Syntactic NFA Minimization via SAT solving

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GOAL

minimal nfas

Given an nfa M with states Q, nfa N with states S is a *union nfa* of M if

- each $s \in S$ accepts $\bigcup\{L(N, q) : q \in X\}$ with $X \subseteq Q$
- $L(N) = L(M)$

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An nfa N is *rpd* if N^R is a partial dfa.
The states of rpd nfas accept pairwise disjoint languages.
All trimmed nfas whose states accept pairwise disjoint languages are rpd.

nfa $N = (Q, \delta, I, F)$ accepting L is **atomic** if the following equivalent statements hold

- $\mathrm{rsc}(N^R)$ is a minimal dfa
- each $q \in Q$ accepts a language from $\mathrm{BLD}(L)$, the closure of $\mathrm{LD}(L)$ under all set-theoretic boolean operations
- each $q \in Q$ accepts a union of congruence classes of the *Nerode left congruence* $\sim_L$:

$$
\begin{aligned}
u \sim_L v &\iff \forall x \in \Sigma^*. u \in x^{-1}L = v \in x^{-1}L \\
&\iff (u^r)^{-1}L^r = (v^r)^{-1}L^r
\end{aligned}
$$

- N is a union nfa of the átomaton $\mathbf{dfa}(L^r)^R$

nfa $N = (Q, \delta, I, F)$ accepting L is **subatomic** if the following equivalent statements hold

- the transition monoid of $\mathrm{rsc}(N^R)$ is isomorphic to $\mathrm{syn}(L^r)$
- each $q \in Q$ accepts a language from BLRD(L), the closure of LD(L) under all boolean operations and right derivatives
- each $q \in Q$ accepts a union of congruence classes of the *syntactic congruence* $\equiv_L$:

$$u \equiv_L v \iff \forall x, y \in \Sigma^*. u \in x^{-1} L y^{-1} = v \in x^{-1} L y^{-1}$$

- N is a union nfa of the syntactic nfa **syn**(L)

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given **dfa**(L) and an rpd nfa $M = (Q, \delta, I, \{q_f\})$ accepting L,

does a union nfa N of M with k states exist?

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We are searching for a union nfa N with states $\mathbf{k}$:

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As a commutative diagram:

$$\begin{tikzcd}
\mathrm{LD}(L) \arrow[d, "G"'] \arrow[dr, "\supseteq"] & \\
\mathbf{k} \arrow[r, "H"'] & Q
\end{tikzcd}$$

Require states to accept their assigned language using the transition relations $R_a \subseteq \mathbf{k} \times \mathbf{k}$ $(a \in \Sigma)$:

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\mathrm{LD}(L) \arrow[d, "G"'] \arrow[dr, "\supseteq"] & \\
\mathbf{k} \arrow[r, "H"'] & Q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Q \arrow[r, "\delta_a"] & Q \\
\mathbf{k} \arrow[u, "H"] \arrow[r, "R_a"'] & \mathbf{k} \arrow[u, "H"']
\end{tikzcd}
$$

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$$N = (\mathbf{k}, (R_a)_{a \in \Sigma}, G[L], H^{-1}[q_f])$$

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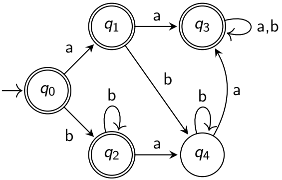
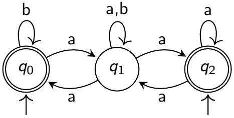
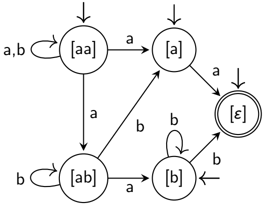
$$L = \{w \in \Sigma^* : |w|_b = 0 \vee |w|_a \neq 1\}$$



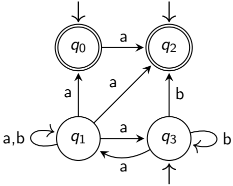
dfa(L)



N



dfa$(L^r)^R \cong$ **syn**(L)



$N_{\text{atm}} = N_{\text{syn}}$

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	$[\varepsilon]$	$[a]$	$[b]$	$[aa]$	$[ab]$
$\varepsilon^{-1}L$	■	■	■	■	
$a^{-1}L$	■	■		■	■
$b^{-1}L$	■		■	■	
$aa^{-1}L$	■	■	■	■	■
$ab^{-1}L$		■		■	■

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	$[\varepsilon]$	$[a]$	$[b]$	$[aa]$	$[ab]$
$\varepsilon^{-1}L$					
$a^{-1}L$					
$b^{-1}L$					
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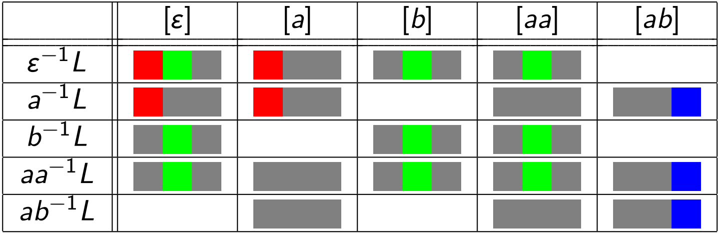
[ε]
[a]
[b]
[aa]
[ab]
ε⁻¹L
a⁻¹L
b⁻¹L
aa⁻¹L
ab⁻¹L

	$[\varepsilon]$	$[a]$	$[b]$	$[aa]$	$[ab]$
$\varepsilon^{-1}L$					
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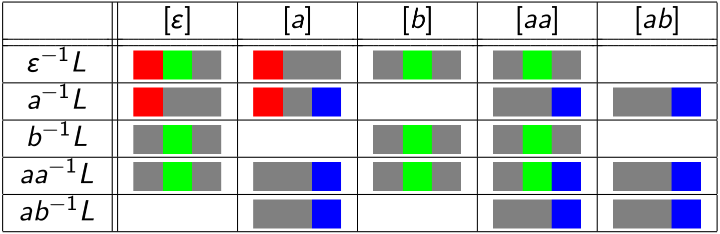

Two possible options for H:

r	$[\varepsilon], [a]$	$[\varepsilon], [a], [aa]$
g	$[\varepsilon], [b], [aa]$	$[\varepsilon], [b], [aa]$
b	$[a], [aa], [ab]$	$[a], [aa], [ab]$

Implementation

size results

k	total	$\mathbf{ns}(L) < \|\mathrm{LD}(L)\|$	$\mathbf{ns}(L) < \mathbf{natm}(L)$	$\mathbf{ns}(L) < \mathbf{nsyn}(L)$	unknown
1	**2**	**1** (50.0%)	**0** (0.0%)	**0** (0.0%)	0
2	**24**	**3** (12.5%)	**0** (0.0%)	**0** (0.0%)	0
3	**1028**	**123** (12.0%)	**0** (0.0%)	**0** (0.0%)	0
4	**56014**	**5911** (10.6%)	**86** (0.15%)	**86** (0.15%)	0
5	**3705306**	**335820** (9.1%)	**13376** (0.36%)	**11122** (0.30%)	0
6	1269000	98376 (7.8%)	5399 (0.43%)	4540 (0.36%)	0
7	1000000	67904 (6.8%)	4159 (0.42%)		93
8	1000000	60497 (6.0%)	4084 (0.41%)		362
9	1000000	55131 (5.5%)	3668 (0.37%)		372
10	1000000	51563 (5.2%)	3598 (0.36%)		318
11	1000000	48070 (4.8%)	3465 (0.34%)		241

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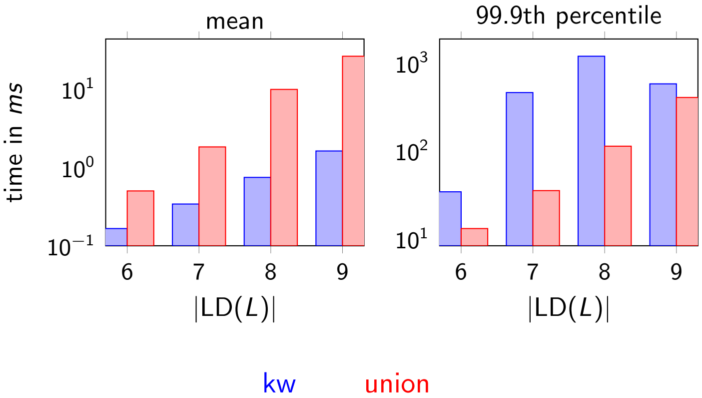
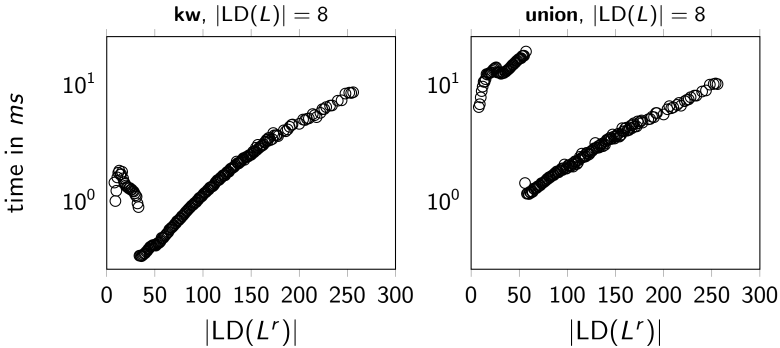
performance results



performance results



Thank you for your attention!