# Logic for Computer Science

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## LECTURE 11:

## UNDECIDABILITY of FIRST-ORDER LOGIC

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# SUMMARY

- Decision problem for first-order logic (ENTSCHEIDUNGSPROBLEM).
- Undecidability of validity and finite satisfiability.
- Tools: tiling problems.
- Undecidability and incompleteness.

# THE DECISION PROBLEM for a theory $\Gamma$ over $\Sigma$.

set of sentences s.t. $\Gamma \models \varphi \Rightarrow \varphi \in \Gamma$.

(ENTSCHEIDUNGSPROBLEM)

INPUT: a sentence $\varphi$ over vocabulary $\Sigma$.

OUTPUT: YES iff $\varphi \in \Gamma$.

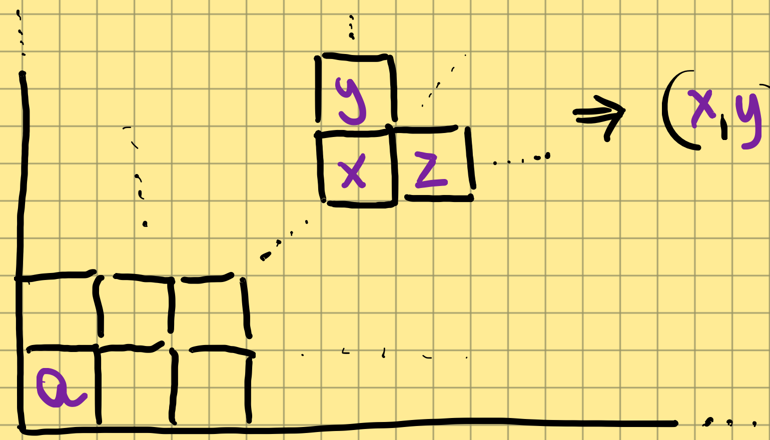
THEOREM (Church '36, Turing '37)
The validity problem for first-order logic is undecidable.
(but recursively enumerable thanks to Gödel's completeness theorem)

THEOREM (Trakhtenbrot '50)
The finite satisfiability problem is undecidable.
(but obviously recursively enumerable)

# INFINITE TILING PROBLEM

Input: A finite set of tiles $T$, an initial tile $a \in T$, vertical and horizontal compatibility relations $V, H \subseteq T \times T$.

Output: YES iff there is an infinite tiling of the quarter plane:




$\Rightarrow (x,y) \in V \;\&\; (x,z) \in H$

The infinite tiling problem is UNDECIDABLE (by reduction from the looping problem for Turing machines)

# TILING $\leq$ SATISFIABILITY

INPUT: Tiling instance $T = \{a_1, \dots, a_m\}$, $a_1 \in T$, $H, V \subseteq T \times T$.

OUTPUT: Formula $\varphi$ s.t. $\boxed{\text{SAT}(\varphi) \text{ iff there is an infinite tiling.}}$

Take $\Sigma = \{\underbrace{< : 2,\ a_1 : 2, \dots, a_m : 2,\ = : 2}_{\text{binary relations}},\ \underbrace{min : 0}_{\text{constant}},\ \underbrace{s : 1}_{\text{unary function}}\}$

Let $\varphi$ say:

1) $<$ is a strict total order with minimal element min.

2) $s(x)$ is the least element larger than $x$: $\forall x. x < s(x) \wedge \forall y. (x < y \to s(x) \leq y)$.

1+2 gives us an infinite chain $min < s(min) < s^2(min) < \cdots$

3) Tiling correct: $\forall x, y. \bigvee_{a_i \in T} \Big( a_i(x,y) \wedge \bigwedge_{a_j \in T \setminus \{a_i\}} \neg a_j(x,y) \wedge \Big( \bigvee_{(a_i, a_j) \in V} a_j(s(x), y) \Big) \wedge \Big( \bigvee_{(a_i, a_j) \in H} a_j(x, s(y)) \Big) \Big)$

4) $a_1(min, min)$.

# WHAT DID WE USE ?

Binary relations: $<, a_1, \dots, a_n, =$. One constant: $min$. One unary function: $s$.

OPTIMISATIONS :

1) $min$ is definable by $\varphi_{min}(x) \equiv \forall y . \neg y < x$.
   Replace $a_1(min, min)$ with $\exists x . \varphi_{min}(x) \wedge a_1(x, x)$.

2) $s$ is definable from $<$ by $\varphi_s(x, y) \equiv x < y \wedge \forall z . x < z \to y \leq z$.
   Replace $a_j(s(x), y)$ with $\exists z . \varphi_s(x, z) \wedge a_j(z, y)$.

3) Alternatively, forget $<$ and require that $s$ is a "successor":
   $\exists x . (\forall y . s(y) \neq x) \wedge (\forall y, z . s(y) = s(z) \to y = z) \wedge \forall y . s(y) \neq y$.

$$x \xrightarrow{s} \xrightarrow{s} \xrightarrow{s} \dots$$

4) Simulate $a_1, \dots, a_n$ with a simple predicate $P$ by "scaling":

$$\boxed{a_5} \rightsquigarrow \begin{array}{|c|c|c|} \hline P & \neg P & P \\ \hline P & P & \neg P \\ \hline \neg P & \neg P & P \\ \hline \end{array}$$

# MINIMAL UNDECIDABLE SIGNATURE

At least:

- one binary relation.
- one binary function.
- two unary functions.

Optimal:

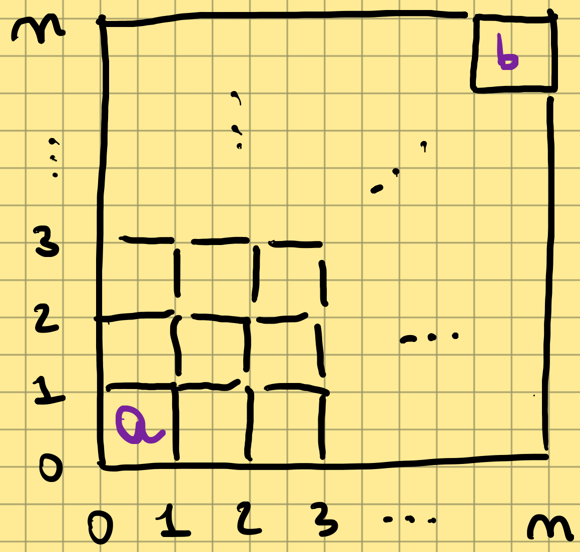
Monadic logic is decidable.

$\left\{ \begin{array}{l} \mathrm{Th}(f:1) \text{ is decidable.} \\ \text{(Ehrenfeucht '59, via tree automata)} \end{array} \right.$

# FINITE TILING PROBLEM

Input: A finite set of tiles $T$, initial and final tiles $a, b \in T$, vertical and horizontal compatibility relations $V, H \subseteq T \times T$.

Output: YES iff there is $m \in \mathbb{N}$ and a tiling of $\{0, \dots, m\} \times \{0, \dots, m\}$:




+ Compatibility conditions

The finite tiling problem is UNDECIDABLE
(by reduction from the halting problem for Turing machines)

# FINITE TILING $\leq$ FINITE SATISFIABILITY

INPUT: Tiling instance $T = \{a_1, \dots, a_m\}$, $a_1, a_2 \in T$, $H, V \subseteq T \times T$.

OUTPUT: Formula $\varphi$ s.t. $\boxed{\text{FINSAT}(\varphi) \text{ iff there is a finite tiling.}}$

Add the constant $max$ to the signature.

1) $<$ is a strict total order with least elem. $min$ and greatest $max$.

2) $S$ is the "looped" successor:

$$\forall x.\ x = max = s(max) \lor x < s(x) \land \forall y. (x < y \to s(x) \leq y).$$

1+2 gives a finite ~~or infinite~~ chain $min < s(min) < \dots < max = s(max)$.

3) Tiling correct: $\forall x, y. \bigvee_{a_i \in T} \Big( a_i(x,y) \land \bigwedge_{a_j \in T \setminus \{a_i\}} \neg a_j(x,y) \land$

$$\Big( x < max \to \Big( \bigvee_{(a_i, a_j) \in V} a_j(s(x), y) \Big) \Big) \land \Big( y < max \to \Big( \bigvee_{(a_i, a_j) \in H} a_j(x, s(y)) \Big) \Big) \Big).$$

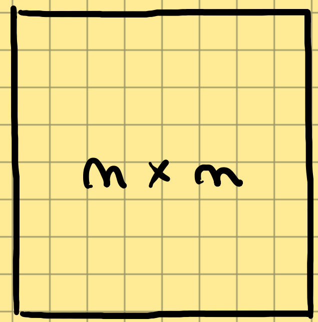
4) $a_1(min, min)$, $a_2(max, max)$.

# VARIANTS of FINITE TILING PROBLEMS

Complexity:

SQUARE

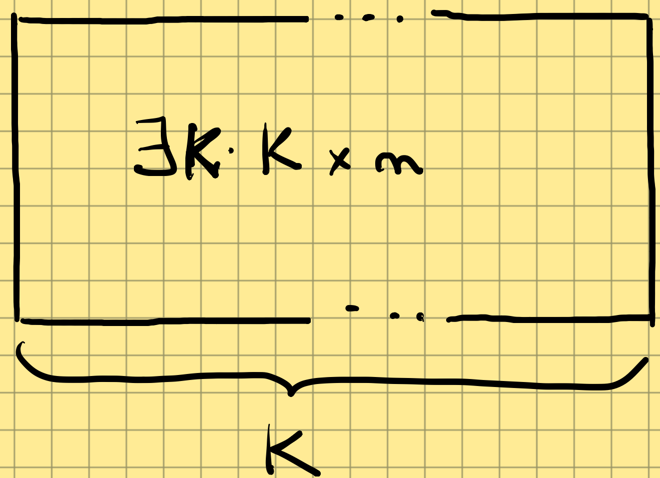
n given in unary




NP-complete

RECTANGLE

n given in unary




PSPACE-complete

# HISTORICAL REMARK on UNDECIDABILITY

- For convincing somebody that a problem is "decidable" it suffices to provide a "method" which intuitively "works".
  → this is what mathematicians have been doing since ever...
- For proving that a problem is undecidable we first need a formal model of computation.
  - Alonzo Church : $\lambda$-calculus.
  - Alan Turing : Turing machines.
  - Kurt Gödel : Partial recursive functions.

  all equivalent (Church-Turing thesis)
- We used a more modern proof based on Tiling problems.

# (UN)DECIDABILITY & (IN)COMPLETENESS

Recall two distinct notions of completeness: ↙ Gödel '29

1) completeness of Hilbert's proof system: $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$.

2) Completeness of a theory $\Gamma$ over $\Sigma$: $\forall \varphi \in \mathrm{Th}(\Sigma)$. $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

$$\boxed{\mathrm{Th}(\Gamma) \text{ complete 2)} + \Gamma \text{ decidable} \Rightarrow \mathrm{Th}(\Gamma) \text{ decidable.}}$$

Proof: To decide $\varphi \overset{?}{\in} \mathrm{Th}(\Gamma)$, look in parallel for a proof of $\Gamma \vdash \varphi$ and a proof of $\Gamma \vdash \neg\varphi$.

By completeness 2), either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$.

By completeness 1), either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$.

$$\boxed{\text{The theory of first-order logic is incomplete}^{*} \text{ 2).}}$$

* no $\varphi$ given