This assignment contains 5 bonus points.

Problem 1) In test II, we had the following problem: Two continuous uniform random variables X and Y have a joint PDF given by

$$\begin{cases} c(x+y), & \text{if } 0 \le x \le 2, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

What is the value of c? Find the marginal PDF of X and Y, i.e., $f_X(x)$ and $f_Y(y)$, Find $E[2X+4Y]$.

(a) Solve the above questions again as similar questions will be on the final. (5 points)

(b) Find $Cov(X,Y)$ (10 points)

Problem 2) pg 290 #10 (15 points)

1)

a) $\int_0^2\int_0^2 c(x+y)\,dy\,dx = 1$

$\int_0^2 [cx + cy]_0^2 \, dx = 1$

$[2cx + cx^2]_0^2 = 1$

$(4c + 4c) - (0+0) = 1$

$8c = 1$

$c = \frac{1}{8}$

$f_X(x) = \int_0^2 f_{XY}(x,y)\,dy$

$= \int_0^2 \frac{1}{8}(x+y)\,dy$

$= \left[\frac{1}{8}\left(xy + \frac{y^2}{2}\right)\right]_0^2$

$= \frac{1}{8}(2x+2) - \frac{1}{8}(x+0)$

$f_X(x) = \frac{1}{4} + \frac{x}{4}$

$f_Y(y) = \int_0^2 f_{XY}(x,y)\,dx$

$= \int_0^2 \frac{1}{8}(x+y)\,dx$

$= \left[\frac{1}{8}(x^2 + xy)\right]_0^2$

$= \frac{1}{8}(4+2y) - \frac{1}{8}(0+0)$

$f_Y(y) = \frac{1}{4} + \frac{y}{4}$

b) $E[2X+4Y] = 2E[X] + 4E[Y]$

$E[X] = \int_0^2 x \cdot f_X(x)\,dx$

$= \int_0^2 \left(\frac{1}{4} + \frac{x}{4}\right)dx = \left[\frac{x^2}{8} + \frac{x^3}{12}\right]_0^2$

$= \left(\frac{4}{8} + \frac{8}{12}\right) - \left(\frac{0}{8} + \frac{0}{12}\right)$

$= \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ $\quad E[X] = \frac{7}{6}$

$$E[Y] = \int_0^2 y \cdot f_Y(y)\,dy$$
$$= \int_0^2 y\left(\frac{1}{4} + \frac{y}{4}\right)dy$$
$$= \int_0^2 \left(\frac{y}{4} + \frac{y^2}{4}\right)dy$$
$$= \left[\frac{y^2}{8} + \frac{y^3}{12}\right]_0^2$$
$$= \left(\frac{4}{8} + \frac{8}{12}\right) - \left(\frac{0}{8} + \frac{0}{12}\right)$$
$$= \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \qquad E[Y] = \frac{7}{6}$$

$$E[2X+4Y] = 2 \cdot \frac{7}{6} + 4 \cdot \frac{7}{6} = \frac{14}{6} + \frac{28}{6} = \frac{42}{6} = 7$$

$$E[2X+4Y] = 7$$

b) $\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y]$

$$E[XY] = \int_0^2 \int_0^2 xy \cdot f_{XY}(x,y)\,dy\,dx \qquad E[X] = \frac{7}{6} \quad E[Y] = \frac{7}{6}$$
$$= \int_0^2 \int_0^2 xy \cdot \frac{1}{8}(x+y)\,dy\,dx$$
$$= \frac{1}{8}\int_0^2 \left[x \int_0^2 (xy + y^2)\,dy\right] dx$$
$$= \frac{1}{8}\int_0^2 \left[x\left(\frac{x}{2}y^2 + \frac{y^3}{3}\right)\right]_0^2 dx$$
$$= \frac{1}{8}\int_0^2 \left[x\left(\frac{x}{2}(2)^2 + \frac{(2)^3}{3}\right) - x\left(\frac{x}{2}(0)^2 + \frac{(0)^3}{3}\right)\right] dx$$
$$= \frac{1}{8}\int_0^2 \left[x\left(2x + \frac{8}{3}\right)\right] dx = \frac{1}{8}\left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^2$$
$$= \frac{1}{8}\left[\left(\frac{2(2)^3}{3} + \frac{8(2)}{3}\right) - \left(\frac{2(0)^3}{3} + \frac{8(0)}{3}\right)\right]$$
$$= \frac{1}{8}\left[\frac{16}{3} + \frac{16}{3}\right] = \frac{32}{24} = \frac{4}{3}$$

$$\mathrm{cov}(X,Y) = \frac{4}{3} - \left(\frac{7}{6} \cdot \frac{7}{6}\right) = \frac{4}{3} - \frac{49}{36} = \frac{48}{36} - \frac{49}{36} = -\frac{1}{36}$$

$$\mathrm{cov}(X,Y) = -\frac{1}{36}$$