Logic for Computer Science

Summer semester
2019-2020

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ORGANIZATION

- WEBSITE of the course: ldem.github.io/logic_course (or google my name)

- This year we have a problem book!

- Please ask questions.

HISTORICAL CONTEXT – ARISTOTLE'S SYLLOGISM

OK

P1 [No homework is fun

P2 [Some reading is homework

C [Some reading is not fun

HISTORICAL CONTEXT – ARISTOTLE'S SYLLOGISM

Some mammals are whales
Some whales are big

Some mammals are big

NO

~~valid~~?

All unicorns are mammals
All mammals are animals

Some unicorns are animals

YES

non-empty models

HISTORICAL CONTEXT

- Leibniz (1646-1716): "quando orientur controversiae sufficiet sedere ad abacos et sibi mutuo dicere: CALCULEMUS"

- Bernard Bolzano (1781-1848), precise handling of quantifiers: continuity of $f : \mathbb{R} \to \mathbb{R}$

$$\forall x . \forall \varepsilon > 0 . \exists \delta > 0 . \forall y . |y - x| \leq \delta \to |f(y) - f(x)| \leq \varepsilon$$

- Augustus de Morgan (1806-1871), eponymous laws:

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \quad \text{and} \quad \overline{A \cap B} = \bar{A} \cup \bar{B}$$

HISTORICAL CONTEXT

– "Mathematical analysis of logic", 1847

- George Boole (1815 – 1864). Birth of MATHEMATICAL LOGIC
  - Mathematical (algebraic) approach to logic.
    → logic became a discipline that could be studied mathematically.
  - Analysis of the syllogism with Boole's algebra

$(\mathbb{B}, +, \cdot, 0, 1)$, $\mathbb{B} = \{0, 1\}$ $(1 \underset{XOR}{+} 1 = 0)$

Every p is q : $pq = p$

Every p is not q : $p(1+q) = p$

Equational reasoning!

- Beyond 3 predicates : $\dfrac{pq = p, \quad qr = q, \quad rs = r}{ps = p}$

HISTORICAL CONTEXT

- Charles S. Pierce (1839-1914): Theory of relations, CNF, DNF, quantifiers Π, Σ – "Algebra of Logic" 1885 decision procedure for propositional logic based on truth tables.

- Gottlob Frege (1848-1925), worked independently. "Begriffsschrift" (= concept script, 1879): relations & quantifiers

- Giuseppe Peano (1858-1932). Symbol for the existential quantifier $\exists$.

- Edward V. Huntington (1874-1952): axiomatisation and naming of "Boolean algebra", albeit with "+" = INCLUSIVE OR.

HISTORICAL CONTEXT

- Bertrand Russell (1872 - 1970): His eponymous paradox (1901) destroyed Frege's formalisation of mathematics:

$$A = \{ B \mid B \notin B \} . \qquad A \in A ?$$

  Fixup: Theory of types.

- Leopold Löwenheim (1878 - 1957). Started model theory

- David Hilbert (1862 - 1943).
  - Introduction of $\forall$ for the universal quantifier.
  - Birth of modern mathematical logic.
  - Logicist program: complete & decidable axiomatisation of mathematics,
    $\longrightarrow$ ENTSCHEIDUNGSPROBLEM (decision problem, 1928)

HISTORICAL CONTEXT

- Alonzo Church (1903-1995). Undecidability of the validity problem in first-order logic, undecidability of first-order arithmetic (1936)
- Alan Turing (1912 - 1954).
  "On computable numbers, with application to the Entscheidungsproblem" (1936). Also
  - MSc: Rediscovery of the central limit theorem (1935)
  - PhD on ordinal logics (1938)
    - invents oracles
- Kurt Gödel (1906-1978).
  - Completeness theorem for Hilbert's proof system (1929)
  - Incompleteness theorems for arithmetic (1931)
  - General recursive functions (1933).
    - American Constit.
    - Time travel
    - Starving

LOGIC in COMPUTER SCIENCE

- Circuits: Claude Shannon (1916-2001) 1938 (Master thesis)
- Intuitionistic logic: The law of excluded middle "$p \vee \neg p$" is not valid (Brouwer, Heyting, Kolmogorov, ...)
- Complexity theory: Cook's theorem SAT is NP-complete.
- Descriptive complexity:
    - Fagin's theorem: NP = $\exists$SO (1974) (Birth of finite model theory)
    - Stockmeyer's theorem: PH = SO (1977)

LOGIC in COMPUTER SCIENCE

- Database theory:

  Codd's theorem FO = Relational Algebra (1970).

- Type theory: Curry - Howard Correspondence:

  Types $\longleftrightarrow$ Propositions
  Programs $\longleftrightarrow$ Proofs
  Execution $\longleftrightarrow$ Proof simplification (cut elimination)

- Artificial intelligence (Knowledge representation)
  - Modal logics, description logics, epistemic logics, ...

LOGIC in COMPUTER SCIENCE

- Temporal logics
  - LTL, CTL, Hennessy-Milner, μ-calculus, ...
  - verification of hardware & software (model checking)
- Logics of programs
  - Hoare logic (program correctness), PDL, ...
- Programming languages

  Simple types, polymorphic types, dependent types, recursive types, intersection types, ...

PROPOSITIONAL LOGIC - SYNTAX

$$\varphi, \psi ::= \bot \mid \top \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg \varphi \mid \varphi \to \psi \mid \varphi \leftrightarrow \psi$$

PROPOSITIONAL LOGIC – SEMANTICS

Truth Tables

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
1	0	0	1
0	1	0	1
1	1	1	1

Propositional variables $Z = \{p, q, \dots\}$

Truth valuation $\rho : Z \to \{0, 1\}$

Extended to all formulas

$$[\![q]\!]_\rho = \rho(q) \ , \ [\![\varphi \wedge \psi]\!]_\rho = F_\wedge([\![\varphi]\!]_\rho, [\![\psi]\!]_\rho) \ , \ \dots$$

PROPOSITIONAL LOGIC – TAUTOLOGIES

φ is a tautology if $\forall \rho \cdot [\![\varphi]\!]_\rho = 1$

satisfiable $\exists \rho \cdot [\![\varphi]\!]_\rho = 1$

SATISFIABILITY PROBLEM (SAT)

Input: φ

Output: is φ satisfiable? NP-complete

TAUTOLOGY PROBLEM (coNP-complete)

PROPOSITIONAL LOGIC – ALTERNATIVE SEMANTICS

- Three-valued logics $\{0, 1/2, 1\}$
  - Heyting-Kleene-Łukasiewicz $\quad 1/2 =$ "unknown" $\approx$ SQL NULL semantics

p	$\neg p$
0	1
1/2	1/2
1	0

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

  - Bochvar: 1/2 propagates and means "fault".
  - Sobociński: 1/2 propagates only if all inputs are 1/2
- Probabilistic logics $\left([0,1], \overset{\max}{\oplus}, \overset{\cdot}{\cdot}\right)$