NORMED ALGEBRAS

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TRANSLATED FROM THE SECOND RUSSIAN EDITION

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DEDICATION

THIS TRANSLATION IS DEDICATED

TO

PROF. DR. MARK ARONOVICH NAIMARK

L.F.B.

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FOREWORD TO THE THIRD AMERICAN EDITION

This third American edition is close to the second Soviet edition which appeared in November 1968.

In comparison with the second (= first revised) American edition, most of the chapters (especially Chapters III and VIII) have been significantly revised and extended by the discussion of new results; the bibliography has also been significantly extended. In the third American edition the term *ring* has been replaced by the now universally used term *algebra*; this accounts for the corresponding change in the title of the book.

In the technical part of the preparation of the third American edition, I was helped by my wife, L. P. Naimark; I use the present occasion to express my profound gratitude.

Inasmuch as the third American edition is coming off the press several years after the second Soviet edition, there have been included in the galley proofs the statement of several of the most essential theoretical results that have been obtained recently, and the bibliography has consequently been extended by several papers; I wish to express my profound gratitude to Dr. A. I. Shtern for the great help rendered in connection with the preparation of this work and in the reading of galley proofs. I also thank Professor L. F. Boron for translating the revised and supplementary material and for editing this American edition.

Moscow, USSR M. A. NAIMARK

March 1972

FOREWORD TO THE SECOND (= FIRST REVISED) AMERICAN EDITION

In this edition we have corrected all misprints and inaccuracies, we have extended the literature list and we have made a number of further references concerning the most essential recent results in the theory of normed rings. Moreover, the last chapter, i.e. Chapter VIII, has been rewritten.

Since it turned out that the theory of Tomita is valid only under the additional assumptions of separability type, the author considered it expedient to give a discussion in Chapter VIII which is closer to the initial simpler theory of von Neumann for the separable case; in this discussion, the articles NAIMARK [3] and NAIMARK and FOMIN [1] were used.

For the convenience of the reader, Appendices II-IV were added to the book – these are necessary for understanding the last chapter.

The author considers it his pleasant duty to express his gratitude to Prof. Leo F. Boron who undertook the task of preparing the revised American edition of this book and to the publisher NOORDHOFF who made possible the appearance of the second edition and enabled the author to introduce the above-mentioned modifications and additions.

Moscow M. A. NAIMARK

August 1963

FOREWORD TO THE SECOND SOVIET EDITION

In this second edition the initial text has been worked over again and improved, certain portions have been completely rewritten; in particular, Chapter VIII has been rewritten in a more accessible form. The changes and extensions made by the author in the Japanese, German, first and second (= first revised) American, and also in the Romanian (lithographed) editions, were hereby taken into account.

Appendices II and III, which are necessary for understanding Chapter VIII, have been included for the convenience of the reader.

The book discusses many new theoretical results which have been developing intensively during the decade after the publication of the first edition. Of course, limitations on the volume of the book obliged the author to make a tough selection and in many cases to limit himself to simply a formulation of the new results or to pointing out the literature. The author was also compelled to make a choice of the exceptionally extensive collection of new works in extending the literature list. Monographs and survey articles on special topics of the theory which have been published during the past decade have been included in this list and in the literature pointed out in the individual chapters.

The author is very much indebted to all his colleagues who pointed out misprints and errors contained in the first edition.

D. A. Raikov and M. G. Sonis carefully edited the manuscript and enabled me by their remarks to improve individual parts of the book; furthermore, M. G. Sonis wrote subsections 3, 5 of §9, subsection 7 of §11, IX in subsection 2, §18, corollaries 1 – 4 to I, subsection 3 – §23 and IV, V, subsection 3, §23.

The author considers it to be his pleasant duty to express his profound gratitude to D. A. Raikov and M. G. Sonis. The author is grateful to A. Z. Rybkin for his careful consideration of the manuscript and expresses his indebtedness to D. P. Zhelobenko for his aid in reading the galley proofs.

Moscow, M. A. NAIMARK

February 1967

FROM THE FOREWORD TO THE FIRST SOVIET EDITION

The theory of normed algebras, although of recent origin, has developed into an extensive branch of functional analysis; it now has numerous applications in various other branches of mathematics.

The first cycle of papers devoted to concrete normed algebras, i.e. algebras of bounded linear operators in Hilbert space, was begun in 1930 by VON NEUMANN [1] and then continued in the works of MURAY and VON NEUMANN [1]. The advantage of considering algebras of operators was already apparent in these works. However, it was the abstract point of view which turned out to be most fruitful; from this viewpoint, the nature of the elements of the algebra does not play any role, so that a normed algebra is simply an arbitrary set of elements which forms an algebra in the algebraic sense and which is, furthermore, provided with a norm which satisfies simple requirements.

This point of view was systematically developed by GELFAND [1 – 7] in his theory of commutative normed algebras. The discovery by Gelfand of the role of maximal ideals, the construction of a compact space of maximal ideals, and the representation of the elements of a semisimple algebra in the form of an algebra of continuous functions on this space were of decisive significance in this connection. Even the first applications showed the power of the theory of normed algebras. Thus, with the aid of normed algebras, a simple proof of the Wiener theorem on trigonometric series was unexpectedly obtained (see WIENER [1]), simple proofs and generalizations of many theorems of Tauberian type were also obtained, and so on.

An essential role in the development of these applications was played by a large cycle of works by SHILOV [1] which are devoted to the investigation of various classes of commutative normed algebras and the structure of ideals in them.

The application of the theory of commutative normed algebras to the theory of locally compact commutative groups which led to the construction of a harmonic analysis on such groups (by GELFAND and M. KREIN [1], M. KREIN [6], and RAIKOV [2 – 5]) and, in particular, to a simple analytic proof of the Pontryagin duality theorem by RAIKOV [4] is especially important.

Another important class of algebras which are no longer commutative, namely algebras with involution (see §10), was considered in the work by GELFAND and NAIMARK [1]. In this work, it was shown that under certain natural conditions, every such algebra can be isomorphically mapped into an algebra of bounded linear operators in Hilbert space such that the involution operation goes over into the operation $A \to A^*$ (where A^* is the adjoint operator of A), and the norm goes over into the operator norm.

Here, an important role was played by the concept of a positive functional, i.e. a linear functional f in the algebra satisfying the condition $f(x^* x) \geqq 0$. The methods worked out in this paper, in particular the concept of a positive functional, were used later in the works of Gelfand and in numerous works of other authors in the investigation of algebras with involution and in the construction of the theory of representations of such algebras; for the particular case of group algebras, these methods were used in the investigation of unitary representations of topological groups.

A second construction of the representations of locally compact groups with the aid of positive definite functions was first given by GELFAND and RAIKOV [2]; in particular, they proved the completeness of the system of all irreducible unitary representations of a locally compact group.

Later, these results of Gelfand and Raikov were repeated in part independently and then developed in the works of GODEMENT [3].

Despite the presence of a large number of results, the theory of normed algebras, especially of non-commutative algebras, cannot be considered complete and many interesting problems in this theory remain open at the present time.

Of special interest is the further development of the theory of characters and harmonic analysis on locally compact groups, constructed in the works of GELFAND and NAIMARK [1 – 8] for the complex classical groups and then carried over, in numerous works of several authors, to other classes of locally compact groups. Furthermore, there remain unsolved a number of problems connected with the decomposition of a given representation of a group or algebra into irreducible representations.

Despite the importance of the theory of normed algebras in many applications and the large number of results which have already been obtained in this theory, there are, at the present time, very few books devoted to this theory. Thus, in the foreign literature there is the book by LOOMIS [1] in which, however, primary attention is given to the theory of commutative and Hilbert algebras and its application to harmonic analysis on locally compact commutative groups and on compact non-commutative groups. Further, some problems in the theory of normed algebras are discussed in the book *Functional Analysis and Semigroups* by Hille.

In the present book, the theory of normed algebras and also of some (commutative as well as non-commutative) topological algebras and its various applications, mainly to the theory of representations of locally compact groups, are discussed. For the convenience of the reader, the first chapter contains the information from functional analysis which will be needed in the remainder of the book.

The author expresses profound gratitude to D. A. RAIKOV who read the book in manuscript form and made a number of valuable observations. The author also expresses his sincere thanks to I. M. GELFAND and G. E. SHILOV for valuable advice.

Moscow M. A. NAIMARK

August 1955