Graviational Waves from Feynman Diagrams

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Preface

This is a Quarto book. I am going to try and hyperlink it as much as possible

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1 Introduction

The detection of gravitational waves by the LIGO and Virgo collaborations in 2016 (2016) has sparked a new era of gravitational wave astronomy. The first detections were of Binary Black Hole () mergers. More recently, Binary Neutron Star () mergers (2017) as well as Neutron Star ()-Black Hole () mergers have been detected (2021). Future detectors will further increase sensitivity and will be able to detect a wide range of astrophysical sources. Studying these gravitational waves signals gives us a very powerful new window into the universe. It allows us to study the properties of neutron stars and black holes, and the physics of compact object mergers, but also gives us a powerful testing apparatus for general relativity.

To detect these faint signals LIGO and Virgo, have been made to be extraordinarily precise instruments. Thus, they demand correspondingly precise theoretical predictions and models. This is not just for comparison's sake, but for detection as well. These faint signals are often buried in the noise of the detector. To counteract this experimental physicists make use of a matched filtering approach, where they try to match the signal to a template. The template is a model of the signal ideally provided by theoretical physicists based on physical theories. The more precise the template, the higher the signal-to-noise ratio, the more probable and precise the detection.

In this thesis we will explore the theoretical landscape surrounding the generation of these templates. We will focus on a nascent subfield where tools originally used for Quantum Field Theory

Abbott, B. P., R. Abbott, T. D. Abbott, et al. 2016. "Observation of Gravitational Waves from a Binary Black Hole Merger." *Physical Review Letters* 116 (6): 061102. https://doi.org/10.1103/PhysRevLett.116.061102.

Abbott, B. P., Rich Abbott, Thomas D. Abbott, et al. 2017. "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral." *Physical Review Letters* 119 (16): 161101. https://doi.org/10.1103/PhysRevLett.119.161101.

Abbott, R., T. D. Abbott, S. Abraham, F. Acernese, K. Ackley, A. Adams, C. Adams, et al. 2021. "Observation of Gravitational Waves from Two Neutron Star–Black Hole Coalescences." *The Astrophysical Journal Letters* 915 (1): L5. https://doi.org/10.3847/2041-8213/ac082e.

() and particle physics are being applied to the study of gravitational waves. Specifically we are interested in the diagrammatic objects that arise when framing the two body problem similarly to particle collisions. We will explain where these tools shine in the broader context of waveform approaches such as Effective One-Body (), Nonrelativistic General Relativity (), Post-Minkowski () and Post-Newtonian () approximations. We will also discuss the challenges that lie ahead in the field.

2 Gravitational Wave Generation

First let us look at where and how gravitional waves are generated. We will look at how General Relativity () predicts that graviational waves exist, and what conditions have to be met for them to become observable. We will also look at how we can detect them, and what we can learn from them.

2.1 Gravitational Waves

The most complete theory of gravity we have right now is that of GR, due to Einstein. It formulates spacetime as a Riemannian manifold with curvature, senstive to mass. Objects then move around in that deformed spacetime encoded in the metric $g_{\mu\nu}$. Objects interact gravitationally when the spacetime they move around in is affected by the curvature caused by other objects. The equation that governs this interaction between mass and curvature is Einstein's field equations,

$$\boxed{R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}}. \tag{2.1}$$

The Left-Hand Side () of this equation, contains several objects that are all only really dependent on the metric $g_{\mu\nu}$. $R_{\mu\nu}$ is the Ricci tensor[1], a contraction of the Riemann curvature tensor $R^{\beta}{}_{\mu\nu\rho}$. This tensor[2] encodes the curvature of spacetime, as it is essentially a measurement of the amount with which covariant derivatives don't commute. When they do, it means that they have collapsed to regular derivatives, and thus the Levi-civita connection[3] must have vanished. This is only possible if the metric is flat $\eta_{\mu\nu}$. The Riemann curvature tensor is exclusively

[1] The Ricci Tensor is given by:

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta}R^{\alpha\mu\beta\nu}$$

[2] The Riemann Tensor is given by:

$$R^{\beta}{}_{\mu\nu\rho} = \Gamma^{\beta}{}_{\mu\nu,\rho} - \Gamma^{\beta}{}_{\mu\rho,\nu} - \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\beta}{}_{\alpha\nu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\rho}.$$

[3] The Levi-Civita Connection is given by:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\Big\{g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}\Big\}$$

made up of the metric, its first, second derivatives and it is linear in the second derivative of the metric. In fact it is the only possible tensor of that sort. The second term on the LHS is the Ricci scalar, a further contraction of the Ricci Tensor: $R = g^{\mu\nu} R_{\mu\nu}$. The final term on the LHS is just a proportional constant factor of the metric, where Λ is called the cosmological constant. It can be measured and has a low known upper bound. It is in part responsible for the expansion of the universe.

The Right-Hand Side () of 2.1 encodes the effect of mass and energy on the metric. $T_{\mu\nu}$ is the stress-energy tensor, dependent on the dynamics of the system. In empty space this term is zero. It is further multiplied by the Gravitational constant G.

Equation 2.1 can be recast in a form where the Ricci scalar has been eliminated,

$$\boxed{R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - T^{\alpha}{}_{\alpha} \frac{g_{\mu\nu}}{2} \right) + \Lambda g_{\mu\nu}} \tag{2.2}$$

Either equation 2.1, 2.2 with $\Lambda = 0$ admit wave solutions. We can see this by looking at these equations in the weak field approximation. Namely, we take the metric to be a Minkowskian background and a small perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.3}$$

with $h_{\mu\nu}$ small. Note that if we restrict ourselves to first order in $h_{\mu\nu}$ then all raising and lowering of indices has to be done with $\eta_{\mu\nu}$, or else we increase the order of the term by 1. Now since the LHS of 2.1 is made up of the Ricci tensor and scalar, let us see how these behave in the weak approximation. Both are in fact made up of Levi-Civita connections, which to first order in $h_{\mu\nu}$ is given by:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \eta^{\alpha\rho} \left\{ h_{\rho\nu,\mu} + h_{\rho\mu,\nu} - h_{\mu\nu,\rho} \right\} + \mathcal{O}\left(h^2\right) \quad . \tag{2.4}$$

Plugging in to the definition of the Ricci tensor we see that the terms with products of the connection vanish, and we are left with the derivative terms:

$$R_{\mu\nu} = \frac{\eta^{\alpha\rho}}{2}\left[h_{\rho\alpha,\mu\nu} + h_{\mu\nu,\rho\alpha} - h_{\mu\alpha,\rho\nu} - h_{\nu\rho,\mu\alpha}\right] + \mathcal{O}(h^2) = R^{(1)}{}_{\mu\nu} + \mathcal{O}(h^2) \tag{2.5}$$

The linearized Ricci scalar is then just $R^{(1)} = \eta^{\mu\nu}R^{(1)}{}_{\mu\nu}$. Regardless of the weak approximation, 2.1 has some gauge freedom. This means that if we solve eq 2.1, it won't be the only possible solution, and in fact we can generate the others by changing coordinated in such a way that the equation isn't effected. To fix this ambiguity we choose a coordinate system ('gauge'), by imposing the harmonic coordinate conditions:

$$g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta} = 0 \tag{2.6}$$

The harmonic coordinate conditions demand the vanishing of the Levi-Civita connection. They have a simplified form in the weak approximation, which we can acces by plugging 2.4 into 2.6, giving:

$$\begin{aligned}
(\eta^{\alpha\beta} + h^{\alpha\beta})\frac{1}{2}(\eta^{\mu\rho} + h^{\mu\rho})\left[h_{\alpha\rho,\beta} + h_{\beta\rho,\alpha} - h_{\alpha\beta,\rho}\right] &= 0 \\
\eta^{\mu\rho}\eta^{\alpha\beta}\left[2h_{\alpha\rho,\beta} - h_{\alpha\beta,\rho}\right] + \mathcal{O}(h^2) &= 0 \\
\eta^{\alpha\beta}h_{\alpha\rho,\beta} &= \frac{1}{2}h_{\alpha\beta,\rho}\eta^{\alpha\beta} + \mathcal{O}(h^2).
\end{aligned}$$

The last equation to first order is called the de Donder gauge and is often written:

$$h_{\mu\nu}{}^{,\mu} = \frac{1}{2}h^{\alpha}{}_{\alpha,\nu}$$

In de Donder gauge we can simplify the terms in 2.5 to:

$$\eta^{\alpha\beta}h_{\alpha\mu,\nu\beta} \approx \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta,\mu\nu},$$

and

$$\eta^{\alpha\beta}h_{\beta\nu,\mu\alpha} \approx \frac{1}{2}h_{\alpha\beta,\mu\nu}.$$

With these two relations the expression for the linearized Ricci tensor 2.5 simplifies to

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}h_{\mu\nu,\alpha\beta} = \frac{1}{2}\Box_{SR}h_{\mu\nu}. \tag{2.7}$$

Where we have defined the Special Relativity () D'Alembertian as: $\Box_{SR} = \eta^{\mu\nu}\partial_\mu\partial_\nu$ We can plug this into 2.2, with $\Lambda = 0$, and up to first order in $h_{\mu\nu}$ we get the linearized Einstein field equations for a system of harmonic coordinates:

$$\Box_{SR} h_{\mu\nu} = -16\pi G \overbrace{(T_{\mu\nu} - \frac{\eta_{\mu\nu}}{2} T^{\alpha}{}_{\alpha})}^{S_{\mu\nu}} \tag{2.8}$$

$$h^{\alpha}{}_{\mu,\alpha} = \frac{1}{2} h^{\alpha}{}_{\alpha,\mu}. \tag{2.9}$$

Where raising and lowering indices has been done with the Minkowski metric. The tensor $S_{\mu\nu}$ encodes the behavior of the source of gravitional waves. One could also plug 2.7 into 2.1, with $\Lambda = 0$, and if we change to the trace reversed perturbation: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^{\alpha}{}_{\alpha}\eta_{\mu\nu}$, we get similar and simpler equations at the cost of a more complex perturbation: [4] [5]

$$\Box_{SR} \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{2.10}$$

$$\bar{h}_{\mu\nu}{}^{,\nu} = 0. \tag{2.11}$$

In this form we can very easily recover the conservation equation for the stress-energy tensor. We take the divergence of 2.10 and use 2.11 to get:

$$T_{\mu\nu}{}^{,\nu} = 0. \tag{2.12}$$

Let us look at what sorts of solutions come out of these equations.

2.2 Homogenous solutions

The simplest first step is to consider the homogeous solution, as all solutions will involve these terms. Setting $T_{\mu\nu} = 0$ or $S_{\mu\nu} = 0$ yields an easily recognisable wave equation.

$$\Box_{SR} h_{\mu\nu} = 0 \tag{2.13}$$

[4] Note that the inverse change of variables is just: $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}^{\alpha}{}_{\alpha}\eta_{\mu\nu}$.

[5] We eliminate the trace of the stress-energy tensor by using: $R = 8\pi G T^{\alpha}{}_{\alpha}$. We can write $R^{(1)} = -\frac{1}{2}\Box_{SR}\bar{h}^{\alpha}{}_{\alpha}$, in de Donder gauge, which is precisely the extra term dropping out of $\Box_{SR} h_{\mu\nu}$ when we express it in terms of $\bar{h}_{\mu\nu}$

The de Donder gauge (2.9) and the remaining gauge freedom[6] restricts the possible forms of this solution to having only helicity ± 2 physically significant components (see Weinberg (1972)). Let us look at its generic form. The metric $h_{\mu\nu}$ ought to be real-valued, thus we seek real solutions of the form

[6] from changes in coordinates such as $x^\mu \to x^\mu + \xi^\mu$ with $\Box_{SR}\xi^\mu = 0$

Weinberg, Steven. 1972. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: Wiley.

$$h_{\mu\nu} = \varepsilon_{\mu\nu} e^{\mathring{\imath} k\cdot x} + {\varepsilon^*}_{\mu\nu} e^{-\mathring{\imath} k\cdot x},$$

where $\varepsilon_{\mu\nu}$ is the polarization tensor and k is the wave vector. The polarization tensor is a symmetric rank-2 tensor, since $h_{\mu\nu}$ is. Additionally we define:

$$k \cdot x \equiv \eta_{\mu\nu} k^\mu k^\nu = k_\mu x^\mu.$$

Substituting into $\Box_{SR} h_{\mu\nu} = 0$ gives $k_\mu k^\mu \equiv k^2 = 0$[7] [8]. From 2.9 we have

[7] assuming non-zero perturbation $h_{\mu\nu}$ of course

[8] this is saying that the wavevector for the wave is null, thus that the wave propagates at the speed of light

$$\varepsilon^\mu{}_\nu k_\mu = \frac{1}{2}\varepsilon^\alpha{}_\alpha k_\nu. \tag{2.14}$$

As said at the beginning of this subsection, we still have some remaining gauge freedom, which we now fix, choosing the following coordinate change $x^\mu \to x^\mu + \zeta^\mu$ where:

$$\zeta^\mu = \mathring{\imath} A^\mu e^{\mathring{\imath} k\cdot x} = -\mathring{\imath} A^{*\mu} e^{-\mathring{\imath} k\cdot x}$$

Imposing this change yields the following modified perturbation:

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \zeta_\mu}{\partial x^\nu} - \frac{\partial \zeta_\nu}{\partial x^\mu} = \varepsilon'_{\mu\nu} e^{\mathring{\imath} k\cdot x} + {\varepsilon'^*}_{\mu\nu} e^{-\mathring{\imath} k\cdot x}.$$

with

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu A_\nu + k_\nu A_\mu \tag{2.15}$$

Equations 2.14 and 2.15 reduce the free components in the polarization tensor to just two. Additionally these equations can conspire to yield a traceless polarisation tensor $\varepsilon^\alpha{}_\alpha = 0$, with $\varepsilon_{0\mu} = 0$ (see Carroll (2019)). This then extends to the metric perturbation, which when imposed to be traceless, becomes equal to its trace-reversed counterpart. This is the so-called transverse traceless gauge:

Carroll, Sean M. 2019. "Spacetime and Geometry: An Introduction to General Relativity." *Higher Education from Cambridge University Press.* https://www.cambridge.org/highereducation/books/spacetime-and-geometry/38ED-ABF9E2BADCE6FBCF2B22DC12BFFE; Cambridge University Press. https://doi.org/10.1017/9781108770385.

$$h_{0\mu} = 0, \quad h^{\alpha}{}_{\alpha} = 0, \quad h_{\mu\nu}{}^{,\nu} = 0.$$

The metric perturbation in this gauge is written as: $h^{TT}{}_{ij}$.

2.3 Inhomogenous solutions

With the homogenous part of 2.8 accounted for, we can now look at the inhomogenous part. The solution in the presence of a source term of 2.8 will be heavily inspired by the analogous problem in electromagnetism. If we define the following retarded Green's function

$$\mathcal{G}_{\text{ret}}(x^\mu - x'^\mu) = -2\pi\, \delta^4((x^\mu - x'^\mu)^2)\Theta(x^0 > x'^0),$$

which satisfies,

$$\Box_{SR}\mathcal{G}_{\text{ret}}(x^\mu - x'^\mu) = \delta^4(x^\mu - x'^\mu).$$

Then, the solution to

$$\Box_{SR} h_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{2.16}$$

is given by:

$$h_{\mu\nu}(x) = 8G \int \mathrm{d}^4x' \; \mathcal{G}_{\text{ret}}(x^\mu - x'^\mu) S_{\mu\nu}(x'), \tag{2.17}$$

since when we plug in 2.17 into 2.16

$$\Box_x h_{\mu\nu} = 8G \int \mathrm{d}^4x' \; ((\underbrace{\Box_x \mathcal{G}_{\text{ret}}(x^\mu - x'^\mu)}_{=-2\pi\,\delta(x^\mu - x'^\mu)})S_{\mu\nu}(x') = -16\pi G S_{\mu\nu}(x).$$

One gets the parallel solution to the trace reversed equation 2.10 by swapping S with T and all h with $\bar{h}$

We can perform the $x'^0 = t'$ integration in 2.17, with the delta function, setting $t' = t - |x^\mu - x'^\mu| = t_r$ the retarded time, i.e:

$$h_{\mu\nu}(x) = 8G \int \mathrm{d}^3\mathbf{y}\, S_{\mu\nu}(t, \mathbf{y})\, \mathrm{d}t' \, \frac{\delta(t' - (t - |\mathbf{x} - \mathbf{y}|))}{2|\mathbf{x} - \mathbf{y}|}$$

$$h_{\mu\nu}(\mathbf{x},t) = 4G\int \mathrm{d}^3\mathbf{y}\,\frac{S_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \tag{2.18}$$

We can interpret the solution at $(\mathbf{x},t)$ above, as the perturbation due to the sumed up contributions of all the sources at the point $(\mathbf{x}-\mathbf{y},t_r)$ on the past light cone. Put differently this will be the gravitational radiation produced by the source $S_{\mu\nu}$. Additionally, the form of the time argument of the source tensor, imposed by the definition of the Green's function, shows that the radiation propagates with velocity $=1=c$. This retarded solution satisfies the harmonic coordinate condition of 2.8, as required. Indeed, we have

$$T_{\mu\nu;\mu} = 0 \;\Rightarrow\; T_{\mu\nu,\mu} + \underbrace{\Gamma\,\Gamma}_{\text{non}-\text{linear}} = 0 \;\Rightarrow\; T_{\mu\nu,\mu} = 0,$$

ignoring non-linearities. Then,

$$S^{\mu\nu}{}_{,\mu} = \partial_\mu\left(T_{\mu\nu} - \frac{\eta^{\mu\nu}}{2}T^\alpha{}_\alpha\right) = -\frac{\eta^{\mu\nu}}{2}T^\alpha{}_{\alpha,\mu}.$$

Also

$$S^\mu{}_\mu = T^\mu{}_\mu - \frac{\delta^\mu_\mu}{2}T^\alpha{}_\alpha = -T^\alpha{}_\alpha.$$

Thus

$$\begin{aligned} S^{\mu\nu}{}_{,\mu} &= \frac{1}{2}S^\alpha{}_{\alpha,\mu}\eta^{\mu\nu} \\ S^\mu{}_{\nu,\mu} &= \frac{1}{2}S^\alpha{}_{\alpha,\nu}. \end{aligned}$$

Then

$$
\begin{aligned}
h^{\mu}{}_{\nu,\mu} &= 8G\frac{\partial}{\partial x^{\mu}}\int \mathrm{d}^4x'\, \mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})S^{\mu}{}_{\nu}(x') \\
&= 8G\int \mathrm{d}^4x'\ \frac{\partial\mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})}{\partial x^{\mu}}S^{\mu}{}_{\nu}(x') \\
&= -8G\int \mathrm{d}^4x'\ \frac{\partial\mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})}{\partial x'^{\mu}}S^{\mu}{}_{\nu}(x') \\
&= \underbrace{8G\mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})\delta^{\mu}_{\nu}(x')}_{=0\ \text{for}\ x=x'} + 8G\int \mathrm{d}^4x'\ \mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})\frac{\partial S^{\mu}{}_{\nu}(x')}{\partial x'^{\mu}} \\
&= 8G\int \mathrm{d}^4x'\ \mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})\frac{1}{2}\frac{\partial S^{\alpha}{}_{\alpha}(x')}{\partial x'^{\mu}} \\
&= \quad \ldots \quad \text{repeat in reverse} \\
&= \frac{\partial}{\partial x^{\mu}}\left\{8G\int \mathrm{d}^4x'\ \mathcal{G}_{\mathrm{ret}}(x^{\mu}-x'^{\mu})\frac{1}{2}S^{\alpha}{}_{\alpha}(x')\right\} \\
&= \frac{1}{2}h^{\alpha}{}_{\alpha,\mu}.\checkmark
\end{aligned}
$$

2.4 Gravitational Wave Sources

Now that we have the general form of the solutions to the linearized Einstein equations, we can proceed to the analysis of the sources of gravitational waves. The first step is to analyse the equations in the frequency domain. We will use the following notation:

$$
\begin{aligned}
\mathcal{F}_t\left[\phi\right](\omega,\mathbf{x}) &= \int \mathrm{d}t\,\phi(t,\mathbf{x})\mathrm{e}^{i\omega t} \\
\mathcal{F}^{-1}_{\omega}\left[\phi\right](t,\mathbf{x}) &= \int \frac{\mathrm{d}\omega}{2\pi}\mathcal{F}_t\left[\phi\right](\omega,\mathbf{x})\mathrm{e}^{-i\omega t}
\end{aligned}
$$

Let us look at the trace reversed solution, as the conservation equation for

$$
\begin{aligned}
\mathcal{F}_t\left[h_{\mu\nu}\right](\omega,\mathbf{x}) &= \int \mathrm{d}t\, h_{\mu\nu}(t,\mathbf{x})\mathrm{e}^{i\omega t} \\
&= 4G\int \mathrm{d}^3\mathbf{y}\ \mathrm{d}t\, \frac{S_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{e}^{i\omega t} \\
&= 4G\int \mathrm{d}^3\mathbf{y}\ \mathrm{d}t_r\, \frac{S_{\mu\nu}(t_r,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{e}^{i\omega t_r}\mathrm{e}^{i\omega|\mathbf{x}-\mathbf{y}|} \\
&= 4G\int \mathrm{d}^3\mathbf{y}\, \frac{\mathcal{F}_{t_r}\left[S_{\mu\nu}\right](\omega,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{e}^{i\omega|\mathbf{x}-\mathbf{y}|}
\end{aligned}
\tag{2.19}
$$

We can now apply various approximations to this form of the perturbation. The first is to consider that we look at the radiation only in the so called *wave zone*, at a distance $r = |\mathbf{x}|$ much larger than the dimensions of the source $R = |\mathbf{y}|_{\max}$. Additionally we assume that $r \gg \frac{1}{\omega}$, i.e long wavelengths don't dominate. Finally we assume that $r \gg \omega R^2$, i.e. the ratio of R to the wavelength is not comparable to the ratio of r to R. Using this approximation we can write:

$$
\begin{aligned}
|\mathbf{x}-\mathbf{y}| &= r\left(1-2\hat{\mathbf{x}}\cdot\mathbf{y}+\frac{\mathbf{x}'^2}{r^2}\right)^{1/2} \\
|\mathbf{x}-\mathbf{y}| &\approx r\left(1-\frac{\hat{\mathbf{x}}\cdot\mathbf{y}}{r}\right) \quad \text{with: } \hat{\mathbf{x}} = \frac{\mathbf{x}}{r}.
\end{aligned}
$$

If we additionally further separate the scales in the following way[9]

$$
r \gg [\frac{1}{\omega}, \omega R^2] \gg R,
$$

Then 2.19 becomes much simpler[10]:

$$
\mathcal{F}_t\left[h_{\mu\nu}\right](\omega,\mathbf{x}) = 4G\frac{\mathrm{e}^{i\omega R}}{R}\int \mathrm{d}^3\mathbf{y}\, \mathcal{F}_t\left[S_{\mu\nu}\right](\omega,\mathbf{y}) \tag{2.20}
$$

Now let us look at the fourier transform of the source term. By definition we have

[9] we just add the condition that $\frac{1}{\omega} \gg R$, that is the source radius is much smaller than the wavelength

[10] the approximations all conspire to be able to neglect the $\mathbf{y}$ dependence of $\mathrm{e}^{i\omega|\mathbf{x}-\mathbf{y}|}$ in the integral

$$\mathscr{F}_t\left[S_{\mu\nu}\right](\omega,\mathbf{y}) = \mathscr{F}_t\left[T_{\mu\nu}\right](\omega,\mathbf{y}) + \tfrac{1}{2}\eta_{\mu\nu}\mathscr{F}_t\left[T^{\alpha}{}_{\alpha}\right](\omega,\mathbf{y})$$

Thus the term to analyse is actually $\mathscr{F}_t\left[T_{\mu\nu}\right]$. We can use the conservation equation 2.12 in fourrier t-space to write:

$$-\mathscr{F}_t\left[T_{i\mu}\right]{}^{,i} = \mathring{\imath}\omega\mathscr{F}_t\left[T_{0\mu}\right] \tag{2.21}$$

This equation becomes algebraic when we further fourrier transorm in $\mathbf{x}$-space:

$$\hat{T}_{\mu\nu}(k^{\alpha}) = \hat{T}_{\mu\nu}(\omega,\mathbf{k}) = \int \mathrm{d}^3\mathbf{y}\,\mathscr{F}_t\left[T_{\mu\nu}\right](\omega,\mathbf{y})\mathrm{e}^{\mathring{\imath}\mathbf{k}\cdot\mathbf{y}}$$

Then the conservation equation becomes:

$$k^{\mu}T_{\mu\nu}(\omega,\mathbf{k}) = 0$$

These four equations enable us to just care about the purely spacelike components of $T_{\mu\nu}$. Let us apply 2.21 to itself to obtain:

$$\mathscr{F}_t\left[T_{ij}\right]{}^{,ij} = -\omega^2\mathscr{F}_t\left[T_{00}\right]$$

which when multiplied by $x_m x_n$ and integrated over $\mathbf{x}$ gives[11]:

$$\int \mathrm{d}\mathbf{x}\,\mathscr{F}_t\left[T_{mn}\right](\omega,\mathbf{x}) = -\frac{\omega}{2}\int \mathrm{d}\mathbf{x}\,x_m x_n\mathscr{F}_t\left[T_{00}\right](\omega,\mathbf{x})$$

Notice the last integral is in fact the fourrier transform of (a third of) the quadrupole moment tensor of the energy denisty[12]. We call it $\hat{q}_{mn}(\omega)$ and we can finally rewrite 2.20 as:

$$\mathscr{F}_t\left[h_{\mu\nu}\right](\omega,\mathbf{x}) = -\frac{2G\omega^2}{3}\frac{\mathrm{e}^{\mathring{\imath}\omega R}}{R}\left(\hat{q}_{mn}(\omega) + \tfrac{1}{2}\eta_{\mu\nu}\hat{q}^{n}{}_{n}(\omega)\right)$$

Going back t-space we have:

[11] two integrations by parts cancels the $x_m x_n$ term in the LHS, since boundary terms are 0 (the source is finite) we have

$$\int \mathrm{d}x\,x_m x_n\mathscr{F}_t\left[T_{ij}\right]{}^{,ij} = \int \mathrm{d}x\,x_m x_n{}^{,ij}\mathscr{F}_t\left[T_{ij}\right]$$

The hessian of $x_m x_n$ is $(\delta^i_m + \delta^i_n)(\delta^j_n + \delta^j_m)$, but since $T_{\mu\nu}$ is symmetric the integral is

$$\int \mathrm{d}\mathbf{y}\,2\mathscr{F}_t\left[T_{mn}\right](\omega,\mathbf{y})$$

[12]

$$q_{mn} = 3\int x_m x_n T_{00}(\omega,\mathbf{x})$$

$$
\begin{aligned}
h_{\mu\nu}(t,\mathbf{x}) &= -\frac{G}{3\pi R}\int \mathrm{d}\omega\,\mathrm{e}^{-\imath\omega(t-R)}\omega^2(\hat{q}_{mn}(\omega)+\tfrac{1}{2}\eta_{\mu\nu}\hat{q}^{n}{}_{n}(\omega)) \\
&= \frac{G}{3\pi R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \mathrm{d}\omega\,\mathrm{e}^{-\imath\omega(t_r)}(\hat{q}_{mn}(\omega)+\tfrac{1}{2}\eta_{\mu\nu}\hat{q}^{n}{}_{n}(\omega)) \\
&= \frac{2G}{3R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q_{mn}(t_r)+\tfrac{1}{2}\eta_{\mu\nu}q^{n}{}_{n}(t_r))
\end{aligned}
$$

This equation has a very nice physical interpretation. The gravitational wave produced by a non-relativistic source is proportional to the second derivative of the quadrupole moment of the its energy denisty at the time where the past light cone of the observer intersects the source (t_r). The nature of gravitational radiation is in stark contrast to the leading electromagnetic contribution which is due the the change in the *dipole* moment of the charge density. The change of the dipole moment can be attributed to the change in center of charge (for Electromagnetism ()), or mass (for GR), and while a center of charge is free to move around, the center of mass (of an isolated) is fixed by the conservation of momentum. The quadrupole moment, on the other hand, is sensitive to the shape of the source, which a gravitional system can modify. Finally the quadrupole radiation is subleading when compared to dipole radiation. Thus on top of the much smaller coupling constant, gravitional radiation is also weakened by this fact, and thus is usually orders of magnitude weaker than electromagnetic radiation.

Thus any object that is modifying its shape is a source of gravitional waves. All orbiting systems therefore are sources of Gravitational Wave ()s. However as said just above, only very important ‘changes in shape’ have a chance to be detectable. These phenomena are what we will explore next.

2.4.1 Compact binaries

How could one construct a very powerful source of GWs? One could take two very massive objects, (such that T_{00} is large) and make them orbit each other. At that point one could hope to detect this orbit if the objects are massive enough and orbiting close enough for the ‘change in shape’ of the system to sizeable.

For these very massive objects to be close enough for a small orbit, they have to be very compact. Assuming that is the case, a funny thing happens, as these objects orbit each other, they emit GWs, and in doing so they lose energy[13]. Thus they slow down, and their orbit shrinks. This continues until the orbit is so small that the objects merge into a single object. Of course this is a very important change of shape and thus we have a constructed (if not all on purpose) a very powerful source of GWs. Such objects are called compact binaries.

[13] of course this happens in every orbiting system just on a time scale that is neglible. Only systems which are massive enough to produce large amounts of radiation actually lose enough energy for it to matter

2.4.2 BH Binaries

Taking the process described above to the limit one could imagine asking for the most dense objects possible to orbit each other. In GR this object is called a black hole (BH). It is a possible solution to the full fat Einstein Field Equations () (2.1), where we consider a static and isotropic universe, with point like mass at its center. Then the solutions to the equations 2.1 have a unique form (Birkhoff and Langer (1923)), called the Schwarzschild solution. The solution is given by:

Birkhoff, George David, and Rudolph Ernest Langer. 1923. *Relativity and Modern Physics.* Cambridge: Harvard University Press; [etc., etc.].

2.4.3 NS Binaries

We can also not go as far as having our compact object be a BH, but can also consider NSs. These object are notheless extremely dense (not infinitely so), and are essentially like big atomic nuclei. Not much is known about them and in fact GW astronomy is could be

2.4.4 Exotic Sources

Cosmic strings, supernovae, and other exotic sources of GWs are also possible.

3 Gravitational Wave Detection

Gravitiional waves were first theorised by Einstein, accompanying his theory of general relativity. However it was not even clear whether this was an artefact of the theory or something real. Of course as we have seen, GWs are predicted to be very weak, and thus there initiall was little hope of ever detecting them. As evidence mounted that GR was indeed a very good theory of gravity, people started to investigate more seriously whether one could detect GWs. The first detectors were resonant antennas,

thowever these have to date, not been sucessful in detecting GWs. The more modern laser based detectors were the final piece of the puzzle. With extremely high precision (the modern measurement accuracy at LIGO is equivalent to measuring the distance to alpha centauri to the width of a human hair), these detectors are able to detect GWs. The first detection was made in 2015, marking the beginning of a new era in astronomy, and maybe physics.

3.1 Laser interferometers

3.1.1 LIGO

3.1.2 Laser Interferometer Space Antenna ()

3.2 matched filtering

The signals measured by the laser interferometers such as LIGO, are a measurement of strain over time. This waveform essentially measures the deformation of spacetime when a GW passes through a detector. While some GW signals coming out of laser interferometers, are clearly identified as such, mainly due to their extraordinary power output, smaller systems and weaker signals are harder to identify. In fact most of the detections have happened under the noise floor of the detectors. How is this possible? Matched filtering and waveform generation. In this chapter we will explain the matched filtering approach and then introduce the methods that are used to generate the waveforms to be matched against. Waveforms for spiraling binaries are the principle tool for comparing data from Gravitational wave detectors to the theory surrounding them. They are the bridge between theory and experiment.

As we have seen, the matched filtering approach is contingent on having a waveform that corresponds to the physical process that is emitting the signal. Usually these processes can be parametrized by a set of parameters, such as in the case of a compact binary, the mass ratio of the binary, the orbit eccentricity, the possible spin etc. The filter waveform must then of course depend on these parameters, be it heuristically or physically (i.e., from first principles).

3.3 Pulsar Timing Arrays

4 Waveform Generation

The matched filtering approach introduced in the previous section is a powerful tool for detecting signals in noise. Additionally, if the matching signal has some physical content, that same content can be expected to describe the emitting object with resonable error. Ideally the matching signal is entirely constructed from first priciples, thus when a match is obtained the physical inputs to the model

The matched filtering approach introduced in the previous section motivates tools for generating waveforms. These can be varied in physical content, but also in practicality. In all cases they need to be able to cover the set of parameters (the parameter space) that one is interested in exploring. Clearly if the waveform has been generated purely heuristically, then the physical content of the detection is close to zero. One can only say one has detected a phenomenon that produces this waveform, but not much more. However, if the input parameters are physically meaningful, like the mass ratio of a binary compact inspiraling system, then the waveform constructed purely from this input, if matched to the signal, tells us that a compact binary inspiraling system with this mass ratio has been detected. However, the waveform generation must be able to cover the whole parameter space (in this case all feasible mass ratios) to be useful, as one does *not* know the parameters of the object one is looking for. Ideally this waveform generation can be done in a way that is computationally efficient, as the number of parameters to explore can be large. In the end, currently, waveform generation is done in a hybrid way, where some waveforms at some parameter space points are generated from first principles, and the rest of the parameter space waveforms are interpolated.

In this chapter we will look at the physically motivated waveform generation tools currently being developed in research. More specifically we are going to explore techniques for compact binary waveforms, as these are currently the objects to which LIGO and others are most sensitive to (see Section 3.1.1) and by far the most common emitters of high intensity GWs. As discussed in Section 2.4.1, orbiting and GW emitting compact objects will necessarily inspiral and merge at somepoint.

For such inspiraling binaries, the waveform has three distinct parts, corresponding to distinct phases of a binary merger: a first inspiral phase, a merger phase where a remnant compact body is produced as a result of the coalescence of the two objects, and a postmerger, or ringdown phase where the remnant still emits gravitational radiation while settling to its new stable configuration. Each phase has specific characteristics in the dynamics of the objects and thus correspondingly different frameworks are aimed at specific portions of the waveforms. Even within one phase, different regimes and thus frameworks exist, additionally dependent to the type/initial state of the two orbiting bodies.

5 Inspiral phase

5.1 PN

5.2 PM

5.3 Gravitational Self-Force ()

5.4 EOB

6 Merger and Ringdown phase

6.1 EOB

7 Scattering amplitudes and Gravitional waves

Throughout we use relativistically natural units, i.e. we do *not* set $\hbar = 1$. In dimensional analysis we can therefore see that $c = 1$ means that $[L][T]^{-1} = 1 \implies [L] = [T]$,

$$E = mc^2 \implies [E] = [m] = [M]$$

and

$$E = \hbar\omega \implies [M] = [\hbar][T]^{-1} \implies [\hbar] = [T][M]$$

. Thus momentum p is in units of $[p] = [M]$ mass and Wavenumber $[\bar{p}] = [\frac{p}{\hbar}] = [T]^{-1}$ is in units of inverse time.

The setup of the Kosower Maybee and O'Connell () framework (Kosower, Maybee, and O'Connell (2019)) is very general, and is aimed at taking the classical limit of a scattering event in an unspecified theory. We will later on apply it to Scalar Quantum Electrodynamics () and gravity. Imagine we want to scatter two particles into each other, obtaining two particles out. In QFT the framework that formalizes scattering of definite states is called the Lehmann Symanzik and Zimmermann ()

Kosower, David A., Ben Maybee, and Donal O'Connell. 2019. "Amplitudes, Observables, and Classical Scattering." *Journal of High Energy Physics* 2019 (2): 137. https://doi.org/10.1007/JHEP02(2019)137.

in state into an at least 2 particle out state. This language

Thus the in state is given by

$$|\psi\rangle_{\text{in}} = \int \mathrm{d}\Phi_2(p_1, p_2)\, \phi_1(p_1)\phi_2(p_2)\mathrm{e}^{\mathring{\imath} b_\mu {p_1}^\mu/\hbar}|p_1, p_2\rangle_{\text{in}} \tag{7.1}$$

The $\mathrm{e}^{\mathring{\imath} b_\mu {p_1}^\mu/\hbar}$ factor encodes the fact that we have translated the wavepacket of particle 1 relative to particle 2 by the impact parameter b .[14] We take it to be perpendicular to the initial

[14] This means that the classical value of momentum $m_i \breve{u}_i^\mu$ is reached in the $\xi \to 0$ limit

momenta p_1, p_2.

The KMOC framework concerns itself with the change of an observable during a scattering event. For such an observable O, its change can be simply obtained by evaluating the difference of the expectation value of the corresponding Hermitian operator, $\mathbb{O}$, between in and out states

$$\Delta O = \langle \text{out}|\mathbb{O}|\text{out}\rangle - \langle \text{in}|\mathbb{O}|\text{in}\rangle$$

In quantum mechanics, the out states are related to the in states by the time evolution operator, i.e. the S-matrix: $|\text{out}\rangle = S|\text{in}\rangle$ and we can write

$$\begin{aligned}
\Delta O &= \langle \text{in}|S^\dagger \mathbb{O} S|\text{in}\rangle - \langle \text{in}|\mathbb{O}|\text{in}\rangle \\
&\overset{S^\dagger S=1}{=} \langle \text{in}|S^\dagger [\mathbb{O}, S]|\text{in}\rangle \\
&\overset{S=1+\mathring{i}qat}{=} \langle \text{in}|[\mathbb{O}, 1+\mathring{i}qat]|\text{in}\rangle - \langle \text{in}|\mathring{i}qat^\dagger [\mathbb{O}, 1+\mathring{i}qat]|\text{in}\rangle \\
&= \langle \text{in}|\mathring{i}[\mathbb{O}, qat]|\text{in}\rangle + \langle \text{in}|qat^\dagger [\mathbb{O}, qat]|\text{in}\rangle \\
&= \Delta O_\text{v} + \Delta O_\text{r}
\end{aligned} \tag{7.2}$$

If we put in the definition of our in state (7.1) we have:

$$\Delta O = \int \mathrm{d}\Phi_4(p_1, p_2, p_1', p_2')\, \phi_1(p_1)\phi_2(p_2)\phi_1^\dagger(p_1')\phi_2^\dagger(p_2')\, \mathrm{e}^{\mathring{i}b_\mu \frac{p_1{}^\mu - p_1'{}^\mu}{\hbar}} \{\mathcal{I}_\text{v}(O) - \mathcal{I}_\text{r}(O)\}$$

Where

$$\begin{aligned}
\mathcal{I}_\text{v}(O) &= \langle p_1' p_2'|\mathring{i}[\mathbb{O}, qat]|p_1 p_2\rangle \\
\mathcal{I}_\text{r}(O) &= \langle p_1' p_2'|qat^\dagger [\mathbb{O}, qat]|p_1 p_2\rangle
\end{aligned}$$

> NB: the notation is slightly different in the (**bernScalarqEDToy2021?**) paper

bernScalarqEDToy2021

Let us first look at the virtual integrand $\mathcal{I}_\text{v}$:

$$
\begin{aligned}
\mathcal{I}_{\mathrm{v}}(O) &= \langle p_1' p_2' | \mathring{i}[\mathbb{O}, qat] | p_1 p_2 \rangle \\
&= \langle p_1' p_2' | \mathring{i}\mathbb{O} qat | p_1 p_2 \rangle - \langle p_1' p_2' | \mathring{i} qat \mathbb{O} | p_1 p_2 \rangle \\
&= \mathring{i} O_{\mathrm{in}'} \langle p_1' p_2' | qat | p_1 p_2 \rangle - \mathring{i} O_{\mathrm{in}} \langle p_1' p_2' | qat | p_1 p_2 \rangle \\
&= \mathring{i} \underset{p-p'}{\Delta O} \ \langle p_1' p_2' | qat | p_1 p_2 \rangle \\
&= \mathring{i} \underset{p-p'}{\Delta O} \ \tilde{\delta}^4(p_1 + p_2 - p_1' - p_2') \mathcal{A}(p_1, p_2 \to p_1', p_2')
\end{aligned}
$$

Note that the amplitude is from in states to in states! Now for the real integrand $\mathcal{I}_{\mathrm{r}}$ we insert a complete set of states :

$$
\begin{aligned}
\mathcal{I}_{\mathrm{r}}(O) &= \langle p_1' p_2' | qat^{\dagger} [\mathbb{O}, qat] | p_1 p_2 \rangle \\
&= \sum_X \int \mathrm{d}\Phi_{2+|X|}(r_1, r_2, X) \, \langle p_1' p_2' | qat^{\dagger} | r_1 r_2 X \rangle \langle r_1 r_2 X | [\mathbb{O}, qat] | p_1 p_2 \rangle \\
&= \sum_X \int \mathrm{d}\Phi_{2+|X|}(r_1, r_2, X) \, \tilde{\delta}^4(p_1 + p_2 - r_1 - r_2 - r_X) \, \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X) \\
&\quad \underset{rX-p}{\Delta O} \ \tilde{\delta}^4(p_1' + p_2' - r_1 - r_2 - r_X) \, \mathcal{A}^*(p_1', p_2' \to r_1, r_2, r_X)
\end{aligned}
$$

For both integrands we can preform some variable changes and eliminate certain delta functions. We introduce momentum shifts $q_i = p_i' - p_i$ and then integrate over q_2 finally relabelling $q_1 \to q$. Thus we have

$$
\begin{aligned}
\Delta O_{\mathrm{v}} = \int \mathrm{d}\Phi_2(p_1, p_2) \ \tilde{\mathrm{d}}^4 q \ \tilde{\delta}(2p_1 \cdot q + q^2) \Theta(p_1^0 + q^0) \, \tilde{\delta}(2p_2 \cdot q - q^2) \Theta(p_2^0 - q^0) \\
\times \phi_1(p_1) \phi_2(p_2) \phi_1^{\dagger}(p_1 + q) \phi_2^{\dagger}(p_2 - q) \, \mathrm{e}^{-\frac{i}{\hbar} b_\mu q^\mu} \\
\times \mathring{i} \underset{q}{\Delta O} \, \mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q)
\end{aligned} \tag{7.3}
$$

$$
\begin{aligned}
\Delta O_{\mathrm{r}} = \sum_X \int \mathrm{d}\Phi_{2+|X|}(r_1, r_2, X)\; \mathrm{d}\Phi_2(p_1, p_2)\; \tilde{\mathrm{d}}^4 q\, \tilde{\delta}(2p_1 \cdot q + q^2)\Theta(p_1^0 + q^0) \\
\times \tilde{\delta}(2p_2 \cdot q - q^2)\Theta(p_2^0 - q^0) \\
\times \phi_1(p_1)\phi_2(p_2)\phi_1^\dagger(p_1 + q)\phi_2^\dagger(p_2 - q)\, \mathrm{e}^{-\frac{i}{\hbar} b_\mu q^\mu} \\
\times \underset{rX-p}{\Delta O}\, \tilde{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - r_X) \\
\times \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X)\mathcal{A}^*(p_1 + q, p_2 - q \to r_1, r_2, r_X)
\end{aligned} \tag{7.4}
$$

8 Classical limit

Since we are concerned with classial observables we need to explore the classical limit of 7.2, i.e. the limit of $\hbar \to 0$. The first target is the wavefunctions.

8.1 Classical limit of wavefunctions

In KMOC framework we are interested in the classical limit of a scattering event. It is then important to understand the precise simplifications this limit yields.

We have multiple conditions on the wavefunctions. The first are those imposed by LSZ reduction. That is,

- Compact support momentum space wavefunction
- Peaked around one value of momenta

Classical limit of the wavefunctions should make sense, thus

- as $\hbar \to 0$ the wavefunction should approach a delta function
- The spread should not be too large as to interact between the two initial states
- The overlap between the wavefunction and its conjugate should be nearly full, since they represent the same particle classically.

Consider for example a nonrelativistic wavefunction:

$$f(\mathbf{p}) = \exp\left(-\frac{\mathbf{p}}{2\hbar m \ell_c/\ell_\omega^2}\right) \overset{\hbar=\ell_c m}{=} \exp\left(-\frac{\mathbf{p}}{2m^2\ell_c^2/\ell_\omega^2}\right)$$

This wavefunction grows sharper in the $\hbar \to 0$ limit.

The Fourier transform of $f(\mathbf{p})$ gives us the position "probability density":

$$\begin{aligned}\mathcal{F}_{\mathbf{p}}^{-1}[f](\mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{p}}{2\pi} \exp\Big(-(\frac{\mathbf{p}}{A})^2\Big) \exp\Big(-\frac{\mathring{\imath}}{\hbar}\mathbf{p}\cdot\mathbf{x}\Big) \\ &= \frac{1}{2\pi} \underbrace{\int \mathrm{d}\mathbf{p} \exp\Big(-\Big(\frac{\mathbf{p}}{A} - \frac{\mathring{\imath}\mathbf{x}A}{2\hbar}\Big)^2\Big)}_{\sqrt{\pi}A} \exp\Big(-\frac{\mathbf{x}^2 A}{4\hbar^2}\Big) \\ &= \frac{\sqrt{2}A}{2\pi} \exp\Big(-\frac{\mathbf{x}^2}{2\ell_\omega^2}\Big)\end{aligned}$$

[15] This wavefunction grows sharper in the $\ell_\omega^2 \to 0$ limit. We must then still have that $\xi = (\frac{\ell_c}{\ell_\omega})^2 \to 0$ when $\hbar \to 0$ as-well. This works if we just directly take the classical limit to be the $\xi \to 0$ limit.

[15] A absorbs the various constants, with $A = \sqrt{2}m\frac{\ell_c}{\ell_\omega}$ and $\mathbf{x}_0$

Going back to the general conditions we want $\phi_i(p_i)$ s.t:

- $\langle p_i^\mu \rangle = \int \mathrm{d}\Phi(p_i)\, p_i^\mu |\phi_i(p_i)|^2 \overset{!}{=} m_i \breve{u}_i^\mu f_{p,i}(\xi)[\hat{}1]$ with $f_{p,i}(\xi) = 1 + \mathcal{O}(\xi^{\beta'})$
- $\sigma^2(p_i)/m_i^2 = \langle (p_i - \langle p_i \rangle)^2 \rangle / m_i^2 = (\langle p_i^2 \rangle - \langle p_i \rangle^2)/m_i^2 = c_\Delta \xi^{\beta}$[16]
- $\breve{u}_i \cdot u_i = 1 + \mathcal{O}(\xi^{\beta''})$[17]

[16] This means that the classical value of momentum $m_i \breve{u}_i^\mu$ is reached in the $\xi \to 0$ limit

[17] This encodes the limit of the spread as $\xi \to 0$. $\langle = |p_i^2| = \rangle m_i^2$ is enforced by the measure $\mathrm{d}\Phi(p)$

Additionally the wavefunction should be Lorentz invariant, thus naively we would have that $\phi(p_i^\mu) = \tilde{\phi}(p_i^2)$ however the integration measure enforces an on shell condition: $m_i^2 = p_i^2$. Thus the wavefunction cannot usefully depend on p_i^2, we need to introduce at least one additional four vector parameter u. The simplest dimensionless combination of parameters it then $\frac{p\cdot u}{m}$. Of course the wavefunction must also depend on ξ and the simplest form of argument will thus be $\frac{p\cdot u}{m\xi}$ so that any p not aligned with u will be strongly suppressed in the $\xi \to 0$ limit.

We now have control over most of the conditions:

- The classical limit is well defined
- The wavefunction spread is controlled
- The arguments of the wavefunction is clear

The last requirement concerns the overlap of ϕ and $\phi^\dagger$ must be $\mathcal{O}(1)$, equivalently and more precisely:

$$\phi^\dagger(p+q) \sim \phi^\dagger(p) \implies \phi^\dagger(p+q) - \phi^\dagger(p) \ll 1 \implies q^\mu \partial_\mu \phi^\dagger(p) \ll 1$$

Making explicit the $\frac{p\cdot u}{m\xi}$ dependence: $\phi(p) = \varphi(\frac{p\cdot u}{m\xi})$ for $\varphi(x)$ a scalar function.

$$\implies \frac{q^\mu u_\mu}{m\xi}\frac{\partial \varphi^\dagger}{\partial x}(\frac{p\cdot u}{m\xi}) \ll 1$$

Thus we require that for a characteristic value of $q = q_0$ we have:

$$\frac{q_0 \cdot u}{m\xi} = \bar{q}_0 \cdot u \frac{\ell_\omega^2}{\ell_c} \ll 1 \iff \bar{q}_0 \cdot u\, \ell_\omega \ll \sqrt{\xi}$$

[18] We now want to examine the classical limit of something like [[KMOC framework#^deltaOv]]. If we consider just the integration over the initial momenta p_i and the initial wave-functions with $\hat{\delta}(2p_i \cdot q + q^2)$, the delta function will smear out to a sharply peaked function whose scale is the same order as the original wavefunctions. As ξ gets smaller, this function will turn back into a delta function
imposed on the q integration.[19] Let us examine this statement more closely:

[18] This enforces the normalisation condition $\breve{u}_i^2 = 1$ in the $\xi \to 0$ limit

[19] Here $\bar{q} = \frac{q}{\hbar}$ is the wavenumber

$$d(m,\xi,u,q) = \int \mathrm{d}\Phi(p)\, \hat{\delta}(2p\cdot q + q^2)\Theta(p^0+q^2)\varphi(\frac{p\cdot u}{m\xi})\varphi^\dagger(\frac{(p+q)\cdot u}{m\xi})$$

^dfunct

This integral must be Lorentz invariant and depends on m, ξ, u, q thus it must manifestly only depend on the following [[math/Lorentz group|Lorentz invariants]]: $u^2, q^2, u\cdot q, \xi$. One of these is not actually a variable as we will normalise $u^2 = 1$. The rest aren't fully dimensionless, and we can render them dimensionless:[^6]

$$[q^2] = [\hbar\bar{q}]^2 = [M]^2 \implies [\ell_c\sqrt{-\bar{q}^2}] = [\frac{\hbar}{m}\sqrt{-\bar{q}^2}] = \frac{[M]}{[M]} = 1$$

$$[u\cdot q] = [M] \implies [\frac{u\cdot\bar{q}}{\sqrt{-\bar{q}^2}}] = [\frac{u\cdot q}{\sqrt{-q^2}}] = \frac{[M]}{[M]} = 1$$

$$[\xi] = 1$$

If we call $\frac{1}{\sqrt{-\bar{q}^2}} = \ell_s$ a scattering length[^7] then our dimensionless ratios are :

$$\frac{\ell_c}{\ell_s} \quad \text{and} \quad \ell_s\, \bar{q}\cdot u$$

The delta function can be rewritten as:

$$\hat{\delta}(2p\cdot q+q^2) = \hat{\delta}(2\hbar m\, u\cdot\bar{q}+\hbar^2\bar{q}^2) = \frac{1}{\hbar m}\hat{\delta}(2\bar{q}\cdot u-\frac{\ell_c}{\ell_s^2}) = \frac{\ell_s}{\hbar m}\hat{\delta}(2\ell_s\,\bar{q}\cdot u-\frac{\ell_c}{\ell_s})$$

8.2 Notation

As we have now seen the phase space integrals over initial momenta, and over the wavefunctions can be readily eliminated in the classical limit, in which case the momenta p_1 and p_2 can be replaced by their classical values. To make explicit that idea we introduce the following notation:

$$\left\langle\!\!\left\langle f(p_1, p_2, \dots) \right\rangle\!\!\right\rangle := \int \mathrm{d}\Phi(p_1)\ \mathrm{d}\Phi(p_2)\, |\phi_1(p_1)|^2 |\phi_2(p_2)|^2\, f(p_1, p_2, \dots)$$

We can now rewrite 7.2:

$$\Delta O \;=\; \left\langle\!\!\left\langle \int \overbrace{\tilde{\mathrm{d}}^4 q\, \tilde{\delta}(2p_1\cdot q+q^2)\Theta(p_1^0+q^0)\tilde{\delta}(2p_1\cdot q+q^2)\Theta(p_2^0+q^0)}^{\mathrm{d}\Psi(q)}\, \mathrm{e}^{-\frac{i}{\hbar}q^\mu b_\mu}\Big(\mathcal{I}'_{\mathrm{v}}(O)+\mathcal{I}'_{\mathrm{r}}(O)\Big) \right\rangle\!\!\right\rangle$$

Where

$$\mathcal{I}'_{\mathrm{v}}(O) = \mathring{i}\,\underset{q}{\Delta}O\,\mathcal{A}(p_1, p_2 \to p_1+q, p_2-q)$$

$$\begin{aligned}\mathcal{I}'_{\mathrm{r}}(O) = \sum_X \int\ &\mathrm{d}\Phi_{2+|X|}(r_1, r_2, X)\underset{rX-p}{\Delta}O\,\tilde{\delta}^{(4)}(p_1+p_2-r_1-r_2-r_X) \\ &\times \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X)\mathcal{A}^*(p_1+q, p_2-q \to r_1, r_2, r_X)\end{aligned}$$

Additionally we want to make clear the dependence on $\hbar$ since we want to eventually take the $\hbar \to 0$ limit. We can change integration variables to $\bar{q} = \frac{q}{\hbar}$ and absorb that $\hbar$ dependence into the redefinition of the integrands:

$$\mathrm{d}\Psi(q) = \hbar^2\, \mathrm{d}\overline{\Psi}(\bar{q}) = \hbar^{\cancelto{2}{4}}\, \tilde{\mathrm{d}}^{4}\bar{q}\, \cancel{\frac{1}{\hbar}} \tilde{\delta}(2p_1 \cdot \bar{q} + \hbar\bar{q}^2)\Theta(p_1^0 + q^0) \cancel{\frac{1}{\hbar}} \tilde{\delta}(2p_1 \cdot \bar{q} + \hbar\bar{q}^2)\Theta(p_2^0 + q^0)$$

$$\begin{aligned}
\overline{\mathcal{I}'_{\mathrm{v}}}(O) &= \hbar^2 \mathcal{I}'_{\mathrm{v}}(O) \\
\overline{\mathcal{I}'_{\mathrm{r}}}(O) &= \hbar^2 \mathcal{I}'_{\mathrm{r}}(O)
\end{aligned}$$

We can finally neatly write:

$$\Delta O = \left\langle\!\!\left\langle \int \mathrm{d}\overline{\Psi}(\bar{q})\, \mathrm{e}^{-\mathring{\imath}\bar{q}^\mu b_\mu} \left(\overline{\mathcal{I}'_{\mathrm{v}}}(O) + \overline{\mathcal{I}'_{\mathrm{r}}}(O) \right) \right\rangle\!\!\right\rangle$$

Now the $\hbar$ dependence is all in the integrands (ignoring the $\hbar\bar{q}^2$ factors in the delta function).

We can now explore the integrands in different theories:

Let us use the [[KMOC framework]] for [[scalar QED]]

$$\Delta O = \left\langle\!\!\left\langle \int \mathrm{d}\overline{\Psi}(\bar{q})\, \exp\left(-\mathring{\imath}\bar{q}^\mu b_\mu \right) \left(\mathcal{I}'_{\mathrm{v}}(O) O + \overline{\mathcal{I}'_{\mathrm{r}}}(O) O \right) \right\rangle\!\!\right\rangle$$

Let us take p_μ^1 as the observable corresponding to the momentum of the first particle.

We then have:

$$\begin{aligned}
\overline{\mathcal{I}'_{\mathrm{v}}}(O) p_\mu^1 &= \hbar^2 \mathring{\imath}\, q\, \mathcal{A}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \\
\overline{\mathcal{I}'_{\mathrm{r}}}(O) p_\mu^1 &= \hbar^2 \;\; \mathrm{d}\Phi_{2+|X|}(r_1, r_2, r_X)\, (r_1^\mu - p_1^\mu)\, \hat{\delta}^{(4)}\left(p_1 + p_2 - r_1 - r_2 - r_X\right) \\
&\quad \times \mathcal{A}\left(p_1, p_2 \to r_1, r_2, r_X\right) \mathcal{A}^*\left(p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q} \to r_1, r_2, r_X\right)
\end{aligned}$$

We can extract $\hbar$ from q and from the amplitude, by extracting each e along with an $\frac{1}{\sqrt{\hbar}}$, thus quartic vertices yield a factor of $\frac{e^2}{\hbar}$ whereas cubic ones yield $\frac{e}{\sqrt{\hbar}}$. If we count the number V_3 of all cubic vertices, and V_4 the number of quartic vertices we have that the number of internal lines is $I = \frac{1}{2}(3V_3 + 4V_4 - E)$. This is because we have $3V_3 + 4V_4$ lines to start with, out of which E are chosen to be external. The remaining $(3V_3 + 4V_4 - E)$ ones are contracted among themselves to form I internal lines. In our case we have $E = 4 + M$ where $M = |X|$ is the number of messenger particles. Using the argument from [[loop counting]] we have that the number of loops of our graph L is given by:

$$\begin{aligned} L = I - V + N =& \frac{1}{2}(3V_3 + 4V_4 - 4 - M) - V_4 - V_3 + 1 \\ =& \frac{1}{2}(V_3 + 2V_4) - 1 - \frac{M}{2} \end{aligned}$$

where N is the number of [[connected topological space|connected]] components ($= 1$ in our case) . Thus we see that the amount of extracted $\hbar$s corresponds directly to the number of loops plus one! We can thus write the amplitude $\mathcal{A}$ as a sum over reduced L-loop amplitudes $\bar{\mathcal{A}}^{(L)}$:

$$\mathcal{A}(p_1, p_2 \to r_1, r_2, X) = \sum_{L=0}^{\infty} \left(\frac{e^2}{\hbar}\right)^{(L+1+\frac{|X|}{2})} \bar{\mathcal{A}}^{(L)}(p_1, p_2 \to r_1, r_2, X)$$

Going back to the integrands we have:

$$\begin{aligned} \mathcal{I}'_{\mathrm{v}}(O)p^1_\mu &= \hbar^3 \mathring{\imath}\, \bar{q} \sum_{L=0}^{\infty} \left(\frac{e^2}{\hbar}\right)^{(L+1)} \bar{\mathcal{A}}^{(L)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \\ &= \mathring{\imath}\, \bar{q} \sum_{L=0}^{\infty} e^{2(L+1)} \hbar^{(2-L)} \bar{\mathcal{A}}^{(L)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \end{aligned}$$

as well as the real kernel:[20]

$$\begin{aligned} \overline{\mathcal{I}'_{\mathrm{r}}}(O)p^1_\mu = \hbar^2 \; & \mathrm{d}\Phi_{|X|}(r_X) \prod_{i=1,2} \tilde{\mathrm{d}}^4 w_i \, \hat{\delta}(2p_i \cdot w_i + w_i^2)\Theta(p_i^0 + w_i^0) \\ & \times w_1^\mu \, \hat{\delta}^{(4)}\left(w_1 + w_2 + r_X\right) \\ & \times \mathcal{A}\left(p_1, p_2 \to p_1 + w_1, p_2 + w_2, r_X\right) \mathcal{A}^*\left(p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q} \to p_1 + w_1, p_2 + w_2, r_X\right) \end{aligned}$$

[20] We changed the integration variable from r_i to $w_i = r_i - p_i$ thus the measure changes:

$$\mathrm{d}\Phi_{2+|X|}(r_1, r_2, X) = \mathrm{d}\Phi_{|X|}(r_X) \prod_{i=1,2} \tilde{\mathrm{d}}^4 w_i \, \hat{\delta}(2p_i \cdot w_i +$$

where we used the same reasoning as for the q_i variable change.

$$= \hbar^2 \; \mathrm{d}\Phi_{|X|}(r_X) \prod_{i=1,2} \hbar^3 \, \tilde{\mathrm{d}}^4 \bar{w}_i \, \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) \Theta(p_i^0 + \hbar \bar{w}_i^0)$$
$$\times \hbar \bar{w}_1^\mu \, \hbar^{-4} \hat{\delta}^{(4)}(\bar{w}_1 + \bar{w}_2 + \frac{1}{\hbar} r_X)$$
$$\times \sum_{L=0}^{\infty} \left(\frac{e^2}{\hbar}\right)^{2L+2+|X|)} \bar{\mathcal{A}}^{(L)}(p_1, p_2 \to p_1 + \hbar \bar{w}_1, p_2 + \hbar \bar{w}_2, r_X)$$
$$\times \bar{\mathcal{A}^*}^{(L)}(p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q} \to p_1 + \hbar \bar{w}_1, p_2 + \hbar \bar{w}_2, r_X)$$

$$= \; \mathrm{d}\Phi_{|X|}(r_X) \prod_{i=1,2} \tilde{\mathrm{d}}^4 \bar{w}_i \, \hat{\delta}(2p_i \cdot \bar{w}_i + \hbar \bar{w}_i^2) \Theta(p_i^0 + \hbar \bar{w}_i^0)$$
$$\times \bar{w}_1^\mu \, \hat{\delta}^{(4)}(\bar{w}_1 + \bar{w}_2 + \frac{1}{\hbar} r_X)$$
$$\times \sum_{L=0}^{\infty} e^{2(2L+2+|X|)} \hbar^{3-2L-|X|} \bar{\mathcal{A}}^{(L)}(p_1, p_2 \to p_1 + \hbar \bar{w}_1, p_2 + \hbar \bar{w}_2, r_X)$$
$$\times \bar{\mathcal{A}^*}^{(L)}(p_1 + \hbar \bar{q}, p_2 - \hbar \bar{q} \to p_1 + \hbar \bar{w}_1, p_2 + \hbar \bar{w}_2, r_X)$$

Schematically we have

$$\overline{\mathcal{I}'_{\mathrm{v}}}(O) p^1_\mu = \sum_{L=0}^{\infty} \mathcal{O}(e^{2(L+1)})$$
$$\overline{\mathcal{I}'_{\mathrm{r}}}(O) O = \sum_{L=0}^{\infty} \mathcal{O}(e^{4(L+1)+2|X|})$$

The contributions from the virtual kernel are lower order for a given loop order. Both kernels contribute together provided that the following equation is verified.

$$L - 1 = 2L' + |X|$$

Where L is the loop count for the virtual kernel and L', $|X|$ are the real kernel loop count and messenger particle count respectively.

Now we see that the leading order contribution [21]to the impulse $\Delta p_1^{\mu,(0)}$ can only be from the virtual kernel at tree level. Thus we have the following equation.

[21] Here the expansion is in powers of the coupling constant, so even though we want $\hbar$s to cancel, the loop order will still affect the order of the contribution through the coupling constant e and the leading order corresponds to e^2

$$\Delta p_1^{\mu,(0)} = \left\langle\!\!\left\langle \int \mathrm{d}\overline{\Psi}(\bar{q}) \, \exp\left(-\mathring{\imath}\bar{q}^{\mu} b_{\mu}\right) \, \overline{\mathcal{I}'_{\mathrm{v}}}(O) p_1^{\mu}, L=0 \right\rangle\!\!\right\rangle$$

And the integrand is given by the tree level 4 point amplitude.

$$\mathcal{I}'_{\mathrm{v}}(O) p_1^{\mu}, L=0 = \mathring{\imath}\, \bar{q}^{\mu}\, e^2 \hbar^2 \bar{\mathcal{A}}^{(0)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q})$$

The amplitude is read off from the single tree level diagram and using [[feynman rules for SQED]].

$$\mathring{\imath}\bar{\mathcal{A}}^{(0)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) = \mathring{\imath}\frac{Q_1 Q_2}{\hbar^2 \bar{q}^2}(4 p_1 \cdot p_2 + \hbar^2 \bar{q}^2)$$

We can now safely take the $\hbar \to 0$ limit as the integrand contains no terms singular in $\hbar$ (the $\frac{1}{\hbar^2}$ is cancelled by the $\hbar^2$ pre-factor). Thus the classical limit is carried out and the final integral to compute is obtained. The integration measure is also simplified in the limit $\lim_{\hbar \to 0} \overline{\Psi}(\bar{q}) = \tilde{\mathrm{d}}^4\bar{q}\, \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(2p_1 \cdot \bar{q})$[22]

$$\Delta p_1^{\mu,(0)} = e^2 Q_1 Q_2 \int \tilde{\mathrm{d}}^4\bar{q}\, \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(2p_1 \cdot \bar{q}) e^{-\mathring{\imath}\bar{q}\cdot b} \bar{q}^{\mu} \frac{4 p_1 \cdot p_2}{\hbar^2 \bar{q}^2}$$

This can be analytically computed (see [[LO impulse SQED]]) to find a closed form for the leading order impulse.

$$\Delta p_1^{\mu,(0)} = -\frac{e^2 Q_1 Q_2}{2\pi} \frac{\gamma}{\sqrt{\gamma^2 - 1}} \frac{b^{\mu}}{b^2}$$

For [[NLO impulse SQED]], using the order equation above we have a 1-loop contribution from the virtual integrand as well as a tree level cut contribution from the real integrand. See [[NLO impulse SQED]] for details.

[22] Compare to

$\overline{\Psi}(\bar{q}) = \tilde{\mathrm{d}}^4\bar{q}\, \hat{\delta}(2p_1 \cdot \bar{q} + \hbar \bar{q}^2) \Theta(p_1^0 + q^0) \hat{\delta}(2p_1 \cdot \bar{q} + \hbar \bar{q}^2) \Theta(p_2^0 -$

The theta functions cancel as $q^0 \to 0$ and p_i becomes classical. And as discussed in [[classical limit of wavefuctions in KMOC]] the integration implicit in the brackets yields new delta functions based on the classical momenta, and with the $\bar{q}^2$s removed

9 Summary

In summary, this book has no content whatsoever.

References