1. (4 points) Prove $\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$

$$\begin{aligned}
\#(A \cup B) &= \#((A \setminus B) \cup (B \setminus A) \cup (A \cap B)) \\
&= \#(A \setminus B) + \#(B \setminus A) + \#(A \cap B) \\
&= \#(A \setminus B) + \#(A \cap B) + \#(B \setminus A) + \#(A \cap B)) - \#(A \cap B) \\
&= \#((A \setminus B) \cup (A \cap B)) + \#(B \setminus A) \cup (A \cap B)) - \#(A \cap B) \\
&= \#(A) + \#(B) - \#(A \cap B)
\end{aligned}$$

For undergrads a Venn diagram proof is also acceptable.

2. (8 points) How many 5 card poker hands have a full house? (two of a kind and three of a kind). Calculate the result theoretically first (4 points). Write a pseudo program to simulate the probability that a random drawn 5-card hand is a full house (4 points).
Note: Another 4 bonus points for both undergrads and grads: run the code and attach a screenshot of the code and result.

```
cards = [None]*52
for i in range(52):
  cards[i] = int(i/4)+1
#print(cards)
loops = 500000
count = 0
hand = [None]*5
for j in range(loops):
  random_numbers = []
  for i in range(5):
    random_numbers.append(random.randint(0, 51))
  hand[0] = cards [random_numbers[0]]
  hand[1] = cards [random_numbers[1]]
  hand[2] = cards [random_numbers[2]]
  hand[3] = cards [random_numbers[3]]
  hand[4] = cards [random_numbers[4]]
  hand = sorted(hand)
  if (hand[0] == hand[1] & hand[2] == hand[3] & hand[3] == hand[4]) | (hand[0] == hand[1] &
hand[1] == hand[2] & hand[3] == hand[4]):
    count+=1
  #print(hand)
print(count/loops)
```

output: 5.2e-05 (Quite small! Value may change from time to time.)

There are $\binom{4}{3} = 4$ different ways to choose 3 cards of the same type and $\binom{4}{2} = 6$ different ways to choose 2 cards of the same type. To get a full house you need to first pick the type of the 3 of a kind, that is $\binom{13}{1} = 13$ different choices, and choose the type of the pair, that is $\binom{12}{1} = 12$ different choices. The order does not count. So there are $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$ different full houses in a 52 cards deck.

Note that there are $\binom{52}{5} = 2598960$ different poker hands, so the probability to being dealt a full house is

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 3744/2598960 \approx 0.14\%$$

3. (6 points) Can 950 balls be put into 100 bins such that no bin has 10 or more balls? If yes, give an example. If no, prove.

   No. Suppose each bin has exactly 9 balls. Then there are 900 balls in the 100 bins. Suppose one more ball is added. Whatever bin it is added to will contain 10 balls.

   However, the problem does not say the balls cannot be divided. If there is a solution that involves cutting the balls in half, read it carefully.

4. (7 points) pg 60, # 30

   Consider the sample space for the hunter's strategy. The events that lead to the correct path are:

   (a) Both dogs agree on the correct path (probability p^2 , by independence).

   (b) The dogs disagree, dog 1 chooses the correct path, and hunter picks dog 1 with probability $\frac{1}{2}$ [probability of this choice $\frac{p(1-p)}{2}$].

   (c) The dogs disagree, dog 2 chooses the correct path, and hunter follows dog 2 [probability $\frac{p(1-p)}{2}$].

   (d) The above events are disjoint, so we can add the probabilities to find that under the hunter's strategy, the probability that he chooses the correct path is

   $$p^2 + \frac{p(1-p)}{2} + \frac{p(1-p)}{2} = p$$

   (e) If the hunter lets just one dog choose the path, this dog will also choose the correct path with probability p. Thus, the two strategies are equally effective.