Logic for Computer Science

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LECTURE 6:

COMPLETENESS for FIRST-ORDER LOGIC

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SEMANTIC VALIDITY

set of formulas over Σ $\qquad$ formula over Σ

$$\Delta \models \varphi$$

LOGICAL CONSEQUENCE

iff

For every structure $\mathbb{A}$ over signature Σ
and for every valuation $\rho : Var \to A$ ($\leftarrow$ domain of $\mathbb{A}$):

$$\mathbb{A}, \rho \models \Delta \text{ implies } \mathbb{A}, \rho \models \varphi .$$

When $\Delta = \phi$, we just write $\models \varphi$: φ is VALID.

HILBERT'S PROOF SYSTEM (first-order logic)

Connectives $\{\rightarrow, \bot, \forall\}$. Signature Σ.

Axioms are all generalizations $\forall x_1, \dots, x_n.\ \varphi$ of instances φ of:

A1: $\varphi \rightarrow \psi \rightarrow \varphi$

A2: $(\varphi \rightarrow \psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \psi) \rightarrow \varphi \rightarrow \theta$

A3: $\neg\neg\varphi \rightarrow \varphi$ (where $\neg\varphi \equiv \varphi \rightarrow \bot$)

MP: $\dfrac{\varphi \rightarrow \psi \qquad \varphi}{\psi}$

the same as in propositional logic

A4: $(\forall x.\ \varphi \rightarrow \psi) \rightarrow (\forall x.\ \varphi) \rightarrow \forall x.\ \psi$

A5: $\varphi \rightarrow \forall x.\ \varphi$ if $x \notin fv(\varphi)$

A6: $(\forall x.\ \varphi) \rightarrow \varphi[x \mapsto t]$ if t is free for x in φ

$\forall$

A7: $x = x$

A8: $x_1 = y_1 \rightarrow \cdots \rightarrow x_n = y_n \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$

A9: $x_1 = y_1 \rightarrow \cdots \rightarrow x_n = y_n \rightarrow R(x_1, \dots, x_n) \rightarrow R(y_1, \dots, y_n)$

"=" is a congruence

SOUND SUBSTITUTIONS

$\forall x . \varphi \models \varphi[x \mapsto t]$ not true in general!

Counterexample: $\underbrace{\forall x . \exists y . x \neq y}_{\text{the model has} \geq 2 \text{ elements}} \not\models \underbrace{\forall y . f(y) \neq y}_{f \text{ has no fixpoints}}$

$\mathbb{A} = (A = \{0, 1\}, f^{\mathbb{A}})$

$f^{\mathbb{A}}(0) = 0$

$f^{\mathbb{A}}(1) = 1$

We require:

t is free for x in φ:
every free variable in t remains free after the substitution in $\varphi[x \mapsto t]$

SOUND GENERALISATIONS

$\varphi \models \forall x \cdot \varphi$ not true in general!

$$x = 0 \not\models \forall x \cdot x = 0$$

Counterexample: $\mathbb{A} = (\{0,1\}, =)$, $\rho(x) = 0$.

$\mathbb{A}, \rho \models x = 0$, but

$\mathbb{A}, \rho \not\models \forall x \cdot x = 0$.

We require:

$x \notin fv(\varphi)$.

FORMAL PROOFs (reminder)

set of formulas

$$\Delta \vdash \varphi$$

iff

$\exists\ \varphi_1, \ldots, \varphi_n \equiv \varphi$ and

each φ_i : either φ_i is $A_1, \ldots, A_9$

or $\varphi_i \in \Delta$,

or $\dfrac{\varphi_k \quad \varphi_j}{\varphi_i}$ by MP $k, j < i$

PROOF EXAMPLE: $\vdash x = y \to y = x$

1. $x = x$ (by A7)
2. $\forall \overset{x}{x_1}, \overset{x}{y_1}, \overset{x}{x_2}, \overset{y}{y_2} .\ x_1 = y_1 \to x_2 = y_2 \to \overset{x_2 = x_1}{R(x_1, x_2)} \to R(y_1, y_2)$ (by A9)
3. $x = x \to x = y \to x = x \to y = x$ (by A6 + MP + 2 multiple times)
4. $x = y \to x = x \to y = x$ (by MP from 1, 2)
5. $x = x \to x = y \to y = x$ (from 3 by propositional reasoning)
6. $x = y \to y = x$ (by MP from 1, 4).

BASIC PROPERTIES of $\vdash$

- **Deduction Theorem**: $\Delta \vdash \varphi \to \psi$ iff $\Delta \cup \{\varphi\} \vdash \psi$

  Proof: literally the same as in propositional logic.

- **Generalisation Theorem**: $\Delta \vdash \forall x . \varphi$ iff $\Delta \vdash \varphi$ $\quad x \notin fv(\Delta)$

  $\Rightarrow$: direct A6 + MP, $\Leftarrow$: induction on proofs + A5 (base) + A4 (step) + MP.

- **Renaming Theorem**: $\Delta \vdash \forall x . \varphi$ implies $\Delta \vdash \forall y . \varphi[x \mapsto y]$

  $y \notin fv(\Delta \cup \{\varphi\})$, y free for x in φ

SOUNDNESS of HILBERT'S SYSTEM

$$\boxed{\Delta \vdash \varphi \text{ implies } \Delta \models \varphi}$$

Proof: by induction on derivations of $\Delta \vdash \varphi$.

The base cases are provided by soundness of each axiom.

E.g.:

A5: $\varphi \to \forall x. \varphi$ we have to prove $\Delta \models \varphi \to \forall x. \varphi$

COMPLETENESS of HILBERT'S SYSTEM

$$\boxed{\Delta \models \varphi \quad \text{implies} \quad \Delta \vdash \varphi}$$

- First proved by Gödel as part of his PhD thesis (1930)
  → using Skolem functions as in Herbrand's theorem (also 1930)
- Milestone result in logic, reproved many times thereafter (Mal'cev, Henkin, ...).
- Semantic completeness (completeness of first-order logic $\models$).

(Not to be confused with another meaning:
Syntactic completeness of Δ: $\forall \varphi$, $\Delta \vdash \varphi$ or $\Delta \vdash \neg\varphi$.
→ Gödel incompleteness theorem (for $\Delta = Th(\mathbb{N})$)
is about this other meaning.)

APPLICATIONS of COMPLETENESS

COMPACTNESS: $\Delta \models \varphi$ implies $\exists \Delta_0 \subseteq_{fin} \Delta \cdot \Delta_0 \models \varphi$

$$\begin{array}{ccc} \Delta \models \varphi & & \exists \Delta_0 \subseteq_{fin} \Delta \cdot \Delta_0 \models \varphi \\ \Downarrow & & \Uparrow \\ \Delta \vdash \varphi & \xrightarrow{\text{proofs are finite}} & \exists \Delta_0 \subseteq_{fin} \Delta \cdot \Delta_0 \vdash \varphi \end{array}$$

VALIDITY / LOGICAL CONSEQUENCE:

Both are semidecidable.

the set of axioms must itself be recursively enumerable

- To check $\Delta \models \varphi$, check $\Delta \vdash \varphi$ instead.
- Guess a proof (finite!) of φ from Δ

COMPLETENESS (alternative formulation)

Γ consistent $\Rightarrow$ Γ satisfiable

$\Gamma \nvdash \bot$; has a model $\mathcal{A} \models \Gamma$

The same as before: $\Delta \models \varphi \Rightarrow \Delta \vdash \varphi$

"$\Downarrow$": Assume $\Delta \models \varphi$. By way of contradiction, assume $\Delta \nvdash \varphi$.

$\Delta \cup \{\neg\varphi\}$ is consistent. If not:

$$\Delta, \neg\varphi \vdash \bot \underset{(DT)}{\Rightarrow} \Delta \vdash \neg\varphi \to \bot \underset{\text{def. } \neg}{\Rightarrow} \Delta \vdash \neg\neg\varphi \underset{A3+MP}{\Rightarrow} \Delta \vdash \varphi.$$

contradiction!

Take $\Gamma := \Delta \cup \{\neg\varphi\}$. By assumption, Γ is satisfiable: $\Delta \not\models \varphi$

contradiction!

"$\Uparrow$": $\Gamma \nvdash \bot \underset{\text{assumption}}{\Rightarrow} \Gamma \not\models \bot \Rightarrow \Gamma \cup \{\neg\bot\}$ has a model $\Rightarrow \Gamma$ has a model

INPUT: Signature Σ, consistent Γ over Σ.

OUTPUT: Model $\mathbb{A}_\Gamma$ for Γ.

COUNTABLE

SENTENCES (no free vars)

- we have only syntax $\Rightarrow$ build a model out of syntax!
- $\mathbb{A}$ must have witnesses : $\mathbb{A} \not\models \forall x. \varphi \Rightarrow \exists a \in A. \mathbb{A}, x{:}a \models \neg\varphi$.

(semantic property)

Γ **SATURATED** if

(syntactic property)

$$\boxed{\begin{gathered}\Gamma \not\vdash \forall x. \varphi(x) \\ \Downarrow \\ \exists a \in \Sigma.\ \Gamma \vdash \neg\varphi[x \mapsto a] \\ \text{for every formula} \\ \varphi(x) \text{ over } \Sigma,\ fv(\varphi) = \{x\}\end{gathered}}$$

SATURATION

$\Gamma \nvdash \forall x.\varphi(x) \Rightarrow \exists a \in \Sigma.\ \Gamma \vdash \neg\varphi[x \mapsto a]$

INPUT: Consistent Δ over Σ.

OUTPUT: Consistent & saturated $\Gamma \supseteq \Delta$ over $\Sigma' \supseteq \Sigma$.

$\Sigma' := \Sigma \cup C$ ← fresh set of constants not in Σ

only f.v.

Enumerate all formulas $\varphi_1(x), \varphi_2(x), \dots$ over Σ'.

$\Gamma := \bigcup_n \Gamma_n$ where $\Gamma_0 = \Delta$ and

fresh constant $\in C$ not previously used

can prove consistent (omitted)

$$\boxed{\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\neg\varphi_n[x \mapsto c_n]\} & \text{if } \Gamma_n \nvdash \forall x.\varphi_n(x) \\ \Gamma_n & \text{otherwise} \end{cases}}$$

SATURATED

$\Gamma \nvdash \forall x.\varphi_n(x) \Rightarrow \Gamma_n \nvdash \forall_n \varphi(x) \Rightarrow \Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n[x \mapsto c_n]\} \Rightarrow$

$\Rightarrow \Gamma_{n+1} \vdash \neg\varphi_n[x \mapsto c_n] \Rightarrow \Gamma \vdash \neg\varphi_n[x \mapsto c_n]$ ✓

THE SYNTACTIC MODEL $\mathbb{A}_\Gamma$

$\Gamma \leftarrow \begin{cases} \text{consistent} & \Gamma \nvdash \bot \\ \& & \\ \text{saturated} & \Gamma \nvdash \forall x.\varphi(x) \Rightarrow \exists a \in A.\, \Gamma \vdash \neg\varphi[x \mapsto a] \end{cases}$

- let $c \sim d$ if $\Gamma \vdash c = d$
- Domain $A = \{[c]_\sim \mid \text{constant } c \in \Sigma\}$

Interpretations:

- constants: $c^{\mathbb{A}_\Gamma} = [c]_\sim \in A$.
- function symbols: $f^{\mathbb{A}_\Gamma}([c]_\sim) = [d]_\sim$ iff $f(c) \sim d$
- relation symbols: $R^{\mathbb{A}_\Gamma}([c]_\sim)$ iff $\Gamma \vdash R(c)$
  (similar checks required)

Must check
1. well-defined (OMITTED)
2. Total:
   $\forall c.\exists d.\ f(c) \sim d$
   by SATURATION!

$\forall x.\neg f(c) = x \vdash \bot$
(just take $x = f(c)$)

$\Rightarrow \Gamma, \forall x.\neg f(c) = x \vdash \bot$

(DT) $\Rightarrow \Gamma \vdash \forall x.\neg f(c) = x \to \bot$

$\Rightarrow \Gamma \vdash \neg \forall x.\neg f(c) = x$

(consistent) $\Rightarrow \Gamma \nvdash \forall x.\neg f(c) = x$

SAT.! $\Rightarrow \Gamma \vdash \neg\neg f(c) = d$ for some $d \in \Sigma$

(A3+MP) $\Rightarrow \Gamma \vdash f(c) = d$

(def $\sim$) $\Rightarrow f(c) \sim d$

BRIDGE LEMMA

$$\mathbb{A}_\Gamma, x_1 : [c_1]_\sim, \dots, x_n : [c_n]_\sim \models \varphi$$
$$\Updownarrow$$
$$\Gamma \vdash \varphi[x_1 \mapsto c_1][x_n \mapsto c_n]$$

(proof by structural induction on φ – OMITTED)

φ sentence (no free variables) $\Rightarrow$ $\mathbb{A}_\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$

$\Rightarrow$ $\mathbb{A}_\Gamma \models \Gamma$

$\Rightarrow$ $\mathbb{A}_\Gamma$ is a model for Γ ! ✓

What if we started from Γ with free variables?

→ replace each free x with a fresh constant c_x : $\hat{\Gamma} := \Gamma[x \mapsto c_x]$

$$\mathbb{A}_\Gamma, \dots x : [c_x]_\sim \dots \models \Gamma \Leftrightarrow \mathbb{A}_{\hat{\Gamma}} \models \hat{\Gamma} \quad ✓$$