Logic for Computer Science

Summer semester
2019 - 2020

LECTURE 9 : Preservation of properties & Ehrenfeucht-Fraïssé games

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SUMMARY

- Relational homomorphisms & preservation of properties.
- Isomorphisms and partial isomorphisms.
- Quantifier rank.
- Elementary equivalence and k-elementary equivalence.
- Ehrenfeucht-Fraïssé games
- Applications to non-definability & non-axiomatisability

RELATIONAL HOMOMORPHISMS

Let $\Sigma = \{R_1 : k_1, \dots, R_m : k_m\}$ be a relational signature (no functions).
Consider two structures $\mathbb{A} = (A, R_1^{\mathbb{A}}, \dots)$, $\mathbb{B} = (B, R_1^{\mathbb{B}}, \dots)$

A RELATIONAL HOMOMORPHISM is a relation $H \subseteq A \times B$
s.t. for every $R_j \in \Sigma$, $(a_1, b_1) \in H, \dots, (a_{k_j}, b_{k_j}) \in H$ implies

$$\text{If } (a_1, \dots, a_{k_j}) \in R_j^{\mathbb{A}} \text{ then } (b_1, \dots, b_{k_j}) \in R_j^{\mathbb{B}}$$

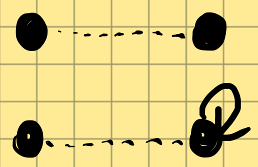
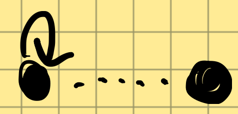
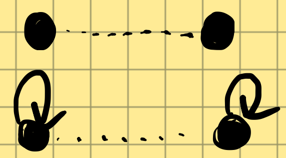
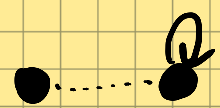
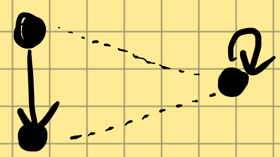
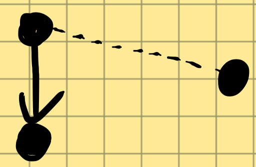
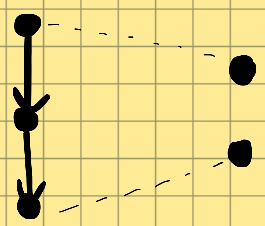
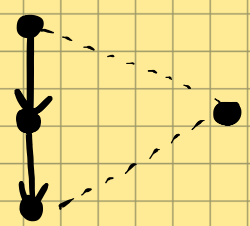
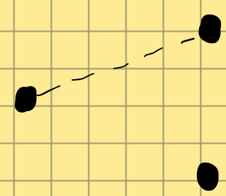
FAITHFUL: $(a_1, \dots, a_{k_j}) \in R_j^{\mathbb{A}}$ iff $(b_1, \dots, b_{k_j}) \in R_j^{\mathbb{B}}$

Special cases of faithful rel. hom.:

- homomorphism (functional)
- isomorphism (bijection)
- endomorphism ($\mathbb{A} = \mathbb{B}$)
- automorphism (endo + iso)

EXAMPLE : GRAPHS

$\Sigma = \{E : 2\}$

	YES	NO
rel. hom.		
faithful		
total		
injective		
surjective		

PRESERVATION of PROPERTIES

A relational homomorphism $H \subseteq A \times B$ preserves $\varphi(x_1, \dots, x_n)$ if $(a_1, b_1) \in H, \dots, (a_n, b_n) \in H$ implies:

$$\text{if } \mathbb{A}, x_1 : a_1, \dots, x_n : a_n \models \varphi \text{ then } \mathbb{B}, x_1 : b_1, \dots, x_n : b_n \models \varphi$$

no negation

THEOREM: A rel. hom. H preserves positive quantifier-free formulas.

PROOF: By structural induction.

Base case: $\varphi \equiv R(x_1, \dots, x_n)$ is preserved by def. of H.

Inductive step:

1) $\varphi \equiv \varphi_1 \wedge \varphi_2$. Assume $\mathbb{A}, \bar{x} : \bar{a} \models \varphi$. Then $\mathbb{A}, \bar{x} : \bar{a} \models \varphi_1$ and $\mathbb{A}, \bar{x} : \bar{a} \models \varphi_2$. By ind. assumption (x 2), $\mathbb{B}, \bar{x} : \bar{b} \models \varphi_1$ and $\mathbb{B}, \bar{x} : \bar{b} \models \varphi_2$, and thus $\mathbb{B}, \bar{x} : \bar{b} \models \varphi$.

2) $\varphi \equiv \varphi_1 \vee \varphi_2$ is analogous.

FURTHER PRESERVATION of PROPERTIES

How to preserve negation $\neg\varphi$? Faithful rel-hom.!

PROOF: Assume NNF. New base case $\varphi \equiv \neg R(x_1, \ldots, x_m)$.

If $\mathbb{A}, \bar{x}:\bar{a} \models \neg R(\bar{x})$, then by faithfulness $\mathbb{B}, \bar{x}:\bar{b} \models \neg R(\bar{x})$.

How to preserve existential properties $\exists x \cdot \varphi$? Total rel. hom.!

PROOF: New ind. case $\varphi \equiv \exists x \cdot \psi$: If $\mathbb{A}, \bar{x}:\bar{a} \models \exists x \cdot \psi$ then there is $a \in A$ s.t. $\mathbb{A}, \bar{x}:\bar{a}, x:a \models \psi$. By totality there is $b \in B$ s.t. $(a,b) \in H$.

By ind. hyp., $\mathbb{B}, \bar{x}:\bar{b}, x:b \models \psi$ and thus $\mathbb{B}, \bar{x}:\bar{b} \models \exists x \cdot \psi$.

How to preserve universal properties $\forall x \cdot \varphi$? Surjective rel. hom!

PROOF: New ind. case $\varphi \equiv \forall x \cdot \psi$: Assume $\mathbb{A}, \bar{x}:\bar{a} \models \forall x \cdot \psi$.

In order to show $\mathbb{B}, \bar{x}:\bar{b} \models \forall x \cdot \psi$, let $b \in B$ arbitrary.

By surjectivity, there is $a \in A$ s.t. $(a,b) \in H$. By def. we have $\mathbb{A}, \bar{x}:\bar{a}, x:a \models \psi$ and by ind. hyp. $\mathbb{B}, \bar{x}:\bar{b}, x:b \models \psi$ as required.

FURTHER PRESERVATION of PROPERTIES

How to preserve equality $\varphi \equiv x_1 = x_2$? Injective rel. hom!

PROOF: New base case $\varphi \equiv x_1 = x_2$: Assume $(a_1, b_1), (a_2, b_2) \in H$ and $\mathbb{A}, x_1 : a_1, x_2 : a_2 \models x_1 = x_2$. By def. $a_1 = a_2$.

By injectivity, $b_1 = b_2$ and thus $\mathbb{B}, x_1 : b_1, x_2 : b_2 \models x_1 = x_2$.

SUMMARY of PRESERVATION of PROPERTIES

$R \subseteq A \times B$

		preserves	
Relational homomorphism	$\longleftrightarrow$	positive quantifier-free	
Faithful	$\longleftrightarrow$	negation	$\neg\varphi$
Total	$\longleftrightarrow$	existential	$\exists x . \varphi$
Surjective	$\longleftrightarrow$	universal	$\forall x . \varphi$
Injective	$\longleftrightarrow$	equality	$x_1 = x_2$

To preserve all formulas of first-order logic we need

rel. hom. faithful + Total + surjective + injective = ?

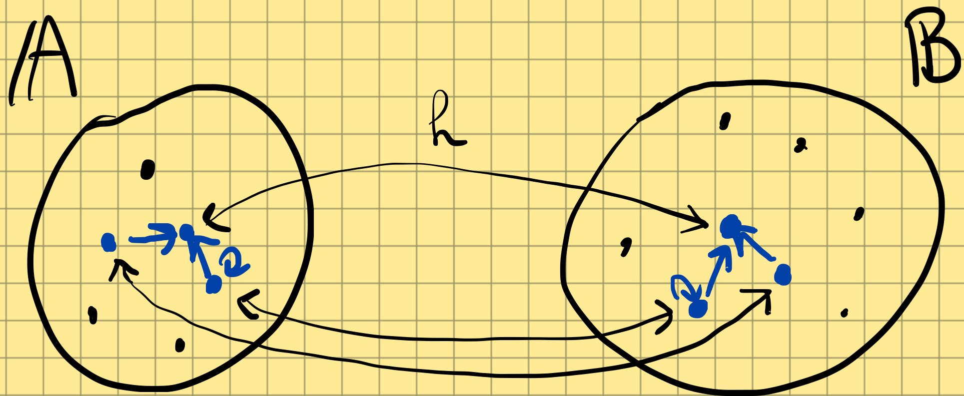
ISOMORPHISM!

ISOMORPHISMS (rel. hom. faithful + total + inj. + surj.)

An *isomorphism* between $\mathbb{A}$ and $\mathbb{B}$ is a bijection $h : A \to B$ s.t. $(a_1, \ldots, a_{k_j}) \in R_j^{\mathbb{A}}$ iff $(h(a_1), \ldots, h(a_{k_j})) \in R_j^{\mathbb{B}}$ for all $R_j \in \Sigma$ and $a_1, \ldots, a_{k_j} \in A$.

$\mathbb{A} \cong_h \mathbb{B}$ if $h : A \to B$ is an isomorphism between $\mathbb{A}$ and $\mathbb{B}$.

A *partial isomorphism* is a partial bijection h from A to B.



MOTIVATION: HOW TO SHOW NON-DEFINABILITY? by a simple formula

and NON-AXIOMATISABILITY over FINITE STRUCTURES?

Problem 1: compactness fails for finite structures
($\Gamma = \{\varphi_{\geq 1}, \varphi_{\geq 2}, \dots\}$ has no finite model while
every finite $\Delta \subseteq_{fin} \Gamma$ has a finite model.)

Problem 2: Skolem-Löwenheim is applicable only to
infinite structures by assumption.

Problem 3: Every class of finite structures is axiomatisable (!)
$\rightarrow$ P2.7.2.

by a simple formula

But not necessarily definable. How to prove it?

SOLUTION: EHRENFEUCHT-FRAÏSSÉ GAMES.

ELEMENTARY EQUIVALENCE

Isomorphism $\mathbb{A} \cong \mathbb{B}$ is the strongest form of equivalence.

A weaker notion is elementary equivalence:

$$\mathbb{A} \equiv \mathbb{B} \quad \text{iff} \quad \forall \varphi \in \mathrm{Th}(\Sigma).\ \mathbb{A} \models \varphi \text{ iff } \mathbb{B} \models \varphi$$

Signature of $\mathbb{A}, \mathbb{B}$

(We have just seen that $\mathbb{A} \cong \mathbb{B}$ implies $\mathbb{A} \equiv \mathbb{B}$.)

strictly weaker!

Example: $(\mathbb{Q}, \leq) \equiv (\mathbb{R}, \leq)$ but $(\mathbb{Q}, \leq) \not\cong (\mathbb{R}, \leq)$

(to be shown later) (why?)

K-ELEMENTARY EQUIVALENCE

Quantifier rank $qr(\varphi)$: max nesting of quantifiers

$qr(R(\overline{x})) = 0$

Example :

$qr(\exists x \exists y . R) = 2$

$qr(\neg\varphi) = qr(\varphi)$

$qr(\varphi \wedge \psi) = qr(\varphi \vee \psi) = \max(qr(\varphi), qr(\psi))$

$qr(\exists x . \varphi) = qr(\forall x . \varphi) = 1 + qr(\varphi)$

$\mathbb{A} \equiv_k \mathbb{B}$ iff $\forall \varphi$ of $qr(\varphi) \leq k$, $\mathbb{A} \models \varphi$ iff $\mathbb{B} \models \varphi$.

k-elementary equivalence

Elementary equivalence $\equiv$ corresponds to $\equiv_\infty$.

EHRENFEUCHT-FRAÏSSÉ GAMES - EXAMPLE

How to show $(\mathbb{R}, \leq) \equiv (\mathbb{Q}, \leq)$?

elementary equivalence

Consider the following turn-based EF-game $\mathcal{G}_\omega((\mathbb{R},\leq),(\mathbb{Q},\leq))$:

$(\bar{a}, \bar{b}) \in \mathbb{R}^n \times \mathbb{Q}^n$ position at round n

Spoiler selects a structure and an element therein

$(\bar{a}a, \bar{b}) \qquad (\bar{a}, \bar{b}b)$

Duplicator selects an element from the other structure

$(\bar{a}a, \bar{b}b) \qquad (\bar{a}a, \bar{b}b)$ position at round $n+1$

Duplicator **survives** round n if $\begin{cases} a_1 \mapsto b_1 \\ \vdots \\ a_n \mapsto b_n \end{cases}$ is a partial isomorphism

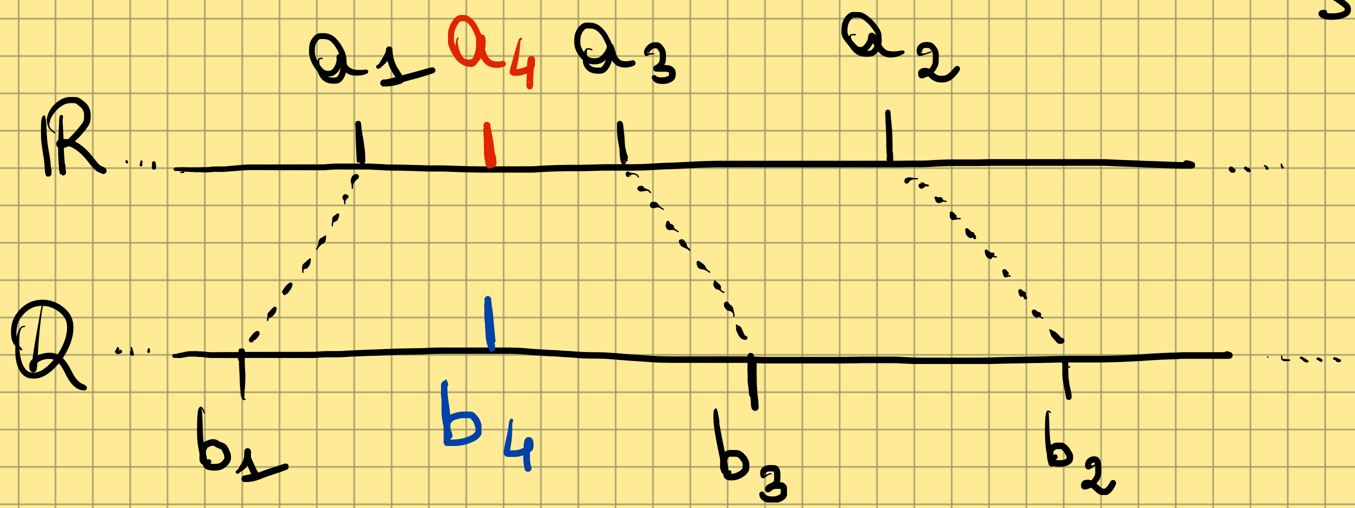
EF-theorem: $(\mathbb{R}, \leq) \equiv (\mathbb{Q}, \leq)$ iff $\forall n$ **Duplicator** survives round n

EHRENFEUCHT-FRAÏSSÉ GAMES - EXAMPLE

How to win $\mathcal{G}_\omega((\mathbb{R},\leq),(\mathbb{Q},\leq))$?

Duplicator maintains the following invariant:

In position $(a_1 \cdots a_n, b_1 \cdots b_n)$: $\underbrace{\forall i,j \cdot a_i \leq a_j \text{ iff } b_i \leq b_j}_{\text{same ORDER TYPE}}$



b_4 exists by density

EHRENFEUCHT-FRAÏSSÉ GAMES

$$
\begin{tikzcd}
 & (\bar{a},\bar{b})\in A^n\times B^n \arrow[dl]\arrow[dr] & \\
a\in A \arrow[d] & & b\in B \arrow[d] \\
b\in B \arrow[dr] & & a\in A \arrow[dl] \\
 & (\bar{a}a,\bar{b}b)\in A^{m+1}\times B^{m+1} &
\end{tikzcd}
$$

Spoiler chooses an element from either A or B

Duplicator chooses an element from the other structure

Duplicator survives the round if $\begin{cases} a_1 \mapsto b_1 \\ \vdots \\ a_m \mapsto b_m \\ a \mapsto b \end{cases}$ is a partial isomorphism.

Duplicator wins: $\mathcal{G}_m(A,\bar{a};B,\bar{b})$ if she survives m rounds.

$\mathcal{G}_\omega(A,\bar{a};B,\bar{b})$ if $\forall m<\omega$ she survives m rounds.

$\mathcal{G}_{\omega+1}(A,\bar{a};B,\bar{b})$ if she survives ω rounds.

EHRENFEUCHT - FRAÏSSÉ THEOREM

Duplicator wins $\mathcal{G}_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$ iff For every $\varphi(x_1, \dots, x_m)$ of $qr(\varphi) \leq m$

$\mathbb{A}, \bar{x}:\bar{a} \models \varphi$ iff $\mathbb{B}, \bar{x}:\bar{b} \models \varphi$

this is the direction we will mostly use

PROOF ($\Rightarrow$) By induction on m. Assume w.l.o.g. φ in PNF.

Base case $m=0$: φ is a boolean combination of atomic formulas and $h(a_i)=b_i$ is a partial isomorphism (faithful + injective + h^{-1} injective)

$\Rightarrow h$ preserves all quantifier-free formulas over $\{x_1, \dots, x_m\}$

Inductive step $m>0$: $\varphi(\bar{x}) \equiv \exists x_{m+1}. \psi(\bar{x}, x_{m+1})$ with $qr(\psi) \leq m-1$.

$\mathbb{A}, \bar{x}:\bar{a} \models \varphi \Rightarrow$ there is $a \in A : \mathbb{A}, \bar{x}:\bar{a}, x_{m+1}:a \models \psi$. Let $b \in B$ be D's reply.

D wins $\mathcal{G}_{m-1}(\mathbb{A}, \bar{a}a; \mathbb{B}, \bar{b}b)$. By ind. hyp., $\mathbb{B}, \bar{x}:\bar{b}, x_{m+1}:b \models \psi$

$\Rightarrow \mathbb{B}, \bar{x}:\bar{b} \models \varphi$. The converse direction and $\varphi \equiv \forall x_{m+1}. \psi$ are similar.

EHRENFEUCHT-FRAÏSSÉ THEOREM

Duplicator wins $\mathcal{G}_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$ iff For every $\varphi(x_1, \dots, x_m)$ of $qr(\varphi) \leq m$
$$\mathbb{A}, \bar{x}:\bar{a} \models \varphi \text{ iff } \mathbb{B}, \bar{x}:\bar{b} \models \varphi$$

PROOF ($\Leftarrow$) omitted (via logical types).

- Can be used to construct distinguishing formulas when Spoiler wins (contrapositive direction); cf. P2.12.10.

ANOTHER EXAMPLE

L_n : linear order with n elements
$1 < 2 < \cdots < n$

$L_{2^m} \equiv_m L_{2^m+1}$: cannot distinguish with φ of $qr(\varphi) \leq m$.

By EF it suffices that Duplicator survives $\mathcal{G}_m(L_{2^m}, L_{2^m+1})$ for m rounds.

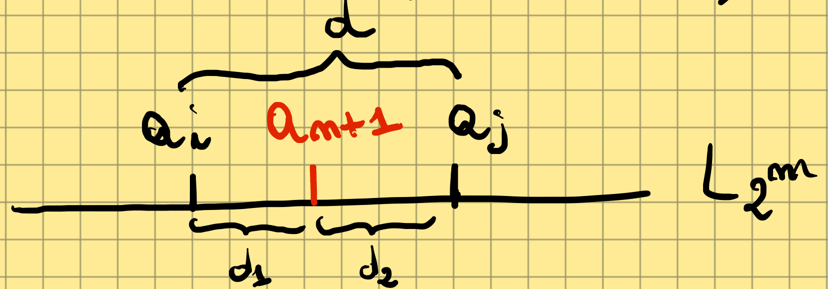
Threshold equivalence : $d_1 \approx_h d_2$ iff either $d_1 = d_2$ or $\underbrace{d_1, d_2 \geq h}_{\text{both large}}$.

Duplicator maintains the invariant : $(a_1, \dots, a_n ; b_1, \dots, b_n)$

for all $i,j \in \{1, \dots, n\}$: 1) $a_i \leq a_j$ iff $b_i \leq b_j$ (order preservation),

2) $d(a_i, a_j) \approx_{2^{m-n}} d(b_i, b_j)$ (threshold distance preservation).

At round $n+1$, let Spoiler play a_{n+1}



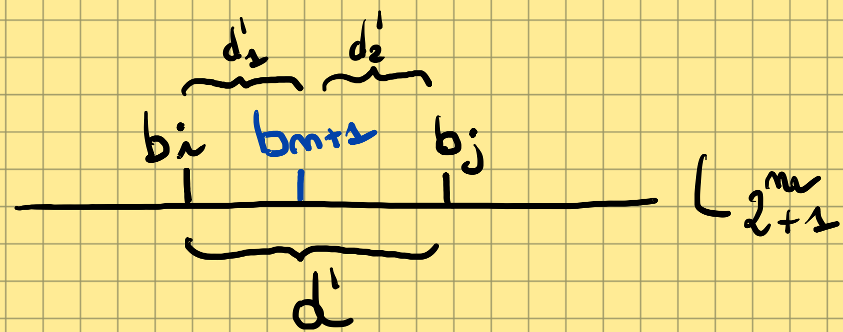
1) $d \leq 2^{m-n} \Rightarrow d = d'$, $d_1' := d_1$, $d_2' := d_2$.

2) $d > 2^{m-n} \Rightarrow d' > 2^{m-n}$.

2.1) $d_1 \leq 2^{m-(n+1)} \Rightarrow d_1' := d_1$.

2.2) $d_2 \leq 2^{m-(n+1)} \Rightarrow d_2' := d_2$.

2.3) $d_1, d_2 > 2^{m-(n+1)} \Rightarrow b_{n+1} := \frac{b_i + b_j}{2}$.



NON-DEFINABILITY of class C (with a single sentence)

Find two sequences of structures $\mathbb{A}_1 \equiv_1 \mathbb{B}_1$, $\mathbb{A}_2 \equiv_2 \mathbb{B}_2$, ...
s.t. $\mathbb{A}_1, \mathbb{A}_2, \ldots \in C$ and $\mathbb{B}_1, \mathbb{B}_2, \ldots \notin C$

PROOF: If φ of $qr(\varphi) = m$ defines $C = Mod(\varphi) = \{\mathbb{A} \mid \mathbb{A} \models \varphi\}$
we would have $\mathbb{A}_m \equiv_m \mathbb{B}_m$, but $\underbrace{\mathbb{A}_m \in C}_{\mathbb{A}_m \models \varphi}$ and $\underbrace{\mathbb{B}_m \notin C}_{\mathbb{B}_m \not\models \varphi}$.

Application: linear orders of even length $\mathbb{A}_m := L_{2^m}$, $\mathbb{B}_m := L_{2^m+1}$

NON-AXIOMATISABILITY of CLASS $\mathcal{C}$ (with a set of sentences)

Find a sequence of structures $\mathbb{A}_1, \mathbb{A}_2, \ldots \in \mathcal{C}$ and a simple structure $\mathbb{B} \notin \mathcal{C}$ s.t. $\mathbb{A}_1 \equiv_1 \mathbb{B}$, $\mathbb{A}_2 \equiv_2 \mathbb{B}$, $\ldots$

PROOF: Assume Γ axiomatises $\mathcal{C} = \mathrm{Mod}(\Gamma) = \{\mathbb{A} \mid \mathbb{A} \models \Gamma\}$.

Consider an arbitrary $\varphi \in \Gamma$. Let $m = qr(\varphi)$.

$\mathbb{A}_m \in \mathcal{C}$, thus $\mathbb{A}_m \models \Gamma$ and in particular $\mathbb{A}_m \models \varphi$.

By assumption, $\mathbb{A}_m \equiv_m \mathbb{B}$, thus $\mathbb{B} \models \varphi$.

By repeating for every $\varphi \in \Gamma$, we have $\mathbb{B} \models \Gamma$.

Thus, $\mathbb{B} \in \mathcal{C}$, which is a contradiction.

SOLVING EF-GAMES on FINITE STRUCTURES in PSPACE

We play $G_m(\mathcal{A}, \mathcal{B})$ for m rounds.

At each round:

Spoiler chooses an element in $\mathcal{A}$ or $\mathcal{B}$ $\forall$

Duplicator chooses an element in the other structure $\exists$

At the last round m we need to check that $h(a_i) = b_i$ is a partial isomorphism $h : A \to B$ (in PTIME).

This gives an algorithm running in APTIME = PSPACE.

(In fact, the problem is PSPACE-complete.)