in Section 1, Tarski expressed the opinion that pseudorecursive varieties are decidable even though not recursive. [18, Section 10] suggests possible interpretations of this at lower complexity (where it should not appear paradoxical) and in abstract settings (where it might appear less parochial). The results of [19, Sections 2,4] augment the discussion.

Any type of §-structured set represents the failure but also (in some informal sense) the near success of the corresponding model of computation. We now propose to accept this result as a valid computation, called *pseudorecursion*, by an expanded model. In

[8] These structures, like pseudorecursive theories, are reminiscent of the Sufi story about blind men investigating an elephant: to the one who found a leg, elephants were treelike, but to the one who found the trunk, snakelike, etc. Each reliably reported a description of a segment of the creature. But they had no encompassing outlook, even if nonvisual, and no coherent account of the whole could emerge.

other words, we expand the model by describing the enlarged range of its computations as the corresponding §-structured sets. (One can always object that any r.e. set can become a §-structure, but the point is to show that, under reasonable restrictions, there is some middle ground.)

A more general notion of computation required by pseudorecursive theories could even lead to a new view of computer architecture, since all physical computing systems are finite approximations of infinistic ones. Here is an example. On the one hand, the definition of nondeterministic computation (as in NP) suggests a structure of independent parallel processing where no interaction occurs between twin processes after they are spawned until one completes its task. On the other hand, actual parallel computers can capitalize on cloned processes, but seek a high degree of interconnectivity because of bounded resources. The results here might offer a loosely coupled yet interdependent model of computer as well as of computation. In addition, initial segments of a sequence of machines approximating the ideal oracular machine required to decide (in the usual Tm sense) a pseudorecursive theory might be realized as a system of communicating expert systems with distinct knowledge domains yet with the ability to compute cooperatively. In this case, the parameter n might have the interpretation of degree of knowledge. An analogy is A. Waibel's "connectionist glue" that uses simple units to glue together complex modular components in neural nets (per Rumelhart [9]; see also [10]).

Of course, this is not very satisfactory: as an extensional programming abstraction, it goes to absurd lengths; and how can one read the speculation on more powerful machines seriously? There is, however, long mathematical tradition of taking the problem to be the solution. Indeed, integer, rational, or complex numbers are just the outcome of asserting that problems such as $0 - 11, 4 \div 7, \sqrt{-1}$ are already solved and studying what that commitment entails. See [19] for more on this. The reader may also wish to compare the notion of "almost decidability" introduced by Millar [8]; although there is no parallel for equational logic, this notion too could be approached not as classification of pathologies but as an extension of computability.

Getting back to mathematics, the following program will provide some legitimacy for §-structures.

Program 4.1.

(1) Prove that logical §-structured sets are compatible with the layered interpretations of [17] in the sense that maps from the equational theories to the §-structures lift the interpretation in a commutative diagram, and the lifted map behaves correctly with the cable maps.
(2) Prove a representation theorem for §-structures that reduces them in some manner to the special case of logical §-structures.
(3) Prove all pseudorecursive theories yield logical §-structures that are not recursive, or classify the exceptions.
(4) Use some cognate of §-structured sets to provide a plausible, usable, convenient abstract extension of pseudorecursive theories.

5. Conclusion

The constructions in [11,16,17,19] of pseudorecursive theories are single-purpose, ad hoc, awkward. The goal here of producing a common abstraction is poorly met by being both too complex and incomplete. The real advance would be a simpler construction or a real-life example. Another resolution would be to find an application for the notion of pseudorecursiveness or, probably better, to find it arising from a known application. Any of these four outcomes would stimulate further study of the notion. By themselves, though, they do not fix the Tarski quandary. But they might justify an intermediate computability and therefore make the middle road more attractive.

Acknowledgements

Andrzej Mostowski put me on the path to pseudorecursiveness by inviting me to create [6], wherein the story began. I am profoundly grateful for his offer and all of his other help, and I regret that the construction was not written down for him to see. This paper was greatly improved by the referee's request for explicit definitions. I also wish to thank Jeff Buckwalter for helping me with the Russian, Ralph McKenzie, who was the hammer in Alfred's hand, and MD, the hammer in the Other hand.

Postlude

Tarski remained adamant that pseudorecursive varieties did not exist until a formal, complete, and rigorous proof was written. He solicited this on many occasions. Tarski overheard my telling John Addison that I had constructed a finitely based pseudorecursive variety. He bristled, "You do not have a proof until you have written it down!" Finally, he invited Ralph McKenzie to dinner and told him the problem. McKenzie predictably sketched a solution within a couple of days. We compared; he agreed his was faulty and mine was sound, which he reported to Tarski.[9] *When I told Tarski that McKenzie would be rock-climbing for several weeks, he gave me a deadline of ten days (met), saying that climbing would not stop Ralph from thinking. "After all," Alfred said, "I found my first good result sitting in a barber's chair!"*

References

[1] S. Crvenković and D. Delić, A variety with locally solvable but globally unsolvable word problems, *Algebra Universalis* **35** (1996), 420–424.

[2] S. Crvenković and I. Dolinka, A variety with undecidable equational theory and solvable word problem, *International Journal of Algebra and Computation* **8** (1998), 625–630.

[3] S. Crvenković, I. Dolinka, and V. Tasić, A locally finite variety of rings with an undecidable equational theory, *Quarterly Journal of Mathematics* **57** (2006), 297–307.

[4] D. Delić, A finitely axiomatizable undecidable equational theory with recursively solvable word problems, *Transactions Amer. Math. Soc.* **352** (2000), 3065–3101.

[5] M. Jackson, On locally finite varieties with undecidable equational theory, *Algebra Universalis* **47** (2002), 1–6.

[6] A.I. Mal'tsev, *The Metamathematics of Algebraic Systems—Collected Papers: 1936–1967*, North-Holland, Amsterdam, 1971. Thirty-four papers in algebra and logic, newly translated, edited, and annotated by Benjamin Wells.

[9] Despite some difficulties with McKenzie's approach, his use of discriminator varieties foreshadowed Delić's results [4].

[7] A. Mekler, E. Nelson, S. Shelah, A variety with solvable, but not uniformly solvable, word problem, *Proc. London Math. Soc.* **66** (1993), 225–256.

[8] T. Millar, Prime models and almost decidability, *J. Symbolic Logic* **51** (1986), 412–420.

[9] D. Rumelhart, personal communication.

[10] A. Waibel, H. Sawai, and K. Shikano, Consonant recognition by modular construction of large phonemic time-delay neural networks (1989), *Readings in Speech Recognition*, A. Waibel and K. Lee, eds, Morgan Kaufmann Publishers, San Francisco, 1990.

[11] B. Wells, *Pseudorecursive varieties and their implications for word problems*, Doctoral dissertation, University of California, Berkeley, 1983.

[12] B. Wells, Finitely based varieties where all word problems are recursively solvable, but not uniformly so, *Abstracts Amer. Math. Soc.* **6** (1985), 487.

[13] B. Wells, Mathematics inspires an extended notion of computation and computers. Abstract and poster session for Computers and Mathematics, Stanford, July 1986.

[14] B. Wells, Pseudorecursiveness: implications for decidability and computability, *J. Symbolic Logic* **51** (1986), 858–859.

[15] B. Wells, New views of pseudorecursiveness, *Abstracts ICM-86* (1986), p. 34.

[16] B. Wells, Pseudorecursive varieties of semigroups—I, *Int. J. of Algebra and Computation* **6** (1996), 457–510.

[17] B. Wells, Pseudorecursive varieties of semigroups—II, *Int. J. of Algebra and Computation* (to appear).

[18] B. Wells, Is there a nonrecursive decidable equational theory?, *Minds and Machines* **12** (2002), 303–326.

[19] B. Wells, Applying, extending, and specializing pseudorecursiveness, *Annals of Pure and Applied Logic* **126** (2004), 225–254.

[20] B. Wells, Hypercomputation by definition, *Theoretical Computer Science* **317** (2004), 191–207.

[21] B. Wells, Erratum to "Applying, extending, and specializing pseudorecursiveness" [*Ann. Pure Appl. Logic* **126** (2004) 225–254], *Annals of Pure and Applied Logic* **128** (2004), 299. Corrections to [19].

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Andrzej Mostowski on the Foundations and Philosophy of Mathematics

Roman MURAWSKI [a,1] and Jan WOLEŃSKI [b]
[a] *Adam Mickiewicz University, Faculty of Mathematics and Comp. Sci., ul. Umultowska 87, 61–614 Poznań, Poland*
rmur@amu.edu.pl
[b] *Jagiellonian University, Institute of Philosophy, ul. Grodzka 52, 31–044 Kraków, Poland*
wolenski@if.uj.edu.pl

The relations between mathematics, its foundations and philosophy are fairly subtle and complicated. One can speak about mathematical foundations of mathematics and philosophical foundations of mathematics. In fact, logicism, formalism and intuitionism can be perceived as foundational projects of investigating mathematics by mathematical means as well as different philosophies answering such questions as, for example, mathematical existence or the epistemological status of deduction. The expression "fundamental problems of mathematic" which comprise both approaches is a convenient label. Mathematicians working in technical problems of the foundations of mathematics have different attitudes related to the question how fundamental problems are mutually related. Some of them entirely disregard philosophical problems of mathematics, other, for example, Brouwer or Hilbert, are inclined to base their metamathematical research on explicitly accepted philosophical assumptions, which essentially influence the further course of technical work, but still other see mathematical and philosophical foundations of mathematics as independent although connected in a way and indispensable for understanding mathematical activity.

The third attitude is characteristic for Polish tradition in logic and the foundations of mathematics. Chwistek and Leśniewski were the only exceptions in this respect. The former proposed a certain version of logicism, but the latter, after the early stage of working on an improvement of Principia Mathematica, began to develop a radical version of finitism and constructivism based on extremely nominalistic presumptions. The considerable rest of Polish logicians and mathematicians represented the view guided by two following principles:

(P1) all commonly accepted mathematical methods should be applied in metamathematical investigations;
(P2) metamathematical research cannot be limited by any a priori accepted philosophical standpoint.

[1] The financial support of the Foundation for Polish Science is acknowledged.

These two principles do not imply that mathematics is free of own genuine philosophical problems or that mathematicians should neglect these problems as exceeding their professional activities. According to Polish school, although formal metamathematical results do not solve philosophical controversies about mathematics, yet the former illuminate the latter.

The tradition of Polish analytic philosophy, originated with Twardowski in Lvov and continued by the Lvov-Warsaw School (see [31] for a comprehensive presentation of this) supplemented (P1) and (P2). According to Twardowski and his students, we must clearly and sharply distinguish world-views and the scientific philosophical work. This idea was particularly stressed by Łukasiewicz, the main architect of the Warsaw school of logic. He regarded various philosophical problems arising in formal sciences as belonging to world-views of mathematicians and logicians, but the work consisting in constructing logical and mathematical systems together with metalogical (metamathematical) investigations constituted for him the subject of logic and mathematics as special sciences. Hence, philosophical views cannot be a stance for measuring the correctness of formal results. Yet philosophy may serve as a source of logical constructions; Łukasiewicz's many-valued logic is a good example in this respect. Since all members of the Warsaw school of logic were Łukasiewicz's students, his opinion about the relation of logic to philosophy became influential.

Due to the above characterized attitude, Polish mathematicians did not accept any of the "big three" in the foundations, that is, logicism, formalism or intuitionism. The Polish view, as we can call it, has a particular and surprising consequence consisting in a freedom of accepting philosophical opinions sometimes being at odds with applied methods. Perhaps Tarski was an extreme example of this practice. He used all admissible mathematical methods in his logical works, in particular infinitary ones, usually associated with Platonism in the philosophy of mathematics, but he contributed to all mentioned grand projects by the idea of logical concepts as invariants (related to logicism), the theory of consequence operations (a component of formalism) and the topological semantics for intuitionistic logic. Tarski himself stressed that this methodological attitude, sometime labelled as "methodological Platonism", became a characteristic feature of Polish school and its essential contribution to metamathematics ([30, p. 713]; page-reference to the reprint):

> As an essential contribution of the Polish school to the development of metamathematics one can regard the fact that from the very beginning it admitted into metamathematical research all fruitful methods, whether finitary or not.

On the other hand, he had explicit sympathies to empiricism, nominalism, reism and finitism (he even called himself "a tortured nominalist"). Due to a strict departure of mathematics and philosophy (philosophical opinions of mathematicians were considered as somehow private in Poland) as well as locating them on different levels, no contradiction occurs in Tarski's position, although we certainly encounter here an example of a cognitive dissonance to some extent. Probably Tarski saw the situation in such a way and perhaps it explains why he usually abstained from a wider elaboration of his philosophical views, at least in his writings. Tarski was more involved in philosophy in oral debates concerning philosophy. It is well documented by the records of discussions between Tarski, Carnap, Quine and Russell in 1940–1941; these protocols were written by Carnap and are deposited in the University of Pittsburgh (see [5]).

Let us turn now to Mostowski's views and opinions concerning the foundations and philosophy of mathematics. The strongest expression of his philosophical position were perhaps these ((a) [10, p. 231]; (b) [12, p. 16]; (c) [12, p. 42]):

> (a) There is no doubt that all mathematical concepts have been developed by abstraction from concepts formed on the basis of a direct experience. But he adds that this statement is not sufficient and the process of abstraction should be deeper analyzed.
>
> (b) Materialistic philosophy has since long been opposed to such attempts [i.e., attempts of treating mathematics solely as a collection of formal axiomatic systems – R. M., J. W.] and has shown the idealistic character both of Hilbert's program which consists in defining the content of mathematics by its axioms and of the neopositivistic program consisting in the explanation of the content of mathematics by an analysis of the language.
>
> (c) Results obtained by mathematical method confirm therefore the assertion of materialistic philosophy that mathematics is in the last resort a natural science, that its notions and methods are rooted in experience and that attempts at establishing the foundations of mathematics without taking into account its originating in natural sciences are bound to fail.

Unfortunately there is a problem of interpretation, because these papers were written in the first half of the fifties and the ideological atmosphere of that time could have had an influence on it (look at a typical slang of this time "confirm therefore the assertion of materialistic philosophy" or "materialistic philosophy shown", required in philosophical remarks at the time). It is not possible now to decide to what extent outside factors influenced the paper. On the other hand the author could restrict himself to purely mathematical issues and avoid entirely any philosophical remarks and declaration. If he did not do it we can treat his remarks as genuine, also because Mostowski's declared sympathies were consistent with materialism. He inherited his general philosophical attitude from Tarski, perhaps also some inclination to empiricism and a respect for nominalism (see [26] for remarks about nominalistic tendencies among Polish logicians). We have also certain evidence that Mostowski sympathized with reism (the view that there are solely individual corporeal things) to some extent (quoted after Kotarbińska [3, p. 73]):

> Please imagine yourself that I heaved a sigh for reism there [that is, at the school on the foundations of mathematics – R. M., J. W.]. Presented ideas were results of speculations, so breakneck, so far elusive to intuition and incomprehensible, that reism appeared as an oasis in which one could rest and to breathe of fresh air.

Mostowski stressed that constructivistic trends in the foundations of mathematics are nearer to the nominalistic philosophy than to the idealistic one (in the platonic sense). This nominalistic character implies that constructivism does not accept the general notions of mathematics as given but try to construct them and implies that mathematical concepts can be identified with their definitions. According to Mostowski, an obvious advantage of nominalism is the fact that several important mathematical theories have been reconstructed in a satisfactory way on a nominalistic basis and those reconstructions have turned out to be equivalent to the classical theories.

On the other hand, Mostowski, respected the principles (P1) and (P2) and freely used infinitary methods and strongly insisted that formal work in mathematics and logic should not be bounded by philosophical assumptions. This position is clearly expressed in [9], a paper devoted to the problem of the classification of logical systems. Mostowski stressed that though the investigations of this problem are purely formal, they nevertheless assume a definite philosophical point of view with respect to logical systems. In particular, he declares that the logical systems are to be considered not as empty schemes devoid of any interpretation and explicitly accepts the objective existence of the mathematical reality populated, e.g., by the set of all integers or the set of all real numbers. The objective existence means here being independent of all linguistic constructions. The aim of logic and mathematics is stated as follows (p. 164; page-reference to the reprint):

> The role of logical and mathematical systems is to describe this reality. Every logical sentence has thus a meaning: it says that the mathematical reality has this or some other property. If the mathematical reality does in fact possess the given property, then the sentence expressing this property is true, otherwise it is false.
>
> [...]
>
> The point of view characterized above allows us to make plausible the existence of undecidable sentences in almost all logical systems. Such sentences evidently exist if it is not possible to prove all the true sentences in the considered system. Now every proof in a logical system consists in a succession of some operations (called the rules of proof) which can be mechanically performed on one or two expressions. The unprovability of certain true sentences can be explained by the fact that the properties of "mathematical reality" are more complicated than the properties which can be established by successive applications of the rules of proof to the axioms.

These rather rich philosophical remarks are immediately frozen by the next remark:

> We do not intend to defend the philosophical correctness or even the philosophical acceptability of the point of view here described. It is evident that it is entirely opposite to the point of view of nominalism and related trends.

Taking into account Mostowski's sympathies, even fairly moderate, to nominalism and empiricism we find in his views the same tension which he attributed to Tarski [20, p. 81]:

> Tarski in oral discussions, has often indicated his sympathies with nominalism. While he never accepted the "reism" of Tadeusz Kotarbiński, he was certainly attracted to it in the early phase of his work. However, the set-theoretical methods that form the basis of his logical and mathematical studies compel him constantly to use the abstract and general notions that a nominalist seeks to avoid. In the absence of more extensive publications by Tarski on philosophical subjects, this conflict appears to have remained unresolved.

It seems, nevertheless, that Mostowski felt himself obliged to a more extensive and systematic treatment of his views in the philosophy of mathematics than it occurred in Tarski's case. It was probably a result of the following view ([8, p. iv]; unfortunately this book was published only in Polish, although English translation was planned – compare, e.g., the back cover of Kuratowski-Mostowski [4] where Mostowski's *Mathematical Logic* is announced as a book in preparation):

> We are not able to deprive logic (independently of how it would be formal) of some, even sub-conscious, philosophical background. A conscious choice in this respect is more difficult, because, in the light of the contemporary discussion on the foundations of mathematics, it is impossible to say with certainty which view, of many competing, is the best or even good.

Mostowski considered this situation as one of the main difficulties encountered by the authors willing to write a book on mathematical logic. Yet he thought that the choice in question exceeds formal logic as such. After 17 years, he said [17, p. 149]:

> We see that the issue between Platonists, formalists and intuitionists is as undecided to-day as it was fifty years ago.

Guided by such evaluations Mostowski tried to intentionally avoid in his textbook any discussion of philosophical problems since they go beyond the limits of formal logic. For working purposes he treated a logical system as a language in which one speaks about sets and relations. An extensionality axiom has been adopted and it has been assumed that elements of the language satisfy principles of the simple theory of types. In Mostowski's opinion such a standpoint is convenient for formal considerations and coincides with views which are more or less consciously adopted by most of mathematicians (what does not necessarily mean that it must be accepted without any doubts by philosophers). However, one can observe that any logical system is interpreted (it speaks about set-theoretical objects). This corresponds with Leśniewski's view (also adopted by Tarski) known as "intutionistic (better 'intuitive') formalism", that mathematics does not consist in purely formal games, because it uses languages equipped with comprehensible meanings, although formalized. This is perhaps the main reason why Mostowski was never especially attracted by formalism.

Mostowski, although sceptical toward prospects of ultimate solutions of principal philosophical controversies, was convinced that mathematical treatment of philosophical problems is illuminating. The following quotation is perhaps characteristic ([12, p. 3]; similar remarks occurs in many other Mostowski's books and articles):

> The present stage of investigations on the foundations of mathematics opened at the time when the theory of sets was introduced. The abstractness of that theory and its departure from the traditional stock of notions which are accessible to experience, as well as the possibility of applying many of its results to concrete classical problems, made it necessary to analyze its epistemological foundations. This necessity became all urgent at the moment when antinomies were discovered, However, there is no doubt that the problem of establishing the foundations of the theory of sets would have been formulated and discussed even if no antinomy had appeared in set theory.

The general philosophical problems stemming from the discussions on the foundations of set theory cover:

(a) the question of the nature of mathematical concepts and its relation to the world;
(b) the nature of mathematical proofs and their correctness.

Mostowski had no illusions that (a) and (b) might be mathematically decided ([12, p. 3]; see also [13] and [11]):

> These problems are of a philosophical nature and we can hardly expect to solve them within the limits of mathematics alone and by applying only mathematical methods.

However, he stressed that these general questions led to more specific ones which are subjected to a formal treatment, namely

(a′) the role and limits of axiomatic method,
(a″) the constructive tendencies,
(b′) the axiomatization of logic,
(b″) the decision problem.

And he ended his report with the following remark [12, p. 42]:

> Thus, as we see, the investigations of the foundations of mathematics are not without importance although they do not stand for a full investigation on the foundations of mathematics.

> Their results are to use for mathematics as well as for philosophy. In this sense they fulfil the tasks assigned to them.

The passing from (a) and (b) to (a′)–(a″) and (b′)–(b″) was related to Mostowski's historical perspective (see [17]) in which the big three in the philosophy of mathematics, that is, logicism, constructivism and formalism, formed at the beginning of the 20th century, have been replaced in the 1930s by three new schools, namely set-theoretical, constructivistic and metamathematical. They are represented by Tarski's semantic theory of truth, Gödel's incompleteness theorems and Heyting's axiomatization of intuitionistic logic, respectively. In general, in the set-theoretical school the semantical properties of expressions of formalized languages are studied, constructivistic investigations are concentrated on various formal (mathematical) conceptual constructions and logical connections between them, and metamathematical investigations are devoted mainly to logical connections "inside" axiomatic systems and to logical properties of them. Clearly, these new paradigms of the foundations of mathematics prefer mathematical methods over philosophical analyses.

Investigating constructivism played a special place in Mostowski's reflection about the foundations in mathematics. For a while he even believed that this direction will give the ultimate base for mathematics in the future [8, p. vi]:

> I am inclined to think that the satisfactory solution of the problem of the foundations of mathematics will follow the line pointed out by constructivism or a view close to it. Yet it is not a sufficient base for writing a textbook of logic at the present moment.

Mostowski abandoned later the idea that constructivism is generally superior over other views, although he saw its advantages in particular cases – for instance, in arithmetic – because it dispenses us with assuming the existence of actual infinity or allows solutions for which nominalism is enough (remember that constructivism and nominalism are closely related to Mostowski's view). In general, Mostowski maintained that finitary, predicative and constructive (these qualifications are related, but not equivalent; we will use "constructive" and "constructivism" as general labels) methods are not sufficient for mathematics (see also [24, pp. 29–32]). However, contrary to many logicians and mathematicians who shared this critical view, Mostowski did not stop at pointing out limitations of constructivism, but very seriously took and examined claims of this program. According to him, constructivism is sometimes more philosophically satisfactory, for example, in the already mentioned case of arithmetic. Applied mathematics is another domain in which constructive approach is very promising. Thus, drawing the exact scope of constructive methods in the framework of classical mathematics constitutes an important task for mathematics as well as for philosophy. This was the background of Mostowski's idea of the degrees of constructivity (see [10,15]) as attached to various mathematical theories (arithmetic of natural numbers, arithmetic of real numbers, theory of real functions or set theory; the axiom that every set is constructible expresses an amount of constructivity, although it is rather weak). A fruitful approach to this question requires a deeper analysis of how classical and constructivist formal schemes are intertranslatable.

Mostowski's works on constructivism entered more deeply into the relation between classical and constructive mathematics than any earlier comparisons of both foundational projects. Mostowski's position can be well-described by a certain distinction introduced by Heyting, namely that of theories of the constructible and constructive theories (in fact, this distinction was motivated by papers of Mostowski and Grzegorczyk delivered at the

Symposium on constructivity in Amsterdam in 1957; see [1,2,15]). A theory of the constructible has three features: (i) a mathematical theory is presupposed in order to define the class of constructible objects; (ii) the notion of constructability is defined, it is not a primitive concept; (iii) a liberty in the choice of a definition of the constructible, although a sufficient correspondence to our intuitive notion of a mathematical construction is required. On the other hand, a constructive theory takes the concept of constructability as primitive and it is obliged to provide its precise characterization, for example, by the postulates generated by intuitionistic logic. Doubtless, Mostowski, strongly influenced by computable analysis of Banach and Mazur (see [6]), developed a theory of the constructible with various degree of constructivity.

Mostowski's understanding of constructivism is well expressed in his following words [15, p. 180]:

> My conception of constructivism will be as naive as possible and will consists in the following. I shall consider theories of real numbers and real functions in which not arbitrary real numbers or real functions are considered but only numbers of functions which belong to a certain class specified in advance. According to the choice of this class, we shall obtain different theories of arithmetic and analysis. Our choice of the initial class will not be arbitrary: we shall try to make the choice so that the elements of the chosen class satisfy certain conditions of calculability or effectivity. We shall start with stringent conditions and then loosen them gradually and we shall see that it is possible in this way to systematize a good deal of older and also of more recent work of constructivists. I shall pay no attention to the way in which the classes just mentioned are defined and shall impose no limitations on methods of proof acceptable in dealing with numbers or functions belonging to these classes. This naive approach to constructivism is certainly objectionable from the constructivist point of view. It does not represent a constructivist development of a branch of mathematics but gives merely a glance of constructivism, so to say, from outside. The value (if any) of such an approach I see in the possibility of reviewing on a common background several of the simplest constructivistic conceptions; but more refined ones an especially those which, like intuitionism, impose restrictions on methods of proof must necessarily be excluded from such a review.

Summarizing Mostowski's attitude toward constructivism, we can say that he looked at this view from the classical point of view. Now it is clear that this understanding of constructivism had to irritate Heyting and provoke him to introducing the mentioned distinction between theories of the constructible and constructive theories. Heyting himself considered constructivism proper (intuitionism) as the position, which accepts only constructive theories. The difference is, then, that whereas Heyting's position was purely constructivistic, Mostowski represented a combination of constructivism and set-theoretical program, which constituted for him a workable basis for mathematically tractable foundations of mathematics. Perhaps a clear distinction between mathematical and philosophical foundations of mathematics is the most important general Mostowski's idea in his approach to foundational problems of formal science.

Another question elaborated by Mostowski concerned the interplay between syntax and semantics, the problem suggested by Gödel's incompleteness theorems and Tarski's theorem on the undefinability of truth; Mostowski was a great expert in these fundamental metamathematical results and refined general semantic method of proofs of them (see [7,14]). From a general and philosophical point of view, he was the first who clearly and mathematically pointed out that semantics requires infinitistic methods, but finitary ones are suitable for syntax. The precise specification of differences between syntactic and semantic formulation of the incompleteness theorems became a by-product of this simple observation, which leads to an important conclusions [17, pp. 42, 50]:

> The interpretation of a language is defined by means of set-theoretical concepts, which gives rise to the close relations between semantics and the set-theoretical, infinitistic philosophy of mathematics; whereas the theory of computability leans toward a more finitistic philosophy. [...]
>
> By way of conclusion, let us try to evaluate Herbrand's and Gentzen's theorems from a more general point of view. There are undoubtedly two opposing trends in the study of the foundations of mathematics: the infinitistic or set-theoretical and the finitistic or arithmetical. Herbrand's and Gentzen's original discoveries belong of course to the second of these trends but the subsequent which has been based on these results has borrowed many ideas from the first. This influence of the set-theoretical approach is clearly visible in Bernays' consistency theorem [if all the axioms are effectively true in a model **M**, the same holds for their logical consequences – R. M., J. W.] in which semantic notions are consciously imitated in finitistic terms. We may say that Herbrand's and Gentzen's methods allow us to make finitistic certain particular cases of set-theoretical constructions.

The phrase "certain particular cases of set theoretical constructions" is here extremely important, because it alludes to limitations of purely syntactic methods as compared with richer semantic procedures.

Mostowski considered also other philosophical aspects of Gödel's incompleteness theorems although he was mainly concerned with their formal shape ([7,14] are among the first expositions of this celebrated results). In particular, he touched the concept of a proof and the idea of a formalization of mathematics. Mostowski declares that he is not going to enter into the discussion of philosophical problems whether questions which are unsolvable today are in fact "essentially undecidable" or not. He sees the source of difficulties here in the fact that we do not have a precise notion of a correct mathematical proof. A notion of a formal proof developed in mathematical logic made it possible to construct and investigate formal systems. It has been believed that such systems encompass the whole of mathematics, i.e., that any intuitively correct mathematical reasoning can be formalized in such systems. But since it is essentially impossible to prove that a given formal system coincides with the intuitive mathematics, hence [14, pp. 3–4]

> there is no immediate connection between the problem of completeness of any proposed formal system and the problem of existence of essentially unsolvable mathematical problems [but] [...] the problem of completeness of formalized systems is [...] important because it makes explicit the degree of difficulty of formalization of intuitive mathematics even if we restrict ourselves to that portion of mathematics which deals with integers. [...] in spite of all efforts of the logicians we are still very far from an exact understanding in what consists the notion of truth in mathematics.

Thus, there is a tension between the research practice of mathematicians and the idea that mathematics can be captured by formalized systems (see [23]). Mostowski says [23, pp. 82, 83, 84] that

> a full formalization of mathematics seems to be nowadays an out-of-date idea. Antinomies in set theory do not frighten any more. Mathematics [...] is being developed not paying any attention to what is happening in its foundations. [...] A mathematical proof is something much more complicated than a simple succession of elementary rules contained in the so called inference rules. [...] Therefore one must necessarily show moderation in stressing the role of logical rules in [mathematical] proofs. [...] The tendency to mechanize mathematical

> reasonings seems to me to be a highly dehumanized activity: as E.L. Post once wrote, the essence of mathematics consists in concepts of truth and meaning.

However, despite these reservations and doubts concerning formalization and its future prospects, Mostowski was definitively optimistic as far as the matter concerns relations between logic and mathematics [23, p. 83]:

> the collaboration of logic and mathematics was fruitful and probably will still bring important results.

In fact it would be rather difficult to expect another attitude by a logician coming from Poland.

Set theory became a favourite field of Mostowski's studies from a philosophical as well as from a mathematical point of view (see [22] as his main contribution to formal set theory). Hence, Mostowski's philosophy of mathematics and his view how the foundations of mathematics should be done, can be additionally illustrated by his remarks about various problems of set theory. He was fully aware that philosophy must enter into set theory and its foundations (see [4, pp. v–vi]). Moreover, studies of the philosophy of set theory are important, because (p. v; these remarks are omitted in English translation of this book, although a general link between philosophy and set theory is indicated):

> There exists so far no comprehensive philosophical discussion of basic assumptions of set theory. The problem whether and to what extent abstract concepts of set theory (and in particular of those parts of it in which sets of very high cardinality are considered) are connected with the basic notions of mathematics being directly connected with the practice has not been clarified so far. Such an analysis is needed because by Cantor, the inventor of set theory, basic notions of this theory were encompassed by a certain mysticism.

One of the most important questions of set theory is the problem on which axioms should set theory be based. There is no absolute freedom in the choice of axioms, but one should choose axioms that guarantee that the theory based on them will have [4, p. v]:

> an essential scientific value, i.e., will be able to serve in the process of getting known the material world, either directly or indirectly via other domains of mathematics for which it will be a tool.

Explaining the foundations of set theory is a challenging task due to its importance for the foundations and the philosophy of mathematics [4, p. vii]:

> The great importance of mathematics as a *tool* for other mathematical fields, including branches of mathematics connected directly with applications, presently prevails, as it seems to be, the importance of investigations in set theory itself.
>
> Investigations on the foundations of set theory play also a great role in the general foundations of mathematics. The analysis of such concepts as consistency of axioms, their independence, categoricity, completeness, effectiveness of proofs – are closely connected with set theory.
>
> In this domain the influence of set theory can be especially strongly seen. In particular, thanks to the definition of a finite set and to the introduction of cardinals, the arithmetic of natural numbers could be founded on a firm basis. Simultaneously new problems connected with the general concept of an infinite set has been established and precisely formulated. This concept has no mystical character any more as it was the case through ages.

As it is to be expected a special attention was paid by Mostowski to controversial axioms and results as the axiom of choice and the continuum hypothesis. He (Kuratowski also shared this attitude) proceeded in a way characteristic for Polish mathematicians, that is, by separating the philosophy behind the axiom of choice on the one hand and its mathematical content and its role in mathematics on the other. This approach was initiated in [27] and summarized in the following manner ([29, p. 95]; this remark was constantly repeated by Sierpiński since the 1920s, see [28, pp. 102–103, for instance]):

> Still, apart from our personal inclination to accept the axiom of choice, we must take into consideration, in any case, its role in the set theory and in the calculus. On the other hand, since the axiom of choice has been questioned by some mathematicians, it is important to know which theorems are proved with its aid and to realize the exact point at which the proof has been based on the axiom of choice; for it has frequently happened that various authors have made use of the axiom of choice in their proofs without being aware of it. And after all, even if no-one questioned the axiom of choice, it would not be without interest to investigate which proofs are based on it and which theorems are proved without its aid – this, as we know, is also done with regards to other axioms.

Mostowski came back to philosophical problems of set theory also later, in particular, when he summarized the place of set theory in mathematics and science (see [24]) or commented Cohen's results on the independence of the axiom of choice and the continuum hypothesis (see [16,18,19,21]). In this context, he considered the following questions: What are sets and how their laws can be discovered? What are in particular sets of reals? Can every set be determined by defining the property of its elements (and consequently, is it identical with this property) or is it a certain abstract object which exists independently of our mental constructions? He concludes [21, p. 177]:

> Unfortunately the problem of truth in mathematics is not simple. Repeat: If sets existed in the same sense as physical objects then we could expect that the truth or falsity of the continuum hypothesis would be ultimately discovered. But if sets are only our own mental construction then the answer to the question whether the continuum hypothesis is true or false can depend on what constructions we will accept as allowed.

Since (see also earlier remarks on the concept of truth in the light of Gödel's results) – according to Mostowski – nothing can be said about the admissibility of platonism in set theory, one does not know whether the question about the truth or falsity of the continuum hypothesis has any sense. On the other hand formal problems concerning its consistency and independence are reasonable and interesting to high degree. The independence results of Cohen (supplementing the earlier results on consistency due to Gödel) do not solve the problem of truth in set theory. Moreover, since the continuum hypothesis and the axiom of choice cannot be decided (on the basis of accepted axioms of set theory), this suggests (see [21, p. 176]) one of the most important arguments against mathematical Platonism. The situation is analogous to the situation in geometry: axiomatic set theory is now in the same situation as axiomatic geometry was after works of Klein and Poincaré which indicated the real meaning of the problem of truth of the parallel axiom. After the results of Cohen various possible but mutually inconsistent axiomatic set theories can be constructed. If this will be the case, then [21, p. 182]

> we shall be forced to admit that in the match between platonism and formalism the latter has again scored one point.

In fact, Mostowski was inclined to a stronger conclusion (see [18,19]). He says that the complicated and not fully clear nature of the concept of set and consequently the possibility of various axiomatizations of set theory imply that – despite of great mathematical and philosophical importance of set theory – there are no chances that it will become a central mathematical discipline. On the one hand most (if not all) mathematical notions can be interpreted within set theory (this remarkable fact requires an explanation), but, on the other hand, we encounter essentially different concepts of set, which are equally suitable as basic for intuitive set theory. Mostowski concludes [18, pp. 94–95]:

> Of course if there are a multitude of set theories then none of them can claim the central place in mathematics. Only their common part could claim such a position; but it is debatable whether this common part will contain all the axioms needed for a reduction of mathematics to set theory.

This concurs with Mostowski's doubts towards the axiomatization of set theory formulated in one of his earlier papers ((a) [12, p. 19]; (b) [24, pp. 28–29]):

> (a) A particularly perturbing fact which calls for explanation is that recently various new axioms have been added to the system of axioms of the theory of sets or the formulations of axioms have been altered; in consequence we have at present to choose between a great many essentially different systems of axioms of the set theory, yet there are no criteria indicating the proper choice among all these numerous systems. He was convinced that the ultimate formulation of axioms of set theory should be proceeded by a discussion of the fundamental assumptions of this theory.
>
> (b) [...] the mere incompleteness of Z-F is not an alarming symptom by itself. What is disturbing is our ignorance of where to look for additional information which would permit us to solve problems which seem very simple and natural but which are nevertheless left open by the axioms of Z-F. We come here very close to fundamental problems of the philosophy of mathematics whose basic question is: what is mathematics about? A formalist would say that it is about nothing; that it is just a game played with arbitrarily selected axioms and rules of proof. The incompleteness of Z-F is thus of no concern for a formalist. Platonists on the contrary believe in the 'objective existence' of mathematical objects. A set-theoretical Platonist believe therefore that we should continue to think more about sets and experiment with them until we finally discover new axioms which, added to Z-F, will permit us to solve all outstanding problems. [...] Whatever the final outcome of the fight between these two opposing trends will be, it is obvious that we should concentrate on the study of concepts which seem perfectly clear and perspicuous to us. In Cantor's time the concept of an arbitrary set seemed to be a very clear concept, but the antinomies proved that this was not so. Today, this concept has been replaced by that of an arbitrary subset of a given set. In addition, the belief that all subsets of a given set form a set is almost universally accepted. However, it is by no means true that these views are shared by all mathematicians. Even Gödel himself, who [...] should be counted among the Platonists, has once expressed the view that the concept of an arbitrary subset of a given set is in need of clarification. [...]. The present writer believes (although he cannot present convincing evidence to support this view) that it is in this direction where the future of set theory lies.

The last quotation with typical reservations and caution expressed by Mostowski suggests that one should return once again to his picture of relations between philosophy and mathematics. Considering the general question whether mathematical objects can be treated as fully defined by appropriate systems of axioms (arithmetic of natural numbers is a good example), Mostowski states that the decision belongs not to mathematics but to philosophy and concludes [12, pp. 15–16, 41–42]:

> The only consistent standpoint, confirming to common sense as well as to mathematical usage, is that according to which the source and ultimate "raison d'être" of the notion of number,

both natural and real, is experience and practical applicability. The same refers to notions of the theory of sets, provided we consider them within rather narrow limits, sufficient for the requirements of the classical branches of mathematics.

If we adopt this point of view, we are bound to draw the conclusion that there exist only one arithmetic of natural numbers, one arithmetic of real numbers and one theory of sets; therefore it is not possible to define these branches of mathematics by systems of axioms which are supposed to establish once and for all their scope and their content.

Systems of axioms play an important role in those theories: they systematize a certain fragment of these theories, namely that which includes out present knowledge; they often facilitate the exposition of a theory and are therefore of didactical value.

Incompleteness results of Gödel showing that natural numbers cannot be fully characterized by a system of axioms and that there exist non-isomorphic models of arithmetic should not lead to pessimistic conclusions because they provide a tool to obtain several independence results.

Similar, and even more difficult problems, arise in the foundations of set theory. Here the main difficulty is the indefiniteness of the notion of an arbitrary set as well as the status of such axioms as the axiom of choice.

[...]

The problem of the foundations of mathematics is not a single concrete mathematical problem which, once solved, may be forgotten. The considerations regarding the foundations of science are just as old as science itself and mathematics is no exception to this rule. For many centuries the essence and content of mathematics have been, and probably will remain also in future, an object of considerations for philosophers. In the course of time mathematics changes and this also necessitates a change of views on its foundations.

[...] An explanation of the nature of mathematics does not belong to mathematics, but to philosophy, and it is possible only within the limits of a broadly conceived philosophical view treating mathematics not as detached from other sciences but taking into account its being rooted in natural sciences, its applications, its associations with other sciences and, finally, its history.

* * *

What conclusions can be drawn from the above analysis of Mostowski's philosophical remarks? First of all one should stress that he was aware of philosophical problems connected with mathematics and its foundations and of their importance and meaning. On the other hand he tried to avoid (with few exceptions) any definite philosophical declarations concentrating instead on strongly mathematical and technical side of issues. If it was necessary then some general philosophical declarations have been made (but, in fact, unwillingly). He was aware of the meaning of results obtained in the foundations of mathematics by mathematical methods for the philosophy of mathematics but simultaneously was convinced that those results cannot give definite solutions to problems of the philosophical nature. Therefore he rather presented various possible solutions instead of making any concrete declarations. Philosophical problems and possible solutions to them were discussed by him on the margin of proper metamathematical and foundational studies, in introductory remarks only and – what is very important – did not influenced the latter. Mostowski strongly avoided philosophical comments and remarks in technical papers. Philosophical perspective on the one hand and metamathematical and foundational one on the other were strictly separated by him. Though some of his results were – as one can suppose – inspired and motivated by philosophical considerations (e.g., independence of definitions of finiteness, constructions that led to the so called today Kleene-Mostowski hierarchy, constructions of models with automorphisms) but he never wrote about that and never formulated them explicitly concentrating on mathematical and mathematical studies. Hence one can only formulate here some hypotheses and conjectures which can be neither confirmed nor rejected. Although Mostowski did not develop a new "ism" in the philosophy of mathematics, his works essentially contributed to this field and can be regarded as paradigmatic cases of a very reasonable interplay of mathematical and philosophical ideas. And he was a perfect (perhaps even the most perfect) example of the Polish attitude to the foundations and philosophy of mathematics.

References

[1] A. Grzegorczyk, Some Approaches to Constructive Analysis, in: *Constructivity in Mathematics. Proceedings of the Colloquium held in Amsterdam, 1957*, A. Heyting, ed., North-Holland Publishing Company, Amsterdam, 1959, pp. 43–61.

[2] A. Heyting, Some Remarks on Intuitionism, in: *Constructivity in Mathematics. Proceedings of the Colloquium held in Amsterdam, 1957*, A. Heyting, ed., North-Holland Publishing Company, Amsterdam, 1957, pp. 69–71.

[3] J. Kotarbińska, Głos w dyskusji [A Contribution to Discussion], *Studia Filozoficzne* **5**(222) (1984), 69–93.

[4] K. Kuratowski and A. Mostowski, *Teoria Mnogości*, Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1952; second edition, Państwowe Wydawnictwo Naukowe, Warszawa, 1966; English translation: *Set Theory*, Polish Scientific Publishers, Warszawa, and North-Holland Publishing Company, Amsterdam, 1967.

[5] P. Mancosu, Harvard 1940–1941: Tarski, Carnap and Quine on a Finitistic Language of Mathematics and Science, *History and Philosophy of Logic* **26** (2005), 327–357.

[6] S. Mazur, *Computable Analysis*, Państwowe Wydawnictwo Naukowe, Warszawa, 1963.

[7] A. Mostowski, O zdaniach nierozstrzygalnych w sformalizowanych systemach matematyki [On Sentences Undecidable in Formalized Systems of Mathematics], *Kwartalnik Filozoficzny* **16** (1946), 223–277.

[8] A. Mostowski, *Logika matematyczna. Kurs uniwersytecki* [Mathematical Logic. University Course], Monografie Matematyczne, Warszawa–Wrocław, 1948.

[9] A. Mostowski, A Classification of Logical Systems, *Studia Philosophica* **4** (1949–1950) 237–274 (published in 1951); reprinted in: [25, vol. II, pp. 154–191].

[10] A. Mostowski, O tzw. konstruktywnych poglądach w dziedzinie podstaw matematyki [On so Called Constructive Views in the Foundations of Mathematics], *Myśl Filozoficzna* **1**(7) (1953), 230–241.

[11] A. Mostowski, Podstawy matematyki na VIII Zjeździe Matematyków Polskich [Foundations of Mathematics at the 8th Congress of Polish Mathematicians], *Myśl Filozoficzna* **2**(12) (1954), 328–330.

[12] A. Mostowski, The Present State of Investigations of the Foundations of Mathematics, *Rozprawy Matematyczne* **9** (1955), 1–48 [co-authors: A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa and R. Sikorski], Państwowe Wydawnictwo Naukowe, Warszawa.

[13] A. Mostowski, Współczesny stan badań nad podstawami matematyki [The Present State of Investigations on the Foundations of Mathematics], *Prace Matematyczne* **1** (1955), 13–55.

[14] A. Mostowski, *Sentences Undecidable in Formalized Arithmetic. An Exposition of the Theory of Kurt Gödel*, North-Holland Publishing Company, Amsterdam, 1957.

[15] A. Mostowski, On Various Degrees of Constructivism, in: *Constructivity in Mathematics. Proceedings of the Colloquium held in Amsterdam, 1957*, A. Heyting, ed., North-Holland Publishing Company, Amsterdam, 1959, pp. 178–194; reprinted in: [25], vol. II, 1979, pp. 359–375.

[16] A. Mostowski, Widerspruchsfreiheit und Unabhängigkeit der Kontinuumhypothese, *Elemente der Mathematik* **19** (1964), 121–125.

[17] A. Mostowski, Thirty Years of Foundational Studies. Lectures on the Development of Mathematical Logic and the Study of the Foundations of Mathematics in 1930–1964, *Societas Philosophical Fennica*, Helsinki, 1965; reprinted in: [25], vol. I, 1979, pp. 1–76.

[18] A. Mostowski, Recent Results in Set Theory, in: *Problems in the Philosophy of Mathematics*, I. Lakatos, ed., North-Holland Publishing Company, Amsterdam, 1967, pp. 82–96 and 105–108.

[19] A. Mostowski, O niektórych nowych wynikach meta-matematycznych dotyczących teorii mnogości [On Some New Metamathematical Results Concerning Set Theory], *Studia Logica* **20** (1967), 99–116.

[20] A. Mostowski, Tarski Alfred, in: *The Encyclopedia of Philosophy*, P. Edwards, ed., vol. 8, The Macmillan Comp. & the Free Press, New York, Collie-Macmillan Ltd. London, 1967, pp. 77–81.

[21] A. Mostowski, Niesprzeczność i niezależność hipotezy kontinuum [Consistency and Independence of the Continuum Hypothesis], *Roczniki Polskiego Towarzystwa Matematycznego, Seria II: Wiadomości Matematyczne* **10** (1968), 175–182.

[22] A. Mostowski, *Constructible Sets with Application*s, PWN-Polish Scientific Publishers, Warszawa, and North-Holland Publ. Comp., Amsterdam, 1969.

[23] A. Mostowski, Matematyka a logika. Refleksje przy lekturze książki A. Grzegorczyka "Zarys arytmetyki teoretycznej" wraz z próba recenzji [Mathematics vs. Logic. Impressions of the Study of "Outline of Theoretical Arithmetic" by A. Grzegorczyk with an Attempt of a Review], *Roczniki Polskiego Towarzystwa Matematycznego, Seria II: Wiadomości Matematyczne* **15** (1972), 79–89.

[24] A. Mostowski, Sets, in: *Scientific Thought. Some Underlying Concepts, Methods and Procedures*, Mouton/Unesco, The Hague, 1972, pp. 1–34.

[25] A. Mostowski, 1979, *Foundational Studies. Selected Works*, vols. I–II, North-Holland Publishing Company, Amsterdam.

[26] R. Murawski, Philosophical Reflection on Mathematics in Poland in the Interwar Period, *Annals of Pure and Applied Logic* **127** (2004), 325–337.

[27] W. Sierpiński, L'axiome de M. Zermelo et son role dans la théorie des ensembles et l'analyse, *Bulletin International de l'Académie des Sciences et des Lettres de Cracovie*, classe de Sciences mathématiques et naturelles, Série A: sciences mathématiques, 1918, pp. 97–152; reprinted in: W. Sierpiński, *Oeuvres choisies*, vol. 2, Państwowe Wydawnictwo Naukowe, Warszawa 1975, 208–255.

[28] W. Sierpiński, *Zarys Teorii Mnogości* (An Outline of Set Theory), Wydawnictwo Kasy im. Mianowskiego, Warszawa, 1923.

[29] W. Sierpiński, *Cardinal and Ordinal Numbers*, Polish Scientific Publishers, Warszawa, 1965.

[30] A. Tarski, Contribution to the Discussion of P. Bernays '*Zur Beurteilung der Situation in der beweistheoretischen Forschung*', *Revue Internationale de Philosophie* **8** (1954), 16–20; reprinted in: A. Tarski, *Collected Papers*, vol. IV, Birkhäuser, Basel, 1986, pp. 711–714.

[31] J. Woleński, *Logic and Philosophy in the Lvov-Warsaw School*, Kluwer Academic Publishers, Dordrecht, 1989.

Section 3

Bibliography of Andrzej Mostowski

This page intentionally left blank

Bibliography of Andrzej Mostowski

Jan ZYGMUNT
Department of Logic and Methodology of Sciences, University of Wrocław
ul. Koszarowa 3, 51-149 Wrocław, Poland
logika@uni.wroc.pl

This is an attempt to set down a complete bibliography of the published works of Andrzej Mostowski, as there has been to date no such attempt[1]. Completeness in this context means that we aim to list all published works by Mostowski and to furnish full and detailed bibliographical information about them.

The bibliography is divided into eight sections: I—papers, II—abstracts, III—books and monographs, IV—problems, V—contributions to discussions, VI—reviews, VII—editorial works, and VIII—miscellanea.

Within each section items are arranged in chronological order and are indexed by a bibliographical sign consisting of two digits and possibly one or two letters, all in square brackets. Digits refer to the year of publication or the year of volume. By means of letters "a", "b", "c", ... an order of published items is established in a given year. The superscript letters refer to sections: abstracts are indicated by a superscript "a", books and monographs by a superscript "b", problems by a superscript "p", contribution to discussion by a superscript "c", reviews by a superscript "r", editorial works by a superscript "e" , and miscellanea by a superscript "m". For example, the first three papers published in 1957 are denoted by [57], [57a], and [57b]; the first two abstracts in 1955 by [55^a] and [55^aa]; the second review in 1957 by [57^ra], and so on.

In general, reprints and translations have not been listed according to the overall chronology, but rather appear directly after the original publication (in chronological order) and are numbered (1), (2), etc. Thus, e.g. [38a](1) refers to the first reprint or translation of [38a]. Some exceptions have been made especially in the section on books and monographs; for substantial and historical reasons three Polish editions of ***Teoria mnogości*** and their two English translations have been listed separately.

Titles of articles are written in italics, titles of books and names of journals in boldface italics, and names of series of books in uninflected typeface. Polish or Russian titles are followed by their English translations in angular parentheses.

Where suitable, additional information has been added in brackets at the end of a citation indicating cross-references, abstracts or reviews. When we refer to a review published in ***The Journal of Symbolic Logic, Mathematical Reviews, Zentralblatt für***

[1] While writing this bibliography we have of course consulted two earlier bibliographies of Mostowski, one of which appeared in the book ***Set Theory and Hierarchy Theory***, Lecture Notes in Mathematics, vol. 537, 1976 (cf. item [76] below) and a second, compiled by Wiktor Marek, which was published in: A. Mostowski, ***Foundational Studies. Selected Works***, volume 1, pp. XI–XIX (cf. [79^b] below). Our bibliography is patterned after Steven Givant's bibliography of Alfred Tarski which appeared in ***The Journal of Symbolic Logic***, vol. 51 no 4 (1986), pp. 913–941.

Mathematik und ihre Grenzgebiete, or ***Jahrbuch über die Fortschritte der Mathematik*** we use abbreviations JSL, MR, ZGM, and JFM respectively. In other cases full names of journals are given. Names of authors of reviews have been inserted parenthetically.

I. Papers

[37] *Abzählbare Boolesche Körper und ihre Anwendung auf die allgemeine Metamathematik*, ***Fundamenta Mathematicae***, vol. 29 (1937), pp. 34–53. [JSL 3, p. 47 (G. Birkhoff); ZMG 16, p. 337 (A. Schmidt); JFM 63.0829.03 (Th. Skolem).]

(1) *Countable Boolean fields and their application to general metamathematics*, in [79$^{\mathrm{b}}$a], pp. 1–17. (English translation of [37] by M. Mączyński.)

[38] *O niezależności definicji skończoności w systemie logiki* < On the independence of the definitions of finiteness in a system of logic >, ***Dodatek do Rocznika Polskiego Towarzystwa Matematycznego*** (supplement to the ***Annales de la Société Polonaise de Mathématique***), vol. 11 (1938), pp. 1–54. [JSL 3, pp. 115–116 (A. Tarski).]

(1) *On the independence of definitions of finiteness in a system of logic*, in [79$^{\mathrm{b}}$a], pp. 18–67. (English translation of [38] by A.H. Lachlan.)

[38a] *Über gewisse universelle Relationen*, ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de mathématique***), vol. 17 (1938), pp. 117–118.

(1) *On some universal relations*, in [79$^{\mathrm{b}}$a], pp. 68–69. (English translation of [38a] by M.J. Mączyński.)

[38b] *Über den Begriff einer endlichen Menge*, ***Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego***, Wydział III Nauk Matematyczno-fizycznych (= ***Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie***, Classe III), vol. 31 (1938), pp. 13–20. [JSL 4, p. 30 (A.A. Fraenkel); ZMG 19, p. 295 (A. Schmidt); JFM 64.0932.02 (Th. Skolem).]

[38c] *Über die Unabhängigkeit des Auswahlaxioms und einiger seiner Folgerungen* (with A. Lindenbaum), ***Sprawozdania z Posiedzeń Towarzystwa Naukowego Warszawskiego***, Wydział III Nauk Matematyczno-fizycznych (= ***Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie***, Classe III), vol. 31 (1938), pp. 27–32. [JSL 4, pp. 30–31 (A.A. Fraenkel); ZMG 19, p. 295 (A. Schmidt); JFM 64.0932.01 (Th. Skolem).]

(1) *On the independence of the axiom of choice and some of its consequences*, [79$^{\mathrm{b}}$a], pp. 70–74 (English translation of [38c] by M.J. Mączyński).

[39] *Boolesche Ringe mit geordneter Basis* (with A. Tarski), ***Fundamenta Mathematicae***, vol. 32 (1939), pp. 69–86. [JSL 4, pp. 124–125 (H. Wallman); ZMG 21, p. 109 (B. Pospíšil); JFM 65.0086.03 (Th. Skolem).]

(1) *Boolean rings with ordered basis*, in [79$^{\mathrm{b}}$a], pp. 75–91. (English translation of [39] by M.J. Mączyński.)

(2) Reprinted in ***Alfred Tarski, Collected Papers. Volume 2, 1935–1944***, ed. by S.R. Givant and R.N. McKenzie, Birkhäuser Verlag, Basel 1986, pp. 531–548.

[39a] *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, ***Fundamenta Mathematicae***, vol. 32 (1939), pp. 201–252. [JSL 4, pp. 129–130 (A.A. Bennett); ZMG 22, pp. 120–121 (L. Egyed); JFM 65.1164.01 (Th. Skolem).]

(1) *On the independence of the well-ordering theorem from the ordering principle*, in [79^{b}], pp. 290–338. (English translation of [39a] by M.J. Mączyński.)

[39b] *Bemerkungen zum Begriff der inhaltlichen Widerspruchsfreiheit*, ***The Journal of Symbolic Logic***, vol. 4 (1939), pp. 113–114. [JSL 5, pp. 115–116 (B. Rosser); MR 1, p. 34 (O. Frink); ZMG 21, pp. 385–386 (H. Hermes); JFM 65.0021.01 (Th. Skolem).]

[45] *Remarques sur la note de M. Sierpiński « Un théorème sur les familles d'ensembles et ses applications »*, ***Fundamenta Mathematicae***, vol. 33 (1945), pp. 7–8. [MR 8, p. 17 (J. Todd); ZMG 60, pp. 127–128 (G. Kurepa).]

[45a] *Axiom of choice for finite sets*, ***Fundamenta Mathematicae***, vol. 33 (1945), pp. 137–168. [JSL 13, pp. 45–46 (L. Kalmár); MR 8, p. 3 (R.M. Martin); ZMG 63, p. 41 (registered).]

[46] *O zdaniach nierozstrzygalnych w sformalizowanych systemach matematyki* < On undecidable propositions in formalized systems of mathematics >, ***Kwartalnik Filozoficzny*** (= ***Revue trimestrielle de philosophie***), vol. 16 (1946), pp. 223–277. [JSL 12, p. 60 (S. Ulam); MR 9, p. 129 (S. Ulam); ZMG 63, p. 41 (registered).]

(1) *On undecidable sentences in formalized mathematical systems*, ***American Mathematical Society Translations***, series 2 vol. 29 (1963), pp. 1–50. (English translation of [46] by A. Robinson.) [JSL 30, p. 111 (registered); MR 27 # 3546 (Editors).]

[46a] *Zarys teorii Galois* < An outline of Galois theory >, supplement to W. SIERPIŃSKI, ***Zasady algebry wyższej*** < Principles of higher algebra >, Monografie Matematyczne, vol. 11, Z subwencji Wydzialu Nauki Ministerstwa Oświaty, Warszawa – Wrocław 1946, pp. 371–428. [MR 8, p. 49 (S. Eilenberg); ZMG 60, p. 30 (registered).]

(1) supplement to the second corrected edition of the same, Monografie Matematyczne, vol. 11, Polskie Towarzystwo Matematyczne, Warszawa–Wrocław 1951, pp. 371–429. [MR 13, p. 311 (registered); ZMG 42, pp. 248–249 (G. Kurepa).]

[47] *On definable sets of positive integers*, ***Fundamenta Mathematicae***, vol. 34 (1947), pp. 81–112. (See [48b].) [JSL 13, pp. 112–113 (D. Nelson); MR 9, pp. 129–130 (M.H.A. Newman); ZMG 31, p. 194 (G. Hasenjaeger).]

[47a] *On absolute properties of relations*, ***The Journal of Symbolic Logic***, vol. 12 (1947), pp. 33–42. (For corrections see [62^{b}], p. 4, footnote 4.) [JSL 13, pp. 46–48 (J.G. Kemeny); MR 10, pp. 93–94 (R.M. Martin); ZMG 29, pp. 100–101 (G. Hasenjaeger).]

[48] *On the principle of dependent choices*, ***Fundamenta Mathematicae***, vol. 35 (1948), pp. 127–130. [JSL 14, p. 257 (W. Ackermann); MR 10, p. 688 (M.H.A. Newman); ZMG 31, p. 289 (W. Ackermann).]

[48a] *Proofs of non-deducibility in intuitionistic functional calculus*, ***The Journal of Symbolic Logic***, vol. 13 (1948), pp. 204–207. [JSL 14, p. 137 (Th. Skolem); MR 10, p. 421 (A. Heyting); ZMG 31, pp. 193–194 (G. Hasenjaeger).]

[48b] *On a set of integers not definable by means of one-quantifier predicates*, ***Annales de la Société Polonaise de Mathématique***, vol. 21 (1948), pp. 114–119. (A sequel to [47].) [JSL 15, p. 135 (M. Davis); MR 10, p. 175 (M.H.A. Newman); ZMG 31, p. 339 (G. Hasenjaeger).]

[48c] *Un théorème sur les nombres* $\cos 2\pi k/n$, ***Colloquium Mathematicum***, vol. 1 (1948), pp. 195–196. [MR 10, p. 432 (G. Whaples).]

[49] *Sur l'interprétation géométrique et topologique des notions logiques*, ***Actes du Xème Congrès International de Philosophie (Amsterdam, 11–18 août 1948)***, vol. 1, fasc. II, North Holland Publishing Company, Amsterdam 1949, pp. 767–769. (See [53c].) [JSL 14, p. 184 (D. Nelson); MR 10, p. 423 (registered); ZMG 31, p. 193 (G. Hasenjaeger).]

[49a] *An undecidable arithmetical statement*, ***Fundamenta Mathematicae***, vol. 36 (1949), pp. 143–164. [JSL 19, pp.118–119 (M. Davis); MR 12, p. 2 (A. Heyting); ZMG 39, p. 8 (G. Hasenjaeger).]

[49b] *La vie et l'oeuvre de S. Dickstein*, ***Prace Matematyczno-Fizyczne***, vol. 47 (1949), pp. VII–XII + 1 plate. [MR 11, p. 573 (registered); ZMG 38, p. 148 (registered).]

(1) *Życie i dzieło Samuela Dicksteina*, ***Wiadomości Matematyczne***, vol. 22 (1979), pp. 285–290. (Polish translation of [49b] by R. Duda, with several notes added by the translator.)

[50] *Kilka refleksji na temat zadań logicznych z "Matematyki"* < *Some reflections on the subject of the logical exercises in* "*Matematyka*" >, ***Matematyka***, vol. 3 no. 3 (10) (1950), pp. 6–11. [JSL 16, p. 228 (J. Kalicki).]

[50a] *A classification of logical systems*, ***Studia Philosophica*** (Poznań), vol. 4 (for 1949–50, pub. 1951; received by the editors on July 26, 1949), pp. 237–274. [JSL 17, pp. 274–275 (D. Nelson); ZMG 45, p. 294 (registered).]

[50b] *Some impredicative definitions in the axiomatic set-theory*, ***Fundamenta Mathematicae***, vol. 37 (for 1950, pub.1951), pp. 111–124. (For corrections see [51c].) [JSL 16, pp. 274–275 (Th. Skolem); MR 12, p. 791 (A. Heyting); ZMG 39, p. 276 (W. Ackermann).]

[51] *On the rules of proof in the pure functional calculus of the first order*, ***The Journal of Symbolic Logic***, vol. 16 (1951), pp. 107–111. [JSL 16, p. 272 (G.D.W. Berry); MR 13, p. 3 (I.L. Novak); ZMG 43, pp. 9–10 (H. Scholz).]

[51a] *Sur un problème de la théorie des groupes et son rapport à la topologie* (with K. Kuratowski), ***Colloquium Mathematicum***, vol. 2 (1951), pp. 212–215. [MR 14, p. 131 (J.L. Tits); ZMG 45, pp. 169–170 (V.S. Krishnan).]

[51b] *Groups connected with Boolean algebras* (*Partial solution of the Problem* P 92), ***Colloquium Mathematicum***, vol. 2 (1951), pp. 216–219. [MR 14, p. 131 (J.L. Tits); ZMG 45, pp. 6–7 (V.S. Krishnan).]

[51c] *Correction to the paper "Some impredicative definitions in the axiomatic set-theory" by Andrzej Mostowski*, ***Fundamenta Mathematicae***, vol. 38 (for 1951, pub. 1952), p. 238. [JSL 18, p. 343 (Th. Skolem); MR 13, p. 715 (registered).]

[52] *On direct products of theories*, ***The Journal of Symbolic Logic***, vol. 17 (1952), pp. 1–31, and Errata, p. iv. [JSL 17, pp. 203–204 (I.L. Novak); MR 13, p. 897 (P. Lorenzen); ZMG 47, pp. 7–8 (J.C. Shepherdson).]

[52a] *On models of axiomatic systems*, ***Fundamenta Mathematicae***, vol. 39 (for 1952, pub.1953), pp. 133–158. [JSL 19, pp. 220–221 (R. McNaughton); MR 14, p. 938 (I. Novak Gál); ZMG 53, pp. 201–202 (G. Hasenjaeger).]

[53] *On a system of axioms which has no recursively enumerable arithmetic model*, ***Fundamenta Mathematicae***, vol. 40 (1953), pp. 56–61. [JSL 23, p. 45 (G. Hasenjaeger); MR 15, p. 667 (I. Novak Gál); ZMG 53, pp. 3–4 (G. Hasenjaeger).]

[53a] *A lemma concerning recursive functions and its applications*, ***Bulletin de l'Académie Polonaise des Sciences***, Classe 3, vol. 1 (1953), pp. 277–280. [JSL 19, pp. 299–300 (R. Péter); MR 15, p. 667 (P. Lorenzen); ZMG 52, pp. 12–13 (W. Markwald).]

(1) *Lémma po rékursivnym funkciám i éé priménéniá*, ***Bullétén' Pol'skoj Akadémii Nauk***, Otd. 3, vol. 1 (1953), pp. 275–279. (Russian translation, in the Russian version of the journal.) [JSL 20, p. 199 (registered).]

[53b] *O tzw. konstruktywnych poglądach w dziedzinie podstaw matematyki < On so called constructive views in the domain of the foundations of mathematics >*, ***Myśl Filozoficzna***, vol. 1 (7) (1953), pp. 230–241.

[53c] *O geometrycznej interpretacji wyrażeń logicznych < A geometrical interpretation of logical formulae* > (with H. Rasiowa), ***Studia Logica***, vol. 1 (1953), pp. 254–275. (With Russian and English summaries. An extended version of [49].) [MR 19, p. 378 (J. Łoś); ZMG 59, pp. 14–15 (Z. Lis).]

[53d] *Undecidability and essential undecidability in arithmetic* (with R.M. Robinson and A. Tarski), part II of ***Undecidable Theories***, by A. Tarski in collaboration with A. Mostowski and R.M. Robinson. Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam 1953, pp. 39–74. (See [53^{b}].)

[54] *Der gegenwärtige Stand der Grundlagenforschung in der Mathematik* (with A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa, R. Sikorski), ***Die Hauptreferate des 8. Polnischen Mathematikerkongresses vom 6. bis 12. September 1953 in Warschau***, Deutscher Verlag der Wissenschaften, Berlin 1954, pp. 11–44. [JSL 21, 372–373 (L. Henkin); MR 16, p. 552 (Editors); ZMG 57, pp. 243–245 (H. Scholz).]

(1) *Sovréménnoé sostoánié isslédovanij po osnovaniám matématiki* (with participation of A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa, R. Sikorski), ***Uspéhi matématičéskih nauk***, vol. 9 no. 3 (1954), pp. 3–38. (Russian translation of [54].) [MR 16, p. 552 (H.B. Curry); ZMG 59, p. 12 (Editors).]

(2) ***The present state of investigations on the foundations of mathematics*** (in collaboration with A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa, and R. Sikorski), Rozprawy Matematyczne (= Dissertationes Mathematicae), vol. 9, (Polska Akademia Nauk, Instytut Matematyczny), Państwowe Wydawnictwo Naukowe, Warszawa 1955, 48 pp. (English translation of [54].) [MR 16, p. 552 (Editors); ZMG 67, p. 248 (registered).]

(3) *Współczesny stan badań nad podstawami matematyki* < *The present state of investigations on the foundations of mathematics* > (with participation of A. Grzegorczyk, S. Jaśkowski, J. Łoś, S. Mazur, H. Rasiowa, and R. Sikorski), ***Prace Matematyczne***, vol. 1 (1955), pp. 13–55. (An expanded version of a report read before the 8th Congress of Polish mathematicians in September 1953; see also [54$^{\mathrm{m}}$], [54$^{\mathrm{m}}$a].) [MR 16, p. 553 (Editors); ZMG 67, p. 248 (registered).]

(4) A Chinese translation of (2), ***Advancement in Mathematics***, vol. 3 (1957), pp. 127–163. [MR 20 # 3784 (Editors).]

(5) *Stadiul actual al cercetărilor de fundamentele matematicii*, ***Logică şi filozofie, orientări în logica modernă şi fundamentele matematicii***, ed. by M. Tirnoveanu and Gh. Enescu, Editura politică, Bucharest 1966, pp. 463–513. (Roumanian translation, with minor changes, of (2).) [JSL 40, p. 286 (registered).]

[55] *A formula with no recursively enumerable model*, ***Fundamenta Mathematicae***, vol. 42 (1955), pp. 125–140. [JSL 23, p. 45 (G. Hasenjaeger); MR 17, p. 225 (G. Kreisel); ZMG 67, pp. 251–252 (G. Hasenjaeger).]

[55a] *Examples of sets definable by means of two and three quantifiers*, ***Fundamenta Mathematicae***, vol. 42 (1955), pp. 259–270. [JSL 25, pp. 364–365 (G. Hasenjaeger); MR 17, pp. 815–816 (P.C. Gilmore); ZMG 66, p. 262 (J.C. Shepherdson).]

[55b] *Contributions to the theory of definable sets and functions*, ***Fundamenta Mathematicae***, vol. 42 (1955), pp. 271–275. [JSL 25, p. 365 (G. Hasenjaeger); MR 17, p. 816 (P.C. Gilmore); ZMG 67, p. 252 (G. Hasenjaeger).]

[55c] *Eine Verallgemeinerung eines Satzes von M. Deuring*, ***Acta Scientarum Mathematicarum*** (Szeged), vol. 16 (1955), pp. 197–203. [MR 17, p. 454 (R. Baer); ZMG 66, p. 290 (F. Kasch).]

(1) *A generalization of a theorem of M. Deuring*, in [79$^{\mathrm{b}}$a], pp. 298–305. (English translation of [55c] by M.J. Mączyński.)

[55d] *Wyznaczanie stopni niektórych liczb algebraicznych* < Determination of the degree of certain algebraic numbers >, ***Prace Matematyczne***, vol. 1 (1955), pp. 239–252. (With Russian and English summaries.) [MR 18 p. 196 (M. Krasner); ZMG 68, p. 32 (L. Holzer).]

[56] *A proof of Herbrand's theorem* (with J. Łoś and H. Rasiowa), ***Journal de mathématique pures et appliquées***, ser. 9, vol. 35 (1956), pp. 19–24. (For addenda see [61g].) [JSL 36, pp. 168–169 (K. Schütte); MR 17, p. 699 (P.R. Halmos); ZMG 73, p. 7 (G. Hasenjaeger).]

[56a] *Development and applications of the "projective" classification of sets of integers*, ***Proceedings of the International Congress of Mathematicians 1954, Amsterdam September 2–September 9***, ed. by J.C.H. Gerretsen and J. de Groot, vol. 3, Erven P. Noordhoff N.V., Groningen, and North-Holland Publishing Co., Amsterdam, 1956, pp. 280–288. [JSL 23, pp. 44–45 (); MR 19, pp. 238–239 (G. Kreisel); ZMG 75, p. 235 (Editors).]

[56b] *Models of axiomatic theories admitting automorphisms* (with A. Ehrenfeucht), ***Fundamenta Mathematicae***, vol. 43 (1956), pp. 50–68. [JSL 31, pp. 644–645 (R.L. Vaught); MR 18, p. 863 (I. Novak Gál); ZMG 73, pp. 7–8 ().]

[56c] *Concerning a problem of H. Scholz*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 2 (1956), pp. 210–214. [Abstract [55[a]a]; JSL 24, p. 241 (W. Ackermann); MR 19, p. 240 (H.B. Curry); ZMG 74, pp. 249–250 (J.C. Shepherdson).]

[56d] *On models of axiomatic set-theory*, ***Bulletin de l'Académie Polonaise des Sciences***, Classe 3, vol. 4 (1956), pp. 663–667. [JSL 32, pp. 531–532 (E. Mendelson); MR 18, p. 711 (P.C. Gilmore); ZMG 75, p. 8 ().]

[57] *L'oeuvre scientifique de Jan Łukasiewicz dans le domaine de la logique mathématique,* ***Fundamenta Mathematicae***, vol. 44 (1957), pp. 1–11. [JSL 22, pp. 387–388 (K. Dürr); MR 19, p. 722 (registered); ZMG 77, p. 242 (Editors).]

[57a] *On a generalization of quantifiers*, ***Fundamenta Mathematicae***, vol. 44 (1957), pp. 12–36. [Abstract [55[a]]; JSL 23, p. 217 (D. Scott); JSL 25, pp. 365–366. (); MR 19, p. 724 (G. Kreisel); ZMG 78, p. 244 (T. Hailperin).]

[57b] *On computable sequences*, ***Fundamenta Mathematicae***, vol. 44 (1957), pp. 37–51. [JSL 25, p. 367 (); MR 19, p. 934 (H.G. Rice); ZMG 79, pp. 247–248 (E. Specker).]

[57c] *On recursive models of formalized arithmetic*, ***Bulletin de l'Académie Polonaise des Sciences***, Classe 3, vol. 5 (1957), pp. 705–710, and p. LXII (Russian summary). [JSL 23, pp. 45–46 (S. Feferman); MR 20 # 7 (G. Kreisel)—see errata to this review in the same volume of MR; ZMG 81, pp. 12–13 (K. Schütte).]

[57d] *Podstawy matematyki wraz z teorią mnogości i logiką matematyczną* < Foundations of mathematics, theory of sets and mathematical logic > (with J. Łoś), ***Bibliografia matematyki polskiej w X-leciu 1944–1954***, Roczniki Polskiego Towarzystwa Matematycznego, seria II, Wiadomości Matematyczne, vol. 2 no. 1, Warszawa 1957, pp. 5–21. [JSL 31, p. 517 (Editors).]

(1) *Foundations of mathematics, theory of sets and mathematical logic* (with J. Łoś), ***Bibliography of Polish mathematics 1944–1954***. Published for the Department of Commerce and the National Science Foundation, Washington, D.C., on the order of Centralny Instytut Informacji Naukowo-technicznej i Ekonomicznej, by Państwowe Wydawinctwo Naukowe, Warsaw 1963, pp. 4–17. (English translation of [57d].) [JSL 3, p. 517 (A. Church).]

[58] *On a problem of W. Kinna and K. Wagner*, ***Colloquium Mathematicum***, vol. 6 (1958), pp. 207–208. [MR 20 # 6980 (F. Bagemihl); ZMG 86, pp. 42–43 (G. Aumann).]

[58a] *The classical and ω-complete arithmetic* (with A. Grzegorczyk and C. Ryll-Nardzewski), ***The Journal of Symbolic Logic***, vol. 23 (for 1958, pub. 1959), pp. 188–206. [JSL 27, pp. 80–81 (S. Feferman); MR 21 # 4908 (A. Rose); ZMG 84, p. 248 (W. Ackermann).]

[58b] *Quelques observations sur l'usage des méthodes non finitistes dans la métamathématique*, ***Le raisonnement en mathématiques et en sciences expérimentales***, Colloques internationaux du Centre National de la Recherche Scientifique, vol. 70, Editions du Centre National de la Recherche Scientifique, Paris 1958, pp. 19–29 and 31–32 (Mostowski's reply to the questions put forward in the discussion by M. Lacombe). [JSL 24, pp. 234–235 (L. Henkin); MR 21 # 4896 (L.N. Gál); ZMG 105, pp. 5–6 (G.H. Müller).]

[58c] *Quelques applications de la topologie à la logique mathématique*, Appendix A in Casimir Kuratowski, ***Topologie***, vol. 1, forth edn., Państwowe Wydawnictwo Naukowe, Varsovie 1958, pp. 470–477. [JSL 36, p. 688 (P. Smith); ZMG 78, p. 146 (E. Pannwitz).]

(1) *Some applications of topology to mathematical logic*, Appendix A in K. Kuratowski, ***Topology***, vol. 1, new edition, revised and augmented, Academic Press New York–London, and PWN—Polish Scientific Publishers, Warszawa 1966, pp. 534–545. (English translation of [58c] by J. Jaworowski.) [ZMG 158, pp. 408–409 (J. Flachsmeyer).]

(2) Russian translation of (1) in K. Kuratovskiĭ, ***Topologija***, vol. 1, translated by M. Ja. Antonovskiĭ, ed. by P.S. Aleksandrov, "Mir" Publishers, Moscow 1966, pp. 543–548. [ZMG 158, p. 409 (Editors).]

(3) Reprint of [58c] in Casimir Kuratowski, ***Topologie, Partie I et II***, Éditions Jacques Gabay, Sceaux 1992. [ZMG 849 # 01044 (registered).]

[59] *Замечания к доказательствам существования стандартных моделей* < *Zaméčaniá k dokazatél'stvam suščéstvovaniá standartnyh modéléj*; Note on the proof of the existence of standard models >. ***Trudy Trét'égo Vsésoúznogo Matématičéskogo Sézda, Moskva, iún'-iúl' 1956***, Volume IV, ***Kratkoé sodéržanié sékcionnyh dokladov, doklady inostrannyh učényh***, Izdatél'stvo Akadémii Nauk SSSR, Moscow 1959, pp. 232–236. (Cf. [56d].) [JSL 32, pp. 531–532 (E. Mendelson); ZMG 135, p. 6 (J. Reichbach).]

[59a] *On various degrees of constructivism*, ***Constructivity in Mathematics. Proceedings of the colloquium held at Amsterdam, 1957***, ed. by A. Heyting, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam 1959, pp. 178–194. [JSL 35, pp. 575–576 (G.S. Boolos); MR 22 # 3688 (G. Kreisel); ZMG 88, p. 250 (E.W. Beth).]

[59b] *A class of models for second-order arithmetic*, ***Bulletin de l'Académie Polonaise des Sciences***, Série des sciences mathématiques, astronomiques et physiques, vol. 7 (1959), pp. 401–404. [JSL 34, pp. 128–129 (H.B. Enderton); MR 22 # 6706 (A. Rose); ZMG 100, p. 11 (W. Ackermann).]

[61] *Formal system of analysis based on an infinistic rule of proof*, ***Infinitistic Methods. Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 2–9 September 1959***, Państwowe Wydawnictwo Naukowe, Warszawa, and Pergamon Press, New York, Oxford, London, and Paris, 1961, pp. 141–166. [JSL 34, pp. 128–129 (H.B. Enderton); MR 34 # 1185 (registered); MR 36 # 3631 (J. Mycielski); ZMG 121, p. 15 (A. Rose); Canadian Mathematical Bulletin, vol. 6 (1963), pp. 304–305 (J. Lam-

bek); Bulletin de la Societé Mathématique de Belgique, vol. 18 (1966), pp. 445–447 (G. Hirsch).]

[61a] *A generalization of the incompleteness theorem*, ***Fundamenta Mathematicae***, vol. 49 (1961), pp. 205–232. [MR 24 # A41 (E.J. Cogan); ZMG 99, pp. 6–7 (A. Oberschelp).]

[61b] *An example of a non-axiomatizable many valued logic*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 7 (1961), pp. 72–76. [JSL 35, p. 143 (L. Hay); MR 30 # 1038 (A. Rose); ZMG 124, p. 248 (J.J.F. Nieland).]

[61c] *Axiomatizability of some many valued predicate calculi*, ***Fundamenta Mathematicae***, vol. 50 (1961), pp. 165–190. [JSL 35, p. 143 (L. Hay); MR 24 # A3060 (G. Kreisel); ZMG 99, pp. 7–8 (A. Oberschelp).]

[61d] *Concerning the problem of axiomatizability of the field of real numbers in the weak second order logic*, ***Essays on the Foundations of Mathematics, dedicated to A.A. Fraenkel on his seventieth anniversary***, edited by Y. Bar-Hillel, E.I.J. Poznanski, M.O. Rabin, and A. Robinson for The Hebrew University of Jerusalem, Magnes Press, Jerusalem 1961, and North-Holland Publishing Company, Amsterdam 1962, pp. 269–286. [JSL 32, pp. 130–131 (R.M. Robinson); MR 28 # 5003 (G. Kreisel); ZMG 168, pp. 249–250 (G. Kreisel).]

[61e] *Definability of sets in models of axiomatic theories* (with A. Grzegorczyk and C. Ryll-Nardzewski), ***Bulletin de l'Académie Polonaise des Sciences***, Série des sciences mathématiques, astronomiques et physiques, vol. 9 (1961), pp. 163–167. [Abstract [58[a]]; JSL 34, p. 126 (A. Lévy); MR 29 # 1138 (P. Axt); ZMG 99, pp. 9–10 (W. Ackermann).]

[61f] *A compact space of models of first order theories* (with A. Ehrenfeucht), ***Bulletin de l'Académie Polonaise des Sciences***, Série des sciences mathématiques, astronomiques et physiques, vol. 9 (1961), pp. 369–373. [JSL 35, pp. 586–587 (G. Fuhrken); MR 26 # 6043 (A. Blake); ZMG 106, p. 5 (K. Schütte).]

[61g] *Addition au travail "A proof of Herbrand's theorem"* (with J. Łoś and H. Rasiowa), ***Journal de mathématique pures et appliquées***, ser. 9, vol. 40 (1961), pp. 129–134. (See [56].) [JSL 36, pp. 168–169 (K. Schütte); MR 28 # 22 (K. Ono); ZMG 199, pp. 7–8 (G. Hasenjaeger).]

(1) *Addition to the paper "A proof of Herbrand's theorem"*, in [79[b]a], pp. 437–441. (English translation of [61g] by W. Marek.)

[62] *Representability of sets in formal systems*, ***Recursive Function Theory***, Proceedings of Symposia in Pure Mathematics, vol. 5, American Mathematical Society, Providence, Rhode Island, 1962, pp. 29–48. [MR 26 # 27 (S. Feferman); ZMG 158, p. 8 (J.N. Crossley), Scripta Mathematica, vol. 28 (1968), pp. 140–141 (E.J. Cogan).]

[62a] *A problem in the theory of models*, ***Bulletin de l'Académie Polonaise des Sciences***, Série des sciences mathématiques, astronomiques et physiques, vol. 10 (1962), pp. 121–126. [JSL 39, p. 600 (D.W. Kueker); MR 32 # 5513 (E. Engeler); ZMG 108, pp. 4–5 (E. Engeler).]

[62b] *L'espace des modèles d'une théorie formalisée et quelques-unes de ses applications*, ***Actes du Colloque de Mathématiques réuni à Clermont à l'Occasion du Tricentenaire de la Mort de Blaise Pascal, 4–8 Juin 1962***, Volume 1, ***Introduction et logique mathématique***, Annales de la Faculté des Sciences de l'Université Clermont, no. 7, Mathématiques, part 1, Clermont-Ferrand 1962, pp. 107–116. [JSL 40, p. 501 (registered); MR 47 # 4785 (J.T. Baldwin).]

[63] *The Hilbert epsilon function in many-valued logics*, ***Proceedings of a Colloquium on Many-valued Logics, Helsinki, 23–26 August, 1962***, Acta Philosophica Fennica, vol. 16 (1963), pp. 169–188. [MR 28 # 2963 (H.J. Keisler); ZMG 121, p. 12 (A. Rose).]

[64] *Widerspruchsfreiheit und Unabhängigkeit der Kontinuumhypothese*, ***Elemente der Mathematik***, vol. 19 (1964), pp. 121–125. (For an extended version see [68a].) [MR 31 # 3332 (R.O. Gandy); ZMG 126, p. 22 (M. Kühnrich).]

[65] *On models of Zermelo-Fraenkel set theory satisfying the axiom of constructibility,* ***Studia logico-mathematica et philosophica, in honorem Rolf Nevanlinna die natali eius septuagesimo 22. X. 1965***; Acta Philosophica Fennica, vol. 18 (1965), pp. 135–144. [JSL 36, p. 542 (F.R. Drake); MR 37 # 58 (A. Lévy); ZMG 143, p. 258 (T. Jech).]

[66] *Mnogości teoria* < Sets, theory of >, ***Wielka Encyklopedia Powszechna PWN***, vol. 7, Państwowe Wydawnictwo Naukowe, Warszawa 1966, pp. 385–387.

[67] *Recent results in set theory*, ***Problems in the Philosophy of Mathematics. Proceedings of the International Colloquium in the Philosophy of Science, London 1965***, vol. 1, ed. by I. Lakatos, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam 1967, pp. 82–96, and *Reply* (to the comments put forward in the discussion by G. Kreisel, A. Robinson and L. Kalmár), pp. 105–108. [JSL 37, pp. 765–766 (T. Jech); ZMG 155, pp. 336–337 (G.T. Kneebone); The British Journal for the Philosophy of Science, vol. 19 no. 2 (1968), pp. 171–173 (A. Sloman); L'âge de la science, no. 3 (1968), pp. 212–216 (R. Martin); Philosophy of Science, vol. 36 (1969), pp. 324–326 (A. Maziarz).]

(1) *O niektórych nowych wynikach meta-matematycznych dotyczących teorii mnogości* < On some new metamathematical results concerning set theory >, ***Studia Logica***, vol. 20 (1967), pp. 99–116. (With Russian and English summaries. A slightly extended version of [67].) [JSL 37, pp. 765–766 (T. Jech); MR 34 # 7357 (from the author's summary); ZMG 301 # 02006 registered).]

(2) Reprinted in ***Mengenlehre***, ed. by U. Felgner, Wissenschaftliche Buchgesellschaft, Darmstad 1979, pp. 276–290. [ZMG 403 # 03035 (registered).]

[67a] *Tarski, Alfred*, ***The Encyclopedia of Philosophy***, ed. by Paul Edwards, The Macmillan Company & The Free Press, New York, and Collier-Macmillan Limited, London, 1967, vol. 8, pp. 77–81. [JSL 35, p. 301 (W. Craig).]

[68] *Craig's interpolation theorem in some extended systems of logic*, ***Logic, Methodology and Philosophy of Science III. Proceedings of the Third International Congress for Logic, Methodology and Philosophy of Science***, ed. by B. van Rootselaar and J.F. Staal, Studies in Logic and the Foundations of Mathematics, North-Holland Pub-

lishing Company, Amsterdam 1968, pp. 87–103. [MR 40 # 4093 (M. Yasugi); ZMG 182, p. 9 (Autorreferat).]

[68a] *Niesprzeczność i niezależność hipotezy continuum* < Consistency and independence of the continuum hypothesis >, ***Wiadomości Matematyczne***, vol. 10 (1968), pp. 175–182. (Extended version of [64].) [MR 39 # 56 (S. Gacsályi).]

[69] *Models of set theory*, ***Aspects of Mathematical Logic***, Centro Internazionale Matematico Estivo (CIME), III Ciclo, Varenna, 9–12 Settembre 1968, ed. by E. Casari, Edizioni Cremonese, Rome, 1969, pp. 65–179. [MR 40 # 7106 (J.R. Shoenfield); ZMG 228 # 02037 (registered).]

[69a] *On ω-models which are not β-models* (with Y. Suzuki), ***Fundamenta Mathematicae***, vol. 65 (1969), pp. 83–93. [MR 40 # 7104 (D.H. Potts); ZMG 294 # 02029 (registered).]

[72] *Models of second order arithmetic with definable Skolem functions*, ***Fundamenta Mathematicae***, vol. 75 (1972), pp. 223–234. (For corrections see [74e].) [JSL 38, pp. 652–653 (S.G. Simpson); MR 47 # 8286 (A. Sochor); ZMG 245 # 02049 (registered).]

[72a] *A transfinite sequence of ω-models*, ***The Journal of Symbolic Logic***, vol. 37 (1972), pp. 96–102. [Abstract [71[a]]; MR 48 # 3721 (W. Schwabhäuser); ZMG 246 # 02038 (registered).]

(1) An extended preprint version of the above, Instytut Matematyki, Uniwersytet Warszawski, Wydawnictwo Uniwersytetu Warszawskiego, 1971, 24 pp.

[72b] *Sets*, ***Scientific thought. Some underlying concepts, methods and procedures***, Mouton/Unesco, Paris 1972, and Mouton, The Hague 1973, pp. 1–34. [ZMG 273 # 00010 (registered).]

(1) *Les ensembles*, ***La Pensée scientifique. Quelques concepts, démarches et méthodes***, Mouton /Unesco, Paris–La Haye–New York 1978, pp. 1–36. (French translation of [72b] by Elliott Cohen.)

[73] *Partial ordering of the family of ω-models*, ***Logic, Methodology and Philosophy of Science IV. Proceedings of the Fourth International Congress for Logic Methodology and Philosophy of Science, Bucharest 1971***, edited by P. Suppes, L. Henkin, A. Joja, and Gr. C. Moisil, Studies in Logic and the Foundations of Mathematics, vol. 74, PWN—Polish Scientific Publishers, Warszawa, and North-Holland Publishing Company, Amsterdam–London, 1973, pp. 13–28. [MR 56 # 8358 (D. Cenzer).]

[73a] *A contribution to teratology*, ***Izbrannye Voprosy Algebry i Logiki: Sbornik posvjascen pamjati A.I. Mal'ceva*** < Selected questions of algebra and logic: Volume dedicated to the memory of A.I. Mal'cev >, ed. by A.I. Shirshov *et al.*, Izdatél'stvo "Nauka", Sibirskoe Otdelenie, Novosibirsk, 1973, pp. 184–196. [MR 48 # 5851 (J.M. Plotkin); ZMG 279 # 02038 (registered).]

[74] *Some problems in the axiomatic theory of classes*, ***Bolletino della Unione Matematica Italiana***, ser. 4, vol. 9, Supplemento al Fasc. 2 (1974), pp. 161–170. [MR 55 # 7775 (M. Boffa); ZMG 313 # 02042 (registered).]

[74a] *Observations concerning elementary extensions of ω-models*, ***Proceedings of the Tarski Symposium***, ed. by L. Henkin *et al.*, Proceedings of Symposia in Pure Mathematics, vol. 25, American Mathematical Society, Providence, Rhode Island, 1974, pp. 349–355. [MR 51 # 12509 (L. Bukovsky); ZMG 329 # 02023 (registered).]

[74b] *Refleksje na temat pewnego wykładu z historii matematyki* < Reflections upon a certain course in the history of mathematics >, ***O nauczaniu historii nauki*** <On teaching the history of science >, ed. by W. Osińska, Monografie z Dziejów Nauki i Techniki, vol. 96, Zakład Narodowy Imienia Ossolińskich, Wrocław–Warszawa 1974, pp. 231–243. (With English and Russian summaries.)

(1) Reprint in ***Wiadomości Matematyczne***, vol. 22 (1979), pp. 65–75. [MR 82h:01056 (S.J. Surma).]

[74c] *Co może maszyna Turinga, czyli o algorytmach (I)* < What a Turing machine is able to do, or on algorithms (I) >, ***Delta. Popularny miesięcznik matematyczno-fizyczny***, nr 10 (1974), pp. 12–14.

[74d] *Czego nie może maszyna Turinga, czyli o algorytmach (II)* < What a Turing machine cannot do, or on algorithms (II) >, ***Delta. Popularny miesięcznik matematyczno-fizyczny***, nr 11 (1974), pp. 6–7.

[74e] *Errata to the paper 'Models of second order arithmetic with definable Skolem functions'*, ***Fundamenta Mathematice***, vol. 84 (1974), p. 173. (See [72].) [MR 50 # 89 (Editors); ZMG 275 # 02047 (registered).]

[75] *An exposition of forcing*, ***Algebra and Logic. Papers from the 1974 Summer Research Institute of the Australian Mathematical Society, Monash University, Australia***, ed. by J.N. Crossley, Lecture Notes in Mathematics, vol. 450, Springer-Verlag, Berlin–Heidelberg–New York, 1975, pp. 220–282. [MR 51 # 7871 (J. Ketonen); ZMG 308 # 02059 (registered).]

[75a] *On extendability of models of ZF set theory to the models of Kelley-Morse theory of classes* (with W. Marek), ***$\models$ ISILC Logic Conference, Proceedings of the International Summer Institute and Logic Colloquium, Kiel 1974***, ed. by G.H. Müller, A. Oberschelp, and K. Potthoff, Lecture Notes in Mathematics, vol. 499, Springer-Verlag, Berlin–Heidelberg–New York, 1975, pp. 460–542. [MR 53 # 7782 (J.K. Truss); ZMG 339 # 02053 (registered).]

[75b] *Travaux de W. Sierpiński sur la théorie des ensembles et ses applications. La théorie générale des ensembles*, in [75^{e}], pp. 9–13. [MR 54 # 2407 (Editors).]

[76] *Two remarks on models of Morse' set theory*, ***Set Theory and Hierarchy Theory. A Memorial Tribute to Andrzej Mostowski, Bierutowice, Poland 1975***. Proceedings of the Second Conference on Set Theory and Hierarchy Theory, ed. by W. Marek, M. Srebrny, and A. Zarach, Lecture Notes in Mathematics, vol. 537, Springer-Verlag, Berlin–Heidelberg–New York, 1976, pp. 13–21. [MR 56 # 11789 (U. Felgner); ZMG 338 # 02036 (registered).]

(1) Reprinted in ***Prace Naukowe Instytutu Matematyki Politechniki Wrocławskiej***, vol. 14, Seria Konferencje, no 1, Politechnika Wrocławska, Wrocław 1977, pp. 5–11. [MR 58 # 5219 (Author's introduction); ZMG 378 # 02030 (registered).]

[76a] *A remark on models of the Gödel-Bernays axioms for set theory*, ***Sets and Classes. On the work of Paul Bernays***, ed. by G.H. Müller, Studies in Logic and the Foundations of Mathematics, vol. 84, North-Holland Publishing Company, Amsterdam–New York–Oxford, 1976, pp. 325–340. [MR 56 # 5293 (M. Benda); ZMG 338 # 02037 (registered).]

[78] *The elementary theory of well-ordering—a metamathematical study* (with J.E. Doner and A. Tarski), ***Logic Colloquium'77***, ed. by A. MacIntyre, L. Pacholski, and J. Paris, Studies in Logic and the Foundations of Mathematics, vol. 96, North-Holland Publishing Company, Amsterdam 1978, pp. 1–54. [Abstract [49$^{\rm a}$]; MR 80d: 03027a (A.B. Slomson); ZMG 461 # 03003 (registered).]

(1) Reprinted in ***Alfred Tarski, Collected Papers. Volume 4, 1958–1971***, ed. by S.R. Givant and R.N. McKenzie, Birkhäuser Verlag, Basel 1986, pp. 437–490.

[84] *Uwagi o rozwiązalności równań w grupach abelowych* < Remarks on solvability of equations in Abelian groups >, ***Wiadomości Matematyczne***, vol. 25 (1984), pp. 177–179. (Paper written probably in the period 1940–1944.) [ZMG 584 # 20041 (St. Balcerzyk).]

II. Abstracts

[38$^{\rm a}$] *Bericht aus einem Satz von Goedel über die Widerspruchsfreiheit des Auswahlaxioms*, ***Rocznik Polskiego Towarzystwa Matematycznego*** (= ***Annales de la Société Polonaise de mathématique***), vol. 17 (1938), p. 115. (Presented by title.)

[48$^{\rm a}$] *Sur une relation projective qui ordonne bien le continu*, ***Colloquium Mathematicum***, vol. 1 (1948), p. 176.

[48$^{\rm a}$a] *Une généralisation des ensembles projectifs*, ***Colloquium Mathematicum***, vol. 1 (1948), pp. 254–255.

[49$^{\rm a}$] *Arithmetical classes and types of well ordered systems* (with A. Tarski), ***Bulletin of the American Mathematical Society***, vol. 55 (1949), p. 65 and p. 1192 (Abstract 78t).

(1) Reprinted in ***Alfred Tarski, Collected Papers. Volume 4, 1958–1971***, ed. by S.R. Givant and R.N. McKenzie, Birkhäuser Verlag, Basel 1986, p. 583.

[49$^{\rm a}$a] *A general theorem concerning the decision problem*, ***Bulletin of the American Mathematical Society***, vol. 55 (1949), pp. 65–66.

[49$^{\rm a}$b] *Undecidability in the arithmetic of integers and in the theory of rings* (with A. Tarski), ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 76.

(1) Reprinted in ***Alfred Tarski, Collected Papers. Volume 4, 1958–1971***, ed. by S.R. Givant and R.N. McKenzie, Birkhäuser Verlag, Basel 1986, pp. 586–587.

[51$^{\rm a}$] *Quelques résultats récents concernant le problème de la décision*, ***Colloquium Mathematicum***, vol. 2 (1951), pp. 309–310.

[55$^{\rm a}$] *O kwantyfikatorach* < On quantifiers >, ***Prace Matematyczne***, vol. 1, p. 425.

(1) *Sur les quantificateurs*, ***Colloquium Mathematicum***, vol. 4 (1957), p. 115. (French translation of [55^{a}].)

[55^{a}a] *O mocach algebr skończonych* < On powers of finite algebras >, ***Prace Matematyczne***, vol. 1, p. 427.

(1) *Sur les puissances des algèbres finies*, ***Colloquium Mathematicum***, vol. 4 (1957), pp. 127–128. (French translation of [55^{a}a].)

[55^{a}b] *O predykatach w ciałach algebraicznie domkniętych* < On predicates in algebraically closed fields >, ***Prace Matematyczne***, vol. 1, p. 431.

(1) *Sur les prédicats dans les corps algébriques clos*, ***Colloquium Mathematicum***, vol. 4 (1957), pp. 133–134. (French translation of [55^{a}b].)

[55^{a}c] *O bazach normalnych dla rozszerzeń nieskończonych* < On normal bases for infinite extensions [of fields] >, ***Prace Matematyczne***, vol. 1, p. 431.

(1) *Sur les bases normales pour les extensions infinies*, ***Colloquium Mathematicum***, vol. 4 (1957), p. 134. (French translation of [55^{a}c].)

[57^{a}] *Sur les problèmes actuels du domaine des fondements des mathématiques*, ***Colloquium Mathematicum***, vol. 4 (1957), p. 118.

[57^{a}a] *Remarques sur les polynômes dans les corps abstraits*, ***Colloquium Mathematicum***, vol. 4 (1957), p. 125 (Presented by title).

[58^{a}] *Representability of sets in models of axiomatic theories* (with C. Ryll-Nardzewski), ***The Journal of Symbolic Logic***, vol. 23 (1958), pp. 458–459.

[60^{a}] *Completeness theorems for some many-valued functional calculi*, ***The Journal of Symbolic Logic***, vol. 25 (1960), pp. 94–95.

[69^{a}] *A theorem on β-models*, ***The Journal of Symbolic Logic***, vol. 34 (1969), pp. 158–159.

[71^{a}] *A transfinite sequence of ω-models*, ***The Journal of Symbolic Logic***, vol. 36 (1971), p. 589.

III. Books and Monographs

[48^{b}] ***Logika matematyczna. Kurs uniwersytecki*** < Mathematical logic. A university course >, Monografie Matematyczne, vol. 18, Warszawa–Wrocław, 1948, VIII + 388 pp. [JSL 14, pp. 189–190 (O.V. Zich); MR 10, p. 229 (H. Hiż); ZMG 34, pp. 147–150 (H. Scholz); Synthese 7 (1948–49), pp. 229–301 (R. Suszko); Ruch Filozoficzny, vol. 16 (1948), p. 103 (J. Słupecki); Matematyka. Czasopismo dla nauczycieli, vol. 2 no. 4 (1949, pp. 34–35 (J. Słupecki); Annales de la Société Polonaise de Mathématiques 22 (1949), pp. 313–315 (H. Greniewski).]

[52^{b}] ***Teoria mnogości*** < Theory of sets > (with K. Kuratowski), Monografie Matematyczne, vol. 27, Polskie Towarzystwo Matematyczne, Warszawa–Wrocław 1952, IX + 311 pp. [MR 14, p. 960 (S. Ulam); ZMG 47, pp. 53–54 (G. Kurepa).]

[52[b]a] ***Sentences Undecidable in Formalized Arithmetic. An exposition of the theory of Kurt Gödel***, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam 1952, VIII + 117 pp. [JSL 19, pp. 119–121 (G. Hasenjaeger); MR 14, p. 3 (A. Robinson); ZMG 47, pp. 9–11 (G.H. Müller); The British Journal for the Philosophy of Science, vol. 3 (1952–3), pp. 364–374 (G. Kreisel); The Mathematical Gazette, vol. 37 (1953), pp. 68–70 (R.L. Goodstein); Nieuw Archief voor Wiskunde, ser. 3 vol. 3 no. 1 (1955), pp. 38–39 (W. Peremans).]

(1) Second printing, 1957.

(2) Reprint, Greenwood Press, Westport, Conn., 1982.

[53[b]] ***Undecidable Theories*** (with A. Tarski and R.M. Robinson), North-Holland Publishing Company, Amsterdam 1953, XII + 98 pp. (From the Preface, p.VIII: "This monograph consists of three papers: *A general method in proofs of undecidability, Undecidability and essential undecidability in arithmetic, Undecidability of the elementary theory of groups*. While the first and the third papers have been written by the undersigned alone, the second paper is a joint work of A. Mostowski, R.M. Robinson, and the undersigned."—The undersigned is Alfred Tarski. See [53d] above.) [JSL 24, pp. 167–169 (M. Davis); MR 15, p. 384 (G. Kreisel); ZMG 55, pp. 4–5 (H. Scholz); Bulletin of the American Mathematical Society, vol. 60 (1954), pp. 570–572 (I.N. Gál); Mathematica Sandinavica, vol. 2 (1954), pp. 167–170 (Th. Skolem); Sûgaku, vol. 6 no. 4 (1955), pp. 239–243 (47–51) (Setsuya Seki)].]

[53[b]a] ***Algebra wyższa. Część pierwsza*** < Higher algebra. First part > (with M. Stark), Biblioteka Matematyczna, vol. 1, Polskie Towarzystwo Matematyczne, Warszawa 1953, VII + 308 pp + errata. [MR 15, p. 594 (A. Zygmund); ZMG 50, p. 248 (C. Kaloujnine).]

[54[b]] ***Algebra wyższa. Część druga*** < Higher algebra. Second part > (with M. Stark), Biblioteka Matematyczna, vol. 3, Państwowe Wydawnictwo Naukowe, Warszawa 1954, VII + 173 pp. [MR 16, p. 104 (A. Zygmund); ZMG 57, p. 11 (L. Kaloujnine).]

[54[b]a] ***Algebra wyższa. Część trzecia*** < Higher algebra. Third part > (with M. Stark), Biblioteka Matematyczna, vol. 4, Państwowe Wydawnictwo Naukowe, Warszawa 1954, VI + 273 pp. + errata. [MR 16, p. 663 (A. Zygmund); ZMG 57, p. 11 (L. Kaloujnine).]

(1) Second edition, 1967, VI + 266 pp. [ZMG 183, p. 21 (Editors).]

[58[b]] ***Elementy algebry wyższej*** < Elements of Higher algebra >, Biblioteka Matematyczna, vol. 16, Państwowe Wydawnictwo Naukowe, Warszawa 1958, 367 pp. + errata. [MR 21 # 1927 (M. Fiedler); ZMG 79, p. 249 (F. Šik); Wiadomości Matematyczne, vol. 3 no 2 (1959), pp. 183–185 (S. Hartman).]

(1) Second edition, corrected and augmented, 1963, [1] + 391 pp. [MR 29 # 111 (H. Minc); ZMG 114, p. 250 (registered).]

(2) Third edition, corrected, 1965, [2] + 390 pp.

(3) Fourth edition, corrected and argmented, 1968, [1] + 397 pp.

(4) Fifth, sixth, seventh, eighth, and ninth edition: 1970, 1972, 1974, 1975, 1977.

[58$^{\mathrm{b}}$a] ***Algebra liniowa*** < Linear algebra > (with M. Stark), Biblioteka Matematyczna, vol. 19, Państwowe Wydawnictwo Naukowe, Warszawa 1958, 188 pp. [MR 21 # 1313 (J.W. Andrushkiw); ZMG 82, pp. 19–20 (L. Holzer).]

(1) Second edition, 1966.

(2) Third, fourth, fifth, and sixth edition: 1968, 1974, 1975,1976.

[62$^{\mathrm{b}}$] ***On Invariant, Dual Invariant and Absolute Formulas***, Rozprawy Matematyczne (= Dissertationes Mathematicae), vol. 29, Państwowe Wydawnictwo Naukowe, Warszawa 1962, 38 pp. [MR 29 # 5721 (H.J. Keisler); ZMG 106, pp. 5–6 (G. Kreisel).]

[64$^{\mathrm{b}}$] ***Introduction to Higher Algebra*** (with M. Stark), International Series of Monographs on Pure and Applied Mathematics, vol. 37, PWN–Polish Scientific Publishers, Warszawa, and Pergamon Press, Oxford–London–New York, 1964, 474 pp. (English translation of [58$^{\mathrm{b}}$](1) by J. Musielak.) [MR 31 # 183 (registered); ZMG 108, p. 251 (Editors).]

[65$^{\mathrm{b}}$] ***Thirty Years of Foundational Studies. Lectures on the development of mathematical logic and the study of the foundations of mathematics in 1930–1964***, Acta Philosophica Fennica, vol. 17 (1965), pp. 1–180. [JSL 33, pp. 111–112 (A. Robinson); MR 33 # 18 (H.B. Curry); ZMG 146, p. 245 (E. Mendelson).]

(1) A corrected reprint of [65$^{\mathrm{b}}$], Basil Blackwell, Oxford 1966, and Barnes & Noble, Inc. New York, 1966, 180 pp. [MR 33 # 5445 (registered); Studia Logica, vol. 22 (1968), pp.169–170 (R. Suszko); Wiadomości Matematyczne, vol. 11 (1969), pp. 358–359 (J. Waszkiewicz).]

(2) Second printing of [65$^{\mathrm{b}}$], 1967.

[66$^{\mathrm{b}}$] ***Teoria mnogości***. < Theory of sets > (with K. Kuratowski), Monografie Matematyczne, vol. 27, Państwowe Wydawnictwo Nukowe, Warszawa 1966, 375 pp. + loose errata. (Second, *completely revised* edition of [52$^{\mathrm{b}}$]. The publisher's note "wydanie drugie rozszerzone" < second, enlarged edition > is highly inaccurate. E.g. from the 1952 edition chapter VI "The consistency and independence of axioms" and an appendix "Paradoxical decomposition of the ball" have been removed, and notation has been changed. For corrections see [69$^{\mathrm{m}}$].) [MR 34 #7379 (B.M. Schein); ZMG 139, p. 246 (registered); Wiadomości Matematyczne, vol. 11 (1969), pp. 153–154 (W. Marek).]

[67$^{\mathrm{b}}$] ***Set Theory*** (with K. Kuratowski), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam, and PWN—Polish Scientific Publishers, Warszawa, 1967, XI + 417 pp. (English translation of [66$^{\mathrm{b}}$] by M. Mączyński.) [JSL 40, pp. 629–630 (H.B. Enderton); MR 37 # 5100 (Editors); ZMG 165, pp. 17–18 (K. Hráček); Philosopy of Science, vol. 38 (1971), pp. 314–315 (J. Woods).]

(1) ***Теория множеств*** < Set theory >, Izdatel'stvo "Mir", Moscow 1970, 416 pp. (Russian translation of [67$^{\mathrm{b}}$] by M.I. Kratko, ed. by A.D. Taĭmanov.) [MR 42 # 2944 (Editors); ZMG 204, p. 313 (registered).]

[67$^{\mathrm{b}}$a] ***Modèles transitifs de la théorie des ensembles de Zermelo-Fraenkel***, Les Presses de l'Université de Montréal, Montréal 1967, 170 pp. [MR 45 # 3197 (A. Levy); ZMG 189, p. 286 (E. Mendelson).]

[69$^{\text{b}}$] ***Constructible Sets with Applications***, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam, and PWN—Polish Scientific Publishers, Warszawa, 1969, IX + 269 pp + loose errata. [JSL 40, pp. 631–632 (P. Smith); MR 41 # 52 (F.R. Drake); ZMG 185, p. 14 (L. Bukovský).]

(1) ***Конструктивные множества и их приложения*** < Constructible sets and their applications >, Izdatel'stvo "Mir", Moscow 1973, 256 pp. (Russian translation of [69$^{\text{m}}$] by M.I. Kratko, N.V. Beljakin and M.K. Valiev, ed. by A.G. Dragalin and A.D. Taimanov.) [MR 49 # 10548 (Editors); ZMG 269 # 02031 (registered).]

[76$^{\text{b}}$] ***Set Theory, with an Introduction to Descriptive Set Theory*** (with K. Kuratowski), Studies in Logic and the Foundations of Mathematics, vol. 86, North-Holland Publishing Company, Amsterdam, New York, Oxford, and PWN—Polish Scientific Publishers, Warszawa, 1976, XIV + 514 pp. (Second, completely revised edition of [67$^{\text{b}}$].) [MR 58 # 5230 (D.W. Bressler); ZMG 0337 # 02034 (registered).]

[78$^{\text{b}}$] ***Teoria mnogości, wraz ze wstępem do opisowej teorii mnogości*** < Set theory, with an introduction to descriptive set theory > (with K. Kuratowski), Monografie Matematyczne, vol. 27, Polskie Wydawnictwo Naukowe, Warszawa 1978, 470 pp. (Third, revised edition of [52$^{\text{b}}$], partially translated from [76$^{\text{b}}$].) [MR 80a: 04001 (Editors); Wiadomości Matematyczne, vol. 23 (1981), pp. 260–262 (L. Pacholski).]

[79$^{\text{b}}$] ***Foundational Studies. Selected Works***, volume 1, edited by K. Kuratowski, W. Marek, L. Pacholski, H. Rasiowa, C. Ryll-Nardzewski, and P. Zbierski. Studies in Logic and the Foundations of Mathematics, vol. 93, North-Holland Publishing Company, Amsterdam and PWN—Polish Scientific Publishers, Warszawa, 1979, XLVI + 635 pp. (1 plate). (The volume contains a very short biography of Mostowski (by W. Marek), a bibliography of his works (also by W. Marek), appreciations of Mostowski's work by: A. Grzegorczyk (recursion theory), W. Guzicki and W. Marek (set theory), L. Pacholski (model theory), C. Rauszer (logical calculi), and P. Zbierski (second order arithmetic), and thirteen Mostowski's papers, some reprinted and some translated into English—in order of appearence in the volume these are: [65$^{\text{b}}$], [69], [39a], [47], [58a], [61], [75], [50b], [56b], [69a], [74a], [49a], [75a].) [MR 81i: 01018a (C. Smoryński); ZMG 425 # 01021 (registered).]

[79$^{\text{b}}$a] ***Foundational Studies. Selected Works***, volume 2, edited by K. Kuratowski, W. Marek, L. Pacholski, H. Rasiowa, C. Ryll-Nardzewski, and P. Zbierski. Studies in Logic and the Foundations of Mathematics, vol. 93, North-Holland Publishing Company, Amsterdam and PWN—Polish Scientific Publishers, Warszawa, 1979, VIII + 605 pp. (The volume contains reprints or English translations of the following papers: [37], [38], [38a], [38c], [39], [45a], [47a], [48], [48a], [48b], [50a], [51], [52], [52a], [53], [53a], [55], [55a], [55b], [55c], [56], [56c], [57a], [57b], [57c], [58], [59a], [61a], [61b], [61c], [61d], [61e], [61f], [61g], [62], [62a], [63], [65], [68], [72], [72a], [73], [73a], [76a], [49$^{\text{a}}$], [74e].) [MR 81i: 01018b (C. Smoryński); ZMG 425 # 01021 (registered).]

IV. Problems

[48$^{\text{p}}$] Problème **P6**, ***Colloquium Mathematicum***, vol. 1 (1948), p. 31. (Also in ***Nowa Księga Szkocka***, problem 11.)

[48$^{\text{p}}$a] Problème **P36**, ***Colloquium Mathematicum***, vol. 1 (1948), p. 239. Errata p. 358.

[51p] Problème **P92** (with C. Kuratowski), ***Colloquium Mathematicum***, vol. 2 (1951), p. 298. (For a partial solution see [51b].)

[51pa] Problème **P95**, ***Colloquium Mathematicum***, vol. 2 (1951), p. 299. (Also in ***Nowa Księga Szkocka***, problem 106.)

V. Contributions to discussions

[58c] Contribution to the discussion of Th. Skolem, *Une relativisation des notions mathématiques fondamentales*. ***Le raisonnement en mathématiques et en sciences expérimentales***, Editions du Centre National de la Recherche Scientifique, Paris 1958, p.18. [JSL 25, pp. 284–285 (M. Davis).]

[58ca] Contribution to the discussion of [58b]. (For bibliographical details see [58b] above.)

[58cb] Contribution to the discussion of L. Nolin, *Sur l'algèbre des prédicats*. ***Le raisonnement en mathématiques et en sciences expérimentales***, Editions du Centre National de la Recherche Scientifique, Paris 1958, p. 37. [JSL 24, p. 235 (L. Henkin).]

[58cc] Contribution to the discussion of R. de Possel et R. Fraïssé, *Hypothèses de la théorie des relations qui permettent d'associer, a un bon ordre d'un ensemble, un bon ordre, défini sans ambiguïté, de l'ensemble de ses parties*. ***Le raisonnement en mathématiques et en sciences expérimentales***, Editions du Centre National de la Recherche Scientifique, Paris 1958, p. 55. [JSL 25, pp. 285–286 (M. Davis).]

[58cd] Contribution to the discussion of H.A. Schmidt, *Un procédé maniable de décision pour la loqique propositionnelle intuitionniste*. ***Le raisonnement en mathématiques et en sciences expérimentales***, Editions du Centre National de la Recherche Scientifique, Paris 1958, p. 65. [JSL 25, p. 286 (M. Davis).]

[67c] Contribution to the discussion of [67]. (For details see [67] above.)

[67ca] *Cohen's independence proof and second order formalization*, contribution to the discussion of Paul Bernays, *What do some recent results in set theory suggest?* ***Problems in the Philosophy of Mathematics, Proceedings of the International Colloquium in the Philosophy of Science, London 1965***, volume 1, ed. by I. Lakatos, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, Amsterdam 1967, pp. 112–115. [JSL 40, p. 499 (registered).]

[75c] Numerous contributions to the discussion on the rise of mathematical logic, in: *Reminiscences of logicians*, reported by J. Crossley, ***Algebra and Logic. Papers from the 1974 Summer Research Institute of the Australian Mathematical Society, Monash University, Australia***, ed. by J.N. Crossley, Lecture Notes in Mathematics, vol. 450, Springer-Verlag, Berlin–Heidelberg–New York, 1975, pp. 220–282.

VI. Reviews

[36r] Review of E. Żyliński, *Der Hilbertsche Formalismus*. I: *Der Formalismus* H_I. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 62 (1936), p. 1048.

[36^{r}a] Review of B. Sobociński, *Aksjomatyzacja implikacyjno-koniunkcyjnej teorii dedukcji* < Die Axiomatisierung der implikativkonjunktiven Deduktonstheorie >. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 62, p. 1056.

[37^{r}] Review of J. Pepis, *O zagadnieniu rozstrzygalności w zakresie węższego rachunku funkcyjnego* < Über das Entscheidungsproblem des engeren logischen Funktionenkalküls >. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 63, p. 823.

[37^{r}a] Review of M. Wajsberg, *Metalogische Beiträge*. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 63, p. 826.

[37^{r}b] Review of W. Hetper, *Podstawy semantyki* < Foundations of semantics >. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 63, p. 827.

[37^{r}c] Review of J. Hirano, *Einige Bemerkungen zum von Neumannschen Axiomensystem der Mengenlehre*. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 63, p. 828.

[38^{r}] Review of J. Kuczyński, *O twierdzeniu Gödla* < Über den Satz von Gödel >. ***The Journal of Symbolic Logic***, vol. 3 (1938), p. 118.

[38^{r}a] Review of A. Church, *The constructive second number class*. ***The Journal of Symbolic Logic***, vol. 3 (1938), p. 168–169.

[38^{r}b] Review of J. Pepis, *Über das Entscheidungsproblem des engeren logischen Funktionenkalküls*. ***Zentralblatt für Mathematik und ihre Grenzgebiete***, vol. 19 (1938), pp. 97–98.

[39^{r}] Review of J. Pepis, *O zagadnieniu rozstrzygalności w zakresie węższego rachunku funkcyjnego* < Über das Entscheidungsproblem des engeren logischen Funktionenkalküls >. ***The Journal of Symbolic Logic***, vol. 4 (1939), p. 93.

[39^{r}a] Review of A.M. Turing, *Systems of logic based on ordinals*. ***The Journal of Symbolic Logic***, vol. 4 (1939), pp. 128–129.

[39^{r}b] Review of A. Robinsohn, *On the independence of the axioms of definiteness (Axiome der Bestimmtheit)*. ***The Journal of Symbolic Logic***, vol. 4 (1939), p. 165.

[39^{r}c] Review of P.G.J. Vredenduin, *A system of strict implication*. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 65, p. 28.

[39^{r}d] Review of C.H. Langford, *A theorem on deducibility for second-order functions*. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 65, p. 28.

[39^{r}e] Review of G.C. Moisil, *Recherches sur le syllogisme*. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 65, p. 29.

[39^{r}f] Review of K. Menger, *A logic of the doubtful. On optative and imperative logic*. ***Jahrbuch über die Fortschritte der Mathematik***, vol. 65, p. 29.

[46^{r}] Review of S. Zaremba, *Uwagi o metodzie w matematyce i fizyce* < Remarks on method in mathematics and physics >. ***The Journal of Symbolic Logic***, vol. 11 (1946), p. 93.

[46$^{\mathrm{r}}$a] Review of L. Chwistek, *Ob aksiomé Cérmélo i éé roli v sovréménnoj matématiké* < Sur l'axiome de Zermelo et son rôle dans les mathématiques contemporaines >. ***The Journal of Symbolic Logic***, vol. 11 (1946), p. 93.

[46$^{\mathrm{r}}$b] Review of L. Chwistek, *Ob osnovah sémantiki* < Sur les fondements de la sémantique >. ***The Journal of Symbolic Logic***, vol. 11 (1946), pp. 93–94.

[46$^{\mathrm{r}}$c] Review of Z. Jordan, *The development of mathematical logic and of logical positivism in Poland between the two wars*. ***The Journal of Symbolic Logic***, vol. 11 (1946), pp. 94–95.

[46$^{\mathrm{r}}$d] Review of F.B. Fitch, *Self-reference in philosophy*. ***The Journal of Symbolic Logic***, vol. 11 (1946), pp. 95–96.

[46$^{\mathrm{r}}$e] Review of F.I. Mautner, *An extension of Klein's Erlanger program: Logic as invariant-theory*. ***The Journal of Symbolic Logic***, vol. 11 (1946), pp. 134–136.

[47$^{\mathrm{r}}$] Review of T. Czeżowski, *Główne zasady nauk filozoficznych* < Fundamental principles of the philosophical sciences >. ***The Journal of Symbolic Logic***, vol. 12 (1947), p. 26.

[47$^{\mathrm{r}}$a] Review of Z. Zawirski, *Geneza i rozwój logiki intuicjonistycznej* < Genesis and development of intuitionistic logic >. ***The Journal of Symbolic Logic***, vol. 12 (1947), pp. 26–27.

[47$^{\mathrm{r}}$b] Review of D.A. Bočvar, *Nékotoryé logičéskié teorémy o normal'nyh množést'vah i prédikatah* < Some logical theorems on the normal sets and predicates >. ***The Journal of Symbolic Logic***, vol. 12 (1947), p. 27.

[47$^{\mathrm{r}}$c] Review of S.C. Kleene, *On the interpretation of intuitionistic number theory*. ***The Journal of Symbolic Logic***, vol. 12 (1947), pp. 91–93.

[47$^{\mathrm{r}}$d] Review of D. Nelson, *Recursive functions and intuitionistic number theory*. ***The Journal of Symbolic Logic***, vol. 12 (1947), pp. 93–94.

[47$^{\mathrm{r}}$e] Review of V.I. Šéstakov, *Prédstavlénié haraktérističéskih funkcij prédloženij posrédstvom wyraženij, réalizuémyh réléjno-kontaktnymi shémami < Representation of characteristic functions by expressions realizable by relay-contact circuits >*. ***The Journal of Symbolic Logic***, vol. 12 (1947), p. 135.

[48$^{\mathrm{r}}$] Review of A. Markov, *Névozmožnost' nékotoryh algorifmov v téorii associativnyh sistém* < Impossibility of certain algorithms in the theory of associative systems >. ***The Journal of Symbolic Logic***, vol. 13 (1948), pp. 52–53.

[48$^{\mathrm{r}}$a] Review of A. Markov, *Névozmožnost' nékotoryh algorifmov v téorii associativnyh sistém* II < Impossibility of certain algorithms in the theory of associative systems II >. ***The Journal of Symbolic Logic***, vol. 13 (1948), p. 53.

[48$^{\mathrm{r}}$b] Review of A. Markov, *O nékotoryh nérazréšimyh problémah kasaúščihsá matric* < On some unsolvable problems concerning matrices >. ***The Journal of Symbolic Logic***, vol. 13 (1948), pp. 53–54.

[48$^{\mathrm{r}}$c] Review of M. Black, *The semantic definition of truth*. ***The Journal of Symbolic Logic***, vol. 13 (1948), pp. 150–151.

[48$^{\mathrm{r}}$d] Review of S. KACZOROWSKI, *Logika matematyczna. Cz. I. Algebra logiki. (Zarys dziejów).* < Mathematical logic. Part I. The algebra of logic (An outline of history) >. ***The Journal of Symbolic Logic***, vol. 13 (1948), p. 167.

[48$^{\mathrm{r}}$e] Review of J. ŁUKASIEWICZ, *O sylogistyce Arystotelesa* < On Aristotle's syllogistic >. ***The Journal of Symbolic Logic***, vol. 13 (1948), p. 167.

[48$^{\mathrm{r}}$f] Review of P.S. NOVIKOV, *O logičéskih paradoksah* < On logical paradoxes >. ***The Journal of Symbolic Logic***, vol. 13 (1948), p. 170.

[49$^{\mathrm{r}}$] Review of S.C. KLEENE, *On the intuitionistic logic*. ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 63.

[49$^{\mathrm{r}}$a] Review of *Filozofia w Stanach Zjednoczonych* 1939–1947 < Philosophy in the United States 1939–1947 >. Therein: H. HIŻ, *Wstęp* < Introduction >; H. HIŻ, *Logika* < Logic >; W. CRAIG, *Teoria indukcji* < Theory of induction >; H.W. JOHNSTONE, *Teoria poznania* < Theory of knowledge >. ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 64.

[49$^{\mathrm{r}}$b] Review of S. JAŚKOWSKI, *Rachunek zdań dla systemów dedukcyjnych sprzecznych* < Un calcul des propositions pour les systèmes déductifs contradictoires >. ***The Journal of Symbolic Logic***, vol. 14 (1949), pp. 66–67.

[49$^{\mathrm{r}}$c] Review of A. MARKOV, *O prédstavlénii rékursivnyh funkcij* < On the representation of recursive functions >. ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 67.

[49$^{\mathrm{r}}$d] Review of L. KALMÁR, *On unsolvable mathematical problems*. ***The Journal of Symbolic Logic***, vol. 14 (1949), pp. 130–131.

[49$^{\mathrm{r}}$e] Review of J. SURÁNYI, *Reduction of the decision problem to formulas containing a bounded number of quantifiers only*. ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 131.

[49$^{\mathrm{r}}$f] Review of K. AJDUKIEWICZ, *Metodologia i metanauka* < Methodology and metascience >. ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 140.

[49$^{\mathrm{r}}$g] Review of A.A. MARKOV, *O zavisimosti axiomy B6 ot drugih aksiom sistémy Bernays'a-Gödel'á* < On the non-independence of axiom B6 from the other axioms of the Bernays-Gödel system >. ***The Journal of Symbolic Logic***, vol. 14 (1949), p. 140.

[49$^{\mathrm{r}}$h] Review of L. HENKIN, *Fragments of propositional calculus*. ***The Journal of Symbolic Logic***, vol. 14 (1949), pp. 197–198.

[49$^{\mathrm{r}}$i] Review of P.S. NOVIKOV, *O klassah régulárnosti* < On regularity classes >. ***The Journal of Symbolic Logic***, vol. 14 (1949), pp. 255–256.

[49$^{\mathrm{r}}$j] Review of P.S. NOVIKOV, *Ob aksiomé polnoj indukcii* < On the axiom of complete induction >. ***The Journal of Symbolic Logic***, vol. 14 (1949), pp. 256–257.

[50$^{\mathrm{r}}$] Review of T. KOTARBIŃSKI, *Przegląd problematyki logiczno-semantycznej* < A review of logico-semantical problems >. ***The Journal of Symbolic Logic***, vol. 15 (1950), p. 65.

[50$^{\mathrm{r}}$a] Review of: H. RASIOWA, *Z dziedziny logiki matematycznej.—I. Klasyczny rachunek zdań* < From the domain of mathematical logic.—I. The classical propositional calculus >; S. GOŁĄB, *O kilku trudnościach związanych z prostymi kwestiami logicznymi* < On some difficulties connected with simple logical questions >; and J. SŁUPECKI, *O własnościach okresu warunkowego* < On properties of the conditional >. ***The Journal of Symbolic Logic***, vol. 15 (1950), pp. 139–140.

[50$^{\mathrm{r}}$b] Review of A. KROKIEWICZ, *O logice stoików* < On the logic of Stoics >. ***The Journal of Symbolic Logic***, vol. 15 (1950), p. 140.

[50$^{\mathrm{r}}$c] Review of B. BRONSTEIN, *Teoria absolutu—Metafizyka jako nauka ścisła* < Theory of the absolute—Metaphysics as an exact science >. ***The Journal of Symbolic Logic***, vol. 15 (1950), p. 206.

[50$^{\mathrm{r}}$d] Review of H. RASIOWA, *Z dziedziny logiki matematycznej. II. Logiki wielowartościowe Łukasiewicza* < From the domain of mathematical logic. II. The many-valued logics of Łukasiewicz >. ***The Journal of Symbolic Logic***, vol. 15 (1950), p. 223.

[50$^{\mathrm{r}}$e Review of B.A. TRAKHTÉNBROT, *Névozmožnost' algorifma dlá problémy razréšimosti na konéčnyh klassah* < Impossibility of an algorithm for the decision problem in finite classes >. ***The Journal of Symbolic Logic***, vol. 15 (1950), p. 229.

[51$^{\mathrm{r}}$] Review of H. WANG, *Remarks on the comparison of axiom systems*. ***The Journal of Symbolic Logic***, vol. 16 (1951), pp. 142–143.

[51$^{\mathrm{r}}$a] Review of A. MARKOV, *Névozmožnost' nékotoryh algoritmov v téorii associativnyh sistém* < Impossibility of certain algorithms in the theory of associative systems >. ***The Journal of Symbolic Logic***, vol. 16 (1951), p. 215.

[52$^{\mathrm{r}}$] Review of A. MARKOV, *Névozmožnost' algorifmov raspoznavaniá nekotoryh svojstv associativnyh sistém* < Impossibility of algorithms for recognizing some properties of associative systems >. ***The Journal of Symbolic Logic***, vol. 17 (1952), pp. 151–152.

[52$^{\mathrm{r}}$a] Review of A. MARKOV, *Ob odnoj nérazréšimoj problémé, kasaûščejsá matric* < On an unsolvable problem concerning matrices >. ***The Journal of Symbolic Logic***, vol. 17 (1952), p. 152.

[52$^{\mathrm{r}}$b] Review of I.I. ŽÉGALKIN (I. Gégalkine), *Probléma razréšimosti na konéčnyh klasah* < Sur le problème de la résolubilité pour les classes finies >. ***The Journal of Symbolic Logic***, vol. 17 (1952), p. 271.

[52$^{\mathrm{r}}$c] Review of G. HASENJAEGER, *Über ω-Unvollständigkeit in der Peano-Arithmetik*. ***The Journal of Symbolic Logic***, vol. 17 (1952), pp. 271–272.

[53$^{\mathrm{r}}$] Review of S.C. KLEENE, *Recursive functions and intuitionistic mathematics*. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 181–182.

[53$^{\mathrm{r}}$a] Review of A.A. MARKOV, *Konstruktivnaá logika* < A constructive logic >. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 257.

[53$^{\mathrm{r}}$b] Review of N.N. VOROB'ÉV, *Konstruktivnoé isčislénié vyskazyvanij s sil'nym otricaniém* < A constructive propositional calculus with strong negation >. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 257–258.

[53$^{\mathrm{r}}$c] Review of N.N. Vorob'év, *Probléma vyvodimosti v konstruktivnom isčislénii vyskazyvanij s sil'nym otricaniém* < The problem of deducibility in the constructive propositional calculus with strong negation >. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 258.

[53$^{\mathrm{r}}$d] Review of A.A. Markov, *Téoriá algorifmov* < The theory of algorithms >. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 340–341.

[53$^{\mathrm{r}}$e] Review of A. Markov, *O nérazréšimyh algorifmičéskih problémah* < On unsolvable algorithmic problems >. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 341.

[53$^{\mathrm{r}}$f] Review of S. Jaśkowski, *O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych* < Sur la conjonction discursive dans le calcul des propositions pour les systèmes contradictoires >. ***The Journal of Symbolic Logic***, vol. 18 (1953), p. 345.

[54$^{\mathrm{r}}$] Review of B.A. Trakhténbrot, *O rékursivnoj otdélimosti* < On recursive separability >. ***The Journal of Symbolic Logic***, vol. 19 (1954), p. 60.

[54$^{\mathrm{r}}$a] Review of P.S. Novikov, *O néprotivoréčivosti nékotoryh položénij déskriptivnoj téorii množéstv* < On the consistency of some theorems of the descriptive theory of sets >. ***The Journal of Symbolic Logic***, vol. 19 (1954), pp. 123–125.

[54$^{\mathrm{r}}$b] Review of N.M. Nagornyj, *K usiléniú téorémy privédéniá téorii algorifmov* < Towards strengthening of the reduction theorem of the theory of algorithms >. ***The Journal of Symbolic Logic***, vol. 19 (1954), p. 218.

[54$^{\mathrm{r}}$c] Review of V.A. Uspénskij, *Téoréma Gëdélá i téoriá algoritmov* < Theorem of Gödel and theory of algorithms >. ***The Journal of Symbolic Logic***, vol. 19 (1954), pp. 218–219.

[54$^{\mathrm{r}}$d] Review of two papers of N.A. Šanin: *O nékotoryh opéraciáh nad logiko-arifmétičéskimi formulami* < Some operations on logico-arithmetical formulas > and *O pogružéniáh klassičéskogo logiko-aritmétičéskogo isčisléniá v konstruktivnoé logiko-aritmétičéskoé isčislénié* < On imbeddings of the classical logico-arithmetical calculus into the constructive logico-arithetical calculus >. ***The Journal of Symbolic Logic***, vol. 19 (1954), pp. 297–298.

[55$^{\mathrm{r}}$] Review of Ú.V. Linnik and N.A. Šanin, *Andréj Andréévič Markov*. (*K pátidésátilétiú so dná roždéniá*) < A.A. Markov. (On his fiftieth birthday) >. ***The Journal of Symbolic Logic***, vol. 20 (1955), p. 73.

[55$^{\mathrm{r}}$a] Review of A.S. Ésénin-Vol'pin, *Nédokazuémost' gipotézy Suslina béz pomošči aksiomy vybora v sistémé aksiom Bérnajsa-Mostovskogo* < The unprovability of Suslin's hypothesis without the aid of the axiom of choice in the system of axioms of Bernays-Mostowski >. ***The Journal of Symbolic Logic***, vol. 20 (1955), pp. 181–182.

[55$^{\mathrm{r}}$b] Review of L. Henkin, *Some interconnections between modern algebra and mathematical logic*. ***The Journal of Symbolic Logic***, vol. 20 (1955), pp. 183–184.

[56^{r}] Review of Ú.T. MÉDVÉDÉV, *O néizomorfnyh rékursivno-péréčislimyh množéstvah* < On nonisomorphic recursively enumerable sets >. ***The Journal of Symbolic Logic***, vol. 21 (1956), pp. 101–102.

[56^{r}a] Review of G.N. POVAROV, *O funkcional'noj razdélimosti bulévyh funkcij* < On functional decomposability of Boolean functions >. ***The Journal of Symbolic Logic***, vol. 21 (1956), p. 102.

[56^{r}b] Review of S.V. JÁBLONSKIJ, *Réalizaciá linéjnoj funkcii v klassé* Π*-shém* < Realization of a linear function in the class of Π-circuits >. ***The Journal of Symbolic Logic***, vol. 21 (1956), p. 102.

[56^{r}c] Review of B.A. TRAKHTÉNBROT, *Tabličnoé prédstavlénié rékursivnyh opératorov* < Matrix representation of recursive operators >. ***The Journal of Symbolic Logic***, vol. 21 (1956), pp. 207–208.

[56^{r}d] Review of Ú.T. MÉDVÉDÉV, *Stépéni trudnosti massovyh problém* < Degrees of difficulty of mass problems >. ***The Journal of Symbolic Logic***, vol. 21 (1956), pp. 320–321.

[56^{r}e] Review of B.S. SODNOMOV, *Néprotivoréčivost'proéktivnoj océnki nékotoryh néeffféktivnyh množéstv* < Consistency of the projective evaluation of non-effective sets >. ***The Journal of Symbolic Logic***, vol. 21 (1956), pp. 406–407.

[56^{r}f] Review of E.W. BETH, *Les fondements logiques des mathématiques*. ***Studia Logica***, vol. 4 (1956), pp. 261–264.

[56^{r}g] Review of H. GRENIEWSKI, *Elementy logiki formalnej* < Elements of formal logic >. ***Studia Logica***, vol. 4 (1956), pp. 272–277.

[57^{r}] Review of V.A. USPÉNSKIJ, *O vyčislimyh opéraciáh* < On calculable operations >. ***The Journal of Symbolic Logic***, vol. 22 (1957), pp. 76–77.

[57^{r}a] Review of G.N. POVAROV, *K izučéniú simmétričéskih Bulévyh funkcij s točki zréniá téorii relějno-kontaktnyh shém* < Concerning the study of symmetric Boolean functions from the point of view of the theory of relay-contact circuits >. ***The Journal of Symbolic Logic***, vol. 22 (1957), p. 99.

[57^{r}b] Review of G.S. CÉJTIN, *Associativnoé isčislénié s nérazréšimoj problémoj ekvivaléntnosti* < An associative calculus with an unsolvable equivalence problem >. ***The Journal of Symbolic Logic***, vol. 22 (1957), pp. 219.

[57^{r}c] Review of G.S. CÉJTIN, *Otnositél'no problémy raspoznavaniá svojstv associativnyh isčislénij* < Concerning the problem of recognizing properties of associative calculi >. ***The Journal of Symbolic Logic***, vol. 22 (1957), pp. 219–220.

[57^{r}d] Review of G. KREISEL, *A variant to Hilbert's theory of the foundations of arithmetic*. ***The Journal of Symbolic Logic***, vol. 22 (1957), pp. 304–306.

[58^{r}] Review of V.A. USPÉNSKIJ, *Vyčislimyé opéracii i ponátié programmy* < Calculable operations and the notion of a program >. ***The Journal of Symbolic Logic***, vol. 23 (1958), p. 49.

[58ra] Review of V.A. USPÉNSKIJ, *Néskol'ko zaméčanij o péréčislimyh množéstvah* < Some notes on recursively enumerable sets >. ***The Journal of Symbolic Logic***, vol. 23 (1958), pp. 49–50.

[58rb] Review of A.A. MUČNIK, *Ob otdélimosti rékursivno-péréčislimyh množestv* < On separability of recursively enumerable sets >. ***The Journal of Symbolic Logic***, vol. 23 (1958), pp. 224–225.

[58rc] Review of V.A. USPÉNSKIJ, *Ob algoritmičéskoj svodimosti* < On algorithmic reducibility >. ***The Journal of Symbolic Logic***, vol. 23 (1958), p. 225.

[59r] Review of two papers by A. MAL'CEV: *Ob odnom obščém métodé polučéniá lokal'nyh téorém téorii grupp* < On a general method for obtaining local theorems in group theory > and *O prédstavléniáh modéléj* < On representations of models > (review co-authored with Leon Henkin), ***The Journal of Symbolic Logic***, vol. 24 (1959), pp. 55–57.

[59ra] Review of A. MAL'CEV, *Kvaziprimitivnyé klassy abstraktnyh algébr* < Quasiprimitive classes of abstract algebras >. ***The Journal of Symbolic Logic***, vol. 24 (1959), p. 57.

[59rb] Review of A. LEVY, *A note on definitions of finiteness*. ***The Journal of Symbolic Logic***, vol. 24 (1959), p. 172.

[60r] Review of A. LEVY, *The indenpendence of various definitions of finiteness*. ***The Journal of Symbolic Logic***, vol. 25 (1960), p. 179.

[62r] Review of T. HAILPERIN, *A complete set of axioms for logical formulas invalid in some finite domains*. ***The Journal of Symbolic Logic***, vol. 27 (1962), pp. 108–109.

[62ra] Review of R. SUSZKO, *Czy logika znów pod znakiem zapytania* < Are there again doubts about the teaching of logic >; K. SZANIAWSKI, *Parę uwag w sprawie niesłusznej decyzji* < Some remarks on an unjust decision >; M. PRZEŁĘCKI, *Jeszcze w sprawie logiki* < Once again concerning logic >; J. WRÓBLEWSKI, *W sprawie wykładu logiki dla prawników* < Concerning the course of logic for the students of law >; J. GREGOROWICZ, *Kilka uwag w sprawie logiki dla prawników* < Some remarks on logic for the students of law>; A. GRZEGORCZYK, *Uwagi o nauczaniu logiki* < Remarks concerning the teaching of logic >; A. PODGÓRECKI, *W sprawie błędów w nauczaniu logiki* < Concerning errors in the teaching of logic >; J. GIEDYMIN, *W sprawie sposobu nauczania logiki* < Concerning ways of the teaching of logic >; T. KOTARBIŃSKI, *Sprawa logiki w przededniu rozstrzygnięć* < Forthcoming decisions about logical studies>; and of K. AJDUKIEWICZ, *W sprawie programów logiki usługowej* < Concerning curricula of logic as an auxiliary subject >. ***The Journal of Symbolic Logic***, vol. 27 (1962), pp. 223–224.

[62rb] Review of B.A. TRAKHTÉNBROT, *Oprédélénié konečnogo množéstva i deduktivnaá népolnota téorii množéstv* < The definition of a finite set and the deductive incompleteness of set theory >. ***The Journal of Symbolic Logic***, vol. 27 (1962), pp. 236–237.

[62$^{\mathrm{r}}$c] Review of P.S. Novikov, *O néprotivoréčivosti nékotoryh logičéskih isčislénij* < On the consitency of some logical calculi >. ***The Journal of Symbolic Logic***, vol. 27 (1962), p. 246.

[62$^{\mathrm{r}}$d] Review of Ú.I. Ánov and A.A. Mučnik, *O suščestvovanii k-značnyh zamknutyh klassov, né iméúščih konéčnogo bazisa* < On the existence of k-valued closed classes not having a finite basis >. ***The Journal of Symbolic Logic***, vol. 27 (1962), p. 247–248.

[63$^{\mathrm{r}}$] Review of R.L. Goodstein, *Models of propositional calculi in recursive arithmetic*. ***The Journal of Symbolic Logic***, vol. 28 (1963), p. 291.

[63$^{\mathrm{r}}$a] Review of V. Vučkovič, *Rekurzivni modeli nekih neklasičnih izkaznih računa* < Rekursive Modelle einiger nichtklassichen Aussagenkalküle >. ***The Journal of Symbolic Logic***, vol. 28 (1963), pp. 291–292.

[63$^{\mathrm{r}}$b] Review of K. Hauschild, *Modelle der Mengenlehre, die aus endlichen Mengen bestehen*. ***Mathematical Reviews*** 26 # 2358.

[63$^{\mathrm{r}}$c] Review of P. Vopěnka, *Models for set theory*. ***Mathematical Reviews*** 26 # 3610.

[63$^{\mathrm{r}}$d] Review of P. Vopěnka, *Construction of models for set theory by the method of ultra-products*. ***Mathematical Reviews*** 26 # 3611.

[63$^{\mathrm{r}}$e] Review of L. Rieger, *On the consistency of the generalized continuum hypothesis*. ***Mathematical Reviews*** 26 # 3617.

[63$^{\mathrm{r}}$f] Review of H.J. Keisler, *Limit ultrapowers*. ***Mathematical Reviews*** 26 # 6054.

[64$^{\mathrm{r}}$] Review of L. Rieger, *Ob algébrah Suslina (S-algébrah) i ih prédstavlénii* < On Suslin algebras and their representations >. ***The Journal of Symbolic Logic***, vol. 29 (1964), pp. 96–97.

[64$^{\mathrm{r}}$a] Review of B.A. Trakhténbrot, *Nékotoryé postroéniá v logiké odnoméstnyh prédikatov* < Certain constructions in the logic of one-place predicates >. ***The Journal of Symbolic Logic***, vol. 29 (1964), p. 100.

[64$^{\mathrm{r}}$b] Review of N.A. Šanin, *O linéjnyh konstruktivnyh funkcionalah v konstruktivnom gil'bértovom prostranstvé* < Über konstruktive lineare Funktionale in einem konstruktiven Hilbertschen Raum >. ***The Journal of Symbolic Logic***, vol. 29 (1964), pp. 107–108.

[64$^{\mathrm{r}}$c] Review of H. Greniewski, 2^{n+1} *wartości logicznych* < 2^{n+1} *logical values* >. ***The Journal of Symbolic Logic***, vol. 29 (1964), pp. 109–110.

[64$^{\mathrm{r}}$d] Review of P.J. Cohen, *A minimal model for set theory*. ***Mathematical Reviews*** 27 # 41.

[64$^{\mathrm{r}}$e] Review of A. Tarski, *Some problems and results relevant to the foundations of set theory*. ***Mathematical Reviews*** 27 # 1382.

(1) Reprinted in ZMG 199, pp. 18–19.

[64$^{\mathrm{r}}$f] Review of L. Rieger, *A contribution to Gödel's axiomatic set theory. III*. ***Mathematical Reviews*** 27 # 3544.

[64[r]g] Review of G.E. MINC and E.P. OREVKOV, *A generalization of the theorems of V.I. Glivenko and G. Kreisel to a class of formulae of the predicate calculus*. ***Mathematical Reviews*** 27 # 4737.

[64[r]h] Review of K. HAUSCHILD, *Ein Beitrag zur Metatheorie der Mengenlehre*. ***Mathematical Reviews*** 27 # 5690.

[64[r]i] Review of A.I. MAL'CEV, *Some problems in the theory of classes of models*. ***Mathematical Reviews*** 27 # 5693.

[64[r]j] Review of P. COHEN, *The independence of the continuum hypothesis*. ***Mathematical Reviews*** 28 # 1118.

[64[r]k] Review of P. COHEN, *The independence of the continuum hypothesis. II*. ***Mathematical Reviews*** 28 # 2962.

[64[r]l] Review of H. LÄUCHLI, *The independence of the ordering principle from a restricted axiom of choice*. ***Mathematical Reviews*** 28 # 5007.

[65[r]] Review of V.F. KOSTYRKO, *Ob ošibké v stat'é I.I. Žégalkina* "*Probléma razréšimosti na konéčnyh klassah*" < On a mistake in the paper of *Žégalkin* "*On the decision problem in finite classes*" >. ***The Journal of Symbolic Logic***, vol. 30 (1965), p. 254.

[65[r]a] Review of P.S. NOVIKOV, *Eléménty matématičéskoj logiki* < Elements of mathematical logic >. ***The Journal of Symbolic Logic***, vol. 30 (1965), pp. 356–357.

[65[r]b] Review of J.R. SHOENFIELD, *The problem of predicativity*. ***Mathematical Reviews*** 29 # 2177.

[65[r]c] Review of P. HÁJEK, *Die durch die schwach inneren Relationen gegebenen Modelle der Mengenlehre*. ***Mathematical Reviews*** 29 # 2179.

[65[r]d] Review of J.D. HALPERN, *The independence of the axiom of choice from the Boolean prime ideal theorem*. ***Mathematical Reviews*** 29 # 2182.

[65[r]e] Review of G. FUHRKEN, *Skolem-type normal forms for first-order languages with a generalized quantifier*. ***Mathematical Reviews*** 29 # 3363.

[65[r]f] Review of R.L. VAUGHT, *The completeness of logic with the added quantifier "there are uncountably many"*. ***Mathematical Reviews*** 29 # 3364.

[65[r]g] Review of R. DOSS, *On Gödel's proof that $V = L$ implies the generalized continuum hypothesis*. ***Mathematical Reviews*** 29 # 3388.

[65[r]h] Review of P. VOPĚNKA, *The indenpendence of the continuum hypothesis*. ***Mathematical Reviews*** 29 # 5742.

[65[r]i] Review of P. VOPENKA and L. BUKOVSKY, *The existence of a PCA-set of cardinal aleph one*. ***Mathematical Reviews*** 29 # 5743.

[65[r]j] Review of Z.I. KOZLOVA, *The axiom of constructivity and multiple separability and inseparability in classes of the analytic hierarchy*. ***Mathematical Reviews*** 30 # 1936.

[65[r]k] Review of A.H. LACHLAN, *Recursive real numbers*. ***Mathematical Reviews*** 30 # 3013.

[65rl] Review of A. GRZEGORCZYK, *A philosophically plausible formal interpretation of intuitionistic logic*. ***Mathematical Reviews*** 30 # 3015.

[66r] Review of B. SOBOCIŃSKI, *On the propositional system A of Vučković and its extension*. ***The Journal of Symbolic Logic***, vol. 31 (1966), p. 118.

[66ra] Review of W.A. POGORZELSKI and J. SŁUPECKI, *O dowodzie matematycznym* < On mathematical proof >. ***The Journal of Symbolic Logic***, vol. 31 (1966), p. 284.

[66rb] Review of J. ŁUKASIEWICZ, *Elementy logiki matematycznej* < Elements of mathematical logic >. ***The Journal of Symbolic Logic***, vol. 31 (1966), p. 284.

[66rc] Review of J. ŁUKASIEWICZ, *Aristotélévskaá sillogistika s točki zréniá sovréménnoj formal'noj logiki* < Aristotle's syllogistic from the standpoint of modern formal logic >. ***The Journal of Symbolic Logic***, vol. 31 (1966), p. 285.

[66rd] Review of P. HÁJEK, *Eine Bemerkung über standarde nichtreguläre Modelle der Mengenlehre*. ***Mathematical Reviews*** 31 # 36.

[66re] Review of S. FEFERMAN, *Some applications of the notions of forcing and generic sets*. ***Mathematical Reviews*** 31 # 1193.

[66rf] Review of H.J. KEISLER, *Complete theories of algebraically closed fields with distinguished subfields*. ***Mathematical Reviews*** 31 # 3331.

[66rg] Review of R. KURATA, *On the existence of a proper complete model of set theory*. ***Mathematical Reviews*** 31 # 4723.

[66rh] Review of E. ELLENTUCK, *The universal properties of Dedekind finite cardinals*. ***Mathematical Reviews*** 31 # 4729.

[66ri] Review of T. NISHIMURA and H. TANAKA, *On two systems for arithmetic*. ***Mathematical Reviews*** 32 # 37.

[66rj] Review of P. VOPENKA, *The limits of sheaves and applications on constructions of models*. ***Mathematical Reviews*** 32 # 53.

[66rk] Review of P. VOPENKA, *On ∇-model of set theory*. ***Mathematical Reviews*** 32 # 54.

[66rl] Review of A.I. MAL'CEV, *Positive and negative enumerations*. ***Mathematical Reviews*** 32 # 1119.

[66rm] Review of W. SADOWSKI, *A proof of axiomatizability of certain n-valued sentential calculi*. ***Mathematical Reviews*** 32 # 2318.

[66rn] Review of J.R. ISBELL and F.B. WRIGHT, *Another equivalent form of the axiom of choice*. ***Mathematical Reviews*** 32 # 3994.

[66ro] Review of S. KACZOROWSKI, *On cardinal numbers*. ***Mathematical Reviews*** 32 # 4018.

[66rp] Review of G. HASENJAGER, *Zur arithmetischen Klassifikation reeller Zahlen*. ***Mathematical Reviews*** 32 # 5506.

[66^{r}q] Review of O.M. SARKOVSKIĬ, *On attracting and attracted sets*. ***Mathematical Reviews*** 32 # 6419.

[66^{r}r] Review of K. HÄRTIG, *Über einen Quantifikator mit zwei Wirkungsbereichen*. ***Mathematical Reviews*** 32 # 7396.

[66^{r}s] Review of M.I. KRATKO, *On a certain class of Post calculi*. ***Mathematical Reviews*** 32 # 7414.

[67^{r}] Review of S.R. KOGALOVSKIĬ, *On finitely reduced classes of models*. ***Mathematical Reviews*** 33 # 47.

[67^{r}a] Review of R.C. LYNDON, *Notes on Logic*. ***Mathematical Reviews*** 33 # 2521.

[67^{r}b] Review of G. KREISEL, *Model theoretic invariants: Applications to recursive and hyperarithmetic operations*. ***Mathematical Reviews*** 33 # 7257.

[67^{r}c] Review of Ja.L. KREĬNIN, *On $C\Phi$-kernels of sets effectively different from Φ-sets*. ***Mathematical Reviews*** 33 # 7263.

[67^{r}d] Review of N. BOURBAKI, *Éléments de mathématique. Fascicule XXII. Livre I: Théorie des ensembles. Chapitre 4: Structures*. ***Mathematical Reviews*** 34 # 2425.

[68^{r}] Review of S.R. KOGALOVSKIĬ, *On higher order logic*. ***Mathematical Reviews*** 35 # 35.

[68^{r}a] Review of A.N. ŠARKOVSKIĬ [O.M. Šarkovs'kiĭ], *Partially ordered system of attracting sets*. ***Mathematical Reviews*** 35 # 311.

[68^{r}b] Review of J. van HEIJENOORT, *From Frege to Gödel. A source book in mathematical logic, 1879–1931*. ***Synthese***, vol.18 (1968), pp. 302–305.

[69^{r}] Review of A.I. MAL'CÉV, *Modél'nyé sootvéctviá* < Model correspondences >. ***The Journal of Symbolic Logic***, vol. 34 (1969), pp. 299–300.

[69^{r}a] Review of A.I. MAL'CÉV, *O malyh modélah* < On small models >. ***The Journal of Symbolic Logic***, vol. 34 (1969), pp. 513–514.

[70^{r}] Review of five papers of P. VOPĚNKA: *Odin métod postroéniá néstandartnoj modéli aksjomatičéskoj téorii množéstv Bérnajsa Gédélá* < A method of constructing a non-standard model for the axiomatic set theory of Barnays-Gödel >; *Postroénié modéléj téorii množéstv métodom spéktra* < Konstruktion von Modellen der Mengenlehre mit der Methode der Spektren >; *Eléméntarnyé ponátiá v téorii množéstv* < Elementare Begriffe in der Mengenlehre >; *Postroénié néstandartnoj nérégulárnoj modéli téorii množéstv* < Konstruktion von nichtregulären Nichtstandardmodellen der Mengenlehre >; *Podmodéli modéléj téorii množéstv* < Teilmodelle von Modellen der Mengenlehre >. ***The Journal of Symbolic Logic***, vol. 35 (1970), pp. 470–472.

[72^{r}] Review of S.R. KOGALOVSKIJ, *K sémantiké teorii tipov* < On the semantics of the theory of types >. ***The Journal of Symbolic Logic***, vol. 37 (1972), pp. 193–194.

[72^{r}a] *Matematyka a logika. Refleksje przy lekturze książki A. Grzegorczyka Zarys arytmetyki teoretycznej wraz z próbą recenzji* < Mathematics and logic. Reflections while reading the book by A. Grzegorczyk *An outline of theoretical arith-*

metic, along with an attempt at a review >. ***Wiadomości Matematyczne***, vol. 15 (1972), pp. 79–89. [MR 58 # 4995 (Editors).]

[73$^{\mathrm{r}}$] Review of A.D. TAJMANOV, *Haraktéristika aksiomatiziruémyh klassov modéléj. I.* < Characterization of axiomatizable classes of models. I. >. ***The Journal of Symbolic Logic***, vol. 38 (1973), pp. 164–165.

[73$^{\mathrm{r}}$a] Review of A.D. TAJMANOV, *Haraktéristika aksiomatiziruémyh klassov modéléj. II.* < Characterization of axiomatizable classes of models. II. >. ***The Journal of Symbolic Logic***, vol. 38 (1973), p. 165.

[73$^{\mathrm{r}}$b] Review of three papers of A.D. TAJMANOV: *Haraktéristika konéčno-aksiomatiziruémyh klassov modéléj; Haraktéristika konéčno aksiomatiziruémyh klassov modéléj*, and *Characterization of finitely axiomatizable classes of models*. ***The Journal of Symbolic Logic***, vol. 38 (1973), pp. 165–166.

[74$^{\mathrm{r}}$] Review of H.B. ENDERTON and H. FRIEDMAN, *Approximating the standard model of analysis*. ***The Journal of Symbolic Logic***, vol. 39 (1974), pp. 600–601.

VII. Editorial Works

[75$^{\mathrm{e}}$] WACŁAW SIERPIŃSKI, ***Oeuvres choisies. Tome II: Théorie des ensembles et ses applications. Travaux des années 1908–1929***. Publié par les soins de Stanisław Hartman, Kazimierz Kuratowski, Edward Marczewski, Andrzej Mostowski et Andrzej Schinzel, PWN—Éditions Scientifiques de Pologne, Warszawa 1975, 780 pp. (See [75b].)

[76$^{\mathrm{e}}$] WACŁAW SIERPIŃSKI, ***Oeuvres choisies. Tome III: Théorie des ensembles et ses applications. Travaux des années 1930–1966***. Publié par les soins de Stanisław Hartman, Kazimierz Kuratowski, Edward Marczewski, Andrzej Mostowski et Andrzej Schinzel, PWN—Éditions Scientifiques de Pologne, Warszawa 1976, 685 pp. + errata.

VIII. Miscellanea

[54$^{\mathrm{m}}$] *Podstawy matematyki na VIII Zjeździe Matematyków Polskich* < Foundations of mathematics at the VIII $^{\mathrm{th}}$ Congress of Polish Mathematicians >, ***Myśl Filozoficzna***, vol. 2 (12) (1954), pp. 328–330. (Popular account of [54]. See also [54$^{\mathrm{m}}$a].)

[54$^{\mathrm{m}}$a] *VIII Zjazd Matematyków Polskich* (*Warszawa, 6–12 września 1953*) < The VIII$^{\mathrm{th}}$ Congress of Polish Mathematicians (Warsaw, 6–12 September 1953) > (with K. Kuratowski), ***Nauka Polska***, vol. 2 no. 1 (1954), pp. 147–151. (Popular account of [54]. See also [54$^{\mathrm{m}}$].)

[56$^{\mathrm{m}}$] *Logika matematyczna na Międzynarodowym Zjeździe Matematyków w Amsterdamie* < Mathematical logic at the International Congress of Mathematicians in Amsterdam >, ***Studia Logica***, vol. 4 (1956), pp. 245–253. [MR 19, p. 237 (J. Łoś).]

[66$^{\mathrm{m}}$] *Przedmowa* < Foreword >, ***Matematyka w świecie współczesnym. Zbiór artykułów z "Scientific American"***, < Contemporary mathematics. A collection of articles from “Scientific American” >, Biblioteka Problemów, vol. 106, Państwowe Wydawnictwo Naukowe, Warszawa 1966, pp. 5–7.

[69$^{\mathrm{m}}$] *List A. Mostowskiego w sprawie błędów w książce: K. Kuratowski i A. Mostowski, Teoria mnogości* < A letter of A. Mostowski to correct errors in the book by K. Kuratowski and A. Mostowski, *Theory of sets* >, ***Wiadomości Matematyczne***, vol. 11 (1969), pp. 152–153. [MR 39 # 5370 (Editors).]

[75$^{\mathrm{m}}$] *Szkoła a uniwersytet* < School vs. university >, ***Wiadomości Matematyczne***, vol. 19 (1975), pp. 51–60.

[79$^{\mathrm{m}}$] *Fragment notatek do wykładu z historii matematyki* < An excerpt from notes to a course in the history of mathematics >, ***Wiadomości Matematyczne***, vol. 22 (1979), pp. 76–77. [MR 82h:01057 (S.J. Surma).]

[79$^{\mathrm{m}}$a] *Fragment książki Logika matematyczna* < An excerpt from the book *Logika matematyczna* >, ***Wiadomości Matematyczne***, vol. 22 (1979), pp. 77–78. [MR 82c:01043 (Editors).]

This page intentionally left blank

Section 4

Memories

This page intentionally left blank

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Warsaw 1957: Memories of Mostowski

J. W. ADDISON
Department of Mathematics, University of California, Berkeley, USA
addison@math.berkeley.edu

During the 1940's various logicians had noticed an analogy between the arithmetical hierarchy of sets of natural numbers (studied by Stephen Kleene and independently by Andrzej Mostowski) and the projective hierarchy of sets of real numbers. Alfred Tarski, for example, had called attention to this in his address at the Princeton Bicentennial Conference in 1946. In 1950 Kleene proved that there exist two disjoint recursively enumerable sets which are not separable by a recursive set, and hence that the first separation principle[1] for the class of recursively enumerable sets fails. Mostowski recognized that this broke the then current analogy since Nikolai Luzin had proved in a fundamental paper in 1927 that two disjoint analytic sets of real numbers are always separable by a Borel set and hence that the first separation principle for analytic sets holds. Mostowski wrote to Kleene about this and when I arrived at the University of Wisconsin as a graduate student in fall 1951 Kleene (who was largely unfamiliar with what the Poles and Russians were calling "descriptive set theory") asked me to study and explain this breakdown.

By shifting from the reals to the irrationals (or equivalently to the Baire space $\mathcal{N}$ of functions from ω into ω) and recognizing that a continuous function from $\mathcal{N}$ into $\mathcal{N}$ is just a function recursive in some element of $\mathcal{N}$ I was able to correct the analogy to one between an effective Borel hierarchy on ω or on $\mathcal{N}$ and the classical Borel hierarchy on $\mathcal{N}$ and one between an effective projective hierarchy on ω or on $\mathcal{N}$ and the classical projective hierarchy on $\mathcal{N}$. The effective Borel hierarchy on ω was just a variant of Kleene's hyperarithmetical hierarchy and the effective projective hierarchy on ω the same as the analytical hierarchy then being developed by Kleene. Indeed I constructed a general theory containing the four hierarchies on ω and on $\mathcal{N}$ as special cases. To present the general theory it was natural to seek a convenient system of notation embracing all of the hierarchies. Recognizing that Σ and Π had been used as quantifiers by some logicians I invented the Σ^t_n and Π^t_n notation, with the t representing the type of variables being quantified and the n representing the number of homogeneous blocks of quantifiers applied to prenex formulas. Lightface Σ and Π would indicate the effective cases (on ω and on $\mathcal{N}$) while boldface $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$ would stand for the other ('fully relativized') extreme. There was the question of whether Σ and Π should represent the outermost homogeneous block or the innermost block. The normal left-to-right reading of most languages as well as the pattern of separation principles in the superscript-0 hierarchies led to the decision to go with the outermost- quantifier notation. Among those I consulted (including Kleene

[1] A class C of subsets of a nonempty set U has the *first separation property relative to* U (as formulated by Kazimierz Kuratowski in 1936) iff for any disjoint sets X, Y in C there exists a superset Z of X which is disjoint from Y and such that Z and $U - Z$ are in C.

and Clifford Spector) the only mention of an important role for the innermost block was given by Kleene, who noted that the inner quantifier seemed to play an important role in his then current work on universal function-quantifier formulas.

When I won a U.S. National Science Foundation Postdoctoral Fellowship for the academic year 1956–1957 it was natural that Kleene should suggest that I spend it in Warsaw with Mostowski. But this was the height of the McCarthy era in the U.S. and the Dean at the University of Michigan where I was teaching refused to bless the proposed study plans for one of their faculty to go to a Communist country and the head of the Mathematics section of the National Science Foundation also declined to write an official blessing! Despite Kleene's worry that I might have trouble finding a job after a year under Communism (after all, the position I held at Michigan had been freed up by the firing of a mathematician thought to have Communist sympathies), I left Michigan and arrangements were made with Mostowski for me to join him. However world events intervened with the Hungarian revolution and some related unsettling events in Poland. For these or perhaps other unknown reasons I was unable to get the necessary visa to go in September as planned. So I spent the fall of 1956 at the Institute for Advanced Study in Princeton where Director J. Robert Oppenheimer apologized to me that they did not have a Mostowski there to serve as my mentor but that he hoped Kurt Gödel would do.

But Mostowski's name did arise during my stay at the Institute. My close friend and fellow IAS member Joseph Shoenfield mentioned to me that Mostowski had told him that Gödel's unpublished proof of the independence of the axiom of choice in type theory was based on a reinterpretation of the propositional connectives. Was this not a striking precursor of the eventual Boolean-valued models approach to independence proofs in set theory? I even went so far as to ask Gödel at one of our weekly sessions to show me his independence proof. He replied by asking me why I wanted to see it. I always later had the feeling that if I could have pulled out an unfinished manuscript with a gap for an unproved lemma and said I thought his proof might give me an idea for proving it, he might have been more forthcoming.

An article in the 8 December 1956 issue of *The New York Times* was headlined "Red Tape on Visas Relaxed by Poles". It noted that: "For years, visas for Polish travel have been extremely difficult to obtain. Those that were granted required months of negotiations." Now "Orbis, the Polish government travel agency, had announced that Americans desiring to travel to Poland would receive either their visas or notice of refusal within eight days after applying". In a short time thereafter our visas were approved and my wife Mary Ann and I embarked on the Queen Mary for Europe. We boarded the Orient Express in Paris and, while finishing up our journey on a train from Prague to Warsaw, we were shocked at what we supposed to be a very friendly border crossing between two fellow Communist countries, Czechoslovakia and Poland: the train was stopped under a bridge lined with armed riflemen while soldiers boarded the train and poked their bayonets under our seats to make sure that no one (still living) was hiding there. We arrived safely in Warszawa on the very day in January when Władysław Gomułka was elected leader.

I soon learned that logic and Mostowski had a status in Poland far higher than one would have expected from the situation in the U.S. Indeed I was astonished already on the train ride into Poland from Czechoslovakia when I mentioned to my working-man compartment-mate the name of Mostowski and found that he had heard of him. And I was stunned to have my arrival as a lowly junior scholar to work under Mostowski heralded with an article on the front page of the major state newspaper, *Trybuna Ludu*, headlined "American Mathematician at Mathematical Institute of the Polish Academy of Science for Studies". The article began with: "The Polish Mathematical School is happy to shine in a well deserved glory. The works of eminent Polish mathematicians Professors Sierpiński, Banach, Kuratowski, Mostowski, and others are being published not only in Poland, but also in a number of other countries." And now there was "the first sign of direct contact [between American and Polish mathematicians] with an arrival from the scientific community of the United States at the Mathematical Institute".

Mostowski turned out to be a wonderful and unusually thoughtful host. He had thought it would be best if Mary Ann and I could live a normal 'Polish' life and he had arranged for us to rent the apartment of a colleague who was soon to leave for a stay in China. We met the colleague and were shown his apartment, but at the last moment the colleague, an ardent Communist, backed out of the deal (apparently for fear of losing status with his neighbors, who might think he was profiteering off of "rich" Americans) to the great embarrassment of Mostowski. In the end we spent our entire stay in Warsaw in Room 317 of the Hotel Bristol. Much of Warsaw was still filled with wrecks of buildings damaged during the war and occupation, but the Bristol had served as headquarters of the German command and so was (aside from the occasional bullet mark) in relatively good shape.

Mostowski arranged for me to share an office with himself and Andrzej Grzegorczyk at the Mathematical Institute at Śniadeckich 8. He said he would send Grzegorczyk to the Hotel Bristol to guide me to the Institute. When I asked him how I would recognize Andrzej (whom I had never met) he answered in his typical charming and amusing manner by saying that it would be very easy to recognize him because he would be the one who looked like a saint! (And sure enough the description fit him to a tee and we spotted him without difficulty.)

The atmosphere in our shared office was very stimulating (especially so since Mostowski liked to keep the window open even in the depth of the Polish winter). He quickly encouraged me to write up some of the part of my dissertation about the analogies between descriptive set theory and recursive function theory. Meanwhile he was busy putting the finishing touches on his famous paper on ω-completeness with Grzegorczyk and Ryll-Nardzewski. As I wrote up "Separation principles in the hierarchies of classical and effective descriptive set theory" for *Fundamenta Mathematicae*, Mostowski and I had intensive discussions about notation. I explained to him the advantages of the sigma-and-pi notation and noted that they were using an older notation in their joint paper. Eventually I convinced him of the advantages of my new notation and he decided to use it in their paper. Grzegorczyk as junior partner was assigned over his objections to redo the whole paper in the sigma-and-pi notation, a formidable and time-consuming task in those days before computers and even before whiteout! I am not sure whether or not Grzegorczyk has ever fully forgiven me for this.

Everyday life in Warsaw was enormously inefficient and waiting in lines was a principal consumer of time. Mary Ann recalls a time when Mostowski arrived late for an appointment because he had been standing in line for hours in a shop to buy a very scarce pound of butter. I often thought later about those Americans who were reluctant to let me come to this Communist society. Little did they realize that perhaps the best way of 'curing' someone with strong Communist leanings would be to let them actually live in Poland for a year! However the Polish people were wonderful and that more than made up for any hardships in the style of living. It was especially nice to be an American, since the people seemed to have a great affection for us. When one was recognized on the street as an American (as I sometimes was because of my white tennis shoes!) strangers would come up, ask if you knew, say, Uncle Staszek in Chicago, and perhaps invite you to come to their home for a meal. Transportation was sometimes a problem. Never in my life have I been in such crowded streetcars! With the enormous crush of people it was almost possible to ride with one's feet off the ground, sandwiched between the bodies of fellow passengers. One day as we approached the stop where we wanted to get off but seemed to have no prospect of getting through the crowds to the door I recalled my days as a boy playing hide-and-seek and suddenly shouted loudly "Olly olly oxen all in free". The crowd in front of us parted like the biblical Red Sea waters (probably thinking I was about to throw up on them) and Mary Ann and I were easily able to leave the streetcar at our stop.

The ever-thoughtful Mostowski arranged for Mary Ann to do research on hedgehog skulls at the Zoological Institute. And he asked in his undergraduate class for volunteers who would like to show the young American couple around Warsaw and Poland. Halina Marcinek and Staszek Klasa volunteered and added enormously to the richness of our stay, culminating in a wonderful trip to Zakopane and the High Tatra.

It was quite a thrill for me to get to know such famous mathematicians as Sierpiński and Kuratowski, whose books I had studied so carefully as a student. (My copy of *Topologie I* was literally coming apart at the seams.) Both were extremely nice to me and I had the pleasure of entertaining Kuratowski at my home in Michigan several years later. He was naturally concerned at that time about the loss of so many Polish scientists to the U.S., a concern that I understood fully and shared, although also seeing the great advantage to U.S. scientific culture of the influx of so many talented people from abroad. Although ironically perhaps the greatest contributor to American mathematical excellence was Hitler, one must also credit Stalin for his share in the process.

There was an impressive amount of research in logic going on at the Institute and the University. Andrzej Ehrenfeucht, a brilliant young graduate student (and also a pretty fair ping-pong player) was frequently in our office. Helena Rasiowa was very friendly and helpful and there were seminars in the Institute, which would on occasion attract visitors like Ryll-Nardzewski. I spoke at a colloquium at the University and was pleased that Sierpiński (who was fluent like many older Poles in French but not in English) understood enough to ask an interesting question about multiple separability. I traveled to give a talk at Toruń and to discuss logic with Jerzy Łoś and Stanisław Jaśkowski. Since Jerzy understood German better than English I delivered the lecture in German. After each of my sentences he would, curiously, give a very much longer Polish 'translation' for the audience. The story given for the added length was that he was filling in the holes in my proofs.

Although the Polish restrictions on visas had, as noted, been eased there were still very few Americans in Warsaw. The Embassy staff, the *New York Times* correspondent, and Mary Ann and I seemed to comprise most of the long-term visitors. This had a few amusing consequences. Mostowski had suggested that we might enjoy lunches in the basement of the Palace of Culture and Science and Mary Ann and I frequently ate there. One day a stranger came up to us and, taking me to be a Pole (for which I was greatly flattered), asked Mary Ann if she was Hungarian. It seems that he thought that the only reason a Pole could be speaking English in Warsaw was that his partner was Hungarian!

And on May Day, amid cries everywhere of “Pierwszy Maja”, Mary Ann and I went down to watch the parade on Marszałkowska Street. But the crowds were many layers deep and we were at the back and couldn’t see very much. Then some people in the crowd heard us speaking English and suddenly there were shouts of “Delegacja, Delegacja” and we were ushered up to the front of the crowd.

Because of the paucity of Americans we were included in many of the American Embassy activities. For example, we were entered in a marvelous class in Polish dances and became quite proficient in the Polonaise, the Mazur, and the Krakowiak. When we went with Embassy staff on trips in the country, where there were almost no cars on the roads, we would always note that not too far behind us was another car . . . presumably of the secret police. One day the American Ambassador drove us to his home for a special luncheon and on another occasion we participated in an Embassy scavenger hunt. Among the assignments was one to count the number of tombstones in the Russian memorial cemetery (which no doubt seemed unduly suspicious to the police and a very bad idea to Mostowski). At one point each team was instructed to drive up to a certain kiosk on a main avenue to receive their next assignment and the mystified police tried driving up themselves to see if they could get a handout. We would occasionally go to the Embassy exchange where we could drink tomato juice, which was not generally available in town, and where the Military Attaché would inquire (unsuccessfully) whether I, as a U.S. Army Reserve officer, would like to earn some retirement points.

All in all it was a wonderful, exciting, and productive semester and I was extremely grateful to Mostowski for making it so special. Years later, in 1975, at his request I arranged for him to come to teach in the Summer Session at Berkeley. For the official appointment he was asked to send me his bibliography. His character and personality shone through in what he sent. He wrote: “I published ca [circa] 100 research papers of which I list below some selected ones which seem to me to be of some importance at the time when they appeared.” He then listed a mere 10 of his papers. Toward the end of his visit he was invited to participate in the legendary Foundations Treasure Hunt in which the logic students and some of the faculty compete in teams of two or three to find a hidden (and, indeed, buried) treasure. They are led to it by a long succession of clues requiring a highly agile mind and body as well as a deep knowledge of the history of logic as they traverse an area covered by a map with a dizzying array of locations named after various logicians. It is a real test and historically the winners have tended to be or to go on to be outstanding logicians. The first American hunt took place in Princeton in 1956 and the winning team consisted of Kleene, Shoenfield, and Mrs. J.C.E. Dekker. It was fitting that in the 1975 Hunt in Berkeley Mostowski led the winning team. Two weeks later an outstanding logician and human being passed away in Vancouver.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Andrzej Mostowski
My Master in Mathematics

Andrzej Jacek BLIKLE
December 28th, 2006

I had two masters in my life, my father and Andrzej Mostowski. The former taught me life, the latter – mathematics.

When in 1956 I graduated from a high school named after Tadeusz Reytan I decided to study electronics at the Warsaw Technical University. Unfortunately – or maybe fortunately – I failed to pass the written exams in mathematics. Of the three problems to be solved I managed to solve only two, and through lack of time did not complete the third. For the two I received the highest mark, "5". For the third I got "0", though on the scale the lowest degree was a "2". As was explained later, "2" was reserved for those who gave the wrong answer, whereas no answer at all got a "0". Based on this somewhat non-classical reasoning my average was thus $(5 + 5 + 0)/3 = 3{,}3$ which closed for me the doors to the Department of Electronics. As a gesture of mercy I was admitted to the Department of Civil Engineering.

Despite these circumstances I did not become a civil engineer. My exam in mathematics by the end of the first term went so well that the examiner started to convince me that I should study the subject. Half a year later, in October 1957, I entered the Faculty of Mathematics and Physics of Warsaw University. It was located in an early nineteenth century building of the Astronomical Observatory between the garden of Łazienki Królewskie and The Botanical Garden in Warsaw. I became a student of the greatest mathematicians from the famous Polish school of mathematics of the 1920's and 1930's: Wacław Sierpiński, Kazimierz Kuratowski, Karol Borsuk, Stanisław Mazur and last but not least – Andrzej Mostowski, the youngest of them.

My first course with Andrzej Mostowski was in linear algebra. I was moved not only by the beauty of presentation but also by the professor's extremely friendly attitude towards students. The doors to his office were always open and I made frequent use of them.

My second course with Andrzej Mostowski – the most important part of my mathematical education – was a two term course in mathematical logic taught according to his book "Logika Matematyczna" (Mathematical Logic) published in Wrocław in 1948 in the series of Monografie Matematyczne (Mathematical Monographs). In the foreword to that book he writes (please excuse my somewhat free translation):

In writing a book on logic one has to overcome difficulties unknown in writing about classical fields of mathematics. In my opinion there are three sources of these difficulties.

The first, undoubtedly, is that mathematical logic is still a very young science with insufficient tradition. Today it is too wide to be presented in one book, and at the same *time we are still not able to distinguish between what is more important and what is less important.*

The source of the second difficulty is the imperfection of the existing logical tools. It is known that the aim of logic is – amongst others – to formalize mathematical reasoning and make it the object of methodological or meta-mathematical investigations. (...). However, the means of formalization of intuitive reasoning known today are either so imprecise that one cannot build upon their base meta-mathematical reasoning, or they lead to such lengthy calculations that one cannot think seriously about their practical applications.

Finally, the third difficulty is the consequence of the fact that we cannot free this logic (whatever when formalized it would be) of some unconscious philosophical background, whereas a conscious choice is very difficult since – in the present state of today's discussion about the foundations of mathematics – one cannot be completely sure which of the conflicting opinions is the best or at least good.

A page later he writes something that shows to what extent that which is evident today was not evident sixty years ago:

Going even further and following the advise of professor Knaster, I have replaced quotation marks commonly used in creating the names of expressions, by preceding expressions with words such as 'proposition', 'variable', 'propositional function' etc. Thus, for instance, I write: 'we substitute the variable x for the variable y in the expression $x^2 - y = 0$*' rather than 'we substitute "x" for "y" in "*$x^2 - y = 0$*" '.*

In late 1950s, when I was following Mostowski's course, logic was regarded as a little applicable branch of mathematics even for a "working mathematician". Only a quarter of a century later it had become a fundamental tool for mathematicians developing the mathematical foundations of computer science. This observation reminds me of the course's last lecture, devoted to general remarks about the philosophy of mathematics and within it, the role of mathematical logic. Among other issues Mostowski mentioned the idea of a three-valued logic as an example of a mathematical extravaganza. He told us a story of a foreign logician who tried to find applications for such a logic, but became mentally ill and was sent to a psychiatric hospital. At that time *p or not p* was undoubtedly true and nobody could think in any other way.

In science, however, it is frequently the case that what is obvious for one generation is not so obvious for the next one. In the late 1970s I came across a problem in the theory of computer programmes which brought my attention to a three-valued logic. The problem was about developing proof rules for so-called *clean total correctness of programs* [1]. A program P is said to be *cleanly totally correct for precondition Pre and postcondition Post* whenever the following is true:

if P gets input data that satisfies *Pre,* then its run terminates properly (cleanly) – i.e. is neither indefinite nor fails to complete – and the output satisfies *Post.*

The proof rules for proving clean total correctness known at that times tacitly assumed – as we always do in classical logic – that each predicate always evaluates to either *true* or *false*. That assumption, however, is rarely satisfied in a programming environment. For instance the predicate

$$a[i] < a[i+1]$$

where a is an array with the index running over the range $[1,100]$ is neither true nor false for $i = 100$ since an attempt to evaluate $a[101]$ results in an interruption of the program's

	Łukasiewicz	Kleene	McCarthy
(1)	*tt* ***implies*** *uu* = *uu*	*tt* ***implies*** *uu* = *uu*	*tt* ***implies*** *uu* = *uu*
(2)	*ff* ***implies*** *uu* = *tt*	*ff* ***implies*** *uu* = *tt*	*ff* ***implies*** *uu* = *tt*
(3)	*uu* ***implies*** *tt* = *tt*	*uu* ***implies*** *tt* = *tt*	*uu* ***implies*** *tt* = *uu*
(4)	*uu* ***implies*** *ff* = *uu*	*uu* ***implies*** *ff* = *uu*	*uu* ***implies*** *ff* = *uu*
(5)	*uu* ***implies*** *uu* = *tt*	*uu* ***implies*** *uu* = *uu*	*uu* ***implies*** *uu* = *uu*

Figure 1. The comparison of three-valued calculi.

execution and generates an error message "index i out of range". In order to tackle this problem in proof theory one must be able to express it in the underlying propositional calculus, i.e. one has to admit that a predicate may evaluate not only to either *true* or *false* but also to same third value – the *undefined*.

Having come to that conclusion at a conference in Oberwolfach, a charming institute in the woods of Black Forest in the southern Germany, I started to look for a suitable three-valued propositional calculus in existing literature. To my surprise I found three such calculi developed by three different authors in three different historical and scientific circumstances: Jan Łukasiewicz (1878–1956), Stephen Cole Kleene (1909–1994) and John McCarthy (b. 1927). Each of these calculi refers to a different understanding of the concept of *undefinedness*. Let me briefly explain the basic philosophical differences between these concepts. This can be done by studying one chosen propositional operator. Let it be ***implies***. Since all the operators coincide with classical ones on classical values, we may restrict our attention to the cases where at least one of the arguments equals *undefined* (Fig. 1). Let *tt, ff* and *uu* denote *true*, *false* and *undefined* respectively.

In the calculus of Łukasiewicz [3] – first published in Lwów in 1920 – *uu* corresponds to the "I don't know" of a philosopher. It should be emphasised that whenever the philosopher of Łukasiewicz does not know whether this or that is true or false, he[1] still assumes that it is either true or false. Moreover, if in a formula there are two or more occurrences of *uu,* then they both denote the same value. For instance his rule (1) derives from the fact that in classical logic *tt* ***implies*** *tt* = *tt* and *tt* ***implies*** *ff* = *ff*.

For Kleene [2], who already knew about computers, *uu* represented an *unfinished computation*, where "unfinished" means either "infinite" (runs forever) or "interrupted before completion". Kleene assumed further that in order to calculate the value of a formula we calculate the values of all its arguments simultaneously (in parallel) and apply the so called *lazy evaluation principle.* The latter means that once we have "enough" information to establish the value of the formula, we stop the calculation. For instance, if during the calculation of *a* ***implies*** *b* we find that the value of *a* is *ff*, or that the value of *b* is *tt*, we stop the calculation, since whatever the value of the other variable, the value of implication is *tt* (cases (2) and (3)). If, however, the value of *a* is *tt*, we have to continue the calculation (case (1)) in order to establish the value of the implication. As we see Kleene and Łukasiewicz differ only in (5). For Kleene none of the computations of *a* and *b* terminates properly, hence also, the computation of the formula does not terminate properly. For Łukasiewicz, although our philosopher does not know the <u>common</u> value of both *uu*'s, whatever the common value is, the formula evaluates to *tt*.

McCarthy [4] was even closer than Kleene to computers and therefore went one step further towards the computational environment. He assumed that our computer had only

[1] At the times of Łukasiewicz the "he or she" formula had not been discovered yet.

one processor (or a finite number of processors, which has the same consequence) and therefore cannot calculate the values of all variables simultaneously. It calculates them one by one e.g. from the left to the right and, just as in both former calculi, applies the lazy evaluation principle. The one-processor assumption is responsible for the difference between Kleene and McCarthy in rule (3). If the calculation of a does not terminate properly, the calculation of the whole formula does not terminate properly either.

The calculus of McCarthy corresponds to the way most computer systems evaluate Boolean expressions. As a consequence, in his calculus some two-valued logic laws are not satisfied. For instance:

a ***and*** b is not equivalent to b ***and*** a

Indeed, *ff* ***and*** $uu = ff$ and uu ***and*** $ff = uu$. This property has practical consequences. For instance, the evaluation of the Boolean expression

$i < 100$ ***and*** $a[i] < a[i+1]$

– e.g. in a loop that searches for the largest element of an array – will always terminate cleanly, since whenever “$i < 100$” evaluates to *ff*, the expression “$a[i] < a[i+1]$” will not be evaluated and therefore will not generate an error message. That is not true, of course, for

$a[i] < a[i+1]$ ***and*** $i < 100$

If $i = 100$ the evaluation of the Boolean expression is interrupted with an error message.

At the end of that story one more remark. The fact that predicates in computer programs are three-valued, does not mean that the *logic of programs* must also be three-valued. Indeed, the metaformula which says that program P is totally correct with clean termination for the precondition *Pre* and postcondition *Post*, or that it is correct in any other sense, is always either true or false. The *tertium non datur* is still there, and with it the memory of Andrzej Mostowski.

References

[1] A. Blikle, The clean termination of iterative programs, *Acta Informatica* **16** (1981), 199–217.

[2] S.C. Kleene, *Introduction to Metamathematics*, North Holland, 1952.

[3] J. Łukasiewicz, *O logice trójwartościowej (About three-valued logic)*, in Polish, Ruch Filozoficzny, vol. 5, Lwów, 1920, pp. 169–171.

[4] J. McCarthy, *A basis for a mathematical theory of computation*, Western Joint Computer Conference, May 1961.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Our Reminiscences of Andrzej Mostowski

Emilio DEL SOLAR [a] and Max DICKMANN [b]
[a] *127 Avenue de Flandre, 75019 Paris, France*
[b] *Equipe de Logique Mathématique (UMR 7056), CNRS – Université Paris VII, and Projet Topologie et Géométrie Algébriques, Institut de Mathématiques de Jussieu (UMR 7586), 2, Place Jussieu – Case 7012, 75251 Paris Cedex 05, France*

We came to Warsaw to pursue post-graduate studies under the direction of Andrzej Mostowski in 1965 (EdS) and in 1967 (MD). We had the chance of being in Warsaw during a particularly active period in the field of mathematical logic, centered around the figure of Mostowski. Besides Polish Ph.D. students and colleagues, a significant number of foreign guests attended Mostowski's seminar in those years; logicians from (at least) the following nations participated: Argentina, Australia, Chile, Czechoslovakia, Great Britain, Japan, Sweden, Soviet Union, USA; not only Mostowski oriented the work of those visitors but, to a greater or lesser extent, he carried out joint research work with several of them.

Though one of us (MD) met Mostowski prior to his arrival in Warsaw—in fact, decided to visit Warsaw as a result of this meeting in Montreal—our appreciation of his rich and attractive personality developed as we came into closer relationship with him.

The first thing one could not fail to be impressed by was his ability as a polyglot, capable of sustaining a fluid conversation in at least four languages simultaneously. One of us (EdS) still recalls with astonishment a seminar where Mostowski spoke in Polish, translating into English immediately after each paragraph, while replying to questions and adding remarks in German and French. At the end he told, "je crois que j'ai même dit une phrase en finnois": shortly before he had lectured at the Vaasa Summer School in Finland (these lectures became his celebrated monograph [1]).

We soon realized that he was a man of a remarkable modesty, a gentle straightness and a charming simplicity, who had, as well, a subtle sense of humor and irony. However, both of us were most deeply impressed by the breath of his culture, reaching far beyond mathematics. A few anecdotes from our personal relationship with Mostowski give, perhaps, most eloquent illustrations of these traits of his personality. Here are some.

— At the end of a meeting in his ample office high up in Warsaw's Pałac Kultury i Nauki (Palace of Culture and Science), the first author distractedly dispossessed Mostowski of his pencil; he said, "Hast du nicht vielleicht einen Bleistift?". To EdS's surprise, who didn't speak German, M. repeated this phrase in French, explaining: "oh, sorry!, this is what Hans Castorp told Mme. Chauchat in Thomas Mann's 'The Magic Mountain' " (cf. [2], p. 426).

— During his first meeting with EdS, Mostowski displays a map of Chile (EdS's country of birth); after a few questions and usual exchanges, the following dialog ensues:

– M: “you do not shed tears?”.
– EdS: “why shall I?”.
– M: “last year, as I showed a map of Czechoslovakia to a Czech visitor—I wanted to spot his place of birth—, he broke in tears. . . ”.

— In a more dramatic tone, one of us (MD) was invited home to dinner by Mostowski and his wife. Late in that unforgettable evening, emotional recollections from war time came to the surface. Visibly moved, M. said: “the war split our lives in the middle; we had two lives: before and after. . . ”. He then recounted his departure from occupied Warsaw for a wandering life in woods and villages, escaping from bombings and combat, a life that lasted many months. He had the choice of taking along either a loaf of bread or the manuscript, then in progress, of the Mostowski–Tarski proof of the decidability of the first-order theory of the ordinals. Bread was more precious and. . . less conspicuous. The manuscript got lost in the fire of Warsaw, but was reconstructed after the war, and posthumously published in [3]. [This recollection also appears in [4], pp. 33–34.]

Only years later, while the first author worked on the review [5] of Mostowski's “Selected Works” [6]—and, for the second author, while reading it—, we came to a better appreciation of the impressive breadth and depth of his intellectual achievement, to which the following paragraphs are devoted.

The formative influences. In various conversations Mostowski spoke about those logicians and mathematicians who had a decisive influence in his formative years:

— Alfred Tarski, actually his thesis advisor, though formally K. Kuratowski was responsible for his work, and
— Adolf Lindenbaum—a victim of the Holocaust—who had a great influence on Mostowski's choice of research subjects before the war. Mostowski considered him “the most lucid mind on the foundations of mathematics” in the late 30s. Together with Gödel[1], Lindenbaum was influential in arising Mostowski's interest on a number of (at that time open) problems involving the axiom of choice, an interest that led to his famous “permutation models”—earlier considered by Fraenkel—a technique he used to prove relative independence results involving the axiom of choice. Many of Mostowski's contributions, now classical, including a joint paper with Lindenbaum from 1938, [7], gave solutions to outstanding problems in this field of research.

Later in life he also came to admire Hermann Weyl as a mathematician and as an intellectual figure—witness a large portrait of Weyl in his office—, with whom he shared the view that mathematics' source of motivation was natural science, in the service of which mathematics ought to be; however, he did not share Weyl's constructivist position on the foundations of mathematics; see Mostowski's rejoinder to the discussion of his article [8].

Mostowski on mathematics and the philosophy of mathematics. On several occasions we had, or witnessed, exchanges where Mostowski stated his viewpoints on mathematics, its philosophy, and its place in science and the world of knowledge.

Mostowski had a “realist” philosophical standpoint on mathematics (we recall him saying “I've never been much of a platonist”). At an early stage of his career M. con-

[1]Mostowski's initial interest on the axiom of choice seems to have come from lectures by Gödel that he attended in Vienna (this information originates in conversations with him).

sidered “constructivism” and “finitism” as philosophically plausible—even attractive—proposals for the foundations of mathematics. Later on in his life, dissatisfaction with the outcome of the constructivist criticisms of classical mathematics led him to consider the latter’s results and the vast range of their applications in science and technology as a sufficient justification for the validity of its methods and procedures; see [9]. In line with this conviction, he was not particularly enthusiastic about the effective reconstruction of time-honored and empirically justified classical results (computer science, a greedy consumer of algorithms constructed from those effective methods, was at that time below the horizon line).

Together with Robinson, Tarski, Malcev and others, Mostowski pioneered the use of infinitistic methods in metamathematics, for example in model theory, and stressed that “infinitistic metamathematics” ought to be considered on an even footing with other branches of mathematics, rather than belonging to a separate area of “foundational studies”. He firmly believed that no mathematical philosophy could ignore the fundamental links of mathematics with the natural sciences (see [8]).

On one occasion, during the 1967 International Congress of Logic, Methodology and Philosophy of Science in Amsterdam, he expressed in private conversation his admiration of B. Russell, who, according to him, “had a strong influence on Polish logic, even on the conception of Leśniewski’s ‘strange systems’ ”.

Mostowski was not particularly attracted to philosophical debate about mathematics. However, by no means he dismissed this type of reflection as a pointless exercise. In his view, the significance of the philosophical debates of the end of the 19th and the beginning of the 20th centuries about the nature and the foundations of mathematics laid more on the influence they exerted upon logic, than on the light they shed on the nature of mathematical entities and the cognitive status of mathematics. For example, he considered that the lasting heritage of the logicist position consisted in the many developments that—willingly or witlessly—this school of thought led to in axiomatic set theory.

Besides the article [9] and lectures I and IX in [1], devoted to subjects in the philosophy of mathematics, some of his positions on these matters only appear occasionally in his written work. A notable case occurs in his paper on generalized quantifiers where he expresses his conviction that “the construction of formal calculi is not the unique, and not even the most important goal of symbolic logic” ([10], p. 12). In this work M. dealt, in addition to the finitary “numerical” quantifiers, with the quantifier “there are countably many”, for which he proved incompleteness results, but not with the quantifier “there are uncountably many”. As he later told us, Keisler’s proof of the completeness of logic with this quantifier came to him, and to many of his colleagues, as a genuine surprise.

We also recall the importance that Mostowski gave to the interaction of logic with “mainstream” mathematics. Faithful to this conviction, he seems to have kept a fluid dialog with working mathematicians, especially in Poland, several of whom became interested in logic through his predicament and did important contributions. If our recollections are right, this was (at least) the case of Ryll-Nardzewski (topology), Łoś (algebra) and Sikorski (real analysis). Another trace of Lindenbaum’s influence, whose early mathematical work was in topology?.[2]

[2]Lindenbaum’s doctoral dissertation on metric spaces was directed by W. Sierpiński.

A brief spotlight on Mostowski's social views. As one could expect from a sophisticated personality having gone through a dense life experience, Mostowski held articulate and informed opinions on society and politics. Though, in accordance with the style of his personality, he was not outspoken about these matters, he did not hide his opinions.

Our conversations on these subjects were in general brief, but his comments revealed an open and progressive trend of thought, one that definitely would have place him in what, by today's Western European terminology (but perhaps not by the Polish one), can be called the progressive/democratic camp.

One of us recalls him expressing his view that, the restoration of Warsaw's historical center immediately after the end of World War II—though very dear in economic terms in an utterly devastated country—was a courageous political decision by the government of the time, insofar it conveyed a strong symbolic message of survival of the nation.

References

[1] A. Mostowski, Thirty Years of Foundational Studies. Lectures on the Development of Mathematical Logic and Studies on the Foundations of Mathematics, 1930–1964, *Acta Philosophica Fennica* **XVII** (1965), 1–180. Reprinted in [6], vol. 1, 1–176.

[2] Th. Mann, *Der Zauberberg*, S. Fischer Verlag, Frankfurt/M, 2003 (Ch. V, section "Walpurgisnacht". As a matter of fact the quoted phrase has, in Mann's novel, implications beyond its mere face value; cf. also Ch. IV, section "Hippe"). First published in 1924.

[3] J.E. Doner, A. Mostowski and A. Tarski, The elementary theory of well-ordering—a metamathematical study, in: *Logic Colloquium 77*, A. Macintyre et al., eds, North-Holland, Amsterdam, 1978, pp. 1–54.

[4] J. Crossley, Reminiscences of logicians, in: *Algebra and Logic, Proceedings 1974 Summer Research Institute, Australian Math. Soc.*, J. Crossley, ed., *Lect. Notes Math.* **450** (1975), 1–62.

[5] S. Berestovoy and E. del Solar-Petit, Review of [6], in: *Encyclopédie Philosophique Universelle*, Presses Universitaires de France, 1992, tome 2, pp. 3574–3575.

[6] A. Mostowski, *Foundational Studies, Selected Works*, 2 vols, PWN Polish Scientific Publishers, Warszawa, and North-Holland Publ. Co., Amsterdam, N. York, Oxford, 1979.

[7] A. Lindenbaum and A. Mostowski, Über die Unabhangigkeit des Auswalhaxioms und einiger seiner Folgerungen, *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Cl. III* **31** (1938), 27–32. Reprinted in English in [6], vol. 2, pp. 70–74.

[8] A. Mostowski, Quelques observations sur l'usage des méthodes non-finitistes dans la metamathématique, in: *Le raisonnement en mathématiques et en sciences expérimentales. Colloque International de Logique Mathématique, 1955*, Colloques Internationaux du CNRS **LXX**, Eds. CNRS (1958).

[9] A. Mostowski, On various degrees of constructivism, in [6], vol. 2, 359–375.

[10] A. Mostowski, On a generalization of quantifiers, *Fundamenta Math.* **44** (1957), 12–36. Reprinted in [6], vol. 2, 311–335.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Andrzej Mostowski: An Appreciation

Solomon FEFERMAN
Departments of Mathematics and Philosophy, Stanford University

A decisively favorable letter from Andrzej Mostowski to Alfred Tarski in the spring of 1956 about my results on the arithmetization of metamathematics came at a crucial time for me to complete my PhD work with Tarski. That was my first personal—albeit indirect—connection with Mostowski, and fifty plus one years later I am still grateful for his intervention.

This is how it came about.[1] After taking various of Tarski's courses and seminars at UC Berkeley beginning in 1949, it became clear to me that logic was my subject and that Alfred Tarski was the one I wanted to work with, if only he would accept me as a PhD student. It took a while to muster the nerve to ask, but I guess he had been waiting for the question, since he readily agreed. I was soon working for him as a course assistant during the days and as a research assistant well into the nights. Tarski also suggested two possible thesis problems, both conjectures of his: one on the representation theorem for a class of cylindric algebras (locally finite infinite dimensional) and the other on a decision procedure for the theory of ordinals under addition. I quickly settled the first problem by adapting Henkin's proof of the completeness of first-order logic, but that didn't satisfy Tarski; he wanted something more algebraic in spirit. Not seeing what more to do with it, I moved on to the second problem, for which I eventually succeeded in getting what I thought was an informative reduction, though not the expected result itself.[2] I was eager to finish my doctoral work and so I proposed putting the two results together for a thesis; but Tarski wouldn't agree, so I continued to go round and round on the ordinals problem without making further progress. In the meantime, my draft board was breathing down my neck, wondering what was taking so long in my graduate studies and why I should continue to get a deferment. It did no good to explain to the board the unpredictable nature of mathematical research in general and research with Tarski in particular, and so in 1953 I was drafted into the US Army for a two-year hitch. Fortunately, this was a period of peace for the US: the Korean War was over and Vietnam was not yet on the radar; it was, however, the height of the Cold War.

After basic training I was posted to the Signal Corps in Fort Monmouth New Jersey, and assigned to a unit doing research on "kill" probabilities for Nike missile battery responses to possible (Soviet, presumably) A-bomb tipped missile attacks on major US cities and military sites. It was unnerving to think about the fact that the chances of a successful defense were never even close to 100%, and so my thoughts dwelt instead on the problems of my growing family and what would happen on my return to graduate school.

[1] Some of this repeats material from the biography that I co-authored with Anita Burdman Feferman, *Alfred Tarski. Life and Logic* [4], pp. 211–213, as well as from my article, "My route to arithmetization" [7].

[2] That work had a direct connection with a singular paper of Mostowski's, to which I will return below.

There were, however, spare moments, and I used whatever opportunity I could to study Kleene's *Introduction to Metamathematics* since I felt I needed to learn more recursion theory and deepen my understanding of Gödel's theorems. As it happens, my previous introduction to the incompleteness theorems was in a course taught the year before by a young assistant professor, Jan Kalicki; the text was Mostowski's book, *Sentences Undecidable in Formalized Arithmetic* [9], in which Mostowski had presented an elegant common generalization of the incompleteness theorems and Tarski's undefinability theorem for truth.[3]

One day in early 1955 I received a distinctively handwritten postcard, inked in black and red with wavy, single and double underlinings, from Alonzo Church, asking if I would review a paper by Hao Wang for *The Journal of Symbolic Logic*. The article concerned a generalization of Bernays' formalized version of Gödel's completeness theorem, to the effect that a recursively axiomatized theory T is interpretable in arithmetic when the consistency of T is adjoined as an axiom. Church, still in Princeton in those days, was the editor of the *JSL* reviews section. I was surprised at the request since I had never had any contact with Church, but guessed that he had been given my name by Dana Scott, who had come to Princeton to complete his graduate studies. Wang's paper, written in a rough-and-ready style far from the standards of Tarskian precision that I had come to accept as the norm, stimulated me to work out a rigorous formulation and proof of the main result, and start to think about the proper treatment of consistency statements in general.

By the time I was released from the army in September 1955, and returned to Berkeley to complete my doctoral studies, I had decided to devote myself to the precise study of the arithmetization of metamathematics in some generality, including both the completeness theorem and the incompleteness theorems. It happened that Tarski was on sabbatical in Europe that year, and he asked Leon Henkin to take over as acting advisor in his absence. Though Henkin's own major interests were in model theory and algebraic logic, he offered me a willing ear and a great deal of encouragement, and with the prod of weekly meetings I soon made significant progress. By the time Tarski returned from his sabbatical in May of 1956, I had a body of work that I was sure was thesis worthy, and presented it to him as such.[4] But to my dismay, instead of pronouncing it "excellent!", he hemmed and hawed. Perhaps he was irked that the subject was not the original one *he* had suggested and not in any of his own main directions of research; instead, it sharpened and generalized the method of arithmetization used to prove Gödel's incompleteness theorems. Perhaps the old rivalry Tarski felt with Gödel over those theorems was awakened. In any event, he decided not to decide on his own whether the work was sufficiently important and instead asked me to send a summary of the results to Andrzej Mostowski in Poland. This took more time and created more tension. To my relief, Mostowski found the results novel and interesting and strongly encouraged Tarski's approval. Mostowski's intervention was decisive, and so Tarski agreed at last to accept the results of my research for the dissertation. But dotting all the i's and crossing all the t's took another year

[3] In November 1953, I was shocked to learn of the death of Kalicki in an automobile accident as he was driving to a logic meeting in Los Angeles. Tarski, his wife Maria, and my fellow student, C.C. Chang, were passengers in the car with him. Kalicki had missed a turn and was thrown out of the car; the others were injured but not badly. The story is told in [4], pp. 204–206.

[4] A summary of that work is given in [7].

before conferral of the PhD, by which time I was installed as an instructor at Stanford University.

To begin at the beginning Tarski-wise, early on in my work with him I had heard Mostowski's name mentioned as his first student. And, more than his name, I soon began to get an appreciation of his varied and incisive research contributions, including the Fraenkel-Mostowski models and their applications, definability hierarchies, absolute properties of relations, and his collaboration with Tarski and Raphael Robinson on undecidable theories. But my first significant engagement with Mostowski's work came via his 1952 paper "On direct products of theories" [10], which showed how to reduce the theory of any direct power M^{I} of a relational structure to the theory of M, and the same for weak direct powers. I thought a generalization of that to ordered powers in some suitable sense could be used to solve the problem Tarski proposed to me on ordinal addition. The background for that was his work with Mostowski on the theory of ordinals under the < relation, reported in their 1949 abstract [13].[5] Among other things they had shown that the theory of $\langle \mathrm{Ord}, < \rangle$ is decidable, and moreover is an elementary extension of $\langle \omega^{\omega}, < \rangle$. Tarski conjectured that the theory of $\langle \mathrm{Ord}, + \rangle$ is decidable and is an elementary extension of $\langle \omega^{\alpha}, + \rangle$ where $\alpha = \omega^{\omega}$. My idea of how to attack this was to treat addition of ordinals in terms of their Cantor normal forms to base ω, which can be interpreted in a weak ordinal power of the natural numbers under addition. In the process of thinking about this, I arrived at the idea of generalized powers M^{J} where J is a structure on an index set I, and showed that the theory of M^{J} is reducible to the theory of M together with the monadic *second order* theory of J (or weak monadic theory in the case of weak powers). Mostowski's results fall out as the special case for J of the form $\langle \mathrm{I}, = \rangle$, whose monadic theories (full and weak) are given by Skolem's classic elimination of quantifiers procedure. Later, in 1956, this led to my collaboration with Bob Vaught on our work [8] on the first order theories of generalized products of structures, that included my results on generalized powers and his solution of the Los conjecture on ordinary Cartesian products. In the meantime, Mostowski's student Andrzej Ehrenfeucht had arrived by specific methods at the solution of the problems about ordinal addition that had sparked my own work in this direction. Though scooped in that respect, I was very satisfied with the many other applications of my work with Vaught detailed in [8].

A couple of years later it was a pleasure for me to finally make the personal acquaintance of Mostowski as well as of Ehrenfeucht, when they came to Berkeley to spend the academic year 1958–59. Early in that year, with my wife Anita and our two little children, we made a memorable excursion with them to San Gregorio beach in our Oldsmobile station wagon. There the two excited Andrzej's could barely contain themselves before jumping into the surf of the Pacific Ocean, oblivious to the extremely chilly waters (usually around 50 °F or 10 °C). In the months that followed—usually at dinners or parties after logic colloquia—Anita and I had more opportunities to get to know Mostowski better and to enjoy his low-keyed wit and charming and gracious personality: a bit of old school gentlemanliness, including the warm handkissing greetings for the ladies. He was always very welcoming, and encouraged us several times to come visit him in Warsaw, but the opportunity to do so did not present itself until fifteen years later. At a certain point in between, though, we asked if Anita's parents might visit him on their return via Poland from a trip to their former homeland in the Ukraine and other parts of Russia.

[5] The details of that remained unpublished until their joint work with John Doner [3] which did not appear until 1978, three years after Mostowski's death.

Mostowski and his wife Maria were most hospitable, inviting Anita's parents to their home for a fine lunch with much interesting conversation reported about life in Poland and the rigors of surviving the war.

Besides the pieces that I have already mentioned, Mostowski published a number of important articles in the 1950s and 1960s that were to have an impact on my own work. Foremost for me among these were his 1957 paper [11] on generalized quantifiers and, a decade later, his contribution to the 1967 Amsterdam Logic and Methodology Congress on counterexamples to Craig's interpolation theorem in various extended systems of logic [12]. The first of these was to have a profound influence on the development of abstract model theory, along with Lindström's theorems characterizing first order logic among abstract logics in general. The interpolation theorem and its variants emerged as a test property for "good" logics, and so it was important to know what lay behind the failure of various specific logics to satisfy the theorem. The significance of Mostowski's work in these directions was signaled in the introduction to the 1985 comprehensive volume *Model-Theoretic Logics* [1] that I edited with Jon Barwise. In the meantime, I had paid my own tribute to Mostowski in that area with the two papers, [5] and [6], in honor of his 60$^{\text{th}}$ birthday.

After 1959, I did not meet Mostowski again until 1967 in Amsterdam, and then four years later in Berkeley, when he came for the Tarski Symposium. But it was in 1973 that we enjoyed the first of two occasions for the most personal contact with him. During the spring of that year I was on sabbatical leave in Paris; that happened also to be designated as Logic Year in Warsaw, and I was invited to take part. Since I was teaching a course in Paris, only a short visit over Easter proved to be possible, but in one respect that turned out to be an inopportune time. The problem was that my luggage was lost on the flight into Warsaw and on arrival no store was open where one could purchase the needed substitutes; that was only my problem, since Anita, who was with me, had wisely not checked her luggage.

Visitors that spring were lodged as guests in private homes or apartments, in our case the home of a dentist's widow. When we moved in to our bedroom there we found that it still had a large leather dental chair square in the middle of it, now functioning as a kind of objet d'art. To complete the "museum", there was a case of dental instruments on display as well. It was on the occasion of that visit to Warsaw that Anita and I first met Maria Mostowska; she and Andrzej invited us to dinner one evening, and instantly took to her with her warmth and graciousness. Life was still not easy in those days, but Maria managed a fine typically Polish meal and we were appreciative of their great hospitality; Andrzej drove us to our lodging afterward. Some twenty years later we returned to Warsaw to do research and interviews for the Tarski biography, and once more Maria was very kind to us and also quite helpful with information.

The last time I saw Mostowski was in February 1974, in Melbourne, Australia, for a month long summer institute on logic and algebra at Monash University. It was during that meeting that an extended conversation conducted by John Crossley was held between Kleene, Mostowski and several younger participants, and recorded as "Reminiscences of Logicians" in [2]. That became famous for Mostowski's stories of his wartime travails, including the one where he had to chose between his notebook and some bread; the bread won out, all his notes were burnt and had later to be more or less reconstructed from memory. Toward the end of the Monash meeting the Crossleys had a party at which there was much drinking of wine and beer, and it was the one time I saw Mostowski lose his composure. There was a swimming pool behind the house and at a certain point Crossley invited his guests to "skinny-dip" along with him. A number of the less inhibited did so, and jumped in and out of the pool and ran around it naked. The laughing and splashing could not be ignored, and I thought that Mostowski was embarrassed. To distract him I tried to make small talk about the excursion we had taken to the southern shore earlier in the week for the amazing sight of thousands of fairy penguins emerging from the water at dusk to feed their chicks.

Mostowski visited Berkeley once more in the final months of his life in the summer of 1975, before going on to Canada. I was in France all that summer and I missed the opportunity of seeing him. It was at a meeting in Clermont-Ferrand that I learned the shocking and sad news of his unexpected death.

Though my contacts with Mostowski were, to my great regret, too few and far between, they were enough to engender my deep appreciation of him both as a truly fine human being and as the author of a body of exceptionally diverse, first class work. I'm gratified to have been able to join in this celebration in his memory.

References

[1] J. Barwise and S. Feferman, eds, *Model-Theoretic Logics*, Springer-Verlag, Berlin, 1985.

[2] J.N. Crossley, ed., Reminiscences of logicians, in: *Algebra and Logic*, Lecture Notes in Mathematics, vol. 450, 1975, pp. 1–62.

[3] J. Doner, A. Mostowski and A.Tarski, The elementary theory of well-ordering—A metamathematical study, in: *Logic Colloquium '77*, A. MacIntyre et al., eds, North-Holland, Amsterdam, 1978, pp. 1–54.

[4] A.B. Feferman and S. Feferman, *Alfred Tarski. Life and Logic*, Cambridge University Press, New York, 2004.

[5] S. Feferman, Two notes on abstract model theory. I. Properties invariant on the range of definable relations between structures, *Fundamenta Mathematicae* **82** (1974), 153–165.

[6] S. Feferman, Two notes on abstract model theory. II. Languages for which the set of valid sentences is semi-invariantly implicitly definable, *Fundamenta Mathematicae* **89** (1975), 111–130.

[7] S. Feferman, My route to arithmetization, *Theoria* **63** (1997), 168–181.

[8] S. Feferman and R.L. Vaught, The first order properties of products of algebraic systems, *Fundamenta Mathematicae* **47** (1959), 57–103.

[9] A. Mostowski, *Sentences Undecidable in Formalized Arithmetic*, North-Holland, Amsterdam, 1952.

[10] A. Mostowski, On direct products of theories, *J. Symbolic Logic* **17** (1952), 1–31.

[11] A. Mostowski, On a generalization of quantifiers, *Fundamenta Mathematicae* **44** (1957), 12–36.

[12] A. Mostowski, Craig's interpolation theorem in some extended systems of logic, in *Logic, Methodology and Philosophy of Science III*, B. van Rootselaar and J.F. Staal, eds, North-Holland, Amsterdam, 1968, pp. 87–103.

[13] A. Mostowski and A. Tarski, Arithmetical classes and types of well-ordered systems (abstract), *Bull. Amer. Math. Soc.* **55**(65) (1949), 1192.

A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Andrzej Mostowski as Teacher and Editor

Einar H. FREDRIKSSON
IOS Press, Amsterdam

These, in addition to some historical regressions, notes cover mainly the period 1964–1975, when the current author studied, or worked as a publisher, with Andrzej Mostowski. It started with a lecture course Mostowski gave during the Summer of 1964 in Vasa, Finland, which I attended together with three other Swedish logicians. In 1965 Mostowski arranged a graduate scholarship for me at Warsaw University, and it is with gratitude to him I look back at two very interesting years there. Mostowski was not only a brilliant lecturer, and an engaging tutor, but was also a father figure, in the broadest sense of the word. In 1969 I commenced work at North-Holland Publishing Co. in Amsterdam as mathematics editor, recommended to the position by Mostowski, and I worked with him from that time onwards as publisher of a book-series and a journal that he edited.

Mostowski's early engagements in Publishing and Library

Mostowski's role as teacher and author has been broadly covered in this volume, and of course also in the Foundational Studies volumes of 1979 [3] and the proceedings of the Logic Colloquium '76 in Oxford [4] during which a commemorative session for him was held. Recollections by students and collaborators go back to the War period by his first PhD students Andrzej Grzegorczyk and Helena Rasiowa. His first textbook (published in Polish, English title "Mathematical Logic, a University Course" [2] appeared in 1948, on the basis of which a whole generation of Polish logicians were brought up. His first English language book "Sentences undecidable in formalized arithmetic; an exposition of the theory of Kurt Gödel" appeared in the series "Studies in Logic and the Foundations of Mathematics", North-Holland, 1952. Compare also Rasiowa, "A Tribute to A. Mostowski" in [4].

Before going to the US for the first time, for the academic year 1948/49 to work with Kurt Gödel, Mostowski attended the 10th International Congress of Philosophy in Amsterdam, 11–18 August, 1948 [1]. It was the first major post-War gathering of mainly European philosophers and logicians, attended also by many of his Polish colleagues including Grzegorczyk and Rasiowa. Alfred Tarski did not attend, and with the exception of Haskell B. Curry, neither did any of the other leading US logic colleagues.

Mostowski was conversant in English, German, French and Russian, apart of course from his Polish mother tongue. Before the War he had published mainly in German and Polish, while in Amsterdam his presentation "Sur l'interpretation geometrique et topologique des notions logiques" in [1] was in French. After 1950 he would publish mainly in English and Polish. For international contacts at this time multilingualism was an asset. The Secretary of the Amsterdam Congress was Evert W. Beth and the publisher of the Congress proceedings was M.D. (Daan) Frank of the small local North-Holland Publishing Co. Before the Congress Beth and Frank had discussed the need for a book series devoted to logic, which would publish monographs and other work in the western World Languages.

At a luncheon meeting in the Amsterdam Vondelpark, organised by Frank during this Congress, several of the main logic speakers were invited to write monographs in their areas of speciality. Thus the book series "Studies in Logic and the Foundations of Mathematics" was born. In addition to Beth, the Dutch logicians L.E.J. Brouwer and A. Heyting were the Editors of the series. Five of the invited authors submitted their manuscripts in time for publication in 1951 (monographs by I.M. Bochenski, H.B. Curry, K. Dürr, A. Robinson and G.H. von Wright) while the above-mentioned monograph by Mostowski appeared in 1952. Since 1948 Mostowski had maintained contacts with Beth – the prime mover behind the series – and with Frank (who also handled the correspondence). Compare the latter's "Twenty-five years of "Studies in Logic" in [4] in connection with the anniversary of the successful series. By now, in 2007, it has published its 150th volume.

Mostowski and Abraham Robinson, who was also a main non-Dutch logic advisor to Frank, each contributed four books to the series. Together with Patrick Suppes they became the prime movers behind the series after the untimely passing of Beth. Mostowski formally became Editor of the series in 1966.

In addition to teaching duties, Mostowski was early involved in journal editing. The editing and reviewing work of others are clearly central activities of any leading scientist, and Mostowski was a prominent editor as well as contributor in both Polish and international journals. He was Editor-in-Chief of the Bulletin of the Polish Academy of Sciences, section Mathematics, Astronomy and Physics (from 1957), member of the Boards of the internationally well established "Fundamenta Mathematicae" (from 1949), "Dissertationes Mathematicae" (from its start in 1952), the international "Journal of Symbolic Logic" published in the U.S.A. for the Association of Symbolic Logic (as well as assuming various positions in the prestigious Association), and he was one of the four founding Editors of "Annals of Mathematical Logic" which was set up in the late sixties to fill a gap between the above JSL and the Studies series. Longer articles could not easily be placed in JSL or other leading journals, and were often too short to be published as monographs in their own rights: AML concentrated on the publication of longer articles and shorter monographs.

The Annals of Mathematical Logic had initially four Editors, in addition to Mostowski: Jerome Keisler, Michael Rabin and Hartley Rogers. It published its first volume in 1970 and has by 2007 (with a title changed to Annals of Pure and Applied Logic) reached Volume 148.

Mostowski met his wife Maria (born Matuszewska) during the War. During the German occupation, she attended the underground university in Warsaw. Before assuming a position as librarian at the State Mathematical Institute in 1952, she had a position at the Mathematics Office of the Warsaw Science Association (Towarzystwo Naukowe Warszawskie). From 1956 Maria Mostowska held the position of Senior Librarian, and from 1964 that of Chief Librarian of the Mathematics Institute of the Polish Academy of Sciences (IMPAN). She retired in 1982 and remained in her position on a part time basis until 2002.

Mostowski and his fellow mathematics professors at the Institute paid much attention to the well being of the library. Through the extended international contact network and skilful management of the library, its holdings of journals and books became up-to-date and outstanding in the country. The IMPAN library was (and is) a central clearing point for mathematics information, also helped by an extensive journal exchange programme involving the mathematics publishing activities undertaken in the same building on Śniadeckich Street.

The Author's Student Experiences

The lectures Mostowski gave in Finland at the Vasa Summer School in June 1964 were advertised also in Sweden. Together with a fellow philosophy and mathematics student at the University of Lund, Håkan Wiberg, we decided to attend this Summer School, where also some leading Finnish logicians including G.H. von Wright and E. Stenius were present. The lectures attended there [5] became a starting point for my interest in mathematical logic and the Polish logic school. Krister Segerberg, one of the other Swedish participants in Vasa, and later Professor in Upsala, had commenced a discussion with Mostowski about the possibility of him spending a year in Warsaw. When realising that this might also be an option for me, Mostowski encouraged me to seriously consider it.

During a visit to Warsaw later that summer, the idea to study at the University of Warsaw and attending courses given by Mostowski was further developed. Contacts between the Nordic countries and Poland had up to then been difficult to maintain. Exchanges, of students or lecturers, were complicated to arrange and riddled by red tape. Land connections between the both geographic areas went either through the German Democratic Republic or the Soviet Union, and it was therefore a great event when, in 1963, the first direct ferry connection was opened between Poland, Sweden and Denmark.

After completing my master degree in mathematics and philosophy at the University of Lund in 1965, I commenced studies in Warsaw in the Autumn of that year. The Professor of Theoretical Philosophy in Lund, Sören Halldén, himself an accomplished logician, supported the idea of a period of post-graduate study. As there were no student exchange programme in place between Poland and Sweden at the time, all arrangements had to be done in Warsaw. Mostowski did so in an admirable fashion: the calm and efficient way he dealt with what were usually heavy bureaucracies in University, Academy, and Poland in general, was truly admirable.

Two of Mostowski's assistants, Janusz Onyszkiewicz and Witek Marek, spoke English. Janusz was given the task of taking care of me during the initial period and also sat next to me and translated into English the main model theory course of the 1965/66 academic year (English title: Relational systems and structures over them). The main book to read was Rasiowa and Sikorski's monumental "Mathematics of Metamathematics" which had appeared in 1963 (ref. [6]). Of key importance were further the weekly seminars in mathematical logic, held at IMPAN and organised by Mostowski. The seminars were mostly held in English, covering the latest results and attracting an international audience. The leading logicians from Warsaw included Grzegorczyk and Rasiowa – sometimes also Jerzy Łoś – while from Wrocław (which had inherited the Lwow cradle of Polish logic from the pre-War period) Jan Mycielski and Czesław Ryll-Nardzewski, as well as a steady stream of top Czech logicians were actively engaged. In one of the semesters I counted participants from 17 countries in this seminar.

During the next academic year, 1966/67, a post-graduate exchange programme was put in place between Poland and Sweden. I became the first post-War Swedish exchange scholar in Poland: Mostowski and the Swedish Embassy in Warsaw had both contributed to this. While we had been only a few foreign students in the Mostowski group the year before (Emilio del Solar Petit from Chile and Yoshindo Suzuki from Japan arrived shortly after me), by 1966 we were quite a large group. The exchange programme between Warsaw and Berkeley/Stanford usually offered two post graduate students from the US to stay in Warsaw during any given academic year at that time (compare B.F. Wells [7]).

During the Summer of 1966 Janusz Onyszkiewicz was instrumental in organising a student group travel to the World Congress of Mathematicians in Moscow. This became a memorable event for logicians in general and the Polish logic school in particular. Paul Cohen from Stanford was there awarded the Field Medal for his work on independence theorems in Set Theory, which contributed to a central role for logic at this time in the mathematics community. Leading Russian logicians and algebra experts like A.A. Mal'tsev gave impressive lectures, and a number of acquaintances with Russian logicians (who had the greatest difficulty to visit even Poland during the Communist years) were maintained or commenced.

Set theoretical independence theorems remained high on the agenda of the Warsaw logic seminar and in the Autumn of 1966 I got from Mostowski the difficult task to present Dana Scott's notes on Boolean-valued models for higher order logic (ref. [8]) at his seminar. These notes caused headaches (not only for me) and the comments in Dana's article in this volume (ref. [9]) seem very adequate.

Towards the end of my student period in Warsaw, many of us joined with Mostowski to attend the Fourth International Conference in Logic, Methodology and Philosophy of Science in Amsterdam (ref. [10]). Mostowski and Tarski played here leading roles, the latter having been a key figure in organising this form of international endeavour – covering a very large scope, within which mathematical logic formed but a part. Mostowski introduced his Warsaw group to a number of important colleagues and I had the first opportunity to meet with Tarski.

During the study period I met my future wife Zosia and we married in Warsaw in 1967. Up to 1969 I regularly visited Warsaw, Mostowski and logic colleagues there. My work came to concentrate on algebraic and Kripke models for some non-classical calculi, and the problems were more in the orbit of Professor Rasiowa who introduced me to Ina Rauszer and George Rosseau; together with Dick de Jongh of the University of Amsterdam they became my study supervisors during the following years.

In March 1968 I stayed in a Warsaw flat with a friend (assistant to a philosophy professor who had lost his position at the University in the political turmoil that month). During a nightly police raid in the flat, where political pamphlets were sought and not found, I was taken for interrogation to Pałac Mostowski (the police headquarters, no relation to Professor Mostowski). Given 24 hours to leave the country, I went in the morning to Professor Mostowski to relate what had happened. He was calm as usual and advised me to do just what they had told me – postponing our discussion on logic matters to some months later. A logic conference organised in Warsaw in August 1968 coincided with the Soviet invasion of Czechoslovakia, and several participants were not able to attend. The next conference of importance to me was the Logic Colloquium '69 in Manchester [4].

At this conference the Dutch publisher North-Holland had a display table, and their representative (also a mathematician but not specialising in logic) took part in our off-lecture activities. At some point he asked a group of us whether there was anyone interested in soliciting for his position as publishing editor in Amsterdam. Most of us had other plans, but as I was to return to Sweden via Amsterdam I expressed interest in visiting his company. There I was well received by the company's owner M.D. Frank and his colleagues. Talking about my background and study with Mostowski, Frank rose in his chair: Mostowski was his old and esteemed friend and advisor, editor of the main logic book series, etc. Frank wrote to Mostowski for a recommendation and I was promptly offered the position at North-Holland.

North-Holland before 1970

Commencing as publishing editor in mathematics, computer science and economics in the autumn of 1969, I started with small programmes in these areas with the exception of logic. North-Holland published at that time only one journal in above areas, Indagationes Mathematicae, for the Dutch Academy of Sciences. The book series "Studies in Logic" and "Contributions to Economic Analysis" (started with Jan Tinbergen as Editor in 1953) were the main outlets, and books had been the only publications of North-Holland outside the company's main areas of physics and biochemistry. In computer science, North-Holland had published the IFIP World Congress proceedings in Munich 1962 and on my desk the galley-proofs of the proceedings of the 1968 World Congress, held in Edinburgh, were waiting to be finalised. My brief was to start new journals in main (sub) areas of the subjects covered by me, and to develop the book programmes.

Among the journal projects already commenced in development before my arrival was "Annals of Mathematical Logic". As mentioned above, the four leading logicians who had agreed to edit the AML had done so under the explicit assumption that the journal would not compete with the Journal of Symbolic Logic – with which they were all associated. The main argument for launching the venture for North-Holland was that shorter monographs could no longer be cost-effectively published in the Studies series.

Mostowski visited Amsterdam in December 1969, Frank explaining to him the reasons for having sold North-Holland to the fellow Amsterdam publishing house Elsevier. The merger between the two companies would be completed in 1970, and Frank would remain with the combined company until 1972 (compare also (ref. [11])). His commitment to logic publishing, initiated in the presence of Mostowski in 1948, would be passed on to me under the guidance of Mostowski. The latter was willing to listen to ambitious publishing plans within the broad scope of mathematics/computer-science/economics, but emphasised that his expert advice would be only in mathematical logic and related areas of pure mathematics.

The broader accent in Studies in Logic, including philosophical logic, had weakened after the passing of Beth in 1964. Frank had a financial share in the Dordrecht D. Reidel Publishing Company, started 1960. D. Reidel had commenced publishing in philosophical logic with the consent of Frank, who thereby accepted the focussing on mathematical logic in the Studies under the main editorship of Mostowski and A. Robinson. (D. Reidel in the early seventies sold out to the Dutch publishing group Kluwer, and D. Reidel Dordrecht became the basis of Kluwer Academic Publishers – KAP – which in turn, some 30 years later, was fused with Springer-Verlag. The earlier D. Reidel Publishing Company has now become Springer-Verlag Dordrecht.)

The reputation North-Holland had been building up in the logic area before 1970 helped in laying contacts in many mathematics areas, in computer science as well as in mathematical and computational economics.

Logic Publishing before 1970

The Studies in Logic series had traditionally contained monographs and conference proceedings; in other words, their contents were on a research level. Textbooks in English could hardly be published out of the Netherlands as that needed strong distribution facilities in English speaking areas. Instead, North-Holland had been experimenting with textbooks outside the Studies series. In 1969 the successful "Models and Ultra-products" by J. Bell & A. Slomson (ref. [12]) was published and a few years later "Introduction to Mathematical Logic" by J. Bell & M. Machover (ref. [13]).

From 1970 fewer titles in philosophical logic would be published in the Studies, as also commented upon in connection with D. Reidel above. At this time another, competing, book series appeared on the horizon, the "Omega" series by Springer-Verlag, and will be commented upon below. That North-Holland had been the only publisher active in the logic field for some 25 years is probably an interesting historic foot-note, but by the late 60-ies the field had grown sufficiently to attract several publishers.

Among classical dossiers passed on by Frank to me were those of A.A. Fränkel and Y. Bar-Hillel with widely known set theory titles, A. Robinson's several important books, and A.M. Turing's collected works which Frank had commenced working on in 1958 after meeting with the late Turing's mother Sara. It would take over 40 years and many meetings before Turing's Collected Works were completed in the early 2000s [14].

Mostowski spent part of the academic year 1969/70 as a Visiting Scholar at All Souls College in Oxford, and in early 1970 I visited him there to give an overview of where we stood with the Studies series and the AML journal. Oxford had become a central place in logic at that time with both Robin Gandy (editor of the Turing work) and Dana Scott there. Mostowski arranged a meeting with Gandy to get a plan set up for the completion of the Turing project. (Gandy's plan for the project from 1970 became a basis for more than 20 years of discussions on its implementation, as also commented upon above.)

A. Robinson had proposed to Frank that the Studies Board (Heyting, Mostowski, Robinson and Suppes) needed strengthening in the area of Model Theory, and that Jerome Keisler would be invited to join. Mostowski strongly supported this and Keisler was invited and accepted to join in 1970.

Mostowski as Editor

By 1970 sixty volumes had been published in the Studies. Many of these had become classics and were repeatedly reprinted or new editions were prepared. All books in the series up to this time had been traditionally type-set in hot lead and printed by impact. Only a few typesetters were able to perform the intricate logic setting work and it was not unusual that the production of a volume in the series could take a year or more. As for proceedings of conferences, here is an example from an event in 1968: the volume "Proof Theory and Intuitionism," edited by A. Kino, J. Myhill, R.E. Vesley (Eds.), appeared in 1970 [15]. The discussions between the publisher and series editors concerned therefore not only new projects being reviewed, but also a large number of projects in progress.

Along with the backdrop of slow editing and production procedures, a steady stream of new book proposals had to be considered by the series Editors. After a volume proposal was accepted for publication, it could take years, sometimes decades, before the book would appear. Criteria for accepting work in the series were sometimes influenced by the publisher's practical and financial considerations. Short monographs, the main fare of the series in the early years, were by 1970 no longer attractive to publish as independent books. Mostowski was actively engaged in the above process both as author and editor. In 1970 he made a detailed review of Rasiowa's manuscript "An Algebraic Approach to Non-Classical Logics" and proposed it for publication in the Studies [16].

In the same year AM was actively engaged in the launch of Annals of Mathematical Logic and within a year it had a sufficient flow of high quality manuscripts to generate a quarterly publication. Around the same time North-Holland/Elsevier-Science was engaged in a number of journal projects which were related to logic and for which advice also was asked from AM. These titles included "Artificial Intelligence", "Discrete Mathematics", "Information Processing Letters", "Journal of Mathematical Economics", "Journal of Pure and Applied Algebra" to mention a few. Regarding the latter a quote from a letter from Mostowski dated 11 December, 1970: "I looked over with interest the first issue of the Journal of Pure and Applied Algebra. It seems to me that it will be an excellent journal. Contrary to what you may think, I was mostly interested in the paper by Green on representations: this is a subject which fascinated me since my early youth although I was never able to do anything in it". It was a privilege for me to be able to ask for advice on a broad range of topics as our publishing programme kept expanding.

A recurring theme in the editorial discussion was the relation to computer science within the Studies; I quote from a letter by Mostowski dated 7 February, 1971, referring to a proposal made earlier by his fellow editor Patrick Suppes: "Should we include books on computer science? I believe that the answer depends on whether the proposed "Computer Science Series" will come into being (separately discussed with Michael Rabin and John Shepherdson, among others; author's note). Should that be the case then I would decidedly be for admitting only books very closely related to logic. . . ." Other themes for discussion between Editors included the boundary with philosophical logic and the inclusion of conference proceedings in the series. Quoting from the same letter: "Concerning Professor Suppes' letters: I am for publishing the Proceedings of the fourth Congress of Logic, Methodology and Philosophy of Science. The two previous Proceedings were very good and I believe we should maintain the tradition of publishing proceedings of these gatherings strange as they are."

A competing book series, to be published by Springer-Verlag, the "Omega" series became an often discussed item with the Editors of the Studies. Sometimes the same authors were invited to contribute to both series, with some irritation and extra correspondence as a result. By 1973 the scope of the activity of the competitor was well known, and both Mostowski and his fellow Editors were sending calming signals. To quote from a letter from A. Robinson of 30 July, 1973: "One becomes more and more aware of the Omega-group. Muller's conception of an international Bourbaki group is very effective, especially as it includes periodic meetings in the Black Forest. Since it seems unlikely that North-Holland/Elsevier will offer free vacations in the Swiss Alps we can only continue in our own small way, relying on our reputation." (Professor Gerd Müller of Heidelberg and the Springer publishing editor Klaus Peters were the initiators of this competitor to the Studies. In hindsight it was probably a good thing for us working on the Studies to face strong competition at the time.)

In the meantime, between 1970 and 1975, a long list of important volumes did appear in the Studies, in total more than twenty! Prominent among them was the graduate text and reference work "Model Theory" by C.C. Chang and H.J. Keisler and the monograph "Non-Standard Analysis" by A. Robinson.

In April 1973 A. Robinson visited Amsterdam, and a first discussion about a "Handbook of Mathematical Logic" was held, at the initiative of the publisher. Both of us were to visit Warsaw and Mostowski the following month and there take the project further. Mostowski, A. Robinson and their fellow Editors were warming to the idea of setting up a large collaborative work covering in review form all major areas of the field. The next step would be to find a General Editor and a workable structure for the project. Soon the name of Jon Barwise was mentioned, and in spring 1974 he finally agreed to take on this work. Neither Mostowski nor A. Robinson were to live to see the publication of this work (ref. [17]).

In autumn of 1974 AM had to deal with an unfortunate matter. Since the Bucharest Congress in 1971 Mostowski had been President of the IULMPS, which would hold its fifth Congress in London, Ontario, in August 1975. Mostowski defended the position that the Proceedings would be published in the Studies series, but was voted down by the local organisers and his Union Council. The Proceedings were to be published in a competing series, by D. Reidel Publishing Company. Other editorial matters were soon in focus, however. A. Heyting had retired as Editor of the Studies in 1972 and was succeeded by his former student Anne Troelstra. Two years later A. Robinson died after a long period of bravely fought illness, remaining active as Editor and revising manuscripts of his own up to the end. Mostowski assumed the central role on the editorial Board, seeking also the appointment of Jon Barwise as the successor of A. Robinson. Mostowski was engaging in editorial matters up to and including his two months stay at Berkeley in the summer of 1975. A copy of his last letter to the publisher, of 7 August, is attached after references. On the way to the London, Ont., Congress, in Vancouver, Andrzej Mostowski passed away two weeks later.

References

[1] E.W. Beth et al., eds, *Proceedings of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948*, North-Holland, Amsterdam, 1949.

[2] A. Mostowski, *Logika Matematyczna*, Warsaw–Wroclaw, 1948.

[3] A. Mostowski, *Foundational Studies, Selected Works, Vols. I and II*, PWN, Warsaw and North-Holland, Amsterdam, 1979.

[4] R.O. Gandy and J.M.E. Hyland, eds., *Logic Colloquium 76*, in: Studies in Logic, Vol. 87, Amsterdam, 1977 (dedicated to the memory of Andrzej Mostowski).

[5] A. Mostowski, Thirty years of foundational studies, in: *Acta Philosophica Fennica*, vol. 17, Helsinki, 1965.

[6] H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, PWN, Warsaw, 1963.

[7] B.F. Wells, *The Prince of Logic*, this volume pp. 421–428.

[8] D.S. Scott, *Boolean-Valued Models for Higher-Order Logic*, Logic Seminar Stanford, Jan. 1966 (mimeographed).

[9] D.S. Scott, *The Algebraic Interpretation of Quantifiers: Intuitionistic and Classical*, this volume pp. 289–312.

[10] B. van Rootselaar and J.F. Staal, eds., *Logic, Methodology, and Philosophy of Science III*, in: Studies in Logic, 1968.

[11] E.H. Fredriksson, ed., *A Century of Science Publishing, A Collection of Essays*, IOS Press, Amsterdam, 2001.

[12] J.L. Bell and A.B. Slomson, *Models and Ultraproducts*, North-Holland, Amsterdam, 1969.

[13] J.L. Bell and M. Machover, *An Introduction to Mathematical Logic*, North-Holland, Amsterdam, 1975.

[14] A.M. Turing, *Collected Works*, in 4 volumes, North-Holland, 1992–2001.

[15] A. Kino, J. Myhill, R.E. Vesley, eds, *Intuitionism and Proof Theory*, in: Studies in Logic, Amsterdam, 1970.

[16] H. Rasiowa, *An Algebraic Approach to Non-Classical Logics*, in: Studies in Logic, Vol. 78, Amsterdam, 1974.

[17] J. Barwise, ed., *Handbook of Mathematical Logic*, in: Studies in Logic, Vol. 90, Amsterdam, 1977.

Berkeley, Calif. ~~June~~ August 7, 1975

Dr. E Fredriksson
North Holland Publishing Co
P.O. Box 103, Amsterdam, The Netherl.

Dear Dr Fredriksson,

The enclosed paper by Hintikka and Rantala is for the Annals. It was received May, 8, refereed by Mr Krynicki from Warsaw and accepted for publication with the consent of all editors. The author's address is: J. Hintikka, 13 Mäntypaadentie, 00830 Helsinki 83, Finland.

I am more and more skeptical about the possibility of having a meeting of editors of the Studies in Logic series in London. Professor Troelstra will not come (his paper is to be read by Mrs Moschovakis); Prof. Keisler wrote to me that he „might come" for the last two days. I will be leaving on Sept. 1. Thus probably only two editors will be present.

In this situation I suggest that you write to the Editors a letter in which you would ask them to express their consent for coopting Prof. Barwise to the editorial committee and invite them to discuss further extensions of the committee in correspondence. From the letters I received and from my conversation with Prof. Suppes I see that Barwise will be accepted unanimously.

I will be leaving Berkeley on August 19 and shall be consecutively in Vancouver (Simon Fraser), Montreal (4135 rue de la Peltrie, c/o. A. Szachański) and in London. From September 2 shall be back in Warsaw.

There is going to be in Poland a logic meeting in which many good people will take part (P. Aczel, S. Grigorieff, J. Crossley, P. Vopěnka, K. Bukowsky, possibly some Russians). What do you think about a publication of the Proceedings as a joint venture of PWN and North Holland? This is a preliminary suggestion. If you were interested we could discuss it in detail later.

With best personal regards,

Andrzej Mostowski.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Mostowski and Czech–Polish Cooperation in Mathematical Logic

P. HÁJEK [a], V.W. MAREK [b] and P. VOPĚNKA [c]
[a] *Institute of Computer Science, Academy of Sciences, Prague, Czech Republic*
hajek@cs.cas.cz
[b] *Department of Computer Science, University of Kentucky, USA*
marek@cs.uky.edu
[c] *Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic*
petr.vopenka@mff.cuni.cz

Czech mathematics was traditionally oriented to geometry and applied mathematics. Between the two world wars also orientation to differential and integral calculus flourished. The approach was conservative, not as in Poland: Polish mathematics accepted the modern set-theoretical and logical directions in the development of mathematics and brought first-class results in them. In Czech mathematics the modern approach was represented only by the famous topological seminar of Professor Eduard Čech at Masaryk University in Brno. During the second world war Czech universities were closed and Czech professors were forbidden to teach. After the second world war the mistrust to mathematical logic and axiomatization of mathematical theories continued.

It was Ladislav S. Rieger (1916–1963) who started Czech work in mathematical logic and become just the founder of it. The thesis [3] is a detailed monograph on L.S. Rieger (in Czech). In 1950 Rieger spent a half-year stay in Warsaw, 1953 he founded his first seminar on mathematical logic and in the same year he has had a lecture on the 8th congress of Polish mathematicians. In 1955 he held a public lecture on basic problems of mathematical logic; in 1959 he actively participated in the Symposium "Infinitistic methods" (in Warsaw) as the only participant from Czechoslovakia and in the same year he defended his DrSc dissertation (habilitation in the German/Russian system of degrees). The referee (opponent) of his dissertation was Andrzej Mostowski, who evaluated Rieger's work highly. Rieger published papers on Gödel's set theory (and its variant for finite sets) in the years 1956, 1958 and 1960.

In 1959, E. Čech informed Vopěnka on Gödel's result on the consistency of the axiom of choice and continuum hypothesis; Vopěnka contacted Rieger and Rieger founded a new set-theoretical seminar devoted to Gödel's result and related topics. Rieger died in 1963 (his monograph on algebraic methods of mathematical logic appeared only posthumously). Rieger's paper "On the consistency of the generalized continuum hypothesis" was published in Rozprawy Matematyczne 31 (1963) 1–45. (That journal has been later renamed "Dissertationes Mathematicae" and is devoted to longer papers.)

Vopěnka continued organizing this seminar and invited some students to participate. The seminar existed till 1973 or so and its members published 92 papers. Their bibliography can be found in [1] and [2]. The following topics were studied in the seminar and in the published papers: Gödel-Bernays set theory GB and its (syntactical) models (interpretations), notably models showing the independence of the continuum hypothesis, also permutation models in the sense of Fraenkel-Mostowski for independence of the axiom of choice; many results of unprovability of particular assertions in GB. Further, Vopěnka's version of boolean-valued models (from 1964, values of formulas being open sets in a topological space), semisets (from 1971) and the theory of semisets. From the participants of the seminar let us name (in alphabetical order) B. Balcar, L. Bukovský, K. Čuda, P. Hájek, K. Hrbáček, T. Jech, J. Mlček, K. Příkrý, A. Sochor, P. Štěpánek.

18 papers from the 92 papers mentioned above were published in the Bulletin de la Academie Polonaise des Sciences and all were presented by Professor Andrzej Mostowski. In the second half of the sixties the possibility of travelling outside Czechoslovakia became more tolerant particularly as far as visiting Poland was concerned. Polish mathematicians could more easily travel to Western countries and were better informed on the development of Foundations in the world. Thus for Czech and Slovak logicians was very useful to visit Poland (Warsaw and also other centers, usually Wrocław) rather frequently, visiting first of all Mostowski and his students, among them Marek, Onyszkiewicz, later also Adamowicz. The visitors very much enjoyed Mostowski's friendliness and admired his knowledge. Czech mathematicians were also happy to meet Professors Grzegorczyk, Mycielski and Ryll-Nardzewski as well as younger colleagues Weglorz, Srebrny, Murawski and others. Czechoslovak and Polish mathematical logicians became good friends, sharing not only mathematical results but also the critical attitude to the totalitarian political system in both countries.

The so-called Prague Spring in the winter of 1967/1968 opened new possibilities for Logic in Czechoslovakia. Those were short-lived. The occupation of Czechoslovakia by Soviet Union and the states of the Warsaw Pact in August 1968 significantly affected the developments of Foundations in Prague, Brno, Košice and the rest of universities in Czechoslovakia. Some members of Vopěnka's seminar (notably T. Jech) emigrated. Those who stayed had problems – Vopěnka's chair of mathematical logic was eliminated and Vopěnka was forbidden to attend conferences and seminars abroad, including those in Poland. It tells something about the way one had to live and do mathematics in those years that several times Vopěnka crossed the Czechoslovak-Polish border (in places where "small visits to places near the frontier" were allowed) and went "incognito" to Warsaw and Wroclaw or to meetings organized by Polish logicians. Hájek's problems were smaller; often he was allowed to attend conferences in Western Europe assuming he was invited and the expenses were paid by the organizers.

In 1973 Vopěnka founded a new seminar on his Alternative set theory with new group of young people. Hájek and Pudlák started a seminar on metamathematics of arithmetic.

The Logic Colloquium in 1980 was scheduled for Prague. Polish mathematicians were invited, W. Marek has been scheduled to speak. The Colloquium was cancelled when the authorities learned that some western logicians prepared protests against imprisoning of the Czech dissident Vaclav Benda. Apparently the political power was afraid that Czechoslovak participants could join the protest (which the totalitarian system could not accept). Hájek was the chairman of the organization committee and after the cancellation he was not permitted to travel to conferences in the West for about three years. The Czech logicians were finally able to organize the Logic Colloquium only in 1998, years after the "Velvet Revolution" disposed of the totalitarian regime. Vopěnka was a honorary chairman of the colloquium and the following was his opening speech: "Ladies and Gentlemen, I am very happy to be able to welcome you to Prague. French historian Ernest Denis once wrote that in Prague every stone tells a story. As you walk across the Charles Bridge, pause to remember Tycho Brahe and Johannes Kepler who used to stroll there over 400 years ago as well as Bernard Bolzano two centuries later. I am sure that you too will fall in love with this old, inspiring, majestic, but also tragic city. These were the words with which I had planned to welcome participants of Logic Colloquium '80 which was cancelled by the communist government. The totalitarian regime was afraid that the participating mathematicians would call for the release of their colleague, mathematician Vaclav Benda, who was serving a 5 year prison term. He was imprisoned for publicly drawing attention to politically motivated prosecution of those opposing the regime. For us, Czech mathematicians, the cancellation meant even deeper isolation from our colleagues abroad. But we never doubted that even though mathematics is very beautiful, freedom is even more so. Logic Colloquium '98 will now commence."

References

[1] P. Hájek, Sets, semisets, models, in: *Axiomatic set theory*, D. Scott, ed., AMS, 1971, pp. 67–82.

[2] P. Hájek, Bibliography of the Prague seminar on foundations of set theory – part II, *Czechoslovak Mathematical Journal* **23** (1973), 521–523.

[3] E. Kozlíková, *Ladislav Svante Rieger 1916/1963*, Thesis, Charles University Prague, 2006.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Reminiscences of Warsaw and Logic, 1964

Peter G. HINMAN
Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA
pgh@umich.edu

I visited Warsaw for the academic year 1964–65 while I was a graduate student at the University of California at Berkeley. The program was sponsored by the US State Department and provided for the placement and upkeep of, as I recall, eight students at similar stages of their careers. In contrast with several other logic students from Berkeley who visited Warsaw around this time, it was not Alfred Tarski who promoted my visit but rather Dana Scott. I had been fairly involved with Tarski a couple of years earlier to the point that he asked me to do my dissertation work with him. In the end I declined and chose rather to work with John Addison, because I wanted to avoid being one of the several Tarski students still trying to finish Ph.D.'s in their seventh, eighth and even tenth years of graduate study. I worried that I would get into trouble by turning Tarski down, but we remained friends and I expect that he wrote something to Mostowski about my arrival.

When I arrived in Warsaw, I was well into research towards my dissertation, but did not have any major results. My first two months were spent with the others in our group in an intensive course in the Polish language. We spent each morning with a rather sympathetic woman who spoke excellent English and each afternoon with a quite stern man who spoke almost no English. It was a good combination, and I learned the language rapidly. I had studied Russian for three years, which was both a benefit and a hindrance: I could easily understand the grammatical structure of the language and many words were identical or similar, but where the languages differ, I often had a hard time replacing my Russian instincts with Polish ones.

I roomed with another member of the group, Mike Iribarne, somewhat uncomfortably with a family that didn't really want a lodger, but needed the extra money. They also didn't speak much English, so at home also I learned Polish by immersion. Mostowski was the faculty member I saw the most of. I listened to his lectures on Set Theory at the University and participated in his seminar at the Mathematical Institute. As I recall we talked privately periodically, but not on a regularly scheduled basis. He was always charming, gracious to me – courtly seems the right term – and made me feel very welcome in both contexts. There was another logic student in our group, Bob Kowalski, but early in the year Bob stopped participating in most of the logical activities for two reasons. His interests were drifting from mainstream logic to computer science, where he eventually made his mark, and more importantly he was spending most of his time with a Polish woman, whom he eventually married during that year in a stereotypical three-day

Polish wedding in her hometown, a very small village some distance from Warsaw. That really felt like a different world.

The seminar was very active that year with many visitors. In an old file I found notes from lectures by Ryll-Nardzewski, Makkai, Příkrý, Grzegorczyk, Szczerba, Onyszkiewicz, Vopěnka and, of course, Mostowski himself. These covered a wide range of topics in Set and Model Theory, although oddly not much in Recursion Theory, my field and also that in which Mostowski made his greatest mark. In particular, there was little knowledge in Warsaw of the esoteric topic of recursion in higher types that was the subject of my dissertation. I gave several talks on the subject, but as I recall I did not manage to stir up a lot of interest. However, I did make progress on my dissertation, proving fairly early in the year one of the central theorems concerning a hierarchy for the 1-section of any normal type-2 object.

Of course, I also remember well many of the other notable logicians and logic sympathizers who were around at that time: Sierpinski, Łoś, Kuratowski, Rasiowa. Sierpinski was quite old and frail, but always to be seen through the open door in the depths of his office. Kuratowski was near retirement, but still very active.

One non-mathematical incident connected with Mostowski sticks in my mind. This was the year of the Free Speech Movement in Berkeley, the beginning of the most intense period of 60's student activism. There was a student strike that shut down the campus for many days. Mostowski, unusually in the Warsaw of the 1960's, had access to Time Magazine, read about the strike, and quizzed me about it several times. He found it hard to understand why the students were so upset, and equally why the University allowed them to have such a large effect. I tried my best to explain what motivated the students, but even though I was largely in sympathy with their goals, I was not radical enough to approve of their methods, so my heart wasn't really in it. Indeed, my time in Warsaw made me something more of a fan of capitalism; seeing the lethargic way that sales clerks, waiters, and others serving the public carried out their duties in a socialist state convinced me of the necessity of economic incentives for performance, something that apparently was totally lacking in the Polish economy.

Life in Warsaw in those times felt drab and confining to a young American. Most buildings, including the Mathematics Department of the University in the Palace of Culture and the Institute were dark and dingy. Although people talked relatively freely in the Institute, even there they tended to avoid political topics. Even a leftist, antiwar protester from Berkeley like me felt moved to frequent the facilities of the US Embassy, and I was adopted by a minor American official who lived across the street from the Embassy. During the year, I made several trips to West Berlin and Vienna, where I recall feeling almost overwhelmed by the profusion of colors and sounds. I also visited Prague, which was, if possible, even grimmer than Warsaw. The joke of the two dogs on the Polish/Czech border seemed entirely apt: the Czech dog boasted that he was lucky to have more to eat than his friend, but the Polish dog was sure he had the better deal because he was allowed to bark.

Spring vacation brought the opportunity to travel with a small group to the Soviet Union: Moscow and Leningrad. This was a very intense trip, which again brought home the differences among the Eastern block countries. Two incidents stand out among many. At a student party in Moscow, I had lively conversations (in my weak Russian and their weak English) with several students, all of whom were desperate to find a way out of the Soviet Union. But one, who had a particularly lovely and charming wife, was so committed to this that he told me in no uncertain terms that if the opportunity came he would leave his wife with no second thoughts. Later, in Leningrad, I was visiting the Anti-religion Museum housed in an old orthodox church. Standing among all of the exhibits which extolled the evil that religion had done for the world, a point of view that agreed well with that of my upbringing and generally my convictions, I found myself so disturbed by the lack of objectivity and sloppy arguments that I gave an impromptu short lecture to the surrounding Russians to the effect that indeed religion had also brought some benefits to mankind. The funniest scene on this trip was right at the beginning as we crossed the Polish-Soviet border. Just before we left Warsaw, I had received a package containing among other things a dozen oranges, worth more than their weight in gold to someone as starved for fresh fruit as I was then. I took them along, but at the border was told that they could not be imported into the Soviet Union. After much wailing, I managed to find out that the fruit could come in, just not the peels. So there we were, in a big immigration shed, surrounded by guards toting machine guns, peeling oranges and dumping them into a plastic bag. We did eat them over the next few days, a bit fermented at the end.

I was tremendously impressed by the Polish spirit that even in a very restrictive and oppressive political and economic environment kept an optimistic view of the future and managed to enjoy life in the present as well. More than most peoples, Poles seem to have learned well the lessons of history; they had seen their country emerge several times from foreign domination and were sure that it would happen again.

Food was another very difficult problem. I grew up in a family that greatly valued good food, and after the dark four years of eating in a college dormitory, I had appreciated through my graduate student years the opportunity to cook good things for myself and my roommates. Living in California also provided the opportunity to sample the products of the nascent wine industry, and the dollar was then so strong against European currencies that even imported wines were at least occasionally in a graduate student budget. All that came to an end in Warsaw. I remember so well the small neighborhood shop that from early in winter offered only the most scrawny of root vegetables and very little meat. And the "supermarket" SuperSam with its shelf after shelf of whatever happened to be available that week – pickles or sugar or flour – packed in featureless packages. Of course, I didn't really have regular access to a kitchen, but these stores set the tone of what food I could get at University cafeterias and town restaurants. Still, for a poor student, a Bar Mleczny was a great place to get some sustenance at very low cost, and Polish vodka was astoundingly cheap and superb.

After leaving Warsaw, I finished my Ph.D. in 1966 and got a job at the University of Michigan. That first year the Department was offered a gift sufficient to support a visiting professor for a year with the stipulation that it be a logician! It wasn't hard to convince the committee, essentially Roger Lyndon, that Mostowski would be the ideal candidate, and I wrote to him in June of 1967 with the invitation – in Polish! I got a reply as soon as the mail service permitted thanking me profusely for both the invitation and the fact that he could reply in his native language. Unfortunately, it turned out that he had too many other obligations to accept the offer; it would have been wonderful to have had him in Ann Arbor for a year.

I next saw Mostowski at the International Congress of Logic, Methodology and Philosophy of Science in Bucharest in 1971. That week in Bucharest made me think of my time in Warsaw in a different light. It was clear from my first day there that the political climate in Romania was incomparably worse than in Poland. There too we stayed with a family that didn't want us, but in this case they took us in not as a free economic decision but because they were forced to. They were sullen and quite unpleasant to us. Restaurants had menus with many items listed, but almost always they offered only one dish, the ubiquitous cevapcici. Waiters, taxi drivers, and many others would always try to cheat you, presumably out of the desperation of their situation. None of this ever happened in Poland.

In 1973 I was invited to the Logic Semester at the Banach Center and spent several weeks in Warsaw giving a series of lectures on material from my book, Recursion-Theoretic Hierarchies, that was in progress. At the end, Andrzej Grzegorczyk, who was the main organizer of the meeting, gave me a present of a small sculpture with two owls and explained that he gave each of the invited speakers a similar sculpture, that everyone else got only one owl, but that he had so enjoyed my talks that he wanted to give me two. I still have and treasure that sculpture.

I had hoped to meet Mostowski again at the next LMPS Congress in London, Ontario in 1975, but sadly he died enroute to that meeting. That fact cast a distinct pall over many of us there.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

From Mathematical Logic, to Natural Language, Artificial Intelligence, and Human Thinking

(A Short Essay in Honour of Andrzej Mostowski)

Robert KOWALSKI
Imperial College London

The Stanford/Berkeley–Warsaw Exchange

I arrived in Warsaw in the summer of 1964, as an exchange student from Stanford University. In addition to two linguistics students, there were two logic students, Peter Hinman from Berkeley and myself. I learned about the Warsaw exchange from Jon Barwise, who had also applied, but was turned down because he was judged to be too young.

Jon and I started our graduate studies together at Stanford in 1963. Jon had just finished his undergraduate studies at Yale University in Connecticut, in three years instead of the usual four. I had just finished my undergraduate studies at the University of Bridgeport, also in Connecticut, but in five years, instead of four, after starting at the University of Chicago and loosing a year between Chicago and Bridgeport.

The academic year in Warsaw did not count towards my academic studies at Stanford. So I was free to take a relaxed approach towards my studies in Warsaw. I took advantage of this freedom, to learn Polish, to meet and spend time with my Polish relations, and to meet and marry my Polish wife.

Although both of my parents were of Polish origin, they were both born in Bridgeport in Connecticut. However my mother returned to Poland with her parents when she was still a child, and both of her brothers were born there. She returned to the United States at the age of seventeen, leaving her family behind.

I attended a primary school, Saint Michael's, attached to a Polish parish. In the first grade, we learned the catechism in Polish: Dlaczego Pan Bóg nas tworzył? (Why did God make us?) And we learned Polish kolędy (Christmas carols). But our Polish lessons were discontinued after the first year, and I didn't learn any more Polish in school afterwards.

Professor Mostowski and his wife invited Peter Hinman and me to his house for tea and cake, shortly after our arrival. They made us feel very much at home. Later I attended his seminar on set theory, where the main topics were Paul Cohen's recent proof of the independence of the continuum hypothesis and large cardinal numbers. Witold Marek was one of the more prominent members of the seminar series and one of the more enthusiastic inventors (or discoverers?) of ever larger cardinals. I also attended seminars by Professors Rasiowa and Grzegorczyk. It was at Professor Rasiowa's seminar that I met my future wife, Danusia.

Logic and Natural Language

Although I studied mathematical logic at Stanford and Warsaw, my interest in logic was not primarily mathematical. I was more interested in the use of logic to improve ordinary human reasoning. My interest in this was wakened by a short exposure to propositional logic in the compulsory, first year, discrete mathematics course at the University of Chicago.

My one year and two months at the University of Chicago were traumatic for me for a number of reasons. Among these was the fact that, at the beginning of the first year, I failed the English placement examination and had to take a non-credit, remedial course in English composition. I finished the year with good A's in all my subjects, except for English, in which I received a poor D. However, I was determined to understand what was wrong with my English and how I could improve it. Eventually, after much independent reading and study, I convinced myself that my problem was that I was treating English as means of self-expression rather than as a medium of communication.

I began to realise that self-expression is a solitary activity, in which a single person attempts to put into words thoughts that are in its own mind. Communication, on the other hand, is a social activity, in which the person attempts to put into words thoughts that it wants to be in another person's mind.

I learned that to communicate effectively, you need to express yourself as clearly and as simply as possible. You need to express yourself clearly, so that your readers (or listeners) understand what you intend and do not understand something else. And you need to express yourself simply, so that your readers do not expend unnecessary effort to extract a useful form of your intended meaning. I eventually convinced myself that both of these characteristics of effective communication have a logical interpretation.

The logic of clarity includes avoiding ambiguity. Pronouns, for example, should have unambiguous referents. Not: "Krysia attended Marysia's logic course. She loved the course." But: "Krysia attended Marysia's logic course. Krysia loved the course." Or: "Krysia attended Marysia's logic course. Marysia loved the course."

Even more obviously, the scope of connectives should be unambiguous. Not: "Krysia will teach logic and Marysia will teach logic or Marysia will teach computing." But: "Krysia and Marysia will teach logic, or Marysia will teach computing." Or: "Krysia will teach logic, and Marysia will teach logic or computing." Similarly, not: "Marysia only teaches logic." (She doesn't sleep, doesn't eat, etc.) But: "Marysia teaches only logic."

The logic of simplicity, on the other hand, is the choice, from among logically equivalent sentences, sentences which are easier for the reader to understand and to use for other purposes. For example, not: "If Krysia teaches logic, then Marysia teaches computing if Basia teaches English." But: "If Krysia teaches logic and Basia teaches English, then Marysia teaches computing." And not "I will attend the logic course or you will not teach it". But I will attend the logic course, if you teach it."

Of course, clarity and simplicity are not enough. Language also needs to be coherent. For example: "Krysia teaches if Marysia studies. Marysia studies if Basia pays. Basia pays." But not the logically equivalent: "Basia pays. Krysia teaches if Marysia studies. Marysia studies if Basia pays."

I was given the opportunity to try out my emerging understanding of these ideas during my stay in Warsaw. Maciej Mączyński, who had visited Stanford the previous year on the same exchange, and who was translating the book "Set Theory with an Introduction to Descriptive Set Theory" by Kuratowski and Mostowski [8], asked me to help him with the translation. We worked out a routine, by means of which Maciej would write the first draft of the translation and I would work on the second draft, improving his English. One day, Maciej informed me that Professor Mostowski wanted to meet me to discuss a problem with the translation.

The problem, it turned out, is how to distinguish between restrictive and non-restrictive relative clauses in English. For example, the relative pronoun "*which*" in the sentence "Let P be a non-empty subset of A *which* contains no first element." introduces a restrictive relative clause. Professor Mostowski maintained that the sentence should be "Let P be a non-empty subset of A *that* contains no first element." He insisted that the correct pronoun for restrictive clauses, as in this sentence, is "*that*", and that the correct pronoun for non-restrictive clauses, as in the sentence "The logic course, *which* was the most enjoyable course Krysia had ever taken, was taught by Marysia." is "*which*".

Professor Mostowski wanted to make sure that I understood the difference between restrictive and non-restrictive clauses: namely that restrictive clauses add extra conditions to the noun phrase which they modify, whereas non-restrictive clauses add extra information which can be expressed equivalently in a separate sentence. Independently of the choice of relative pronoun, non-restrictive clauses should be set off from the main clause by commas, but restrictive clauses should not. Moreover, "*which*", rather than "*that*", is the correct pronoun to be used for non-restrictive clauses.

When I had satisfied him that I understood the difference between the two kinds of relative clauses and that I knew the rule about commas, he accepted my assurances that in the case of restrictive clauses "*that*" and "*which*" are interchangeable. What I didn't know then, but do know now, is that he was right about the distinction between "*that*" and "*which*" in traditional English, but that I was also right that by the 1960s the traditional distinction had largely fallen out of use.

Back in the USA and Beyond

I left Poland with a better knowledge of both Polish and English and with my newly wed Polish wife. But I was disillusioned, not so much with mathematical logic, but with mathematics more generally; and when I returned to Stanford I couldn't settle down to my studies. I left Stanford with enough credits for a Master's degree, and got a job teaching mathematics at the Inter-American University in San Juan, Puerto Rico.

I soon realised, however, that I wasn't going to accomplish very much without a Ph.D., and I left Puerto Rico after only a year, to study at the University of Edinburgh. My Ph.D, which I completed in 1970, was in the field of automated theorem-proving. This led to my work on logic programming [3], in which an appropriately restricted resolution theorem-prover treats logical implications as goal-reduction procedures. I collaborated with Alain Colmerauer in Marseille, who developed the programming language, Prolog, based on this idea. My book "Logic for Problem Solving" [4], published in 1979, aimed to be "an introduction to logic, the theory of problem-solving, and computer programming", all in one.

I left Edinburgh in 1975, joining the Department of Computing at Imperial College in London, working in the area of logic for artificial intelligence. One of the main topics of my research, developed with Marek Sergot, was legal reasoning. We applied logic programming and its extensions to the representation of the 1981 British Nationality Act [9]. Another topic, also developed with Marek, was the event calculus [5], a logic programming representation of causal reasoning, which has been used for such applications as the formalization of tense and aspect in natural language.

I also worked on the development of abductive logic programming [2], which combines predicates defined by logic programs with undefined (or abducible) predicates constrained by integrity constraints. This work gave rise to two further developments. One was the development of an argumentation theory, which led to the demonstration that most logics for default reasoning can be regarded as special cases of assumption-based argumentation [1].

The other development was the embedding of abductive logic programming as the thinking component of an intelligent agent interacting with a changing environment. Working with Fariba Sadri, I developed an agent model [6] in which beliefs are represented by logic programs and goals are represented by integrity constraints. The goals can include maintenance goals, achievement goals, prohibitions, and condition–action rules. Observations and actions are represented by abducible predicates.

I took early retirement in 1999, and became an Emeritus Professor and Senior Research Fellow, so that I could change the focus of my research and return my attention to the original concern that attracted me to logic in the first place, to improve the quality of human thinking [7]. I am happy to acknowledge that my studies of mathematical logic and my work in computational logic have been a useful detour, helping me to develop the tools and techniques needed for this purpose.

References

[1] A. Bondarenko, P.M. Dung, R. Kowalski and F. Toni, An Abstract Argumentation-theoretic Approach to Default Reasoning, *Journal of Artificial Intelligence* **93**(1–2), 1997, 63–101.

[2] T. Kakas, K. Kowalski and F. Toni, Abductive Logic Programming, *Journal of Logic and Computation* **2**(6) (1992), 719–770.

[3] R. Kowalski, *Predicate Logic as Programming Language*, in: Proceedings IFIP Congress, Stockholm, North Holland Publishing Co., 1974, pp. 569–574.

[4] R. Kowalski, *Logic for Problem Solving*, North Holland Elsevier, 1979, 287 pages.

[5] R. Kowalski and M. Sergot, *A Logic-based Calculus of Events*, in: New Generation Computing, Vol. 4, No. 1, February 1986, pp. 67–95.

[6] R. Kowalski and F. Sadri, From Logic Programming towards Multi-agent Systems, *Annals of Mathematics and Artificial Intelligence* **25** (1999), 391–419.

[7] R. Kowalski, *The Logical Way to be Artificially Intelligent*, Proceedings of CLIMA VI, F. Toni and P. Torroni, eds, Springer Verlag, LNAI, 2006, pp. 1–22.

[8] K. Kuratowski and A. Mostowski, *Set Theory with and Introduction to Descriptive Set Theory*, PWN Polish Scientific Publishers, Warszawa, and North-Holland, Amsterdam, 1976.

[9] M. Sergot, F. Sadri, R. Kowalski, F. Kriwaczek, P. Hammond and T. Cory, *The British Nationality Act as a Logic Program*, in: CACM, vol. 29, no. 5, 1986, pp. 370–386.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

A Very Personal Recollection

Hidemitsu SAYEKI *

It was October 1960, when I met Mostowski for the first time on the 9th floor of the PKiN which lifts up its tall spire to the skies with a majestic air in the center of Warsaw. At that time I had just graduated from the mathematics, and physics departments of Tokyo University. In those days there were no professors of set theory or mathematical logic, and no courses on the foundations of mathematics were given at Tokyo University. But I dipped into newly arrived Fundamenta Mathematicae in the library and was impressed by vol. 33 which opens with a dedication to the victims of war. The fact that a school had arisen from the remains of a ruined city despite such suffering appealed to me. I was gradually tempted to pursue work in the foundations of mathematics at the Polish school. But in the 1950's there were no means for a student to take passage from Japan to a country with a communist regime. While studying at the department of physics, I was forced to face serious difficulties in the foundations of quantum mechanics. I wondered if the foundations of mathematics could be separated from those of physics. When I stepped into the labyrinth of quantum mechanics and realized the "unreasonable effectiveness" of mathematics in physics, my interest in the foundation of mathematics was revived and my desire to visit Warsaw grew even more. I had been most encouraged by the booklet "The present state of investigations on the foundations of mathematics" in which Mostowski wrote that "attempts at establishing the foundations of mathematics without taking into account its origins in natural sciences are bound to fail."[1] Finally, in 1960 I was fortunate enough to find a way out of my country.

I was rather tense when I entered his room. The excitement to see a great logician made me nervous. I conceived that he was sitting amongst vast heaps of papers and books in his office. Contrary to my expectations the professor whom I held in very high esteem sat at an empty desk in a spacious room where an extensive view of the city spread through wide open windows. He was very open to this student he was meeting for the first time: I was warmly greeted. He listened to my aspirations spoken in my poor Polish. He asked about my interests and books I had read. My knowledge was very poor. I had barely read Kleene's Introduction to Metamathematics, in which I learned Gödel's Incompleteness Theorem, but at that time in Japan the study of the foundation was limited to Gentzen-type proof theory on the line of Hilbert's program under the leadership of G. Takeuti. I knew Gödel's proof of the relative consistency of GCH and AC through a translation by a philosopher-historian, but it had not become a main topic of active mathematicians there.

My life in Warsaw started by attending his course on the foundations of mathematics, struggling with Polish, while reading a lot of literature including Mostowski's book "Sentences Undecidable in Formalized Arithmetic", which he recommended to me first.

*Université de Montréal. E-mail: sayeki@dms.umontreal.ca
[1] Rozprawy Matematyczne IX, 1955.

As for Gödel's Theorem, I found his book was clearer and had more insight into the nature of thought than Kleene's strictly formalistic proof. From it I learned non-constructive reasoning in the foundational study for the first time.

A few months later, in January 1961, I was invited to Mostowski's home. It was the coldest day I had experienced until then. When I arrived, he said, looking at a thermometer suspended outside his window, that it was 25 degrees below zero. In a cozy room that I have visited several times since then, I had a chance to listen to his philosophy and to talk about my academic interests. He insisted that our investigation must consist of the free use of all fruitful mathematical methods which include non-constructive ones, and that the set theoretical approach was essential. He also said that mathematical notions and methods are rooted in human experience, so that the investigation of the foundations of mathematics should have an empirical basis. He encouraged me to seek applications of foundational problems in natural sciences. I understood later that his thoughts aligned with Tarski's semantical realism and Karl Popper's scientific rationalism. When I ventured to mention an unfounded suspicion on the Church's Thesis, he said he supported this thesis, but any result on the basis of this hypothesis – unsolvability – should stand the test.

One of my interests in mathematical logic was in many-valued logic related to quantum logic, with the intent of interpreting the mystery of quantum. When I mentioned it, he advised me to work on continuous valued logic and its model theory being developed by C.C. Chang. At that time Mostowski had just worked out 2 papers, applying his generalized quantifiers, on axiomatizability of many-valued logics in which the set of truth values is a compactly ordered set.

In 1960, Warsaw had not been completely reconstructed yet from the horrible ravages of the uprising in the final stages of the war. There were still heaps of rubble up and down the street. It was just like the familiar sights of Tokyo a decade before. But Warsaw was very much alive. Despite the low standard of living, I encountered everywhere people with intelligence, liveliness and a sense of humor. In those days in Japan it was widely believed that Poland was totally deprived of freedom and that a narrow ideology was forced upon all intellectuals, though I never took it seriously. In fact I saw in 1960 how different it actually was, at least in mathematical circles. I never met a logician persecuted for his ideology. But I later learned that it was not the same in physics. According to Leopold Infeld, a physicist who returned to Poland from Canada in 1950, Einstein was labeled an idealist and the theory of relativity was oppressed; another physicist was forced to commit a "self-criticism" of his ideology before the public.[2] It goes without saying that toward the end of communist regime, many Polish logicians were sacrificed for their political views. But in the 1960's, as far as I know, every logician was allowed to express his philosophical view, such as platonism, constructivism or intuitionism. It is remarkable that during that period the Polish school of logic did not suffer any ideological oppression, and maintained a leading role in foundational study. I think that Mostowski's strong and righteous character with his firm faith in methodological tradition relating to Tarski's scientific realism must have contributed to this.

Once I saw Mostowski sitting on a small chair in the lobby of the National Bank in the center of Warsaw. He was waiting to meet someone for the first time. On his lap he held Sierpinski's book "Teoria Liczb" (Number Theory). I wondered if he was working

[2]Leopold Infeld, "Szkice z przeszłości" Warsaw, 1964.

on a topic related to number theory. He said this would make it easier for the person meeting him to spot the professor of mathematics. I was there to arrange my trip abroad. I had a little problem: I had failed to collect all the documents necessary for processing. A friend who had dealt with bureaucrats many times before had suggested that I bluff my way through. When I explained the situation to Mostowski, he said "Tarski says what it is, is, and what is not, is not. The best way is to tell just as it is." I followed his advice, and the procedure went through without any trouble.

Around that time I became aware that we were in the middle of an upheaval in the study of the foundation of mathematics. With the new method of ultraproducts, nonstandard analysis was emerging, and the theory of models was making remarkable progress. It was followed by the discovery of Paul Cohen in set theory. In the spring of 1963 I heard from Mostowski about the news of independence proof of the continuum hypothesis and the axiom of choice. Nobody knew at that stage if it was confirmed. I spent the summer of 1963 in Japan, studying a preprint of Cohen's work.

When I returned to Warsaw in October, Mostowski was kind enough to lend me the notes he took at Paul Cohen's talk held on Independence Day at Berkeley. (In those days there was no copying device available in Poland.) His notes had some interesting comments in the margins and was very useful. It was apparent that Mostowski's work on the permutation model played an important role in the new result. When I mentioned his contribution, he told me the following story. In 1937 he attended Gödel's lecture in Vienna about his proof of consistency of the axiom of choice. After he proved the independence of the axiom of choice for the system with individuals, Mostowski was concerned about the same problem for the Zermelo-Fraenkel system. But he was told that Gödel already had a proof of it. So Mostowski abandoned his pursuit of that problem, although Gödel never published his result. Nearly 40 years later I found another side to this episode in a Gödel's letter to W. Rautenberg in 1967.[3]

Then came a surge of independence results using the method of forcing in set theory. Mostowski's idea of the permutation model was applied to generic models by many authors. However, he never mentioned it when he talked about these results. Generally, he didn't cite his name even when he used his own achievements in his lectures. He was extremely modest and humble in manner. At every seminar at the Mathematical Institute in Warsaw it was customary to circulate a sheet of paper asking the attendants to sign up. Those who were present wrote their names one after another from the top left of the sheet. When the sheet came to me, I found always Mostowski's signature in his distinctive hand on the bottom right of the sheet so that nobody could sign below his name.

In the summer of 1964 I attended the International Congress in Jerusalem, and witnessed the famous declaration by Abraham Robinson as formalist. His position was based on such points as, infinite totalities do not exist either really or ideally, but we should do mathematics as if infinite totalities really existed. After returning to Warsaw I asked Mostowski for his opinion about it. He said that the epistemology depends on our notion of the sets and so it must be decided by future developments of set theory. Just before the Congress Mostowski spent that summer in Finland, where he gave a series of lectures "Thirty years of foundational studies".[4] When this lecture was published in 1966, he gave me a copy of this book with the author's signature. There I read:

[3] Kurt Gödel, "Collected Works", Vol. V, pp. 179–183.

[4] Acta Philosophica Fennica, Fasc XVII, Oxford, 1966. Also in "Foundational Studies, Mostowski's Selected Works", Vol. I, PWN-North Holland, 1979.

> The rate of development of these domains is presently so rapid that many new excellent results will certainly appear before lectures will come to the hands of prospective readers. Let us hope that these new results will not only bring new interesting insights into the details but also allow us to form a sound judgement about the outstanding problems in the philosophy of mathematics which have been waiting so long for a final solution.

I understood here was his answer to my question. This book was an unparalleled, exhaustive survey of the investigations into the foundation of mathematics. He also wrote there these suggestive words:

> I do not believe that it is worth-while to reconstruct classical theories in constructive terms. No particularly interesting results have been obtained and hardly anybody believes that the cumbersome theories obtained in this way will really replace the elegant classical theories. I am inclined to believe that there are branches of mathematics which simply are not susceptible to finitistic treatment.

In 1965 Mostowski gave another very interesting talk, “Recent results in set thory” at the Colloquium in London.[5] In this talk he surveyed the results on various axioms of infinity and the relative consistency and independence of such hypotheses discovered since Cohen. Then he discussed the comments given by G. Kreisel, A. Robinson and L. Kalmár. He agreed with A. Robinson that the importance of these development depended on the analysis of the role of ZF axioms and the most probable course of set theory in the future is a bifurcation or even a multifurcation. No doubt this did not mean his position was so-called Glib Formalism which says that set theory is just a matter of determining which conclusions do and don’t follow from which assumptions.[6] In fact in reply to Kreisel he said:

> [Kreisel’s] statement about the intuitive evidence of the axiom of choice seems to me rather bold in view of the paradoxical sets whose existence is provable with the help of this axiom.

He concluded:

> We shall have in the future essentially different intuitive notions of sets just as we have different notions of space. There will be a common part of these various axiomatic systems and axioms belonging to the common part will describe the most primitive parts of set theory which are needed in the exposition of mathematical theories perhaps including the category theory.

This insight into the future of set theory provided a guideline for the investigation of set theory in the Post-Cohen era.

After teaching for one year in Japan, I arrived in 1966 at the Université de Montréal as a postdoctoral fellow on the recommendation of Mostowski. In Montréal I met Mrs. Krystyna Szachanska, Mostowski’s sister who was working in the Department of Physiology of the Université de Montréal. Our contact continued until 1993 when she passed away in Ottawa. She told me of her experiences, and of the unspeakable hardship that she and her brother went through during the Nazi occupation, though Mostowski didn’t talk to me much about it.[7] I was overwhelmed by the thought that the country in which I spent my childhood was an accomplice to such atrocities.

[5] Published in “Problems in the Philosophy of Mathematics”, ed. Imre Lakatos, North Holland, 1967.

[6] Penelope Maddy,“ Naturalism in Mathematics”, Oxford, 1997.

[7] He talked later about the suffering he had to endure, and about his works during that time. “Reminiscences of Logicians”, in “Algebra and Logic”, Lecture Notes in Mathematics 450, Springer, 1975, pp. 1–62.

In 1966, 6 months after my arrival in Montréal, the "Séminaire de Mathématiques Supérieures" on mathematical logic was held for 5 weeks at the Université de Montréal. Mostowski was invited as a main guest and gave a series of 20 lectures entitled "Les problèmes métamathématiques de la théorie des ensembles". There he presented a topological interpretation of generic sets which was one of the earliest results to interpret the notion of forcing. The lectures were very well organized and lucid. I was impressed, as always, by the clarity of his thinking. He devoted his energy to this lecture. During the seminar he stayed at his sister's home. She later told me that she was struck with wonder by his enthusiasm for his work. Mostowski declined to attend some social events, but always gave us time to answer our questions. He prepared the text in English but the lectures were given in French. After the seminar we edited his lecture on the basis of notes taken by Marc Venne and published them as "Modèles transitifs de la théorie des ensembles de Zermelo-Fraenkel" from Les Presse de l'Université de Montréal in 1967. The responsibility of some errors in the published note is, of course, entirely ours.

Several times I bothered him by writing to ask questions. Despite his very busy life, he always replied promptly, even to a trivial matters. He wrote in Polish when he was in Warsaw, but from Paris he wrote in French, and from the United States, in English. He was not only very prudent in mathematics, but also very sensitive in language. I am grateful to him also for many corrections of my Polish as well as English text. As a weakness common to Japanese, I often mixed the pronunciation and even spelling of the consonants "r" and "l". At an international conference, when he defined some operations with prefix r (from right) and with prefix l (from left), he gave a remark that "Japanese mathematicians should pay special attention to this distinction." He was interested in Japanese and Chinese, and particularly in Kanji. More than once he told me that the next century would be the century of China. He showed much interest when I told him that the syntax of Japanese language is similar to inverse Polish notation, i.e., its structure is not of the scheme with S+V as European languages, but a tree with V as a trunk. Mrs. Mostowska too always warmly helped me. They both had a variety of interests. In their sitting room there was a bookcase stuffed with mysteries by Agatha Christie, one of which Mrs. Mostowska once lent me. During our conversations whenever I had difficulty explaining something in Polish, he often asked me to draw a picture, and saved me courteously.

The last letter I received from him was stamped on August 14, 1975 in Berkeley. It was written in English:

> This is just to let you know that I am planning to spend a couple of days in Montreal before the opening of the Congress in Ontario.[8] I shall arrive around August 20 and will be staying with my sister. It would be nice if we could meet one day.

I received this letter on August 19th. Since I had many questions to ask him, I awaited his arrival with much anticipation. I phoned Mrs. Szachanska for the details of his schedule. Then next day I prepared all day for my meeting with him. Late evening, Mrs. Szachanska called me and said, "He fell ill, ..., very ill, ..." Her voice had such

[8]Congress of the International Union of History and Philosophy of Science which opened on August 27th in London, Ontario. Mostowski was going to attend it as the president of the Division of Logic, Methodology and Philosophy of Science.

a serious tone that I was utterly confused. Two days later, Professor Mostowski died in Vancouver without regaining consciousness.

On August 26th his ashes were returned to the earth by Mrs. Mostowska in Laurentian Park, some 30 kilometers north of Quebec City, where he once enjoyed its beauty and serenity. This was in accordance with his wishes expressed during his life. It was his desire to have his ashes dispersed in the land where his life ended. I was honored to accompany with Mrs. Szachanska to the house of Professor Matuszewki, brother of Mrs. Mostowska in St. Foy and there bid my last respects to him.

Since his passing, many events challenging to the foundations of mathematics took place. The proof of Four-Color Conjecture raised a question of the mathematical legitimacy of computer usage. Categorical foundation based on intuitionism exerts a growing influence in the study of foundations of mathematics. Exploration of quantum computers demands a fresh reflection on the Church-Turing Thesis. Moreover the development of cosmology might be changing our world view radically. Experimental tests of John Bell's inequalities urge serious contemplation of what physicists have long believed. Everett's thesis[9] drew the attention of physicists only in the 1970's. It is extremely regrettable that we could not see Mostowski contribute in these developments. I can only figure his position to myself. A portrait can only capture an image of a person at a point in time; this is a portrait of Mostowski taken from my personal experience at a specific period more than 30 years ago.

Although Mostowski confined his work mostly to purely mathematical problems, he was always aware of the philosophical aspects of the foundations of mathematics. He regarded mathematics as not detached from other sciences. This attitude toward the reality "out there", as well as his extensive interests in every aspect of foundational study, made him conspicuous among logicians. Maybe, it is not too far-fetched to say that Mostowski did mathematics as a scientist. But if mathematics is a quasi-empirical science, then, as Lakatos indicated, the difference between mathematics and science, if any, must be in the nature of potential falsifiers. What is the nature of mathematics, i.e., what is the nature of potential falsifiers of mathematical theories? Informal arithmetics of natural numbers? No formalist theory can avoid infinite regress. I think that Mostowski was interested in intuitionism not for providing foundations, but for providing falsifiers.

A. Robinson presented the infinitesimals in a formal system justifiable only on the basis of the set theory. If infinite totalities do not exist either really or ideally, why is it necessary to assume "as if they existed"? This question is unavoidable for anyone who believes that mathematics is not a mere game of meaningless terms. Most of physicists believe that physics ends at the smallest distance, Planck's length, and they take great pain to avoid infinity when it appears in a theory or in a calculation. In quantum mechanics a formidable technique is adopted for the elimination. But wherever they go, infinity follows. "It always bothers me", wrote Feynman, "that it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space, and no matter how tiny a region of time."[10] Now general relativity advocates the existence of black hole in nature. And the center of a black hole is a real singularity where infinity lives. A suggested duality shows that the physics inside the Planck length can be identical to the physics beyond the Planck distance.

---

[9]Hugh Everett III, "The theory of the universal wave function", PhD Thesis, Princeton University, 1957.

[10]Richard Feynman, "The Character of Physical Law", MIT, 1967.

I think Mostowski agreed with Galileo's statement that this great book (the universe) is written in the language of mathematics. Perhaps he believed that mathematical existence is equivalent to physical existence. The physicist looks at the world from a frog's point of view; Mostowski looked at it from a bird's perspective.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

The Prince of Logic

Benjamin WELLS
Department of Mathematics, University of San Francisco
wells@usfca.edu

Tarski and Mostowski

When Alfred Tarski learned that I wished to participate in the Séminaire de Mathématiques Supérieures at the University of Montreal in summer 1966, he was enthusiastic. This summer school would be devoted to mathematical logic, and his Berkeley colleague and longtime collaborator Leon Henkin would be teaching there. The organizer was Aubert Daigneault, and other past visitors to Berkeley, including Roland Fraïssé and Léon LeBlanc were lecturing. Simon Kochen, one of the codiscoverers of the decidability of the elementary theory of p-adic fields, which I had been studying, was on the faculty[1]. But most of all, Andrzej Mostowski was coming from Warsaw. I had not met Mostowski, but many of Tarski's students had, and some had visited him for a year in Poland. He was Tarski's first doctoral student, and (as time would tell) I was his last[2]. More than his contributions to many areas of mathematics and logic, more than his collaboration with Tarski on books and other projects, Mostowski's personal connection with his teacher drove Tarski's enthusiasm. That had a lot to do with my acceptance to the seminar.

A Personal Title

The title of this reminiscence is my own title for Mostowski. It has two sources: as Tarski's first student, he might naturally be viewed as the prince of the Tarskian kingdom, not a far-fetched analogy. In addition, I was told by someone in the Warsaw logic community that Mostowski was a Polish aristocrat, a "książę … prince" (or duke). Others can better verify and disambiguate the latter assertion. For the former, I observed that Tarski never spoke so frequently or so highly of any student as of Mostowski. And it was to that princely court in Warsaw that he sent many students to refine their mathematics and their logic, and maybe their character.

[1] I would have the pleasure of briefing him in Montreal on Paul Cohen's sketch of a proof of this theorem.

[2] Both statements require a gloss. Mostowski was formally Kazimierz Kuratowski's doctoral student, just as Tarski was Stanisław Leśniewski's student on record, not Tadeusz Kotarbiński's. At the respective times, neither Tarski nor Kotarbiński held academic positions qualifying him to direct doctoral work officially. Mostowski is certainly recognized as Tarski's first student (see for example [3, p. 483], [4, p. 866], [2, p. 385]); Tarski's teachers are a more problematic situation (see [2, pp. 39–42]). As for Tarski's last student, Judith Kan Ching Ng was officially Ralph McKenzie's student when she finished her dissertation, but Tarski directed all of Judy's work, and fully approved her results after the date of my thesis, which was indeed the last one he signed (see [4, p. 867], [2, p. 386]). For more on the date of my thesis, see [7].

Mostowski in Montreal

Max and Ana Dickmann and I drove across the US and Canada to Montreal. He was an Argentine logician studying at Berkeley who would later work in Europe, with a brief spell in Chile during the Allende period [1].

At the first meeting of the seminar, the lecturers were introduced, and in addition to Mostowski, his student Janusz Onyszkiewicz[3] (JO) was also introduced. As a member of an immersive Russian dormitory at MIT, I had encountered a number of Soviet visitors, and it seemed natural to engage this Pole, even though I knew no Polish. No fear, he spoke perfect English. I showed Janusz to his dormitory room and helped him settle in. Later I introduced him to Max and Ana; they became fast friends.

At the welcoming reception, I approached Mostowski, who assumed I spoke French but smoothly passed to flawless English when he learned who I was and that francophone I was not. Tarski had alerted him I would be there, of course. I complimented him on the clarity of his French lecture, for I could not follow the Canadians' but his was easy for me to understand. He asked me about my interests. I told him that inasmuch as the decidability of real closed fields ought to have some impact on classical mechanics, I thought that the decidability of p-adic fields might have an impact on modern physics. This was an unsubstantiated view—perhaps insubstantial one—but I had picked up the notion of a connection of the fields[4] from a physics paper and dreamed of an application of decidability. In fact, I had little more to say about it than that, and no one had ever desired more (although at a Tarski party I became uncomfortable when the brilliant Fred Galvin seemed too interested). Mostowski smiled and asked directly, "Which p?" I fell speechless; truly, the possible relevance of this obvious question had not even occurred to me, and I felt inadequate to give any answer. Of course, that was not his purpose, and he easily moved the conversation along to something else. But that is when I lost my glibness on this topic.

To Warsaw

Near the end of the seminar, I learned that the Dickmanns were planning to visit Warsaw, partly for logic, partly for communism, partly because of the friendship they had struck up with JO. I had ignored all three for several weeks, and felt envy at this plan. It drove me to consider going myself, and then to express this to Tarski. He wrote me a stern letter, stating that, despite the scientific and personal advantages of a visit to Poland, a change of scene alone would not solve my perennial procrastination on research—nevertheless, he would aid my travel plans as much as he could.

The wish to spend 1966–67 in Poland was ill-formed, but it did disclose the sort of preparations required. It was my good fortune during that year to join a graduate student exchange program at Stanford, study Polish on a scholarship to the University of Colorado in Boulder, survive the termination of my NSF fellowship, and travel to Warsaw in August for the academic year 1967–68. Tarski had arranged sponsorship by

[3] Janusz completed his doctorate in 1966 at the University of Warsaw (UW). At the time I met him, he was an asystent (assistant professor) and later tutor, adjunkt, lecturer, and docent (associate professor, and more) at UW. In 1991 he was awarded an honorary doctorate by the University of Leeds.

[4] The multiple meanings, maybe even a pun, are intentional.

Mostowski at the University of Warsaw (Universytet Warszawski—UW). As I prepared to go to Poland, Tarski often referred to putting me under Mostowski's mathematical care and more.

And so, by never saying no (because I was never asked) I became a Stanford exchange student in Warsaw, living first in Mokotów and then in Wola across from the main railroad station. It turned out that the Dickmanns also had to delay a year, and now they lived not far from me. Max became disenchanted with Poland and took a research position at Aarhus in the middle of the year. Till then, we saw each other weekly at math venues.

Mostowski[5] chaired the UW Katedra Algebry, or Department of Algebra. His UW office and department were located in the Palace of Culture and Science (Pałac Kultury i Nauki—PKiN), the Stalinist wedding-cake building imposed on Warsaw (Prague hid theirs in a canyon). I shared an office on a higher floor with Diane Resek and George Wilmers. Diane was Henkin's student and the only other Stanford exchange student in logic. Math lectures were at PKiN, Polish lessons at the main UW campus, on Krakowskie Przedmieście. In addition, we were also participants in the seminar Mostowski conducted at the Mathematical Institute of the Polish Academy of Sciences (Instytut Matematyczne Polskiej Akademii Nauk—IMPAN), where his wife served as librarian and gracious hostess, especially for all befuddled foreign graduate students. More to the point, they frequently hosted the luminaries of Eastern European logic there, including A.A. Markov and B.A. Trakhtenbrot.

Detour to Amsterdam

Mostowski and Tarski both attended the Third International Congress for Logic, Methodology, and the Philosophy of Science in 1967 in Amsterdam. At the last minute, so did I, traveling by train with Einar Fredriksson and Yoshindo Suzuki[6], just three weeks after I had arrived in Poland. I dined with Tarski and Mostowski, and Tarski introduced me to Yuri Ershov[7], Henry Hiż, and Zdzisław Pawlak, as well as to a number of his own students I had not yet met, such as C.C. Chang and Solomon Feferman. I also met others interested in the metamathematics of fields, including a student of Y. Bar-Hillel whom I would see again in London at Christmas. On returning to Warsaw, I made my first of many visits to the main police station at Pałac Mostowskich (Palace of the Mostowskis, no relation known to me) for an updated visa. Despite trepidations based on events to unfold below, padded doors, and dominating officials, the bureaucracy always seemed to be on my side here. With no loss of Polish fluency (in fact, it always seemed better after I returned to Poland), I resumed classes and lectures.

[5] In Warsaw, he was frequently called Andrzej Mostowski Starszy (the Elder), for there was another academic by that name. It was a misnomer: although 53, he looked younger and he acted younger.

[6] Einar was the force that drafted me into this trip; he was down to the formal details in completing his doctorate under Mostowski's direction. For accounts of two Warsaw mathematical adventures involving Yoshindo and Emilio del Solar, another Mostowski student, please see [6, p. 229].

[7] Ershov was A.I. Mal'tsev's student and yet another decider of p-adic fields.

Math in Warsaw

In early fall, Mostowski asked me if I would like to talk in the UW logic colloquium. I readily agreed, largely repeating a lecture I gave for Robert Vaught's seminar on set theory at Berkeley the year before. It covered reflection principles, and I showed the improvement I had made on a small point in Vaught's seminal paper. More importantly, I prefaced my talk with a short speech in Polish, which delighted Mostowski and the local audience.

With vague memories and sketchy notes of multiple lectures as evidence, my logical year seemed to flow evenly, punctuated by Christmas in England and Easter in the Soviet Union. A strong Polish logic community was resident or visiting in Warsaw during my tenure. Mostowski's first two students, Helena Rasiowa and Andrzej Grzegorczyk, and Tarski's student Wanda Szmielew (all from 1950) were bold presences. There were waves of younger logicians, especially from Wrocław (and from Prague), that rolled through Warsaw during the year. I met numerous UW and Politechnika grad students and a few undergraduate mathematics students. There was also a contingent of logic participants beyond the ranks of mathematicians. There were two in this group that I saw often. Antoni Moniuszko—a physician, paint manufacturer, and artist—was an amateur enthusiast for logic, especially set theory. He would hire tutors, including Stanisław Mrówka and Wiktor Marek[8]. The second had a professional interest in logic, but styled himself as a computer scientist at a time when that was a barely recognizable term. He was Andrzej (Jacek) Blikle, the eponymous confectionary scion (and now its president), who would later become a professor of computer science at PAN[9].

I had almost no mathematical conversation with JO, but we shared an ardor for caving and climbing. I was able to join the Warsaw Speleoklub on a number of outings and eventually became the first international member (although somewhat off the books). Besides some short technical climbs with JO, he led me halfway down Śnieżna to remove protection from a previous expedition. At the time it was the world's fourth deepest cave, and JO had been a party to the first full descent. Mostowski disapproved of climbing and caving as irresponsible; in that, he contrasted with Tarski who enjoyed others' adventures as much as his own in his beloved Tatra Mountains.

A Tide in the Affairs of Students

On January 30, 1968, I met an American theater student at a party. She wanted to leave right then to attend the last performance of Adam Mickiewicz's play *Dziady* before it

[8]Marek reports: "I did it for a couple of years. The meetings were always exactly the same: we would go through the first chapter of Kuratowski-Mostowski (*Teoria Mnogości—Set Theory*). A domestic would cart in a table with an excellent dinner. My plate would have an envelope underneath (with a banknote). The thing would be repeated in 4 weeks." It is amusing that Mrówka was Kuratowski's student and Marek was Mostowski's. Marek continues: "Sometime in 1970 Moniuszko decided to paint my portrait. I sat for him, and he painted two pieces. He gave me one; it is still in my home."

[9]Marek reports: "He was the first in our seminar who told Mostowski that he was in Computer Science, not Math, but needed logic, and so he attended. There was an entire group of theoreticians of CS who were close to us logicians: Antoni Mazurkiewicz, Jozef Winkowski, Jacek Blikle, and the late Zdzisław Pawlak. They created the CS Institute of the Academy (Instytut Podstaw Informatyki PAN, or IPIPAN), which at first was called the Computational Center of PAN."

was closed by the censors. I was happy to join her in crashing the box office (with their obvious collusion) and seeing this historic anti-Russian/anti-Soviet play. As we left, we noticed a picket line forming outside the theater. They carried a placard: "Żądamy Dalszych Przedstawień" (we demand further performances). I recognized several students who hailed us to join in. Given a background in the Free Speech Movement, I thought that was a good idea, and my companion agreed.

After marching around the theater complex, we trooped along Trębacka to the Mickiewicz monument on Krakowskie Przedmieście, which several students scaled to hang the placard. It was by then obvious that numerous plainclothes men and women were accompanying us. As the students dispersed, we watched a man help a woman climb the monument to remove the sign. We wandered on down the street, eventually turning right, probably on Królewska. Soon after, a police car pulled up and grabbed two students walking in front of us. Then to our surprise it was our turn, despite our sense of insulation. Two cops ushered me into the car, making a happy discovery in my passport; they gloated in English, "You are American—very good!" Perhaps they thought they had captured a foreign ringleader of the protest. My companion managed to evade them, run across the street, and jump into a taxi whose driver miraculously spoke English, for she knew little Polish. I was taken to the police station on Wilcza, where I joined many companions from the picket line. I was the last to be interviewed. It seemed appropriate to feign no Polish, and a friend acted as interpreter. The police asked me why I was at the station. I told them that I was innocently walking down the street when I was swept up in the dragnet. They released me immediately[10].

Mostowski was my host of record with UW and with the police. He was informed of my apparent involvement in the *Dziady* protest by authorities, as was our Polish exchange supervisor and our Stanford student leader. The only thing said was an encouragement to be more circumspect about what I did. That was gentler than I expected. It occurred to me that Mostowski, Tarski, and Stanford would not be pleased if I were expelled. In fact, I became somewhat paranoid for several weeks, but that passed in early March.

On March 7, 1968, I was stopped in front of UW by some art students I knew[11]. They invited me to a photo session for an April Fool's article in *Zwierciadło*, a popular monthly women's magazine. The feature, entitled "Mini-moda Powraca" (the mini style returns), was based on our rolling up our pants-legs and revealing our socks at the Academy of Fine Arts across the street from UW. One of the results appears in the Fefermans' book [2, p. 326]. But that marked the end of normalcy at UW for many months, probably longer.

In the morning of March 8, while I watched fascinated from an upstairs window at the rear of the campus, more than 15 busloads of goons with truncheons and nylon raincoats trooped by on their way to assault students gathered nearer the front gate. This was the beginning of protests, strikes, violence, arrests, provocation, staged politics, and rising antisemitism (typically couched as antizionism) that spread to universities across

[10] As you read this (and I reread it), it may appear that I was set up. I don't think so now, but it would have fit well with my thinking then.

[11] I met a great number of interesting, involved people in Warsaw, Kraków, and Poznań, but typically through cavers, other Stanford students, Americans, and not so much through logicians besides JO. These acquaintances included artists, writers, actors, hippies, doctors, journalists, filmmakers, rock stars, and of course other students. JO introduced me to young logicians from Wrocław, Prague, and elsewhere, often in the apartment he shared with Janusz Rzepecki (Rzepa).

Poland. Although I did not hear of direct repercussions on my Jewish friends and acquaintances, I know one left Warsaw earlier for Paris, and others left during the following months in the exodus that would reduce the Jewish population of Poland to an even smaller fraction[12].

I did not participate actively in the March protests, although my impression was that JO did have a role[13]. He never said anything explicitly, but his ability to read the situation seemed uncanny, from my naive perspective, and proved invariably correct. There were, however, a number of different groups closer to the action that I hung out with; my own paranoia was washed away by their enthusiasm, despair, honesty, and courage.

Reporting Progress to Tarski

In late March, Mostowski invited me to his home in the Sadyba area. After discussing the value of my tenure as an exchange student, he asked me whether I had accomplished any results that he could convey to Alfred. He made it clear that neither of us should report empty handed. I had to admit that despite good ideas, good talks, good discussions, I had no perceivable results. He had known as much. He then suggested that I might like to undertake a project that could be useful to both of us and that Tarski would endorse. He asked, "How would you like to translate some papers in logic into English?" He knew that I was familiar with Russian and now read some Polish. He gave me a choice: translate Łukasiewicz from Polish or Mal'tsev from Russian. The former would be a work of love leading to a monograph published by IMPAN, and the latter would be a "yellow book" published by North-Holland Publishing Co., likely to offer compensation and wider circulation. I immediately chose Mal'tsev, for I had already worked on several papers for Tarski (who could read Russian in any case) and reviewed them for the *Journal of Symbolic Logic*. Mostowski described the proposed book: it would be a collection of Mal'tsev's writing on the metamathematics of algebraic systems.

In May, Mostowski invited me to a luncheon at PKiN with the North-Holland owner-publisher, M. D. Frank, who had traveled from Amsterdam for a book fair. At this meeting, Mr. Frank pressed me for my requirements, and because I was totally inexperienced in such negotiations, I said nothing. By standing mute, I was presented three successively better deals, with Mr. Frank finally stating, "This is my last offer!" I took it. I believe Mostowski was mightily amused and no little relieved that an agreement had been reached. Later he supplied me with a list of papers proposed by Ershov, most of which I accepted. I have always regretted that the Mal'tsev Conditions paper was not among them, and that I decided independently not to include it. By a pleasant turn of fate, Einar Fredriksson joined North-Holland and became the book's editor. Scrutinizing and correcting all the math led me to the topic of my dissertation for Tarski and to several papers, including the one printed here. My gratitude to Andrzej Mostowski for setting me on this path is unbounded.

[12] I saw Ida Kamińska's last performance in Brecht's *Mother Courage* at the Warsaw Jewish State Theater; she soon left the country and never returned to perform for the theater, now named for her.

[13] See [5]. JO's political activity would be more public in years to come, first on my own TV as the press spokesman for Solidarność (Solidarity), and later as Minister of Defense (twice), party chief, member of the Polish and European Parliaments, and most recently as a vice-president of the latter.

After Math

A highlight of late spring was the thesis defense by Mostowski's latest graduating student, Wiktor Marek; it went superbly and ended with a grand party. Soon the academic year and my visa ended. I set out on travels through Europe, arranging to pick up a return Polish visa in Prague for an IMPAN model theory conference at Mostowski's invitation. On August 21, as I hitchhiked to the Austrian-Czech border with a climber from Santa Barbara, we were informed that the border had been closed that morning by the Soviet invasion. We persevered, getting halfway through the buffer zone before Czech guards persuaded us to turn back—they were confused and scared by what they had seen in their own towns that morning. Our border adventure was shown on Austrian national TV that night.

By the time I did make it back to Warsaw, the meeting was over, but Mostowski was glad to see me and hear of my experiences with the situation in Czechoslovakia. Judy Ng had visited Warsaw in the fall while I was there. She had been attracted to Poland, and she had now arrived for 1968-69, also sponsored by Mostowski.

After I left Warsaw in September 1968, I would have no contact with Mostowski till we attended the 1971 Tarski Symposium in Berkeley celebrating Tarski's 70th birthday. At the banquet, teacher and student were both charmed by an ensemble of grad students that I led in singing the Polish birthday song, "Sto Lat" (Live a hundred years!), which I accompanied with tonette (a plastic training recorder). There would be no next meeting: in 1975 Tarski told me that Mostowski had died unexpectedly in Canada, a loss for all of us, but especially for Tarski and for Polish logic.

Urbane and Aristocratic

Andrzej Mostowski first greeted me in Montreal with his characteristic cocked-arm handshake. Whether approaching or approached, whether with man or with woman, he would draw his right arm back, his elbow bent, then at the first sign of movement of the other party's hand, he would immediately extend his own for handshaking, or in the usual case with a woman who would understand the gesture, for kissing her hand. Later, I would witness this act many times, always marveling at how smoothly and warmly he accomplished it. I watched a local Polish restaurateur accost my wife to kiss her hand, thinking that Professor Mostowski would never have given discomfort. This is but one example of how he portrayed his inner manner in his outer. I use "aristocratic" advisedly, with the force of grand and stylish. For "urbane," I choose the meaning of refined, unaffected expression. The tension between these is not lost on me, but it was obliterated in him. He was a light and a delight. As friends of mine sing, "manner is the touchstone of the heart, the soul made visible."[14]

References

[1] John L. Bell, London, 1968-73, in: *Perpetual Motion: My First Thirty Years*, available electronically from: http://publish.uwo.ca/~jbell/Lon.pdf.

[14] "You Can Tell," copyright ©1975 by Consortium of the Arts.

[2] Anita Burdman Feferman and Solomon Feferman, *Alfred Tarski: Life and Logic*, Cambridge Univ. Press, New York, 2004.

[3] Leon Henkin, John Addison, C. C. Chang, William Craig, Dana Scott, and Robert Vaught, eds, *Proceedings of the Tarski Symposium*, Amer. Math. Soc., Providence, Rhode Island, 1974.

[4] Wilfrid Hodges, Alfred Tarski, *J. Symbolic Logic* **51** (1986), 866–888.

[5] http://ludzie.wprost.pl/sylwetka/?O=33113.

[6] Benjamin Wells, Applying, Abstracting, Extending, and Specializing Pseudorecursiveness, *Annals of Pure and Applied Logic* **126** (2004), 225–254.

[7] Benjamin Wells, The Triumph of the One, in: Reza Sarhangi and Carlo Sequin, eds, *Bridges: Mathematical Connections in Art, Music, and Science* (*Proceedings of Bridges 2004*), 2004.

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Remembrances of Professor Andrzej Mostowski

Kazimierz WIŚNIEWSKI
Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80–952 Gdańsk, Poland
kwisniew@math.univ.gda.pl

I shall commence my remembrances at the time when I was beginning studying at the end of the 50s. At once, I was lucky enough to come across two distinguished personalities. The first was Karol Borsuk. He was a wonderful lecturer of multidimensional analytic geometry. Later I listened to his monographic lecture on geometric topology (in Borsuk's sense) and participated in the proceedings of a seminar devoted to this subject. The other personality that fascinated me was Andrzej Mostowski. His beautiful lecture on algebra, modeled on a course book written by himself in collaboration with M. Stark, was entrancing and given in an extremely proper Polish. The first time when I met Professor Mostowski directly was at the exam. The Professor questioned one of my answers. He must have been tired as I was the last person to take the exam. I decided to defend my position and was able to do so with good results. The last question took me by surprise. It was about my participation at the lectures. Indeed, I didn't stand out. This event became the point of origin for closer contacts with the Professor. Back then, he was the head of the Department of Algebra at the Institute of Mathematics of the University of Warsaw and he devoted a great deal of time to instruction and training the University algebra staff, which was almost non-existent in those days. Together with M. Stark, he wrote an entire series of modern (for those times) course books on algebra. He was also leading algebra seminars. The topic of one of those was differential algebra – a very modern topic at the time. I was quite active in the meetings of the seminar. Other participants included A.S. Birula-Białynicki, J. Browkin, and J. Brzeziński. We were presenting the works of Kaplansky, Ritt and others. The subjects of these works were differential modified Galois theory. By the way, Mostowski enjoyed giving lectures on the Galois theory a great deal. It was one of the spheres of the Professor's activity. I found about the other one, probably dearer to his heart, at a course lecture on mathematical logic. One might have expected it to be modeled on the monograph "Mathematical logic" that Professor wrote in the 40ies, but the lecture was very different, led as usual with zest. I was most impressed by the proof of Gödel's completeness theorem presented in a very clear way. What is interesting is the fact that the Author didn't wish for such a beautifully written book to be translated into any other language in spite of many such offers presented to him. He claimed it was outdated. At one of his lectures he confessed that he had a beautiful leather-bound notebook where He would make notes. Before his obligatory departure from Warsaw, after the fall of the Warsaw Uprising, he was forced to choose between a loaf of bread and the notebook. He chose the bread. He never reconstructed his notes after that. Also lost became a large portrait of Giuseppe Peano, which he regretted very much.

While giving lectures, the Professor was also running a seminar, the subject of which was Gentzen's calculus. A great deal of time was devoted at the seminar to consistency proof for Peano arithmetic (with the ω-rule) iterated by induction to the first epsilon number. The seminar work was very intensive. Some of the participants were A. Grzegorczyk, L.W. Szczerba, and A. Salwicki. At the Department of Algebra, at the very beginning almost no one, beside its Boss (i.e. Mostowski), took interest in the foundations of mathematics. With the time, however, there would appear more and more people interested in this area. These were: J. Onyszkiewicz, W. Marek, M. Jaegermann, P. Zbierski, and W. Guzicki.

The Professor made his office available to us. Perhaps he regretted his decision afterwards. That's because the office, after some time, took on a role of a club of all sorts, where we could spend time free of our teaching duties. Ordinarily, it would look like this: One of us (W. Marek) sat on the Boss's desk and dangling his feet smoked an enormous pipe; the others, smoking cigarettes were engaged in lively discussions in small groups. When the Professor entered the office, he would run towards the window, covered in clouds of smoke, to let some air in. Marek introduced the habit of smoking pipes. Onyszkiewicz and the algebraist M. Bryński, joined him. I also was a member of the society. Once the Professor, addressing me, told me that I looked like his uncle Franciszek. I will admit that we used the office from time to time, not in accordance with its purpose, for playing cards. The Boss pretended not to see that. One needs to say that the relations at our Department were family-like. The great kindness of the Boss and friendships between the employees contributed to the atmosphere of pleasurable work. We were also on very friendly terms with the colleagues from the Department of Logic (including C. Rauszer, A. Wasilewska, and K. Dałek). That Department was directed by Prof. H. Rasiowa – a long-time Dean of the Faculty of Mathematics and Mechanics of the University of Warsaw, and a former student of our Professor. Once a week the Boss would take us to the Mathematical Institute of the Polish Academy of Sciences. The distance was not great, but being driven in the Boss's car *warszawa*[1], crammed like sardines, was great fun. With time, the scholarly life moved to the Mathematical Institute of the Academy and there was no need to transport the seminar participants. I think the warszawa did not exist then anymore. The meetings were taking place on Friday afternoons. We were always up to date on the newest results achieved by mathematicians abroad. Mathematicians from other institutions of higher education were often invited to the meetings. Some colleagues from Wrocław visited very often, sometimes every week (L. Pacholski, B. Węglorz, and J. Waszkiewicz). Longer-term visitors from abroad were usually present, too. I recall from the latter group Japanese: H. Sayeki, and Y. Suzuki, the Czechs: M. Benda, and K. Prikry, E. Fredriksson from Sweden, and K. Durr from Germany. The Czech and Slovak scientists (P. Vopenka, P. Hajek, L. Bukovsky and others) having lost their scientific leader L. Rieger (also a longer-term visitor of the Professor) found support in Professor Mostowski. Some of the Boss's long-term visitors learned Polish. H. Sayeki basically perfected it (a non-small feat for a Japanese). When, after many years, I met him in Gdańsk, we could speak Polish fluently. Following the meetings, the Professor would lock himself with one of the seminar participants in his office and lead long con-

[1]Eds: This exquisite automobile was, in fact, made on license from Russian *pobieda*, which, in turn, was a copy of a pre WW II German Opel.

versations. These conversations included words of approval and encouragement, but also friendly yet bitter comments. Many of us owe a great deal to these meetings.

I am not going to try and describe the lectures given at the seminar. Their topics touched on many branches such as theory of models, recursion theory, set theory and its restricted version known as second-order arithmetic. At the end of the first half of the 60s a piece of news from across the ocean reached Warsaw that American logician P.J. Cohen had finally found a solution to a very old problem: the continuum hypothesis. In order to do that, he employed a created a novel technique, called forcing. I do not know how we learned about it in Warsaw; I was serving in the army at the time. When I came back to work, everyone was keen on studying forcing. The first forcing-based result in Poland was obtained by M. Jaegermann. He showed that in the proof of equivalence of two definitions of continuity of functions (Heine and Cauchy continuities) it is necessary to employ the Axiom of Choice. The problem of this equivalence without the Axiom of Choice was posed by W. Sierpiński in 1918. Later no, using forcing, Marek and Onyszkiewicz presented a number of results regarding various independences Eventually, Marek wrote a dissertation on the forcing method and possibilities of transforming of models of set theory with individuals to models of the theory without individuals.

In Warsaw, the axiom of choice was studied from its invention. W. Sierpiński was long-interested in the Axiom of Choice and his interests were passed on to his students and other Warsaw mathematicians. Certainly the studies of the Axiom of Choice were important for the Warsaw Mathematics School and involved such mathematicians as A. Tarski and A. Lindenbaum – the teachers of Professor Mostowski.

It comes as no surprise, then, that Mostowski was interested in the Axiom of Choice. In his investigations, using certain ideas of Fraenkel, he created the Fraenkel-Mostowski method of construction of models for set theory with individuals. The method requires an infinite set A (of individuals), a subgroup G of the symmetric group of A, and an ideal I of subsets of A. A permutation of A extends to an automorphism of the entire class of sets built on the basis of the set of atoms A. A set x is said to be symmetric (with respect to G and I) if for some Z from I, all permutations from G whose fixed points contained Z, preserve x. The hereditarily symmetric sets belong to the desired model. The properties of the permutation group imply truthfulness or falsehood of certain sentences in the model. Professor Mostowski used this method in a number of arguments. An example of this is a paper, which deals with various versions of the axiom of choice for families of finite sets.

To make this part more precise, let us introduce some terminology. By $[n]$, where n is a natural number we mean the following set-theoretical statement: *For each family of n-element sets there exists a choice function*. Next, when we write $[Z]$, for a finite set Z of natural numbers, we mean the conjunction of $[n]$ for n belonging to Z. In his classical work on finite Axioms of Choice Professor Mostowski formulates a certain condition (D), and proves, that it is sufficient for the implication $[Z] \to [n]$ to be a theorem of set theory ZF. The condition (D) is expressed in group-theoretical terms and states: *Every subgroup G of the symmetric group of an n-element set without fixed points contains a subgroup H such that there is a finite number of proper subgroups $K_1, K_2, \ldots, K_r$ of H such that the sum $[H : K_1] + [H : K_2] + \cdots + [H : K_r]$ belongs to the set Z*. The group used in the construction of the appropriate model of set theory with individuals model is the weak countable power of the permutation group G. This power can be viewed as a permutation group of a countable set A (the set of individuals). The role of the ideal I is played by the ideal of finite subsets of A.

I have studied similar subjects for axioms of choice for some families of n-element sets. Namely, let $[n]'$ denotes the sentence *"For every linearly orderable family of n-element sets there exists a choice function"* and let $[n]''$ denote the sentence *"For every countable family of n-element sets there is a choice function"*. I proved the following theorem: *Let G be a finite group, let $Q(G)$ be the additive semi-group of positive integers generated by indices of proper subgroups of the G, and let $R(G)$ be the additive semi-group of positive integers generated by orders of non-trivial elements of the group G. Then there exists a model of set theory with individuals in which sentences $[n]'$ for $n \notin Q(G)$ are true, and sentences $[n]$ for $n \notin R(G)$ and $[n]''$ for $n \in Q(G)$ are false.* The proof of this theorem is a modification of the original Mostowski argument. The group G can be represented as the permutation group of the collection of cosets of its proper subgroups. The following proposition is a corollary of the above theorem. *If for every prime number p there exist infinitely many prime numbers of the form $(p^q - 1)/(p - 1)$, then there exists a model of set theory with individuals in which every sentence $[n]'$ is true and every sentence $[n]$ for $n \geqslant 2$ is false.* (The famous conjecture on the existence of infinitely many Mersenne primes is a particular case of the conjecture stated in the assumption of the proposition.)

Let me also mention a certain Katetov's theorem, which was mistakenly called by A. Mąkowski and myself Abian's theorem. The theorem is stated as follows: *Let f be a function of a set E into itself. Then the function f has no fixed points if and only if E can be divided into three mutually disjoint subsets, which are disjoint with their images with respect to f*. It is of independent interest that the above theorem is equivalent to the following version of the axiom of choice: *every family of finite sets possesses a choice function.*

Professor Mostowski was well known for his proficiency in a number of languages. It is said that after one of his lectures at the University of Montreal he was bombarded with questions in various languages, which he easily kept answering. Sayeki's[2] question in Polish caused consternation, but of course Sayeki got his answer as well. Afterwards, Mostowski received congratulations for his proficiency in Japanese. I have always admired Professor Mostowski's lectures, their content's depth and the clarity of presentation. Particularly memorable were moments when he was speaking English and writing in French or German. It should be noted that he always wrote beautifully on the blackboard, economically using all its space.

Professor Mostowski enjoyed great respect among his students and was for them a real authority (both scientific and moral). I am going to describe an event that I accidentally witnessed. It was happening on the 9th floor of the Palace of Culture and Science in Warsaw, where the Institute of Mathematics was then located at the time. A couple of students were chasing one another around a pillar next to the lifts. Out of the blue one of them said, "Stary Most (Old Mostowski) said..."[3] The lift's door opened, the Professor emerged and asked, "What did Stary Most say?" He didn't receive an answer from the embarrassed students. There is an additional need for an explanation here. There were two Mostowskis at the Institute back then, the younger one was called AW (Andrzej Włodzimierz).

[2]Eds: Professor H. Sayeki of the Department of Mathematics of the Montreal University is Japanese.
[3]Eds: A play of words, "Most" means "bridge" in Polish.

Later, when many, many years had passed, at a meeting of the alumni of my class we were talking a lot about our, sadly, late professors. The tales most often included the name Mostowski. He had the deserved gratitude of my friends, even though for a number of them their direct contacts with him had been restricted to their first year there (when Professor Mostowski taught them Algebra.) One of my fellow ex-students (A.J. Blikle) had acquired, from a janitor who had been tidying the Professor's office after his passing away, the notebook with the Professor's notes. There were some remarks on our presentations, regarding both their content and the methods of presentation. One of those criticized me heavily. The Professor wrote, *He keeps drawing some weird stuff. I can't understand a thing.*

I left Warsaw and thus Professor Mostowski group in 1970. There were a couple of reasons. The most crucial one was my daughter's illness (which required for her to change climate) and service in the army, which over the course of two years took me two days of every week. I went to Gdańsk. I commenced work at the newly opened Gdańsk University. Originally a school with a mission restricted to teaching high-school teachers, it was transformed into a university. A great deal of organizational work and especially teaching was required. I served as a director of the Institute of Mathematics of the Gdańsk University at the worst possible moment – during the Martial Law 1981–1983. It was not a good time for an academy. Aside from these duties, I was trying to instill into my students an interest in the foundation of mathematics. Of my many students in Gdańsk the strongest one was Jan Tryba (unfortunately died prematurely), who received his doctoral degree from W. Guzicki.

This page intentionally left blank

Name Index

This page intentionally left blank

Andrzej Mostowski and Foundational Studies
A. Ehrenfeucht, V.W. Marek and M. Srebrny (Eds.)
IOS Press, 2008

Author Index

This page intentionally left blank