Logic for Computer Science

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LECTURE 8: COMPACTNESS, SKOLEM-LÖWENHEIM, and applications

Lectures: LORENZO CLEMENTE
Tutorials: DARIA WALUKIEWICZ, JACEK CHRZASZCZ, JĘDRZEJ KOŁODZIEJSKI
Labs: DARIA, JACEK + PIOTR HOFMAN

PLAN

- Compactness theorem for first-order logic.
- Applications of compactness:
  - Arbitrary large infinite models.
  - Construction of total orders.
  - Bonus: Non-standard models of the reals (infinitesimals).
  - Non-axiomatisability of well-orderings.
  - More applications in the tutorials.
- Skolem-Löwenheim theorem.
- Applications of Skolem-Löwenheim:
  - Hessenberg theorem.
  - More applications in the tutorials.

COMPACTNESS THEOREM for FIRST-ORDER LOGIC

Two equivalent formulations:

1) $\Gamma \models \varphi$ implies there is a finite $\Delta \subseteq_{fin} \Gamma$ s.t. $\Delta \models \varphi$.

2) Finite satisfiability implies satisfiability:

$$\left(\text{for every finite } \Delta \subseteq_{fin} \Gamma .\ SAT(\Delta)\right) \Rightarrow SAT(\Gamma).$$

Proof of 1): $\Gamma \models \varphi \Rightarrow \underbrace{\Gamma \vdash \varphi}_{\text{proof in Hilbert's system}} \Rightarrow \exists \Delta \subseteq_{fin} \Gamma . \Delta \vdash \varphi \Rightarrow \exists \Delta \subseteq_{fin} \Gamma . \Delta \models \varphi$

by completeness

proof in Hilbert's system

proofs are finite!

soundness

EXAMPLE APPLICATION of COMPACTNESS (1)

If Γ has arbitrarily large models, then it has an infinite model.

Proof: $\Delta = \Gamma \cup \{\varphi_1, \varphi_2, \dots\}$, $\varphi_n \equiv \exists x_1, \dots, x_n. \bigwedge_{i \neq j} x_i \neq x_j$ (there are at least n elements)

Claim: Δ is finitely satisfiable.

Indeed, every finite $\Delta' \subseteq_{fin} \Delta$ is a subset of $\underbrace{\Gamma \cup \{\varphi_1, \dots, \varphi_n\}}_{\text{satisfiable by assumption!}}$

By compactness, Δ has a model $\mathbb{A} \models \Delta$.

In particular, $\mathbb{A} \models \Gamma$ and $\forall n \in \mathbb{N}.\ \mathbb{A} \models \varphi_n$.

$\Rightarrow$ Γ has an infinite model.

EXAMPLE APPLICATION of COMPACTNESS (2)

Every set A can be totally ordered.

binary relation.

Proof: Signature $\Sigma = \{ < : 2 \} \cup A$

$$\Gamma = \{ \forall x . \neg x < x, \qquad \text{(irreflexive)}$$
$$\forall x,y,z . \; x<y \wedge y<z \rightarrow x<z, \qquad \text{(transitive)}$$
$$\forall x,y . \; x<y \vee y<x \} \cup \qquad \text{(total)}$$
$$\cup \{ a \neq b \mid a,b \in A \}$$

Γ is finitely satisfiable (every finite set can be totally ordered).
By compactness, Γ has a model $\mathbb{B} \models \Gamma$ with domain $B \supseteq A$.

Define $a_1 \sqsubset a_2$ iff $\mathbb{B} \models a_1 < a_2$.
Then, "$\sqsubset$" is a total ordering of A.

PROPERTIES DEFINABLE by ONE SENTENCE of FIRST-ORDER LOGIC

$\approx$ finitely many sentences

- The model has $\geq n$ elements $\varphi_{\geq n}$.
- "$<$" is a strict total order.
- Monoids: $(M, \cdot, 1)$, $\cdot : M \times M \to M$, $1 \in M$:

  Associativity: $\forall x, y, z.\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$.

  Unit: $\forall x.\ x \cdot 1 = 1 \cdot x = 1$.
- Isomorphic to a fixed finite structure $\mathbb{A} = (A, a, b, R = \{(a,b), (b,a)\})$

$$\varphi_{\mathbb{A}} \equiv \exists x, y.\ \forall z.\ (z = x \vee z = y) \wedge x \neq y \wedge R(x,y) \wedge R(y,x) \wedge \neg R(x,x) \wedge \neg R(y,y).$$

characteristic sentence (c.f. P2.1.8)

In terms of DBs: can encode the DB in the query.

PROPERTIES AXIOMATISABLE by a SET of SENTENCES of FIRST-ORDER LOGIC

- The model is infinite $\{\varphi_{\geq 1}, \varphi_{\geq 2}, \dots\}$.
- Propositional logic tautologies $\{\top, p \vee \neg p, p \to p, p \to q \to p, \dots\}$.
- First-order logic tautologies $\{\forall x. x = x, \forall x, y. x = y \to f(x) = f(y), \dots\}$.
  (over a fixed signature)

- All classes of finite structures are axiomatisable (!).

Proof:

Let $\mathbb{A} = \{\mathbb{A}_0, \mathbb{A}_1, \dots\}$ be a (isomorphism-closed) collection of finite structures over Σ.

Let $\mathbb{B} = \{\mathbb{B}_0, \mathbb{B}_1, \dots\}$ be the finite structures over Σ not in $\mathbb{A}$.

Let $\varphi_{\mathbb{B}_i}$ be the characteristic sentence of $\mathbb{B}_i$.

$\mathbb{A}$ is axiomatised by $\{\neg \varphi_{\mathbb{B}_0}, \neg \varphi_{\mathbb{B}_1}, \dots\}$.

EXAMPLE APPLICATION of COMPACTNESS (3)

A well-order is a total order $<$ with no infinite decreasing chain $a_0 > a_1 > \cdots$

Equivalently, every decreasing chain is finite.

$$\boxed{\text{The class of well-orderings cannot be axiomatised in FO}}$$

Proof: By way of contradiction, let Γ axiomatise that $<$ is well-order.

Add constants $c_1, c_2, \ldots$

Consider $\Delta = \Gamma \cup \{c_1 > c_2, c_2 > c_3, \ldots\}$ $\leftarrow$ finitely satisfiable

(there are well-orders with decreasing chains of len. $1, \ldots, n$).

By compactness, Δ has a model $\mathbb{A} = (A, <^{\mathbb{A}}, c_1^{\mathbb{A}}, c_2^{\mathbb{A}}, \ldots)$

but $c_1^{\mathbb{A}} > c_2^{\mathbb{A}} > c_3^{\mathbb{A}} > \cdots$ is an infinite decreasing chain.

Contradiction!

SKOLEM-LÖWENHEIM THEOREM

If Γ has an *infinite* model,

then it has a model of each infinite cardinality $\kappa \geq |\Sigma|$. (Σ: signature of Γ)

Proof: D set of fresh constants of cardinality $|D| = \kappa$.

$\Delta = \Gamma \cup \{ c \neq d \mid c, d \in D \}$, new signature $\Sigma_0 = \Sigma \cup D$.

By compactness, Δ has a model (satisfiable)

By soundness, $\Delta \nvdash \bot$ (consistent).

By the saturation construction (c.f. proof of completeness),

Δ has a model $\mathcal{A}$ of size $|\mathcal{A}| \leq |\Sigma_0| = \kappa$

because elements in the syntactic model are equivalence classes of constants from $\underbrace{\Sigma_0}_{\kappa} \cup \underbrace{C}_{\kappa}$

By def. of Δ, $|\mathcal{A}| \geq \kappa$

$\Longrightarrow |\mathcal{A}| = \kappa$.

APPLICATION of SKOLEM-LÖWENHEIM

HESSENBERG'S THEOREM

$\kappa = \kappa \times \kappa$ for every infinite cardinal κ.

Proof: $\kappa \leq \kappa \times \kappa$ is obvious. We show $\kappa \times \kappa \leq \kappa$.

Let A be a set of cardinality $|A| = \kappa$.

Consider the signature $\Sigma = \{ f : 2 \}$ binary function symbol

Consider the sentence "f is injective":

$$\varphi \equiv \forall x_0 x_1 y_0 y_1 . \; f(x_0, y_0) = f(x_1, y_1) \rightarrow x_0 = x_1 \wedge y_0 = y_1$$

1) $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}| \Rightarrow \varphi$ has an infinite model.

2) By Skolem-Löwenheim, $\mathbb{A} \models \varphi$ for some $\mathbb{A} = (A, f^{\mathbb{A}})$ of cardinality $|A| = \kappa$ where $f^{\mathbb{A}} : A \times A \rightarrow A$ is the required injection.

LIMITATIONS of COMPACTNESS

1. No compactness for finite models:

$$\underbrace{\Gamma \models_{fin} \varphi}_{\text{every finite model of } \Gamma \text{ is a model of } \varphi.} \quad \text{implies} \quad \exists \Delta \subseteq_{fin} \Gamma .\ \Delta \models_{fin} \varphi .$$

Counterexample: $\Gamma = \{ \varphi_{\geq 1}, \varphi_{\geq 2}, \dots \}$, $\varphi \equiv \bot$.

$\Gamma \models_{fin} \varphi$ holds (no finite model satisfies Γ).

There is no finite $\Delta \subseteq_{fin} \Gamma$ s.t. $\Delta \models_{fin} \bot$.

(any such Δ has a finite model).

2. Cannot use compactness to show non-definability by a simple formula (e.g., infinite or even cardinality)

axiomatisable $\{\neg\varphi_{=1}, \neg\varphi_{=3}, \dots\}$

→ solved with Ehrenfeucht-Fraïssé games.

NON STANDARD MODELS of the REALS

Let $\mathrm{Th}(\mathbb{R})$ be the first-order theory of the reals $(\mathbb{R}, 0, 1, +, \cdot)$ standard model

$$= \{ \varphi \text{ sentence over } \Sigma = \{0, 1, +, \cdot\} \mid \mathbb{R} \models \varphi \}$$

There exist non-standard models of the reals
(non-standard analysis)

Proof: Add a new constant ∞ to the signature.

Let $\Gamma = \mathrm{Th}(\mathbb{R}) \cup \{ 1 < \infty,\ 1+1 < \infty,\ \dots \}$

"∞ is larger than every standard real".

Γ is finitely satisfiable (interpret ∞ sufficiently large)

By compactness, Γ has a model $(\mathbb{R}^*, 0, 1, +, \cdot, \infty)$

Moreover, $\mathrm{Th}(\mathbb{R}) \subseteq \Gamma \subseteq \mathrm{Th}(\mathbb{R}^*)$: even a conservative extension!

$\mathbb{R}^*$ is non-standard : ∞ is larger than any standard real.